Hitchin Connections for Various Families of Kähler Structures

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Preface

The following dissertation presents the work that I have done during my four years as a PhD student at the Centre for Quantum Geometry of Moduli Spaces (QGM) at the Department of Mathematics, Aarhus University. The subject of my studies has been mathematical quantization and after having studied various aspect in the first part, I have later focused on geometric quantization and more specifically the Hitchin connection. This has resulted in the paper [AR16] together with my advisor Jørgen Ellegaard Andersen some of his earlier work, but also in unpublished work, that we will hopefully be able to finish in the nearest future [AR18].

I would like to say a big thanks to my advisor Jørgen Ellegaard Andersen, both for his many ideas, inspirational enthusiasm and also great patience and trust in my abilities.

Furthermore I would like to thank the everyone at QGM, and also a special thanks to Niels Leth Gammelgaard, who was my co-supervisor for the first part of PhD, and helped me a lot in my beginning years as PhD student.

Likewise I thank friends an fellow students, who have supported me through the years

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Contents

Abstract

In this thesis we study aspects of the mathematical formulation of quantization and more specifically geometric quantization. Our main objective is the construction of a *Hitchin connection* in settings, that generalise the constructions of Andersen in [And12], which again was a generalisation of the original work by Hitchin [Hit90] studying the case of the moduli space of flat connections on a surface.

We review the construction by Andersen and this Author in [AR16], where we succeeded in significantly weakening the so called *rigidity* condition on the family of complex structures, which was required for Andersens original construction to work. We also include calculations of the curvature in this so-called *weakly restricted* case.

Afterwards we continue with new work joint with Andersen, where we construct a Hitchin connection for a general family of Kähler structures under certain cohomological conditions. Under similar conditions, we can even state the uniqueness of such a connection, and proof that this condition is also necessary for the existence of such a Hitchin connection. This is at the moment work in progress, which we expect to publish ultimo 2018 [AR18].

Besides stating and proving these results, we introduce the context by going through some basics of complex geometry, quantization and review the original moduli space case studied by Hitchin.

Resumé

I denne afhandling, studerer vi aspekter indenfor den matematiske formulering af kvantisering og mere specifikt geometrisk kvantisering. Vores hovedmål er at konstruere en *Hitchin konnektion* i et setup, der generaliserer Andersens konstruktion fra [And12], som igen var en generalisering a Hitchins oprindelig studier i [Hit90], hvor han betragtede modulirummet af flade konnektioner på en flade.

Vi gennemgår konstruktionen af Andersen og denne forfatter i [AR16], hvor vi lykkedes med at svække den såkaldte *rigidity* betingelse på familien af komplekse strukturer, som var en vigtig betingelse for Andersens originale konstruktion. Vi inkluderer også udregninger af krumningen i dette såkaldte *weakly restricted* tilfælde.

Bagefter fortsætter vi med at gennemgå nyt arbejde, lavet i samarbejde med Andersen, hvor vi konstruerer en Hitchin konnektion for en general familie af Kähler strukturer under visse kohomologiske betingelser. Under lignende betingelser, kan vi endda vise at Hitchin konnektionen er entydig, og endvidere bevise at betingelsen er nødvendig for at sådan en Hitchin konnektion kan eksistere. Dette er i øjeblikket igangværende arbejde, som vi forventer at publicere ultimo 2018 [AR18].

Udover at vise disse resultater, vil vi introducere konteksten ved at gennem noget af den grundlæggende teori om kompleks geometri og kvantiseringer samt gennemgå det oprindelig modulirums tilfælde, som Hitchin studerede.

Introduction

The following work takes place in the subject of mathematical quantization, which relates to problems on the border between physics and mathematics. In the beginning of the 20th century physicists discovered that the laws of classical Newtonian mechanics fail to describe events happening at the level of particles. This gave rise to the theory of quantum mechanics, which was since developed and widely accepted during the 1920's. Ever since, mathematicians have tried to come up with a mathematically sound way of producing a quantum mechanical description for a physical system from its classical description, a so-called quantization scheme.

It has been shown, that there can not exist a full quantization in the sense that the physicists wanted, but quantum mechanics has, however, been hugely succesful in describing and predicting events studied in experiments. Thus it has been a great field of mathematical research to construct and study different approaches to quantization.

We will in this thesis primarily study the approach called geometric quantization, that takes its starting point from a classical theory encoded through its Hamiltonian description by the phase space given by a symplectic manifold (M, ω). To this geometric quantization aims to associate a corresponding Hilbert space of quantum states $\mathcal{H}$, and to classical observables given by functions on phase space it should assign a self adjoint operator on $\mathcal{H}$. All of this should be done in a way, such that the assignment satisfies certain conditions, which we will later describe in detail in chapter 2.

The first step is prequantization, which is a purely geometric construction, that almost satisfies all the criteria of a quantization, except it doesn't reproduce so-called canonical quantization of $\mathbb{R}^n$. It produces a state space as sections of a line bundle, but this space depends on twice as many variables as desired. To deal with this, we need to introduce a polarization on M and only consider polarized sections of the line bundle. This gives the desired quantization, except we have to weaken the so-called commutation relation to only hold asymptotically.

We will only work with polarizations coming from compatible Kähler structures. The problem is that different choices of Kähler structure give rise to different quantizations, and since the Kähler structure is complementary to the physical theory, we have no good reason to choose one over another. Thus, we need a way of relating different choices.

If we assume that the quantum spaces H_σ constitute the fibers of a vector bundle $H \to \mathcal{T}$, where $\mathcal{T}$ parametrizes Kähler structures, and we can find a connection in this bundle, then we can relate the quantum spaces arising from different complex structures by parallel transport along a curve connecting them. We do, however, need the connection to be flat in order for this to be independent of the curve, such that we get a canonical isomorphism of the fibers.

Now we leap forward to the study of Chern-Simons theory, where the moduli space $\mathcal{M}$ of flat SU(n)-connections on a genus $g \geq 2$ surface Σ arises as the classical solutions to the Euler-Lagrange equations of the Chern-Simons action functional. The space comes naturally equipped with the Seshadri-Atiyah-Bott-Goldman symplectic form ω [AB83, NS64, NS65, Gol84], and Teichmüller space $\mathcal{T}(\Sigma)$ of the surface parametrizes almost complex structures J_σ making $M_\sigma = (M, \omega, J_\sigma)$ Kähler for each $\sigma \in \mathcal{T}$. Furthermore, $\mathcal{M}$ admits a Hermitian line bundle $\mathcal{L}$ with a compatible connection of curvature given by the symplectic form. Now for each level $k \in \mathbb{N}$, we get the space of quantum states $H_J^{(k)} = H^0(M_\sigma, \mathcal{L}^k)$ as the holomorphic sections of the k'th tensor power of this line bundle with respect to the Kähler structure J_σ, and these form the fibers of the so-called Verlinde Bundle, which is a vector bundle over Teichmüller space. It was this quantization, that Hitchin studied in the paper [Hit90], and for which he showed that there exists a natural projectively flat connection, which we call the Hitchin connection.

In [And12], Andersen gave a purely geometrical construction of a Hitchin connection on any compact prequantizable symplectic manifold satisfying certain simple topological restrictions, but with one rather restrictive condition on the familiy of complex structures, namely that it satisfies the so-called *rigidity* condition. This means that the corresponding deformations of the metric is by the real part of a global holomorphic symmetric tensor, as we recall in details below. Importantly, Andersens abstract construction applied to the original case studied by Hitchin, and in this case reproduced the connection constructed by Hitchin.

It has, of course, been of great interest to try to get rid of the rigidity condition and construct a Hitchin connection in a more general setting. In the paper [AR16], Andersen and I succeeded in weakening this criterion considerably, only demanding that the family was *weakly restricted*, a notion we will introduce below. We go through the construction and proofs in detail in chapter 6.

We have since then, continued the work in trying to understand exactly when it is possible to define a Hitchin connection, and what the ambiguity in choosing one is, if it exists. Furthermore we have worked on computing its curvature, and to show if it is projectively flat or not. This is still ongoing work to appear in the article [AR18]. In this, we show that a Hitchin connection exists when certain cohomological restrictions are fulfilled, and that a Hitchin connection of that form only exists under these restrictions. Furthermore, we can give similar cohomological restrictions ensuring uniqueness and projective flatness of said connection.

Let us now briefly introduce the mathematical setting, so we can rigorously describe the results.

We let (M, ω) be a symplectic manifold and let $\mathcal{T}$ be a complex manifold parametrizing a holomorphic family $J \colon \mathcal{T} \to C^\infty(M, \mathrm{End}(TM))$ of complex structures, which are all Kähler with respect to ω. For any point $\sigma \in \mathcal{T}$ we will write M_σ, when we refer to the complex manifold (M, J_σ).

We will consider the variation of the family J along a real vector field V on $\mathcal{T}$, which we denote $V[J]$. We consider the splitting of $V = V' + V''$ into types on $\mathcal{T}$ and we consider the symmetric bi-vector field $G(V') = V'[J] \cdot \tilde{\omega}$, where $\tilde{\omega}$ is the bivector field, inverse to ω. We think of G as a one form on $\mathcal{T}$ with coefficients in bi-vector fields and as such, we write $G(V) = G(V')$. Observe that if g is the corresponding family of Kähler metrics parametrized by $\mathcal{T}$, then we have that

$$V[g] = G(V) + \overline{G(V)}.$$

The assumption that the family J is rigid, says that $G(V)$ defines a holomorphic section $G(V)_\sigma \in H^0(M_\sigma, S^2(T'M_\sigma))$ at all points $\sigma \in \mathcal{T}$, which as mentioned was the case for the setting studied in [Hit90], but this is still a very restrictive condition.

In [AR16], we weakened the rigidity criterion by adding the possibility of varying the bi-vector field $G(V)_\sigma$ by adding a term of the form $\bar{\partial}\beta(V)_\sigma \cdot \tilde{\omega}$ for an arbitrary vector field $\beta(V)_\sigma \in C^\infty(M_\sigma, T'M_\sigma)$.

Definition 1 (Weakly restricted)**.** *We call the family J* weakly restricted *if there exist a one form β on $\mathcal{T}$ with values in $C^\infty(M_\sigma, T'M_\sigma)$ at each point $\sigma \in \mathcal{T}$, such that for all vector fields V along $\mathcal{T}$ and all $\sigma \in \mathcal{T}$, there exist $G_\beta(V)_\sigma \in H^0(M, S^2(T'M_\sigma))$ such that*

$$G_\beta(V)_\sigma \cdot \omega = V'[J]_\sigma + \bar{\partial}\beta(V)_\sigma.$$

It is, of course, interesting to investigate when we can solve the weakly restricted criterion. We explain this in section 3.5, but In general, we can solve the equation (1) if the cohomology class of $V'[J]_\sigma$ is contained in the image of the map

$$\cdot\omega\colon H^0(M_J, S^2(T'M_J)) \to H^1(M_J, T'M_J)_\omega,$$

where $H^1(M_J, T'M_J)_\omega$ is defined to be be the symmetric part of this cohomology. In particular, it is clear that this is always possible, if the map is surjective. A particularly simple case of this is when

$$H^1(M_J, T'M_J)_\omega = 0.$$

From this it follows that our construction provide a partial connection on the space of all complex structures compatible with the symplectic form on ω, but If M is compact, we see that this partial connection is defined on a subspace of finite co-dimension of the tangent space to the space of all complex structures compatible with ω.

To state our theorems we recall the setup in geometric quantization. Let (M, ω) be a symplectic manifold and assume that (M, ω) admits a prequantum line bundle $(\mathcal{L}, h, \nabla)$. Let $\mathcal{T}$ be a complex manifold parametrizing a holomorphic family of complex structures J making (M, ω, J_σ) Kähler for each $\sigma \in \mathcal{T}$. Now for each $\sigma \in \mathcal{T}$ we consider the quantum space at level $k \in \mathbb{N}$, which is the subspace $H^{(k)}_\sigma$ of the prequantum space $\mathcal{H}^k = C^\infty(M, \mathcal{L}^k)$ consisting of holomorphic sections

$$H^{(k)}_\sigma = H^0(M_\sigma, \mathcal{L}^k) \subset \mathcal{H}^k.$$

We will in the following assume that these quantum spaces form a smooth subbundle $H^{(k)}$ of the trivial bundle

$$\hat{\mathcal{H}}^k = \mathcal{T} \times \mathcal{H}^k.$$

Now we let ∇^T denote the trivial connection on $\hat{\mathcal{H}}^k$ and consider a connection of the form

$$\boldsymbol{\nabla}_V = \nabla^T_V + u(V), \tag{1}$$

where $u \in \Omega^1(\mathcal{T}, \mathcal{D}(M, \mathcal{L}^k))$ is a one-form on $\mathcal{T}$ with values in the space of differential operators on sections of $\mathcal{L}^k$. Our goal is to construct a u, such that $\boldsymbol{\nabla}$ preserves the quantum spaces $H^{(k)}_\sigma$ inside each fibre of $\hat{\mathcal{H}}^k$.

Definition 2 (Hitchin connection)**.** *A Hitchin connection in the bundle $\hat{\mathcal{H}}^k$ is a connection of the form (1), that preserves the subspaces $H^{(k)}_\sigma$ inside each fibre of $\hat{\mathcal{H}}^k$.*

We can now state the main theorem proven in [AR16].

Theorem 3 (Hitchin connection for weakly restricted families)**.** *Let (M, ω) be a symplectic manifold with a prequantum line bundle $\mathcal{L}$. Assume that M has first Chern class of the form $c_1(M, \omega) = n\left[\frac{\omega}{2\pi}\right]$ for some integer $n \in \mathbb{Z}$ and such that $H^1(M, \mathbb{R}) = 0$. Furthermore, let $J \colon \mathcal{T} \to C^\infty(M, \mathrm{End}(TM))$ be a weakly restricted, holomorphic family of Kähler structures on M, parametrized by a complex manifold $\mathcal{T}$, and assume that the family admits a family of Ricci potentials F. Then there exists a Hitchin connection $\boldsymbol{\nabla}$ in the bundle $\hat{\mathcal{H}}^k$ over $\mathcal{T}$, given by the expression*

$$\boldsymbol{\nabla}_V = \nabla^T_V + u(V),$$

where

$$\begin{aligned} u(V) = \frac{1}{2(2k+n)}(&\Delta_{G_\beta(V)} + 2\nabla_{G_\beta(V)\cdot dF} - i(2k+n)\nabla_{\beta(V)} \\ &+ 4kV'[F] - 2ikdF \cdot \beta(V) - ik\delta(\beta(V)) + 2k(k+n)\varphi(V) + ik\psi(V)), \end{aligned}$$

and $\varphi(V), \psi(V) \in C^\infty(M)$ are smooth functions, satisfying

$$\bar\partial\varphi(V) = \omega \cdot \beta(V) \quad \text{and} \quad \bar\partial\psi(V) = \Omega(V),$$

where $\Omega(V) \in \Omega^1(M)$ is given by

$$\Omega(V) = -\delta(G_\beta(V)) \cdot \omega + \delta(\bar\partial\beta(V)) - 2dF \cdot G_\beta(V) \cdot \omega + 2\bar\partial\beta(V) \cdot dF + 4i\bar\partial V'[F].$$

For remarks on the details and special cases of this theorem see section 6.2, where the theorem is proved and the remarks following the theorem.

It still remains to investigate the curvature of this connection further, since a projectively flat connection gives a canonical identification of the projectivised quantum spaces, if $\mathcal{T}$

is simply connected, and thus proves that the quantization is independent of the arbitrary choice of a distinct Kähler polarization.

Projective flatness was proved by Hitchin in [Hit90] and Axelrod, Della Pietra and Witten in [ADPW91] in the original moduli space setting, and by a purely differential geometric argument, Andersen and Gammelgaard proved that the Hitchin connection constructed in [And12] is also projectively flat if all complex structures in the family have zero-dimensional symmetry group. This is, especially, valid in the case of the moduli spaces of flat connection on a closed oriented surface.

In the case of weakly restricted families, we already know, that we cannot prove projective flatness in general due to the No-Go Theorem of [GM00], see section 9.1 for the proof of this. We would, however, still like to calculate a direct expression for the curvature given by differential operators, and understand the conditions under which the connection is indeed projectively flat. We have attacked this problem using different approaches, and the best result so far is by a direct computation using differential operators in the same spirit as Andersen and Gammelgaard in [AG14]. The full result can be seen in section 8.4, but we include here the condition for projective flatness.

Corollary 4. *The Hitchin connection given in theorem 3 is projectively flat if and only if*

$$0 = \Gamma_3(V,W) - \Gamma_3(W,V)$$

$$\begin{aligned} 0 = &-i\nabla_{\beta(V)}G_\beta(W) - i\,\mathcal{S}(G_\beta(V)\cdot\nabla\beta(W)) \\ &+ i\nabla_{\beta(W)}G_\beta(V) + i\,\mathcal{S}(G_\beta(W)\cdot\nabla\beta(V)) \\ &+ V[G_\beta(W)] - W[G_\beta(V)] \end{aligned}$$

$$\begin{aligned} 0 = &\Delta_{G_\beta(V)}\delta G_\beta(W) - \Delta_{G_\beta(W)}\delta G_\beta(V) \\ &+ 2\nabla^2_{G_\beta(V)}(G_\beta(W)\cdot dF) - 2\nabla^2_{G_\beta(W)}(G_\beta(V)\cdot dF) \\ &+ 4\big[G_\beta(V)\cdot dF, G_\beta(W)\cdot dF\big] \\ &+ 2(\big[\delta G_\beta(V), G_\beta(W)\cdot dF\big] - \big[\delta G_\beta(W), G_\beta(V)\cdot dF\big]) \end{aligned}$$

$$\begin{aligned} 0 = &-i\nabla^2_{G_\beta(V)}\beta(W) + i\nabla^2_{G_\beta(W)}\beta(V) \\ &+ 4V[G_\beta(W)]\cdot dF - 4W[G_\beta(V)]\cdot dF \\ &+ 4G_\beta(W)\cdot dV[F] - 4G_\beta(V)\cdot dW[F] \\ &+ 2\delta V[G_\beta(W)] - 2\delta W[G_\beta(V)] \\ &- 2i\big[G_\beta(V)\cdot dF, \beta(W)\big] + 2i\big[G_\beta(W)\cdot dF, \beta(V)\big] \\ &- i\big[\delta G_\beta(V), \beta(W)\big] + i\big[\delta G_\beta(W), \beta(V)\big] \end{aligned}$$

$$\begin{aligned} 0 = &-4\big[\beta(V), \beta(W)\big] \\ &- 2iV[\beta(W)] + 2iW[\beta(V)] \end{aligned}$$

We have also added a uniqueness result for the Hitchin connection in the weakly restricted case, utilising some of the techniques, that we use in our new work on general families of Kähler structures.

Theorem 5 (Uniqueness of the Hitchin connection)**.** *Assume the setup of Theorem 3. Furthermore assume that the contraction map*

$$\omega\cdot : H^0(M, S^2(T'M_\sigma)) \to H^1(M, T'M_\sigma)$$

is injective and $H^0(M, T'M_\sigma) = 0$*. Then any Hitchin connection*

$$\tilde{\boldsymbol{\nabla}} = \nabla^T + \tilde{u}$$

in $\mathcal{H}^k$ *of order at most 2 with*

$$\tilde{u}(V) = \sum_{i=0}^{2} \nabla^{(i)}_{G_i(V)}$$

is unique and thus $\tilde{u} = u$*, where* u *is given by the expression in Theorem 6.11.*

Besides the example on the no-go theorem, we include two examples in chapter 11 that illustrate how our construction is applicable. We show that our constructions can be used on certain open subsets of the entire family of all complex structures on $\mathbb{R}^{2n}$ with the standard symplectic structure and the entire family of all complex structures on $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ compatible with the standard symplectic structure.

Furthermore, our construction also applies to certain open subsets of the entire family of all complex structures on co-adjoint orbits and on the moduli spaces of flat $\mathrm{SU}(n)$-connections on a surface of genus $g \geq 2$, possibly with fixed central holonomy around a point on the surface, when equipped with the Seshadri-Atiyah-Bott-Goldman symplectic structure.

We conclude the introduction by discussing our current research. For this, we consider the same general setup as above, except for the restrictions on the family of Kähler structures. Inspired by our use of symbols to describe the curvature, we started investigating some general properties of differential operators given in terms of their symbols, and some of these results are described in chapter 5.

Now instead of just looking, at the equation $G_\beta(V)_\sigma \cdot \omega = V'[J]_\sigma + \bar{\partial}\beta(V)_\sigma$, or even the more restrictive rigid case with $\beta = 0$, we realised that it was very interesting to look at a contraction with a linear combination of the symplectic form ω and the ricci form ρ. Furthermore we consider the contraction with symmetric n-tensors of arbitrary degree and not just bi-vector fields as in the weakly restricted or rigid case.

Definition 6. *For any* $n \geq 1$*, we let*

$$\Psi^{(n)} \colon H^0(M_\sigma, \mathcal{S}^n(T'M_\sigma)) \to H^1(M_\sigma, S^{n-1}(T'M_\sigma)),$$

be given by

$$\Psi^{(n)}_Z(G_n) = i(kn\iota_Z\omega + (n-1)\iota_Z\rho) \cdot G_n,$$

where $Z \in C^\infty(M, TM)$*. We will need the kernel and image of these maps and name them*

$$K^{(n)} := \ker(\Psi^{(n)}) \colon H^0(M_\sigma, \mathcal{S}^n(T'M_\sigma)) \to H^1(M_\sigma, S^{n-1}(T'M_\sigma))$$
$$I^{(n)} := \mathrm{Im}(\Psi^{(n)}) \colon H^0(M_\sigma, \mathcal{S}^n(T'M_\sigma)) \to H^1(M_\sigma, S^{n-1}(T'M_\sigma)).$$

We start by considering any differential operator D, which can be written on the form

$$D = \sum_{i=0}^{N} \nabla^{(i)}_{G_i}.$$

Now we know from lemma 6.2, which is also the starting point of our earlier constructions of the Hitchin connection, that the condition for being a Hitchin connection is in fact a condition on the derivative $\nabla^{0,1} Ds$ of a differential operator for any holomorphic section s. Thus we tried to find a way of computing this in general, as it is done for Δ_G in lemma 6.3, and it turned of that it was possible and very useful to do this calculation modulo differential operators of degree $n-2$ or less. Using this calculation we have proved the following theorems.

Theorem 7. *Let (M, ω) be a symplectic manifold with a prequamtum line bundle $\mathcal{L}$ and let $J\colon \mathcal{T} \to C^\infty(M, \mathrm{End}(TM))$ be a family of Kähler structures on M, parametrized by a complex manifold $\mathcal{T}$. If $H^1(M, \mathbb{R}) = 0$, then an order 2 Hitchin connection $\boldsymbol{\nabla} = \nabla^T + u$ with*

$$u(V) = \sum_{i=0}^{2} \nabla^{(i)}_{G_i(V)}$$

on $\hat{\mathcal{H}}^k$ exists if and only if

$$[V'[J]] \in I^{(2)}$$

for all $V \in C^\infty(\mathcal{T}, T\mathcal{T})$. If so, a Hitchin connection is given by

$$\begin{aligned} \boldsymbol{\nabla}_V &= \nabla^T_V + u(V) \\ u(V) &= \nabla^2_{G_2(V)} + \nabla_{\delta(G_2(V))} - \nabla_{\beta(V)} - \psi(V), \end{aligned}$$

where

$$V'[J] = \Psi^{(n)}(2iG_2(V)) + \bar{\partial}(2i\beta(V)),$$

and $\psi(V)$ is a function, such that

$$\bar{\partial}\psi(V) = ik\iota_{\delta(G_2(V))-\beta(V)}\omega.$$

Furthermore, if the Hitchin connection exists, it is projectively flat, if $K^{(n)}_\sigma = 0$ for $n = 1, 2, 3$ and all $\sigma \in \mathcal{T}$.

Theorem 8. *Let (M, ω) be a symplectic manifold with a prequamtum line bundle $\mathcal{L}$ and let $J\colon \mathcal{T} \to C^\infty(M, \mathrm{End}(TM))$ be a family of Kähler structures on M, parametrized by a complex manifold $\mathcal{T}$.*

If $K^{(n)}_\sigma = 0$ for all $n \geq 2$ and all $\sigma \in \mathcal{T}$, then for any Hitchin connection $\boldsymbol{\nabla} = \nabla^T + u$ with

$$u(V) = \sum_{i=0}^{N} \nabla^{(i)}_{G_i(V)}$$

on $\hat{\mathcal{H}}^k$, the order $N \leq 2$.

Furthermore if $H^0(M, T'M) = 0$ and a Hitchin connection of this form exists on $\hat{\mathcal{H}}^k$, it is unique up to addition of a scalar and projectively flat.

We are still working on expanding this theory to an even more general setting, where we can give an “if and only if” statement on the existence of a Hitchin connection of any order. Also, we are working on examples, where our new theorem applies.

Chapter 1

Complex Differential Geometry

We will, in the following chapter, review the most important background material from complex differential geometry, that will be used throughout the report. We assume knowledge of this part and include as a reminder to introduce the setting and to fix notation and conventions for the rest of the thesis.

Definition 1.1. Let M be a smooth manifold. Then an almost complex structure on M is a smooth family of endomorphisms

$$J_p \colon T_pM \to T_pM, \quad \text{such that} \quad J_p^2 = -\operatorname{Id}, \forall p \in M.$$

That is, J is a smooth section of the endomorphism bundle $\operatorname{End}(TM)$ over M, with $J^2 = -\operatorname{Id}$. A manifold M with an almost complex structure J will be called an almost complex manifold, and we denote this M_J.

A complex structure turns the tangent bundle into a complex vector space, since multiplication with i is given by J, and thus an almost complex manifold must have even real dimension $2m$.

We are interested in complex structures on a symplectic manifold (M, ω), and we say that a complex structure is compatible with ω, if the bilinear form g on the tangent bundle given by

$$g(X, Y) = \omega(X, JY) \tag{1.1}$$

for vector fields X and Y on M, defines a Riemannian metric, i.e. g is symmetric and positive definite. We also call the triple (J, g, ω) compatible and remark that two of the structures in a compatible triple uniquely determines the other two by (1.1). The symmetry of g is equivalent to J-invariance of ω, and by the relation between ω and g, it follows that g is J-invariant as well, which is also the criterion for g being *Hermitian*.

We will often need contraction of tensors as in the compatibility criterion above, so we will introduce some notation for this. For contraction of tensors placed next to each other, we will use a dot, so the above compatibility will be denoted by

$$g = \omega \cdot J,$$

but for some of the more complicated expressions, we can't always indicate contraction by placing the tensors next to each other. For this, we will use abstract index notation to denote the entries of the tensor, and following the Einstein summation convention, repeated indices are contracted. In the same spirit, we use substript indices for covariant entries and superscript indices for contravariant entries. The contraction correspond to a summation in local coordinates, but our indices only name the entries of the tensor and do not represent a choice of local coordinates. Writing the expression from before in abstract index notation, would look like

$$g_{ab} = \omega_{au} J^u_b,$$

and we remark, that we try to use the letters a, b, c, d for entries, that are not contracted, and letters u, v, w, x, y for contracted indices.

We will need the inverses of the metric and the symplectic form. These are the symmetric bivector field $\tilde{g}$ and antisymmetric bivector field $\tilde{\omega}$, such that

$$g \cdot \tilde{g} = \tilde{g} \cdot g = \mathrm{Id} \quad \text{and} \quad \omega \cdot \tilde{\omega} = \tilde{\omega} \cdot \omega = \mathrm{Id}\,.$$

These exist, since the metric and symplectic forms are nondegenerate. We will sometimes use g and $\tilde{g}$ to either lower or raise an index by contraction. It is also useful to record, that the relation between g and ω implies that

$$\tilde{g} = -J \cdot \tilde{\omega}.$$

1.1 Poisson Structure

Related to a symplectic structure, we also get a Poisson structure on the algebra of smooth functions $C^\infty(M)$, that is a Poisson bracket

$$\{\cdot, \cdot\} : \mathbb{C}^\infty \times C^\infty \to C^\infty(M),$$

which is a skewsymmetric, bilinear map satisfying the Leibniz rule

$$\{f, gh\} = \{f, g\}\, g + g\, \{f, h\}$$

and the Jacobi identity

$$\{f\,\{g, h\}\}\,\{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

The Poisson structure associated to the symplectic structure, is given directly using $\tilde{\omega}$, which is therefore also known as the Poisson tensor, by

$$\{f, g\} = df{\cdot}\tilde{\omega},$$

but is more often seen in the more indirect notations

$$\{f, g\} = -\omega(X_f, X_g),$$

where $X_f = \iota_\omega^{-1}(df) = df{\cdot}\tilde{\omega}$ is the Hamiltonian vector field associated to the function f. We recall also the identity between the commutator of Hamiltonian vector fiels and the Poisson bracket of the functions

$$[X_f, X_g] = X_{\{f,g\}}.$$

1.2 Splitting of the Tangent Bundle

An almost complex structure J on a manifold M, extended linearly to the complexified tangent bundle, induces a splitting

$$TM_{\mathbb{C}} = T'M_J \oplus T''M_J,$$

into the holomorphic and anti-holomorphic parts, i.e. the i and $-i$ eigenspaces og J. We have projections on the subbundles

$$\pi_J^{1,0} \colon TM_{\mathbb{C}} \to T'M_J, \quad \text{given by} \quad \pi_J^{1,0} = \frac{1}{2}(Id - iJ)$$
$$\pi_J^{0,1} \colon TM_{\mathbb{C}} \to T''M_J, \quad \text{given by} \quad \pi_J^{0,1} = \frac{1}{2}(Id + iJ).$$

When we have a compatible triple, the Riemannian metric also induces a Hermitian structure h on the the holomorphic tangent bundle $T'M_J$ given by

$$h(X,Y) = g(X,\bar{Y}),$$

where the bar denotes conjugation with respect to J.

Likewise we denote the splitting of a vector field X on M_J by $X = X'_J + X''_J$. Similarly the complexified cotangent bundle splits into

$$TM^*_{\mathbb{C}} = T'M^*_J \oplus T''M^*_J,$$

such that $T'M^*_J$ is the subbundle of 1-forms that vanish on $T''M_J$, and $T''M^*_J$ is the subbundle of forms that vanish on $T'M_J$.

These splittings induce splittings of all bundles of tensorpowers of $TM_{\mathbb{C}}$ and thus on the exterior algebra and the forms on M. We first define (p,q)-forms.

Definition 1.2 (Forms of type (p,q))**.** The differential forms of type (p,q) are the sections

$$\Omega_J^{p,q}(M) = C^\infty(M, \Lambda^{p,q}(TM^*_J)),$$

where we define

$$\bigwedge^{p,q} TM^*_J = \bigwedge^p T'M^*_J \otimes \bigwedge^q T''M^*_J.$$

With this definition the space of k-forms on M splits as the direct sum

$$\Omega^k(M) = \bigoplus_{p+q=k} \Omega_J^{p,q}(M),$$

and for pairs (p,q) with $p+q=k$, we get projections

$$\pi_J^{p,q} \colon \Omega^k(M) \to \Omega_J^{p,q}(M),$$

and composing the usual exterior derivative with these projection maps, we get the differential operators ∂ and $\bar{\partial}$ on forms of type (p,q)

$$\partial_J \colon \Omega_J^{p,q}(M) \to \Omega_J^{p+1,q}(M) \quad \text{given by} \quad \partial_J := \pi_J^{p+1,q} \circ d$$
$$\bar{\partial}_J \colon \Omega_J^{p,q}(M) \to \Omega_J^{p,q+1}(M) \quad \text{given by} \quad \partial_J := \pi_J^{p,q+1} \circ d.$$

The metric and the symplectic forms are both bilinear, and since they are J-invariant, it follows that the $(2,0)$ and $(0,2)$ parts vanish, since

$$\omega(X',Y') = \omega(JX',JY') = -\omega(X',Y')$$

and similarly for g and for two anti holomorphic vector fields. Thus we see that they are both type $(1,1)$. We will use this fact extensively in later computations, where we will see ω vanish, when it is contracted with two vector fields of type $(1,0)$.

Another construction, that we will need later, is the canonical line bundle of M_J, which is just defined as the top exterior power of the holomorphic tangent bundle

$$K_J = \bigwedge^m T'M_J.$$

The hermitian structure on $T'M_J$ induced by g again induces a hermitian structure in the canonical line bundle, which we will also just denote by h.

1.3 Complex Manifolds

A complex structure on a manifold is defined similarly to a smooth structure, except that the charts should be maps from subsets of $\mathbb{C}^n$ with biholomorphic transition functions. Given charts with complex coordinates $z^k = x^k + iy^k$, we get a canonical almost complex structure on the tangent bundle, defined locally on the coordinate vector fields by

$$J\left(\frac{\partial}{\partial x^k}\right) = \frac{\partial}{\partial y^k}, \quad \text{and} \quad J\left(\frac{\partial}{\partial y^k}\right) = -\frac{\partial}{\partial x^k}.$$

The biholomorphicity of the charts, ensures that the coordinate functions satisfies the Cauchy-Riemann equations, and using this we get that J is well-defined on overlaps of charts and thus well-defined globally.

An almost complex structure coming from a complex structure is called *integrable*, and these will be of particular importance to us. The following proposition states equivalent conditions for integrability.

Proposition 1.3. *The following statements about an almost complex structure J on M are equivalent.*

1. *J is integrable*

2. *The Nijenhuis tensor, given on vector fields X and Y on M by*

$$N_J(X,Y) = [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY],$$

vanishes on M.

3. *The exterior differential splits as*

$$d = \partial_J + \bar{\partial}_J$$

4. The square of the delbar operator vanishes, i.e.

$$\bar{\partial}_J^2 = 0.$$

As it is seen from the proposition, the differential operator $\bar{\partial}_J$ for an integrable almost complex structure is a cochain map, and for each non-negative integer p, we get a cochain complex

$$\Omega_J^{p,0}(M) \xrightarrow{\bar{\partial}_J} \Omega_J^{p,1}(M) \xrightarrow{\bar{\partial}_J} \Omega_J^{p,2}(M) \xrightarrow{\bar{\partial}_J} \cdots,$$

and the cohomology of this complex is called the dolbeault cohomology of M and is denoted

$$H_J^{p,q}(M, \mathbb{C}).$$

We will be especially interested in manifolds, that have all these different structures, and we will call these Kähler manifolds.

Definition 1.4. A Kähler manifold is a smooth manifold M equipped with a symplectic form ω, a Riemannian metric g and an integrable almost complex structure J, such that the triple (J, g, ω) is compatible.

On a Kähler manifold we will also call ω the Kähler form and g the Kähler metric.

1.4 Connections and Curvature

We will define connections on vector bundles in the form of covariant derivatives. The definition is equivalent for $\mathbb{R}$ respectively $\mathbb{C}$ vector bundles, and we will not worry about the distinction in the following. Afterwards we will define the curvature of a connection and then look at the special case of the Levi-Civita connection on a Riemannian manifold.

Definition 1.5 (Connection)**.** Let $\pi\colon E \to M$ be a vector bundle. A connection on E is a map

$$\nabla\colon \mathfrak{X}(M) \times C^\infty(M, E) \to C^\infty(M, E)$$

satisfying the following properties

(i) $\nabla_{fX} s = f \nabla_X s$ for all $f \in C^\infty(M), X \in \mathfrak{X}(M)$ and $s \in C^\infty(M, E)$,

(ii) $\nabla_{X_1+X_2} s = \nabla_{X_1} s + \nabla_{X_2} s$ for all $X_1, X_2 \in \mathfrak{X}(M)$ and $s \in C^\infty(M, E)$,

(iii) $\nabla_X(s_1 + s_2) = \nabla_X s_1 + \nabla_X s_2$ for all $X \in \mathfrak{X}(M)$ and $s_1, s_2 \in C^\infty(M, E)$,

(iv) $\nabla_X(fs) = (Xf)s + f\nabla_X s$ for all $f \in C^\infty(M), X \in \mathfrak{X}(M)$ and $s \in C^\infty(M, E)$.

It turns out, that the value of $(\nabla_X s)(p)$ does not depend on the vector field X but only on the value of X at p, i.e. on the vector X_p, so the definition is equivalent to having a map

$$\nabla\colon TM \times C^\infty(M, E) \to C^\infty(M, E)$$

with corresponding properties.

Definition 1.6 (Curvature)**.** The curvature of a connection ∇ on a vectorbundle $E \to M$ is the 2-form with values in the endomorphism bundle $\mathrm{End}(E)$ given by

$$F_\nabla(X,Y)s = \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s,$$

for vector fields X, Y on M and a section $s \in C^\infty(M,E)$. We call the connection *flat*, if the curvature is zero.

On a Riemannian manifold we have the unique Levi-Civita connection ∇, which is the unique torsion-free metric connection, compatible with the Riemannian metric. Here compatible with the metric means

$$\nabla g = 0$$

or equivalently

$$\nabla_X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

and torsion-free means, that the torsion

$$\nabla_X Y - \nabla_Y X - [X,Y] = 0.$$

We note that on a Kähler manifold we also have

$$\nabla J = \nabla\omega = 0.$$

These relations will be useful in later computations, and another important fact that follows is

$$\nabla_X(JY) = J\nabla_X(Y) \tag{1.2}$$

for any vector fields X, Y on M, and therefore ∇ preserves $T'M_J$ and $T''M_J$, and thus it induces a connection on $T'M_J$.

By the Kähler curvature, we will refer to the curvature of the Levi-Civita connection, that we will denote by R, which in abstract index notation will be R^d_{abc}. Sometimes it will be useful for us to define a purely covariant version, which is the curvature tensor, obtained by lowering an index using the metric

$$R_{abcd} = R^u_{abc} g^u_d.$$

Proposition 1.7. *The Riemann curvature tensor has the following symmetries*

(i) $R_{abcd} = -R_{bacd}$

(ii) $R_{abcd} = -R_{abdc}$

(iii) $R_{abcd} = R_{cdab}$

(iv) $R_{abcd} + R_{cabd} + R_{bcad} = 0.$

The equality (iv) is called the first Bianchi identity.

By (1.2) it follows immediately, that $R(X,Y)JZ = JR(X,Y)Z$, and by using (vi) and by combining with J-invariance of the metric and symmetry of the curvature tensor, we get that the form part of R is also J-invariant, that is $R(JX,JY) = R(X,Y)$. This follows by the little calulation

$$J_a^u J_b^v R_{uvcd} = R_{cduv} J_a^u J_b^v = R_{cdu}^w J_a^u g_{wv} J_b^v = R_{cda}^w J_w^u g_{uv} J_b^v = R_{cda}^w J g_{wb} = R_{cdab} = R_{abcd},$$

and by raising the last index again, this holds for the curvature as well as the curvature tensor. Just as for ω and g it follows, that the Kähler curvature is of type $(1,1)$, which we also use repeatedly to simplify expressions in calculation later. Since $R(X,Y)$ commutes with J, we get that the endomorphism part preserves type, such that $R(X,Y)$ takes values in $\operatorname{End}(T'M) \oplus \operatorname{End}(T''M)$.

Lastly we define the *ricci curvature tensor* as

$$r(X,Y) = \operatorname{tr}(Z \mapsto R(Z,X)Y) \quad \text{or in index notation} \quad r_{ab} = R_{wab}^w.$$

Again by using the symmetries etc from above, it follows that r is a symmetric, J-invariant bilinear form. We also define the *ricci form*, which is the skew-symmetric $(1,1)$-form given by

$$\rho(X,Y) = r(JX,Y).$$

Again skew-symmetry follows from J-invariance and symmetry of r, and the type decomposition as for ω and g.

1.4.1 Divergence

We will also encounter the notion of divergence $\delta(X)$ of a vector field X, which is originally defined only using the volume form ω^n and the Lie derivative by the formula

$$\mathcal{L}_X \omega^n = (\delta X)\omega^n.$$

We will, however, not see the divergence on this form, but it will arrise, when we calculate certain expressions involving the trace and the Levi-Civita connection, since on a Kähler manifold the divergence can be calculated by the expression

$$\delta X = \operatorname{tr} \nabla X = \nabla_u X^u.$$

Likewise we use the generalisation of this expression to tensors of higher order, which we define for vector fields $X_1, \dots, X_n$, as

$$\delta(X_1 \otimes \ldots \otimes X_n) = \delta(X_1) X_2 \otimes \ldots \otimes X_n + \sum_j X_2 \otimes \ldots \otimes \nabla_{X_1} X_j \otimes \ldots \otimes X_n,$$

and we remark, that this is really just taking the induced connection on the n'th tensor power of the tangent bundle followed by the trace of the first entry, and that it defines a map

$$\delta \colon C^\infty(M, TM^n) \to C^\infty(M, TM^{n-1}).$$

For our new result in section 5, we need a similar operator, but taking traces of several entries in a specific pattern. We define this operator $\tilde{\delta}$ inductively using the divergence.

Definition 1.8. For $n \geq 2$, we define the map

$$\tilde{\delta}\colon C^\infty(M, TM^n) \to C^\infty(M, TM^{n-1})$$

inductively by

$$\tilde{\delta}(G_n) = \begin{cases} \delta(G_n) & n = 2 \\ \delta(G_n) + X_1 \otimes \tilde{\delta}(G_{n-1}), & n \geq 3, \end{cases}$$

and then expanding by linearity to any n-tensor.

1.5 First Chern Class

We will not go through the theory of introducing Chern classes, but we state a couple of results that we will need on line bundles. First of all, the only non trivial Chern class on a complex line bundle $\mathcal{L}$ is the first Chern class $c_1(\mathcal{L})$, which is an element of the second cohomology with integral coefficients, $c_1(\mathcal{L}) \in H^2(M, \mathbb{Z})$. It turns out that the first Chern class is a complete invariant for isomorphism classes of line bundles.

We will be referring to the first Chern class of a symplectic manifold (M, ω) in the following, and to give sense to this, we will have to associate a line bundle to (M, ω). On a Kähler manifold (M, ω, J), we have the canonical line bundle K_J as described earlier, and we will define

$$c_1(M, \omega, J) = -c_1(M, K_J).$$

This is, however, apriori depended on the almost complex structure J, but since we have an injective map

$$J \mapsto \omega \cdot J$$

from the space of compatible almost complex structures to the space of Riemannian metrics on M, which is convex, it can be shown using a retraction, that the space of compatible almost complex structures is contractible. Now, since the first Chern class is integral, we get that any choice of a compatible almost complex structure will give the same Chern class, so we can define $c_1(M, \omega) = c_1(M, \omega, J)$ for any J.

We will also be referring to the real first Chern class of a complex line bundle $\tilde{c}_1(\mathcal{L})$, but this just given as viewing the element the first Chern class $c_1(\mathcal{L})$ as an element of $H^2(M, \mathbb{R})$ under the natural homomorphism $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$. Here we will recall that the real first Chern class can be calculated using any connection ∇ on $\mathcal{L}$ by the formula

$$\tilde{c}_1(\mathcal{L}) = \frac{i}{2\pi}[F_\nabla] = \frac{1}{2\pi}[\rho],$$

where F_∇ as usual denotes the curvature and of the connection and ρ the Ricci connection.

1.6 Hodge Theory and Ricci Potentials

Now we will review some powerful theorems from Hodge theory, that we can utilize in the case, when M is a compact Kähler manifold. So let us assume this for the rest of this section, and let us introduce the framework.

Firstly we define the Hodge on forms. This can be defined directly by its value on a basis, but we state it here implicitly as follows.

Definition 1.9 (Hodge star)**.** For $0 \leq k \leq n$, the Hodge star operator $*$ is the unique vector bundle isomorphism

$$*\colon \bigwedge^k TM^* \to \bigwedge^{n-k} TM^*,$$

which satisfies the relation

$$\alpha \wedge *\beta = g(\alpha, \beta)\frac{\omega}{m!} \tag{1.3}$$

for $\alpha, \beta \in \bigwedge^k TM^*$, where g is the pointwise inner product on forms, which is induced by the Kähler metric.

We also define an inner product on forms by integrating the expression (1.3) over the manifold, i.e.

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta,$$

and using the Hodge star and the differentials, we can define the adjoint d^* of d, i.e. the operator such that $\langle d\alpha, \beta \rangle = \langle \alpha, d^*\beta \rangle$, which on a compact oriented Riemannian manifold is given by

$$d^* = - * d *\colon \quad \Omega^k(M) \to \Omega^{k-1}(M),$$

and completely analogous for ∂ and $\bar{\partial}$. For each of these we have a Laplacian, and we denote these by

$$\Delta = dd^* + d^*d, \qquad \Box = \partial\partial^* + \partial^*\partial, \qquad \bar{\Box} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.$$

Form in the kernel of one of the Laplace operators are called harmonic, with respect to this Laplacian, and on Kähler manifolds these are equal for each of the above.

Theorem 1.10. *On a Kähler manifold, we have that the Laplacians are related by*

$$\Delta = 2\Box = 2\bar{\Box},$$

and preserves the type decomposition. Hence the harmonic k-forms decompose, and we have

$$\{\alpha \in \Omega^k(M) \mid \Delta\alpha = 0\} = \mathcal{H}^k_\Delta = \bigoplus_{p+q=k} \mathcal{H}^{p,q}_\Delta.$$

Furthermore, any harmonic form is d-closed, and so it defines an element in de Rham cohomology, and we even have the following theorem, for which a proof can be found in [Wel08].

Theorem 1.11 (Hodge decomposition)**.** *On a compact, Kähler manifold every de Rham cohomology class $[\alpha]$ has a unique harmonic representative α^H, and we have the following isomorphisms*

$$H^k(M, \mathbb{C}) \cong \mathcal{H}^k_\Delta \cong \bigoplus_{p+q=k} \mathcal{H}^{p,q}_\Delta \cong \bigoplus_{p+q=k} H^{p,q}(M, \mathbb{C}).$$

We will need the following global $\partial\bar{\partial}$-lemma for $(1,1)$ forms, which is also a result in Hodge theory.

Proposition 1.12. *On a compact Kähler manifold M, any exact (real) form $\alpha \in \Omega^{1,1}(M)$, there exist a (real) function $F \in C^\infty(M)$, such that*

$$\alpha = 2i\partial\bar{\partial}F.$$

We recall that the ricci form ρ has type $(1,1)$, and furthermore is real and closed. Thus it defines an element in cohomology and by Theorem 1.11 it has a harmonic representative ρ^H, and now the difference $\rho - \rho^H$ is a real, exact $(1,1)$-form, for which proposition 1.12 applies to give a real function F, such that

$$\rho = \rho^H + 2i\partial\bar{\partial}F.$$

We call F a Ricci potential, and we observe that it is only unique up to addition of a constant, but since M is compact, we can choose an F, such that the average

$$\int_M F\omega^n = 0,$$

which then determines F uniquely.

CHAPTER 2

Quantization

In this chapter we will introduce the mathematical theory of quantization and explain some of the problems that arise, when we try to define a matematically rigid theory of quantization (see e.g. [Wel08] [AE05]). One of the main points for us later, is the need to choose a polarization, when we do geometric quantization. In our case this will be a Kähler structure compatible with the symplectic form on the manifold, but this choice is not canonical and is auxiliary to the physical theory, and therefore we would suspect that the theory should be independent of this choice. The Hitchin connection aims to relate these different choices.

Before we get this, let us start from beginning. A quantization scheme is in the simplest form, a way to pass from classical mechanics to quantum mechanics. That is, to a system in classical mechanic, in the form of a phase space consistenting of a symplectic $2n$ dimensional manifold (M, ω), it assigns a corresponding Hilbert space $\mathcal{H}$ of quantum states, and to a classical observable given by a smooth function $f \in C^\infty(M)$ it assigns a self adjoint operater Q_f on $\mathcal{H}$, where this assignment satisfies a number of assumptions.

This proces should, ideally, be done in a way, such that we can retrieve the classical system, when we go to macroscopic scales, where classical Newtonian mechanics describe the system. This is the so-called classical or correspondance limit, which arises as Planck's constant $\hbar \to 0$, since the contribution by terms involving $\hbar$ becomes so small, that they can effectively be taken to be zero on the macroscopic level. It does, however, turn out that there exist quantum systems with no classical counterpart and also many different quantum systems, which reduce to the same classical system. This leaves the question of which quantization to choose and how to relate different quantizations.

2.1 Quantization Axioms

Let us go into more mathematical detail and describe the original concept of quantization, which we refer to as canonical quantization. The description goes back to Weyl, Von Neumann and Dirac, and takes place in the setting, where the classical phase space is given by $\mathbb{R}^{2n}$ with the standard symplectic form ω, coordinates $\mathbf{q} = (q_1, \dots, q_n)$ for position and $\mathbf{p} = (p_1, \dots, p_n)$ for momentum, and with the observables being real valued smooth functions $C^\infty(\mathbb{R}^{2n}, \mathbb{R})$ on the phase space. In this case the Hilbert space of quantum states is given by $\mathcal{H} = L^2(\mathbb{R}^n, d\mathbf{q})$

and for each $f \in C^\infty(\mathbb{R}^{2n}, \mathbb{R})$ the assignment of a quantum observable Q_f, which a self-adjoint operator on $\mathcal{H}$, should satisfy the following axioms.

(Q1) The assignment $f \mapsto Q_f$ is linear.

(Q2) $Q_1 = \mathbb{1}$ where 1 is the constant function 1, and $\mathbb{1}$ is the identity operator.

(Q3) The equation $Q_{\varphi \circ f} = \varphi(Q_f)$ should hold for any function $\varphi \colon \mathbb{R} \to \mathbb{R}$, as long as both sides of the equation are defined.

(Q4) The quantum observables corresponding to position and momentum must be given by

$$Q_{q_i}\psi = q_i\psi, \quad \text{and} \quad Q_{p_i}\psi = -i\hbar\frac{\partial \psi}{\partial q_i}, \quad \text{for } \psi \in \mathcal{H} = L^2(\mathbb{R}^n, d\mathbf{q}).$$

Furthermore, an important theorem of Stone and von Neumann, shows that the momentum and position operators, acting on the Hilbert space, satisfy the commutation relations

$$\left[Q_{q_j}, Q_{q_k}\right] = \left[Q_{p_j}, Q_{p_k}\right] = 0, \quad \text{and} \quad \left[Q_{q_j}, Q_{p_k}\right] = i\hbar\delta_{jk}\mathbb{1},$$

and that there are no subspaces $\mathcal{H}_0 \subseteq \mathcal{H}$ other than $\{0\}$ and $\mathcal{H}$ itself, that are stable under the action of all of all these operators.

If we consider $f \in \mathbb{R}[\mathbf{q}]$ to be a polynomial in the position variables, and likewise $g \in \mathbb{R}[\mathbf{p}]$ a polynomial in the momentum variables, then (Q1) and (Q3) together with the commutation relations imply

(Q5) the commutator of f and g satisfies

$$\left[Q_f, Q_g\right] = i\hbar Q_{\{f,g\}},$$

where $\{f, g\}$ is the poisson bracket

$$\{f, g\} = \sum_i \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$$

induced from the standard symplectic form, and again from physical considerations it turns out that another desirable criterium for a quantization is that (Q5) holds for all quantizable functions f and g, since this is related to the Heisenberg uncertainty principle.

This leads us to the point, where the inconsistencies in the theory arrise. We would like to define a quantization Q satisfying all the rules (Q1)-(Q5) on the largest possible subspace $\mathcal{O} \subseteq C^\infty(\mathbb{R}^n, \mathbb{R})$ of observables, and at least containing the coordinate functions. It can however be shown using only polynomials of degree 4 or less and simple computations, that any three of the axioms (Q1), (Q3), (Q4) and (Q5) taken together are inconsistent.

The two classical approaches to deal with these problems are to either restrict the set $\mathcal{O}$ of observables even further, or to weaken the criterion (Q5) in such a way, that this should only hold asymptotically. We will not consider this second option, since the first will limit the functions we can quantize too drastically, but will consider two different quantization schemes utilising the second approach.

2.2 Geometric Quantization

Let us move on from the case of $\mathbb{R}^{2n}$ to consider a general symplectic manifold (M, ω), and let us restate the axioms, as they should hold in this general setup.

Definition 2.1. A quantization of a symplectic manifold (M, ω) is an assignment of a separable Hilbert space $\mathcal{H}$ and a map $Q : f \mapsto Q_f$ from a subalgebra $\mathcal{O} \subseteq C^\infty(M)$, with the Poisson bracket associated to ω, into the self-adjoint linear operators on $\mathcal{H}$ satisfying

(Q1) The map $f \mapsto Q_f$ is linear.

(Q2) $Q_1 = \mathbb{1}$ where 1 is the constant function 1, and $\mathbb{1}$ is the identity operator on $\mathcal{H}$.

(Q3) For all $f, g \in \mathcal{O}$ the commutator satisfies

$$\left[Q_f, Q_g\right] = i\hbar Q_{\{f,g\}},$$

(Q4) The assignment should be functorial in the sense that for any symplectomorphism $\varphi \colon (M, \omega) \to (\tilde{M}, \tilde{\omega})$, there should exist a unitary operator U_φ from $\mathcal{H} \to \tilde{\mathcal{H}}$, such that

$$Q_{\tilde{f}\circ\varphi} = U_\varphi^* \tilde{Q}_{\tilde{f}} U_\varphi \quad \forall \tilde{f} \in \tilde{\mathcal{O}}.$$

(Q5) For $\mathbb{R}^{2n}$ with the standard symplectic form, we should recover canonical quantization.

We remark, also, that a special case of (Q4) is when there is a group of symplectomorphisms acting on (M, ω), then this shows that the quantization map is essentially G-invariant.

As mentioned, we can unfortunately not construct a quantization, that fullfils all of the above on the same time, so even though we introduce prequantization, as in the following, such (Q1) -(Q4) is satisfied and only (Q5) fails, we will skip this part and weaken (Q3). The way we do this, is such that this condition only holds asymptotically as $\hbar \to 0$. That is

($\tilde{Q}3$) For all $f, g \in \mathcal{O}$ the commutator satisfies

$$\left[Q_f, Q_g\right] = i\hbar Q_{\{f,g\}} + O(\hbar^2) \quad \text{as} \quad \hbar \to 0.$$

2.2.1 Prequantization

We will continue to introduce the machinery of geometric quantization, which originally was introduced by Kostant [Kos73] and Souriau [Sou70]. This is a two step proces, and the first step is the prequantization. Here we construct a Hilbert space of quantum observables as sections of tensor powers of a so called *prequantum line bundle* over the phase space (M, ω).

Definition 2.2 (Prequantum line bundle)**.** A prequantum line bundle over the symplectic manifold (M, ω) is a triple $(\mathcal{L}, h, \nabla)$ consisting of a line bundle $\mathcal{L}$ over M with a Hermitian metric h and a compatible connection ∇, such that the curvature of ∇ is related to the symplectic form by the relation

$$F_\nabla = -i\omega. \tag{2.1}$$

We call (M, ω) *prequantizable*, if there exist a prequantum line bundle over it.

By looking at the real first chern class $\tilde{c_1}(\mathcal{L})$, it is seen that the condition (2.1) is actually a restrictive condition, since $\tilde{c_1}(\mathcal{L}) = \frac{i}{2\pi}[F_\nabla] = [\frac{\omega}{2\pi}]$, which is not the case for all symplectic manifolds. It is however true, that (M, ω) is prequantizable, precisely when

$$\left[\frac{\omega}{2\pi}\right] \in \operatorname{Im}(H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{R})),$$

and we call this the *prequantum condition*. Now given a prequantum line bundle over M, we can define the *prequantum space*. Here we remark, that a prequantum structure on a line bundle induces a prequantum structure on any tensorpower of the bundle, and we will also use h and ∇ for the induced metric and connection.

Definition 2.3 (Prequantum space)**.** The prequantum space of level $k \in \mathbb{N}$ is the infinite dimensional vector space of sections of the k'th tensor power of $\mathcal{L}$

$$\mathcal{H}^k = C^\infty(M, \mathcal{L}^k).$$

More precisely, we actually consider the L^2-completion with respect to the Hermitian inner product on compactly supported sections, given by

$$\langle s_1, s_2 \rangle = \frac{1}{n!}\int_M h(s_1, s_2)\omega^n.$$

We will not distinguish between $\mathcal{H}^k$ and the completion in the following.

Next we define the prequantum map, sending a function $f \in C^\infty(M)$ to an operator on $\mathcal{H}^k$ by the expression

$$P_f^{(k)} = -\frac{i}{k}\nabla_{X_f} + f,$$

and it is checked directly, that the relations (Q1), (Q2) and (Q4) hold. We don't get (Q3) in the explicit form as above, since we have changed to a discretized version where we think of $\frac{1}{k}$ as a substitute for $\hbar$, since this the form, we will use in our final form of geometric quantization. Thus the commutation relations takes the form

$$[P_f^{(k)}, P_g^{(k)}] = \frac{i}{k}P_{\{f,g\}}^{(k)}.$$

Prequantization does, however, not reproduce canonical quantization, when used on $\mathbb{R}^{2n}$, which can be seen directly since the operators do not act on $L^2(\mathbb{R}^n)$ but on $L^2(\mathbb{R}^{2n})$, so we have in a sense produced wave functions depending on twice a many variable, as they should in accordance to the physical theory.

2.2.2 Polarization

To deal with the problem, that prequantization produces a Hilbert space of twice the desired dimension, we introduce a so-called polarization to restrict to half of the varibles. We will focus solely on complex polarization and specifically on Kähler polarization, since this part of the theory is rich on examples, and the original case where Hilbert constructed his connection also takes place in this setting.

Definition 2.4 (Polarization)**.** Let (M, ω) be a symplectic manifold, then a complex polarization $\mathcal{P}$ is a distribution of the complexified tangent bundle $TM_{\mathbb{C}}$, satisfying

(i) $\mathcal{P}$ is Lagrangian, i.e. $\mathcal{P} = \{X \in TM_{\mathbb{C}} \mid \omega(X, Y) = 0, \forall Y \in \mathcal{P}\}$.

(ii) $\mathcal{P}$ is involutive, i.e. $\left[X, Y\right] \in \mathcal{P}, \forall X, Y \in \mathcal{P}$.

(iii) $\dim(\mathcal{P}_x \cap \bar{\mathcal{P}}_x \cap T_x M)$ is constant (i.e. independent of $x \in M$).

We will immediately restrict out attention to Kähler polarization, which we define.

Definition 2.5 (Kähler Polarization)**.** A Kähler polarization $\mathcal{P}$ on a symplectic manifold (M, ω), is a complex polarization, if the associated Hermitian form on $\mathcal{P}$ defined by $h(X, Y)$ is positive definite.

This is called Kähler, since we can define a Kähler structure J on (M, ω) by letting $\mathcal{P} = T''M_J$ be the $-i$ eigenspace and likewise $\mathcal{P} = T'M_J$ be the i eigenspace. Using involutiveity of $\mathcal{P}$ it follows that the so defined almost complex structure is integrable. Moreover, we get that $g(X, Y) = \omega(X, IY)$ defines Riemannian metric, such that (M, ω, J) is Kähler. Conversely we get a polarization on any Kähler manifold by letting $\mathcal{P} = T''M_J$, and the polarizations we will study all arrise from a a Kähler structure on the manifold in this way.

Given a Kähler polarization, the line bundles $\mathcal{L}^k$ get induced complex structures given by $\bar{\partial} = \nabla^{(0,1)}$, since ω has type $(1, 1)$ with respect to J, and the prequantum condition thus ensures that $(F_\nabla)^{(0,2)} = 0$. This means we can choose the quantum space to be the holomorphic sections

$$H_J^{(k)} = H^0(M_\sigma, \mathcal{L}^k) = \left\{ s \in \mathcal{H}^{(k)} \mid \nabla_\sigma^{0,1} s = 0 \right\}.$$

This is a subspace of the prequantum space $\mathcal{H}^k$, and it is finite dimensional, if M is compact. This way we get the restriction of the space of quantum observables, that we need to make (Q5) hold.

2.2.3 Kähler Quantization

The last step in the geometric quantization is utilize the Kähler polarization and restrict the prequantum operators to act on the subspace $H_\sigma^{(k)}$. The operator P_k^f does, however, not preserve the quantum spaces $H_J^{(k)}$, unless

$$[X_f, T''M_J] \subseteq T''M_J,$$

and this gives a very limited amount of quantizable functions, so we will instead us the fact that $H_\sigma^{(k)}$ is a closed subspace of $\mathcal{H}^k$ (see [Woo92]). Thus, we have a projection map $\pi_\sigma^{(k)} : \mathcal{H}^k \to H_\sigma^{(k)}$, and we can define the quantum operator by taking the prequantum operator and projecting the result back on $H_\sigma^{(k)}$, that is

$$Q_{f,J}^{(k)} = \pi_J^{(k)} \circ P_f^{(k)}.$$

We might lose the subscript J, when the relation to the complex structure is clear or irrelevant. With this construction we lose the commutation relation ($Q3$), but we do get the asymptotic relation ($\tilde{Q}3$), which explicitly stated is

$$\left\| [Q^{(k)}_{f,J}, Q^{(k)}_{g,J}] - \frac{i}{k} Q^{(k)}_{\{f,g\},J} \right\| = O(k^{-2}) \quad \text{as} \quad k \to \infty,$$

with respect to the operator norm on the quantum subspace. The proof of this relies on the theorem of Tuynman [Tuy87], which states that the quantum operators can be represented as Toeplitz operators.

Theorem 2.6 (Tuynman)**.** *For any $f \in C^\infty(M)$, the quantum operator satisfy*

$$Q^{(k)}_{f,J} = T^{(k)}_{f+\frac{1}{2k}\Delta_J f, J},$$

where Δ_J is the Laplacian on (M_J, ω).

When combined with the theorem of Bordemann, Meinrenken and Schlichenmaier on the assymptotic behaviour of the commutator of Toeplitz operators [BMS94], which we will get back to after introducing Toeplitz operators in section 2.4, the commutation relation follows quite easily.

2.3 Deformation Quantization

Another approach to quantization is to completely avoid the construction of the Hilbert space of quantum states, and instead construct a non-commutative deformation of the algebra of classical observables. To be more precise, let

$$C^\infty_h(M) = C^\infty(M)[[h]]$$

be the space of formal power series in the variable h with coefficients in $C^\infty(M)$.

Definition 2.7 (Formal Deformation Quantization)**.** A formal deformation quantization of a symplectic manifold (M, ω) is an associative $\mathbb{C}[[h]]$ bilinear product $*$ on the algebra $C^\infty_h(M)$ given by

$$f * g = \sum_{l}^{\infty} C^{(l)}(f,g) h^j,$$

such that $C^{(l)} \colon C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ are bilinear operators satisfying

(DQ1) $C^{(0)}(f,g) = fg$

(DQ2) $C^{(1)}(f,g) - C^{(1)}(g,f) = i\{f,g\}$.

Furthermore the product is said to be differential, if the $C^{(l)}$ are bidifferential operators, and the product is called normalised if we have

(DQ3) $C^{(l)}(1,f) = C^{(l)}(f,1) = 0$ for all $j \geq 1$.

Lastly, the star product should be invariant under the symplectic action of a symmetry Γ in the sense that

$$\gamma^*(f * g) = \gamma^*(f) * \gamma^*(g)$$

for all $\gamma \in \Gamma$ and $f, g \in C^\infty$.

We will not dive further into the general theory of deformation quantization and the existence and equivalence of star products, but refer to the work of Kontsevich [Kon03], where he shows that any Poisson manifold admit a deformation quantization and also gives a classifications of star products on these.

We will, however, study the Berezin-Teoplitz deformation in more detail, since this helps to establish a connection between deformation quantization and geometric quantization.

2.4 Berezin-Toeplitz Deformation Quantization

We remain in the setup from section 2.2.3 and we will see how this gives rise to the Berezin-Toeplitz Deformation Quantization. That is we have a Kähler manifold (M, ω, J) with a prequantum line bundle $\mathcal{L} \to M$ and the projection $\pi_J^{(k)} : \mathcal{H}^k \to H_J^{(k)}$.

Definition 2.8. For $f \in C^\infty(M)$, the Toeplitz operator $T_{f,J}^{(k)} : \mathcal{H}^k \to H_J^{(k)}$ at level k is given by

$$T_{f,J}^{(k)} = \pi_J^{(k)} \circ M_f,$$

where $M_f \in \mathrm{End}(\mathcal{H}^k)$ is the multiplication operator associated to f. .

The Toeplitz operators are elements in $\mathrm{Hom}(\mathcal{H}^k, H^{(k)})$ and as such restrict to elements in $\mathrm{End}(H^{(k)})$, but even though $M_f : \mathcal{H}^k \to \mathcal{H}^k$ is trivially an algebra homomorphism, pre-composing with the projection changes this, and in general

$$T_{f,J}^{(k)} T_{g,J}^{(k)} = \pi_J^{(k)} \circ M_f \circ \pi_J^{(k)} \circ M_g \neq \pi_J^{(k)} \circ M_{fg} = T_{fg,J}^{(k)}.$$

It has, however been proven by Schlichenmaier in [Sch96] that there is an asymptotic expansion of such a product on a compact Kähler manifold.

Theorem 2.9 (Schlichenmaier)**.** *For any $f, g \in C^\infty(M)$ on a compact Kähler manifold (M, ω, J), the product of their Toeplitz operators have the asymptotic expansion*

$$T_{f,J}^{(k)} T_{g,J}^{(k)} \sim \sum_{l=0}^{\infty} T_{C_J^{(l)}(f,g),J}^{(k)} k^{-l},$$

where the $C_J^{(l)}(f, g), J \in C^\infty(M)$ are uniquely determined. More precisely

$$\left\| T_{f,J}^{(k)} T_{g,J}^{(k)} - \sum_{l=0}^{N} T_{C_J^{(l)}(f,g),J}^{(k)} k^{-l} \right\| = O\left(\frac{1}{N-1}\right),$$

for any $N \in \mathbb{N}$.

Furthermore, the operators $C_J^{(l)}(f, g), J$ actually define a star product, which gives the Berezin-Toeplitz deformation quantization. This was also proved by Schlichenmaier, and further it was proved to be with seperation of variables in joint work with Karabegov [KS01].

Theorem 2.10 (Karabegov and Schlichenmaier)**.** *The product* $*^{BT}$ *defined by*

$$f *^{BT} g = \sum_{l=0}^{N} (-1)^l C_J^{(l)}(f,g) h^l,$$

where $f, g \in C^\infty(M)$ *and* $C_J^{(l)}(f,g)$ *are defined to by the expansion in theorem 2.9, is a differential deformation quantization of the compact Kähler manifold* (M,ω)*, which is called the Berezin-Toeplitz deformation quantization.*

Besides constructing the Berezin-Toeplitz deformation quantization, the Toeplitz operators are, as mentioned, used to show that geometric quantization with Kähler polarization satisfies the asymptotic commutation relation. Before getting to this, let us state the following theorem by Bordemann, Meinrenken and Schlichenmaier [BMS94] proving that the Toeplitz opeator of function represent the function faithfully.

Theorem 2.11 (Bordemann, Meinrenken and Schlichenmaier)**.** *For any* $f \in C^\infty(M)$*, the Toeplitz operators satisfy*

$$\lim_{k\to\infty} \left\| T_{f,J}^{(k)} \right\| = \sup_{x\in M} |f(x)|,$$

and this limit is approached from below.

In the same paper, they show the following asymptotic result on the commutator of the Toeplitz operators.

Theorem 2.12. *The Toeplitz operators satisfy*

$$\left\| \left[T_{f,J}^{(k)}, T_{g,J}^{(k)}\right] - \frac{i}{k} T_{\{f,g\},J}^{(k)} \right\| = O\left(k^{-2}\right) \quad as \quad k \to \infty,$$

for any $f, g \in C^\infty(M)$.

Now it follows from Tuynman's Theorem 2.6 and Theorem 2.11, as we stated earlier, that we have the following theorem.

Theorem 2.13. *The quantum operators on a compact kähler manifold* (M,ω,I) *satisfy*

$$\left\| \left[Q_{f,J}^{(k)}, Q_{g,J}^{(k)}\right] - \frac{i}{k} Q_{\{f,g\},J}^{(k)} \right\| = O(k^{-2}) \quad as \quad k \to \infty,$$

for any smooth functions $f, g \in C^\infty(M)$.

2.5 Hitchin Connections

We have now gone through the basics of quantization, and it was evident, that we needed not only the initial physical information given in terms of a symplectic manifold (M,ω), but we also had to choose a compatible Kähler structure to construct the spaces $H_J^{(k)}$ of quantum states, which appear both in the geometric quantization and the formal Berezin-Toeplitz quantization. Unfortunately, there is no physical motivation for choosing one such Kähler structure over another, so we need a way of relating the different choices of complex structures.

Moreover the symmetry group will in general act by symplectomorphisms but not by automorphisms of the Kähler Structure, so it will act on the space of complex structures, and as such it will permute the quantum spaces associated to different Kähler structures.

If we assume that the spaces $H_J^{(k)}$ form a vector bundle over the space of Kähler structures, and we can find connection in this bundle, then we can relate different complex structures by parallel transport along a curve connecting them. We do, however, need the connection to be flat in order for this to be independent of the curve, such that we get an isomorphism of the fibers, i.e. the different quantum state spaces. We would also like the quantization to respect symmetries, and thus the connection should be invariant under the action of the symmetry group.

This is exactly the idea behind the Hitchin connection, and in chapter 3 we will introduce families of Kähler structures and work with different conditions on these under which, we can construct such a connection. It was, as mentioned in the introduction, initially constructed by Hitchin on the moduli space of flat $\mathrm{SU}(n)$-connection on a Riemann surface Σ with complex structures parametrized by Teichmüller space. We will review this setting in chapter 4, before we continue with the general setup and our generalisations of Andersens construction of the Hitchin connection in this setup.

Chapter **3**

Families of Kähler Structures

Before we get to the construction of the Hithin connection in the next chapter, we explore the properties of families of Kähler structures on a symplectic manifold (M, ω). We start out with a smooth family of integrable almost complex structures, all compatible with ω, giving us a family of Kähler structures on M. For each complex structure we get a splitting of the tangent bundle as described in section 1.2.

After establishing the basic relations, we introduce further restrictions on the family, namely that the family is holomorphic and rigid or weakly restricted.

3.1 Smooth Families of Kähler Structures

We let $\mathcal{T}$ be a smooth manifold and (M, ω) a symplectic manifold. Then we say that $\mathcal{T}$ smoothly parametrizes a family of almost complex structures on (M, ω), if there exist a smooth map

$$J \colon \mathcal{T} \to C^\infty(M, \mathrm{End}(TM)) \quad \text{mapping} \quad \sigma \mapsto J_\sigma$$

such that J_σ is an almost complex structure for each $\sigma \in \mathcal{T}$. We say that J is smooth, when it defines a smooth section of the pullback bundle $\pi_M^*(\mathrm{End}(TM))$, where $\pi_M \colon \mathcal{T} \times M \to M$ is the projection on M.

We will look at the case where all J_σ are integrable and compatible with the symplectic structure, such that (M, ω, J_σ) is kähler for each $\sigma \in \mathcal{T}$.

As described earlier, each complex structure gives a splitting of the complexified tangent bundle, so now we have a splitting

$$TM_{\mathbb{C}} = T'M_\sigma \oplus T''M_\sigma,$$

for each $\sigma \in \mathcal{T}$, into the holomorphic and anti-holomorphic parts, i.e. the i and $-i$ eigenspaces of J_σ.

Now we want to investigate the variation of the family J. More precisely we will differentiate along a vectorfield V on $\mathcal{T}$. This derivative is again a map into the endomorphism bundle which we denote by

$$V[J] \colon \mathcal{T} \to C^\infty(M, \mathrm{End}(TM)).$$

Differentiating the equality $J^2 = -1$ and using the Leibniz rule, we get

$$JV[J] = -V[J]J,$$

which shows that $V[J]_\sigma$ interchanges types on $TM_\mathbb{C}$, sending i eigenvectors to $-i$ eigenvectors and the other way around. Thus $V[J]$ decomposes as

$$V[J] = V[J]' + V[J]''$$

where the two components

$$\begin{aligned} V[J]'_\sigma &\in C^\infty(M, T''M_\sigma^* \otimes T'M_\sigma) \quad \text{and} \\ V[J]''_\sigma &\in C^\infty(M, T'M_\sigma^* \otimes T''M_\sigma) \end{aligned}$$

are each others conjugates. Now since contraction in the first entry of ω defines an isomorphism $i_\omega \colon TM_\mathbb{C} \to TM_\mathbb{C}^*$, we can get any element in

$$C^\infty(M, \mathrm{End}(TM_\mathbb{C})) = C^\infty(M, TM_\mathbb{C} \otimes TM_\mathbb{C}^*)$$

by contraction with a bivector field. We let $\tilde{G}(V) \colon \mathcal{T} \to C^\infty(M, TM_\mathbb{C} \otimes TM_\mathbb{C})$ be such that the equality

$$V[J] = \tilde{G}(V) \cdot \omega$$

holds at each $\sigma \in \mathcal{T}$ and for each vector field V on $\mathcal{T}$. We get another expression for $\tilde{G}(V)$ by differentiating the equality $\tilde{g} = -J \cdot \tilde{\omega}$ along V, namely

$$V[\tilde{g}] = -V[J] \cdot \tilde{\omega} = -\tilde{G}(V). \tag{3.1}$$

This is again using the Leibniz rule, and that $\tilde{\omega}$ does not depend on σ, so the derivative along any vector field V vanishes. Since $\tilde{g}$ is symmetric, (3.1) implies that $\tilde{G}(V)$ is also symmetric.

Looking at the types of $V[J]$ and ω, we see that $\tilde{G}(V)$ has no $(1,1)$-part, and so we get a decomposition of $\tilde{G}(V)$ into

$$\tilde{G}(V) = G(V) + \bar{G}(V),$$

where

$$G(V)_\sigma \in C^\infty(M, S^2(T'M_\sigma)) \quad \text{and} \quad \bar{G}(V)_\sigma \in C^\infty(M, S^2(T''M_\sigma)).$$

Observe also that $\bar{G}(V)$ is actually the conjugate of $G(V)$, since $\tilde{G}(V)$ is real and thus its own conjugate. We can also express the variation of the kähler metric in terms of $\tilde{G}(V)$ by differetiating the compatibility condition $g = \omega \cdot J$, getting

$$\begin{aligned} V[g] &= \omega \cdot V[J] \\ &= \omega \cdot \tilde{G}(V) \cdot \omega \\ &= g \cdot J \cdot \tilde{G}(V) \cdot g \cdot J \\ &= -g \cdot J \cdot (G(V) + \bar{G}(V)) \cdot J \cdot g \\ &= -i^2 g \cdot G(V) \cdot g - (-i)^2 g \cdot \bar{G}(V) \cdot g \\ &= g \cdot \tilde{G}(V) \cdot g. \end{aligned}$$

This also shows that $V[g]_\sigma \in C^\infty(M, S^2(T'M_\sigma^*) \oplus S^2(T''M_\sigma^*))$. One more relation about variations, that we will need, is the variation of the Levi-Civita connection ∇^g, here superscripted with g to denote that it is the connection related to the metric g. We will not calculate this here, but just state the result, which is given implicitly in ([Bes87] Theorem 1.174) by the formula

$$2g(V[\nabla^g]_X Y, Z) = \nabla_X(V[g](Y,Z)) + \nabla_Y(V[g])(X,Z) - \nabla_Z(V[g])(X,Y)$$

for vector fields X, Y, Z on M. To give an explicit expression, we use the above result and write it in the index notation as

$$2V[\nabla^g]^c_{ab} = \nabla_a \tilde{G}(V)^{cu} g_{ub} + g_{au}\nabla_b \tilde{G}(V)^{uc} - g_{au}\tilde{g}^{cw}\nabla_w \tilde{G}(V)^{uv} g_{vb}. \tag{3.2}$$

3.2 The Canonical Line Bundle of a Family

Our purpose in this chapter is to derive an expression for the variation of the Ricci form ρ. This will not seem apparent from the beginning, but we will construct a certain line bundle over the product manifold $\mathcal{T} \times M$ and consider the induced connection in this bundle. The Ricci form will appear in an expression for the curvature in some directions on $\mathcal{T} \times M$ and using the Bianchi identity, we will get a very useful relation.

First we consider the vector bundle $\hat{T}'M \to \mathcal{T} \times M$, where the fibers are just the holomorphic tangent spaces of M, that is $\hat{T}'M_{(\sigma,p)} = T'_pM_\sigma$. So the point $\sigma \in \mathcal{T}$ determines the splitting of the bundle $TM_{\mathbb{C}}$, and the point $p \in M$ chooses the fiber T'_pM_σ of the subbundle $T'M_\sigma$. We use the hat in the notation to denote, that we look at the bundle over the product $\mathcal{T} \times M$, and similarly we will use a hat on the differential $\hat{d}$ and connection $\hat{\nabla}$ on this bundle.

We notice that the Kähler metric induces a Hermitian structure $\hat{h}$ on $\hat{T}'M$, as we have already described for fixed $\sigma \in \mathcal{T}$. Now the Kähler and Hermitian metric just varies with σ as well. We construct a connection on $\hat{T}'M$ in two steps. We first notice that along vector fields on $M = \{\sigma\} \times M$, we can use the Levi-Civita Connection on the bundle $T'M_\sigma$, which gives us a partial connection on $\hat{T}'M$ compatible with the Hermitian structure.

Now we think of a section $Z \in C^\infty(\mathcal{T} \times M, \hat{T}'M)$ as a smooth family of sections of the holomorphic tangent bundles, and we let V be a vector field on $\mathcal{T} = \mathcal{T} \times \{p\}$. Since each of the holomorphic tangent spaces sits inside the larger complexified tangent bundle $T'M_\sigma \subset TM_{\mathbb{C}}$, we can think of Z as a smooth family of vector fields in this bundle. Since $TM_{\mathbb{C}}$ is unchanged along V, we can take the variation of Z along V in $TM_{\mathbb{C}}$, and then project the result back onto the holomorphic subbundle. This defines a connection along directions on $\mathcal{T}$, that is

$$\hat{\nabla}_V Z = \pi^{1,0} V[Z].$$

We still want this connection to be compatible with the Hermitian structure, so we check this by calculation. Let V be a vector field on $\mathcal{T}$ and X, Y sections of $\hat{T}'M$, then we have

$$\begin{aligned} V[\hat{h}(X,Y)] = V[g(X,\bar{Y})] &= V[g](X,\bar{Y}) + g(V[X],\bar{Y}) + g(X,\overline{V[Y]}) \\ &= g(V[X],\bar{Y}) + g(X,\overline{V[Y]}) \\ &= h(\hat{\nabla}_V X, Y) + h(X, \hat{\nabla}_V Y), \end{aligned}$$

since the $(1,1)$-part of $V[g]$ vanishes as shown earlier. This is exactly the condition, that $\hat{\nabla}$ is compatible with the Hermitian structure. In this way we have constructed a connection on all of $\hat{T}'M$ compatible with the Hermitian structure.

Next we consider the top exterior power, which as in the case of $T'M_\sigma$ gives a line bundle, and we will call this the *canonical line bundle of the family* of complex structures, and we will denote this

$$\hat{K} = \bigwedge^m \hat{T}'M^* \to \mathcal{T} \times M.$$

Just as for the normal canonical line bundle, we get an induced Hermitian structure and a compatible connection on $\hat{K}$.

Now we want to give an expression for the curvature of $\hat{\nabla}^K$, but before we are ready to do that, we will introduce some notation.

Definition 3.1. For vectorfields V, W on $\mathcal{T}$ we define $\Theta \in \Omega(\mathcal{T}, S^2(TM))$ by

$$\Theta(V, W) = S(\tilde{G}(V) \cdot \omega \cdot \tilde{G}(W)),$$

where S denotes symmetrization. Furthermore, we define the metric trace of Θ to be

$$\theta(V, W) = -\frac{1}{4} g(\Theta(V, W)) = -\frac{1}{4} g_{uv} \Theta(V, W)^{uv},$$

and we observe that $\theta \in \Omega^2(\mathcal{T}, C^\infty(M))$.

The following proposition states a couple of basic properties of Θ.

Proposition 3.2. *Let V, W be vector fields on $\mathcal{T}$. Then we have*

$$\begin{aligned} W[G(V)] &= iS(G(V) \cdot \omega \cdot \bar{G}(W)) - \pi^{2,0}(WV[\tilde{g}]) \qquad (3.3) \\ d_{\mathcal{T}} G &= -i\Theta, \end{aligned}$$

where we view G as a 1-form in $\Omega^1(\mathcal{T}, S^2(T'M))$.

Proof. The proof of (3.3) is the following calculation, where we use, that we can write $G(V) = \pi^{2,0}(\tilde{G}(V)) = (\pi^{1,0} \otimes \pi^{1,0})(\tilde{G}(V))$. Then we let V, W be vector fields on $\mathcal{T}$ and calculate the variation using the Leibniz rule

$$\begin{aligned} W[G(V)] &= W[(\pi^{1,0} \otimes \pi^{1,0})(\tilde{G}(V))] \\ &= (W[\pi^{1,0}] \otimes \pi^{1,0})(\tilde{G}(V)) + (\pi^{1,0} \otimes W[\pi^{1,0}])(\tilde{G}(V)) + (\pi^{1,0} \otimes \pi^{1,0})(W[\tilde{G}(V)]) \\ &= -\frac{i}{2}(W[J_\sigma] \otimes \pi^{1,0})(\tilde{G}(V)) - \frac{i}{2}(\pi^{1,0} \otimes W[J_\sigma])(\tilde{G}(V)) - \pi^{2,0}(WV[\tilde{g}]) \\ &= -\frac{i}{2}\tilde{G}(W) \cdot \omega \cdot G(V) + \frac{i}{2} G(V) \cdot \omega \cdot \tilde{G}(W) - \pi^{2,0}(WV[\tilde{g}]) \\ &= -\frac{i}{2}\bar{G}(W) \cdot \omega \cdot G(V) + \frac{i}{2} G(V) \cdot \omega \cdot \bar{G}(W) - \pi^{2,0}(WV[\tilde{g}]) \\ &= iS(G(V) \cdot \omega \cdot \bar{G}(W)) - \pi^{2,0}(WV[\tilde{g}]), \end{aligned}$$

where we have used symmetries and that ω is type $(1,1)$. Now for commuting vector fields V and W, we get by the invariant formula for the exterior derivative that

$$\begin{aligned}(d_{\mathcal{T}}G)(V,W) &= V[G(W)] - W[G(V)] \\ &= i(S(G(W)\cdot\omega\cdot\bar{G}(V)) - S(G(V)\cdot\omega\cdot\bar{G}(W))) \\ &= -i(S(\bar{G}(V)\cdot\omega\cdot G(W)) + S(G(V)\cdot\omega\cdot\bar{G}(W))) \\ &= -iS(\tilde{G}(V)\cdot\omega\cdot\tilde{G}(W)) \\ &= -i\Theta(V,W).\end{aligned}$$

Thus we have shown that Θ is exact. □

Now we are ready to state the proposition about the curvature of $\hat{\nabla}^K$. The proof of this can be found in [AGL12], where it is carried out in full detail.

Proposition 3.3. *Given vector fields X, Y on M and V, W on $\mathcal{T}$, the curvature of $\hat{\nabla}^K$ is given by*

$$\begin{aligned}F_{\hat{\nabla}^K}(X,Y) &= i\rho(X,Y), \\ F_{\hat{\nabla}^K}(V,X) &= \frac{i}{2}\delta\tilde{G}(V)\cdot\omega\cdot X, \\ F_{\hat{\nabla}^K}(V,W) &= i\theta(V,W).\end{aligned}$$

Applying the Bianchi identity and the results of proposition 3.3 gives us the desired result about the variation of the the Ricci form.

Proposition 3.4. *The variation of the Ricci form along a vector field V on $\mathcal{T}$ is given by*

$$V[\rho] = \frac{1}{2}d(\delta\tilde{G}(V)\cdot\omega).$$

Proof. Let X, Y be commuting vector fields on M and V a vector field on $\mathcal{T}$. Then the Bianchi identity for $\hat{\nabla}^K$ gives

$$\begin{aligned}0 &= V[F_{\hat{\nabla}^K}(X,Y)] + X[F_{\hat{\nabla}^K}(Y,V)] + Y[F_{\hat{\nabla}^K}(V,X)] \\ &= iV[\rho(X,Y)] - X\left[\frac{i}{2}\delta\tilde{G}(V)\cdot\omega\cdot Y\right] + Y\left[\frac{i}{2}\delta\tilde{G}(V)\cdot\omega\cdot X\right] \\ &= iV[\rho(X,Y)] - \frac{i}{2}d(\delta\tilde{G}(V)\cdot\omega)(X,Y),\end{aligned}$$

where the last equality again follows by the invariant formula for the exterior derivative, since X and Y were chosen to commute. Now isolating $V[\rho(X,Y)]$ gives the desired equality. □

3.3 Holomorphic Families of Kähler Structures

Now we introduce some extra structure. We assume that $\mathcal{T}$ is a complex manifold, and then we can require, that the the family J of holomorphic structures define a holomorphic map from $\mathcal{T}$ to the space of holomorphic structures. This is defined as follows.

Definition 3.5. Let $\mathcal{T}$ be a complex manifold and $J\colon \mathcal{T} \to C^\infty(M, \mathrm{End}(TM))$ a family of integrable almost complex structures on M. Then J is holomorphic if

$$V'[J] = V[J]' \quad \text{and} \quad V''[J] = V[J]''$$

for any vector field V on $\mathcal{T}$.

Now let us denote the integrable almost complex structure on $\mathcal{T}$ by I. Then we get an almost complex structure $\hat{J}$ on the product manifold $\mathcal{T} \times M$ defined by

$$\hat{J}(V \oplus X) = IX \oplus J_\sigma X, \quad \text{for} \quad V \oplus X \in T_{(\sigma,p)}(\mathcal{T} \times M).$$

It can be shown that holomorphicity of the family of complex structure as defined above is equivalent to the $\hat{J}$ being an integrable almost complex structure on $\mathcal{T} \times M$ (see [AGL12]). This is shown by using the criterion of the vanishing of the Nijunhuis tensor.

A useful consequnce of holomorphicity is that

$$\tilde{G}(V') = V'[J] \cdot \tilde{\omega} = V[J]' \cdot \tilde{\omega} = G(V),$$

and similarly $\tilde{G}(V'') = \bar{G}(V)$.

3.4 Rigid Families of Kähler Structures

Now we introduce a further restriction on the family of complex structures, which is that $G(V)$ is a holomorphic section of $S^2(T'M)$ for each vector field V on $\mathcal{T}$.

Definition 3.6 (Rigid)**.** A family of Kähler structures is called rigid if

$$\nabla^{0,1}_\sigma G(V)_\sigma = 0 \tag{3.4}$$

for all vector fields V on $\mathcal{T}$ and for all points $\sigma \in \mathcal{T}$.

This is a crucial assumption in the construction of the Hitchin connection in [And12]. It is also used to show the following result, which is used in the calculation of the curvature of the Hitchin connection in [AG14].

Proposition 3.7. *Given a rigid family of complex structures, the associated bivector fields satisfy the following symmetry property*

$$S(G(V) \cdot \nabla G(W)) = S(G(W) \cdot \nabla G(V))$$

for any vector fields V and W on $\mathcal{T}$.

Proof. We let V and W be commuting vector fields on $\mathcal{T}$, and then we differentiate the identity (3.4) along W using the Leibniz rule. We do the calculation using index notation

$$\begin{aligned} 0 &= W[\nabla_{a''} G(V)^{bc}] = W[(\pi^{0,1})^u_a \nabla_u G(V)^{bc}] \\ &= W[\pi^{0,1}]^u_a \nabla_u G(V)^{bc} + \nabla_{a''} W[G(V)]^{bc} + W[\nabla]^b_{a''u} G(V)^{uc} + W[\nabla]^c_{a''u} G(V)^{bu}. \end{aligned}$$

We calculate expressions for each of these terms

$$
\begin{aligned}
2W[\pi^{0,1}]^u_a \nabla_u G(V)^{bc} &= iW[J]^u_a \nabla_u G(V)^{bc} = i\tilde{G}(W)^{uv}\omega_{va}\nabla_u G(V)^{bc} \\
&= i\tilde{G}(W)^{uv} J^w_v g_{wa} \nabla_u G(V)^{bc} = -G(W)^{uv} g_{va} \nabla_u G(V)^{bc} \\
&= -g_{av} G(W)^{vu} \nabla_u G(V)^{bc},
\end{aligned}
$$

where the second to last equality follows, since $i\bar{G}(W)^{uv} J^w_v g_{wa} \nabla_u G(V)^{bc} = 0$ by rigidity. For the next term we use the expression (3.3) and get

$$
\begin{aligned}
&2\nabla_{a''} W[G(V)]^{bc} \\
=&\nabla_a(-i\bar{G}(W)^{bu}\omega_{uv}G(V)^{vc} + iG(V)^{bu}\omega_{uv}\bar{G}(W)^{vc} - 2(\pi^{2,0})^{bc}_{uv}(WV[\tilde{g}^{uv}])) \\
=&\nabla_a(-i\bar{G}(W)^{bu}J^w_u g_{wv}G(V)^{vc} + iG(V)^{bu}J^w_u g_{wv}\bar{G}(W)^{vc} - 2(\pi^{2,0})^{bc}_{uv}(WV[\tilde{g}^{uv}])) \\
=&\nabla_a(-\bar{G}(W)^{bu}g_{uv}G(V)^{vc} - G(V)^{bu}g_{uv}\bar{G}(W)^{vc} - 2(\pi^{2,0})^{bc}_{uv}(WV[\tilde{g}^{uv}])) \\
=&-\nabla_a(\bar{G}(W)^{bu})g_{uv}G(V)^{vc} - G(V)^{bu}g_{uv}\nabla_a(\bar{G}(W)^{vc}) - 2\nabla_a((\pi^{2,0})^{bc}_{uv}(WV[\tilde{g}^{uv}])).
\end{aligned}
$$

For the last two terms we use the expression for the variation of the Levi-Civita connection (3.2), that we stated earlier and by using symmetries and type decompositions, we get

$$
\begin{aligned}
&2W[\nabla]^b_{a''u}G(V)^{uc} \\
=&\nabla_{a''}(\tilde{G}(W)^{bv})g_{vu}G(V)^{uc} + g_{a''v}\nabla_u(\tilde{G}(W)^{vb})G(V)^{uc} - g_{a''v}\tilde{g}^{bw}\nabla_w(\tilde{G}(W)^{vx})g_{xu}G(V)^{uc} \\
=&\nabla_{a''}(\bar{G}(W)^{bv})g_{vu}G(V)^{uc} + g_{a''v}\nabla_u(G(W)^{vb})G(V)^{uc} - g_{a''v'}\tilde{g}^{bw}\nabla_w(\tilde{G}(W)^{v'x''})g_{x''u'}G(V)^{u'c'} \\
=&\nabla_a(\bar{G}(W)^{bv})g_{vu}G(V)^{uc} + g_{av}\nabla_u(G(W)^{vb})G(V)^{uc},
\end{aligned}
$$

where the last equality follows, since there is no $(1,1)$ part of $\tilde{G}(W)$. The last term follows in the same way and gives

$$
2W[\nabla]^c_{a''u}G(V)^{bu} = \nabla_a(\bar{G}(W)^{cv})g_{vu}G(V)^{bu} + g_{av}\nabla_u(G(W)^{vc})G(V)^{bu}.
$$

Now we are ready to combine all of these expressions, and we get

$$
\begin{aligned}
0 =& -g_{av}G(W)^{vu}\nabla_u G(V)^{bc} \\
&\cancel{-\nabla_a(\bar{G}(W)^{bu})g_{uv}G(V)^{vc}}\cancel{-G(V)^{bu}g_{uv}\nabla_a(\bar{G}(W)^{vc})} - 2\nabla_a((\pi^{2,0})^{bc}_{uv}(WV[\tilde{g}^{uv}])) \\
&\cancel{+\nabla_a(\bar{G}(W)^{bv})g_{vu}G(V)^{uc}} + g_{av}\nabla_u(G(W)^{vb})G(V)^{uc} \\
&\cancel{+\nabla_a(\bar{G}(W)^{cv})g_{vu}G(V)^{bu}} + g_{av}\nabla_u(G(W)^{vc})G(V)^{bu} \\
=& \, g_{av}\nabla_u(G(W)^{vb})G(V)^{uc} + g_{av}\nabla_u(G(W)^{vc})G(V)^{bu} \\
&- g_{av}G(W)^{vu}\nabla_u G(V)^{bc} - 2\nabla_a((\pi^{2,0})^{bc}_{uv}(WV[\tilde{g}^{uv}])).
\end{aligned}
$$

Now we raise the index a, and we add the term $G(V)^{au}\nabla_u G(W)^{bc}$ on both sides of the equation, which gives us

$$
\begin{aligned}
G(V)^{au}\nabla_u G(W)^{bc} =& \, G(V)^{au}\nabla_u G(W)^{bc} + \nabla_u(G(W)^{ab})G(V)^{uc} + \nabla_u(G(W)^{ac})G(V)^{bu} \\
&- G(W)^{au}\nabla_u G(V)^{bc} - 2\tilde{g}^{aw}\nabla_{w''}((\pi^{2,0})^{bc}_{uv}(WV[\tilde{g}^{uv}])).
\end{aligned}
$$

Writing out $6S(G(V)\cdot\nabla G(W))$ and using symmetry gives exactly two of each of the first three terms on the right hand side. Now reordering the terms gives us

$$\begin{aligned}&3S(G(V)\cdot\nabla G(W))^{abc}\\=&G(V)^{au}\nabla_u G(W)^{bc}+G(W)^{au}\nabla_u G(V)^{bc}+2\tilde{g}^{aw}\nabla_{w''}((\pi^{2,0})^{bc}_{uv}(WV[\tilde{g}^{uv}])),\end{aligned}$$

which is symmetric in V and W, since these were chosen to be commutative. □

3.5 Weakly Restricted Families of Kähler Structures

The assumption that the *family* J is rigid is, of course, a very restrictive condition, however, it is satisfied in the setting, in which Hitchin initially introduced his connection. This was the case of Teichmüller space parametrizing Kähler structures for the Seshadri-Atiyah-Bott-Goldman symplectic form [AB83, NS64, NS65, Gol84] on the moduli spaces of flat $\mathrm{SU}(n)$ connections on a genus g surface [Hit90].

The main object of my research has been to find ways to weaken this criterion, and in the paper [AR16] Andersen and I weakened the criterion by adding the possibility of varying the bi-vector field $G(V)$ by adding a term of the form $\bar{\partial}\beta(V)\cdot\tilde{\omega}$ for an arbitrary vector field $\beta(V)_\sigma\in C^\infty(M_\sigma,T'M_\sigma)$.

Definition 3.8 (Weakly restricted)**.** We call the family J *weakly restricted* if there exists a one form β on $\mathcal{T}$ with values in $C^\infty(M_\sigma,T'M_\sigma)$ at each point $\sigma\in\mathcal{T}$, such that for all vector fields V along $\mathcal{T}$ and all $\sigma\in\mathcal{T}$, there exists $G_\beta(V)_\sigma\in H^0(M,S^2(T'M_\sigma))$ such that

$$G_\beta(V)_\sigma\cdot\omega=V'[J]_\sigma+\bar{\partial}\beta(V)_\sigma. \tag{3.5}$$

It is clear that a rigid family is also weakly restricted, since $G_\beta(V)=G(V)$ with $\beta(V)=0$ obviously fulfils the weak rigidity condition.

Furthermore it is, of course, interesting to investigate when we can solve the weakly restricted criterion. We let $\mathcal{C}_\omega$ be the space of all complex structures on M compatible with the symplectic form ω and let $J\in\mathcal{C}_\omega(M)$. Then we have that

$$T_J\mathcal{C}_\omega=\ker(\bar{\partial}_J\colon\Omega^{0,1}(M,T'M_J)_\omega\to\Omega^{0,2}(M,T'M_J)),$$

where

$$\Omega^{0,1}(M,T'M_J)_\omega=\{V_J\in\Omega^{0,1}(M,T'M_J)\mid\omega(V_J\cdot,J\cdot)=\omega(\cdot,JV_J\cdot)\},$$

which is the same as stating that V_J is symmetric with respect to the Kähler metric g_J associated to ω and J. Thus, we see that given $V_J\in T_J\mathcal{C}_\omega$, we can solve the weakly restricted condition, e.g. find $\beta(V)$, whenever we have

$$V_J\in H^0(M_J,S^2(T'M_J))\cdot\omega+\mathrm{Im}(\bar{\partial}_J\colon C^\infty(M,T'M_J)_\omega\to\Omega^{0,1}(M,T'M_J)_\omega).$$

where

$$C^\infty(M,T'M_J)_\omega=\{X\in C^\infty(M,T'M_J)\mid\bar{\partial}(i_X\omega)=0\}.$$

If the map

$$\cdot\omega\colon H^0(M_J,S^2(T'M_J))\to H^1(M_J,T'M_J)_\omega$$

is surjective, this is always possible. Here $H^1(M_J, T'M_J)_\omega$ is defined in analogy with (3.5), namely to be the symmetric part of this cohomology.

A particular simple case, where we can always solve (3.5) is of course if

$$H^1(M_J, T'M_J)_\omega = 0.$$

In general, we can solve the equation (3.5) if the cohomology class of $V'[J]_\sigma$ is contained in the image of $\cdot\omega$ in (3.5). Thus our construction will only provide a partial connection on the space of all complex structures compatible with the symplectic form on ω. If M is compact, we see that this partial connection is defined on a subspace of finite co-dimension of the tangent space to the space of all complex structures compatible with ω.

3.6 Families of Ricci Potentials

Just as we have families of Kähler structures, we need the notion of a *family of Ricci potentials*, which we will define to be a smooth assignment of a Ricci potential to each Kähler structure. That is, assume that we have a symplectic manifold (M, ω) with a family of Kähler structures J parametrized by $\mathcal{T}$. Then we call a smooth map

$$F \colon \mathcal{T} \to C^\infty(M)$$

a family of Ricci potentials if it satisfies

$$\rho_\sigma = \rho_\sigma^H + 2i\partial_\sigma\bar{\partial}_\sigma F_\sigma. \tag{3.6}$$

Now assume that M is compact, such that we for each $\sigma \in \mathcal{T}$ get a Ricci potential F_σ as we saw in section 1.6, and by choosing the unique F_σ with zero average, we get a smooth function

$$\tilde{F} \in C^\infty(\mathcal{T} \times M),$$

which we can also view as a smooth function $F \colon \mathcal{T} \to C^\infty(M)$ satisfying (3.6). Now if we assume that the real first chern class is represented by

$$c_1(M, \omega) = n\left[\frac{\omega}{2\pi}\right],$$

for some integer $n \in \mathbb{Z}$, and since we know that it is also represented by

$$c_1(M, \omega) = \left[\frac{\rho}{2\pi}\right],$$

we get that $[\rho] = [n\omega]$, and since ω is harmonic, we get that

$$\rho_\sigma = n\omega + 2i\partial_\sigma\bar{\partial}_\sigma F_\sigma.$$

Chapter **4**

The Moduli Space of Flat SU(n)-connections

We include this chapter on the special example of the moduli space of flat SU(n)-connections, since this first of all is an interesting and well studied example where our constructions of the Hitchin connection in the following chapters apply, and secondly because of its historic significance in the development of the theory and relation to physics.

4.1 Chern-Simons Theory and the Moduli Space

Let us first briefly introduce classical Chern-Simons theory so as to motivate, how the moduli space arrises from this theory. For this introduction, we will follow the formulation of Freed (see [Fre95]).

We start by the description of Chern-Simons theory as what Freed calls a *local Lagrangian field theory*. Here space is modelled by a closed oriented surface Σ, and spacetime is modelled by a compact oriented three-manifold X with boundary $\partial X = \Sigma$. The fields in the theory are connections on principal SU(n)-bundles $P \to X$, and for a connection B on $P_{|\Sigma}$ we associate a Hermitian line, i.e. a 1 dimensional complex vector space with a Hermitian structure,

$$B \mapsto \mathcal{L}_B. \tag{4.1}$$

If the boundary $\Sigma = \emptyset$ we define this Hermitian line to be $\mathbb{C}$, and we call these Hermitian lines the *Chern-Simons lines*. For each connection A on the entire space X we associate a unitary element

$$A \mapsto e^{2\pi i S(A)} 1_{A|\Sigma} \in \mathcal{L}_{A|\Sigma},$$

where $1_{A|\Sigma}$ is a basis element of unit length, and S is the action functional, which for closed X is given by

$$S(A) = \int_X \langle A \wedge F_A \rangle - \frac{1}{6} \langle A \wedge [A \wedge A] \rangle,$$

where $\langle \cdot, \cdot \rangle \colon \mathfrak{su}(n) \times \mathfrak{su}(n) \to \mathbb{R}$, is a suitably normalized Ad-invariant symmetric bilinear form and $F_A = dA + \frac{1}{2}[A \wedge A]$ is the curvature of A. The normalization of $\langle \cdot , \cdot \rangle$ ensures that S descends to an action functional

$$S : \mathcal{A}/\mathcal{G} \to \mathbb{R}/\mathbb{Z},$$

where $\mathcal{A}$ is the space of connections on principal bundles $P \to X$, and $\mathcal{G}$ is the group of gauge transformations.

Since all principal $\mathrm{SU}(n)$-bundles over compact orientable three-manifolds are topologically trivial, we get the identification

$$\mathcal{A} \simeq \Omega^1(X, \mathfrak{su}(n)),$$

which shows that $\mathcal{A}$ is an infinite dimensional smooth manifold. It can further be shown, that the action functional S is smooth, and therefore it makes sense to investigate the Euler-Lagrange equation.

In doing so, it turns out that the space of classical solutions is given by the *moduli space of flat connections on* Σ

$$\mathcal{M} = \mathcal{F}/\mathcal{G},$$

where $\mathcal{F}$ is the space of flat connections on principal bundles over Σ. The moduli space is often identified, or even defined, by its equivalent description as a character variety, and we get the isomorphism through the map

$$\mathrm{Hol} : \mathcal{M} \xrightarrow{\sim} \mathrm{Hom}(\pi_1(\Sigma)), \mathrm{SU}(n))/\,\mathrm{SU}(n),$$

which for each flat connection associates its holonomy representation. Also we remark that we have implicit that $\pi_1(\Sigma) = \pi_1(\Sigma, p)$ for some fixed basepoint, and if Σ has a boundary we will assume $p \in \partial\Sigma$. We remember that for any connection in a principal bundle $P \to \Sigma$, a loop in Σ lifts to a unique horizontal loop in P, and since the start and endpoint are in the same fiber, they are associated via the action of an element in $\mathrm{SU}(n)$. This construction is the holonomy represention of the connection in that fiber. Now for a flat connection, the parallel transport depends only on the homotopy type of the loop, so the map restricts to give a homomorphism from $\pi_1(\Sigma)$ to $\mathrm{SU}(n)$. Also, choosing a different point in the fiber, would give a conjugate homomorphism, so the map is well-defined from $\mathcal{F} \to \mathrm{Hom}(\pi_1(\Sigma), \mathrm{SU}(n)/\,\mathrm{SU}(n)$ and since gauge equivalent connections have conjugate holonomy representations the map restricts to a map from M, which is an isomorphism.

Through this map, we can define a notion of *irreducibility* of a connection. A connection B is said to be irreducible if its associated unitary representation on $\mathbb{C}^N$ is irreducible. It turns out that $\mathcal{M}$ is singular, but throwing away the reducible representations and looking only at the space of *irreducible flat connections modulo gauge equivalence*

$$\mathcal{M}^* = \mathcal{F}^{\mathrm{Irr}}/\mathcal{G} = \mathrm{Hom}^{\mathrm{Irr}}(\pi_1(\Sigma), \mathrm{SU}(n))/\,\mathrm{SU}(n),$$

we get a smooth manifold, which is an open dense subset of $\mathcal{M}$.

As mentioned in the introduction, we might also consider only connection with fixed central holonomy around a point on the surface. In this case we consider a compact, smooth and orientable surface Σ of genus $g \geq 2$ with a single boundary component. Let $\gamma \in \pi_1(\Sigma)$ be the class of a curve homotopic to the boundary of Σ. Also for $d \in \mathbb{Z}/n\mathbb{Z}$, let $Z = e^{2\pi i d} I \in \mathrm{SU}(n)$ be a generator of the center of $\mathrm{SU}(n)$, and define

$$\mathrm{Hom}_d(\pi_1(\Sigma), \mathrm{SU}(n)) = \{\rho \in \mathrm{Hom}(\pi_1(\Sigma), \mathrm{SU}(n)) \mid \rho(\gamma) = Z\}\,.$$

Since Z is central, this subspace is preserved by the conjugation action, and since every element in $\mathrm{Hom}_d(\pi_1(\Sigma), \mathrm{SU}(n))$ can be shown to be irreducible, it follows that

$$M^d = \mathrm{Hom}_d(\pi_1(\Sigma), \mathrm{SU}(n)) / \mathrm{SU}(n)$$

is a compact, smooth manifold.

4.2 The Tangent Bundle and Symplectic Structure

For the moduli space to fit into the theory of quantization, we of course need a symplectic structure, and it turns out this can be defined in a natural way. Let us first introduce the tangent bundle. Let $\mathrm{Ad}_P \to \Sigma$ be the adjoint bundle associated to a principal bundle $P \to \Sigma$ and recall that any connection A on P induces a connection ∇^A on Ad_P. If A is flat, then so is ∇^A. Hence we get a cochain complex $\left(\Omega^\bullet_\Sigma(\mathrm{Ad}_P), \nabla^A\right)$, and we let $H^\bullet(\Sigma, \mathrm{Ad}^A_P)$ be the associated cohomology groups. At an irreducible point $[A]$ we have

$$T_{[A]}\mathcal{M}^* \simeq H^1(\Sigma, \mathrm{Ad}^A_P).$$

For closed 1-forms $\alpha, \beta \in \Omega^1(\Sigma, \mathrm{Ad}^A_P)$ representing tangent vectors at $[A] \in M^*$, we get a 2-form on Σ by the pairing $-\operatorname{Tr}(\alpha \wedge \beta)$, and we let

$$\omega([\alpha], [\beta]) = -\int_\Sigma \operatorname{Tr}(\alpha \wedge \beta).$$

This defines a symplectic form on M^*, which we will refer to as the Seshadri-Atiyah-Bott-Goldman symplectic form, since work on and construction of it was done in various settings by these people [AB83, NS64, NS65, Gol84].

Likewise, we get a symplectic structure on the moduli space $\mathcal{M}^d$.

4.3 Teichmüller Space and Kähler Structure

The next component we need, is a compatible Kähler structure on the moduli space. To construct this, we remember that the complex structures on the surface Σ are identified with points in *Teichmüller space* $\mathcal{T}(\Sigma)$, which is given by

$$\mathcal{T}(\Sigma) = \mathcal{C}(\Sigma) / \mathrm{Diff}_0(\Sigma),$$

where $\mathcal{C}(\Sigma)$ is the space of conformal equivalence classes of Riemannian metrics on Σ, $\mathrm{Diff}(\Sigma)$ is the group of orientation preserving diffeomorphisms fixing the boundary, and $\mathrm{Diff}_0(\Sigma)$ is the subgroup of diffeomorphisms that are isotopic to the identity. It is, furthermore, well-known that $\mathcal{T}(\Sigma)$ is a contractible space and carries a natural complex structure.

Now, let us explain how $\mathcal{T}(\Sigma)$ parametrizes Kähler structures on the moduli space. For $\sigma \in \mathcal{T}$ we get a Kähler structure on Σ and a corresonding Hodge star operater

$$*_\sigma \colon \Omega^1(\Sigma) \to \Omega^1(\Sigma).$$

This extends to the cochain complex $\Omega^\bullet_\Sigma(\mathrm{Ad}_P)$. Now, if A is an irreducible flat connection, then by Hodge theory, we have an identification of the tangent space with the harmonic forms

$$T_{[A]}\mathcal{M}^* \simeq H^1(\Sigma, \mathrm{Ad}_P^A) \simeq \ker(\nabla^A) \cap \ker(*\nabla^A *),$$

and since $*_\sigma$ satisfies $*_\sigma^2 = -1$, we can define an almost complex structure J_σ through

$$J_\sigma[\alpha] = [- *_\sigma (\alpha)],$$

where α is the unique harmonic representative of its cohomology class. Through work of Narasimhan and Seshadri [NS64], this is known to induce an integrable almost complex structure on $\mathcal{M}^*$. In fact the choice of complex structure on Σ allows us to identify the whole moduli space of flat connections $\mathcal{M}$ with the *moduli space of semistable holomorphic rank n vector bundles of trivial determinant*. We denote this moduli space by $\mathcal{N}_\sigma$.

The subset $\mathcal{M}^* \subset \mathcal{N}_\sigma$ is the smooth part, and if we denote the corresponding complex manifold by $\mathcal{M}^*_\sigma$ then $(\mathcal{M}^*_\sigma, \omega)$ will in fact be a be Kähler manifold.

Moreover, the complex structures preserve the subspaces M^d, such that these become compact Kähler manifolds.

4.4 Quantization of Moduli Spaces

The last ingredient in the quantization data is a prequantum line bundle, and this is given by the Chern-Simons lines described earlier. We refer to following theorem of Freed.

Theorem 4.1 (Freed)**.** *The Chern-Simons lines* (4.1) *descends to a Prequantum line bundle*

$$\mathcal{L}_{CS} \to \mathcal{M}^*,$$

on $\mathcal{M}^*$ with the Seshadri-Atiyah-Bott-Goldman symplectic form ω.

The content of theorem 4.1 can be found in proposition 3.17, in [Fre95] where it is also discussed how to realize the moduli space as a *symplectic quotient*. In the same way, we get a prequantum line bundle in the case of the moduli space M^d.

To summarize, we started from Chern-Simons theory and as a description of the space of classical solutions, we got the moduli space of flat connections, which we have seen is a symplectic manifold with a prequantum line bundle given by the Chern-Simons lines central to the Lagrangian formulation of Chern-Simons theory. This constitutes the *pre-quantization data*, as it was described in section 2.2.1.

Furthermore, we have a family of Kähler structures parametrized by Teichmüller space, and for each Kähler structure we get the space of Holomorphic sections

$$\mathcal{V}^{(k)}_\sigma = H^0\left(\mathcal{M}^*_\sigma, \mathcal{L}^k_{CS}\right).$$

These constitue the fibers in the so-called *Verlinde bundle*

$$\mathcal{V}^{(k)} \to \mathcal{T}.$$

which turns out to be finite-dimensional. This was, as mentioned, the bundle which Hitchin studied in [Hit90], and he proved that it admitted a projectively flat connection. In [ADPW91] a similar construction is carried out, and Andersen have proven the following theorem in [And12].

Theorem 4.2 (Andersen)**.** *The construction provided by Hitchin in [Hit90] is equivalent to the construction by Axelrod, Della Pietra, and Witten provided in [ADPW91].*

In chapter 6 we will return to Andersens paper [And12], wherein the existence of a projectively flat connection on $\mathcal{V}^{(k)} \to \mathcal{T}$ is explained and generalized to rigid families of Kähler structures as introduced in section 3.4, and we will furthermore give the construction in the weakly restricted case, which was done in [AR16]. In chapter 7 we will discuss the general case of any family of Kähler structures.

Returning again to the case in which Σ is a compact surface of genus $g \geq 2$ with a single boundary component, the space $\mathcal{M}^d$ is a compact Kähler manifold and carries in the same way a prequantum line bundle. This space fullfils all the assumptions of theorem 6.7, and further all complex structures in the family have zero-dimensional symmetry group, such that Andersen and Gammelgaards calculation of the curvature in [AG14] shows that it is projectively flat.

Building on the former results, Andersen and his former PhD student N. S. Poulsen are able to explicitly calculate the curvature of the Hitchin connection on the moduli space in the paper [AP16]. As it is projectively flat, the curvature form is a 2-form tensored together with the identity endomorphism. They get the following expression for the 2-form

Theorem 4.3 (Andersen & Poulsen)**.**

$$F_{\nabla^H} = \frac{ik(N^2-1)}{12(k+N)\pi}\omega_{\mathcal{T}},$$

where $\omega_{\mathcal{T}}$ is the Weil-Petersson form.

4.5 The Mapping Class Group

Let us finish the discussion by briefly mentioning the natural group of symmetries on the moduli space. We remember that $\mathrm{Diff}(\Sigma)$ is the group of orientation preserving diffeomorphisms fixing the boundary, and $\mathrm{Diff}_0(\Sigma)$ is the subgroup of diffeomorphisms that are isotopic to the identity, and we define the mapping class group of Σ to be the quotient

$$\Gamma(\Sigma) = \mathrm{Diff}(\Sigma)/\,\mathrm{Diff}_0(\Sigma).$$

An element φ of the mapping class group is a diffeomorphism of Σ, and so it induces an isomorphism on the fundamental group $\pi_1(\Sigma, p) \to \pi_1(\Sigma, \varphi(p))$. Now any path from p to $\varphi(p)$ induces an isomorphism $\pi_1(\Sigma, \varphi(p)) \to \pi_1(\Sigma, p)$, and postcomposing with this gives an automorphism of $\pi_1(\Sigma, p)$. This is not independent of the choice of path from p to $\varphi(p)$, but different choices give conjugate automorphisms, and so this gives a well-defined action of the mapping class group on the moduli space.

If Σ has boundary, we have assumed $p \in \partial\Sigma$, and so each $\varphi \in \mathrm{Diff}(M)$ directly induce a automorphism of $\pi_1(\Sigma, p)$ only depending on the isotopi class of φ, and this action commutes with the conjugation action of $\mathrm{SU}(n)$.

Furthermore, the mapping class group action preserves the symplectic structure, and the map $J\colon \mathcal{T}(\Sigma) \to C^\infty(\mathcal{M},^* \mathrm{End}(T\mathcal{M}^*))$ defining Kähler structures on $\mathcal{M}^*$ is likewise equivariant with respect to the of $\Gamma(\Sigma)$.

Chapter 5

Differential Operators and Symbols

In this section we will investigate some general properties and equalities on differential operators and their symbols. These will be paramount in our study of the Hitchin connection in chapter 7.

We first consider an arbitrary vector bundle $E \to M$ with a connection ∇. We will derive an identity on the n'th order differential operator with principal symbol G_n, which we define inductively as in [AGL12] by

$$\begin{aligned}\nabla^n_{X_1 \otimes \ldots \otimes X_n} s &= \nabla_{X_1} \nabla^{n-1}_{X_2 \otimes \ldots \otimes X_n} s - \nabla^{n-1}_{\nabla_{X_1}(X_2 \otimes \ldots \otimes X_n)} s \\ &= \nabla_{X_1} \nabla^{n-1}_{X_2 \otimes \ldots \otimes X_n} s - \sum\nolimits_j \nabla^{n-1}_{X_2 \otimes \ldots \otimes \nabla_{X_1} X_j \otimes \ldots \otimes X_n} s,\end{aligned}$$

with the obvious induction start given by the covariant derivative, and with the covariant derivative on tensor powers of the tangent bundle being the one induced via the Leibniz rule by the covariant derivative on the tangent bundle. It can be verified directly that this expression is tensorial in the vector fields, so that we get a map

$$\nabla^n \colon C^\infty(M, TM^n) \to \mathcal{D}^n(M, E),$$

where $\mathcal{D}^n(M, E)$ is the space of differential operators of order at most n on the bundle E. For any tensor field $G_n \in C^\infty(M, TM^n)$, the symbol of $\nabla^n_{G_n}$ is given by the symmetrization $\mathcal{S}(T_n) \in C^\infty(M, S^n(TM))$ of G_n. Given any operator $D \in \mathcal{D}^n(M, E)$, with principal symbol $\sigma_P(D) = S_n \in C^\infty(M, S^n(TM))$, the operator $D - \nabla^n_{S_n}$ is of order at most $n-1$, since the symbol of order n obviously vanishes. Inductively, it follows that the operator D can be written uniquely in the form

$$D = \nabla^n_{S_n} + \nabla^{n-1}_{S_{n-1}} + \cdots + \nabla_{S_1} + S_0, \tag{5.1}$$

where $S_d \in C^\infty(M, S^d(TM))$ is called the *symbol of order* d and gives rise to a map from the space of all differential operators

$$\sigma_d \colon \mathcal{D}(M, E) \to C^\infty(M, S^d(TM)).$$

Any finite order differential operator on E is uniquely determined by these symbols. In fact, through the expression (5.1), a choice of symbols specifies a differential operator on any vector bundle with connection, and in particular on functions.

5.1 Differential Operators on the Bundle of Quantum Spaces

Now we will restrict ourselves to look at the case of a symplectic manifold (M, ω) with a pre-quantum line bundle $\mathcal{L}$ as defined in section 2.2.1, and we will investigate what happens, when we differentiate $\nabla^n_{G_n} s$ in the $(0,1)$-direction for any holomorphic section $s \in H^0(M_\sigma, \mathcal{L}^k)$.

To state the result, we define, for an n-tensor $G_n = X_1 \otimes \ldots \otimes X_n \in TM^{\otimes n}$ and a k-form Ω^k for any $k \leq n$, the trace tr as follows.

$$\operatorname{tr}(\Omega^k, G_n) = \sum_{\substack{i_1 < \cdots < i_k \\ j_1 < \cdots < j_{n-k} \\ i_\nu \neq j_\eta}} \Omega^k(X_{i_1}, \ldots, X_{i_n}) X_{j_1} \otimes \ldots \otimes X_{j_{n-k}}.$$

We expand the definition by linearity to any tensor in $TM^{\otimes n}$, and now we can state the proposition. We will in the following do calculations modulo $\mathcal{D}^{n-2}(M, \mathcal{L}^k)$, and by this we simply mean that the equations hold up to additions of differential operators of order $n-2$ or lower.

Proposition 5.1. *For any tensor $G_n \in C^\infty(M_\sigma, \mathcal{S}^n(T'M_\sigma))$, any holomorphic section $s \in H^0(M_\sigma, \mathcal{L}^k)$ and any vector field $Z \in C^\infty(M_\sigma, T''M_\sigma)$, we get the identity*

$$\nabla^{0,1}_Z \nabla^n_{G_n} s = \begin{cases} \nabla_{\nabla^{0,1}_Z G_n} s - ik\omega(Z, G_1)s, & n = 1 \\ \nabla^n_{\nabla^{0,1}_Z G_n} s - \nabla^{n-1}_{ik \operatorname{tr}(\iota_Z \omega, G_n) + \operatorname{tr}(\iota_Z R, G_n)} s \mod \mathcal{D}^{n-2}(M, \mathcal{L}^k), & n \geq 2. \end{cases}$$

Here R denotes the curvature of the connection in tensor powers of the tangent bundle, induced by the Levi Civita connection.

Proof. To begin the proof, we notice that any n-tensor can be written in the form

$$G_n = \sum_i X^i_1 \otimes \ldots \otimes X^i_n,$$

and by linearity, it is enough to show the result for $G_n = X_1 \otimes \ldots \otimes X_n$. The proof is done by induction in n, so let us first establish the result for $n = 1$, which basically follows directy from the definition of the curvature

$$F_\nabla(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$

and the fact that the torsion

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

is 0 for the Levi Civita connection, and lastly that the Levi Civita connection preserves types on the tangent bundle. Using this we calculate

$$\begin{aligned} \nabla^{0,1}_Z \nabla_{G_1} s &= -ik\omega(Z, G_1)s + \nabla_{[Z,G_1]} s + \cancel{\nabla_{G_1} \nabla^{0,1}_Z s} \\ &= -ik\omega(Z, G_1)s + \nabla_{\nabla^{0,1}_Z G_1} s - \cancel{\nabla_{\nabla_{G_1} Z} s} \\ &= \nabla_{\nabla^{0,1}_Z G_1} s - ik\omega(Z, G_1)s. \end{aligned}$$

This establishes the result for $n = 1$, which will be used several times in the following. Since $n = 1$ is different, we will start the induction with $n = 2$, and in the following the calculations are done modulo $\mathcal{D}^{n-2}(M, \mathcal{L}^k)$, so we immidiately discard terms of order $n - 2$ or lower.

$$\begin{aligned}
\nabla^{0,1}_Z \nabla^2_{G_2} s =& \nabla^{0,1}_Z \nabla_{X_1} \nabla_{X_2} s - \nabla^{0,1}_Z \nabla_{\nabla_{X_1} X_2} s \\
=& - ik\omega(Z, X_1) \nabla_{X_2} s + \nabla_{[Z,X_1]} \nabla_{X_2} s + \nabla_{X_1} \nabla^{0,1}_Z \nabla_{X_2} s \\
& - \nabla_{\nabla^{0,1}_Z \nabla_{X_1} X_2} s \\
=& \nabla_{-ik\omega(Z,X_1)X_2} s + \nabla_{\nabla^{0,1}_Z X_1} \nabla_{X_2} s - \nabla^{0,1}_{\nabla_{X_1} Z} \nabla_{X_2} s \\
& + \nabla_{X_1} \nabla_{\nabla^{0,1}_Z X_2} s + \nabla_{-ik\omega(Z,X_2)X_1} s \\
& - \nabla_{\nabla^{0,1}_Z \nabla_{X_1} X_2} s \\
=& \nabla_{\nabla^{0,1}_Z X_1} \nabla_{X_2} s + \nabla_{X_1} \nabla_{\nabla^{0,1}_Z X_2} s \\
& + \nabla_{-ik\omega(Z,X_1)X_2} s + \nabla_{-ik\omega(Z,X_2)X_1} s \\
& - \nabla_{\nabla^{0,1}_Z \nabla_{X_1} X_2} s - \nabla_{\nabla^{0,1}_{\nabla_{X_1} Z} X_2} s \\
=& \nabla^2_{\nabla^{0,1}_Z G_2} s - \nabla_{ik \operatorname{tr}(\iota_Z \omega, G_2)} s \\
& + \nabla_{\nabla_{X_1} \nabla^{0,1}_Z X_2} s - \nabla_{\nabla^{0,1}_Z \nabla_{X_1} X_2} s + \nabla_{\nabla_{\nabla^{0,1}_Z X_1} X_2} s - \nabla_{\nabla^{0,1}_{\nabla_{X_1} Z} X_2} s \\
=& \nabla^2_{\nabla^{0,1}_Z G_2} s - \nabla_{ik \operatorname{tr}(\iota_Z \omega, G_2)} s \\
& + \nabla_{\nabla_{X_1} \nabla^{0,1}_Z X_2} s - \nabla_{\nabla^{0,1}_Z \nabla_{X_1} X_2} s - \nabla_{\nabla_{[X_1,Z]} X_2} s \\
=& \nabla^2_{\nabla^{0,1}_Z G_2} s - \nabla_{ik \operatorname{tr}(\iota_Z \omega, G_2)} s + \nabla_{ikR(X_1,Z)X_2 s} \\
=& \nabla^2_{\nabla^{0,1}_Z G_2} s - \nabla_{ik \operatorname{tr}(\iota_Z \omega, G_2) + \operatorname{tr}(\iota_Z R, G_2)} s
\end{aligned}$$

where we have used, that by defintion

$$\nabla_{\nabla^{0,1}_Z X_1} \nabla_{X_2} s + \nabla_{X_1} \nabla_{\nabla^{0,1}_Z X_2} s = \nabla^2_{\nabla^{0,1}_Z G_2} s + \nabla_{\nabla_{\nabla^{0,1}_Z X_1} X_2} s + \nabla_{\nabla_{X_1} \nabla^{0,1}_Z X_2} s.$$

Now we continue with the proof for general n, which uses the induction step and Leibniz rule for the covariant derivative, but othervise follows the same structure as the $n = 2$ case. In the follwing we will write $G_{n-1} = X_2 \otimes \ldots \otimes X_n$.

$$\begin{aligned}
\nabla^{0,1}_Z \nabla^n_{G_n} s =& \nabla^{0,1}_Z \nabla_{X_1} \nabla^{n-1}_{G_{n-1}} s - \nabla^{0,1}_Z \nabla^{n-1}_{\nabla_{X_1} G_{n-1}} s \\
=& - ik\omega(Z, X_1) \nabla^{n-1}_{G_{n-1}} s + \nabla_{[Z,X_1]} \nabla^{n-1}_{G_{n-1}} s + \nabla_{X_1} \nabla^{0,1}_Z \nabla^{n-1}_{G_{n-1}} s \\
& - \nabla^{n-1}_{\nabla^{0,1}_Z \nabla_{X_1} G_{n-1}} s \\
=& - \nabla^{n-1}_{ik\omega(Z,X_1)G_{n-1}} s + \nabla_{\nabla^{0,1}_Z X_1} \nabla^{n-1}_{G_{n-1}} s - \nabla^{0,1}_{\nabla_{X_1} Z} \nabla^{n-1}_{G_{n-1}} s \\
& + \nabla_{X_1} \nabla^{n-1}_{\nabla^{0,1}_Z G_{n-1}} s - \nabla_{X_1} \nabla^{n-2}_{ik \operatorname{tr}(\iota_Z \omega, G_{n-1})} s - \nabla_{X_1} \nabla^{n-2}_{\operatorname{tr}(\iota_Z R, G_{n-1})} s \\
& - \nabla^{n-1}_{\nabla^{0,1}_Z \nabla_{X_1} G_{n-1}} s
\end{aligned}$$

$$
\begin{aligned}
=&\nabla_{\nabla^{0,1}_Z X_1}\nabla^{n-1}_{G_{n-1}}s+\nabla_{X_1}\nabla^{n-1}_{\nabla^{0,1}_Z G_{n-1}}s\\
&-\nabla^{n-1}_{ik\omega(Z,X_1)G_{n-1}}s-\nabla_{X_1}\nabla^{n-2}_{ik\,\mathrm{tr}(\iota_Z\omega,G_{n-1})}s\\
&-\nabla^{n-1}_{\nabla^{0,1}_{\nabla_{X_1}Z}G_{n-1}}s-\nabla^{n-1}_{\nabla^{0,1}_Z\nabla_{X_1}G_{n-1}}s-\nabla_{X_1}\nabla^{n-2}_{\mathrm{tr}(\iota_Z R,G_{n-1})}s\\
=&\nabla^n_{\nabla^{0,1}_Z G_n}s-\nabla^{n-1}_{ik\,\mathrm{tr}(\iota_Z\omega,G_n)}s\\
&+\nabla^{n-1}_{\nabla_{X_1}\nabla^{0,1}_Z G_{n-1}}s-\nabla^{n-1}_{\nabla^{0,1}_Z\nabla_{X_1}G_{n-1}}s+\nabla^{n-1}_{\nabla_{\nabla^{0,1}_Z X_1}G_{n-1}}s-\nabla^{n-1}_{\nabla^{0,1}_{\nabla_{X_1}Z}G_{n-1}}s\\
&-\nabla_{X_1}\nabla^{n-2}_{\mathrm{tr}(\iota_Z R,G_{n-1})}s\\
=&\nabla^n_{\nabla^{0,1}_Z G_n}s+\nabla^{n-1}_{-ik\,\mathrm{tr}(\iota_Z\omega,G_n)}s\\
&-\nabla^{n-1}_{\mathrm{tr}(\iota_Z R(X_1),G_{n-1})}-\nabla_{X_1}\nabla^{n-2}_{\mathrm{tr}(\iota_Z R,G_{n-1})}s\\
=&\nabla^n_{\nabla^{0,1}_Z G_n}s-\nabla^{n-1}_{ik\,\mathrm{tr}(\iota_Z\omega,G_n)}s-\nabla^{n-1}_{\mathrm{tr}(\iota_Z R,G_n)}s,
\end{aligned}
$$

where we have used, that by defintion

$$
\nabla_{\nabla^{0,1}_Z X_1}\nabla^{n-1}_{G_{n-1}}s+\nabla_{X_1}\nabla^{n-1}_{\nabla^{0,1}_Z G_{n-1}}s=\nabla^n_{\nabla^{0,1}_Z G_n}s+\nabla^{n-1}_{\nabla_{\nabla^{0,1}_Z X_1}G_{n-1}}s+\nabla^{n-1}_{\nabla_{X_1}\nabla^{0,1}_Z G_{n-1}}s.
$$

□

We would like to improve the equality a little more when possible, and the following lemma shows, that we can substitute the term involving the trace of the Levi Civita curvature with an exact term and a trace over the ricci form, given that G_n is actually a holomorphic section. To state this lemma intrinsically, we define the following variation of the trace from above. For an n-tensor $G_n = X_1 \otimes \ldots \otimes X_n \in TM^{\otimes n}$ and a k-form Ω^k for any $k \leq n-1$, the trace $\tilde{\mathrm{tr}}$ is given by

$$
\tilde{\mathrm{tr}}(\Omega^k, G_n) = \mathrm{tr}(\Omega^k, X_1 \otimes \ldots \otimes X_{n-1})X_n,
$$

and again expanding by linearity.

Lemma 5.2. *For $n \geq 2$, we get for any holomorphic tensor $G_n \in H^0(M_\sigma, T'M_\sigma^{\otimes n})$ and any vector field $Z \in C^\infty(M_\sigma, T''M_\sigma)$, the identity*

$$
\nabla^{0,1}_Z \tilde{\delta}(G_n) = -\tilde{\mathrm{tr}}(i\iota_Z\rho, G_n) + \mathrm{tr}(\iota_Z R, G_n),
$$

The equation expands by linearity to any $G_n \in H^0(M_\sigma, \mathcal{S}^n(T'M_\sigma))$.

Proof. The proof is inductive starting at $n=2$, but firstly we will start with the following little calculation, which we will use several times. Let $X \in H^0(M_\sigma, T'M)$ be a holomorphic vector field, then

$$\begin{aligned}
\nabla_Z^{0,1}\delta(X) &= \nabla_Z^{0,1}\operatorname{tr}(\nabla X) \\
&= \operatorname{tr}(\nabla_Z^{0,1}\nabla X) \\
&= \operatorname{tr}(R(Z,\cdot)X) \\
&= -\operatorname{tr}(R(\cdot,Z)X) \\
&= -r(Z,X) \\
&= -r(JZ,JX) \\
&= -ir(JZ,X) \\
&= -i\rho(Z,X).
\end{aligned}$$

Now as in proposition 1.3, we will assume $G_n = X_1 \otimes \ldots \otimes X_n$ expressed locally by holomorphic vector fields $X_1, \ldots, X_n$, and it is enough to proof the lemma in this case, since it just extends by linearity. For $n = 2$, the calculation goes as follows

$$\begin{aligned}
\nabla_Z^{0,1}\delta(G_2) &= \nabla_Z^{0,1}\delta(X_1)X_2 + \nabla_Z^{0,1}\nabla_{X_1}X_2 \\
&= -i\rho(Z,X_1)X_2 + \nabla_Z^{0,1}\nabla_{X_1}X_2 - \nabla_{X_1}\nabla_Z^{0,1}X_2 - \nabla_{[Z,X_1]}X_2 \\
&= -i\rho(Z,X_1)X_2 + R(Z,X_1)X_2 \\
&= -\operatorname{tr}(i\iota_Z\rho, X_1)X_2 + \operatorname{tr}(\iota_Z R, G_2),
\end{aligned}$$

where we in the second equality, we have subtracted two terms, that are zero because of holomorphicity, to get the curvature term. Now to shorten notation we write
$X_2 \otimes \ldots \otimes X_n = G_{n-1}$, and assuming the lemma holds for $n-1$, we similarly calculate for n, that

$$\begin{aligned}
\nabla_Z^{0,1}\tilde{\delta}(G_n) =& \nabla_Z^{0,1}\delta(X_1)G_{n-1} + \nabla_Z^{0,1}\nabla_{X_1}G_{n-1} + X_1 \otimes \nabla_Z^{0,1}\tilde{\delta}(G_{n-1}) \\
=& -i\rho(Z,X_1)G_{n-1} \\
&+ \nabla_Z^{0,1}\nabla_{X_1}G_{n-1} - \nabla_{X_1}\nabla_Z^{0,1}G_{n-1} - \nabla_{[Z,X_1]}G_{n-1} \\
&- X_1 \otimes \operatorname{tr}(i\iota_Z\rho, X_2 \otimes \ldots \otimes X_{n-1})X_n + X_1 \otimes \operatorname{tr}(\iota_Z R, G_{n-1}) \\
=& -\operatorname{tr}(i\iota_Z\rho, X_1 \otimes \ldots \otimes X_{n-1})X_n \\
&+ R(Z,X_1)G_{n-1} + X_1 \otimes \operatorname{tr}(\iota_Z R, G_{n-1}) \\
&= -\operatorname{tr}(i\iota_Z\rho, X_1 \otimes \ldots \otimes X_{n-1})X_n + \operatorname{tr}(\iota_Z R, G_n) \\
&= -\tilde{\operatorname{tr}}(i\iota_Z\rho, G_n) + \operatorname{tr}(\iota_Z R, G_n),
\end{aligned}$$

which finishes the proof. □

We will often use these two results together and combine them in a corollary. Before we state this, we will introduce the following maps on cohomology, which together with the previous results, will play an important role in the existence theorems for the Hitchin connection in chapter 7.

Definition 5.3. For any $n \geq 1$, we let

$$\Psi^{(n)}\colon H^0(M_\sigma, \mathcal{S}^n(T'M_\sigma)) \to H^1(M_\sigma, S^{n-1}(T'M_\sigma)),$$

be given by

$$\Psi^{(n)}_Z(G_n) = i(kn\iota_Z\omega + (n-1)\iota_Z\rho) \cdot G_n,$$

where $Z \in C^\infty(M, TM)$. We will need the kernel and image of these maps and name them

$$K^{(n)} := \ker(\Psi^{(n)})\colon H^0(M_\sigma, \mathcal{S}^n(T'M_\sigma)) \to H^1(M_\sigma, S^{n-1}(T'M_\sigma))$$
$$I^{(n)} := \mathrm{Im}(\Psi^{(n)})\colon H^0(M_\sigma, \mathcal{S}^n(T'M_\sigma)) \to H^1(M_\sigma, S^{n-1}(T'M_\sigma)).$$

Corollary 5.4. *For $n \geq 2$, for any symmetric holomorphic tensor $G_n \in C^\infty(M_\sigma, \mathcal{S}^n(T'M_\sigma))$, any holomorphic section $s \in H^0(M_\sigma, \mathcal{L}^k)$ and any vector field $Z \in C^\infty(M_\sigma, T''M_\sigma)$, we get the identity*

$$\nabla^{0,1}_Z(\nabla^n_{G_n} + \nabla^{n-1}_{\tilde{\delta}(G_n)})s = \nabla^n_{\nabla^{0,1}_Z G_n} s - \nabla^{n-1}_{\Psi^{(n)}(G_n)} \quad \mathrm{mod}\ \mathcal{D}^{n-2}(M, \mathcal{L}^k)$$

Proof. Like earlier we calculate for $G_n = X_1 \otimes \ldots \otimes X_n = X_1 \otimes G_{n-1} \in H^0(M_\sigma, \mathcal{S}^n(T'M_\sigma))$ and get the general result by expanding by linearity. By combining Proposition 1.3 and Lemma 5.2, we calculate directly the following, where the only extra ingredient is to apply the symmetry in the fifth equality.

$$\begin{aligned}
&\nabla^{0,1}_Z(\nabla^n_{G_n} + \nabla^{n-1}_{\tilde{\delta}(G_n)})s \\
&= \nabla^n_{\nabla^{0,1}_Z G_n} s - \nabla^{n-1}_{ik\,\mathrm{tr}(\iota_Z\omega, G_n) + \mathrm{tr}(\iota_Z R, G_n)} s + \nabla^{n-1}_{\nabla^{0,1}_Z \tilde{\delta}(G_n)} s \\
&= \nabla^n_{\nabla^{0,1}_Z G_n} s + \nabla^{n-1}_{-ik\,\mathrm{tr}(\iota_Z\omega, G_n) - \mathrm{tr}(\iota_Z R, G_n)} s + \nabla^{n-1}_{-\tilde{\mathrm{tr}}(i\iota_Z\rho, G_n) + \mathrm{tr}(\iota_Z R, G_n)} s \\
&= \nabla^n_{\nabla^{0,1}_Z G_n} s + \nabla^{n-1}_{-ik\,\mathrm{tr}(\iota_Z\omega, G_n) - \tilde{\mathrm{tr}}(i\iota_Z\rho, G_n)} s \\
&= \nabla^n_{\nabla^{0,1}_Z G_n} s - \nabla^{n-1}_{ik\,\mathrm{tr}(\iota_Z\omega, G_n) + \mathrm{tr}(i\iota_Z\rho, X_1 \otimes \ldots \otimes X_{n-1})X_n} \\
&= \nabla^n_{\nabla^{0,1}_Z G_n} s - \nabla^{n-1}_{ikn\iota_Z\omega \cdot G_n + i(n-1)\iota_Z\rho \cdot G_n} \\
&= \nabla^n_{\nabla^{0,1}_Z G_n} s - \nabla^{n-1}_{i(kn\iota_Z\omega + (n-1)\iota_Z\rho) \cdot G_n} \\
&= \nabla^n_{\nabla^{0,1}_Z G_n} s - \nabla^{n-1}_{\Psi^{(n)}_Z(G_n)} \quad \mathrm{mod}\ \mathcal{D}^{n-2}(M, \mathcal{L}^k).
\end{aligned}$$

□

5.2 Sections vanishing up to order $N-1$

In this section we will state a general result, that for any point $p \in M$ and any integer N, we can find a level k and a subspace of $H^0(M_\sigma, \mathcal{L}^k)$ which span the fibers of the bundle of N jets of holomorphic sections at p. By evaluating a differential operator D at all these sections we will be able to get equalities on the symbols from equalities that hold for each holomorphic section. We formulate this in the following.

Proposition 5.5. *Let $p \in M$, $N \geq 0$ and $d_N = \dim(S^N(T'M_p))$. Also let $\{e_1, \dots e_{d_N}\}$ be a basis for $S^N(T_p'M_\sigma)$. Then there exists a level k, and a set $\{s_1, \dots, s_{d_N}\} \subseteq H^0(M_\sigma, \mathcal{L}^k)$ of holomorphic sections, such that*

$$(\nabla_{e_i}^N s_j)(p) = \delta_{ij},$$

and such that for any $0 \leq l \leq N$, the lower order derivatives

$$(\nabla^l s_j)(p) = 0$$

for all $j = 0, \dots, d_N$.

Furthermore for compact M, we can choose the level k independent of the point p.

This is a result from the theory of jets of sections, and the proof follows the same idea, which is applied in the proof of Kodairas Embedding Theorem. We will need it to proof the following theorem.

Theorem 5.6. *Assume that $D^{(1)}$ and $D^{(2)}$ are differential operators of the form*

$$D^{(j)} = \sum_{i=0}^{N} \nabla_{G_i^{(j)}}^{(i)}, \quad j = 1, 2,$$

such that for all $k \geq 0$ and for all holomorphic sections $s \in H^0(M_\sigma, \mathcal{L}^k)$

$$D^{(1)} s = D^{(2)} s.$$

Then we get equality of the principal symbols

$$G_N^{(1)} = G_N^{(2)}.$$

Proof. At any point $p \in M$, find a level k and sections $\{e_1, \dots e_{d_N}\}$ as in proposition 5.5. The theorem follows directly by evaluating the $D^{(j)}$ on all of these sections. It is clear that we can write the symbol $G_N^{(j)}$ in terms of the basis

$$G_N^{(j)} = \sum_{i=1}^{l_d} c_i^{(j)} e_i, \quad j = 1, 2.$$

Using proposition 5.5, it follows that

$$(D^{(j)} s_i)(p) = c_i^{(j)}, \quad j = 1, 2,$$

and thus

$$G_N^{(1)} = G_N^{(2)}.$$

□

5.3 Differential Operators Preserving the Subspace of Holomorphic sections

Now we can combine our results from above to proof the following main theorem of this chapter.

Theorem 5.7. *If $K^{(n)} = 0$ for all $n \geq l$, then any operator of the form*

$$D = \sum_{i=0}^{N} \nabla^{(i)}_{G_i} \tag{5.2}$$

which maps $H^0(M_\sigma, \mathcal{L}^k)$ to itself for all k, must be of order at most most $l-1$. Moroever the symbol of order $l-1$ of D must be holomorphic.

Proof of Theorem 5.7. Consider a differential operator D of order N, which we can write on the form

$$D = \sum_{i=0}^{N} \nabla^{(i)}_{G_i}, \tag{5.3}$$

where $G_i \in C^\infty(M_\sigma, S^i(T'M_\sigma))$. We assume that D maps $H^0(M_\sigma, \mathcal{L}^k)$ to itself for all k. In other words the derivative

$$\nabla^{0,1} Ds = 0, \quad \forall\, s \in H^0(M_\sigma, \mathcal{L}^k).$$

On the other hand, using Proposition 5.1 on derivatives of differential operators, we get for any vector field $Z \in C^\infty(M_\sigma, T''M_\sigma)$, that when calculating modulo $\mathcal{D}^{N-2}(M, \mathcal{L}^k)$, we have

$$\begin{aligned} 0 = \nabla^{0,1}_Z Ds &= \nabla^N_{\nabla^{0,1}_Z G_N} s + \nabla^{N-1}_{\nabla^{0,1}_Z G_{N-1}} s \\ &\quad - \nabla^{N-1}_{ik\,\mathrm{tr}(\iota_Z\omega, G_N) + \mathrm{tr}(\iota_Z R, G_N)} s \mod \mathcal{D}^{N-2}(M, \mathcal{L}^k). \end{aligned}$$

Now by Theorem 5.6 we get equality of the principal symbol of the left hand side, which is zero, with the principal symbols of the right hand side, since this is still an operator of the form (5.3). That is, we have

$$\nabla^{0,1}_Z G_N = 0,$$

and thus, since this show that G_n is holomorphic, we can also use corollary 5.4 in combination with Proposition 5.1 to get

$$\begin{aligned} \nabla^{N-1}_{\nabla^{0,1}_Z \tilde{\delta}(G_N)} s &= \nabla^{0,1}_Z (\nabla^{N-1}_{\tilde{\delta}(G_N)}) s \\ &= \nabla^{0,1}_Z (D + \nabla^{N-1}_{\tilde{\delta}(G_N)}) s \\ &= \nabla^N_{\nabla^{0,1}_Z G_N} s + \nabla^{N-1}_{\nabla^{0,1}_Z G_{N-1}} s - \nabla^{N-1}_{\Psi^{(N)}_Z (G_N)} s \\ &= \nabla^{N-1}_{\nabla^{0,1}_Z G_{N-1}} s - \nabla^{N-1}_{\Psi^{(N)}_Z (G_N)} s \mod \mathcal{D}^{N-2}(M, \mathcal{L}^k), \end{aligned}$$

where $\Psi^{(N)}$ is defined in defintion 5.3. Now the principal symbol is of order $N-1$, and again applying theorem 5.6 we get the equation

$$\Psi^{(N)}_Z (G_N) = \nabla^{0,1}_Z (G_{N-1} - \tilde{\delta}(G_N)).$$

This shows that $\Psi^{(N)}(G_N)$ is exact, and thus 0 on cohomology. Now if $K^{(N)} = 0$, we get by injectivity of $\Psi^{(N)}$ that $G_N = 0$, which again gives

$$0 = \nabla^{0,1}_Z(G_{N-1} + \tilde{\delta}(G_N)) = \nabla^{0,1}_Z(G_{N-1}).$$

Continuing this procedure inductively gives us $G_n = 0$ for all $n = l, \dots, N$ and also $\nabla^{0,1}_Z(G_{l-1}) = 0$, and we conclude that that D is of order at most l and that the symbol of order $l-1$ is holomorphic. □

We also state the following immidiate corollary, which handles some special and useful cases.

Corollary 5.8. *If $K^{(n)} = 0$ for all $n \geq 1$, then any D as in Theorem 5.7 must be multiplication by a holomorphic function.*

Furthermore if $H^0(M_\sigma, T'M_\sigma) = 0$ we get automatically that $K^{(1)} = 0$, and that all holomorphic functions are constant, implying that D vanish projectively.

Proof of corollary 5.8. If $K^{(n)} = 0$ for all $n \geq 1$, Theorem 5.7 shows that D must be an operator of order 0, i.e. a function, and since the principal symbol is holemorphic, this means that D is a holomorphic function.

If $H^0(M_\sigma, T'M_\sigma) = 0$, it is clear that $K^{(1)} = 0$, since $\Psi^{(n)}$ is a map from the zero set. Also if $H^0(M_\sigma, T'M_\sigma) = 0$, all holomorphic functions on M are constant. □

CHAPTER 6

Hitchin Connections for Restricted Families of Kähler Structures

In this section we go through two explicit constructions of the Hitchin connection in Geometric Quantization. First we review the construction in the rigid setting, which was carried out in [And12] and for which further work was done on calculating the curvature in [AG14]. Afterwards we will carry out the construction under the weakly restricted assumption, which was developed by Andersen and Rasmussen in [AR16] expanding on the same general idea of construction, but loosening the restriction on the family of complex structures considerably.

To emphasise where the weakly restricted case differs from the rigid case, we will go through the first part of the construction without applying either of these condition. This way we go through as much of the theory as possible in the most general setup, and afterwards we add a section showing the last part of the construction in each of the cases.

Let us briefly sum up the setup. We let (M, ω) be a symplectic manifold with a prequantum line bundle $\mathcal{L}$. Assume (M, ω) has real first chern class $c_1(M, \omega) = n\left[\frac{\omega}{2\pi}\right]$ for some integer $n \in \mathbb{Z}$, and let $\mathcal{T}$ be a complex manifold parametrizing a holomorphic family of integrable almost complex structures J making (M, ω, J_σ) kähler for each $\sigma \in \mathcal{T}$.

Now for each $\sigma \in \mathcal{T}$ we consider the quantum space at level $k \in \mathbb{N}$, which is the finite dimensional subspace $H_\sigma^{(k)}$ of the prequantum space $\mathcal{H}^{(k)} = C^\infty(M, \mathcal{L}^k)$ as defined in chapter 2. We will assume that these quantum spaces form a smooth finite rank subbundle $\hat{H}^{(k)}$ of the trivial bundle

$$\hat{\mathcal{H}}^{(k)} = \mathcal{T} \times \mathcal{H}^{(k)}.$$

Now we let ∇^T denote the trivial connection on $\hat{\mathcal{H}}^{(k)}$, and then we consider a connection of the form

$$\boldsymbol{\nabla}_V = \nabla_V^T + u(V),$$

where $u \in \Omega^1(\mathcal{T}, \mathcal{D}(M, \mathcal{L}^k))$ is a one-form on $\mathcal{T}$ with values in the space of differential operators on sections of $\mathcal{L}^k$. Our goal is to construct a u, such that $\boldsymbol{\nabla}$ preserves the quantum spaces $H_\sigma^{(k)}$ inside each fiber of $\hat{\mathcal{H}}^{(k)}$.

Definition 6.1 (Hitchin Connection)**.** A Hitchin connection on the bundle $\hat{\mathcal{H}}^{(k)}$ is a connection $\boldsymbol{\nabla} = \nabla^T + u$, that preserves the subspaces $H^{(k)}_\sigma$ inside each fiber of $\hat{\mathcal{H}}^{(k)}$. That is such that $\boldsymbol{\nabla} s \in H^{(k)}_\sigma$ for each section $s \in H^{(k)}_\sigma$.

To construct a Hitchin connection, our approach is always the same, namely to construct a u, which satisfies the condition in the following lemma.

Lemma 6.2. *The connection $\boldsymbol{\nabla} = \nabla^T + u$ is a Hitchin connection if and only if*

$$\nabla^{0,1}_\sigma u(V)s = \frac{i}{2} V'[J_\sigma] \cdot \nabla^{1,0}_\sigma s, \tag{6.1}$$

for any holomorphic section $s \in H^{(k)}_\sigma$ and any smooth vector field V on $\mathcal{T}$.

Proof. By assumption we need to have

$$\begin{aligned} 0 &= \nabla^{0,1}_\sigma(\boldsymbol{\nabla}_V s) \\ &= \nabla^{0,1}_\sigma V[s] + \nabla^{0,1}_\sigma u(V)s. \end{aligned}$$

Now by differentiating $\nabla^{0,1}_\sigma s = 0$ along V we get

$$\begin{aligned} 0 = V[\nabla^{0,1}_\sigma s] &= V\left[\frac{1}{2}(Id + iJ_\sigma)\cdot \nabla s\right] \\ &= \frac{i}{2} V[J_\sigma] \cdot \nabla^{1,0}_\sigma s + \nabla^{0,1}_\sigma V[s] \end{aligned}$$

Combining the above expressions, we get the equation (6.1). □

The construction of the connection is carried out through a number of lemmas. We will, as mentioned, start in the most general setting and then add the assumptions in the lemmas, when we need them. Firstly we just assume that we have a symplectic manifold (M,ω) with a prequantum line bundle $\mathcal{L}$, a family of Kähler structures J, and that we have an arbitrary symmetric bivector field

$$G \in C^\infty(M_\sigma, S^2(T'M_\sigma)).$$

Then we get a linear bundle map

$$G\colon T'M^*_\sigma \to T'M_\sigma,$$

given by contracting with one of the entries of G. Using this we construct an operator Δ_G on $\mathcal{H}^{(k)}$ given by

$$\begin{aligned} \Delta_G\colon \mathcal{H}^{(k)} = C^\infty(M,\mathcal{L}^k) &\xrightarrow{\nabla^{1,0}_\sigma} C^\infty(M, T'M^*_\sigma \otimes \mathcal{L}^k) \\ &\xrightarrow{G\otimes \mathrm{Id}} C^\infty(M, T'M_\sigma \otimes \mathcal{L}^k) \\ &\xrightarrow{\nabla^{1,0}_\sigma \otimes \mathrm{Id} + \mathrm{Id}\otimes \nabla^{1,0}_\sigma} C^\infty(M, T'M^*_\sigma \otimes T'M_\sigma \otimes \mathcal{L}^k) \\ &\xrightarrow{\mathrm{tr}} C^\infty(M, T'M_\sigma \otimes \mathcal{L}^k). \end{aligned}$$

In abstract tensor notation we can write this in the short form

$$\Delta_G s = \nabla_{u'} G^{u'v'} \nabla_{v'} s,$$

where the outer connection is the connection in the tensor product $T'M_\sigma \otimes \mathcal{L}^k$, which is given exactly as described above by the Leibniz rule.

Before proceeding we observe, that Δ_G is a second order differential operator of order 2, and as such we can of course write Δ_G on the form (5.1) as in chapter 5. We will need this other description later on, so we calculate for $G = X \otimes Y$, that

$$\begin{aligned}\Delta_{X\otimes Y}s &= \nabla_X\nabla_Y s + \nabla_{\delta(X)Y}s \\ &= \nabla_X\nabla_Y s - \nabla_{\nabla_X Y}s + \nabla_{\delta(X)Y}s + \nabla_{\nabla_X Y}s \\ &= \nabla^2_{X\otimes Y}s + \nabla_{\delta(X\otimes Y)}s,\end{aligned}$$

and by linearity, we get for any G that $\Delta_G = \nabla^2_G + \nabla_{\delta G}$.

We will, however, continue our calculations using the index notation in the following, where this first lemma is the crucial common ingredient in the constructions.

Lemma 6.3. *Let $G \in H^0(M_\sigma, S^2(T'M_\sigma))$ be any holomorphic bi-vector field on (M, ω_σ), then we have for any section $s \in H^{(k)}_\sigma$*

$$\nabla^{(0,1)}\Delta_G s = -2ik\omega \cdot G \cdot \nabla^{(1,0)}s - i\rho \cdot G \cdot \nabla^{(1,0)}s - ik\omega \cdot \delta(G)s. \tag{6.2}$$

Proof. The proof is a calculation that mainly uses the trick of commuting two covariant derivatives to get one term that disappears because of type considerations plus a curvature term. We will write out the proof using abstract tensor notation, which highlights contraction of terms. So for $s \in H^{(k)}_\sigma$ we get that

$$\begin{aligned}\nabla^{(0,1)}\Delta_G s &= \nabla_{a''}\nabla_{u'}G^{u'v'}\nabla_{v'}s \\ &= \nabla_{u'}\nabla_{a''}G^{u'v'}\nabla_{v'}s + [\nabla,\nabla]_{a''u'}G^{u'v'}\nabla_{v'}s \\ &= \nabla_{u'}G^{u'v'}\nabla_{a''}\nabla_{v'}s + [\nabla,\nabla]_{a''u'}(G^{u'v'})\nabla_{v'}s + G^{u'v'}[\nabla,\nabla]_{a''u'}\nabla_{v'}s \\ &= \nabla_{u'}G^{u'v'}[\nabla,\nabla]_{a''v'}s + R^w_{a''wu'}G^{u'v'}\nabla_{v'}s - ik\omega_{a''u'}G^{u'v'}\nabla_{v'}s \\ &= -ik\nabla_{u'}G^{u'v'}\omega_{a''v'}s - R^w_{wa''u'}G^{u'v'}\nabla_{v'}s - ik\omega \cdot G \cdot \nabla s \\ &= -ik\nabla_{u'}(G^{u'v'})\omega_{a''v'}s - ikG^{u'v'}\omega_{a''v'}\nabla_{u'}s - r_{a''u'}G^{u'v'}\nabla_{v'}s - ik\omega \cdot G \cdot \nabla s \\ &= -ik\omega_{a''v'}\delta(G)^{v'}s - ik\omega \cdot G \cdot \nabla s - J^x_a r_{xy} J^y_{u'}G^{u'v'}\nabla_{v'}s - ik\omega \cdot G \cdot \nabla s \\ &= -2ik\omega \cdot G \cdot \nabla s - ik\omega \cdot \delta(G)s - i\rho_{au'}G^{u'v'}\nabla_{v'}s \\ &= -2ik\omega \cdot G \cdot \nabla s - i\rho \cdot G \cdot \nabla s - ik\omega \cdot \delta(G)s.\end{aligned}$$

□

Now we will start applying more assumptions to continue the construction.

Corollary 6.4. *Consider the situation as in Lemma 6.3 and assume that the family admits a family of Ricci potentials F. Then we get*

$$\nabla^{(0,1)}\Delta_G s = -i(2k+n)\omega \cdot G \cdot \nabla^{(1,0)}s + 2(d\bar\partial_\sigma F_\sigma) \cdot G \cdot \nabla^{(1,0)}s - ik\omega \cdot \delta(G)s. \tag{6.3}$$

Proof. The proof follows directly by inserting the expression for the Ricci form given in terms of the family of Ricci potentials F_σ, i.e. $\rho_\sigma = n\omega_\sigma + 2id\bar\partial_\sigma F_\sigma$, in (6.2). □

To get rid of the second term here, we can use the following lemma.

Lemma 6.5. *Under the same assumptions as above, we have that*

$$\nabla^{(0,1)}(\nabla_{G\cdot dF}s) = -ik\omega \cdot G \cdot dFs - (d\bar{\partial}_\sigma F_\sigma) \cdot G \cdot \nabla^{(1,0)}s, \tag{6.4}$$

and thus we get the equality

$$\nabla^{(0,1)}(\Delta_G s + 2\nabla_{G\cdot dF}s) = -i(2k+n)\omega \cdot G \cdot \nabla^{(1,0)}s - ik\omega \cdot \delta(G)s - 2ik\omega \cdot G \cdot dFs. \tag{6.5}$$

Proof. Again the proof is a calculation. For $s \in H_\sigma^{(k)}$, we have that

$$\begin{aligned}
\nabla^{(0,1)}(\nabla_{G\cdot dF}s) &= \nabla_{a''}G^{u'v'}dF_{v'}\nabla_{u'}s \\
&= G^{u'v'}dF_{v'}\nabla_{a''}\nabla_{u'}s + G^{u'v'}\nabla_{a''}(dF_{v'})\nabla_{u'}s \\
&= G^{u'v'}dF_{v'}[\nabla,\nabla]_{a''u'}s + \nabla_{a''}(dF_{u'})G^{u'v'}\nabla_{v'}s \\
&= -ik\omega_{a''u'}G^{u'v'}dF_{v'}s + (\bar{\partial}dF) \cdot G \cdot \nabla s \\
&= -ik\omega \cdot G \cdot dFs + (\bar{\partial}\partial F) \cdot G \cdot \nabla s \\
&= -ik\omega \cdot G \cdot dFs - (\partial\bar{\partial}F) \cdot G \cdot \nabla s.
\end{aligned}$$

Now (6.5) follows by combining equations (6.3) and (6.4). □

6.1 The Rigid Case

In this section we continue with the construction in the rigid case. We don't include the proof of the following proposition, but it is a special case of Proposition 6.9, when $\beta(V) = 0$. For interest in this specific proof, it can be found in [AG14].

Proposition 6.6. *Consider the situation as in corollary 6.4, and assume that the family of holomorphic structures J_σ is rigid. Furthermore assume that $H^1(M,\mathbb{R}) = 0$, that $c_1(M,\omega) = n\left[\frac{\omega}{2\pi}\right]$ and that the family admits a family of Ricci potentials F, but that non of the complex structures admit non-constant holomorphic functions on M. Then we have*

$$\delta(G(V)) \cdot \omega + 2dF \cdot G(V) \cdot \omega = 4i\bar{\partial}V'[F].$$

We can now insert this result in the expression from Lemma 6.5 and get

$$\begin{aligned}
&\nabla^{(0,1)}(\Delta_{G(V)}s + 2\nabla_{G(V)\cdot dF}s) \\
&\quad = -i(2k+n)\omega \cdot G(V) \cdot \nabla^{(1,0)}s - ik\omega \cdot \delta(G(V))s - 2ik\omega \cdot G(V) \cdot dFs \\
&\quad = i(2k+n)(G(V)\cdot\omega) \cdot \nabla^{(1,0)}s + ik(\delta(G(V))\cdot\omega + 2dF \cdot G(V)\cdot\omega)s \\
&\quad = i(2k+n)V'[J] \cdot \nabla^{(1,0)}s - 4k\bar{\partial}V'[F]s.
\end{aligned}$$

From this follows immediately that

$$\nabla^{(0,1)}(\Delta_{G(V)} + 2\nabla_{G(V)\cdot dF} + 4kV'[F])s = 2(2k+n)\frac{i}{2}V'[J] \cdot \nabla^{(1,0)}s,$$

and we see that dividing with $2(2k+n)$ gives a $u(V)$ that satisfies the condition in Lemma 6.2. Thus we have proved the following theorem.

Theorem 6.7 (Hitchin Connection in the Rigid Setting)**.** *Let (M, ω) be a symplectic manifold with a prequamtum line bundle $\mathcal{L}$ and $H^1(M, \mathbb{R}) = 0$. Assume that M has first chern class $c_1(M, \omega) = n\left[\frac{\omega}{2\pi}\right]$ for some integer $n \in \mathbb{Z}$. Furthermore let $J : \mathcal{T} \to C^\infty(M, \mathrm{End}(TM))$ be a rigid, holomorphic family of Kähler structures on M, parametrized by a complex manifold $\mathcal{T}$, and assume that the family admits a family of Ricci potentials F, but that non of the complex structures admit non-constant holomorphic functions on M. Then there exits a Hitchin connection $\boldsymbol{\nabla}$ in the bundle $\hat{\mathcal{H}}^{(k)}$ over $\mathcal{T}$, given by the expression*

$$\boldsymbol{\nabla}_V = \nabla^T_V + u(V),$$
$$u(V) = \frac{1}{2(2k+n)}(\Delta_{G(V)} + 2\nabla_{G(V)\cdot dF} + 4kV'[F]).$$

6.2 The Weakly Restricted Case

Now, instead of the rigid condition, we impose the weakly restricted condition and continue the construction in the case where the family of complex structures is holomorphic. The following lemma and proposition are the key components.

Lemma 6.8. *Consider the situation as in corollary 6.4, and assume that the family of holomorphic structures J_σ is weakly restricted. Furthermore assume that $c_1(M, \omega) = n\left[\frac{\omega}{2\pi}\right]$ and that the family admits a family of Ricci potentials F.*

Let $G_\beta(V)$ and $\beta(V)$ be bivector- and vector fields associated to the the family J_σ satisfying (3.5). Then we have that the 1-form

$$\Omega(V) = -\delta(G_\beta(V))\cdot\omega + \delta(\bar{\partial}\beta(V)) - 2dF\cdot G_\beta(V)\cdot\omega + 2\bar{\partial}\beta(V)\cdot dF + 4i\bar{\partial}V'[F]$$

is closed and of type $(0,1)$.

Proof. We start with the equation $\rho_\sigma = n\omega + 2id\bar{\partial}_\sigma F_\sigma$. Differentiating this equation along V' we get

$$\begin{aligned} V'[\rho] &= 2idV'[\bar{\partial}]F + 2id\bar{\partial}V'[F] \\ &= -dV'[J]\cdot dF + d(2i\bar{\partial}V'[F]) \\ &= -d(G_\beta(V)\cdot\omega)\cdot dF + d\bar{\partial}\beta(V)\cdot dF + d(2i\bar{\partial}V'[F]) \\ &= -d(dF\cdot G_\beta(V)\cdot\omega) + d(\bar{\partial}\beta(V)\cdot dF) + d(2i\bar{\partial}V'[F]). \end{aligned}$$

Now we use Proposition 3.4, which gives us

$$2V'[\rho] = d\delta(G(V)\cdot\omega) = d\delta(V'[J]) = d\delta(G_\beta(V))\cdot\omega - d\delta(\bar{\partial}\beta(V)).$$

Inserting this on the left hand side of the above equation, we get that

$$\begin{aligned} 0 \quad = \quad & -d(\delta(G_\beta(V))\cdot\omega) + d\delta(\bar{\partial}\beta(V)) - d(2dF\cdot G_\beta(V)\cdot\omega) \\ & +d(2\bar{\partial}\beta(V)\cdot dF) + d(4i\bar{\partial}V'[F]) = d\Omega(V), \end{aligned}$$

which exactly states that $\Omega(V)$ is closed. By checking each term, it is also seen directly to be of type $(0,1)$. ☐

Proposition 6.9. *Consider the setup of Lemma 6.8, and furthermore assume that $H^1(M,\mathbb{R}) = 0$. Then there exists $\psi(V) \in C^\infty(M)$, such that*

$$\begin{aligned}&\delta(G_\beta(V)) \cdot \omega + 2dF \cdot G_\beta(V) \cdot \omega \\ &= 4i\bar{\partial}V'[F] + 2\bar{\partial}(dF \cdot \beta(V)) + in\omega \cdot \beta(V) + \bar{\partial}\delta(\beta(V)) - \bar{\partial}\psi(V).\end{aligned}$$

If none of the complex structures admits non-constant holomorphic functions on M, which is true for instance if M is compact, we get that $\psi(V) = 0$.

Proof. We know that $\Omega(V)$ is closed and of type $(0,1)$, and since we have assumed $H^1(M,\mathbb{R}) = 0$, it is exact. Thus there exist a function $\psi(V) \in C^\infty(M)$, such that $\Omega(V) = \bar{\partial}\psi(V)$.

Observe that if none of the complex structures admits non-constant holomorphic functions on M, we get that $\psi(V) = 0$, since the equation $d\psi(V) = \bar{\partial}\psi(V)$ shows that $\psi(V)$ is anti-holomorphic.

Combining expressions we get that

$$\delta(G_\beta(V)) \cdot \omega + 2dF \cdot G_\beta(V) \cdot \omega = 4i\bar{\partial}V'[F] + 2dF \cdot \bar{\partial}\beta(V) + \delta(\bar{\partial}\beta(V)) - \bar{\partial}\psi(V).$$

Now we only need to rewrite the term $\delta(\bar{\partial}\beta(V))$, and this is again done by an application of commuting covariant derivatives, which goes as follows

$$\begin{aligned}\delta(\bar{\partial}\beta(V)) &= \nabla_{u'}\nabla_{a''}\beta(V)^{u'} \\ &= [\nabla,\nabla]_{u'a''}\beta(V)^{u'} + \nabla_{a''}\nabla_{u'}\beta(V)^{u'} \\ &= R^{u'}_{u'a''v'}\beta(V)^{v'} + \bar{\partial}\delta(\beta(V)) \\ &= r_{a''v'}\beta(V)^{v'} + \bar{\partial}\delta(\beta(V)) \\ &= J^u_{a''}r_{uw}J^w_{v'}\beta(V)^{v'} + \bar{\partial}\delta(\beta(V)) \\ &= i\rho_{a''v'}\beta(V)^{v'} + \bar{\partial}\delta(\beta(V)) \\ &= i\rho \cdot \beta(V) + \bar{\partial}\delta(\beta(V)) \\ &= in\omega \cdot \beta(V) - 2\partial\bar{\partial}F \cdot \beta(V) + \bar{\partial}\delta(\beta(V) \\ &= in\omega \cdot \beta(V) + 2\bar{\partial}dF \cdot \beta(V) + \bar{\partial}\delta(\beta(V).\end{aligned}$$

Inserting this above and rewriting

$$\bar{\partial}dF \cdot \beta(V) + dF \cdot \bar{\partial}\beta(V)) = \bar{\partial}(dF \cdot \beta(V)),$$

we obtain the desired equation, which completes the proof. □

We can now insert this in the expression from Lemma 6.5 and we get that

$$\begin{aligned}&\nabla^{(0,1)}(\Delta_{G_\beta(V)}s + 2\nabla_{G_\beta(V)\cdot dF}s) \\ &= -i(2k+n)\omega \cdot G_\beta(V) \cdot \nabla^{(1,0)}s - ik(\omega \cdot \delta(G_\beta(V))s + 2\omega \cdot G_\beta(V) \cdot dFs) \\ &= i(2k+n)(G_\beta(V) \cdot \omega) \cdot \nabla^{(1,0)}s + ik(\delta(G_\beta(V)) \cdot \omega + 2dF \cdot G_\beta(V) \cdot \omega)s \\ &= i(2k+n)(V'[J] + \bar{\partial}\beta(V)) \cdot \nabla^{(1,0)}s \\ &+ ik(4i\bar{\partial}V'[F] + 2\bar{\partial}(dF \cdot \beta(V)) + in\omega \cdot \beta(V) + \bar{\partial}\delta(\beta(V)) - \bar{\partial}\psi(V))s \\ &= 2(2k+n)\frac{i}{2}V'[J] \cdot \nabla^{(1,0)}s + i(2k+n)\bar{\partial}\beta(V) \cdot \nabla^{(1,0)}s \\ &- 4k\bar{\partial}V'[F]s + 2ik\bar{\partial}(dF \cdot \beta(V))s - kn\omega \cdot \beta(V)s + ik\bar{\partial}\delta(\beta(V))s - ik\bar{\partial}\psi(V)s.\end{aligned}$$

Now we only need one last lemma to get rid of the last first-order term on the right side.

Lemma 6.10. *For any family of vector fields $\beta(V) \in C^\infty(M_\sigma, T'M_\sigma)$ and a holomorphic section $s \in H^k_\sigma$, we have that*

$$\nabla^{(0,1)}\nabla_{\beta(V)}s = \bar{\partial}\beta(V) \cdot \nabla^{(1,0)}s - ik\omega \cdot \beta(V)s.$$

Proof. The result follows directly by the following calculation

$$\begin{aligned}
\nabla^{(0,1)}\nabla_{\beta(V)}s &= \bar{\partial}\beta(V) \cdot \nabla^{(1,0)}s + \beta(V)^{u'}\nabla_{a''}\nabla_{u'}s \\
&= \bar{\partial}\beta(V) \cdot \nabla^{(1,0)}s + \beta(V)^{u'}[\nabla, \nabla]_{a''u'}s \\
&= \bar{\partial}\beta(V) \cdot \nabla^{(1,0)}s - \beta(V)^{u'}ik\omega_{a''u'}s \\
&= \bar{\partial}\beta(V) \cdot \nabla^{(1,0)}s - ik\omega \cdot \beta(V)s.
\end{aligned}$$

□

Now using this lemma we get that

$$\begin{aligned}
&\nabla^{(0,1)}(\Delta_{G_\beta(V)}s + 2\nabla_{G_\beta(V)\cdot dF}s - i(2k+n)\nabla_{\beta(V)}s) \\
&\quad = 2(2k+n)\frac{i}{2}V'[J] \cdot \nabla^{(1,0)}s - 4k\bar{\partial}V'[F]s \\
&\quad + 2ik\bar{\partial}(dF \cdot \beta(V))s - 2k(k+n)\omega \cdot \beta(V)s + ik\bar{\partial}\delta(\beta(V))s - ik\bar{\partial}\psi(V)s,
\end{aligned}$$

and now by moving all the 0'th order terms to the left side, we get the desired result. Here $\varphi(V) \in C^\infty(M)$ is a smooth function, such that $\bar{\partial}\varphi(V) = \omega \cdot \beta(V)$, and thus we get that

$$\begin{aligned}
&\nabla^{(0,1)}(\Delta_{G_\beta(V)}s + 2\nabla_{G_\beta(V)\cdot dF}s - i(2k+n)\nabla_{\beta(V)}s \\
&\quad + 4kV'[F]s - 2ikdF \cdot \beta(V)s - ik\delta(\beta(V))s + 2k(k+n)\varphi(V)s + ik\psi(V)s) \\
&\quad = 2(2k+n)\frac{i}{2}V'[J] \cdot \nabla^{(1,0)}s,
\end{aligned}$$

and at last we see, that we get a Hitchin connection by setting

$$\begin{aligned}
u(V) = \frac{1}{2(2k+n)}&(\Delta_{G_\beta(V)} + 2\nabla_{G_\beta(V)\cdot dF} - i(2k+n)\nabla_{\beta(V)} \\
&+ 4kV'[F] - 2ikdF \cdot \beta(V) - ik\delta(\beta(V)) + 2k(k+n)\varphi(V) + ik\psi(V)),
\end{aligned}$$

and thus we have proved the following theorem.

Theorem 6.11 (Hitchin connection for weakly restricted families)**.** *Let (M, ω) be a symplectic manifold with a prequantum line bundle $\mathcal{L}$. Assume that M has first Chern class of the form $c_1(M, \omega) = n\left[\frac{\omega}{2\pi}\right]$ for some integer $n \in \mathbb{Z}$ and such that $H^1(M, \mathbb{R}) = 0$. Furthermore, let $J\colon \mathcal{T} \to C^\infty(M, \mathrm{End}(TM))$ be a weakly restricted, holomorphic family of Kähler structures on M, parametrized by a complex manifold $\mathcal{T}$, and assume that the family admits a family of Ricci potentials F. Then there exists a Hitchin connection $\boldsymbol{\nabla}$ in the bundle $\hat{\mathcal{H}}^{(k)}$ over $\mathcal{T}$, given by the expression*

$$\boldsymbol{\nabla}_V = \nabla^T_V + u(V)$$

where

$$u(V) = \frac{1}{2(2k+n)}(\Delta_{G_\beta(V)} + 2\nabla_{G_\beta(V)\cdot dF} - i(2k+n)\nabla_{\beta(V)} \\ + 4kV'[F] - 2ikdF\cdot\beta(V) - ik\delta(\beta(V)) + 2k(k+n)\varphi(V) + ik\psi(V)),$$

and $\varphi(V), \psi(V) \in C^\infty(M)$ *are smooth functions, satisfying*

$$\bar\partial\varphi(V) = \omega\cdot\beta(V) \quad \textit{and} \quad \bar\partial\psi(V) = \Omega(V),$$

where $\Omega(V) \in \Omega^1(M)$ *is given by*

$$\Omega(V) = -\delta(G_\beta(V))\cdot\omega + \delta(\bar\partial\beta(V)) - 2dF\cdot G_\beta(V)\cdot\omega + 2\bar\partial\beta(V)\cdot dF + 4i\bar\partial V'[F].$$

There are several remarks to be made on this construction, and for clarity we choose to state them as bullet points.

Remark 6.12. If none of the complex structures admit non-constant holomorphic functions on M we get that $\psi(V) = 0$.

Remark 6.13. When M is compact, Hodge theory will provide us with a family of Ricci potentials and of course there will in that case only be constant holomorphic functions globally on M. So we don't need the assumption, that such a family of Ricci potentials exist, and we automatically get $\psi(V) = 0$ from remark 6.12. Thus in this case we can reduce the assumptions in Theorem 6.11 to the two cohomological restrictions

$$H^1(M,\mathbb{R}) = 0 \quad \text{and} \quad c_1(M,\omega) = n\left[\frac{\omega}{2\pi}\right].$$

Remark 6.14. We expect that the Fano type condition

$$c_1(M,\omega) = n\left[\frac{\omega}{2\pi}\right]$$

can be removed by doing metaplectic correction as considered in [AGL12].

Remark 6.15. The condition $H^1(M,\mathbb{R}) = 0$ is only used to ensure that the closed 1-form $\Omega(V)$ is exact, such that $\psi(V)$ exists. In other words, if we already have a $\bar\partial$-primitive for $\Omega(V)$ for all V, this assumption can also be ignored. See details in the proof of Proposition 6.9.

Remark 6.16. When $\beta(V) = 0$, the family is rigid and this new Hitchin connection restricts to the Hitchin connection in [And12].

Remark 6.17. We stress that we do not need that $\mathcal{T}$ is a complex manifold, in fact, we have a complete analog of Theorem 6.11, when $\mathcal{T}$ has no complex structure, and we go through this case in section 6.4 below.

6.3 Uniqueness of the Hitchin connection

In this section, we apply some of our results on differential operators and the techniques, which we will be using in chapter 7, to prove a uniqueness result for Hitchin connections of order at most 2 in the setting of Theorem 6.11.

Theorem 6.18 (Uniqueness of the Hitchin connection)**.** *Assume the setup of Theorem 6.11. Furthermore assume that the contraction map*

$$\omega\cdot : H^0(M, S^2(T'M_\sigma)) \to H^1(M, T'M_\sigma)$$

is injective and $H^0(M, T'M_\sigma) = 0$. Then any Hitchin connection

$$\tilde{\boldsymbol{\nabla}} = \nabla^T + \tilde{u}$$

in $\hat{\mathcal{H}}^k$ of order at most 2 with

$$\tilde{u}(V) = \sum_{i=0}^{2} \nabla^{(i)}_{G_i(V)}$$

is unique and thus $\tilde{u} = u$, where u is given by the expression in Theorem 6.11.

Proof. Assume that we have such a connection. By Lemma 6.2 we must have for all k, that

$$\nabla^{0,1}_\sigma \tilde{u}(V)s = \nabla_{\frac{i}{2}V'[J_\sigma]}s \tag{6.6}$$

for all $s \in H^0(M_\sigma, L^k)$ and all $\sigma \in \mathcal{T}$. By the right hand side of this equation, we see that only the symbol of order 1 for $\nabla^{0,1}_Z u(V)s$ is non-zero, so arguing as in the proof of Theorem 5.7, we get that

$$\begin{aligned}\nabla_{\frac{i}{2}V'[J](Z)}s = \nabla^{0,1}_Z \tilde{u}(V)s &= \nabla^2_{\nabla^{0,1}_Z G_2}s + \nabla_{\nabla^{0,1}_Z G_{N-1}}s \\ &\quad - \nabla_{ik\,\mathrm{tr}(\iota_Z\omega, G_2) + \mathrm{tr}(\iota_Z R, G_2)}s \mod \mathcal{D}^0(M, \mathcal{L}^k),\end{aligned}$$

where the equation for the symbol of order 2 gives us

$$\nabla^{0,1}_Z G_2(V) = 0.$$

Now to fit the situation of our construction in Theorem 6.11, we rewrite

$$\tilde{u}(V) = \sum_{i=0}^{2} \nabla^{(i)}_{G_i(V)} = \Delta_{G_2(V)} + \nabla_{\tilde{\beta}(V)} + \tilde{\psi}(V).$$

We can obviously do this by letting $\tilde{\beta}(V) = G_1(V) - \delta(G_2(V))$ and $\tilde{\psi}(V) = G_0(V)$.

Continuing our analysis using corollary 6.4, we get for any section $s \in H^0(M_\sigma, L^k)$ and any vector field $Z \in C^\infty(M, T''M_\sigma)$, that

$$\begin{aligned}\nabla^{0,1}_Z \tilde{u}(V)s = &- i(2k+n)\nabla_{\iota_Z\omega\cdot G_2(V)}s + 2\nabla_{\iota_Z(d\bar{\partial}_\sigma F_\sigma)\cdot G_2(V)}s + \nabla_{[Z,\tilde{\beta}(V)]}s \\ &- ik\omega(Z, \delta(G_2(V)))s - ik\omega(Z, \tilde{\beta}(V))s + Z[\tilde{\psi}(V)]s.\end{aligned}$$

So we see by theorem 5.6 and equation (6.6) that

$$\frac{i}{2}V'[J_\sigma](Z) = -i(2k+n)\iota_Z\omega \cdot G_2(V) + 2\iota_Z(d\bar{\partial}_\sigma F_\sigma)\cdot G_2(V) + [Z, \tilde{\beta}(V)].$$

The last term just gives $[Z, \tilde{\beta}(V)] = Z[\tilde{\beta}(V)] = \bar{\partial}(\tilde{\beta}(V))(Z)$, and since the equation holds for all such Z, we get

$$\frac{i}{2}V'[J_\sigma] = -i(2k+n)\omega \cdot G_2(V) + 2(d\bar{\partial}_\sigma F_\sigma) \cdot G_2(V) + \bar{\partial}\tilde{\beta}(V). \tag{6.7}$$

Since $G_2(V)$ is holomorphic, we see that

$$2(d\bar{\partial}_\sigma F_\sigma) \cdot G_2(V) = -2\bar{\partial}_\sigma(\partial_\sigma F_\sigma \cdot G_2(V)).$$

Since the map

$$\omega : H^0(M, S^2(T'M_\sigma) \to H^1(M, T'M_\sigma)$$

is injective, we see from equation (6.7), that first of all $G_2(V)$ is uniquely determined by projecting this equation onto $H^1(M, T'M_\sigma)$. Moreover, we see that then $\tilde{\beta}(V)$ is uniquely determined by equation (6.7), since $H^0(M, T'M_\sigma) = 0$.

Now, we can also conclude that

$$\nabla^{0,1}_Z \tilde{u}(V) - \nabla_{\frac{i}{2}V'[J_\sigma](Z)} s = -ik\omega(Z, \delta(G_2(V)))s - ik\omega(Z, \tilde{\beta}(V))s + Z[\tilde{\psi}(V)]s$$

for all $Z \in C^\infty(M, T''M_\sigma)$, but this must be the zero operator by equation (6.7). Thus we must have that

$$-ik\omega \cdot \delta(G_2(V)) - ik\omega \cdot \tilde{\beta}(V) + \bar{\partial}\tilde{\psi}(V) = 0.$$

Now we see that $\tilde{\psi}(V)$ is uniquely determined by this equation, since $H^0(M, T'M_\sigma) = 0$ implies that M has only constant holomorphic functions. This way we have concluded that all the symbols and thus the operator $\tilde{u}$ is determined uniquely, and thus it must be the u constructed in Theorem 6.11. □

6.4 Hitchin connection for smooth families of complex structures

We have in the above constructions of the Hitchin connection assumed that the family of complex structures is holomorphic. We can however go through the construction without assuming that $\mathcal{T}$ is a complex manifold. We have not used holomorphicity of the family before assuming it to be weakly restricted, so instead of differentiation along V' in Lemma 6.8, we instead differentiate along V.

Doing this we get that the form

$$\begin{aligned} &-\delta(G_\beta(V)) \cdot \omega - 2dF \cdot G_\beta(V) \cdot \omega + \delta(\bar{\partial}\beta(V)) + 2\bar{\partial}\beta(V) \cdot dF + 4i\bar{\partial}V[F] \\ &-\delta(\bar{G}(V)) \cdot \omega - 2dF \cdot \bar{G}(V) \cdot \omega \end{aligned}$$

is closed and hence exact. It is however no longer of type $(0,1)$, but it splits into a $(1,0)$ and a $(0,1)$ part, which come as ∂ and $\bar{\partial}$ of a function $\tilde{\psi}(V) \in C^\infty(M)$. Both of the new terms are of type $(1,0)$, so we get similarly as above

$$\bar{\partial}\tilde{\psi}(V) = -\delta(G_\beta(V)) \cdot \omega + \delta(\bar{\partial}\beta(V)) - 2dF \cdot G_\beta(V) \cdot \omega + 2\bar{\partial}\beta(V) \cdot dF + 4i\bar{\partial}V[F].$$

Arguing as in the proof of Proposition 6.9, we get that

$$\begin{aligned} &\delta(G_\beta(V)) \cdot \omega + 2dF \cdot G_\beta(V) \cdot \omega \\ &= 4i\bar{\partial}V[F] + 2\bar{\partial}(dF \cdot \beta(V)) + in\omega \cdot \beta(V) + \bar{\partial}\delta(\beta(V)) - \bar{\partial}\psi(V). \end{aligned}$$

Now going through the construction of the Hitchin Connection as above, still assuming weakly restricted but without holomorphicity of the family of Kähler structures, we get the following theorem.

Theorem 6.19 (Hitchin Connection for smooth $\mathcal{T}$)**.** *Consider the same setup as in Theorem 6.11, except the manifold $\mathcal{T}$ is only assumed to be smooth and the assumption of holomorphicity of the family J is dropped. Then there exists a Hitchin connection $\boldsymbol{\nabla}$ in the bundle $\hat{\mathcal{H}}^{(k)}$ over $\mathcal{T}$, given by the expression*

$$\boldsymbol{\nabla}_V = \nabla^T_V + u(V),$$

where

$$u(V) = \frac{1}{2(2k+n)}(\Delta_{G_\beta(V)} + 2\nabla_{G_\beta(V)\cdot dF} - i(2k+n)\nabla_{\beta(V)} + 4kV[F] - 2ikdF\cdot\beta(V) - ik\delta(\beta(V)) + 2k(k+n)\varphi(V) + ik\tilde{\psi}(V)),$$

where $\varphi(V)$ is defined as in Theorem 6.11 and $\tilde{\psi}(V) \in C^\infty(M)$ satisfies

$$\bar{\partial}\tilde{\psi}(V) = \tilde{\Omega}(V),$$

and $\tilde{\Omega}(V) \in \Omega^1(M)$ is the closed and hence exact 1-form

$$\tilde{\Omega}(V) = -\delta(G_\beta(V))\cdot\omega + \delta(\bar{\partial}\beta(V)) - 2dF\cdot G_\beta(V)\cdot\omega + 2\bar{\partial}\beta(V)\cdot dF + 4i\bar{\partial}V[F].$$

CHAPTER **7**

Hitchin Connections for General Families of Kähler Structures

This chapter is dedicated to show some new results on the existence and uniqueness of a Hitchin connection for general families of Kähler strutures without assuming any restrictions on the family itself, but giving conditions on cohomology involving the map $\Psi^{(n)}$ defined in chapter 5.

7.1 Existence of a Hitchin Connection

We assume the same setup as in chapter 6 and go straight to the theorems.

Theorem 7.1. *Let (M, ω) be a symplectic manifold with a prequamtum line bundle $\mathcal{L}$ and let $J\colon \mathcal{T} \to C^\infty(M, \mathrm{End}(TM))$ be a family of Kähler structures on M, parametrized by a complex manifold $\mathcal{T}$. If $H^1(M, \mathbb{R}) = 0$, then an order 2 Hitchin connection $\boldsymbol{\nabla} = \nabla^T + u$ with*

$$u(V) = \sum_{i=0}^{2} \nabla^{(i)}_{G_i(V)}$$

on $\hat{\mathcal{H}}^k$ exists if and only if

$$[V'[J]] \in I^{(2)}$$

for all $V \in C^\infty(\mathcal{T}, T\mathcal{T})$. If so, a Hitchin connection is given by

$$\begin{aligned} \boldsymbol{\nabla}_V &= \nabla^T_V + u(V) \\ u(V) &= \nabla^2_{G_2(V)} + \nabla_{\delta(G_2(V))} - \nabla_{\beta(V)} - \psi(V), \end{aligned}$$

where

$$V'[J] = \Psi^{(n)}(2iG_2(V)) + \bar{\partial}(2i\beta(V)),$$

and $\psi(V)$ is a function, such that

$$\bar{\partial}\psi(V) = ik\iota_{\delta(G_2(V))-\beta(V)}\omega.$$

Furthermore, if the Hitchin connection exists, it is projectively flat, if $K^{(n)}_\sigma = 0$ for $n = 1, 2, 3$ and all $\sigma \in \mathcal{T}$.

Proof of Theorem 7.1. Assume that $\boldsymbol{\nabla} = \nabla^T + u$ is a hitchin connection on $H^{(k)}$ of order at most 2. Then we can write

$$u(V) = \sum_{i=0}^{2} \nabla^{(i)}_{G_i(V)},$$

and since $\boldsymbol{\nabla}$ is a Hitchin connection, we know by Lemma 6.2, that for any $Z \in C^\infty(M, T''M)$, we have

$$\nabla^{0,1}_Z u(V)s = \nabla_{\frac{i}{2}V'[J](Z)} s.$$

By the right hand side of this equation, we see that only the symbol of order 1 for $\nabla^{0,1}_Z u(V)s$ is non-zero, so arguing as in the proof of Theorem 5.7, we get that

$$\begin{aligned}\nabla_{\frac{i}{2}V'[J](Z)} s = \nabla^{0,1}_Z u(V)s &= \nabla^2_{\nabla^{0,1}_Z G_2} s + \nabla_{\nabla^{0,1}_Z G_{N-1}} s \\ &\quad - \nabla_{ik\,\mathrm{tr}(\iota_Z\omega, G_2) + \mathrm{tr}(\iota_Z R, G_2)} s \mod \mathcal{D}^0(M, \mathcal{L}^k).\end{aligned}$$

where the equation for the symbol of order 2 gives us

$$\nabla^{0,1}_Z G_2(V) = 0,$$

which again means we can apply corollary 6.4 as well, such that we get

$$\begin{aligned}\nabla_{\frac{i}{2}V'[J](Z) + \nabla^{0,1}_Z \delta(G_2(V))} s &= \nabla^{0,1}_Z (\nabla_{\frac{i}{2}V'[J](Z) + \delta(G_2(V))}) s \\ &= \nabla^{0,1}_Z (u(V) + \nabla_{\delta(G_2(V))}) s \\ &= \nabla^2_{\nabla^{0,1}_Z G_2(V)} s + \nabla_{\nabla^{0,1}_Z G_1} s - \nabla_{\Psi^{(2)}_Z(G_2(V))} s \mod \mathcal{D}^0(M, \mathcal{L}^k),\end{aligned}$$

For the symbol of order $n = 1$, this gives us the new equation

$$\frac{i}{2} V'[J](Z) + \nabla^{0,1}_Z \delta(G_2(V)) = \nabla^{0,1}_Z G_1 - \Psi^{(2)}_Z(G_2(V))$$

and rewriting this gives

$$V'[J] = \Psi^{(2)}_Z(2iG_2(V)) + \nabla^{0,1}_Z(-2iG_1(V) + \delta(2iG_2(V))),$$

which proves that $[V'[J]] \in I^{(2)}$.

To prove the other implication, let $V \in C^\infty(\mathcal{T}, T\mathcal{T})$ and assume that $[V'[J]] \in I^{(2)}$. Now we can choose $G_2(V) \in H^0(M, \mathcal{S}^2(T'M))$ and $\beta(V) \in C^\infty(M, T'M)$, such that

$$V'[J] = \Psi^{(n)}(2iG_2(V)) + \bar{\partial}(2i\beta(V)).$$

Now we need to redo the calculation of $\nabla^{0,1}_Z(\nabla^2_{G_2(V)} + \nabla_{\delta(G_2(V))})s$ for a section $s \in H^0(M_\sigma, \mathcal{L}^k)$, since we need to have an exact result including constant terms. This is, however, exactly the result of Lemma 6.3, since we remember that

$$\Delta_{G_2(V)} = \nabla^2_{G_2(V)} + \nabla_{\delta(G_2(V))},$$

so rewriting this result a bit to fit the notation of this chapter, we get

$$\begin{aligned}&\nabla^{0,1}_Z(\nabla^2_{G_2(V)} + \nabla_{\delta(G_2(V))})s \\ =&\nabla^{0,1}_Z(\Delta_{G_2(V)})s \\ =&\nabla_{-2ik\iota_Z\omega\cdot G_2(V)} s + \nabla_{-i\iota_Z\rho\cdot G_2(V)} s - ik\omega(Z, \delta(G_2(V))) \\ =&\nabla_{\frac{i}{2}\Psi^{(2)}_Z(2iG_2(V))} s - ik\omega(Z, \delta(G_2(V)))s.\end{aligned}$$

We remember that Lemma 6.10 shows that

$$\nabla^{0,1}_Z(\nabla_{\beta(V)})s = -ik\omega(Z,\beta(V))s + \nabla_{\nabla^{0,1}_Z\beta(V)}s = -ik\omega(Z,\beta(V))s + \nabla_{(\bar\partial\beta(V))(Z)}s,$$

and combining with the above we get

$$\begin{aligned}&\nabla^{0,1}_Z(\nabla^2_{G_2(V)} + \nabla_{\delta(G_2(V))} - \nabla_{\beta(V)})s\\ =&\nabla_{\frac{i}{2}(\Psi^{(2)}_Z(2iG_2(V))+\bar\partial(2i\beta(V))(Z))}s - ik\omega(Z,\delta(G_2(V)))s + ik\omega(Z,\beta(V))s\\ =&\nabla_{\frac{i}{2}V'[J](Z)}s + ik\iota_{\delta(G_2(V))-\beta(V)}\omega(Z)s.\end{aligned}$$

This is almost the requirement for a differential operator from Lemma 6.2. We do however need to find a function $\psi(V)$, such that $\bar\partial\psi(V) = ik\iota_{\delta(G_2(V))-\beta(V)}\omega$. Given this, we get a Hitchin connection by letting

$$u(V) = \nabla^2_{G_2(V)} + \nabla_{\delta(G_2(V))} - \nabla_{\beta(V)} - \psi(V).$$

We observe that $\iota_{\delta(G_2(V))-\beta(V)}\omega \in \Omega^{0,1}(M)$, and since $H^1(M) = 0$, it is enough to show $\bar\partial\iota_{\delta(G_2(V))-\beta(V)}\omega = 0$. To do this we use that

$$\bar\partial\nabla^{0,1}(\nabla^2_{G_2(V)} + \nabla_{\delta(G_2(V))} - \nabla_{\beta(V)}) = \bar\partial^2(\nabla^2_{G_2(V)} + \nabla_{\delta(G_2(V))} - \nabla_{\beta(V)}) = 0,$$

and thus it is enough to show that the other term on the right side of the equation vanishes, that is

$$\bar\partial\nabla_{\frac{i}{2}V'[J]} = 0.$$

On $\Omega^{0,1}(M)$, the $\bar\partial$-operator is just the antisymmetrization of $\nabla^{0,1}$. We know already that

$$\nabla^{0,1}\nabla_{V'[J]}s = \nabla_{\nabla^{0,1}V'[J]}s + ik\iota_{V'[J]}\omega s.$$

The antisymmetrization $\mathcal{A}$ of each of these terms are zero. Firstly

$$\mathcal{A}(\nabla_{\nabla^{0,1}V'[J]}) = \nabla_{\bar\partial V'[J]},$$

and Hitchin already showed that $\bar\partial V'[J] = 0$ in general (See [Hit90] eq. 1.12). To see that the second term is zero, we remark that

$$\omega(JX, JY) = \omega(X,Y),$$

so differentiating this equation along V', gives

$$\omega(V'[J]X, JY) + \omega(JX, V'[J]Y) = 0.$$

Now choose two vector fields $Z_1, Z_2 \in C^\infty(M, T''M_\sigma)$, such that $JZ_j = -iZ_j$ for $j = 1,2$. This gives us

$$\begin{aligned}0 &= \omega(V'[J]Z_1, Z_2) + \omega(Z_1, V'[J]Z_2)\\ &= \omega(V'[J]Z_1, Z_2) - \omega(V'[J]Z_2, Z_1) = \mathcal{A}(\nabla^{0,1}\iota_{V'[J]})(Z_1,Z_2),\end{aligned}$$

and since Z_1, Z_2 was chosen randomly, this proves the claim.

That the connection is projectively flat if $K^{(n)} = 0$ for $n = 1, 2, 3$ follows directly from Theorem 5.7, since the curvature is apriori a differential operator of order at most 3 that preserves the subspace of holomorphic sections $H^0(M_\sigma, \mathcal{L}^k)$. Furthermore the curvature is the commutator of differential operators on the form (5.2) as in this theorem, and thus it is on this form itself, so the theorem applies. □

Finally we present one very general existence result.

Theorem 7.2. *Let (M, ω) be a symplectic manifold with a prequamtum line bundle $\mathcal{L}$ and let $J\colon \mathcal{T} \to C^\infty(M, \mathrm{End}(TM))$ be a family of Kähler structures on M, parametrized by a complex manifold $\mathcal{T}$.*

If $K_\sigma^{(n)} = 0$ for all $n \geq 2$ and all $\sigma \in \mathcal{T}$, then for any Hitchin connection $\boldsymbol{\nabla} = \nabla^T + u$ with

$$u(V) = \sum_{i=0}^{N} \nabla^{(i)}_{G_i(V)}$$

on $\hat{\mathcal{H}}^k$, the order $N \leq 2$.

Furthermore if $H^0(M, T'M) = 0$ and a Hitchin connection of this form exists on $\hat{\mathcal{H}}^k$, it is unique up to addition of a scalar and projectively flat.

Proof of Theorem 7.2. Let $\boldsymbol{\nabla} = \nabla^T + u$ be a Hitchin connection on $\hat{\mathcal{H}}^k$ with

$$u(V) = \sum_{i=0}^{N} \nabla^{(i)}_{G_i(V)}.$$

Again we know from Lemma 6.2 that

$$\nabla^{0,1}_Z u(V)s = \nabla_{\frac{i}{2}V'[J](Z)}s.$$

For $i \geq 3$, we can argue exactly as in the proof of 5.7, since the symbols of $\nabla^{0,1}_Z u(V)$ of order i and $i-1$ vanish. This shows that $\boldsymbol{\nabla}$ can be of order at most 2.

Now assume $H^0(M, T'M) = 0$, and assume we have another Hitchin connection

$$\tilde{\boldsymbol{\nabla}} = \nabla^T + \tilde{u}(V)$$

with $\tilde{u}$ on the same form. Now the difference $D = \boldsymbol{\nabla} - \tilde{\boldsymbol{\nabla}} = u(V) - \tilde{u}(V)$ is a differential operator, again of the same form, which preserves $H^{(k)}$, and since $K^{(n)} = 0$ for all $n \geq 1$, we get from Corollary 5.8 that D vanish projectively.

We get immediately from Theorem 7.1 that $\boldsymbol{\nabla}$ is projectively flat, since it is of order 2. □

Chapter 8

Curvature of the Hitchin Connection in the Weakly Restricted Case

In this section, we continue the study of the Hitchin connection in the weakly restricted case, which Andersen and I constructed in [AR16]. In this paper, we did however not include calculations of the curvature. We realised, by applying a no-go theorem from [GM00], that the connection could not be projectively flat in general, see section 9.1 for the proof. As mentioned in the article, it is however still interesting to calculate the curvature and investigate, what constitutes the obstruction to the projective flatness.

We have spend quite some time investigating the curvature using different approaches. The best result so far, is a direct calculation of the symbols, which gives a rather simple expression for the third and second order symbols, but still leaves quite a big expression for the first order symbol. By splitting each of these symbols into orders of $(2k+n)$, we however get som simplification, and since each symbol of order $1, 2$ and 3 vanish for all k if and only if the curvature is projectively flat, we get a series of equation, each of which vanish if and only if the curvature is projectively flat.

8.1 Symmetry Property

As described earlier, Andersen and Gammalgaard proved in [AG14] that the Hitchin connection associated to a rigid family is projectively flat, if each complex structure in the family have zero-dimensional symmetry group. Their proof is done by a lengthy direct computation of the symbols of commutators of the differential operators in the explicit expression for the connection, followed by realising that different leftover terms appear as derivatives of certain forms.

The first thing to notice is that the the Hitchin connection in both the rigid and weakly restricted case, only have one term of order 2, namely $\Delta_{G(V)}$ and $\Delta_{G_\beta(V)}$ respectively, and thus the principal symbol of the curvature $F_{\boldsymbol{\nabla}}(V, W)$ only depends on the commutator $[\Delta_{G(V)}, \Delta_{G(W)}]$. We show below that the vanishing of this principal symbal is equivalent to the symmetry of the expression $\mathcal{S}(G_\beta(V)\cdot\nabla G_\beta(W))$, which in the rigid case is proven to hold in the Proposition 3.7. The proof relies heavily on $G(V)$ being the $(2,0)$ part of $\widetilde{G}(V)$, which we can not expect to replicate in the weakly restricted case. Instead we will just state that the Hitchin connection in the weakly restricted case can only be flat, if

$$\mathcal{S}(G_\beta(V)\cdot\nabla G_\beta(W)) = \mathcal{S}(G_\beta(W)\cdot\nabla G_\beta(V)), \tag{8.1}$$

and thus we will assume this in the following proposition, which holds without assumptions in the rigid case.

Proposition 8.1. *If the principal symbols for the Hitchin connection for a weakly restricted family of complex structures J has the symmetry property (8.1), then the four two forms Γ_j on $\mathcal{T}$, with values in symmetric contravariant tensors on M, defined by*

$$\begin{aligned}
\Gamma_3(V,W) &= \mathcal{S}(G_\beta(V)\cdot\nabla G_\beta(W))\\
\Gamma_2(V,W) &= \Delta_{G_\beta(V)}G_\beta(W) + 2\,\mathcal{S}(G_\beta(V)\cdot\nabla\delta G_\beta(W))\\
\Gamma_1(V,W) &= 2\Delta_{G_\beta(V)}\delta G_\beta(W) + G_\beta(V)\cdot d\delta\delta G_\beta(W) + \nabla_w(G_\beta(V)^{uv})\nabla_u\nabla_v(G_\beta(W)^{wa})\\
\Gamma_0(V,W) &= \Delta_{G_\beta(V)}\delta\delta G_\beta(W) + \nabla_w(G_\beta(V)^{uv})\nabla_u\nabla_v\delta G_\beta(W)^w,
\end{aligned}$$

are all symmetric in the vector fields V and W on $\mathcal{T}$.

Proof. The idea, which is explained in the similar proposition in [AG14], is that taking the divergence of the contravariant trivector field part of Γ_3 doesn't change the symmetry property, and thus gives a new symmetric two form with values in symmetric bivector fields and so on.

Firstly lets us write out the symmetrization defining Γ_3.

$$\begin{aligned}
\Gamma_3(V,W) &= \mathcal{S}(G_\beta(V)\cdot\nabla G_\beta(W))^{abc}\\
&= \frac{1}{3}\left(G_\beta(V)^{av}\nabla_v G_\beta(W)^{bc} + G_\beta(V)^{bv}\nabla_v G_\beta(W)^{ac} + G_\beta(V)^{cv}\nabla_v G_\beta(W)^{ab}\right),
\end{aligned}$$

where we have gathered the 6 initial terms in pairs by the symmetry of $G_\beta(W)$. Now we take the divergence of this expression.

$$\begin{aligned}
\delta(\Gamma_3(V,W)) &= \frac{1}{3}\delta\left(G_\beta(V)^{av}\nabla_v G_\beta(W)^{bc} + G_\beta(V)^{bv}\nabla_v G_\beta(W)^{ac} + G_\beta(V)^{cv}\nabla_v G_\beta(W)^{ab}\right)\\
&= \frac{1}{3}(\Delta_{G_\beta(V)}G_\beta(W)\\
&\quad + \nabla_u G_\beta(V)^{bv}\nabla_v G_\beta(W)^{uc} + G_\beta(V)^{bv}\nabla_u\nabla_v G_\beta(W)^{uc}\\
&\quad + \nabla_u G_\beta(V)^{cv}\nabla_v G_\beta(W)^{ub} + G_\beta(V)^{cv}\nabla_u\nabla_v G_\beta(W)^{ub})\\
&= \frac{1}{3}(\Delta_{G_\beta(V)}G_\beta(W)\\
&\quad + \nabla_u G_\beta(V)^{bv}\nabla_v G_\beta(W)^{uc} + \nabla_u G_\beta(V)^{cv}\nabla_v G_\beta(W)^{ub}\\
&\quad + G_\beta(V)^{bv}\nabla_u\nabla_v G_\beta(W)^{uc} + G_\beta(V)^{cv}\nabla_u\nabla_v G_\beta(W)^{ub})\\
&= \frac{1}{3}(\Delta_{G_\beta(V)}G_\beta(W)\\
&\quad + \nabla_u G_\beta(V)^{bv}\nabla_v G_\beta(W)^{uc} + \nabla_u G_\beta(V)^{cv}\nabla_v G_\beta(W)^{ub}\\
&\quad + \cancel{G_\beta(V)^{bv}[\nabla,\nabla]_{uv}G_\beta(W)^{uc} + G_\beta(V)^{cv}[\nabla,\nabla]_{uv}G_\beta(W)^{ub}}\\
&\quad + G_\beta(V)^{bv}\nabla_v\nabla_u G_\beta(W)^{uc} + G_\beta(V)^{cv}\nabla_v\nabla_u G_\beta(W)^{ub})
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{3}(\Delta_{G_\beta(V)} G_\beta(W) + 2\,\mathcal{S}(G_\beta(V)\cdot\nabla\delta G_\beta(W)) \\
&+ \nabla_u G_\beta(V)^{bv}\nabla_v G_\beta(W)^{uc} + \nabla_u G_\beta(V)^{cv}\nabla_v G_\beta(W)^{ub}) \\
&= \frac{1}{3}(\Gamma_2(U,V) + \nabla_u G_\beta(V)^{bv}\nabla_v G_\beta(W)^{uc} + \nabla_u G_\beta(V)^{cv}\nabla_v G_\beta(W)^{ub}),
\end{aligned}
$$

where the remaining term is clearly seen to be symmetric in V and W, and since we know $\delta(\Gamma_3(V,W))$ is symmetric, we get as desired that Γ_2 is also symmetric. In the computation, we have used that the Levi-Civita Curvature is of type $(1,1)$, and since G_β is always of type $(2,0)$, all the curvature terms contracted with these vanish.

Now we continue using the exact same idea and take the divergence of Γ_2. In the following we will not write out all the commutators that disappear.

$$
\begin{aligned}
&\delta(\Gamma_2(V,W)) \\
&= \delta(\Delta_{G_\beta(V)} G_\beta(W) + 2\,\mathcal{S}(G_\beta(V)\cdot\nabla\delta G_\beta(W))) \\
&= \nabla_w\nabla_u G_\beta(V)^{uv}\nabla_v G_\beta(W)^{wa} \\
&+ \nabla_w G_\beta(V)^{wv}\nabla_v\delta(G_\beta(W))^a \\
&+ \nabla_w G_\beta(V)^{av}\nabla_v\delta(G_\beta(W))^w \\
&= \nabla_u G_\beta(V)^{uv}\nabla_v\delta G_\beta(W)^a + \nabla_w(G_\beta(V)^{uv})\nabla_u\nabla_v G_\beta(W)^{wa} + \nabla_w(\delta G_\beta(V))^v\nabla_v G_\beta(W)^{wa} \\
&+ \Delta_{G_\beta(V)}\delta(G_\beta(W)) \\
&+ \nabla_w(G_\beta(V)^{av})\nabla_v\delta(G_\beta(W))^w + G_\beta(V)^{av}\nabla_v\nabla_w\delta(G_\beta(W))^w \\
&= 2\Delta_{G_\beta(V)}\delta(G_\beta(W)) + \nabla_w(G_\beta(V)^{uv})\nabla_u\nabla_v G_\beta(W)^{wa} + G_\beta(V)\cdot d\delta\delta(G_\beta(W)) \\
&+ \nabla_w(\delta G_\beta(V))^v\nabla_v G_\beta(W)^{wa} + \nabla_v\delta(G_\beta(W))^w\nabla_w(G_\beta(V)^{va}) \\
&= \Gamma_1(V,W) + \nabla_w(\delta G_\beta(V))^v\nabla_v G_\beta(W)^{wa} + \nabla_v\delta(G_\beta(W))^w\nabla_w(G_\beta(V)^{va}),
\end{aligned}
$$

and again the leftover term is clearly seen to be symmetric. We continue the procedure one last time and calculate

$$
\begin{aligned}
&\delta(\Gamma_1(V,W)) \\
&= \delta(2\Delta_{G_\beta(V)}\delta(G_\beta(W)) + \nabla_w(G_\beta(V)^{uv})\nabla_u\nabla_v G_\beta(W)^{wa} + G_\beta(V)\cdot d\delta\delta(G_\beta(W))) \\
&= 2\nabla_z\nabla_u G_\beta(V)^{uv}\nabla_v\delta G_\beta(W)^z \\
&+ \nabla_z\nabla_w(G_\beta(V)^{uv})\nabla_u\nabla_v G_\beta(W)^{wz} \\
&+ \nabla_z G_\beta(V)^{zu}\nabla_u\delta\delta G_\beta(W)) \\
&= 2\Delta_{G_\beta(V)}\delta\delta G_\beta(W) + 2\nabla_z(\delta G_\beta(V))^v\nabla_v\delta(G_\beta(W))^z + 2\nabla_z(G_\beta(V)^{uv})\nabla_u\nabla_v\delta G_\beta(W)^z \\
&+ \nabla_z(\nabla_w G_\beta(V)^{uv})\nabla_u\nabla_v G_\beta(W)^{wz} + \nabla_w(G_\beta(V)^{uv})\nabla_u\nabla_v\delta G_\beta(W)^w \\
&+ \Delta_{G_\beta(V)}\delta\delta G_\beta(W) \\
&= 3\Delta_{G_\beta(V)}\delta\delta G_\beta(W) + 3\nabla_w(G_\beta(V)^{uv})\nabla_u\nabla_v\delta G_\beta(W)^w \\
&+ 2\nabla_z(\delta G_\beta(V))^v\nabla_v\delta(G_\beta(W))^z + \nabla_z(\nabla_w G_\beta(V)^{uv})\nabla_u\nabla_v G_\beta(W)^{wz} \\
&= 3\Gamma_0(V,W) + 2\nabla_z(\delta G_\beta(V))^v\nabla_v\delta(G_\beta(W))^z + \nabla_z(\nabla_w G_\beta(V)^{uv})\nabla_u\nabla_v G_\beta(W)^{wz},
\end{aligned}
$$

which finishes the proof. □

8.2 Commutators of Differential Operators

Our explicit description of the Hitchin connection in the weakly restricted case, as in the rigid case, is in terms of differential operators, and the direct calculation of the curvature is a very long computation involving commutators of various differential operators on sections of the prequantum line bundle $\mathcal{L}$ and its tensor powers. Therefore, we divide the calculation by writing each operator in the form of (5.1), and using the following general formulas for the curvature of differential operators on this form in terms of their symbols.

This is the idea, that Andersen and Gammelgaard used to calculated the curvature of the Hitchin and the Hitchin-Witten connection in [AG14], and since the following commutators are calculated for the generel case, we can use these results directly. We will not go through the explicit calculation but state the lemmas.

Firstly the commutator of two first order operators is of course given directly by the definition of the curvature of the line bundle $\mathcal{L}^k$, i.e.

$$[\nabla_X, \nabla_Y]s = \nabla_{[X,Y]}s - ik\omega(X,Y)s$$

for any vector fields X, Y on M and any smooth section $s \in \mathcal{H}^{(k)}$. However this quickly becomes more complicated, when we introduce second-order operators. The following lemma gives the commutator for a second order and a first order operator.

Lemma 8.2. *On any symplectic manifold (M,ω) with a Kähler structure J, any vector field X on M, and any symmetric bivector field $B \in C^\infty(M, S^2(TM))$, we have the symbols*

$$\begin{aligned}
\sigma_2\big[\nabla^2_B, \nabla_X\big] &= 2\,\mathcal{S}(B\cdot\nabla X) - \nabla_X B \\
\sigma_1\big[\nabla^2_B, \nabla_X\big] &= \nabla^2_B X - 2ikB\cdot\omega\cdot X + B^{uv}R^a_{wuv}X^w \\
\sigma_0\big[\nabla^2_B, \nabla_X\big] &= -ik\,\omega(B\cdot\nabla X)
\end{aligned}$$

for the commutator of the operators ∇_X and ∇^2_B acting on $\mathcal{H}^{(k)}$.

The proof is done in [AG14], and is done by direct computation and then extracting the value of the symbols. The next lemma gives the commutator for two second-order operator, which is vastly more complicated calculation, which is however done in a similar fashion.

Lemma 8.3. *On any symplectic manifold (M,ω) with a Kähler structure J, for any symmetric bivector fields $A, B \in C^\infty(M, S^2(TM))$, we have the symbols*

$$\begin{aligned}
\sigma_3\big[\nabla^2_A, \nabla^2_B\big] &= 2\,\mathcal{S}(A\cdot\nabla B) - 2\,\mathcal{S}(B\cdot\nabla A) \\
\sigma_2\big[\nabla^2_A, \nabla^2_B\big] &= \nabla^2_A B - \nabla^2_B A - 4ik\,\mathcal{S}(A\cdot\omega\cdot B) \\
&\quad + 2\,\mathcal{S}(A^{xy}R^a_{uxy}B^{ub}) - 2\,\mathcal{S}(B^{uv}R^a_{xuv}A^{xb}) \\
\sigma_1\big[\nabla^2_A, \nabla^2_B\big] &= -2ikA^{xy}\omega_{yu}\nabla_x(B^{ua}) + 2ikB^{uv}\omega_{vx}\nabla_u(A^{xa}) \\
&\quad - A^{xy}\nabla_x(R^a_{yuv})B^{uv} + B^{uv}\nabla_v(R^a_{uxy})A^{xy} \\
&\quad - \frac{4}{3}A^{xy}R^a_{xuv}\nabla_y B^{uv} + \frac{4}{3}B^{uv}R^a_{uxy}\nabla_v A^{xy}
\end{aligned}$$

$$\sigma_0\big[\nabla^2_A, \nabla^2_B\big] = \frac{ik}{2} A^{xy} J^j_y R_{xuvj} B^{uv} - \frac{ik}{2} B^{uv} J^j_v R_{uxyj} A^{xy}$$

for the commutator of the operators ∇^2_A *and* ∇^2_B *acting on* $\mathcal{H}^{(k)}$.

8.3 Commutators in The Weakly Restricted Case

In this section we use the lemmas from the previous section to calculate the symbols for each of the commutators that appears in the curvature of the Hitchin connection in the weakly restricted case. Remember that we have the expression

$$\boldsymbol{\nabla}_V = \nabla^T_V + u(V),$$

where we rewrite

$$u(V) = \frac{1}{2(2k+n)}(\nabla^2_{s_2(V)} + \nabla_{s_1(V)} + s_0(V)),$$

where

$$\begin{aligned} s_2(V) &= G_\beta(V) \\ s_1(V) &= \delta G_\beta(V) + 2G_\beta(V)\cdot dF - i(2k+n)\beta(V) \\ s_0(V) &= 4kV'[F] - 2ikdF\cdot\beta(V) - ik\delta(\beta(V)) + 2k(k+n)\varphi(V) + ik\psi(V) \end{aligned}$$

Now to calculate the curvature, we want to find an expression for

$$[\nabla^T_V + u(V), \nabla^T_W + u(W)] = [\nabla^T_V, \nabla^T_W] + [\nabla^T_V, u(W)] - [\nabla^T_W, u(V)] + [u(V), u(W)],$$

and we split this computation into parts. The commutator of the trivial connection is zero, so we can forget about that one, and we carry on to calculate the next two commutators. These are fairly simple, and the following lemma gives the result.

Lemma 8.4. *For any weakly restricted family of Kähler structures, the symbols of*

$$C_0 = 2(2k+n)([\nabla^T_V, u(W)] - [\nabla^T_W, u(V)])$$

acting on $\mathcal{H}^{(k)}$*, are given as follows*

$$\begin{aligned} \sigma_2(C_0) =& V[G_\beta(W)] - W[G_\beta(V)] \\ \sigma_1(C_0) =& 2V[G_\beta(W)]\cdot dF + 2G_\beta(W)\cdot dV[F] + \delta V[G_\beta(W)] - i(2k+n)V[\beta(W)] \\ & -2W[G_\beta(V)]\cdot dF - 2G_\beta(V)\cdot dW[F] - \delta W[G_\beta(V)] + i(2k+n)W[\beta(V)] \\ \sigma_0(C_0) =& ik(V[\psi(W)] - W[\psi(V)]) \\ & +2k(k+n)(V[\varphi(W)] - W[\varphi(V)]) \\ & -ik(V[\delta(\beta(W))] - W[\delta(\beta(V))]) \\ & -2ik(dV[F]\cdot\beta(W) - dF\cdot V[\beta(W)] - dW[F]\cdot\beta(V) + dF\cdot W[\beta(V)]) \\ & +4k(V[W'[F]] - W[V'[F]]) \end{aligned}$$

for any vector fields V *and* W *on* $\mathcal{T}$.

Proof. We start by simply calculating the following commutators

$$\big[\nabla^T_V, \nabla_{s_2(W)}\big] = \nabla^2_{V[G_\beta(W)]}$$

$$
\begin{aligned}
\left[\nabla^T_V, \nabla_{s_1(W)}\right] &= \nabla_{\delta V[G_\beta(W)] + 2V[G_\beta(W)]\cdot dF + 2G_\beta(W)\cdot dV[F] - i(2k+n)V[\beta(W)]} \\
\left[\nabla^T_V, s_0(W)\right] &= ikV[\psi(W)] + 2k(k+n)V[\varphi(W)] - ikV[\delta(\beta(W))] \\
&\quad - 2ik(dV[F]\cdot\beta(W) - dF\cdot V[\beta(W)]) + 4kV[W'[F]]
\end{aligned}
$$

It is clear that the entries of $-[\nabla^T_W, u(V)]$ is calculated completely equivalently, and we can easily extract the symbols from the above to get the expression claimed in the lemma. □

Things get a lot more complicated, when we start calculating $[u(V), u(W)]$, and this calculation will rely heavily on Lemma 8.2 and 8.3. Due to the length of the expressions, we have again split the calculation into a series of smaller lemmas.

Lemma 8.5. *For any weakly restricted family of Kähler structures, the symbols of*

$$
C_1 = [\nabla^2_{s_2(V)}, \nabla^2_{s_2(W)}] = [\nabla^2_{G_\beta(V)}, \nabla^2_{G_\beta(W)}]
$$

acting on $\mathcal{H}^{(k)}$, are given as follows

$$
\begin{aligned}
\sigma_3(C_1) =& 2\,\mathcal{S}(G_\beta(V)\cdot\nabla G_\beta(W)) - 2\,\mathcal{S}(G_\beta(W)\cdot\nabla G_\beta(V)) \\
\sigma_2(C_1) =& \nabla^2_{G_\beta(V)} G_\beta(W) - \nabla^2_{G_\beta(W)} G_\beta(V) \\
\sigma_1(C_1) =& 0 \\
\sigma_0(C_1) =& 0
\end{aligned}
$$

for any vector fields V and W on $\mathcal{T}$.

Proof. By Lemma 8.3 we get that

$$
\begin{aligned}
\sigma_3(C_1) &= 2\,\mathcal{S}(G_\beta(V)\cdot\nabla G_\beta(W)) - 2\,\mathcal{S}(G_\beta(W)\cdot\nabla G_\beta(V)) \\
\sigma_2(C_1) &= \nabla^2_{G_\beta(V)} G_\beta(W) - \nabla^2_{G_\beta(W)} G_\beta(V) - \cancel{4ik\,\mathcal{S}(G_\beta(V)\cdot\omega\cdot G_\beta(W))} \\
&\quad + \cancel{2\mathcal{S}(G_\beta(V)^{xy} R^a_{uxy} G_\beta(W)^{ub})} - \cancel{2\mathcal{S}(G_\beta(W)^{uv} R^a_{xuv} G_\beta(V)^{xb})} \\
&= \nabla^2_{G_\beta(V)} G_\beta(W) - \nabla^2_{G_\beta(W)} G_\beta(V) \\
\sigma_1(C_1) &= \cancel{-2ikG_\beta(V)^{xy}\omega_{yu}\nabla_x(G_\beta(W)^{ua})} + \cancel{2ikG_\beta(W)^{uv}\omega_{vx}\nabla_u(G_\beta(V)^{xa})} \\
&\quad - \cancel{G_\beta(V)^{xy}\nabla_x(R^a_{yuv})G_\beta(W)^{uv}} + \cancel{G_\beta(W)^{uv}\nabla_v(R^a_{uxy})G_\beta(V)^{xy}} \\
&\quad - \cancel{\tfrac{4}{3}G_\beta(V)^{xy}R^a_{xuv}\nabla_y G_\beta(W)^{uv}} + \cancel{\tfrac{4}{3}G_\beta(W)^{uv}R^a_{uxy}\nabla_v G_\beta(V)^{xy}} \\
&= 0 \\
\sigma_0(C_1) &= \cancel{\frac{ik}{2}G_\beta(V)^{xy}J^j_y R_{xuvj} G_\beta(W)^{uv}} - \cancel{\frac{ik}{2}G_\beta(W)^{uv}J^j_v R_{uxyj} G_\beta(V)^{xy}} \\
&= 0,
\end{aligned}
$$

where the cancellations happen, because both the Levi-Civita curvature, and the symplectic form are type $(1,1)$. □

Lemma 8.6. *For any weakly restricted family of Kähler structures, the symbols of*

$$C_2 = [\nabla^2_{s_2(V)}, \nabla_{s_1(W)}] - [\nabla^2_{s_2(W)}, \nabla_{s_1(V)}]$$

acting on $\mathcal{H}^{(k)}$*, are given as follows*

$$\begin{aligned}
\sigma_2(C_2) &= 6\Gamma_3(V,W)\cdot dF - 6\Gamma_3(W,V)\cdot dF \\
&\quad + 2(\mathcal{S}(G_\beta(V)\cdot\nabla(\delta G_\beta(W))) - \mathcal{S}(G_\beta(W)\cdot\nabla(\delta G_\beta(V)))) \\
&\quad - i2(2k+n)(\mathcal{S}(G_\beta(V)\cdot\nabla\beta(W)) - \mathcal{S}(G_\beta(W)\cdot\nabla\beta(V))) \\
&\quad - \nabla_{\delta G_\beta(W)}G_\beta(V) + \nabla_{\delta G_\beta(V)}G_\beta(W) \\
&\quad + i2(2k+n)(\nabla_{\beta(W)}G_\beta(V) - \nabla_{\beta(V)}G_\beta(W)) \\
\sigma_1(C_2) &= \nabla^2_{G_\beta(V)}\delta G_\beta(W) - \nabla^2_{G_\beta(W)}\delta G_\beta(V) \\
&\quad + 2(\nabla^2_{G_\beta(V)}(G_\beta(W)\cdot dF) - \nabla^2_{G_\beta(W)}(G_\beta(V)\cdot dF)) \\
&\quad - i(2k+n)(\nabla^2_{G_\beta(V)}\beta(W) - \nabla^2_{G_\beta(W)}\beta(V)) \\
\sigma_0(C_2) &= 0
\end{aligned}$$

for any vector fields V *and* W *on* $\mathcal{T}$*.*

Proof. By Lemma 8.2 we get that the second order symbol is given by

$$\begin{aligned}
\sigma_2(C_2) &= 2\,\mathcal{S}(G_\beta(V)\cdot\nabla(\delta G_\beta(W) + 2G_\beta(W)\cdot dF - i(2k+n)\beta(W)) \\
&\quad - 2\,\mathcal{S}(G_\beta(W)\cdot\nabla(\delta G_\beta(V) - 2G_\beta(V)\cdot dF + i(2k+n)\beta(V)) \\
&\quad - \nabla_{(\delta G_\beta(W)+2G_\beta(W)\cdot dF - i(2k+n)\beta(W)}G_\beta(V) \\
&\quad + \nabla_{(\delta G_\beta(V)-2G_\beta(V)\cdot dF - i(2k+n)\beta(V)}G_\beta(W) \\
&= 4(\mathcal{S}(G_\beta(V)\cdot\nabla(G_\beta(W)\cdot dF)) - \mathcal{S}(G_\beta(W)\cdot\nabla(G_\beta(V)\cdot dF))) \\
&\quad - 2(\nabla_{G_\beta(W)\cdot dF}G_\beta(V) - 2\nabla_{G_\beta(V)\cdot dF}G_\beta(W)) \\
&\quad + 2(\mathcal{S}(G_\beta(V)\cdot\nabla(\delta G_\beta(W))) - \mathcal{S}(G_\beta(W)\cdot\nabla(\delta G_\beta(V)))) \\
&\quad - i2(2k+n)(\mathcal{S}(G_\beta(V)\cdot\nabla\beta(W)) - \mathcal{S}(G_\beta(W)\cdot\nabla\beta(V))) \\
&\quad - \nabla_{\delta G_\beta(W)}G_\beta(V) + \nabla_{\delta G_\beta(V)}G_\beta(W) \\
&\quad + i2(2k+n)(\nabla_{\beta(W)}G_\beta(V) - \nabla_{\beta(V)}G_\beta(W)) \\
&= 6\,\mathcal{S}(G_\beta(V)\cdot\nabla(G_\beta(W)))\cdot dF - 6\,\mathcal{S}(G_\beta(W)\cdot\nabla(G_\beta(V)))\cdot dF \\
&\quad + 2(\mathcal{S}(G_\beta(V)\cdot\nabla(\delta G_\beta(W))) - \mathcal{S}(G_\beta(W)\cdot\nabla(\delta G_\beta(V)))) \\
&\quad - i2(2k+n)(\mathcal{S}(G_\beta(V)\cdot\nabla\beta(W)) - \mathcal{S}(G_\beta(W)\cdot\nabla\beta(V))) \\
&\quad - \nabla_{\delta G_\beta(W)}G_\beta(V) + \nabla_{\delta G_\beta(V)}G_\beta(W) \\
&\quad + i2(2k+n)(\nabla_{\beta(W)}G_\beta(V) - \nabla_{\beta(V)}G_\beta(W))
\end{aligned}$$

$$\begin{aligned}
&= 6\Gamma_3(V,W)\cdot dF - 6\Gamma_3(W,V)\cdot dF \\
&+ 2(\mathcal{S}(G_\beta(V)\cdot\nabla(\delta G_\beta(W))) - \mathcal{S}(G_\beta(W)\cdot\nabla(\delta G_\beta(V)))) \\
&- i2(2k+n)(\mathcal{S}(G_\beta(V)\cdot\nabla\beta(W)) - \mathcal{S}(G_\beta(W)\cdot\nabla\beta(V))) \\
&- \nabla_{\delta G_\beta(W)} G_\beta(V) + \nabla_{\delta G_\beta(V)} G_\beta(W) \\
&+ i2(2k+n)(\nabla_{\beta(W)} G_\beta(V) - \nabla_{\beta(V)} G_\beta(W))
\end{aligned}$$

Likewise, and using type considerations, we get

$$\begin{aligned}
\sigma_1(C_2) &= \nabla^2_{G_\beta(V)}(\delta G_\beta(W) + 2G_\beta(W)\cdot dF - i(2k+n)\beta(W)) \\
&- \nabla^2_{G_\beta(W)}(\delta G_\beta(V) - 2G_\beta(V)\cdot dF + i(2k+n)\beta(V)) \\
&- \cancel{2ikG_\beta(V)\cdot\omega\cdot(\delta G_\beta(W) + 2G_\beta(W)\cdot dF - i(2k+n)\beta(W))} \\
&+ \cancel{2ikG_\beta(W)\cdot\omega\cdot(\delta G_\beta(V) - 2G_\beta(V)\cdot dF - i(2k+n)\beta(V))} \\
&+ \cancel{G_\beta(V)^{uv}R^a_{wuv}(\delta G_\beta(W) + 2G_\beta(W)\cdot dF - i(2k+n)\beta(W))^w} \\
&- \cancel{G_\beta(W)^{uv}R^a_{wuv}(\delta G_\beta(V) - 2G_\beta(V)\cdot dF - i(2k+n)\beta(V))^w} \\
&= \nabla^2_{G_\beta(V)}\delta G_\beta(W) - \nabla^2_{G_\beta(W)}\delta G_\beta(V) \\
&+ 2(\nabla^2_{G_\beta(V)}(G_\beta(W)\cdot dF) - \nabla^2_{G_\beta(W)}(G_\beta(V)\cdot dF) \\
&- i(2k+n)(\nabla^2_{G_\beta(V)}\beta(W) - \nabla^2_{G_\beta(W)}\beta(V)),
\end{aligned}$$

and lastly

$$\begin{aligned}
\sigma_0(C_2) &= \cancel{-ik\,\omega(G_\beta(V)\cdot\nabla(\delta G_\beta(W) + 2G_\beta(W)\cdot dF - i(2k+n)\beta(W)))} \\
&\quad \cancel{+ ik\,\omega(G_\beta(W)\cdot\nabla(\delta G_\beta(V) - 2G_\beta(V)\cdot dF - i(2k+n)\beta(V)))} \\
&= 0.
\end{aligned}$$

□

Lemma 8.7. *For any weakly restricted family of Kähler structures, the symbols of*

$$C_3 = [\nabla_{s_1(V)}, \nabla_{s_1(W)}]$$

acting on $\mathcal{H}^{(k)}$, are given as follows

$$\begin{aligned}
\sigma_1(C_3) &= \big[\delta G_\beta(V), \delta G_\beta(W)\big] + 4\big[G_\beta(V)\cdot dF, G_\beta(W)\cdot dF\big] - (2(2k+n))^2\big[\beta(V),\beta(W)\big] \\
&+ 2(\big[\delta G_\beta(V), G_\beta(W)\cdot dF\big] - \big[\delta G_\beta(W), G_\beta(V)\cdot dF\big]) \\
&- i(2k+n)(\big[\delta G_\beta(V),\beta(W)\big] - \big[\delta G_\beta(W),\beta(V)\big]) \\
&- i2(2k+n)(\big[G_\beta(V)\cdot dF,\beta(W)\big] - \big[G_\beta(W)\cdot dF,\beta(V)\big]) \\
\sigma_0(C_3) &= 0,
\end{aligned}$$

for any vector fields V and W on $\mathcal{T}$.

Proof. By using the definition of the curvature, we get

$$
\begin{aligned}
\sigma_1(C_3) &= \big[s_1(V), s_1(w)\big] \\
&= \big[\delta G_\beta(V), \delta G_\beta(W)\big] + 4\big[G_\beta(V)\cdot dF, G_\beta(W)\cdot dF\big] - (2(2k+n))^2\big[\beta(V), \beta(W)\big] \\
&+ 2(\big[\delta G_\beta(V), G_\beta(W)\cdot dF\big] - \big[\delta G_\beta(W), G_\beta(V)\cdot dF\big]) \\
&- i(2k+n)(\big[\delta G_\beta(V), \beta(W)\big] - \big[\delta G_\beta(W), \beta(V)\big]) \\
&- i2(2k+n)(\big[G_\beta(V)\cdot dF, \beta(W)\big] - \big[G_\beta(W)\cdot dF, \beta(V)\big])
\end{aligned}
$$

$$
\begin{aligned}
\sigma_0(C_3) &= -ik\omega(s_1(V), s_1(w)) \\
&= -ik\omega(\delta G_\beta(V), \delta G_\beta(W)) + 4 - ik\omega(G_\beta(V)\cdot dF, G_\beta(W)\cdot dF) + ik(2(2k+n))^2\omega(\beta(V), \beta(W)) \\
&- 2ik(\omega(\delta G_\beta(V), G_\beta(W)\cdot dF) - \omega(\delta G_\beta(W), G_\beta(V)\cdot dF)) \\
&- k(2k+n)(\omega(\delta G_\beta(V), \beta(W)) - \omega(\delta G_\beta(W), \beta(V))) \\
&- 2k(2k+n)(\omega(G_\beta(V)\cdot dF, \beta(W)) - \omega(G_\beta(W)\cdot dF, \beta(V))) \\
&= 0,
\end{aligned}
$$

since all the terms going into the symplectic form are of type $(1,0)$. □

Lemma 8.8. *For any weakly restricted family of Kähler structures, the symbols of*

$$
\begin{aligned}
C_4 &= [\nabla_{s_2(V)}, \nabla_{s_0(W)}] - [\nabla_{s_2(W)}, \nabla_{s_0(V)}] \\
C_5 &= [\nabla_{s_1(V)}, \nabla_{s_0(W)}] - [\nabla_{s_1(W)}, \nabla_{s_0(V)}] \\
C_6 &= [\nabla_{s_0(V)}, \nabla_{s_0(W)}]
\end{aligned}
$$

acting on $\mathcal{H}^{(k)}$, are given as follows

$$
\begin{aligned}
\sigma_0(C_4) &= 4k(\nabla^2_{G_\beta(W)}V'[F] - \nabla^2_{G_\beta(V)}W'[F]) \\
&- 2ik(\nabla^2_{G_\beta(W)}dF\cdot\beta(V) - \nabla^2_{G_\beta(V)}dF\cdot\beta(W)) \\
&- ik(\nabla^2_{G_\beta(W)}\delta(\beta(V)) - \nabla^2_{G_\beta(V)}\delta(\beta(W))) \\
&+ 2k(k+n)(\nabla^2_{G_\beta(W)}\varphi(V) - \nabla^2_{G_\beta(V)}\varphi(W)) \\
&+ ik(\nabla^2_{G_\beta(W)}\psi(V) - \nabla^2_{G_\beta(V)}\psi(W))
\end{aligned}
$$

$$
\begin{aligned}
\sigma_0(C_5) &= 4k(\nabla_{\delta G_\beta(V)}W'[F] - \nabla_{\delta G_\beta(W)}V'[F]) \\
&- 2ik(\nabla_{\delta G_\beta(V)}dF\cdot\beta(W) - \nabla_{\delta G_\beta(W)}dF\cdot\beta(V)) \\
&- ik(\nabla_{\delta G_\beta(V)}\delta(\beta(W)) - \nabla_{\delta G_\beta(W)}\delta(\beta(V))) \\
&+ 2k(k+n)(\nabla_{\delta G_\beta(V)}\varphi(W) - \nabla_{\delta G_\beta(W)}\varphi(V)) \\
&+ ik(\nabla_{\delta G_\beta(V)}\psi(W) - \nabla_{\delta G_\beta(W)}\psi(V)) \\
&+ 8k(\nabla_{G_\beta(V)\cdot dF}W'[F] - \nabla_{G_\beta(W)\cdot dF}V'[F]) \\
&- 4ik(\nabla_{G_\beta(V)\cdot dF}dF\cdot\beta(W) - \nabla_{G_\beta(W)\cdot dF}dF\cdot\beta(V)) \\
&- 2ik(\nabla_{G_\beta(V)\cdot dF}\delta(\beta(W)) - \nabla_{G_\beta(W)\cdot dF}\delta(\beta(V))) \\
&+ 4k(k+n)(\nabla_{G_\beta(V)\cdot dF}\varphi(W) - \nabla_{G_\beta(W)\cdot dF}\varphi(V)) \\
&+ 2ik(\nabla_{G_\beta(V)\cdot dF}\psi(W) - \nabla_{G_\beta(W)\cdot dF}\psi(V))
\end{aligned}
$$

$$
\begin{aligned}
&- i4k(2k+n)(\nabla_{\beta(V)}W'[F] - \nabla_{\beta(V)}V'[F]) \\
&+ 2k(2k+n)(\nabla_{\beta(V)}dF\cdot\beta(W) - \nabla_{\beta(V)}dF\cdot\beta(V)) \\
&+ k(2k+n)(\nabla_{\beta(V)}\delta(\beta(W)) - \nabla_{\beta(V)}\delta(\beta(V))) \\
&- i2k(k+n)(2k+n)(\nabla_{\beta(V)}\varphi(W) - \nabla_{\beta(V)}\varphi(V)) \\
&- k(2k+n)(\nabla_{\beta(V)}\psi(W) - \nabla_{\beta(V)}\psi(V))
\end{aligned}
$$

$$
\sigma_0(C_6) = 0
$$

for any vector fields V and W on $\mathcal{T}$.

Proof. For these commutators, we simply get zeroth order terms given by the second, respectively first, order operator acting on the zeroth order term. Also, the commutator of the two first order terms simply vanish. □

8.4 Curvature in the Weakly Restricted Case

Now we have calculated each of the terms constituting the commutator $\big[u(V), u(W)\big]$, and we are ready to combine all the results to get an explicit expression for the curvature

Theorem 8.9. *The symbols of the curvature $F_{\boldsymbol{\nabla}}$ of the Hitchin connection in the weakly restricted setting, acting on $\mathcal{H}^{(k)}$, is given by*

$$
\sigma_i(F_{\boldsymbol{\nabla}}(V,W)) = \frac{1}{(2k+n)^2}\sigma_i(C),
$$

where C is given by

$$
C = (2(2k+n))^2\big[u(V), u(W)\big] = 2(2k+n)C_0 + \sum_{i=1}^{6} C_i,
$$

and

$$
\sigma_3(C) = 2\Gamma_3(V,W) - 2\Gamma_3(W,V)
$$

$$
\begin{aligned}
\sigma_2(C) = {} & 6\Gamma_3(V,W)\cdot dF - 6\Gamma_3(W,V)\cdot dF \\
& + \Gamma_2(V,W) - \Gamma_2(W,V) \\
& + 2(2k+n)(-i\nabla_{\beta(V)}G_\beta(W) - i\,\mathcal{S}(G_\beta(V)\cdot\nabla\beta(W)) \\
& \qquad + i\nabla_{\beta(W)}G_\beta(V) + i\,\mathcal{S}(G_\beta(W)\cdot\nabla\beta(V)) \\
& \qquad + V[G_\beta(W)] - W[G_\beta(V)]).
\end{aligned}
$$

$$
\begin{aligned}
\sigma_1(C) = {} & \nabla^2_{G_\beta(V)}\delta G_\beta(W) - \nabla^2_{G_\beta(W)}\delta G_\beta(V) \\
& + 2\nabla^2_{G_\beta(V)}(G_\beta(W)\cdot dF) - 2\nabla^2_{G_\beta(W)}(G_\beta(V)\cdot dF) \\
& + \big[\delta G_\beta(V), \delta G_\beta(W)\big] \\
& + 4\big[G_\beta(V)\cdot dF, G_\beta(W)\cdot dF\big] \\
& + 2(\big[\delta G_\beta(V), G_\beta(W)\cdot dF\big] - \big[\delta G_\beta(W), G_\beta(V)\cdot dF\big])
\end{aligned}
$$

$$
\begin{aligned}
&+(2k+n)(-i\nabla^2_{G_\beta(V)}\beta(W)+i\nabla^2_{G_\beta(W)}\beta(V)\\
&\qquad +4V[G_\beta(W)]\cdot dF-4W[G_\beta(V)]\cdot dF\\
&\qquad +4G_\beta(W)\cdot dV[F]-4G_\beta(V)\cdot dW[F]\\
&\qquad +2\delta V[G_\beta(W)]-2\delta W[G_\beta(V)]\\
&\qquad -2i\big[G_\beta(V)\cdot dF,\beta(W)\big]+2i\big[G_\beta(W)\cdot dF,\beta(V)\big]\\
&\qquad -i\big[\delta G_\beta(V),\beta(W)\big]+i\big[\delta G_\beta(W),\beta(V)\big])\\
&+(2k+n)^2(-4\big[\beta(V),\beta(W)\big]\\
&\qquad -2iV[\beta(W)]+2iW[\beta(V)])
\end{aligned}
$$

$$
\begin{aligned}
\sigma_0(C)=&ik(V[\psi(W)]-W[\psi(V)])\\
&+2k(k+n)(V[\varphi(W)]-W[\varphi(V)])\\
&-ik(V[\delta(\beta(W))]-W[\delta(\beta(V))])\\
&-2ik(dV[F]\cdot\beta(W)-dF\cdot V[\beta(W)]-dW[F]\cdot\beta(V)+dF\cdot W[\beta(V)])\\
&+4k(V[W'[F]]-W[V'[F]])\\
&\quad +4k(\nabla^2_{G_\beta(W)}V'[F]-\nabla^2_{G_\beta(V)}W'[F])\\
&\quad -2ik(\nabla^2_{G_\beta(W)}dF\cdot\beta(V)-\nabla^2_{G_\beta(V)}dF\cdot\beta(W))\\
&\quad -ik(\nabla^2_{G_\beta(W)}\delta(\beta(V))-\nabla^2_{G_\beta(V)}\delta(\beta(W)))\\
&\quad +2k(k+n)(\nabla^2_{G_\beta(W)}\varphi(V)-\nabla^2_{G_\beta(V)}\varphi(W))\\
&\quad +ik(\nabla^2_{G_\beta(W)}\psi(V)-\nabla^2_{G_\beta(V)}\psi(W))\\
&\quad +4k(\nabla_{\delta G_\beta(V)}W'[F]-\nabla_{\delta G_\beta(W)}V'[F])\\
&\quad -2ik(\nabla_{\delta G_\beta(V)}dF\cdot\beta(W)-\nabla_{\delta G_\beta(W)}dF\cdot\beta(V))\\
&\quad -ik(\nabla_{\delta G_\beta(V)}\delta(\beta(W))-\nabla_{\delta G_\beta(W)}\delta(\beta(V)))\\
&\quad +2k(k+n)(\nabla_{\delta G_\beta(V)}\varphi(W)-\nabla_{\delta G_\beta(W)}\varphi(V))\\
&\quad +ik(\nabla_{\delta G_\beta(V)}\psi(W)-\nabla_{\delta G_\beta(W)}\psi(V))\\
&\quad +8k(\nabla_{G_\beta(V)\cdot dF}W'[F]-\nabla_{G_\beta(W)\cdot dF}V'[F])\\
&\quad -4ik(\nabla_{G_\beta(V)\cdot dF}dF\cdot\beta(W)-\nabla_{G_\beta(W)\cdot dF}dF\cdot\beta(V))\\
&\quad -2ik(\nabla_{G_\beta(V)\cdot dF}\delta(\beta(W))-\nabla_{G_\beta(W)\cdot dF}\delta(\beta(V)))\\
&\quad +4k(k+n)(\nabla_{G_\beta(V)\cdot dF}\varphi(W)-\nabla_{G_\beta(W)\cdot dF}\varphi(V))\\
&\quad +2ik(\nabla_{G_\beta(V)\cdot dF}\psi(W)-\nabla_{G_\beta(W)\cdot dF}\psi(V))\\
&\quad -i4k(2k+n)(\nabla_{\beta(V)}W'[F]-\nabla_{\beta(V)}V'[F])\\
&\quad +2k(2k+n)(\nabla_{\beta(V)}dF\cdot\beta(W)-\nabla_{\beta(V)}dF\cdot\beta(V))\\
&\quad +k(2k+n)(\nabla_{\beta(V)}\delta(\beta(W))-\nabla_{\beta(V)}\delta(\beta(V)))\\
&\quad -i2k(k+n)(2k+n)(\nabla_{\beta(V)}\varphi(W)-\nabla_{\beta(V)}\varphi(V))\\
&\quad -k(2k+n)(\nabla_{\beta(V)}\psi(W)-\nabla_{\beta(V)}\psi(V)).
\end{aligned}
$$

Proof. We first observe, that by choosing commuting vector fields V and W, we get the curvature as

$$F_{\boldsymbol{\nabla}}(V,W) = \big[u(V), u(W)\big],$$

and by writing out this commutator, we see that we get exactly all the terms calculated in the C_i's the previous lemmas, as we have stated in the theorem.

Now we combine all the calculated terms from Lemmas 8.4, 8.5, 8.6, 8.7 and 8.8. Firstly there is only one contribution to the principal symbol, which is from C_1, and are given by

$$\begin{aligned}\sigma_3(C) = \sigma_3(C_1) &= 2\,\mathcal{S}(G_\beta(V)\cdot\nabla G_\beta(W)) - 2\,\mathcal{S}(G_\beta(W)\cdot\nabla G_\beta(V)) \\ &= 2\Gamma_3(V,W) - 2\Gamma_3(W,V).\end{aligned}$$

We continue with the second order symbol, which get contributions from C_0, C_1 and C_2 and furthermore gather the terms in powers of $2k+n$, which we will elaborate on later.

$$\begin{aligned}\sigma_2(C) =& 2(2k+n)\sigma_2(C_0) + \sigma_2(C_1) + \sigma_2(C_2) \\ =& 2(2k+n)(V[G_\beta(W)] - W[G_\beta(V)]) \\ &+ \nabla^2_{G_\beta(V)}G_\beta(W) - \nabla^2_{G_\beta(W)}G_\beta(V) \\ &+ 6\Gamma_3(V,W)\cdot dF - 6\Gamma_3(W,V)\cdot dF \\ &+ 2(\mathcal{S}(G_\beta(V)\cdot\nabla(\delta G_\beta(W))) - \mathcal{S}(G_\beta(W)\cdot\nabla(\delta G_\beta(V)))) \\ &- i2(2k+n)(\mathcal{S}(G_\beta(V)\cdot\nabla\beta(W)) - \mathcal{S}(G_\beta(W)\cdot\nabla\beta(V))) \\ &- \nabla_{\delta G_\beta(W)}G_\beta(V) + \nabla_{\delta G_\beta(V)}G_\beta(W) \\ &+ i2(2k+n)(\nabla_{\beta(W)}G_\beta(V) - \nabla_{\beta(V)}G_\beta(W)) \\ =& \nabla^2_{G_\beta(V)}G_\beta(W) + \nabla_{\delta G_\beta(V)}G_\beta(W) + 2\,\mathcal{S}(G_\beta(V)\cdot\nabla(\delta G_\beta(W))) \\ &- \nabla^2_{G_\beta(W)}G_\beta(V) - \nabla_{\delta G_\beta(W)}G_\beta(V) - 2\,\mathcal{S}(G_\beta(W)\cdot\nabla(\delta G_\beta(V))) \\ &+ 6\Gamma_3(V,W)\cdot dF - 6\Gamma_3(W,V)\cdot dF \\ &+ 2(2k+n)(-i\,\mathcal{S}(G_\beta(V)\cdot\nabla\beta(W)) + i\,\mathcal{S}(G_\beta(W)\cdot\nabla\beta(V)) \\ &\qquad + i\nabla_{\beta(W)}G_\beta(V) - i\nabla_{\beta(V)}G_\beta(W) \\ &\qquad + V[G_\beta(W)] - W[G_\beta(V)]) \\ =& 6\Gamma_3(V,W)\cdot dF - 6\Gamma_3(W,V)\cdot dF \\ &+ \Gamma_2(V,W) - \Gamma_2(W,V) \\ &+ 2(2k+n)(-i\nabla_{\beta(V)}G_\beta(W) - i\,\mathcal{S}(G_\beta(V)\cdot\nabla\beta(W)) \\ &\qquad + i\nabla_{\beta(W)}G_\beta(V) + i\,\mathcal{S}(G_\beta(W)\cdot\nabla\beta(V)) \\ &\qquad + V[G_\beta(W)] - W[G_\beta(V)]).\end{aligned}$$

For the first order symbol we get contributions from C_0, C_1, C_2 and C_3.

$$\begin{aligned}
\sigma_1(C) =& 2(2k+n)\sigma_1(C_0) + \sigma_1(C_1) + \sigma_1(C_2) + \sigma_1(C_3) \\
=& 2(2k+n)(2V[G_\beta(W)] \cdot dF - 2W[G_\beta(V)] \cdot dF) \\
&+ 2(2k+n)(2G_\beta(W) \cdot dV[F] - 2G_\beta(V) \cdot dW[F]) \\
&+ 2(2k+n)(\delta V[G_\beta(W)] - \delta W[G_\beta(V)]) \\
&- i2(2k+n)^2(V[\beta(W)] - W[\beta(V)]) \\
&+ \nabla^2_{G_\beta(V)}\delta G_\beta(W) - \nabla^2_{G_\beta(W)}\delta G_\beta(V) \\
&+ 2(\nabla^2_{G_\beta(V)}(G_\beta(W) \cdot dF) - \nabla^2_{G_\beta(W)}(G_\beta(V) \cdot dF)) \\
&- i(2k+n)(\nabla^2_{G_\beta(V)}\beta(W) - \nabla^2_{G_\beta(W)}\beta(V)) \\
&+ \big[\delta G_\beta(V), \delta G_\beta(W)\big] + 4\big[G_\beta(V) \cdot dF, G_\beta(W) \cdot dF\big] - (2(2k+n))^2\big[\beta(V), \beta(W)\big] \\
&+ 2(\big[\delta G_\beta(V), G_\beta(W) \cdot dF\big] - \big[\delta G_\beta(W), G_\beta(V) \cdot dF\big]) \\
&- i(2k+n)(\big[\delta G_\beta(V), \beta(W)\big] - \big[\delta G_\beta(W), \beta(V)\big]) \\
&- i2(2k+n)(\big[G_\beta(V) \cdot dF, \beta(W)\big] - \big[G_\beta(W) \cdot dF, \beta(V)\big]) \\
=& + \nabla^2_{G_\beta(V)}\delta G_\beta(W) - \nabla^2_{G_\beta(W)}\delta G_\beta(V) \\
&+ 2\nabla^2_{G_\beta(V)}(G_\beta(W) \cdot dF) - 2\nabla^2_{G_\beta(W)}(G_\beta(V) \cdot dF) \\
&+ \big[\delta G_\beta(V), \delta G_\beta(W)\big] \\
&+ 4\big[G_\beta(V) \cdot dF, G_\beta(W) \cdot dF\big] \\
&+ 2(\big[\delta G_\beta(V), G_\beta(W) \cdot dF\big] - \big[\delta G_\beta(W), G_\beta(V) \cdot dF\big]) \\
+(2k+n)&(-i\nabla^2_{G_\beta(V)}\beta(W) + i\nabla^2_{G_\beta(W)}\beta(V) \\
&+ 4V[G_\beta(W)] \cdot dF - 4W[G_\beta(V)] \cdot dF \\
&+ 4G_\beta(W) \cdot dV[F] - 4G_\beta(V) \cdot dW[F] \\
&+ 2\delta V[G_\beta(W)] - 2\delta W[G_\beta(V)] \\
&- 2i\big[G_\beta(V) \cdot dF, \beta(W)\big] + 2i\big[G_\beta(W) \cdot dF, \beta(V)\big] \\
&- i\big[\delta G_\beta(V), \beta(W)\big] + i\big[\delta G_\beta(W), \beta(V)\big]) \\
+(2k+n)^2&(-4\big[\beta(V), \beta(W)\big] \\
&- 2iV[\beta(W)] + 2iW[\beta(V)]).
\end{aligned}$$

Lastly we gather all the terms for the zeroth order symbol. This time we get contribution from C_0, C_4 and C_5, and we have done nothing, except add the terms together and we get the expression in the theorem from

$$\sigma_0(C) = \sigma_0(C_0) + \sigma_0(C_4) + \sigma_0(C_5).$$

This completes the proof. □

8.5 Projective Flatness

The expressions from Theorem 8.9 are clearly long and complicated, but we are still able to extract some simpler information from them.

First of all, as mentioned in the beginning of the chapter, the first criterion for projective flatness is that

$$0 = \Gamma_3(V, W) - \Gamma_3(W, V). \tag{8.2}$$

Given that this is fulfilled, we can apply Proposition 8.1 to get

$$0 = \Gamma_2(V, W) - \Gamma_2(W, V),$$

which reduces the criterion on the second order symbol to

$$\begin{aligned} 0 = &- i\nabla_{\beta(V)} G_\beta(W) - i\,\mathcal{S}(G_\beta(V)\cdot\nabla\beta(W)) \\ &+ i\nabla_{\beta(W)} G_\beta(V) + i\,\mathcal{S}(G_\beta(W)\cdot\nabla\beta(V)) \\ &+ V[G_\beta(W)] - W[G_\beta(V)], \end{aligned} \tag{8.3}$$

which is seen to be a symmetry property as well. We have, however, not found any other criterion that gives vanishing of these. Now continuing to the first order symbol, we will use that we have an expression like

$$(2k+n)^0 a_0 + (2k+n)^1 a_1 + (2k+n)^2 a_2 = 0,$$

that holds for all levels k, and thus we must have $a_0 = a_1 = a_2 = 0$. This splits the equation on the first order symbol into three seperat equations, which gives us more precise information on which terms should vanish.

$$\begin{aligned} 0 = &\nabla^2_{G_\beta(V)}\delta G_\beta(W) - \nabla^2_{G_\beta(W)}\delta G_\beta(V) \\ &+ 2\nabla^2_{G_\beta(V)}(G_\beta(W)\cdot dF) - 2\nabla^2_{G_\beta(W)}(G_\beta(V)\cdot dF) \\ &+ \big[\delta G_\beta(V), \delta G_\beta(W)\big] \\ &+ 4\big[G_\beta(V)\cdot dF, G_\beta(W)\cdot dF\big] \\ &+ 2(\big[\delta G_\beta(V), G_\beta(W)\cdot dF\big] - \big[\delta G_\beta(W), G_\beta(V)\cdot dF\big]) \\ = &\Delta_{G_\beta(V)}\delta G_\beta(W) - \Delta_{G_\beta(W)}\delta G_\beta(V) \\ &+ 2\nabla^2_{G_\beta(V)}(G_\beta(W)\cdot dF) - 2\nabla^2_{G_\beta(W)}(G_\beta(V)\cdot dF) \\ &+ 4\big[G_\beta(V)\cdot dF, G_\beta(W)\cdot dF\big] \\ &+ 2(\big[\delta G_\beta(V), G_\beta(W)\cdot dF\big] - \big[\delta G_\beta(W), G_\beta(V)\cdot dF\big]) \end{aligned} \tag{8.4}$$

$$\begin{aligned} 0 = &-i\nabla^2_{G_\beta(V)}\beta(W) + i\nabla^2_{G_\beta(W)}\beta(V) \\ &+ 4V[G_\beta(W)]\cdot dF - 4W[G_\beta(V)]\cdot dF \\ &+ 4G_\beta(W)\cdot dV[F] - 4G_\beta(V)\cdot dW[F] \\ &+ 2\delta V[G_\beta(W)] - 2\delta W[G_\beta(V)] \\ &- 2i\big[G_\beta(V)\cdot dF, \beta(W)\big] + 2i\big[G_\beta(W)\cdot dF, \beta(V)\big] \\ &- i\big[\delta G_\beta(V), \beta(W)\big] + i\big[\delta G_\beta(W), \beta(V)\big] \end{aligned}$$

$$0 = -4\big[\beta(V), \beta(W)\big] - 2iV[\beta(W)] + 2iW[\beta(V)]. \tag{8.5}$$

We see, for instance, that the last equation gives us an identity only involving β and the vector fields V and W. We could hope to find some way to express these expressions, such that they simplify further or even prove that they hold given some general condition, but this remains an open problem.

We state the above in a corollary.

Corollary 8.10. *The Hitchin connection in the weakly restricted case is projectively flat if and only if equations* (8.2)*,* (8.3)*,* (8.4) *and* (8.5) *hold.*

Chapter 9

Obstructions for Projective Flatness

We have already seen the explicit calculations for the curvature of the Hitchin connection in the rigid setting in chapter 8, and we have stated the obstruction to being projectively flat explicitly in terms of differential operators. Here we approach the problem from a different viewpoint and state a theorem of Ginzburg and Montgomery, that gives a more abstract condition on the symplectic manifold and space of compatible complex structures under which no projectively flat connection can exist on the bundle of quantum spaces. It turns out that the Hitchin connection constructed in Theorem 6.11 fullfils the conditions, and thus it cannot be projectively flat in general. Explicitly we show in the case of the coadjoint orbit, that we get a Hitchin connection that cannot be projectively flat.

9.1 The no-go theorem and projective flatness

Let us begin with briefly recalling the work [GM00], in which Ginzburg and Montgomery show a no-go theorem, stating conditions under which no natural projectively flat connection can exist on the vector bundle of quantizations.

We will introduce some notation to state the theorem. We have as usual a symplectic manifold (M, ω). We let $\operatorname{Ham}(M, \omega)$ be the group of hamiltonian symplectomorphisms of M, and $\mathcal{G}$ be the group of diffeomorphisms of the unit circle bundle U of $\mathcal{L}$ which preserve the connection form. Lastly we let $\mathcal{G}_0$ be the connected component of $\mathcal{G}$ containing the identity. We consider a compatible complex structure $J_0 \in \mathcal{C}_\omega(M)$ and assume that there exist a neighbourhood $\mathcal{C}^0_\omega(M)$ of J_0, such that $H^{(k)}|_{\mathcal{C}^0_\omega(M)}$ is a vector bundle over $\mathcal{C}^0_\omega(M)$.

Theorem 9.1 (Ginzburg and Montgomery)**.** *Assume that there exist a complex structure J_0 with stabilizer G_{J_0} in $\operatorname{Ham}(M, \omega)$ of positive dimension, and that the infinitesimal representation of G_{J_0} on $H^{(k)}_{J_0}$ is non-trivial. Then there is no projectively flat connection on $H^{(k)}|_{\mathcal{C}^0_\omega(M)}$, which is invariant under the $\mathcal{G}_0$ local action.*

We will now proceed with the mentioned example, where we can apply our construction for a certain small enough neighbourhood of a particular J_0 with such a symmetry group.

Let G be a compact simple and simply-connected Lie group. We are going to consider a co-adjoint orbit M in $\mathfrak{g}^*$. On M we are going to consider the Kirillov-Kostant symplectic structure (see e.g. [Woo92]). Furthermore, we have the natural G-invariant complex structure

J_0 on M coming from the identification

$$M = G^{\mathbb{C}}/P,$$

where P is a parabolic subgroup determined by M. It is well known that (M, J_0) is rigid and that there exist a small enough neighbourhood $\mathcal{C}^0_\omega(M)$ of J_0 such that

$$H^1(M_J, T'M_J) = 0$$

for all complex structures $J \in \mathcal{C}^0_\omega(M)$.

We now want to determine $\beta(V)_J$ uniquely for all $J \in \mathcal{C}^0_\omega(M)$ and all $V \in T_J\mathcal{C}^0_\omega(M)$ solving

$$V'[J]_J = -\bar{\partial}_J \beta(V)_J.$$

This we can do uniquely by the above vanishing of $H^1(M_J, T'M_J)$ and if we impose suitable conditions on $\beta(V)_J$. One possibility is to require that $\beta(V)_J$ is orthogonal to all holomorphic vector fields on (M, J). Another way could be to require special evaluation properties of $\beta(V)_J$ at various points on M. Furthermore, we can determine a smooth family of Ricci potentials, by picking, for each complex structure $J \in \mathcal{C}^0_\omega(M)$, the unique potential with zero average. Hence, since M is simply connected, there is a unique prequantum line bundle $(\mathcal{L}, h, \nabla)$ with curvature $-i\omega$. Thus we satisfy all assumptions of Theorem 6.11 and so we get the following corollary.

Corollary 9.2. *For the coadjoint orbit M, we get a Hitchin connection in the bundle $H^{(k)}$ over the subspace $\mathcal{C}^0_\omega(M)$. This connection is invariant under the local action of the group of bundle automorphisms of the prequantum line bundle $(\mathcal{L}, h, \nabla)$ covering the symplectomorphism group of (M, ω).*

We see that this connection therefore satisfies all the requirements of Ginzburg and Montgomery's Theorem 9.1 above, and thus this connection cannot be projectively flat over $\mathcal{C}^0_\omega(M)$.

Chapter **10**

Pullbacks of the Hitchin Connection

We will now present a way to construct a Hitchin connection on a nice enough connected neighbourhood $C^0_\omega(M)$ of a rigid family of complex structures $\mathcal{T}$. This gives, in the case of $(\mathbb{P}^n, \omega_{FS})$, an alternative construction of a connection, to the weakly restricted construction for the case of a coadjoint orbit presented above. It follows by projective flatness of the pullback connection, that these two constructions actually gives different connections on $(\mathbb{P}^n, \omega_{FS})$.

10.1 General Scheme for Pullbacks of The Hitchin Connection

Let us consider a symplectic manifold (M, ω), and assume that we have a rigid subfamily $\mathcal{T} \subseteq \mathcal{C}_\omega(M)$ of all the complex structures compatible with ω.

Furthermore we assume, that we have some connected subspace $\mathcal{C}^0_\omega(M)$, on which we can find a map $\Phi\colon \mathcal{C}^0_\omega(M) \to \mathrm{Diff}(M)$ denoted $J \mapsto \Phi_J$, such that for each $J \in \mathcal{C}^0_\omega(M)$ there exists a $J' \in \mathcal{T}$ with

$$\Phi_J^*(J') = J \quad \text{and} \quad \Phi_{|\mathcal{T}} = \mathrm{Id}\,.$$

That is Φ_J gives a biholomorphism from M with the complex structure J to M with the complex structure J' from the rigid family $\mathcal{T}$.

Now for each J we can consider the pullback bundle $\Phi_J^*\mathcal{L} \to M$, which is naturally isomorphic to $\mathcal{L}$ itself, since Φ_J is isotopic to the identity for all $J \in \mathcal{C}^0_\omega(M)$.

Choosing a holomorphic isomorphism $\widetilde{\Psi}_J\colon \mathcal{L} \to \Phi_J^*\mathcal{L}$ we get the following commutative diagram.

$$\begin{array}{ccccc} \mathcal{L} & \xrightarrow{\widetilde{\Psi}_J} & \Phi_J^*\mathcal{L} & \xrightarrow{p} & \mathcal{L} \\ \downarrow & & \downarrow & & \downarrow \\ M & \xrightarrow[\mathrm{Id}]{} & M & \xrightarrow[\Phi_J]{} & M \end{array}$$

where p is the map given canonically in the construction of the pullback bundle. Composing the maps in the top of the diagram, we get an induced endomorphism on $\mathcal{L}$ given by $\Psi_J = p \circ \widetilde{\Psi}_J$. We need to fix Ψ_J uniquely up to the action of the automorphism group of the line bundle, $\operatorname{Aut}(\mathcal{L}) = C^*$. To do this we seek a section Ψ of the bundle

$$\mathcal{L}(M) = \left\{(J, \Psi) \in \mathcal{C}_\omega^0(M) \times \operatorname{Hom}(\mathcal{L}, \mathcal{L}) \mid \Psi\colon (L, J) \to (L, J') \text{ holo. for some } J' \in \mathcal{T}\right\}$$

over $\mathcal{D}(M)$, where

$$\mathcal{D}(M) = \left\{(J, \Phi) \in \mathcal{C}_\omega^0(M) \times \operatorname{Diff}(M) \mid J = \Phi^*(J') \text{ for some } J' \in \mathcal{T}\right\},$$

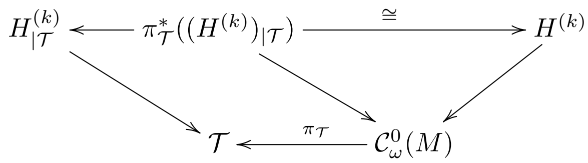
which in turn is a bundle over $\mathcal{C}_\omega^0(M)$. If we have one point $x \in M$, which is fixed for all Φ_J, $J \in \mathcal{C}_\omega^0(M)$, then we can fix the ambiguity by requiring that

$$(\Psi_J)_x = \operatorname{Id}\colon \mathcal{L}_x \to \mathcal{L}_x,$$

and hence get the required section Ψ. Let us now assume we have a map

$$\pi_{\mathcal{T}}\colon \mathcal{C}_\omega^0(M) \to \mathcal{T},$$

which is compatible with some Φ. Then any section Ψ as above will induce an isomorphism of the bundles, such that we get the following commutative diagram



We know from that we get a projectively flat connection on the bundle of quantum states $(H^{(k)})_{|\mathcal{T}}$ restricted to the rigid part $\mathcal{T}$ of the compatible complex structures, since this space fulfils all the requirements from [AG14]. Now this connection induces a projectively flat connection on the pullback bundle, which then gives a projectively flat connection on $H^{(k)}$.

Here we have used that

$$\pi^*((H^{(k)})_{|\mathcal{T}})_J = H^0(M_J, \Phi_J^* \mathcal{L}^k),$$

giving us the isomorphism on each fiber, and since the diagram 10.1 commutes, an isomorphism on the level of bundles is obtained.

10.2 Pullback of the Hitchin connection on $\mathbb{P}^n$

Let us restrict to an example, where we can actually construct a Φ fulfilling the requirements above. We consider $M = \mathbb{P}^n$ as a symplectic manifold equipped with the Fubini-Study symplectic form ω but without the normal structure as a complex manifold. We do, however, denote the almost complex structure coming from the normal structure on $\mathbb{P}^n$ by J_0.

On M there exist a neighbourhood $\mathcal{C}_\omega^0(M)$ around J_0 such that every complex structure $J' \in \mathcal{C}_\omega^0(M)$ is biholomorphic to J_0, and thus we can choose the subfamily $\mathcal{T} \subseteq \mathcal{C}_\omega(M)$ to just be the pointset $\mathcal{T} = \{J_0\}$. This way we have $\mathcal{D}(M)_J \neq \emptyset$ for all $J \in \mathcal{C}_\omega^0(M)$.

It is known that $\mathbb{P}^n$ has the property, that we can choose a set of $n+1$ points

$$X = \{x_0, x_1, \ldots, x_n\} \subseteq \mathbb{P}^n,$$

such that any set of lifts of these to $\mathbb{C}^{n+1}$ is a basis. Furthermore, we have that for any such set

$$Y = \{y_0, y_1, \dots, y_n\} \subseteq \mathbb{P}^n,$$

there exists a unique $\Phi_0 \in \mathrm{Aut}(\mathbb{P}^n)$, such that

$$\Phi_0(y_i) = x_i.$$

This means that we can for any $J \in \mathcal{C}^0_\omega(M)$ determine a unique biholomorphism

$$\Phi_J \colon (M, J) \to (M, J_0) \quad \text{with} \quad \Phi_J(x_i) = x_i$$

This way, we can define a section $\Phi \in C^\infty(\mathcal{C}^0_\omega(M), D(M))$.

Note that we have $\Phi_{J_0} = \mathrm{Id}$, and therefore Φ and $\mathcal{T} = \{J_0\}$ fullfils the requirements outlined above. Thus we get a flat connection in $H^{(k)}$ over the entire space $\mathcal{C}^0_\omega(M)$. This connection does however not have the symmetry required by Theorem 9.1, and does not agree with the connection obtained in Corollary 9.2, since that connection is not projectively flat.

Chapter 11

Examples of Weakly Restricted Families

In this chapter, we give a number of examples, where we can solve the weakly restricted criterion for open subsets of the entire family of complex structures on a given symplectic manifold, such that we can apply theorem 6.11 to get a Hitchin connection on such subspaces of all complex structures on the given symplectic manifold.

11.1 Euclidean space $\mathbb{R}^{2n}$

The first example we consider is $M = \mathbb{R}^{2n}$ with the standard symplectic structure. We call the standard complex structure J_0 and let $\mathcal{C}_\omega^0(M)$ be an open and small enough neighbourhood of J_0, such that $H^{(k)} |_{\mathcal{C}_\omega^0(M)}$ is a vector bundle over $\mathcal{C}_\omega^0(M)$.

Furthermore we choose $\mathcal{C}_\omega^0(M)$ such that

$$H^1(M_J, T'M_J) = 0$$

for all complex structures $J \in \mathcal{C}_\omega^0(M)$. This means that we have a solution to the weakly restricted criterion with $G_\beta(V) = 0$ and $\beta(V)_\sigma$ a solution to

$$V'[J]_\sigma = -\bar{\partial}_\sigma \beta(V)_\sigma$$

for all vector fields V and points σ on $\mathcal{C}_\omega^0(M)$. We will need a smooth family of β's, which we can assume exists by choosing a suitable $\mathcal{C}_\omega^0(M)$.

In this case the functions $\varphi(V)$ and $\psi(V)$ in the expression of the Hitchin connection from Theorem 6.11 can be calculated explicitly by curve integrals (depending of course on a choice of base point in $\mathcal{C}_\omega^0(M)$) of the $\bar{\partial}$-exact forms that they are related to by definition.

11.2 Symplectic Tori

Now let us consider the symplectic torus $M = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ with the standard symplectic structure ω. In this case, it is not true that the moduli space of complex structures is locally a point. We consider the usual moduli space of linear complex structures compatible with the standard symplectic structure, which is the moduli space of principal polarised abelian varieties. In fact, the space of all linear complex structures on $\mathbb{R}^{2n}$ compatible with ω can be identified with

$$\mathbb{H} = \{Z \in M_{n,n}(\mathbb{C}) \mid Z = Z^t, \mathrm{Im}(Z) > 0\}.$$

We denote the complex structure corresponding to $Z \in \mathbb{H}$ by J_Z. It is easy to check that the map

$$\cdot\omega\colon H^0(M_{J_Z}, S^2(T'M_{J_Z})) \to H^1(M_{J_Z}, T'M_{J_Z})_\omega$$

is surjective for all $Z \in \mathbb{H}$. Recall that for any $J \in C_\omega(M)$, there exists a unique $Z \in \mathbb{H}$, for which there exists a unique biholomophism

$$\Phi_J\colon (M, J) \to (M, J_Z),$$

which induces the identity on $H^1(M, \mathbb{Z})$ and which preserves $0 \in M$. Then we also have that

$$\cdot\omega\colon H^0(M_J, S^2(T'M_J)) \to H^1(M_J, T'M_J)_\omega \tag{11.1}$$

is surjective for all complex structures $J \in \mathcal{C}_\omega(M)$, and furthermore this gives us a natural projection map

$$\pi\colon \mathcal{C}_\omega(M) \to \mathbb{H}.$$

We now fix a prequantum line bundle $(\mathcal{L}, h, \nabla)$ over (M, ω). Consider the bundle of quantum spaces $\tilde{H}^{(k)} \to \mathbb{H}$, with its usual Hitchin connection (see e.g. [Hit90, And05]) and further the pullback

$$\pi^* \tilde{H}^{(k)} \to \mathcal{C}_\omega(M).$$

Since each Φ_J induces the identity on the first homology, we see that $\Phi_J^* \mathcal{L} \cong \mathcal{L}$ and as we furthermore have $\Phi_J(0) = 0$, we can find a section Ψ as discussed in section 10.1, which induces an isomorphism of the quantum bundles

$$\Psi^*\colon \pi^* \tilde{H}^{(k)} \to H^{(k)}.$$

We now pull back the Hitchin connection in $\tilde{H}^{(k)}$ to $\pi^* \tilde{H}^{(k)}$ and push it to $H^{(k)}$ by this isomorphism, to get a projectively flat connection. Again by the no-go Theorem 9.1, this connection cannot be natural, which is also clear from its construction.

We will now show that our construction of the Hitchin connection in the weakly restricted setting applies to provide a construction of a natural partial connection in $H^{(k)}$ over $\mathcal{C}_\omega(M)$. Since we have that (11.1) is surjective, we see that the equation (3.5) can be solved for all $J \in \mathcal{C}^0_\omega(M)$ and all tangent vectors $V \in T_J\mathcal{C}^0_\omega(M)$. For any choice of solution to this equation, we get a $\bar\partial$-closed form $\omega \cdot \beta_J(V)$. However, the map

$$\omega\cdot\colon H^0(M_J, T'M_J) \to H^{0,1}(M)$$

is an isomorphism for all $J \in \mathcal{C}_\omega(M)$, so we can uniquely determine $\beta_J(V)$ as a solution to (3.5), by requiring that $\omega \cdot \beta_J(V) = 0$ in $H^{0,1}(M)$. This in turn means that we can indeed find a unique solution to the equation $\bar\partial\varphi(V) = \omega \cdot \beta_J(V)$ of zero average.

We now consider the linear map

$$[\Omega]_J\colon T_J\mathcal{C}^0_\omega(M) \to H^{0,1}(M_J).$$

We see that we can apply our Hitchin connection construction along the distribution

$$\ker[\Omega] \subset T\mathcal{C}^0_\omega(M),$$

simply by just choosing the $\psi(V)$ with zero average which solves

$$\bar{\partial}\psi(V) = \Omega(V).$$

11.3 The Moduli Space of Flat SU(n)-Connections

The last example we will consider is the moduli spaces $\mathcal{M}$ of flat SU(n)-connections on a surface of genus $g \geq 2$ possibly with central holonomy around a point on the surface, which we have already discussed in chapter 4. We have again the Narasimhan-Goldman-Atiyah-Bott symplectic form ω on (the smooth part of) $\mathcal{M}$, and we further have that the Chern-Simons functional induces the Chern-Simons line bundle $(\mathcal{L}_{CS}, h, \nabla)$ over $(\mathcal{M}, \omega)$.

We first consider the usual family of complex structures J parametrized by Teichmüller space $\mathcal{T}$. In this situation Hitchin has proved [Hit90] that the map

$$\cdot\omega \colon H^0(\mathcal{M}_J, S^2(T'\mathcal{M}_J)) \to H^1(\mathcal{M}_J, T'\mathcal{M}_J)_\omega$$

is surjective for all $J \in \mathcal{T}$. We define $\mathcal{C}^0_\omega(\mathcal{M})$ to be the maximal connected subspace of all complex structures on $\mathcal{M}$, which is compatible with ω and for which there exist a unique $J' \in \mathcal{T}$ and a unique biholomophism

$$\Phi_J \colon (\mathcal{M}, J) \to (\mathcal{M}, J'),$$

with the property that it varies smoothly with $J \in \mathcal{C}^0_\omega(\mathcal{M})$ and $\Phi_J = \mathrm{Id}$ for all $J \in \mathcal{T}$. Now we see as above that (3.5) can always be solved and since there are no holomorphic vector fields on $(\mathcal{M}, J)$ for all $J \in \mathcal{C}^0_\omega(\mathcal{M})$, we get a unique $\beta_J(V)$ with the needed properties for all $J \in \mathcal{C}^0_\omega(\mathcal{M})$ and $V \in T_J\mathcal{C}^0_\omega(\mathcal{M})$. Then we have that Theorem 6.11 applies and we we get a Hitchin connection in $H^{(k)}$ over all of $\mathcal{C}^0_\omega(\mathcal{M})$. We have here normalized $\psi(V)$ and $\varphi(V)$ by requiring that they have zero average over $\mathcal{M}$.

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