Knapp-Stein Operators and Fourier Transformations

PhD Thesis

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Resumé

I den følgende afhandling analyserer vi visse Knapp-Stein sammenfletningsoperatorer via distributionsteori på Liegrupper. Sådan en operator virker ved foldning med en distribution i enten det kompakte eller det ikkekompakte billede og vi undersøger om distributionen er positivt definit. I bekræftende fald kan operatoren bruges til at konstruere nye invariante indre produkter og dermed nye repræsentationer.

På en vilkårlig kompakt Liegruppe K giver vi en Bochners Sætning, der bestemmer hvornår en distribution (med operatorværdier) er positivt definit. På Heisenberggruppen $\overline{N}$ konstruerer vi en Fouriertransformation, der bestemmer hvornår en distribution er positivt definit på et bestemt $*$-ideal i algebraen af Schwartzfunktioner på $\overline{N}$. Konstruktionen bruger teorien om tensorprodukter af topologiske vektorrum: Givet to hypokontinuerte bilineære afbildninger har vi brug for at vide, at deres “tensor produkt” er hypokontinuert. Vi beviser sådan en sætning i tilfældet hvor nogle af de involverede rum er af type $\mathcal{F}$ eller $\mathcal{DF}$.

Til sidst udregner vi Knapp-Stein-kernerne og analyserer deres Fouriertransformerede i tre tilfælde: Først betragter vi $\mathrm{SL}(d, \mathbb{R})$ med en stor parabolsk undergruppe “i midten” og repræsentationer induceret fra karaktererne i $\overline{N}$-billedet. Dernæst betrager vi den samme repræsentation i tilfældet $d = 3$ på K, og til sidst betragter vi $\widetilde{\mathrm{SL}}(3, \mathbb{R})$ i det kompakte billede, hvor $K = \mathrm{SU}(2)$, og vi inducerer fra den naturlige repræsentation af $M \subseteq \mathrm{SU}(2)$. For et bestemt valg af parametre giver dette en ny og eksplicit konstruktion af Torassorepræsentationen.

Abstract

In the following thesis we analyse certain Knapp-Stein intertwining operators via distribution theory on Lie groups. Such an operator acts as convolution with a distribution in either the compact or the noncompact picture, and we consider the question of whether or not the distribution is positive definite. In the affirmative case, the operator can be used to construct new invariant inner products and thus new representations.

On an arbitrary compact Lie group K we give a Bochner's Theorem which determines when an (operator-valued) distribution is positive definite. On the Heisenberg group $\overline{N}$ we construct a Fourier transform which determines when a distribution is positive definite on a certain $*$-ideal in the algebra of Schwartz functions on $\overline{N}$. The construction uses the theory of tensor products of topological vector spaces. Namely, given two hypocontinuous bilinear maps we need to know that their "tensor product" is hypocontinuous. We prove such a theorem in the case where some of the involved spaces are type $\mathcal{F}$ or $\mathcal{DF}$.

Lastly, we compute concretely the Knapp-Stein kernels and analyse their Fourier transform in three cases: First we consider $\mathrm{SL}(d,\mathbb{R})$ with a big "middle parabolic" and representations induced from characters in the $\overline{N}$-picture. Second, we consider the same representation in the case $d=3$ on K, and thirdly we consider $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ in the compact picture where $K = \mathrm{SU}(2)$ and we induce from the natural representation of $M \subseteq \mathrm{SU}(2)$. For a certain choice of parameters this gives a new and explicit construction of the Torasso representation.

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Introduction

It is an interesting problem with applications in physics to determine, given a group of symmetries G, its unitary dual, i.e., to determine all the irreducible unitary representations of G. We typically think of the symmetries as falling into continuous or even smooth families so the usual case is the one where G is a Lie group. In this thesis we work in the context of G a semisimple Lie group. Then it is known where to look for the unitary dual: By Langlands classification, any irreducible admissible representation of G can be found as a quotient of a representation in the family of representations $\{U(S,\varepsilon,\lambda)\}_{S,\varepsilon,\lambda}$ ([21, Thm. 8.54]). A unitary irreducible representation is admissible so the only thing missing from determining the unitary dual is to determine which $U(S,\varepsilon,\lambda)$ that admits a unitary quotient.

The construction is roughly as follows (cf. [21, Ch. VII]): We have Iwasawa decompositions $G = KAN$ into subgroups K, A, N where $K \subseteq G$ is compact, and with $M = Z_K(\mathfrak{a})$ we have a minimal parabolic MAN. Then parabolic induction from a representation $\varepsilon \in \widehat{M}$ and depending on $\lambda \in \mathfrak{a}'_{\mathbb{C}}$ gives a representation — together such representations make up the nonunitary principal series. Any subgroup that contains a conjugate of MAN is called a parabolic subgroup. Any such subgroup S admits a Langlands decomposition $S = M_S A_S N_S$ just as the minimal parabolic. Then again parabolic induction gives us representations $U(S,\varepsilon,\lambda)$ depending on $\varepsilon \in \widehat{M_S}$, $\lambda \in \mathfrak{a}'_{\mathbb{C}}$.

The Langlands classification then says more specifically that we must look for irreducible admissible representations as irreducible quotients of $U(S,\varepsilon,\lambda)$ where $S \supseteq MAN$ and where $\operatorname{Re}\lambda$ is big enough. For two different parabolics S, S' with $M_S = M_{S'}$, $A_S = A_{S'}$ there is a formal intertwiner

$$A(S', S, \varepsilon, \lambda) : U(S,\varepsilon,\lambda) \to U(S',\varepsilon,\lambda)$$

defined on the smooth vectors and the subquotient can then be found as the image of $A(\overline{S}, S, \varepsilon, \lambda)$ where $\overline{S} = \Theta(S)$ for $\Theta : G \to G$ the Cartan involution.

When looking for *unitary* irreducible representations it is convenient to look at slightly different intertwiners. For $w \in N_K(\mathfrak{a})$ we have the unbounded Knapp-Stein intertwiner

$$A_S(w,\varepsilon,\lambda) : U(S,\varepsilon,\lambda) \to U(S, w\varepsilon, w\lambda)$$

defined on the smooth vectors where $N_K(\mathfrak{a})$ acts on $\widehat{M}$ and $\mathfrak{a}'_{\mathbb{C}}$ via conjugation.

Let us fix a parabolic S and let its Langlands decomposition be MAN. Write $P(\varepsilon, \lambda) = U(S, \varepsilon, \lambda)$ and let us also fix w and use $T_\varepsilon^\lambda = A_S(w, \varepsilon, \lambda)$. Owing to the decomposition $G = KAN$, $P(\varepsilon, \lambda)$ can be realised as a space of functions on K and owing to the decomposition of almost all of G into $\overline{N}MAN$, $\overline{N} = \Theta(N)$, $P(\varepsilon, \lambda)$ can also be realised as a space of functions on $\overline{N}$.

We consider certain w so that T_ε^λ in either picture is given by convolution with the distribution

$$x \mapsto a(w^{-1}x)^{\lambda-\rho}\varepsilon(m(w^{-1}x)) \tag{1}$$

where a, m refer to the projections in the $\overline{N}MAN$-decomposition. It is known ([21, Thm. 8.38]) that the operator T_ε^λ makes sense defined on the smooth vectors for $\operatorname{Re}\lambda$ big and also that it admits a meromorphic extension to the entirety of $\mathfrak{a}'_{\mathbb{C}}$. Correspondingly, we have a meromorphic family of distributions defined by Eq. (1).

There is a G-invariant sesquilinear pairing

$$\begin{aligned} P(\lambda, \varepsilon) \times P(-\overline{\lambda}, \varepsilon) &\to \mathbb{C} \\ (\varphi, \psi) &\mapsto (\varphi|\psi) \end{aligned}$$

which when the representation space of $P(\lambda, \varepsilon)$ and $P(-\overline{\lambda}, \varepsilon)$ is realised as functions on $\overline{N}$ or K is given as the usual sesquilinear pairing over these groups:

$$(\varphi|\psi) = \int_K (\varphi(k)|\psi(k))\, dk = \int_{\overline{N}} (\varphi(n)|\psi(n))\, dn.$$

So that when $w\lambda = -\overline{\lambda}$ one obtains a G-invariant sesquilinear pairing on the smooth vectors $C^\infty P(\lambda, \varepsilon)$ via T_ε^λ:

$$\begin{aligned} C^\infty P(\lambda, \varepsilon) \times C^\infty P(\lambda, \varepsilon) &\to \mathbb{C} \\ (\varphi, \psi) &\mapsto (T_\varepsilon^\lambda \varphi|\psi). \end{aligned}$$

When this pairing is positive, i.e., when $(T_\varepsilon^\lambda \varphi|\varphi) \geq 0$ for all φ, we have a G-invariant pseudo-inner product which enables us to construct a quotient representation of $P(\varepsilon, \lambda)$ in the usual way. This is a way to construct the so-called "complementary series" of representations.

We are interested in determining for which λ and ε the above pairing is positive for a few choices of G and MAN. It is clear that this problem is related to the question of whether or not the distribution in Eq. (1) is positive definite as a distribution on $\overline{N}$ or on K. In fact, in the compact picture this is an equivalent problem since the smooth vectors of $P(\lambda, \varepsilon)$ are the smooth functions in the compact realisation of $P(\lambda, \varepsilon)$. In the noncompact picture we can only say that it is necessary that the distribution is positive definite for the pairing to be positive. For the cases that we consider it is already known for what parameters the pairing is positive but what is new here is the explicit calculations of the kernels and their analysis through Fourier analysis on $\overline{N}$ and K.

The problem of telling whether a distribution on $\mathbb{R}^d$ is positive definite is solved by Bochner's Theorem: Such a distribution f is positive definite if and only if it is tempered and its Fourier transform $\widehat{f}$ is positive, i.e., if and only if $\widehat{f}$ is a tempered positive measure. For a general Lie group there is no analogous result, except in the case of a compact Lie group K for which the Fourier transform of a distribution is a collection of matrices. Bochner's Theorem in this case says that the distribution is positive definite if and only if all these matrices are positive. Theorems 2.3.4 and 2.3.7 give the Bochner's Theorem on a compact Lie group in the scalar and operator-valued case. In general, the contents of Chapter 2 perhaps except for the operator-valued cases is known but I have not seen it collected anywhere.

We will consider the case where $\overline{N}$ is a Heisenberg group and as one of the simplest non-Abelian Lie groups this group would be a good candidate for a generalisation of Bochner's Theorem. The unitary dual of the Heisenberg group H in d dimensions can be split into two families: There are the infinite dimensional representations parametrised by an $h \in \mathbb{R}^* = \mathbb{R} \setminus 0$ and there are the characters parametrised by $\mathbb{R}^d \times \mathbb{R}^d$. The first gives rise to a continuous linear Fourier transform

$$\mathcal{S}(H) \to \mathcal{E}(\mathbb{R}^*, \mathcal{L}_0)$$

where $\mathcal{L}_0 \subseteq \mathcal{L} := \mathcal{L}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$ is the subspace of operators with kernels in $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$, considered with the topology from this space. The second gives rise to a continuous linear Fourier transform

$$\mathcal{S}(H) \to \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d).$$

Both take the convolution to pointwise multiplication and the first comes with a Fourier Inversion Theorem so that it is injective. One can use the Inversion Theorem to define an appropriate Fourier transform

$$\mathcal{S}'(H) \to \mathcal{D}'(\mathbb{R}^*, \mathcal{L})$$

which is not injective. Its kernel is the space $\mathrm{Pol}_c(H) \subseteq \mathcal{S}'(H)$ of distributions that are polynomial along the center. This suggests considering the $*$-ideal $\mathcal{S}^\infty(H)$ in the $*$-algebra $\mathcal{S}(H)$ consisting of functions on which $\mathrm{Pol}_c(H)$ vanish. The Fourier transform on these functions becomes a $*$-algebra isomorphism

$$\mathcal{S}^\infty(H) \to \mathcal{S}_0(\mathbb{R}, \mathcal{L}_0)$$

to a certain multiplication algebra. The consideration of the "Lizorkin space" $\mathcal{S}^\infty(H)$ and its Fourier transform is new.

In order to work efficiently with these spaces it is convenient to be well-versed in tensor product theory of locally convex spaces. I have included an appendix that outlines the general theory of topological vector spaces for convenience and Chapter 4 is dedicated to a certain problem within this theory: Given locally convex spaces X, Y, Z, E, F, G and hypocontinuous bilinear maps $M : X \times Y \to Z$, $B : E \times F \to G$ is the "tensor product"

$$M \mathbin{\widehat{\otimes}} B : X \mathbin{\widehat{\otimes}} E \times Y \mathbin{\widehat{\otimes}} F \to Z \mathbin{\widehat{\otimes}} G$$

hypocontinuous? Here $\widehat{\otimes}$ denotes taking the completed projective tensor product but the problem could be stated with other topologies or with other regularities of M, B. An instance of this problem is the case where M is convolution of scalar-valued functions on a Lie group so $M \widehat{\otimes} B$ is a corresponding convolution of vector-valued functions. This is considered in [4], for instance, where the focus is on proving continuity of $M \widehat{\otimes} B$. We prove a somewhat general result Theorem 4.1.8 which says that the tensor product is hypocontinuous if, in addition to some reasonable assumptions on the involved spaces, we assume that X, E are both $\mathcal{F}$ or both $\mathcal{DF}$. This result would be new, but after proving it I discovered it in [3].

Among the distributions in $\mathrm{Pol}_c(H)$ one can determine the positive definite distributions with the help of the character Fourier transform $\mathcal{S}(H) \to \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ which has a transpose $\mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d) \to \mathcal{S}'(H)$ which actually maps into $\mathrm{Pol}_c(H)$ and takes a positive definite distribution on $\mathbb{R}^d \times \mathbb{R}^d$ to a positive definite distribution on H. Unfortunately, the full Bochner's Theorem still eludes me and at the end I only achieve necessary conditions for positive definiteness.

At the end of the thesis we use the Fourier transform on K and $\overline{N}$ to say determine positive definiteness of the Knapp-Stein kernel distributions. Considering $G = \mathrm{SL}(d, \mathbb{R})$ split into blocks of sizes $1, d-2, 1$ with the middle parabolic consisting of matrices of the form

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

the group $\overline{N}$ is exactly the $d-2$-dimensional Heisenberg group. When inducing from certain characters of the rather large M (in the $d > 3$-case it contains $\mathrm{SL}(d-2, \mathbb{R})$) we obtain explicitly the Knapp-Stein kernel as a distribution

$$\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \ni (x, y, z) \mapsto |z - xy|_\varepsilon^{\lambda - 1} |z|_\varepsilon^{\overline{\lambda} - 1}$$

where $xy \in \mathbb{R}$ is the inner product of $x, y \in \mathbb{R}^d$. This family of distributions has been considered in [19] where among other things it is proven (for $d = 3$) that $T_\varepsilon^\lambda \varphi$ is a solution to the equation

$$(XY + YX)\psi = 0$$

for $\psi \in \mathcal{S}(H)$ where X, Y, Z are the usual basis for the Lie algebra of H, $[X, Y] = Z$.

We compute the Fourier transform and find, in the $d = 3$ case, necessary conditions for the existence of the complementary series, cf. Theorem 5.1.5. This agrees with other methods, for instance [36, Thm. 4.1]. The case of $\mathrm{Re}\,\lambda = \frac{1}{2}$ is special because in this case the Fourier transform is a generalised family of rank-one operators given by projection onto the function $|x|_\varepsilon^{-\frac{1}{2}}$. This special case is also simple in the $d > 3$-case and we suggest a conjecture in Theorem 5.1.7.

We also carry out the complete analysis of the same Knapp-Stein in the case $d = 3$ in the compact picture. Then $K = \mathrm{SO}(3)$ has a convenient Euler angle parametrisation (ψ, θ, φ) in which the kernel becomes

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R} \ni (\psi, \theta, \varphi) \mapsto |\sin \psi|_{\varepsilon}^{\lambda-1} |\sin \theta|^{2\operatorname{Re}\lambda - 1} |\sin \varphi|_{\varepsilon}^{\overline{\lambda}-1}$$

Fourier transformation on K then gives us a collection of matrices which we analyse to obtain necessary and sufficient conditions for positive definiteness, cf. Theorem 6.2.2. Here we can see again that the case $\operatorname{Re}\lambda = \frac{1}{2}$ is special because the middle term disappears.

Lastly, we consider the case of $G = \widetilde{\mathrm{SL}}(3, \mathbb{R})$ which we only analyse in the compact picture. Then $K \cong \mathrm{SU}(2)$ and M is the quaternion group. There is a single 2-dimensional irreducible representation of M that doesn't factor through the M of $\mathrm{SL}(3, \mathbb{R})$ and we induce from this representation. Using again the Euler angles we carry out the Fourier transform and obtain again necessary and sufficient conditions for positive definiteness, cf. Theorem 6.3.4. The case of $\lambda = \frac{1}{2}$ is a special isolated representation which we consider in Theorem 6.3.12. This representation is the famous Torasso representation constructed in [38]. The construction here can be regarded as the integral operator counterpart to the differential operator construction of the same representation in [24].

Analysis in the noncompact picture could perhaps be carried out but the calculations in Appendix B.2 suggests that the Knapp-Stein kernel (which is now operator-valued) has a very complicated Fourier transform: The kernel in this case not only depends on the sign of $z - xy$ and the sign of z but it also depends on the sign of y. The Fourier transform can be calculated in principle but the expression on the Fourier side needs to be simplified before analysis is tenable.

Symbols

$\mathcal{O}_M(N)$	Multiplier space for $\mathcal{S}(N)$ (page 30).
$\mathcal{O}_M(D)$	Multiplier space for $\mathcal{S}(D)$ (page 37).
$\mathcal{O}_M(\mathbb{R} \setminus 0)$	Multiplier space for $\mathcal{S}_0(\mathbb{R})$ (page 66).
$\mathcal{S}_d$	Schwartz space on $\mathbb{R}^d$ (page 117).
$\mathcal{L}^d$	$\mathcal{L}(\mathcal{S}_d, \mathcal{S}'_d)$, isomorphic to $\mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$.
$\mathcal{L}^d_0$	Subspace of $\mathcal{L}$ with Schwartz kernels, isomorphic to $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$.
$\mathcal{S}(\mathbb{R}^d)_c$	Schwartz space convolution algebra (page 23).
$\mathcal{S}(\mathbb{R}^d)_m$	Schwartz space multiplication algebra (page 23).
$\mathcal{S}^\infty(H)$	Heisenberg group Lizorkin space (page 31).
$\mathcal{S}^N(H)$	Heisenberg group Lizorkin-like space (page 32).
$L^1(\mathcal{H})$	Trace-class operators on a Hilbert space $\mathcal{H}$.
$(\cdot\|\cdot)$	Sesquilinear pairing.
$\langle \cdot, \cdot \rangle$	Dual, bilinear pairing, or the pairing of $\mathcal{D}(U, E)$ with $\mathcal{D}(U)$.

Chapter 1

Knapp-Stein Intertwiners for $\mathrm{SL}(d,\mathbb{R})$

In this chapter we introduce the representation theoretic background for the thesis. For the group $\mathrm{SL}(d,\mathbb{R})$ we spell out the Iwasawa and the $\overline{N}MAN$-decomposition when MAN is the minimal parabolic. We are interested in the case where $\overline{N}$ is a Heisenberg group so we consider also the case where MAN is a "middle parabolic" with a big middle block. We shall also consider the double covering group $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ so we introduce some of the notation from [38]. We then briefly introduce the principal series representations and the Knapp-Stein intertwiners. For motivation we also consider the $\mathrm{SL}(2,\mathbb{R})$-case. The material in this chapter is well-known and can be found in [21].

1.1 Minimal Parabolic for $\mathrm{SL}(d,\mathbb{R})$

We will go through the root system theory for $\mathrm{SL}(d,\mathbb{R})$ in order to introduce the relevant notation. Let $G = \mathrm{SL}(d,\mathbb{R})$ be the group of matrices with determinant 1. Its Lie algebra is $\mathfrak{g} = \mathfrak{sl}(d,\mathbb{R})$, the Lie algebra of matrices with trace 0.

Cartan Decomposition. We have the Cartan involution $\Theta : G \to G$, $g \mapsto g^{-t} := (g^t)^{-1}$ where ^{t} denotes transpose. Its derivative $\theta = d\Theta : \mathfrak{g} \to \mathfrak{g}$ is given by $X \mapsto -X^t$ is the Cartan involution on the Lie algebra level. These maps give us a subgroup K of fixed elements of Θ and a subalgebra $\mathfrak{k}$ of fixed elements of θ. The Lie algebra $\mathfrak{k}$ is the Lie algebra corresponding to K. Here we have $K = \mathrm{SO}(d)$ and $\mathfrak{k} = \mathfrak{so}(d)$. On the complexification $\mathfrak{g}_{\mathbb{C}}$ we have the inner product

$$(X|Y) := \operatorname{tr} XY^*$$

which restricts to a real inner product on $\mathfrak{g}$ for which θ is self-adjoint. Note also $\theta^2 = 1$ so $\mathfrak{g}$ splits into an orthogonal sum of eigenspaces for θ,

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}.$$

which also gives a decomposition for G which is not so important for our purposes so we will omit it.

Iwasawa Decomposition. We take $\mathfrak{a}$ to be the subalgebra of diagonal matrices in $\mathfrak{g}$. Then $\mathfrak{a}$ is a maximal Abelian subspace of $\mathfrak{p}$ that acts symmetrically on $\mathfrak{g}$ which then splits into $\mathfrak{a}$-eigenspaces. For each $\lambda \in \mathfrak{a}'$ let

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \,|\, [H,X] = \lambda(H)X \,\forall H \in \mathfrak{a}\}.$$

The nonzero λ for which $\mathfrak{g}_\lambda \neq 0$ are the *roots* of this system. Let E_{ij} be the matrix with 0 at every entry except the (i,j)'th one where there is a 1. Let $e_j \in \mathfrak{a}'$ be the functional that gives the j'th diagonal entry of a matrix in $\mathfrak{a}$. Then we find

$$\mathfrak{g}_{e_i - e_j} = \mathrm{span}(E_{ij})$$

and

$$\mathfrak{g}_0 = \mathfrak{a}.$$

We can partition the roots into positive and negative ones. Mostly we will do this by declaring that a root $e_i - e_j$ is positive if $i < j$ and negative if $i > j$ but it will also become convenient to switch this around later. Then it becomes the case that if λ, μ are positive then $\lambda + \mu$ is positive also if it is a root. We collect all the positive roots spaces and the negative root spaces

$$\mathfrak{n} = \sum_{\lambda > 0} \mathfrak{g}_\lambda, \quad \overline{\mathfrak{n}} = \sum_{\lambda < 0} \mathfrak{g}_\lambda.$$

Then $\mathfrak{n}$ resp. $\overline{\mathfrak{n}}$ consist of all strictly upper resp. lower triangular matrices in $\mathfrak{g}$. Then we have the Iwasawa Decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$$

which on the Lie group level gives rise to

$$G = KAN.$$

We use the notation

$$g = \kappa(g)\alpha(g)\nu(g)$$

so give the latter decomposition of an element $g \in G$. In this case this is the decomposition gotten out of the Gram-Schmidt Orthogonalisation Process. Indeed, the columns of g is a basis of $\mathbb{R}^n$ which then through this process gives an orthogonal basis which are exactly the columns of $\kappa(g)$. Going through the process carefully one can determine $\alpha(g)$ and $\nu(g)$, too.

$\overline{N}MAN$**-Decomposition.** The general theory gives us another decomposition, namely

$$G = \overline{N}MAN.$$

Here $M = Z_K(\mathfrak{a})$. It says that for almost every $g \in G$ (in the sense that the dimension of the complement has lower dimension of G) there is a decomposition

$$g = \overline{n}(g)m(g)a(g)n(g).$$

We will need an explicit determination of the middle part. Note that in our case AM is simply the subgroup consisting of all diagonal elements in G.

For a matrix $A \in M_d(\mathbb{C})$ and $1 \le l \le d$ let $A[l] \in M_l(\mathbb{C})$ be the l'th principal submatrix, i.e., $A[l]_{ij} = A_{ij}$, $1 \le i, j \le l$. The LDU-decomposition of A is a decomposition

$$A = LDU$$

where L resp. U is lower resp. upper triangular with 1's on the diagonal and where D is a diagonal matrix. This decomposition doesn't always exist. One gets

Proposition 1.1.1. *An invertible matrix A has an LDU-decomposition if and only if* $\det A[l] \neq 0$ *for all l, $1 \le l \le d$. In this case the diagonal components D_l, $1 \le l \le d$ of D are given by*

$$D_l = \frac{\det A[l]}{\det A[l-1]}.$$

Proof. We can decompose each matrix in block matrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad L = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix}, \quad D = \begin{pmatrix} D_{11} & 0 \\ 0 & D_{22} \end{pmatrix}, \quad U = \begin{pmatrix} U_{11} & U_{21} \\ 0 & U_{22} \end{pmatrix}.$$

Let us make the decomposition such that $A_{11} = A[l]$. Then $A = LDU$ implies

$$A[l] = A_{11} = L_{11} D_{11} U_{11}$$

which is the LDU-decomposition of $A[l]$. Taking determinants one gets

$$\det A[l] = \det D[l] = \prod_{1 \le j \le l} d_j$$

Consequently,

$$d_l = \frac{\det A[l]}{\det A[l-1]}.$$

In order to give existence we only give a brief argument. One goes forward iteratively and by taking principal submatrices it is always only necessary to look at the above blocks where L_{11}, U_{11}, D_{11} have been determined (in the start of the iteration one sets $L_{11} = U_{11} = 1$), $L_{22} = U_{22} = 1$ and we only need to determine *vectors* L_{21} and U_{21}. Then can take

$$\begin{aligned} L_{21} &= A_{21}(L_{11}D_{11})^{-1} \\ U_{12} &= (L_{11}D_{11})^{-1}A_{12}. \end{aligned}$$ □

For G, we get

Proposition 1.1.2. *For every element $g \in G$ with* $\det g[l] \neq 0$*, $1 \le l \le d$, we have*

$$g = \overline{n}(g) m(g) a(g) n(g)$$

where $\overline{n}(g) \in \overline{N}$, $m(g) \in M$, $a(g) \in A$, $n(g) \in N$ and where the diagonal elements $m(g)_l$, $a(g)_l$ of $m(g)$, $a(g)$ are given by

$$a(g)_l = \frac{|\det g[l]|}{|\det g[l-1]|},$$
$$m(g)_l = \frac{\operatorname{sgn}\det g[l]}{\operatorname{sgn}\det g[l-1]}.$$

Parametrisation of $\mathfrak{a}'_{\mathbb{C}}$. We call the roots

$$\alpha_j = e_j - e_{j+1}$$

simple because they are the only positive roots that cannot be written as a linear combination of the other positive roots. They span the entirety of $\mathfrak{a}'_{\mathbb{C}}$ but we will use a different basis still. Since $(\cdot|\cdot)$ is an inner product on $\mathfrak{a}$ it is in particular nondegenerate so it induces an isomorphism $\mathfrak{a} \cong \mathfrak{a}'$ which allows us to transfer the inner product to $\mathfrak{a}'$. Explicitly, we need to determine for each $\lambda \in \mathfrak{a}'$ the unique $H_\lambda \in \mathfrak{a}$ such that

$$(H|H_\lambda) = \lambda(H).$$

Then

$$(\lambda|\mu) = (H_\lambda|H_\mu) = \lambda(H_\mu) = \mu(H_\lambda).$$

One finds easily that $H_{\alpha_j} = E_{jj} - E_{j+1,j+1}$. This means

$$(\alpha_j|\alpha_l) = \begin{cases} 2 & j = l, \\ -1 & |j-l| = 1, \\ 0 & \text{Otherwise.} \end{cases}$$

We introduce the *fundamental weights* $\delta_j \in \mathfrak{a}'$ given uniquely by

$$(\delta_j|\alpha_l) = \delta_{jl}$$

where δ_{jl} is the Kronecker delta. This implies

$$\delta_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$$
$$\alpha_l = \frac{1}{2}\alpha_{l-1} + \alpha_l + \frac{1}{2}\alpha_{l+1}, \quad 2 \le l \le d-2$$
$$\alpha_{d-1} = \frac{1}{3}\alpha_{d-2} + \frac{2}{3}\alpha_{d-1}.$$

Perhaps nicer is the formula

$$\delta_l = \sum_{j=1}^{l} e_j$$

which holds because

$$\left(\sum_{j=1}^{l} e_j\right)(H_{\alpha_k}) = \delta_{lk}.$$

Why is this a better basis for our purposes? This essentially comes down to the fact that the interaction with $a(g)$ is better. Note that the exponential gives a bijection $\mathfrak{a} \to A$ which allows us to define a pairing

$$\begin{aligned} A \times \mathfrak{a}'_{\mathbb{C}} &\to \mathbb{C} \\ (\exp H, \lambda) &\mapsto \exp(\lambda(H)). \end{aligned}$$

We denote the pairing of $a \in A$ and $\lambda \in \mathfrak{a}'_{\mathbb{C}}$ by a^λ. Then we obviously have

$$\begin{aligned} a^0 &= e^\lambda = 1 \\ a^{\lambda+\mu} &= a^\lambda a^\mu \\ (a_1 a_2)^\lambda &= a_1^\lambda a_2^\lambda. \end{aligned}$$

We find then

$$a^{\lambda\delta_l} = \prod_{j=1}^{l} a_j^\lambda$$

so that

$$a(g)^{\lambda\delta_l} = \prod_{j=1}^{l} \frac{|\det g[j]|^\lambda}{|\det g[j-1]|^\lambda} = |\det g[l]|^\lambda.$$

We identify $\mathfrak{a}'_{\mathbb{C}} \cong \mathbb{C}^{n-1}$ by $\lambda \in \mathbb{C}^{n-1}$ corresponding to $\sum_{j=1}^{n} \lambda_j \delta_j$. With this notation,

$$a(g)^\lambda = \prod_{j=1}^{n} |\det g[j]|^{\lambda_j}.$$

The Case of $\mathrm{SL}(2, \mathbb{R})$**.** It is convenient to first introduce notation on $\mathrm{SL}(2, \mathbb{R})$ and then using injections into $\mathrm{SL}(d, \mathbb{R})$ to get notation in general. The highlighted elements of $\mathfrak{sl}(2, \mathbb{R})$ are then

$$X := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad W := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and in the case $d = 2$, $\mathfrak{n} = \mathrm{span}(X)$, $\overline{\mathfrak{n}} = \mathrm{span}(Y)$, $\mathfrak{a} = \mathrm{span}(a)$, $\mathfrak{k} = \mathrm{span}(W)$ so that we have parametrisations of the corresponding subgroups $N, \overline{N}$ and K,

$$\begin{aligned} n(t) &:= \exp(tX) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \overline{n}(t) := \exp(tY) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \\ k(t) &:= \exp(tW) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \end{aligned}$$

for $t \in \mathbb{R}$. For A, I choose a slightly different convention

$$a(r) := \exp(\log(r)H) = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$$

for $r > 0$. So that while $n, \overline{n}, k$ are homomorphisms from $(\mathbb{R}, +)$, a is a homomorphism from $((0,\infty), \cdot)$.

Two elements, both in K, are worth singling out, namely

$$m = k(\pi) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad w = k\left(\frac{\pi}{2}\right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

For $1 \leq j \leq l \leq d$, there are natural injective Lie algebra homomorphisms $d\iota_{jl} : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{sl}(d,\mathbb{R})$ given by inclusion in the (j,l) entries

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto aE_{jj} + bE_{jl} + cE_{lj} + dE_{ll} \tag{1.1}$$

The integrated map exists $\iota_{jl} : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(d,\mathbb{R})$ as an injective Lie group homomorphism and it is again given by Eq. (1.1). Now, we have the corresponding notation in $\mathrm{SL}(d,\mathbb{R})$:

$$X_{jl} = d\iota_{jl}(X), Y_{jl} = d\iota_{jl}(Y), H_{jl} = d\iota_{jl}(H), W_{jl} = d\iota_{jl}(H),$$
$$n_{jl}(t) = \iota_{jl}n(t), \overline{n}_{jl}(t) = \iota_{jl}\overline{n}(t), a_{jl}(r) = \iota_{jl}a(r), k_{jl}(t) = \iota_{jl}k(t)$$

and

$$m_{jl} = \iota_{jl}(m),$$
$$w_{jl} = \iota_{jl}(w).$$

The Subgroup M. We return to the case of a general d. $M = Z_K(\mathfrak{a})$ is the subgroup of diagonal matrices with entries $\{-1, 1\}$. It is generated by the diagonal matrices m_{jl}, $1 \leq j < l \leq d$. Actually, it is generated by the elements $m_j := m_{j,j+1}$, $j = 1, 2, \ldots, d-1$. Any irreducible representation σ of M is one-dimensional and it is uniquely given by

$$\varepsilon_j = \sigma(m_j).$$

Thus $\widehat{M} \cong \{-1, 1\}^{d-1}$. We will make the isomorphism explicit: Introduce irreducible representations $e_j : M \to \mathbb{C}$, $d \mapsto d_j$ (j'th diagonal component) then with $\delta_l = e_1 \otimes e_2 \cdots \otimes e_l$ we have

$$\delta_l(m_j) = \begin{cases} -1 & \text{if } j = l, \\ 1 & \text{if } j \neq l. \end{cases}$$

We can then introduce for every $\varepsilon \in \{-1, 1\}^{d-1}$,

$$\sigma_\varepsilon = \bigotimes_{\{j \mid \varepsilon_j = -1\}} \delta_j$$

and obtain

$$\sigma_\varepsilon(m_j) = \varepsilon_j.$$

Also,

$$\delta_l(m(g)) = \prod_{j=1}^{l} \frac{\operatorname{sgn}\det g[j]}{\operatorname{sgn}\det g[j-1]} = \operatorname{sgn}\det g[l]$$

so that

$$\sigma_\varepsilon(m(g)) = \prod_{\{j|\varepsilon_j=-1\}} \operatorname{sgn}\det g[j] = \prod_{j=1}^{n} |\det g[j]|_{\varepsilon_j}^{0}$$

where we use, for $x \in \mathbb{R}$,

$$|x|_\varepsilon^\lambda := \begin{cases} |x|^\lambda & \text{if } \varepsilon = +1, \\ \operatorname{sgn}(x)|x|^\lambda & \text{if } \varepsilon = -1. \end{cases}$$

Now, using both the identifications $\mathfrak{a}'_\mathbb{C} \cong \mathbb{C}^{d-1}$ and $\widehat{M} \cong \{-1,1\}^{d-1}$ we have

$$a(g)^\lambda \varepsilon(m(g)) = \prod_{j=1}^{n} |\det g[j]|_{\varepsilon_j}^{\lambda_j}$$

Action of W on $\mathfrak{a}'_\mathbb{C}$ and $\widehat{M}$. The Weyl group W is the group of linear transformations on $\mathfrak{a}'$ generated by the root reflections s_α,

$$s_\alpha(\varphi) = \varphi - 2\frac{(\varphi|\alpha)}{|\alpha|^2}\alpha.$$

It is isomorphic to $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ where $w \in N_K(\mathfrak{a})$ acts on $\mathfrak{a}'$ by

$$[w\lambda](H) := \lambda(w^{-1}Hw).$$

The group W is generated by the root reflections s_{α_j} corresponding to the simple roots. The group $N_K(\mathfrak{a})$ is generated by the elements w_{jl}, m_{jl}. So then $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ is generated by $w_{jl}Z_K(\mathfrak{a})$. One notes that if $j < k < l$ then

$$w_{jl} = w_{jk}w_{kl}w_{jk}^{-1}.$$

This can be proven inside $\mathrm{SL}(3,\mathbb{R})$ by the corresponding identity

$$w_{13} = w_{12}w_{23}w_{12}^{-1}.$$

It follows that $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ is generated by the simple elements $w_jZ_K(\mathfrak{a})$ where $w_j = w_{j,j+1}$. Actually, the $w_jZ_K(\mathfrak{a})$ corresponds to the root reflection s_{α_j}. Indeed, w_j acts on $\mathfrak{a}$ by transposing the j'th and the $(j+1)$'th coordinate so $w_je_j = e_{j+1}$, $w_je_{j+1} = e_j$ and $w_je_l = e_l$ for $l \neq j, j+1$. This is exactly how s_{α_j} acts.

The *length* of an element $w \in W$ is the minimal number of factors when w is written as a product of simple reflections. It is part of the general theory that a root system has a unique *longest element* $w \in W$ which as it turns out is given by $w\Pi = -\Pi$ where $\Pi \subseteq \Delta$ is the set of simple roots. The longest element will have length $|\Delta_+| = \frac{d(d-1)}{2}$. For $d = 2, 3$ the longest element is w_{1d}. In general,

$$w_{1d} = w_1 w_2 \cdots w_{d-1} w_{d-2} \cdot w_1$$

and it turns out that the length of w_{1d} is $2d-3$. One can also see that the length of w_{jl} is $2(l-j)-1$. Anyway, this means that the w_{1d} is not the longest element for $d > 3$.

The longest Weyl group element $w_0 \in W$ has a pleasant action on the δ_l's. Indeed, the element

$$w_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^n i^{n(n+1)} & 0 & \cdots & 0 & 0 \end{pmatrix} \in N_K(\mathfrak{a})$$

corresponds to the action $E_{jj} \mapsto E_{d-j,d-j}$ on $\mathfrak{a}$ so that $w_0 e_j = e_{d-j}$ so $w_0 \alpha_j = -\alpha_{d-j}$. Since w_0 sends every simple root to minus a simple root it must be the longest element. We find

$$(w_0 \delta_j | \alpha_l) = -(\delta_j | \alpha_{d-l}) = -\delta_{j,d-l} = -\delta_{d-j,l}$$

so that $w_0 \delta_j = -\delta_{d-j}$.

There is also an action of $N_K(\mathfrak{a})$ on $M = Z_K(\mathfrak{a})$ which induces an action on $\widehat{M}$ given by

$$[w\sigma](m) = \sigma(w^{-1} m w)$$

for $\sigma \in \widehat{M}$. Since $w_0^{-1} m_j w_0 = m_{n-j}$ we find in terms of $\widehat{M} \cong \{-1, 1\}^{d-1}$ that

$$[w_0 \varepsilon]_j = \varepsilon_{n-j}.$$

In terms of $\mathfrak{a}'_{\mathbb{C}} \cong \mathbb{C}^{d-1}$ we have

$$[w_0 \lambda]_j = -\lambda_{n-j}.$$

1.2 Middle Parabolic in $\mathrm{SL}(d,\mathbb{R})$

When $d = 2, 3$ and MAN is the minimal parabolic, $\overline{N}$ is isomorphic to a Heisenberg group. This is not the case when $d > 3$. But we get $\overline{N}$ is isomorphic to a Heisenberg group if we consider a different parabolic. We split an element $g \in G$ into $1 \times (d-2) \times 1$ blocks so

$$g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}.$$

where $g_{11}, g_{33} \in M_1(\mathbb{R})$, $g_{22} \in M_{d-2}(\mathbb{R})$. We take a parabolic subgroup $P \subseteq G$ consisting of the elements with $g_{21} = 0$, $g_{31} = 0$, $g_{32} = 0$. Then this parabolic has a decomposition $P = MAN$. Here M, A, N are new groups, we will use $M_\mathfrak{p}$, $A_\mathfrak{p}$, $N_\mathfrak{p}$, etc. for the groups and Lie algebras above.

The Langlands Decomposition of P. In the parlance of [21, Prop. 5.23] we take $\Pi_P = \{\alpha_2, \dots, \alpha_{n-2}\}$. Then the Lie algebra $\mathfrak{a}$ of A is the subalgebra $\mathfrak{a} \subseteq \mathfrak{a}_{\mathfrak{p}}$ consisting of the elements orthogonal to the elements in Π_P which implies that $\mathfrak{a} = \mathrm{span}(H_1, H_2)$ where

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{d-2} I_{d-2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{d-2} I_{d-2} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Note that e_l restricted to $\mathfrak{a}$ is the same functional for $2 \leq l \leq d-2$. We have two simple roots

$$\alpha_1 = e_1 - e_l$$
$$\alpha_2 = e_l - e_d$$

where $2 \leq l \leq d-2$. The corresponding root spaces are

$$\mathfrak{g}_{\alpha_1} = \mathrm{span}\{E_{1l} \,|\, 2 \leq l \leq d-2\} \mathfrak{g}_{\alpha_2} = \mathrm{span}\{E_{jd} \,|\, 2 \leq j \leq d-2\}.$$

The only other positive root is $\alpha_1 + \alpha_2$ with root space

$$\mathfrak{g}_{\alpha_1+\alpha_2} = \mathrm{span}(E_{1d}).$$

Then we have

$$\mathfrak{n} = \mathfrak{g}_{\alpha_1} + \mathfrak{g}_{\alpha_2} + \mathfrak{g}_{\alpha_1+\alpha_2} = \left\{ \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \right\}$$
$$\overline{\mathfrak{n}} = \mathfrak{g}_{-\alpha_1} + \mathfrak{g}_{-\alpha_2} + \mathfrak{g}_{-\alpha_1-\alpha_2} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix} \right\}$$

Lastly we get

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \cong \mathfrak{sl}(d-2, \mathbb{R}).$$

Since A is the analytic subgroup with Lie algebra $\mathfrak{a}$ we find

$$A = \left\{ \begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 I & 0 \\ 0 & 0 & s_3 \end{pmatrix} \middle| \, s_j > 0, s_1 s_2^{d-2} s_3 = 1 \right\}$$

N is the analytic subgroup with Lie algebra $\mathfrak{n}$ and M_0 is the analytic subgroup with Lie algebra $\mathfrak{m}$ and $M = Z_K(\mathfrak{a}) M_0$. We find

$$Z_K(\mathfrak{a}) = S(O(1) \times O(d-2) \times O(1)),$$
$$M_0 = 1 \times \mathrm{SL}(d-2, \mathbb{R}) \times 1$$

so that

$$M = S(O(1) \times \mathrm{SL}_{\pm}(d-2, \mathbb{R}) \times O(1)).$$

The $\overline{N}MAN$-Decomposition.

Proposition 1.2.1. *For every $g \in \mathrm{SL}(d,\mathbb{R})$ written as*

$$g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$$

with $g_{11} \neq 0$ and $g_{22} - \frac{g_{21}g_{12}}{g_{11}} \in M_{d-2}(\mathbb{R})$ invertible we can write

$$g = \overline{n}(g)m(g)a(g)n(g)$$

where $\overline{n}(g) \in \overline{N}$, $n(g) \in N$ and

$$m(g) = \begin{pmatrix} \operatorname{sgn} g_{11} & 0 & 0 \\ 0 & \frac{g_{22}-\frac{g_{21}g_{12}}{g_{11}}}{\left|\det(g_{22}-\frac{g_{21}g_{12}}{g_{11}})\right|^{1/(d-2)}} & 0 \\ 0 & 0 & \operatorname{sgn} g_{11} \operatorname{sgn}\det(g_{22} - \frac{g_{21}g_{12}}{g_{11}}) \end{pmatrix} \in M$$

and

$$a(g) = \begin{pmatrix} |g_{11}| & 0 & 0 \\ 0 & |\det(g_{22} - \frac{g_{21}g_{12}}{g_{11}})|^{1/(d-2)} I & 0 \\ 0 & 0 & |g_{11}|^{-1}|\det(g_{22} - \frac{g_{21}g_{12}}{g_{11}})|^{-1} \end{pmatrix} \in A.$$

Proof. Writing out $g = \overline{n}maan$ using the block notation gives us for the four upper left blocks

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} m_1a_1 & m_1a_1n_{12} \\ \overline{n}n_{12}m_1a_1 & \overline{n}n_{12}m_1a_1n_{12} + m_2a_2 \end{pmatrix}.$$ □

Parametrisation of $\mathfrak{a}'_{\mathbb{C}}$. Again we use “fundamental weights” $\delta_1 = e_1$ and

$$\delta_2 = \sum_{j=1}^{d-1} e_j$$

so that $\delta_j(H_l) = \delta_{jl}$. Note that an arbitrary element a of A is

$$a = \begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 I & 0 \\ 0 & 0 & s_3 \end{pmatrix} = e^{\log(s_1)H_1} e^{-\log(s_3)H_3}$$

since $s_1 s_2^{d-2} s_3 = 1$. So we find

$$a^{\lambda_1\delta_1+\lambda_2\delta_2} = s_1^{\lambda_1} s_3^{-\lambda_3}$$

so with $\lambda = \lambda_1\delta_1 + \lambda_2\delta_2$ we have

$$a(g)^\lambda = |g_{11}|^{\lambda_1+\lambda_2}|\det(g_{22} - \frac{g_{21}g_{12}}{g_{11}})|^{\lambda_2}.$$

Actually, according to [26, p. 475] we have

$$\det(g_{22} - \frac{g_{21}g_{12}}{g_{11}}) = \det(g_{22})(1 - \frac{g_{12}g_{22}^{-1}g_{21}}{g_{11}})$$

so

$$a(g)^\lambda = |g_{11}|^{\lambda_1}|\det g_{22}|^{\lambda_2}|g_{11} - g_{12}g_{22}^{-1}g_{21}|^{\lambda_2}$$

Parametrisation of $\widehat{M}$. We single out the element

$$m_{1d} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

For any unitary irreducible representation σ of M, $\sigma(m_{1d})$ must be scalar by Schur's Lemma and $\sigma(m_{1d})^2 = 1$ so $\sigma(m_{1d}) \in \{-1, 1\}$. Also, the map

$$\mathrm{SL}_{\pm}(d-2,\mathbb{R}) \ni m \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \det m \end{pmatrix}$$

is an injective homomorphism so we conclude that $\widehat{M} \cong \{-1,1\} \times \mathrm{SL}_{\pm}(d-2,\mathbb{R})^{\wedge}$.

For our purposes we single out the representation $m \mapsto |\det m|^0_{\varepsilon_2}$ of $\mathrm{SL}_{\pm}(d-2,\mathbb{R})$ for a sign $\varepsilon_2 \in \{-1,1\}$ which combined with a choice of sign $\varepsilon_1\varepsilon_2$ for $\sigma(m_{1d})$ gives us four representations $\sigma_{\varepsilon_1,\varepsilon_2}$. This somewhat arbitrary choice of sign gives us that

$$\sigma_\varepsilon(m_{1l}) = \varepsilon_1$$
$$\sigma_\varepsilon(m_{jd}) = \varepsilon_2$$

for $2 \le j, l \le d-1$. Here m_{jl} are defined as above, i.e., m_{jl} is a diagonal matrix with entries 1 except for at the j'th and l'th index where the entry is -1.

Now one finds

$$\sigma_\varepsilon(m(g)) = |g_{11}|^0_{\varepsilon_1\varepsilon_2} |\det(g_{22} - \frac{g_{21}g_{12}}{g_{11}})|^0_{\varepsilon_2} = |g_{11}|^0_{\varepsilon_1} |\det g_{22}|^0_{\varepsilon_2} |g_{11} - g_{12}g_{21}|^0_{\varepsilon_2}$$

So if we use the shorthand ε for σ_ε we have

$$a(g)^\lambda \varepsilon(m(g)) = |g_{11}|^{\lambda_1}_{\varepsilon_1} |\det g_{22}|^{\lambda_2}_{\varepsilon_2} |g_{11} - g_{12}g_{21}|^{\lambda_2}_{\varepsilon_2}.$$

The Group $N_K(\mathfrak{a})$. For an element $w \in N_K(\mathfrak{a})$ one finds that the action on $\mathfrak{a}$ must permute the entries, i.e., for every such w there is a permutation σ such that

$$w \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} w^{-1} = \begin{pmatrix} h_{\sigma(1)} & 0 & 0 \\ 0 & h_{\sigma(2)} & 0 \\ 0 & 0 & h_{\sigma(3)}. \end{pmatrix}$$

When $d = 3$ all permutations are possible but when $d > 3$ only $\sigma = (1\,3)$ and the identity are possible. Indeed, it is clear that the element

$$w_{1d} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

realises $\sigma = (1\,3)$ and writing the above out for $\sigma = (1\,2)$ implies that w has the form

$$w = \begin{pmatrix} 0 & w_{12} & 0 \\ w_{21} & 0 & 0 \\ 0 & 0 & w_{13}^2 \end{pmatrix}.$$

But $w \in K$ then implies that $w_{21}w_{21}^t = I$ which is not possible when $d > 3$. The same argument can be give for $\sigma = (2\,3)$ and so since $N_K(\mathfrak{a})$ is a group one can always reduce to these two cases so all other permutations except $(1\,3)$ and the identity are impossible as well.

It follows that $N_K(\mathfrak{a})$ consists of $w_{1d}z$ and z where $z \in Z_K(\mathfrak{a})$. Now, $N_K(\mathfrak{a})$ acts on $\mathfrak{a}'_{\mathbb{C}}$ and $\widehat{M}$ and we particularly find

$$\begin{aligned} w_{1d}\delta_1 &= -\delta_2 \\ w_{1d}\delta_2 &= -\delta_1 \end{aligned}$$

so that in the isomorphism $\mathfrak{a}'_{\mathbb{C}} \cong \mathbb{C}^2$ we have

$$w_{1d}(\lambda_1, \lambda_2) = (-\lambda_2, \lambda_1).$$

Also, $w_{1d}^{-1} m_{1l} w_{1d} = m_{ld}$ so that for our chosen representations

$$w_{1d}(\varepsilon_1, \varepsilon_2) = (\varepsilon_2, \varepsilon_1).$$

1.3 The Double Cover of SL$(3,\mathbb{R})$

We will also have occasion to consider the double covering group $\widetilde{G}$ of SL$(3,\mathbb{R})$. Then the maximal compact subgroup $\widetilde{K}$ is isomorphic to the group SU(2). In $\mathfrak{su}(2)$ we have elements

$$\widetilde{W}_{12} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \widetilde{W}_{23} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \widetilde{W}_{13} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

and the derivative of the double covering $\widetilde{G} \to G$ is the Lie algebra homomorphism given by $\widetilde{W}_{jl} \mapsto 2W_{jl}$. We introduce the one-parameter subgroups

$$\widetilde{k}_{jl}(t) = \exp t\widetilde{W}_{jl}$$

so that

$$\widetilde{k}_{12}(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}, \quad \widetilde{k}_{23}(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad \widetilde{k}_{13}(t) = \begin{pmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{pmatrix}.$$

The group $\widetilde{M} = Z_{\widetilde{K}}(\mathfrak{a})$ is the group generated by $\widetilde{m}_{jl} = \widetilde{k}_{jl}\big(\frac{\pi}{2}\big)$ given by

$$\widetilde{m}_{12} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \widetilde{m}_{23} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \widetilde{m}_{13} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Indeed, $\widetilde{k}_{jl}(t) \in \widetilde{M}$ if and only if $k_{jl}(2t) \in M$. Then $\widetilde{M}$ is a group of order 8. It also includes the only central element that isn't e, namely

$$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \widetilde{m}_{jl}^2.$$

The same argument tells us that $W = N_K(\mathfrak{a})$ is generated by $\widetilde{m}_{jl}$ and $\widetilde{w}_{jl} = \widetilde{k}_{jl}(\frac{\pi}{4})$ given by

$$\widetilde{w}_{12} = \begin{pmatrix} e^{i\frac{\pi}{4}} & 0 \\ 0 & e^{-i\frac{\pi}{4}} \end{pmatrix}, \quad \widetilde{w}_{23} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \widetilde{w}_{13} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

Besides the representations of $\widetilde{M}$ that factor through M, $\widetilde{M}$ has also a 2-dimensional irreducible representation, namely the one it inherits from the action of SU(2) on $\mathbb{C}^2$.

Some computations of the $\overline{N}MAN$-decomposition are included in Appendix B.

1.4 Parabolically Induced Representations

The following applies in general when G is a linear connected reductive group, MAN is a parabolic subgroup and $\overline{N} = \Theta N$, cf. [21, Ch. VII]. For our purposes it is primarily relevant for MAN the middle parabolic from the previous section. We will also need the following when G is the double covering $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ but it still applies, cf. [40, 5.2].

Pick $\varepsilon \in \widehat{M}$ and $\lambda \in \mathfrak{a}'_{\mathbb{C}}$. Then we have a representation $P(\varepsilon, \lambda)$ as follows. The Hilbert space is the space of measurable functions $F : G \to \mathcal{H}_\varepsilon$ for which

$$F(gman) = a^{-\lambda-\rho}\varepsilon(m)^{-1}F(g)$$

and for which

$$\int_K |F(k)|^2\, dk < \infty.$$

Here $\rho = \sum_j \delta_j$ is the half-sum of the positive roots. The inner product is given by

$$(F|F') = \int_K (F(k)|F'(k))\, dk.$$

The action of G on $P(\varepsilon, \lambda)$ is simply the left translation action.

We can realise this representation solely as functions on K using the Iwasawa decomposition to pass to the full picture. Then the action becomes

$$[P_{\varepsilon,\lambda}(g)F](k) = \alpha(g^{-1}k)^{\lambda-\rho}F(\kappa(g^{-1}k))$$

which is in general a relatively complicated expression. The upside to this realisation is that the smooth vectors of the representation are simply the smooth functions $G \to \mathcal{H}_\varepsilon$. This will be useful because the smooth vectors are the domain of the Knapp-Stein intertwiners.

We can also realise this representation as functions on $\overline{N}$ using the $\overline{N}MAN$-decomposition to pass to the full picture. The space is then the L^2-functions $\overline{N} \to \mathcal{H}_\varepsilon$ with regards to the measure $\alpha(\overline{n})^{2\,\mathrm{Re}\,\lambda}d\overline{n}$. The action is

$$[P_{\varepsilon,\lambda}(g)F](\overline{n}) = a(g^{-1}\overline{n})^{\lambda-\rho}\varepsilon(m(g^{-1}\overline{n}))^{-1}F(\overline{n}(g^{-1}\overline{n})).$$

This can be made relatively explicit but we lose the easy characterisation of the smooth vectors.

Knapp-Stein Intertwiners. Consider the case where $\overline{N}\cap w^{-1}Nw = \overline{N}$. This is the case for $d = 3$ with the w chosen as above. Then we have intertwiners $T(\varepsilon,\lambda) : P(\varepsilon,\lambda) \to P(w\varepsilon, w\lambda)$ given by

$$\begin{aligned}[T(\varepsilon,\lambda)F](x) &= \int_{\overline{N}} a(w^{-1}\overline{n})^{\lambda-\rho}\varepsilon(m(w^{-1}\overline{n}))F(x\overline{n})\,d\overline{n}\\ &= \int_K a(w^{-1}k)^{\lambda-\rho}\varepsilon(m(w^{-1}k))F(xk)\,dk.\end{aligned}$$

The integrals do not converge for all choices of F and λ. But when F is a smooth vector the integrals converge for sufficiently large λ and the integrals can be extended by analytic continuation.

We see that the above expressions give $T(\varepsilon,\lambda)$ as convolutions operators on the right but or alternatively on the left on the functions $\widetilde{F}(x) = F(x^{-1})$ with kernels

$$\begin{aligned}\overline{n} &\mapsto a(w^{-1}\overline{n})^{\lambda-\rho}\varepsilon(m(w^{-1}\overline{n}))\\ k &\mapsto a(w^{-1}k)^{\lambda-\rho}\varepsilon(m(w^{-1}k))\end{aligned}$$

over $\overline{N}$ or K.

Duality. Suppose that π, π' are representations of a topological group M. We say that π, π' form a *dual pair* if there is a continuous sesquilinear form $(\cdot|\cdot)$ on $\mathcal{H}_\pi \times \mathcal{H}_{\pi'}$ which is invariant in the sense that

$$(\pi(m)v|\pi'(m)v') = (v|v')$$

for all $m \in M$, $v \in \mathcal{H}_\pi$, $v' \in \mathcal{H}_{\pi'}$. Continuity means that there is a $C > 0$ such that

$$|(v|v')| \leq C\|v\| \cdot \|v'\|.$$

Normalizing $(\cdot|\cdot)$ preserves the invariance so we shall assume that $C = 1$. An example then of a dual pair is any unitary representation paired with itself.

Suppose that σ, σ' form a dual pair of representations of M with invariant sesquilinear form $(\cdot|\cdot)$. Then there is a natural candidate for an invariant sesquilinear form on $P(\sigma,\lambda) \times P(\sigma',\lambda')$, namely

$$(F|F') = \int_K (F(k)|F'(k))\,dk.$$

We find that

Proposition 1.4.1. *If σ, σ' is a dual pair of representations of M then $P(\sigma,\lambda)$, $P(\sigma', -\overline{\lambda})$ forms a dual pair of representations of G. The invariant form is given by*

$$(F|F') = \int_K (F(k)|F'(k))\,dk = \int_{\overline{N}} (F(\overline{n})|F'(\overline{n}))\,d\overline{n}$$

for $F \in P(\sigma,\lambda)$, $F' \in P(\sigma', -\overline{\lambda})$.

Proof. The continuity follows from Cauchy-Schwarz:

$$|(F|F')| \leq \int_K \|F(k)\| \cdot \|F'(k)\| \, dk \leq \|F\| \cdot \|F'\|.$$

Since the form is continuous it is enough to check invariance on continuous F, F'. Since $(\cdot|\cdot)$ is M-invariant,

$$K \ni k \mapsto (F(k)|F'(k))$$

is right invariant under $M \cap K$ and continuous so [21, Ch. VII, §2, (2)] gives us for any $g \in G$,

$$\int_K (F(k)|F'(k)) \, dk = \int_K \alpha(g^{-1}k)^{-2\rho}(F(\kappa(g^{-1}k))|F'(\kappa(g^{-1}k))) \, dk.$$

Note that we have for any $x \in G$,

$$\begin{aligned}(F(x)|F'(x)) &= (\alpha(x)^{-\lambda-\rho}\sigma(m)^{-1}F(\kappa(x))|\alpha(x)^{\overline{\lambda}-\rho}\sigma'(m)^{-1}F'(\kappa(x))) \\ &= \alpha(x)^{-2\rho}(F(\kappa(x))|F'(\kappa(x)))\end{aligned}$$

owing to the invariance of $(\cdot|\cdot)$. Now, [21, 5.25] says that for continuous F, F' we have

$$\int_K (F(k)|F'(k)) \, dk = \int_{\overline{N}} (F(\kappa(\overline{n}))|F'(\kappa(\overline{n})))\alpha(\overline{n})^{-2\rho} \, d\overline{n} = \int_{\overline{N}} (F(\overline{n})|F'(\overline{n})) \, d\overline{n}.$$

We get the equality

$$\int_K (F(k)|F'(k)) \, dk = \int_{\overline{N}} (F(\overline{n})|F'(\overline{n})) \, d\overline{n}$$

for all F, F' by approximation because $(F, F') \mapsto \int_{\overline{N}}(F, F')$ is also continuous:

$$\int_{\overline{N}} |(F, F')| \leq \int_{\overline{N}} \|F(\overline{n})\| \alpha(\overline{n})^{\operatorname{Re}\lambda} \alpha(\overline{n})^{-\operatorname{Re}\lambda} \|F'(\overline{n})\| \, d\overline{n} \leq \|F\| \cdot \|F'\|. \qquad \square$$

Knapp-Stein Kernel for the Middle Parabolic. With $w = w_{1d}$ and

$$\overline{n}(x, y, z) = \begin{pmatrix} 1 & 0 & 0 \\ x & I & 0 \\ z & y & 0 \end{pmatrix},$$

we find

$$w^{-1}\overline{n}(x, y, z) = \begin{pmatrix} -z & -y & -1 \\ x & I & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

so for $\overline{n} = \overline{n}(x, y, z)$,

$$a(w^{-1}\overline{n})^{\lambda-\rho}\varepsilon(m(w^{-1}\overline{n})) = |-z|_{\varepsilon_1}^{\lambda_1-1}|-z+yx|_{\varepsilon_2}^{\lambda_2-1}$$

1.5 The $\mathrm{SL}(2,\mathbb{R})$-Case

We will briefly touch upon $\mathrm{SL}(2,\mathbb{R})$ as a motivating example. This is the above with $d = 2$ so we have the parabolic series with a choice of sign $\varepsilon = \varepsilon_1$ and a choice of scalar $\lambda = \lambda_1$. Also, $\overline{N}$ is isomorphic with $\mathbb{R}$ through $t \mapsto \overline{n}(t)$ and the Haar measure is simply the Lebesgue measure. The measure $\alpha(\overline{n})^{2\,\mathrm{Re}\,\lambda}$ can be given explicitly as follows: When $\overline{n}(t) = kan$ in the $G = KAN$-decomposition we find

$$\overline{n}(t)^t\overline{n}(t) = n^t a^2 n$$

which is the LDU-decomposition so since

$$\overline{n}(t)^t\overline{n}(t) = \begin{pmatrix} 1+t^2 & t \\ t & 1 \end{pmatrix}$$

one finds

$$a^2 = \begin{pmatrix} 1+t^2 & 0 \\ 0 & (1+t^2)^{-1} \end{pmatrix}$$

so that $\alpha(\overline{n}(t))^{2\,\mathrm{Re}\,\lambda} = (1+t^2)^{\mathrm{Re}\,\lambda}$. One can also figure out the action of G on $P(\varepsilon,\lambda)$ explicitly.

We find

$$w^{-1}\overline{n}(x)^{-1} = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}$$

so that the Knapp-Stein kernel in the noncompact picture is

$$a(w^{-1}\overline{n}(x)^{-1})^{\lambda-\rho}\varepsilon(m(w^{-1}\overline{n}(x)^{-1})) = |x|_\varepsilon^{\lambda-1}.$$

Now, K is isomorphic to $\mathbb{R}/2\pi\mathbb{Z}$ through $\theta \mapsto k(\theta)$ where

$$k(\theta) = \exp\theta W = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and we take the Haar measure on K to be the normalised Lebesgue measure. Note that

$$w = w_0 = k(\frac{\pi}{2})$$

We find

$$w^{-1}k(\theta)^{-1} = k(-\theta - \frac{\pi}{2}) = \begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$$

so that the Knapp-Stein kernel in the compact picture is

$$a(w^{-1}k(\theta)^{-1})^{\lambda-\rho}\varepsilon(m(w^{-1}k(\theta)^{-1})) = |\sin\theta|_\varepsilon^{\lambda-1}.$$

Analysis on $\overline{N}$. One readily obtains

$$\frac{d}{dx}|x|_\varepsilon^\lambda = \lambda|x|_{-\varepsilon}^{\lambda-1}$$

which gives the extension of the analytic map $\lambda \to |x|_\varepsilon^\lambda$ from $\{z \mid \operatorname{Re} z > -1\} \to \mathcal{S}'(\mathbb{R})$ to $\mathbb{C} \setminus (-\mathbb{N}) \to \mathcal{S}'(\mathbb{R})$. As in [11, p. 173] we consider the following normalisations:

$$\chi_+^\lambda := \frac{2^{-\frac{1}{2}\lambda}}{\Gamma\left(\frac{\lambda+1}{2}\right)}|x|_+^\lambda$$
$$\chi_-^\lambda := \frac{2^{-\frac{1}{2}\lambda}}{\Gamma\left(\frac{\lambda+2}{2}\right)}|x|_-^\lambda$$

The takeaway is

Proposition 1.5.1. *The family $(\chi_\varepsilon^\lambda)_{\lambda\in\mathbb{C}}$ is a family of homogeneous tempered distributions on $\mathbb{R}$ of degree λ that have parity ε. The map $\lambda \mapsto \chi_\varepsilon^\lambda$ is analytic and χ_ε^λ is nonzero for all λ. Also, when $n \in \mathbb{N}_0$,*

$$\chi_\varepsilon^{-n} \quad \textit{is propositional to} \quad \begin{cases} \delta_0^{(n-1)} & \textit{for } (-1)^n = -\varepsilon, \\ x^{-n} & \textit{for } (-1)^n = \varepsilon. \end{cases}$$

One finds the identities

$$\partial^2\chi_+^\lambda = \lambda\chi_+^{\lambda-2},$$
$$\partial^2\chi_-^\lambda = (\lambda-1)\chi_-^{\lambda-2}.$$

We normalise T_ε^λ such that

$$T_\varepsilon^\lambda\varphi = \chi_\varepsilon^{\lambda-1} * \varphi$$

for $\varphi \in \mathcal{S}(\mathbb{R})$. In order to figure out if T_ε^λ is positive definite on $\mathcal{S}(\mathbb{R})$ we can take the Fourier Transform.

Proposition 1.5.2. *The Fourier transform of χ_ε^λ is $c(\varepsilon)\chi_\varepsilon^{-1-\lambda}$ where*

$$c(+) = 1,$$
$$c(-) = i.$$

One easily sees that

Proposition 1.5.3. *χ_ε^λ is real if and only if $\lambda \in \mathbb{R}$. χ_ε^λ is positive if and only if $\lambda \geq -1$ and $\varepsilon = +$.*

Proof. It is readily apparent that χ_+^λ is positive when $\lambda \geq -1$.

Suppose that χ_ε^λ is positive for some $\lambda \in \mathbb{R}$. Because of the parity, $\varepsilon = +$. Then we must have

$$0 \leq \langle |x|_+^\lambda, e^{-x^2}\rangle = \Gamma(\frac{\lambda+1}{2})$$

but also

$$0 \leq \langle |x|_+^\lambda, x^2 e^{-x^2} \rangle = \frac{\lambda+1}{2}\Gamma(\frac{\lambda+1}{2}).$$

These two combined implies that $\lambda \geq -1$. □

By Bochner's Theorem this implies

Proposition 1.5.4. *χ_+^λ is positive definite if and only if $\lambda \leq 1$.*

Consequences for the Representation Theory. The Knapp-Stein operator in the compact picture is defined on the set of smooth functions $\varphi : K \to \mathbb{C}$ for which $\varphi(\theta+\pi) = \varepsilon\varphi(\theta)$. It is given by convolution with $|\sin\theta|_\varepsilon^{\lambda-1}$. One notes that this is a function in $L^1(K)$ when $\operatorname{Re}\lambda > 0$ so by the Young Inequality, T_ε^λ is continuous $L^2(K) \to L^2(K)$ for these λ. It follows that for any $\varphi \in C^\infty(P_\varepsilon^\lambda)$, the pairing

$$(T_\varepsilon^\lambda \varphi|\varphi)$$

can be approximated by $(T_\varepsilon^\lambda \varphi_n|\varphi_n)$ where $\varphi_n \in \mathcal{S}(\mathbb{R})$ in the noncompact picture. We can even take φ_n to have compact support on $\overline{N}$. So one obtains:

Proposition 1.5.5. *The operator $T_\varepsilon^\lambda : C^\infty(P_\varepsilon^\lambda) \to C^\infty(P_\varepsilon^{-\lambda})$ is positive in the sense that*

$$(T_\varepsilon^\lambda \varphi, \varphi) \geq 0$$

for $\varphi \in C^\infty(P_\varepsilon^{-\lambda}) \setminus 0$ if $0 < \lambda \leq 1$. If the operator is positive then $-1 \leq \lambda \leq 1$.

Proof. For the "only if" part of the statement one switches what root is considered positive, i.e., we realise the representation on N instead of on $\overline{N}$. The Knapp-Stein kernel on N is then given by $|x|_\varepsilon^{-\lambda-1}$ so the previous analysis gives us that we must have $-\lambda \leq 1$. □

Analysis on K. On K the Knapp-Stein kernel is given by convolution with $|\sin\theta|_\varepsilon^{\lambda-1}$. One should make the same normalisation as in the compact picture but it is simpler to consider the normalisation

$$S_\varepsilon^\lambda = \frac{1}{\Gamma(\lambda+1)}|\sin\theta|_\varepsilon^\lambda.$$

Then we have $\lambda \mapsto S_\varepsilon^\lambda$ is analytic and the extension is given by $D_\lambda S_\varepsilon^\lambda = S_\varepsilon^{\lambda-2}$ where

$$D_\lambda = \frac{d^2}{d\theta^2} + \lambda^2.$$

It is easy to see that

$$S_\varepsilon^{-1} = D_1 S_\varepsilon^1 = (1+\varepsilon)(\delta_\pi - \delta_0),$$
$$S_\varepsilon^{-2} = D_0 S_\varepsilon^0 = (1-\varepsilon)(\delta_\pi - \delta_0)$$

which allows us to calculate the values at $-\mathbb{N}_0$ recursively. Actually, on $C^\infty(P_\varepsilon^\lambda)$, $\delta_\pi = \varepsilon\delta_0$ so acting there,

$$S_\varepsilon^{-1} = (1+\varepsilon)(\varepsilon-1)\delta_0 = 0,$$
$$S_\varepsilon^{-2} = (1-\varepsilon)(\varepsilon-1)\delta_0 = -(\varepsilon-1)^2\delta_0.$$

The Fourier Transform of S_ε^λ is

$$\widehat{S_\varepsilon^\lambda}(n) = \frac{1}{\Gamma(\lambda+1)}\int_{-\pi}^{\pi}|\sin\theta|_\varepsilon^\lambda e^{in\theta}\,d\theta = \frac{1+(-1)^n\varepsilon}{2}\frac{i}{2^\lambda\Gamma(\frac{\lambda}{2}+1+\frac{n}{2})\Gamma(\frac{\lambda}{2}+1-\frac{n}{2})}$$

The result remains the same:

Proposition 1.5.6. *T_ε^λ is positive if and only if $-1 \le \lambda \le 1$.*

Chapter 2

$*$-Algebras

In this chapter we consider topological $*$-algebras. The problem of finding a new inner product for a parabolically induced representation is related to the problem of determining when a linear functional f on a $*$-algebra A is positive, i.e., when $\langle f, aa^* \rangle \geq 0$ for all $a \in A$. In our case we would like A to be a convolution algebra of functions on K or $\overline{N}$. We will consider examples where A is the convolution algebra $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{D}(G)$ or $\mathcal{S}(N)$ for an arbitrary Lie group G or a simply connected nilpotent Lie group N. In order to avoid confusion we will call the (continuous) positive linear functionals on such algebras positive definite distributions. It is hard, even on $\mathbb{R}^d$, to see directly when a distribution is positive definite. On the other hand we will also consider multiplication algebras $\mathcal{S}(\mathbb{R}^d)$ or $\mathcal{E}(\mathbb{R}^*, \mathcal{L}_0)$ on which the positive linear functionals are simply the positive distributions — i.e., the problem of determining the positive linear functionals is easy. On $\mathbb{R}^d$ Bochner's Theorem relates the positive definite distributions on $\mathbb{R}^d$ with the positive distributions. What is happening is that the Fourier transform gives a $*$-algebra isomorphism of $\mathcal{S}(\mathbb{R}^d)$ considered as a convolution algebra with $\mathcal{S}(\mathbb{R}^d)$ considered as a multiplication algebra. It would be extraordinarily satisfying to have a corresponding classification for all Lie groups but such a classification is not known. The only other case where it is known is for the compact Lie groups where we have an isomorphism of $\mathcal{E}(K)$ with a certain space of matrices. We will consider this case in detail as well.

Definition. For us, a topological **-algebra* A is a topological vector space over $\mathbb{C}$ equipped with a separately continuous bilinear operation

$$A \times A \ni (a, a') \mapsto aa' \in A$$

and a continuous antilinear operation

$$A \ni a \mapsto a^* \in A$$

satisfying

$$(aa')a'' = a(a'a'')$$
$$(ab)^* = b^*a^*$$

A *Fréchet* $*$-algebra is a topological $*$-algebra for which the underlying topological vector space is a Fréchet space. Note that this entails the bilinear operation $*$ is jointly continuous. We will not go into great depth about topological $*$-algebras. For a more thorough treatment consider [10].

An *approximate identity* in a topological $*$-algebra A is a net $(a_\lambda)_{\lambda\in I}$ such that

$$a_\lambda a \to a$$

as $\lambda \to \infty$.

Positive Linear Functionals. A *positive linear functional* on a topological $*$-algebra A is an element $f \in A'$ that satisfies

$$\langle f, aa^* \rangle \geq 0$$

for all $a \in A$.

Remark 2.0.1. In general, a positive linear functional is allowed to be discontinuous. We consider only continuous ones.

When we have such a positive linear functional we can construct a pseudo inner product on A by

$$A \times A \ni (a, b) \mapsto \langle f, a * b^* \rangle.$$

This is the idea behind the so-called GNS-construction.

There is a natural $*$-operation on A', namely the one given by

$$\langle f^*, a \rangle = \overline{\langle f, a^* \rangle}$$

for $f \in A'$, $a \in A$. An element $f \in A'$ such that $f^* = f$ is said to be *self-adjoint*. We single out [10, Lemma 12.3 (1)]:

Proposition 2.0.2. *Suppose that $f \in A'$ is a positive linear functional on a topological $*$-algebra A. Then*

$$\langle f, xy^* \rangle = \overline{\langle f, yx^* \rangle}.$$

If A admits an approximate identity then f is self-adjoint.

Proof. The first identity is purely algebraic and arises by polarisation: Indeed, let $Q(x) = \langle f, xx^* \rangle$ then $Q(x) \geq 0$ and $Q(i^k x) = Q(x)$ so

$$4\langle f, xy^* \rangle = \sum_{k=0}^{3} i^k Q(x + i^k y) = \overline{\sum_{k=0}^{3} (-i)^k Q(y + (-i)^k x)} = 4\overline{\langle f, yx^* \rangle}.$$

If $(x_\lambda)_\lambda$ is an approximate identity we get

$$\langle f, y^* \rangle = \lim_\lambda \langle f, x_\lambda y^* \rangle = \lim_\lambda \overline{\langle f, yx_\lambda^* \rangle} = \overline{\langle f, y \rangle}$$

so that f is self-adjoint. $\square$

Example: Schwartz Functions on $\mathbb{R}^d$. On $\mathcal{S}(\mathbb{R}^d)$ we have two bilinear operations $*$ and $\cdot$ given by

$$\varphi * \psi(x) = \int_{\mathbb{R}^d} \varphi(y)\psi(x-y)\,dy$$
$$\varphi \cdot \psi(x) = \varphi(x)\psi(x)$$

and involutions * and $\bar{\cdot}$ given by

$$\varphi^*(x) = \overline{\varphi(-x)}$$
$$\overline{\varphi}(x) = \overline{\varphi(x)}.$$

Proposition 2.0.3. *Both* $(\mathcal{S}(\mathbb{R}^d), *, ^*)$ *and* $(\mathcal{S}(\mathbb{R}^d), \cdot, \bar{\cdot})$ *are Fréchet $*$-algebras admitting approximate identities.*

Proof. This is more or less obvious. Continuity of $\cdot$ is obvious because of Leibniz' rule and continuity of $*$ then follows since the Fourier transform is a homeomorphism that takes $*$ to $\cdot$.

As for approximate identities, when the bilinear operation is $\cdot$ we can choose $\varphi_n \in \mathcal{S}(\mathbb{R}^d)$ such that $1 \geq \varphi_n \geq 0$ and $\varphi_n(x) = 1$ for $|x| \leq n$. Then $P\varphi_n\psi \to P\psi$ uniformly for any polynomial P and

$$P(\varphi_n\psi)^{(k)} = \sum_{|l|\leq|k|} \binom{k}{l} P\varphi_n^{(l)}\psi^{(k-l)}$$

goes uniformly to $P\psi^{(k)}$ because $\varphi_n^{(l)}(x) = 0$ for $|x| \leq n$ when $l \neq 0$.

Again by Fourier transformation we get an approximate identity for convolution as well. □

For brevity we denote $\mathcal{S}(\mathbb{R}^d)_c$ for the former $*$-algebra and $\mathcal{S}(\mathbb{R}^d)_m$ for the latter.

A function $\varphi : \mathbb{R}^d \to \mathbb{C}$ is said to be *positive definite* if $(\varphi(x_i - x_j))_{i,j}$ is a positive matrix for any choice of points $x_1, \ldots, x_n \in \mathbb{R}^d$, i.e.,

$$\sum_{ij} \varphi(x_i - x_j) z_i \overline{z_j} \geq 0$$

for all $z_1, \ldots, z_n \in \mathbb{C}$.

A function $f : \mathbb{R}^d \to \mathbb{C}$ can induce a tempered distribution by the pairing

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^d} f(x)\varphi(x)\,dx.$$

Proposition 2.0.4. *The the positive linear functionals in $\mathcal{S}'(\mathbb{R}^d)_c \cap C(\mathbb{R}^d)$ are exactly the positive definite functions in $\mathcal{S}'(\mathbb{R}^d)_c \cap C(\mathbb{R}^d)$.*

Proof. It is known from [9, Prop. 3.35] or from [35, Ch. VII, §9] that a continuous function $f : \mathbb{R}^d \to \mathbb{C}$ is positive definite if and only if

$$\langle f, \varphi * \varphi^* \rangle \geq 0$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Since $\mathcal{D}(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$ we have the result. □

It is customary to call all the positive functionals in $\mathcal{S}'(\mathbb{R}^d)_c$ *positive definite* distributions. We will use this nomenclature to avoid mistaking the positive distributions on the algebra for the distributions that are positive in the following sense:

Suppose that $U \subseteq \mathbb{R}^d$ is open. We say that a function $\varphi : U \to \mathbb{C}$ is *positive* if its values are positive, i.e., if

$$\varphi(x) \geq 0$$

for all $x \in U$. Positivity extends as a concept to $\mathcal{D}'(U)$: A distribution $f \in \mathcal{D}'(U)$ is said to be *positive* if for any positive $\varphi \in \mathcal{D}(U)$ we have $\langle f, \varphi \rangle \geq 0$. Then it is easily seen that

Proposition 2.0.5. *A continuous function $f \in C(U)$ is positive as a distribution if and only if it is positive as a functions.*

In contrast to positivity of a linear functional on $\mathcal{S}(\mathbb{R}^d)_c$, it is easy to see when a linear functional on $\mathcal{S}(\mathbb{R}^d)_m$ is positive. Indeed,

Proposition 2.0.6. *The positive linear functionals on $\mathcal{S}(\mathbb{R}^d)_m$ are exactly the positive tempered distributions in $\mathcal{S}'(\mathbb{R}^d)$.*

Proof. A linear functional $f \in \mathcal{S}'(\mathbb{R}^d)_m$ is positive if and only if

$$\langle f, |\varphi|^2 \rangle \geq 0$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. By the argument presented in [35, p. 277] this is equivalent to $\langle f, \varphi \rangle \geq 0$ for all positive $\varphi \in \mathcal{S}(\mathbb{R}^d)$. □

A continuous linear map $\Phi : A \to B$ between topological ∗-algebras is said to be a *homomorphism* if it preserves the ∗-algebra structures, i.e., if

$$\Phi(a * a') = \Phi(a) * \Phi(a')$$
$$\Phi(a^*) = \Phi(a)^*.$$

If Φ is bijective then we say it is an *isomorphism* and that A, B are *isomorphic*. It is well-known that

Theorem 2.0.7. *The Fourier transform is an isomorphism of $\mathcal{S}(\mathbb{R}^d)_c$ to $\mathcal{S}(\mathbb{R}^d)_m$.*

Corollary 2.0.8 (Bochner's Theorem)**.** *A distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ is positive definite if and only if $\widehat{f}$ is positive. A function $f \in \mathcal{S}'(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ is positive definite (as a function) if and only if $\widehat{f}$ is a positive function.*

Remark 2.0.9. Positive distributions are the same as positive measures, cf. [35, Ch. I, §4, Thm. V]

2.1 Convolution Algebras on Groups

Example: Functions on a Locally Compact Group. When G is a locally compact topological group (i.e., a topological group that is Hausdorff and covered by compact neighbourhoods) it is well-known that $L^1(G)$ admits the structure of a Banach $*$-algebra. Indeed, one takes

$$f * g(x) = \int_G f(y)g(y^{-1}x)\,dy \tag{2.1}$$

$$f^*(x) = \overline{f(x^{-1})} \tag{2.2}$$

where dy is a Haar measure on G. This situation is covered in [9, §3.3]. The algebra $L^1(G)$ admits approximate identities. A function $f \in L^\infty(G)$ gives rise to a continuous functional by

$$\langle f, \varphi\rangle = \int_G f(x)\varphi(x)\,dx. \tag{2.3}$$

A function $f : G \to \mathbb{C}$ is said to be positive definite if $\sum_{ij} c_i\overline{c_j} f(x_j^{-1}x_i) \geq 0$ for all $c_1, \ldots, c_n \in \mathbb{C}$, $x_1, \ldots, x_n \in G$. Such a function is necessarily self-adjoint $f^* = f$ and it is bounded;

$$|f(x)| \leq f(0)$$

cf. [9, p. 91]. We have [9, Prop. 3.35]:

Proposition 2.1.1. *A bounded function $f \in C(G)$ gives rise to a positive linear functional if and only if it is positive definite.*

Example: Functions on a Lie Group. Generalising the above to a unimodular Lie group G it is natural to consider $\mathcal{D}(G) = C_c^\infty(G)$. Again we define convolution and involution by Eqs. (2.1) and (2.2). Note that $\mathcal{D}(G)$ is given the natural $\mathcal{LF}$-topology. Then [4, Prop. 3.1] tells us that

$$* : \mathcal{D}(G) \times \mathcal{D}(G) \to \mathcal{D}(G)$$

is hypocontinuous. Continuity of the involution is obvious since $x \mapsto x^{-1}$ is a diffeomorphism.

Note that when $X : C^\infty(G) \to C^\infty(G)$ is a left-invariant vector field on G, i.e., $XL_g = L_gX$ for all $g \in G$ (where $L_g\varphi(x) = \varphi(g^{-1}x)$ is the canonical left representation) then

$$X(\varphi * \psi) = \int_G \varphi(g) X L_g \psi\,dg = \varphi * X\psi.$$

A linear map $A : \mathcal{D}(G) \to E$ is continuous if and only if for each compact $K \subseteq G$, $A : \mathcal{D}_K(G) \to E$ is continuous. This latter map is continuous if and only if when $\varphi_\lambda \in \mathcal{D}_K(G)$ is such that $X_1X_2\cdots X_k\varphi_\lambda \to 0$ uniformly for every choice of left-invariant vector fields $X_1, \ldots, X_k$ then $A\varphi_\lambda \to 0$.

Proposition 2.1.2. *For every Lie group G, $\mathcal{D}(G)$ is an $\mathcal{LF}$ topological $*$-algebra that admits approximate identities.*

Proof. We only need to see that there are approximate identities. We mimic the construction in [9, Prop. 2.44]: For every neighbourhood U of $e \in G$ we can choose $\psi_U \in \mathcal{D}(G)$ such that $\psi_U \geq 0$, $\operatorname{supp} \psi_U \subseteq U$ and $\int_G \psi_U = 1$. Then the argument from [9, Prop. 2.44] tells us that for every $\varphi \in \mathcal{D}(G)$ as U shrinks to $\{e\}$, $\psi_U * \varphi \to \varphi$ uniformly (seeing as every $\varphi \in \mathcal{D}(G)$ is uniformly continuous). As U shrinks we can take it to be contained in the same compact $K \subseteq G$ so that $\operatorname{supp} \psi_U * \varphi \subseteq K \operatorname{supp} \varphi$ so in order to see that $\psi_U * \varphi \to \varphi$ in $\mathcal{D}(G)$ it suffices to see that $\psi_U * \varphi \to \varphi$ in $\mathcal{D}_{K \operatorname{supp} \varphi}(G)$. This is the case since for all left-invariant vector fields $X_1, \ldots, X_k$,

$$X_1 \cdots X_k(\psi_U * \varphi) = \psi_U * (X_1 \cdots X_k \varphi) \to X_1 \cdots X_k \varphi$$

uniformly as U shrinks. □

This means in particular that positive linear functionals on $\mathcal{D}(G)$ are self-adjoint, cf. Theorem 2.0.2. A continuous function $f \in C(G)$ gives rise to an element $f \in \mathcal{D}'(G)$ by Eq. (2.3). It still makes sense to say that $f \in C(G)$ is positive definite. We find

Proposition 2.1.3. *$f \in C(G)$ is positive definite if and only if f induces a positive linear functional on $\mathcal{D}(G)$.*

Proof. $\mathcal{D}(G) \subseteq L^1(G)$ so a positive definite function must give rise to a positive linear functional. On the other hand, if f gives rise to a positive linear functional we can repeat the argument in [9, Prop. 3.35] using an approximate identity $(\psi_U)_U$ in $\mathcal{D}(G)$ instead of $C_c(G)$. □

Regularisation of Distributions. We will consider regularisations as in [35, Ch. VI, §4]. Let G be any connected Lie group; for convenience suppose that it is unimodular. First for any function $\varphi : G \to \mathbb{C}$, let

$$\widetilde{\varphi}(x) := \varphi(x^{-1}).$$

Then $\varphi \mapsto \widetilde{\varphi}$ is a continuous linear map $\mathcal{D}(G) \to \mathcal{D}(G)$. There is a corresponding continuous linear map $\mathcal{D}'(G) \to \mathcal{D}'(G)$ given by

$$\langle \widetilde{f}, \varphi \rangle := \langle f, \widetilde{\varphi} \rangle.$$

One finds

$$\varphi * \psi(x) = \langle \widetilde{\varphi}, R_x \psi \rangle = \langle \widetilde{\psi}, L_{x^{-1}} \varphi \rangle$$

which leads to the definitions

$$\begin{aligned} f * \varphi(x) &:= \langle \widetilde{f}, R_x \varphi \rangle \\ \varphi * f(x) &:= \langle \widetilde{f}, L_{x^{-1}} \varphi \rangle \end{aligned} \tag{2.4}$$

for $f \in \mathcal{D}'(G)$, $\varphi \in \mathcal{D}(G)$. We have

Proposition 2.1.4. *The convolution is hypocontinuous*

$$\mathcal{D}'(G) \times \mathcal{D}(G) \to \mathcal{E}(G)$$
$$\mathcal{D}(G) \times \mathcal{D}'(G) \to \mathcal{E}(G).$$

Proof. It is proven in [8, Prop. 6] that convolution is separately continuous (with a slightly different convention for the convolution). Since $\mathcal{LF}$-spaces are barreled, the map is hypocontinuous. □

Now, when $(\varphi_n)_n \subseteq \mathcal{D}(G)$ is an approximate identity, $\varphi_n \to \delta_e$ as distributions; indeed,

$$\langle \varphi_n, \psi \rangle = \varphi_n * \widetilde{\varphi}(e) \to \widetilde{\varphi}(e) = \langle \delta_e, \varphi \rangle$$

and the convergence is uniform as ψ varies over a bounded set because $*$ is hypocontinuous on $\mathcal{D} \times \mathcal{D}$.

It is convenient to note that

$$\langle \varphi * f, \psi \rangle = \langle f, \widetilde{\varphi} * \psi \rangle$$
$$\langle f * \varphi, \psi \rangle = \langle f, \psi * \widetilde{\varphi} \rangle$$

It follows that $\mathcal{E}(G)$ is dense in $\mathcal{D}'(G)$ as usual. More interesting, however, is

Proposition 2.1.5. *Any positive definite distribution in $\mathcal{D}'(G)$ is the limit of positive definite smooth functions in $\mathcal{E}'(G)$.*

Proof. Suppose that $f \in \mathcal{D}'(G)$ is positive. Then f is the limit of functions of the form $\alpha^* * f * \alpha$ for $\alpha \in \mathcal{D}(G)$. These are positive definite; indeed,

$$\langle \alpha^* * f * \alpha, \varphi * \varphi^* \rangle = \langle f, (\overline{\alpha} * \varphi) * (\overline{\alpha} * \varphi)^* \rangle \geq 0$$

for all φ. □

The Operator-Valued Case. The details in this paragraph depends on the theory of tensor products, cf. Appendix A.3. We will also take the tensor product of bilinear maps, cf. Chapter 4.

Suppose that $\mathcal{H}$ is a Hilbert space with inner product $(\cdot|\cdot)$. Then a map $\Phi : G \to \mathcal{L}\mathcal{H}$ is said to be *positive definite* if

$$\sum_{jl} (\Phi(x_j x_l^{-1}) u_l | u_j) \geq 0$$

for all choices of $x_1, \dots, x_n \in G$ and $u_1, \dots, u_n \in \mathcal{H}$. This is the definition given in [27]. Just as above we have

Proposition 2.1.6. *A continuous $\Phi \in C(G, \mathcal{L}\mathcal{H})$ is positive definite if and only if*

$$\int_G (\Phi(xy^{-1})\varphi(y)|\varphi(x))\, dy\, dx \geq 0$$

for all $\varphi \in \mathcal{D}(G, \mathcal{H})$.

Now, tensor product of convolution with the application map $\mathcal{L}\mathcal{H} \times \mathcal{H} \to \mathcal{H}$ gives us a vector-valued hypocontinuous convolution

$$\mathcal{D}'(G, \mathcal{L}\mathcal{H}) \times \mathcal{D}(G, \mathcal{H}) \to \mathcal{E}(G, \mathcal{H}) \to \mathcal{D}'(G, \mathcal{H})$$

cf. Theorem 4.1.1. Denote by $(\cdot|\cdot)$ the sesquilinear pairing of $\mathcal{D}'(G, \mathcal{H})$ with $\mathcal{D}(G, \mathcal{H})$ given by the tensor product of the sesquilinear pairings $\mathcal{D}'(G) \times \mathcal{D}(G) \to \mathbb{C}$ and $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$. Then a distribution $F \in \mathcal{D}'(G, \mathcal{L}\mathcal{H})$ is said to be *positive definite* if $(F * \varphi|\varphi) \geq 0$ for all $\varphi \in \mathcal{D}'(G, \mathcal{H})$. This is consistent with the definition of positive definiteness of a function above.

We will find that there is a connection to a $*$-algebra just as in the scalar case. Indeed, $\mathcal{D}(G, L^1(\mathcal{H}))$ ($L^1(\mathcal{H})$ are the trace-class operators with trace-norm $\|\cdot\|_1$) becomes a $*$-algebra with operations (note $\|TS\|_1 \leq \|T\|_1\|S\|_1$ so the convolution here is the tensor product of the convolution with the continuous composition on $L^1(\mathcal{H})$ so we can apply Theorem 4.1.1)

$$\Phi * \Psi(x) = \int \Phi(y)\Psi(y^{-1}x)\, dy$$
$$\Phi^*(x) = \Phi(x^{-1})^*.$$

According to [34, Prop. 22, Cor. 3 (p 104)],

$$\mathcal{D}(G, L^1(\mathcal{H}))' \cong \mathcal{D}'(G, \mathcal{L}\mathcal{H}),$$

the isomorphism given on simple tensors by

$$\langle F, \varphi \otimes T\rangle = \mathrm{tr}[F(\varphi)T].$$

The element in $\mathcal{D}(G, L^1(\mathcal{H}))'$ corresponding to $F \in C(G, \mathcal{L}\mathcal{H})$ is given by

$$\langle F, \Phi\rangle = \int_G \mathrm{tr}\, F(x)\Phi(x)\, dx.$$

Proposition 2.1.7. *$F \in C(G, \mathcal{L}\mathcal{H})$ is positive definite if and only if $\widetilde{F}$ induces a positive linear functional on $\mathcal{D}(G, L^1(\mathcal{H}))$.*

Here and elsewhere, $\widetilde{F}(x) = F(x^{-1})$.

Proof. If we take $(e_j)_j$ to be an orthonormal basis for $\mathcal{H}$ we get for any $\Phi \in \mathcal{D}(G, L^1(\mathcal{H}))$,

$$\langle F, \Phi * \Phi^*\rangle = \sum_j (\widetilde{F} * \Phi_j|\Phi_j)$$

where $\Phi_j(x) = \Phi(x)e_j$. □

Example: The Schwartz Functions on a Nilpotent Group. For a connected, simply connected, nilpotent Lie group N we can consider the Schwartz functions $\mathcal{S}(N)$ on N, cf. [7, A.2]. We will not go into detail about the exact definition of $\mathcal{S}(N)$ but it basically arises by taking global *polynomial* coordinates on N and taking the Schwartz functions on N to be those that are Schwartz functions in the coordinates. Again we define the structure of $\mathcal{S}(N)$ as a $*$-algebra by Eqs. (2.1) and (2.2).

Proposition 2.1.8. *For every connected, simply connected, nilpotent Lie group N, $\mathcal{S}(N)$ is a Fréchet $*$-algebra that admits an approximate identity.*

Proof. Since $\mathcal{S}(N) \subseteq L^1(N)$, the integral defining the convolution exists. As in the proof of [7, Theorem A.2.5], we take some norm on the Lie algebra of N and transfer it to the group by the exponential map. Then we have

$$\|x^{-1}\| = \|x\|$$

and as noted in the aforementioned proof, if N has dimension n, there is some $c > 0$ such that

$$\|xy\| \leq c(1 + \|x\| + \|y\|)^n$$

for all $x, y \in N$. We introduce

$$p(x) := (1 + \|x\|^2)$$

and as a consequence of the above inequality, there is some $C > 0$ such that

$$p(xy) \leq Cp(x)^n p(y)^n$$

for all $x, y \in N$.

The topology on $\mathcal{S}(N)$ is given by the norms $\|p^m \cdot X_L^\alpha \varphi\|_{m,\alpha}$ where

$$X_L^\alpha = (X_1^{\alpha_1})_L \cdots (X_1^{\alpha_n})_L$$

for a basis $X_1, \dots X_n$ of the Lie algebra of N where X_L is the left invariant vector field associated to an element X of the Lie algebra. Then we have

$$\begin{aligned} \|p^m \cdot X_L^\alpha(\varphi * \psi)\|_\infty &\leq C^m \sup_x \int_N p(y)^{mn} |\varphi(y)| p(y^{-1}x)^{mn} X_L^\alpha |\psi(y^{-1}x)| \, dy \\ &\leq C^m \|p^{mn}\varphi\|_{L^2} \|p^{mn} X_L^\alpha \psi\|_{L^2}. \end{aligned}$$

The topology on $\mathcal{S}(N)$ is also generated by the norms $\|p^m X_L^\alpha \varphi\|_{L^2}$ so we conclude that convolution maps $\mathcal{S}(N) \times \mathcal{S}(N)$ into $\mathcal{S}(N)$ and that it is jointly continuous.

As for the involution note that $p(x^{-1}) = p(x)$ and $X_L \varphi^* = (X_R \varphi)^*$ where

$$X_R \varphi(g) = \frac{d}{dt}\Big|_{t=0} \varphi(\exp(-tX)g)$$

for every $X \in \mathfrak{n}$ so $X_L^\alpha \varphi^* = (X_R^\alpha \varphi)^*$. We get

$$\|p^m X_L^\alpha \varphi^*\|_\infty = \sup_x p(x)^m |X_R^\alpha \varphi(x^{-1})| = \|p^m X_R^\alpha \varphi\|_\infty.$$

The topology on $\mathcal{S}(N)$ is also generated by the norms $\varphi \mapsto \|p^m X_R^\alpha \varphi\|_{m,\alpha}$ so the involution is also continuous.

For an approximate identity we take $\varphi_U \in \mathcal{D}(N)$ such that $\varphi_U \geq 0$, $\int_N \varphi_U = 1$, $\operatorname{supp} \varphi_U \subseteq U$ for every neighbourhood U of e. Then

$$\varphi_U * \psi - \psi = \int_U \varphi_U(x)(L_x\varphi - \varphi)\, dx.$$

According to [7, Theorem A.2.6], $x \mapsto L_x\varphi$ is continuous so if $\|\cdot\|$ is a continuous norm on $\mathcal{S}(N)$ and if $\varepsilon > 0$, there is a neighbourhood U of e such that $\|L_x\varphi - \varphi\| \leq \varepsilon$ for all $x \in U$. Then

$$\|\varphi_U * \psi - \psi\| \leq \varepsilon$$

demonstrating that $\varphi_U * \psi \to \psi$ as U shrinks to $\{1\}$. □

The continuous linear functionals on $\mathcal{S}(N)$ are the *tempered distributions* on N. By using the polynomial coordinates we define the subspace of *functions that grow slowly* $\mathcal{O}_M(N) \subseteq \mathcal{E}(N)$ to be the functions that become $\mathcal{O}_M(\mathbb{R}^n)$ in the polynomial coordinates (i.e., have polynomial growth; cf. [35, Ch. VII, §5]). Explicitly, $\varphi \in \mathcal{E}(N)$ is in $\mathcal{O}_M(N)$ if and only if $X_L^\alpha\varphi$ (or $X_R^\alpha\varphi$) has polynomial growth for all α. A net $\varphi_\lambda \in \mathcal{O}_M(N)$ converges to 0 if and only if $\psi \cdot X_L^\alpha\varphi$ (or $\psi \cdot X_R^\alpha\varphi$) converges uniformly to 0 for all α and for all $\psi \in \mathcal{S}(N)$. Then it is pretty clear that we have a continuous linear injection $\mathcal{O}_M(N) \to \mathcal{S}'(N)$ by way of Eq. (2.3).

Remark 2.1.9. In order to justify the notation, $\mathcal{O}_M(N)$ should be *defined* as the multiplier space of $\mathcal{S}(N)$, i.e., it should be defined as the set of functions $f \in \mathcal{E}(N)$ such that $f \cdot \varphi \in \mathcal{S}(N)$ for all $\varphi \in \mathcal{S}(N)$. This also explains where the topology comes from; $\mathcal{O}_M(N)$ inherits the topology from $\mathcal{L}(\mathcal{S}(N))$. Since everything is simply done in global coordinates, it is the case that the multiplier space is simply the space of functions with polynomial growth defined above because this is the state of things on $\mathbb{R}^d$. Schwartz remarks this (on $\mathbb{R}^d$) but does not prove it — the only proof that I have found is [17, Ch. 4, §11, Prop. 5].

Of course, even a continuous function of polynomial growth induces a tempered distribution through Eq. (2.3). Since $\mathcal{D}(N)$ is dense in $\mathcal{S}(N)$ we have

Proposition 2.1.10. *A function $f \in C(N)$ of polynomial growth induces a positive linear functional on $\mathcal{S}(N)$ if and only if it is a positive definite function.*

Regularisation of Tempered Distributions. We continue the investigation of Section 2.1 in the context of nilpotent groups. This will have important consequences on the Heisenberg group. The definition Eq. (2.4) still makes sense when $G = N$ and $f \in \mathcal{S}'(N)$, $f \in \mathcal{S}(N)$ since according to [7, Thm. A.2.6], $x \mapsto L_x\varphi$ and $x \mapsto R_x\varphi$ are smooth $N \mapsto \mathcal{S}(N)$.

A priori, convolution is then a bilinear map into $\mathcal{E}(N)$.

Proposition 2.1.11. *The convolution is hypocontinuous*

$$\mathcal{S}'(N) \times \mathcal{S}(N) \to \mathcal{O}_M(N)$$
$$\mathcal{S}(N) \times \mathcal{S}'(N) \to \mathcal{O}_M(N)$$

Proof. Consider p as in Theorem 2.1.8. When $f \in \mathcal{S}'(N)$ there is a $c > 0$ such that

$$|\langle \widetilde{f}, \varphi \rangle| \le c \max_{|\alpha| \le N} \|p^N \cdot X_R^\alpha \varphi\|_\infty$$

so that

$$|f * \varphi(x)| \le c \max_{|\alpha| \le N} \|p^N \cdot X_R^\alpha R_x \varphi\|_\infty.$$

Here,

$$\begin{aligned} \|p^N X_R^\alpha R_x \varphi\|_\infty &= \sup_y p^N(y) |X_R^\alpha \varphi(yx)| = \sup_y p^N(yx^{-1}) |X_R^\alpha \varphi(y)| \\ &\le C p(x)^{nN} \|p^{nN} X_R^\alpha \varphi\|_\infty. \end{aligned}$$

It follows that $f * \varphi$ has polynomial growth and that $X_L^\alpha(f * \varphi) = f * X_L^\alpha \varphi$ has polynomial growth for all α so that $f * \varphi \in \mathcal{O}_M(N)$. Since the norms of φ figure explicitly we also find that $\varphi \mapsto f * \varphi$ is continuous. As for continuity in f, suppose that $f_\lambda \to 0$. The above calculation tells us that when $\psi \in \mathcal{S}(N)$,

$$B = \{\psi(x) R_x \varphi \mid x \in N\}$$

is a bounded subset of $\mathcal{S}(N)$ so that $\widetilde{f_\lambda}|_B \to 0$ uniformly. But then

$$\psi(x) f_\lambda * \varphi(x) = \langle \widetilde{f_\lambda}, \psi(x) R_x \varphi \rangle \to 0$$

uniformly in x. Since φ is arbitrary we can replace it by $X_L^\alpha \varphi$ and obtain

$$\psi X_L^\alpha (f_\lambda * \varphi) = \psi(f_\lambda * X_L^\alpha \varphi) \to 0$$

uniformly. □

Just as before, when $(\varphi_n)_n \subseteq \mathcal{S}(N)$ is an approximate identity, $\varphi_n \to \delta_e$ in terms of tempered distributions. So we have $\mathcal{O}_M(N)$ is dense in $\mathcal{S}'(N)$. Again,

Proposition 2.1.12. *A positive definite tempered distribution in $\mathcal{S}'(N)$ is a limit of positive definite smooth functions in $\mathcal{O}_M(N)$.*

Example: The Lizorkin Space on the Heisenberg Group Let H be the Heisenberg group considered as $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ with composition

$$(p, q, t)(p', q', t') = (p + p', q + q', t + t' + pq').$$

Then we have

$$\mathcal{S}(H) = \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}).$$

Let $\mathcal{S}^\infty(H)$ be the closed subspace of $\mathcal{S}(H)$ consisting of φ such that

$$\int_{-\infty}^{\infty} \varphi(x, y, t) t^n \, dt = 0$$

for all $x, y \in \mathbb{R}^d$ and $n \in \mathbb{N}_0$.

Proposition 2.1.13. *$\mathcal{S}^\infty(H)$ is a $*$-ideal in $\mathcal{S}(H)$.*

A $*$-*ideal* in a $*$-algebra A is a subspace $I \subseteq A$ which is stable under the involution and which satisfies $a * x \in I$ and $x * a \in I$ for all $a \in A$, $x \in I$.

Proof. We must show that $\varphi * \psi \in \mathcal{S}^\infty(H)$ when $\varphi \in \mathcal{S}(H)$ and $\psi \in \mathcal{S}^\infty(H)$ and that $\varphi^* \in \mathcal{S}^\infty(H)$ when $\varphi \in \mathcal{S}^\infty(H)$. We start with the first claim. Since $\mathcal{S}^\infty(H)$ is closed and $\mathcal{D}(H)$ is dense in $\mathcal{S}(H)$ it is enough to assume that φ has compact support. In this case (cf. [30, Theorem 3.27]),

$$\varphi * \psi = \int_H \varphi(x) L_x \psi \, dx$$

as a vector-valued integral so again since $\mathcal{S}^\infty(H)$ is closed it is enough to see that $L_x\psi \in \mathcal{S}^\infty(H)$ for all x:

$$\begin{aligned}
\int L_{(p',q',t')^{-1}}\varphi(p,q,t)t^n \, dt &= \int \varphi(p'+p, q'+q, t'+t+p'q)t^n \, dt \\
&= \int \varphi(p'+p, q'+q, t)(t-t'-p'q)^n \, dt \\
&= \sum_{a,b,c} \binom{n}{a,b,c} (-t')^b(-p'q)^c \int \varphi(p'+p, q'+q, t)t^a \, dt \\
&= 0.
\end{aligned}$$

As for the involution,

$$\begin{aligned}
\int \varphi^*(p,q,t)t^n \, dt &= \int \overline{\varphi(-p,-q,pq-t)}t^n \, dt = \int \overline{\varphi(-p,-q,t)}(pq-t)^n \, dt \\
&= \sum_a \binom{n}{a} (pq)^{n-a}(-1)^a \int \overline{\varphi(-p,-q,t)}t^a \, dt = 0
\end{aligned}$$ □

Let $\mathrm{Pol}_c(H) \subseteq \mathcal{S}'(H)$ be the subspace of tempered distributions that are polynomial along the center; explicitly $\mathrm{Pol}_c(H)$ consists of

$$f(p,q,t) = \sum_{n=0}^{N} f_n(p,q)t^n \tag{2.5}$$

where $f_n \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ and $N \in \mathbb{N}_0$. Let

$$\mathcal{S}^N(H) = \{\varphi \in \mathcal{S}(H) \,|\, \forall n \leq N, \int_{-\infty}^{\infty} \varphi(\cdot,\cdot,t)t^n \, dt = 0\}.$$

Proposition 2.1.14. *The subspace $\mathcal{S}^N(H)$ is a $*$-ideal of $\mathcal{S}(H)$ for all N. The distributions $f \in \mathcal{S}'(H)$ for which $f|_{\mathcal{S}^N(H)} = 0$ are exactly the distributions with a representation Eq. (2.5).*

Proof. The first part follows from the proof of Theorem 2.1.13. As for the second part, suppose that $f|_{\mathcal{S}^N(H)} = 0$. Let $\varphi_n \in \mathcal{S}(N)$ such that

$$\langle \varphi_n(t), t^m \rangle = \delta_{nm}.$$

Then pairing f with φ_n in the third coordinate gives us distributions $f_n \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$. For $\varphi \in \mathcal{S}(H)$ let

$$\psi(p,q,t) = \varphi(p,q,t) - \sum_{n=0}^{N} \langle \varphi(p,q,s), s^n \rangle \varphi_n(t).$$

Then we have $\psi \in \mathcal{S}^N(H)$ so

$$\langle f, \varphi \rangle = \sum_{n=0}^{N} \langle f_n(p,q) t^n, \varphi(p,q,t) \rangle. \qquad \square$$

The fact that positive linear functionals can be approximated by positive definite functions implies that not many elements in $\mathrm{Pol}_c(H)$ can be positive:

Theorem 2.1.15. *For an element $f \in \mathrm{Pol}_c(H)$ written as Eq. (2.5) to be positive it is necessary that $N = 0$.*

Proof. Indeed, for such an f we have $f|_{\mathcal{S}^N(H)} = 0$. Now, f is approximated by $\alpha^* * f * \alpha$ for $\alpha \in \mathcal{S}(H)$. If $\varphi \in \mathcal{S}^N(H)$ we use the ideal property to see

$$\langle \alpha^* * f * \alpha, \varphi \rangle = \langle f, \overline{\alpha} * f * \widetilde{\alpha} \rangle = 0$$

so that $\alpha^* * f * \alpha|_{\mathcal{S}^N(H)} = 0$. It follows that

$$\alpha^* * f * \alpha(p,q,t) = \sum_{n=0}^{N} \varphi_n(p,q) t^n$$

where $\varphi_n \in \mathcal{E}(\mathbb{R}^d \times \mathbb{R}^d)$. But $\alpha^* * f * \alpha$ is positive definite so it must be bounded. This can only be satisfied with $N = 0$. $\square$

2.2 Multiplication Algebras

Example: Smooth Families of Operators. On the Fourier hand side we will end up considering $\mathcal{S}_0(\mathbb{R}, \mathcal{L}_0^d)$ consisting of all $\Phi \in \mathcal{S}(\mathbb{R}, \mathcal{L}_0)$ with

$$\Phi^{(n)}(0) = 0$$

for all n. Recall that $\mathcal{L}_0^d \subseteq \mathcal{L}^d = \mathcal{L}(\mathcal{S}_d, \mathcal{S}_d')$ are the operators with kernels in $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$. Then $\mathcal{S}_0(\mathbb{R}, \mathcal{L}_0^d)$ is a Fréchet $*$-algebra with algebraic operations given by

$$[\Phi\Psi](h) = \Phi(h) \circ \Psi(h), \qquad (2.6)$$
$$\Phi^*(h) = \Phi(h)^*.$$

It is also equipped with a continuous bilinear form and a continuous inner product

$$\begin{aligned}\langle \Phi, \Psi \rangle &:= \int_{-\infty}^{\infty} |h|^d \operatorname{tr}[\Phi(h)\Psi(h)]\, dh \\ (\Phi|\Psi) &:= \langle \Phi, \Psi^* \rangle\end{aligned} \tag{2.7}$$

The form $\langle \cdot, \cdot \rangle$ defined the continuous linear injection $\mathcal{S}_0(\mathbb{R}, \mathcal{L}_0) \to \mathcal{S}_0(\mathbb{R}, \mathcal{L}_0)'$. There is a natural identification $\mathcal{S}_0(\mathbb{R}, \mathcal{L}_0^d)' \cong \mathcal{S}_0'(\mathbb{R}, \mathcal{L}^d)$ determined by

$$\operatorname{tr}[F(\chi)T] = \langle F, \chi \otimes T \rangle$$

for $\chi \in \mathcal{S}_0(\mathbb{R})$, $T \in \mathcal{L}_0^d$, cf. [39, Prop. 50.7].

There is a natural notion of positivity on $\mathcal{S}_0'(\mathbb{R}, \mathcal{L})$. First, an operator $T \in \mathcal{L}^d$ is *positive* ($T \geq 0$) if

$$(T\varphi|\varphi) = \langle T\varphi, \overline{\varphi} \rangle \geq 0$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then also $\mathcal{S}_0(\mathbb{R}, \mathcal{L}_0)$ admits a notion of positivity: An element $\Phi \in \mathcal{S}_0(\mathbb{R}^*, \mathcal{L}_0)$ is *positive* ($\Phi \geq 0$) if it is pointwise positive, i.e., $\Phi(h) \geq 0$ for all $h \in \mathbb{R}$. The notion of positivity naturally extends to $\mathcal{S}_0'(\mathbb{R}, \mathcal{L})$: A distribution $F \in \mathcal{S}_0'(\mathbb{R}, \mathcal{L})$ is *positive* ($F \geq 0$) if $F(\chi) \geq 0$ for all $\chi \in \mathcal{S}_0(\mathbb{R})$, $\chi \geq 0$.

The following is a useful result:

Lemma 2.2.1. *An linear map $T \in \mathcal{L}^d$ is positive if and only if it is positive as a member of $(\mathcal{L}_0^d)'$, i.e.,*

$$\langle T, S \rangle = \operatorname{tr}[TS] \geq 0$$

for all positive $S \in \mathcal{L}_0^d$, $S \geq 0$.

Proof. When S has the form $S(x, y) = \varphi(x)\varphi(y)$ then $S \geq 0$ and

$$\operatorname{tr}[TS] = (T\varphi|\varphi).$$

This shows that if T is positive as a functional then it is also positive as an operator. On the other hand we need to approximate an arbitrary $S \in \mathcal{L}_0^d$, $S \geq 0$. We can assume that S has finite rank and then its positivity will imply that $S\varphi = 0$ if φ is orthogonal to the image of S. But then S is practically an operator on a finite-dimensional space and we can split it out in eigenvectors (in $\mathcal{S}_d$). □

Just as for $\mathcal{S}(\mathbb{R}^d)_m$, it is easy to determine the positive linear functionals:

Proposition 2.2.2. *A linear functional $F \in \mathcal{S}_0(\mathbb{R}, \mathcal{L}_0)_m'$ is positive if and only if it is positive as an element of $\mathcal{S}_0'(\mathbb{R}, \mathcal{L})$.*

Proof. If F is a positive linear functional then

$$0 \leq \langle F, (\chi \otimes T)(\chi \otimes T)^* \rangle = \langle F(|\chi|^2), TT^* \rangle$$

for all $\chi \in \mathcal{S}_0(\mathbb{R})$, $T \in \mathcal{L}_0^d$. The argument of [35, p. 277] means that any positive $\eta \in \mathcal{S}_0(\mathbb{R})$ can be approximated by elements of the form $|\chi|^2$ and any $S \in \mathcal{L}_0^d$,

$S \geq 0$ can be approximated by TT^* by using the arguments of the lemma. So we conclude that if F is a positive linear functional then it is positive as an operator-valued distribution.

On the other hand, suppose F is positive as an element of $\mathcal{S}_0'(\mathbb{R}^*, \mathcal{L}^d)$. Given $\Psi \in \mathcal{S}_0(\mathbb{R}, \mathcal{L}_0)$ we can approximate Ψ by

$$\Psi_N = \sum_{j=1}^{N} \xi_j^{(N)} \otimes \varphi_j^{(N)} \otimes \psi_j^{(N)}.$$

Using a Gram-Schmidt Orthogonalisation Process we can assume that

$$(\psi_j^{(N)}|\psi_l^{(N)}) = \delta_{jl}.$$

Then

$$\begin{aligned}\langle F, \Psi_N \cdot \Psi_N^* \rangle &= \sum_{j,l=1}^{N} (\psi_j^{(N)}|\psi_l^{(N)}) \langle F, \xi_j^{(N)} \overline{\xi_l^{(N)}} \otimes \varphi_j^{(N)} \otimes \overline{\varphi_l^{(N)}} \rangle \\ &= \sum_{j=1}^{N} \langle F, |\xi_j^{(N)}|^2 \otimes \varphi_j^{(N)} \otimes \overline{\varphi_j^{(N)}} \rangle = \sum_j (F(|\xi_j|^2)\varphi_j^{(N)}|\varphi_j^{(N)}) \geq 0.\end{aligned}$$

Taking the limit as $N \to \infty$ we obtain $\langle F, \Psi \cdot \Psi^* \rangle \geq 0$. □

Other Multiplication Algebras. We could also have considered $\mathcal{D}(\mathbb{R}^*, \mathcal{L}_0)$ or $\mathcal{E}(\mathbb{R}^*, \mathcal{L}_0)$. The latter is ill suited because it is too big so that the distribution space becomes small. The former gives us some of the same analysis but it is complicated by the fact that $\mathcal{D}(\mathbb{R}^*)$ is not a Fréchet space so we have omitted it.

2.3 Fourier Theory on a Compact Lie Group

Suppose that K is a compact connected Lie group. We have as above the Fréchet $*$-algebra $\mathcal{E}(K) = \mathcal{D}(K)$. There is a natural Bochner theorem in this setting. We use the notation as in [37] with $K = G$: One choose a maximal toral subgroup $T \subseteq G$ and denote by $\mathfrak{t}$ its Lie algebra. One lets $(\cdot|\cdot)$ denote an $\mathrm{Ad}\, K$-invariant inner product with corresponding norm $|\cdot|$ by which forms on $\mathfrak{g}$ can be identified with elements of $\mathfrak{g}$. Under this identification, D denotes the set of all dominant K-integral forms, i.e., all elements $\lambda \in \mathfrak{t}$ for which

$$(\lambda|H) \in 2\pi\mathbb{Z}$$

for all $H \in \mathfrak{t}$ and

$$(\lambda|s) \geq 0$$

for every simple root s of $\mathfrak{g}$. Every element $\lambda \in D$ corresponds uniquely to an irreducible representation $\pi_\lambda : K \to \mathcal{L}(\mathcal{H}_\lambda)$. Here the dimension $d(\lambda)$ of $\mathcal{H}_\lambda$ if of course finite. Denote by $\|A\|_2 = \mathrm{tr}[AA^*]^{1/2}$ the Hilbert-Schmidt norm of any matrix A which is a norm on any $\mathcal{L}(\mathcal{H}_\lambda)$. One denotes by $\mathcal{S}(D)$ the collection of "maps" F for which $F(\lambda) \in \mathcal{L}(\mathcal{H}_\lambda)$ for every λ and for which

$$q_n(F) := \max_\lambda |\lambda|^n \|F(\lambda)\|_2$$

is finite for every n. Then $\mathcal{S}(D)$ is topologised by the seminorms q_n and it becomes a Fréchet space. Actually, $\mathcal{S}(D)$ is a Fréchet $*$-algebra with algebraic operations

$$\begin{aligned} F \cdot G(\lambda) &:= F(\lambda) \circ G(\lambda), \\ F^*(\lambda) &:= F(\lambda)^*. \end{aligned}$$

To every $\varphi \in \mathcal{E}(K)$ one can associate its Fourier transform $\widehat{\varphi}$ given by

$$\widehat{\varphi}(\lambda) = \int_K \varphi(x)\pi_\lambda(x)\,dx$$

We have from [37, Theorem 4] that

Theorem 2.3.1. *The Fourier transform is a topological $*$-isomorphism*

$$\mathcal{E}(K) \to \mathcal{S}(D).$$

In order to define the Fourier transform of distributions, let $\mathcal{S}(D) \to \mathcal{E}(K)$, $F \mapsto \widehat{F}$ be the map given by

$$\widehat{F}(x) = \sum_\lambda d(\lambda)\,\mathrm{tr}\,F(\lambda)\pi_\lambda(x).$$

This is not quite the inverse Fourier transform but we have

$$(\widehat{\varphi})\widehat{\ }(x) = \varphi(x^{-1}).$$

For $F, G \in \mathcal{S}(D)$ let

$$\begin{aligned} \langle F, G\rangle &= \sum_\lambda d(\lambda)\,\mathrm{tr}[F(\lambda)G(\lambda)] \qquad (2.8) \\ (F|G) &= \langle F, G^*\rangle. \end{aligned}$$

The continuous bilinear form $\langle\cdot,\cdot\rangle$ allows us to define an injective continuous linear map $\mathcal{S}(D) \to \mathcal{S}(D)'$ just as the corresponding forms on K

$$\begin{aligned} \langle\varphi, \psi\rangle &= \int \varphi(x)\psi(x)\,dx \qquad (2.9) \\ (\varphi|\psi) &= \langle\varphi, \overline{\psi}\rangle \end{aligned}$$

allows us to define an injective continuous linear map $C(K) \to \mathcal{E}'(K)$ (note that $\overline{\varphi}$ is *not* the involution on $\mathcal{E}(K)$).

It is easy to see that we get

$$\langle\widehat{\varphi},\Phi\rangle = \langle\varphi,\widehat{\Phi}\rangle$$

for $\varphi\in\mathcal{E}(K)$, $\Phi\in\mathcal{S}(D)$ and this allows us to define the Fourier transform of distributions by simply replacing φ by a distribution $f\in\mathcal{E}'(K)$. Naturally, we have the Plancherel Identity

$$(\widehat{\varphi}|\widehat{\psi}) = (\varphi|\psi)$$

as well. Since $\widehat{\Phi^*} = \widehat{\Phi}^*$ (for $\Phi\in\mathcal{S}(D)$) this even extends to $\varphi = f\in\mathcal{E}'(K)$ and $\varphi\in\mathcal{E}(K)$.

Positive Functionals. Owing to the fact that D is discrete, the dual $\mathcal{S}(D)'$ is actually known explicitly. Define $\mathcal{O}_M(D)$ to be the set of "maps" F for which $F(\lambda)\in\mathcal{L}(\mathcal{H}_\lambda)$ for every $\lambda\in D$ and for which there is some n such that

$$\max_\lambda|\lambda|^{-n}\|F(\lambda)\|_2<\infty.$$

It is not difficult to see that every $F\in\mathcal{O}_M(D)$ determines an element of $\mathcal{S}(D)'$ by way of Eq. (2.8), cf. [37, Lemma 1.3].

Remark 2.3.2. To become worthy of the notation $\mathcal{O}_M(D)$ we should define it as the space of maps F for which $F\cdot\Phi\in\mathcal{S}(D)$ for every $\Phi\in\mathcal{S}(D)$. This should come to the same thing.

For every $A\in\mathcal{L}(\mathcal{H}_\lambda)$ there is a natural element $\Phi_A\in\mathcal{S}(D)$ defined by

$$\Phi_A(\mu) = \begin{cases} A & \mu=\lambda \\ 0 & \mu\neq\lambda. \end{cases}$$

Then if $F\in\mathcal{S}(D)'$, $\mathcal{L}(\mathcal{H}_\lambda)\ni A\mapsto\langle F,\Phi_A\rangle$ is continuous so that there is some $F(\lambda)\in\mathcal{L}(\mathcal{H}_\lambda)$ such that

$$\langle F,\Phi_A\rangle = d(\lambda)\operatorname{tr}F(\lambda)A$$

for every $A\in\mathcal{L}(\mathcal{H}_\lambda)$. We must have

$$|\langle F,\Phi\rangle|\le C\max_{n\le N}q_n(\Phi)$$

for some $C>0$ and $N\in\mathbb{N}_0$. So that

$$d(\lambda)\|F(\lambda)\|_2^2 = |\langle F,\Phi_{F(\lambda)^*}\rangle|\le C\max_{n\le N}|\lambda|^n\|F(\lambda)\|_2$$

Note that $d(\lambda)$ grows no faster than $|\lambda|^m$ for some power m so we have proven

Proposition 2.3.3. *$\mathcal{S}(D)'$ can be identified with $\mathcal{O}_M(D)$.*

When $f\in\mathcal{E}'(K)$ we can actually pair it with $\pi_\lambda\in\mathcal{E}(K,\mathcal{L}(\mathcal{H}_\lambda))\cong\mathcal{E}(K)\widehat{\otimes}\mathcal{L}(\mathcal{H}_\lambda)$ and obtain

$$\widehat{f}(\lambda)\in\mathcal{L}(\mathcal{H}_\lambda).$$

This is then the Fourier transform $\mathcal{E}(K)\to\mathcal{O}_M(D)$ explicitly.

Theorem 2.3.4 (Bochner's Theorem)**.** *A distribution $f \in \mathcal{E}'(K)$ is positive definite if and only if $\widehat{f}(\lambda)$ is positive for every λ.*

Proof. $F \in \mathcal{O}_M(D) \cong \mathcal{S}(D)'$ is positive if and only if

$$0 \le \langle F, \Phi\Phi^* \rangle = \sum_\lambda d(\lambda) \operatorname{tr} F(\lambda)\Phi(\lambda)\Phi(\lambda)^*.$$

By taking $\Phi = \Phi_A$ we find this is the case if and only if

$$\operatorname{tr} F(\lambda) A A^* \ge 0$$

for every λ and every A which since every positive matrix is of the form AA^* is equivalent to $\operatorname{tr} F(\lambda)B \ge 0$ for every positive B which is equivalent to positivity of $F(\lambda)$. □

Operator-Valued Case. When E is a locally convex space we can take the tensor product and obtain the vector-valued Fourier transform

$$\mathcal{E}(K, E) \cong \mathcal{E}(K) \mathbin{\widehat{\otimes}} E \to \mathcal{S}(D) \mathbin{\widehat{\otimes}} E.$$

We are primarily interested in the case $E = L^1(\mathcal{H})$ for some Hilbert space so we will take E to be a complete normed space with norm $\|\cdot\|$. First we will achieve explicit characterisations of $\mathcal{S}(D) \mathbin{\widehat{\otimes}} E$ and of

$$(\mathcal{S}(D) \mathbin{\widehat{\otimes}} E)' \cong \mathcal{S}'(D) \mathbin{\widehat{\otimes}} E'$$

Define $\mathcal{S}(D, E)$ to be the set of "maps" Φ such that $\Phi(\lambda) \in \mathcal{L}\mathcal{H}_\lambda \mathbin{\widehat{\otimes}} E$ and such that

$$q_n(\Phi) := \max_\lambda |\lambda|^n \|\Phi(\lambda)\|_\pi < \infty.$$

Here $\|\cdot\|_\pi$ denotes the projective tensor product norm of $\|\cdot\|_1$ and $\|\cdot\|$. Note that when $A \in \mathcal{L}\mathcal{H}_\lambda$ we have $\|A\|_2 \le \|A\|_1 \le \sqrt{d(\lambda)}\|A\|_2$ so that the norm $\|\cdot\|_2$ in the definition of $\mathcal{S}(D)$ can safely be replaced by $\|\cdot\|_1$.

$\mathcal{S}(D, E)$ is a Fréchet space: If $(\Phi_n)_n$ is a Cauchy sequence then $(\Phi_n(\lambda))_n$ is Cauchy so it has a pointwise limit $\Phi(\lambda)$. For every $\varepsilon > 0$ we have some N such that

$$|\lambda|^l \|\Phi_n(\lambda) - \Phi_m(\lambda)\|_\pi \le \varepsilon$$

for all λ and for all $n, m \ge N$. Taking the limit as $n \to \infty$ we have $q_l(\Phi - \Phi_m) \le \varepsilon$ for $m \ge N$. It follows that $\Phi_n \to \Phi$.

Proposition 2.3.5. *$\mathcal{S}(D, E)$ is linearly isomorphic to $\mathcal{S}(D) \mathbin{\widehat{\otimes}} E$.*

Proof. Consider the natural bilinear map $\Lambda : \mathcal{S}(D) \times E \to \mathcal{S}(D, E)$, $(\Phi, e) \mapsto \Phi \otimes e$ where $(\Phi \otimes e)(\lambda) = \Phi(\lambda) \otimes e$. Then we have

$$q_n(\Lambda(\Phi \otimes e)) = \max_\lambda |\lambda|^n \|\Phi(\lambda) \otimes e\|_\pi = q_n(\Phi)\|e\|$$

so that the bilinear map is jointly continuous. Thus we have a continuous linear map $\mathcal{S}(D) \mathbin{\widehat{\otimes}} E \to \mathcal{S}(D, E)$. The claim is that it is an isomorphism. The set of $\Phi \in \mathcal{S}(D, E)$ for which $\Phi(\lambda)$ is nonzero only for finitely many λ is dense and each such element can be approximated by Φ's with $\Phi(\lambda) \in \mathcal{L}\mathcal{H}_\lambda \otimes E$ which is in the image of Λ. So we have the image of Λ which we will also denote by $\mathcal{S}(D) \otimes E$ is dense.

We have a trilinear map

$$\begin{aligned} \mathcal{L}\mathcal{H}_\lambda \times \mathcal{L}\mathcal{H}_\lambda \times E &\to E \\ (T, S, e) &\mapsto \operatorname{tr}[TS]e \end{aligned}$$

which satisfies

$$\|\operatorname{tr}[TS]e\| \le \|T\|_1 \|S\|_1 \|e\| = \|T\|_1 \|S \otimes e\|_\pi.$$

So we get a jointly continuous bilinear map $B_\lambda : \mathcal{L}\mathcal{H}_\lambda \times \mathcal{L}\mathcal{H}_\lambda \mathbin{\widehat{\otimes}} E \to E$ which satisfies

$$\|B_\lambda(T, u)\| \le \|T\|_1 \|u\|_\pi.$$

This allows us to define the "application" map (we are trying to prove $\mathcal{S}(D, E) \cong \mathcal{S}(D) \mathbin{\widehat{\otimes}} E \cong \mathcal{L}(\mathcal{S}'(D), E)$)

$$\mathcal{S}'(D) \times \mathcal{S}(D, E) \to E$$

by using the association $\mathcal{S}'(D) \simeq \mathcal{O}_M(D)$

$$F \cdot \Phi = \sum_\lambda d(\lambda) B(F(\lambda), \Phi(\lambda)).$$

If $H \subseteq \mathcal{S}'(D)$ is an equicontinuous subset then there is some n and C such that

$$\max_\lambda |\lambda|^{-n} \|F(\lambda)\|_1 \le C$$

for all $F \in H$. Then

$$\begin{aligned} \|F \cdot \Phi\| &\le \sum_\lambda d(\lambda) \|F(\lambda)\|_1 \|\Phi(\lambda)\|_\pi \\ &\le C c q_{n'}(\Phi) \end{aligned}$$

for some appropriate $n' \ge n$. This demonstrates that the application map is separately continuous in $\mathcal{S}(D, E)$. As for $\mathcal{S}'(D)$, if $\Phi \in \mathcal{S}(D) \otimes E$, Φ can be written as $\sum_j \varphi_j \otimes e_j$ so

$$F \cdot \Phi = \sum_j \sum_\lambda d(\lambda) \operatorname{tr} F(\lambda) \varphi_j(\lambda) e_j = \sum_j \langle F, \varphi_j \rangle e_j$$

which gives us separate continuity in $\mathcal{S}'(D)$ is the element in $\mathcal{S}(D, E)$ is held fixed in $\mathcal{S}(D) \otimes E$. Note that $\mathcal{S}(D) \otimes E$ is sequentially dense in $\mathcal{S}(D, E)$ and $\mathcal{S}(D)$ is a nuclear Fréchet space so it is Montel (cf. [39, Prop. 50.2, Corollary 3]) so that

$\mathcal{S}'(D)$ is also Montel so in particular it is barreled. Approximating an element of $\mathcal{S}(D, E)$ by elements in $\mathcal{S}(D) \otimes E$ gives us then a sequence of continuous operators $\mathcal{S}'(D) \to E$ which is pointwise bounded so it is an equicontinuous sequence. The limit operator is then continuous which gives us separate continuity in both variables. Since both $\mathcal{S}'(D)$ and $\mathcal{S}(D, E)$ are barreled, it is hypocontinuous which means that we have a continuous linear map

$$\mathcal{S}(D, E) \to \mathcal{L}_b(\mathcal{S}'(D), E) \cong \mathcal{S}(D) \mathbin{\widehat{\otimes}} E$$

which is actually an inverse of the map we started out defining $\mathcal{S}(D) \mathbin{\widehat{\otimes}} E \to \mathcal{S}(D, E)$. □

Just as before we define $\mathcal{O}_M(D, E)$ to be the set of "maps" F such that $F(\lambda) \in \mathcal{L}(\mathcal{H}_\lambda) \mathbin{\widehat{\otimes}} E$ for every $\lambda \in D$ and

$$\max_\lambda |\lambda|^{-n} \|F(\lambda)\|_\pi < \infty$$

for some n. Any element $F \in \mathcal{O}_M(D, E')$ induces a continuous linear functional on $\mathcal{S}(D, E)$ by

$$\langle F, \Phi \rangle = \sum_\lambda d(\lambda) \langle F(\lambda), \Phi(\lambda) \rangle.$$

Employing the same proof as before we have

Proposition 2.3.6. *$\mathcal{S}(D, E)'$ is identified with $\mathcal{O}_M(D, E)$.*

Now we return to the case $E = L^1(\mathcal{H})$. Then $E' = \mathcal{L}\mathcal{H}$. Concretely, the Fourier transform $\mathcal{E}'(K, \mathcal{L}\mathcal{H}) \to \mathcal{O}_M(D, \mathcal{L}(\mathcal{H}))$ has the following description: A distribution $f \in \mathcal{E}'(K, \mathcal{L}\mathcal{H})$ is a linear map $\mathcal{E}(K) \to \mathcal{L}\mathcal{H}$ which can be paired with $\pi_\lambda \in \mathcal{E}(K, \mathcal{L}\mathcal{H}_\lambda)$ to produce

$$\widehat{f}(\lambda) = \langle f, \pi_\lambda \rangle \in \mathcal{L}\mathcal{H}_\lambda \mathbin{\widehat{\otimes}} \mathcal{L}\mathcal{H} \cong \mathcal{L}(\mathcal{H}_\lambda \mathbin{\widehat{\otimes}} \mathcal{H}).$$

Just as before we find

Theorem 2.3.7 (Bochner's Theorem)**.** *A distribution $f \in \mathcal{E}'(K, \mathcal{L}\mathcal{H})$ is positive definite if and only if $\widehat{f}(\lambda)$ is positive for all λ.*

Proof. An $F \in \mathcal{O}_M(D, L^1(\mathcal{H}))$ is a positive functional on $\mathcal{S}(D, L^1(\mathcal{H}))$ if and only if

$$0 \leq \sum_\lambda d(\lambda) \operatorname{tr} F(\lambda)\Phi(\lambda)\Phi(\lambda)^* \geq 0$$

for all $\Phi \in \mathcal{S}(D, L^1(\mathcal{H}))$. This is equivalent to $F(\lambda)$ being positive for all λ. □

Chapter 3

Vector-Valued Distribution Theory

In this chapter we consider distributions that have vector values. These have been studied by Schwartz in [33, 34]. This inquiry is motivated by the following: It is well-known that the Fourier transformation on the Heisenberg group takes a function to a family (i.e., a function) of operators on the dual. Then a generalised function (i.e., a distribution) should be taken to a generalised family of operators (i.e., an operator-valued distribution). This chapter does take us on a bit of a detour and its results are not strictly speaking necessary for the analysis of the Knapp-Stein, but it features a very general structure theorem for distributions with punctual support in Theorem 3.1.7. This interesting theorem I have not seen anywhere.

In this chapter is also included the easy characterisation of vector-valued homogeneous distributions on the line and also a proof of the smooth action of the general linear group $\mathrm{GL}(d, \mathbb{R})$ on $\mathcal{S}(\mathbb{R}^d)$ which will be used extensively later. The characterisation of homogeneous distributions is simply a generalisation of the arguments in [11, Ch. I, 3.11].

General Definitions. Suppose that $U \subseteq \mathbb{R}^d$, that E is a topological vector space and that $\mathcal{H}(U) \subseteq \mathcal{E}(U)$ is some vector subspace equipped with some topology finer than the topology on $\mathcal{E}(U)$. Then we define

$$\mathcal{H}'(U, E) := \mathcal{L}_b(\mathcal{H}(U), E).$$

In this way we define $\mathcal{D}'(U, E)$, $\mathcal{E}'(U, E)$, $\mathcal{S}'(\mathbb{R}^d, E)$, $(\mathcal{S}^\infty)'(U, E)$, $\mathcal{S}_0'(U, E)$. The continuous inclusion

$$\mathcal{D}(U) \subseteq \mathcal{E}(U)$$

gives rise to a continuous injection

$$\mathcal{E}'(U, E) \subseteq \mathcal{D}'(U, E)$$

and the continuous inclusions

$$\mathcal{D}(\mathbb{R}^d) \subseteq \mathcal{S}(\mathbb{R}^d) \subseteq \mathcal{E}(\mathbb{R}^d)$$

gives rise to continuous injections

$$\mathcal{E}'(\mathbb{R}^d, E) \subseteq \mathcal{S}'(\mathbb{R}^d, E) \subseteq \mathcal{D}'(\mathbb{R}^d, E).$$

Note that here we also use density of the spaces in question.

Remark 3.0.1. For this chapter we use the norms introduced in Appendix A.2.

Support of a Vector-Valued Distribution. For every $V \subseteq U$ the continuous inclusion $\mathcal{D}(V) \to \mathcal{D}(U)$ induces a continuous transpose $\mathcal{D}'(U, E) \to \mathcal{D}'(V, E)$, $T \mapsto T|_{\mathcal{D}(V)}$. For every $T \in \mathcal{D}'(U, E)$ the *support* $\operatorname{supp} T$ of T is the complement in U to the union of all the open V such that $T|_V = 0$. We will say that T has *compact support* when $\operatorname{supp} T$ is compact.

Proposition 3.0.2. *The elements of $\mathcal{D}'(U, E)$ with compact support is contained in $\mathcal{E}'(U, E)$.*

Proof. Let $T \in \mathcal{D}'(U, E)$, $K = \operatorname{supp} T$. Then there is some $\varphi \in \mathcal{D}(U)$ such that $\varphi = 1$ on K. The map $\mathcal{E}(U) \ni \psi \mapsto T(\varphi\psi) \in E$ is a continuous extension of T. □

Remark 3.0.3. Unlike the scalar case, the subspace of distributions with compact support is not identical to $\mathcal{E}'(U, E)$. Indeed, one can consider $E = \mathbb{R}^{\mathbb{N}}$ with $e_n \in E$ being the element defined by $(e_n)_j = \delta_{nj}$. We then have the distribution

$$\mathcal{E}(U) \ni \varphi \mapsto \sum_{x \in \mathbb{N}} \varphi(x) e_x.$$

This is continuous because E is given the product topology and composition with the projection onto the n'th factor gives $\varphi \mapsto \varphi(n)$ which is continuous. But the support is seen to be $\mathbb{N}$ which is noncompact.

However, we have

Proposition 3.0.4. *Suppose that E is a topological vector space with a continuous norm. Then the elements of $\mathcal{E}'(U, E)$ have compact support.*

Proof. Let $q = \|\cdot\|$ be a continuous norm on E and let $T \in \mathcal{E}'(U, E)$. Then we have

$$\|T\varphi\| \leq C p_{q,n,K}(\varphi)$$

for some compact $K \subseteq U$ and some $n \in \mathbb{N}_0$. This implies that T has support contained in K. □

Remark 3.0.5. The spaces E satisfying the condition of the last proposition include the normed spaces, naturally. The space $\mathbb{R}^{\mathbb{N}}$ is characteristic for not satisfying this condition. Indeed, if p is a continuous seminorm on $\mathbb{R}^{\mathbb{N}}$ by there is some N such that $|x_j| \leq r_j$ for $j = 1, \ldots, N$ implies $p(x) \leq 1$. This in turn implies that $p(x) \leq \max_{j \leq N} |x_j|$ for all x so that p has a huge kernel. One can see, cf. [18, Theorem 7.2.7], that a Fréchet space has a continuous norm if and only if it does not contain a subspace linearly homeomorphic to $\mathbb{R}^{\mathbb{N}}$.

As a consequence we have the well-known fact that $\mathcal{E}'(U) = \mathcal{E}'(U, \mathbb{C})$ is identified with the space of distributions in $\mathcal{D}'(U)$ having compact support.

Order of a Vector-Valued Distribution. For every $m \in \mathbb{N}_0$ we have the space $\mathcal{D}^m(U)$ which contains $\mathcal{D}(U)$ as a dense subspace. We will say that $T \in \mathcal{D}(U, E)$ has *finite order* if T is continuous in the topology inherited from $\mathcal{D}^m(U)$ for some m. This is equivalent to T admitting a continuous extension $\mathcal{D}^m(U) \to \widehat{E}$ where $\widehat{E}$ is the completion of E. The *order* is then the smallest m for which the above is true.

In the case that E is locally convex this translates to a definition analogous to the one for scalar-valued distributions in e.g. [16, Definition 2.1.1]. In the locally convex case, $T \in \mathcal{D}'(U, E)$ will have finite order if and only if there is an $m \in \mathbb{N}_0$ such that for all continuous seminorms p on E and for all compact $K \subseteq U$ there is a $C > 0$ such that

$$p(T\varphi) \leq C\|\varphi\|_m \tag{3.1}$$

for all $\varphi \in \mathcal{D}_K(U)$. The *order* of T is then the smallest m such that this is true.

Remark 3.0.6. Unlike the scalar case, not all elements of $\mathcal{E}'(U, E)$ have finite order. Indeed, we see that

$$\mathcal{E}(\mathbb{R}) \ni \varphi \mapsto \sum_n \varphi^{(n)}(0) e_n$$

is an element in $\mathcal{E}'(\mathbb{R}, \mathbb{R}^{\mathbb{N}})$ of infinite order. Actually, this distribution has compact support so not even this is sufficient for having finite order.

We say that $T \in \mathcal{D}'(U, E)$ is *locally of finite order* if $T|_V$ has finite order for each open and bounded $V \subseteq U$.

Proposition 3.0.7. *Suppose that E is a normed space. Then every element of $T \in \mathcal{D}'(U, E)$ has finite order.*

Proof. This is obvious since for every compact $K \subseteq U$ we have Eq. (3.1) when p is a norm that defines the topology on E. Every seminorm on E is equivalent to p so T restricted to the interior of K has finite order. □

Every normed space is a $\mathcal{DF}$-space. Schwartz proves in general that

Proposition 3.0.8. *Suppose that E is a quasi-complete $\mathcal{DF}$-space. Then every element of $\mathcal{D}'(U, E)$ is locally of finite order.*

Proof. [33, Prop. 23, Corollaire 2]. □

Using tensor-product methods we can a similar result:

Proposition 3.0.9. *Suppose that F is a Fréchet space. Then every element of $\mathcal{E}'(U, F')$ has compact support and finite order.*

Proof. According to [39, Prop. 50.7] we have $\mathcal{E}'(U, F') \cong \mathcal{E}(U, F)'$. If $T \in \mathcal{E}(U, F)'$ then there is some compact $K \subseteq U$ and a continuous seminorm q on F such that

$$|\langle T, \Phi \rangle| \leq C p_{q,n,K}(\Phi).$$

The element $S \in \mathcal{E}'(U, F')$ corresponding to T is $\langle S\varphi, f\rangle = T(\varphi \otimes f)$. We have

$$p_{q,n,K}(\varphi \otimes f) = \max_{|\alpha| \le n} \sup_{x \in K} p(\partial^\alpha \varphi(x) f) \le \|\varphi\|_{n,K} p(f).$$

It is enough to consider on F' the continuous seminorms p_B for $B \subseteq F$ bounded given by

$$p_B(f) = \sup_{\varphi \in B} |\langle f, \varphi\rangle|.$$

Then we have

$$p_B(S\varphi) \le C\|\varphi\|_{K,n} \sup_{f \in B} p(f).$$

This demonstrates that the order is at most n and that the support of S is contained in K. □

3.1 Structure Theorem for Distributions with Punctual Support

The following theorem is well-known to be true for $E = \mathbb{C}$:

Theorem 3.1.1. *Suppose that $T \in \mathcal{D}'(U, E)$ has support $\{x\}$ and order n. Then it has the form*

$$T(\varphi) = \sum_{j=1}^{n} \varphi^{(j)}(x) e_j$$

for some $N \in \mathbb{N}$ and $e_j \in E$.

Proof. We can follow [16, Theorem 2.3.4] basically without change. For this we need the following theorem. □

Theorem 3.1.2. *Suppose that $T \in \mathcal{D}'(U, E)$ has compact support and finite order n. Then if $\varphi \in \mathcal{D}(U)$ such that $\partial^\alpha \varphi(x) = 0$ for all $x \in \operatorname{supp} T$ and $|\alpha| \le n$ then $T(\varphi) = 0$.*

Proof. The proof in [16, Theorem 2.3.3] can be brought over without change. □

This argument is present in basically the same form in [35, Ch. III, §10, Thm. XXXV] and in [17, Ch. 4, §4, Prop. 5]. We will carry out a refactoring of this argument which will enable us to generalise Theorem 3.1.1 to Theorem 3.1.7. The unstated proposition that is being proven on the way in the previous proof is

Lemma 3.1.3. *Suppose that E is a quasi-complete locally convex space. For $N \in \mathbb{N}_0 \cup \{\infty\}$ the closure of*

$$\{\varphi \in \mathcal{D}^N(\mathbb{R}^d, E) \,|\, \operatorname{supp} \varphi \subseteq \mathbb{R}^d \setminus 0\}$$

in $\mathcal{D}^N(\mathbb{R}^d, E)$ is exactly

$$\{\varphi \in \mathcal{D}^N(\mathbb{R}^d, E) \,|\, \varphi^{(n)}(0) = 0 \text{ for all } |n| \le N\}.$$

This is a remark (for $E = \mathbb{C}$) made during the proof of the structure theorem in [11, Ch. II, 4.5 Theorem] but it is actually made explicit in [39, Lemma 24.1] without a detailed proof.

Remark 3.1.4. Here and other places we shall abuse notation slightly. When $N = \infty$, $\varphi^{(n)}(0) = 0$ for $|n| \leq N$ means that $\varphi^{(n)}(0) = 0$ for all n. It does not make sense in this case to put $n = \infty$!

Proof. If a function $\varphi \in \mathcal{D}^N(\mathbb{R}^d, E)$ satisfies $\operatorname{supp}\varphi \subseteq \mathbb{R} \setminus 0$ then $\varphi^{(n)}(0) = 0$ for all $|n| \leq N$ and each $\delta_0^{(n)}$ is continuous on $\mathcal{D}^N(\mathbb{R}, E)$ for $|n| \leq N$ so we have one inclusion. For the other, it is necessary to prove that any $\varphi \in \mathcal{D}^N(\mathbb{R}, E)$ satisfying $\varphi^{(n)}(0) = 0$ for all $|n| \leq N$ is a limit of functions with support in $\mathbb{R} \setminus 0$. Take $\chi \in \mathcal{D}(\mathbb{R}^d)$ such that $\chi(x) = 1$ for $|x| \leq \frac{1}{2}$ and $\chi(x) = 0$ for $|x| \geq 1$. Let $\chi_k(x) = \chi(kx)$. Then we prove that $\chi_k\varphi \to 0$ as $k \to \infty$. This will suffice because then $(1 - \chi_k)\varphi \to \varphi$. First note that $\chi_k\varphi \to 0$ uniformly. Indeed if q is some continuous seminorm on E and $\varepsilon > 0$ is given, by continuity there is some $\delta > 0$ such that $q(\varphi(x)) \leq \varepsilon$ for $|x| \leq \delta$. We have $\chi_k(x) = 0$ if $|x| \geq 1/k$ so when $\delta > 1/k$ we have $\|\chi_k\varphi\|_q \leq \varepsilon$.

Next we consider the derivatives. By Leibniz' rule (Theorem A.1.10),

$$(\chi_k\varphi)^{(n)} = \sum_{0 \leq m < n} \binom{n}{m} \chi_k^{(n-m)} \varphi^{(m)} + \chi_k \varphi^{(n)}.$$

For $|n| \leq N$ we have $\varphi^{(n)}(0) = 0$ so by the above argument $\chi_k\varphi^{(n)} \to 0$ uniformly. As for the other terms,

$$\chi_k^{(n-m)}(x) = k^{|n|-|m|}\chi^{(n-m)}(kx)$$

is equal to 0 for $|x| \leq \frac{1}{2k}$ and for $|x| \geq \frac{1}{k}$ if $m < n$. So we have

$$\|\chi_k^{(n-m)}\varphi^{(m)}\|_q \leq k^{|n|-|m|}\|\chi^{(n-m)}\|_\infty \sup_{\frac{1}{2k} \leq |x| \leq \frac{1}{k}} q(\varphi^{(m)}(x)).$$

The Taylor expansion (Theorem A.1.12) of $\varphi^{(m)}$ around 0 of degree $|n|-|m|-1 \in \mathbb{N}_0$ is

$$\begin{aligned} \varphi^{(m)}(x) = &\sum_{l \leq |n|-|m|-1} \frac{\varphi^{(m+l)}(0)}{l!} x^l \\ &+ \sum_{|l|=|n|-|m|} \frac{|n|-|m|}{l!} x^l \int_0^1 \varphi^{(l+m)}(tx)(1-t)^{|n|-|m|-1}\, dt. \end{aligned}$$

Note that $|l| \leq |n| - |m| - 1$ implies $|m + l| \leq |n| - 1 \leq N$ so $\varphi^{(m+l)}(0) = 0$. This means the first terms disappear so we have

$$\begin{aligned} q(\varphi^{(m)}(x)) &\leq \sum_{|l|=|n|-|m|} \frac{|n|-|m|}{l!} |x^l| \int_0^1 (1-t)^{|n|-|m|-1} q(\varphi^{(l+m)}(tx))\, dt \\ &\leq |x|^{|n|-|m|} \sum_{|l|=|n|} \frac{1}{l!} \sup_{|y| \leq |x|} q(\varphi^{(l)}(y)) \end{aligned}$$

So we get

$$\sup_{\frac{1}{2k}\leq|x|\leq\frac{1}{k}} q(\varphi^{(m)}(x)) \leq k^{|m|-|n|} \sum_{|l|=|n|} \sup_{|x|\leq\frac{1}{k}} |\varphi^{(l)}(x)|$$

and

$$\|\chi_k^{(n-m)}\varphi^{(m)}\|_q \leq \|\chi^{(n-m)}\|_\infty \sup_{|x|\leq\frac{1}{k}} \sum_{|l|=|n|} q(\varphi^{(l)}(x)).$$

By continuity of $\varphi^{(l)}$ and since $\varphi^{(l)}(0) = 0$ this converges to 0 as $k \to \infty$. □

We will need a similar result for other spaces. Using that $\mathcal{D}(U, E)$ is dense in $\mathcal{E}(U, E)$ and $\mathcal{D}(\mathbb{R}^d, E)$ is dense in $\mathcal{S}(\mathbb{R}^d, E)$ we get

Proposition 3.1.5. *Suppose that E is a quasi-complete locally convex space. The closure of the set of $\varphi \in \mathcal{E}(U, E)$ with $\operatorname{supp}\varphi \subseteq \mathbb{R}^d \setminus 0$ is exactly the subspace of φ that vanish to all degrees at 0.*

Proposition 3.1.6. *Suppose that E is a quasi-complete locally convex space. The closure of the set of $\varphi \in \mathcal{S}(\mathbb{R}^d, E)$ with $\operatorname{supp}\varphi \subseteq \mathbb{R}^d \setminus 0$ is exactly $\mathcal{S}_0(\mathbb{R}^d, E)$, i.e., the subspace of functions that vanish to all degrees at 0.*

Theorem 3.1.7. *Suppose that $T \in \mathcal{D}'(\mathbb{R}^d, E)$ has support equal to $\{0\}$. Then there are elements $e_n \in E$ such that*

$$T = \sum_{n\in\mathbb{N}_0^d} \delta_0^{(n)} \otimes e_n. \tag{3.2}$$

Proof. By the previous theorem the kernel of T must contain

$$N = \bigcap_n \ker \delta_0^{(n)}.$$

There is a continuous linear map $\Phi : \mathcal{D}(\mathbb{R}^d) \to \mathbb{C}^{\mathbb{N}_0^d}$ given by $\Phi(\varphi) = (\varphi^{(n)}(0))_n$. Restricted to $\mathcal{D}_K(\mathbb{R}^d)$ the map is surjective by Borel's lemma ([39, Thm. 38.1]) and both $\mathcal{D}_K(\mathbb{R}^d)$ and $\mathbb{C}^{\mathbb{N}_0^d}$ are Fréchet spaces so the restriction is an open map cf. [30, Thm. 2.11]. Since a local basis for the topology on $\mathcal{D}(\mathbb{R}^d)$ is given by the absolutely convex sets that intersect with $\mathcal{D}_K(\mathbb{R}^d)$ to a 0-neighbourhood in $\mathcal{D}_K(\mathbb{R}^d)$ (cf. [28, Ch. V, §2, Prop. 4]) we find that Φ must be open as well. But then Φ induces an isomorphism $\widetilde{\Phi} : \mathcal{D}(\mathbb{R}^d)/N \to \mathbb{C}^{\mathbb{N}_0^d}$, cf. [18, Prop. 4.2.4]. Since the kernel of T contains N it induces a continuous linear map $\widetilde{T} : \mathcal{D}(\mathbb{R}^d)/N \to E$ ([18, Prop. 4.1.2]). Now, $\widetilde{T} \circ \widetilde{\Phi}^{-1}$ is given uniquely by its values on the standard basis vectors s_n in $\mathbb{C}^{\mathbb{N}_0^d}$. Then we set

$$e_n = \widetilde{T}(\widetilde{\Phi})^{-1}(s_n).$$

When $\varphi \in \mathcal{D}(\mathbb{R}^d)$ we then find that

$$T(\varphi) = \widetilde{T}[\varphi] = \widetilde{T}(\widetilde{\Phi})^{-1}\widetilde{\Phi}[\varphi] = \widetilde{T}(\widetilde{\Phi})^{-1}(\varphi^{(n)}(0))_n = \sum_n \varphi^{(n)}(0)e_n.$$

The equality Eq. (3.2) is an equality in $\mathcal{D}'(U, E)$ so we need to see that the convergence in the sum is uniform on bounded subsets of $\mathcal{D}(\mathbb{R})$. The family

$$\{\sum_{|n|\leq N} \delta_0^{(n)} \otimes e_n \,|\, N \in \mathbb{N}_0\} \cup \{T\}$$

is equicontinuous because restricted to each $\mathcal{D}_K(\mathbb{R}^d)$ it is equicontinuous because of the Uniform Boundedness Principle (cf. [28, Ch. V, §2, Prop. 5] and [30, Thm. 2.6] — pointwise we have convergence so we have boundedness since it is a sequence). Now, on an equicontinuous set of linear maps the topology of pointwise convergence and compact convergence coincide, cf. [39, Proposition 32.5]. Since $\mathcal{D}(\mathbb{R}^d)$ is a Montel space, this topology coincides with the topology of bounded convergence. □

Now a natural generalization presents itself: It is possible to define, in general, $\mathcal{D}'(M, E)$ for any smooth manifold M. When $N \subseteq M$ is an (embedded) submanifold and $T \in \mathcal{D}'(M, E)$ has support contained in N it is natural to ask whether or not there exists an $S \in \mathcal{D}'(N, E)$ such that $T(\varphi) = S(\varphi|_N)$ for all $\varphi \in \mathcal{D}(M)$. We will give a simple theorem with an affirmative answer:

Corollary 3.1.8. *Suppose that $T \in \mathcal{D}'(\mathbb{R}^d, E)$ has support contained in $\mathbb{R}^k \times 0$ for some $k \leq d$. Then there is a decomposition*

$$T = \sum_n T_n \otimes \delta_0^{(n)}$$

for some $T_n \in \mathcal{D}'(\mathbb{R}^k, E)$.

This corollary is of course well-known in the $E = \mathbb{C}$-case.

Proof. We simply note that

$$\mathcal{D}'(\mathbb{R}^d, E) \cong \mathcal{D}'(\mathbb{R}^{d-k}) \,\widehat{\otimes}\, \mathcal{D}'(\mathbb{R}^k) \,\widehat{\otimes}\, E \cong \mathcal{D}'(\mathbb{R}^{d-k}, \mathcal{D}'(\mathbb{R}^k, E)).$$

That T has support in $\mathbb{R}^k \times 0$ means that the corresponding element $\widetilde{T}$ in $\mathcal{D}'(\mathbb{R}^{d-k}, \mathcal{D}'(\mathbb{R}^k, E))$ has support at $\{0\}$. The previous theorem implies that

$$\widetilde{T} = \sum_n \delta_0^{(n)} \otimes T_n$$

for some $T_n \in \mathcal{D}'(\mathbb{R}^k, E)$. □

3.2 Homogeneous Vector-Valued Distributions

Suppose that E is a locally convex space and let X be either $\mathbb{R}^d$ or $\mathbb{R}^d \setminus 0$. An element $\Phi \in \mathcal{E}(X, E)$ is said to be *homogeneous of degree* $\lambda \in \mathbb{C}$ if for all $s > 0$ and we have

$$\Phi(sx) = s^\lambda \Phi(x)$$

for all x. We introduce $\delta_s : \mathcal{E}(X) \to \mathcal{E}(X)$ given by $[\delta_s \varphi](x) = \varphi(sx)$. Then homogeneity of Φ means $[\delta_s \otimes I]\Phi = s^\lambda \Phi$. One finds

Lemma 3.2.1. *For every $\varphi \in \mathcal{D}(X)$, $\mathbb{R}^* \ni s \mapsto \delta_s\varphi \in \mathcal{D}(X)$ is differentiable with derivative at s equal to $\frac{1}{s}x \cdot \nabla\delta_s\varphi$.*

Remark 3.2.2. We use the shorthand

$$x \cdot \nabla = \sum_{j=1}^{d} x_j \partial_j.$$

On the distributions it acts as

$$\langle x \cdot \nabla T, \varphi \rangle = -\sum_j \langle T, \partial_j x_j \varphi \rangle = -d\langle T, \varphi \rangle - \langle T, x \cdot \nabla\varphi \rangle \tag{3.3}$$

by the chain rule.

Proof. Note first that $s \mapsto \delta_s\varphi$ is continuous. Indeed, for $s \in B \subseteq \mathbb{R}^*$ bounded, $\operatorname{supp} \delta_s\varphi \subseteq B^{-1} \operatorname{supp} \varphi$ so in proving $\delta_t\varphi \to \delta_s\varphi$ for $t \to s$ we can work inside $\mathcal{D}_K(X)$ for $K = B^{-1} \operatorname{supp} \varphi$ where B is a bounded neighbourhood of s. We have φ is uniformly continuous so given $\varepsilon > 0$ we can find $r > 0$ such that $|x - y| \le r$ implies $|\varphi(x) - \varphi(y)| \le \varepsilon$. Let $R > 0$ such that $K \subseteq \overline{B}(0, R)$. Then for every $x \in X$ we have either $|x| > R$ so that when $t \in B$, $\delta_t\varphi(x) = 0 = \delta_s\varphi(x)$ or we have $|x| \le R$ so that when $t \in B$, $|t - s| \le r/R$, $|tx - sx| \le r$ so $|\varphi(tx) - \varphi(sx)| \le \varepsilon$.

In short, we have proven that as $t \to s$, $\delta_t\varphi \to \delta_s\varphi$ uniformly. Note that $\partial_j\delta_s\varphi = s\delta_s\partial_j\varphi$ so one finds

$$\partial_j\delta_t\varphi - \partial_j\delta_s\varphi = t(\delta_t\partial_j\varphi - \delta_s\partial_j\varphi) + (t - s)\delta_s\partial_j\varphi$$

goes uniformly to 0 as $t \to s$. This completes the argument that $s \mapsto \delta_s\varphi$ is *continuous*.

As for the rest, we start with

$$\frac{d}{dt}\varphi(tx) = \sum_j x_j\partial_j\varphi(tx) = \frac{1}{t}x \cdot \nabla\delta_t\varphi(x)$$

so pointwise we have

$$\frac{\delta_{s+h}\varphi - \delta_s\varphi}{h} - \frac{1}{s}x \cdot \nabla\delta_s\varphi = \frac{1}{h}\int_s^{s+h} \frac{1}{t}x \cdot \nabla\delta_t\varphi - \frac{1}{s}x \cdot \nabla\delta_s\varphi \, dt.$$

We can take care of the supports as before so that it is only necessary to work within $\mathcal{D}_K(X)$. It is clear that the norms of $\mathcal{D}_K(X)$ is sublinear with regards to the integral above so it is only necessary to see that

$$t \mapsto \frac{1}{t}x \cdot \nabla\delta_t\varphi$$

is continuous. But this follows from the above. □

Then one has

Proposition 3.2.3. *An element $\Phi \in \mathcal{E}(X, E)$ is homogeneous of degree λ if and only if the corresponding $\Phi \in \mathcal{D}'(X, E)$ satisfies*

$$\langle T, \delta_s \varphi \rangle = s^{\lambda+d} \langle T, \varphi \rangle$$

for all $s > 0$.

Proof. For a general $\varphi \in \mathcal{E}(X)$ and $\psi \in \mathcal{D}(X)$ we have by substitution

$$\langle \varphi, \delta_s \psi \rangle = s^{-d} \langle \delta_s^{-1} \varphi, \psi \rangle.$$

Since δ_s is continuous this extends to the vector-valued case $\Phi \in \mathcal{E}(X, E)$ and $\varphi \in \mathcal{D}(X)$:

$$\langle \Phi, \delta_s \varphi \rangle = s^{-d} \langle [\delta_s^{-1} \otimes I] \Phi, \psi \rangle.$$

Indeed, one simply approximates Φ by simple tensors.

Homogeneity of Φ as a function is equivalent to $[\delta_s^{-1} \otimes I]\Phi = s^{-\lambda}\Phi$ so we are done. □

This suggests the general definition: An element $T \in \mathcal{D}'(X, E)$ is said to be *homogeneous of degree* $\lambda \in \mathbb{C}$ if

$$\langle T, \delta_s \varphi \rangle = s^{-\lambda-d} \langle T, \varphi \rangle. \tag{3.4}$$

Just as in the scalar case there is an equivalent formulation by using the Euler operator $x \cdot \nabla$.

Proposition 3.2.4. *An element $T \in \mathcal{D}'(X, E)$ is homogeneous of degree $\lambda \in \mathbb{C}$ if and only if*

$$[x \cdot \nabla] T = \lambda T.$$

Proof. Indeed, differentiating Eq. (3.4) and using the lemma above we have

$$\frac{1}{s} \langle T, x \cdot \nabla \delta_s \varphi \rangle = -(\lambda + d) s^{-\lambda-d-1} \langle T, \varphi \rangle.$$

Using Eq. (3.3) this is

$$d\frac{1}{s} \langle T, \varphi \rangle + \frac{1}{s} \langle x \cdot \nabla T, \varphi \rangle = (\lambda + d) s^{-\lambda-d-1} \langle T, \varphi \rangle.$$

Evaluating at $s = 1$ we have

$$\langle x \cdot \nabla T, \varphi \rangle = \lambda \langle T, \varphi \rangle$$

as we want.

As for the other direction, we see that $s \mapsto \langle T, \delta_s \varphi \rangle$ solves the differential equation $y' = -(\lambda + d) y / x$ which has exactly $c s^{\lambda+d}$ as a solution. It follows that T is homogeneous of degree λ. □

It will also be convenient to have the concept of *parity*. A function $\Phi \in \mathcal{E}(X,E)$ is even if $\Phi(-x) = \Phi(x)$ and odd if $\Phi(-x) = -\Phi(x)$. In short, Φ has parity $\varepsilon \in \{-1,1\}$ if $\Phi(-x) = \varepsilon\Phi(x)$. This concept generalises naturally to $\mathcal{D}'(X,E)$: Let $\iota : \mathcal{D}(X) \to \mathcal{D}(X)$ be the continuous linear map defined by $\iota\varphi(x) = \varphi(-x)$. Then $F \in \mathcal{D}'(X,E)$ has parity ε if

$$\langle F, \iota\varphi\rangle = \varepsilon\langle F, \varphi\rangle,$$

i.e., if $F \circ \iota = \varepsilon F$. Again, a distribution is even if it has parity $+$ and odd if it has parity $-$.

The Scalar Case. Let $H^\lambda(X) \subseteq \mathcal{D}'(X)$ be the vector space of homogeneous distributions of degree λ. Let us review the homogeneous distributions in the scalar case for $d = 1$. Consider the functions on $\mathbb{R}$ defined by

$$x_+^\lambda(t) = \begin{cases} t^\lambda, & t > 0 \\ 0, & t \leq 0. \end{cases}$$

$$x_-^\lambda(t) = \begin{cases} 0, & t \geq 0 \\ (-t)^\lambda, & t < 0. \end{cases}$$

Their restrictions to $\mathbb{R}^*$ are smooth and so for each $\lambda \in \mathbb{C}$ we have homogeneous distributions $x_\varepsilon^\lambda \in \mathcal{D}'(\mathbb{R}^*)$. It is even the case that $(x_\varepsilon^\lambda)_\lambda$ is an analytic family, i.e., the map $\lambda \mapsto x_\varepsilon^\lambda$ is analytic. It is classical and follows from the same argument as given below that

$$H^\lambda(\mathbb{R}^*) = \operatorname{span}(x_+^\lambda, x_-^\lambda).$$

Now, for $X = \mathbb{R}$ the x_ε^λ's are only smooth functions for $\operatorname{Re}\lambda > 0$ but they are still locally integrable for $\operatorname{Re}\lambda > -1$. Also it is still the case that

$$\{\operatorname{Re}\lambda > -1\} \ni \lambda \mapsto x_\varepsilon^\lambda \in \mathcal{D}'(\mathbb{R})$$

is analytic and we have

$$\frac{d}{dx}x_\varepsilon^\lambda = \varepsilon\lambda x_\varepsilon^{\lambda-1}.$$

This equation allows us to analytically extend x_ε^λ to $\lambda \in \mathbb{C} \setminus (-\mathbb{N})$ and we have again

$$H^\lambda(\mathbb{R}) = \operatorname{span}(x_+^\lambda, x_-^\lambda)$$

for $\lambda \in \mathbb{C}\setminus(-\mathbb{N})$. Indeed, when $T \in \mathcal{D}'(\mathbb{R})$ is homogeneous of degree $\lambda \in \mathbb{C}\setminus(-\mathbb{N})$ we have $T|_{\mathbb{R}^*} = c_+x_+^\lambda + c_-x_-^\lambda$ for some c_+, c_- so that $T - c_+x_+^\lambda - c_-x_-^\lambda$ is supported at 0 so it is a sum of Dirac deltas. But the Dirac deltas are homogeneous of degree $-1, -2, \ldots$ so the sum is 0.

The singularities are simple poles; with $\chi_\varepsilon^\lambda = \frac{x_\varepsilon^\lambda}{\Gamma(\lambda+1)}$ the above equation gives us

$$\chi_\varepsilon^\lambda = \varepsilon^n \frac{d^n}{dx^n}\chi_\varepsilon^{\lambda+n}.$$

One finds by the Fundamental Theorem of Analysis that

$$\frac{d^n}{dx^n}\chi_\varepsilon^0 = \varepsilon\delta_0^{(n-1)}$$

so that

$$\chi_\varepsilon^{-n} = \varepsilon^{n-1}\delta_0^{(n-1)}$$

for all $n \in \mathbb{N}$.

We have

$$\operatorname{Res}_{\lambda=-n} x_\varepsilon^\lambda = \chi_\varepsilon^{-n} = \varepsilon^{n-1}\delta_0^{(n-1)}$$

and as such this is the obstruction to extending x_ε^λ to $\lambda = -n$. We can reparametrise

$$|x|_\varepsilon^\lambda = x_+^\lambda + \varepsilon x_-^\lambda$$

and use that when n is odd, $\chi_+^{-n} = \chi_-^{-n}$ so that the limit $|x|_-^{-n}$ exists. A convenient name for this distribution is x^{-n} because it corresponds with this reciprocal inverse on $\mathbb{R}^*$. Likewise, for n even, $\chi_+^{-n} = -\chi_-^{-n}$ so the limit $|x|_+^{-n}$ exists and we will also call it x^{-n}. One has then

$$x^{-n}|_{\mathbb{R}\setminus 0} = x_+^{-n} + (-1)^n x_-^{-n}.$$

Note that then x^{-n} is odd when n is odd and even when n is even. Also we have $\delta_0^{(n-1)}$ is homogeneous of degree $-n$ and it is even when n is odd and odd when n is even.

As for extending x_ε^{-n} from $\mathbb{R} \setminus 0$ to $\mathbb{R}$ this is possible. Indeed, we can simply subtract the singular part and consider

$$X_\varepsilon^{-n} = \lim_{\lambda\to -n}(x_\varepsilon^\lambda - \frac{1}{\lambda+n}\varepsilon^{n-1}\delta_0^{(n-1)}).$$

Then $X_\varepsilon^{-n} \in \mathcal{D}'(\mathbb{R})$ satisfies $X^{-n}|_{\mathbb{R}\setminus 0} = x_\varepsilon^{-n}$. While this can seem the natural choice, X_ε^{-n} is of course only unique up to addition of $\delta_0^{(k)}$'s. Unfortunately, X_ε^{-n} is *not* homogeneous. Indeed, by continuity of $x\frac{d}{dx}$,

$$x\frac{d}{dx}X_\varepsilon^{-n} = \lim_{\lambda\to -n}(\lambda x_\varepsilon^\lambda - \frac{1}{\lambda+n}(-n)\varepsilon^{n-1}\delta_0^{(n-1)}) = -nX_\varepsilon^{-n} + \varepsilon^{n-1}\delta_0^{(n-1)}.$$

Adding Dirac deltas to X_ε^{-n} does not change the fact that it is not homogeneous but we have

$$x^{-n} = X_+^{-n} + (-1)^n X_-^{-n}$$

is homogeneous. The other linear combination $X_+^{-n} - (-1)^n X_-^{-n}$ is not homogeneous so we arrive at

Proposition 3.2.5. *Consider the restriction*

$$H^\lambda(\mathbb{R}) \to H^\lambda(\mathbb{R}\setminus 0).$$

For any $\lambda \in \mathbb{C} \setminus (-\mathbb{N})$, it is an isomorphism. For $\lambda = -n \in -\mathbb{N}$, the image is one-dimensional and spanned by

$$x^{-n} = x_+^{-n} + (-1)^n x_-^{-n}.$$

With this in hand we can classify all the homogeneous distributions:

Theorem 3.2.6. *For $\lambda \in \mathbb{C} \setminus (-\mathbb{N})$, with an abuse of notation*

$$H^\lambda(\mathbb{R} \setminus 0) = \operatorname{span}(x_+^\lambda, x_-^\lambda) = H^\lambda(\mathbb{R}).$$

For $\lambda = -n \in \mathbb{N}$ we have

$$H^{-n}(\mathbb{R} \setminus 0) = \operatorname{span}(x_+^{-n}, x_-^{-n})$$

and

$$H^{-n}(\mathbb{R}) = \operatorname{span}(x^{-n}, \delta_0^{(n-1)}).$$

Remark 3.2.7. The argument given for this in [11, p. 81] has a small flaw in the case $\lambda = -n$: The problem lies in the sentence "We shall assume that $f(x)$ is even for even n and odd for odd n". This assumption cannot be allowed but we can split f into an even part and an odd part and then we need to argue that that the wrong parity cannot exist.

Proof. The only thing that has not been commented on is the last bit. Consider a homogeneous distribution $T \in \mathcal{D}'(\mathbb{R})$ of degree $-n$. Then T is uniquely a sum $T_1 + T_2$ of distributions where T_1 is of parity $(-1)^n$ and T_2 is of parity $-(-1)^n$. By the previous proposition, T_2 restricts to 0 in $\mathbb{R} \setminus 0$ so T_2 is a sum of Dirac deltas. Since T_2 is also homogeneous of degree $-n$ we must have that it is a scalar multiple of $\delta_0^{(n-1)}$. As for T_1 it must restrict to cx^{-n} for the parity to hold. Then $T_1 - cx^{-n}$ has support at 0 so it is a sum of Dirac deltas. But the only Dirac delta of the correct degree is $\delta_0^{(n-1)}$ and even this will not do because it has the wrong parity. It follows that $T_1 = cx^{-n}$. Thus we are done. □

The Vector-Valued Case. We will use the notation $H^\lambda(X, E) \subseteq \mathcal{D}'(X, E)$ for the vector-valued distributions homogeneous of degree λ.

Proposition 3.2.8. *For any topological vector space E and any $\lambda \in \mathbb{C}$ we have*

$$H^\lambda(\mathbb{R}^*, E) = H^\lambda(\mathbb{R}^*) \otimes E.$$

The proof hinges on the following generalisation of a result in the scalar case.

Lemma 3.2.9. *Suppose that $U \subseteq \mathbb{R}$ is an open interval and suppose that $T \in \mathcal{D}'(U, E)$ satisfies $T' = 0$. Then $T = 1_U \otimes e$ for some $e \in E$.*

Proof. The proof is basically the same as in the scalar case. That $T' = 0$ implies that the kernel of 1_U is contained in the kernel of T. We get

$$\mathbb{C} \cong \mathcal{D}(U)/\ker 1_U \to \mathcal{D}(U)/\ker T \to E$$

is a linear map. Therefore, it must have the form $1 \mapsto e$ for some $e \in E$. An arbitrary element $\varphi \in \mathcal{D}(U)$ corresponds in $\mathbb{C}$ to $\int \varphi$ which is mapped to, on the one hand, $\int \varphi \cdot e$, and on the other, $T\varphi$. □

Proof of Theorem 3.2.8. Suppose that $T \in \mathcal{D}'(\mathbb{R}^*, E)$ is homogeneous of degree $\lambda \in \mathbb{C} \setminus (-\mathbb{N})$. Then $|x|^{-\lambda}T$ is also a distribution and using the chain rule,

$$x\frac{d}{dx}(|x|^{-\lambda}T) = 0$$

so that $|x|^{-\lambda}T$ is constant on connected components, i.e., there are $e_\pm \in E$ such that

$$|x|^{-\lambda}T = x_+^0 e_+ + x_-^0 e_-.$$

This implies

$$T = x_+^\lambda e_+ + x_-^\lambda e_-. \qquad \square$$

Proposition 3.2.10. *For any topological vector space E and any $\lambda \in \mathbb{C}$*

$$H^\lambda(\mathbb{R}, E) = H^\lambda(\mathbb{R}) \otimes E.$$

Proof. Suppose that $T \in H^\lambda(\mathbb{R}, E)$. First consider the case for $\lambda \in \mathbb{C} \setminus (-\mathbb{N})$. Then, naturally, $T|_{\mathbb{R}\setminus 0}$ is also homogeneous of degree λ so that by the previous proposition,

$$T = T_1 \otimes e_1 + T_2 \otimes e_2$$

for $T_j \in H^\lambda(\mathbb{R} \setminus 0)$ and $e_j \in E$. Note that we have used here that $H^\lambda(\mathbb{R} \setminus 0)$ is 2-dimensional. Both T_1 resp. T_2 have extensions $\widetilde{T}_1$ resp. $\widetilde{T}_2$ to $\mathbb{R}$. But then

$$T - \widetilde{T}_1 \otimes e_1 - \widetilde{T}_2 \otimes e_2$$

has support $\{0\}$ so Theorem 3.1.7 tells us that

$$T = \widetilde{T}_1 \otimes e_1 + \widetilde{T}_2 \otimes e_2 + \sum_{n=0}^{\infty} \delta_0^{(n)} \otimes x_n$$

for some $x_n \in E$. But since T, T_1, T_2 are homogeneous of degree λ we get

$$0 = (x\frac{d}{dx} - \lambda)\sum_{n=0}^{\infty} \delta_0^{(n)} \otimes x_n = -\sum_{n=0}^{\infty}(n+1+\lambda)\delta_0^{(n)} \otimes x_n$$

since $\delta_0^{(n)}$ is homogeneous of degree $-(n+1)$. It is possible to exhibit functions in $\mathcal{D}(\mathbb{R})$ with $(\varphi^{(n)}(0))_n$ as any given sequence so this implies that

$$(n+1+\lambda)x_n = 0$$

for all $n \in \mathbb{N}_0$. Since $\lambda \notin -\mathbb{N}$ we must have $x_n = 0$ for all n so that

$$T = \widetilde{T}_1 \otimes e_1 + \widetilde{T}_2 \otimes e_2.$$

Now suppose that $\lambda = -n \in -\mathbb{N}$. Then $T = T_1 + T_2$ uniquely where $T_1(-x) = (-1)^n T_1(x)$ and $T_2(-x) = -(-1)^n T_2(x)$. The previous theorem together with the parity informs us that

$$T_1|_{\mathbb{R}\setminus 0} = x^{-n}|_{\mathbb{R}\setminus 0} \otimes e_1$$

so that $T_1 - x^{-n} \otimes e_1$ has support $\{0\}$ so it is a sum of Dirac deltas. The argument before gives us that because of the homogeneity

$$T_1 - x^{-n} \otimes e_1 = \delta_0^{(n-1)} \otimes y$$

but $\delta_0^{(n-1)}$ has parity $-(-1)^n$ so $y = 0$. So $T_1 = x^{-n} \otimes e_1$. As for T_2 we must have

$$T_2|_{\mathbb{R}\setminus 0} = |x|^{-n}_{-(-1)^n} \otimes e_2$$

and we need to say here that $e_2 = 0$. Indeed, if $e_2 \neq 0$ then $|x|^{-n}_{-(-1)^n} \otimes e_2$ cannot be extended to a homogeneous distribution of degree $-n$ on $\mathbb{R}$. This follows just as in the scalar case: We have a continuous bilinear map

$$\mathcal{D}'(\mathbb{R}) \times E \to \mathcal{D}'(\mathbb{R}, E)$$

so we see that

$$\lambda \mapsto x_\varepsilon^\lambda \otimes e \in \mathcal{D}'(\mathbb{R}, E)$$

is analytic with residues

$$\operatorname{Res}_{\lambda=-n} x_\varepsilon^\lambda \otimes e = \varepsilon^{n-1} \delta_0^{(n-1)} \otimes e.$$

Removing the singularity, it is possible to find an extension of $|x|^{-n}_{-(-1)^n} \otimes e_2$ to $\mathbb{R}$ unique up to Dirac deltas because of Theorem 3.1.7. But just as in the scalar case we see that none of these will be homogeneous.

This all implies that $e_2 = 0$ so T_2 is a sum of Dirac deltas. Considering the homogeneity we must have

$$T_2 = \delta_0^{(n-1)} \otimes e_3$$

which concludes the proof. □

3.3 The Representation of $\mathrm{GL}(d, \mathbb{R})$ on $\mathcal{S}_d$

Action of $\mathrm{GL}(d, \mathbb{R})$ on $\mathcal{S}_d$. For $g \in \mathrm{GL}(d, \mathbb{R})$ and $\varphi \in \mathcal{S}_d$ we define $g \cdot \varphi : \mathbb{R}^d \to \mathbb{C}$ by

$$[g \cdot \varphi](x) = \varphi(g^{-1}x).$$

Lemma 3.3.1. *For all $g \in \mathrm{GL}(d, \mathbb{R})$, $\varphi \in \mathcal{S}_d$, $g\varphi \in \mathcal{S}_d$. Also, the linear map $\varphi \mapsto g\varphi$ is continuous $\mathcal{S}_d \to \mathcal{S}_d$.*

The n'th partial derivatives of φ at a point x can be understood as an n-linear map $D^n\varphi_x : (\mathbb{R}^d)^n \to \mathbb{C}$. That φ is a Schwartz function simply means that

$$\sup_x |x|^m |D^n \varphi_x(v_1, \ldots, v_n)| < \infty$$

for all m, n and all vectors $v_1, \ldots, v_n$.

Proof. The chain rule implies that

$$\begin{aligned} D^n(g\varphi)_x(v_1,\dots,v_n) &= D^n\varphi_{g^{-1}x}(g^{-1}v_1,\dots,g^{-1}v_n) \\ &= \sum_k \prod_i (g^{-1}v_i|e_{k_i}) D^n\varphi_{g^{-1}x}(v_{k_1},\dots,v_{k_n}) \end{aligned}$$

where $(e_i)_i$ is some orthonormal basis of $\mathbb{R}^d$. It follows that it is enough to see that $|x|^m/|g^{-1}x|^m$ is bounded as $x \in \mathbb{R}^d$ varies. Indeed, if $\|g\|$ is the operator norm of g then $|gx| \le \|g\| \cdot |x|$ so

$$|x| \le \|g\| \cdot |g^{-1}x|.$$

Using this we find

$$\sup_x |x|^m |D^n\varphi_x(v_1,\dots,v_n)| \le \|g\| \sum_k \prod_i |(g^{-1}v_i|e_{k_i})| \sup_x |x|^m |D^n\varphi_x(e_{k_1},\dots,e_{k_n})|$$

which gives us what we want. □

Proposition 3.3.2. *The action of* GL$(d,\mathbb{R})$ *on* $\mathcal{S}_d$ *is strongly continuous, i.e.,* GL$(d,\mathbb{R}) \ni g \mapsto g\varphi \in \mathcal{S}_d$ *is continuous.*

Proof. We need to see that

$$|x|^m D^n(g\varphi)_x(v_1,\dots,v_n) \to |x|^m D^n\varphi_x(v_1,\dots,v_n)$$

uniformly in $x \in \mathbb{R}^d$ as $g \to 1$. We have

$$\begin{aligned} &|x|^m D^n(g\varphi)_x(v_1,\dots,v_n) \\ &\quad = \frac{|x|^m}{|g^{-1}x|^m} \sum_k \prod_i (g^{-1}v_i, e_{k_i}) |g^{-1}x|^m D^n\varphi_{g^{-1}x}(e_{k_1},\dots,e_{k_n}). \end{aligned}$$

It is clear that the sum goes uniformly to $|x|^m D^n\varphi_x(v_1,\dots,v_n)$ as $g \to \mathrm{Id}$ (indeed, this function vanishes at infinity and for x bounded we can make $g^{-1}x$ close to x). Also, we have that

$$\left| \frac{|x|}{|g^{-1}x|} - 1 \right| \le \frac{|x - g^{-1}x|}{|g^{-1}x|} \le \frac{\|1-g^{-1}\| |x|}{\frac{|x|}{\|g\|}} = \|g\| \cdot \|1-g^{-1}\|$$

so we are in the situation where we have $f_\lambda \to f$, $g_\lambda \to g$ uniformly, $\|f_\lambda\| \le c$ for all λ and g is bounded. Then $f_\lambda g_\lambda \to fg$ uniformly. Indeed,

$$\|f_\lambda g_\lambda - fg\|_\infty \le \|f_\lambda(g_\lambda - g)\|_\infty + \|(f_\lambda - f)g\|_\infty \le c\|g_\lambda - g\|_\infty + \|f_\lambda - g\|_\infty \|g\|_\infty \to 0$$

as $\lambda \to \infty$. □

So we have a strongly continuous representation

$$\mathrm{GL}(d,\mathbb{R}) \to \mathcal{L}(\mathcal{S}_d).$$

Proposition 3.3.3. *For all $\varphi \in \mathcal{S}_d$, the map $g \mapsto g\varphi$ is smooth $\mathrm{GL}(d,\mathbb{R}) \to \mathcal{S}_d$.*

Proof. According to [39, Exercise 36.4], since $\mathcal{S}_d$ is a Montel space, it is enough to see that for all $f \in \mathcal{S}'_d$,

$$g \mapsto \langle f, g\varphi\rangle$$

is smooth. Consider first when $f \in \mathcal{S}_d$. Let $\Phi(g) = \int f(x)\varphi(gx)\,dx$. Then Φ is clearly smooth, and the derivative is

$$D_g\Phi(h) = \int f(x) D\varphi_{gx}(hx)\,dx.$$

Now suppose that $f \in \mathcal{S}'_d$ and let $f_\lambda \in \mathcal{S}_d$ such that $f_\lambda \to f$. Let $\Phi_\lambda(g) = \langle f_\lambda, g^{-1}\varphi\rangle$ and $\Phi(g) = \langle f, g^{-1}\varphi\rangle$. Let $g_0 \in \mathrm{GL}(d,\mathbb{R})$ and let K be some compact neighbourhood of g_0. Then $B = \{g^{-1}\varphi\}_{g\in K}$ is compact by the previous proposition so it is bounded which implies that $f_\lambda|_B \to f|_B$ uniformly. It follows that $\Phi_\lambda|_K \to \Phi_\lambda$ uniformly, and by the above calculations $g \mapsto D(\Phi_\lambda)_g(h)$ converges uniformly on K (there is some polynomial contribution from hx but this is not a problem since $P \cdot f_\lambda \to P \cdot f$ for all polynomials P) to some function Ψ given by

$$\Psi(g) = \int f(x) D\varphi_{gx}(h)\,dx.$$

It follows that Φ is differentiable (initially on K), and that

$$D\Phi_g(h) = \int f(x) D\varphi_{gx}(h)\,dx.$$

Now, this argument can be repeated for all partial derivatives so we find that Φ is actually smooth. □

Corollary 3.3.4. *The map $\mathrm{GL}(d,\mathbb{R}) \to \mathcal{L}(\mathcal{S}_d)$ is smooth.*

Proof. Suppose $g \in \mathrm{GL}(d,\mathbb{R})$, and let $L_h : \mathcal{S}_d \to \mathcal{S}_d$ be defined by

$$[L_h\varphi](x) = D\varphi_{gx}(h).$$

Then we have

$$\frac{(g+h)^{-1}\varphi - g^{-1}\varphi - L_h\varphi}{\|h\|} \to 0$$

as $h \to 0$ for all φ. Banach-Steinhaus ([39, Thm. 33.1, Corollary]) turns this convergence into uniform convergence on all compacts in $\mathcal{S}_d$ which is a Montel space ([39, Prop. 34.4]) so we have uniform convergence on all bounded sets. Note that $h \mapsto L_h$ is linear $M(d,\mathbb{R}) \to \mathcal{L}(\mathcal{S}_d)$ which then becomes the differential. □

Chapter 4

Tensor Products of Bilinear Maps

In this chapter we consider the problem of taking the tensor product of hypocontinuous bilinear maps. This is a problem that was considered by Schwartz in [34] and we will see that it is also related to Grothendieck's "Problème des topologies". The solution that we will use is Theorem 4.1.8 which uses that Grothendieck solved the problem of topologies for $\mathcal{F}$- and $\mathcal{DF}$-spaces. After discovering this theorem, I found the same theorem in [3] so it is not quite new.

This chapter also sees extensive analysis of the space $\mathcal{S}_0(\mathbb{R}, E)$ and its multiplier space $\mathcal{S}_M(\mathbb{R} \setminus 0)$. This analysis will be necessary for the Fourier transform of Lizorkin functions on the Heisenberg group.

Tensor Products of Bilinear Maps. In this paragraph we will consider the following problem: Let $M : X \times Y \to Z$ and $B : E \times F \to G$ be separately continuous bilinear maps. Then we have a natural induced bilinear map

$$\begin{aligned} M \otimes B : X \otimes E \times Y \otimes F &\to Z \otimes G \\ (x \otimes e, y \otimes f) &\mapsto M(x, y) \otimes B(e, f). \end{aligned}$$

The problem is then to find conditions under which this map extends to a separately continuous bilinear map

$$M \mathbin{\widehat{\otimes}} B : X \mathbin{\widehat{\otimes}} E \times Y \mathbin{\widehat{\otimes}} F \to Z \mathbin{\widehat{\otimes}} G.$$

If this map exists we shall say that the tensor product of M and B exists, and we will call this map the *tensor product* of M and B.

One easy example is the following:

Proposition 4.0.1. *Suppose that $M : X \times Y \to Z$ and $B : E \times F \to G$ are* continuous *bilinear maps. Then the tensor product of M and B exists and it is continuous.*

Proof. Using that any convex neighbourhood of zero in the projective tensor product contains $\operatorname{acx}(U \otimes V)$ for some neighbourhoods of zero U, V (here $U \otimes V =$

$\{u \otimes v \mid u \in U, v \in V\}$; it is not a subspace), it is not hard to see that $M \otimes B$ is continuous (at $(0,0)$ at least but this becomes continuity everywhere, cf. [22, §15, 14. (1)]). Continuous bilinear maps can always be extended because they become uniformly continuous. □

Alternative Proof. By using the universal properties and taking tensor product we get an induced linear continuous map

$$(X \mathbin{\widehat{\otimes}} Y) \mathbin{\widehat{\otimes}} (E \mathbin{\widehat{\otimes}} F) \to Z \mathbin{\widehat{\otimes}} G.$$

But the projective tensor product of locally convex spaces is associative (we have an explicit construction of the continuous seminorms) and it is clearly commutative so we can arrange the factors so it suits us. □

In general, however, we need to take tensor products of bilinear maps that are not continuous. As an example, consider the following:

Theorem 4.0.2. *Suppose that Λ is a hypocontinuous bilinear map $E \times F \to G$. If $\varphi \in \mathcal{E}(U, E)$ and $\psi \in \mathcal{E}(U, F)$ then $\Lambda(\varphi, \psi)$ defined by*

$$\Lambda(\varphi, \psi)(x) = \Lambda(\varphi(x), \psi(x))$$

is a smooth function $U \to G$. This way Λ induces a hypocontinuous bilinear map

$$\begin{aligned} \mathcal{E}(U, E) \times \mathcal{E}(U, F) &\to \mathcal{E}(U, G) \\ (\varphi, \psi) &\mapsto \Lambda(\varphi, \psi). \end{aligned}$$

If $\psi \in \mathcal{D}(U, F)$ then $\Lambda(\varphi, \psi) \in \mathcal{D}(U, G)$ and we obtain a separately continuous bilinear map

$$\mathcal{E}(U, E) \times \mathcal{D}(U, F) \to \mathcal{D}(U, G).$$

The first claim is an example of taking tensor products because for all complete Hausdorff locally convex spaces E (cf. [39, p. 533]),

$$\mathcal{E}(U, E) \cong \mathcal{E}(U) \mathbin{\widehat{\otimes}} E.$$

Actually, the second claim is also an example of taking tensor product (albeit using a different tensor product) because if E is an $\mathcal{F}$-space then (cf. [15, Ch. II, §3, no. 3, Examples 4.])

$$\mathcal{D}(U, E) \cong \mathcal{D}(U) \mathbin{\overline{\otimes}} E.$$

So in the case where E, F, G are $\mathcal{F}$-spaces the constructed map is

$$\mathcal{E}(U) \mathbin{\overline{\otimes}} E \times \mathcal{D}(U) \mathbin{\overline{\otimes}} F \to \mathcal{D}(U) \mathbin{\overline{\otimes}} G.$$

which is the tensor product of the multiplication $\mathcal{E}(U) \times \mathcal{D}(U) \to \mathcal{D}(U)$ with the given bilinear map $E \times F \to G$.

Proof. According to Theorem A.1.10, $\Lambda(\varphi,\psi) \in \mathcal{E}(U,G)$ for $\varphi \in \mathcal{E}(U,E)$, $\psi \in \mathcal{E}(U,F)$ and

$$\Lambda(\varphi,\psi)^{(n)} = \sum_{|k|\leq|n|} \binom{n}{k} \Lambda(\varphi^{(k)}, \psi^{(n-k)}).$$

If $B \subseteq \mathcal{E}(U,F)$ is bounded then for any compact $K \subseteq U$ and $k \in \mathbb{N}_0^d$,

$$\bigcup_{\psi\in B} \varphi^{(k)}(K)$$

is bounded in F so we conclude that $\Lambda(\varphi_\lambda,\psi)^{(n)} \to 0$ uniformly on K and uniformly in $\psi \in B$ as $\varphi_\lambda \to 0$, proving hypocontinuity with regards to the bounded subsets of $\mathcal{E}(U,F)$. Likewise for the other factor.

It is clear that $\operatorname{supp}\Lambda(\varphi,\psi) \subseteq \operatorname{supp}\varphi \cap \operatorname{supp}\psi$ so the map

$$\mathcal{E}(U,E) \times \mathcal{D}(U,F) \to \mathcal{D}(U,G)$$

is well-defined. If one fixes $\varphi \in B \subseteq \mathcal{E}(U,E)$ for B bounded then it is enough to see that the restricted maps $\mathcal{D}_K(U,F) \to \mathcal{D}(U,G)$ is equicontinuous for every compact $K \subseteq U$ as φ varies over B. But this factors through $\mathcal{D}_K(U,G)$ and the corresponding family is equicontinuous for the same reason as above. If one fixes some bounded $B \subseteq \mathcal{D}(U,F)$ then $B \subseteq \mathcal{D}_K(U,F)$ for some compact $K \subseteq U$ and so if $\varphi_\lambda \to 0$ then in particular all derivatives of φ_λ converges uniformly to 0 on K and so all derivatives of $\Lambda(\varphi_\lambda,\psi)$ converges to 0 in $\mathcal{D}_{\operatorname{supp}\psi}(U,G)$ uniformly as $\psi \in B$. □

4.1 Solutions

Schwartz' Solution. Laurent Schwartz has a partial solution to the problem of taking the tensor product of two bilinear maps. The result in [31, Lecture 14] is

Theorem 4.1.1. *Suppose X, Y, Z are nuclear complete Hausdorff locally convex spaces with nuclear strong duals. Let $M : X \times Y \to Z$ be a hypocontinuous bilinear map. Let E, F, G be Banach spaces and suppose that $B : E \times F \to G$ is a continuous bilinear map. Then the tensor product of M and B exists and is hypocontinuous.*

Using analogous methods we prove

Theorem 4.1.2. *Suppose X, Y, Z are nuclear complete Hausdorff locally convex spaces, and that E, F, G are complete locally convex spaces. Suppose either that (1) E is nuclear and Y is barreled or that (2) F is nuclear and X is barreled. Suppose that $M : X \times Y \to Z$ is a hypocontinuous bilinear map and $B : E \times F \to G$ is a continuous bilinear map. Then the tensor product of M and B exists and is separately continuous.*

Under the assumptions of the theorem and according to [39, Proposition 50.4] we need to define a bilinear map

$$\mathcal{L}_\varepsilon(X', E) \times \mathcal{L}_\varepsilon(Y', F) \to \mathcal{L}_\varepsilon(Z', G).$$

Note that B induces $B : E\widehat{\otimes}F \to G$ so we need to define a map into $\mathcal{L}_\varepsilon(Z', E\widehat{\otimes}F)$. In the case E is nuclear we have

$$E \mathbin{\widehat{\otimes}} F \cong \mathcal{L}_\varepsilon(E', F)$$

so we need a map into $\mathcal{L}_\varepsilon(Z', \mathcal{L}_\varepsilon(E', F))$. We can also replace $\mathcal{L}_\varepsilon(X', E) \cong \mathcal{L}_\varepsilon(E', X)$.

We have a natural transpose $M' : Z' \to \mathcal{B}(X, Y)$, $z' \mapsto z' \circ M$. Actually, in the case where E is nuclear it is convenient to think of this map as

$$M' : Z' \to \mathcal{L}_b(X, Y')$$

given by

$$\langle [M'z'](x), y\rangle := \langle z', M(x, y)\rangle.$$

We find

Lemma 4.1.3. *The map $M' : Z' \to \mathcal{L}_b(X, Y')$ is well-defined and continuous when M is hypocontinuous. Also, M' takes equicontinuous subsets to equicontinuous subsets.*

Proof. Since Y' is given the strong topology we need hypocontinuity of M to get the continuity of $M'z'$ but otherwise it's pretty clear. As for continuity of $z' \mapsto M'z'$ this relies on the fact that M takes products of bounded subsets to bounded subsets which is also a property of hypocontinuity bilinear maps.

If $H \subseteq Z'$ is equicontinuous then $M'H$ is equicontinuous because of the hypocontinuity of M. □

Now it is natural to define

$$\mathcal{L}_\varepsilon(E', X) \times \mathcal{L}_\varepsilon(Y', F) \to \mathcal{L}_\varepsilon(Z', \mathcal{L}_\varepsilon(E', F))$$

taking $(T, S) \mapsto \Phi_{T,S}$ where $\Phi_{T,S}z' = S \circ M'z' \circ T$.

Lemma 4.1.4. *Suppose that Y is barreled. Then $(T, S) \mapsto \Phi_{T,S}$ is hypocontinuous with regards to the bounded subsets of $\mathcal{L}_\varepsilon(E', X)$ and the equicontinuous subsets of $\mathcal{L}(Y', F)$.*

Proof. The previous proposition tells us that $\Phi_{T,S}$ is continuous. When Y is barreled we are actually looking at

$$\mathcal{L}_\varepsilon(E', X) \times \mathcal{L}_b(Y', F) \to \mathcal{L}_\varepsilon(Z', \mathcal{L}_\varepsilon(E', F)).$$

Suppose that $B \subseteq \mathcal{L}_\varepsilon(E', X)$ is bounded and that $S_\lambda \in \mathcal{L}_b(Y', F)$, $S_\lambda \to 0$. Then we need $S_\lambda \circ M'H \circ B \to 0$ uniformly in $\mathcal{L}_\varepsilon(E', F)$ for any equicontinuous $H \subseteq Z'$.

But given an equicontinuous $H_1 \subseteq E'$, $B(E') \subseteq X$ is bounded so since $M'H$ is equicontinuous, $M'H \circ B(E')$ is bounded on which $S_\lambda \to 0$ uniformly.

On the other hand, suppose that $B \subseteq \mathcal{L}_b(Y', F)$ is equicontinuous and that $T_\lambda \in \mathcal{L}_\varepsilon(E', X)$, $T_\lambda \to 0$. Then we need $B \circ M'H \circ T_\lambda \to 0$ uniformly in $\mathcal{L}_\varepsilon(E', F)$. Given equicontinuous $H_1 \subseteq E'$, $T_\lambda|_{H_1} \to 0$ uniformly and $B \circ M'H$ is equicontinuous so we get the result. □

Composing with B is continuous so in the case where E is nuclear and Y is barreled we have a separately continuous bilinear map

$$\mathcal{L}_\varepsilon(E', X) \times \mathcal{L}_\varepsilon(Y', F) \to \mathcal{L}_\varepsilon(Z', G).$$

Proof of the Theorem. We only need to check that the defined map is actually an extension. We will assume that we are in the situation (1). If $x \otimes e \in X \widehat{\otimes} E$ it induces $x \otimes e \in \mathcal{L}(E', X)$ by $e' \mapsto \langle e', e\rangle x$. If $y \otimes f \in Y \widehat{\otimes} F$ is induces $y \otimes f \in \mathcal{L}(Y', F)$ by $y' \mapsto \langle y', y\rangle f$. The element in $\mathcal{L}(Z', \mathcal{L}(E', F))$ is

$$z' \mapsto (e' \mapsto \langle e', e\rangle\langle z', M(x, y)\rangle f.$$

Note that

$$e' \mapsto \langle e', e\rangle\langle z', M(x, y)\rangle f$$

corresponds to $\langle z', M(x, y)\rangle e \otimes f$ in $E \widehat{\otimes} F$ so after applying B the entire thing must correspond to

$$M(x, y) \otimes B(e, f).$$ □

Relaxation of Schwartz' Conditions. Even though we have in some ways generalized the result by Schwartz this is not enough for our purposes since B is assumed to be continuous. Next we will present a result for the case where M and B are merely hypocontinuous.

Theorem 4.1.5. *Suppose that $M : X \times Y \to Z$ and $B : E \times F \to G$ are hypocontinuous. Then there is a bilinear map*

$$X \otimes_i E \times \mathcal{L}_\varepsilon(Y', F) \to \mathcal{L}_\varepsilon(Z', G)$$

which coincides on the tensors with the natural tensor product of M and B. This map is hypocontinuous with regards to subsets $\overline{\text{acx}}(A \otimes B)$ of $X \otimes E$ where $A \subseteq X$, $B \subseteq E$ are bounded.

For the proof we will need some lemmata:

Lemma 4.1.6. *Suppose that $M : X \times Y \to Z$ is hypocontinuous. Then M induces a continuous linear map*

$$X \to \mathcal{L}_b(Y, Z)$$
$$x \mapsto M(x, \cdot)$$

Proof. [5, III §5 Proposition 3] has this as an equivalent formulation of hypocontinuity. □

Lemma 4.1.7. *Suppose E, F are locally convex topological vector spaces. The transpose is a continuous linear map*

$$\mathcal{L}_b(E,F) \to \mathcal{L}_\varepsilon(F',E')$$
$$T \mapsto T'$$

which takes equicontinuous subsets to equicontinuous subsets.

Proof. [5, III §5 Proposition 9] informs us that the composition

$$\mathcal{L}_b(E,F) \times \mathcal{L}_b(F,\mathbb{C}) \to \mathcal{L}_b(E,\mathbb{C})$$

is hypocontinuous with regards to the bounded subsets of $\mathcal{L}_b(E,F)$ and the equicontinuous subsets of $F' = \mathcal{L}_b(F,\mathbb{C})$. Because of [5, III §5 Proposition 3], we have the first claim. As for the second, suppose that $H \subseteq \mathcal{L}_b(E,F)$ is equicontinuous and let $V \subseteq \mathbb{C}$ be a neighbourhood of 0, $B \subseteq E$ be bounded. Then $H(B) = \bigcup_{h \in H} h(B)$ is also bounded and so there is a neighbourhood of 0 in F' given by all the functionals mapping $H(B)$ to V. H' maps this neighbourhood to the neighbourhood of 0 in E' consisting of functionals mapping B to V. □

Proof of the Theorem. We can define

$$X \otimes_i E \times \mathcal{L}_\varepsilon(Y',F) \to \mathcal{L}_\varepsilon(Z',G)$$

by

$$(x \otimes e, T) \mapsto B(e,\cdot) \circ T \circ M(x,\cdot)'.$$

This is separately continuous in x, e, T because the composition

$$\mathcal{L}_\varepsilon(Z',Y') \times \mathcal{L}_\varepsilon(Y',F) \times \mathcal{L}_b(F,G)$$

is separately continuous. Furthermore, suppose that $A \subseteq X$ and $C \subseteq E$ are bounded. Then A through M induces an equicontinuous subset of $\mathcal{L}_b(Y,Z)$ which in turn is mapped to an equicontinuous subset of $\mathcal{L}_\varepsilon(Z',Y')$, and C through B is mapped to an equicontinuous subset of $\mathcal{L}_b(F,G)$. Now, the hypocontinuity of the composition (twice) gives us that the map above is hypocontinuous with regards to $A \otimes B \subseteq X \otimes_i E$. The closed convex hull of an equicontinuous set of linear maps is equicontinuous, cf. [39, Proposition 32.2], [39, Proposition 32.4] so we have the wanted hypocontinuity.

The defined map coincides with the natural tensor product of the bilinear maps. Indeed, the element corresponding to $y \otimes f$ in $\mathcal{L}(Y',F)$ is the map $y' \mapsto \langle y, y' \rangle f$ so $(x \otimes e, y \otimes f)$ is mapped to the element of $\mathcal{L}(Z',G)$ given by

$$z' \mapsto B(e,\cdot)(y \otimes f)M(x,\cdot)'z' = B(e,f)\langle M(x,y), z' \rangle$$

which is exactly the element induced by $M(x,y) \otimes B(e,f)$. □

It is clear that under the assumptions above we can always extend to

$$X \mathbin{\overline{\otimes}} E \times \mathcal{L}_\varepsilon(Y', F) \to \mathcal{L}_\varepsilon(Z', G)$$

which will always be continuous in the first variable if the other variable is held fixed. But it is not clear in general that we will get continuity in the second variable as the first is held fixed. In general as the first variable is held fixed, the map in the second variable will be the pointwise limit of continuous maps. The usual way of ensuring continuity is to arrange that the family of approximating maps is equicontinuous. This will be satisfied in the above context if we can somehow show that any element of $X \mathbin{\overline{\otimes}} E$ is in $\overline{\mathrm{acx}}(A \otimes C)$ for $A \subseteq X$, $C \subseteq E$ bounded where the closure is now taken in $X \mathbin{\overline{\otimes}} E$. This is a weak form of Grothendieck's "Probléme des topologies": Given a bounded subset of $X \otimes_i E$ (or, usually, $X \otimes_\pi E$) Grothendieck asks if it is possible to find bounded subsets A, C as above such that the given bounded subset if contained in $\overline{\mathrm{acx}}(A \otimes C)$. This problem has not been solved in general, but there are two natural cases where we can answer in the affirmative, namely if both X, E are $\mathcal{F}$-spaces (technically, only a partial answer) or if both X, E are $\mathcal{DF}$-spaces.

Theorem 4.1.8. *Suppose that X, Y, Z are nuclear complete spaces and that E, F, G are complete locally convex spaces. Suppose that X, E are both $\mathcal{F}$-spaces or both $\mathcal{DF}$-spaces. Suppose that $M : X \times Y \to Z$ resp. $B : E \times F \to G$ is a bilinear map hypocontinuous with regards to the bounded subsets of X resp. E. Their tensor product exists*

$$X \mathbin{\widehat{\otimes}} E \times Y \mathbin{\widehat{\otimes}} F \to Z \mathbin{\widehat{\otimes}} G$$

and it is hypocontinuous with respect to the bounded subsets of $X \mathbin{\widehat{\otimes}} E$.

If both X, E are barreled then the map is in fact hypocontinuous.

Proof. Considering [18, 11.1.6] and [18, 15.7.7] any separately continuous bilinear form on $X \times E$ is continuous so $X \otimes_i E = X \otimes_\pi E$ as topological spaces. According to [39, Proposition 50.4] we have

$$Y \mathbin{\widehat{\otimes}} F \cong \mathcal{L}_\varepsilon(Y', F)$$
$$Z \mathbin{\widehat{\otimes}} F \cong \mathcal{L}_\varepsilon(Z', G)$$

so the previous theorem gives us bilinear

$$X \otimes_\pi E \times Y \mathbin{\widehat{\otimes}} F \to Z \mathbin{\widehat{\otimes}} G$$

which is hypocontinuous with regards to $\overline{\mathrm{acx}}(A \otimes C)$ for $A \subseteq X$, $C \subseteq E$ bounded. Now, [15, Ch. II, §3, no. 1, Proposition 12] informs us that any bounded subset of $X \mathbin{\widehat{\otimes}} E$ is contained in such a set (when the closure is taken in $X \mathbin{\widehat{\otimes}} E$) so in particular the sets cover $X \mathbin{\widehat{\otimes}} E$.

The hypocontinuity now ensures that the extensions to $X \mathbin{\widehat{\otimes}} E \times Y \mathbin{\widehat{\otimes}} F$ is continuous in the second variable and we get hypocontinuity with regards to the bounded subsets of $X \mathbin{\widehat{\otimes}} E$. As for the bounded subsets of $Y \mathbin{\widehat{\otimes}} F$, it is enough to remark that $X \mathbin{\widehat{\otimes}} E$ is barreled. This is the case if both X, E are barreled according to [18, 15.6.6] and [18, 15.6.8]. □

4.2 Applications

We consider some application of Theorem 4.1.8. We have already seen some applications in Chapter 2 and we will see more in Chapter 5. This section outlines some results that can be used in an alternative approach to the Fourier theory on the Heisenberg group than what will be presented in Chapter 5. This approach was ultimately abandoned but it is provided here nevertheless for elucidation of Theorem 4.1.8 through examples.

In the context of the Fourier Theory on the Heisenberg group H_d we are looking at $\mathcal{D}'(\mathbb{R}^*, \mathcal{L})$ where $\mathcal{L} = \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$. We want to take the trace of elements in $\mathcal{D}'(\mathbb{R}^*, \mathcal{L})$ but not all elements of $\mathcal{L}$ are trace-class. The trace-class operators are $L^1(\mathcal{S}'_d) = \mathcal{S}'_d \overline{\otimes} \mathcal{S}_d$ and $L^1(\mathcal{S}_d) = \mathcal{S}_d \overline{\otimes} \mathcal{S}'_d$ giving rise to generalised families of trace-class operators $\mathcal{D}'(\mathbb{R}^*, L^1(\mathcal{S}'_d))$ and $\mathcal{D}'(\mathbb{R}^*, L^1(\mathcal{S}_d))$. As an analogue of the fact that the composition of Hilbert-Schmidt operators give trace-class operators we have

Proposition 4.2.1. *Suppose that E, F are complete barreled locally convex spaces and E', F' are nuclear complete spaces. Suppose either (1) that E is a $\mathcal{F}$-space and F is a $\mathcal{DF}$-space or (2) that E is a $\mathcal{DF}$ space and F is a $\mathcal{F}$-space. Then the composition*

$$\mathcal{L}(F, E) \times \mathcal{L}(E, F) \to \mathcal{L}(E)$$

factors

$$\mathcal{L}(F, E) \times \mathcal{L}(E, F) \to E \overline{\otimes} E'.$$

This bilinear map is hypocontinuous.

Proof. We know cf. [39, Proposition 50.5] that $\mathcal{L}(E, F) \cong F \widehat{\otimes} E'$ and $\mathcal{L}(F, E) \cong E \widehat{\otimes} F'$. So we want to use the previous theorem to extend

$$\begin{aligned} E \otimes_\pi F' \times F \otimes_\pi E' &\to E \overline{\otimes} E' \\ (e \otimes f', f \otimes e') &\mapsto \langle f', f \rangle e \otimes e' \end{aligned}$$

to the completions. Here we are taking the tensor product of the bilinear maps

$$\begin{aligned} F' \times F &\to \mathbb{C} \\ (f', f) &\mapsto \langle f', f \rangle \end{aligned}$$

and

$$\begin{aligned} E \times E' &\to E \overline{\otimes} E' \\ (e, e') &\mapsto e \otimes e' \end{aligned}$$

that are both hypocontinuous because all the involved spaces are barreled (even E', F' are barreled under the conditions of the theorem, cf. the proof of [39, Proposition 50.5]). We see that one of the factors in each of the tensor products is nuclear and that in the case (1), E', F are both $\mathcal{F}$-spaces and in the case (2), E', F are both $\mathcal{DF}$-spaces. □

Corollary 4.2.2. *The hypocontinuous compositions*

$$\begin{aligned}\mathcal{L} \times \mathcal{L}_0 &\to \mathcal{L}(\mathcal{S}'_d)\\ \mathcal{L}_0 \times \mathcal{L} &\to \mathcal{L}(\mathcal{S}_d)\\ (T, S) &\mapsto TS\end{aligned}$$

factor through the trace-class operators so we obtain hypocontinuous

$$\begin{aligned}\mathcal{L} \times \mathcal{L}_0 &\to L^1(\mathcal{S}'_d)\\ \mathcal{L}_0 \times \mathcal{L} &\to L^1(\mathcal{S}_d).\end{aligned}$$

Proof. Use the previous proposition combined with the fact that $\mathcal{L}, \mathcal{L}_0$ are nuclear, barreled, complete and that $\mathcal{L}$ is $\mathcal{DF}$ and $\mathcal{L}_0$ is $\mathcal{F}$. □

As an extension of Theorem 4.0.2 we have

Proposition 4.2.3. *Suppose that E, F, G are complete locally convex spaces and in addition that E is an $\mathcal{F}$-space. Suppose that $\Lambda : E \times F \to G$ is a hypocontinuous bilinear map. As an extension of the induced separately continuous bilinear map*

$$\mathcal{E}(U, E) \times \mathcal{E}(U, F) \to \mathcal{E}(U, G)$$

we have a hypocontinuous bilinear map

$$\begin{aligned}\mathcal{E}(U, E) \times \mathcal{D}'(U, F) &\to \mathcal{D}'(U, G)\\ \mathcal{D}'(U, E) \times \mathcal{E}(U, F) &\to \mathcal{D}'(U, G).\end{aligned}$$

Proof. One simply uses Theorem 4.1.8 in the type $\mathcal{F}$ case. Note that $\mathcal{E}(U)$ and $\mathcal{D}'(U)$ are both nuclear complete spaces. □

Corollary 4.2.4. *The space $\mathcal{D}'(\mathbb{R}^*, \mathcal{L})$ is a two-sided module over $\mathcal{E}(\mathbb{R}^*, \mathcal{L}_0)$, the multiplication being an extension of the multiplication on $\mathcal{E}(\mathbb{R}^*, \mathcal{L}_0)$. In fact, we have hypocontinuous bilinear maps*

$$\begin{aligned}\mathcal{E}(\mathbb{R}^*, \mathcal{L}_0) \times \mathcal{D}'(\mathbb{R}^*, \mathcal{L}) &\to \mathcal{D}'(\mathbb{R}^*, L^1(\mathcal{S}'_d))\\ \mathcal{D}'(\mathbb{R}^*, \mathcal{L}) \times \mathcal{E}(\mathbb{R}^*, \mathcal{L}_0) &\to \mathcal{D}'(\mathbb{R}^*, L^1(\mathcal{S}_d))\end{aligned}$$

so that the image of the multiplication consists of generalised families of trace-class operators. The pointwise conjugate transpose $\mathcal{E}(\mathbb{R}^*, \mathcal{L}_0) \to \mathcal{E}(\mathbb{R}^*, \mathcal{L}_0)$ extends to $\mathcal{D}'(\mathbb{R}^*, \mathcal{L}_0) \to \mathcal{D}'(\mathbb{R}^*, \mathcal{L}_0)$. The pointwise trace $\mathcal{E}(\mathbb{R}^*, \mathcal{L}_0) \to \mathcal{E}(\mathbb{R}^*)$ extends to continuous linear maps

$$\begin{aligned}\mathcal{D}'(\mathbb{R}^*, L^1(\mathcal{S}'_d)) &\to \mathcal{D}'(\mathbb{R}^*)\\ \mathcal{D}'(\mathbb{R}^*, L^1(\mathcal{S}_d)) &\to \mathcal{D}'(\mathbb{R}^*).\end{aligned}$$

that satisfy

$$\begin{aligned}\operatorname{tr}[\Phi F] &= \operatorname{tr}[F\Phi]\\ \operatorname{tr}[F^*] &= \overline{\operatorname{tr} F}.\end{aligned}$$

Corollary 4.2.5. *There is a hypocontinuous multiplication*

$$\begin{aligned}\mathcal{D}'(\mathbb{R}^*, \mathcal{L}) \times \mathcal{D}(\mathbb{R}^*, \mathcal{L}_0) &\to \mathcal{E}'(\mathbb{R}^*, L^1(\mathcal{S}'_d))\\ \mathcal{D}(\mathbb{R}^*, \mathcal{L}_0) \times \mathcal{D}'(\mathbb{R}^*, \mathcal{L}) &\to \mathcal{E}'(\mathbb{R}^*, L^1(\mathcal{S}_d)).\end{aligned}$$

which is compatible with the multiplication on $\mathcal{E}(\mathbb{R}^*, \mathcal{L}_0)$.

Proof. For each compact $K \subseteq \mathbb{R}^*$ we can use Theorem 4.1.8 to get hypocontinuous bilinear maps

$$\begin{aligned}\mathcal{D}'(\mathbb{R}^*, \mathcal{L}) \times \mathcal{D}_K(\mathbb{R}^*, \mathcal{L}_0) &\to \mathcal{E}'(\mathbb{R}^*, L^1(\mathcal{S}'_d))\\ \mathcal{D}_K(\mathbb{R}^*, \mathcal{L}_0) \times \mathcal{D}'(\mathbb{R}^*, \mathcal{L}) &\to \mathcal{E}'(\mathbb{R}^*, L^1(\mathcal{S}_d)).\end{aligned}$$

Indeed, $\mathcal{D}_K(\mathbb{R}^*)$ and $\mathcal{L}_0$ are $\mathcal{F}$-spaces and $\mathcal{D}_K(\mathbb{R}^*, \mathcal{L}_0) \cong \mathcal{D}_K(\mathbb{R}^*) \widehat{\otimes} \mathcal{L}_0$. Also, the multiplication $\mathcal{D}'(\mathbb{R}^*) \times \mathcal{D}(\mathbb{R}^*) \to \mathcal{E}'(\mathbb{R}^*)$ is hypocontinuous. Since $\mathcal{D}(\mathbb{R}^*, \mathcal{L}_0)$ is per definition the inductive limit of the spaces $\mathcal{D}_K(\mathbb{R}^*, \mathcal{L}_0)$ we get a separately continuous bilinear map which is actually hypocontinuous: Any bounded subset of $\mathcal{D}(\mathbb{R}^*, \mathcal{L}_0)$ is in some $\mathcal{D}_K(\mathbb{R}^*, \mathcal{L}_0)$ and a family of linear maps on $\mathcal{D}(\mathbb{R}^*, \mathcal{L}_0)$ is equicontinuous if and only if each family of restrictions to $\mathcal{D}_K(\mathbb{R}^*, \mathcal{L}_0)$ is equicontinuous for each compact $K \subseteq \mathbb{R}^*$, cf. [28, Ch. V, §2, Proposition 5]. □

4.3 Multipliers on $\mathcal{S}_0(\mathbb{R})$

This section provides a result akin to Theorem 4.0.2 in the case where multiplication $\mathcal{E} \times \mathcal{E} \to \mathcal{E}$ is replaced with multiplication $\mathcal{O}_M \times \mathcal{S}_0 \to \mathcal{S}_0$. In the end it didn't seem possible to prove Theorem 4.3.10 without using Theorem 4.1.8 despite the fact that the bilinear map can be written down explicitly and is not defined by continuity.

Multiplication Operators on $\mathcal{S}_0(\mathbb{R}, E)$. Analogously with [32, p. 97, 4°] we introduce the multiplier space

$$\begin{aligned}&\mathcal{O}_M(\mathbb{R} \setminus 0, E)\\ &\quad = \{\varphi \in \mathcal{E}(\mathbb{R} \setminus 0, E) \mid \varphi \cdot \alpha^{(n)} \text{ is bounded for all } \varphi \in \mathcal{S}_0(\mathbb{R}) \text{ and } n \in \mathbb{N}_0\}\end{aligned}$$

which is then given the topology such that $\alpha_\lambda \to 0$ if and only if $\varphi\alpha_\lambda^{(n)} \to 0$ uniformly for all $\varphi \in \mathcal{S}_0(\mathbb{R})$. We use the shorthand $\mathcal{O}_M(\mathbb{R} \setminus 0, \mathbb{C}) = \mathcal{O}_M(\mathbb{R} \setminus 0)$. One finds

Lemma 4.3.1. *Suppose that E is locally convex. For $\alpha : \mathbb{R} \setminus 0 \to E$ to be in $\mathcal{O}_M(\mathbb{R} \setminus 0, E)$ it is necessary and sufficient that $\langle \alpha, e' \rangle \in \mathcal{O}_M(\mathbb{R} \setminus 0)$ for all $e' \in E'$.*

Proof. It is clearly necessary. The fact that it is sufficient follows from the fact that in locally convex spaces weak smoothness is the same as smoothness (Theorem A.1.7) and the fact that in locally convex spaces weak boundedness is equivalent to boundedness (Mackey's Theorem; [39, Thm. 36.2]). □

Lemma 4.3.2. *Suppose that E is a complete locally convex space. Suppose that $\varphi \in \mathcal{E}(\mathbb{R} \setminus 0, E)$ such that*

$$\lim_{x \to 0} \varphi^{(n)}(x) = 0$$

for all n. Then there is an element $\overline{\varphi} \in \mathcal{E}(\mathbb{R}, E)$ uniquely given by $\overline{\varphi}|_{\mathbb{R}\setminus 0} = \varphi$. This element satisfies

$$\overline{\varphi}^{(n)}(0) = 0.$$

for all n.

Proof. Define $\overline{\varphi}^{(n)}$ as above. Then it is clear that $\overline{\varphi}^{(n)}$ is a continuous function for all n. The only thing that needs to be proven is that it is in fact differentiable at 0 with derivative 0, i.e., we want to prove that

$$\frac{\overline{\varphi}^{(n)}(h) - \overline{\varphi}^{(n)}(0)}{h - 0} = \frac{\varphi^{(n)}(h)}{h} \to 0$$

as $h \to 0$. Let $V \subseteq E$ be a neighbourhood of 0. Since E is locally convex we can take V to be closed and convex. Since $\lim_{x\to 0} \varphi^{(n+1)}(x) = 0$ there is some $\delta > 0$ such that $\varphi^{(n+1)}(x) \in V$ when $|x| \leq \delta$. Then Theorem A.1.3 implies that when $|h| \leq \delta$,

$$\frac{\varphi^{(n)}(h)}{h} \in V.$$

□

Proposition 4.3.3. *Let $\iota : \mathcal{E}(\mathbb{R} \setminus 0, E) \to \mathcal{E}(\mathbb{R} \setminus 0, E)$ be the inversion*

$$[\iota\varphi](x) = \varphi(1/x)$$

Then ι induces an isomorphism

$$\mathcal{S}_0(\mathbb{R}, E) \to \mathcal{S}_0(\mathbb{R}, E)$$

where

$$[\iota\varphi]^{(n)}(0) = 0$$

for all n and $\varphi \in \mathcal{S}_0(\mathbb{R}, E)$. Also, ι restricts to an isomorphism

$$\mathcal{O}_M(\mathbb{R} \setminus 0, E) \to \mathcal{O}_M(\mathbb{R} \setminus 0, E)$$

Proof. There are polynomials P_j^n such that for $n \geq 1$,

$$\left(\frac{d}{dx}\right)^n \varphi\left(\frac{1}{x}\right) = \sum_{j=1}^{n} P_j^n\left(\frac{1}{x}\right)\varphi^{(j)}\left(\frac{1}{x}\right)$$

Indeed, differentiating shows that

$$P_1^1(x) = -x^2$$

and

$$\begin{aligned} P_1^{n+1}(x) &= -x^2(P_1^n)'(x),\\ P_j^{n+1}(x) &= -x^2\big((P_j^n)'(x) + P_j^{n-1}(x)\big), \quad 2 \le j \le n,\\ P_{n+1}^{n+1}(x) &= -x^2 P_n^n(x). \end{aligned}$$

For $\varphi \in \mathcal{S}_0(\mathbb{R}, E)$ we find

$$\lim_{x\to 0}[\iota\varphi]^{(n)}(x) = \lim_{x\to\infty}\sum_{j=1}^{n} P_j(x)\varphi^{(j)}(x) = 0$$

for all n. Then Theorem 4.3.2 tells us that $\iota\varphi$ extends to a smooth function on $\mathbb{R}$ with derivatives 0 at 0. Also, when p is a continuous seminorm on E,

$$\sup_x |x|^m p([\iota\varphi]^{(n)}(x)) \le \sum_{j=1}^{n} \sup_x \frac{p(P_j^n(x)\varphi^{(j)}(x))}{|x|^m}$$

Let $\psi(x) = P_j^n(x)\varphi^{(j)}(x)$ then Leibniz rule implies that $\psi^{(n)}(0) = 0$ for all n so that ψ is equal to its remainder term in the Taylor expansion (of any degree). Then [20, A.4.2] informs us that

$$p(\psi(x)) \le \frac{|x|^m}{m!}\sup_{0\le t\le 1} p(\psi^{(m)}(tx))$$

so

$$\sup_x |x|^m p([\iota\varphi]^{(n)}(x)) \le \frac{1}{m!}\sum_{j=1}^{n}\sup_x p(\partial_x^m(P_j^n(x)\varphi^{(j)}(x))).$$

If the differentiation is carried out we will have continuous seminorms on $\mathcal{S}(\mathbb{R}, E)$ so we have not only proven that $\iota\varphi \in \mathcal{S}(\mathbb{R}, E)$ we have also proven that ι is continuous $\mathcal{S}_0(\mathbb{R}, E) \to \mathcal{S}_0(\mathbb{R}, E)$.

When $\alpha \in \mathcal{O}_M(\mathbb{R}\setminus 0, E)$ then $\varphi\alpha$ is bounded for all $\varphi \in \mathcal{S}_0(\mathbb{R})$ so $[\iota\varphi][\iota\alpha]$ is bounded for all φ so since ι is surjective we find $\iota\alpha \in \mathcal{O}_M(\mathbb{R}\setminus 0, E)$. If $\alpha_\lambda \to 0$ then $\varphi\alpha_\lambda \to 0$ uniformly so $[\iota\varphi][\iota\alpha_\lambda] \to 0$ uniformly so we get continuity by the surjectivity of ι again. □

Next let us characterise $\mathcal{O}_M(\mathbb{R}\setminus 0)$. First recall that the multiplier space $\mathcal{O}_M(\mathbb{R})$ of $\mathcal{S}(\mathbb{R})$ has a handy characterisation:

Proposition 4.3.4. *The space of smooth functions $\alpha : \mathbb{R}^d \to \mathbb{C}$ such that $\varphi\alpha$ is bounded for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$ is exactly the space of smooth functions α for which every derivative $\alpha^{(n)}$ is bounded by some polynomial.*

Proof. [17, 4, §11, Proposition 5]. □

Proposition 4.3.5. *The space $\mathcal{O}_M(\mathbb{R}\setminus 0)$ is the space of smooth functions α on $\mathbb{R}\setminus 0$ for which for every j there is an $m \in \mathbb{N}_0$ and a $c > 0$ such that*

$$|\alpha^{(j)}(t)| \le ct^{-m}$$

for small t *and*

$$|\alpha^{(j)}(t)| \leq ct^m$$

for large t.

Furthermore, each element of $\mathcal{O}_M(\mathbb{R}\setminus 0)$ *defines a unique continuous linear operator on* $\mathcal{S}_0(\mathbb{R})$ *by multiplication and in fact* $\mathcal{O}_M(\mathbb{R}\setminus 0)$ *has exactly the topology inherited from* $\mathcal{L}(\mathcal{S}_0(\mathbb{R}))$.

Proof. First note that the conditions are necessary: Suppose that $\alpha \in \mathcal{O}_M(\mathbb{R}\setminus 0)$. Then we may write $\alpha = \alpha_0 + \alpha_\infty$ where α_0 is supported around 0 and α_∞ is supported around ∞. Then $\iota\alpha_0, \alpha_\infty \in \mathcal{O}_M(\mathbb{R})$ so they have polynomial growth by the previous proposition. Also, now

$$\alpha\varphi = \iota([\iota\alpha_0][\iota\varphi]) + \alpha_\infty\varphi$$

shows that $\varphi \mapsto \alpha\varphi$ is continuous since we already know that $\mathcal{O}_M(\mathbb{R})$ acts continuously on $\mathcal{S}(\mathbb{R})$. The topology on $\mathcal{O}_M(\mathbb{R}\setminus 0)$ is exactly the topology from $\mathcal{L}_b(\mathcal{S}_0(\mathbb{R}))$: If $\alpha_\lambda \to 0$ then $\alpha_\lambda\varphi \to 0$ for every φ. Note that $\mathcal{S}_0(\mathbb{R})$ is barreled because it is a Fréchet space so [39, Thm. 33.1, Corollary] tells us that $\alpha_\lambda \to 0$ uniformly on compact subsets. But since $\mathcal{S}_0(\mathbb{R})$ is a closed subspace of a Montel space $\mathcal{S}(\mathbb{R})$ we find that the convergence is uniform on all bounded subsets. On the other hand, it is clear that the topology on $\mathcal{O}_M(\mathbb{R}\setminus 0)$ is courser than the topology on $\mathcal{L}_b(\mathcal{S}_0(\mathbb{R}))$.

Next, we show that the conditions are sufficient and furthermore that they imply that we have a continuous linear operator on $\mathcal{S}_0(\mathbb{R})$. For every $\varphi \in \mathcal{S}_0(\mathbb{R})$, the function $\alpha\varphi$ extends to a smooth function on $\mathbb{R}$. Indeed, because of Theorem 4.3.2 it suffices to show that for each n the limit of $(\alpha\varphi)^{(n)}(t)$ as $t \to 0$ exists and is 0. Leibniz' rule tells us that it is sufficient to consider the limit of $\alpha^{(j)}(t)\varphi^{(n)}(t)$ as $t \to 0$. By assumption there is some m and c such that

$$|\alpha^{(j)}(t)| \leq ct^{-m}$$

for small t. Then

$$\alpha^{(j)}(t)\varphi^{(n)}(t) = (t^m\alpha^{(j)}(t)) \cdot (\frac{\varphi^{(n)}}{t^m}).$$

The first factor is bounded for small t by assumption. As for the second, l'Hôspital's rule tells us that

$$\lim_{t\to 0} \frac{\varphi^{(n)}(t)}{t^m} = \lim_{t\to 0} \frac{\varphi^{(n+m)}(t)}{m!} = 0.$$

Not only have we shown that $\alpha\varphi$ extends to a smooth function on $\mathbb{R}$ we have also proven that all its derivatives at 0 vanish. Since α also has polynomial growth at infinity we have that $\alpha\varphi \in \mathcal{S}(\mathbb{R})$ so $\alpha\varphi \in \mathcal{S}_0(\mathbb{R})$. □

Corollary 4.3.6. $\mathcal{O}_M(\mathbb{R}\setminus 0)$ *is nuclear.*

Proof. Indeed, $\mathcal{L}_b(\mathcal{S}_0(\mathbb{R}), \mathcal{S}_0(\mathbb{R}))$ is nuclear according to [15, Ch. II, §2, no. 1, Thm. 9, Cor. 3] so this follows from [39, Proposition 50.1] (a subspace of a nuclear space is nuclear). □

Proposition 4.3.7. *Suppose that E is locally convex. The topology on $\mathcal{O}_M(\mathbb{R}\setminus 0, E)$ is exactly the topology inherited from $\mathcal{L}_\varepsilon(E', \mathcal{O}_M(\mathbb{R}\setminus 0))$. It follows that the topology induced on the tensor product $\mathcal{O}_M(\mathbb{R}\setminus 0)\otimes E$ is the ε-topology. In fact we find*

$$\mathcal{O}_M(\mathbb{R}\setminus 0, E) \cong \mathcal{O}_M(\mathbb{R}\setminus 0)\,\widehat{\otimes}\, E.$$

Proof. The map $\mathcal{O}_M(\mathbb{R}\setminus 0, E)\to \mathcal{L}_\varepsilon(E', \mathcal{O}_M(\mathbb{R}\setminus 0))$ is the map that associates to each α the operator $e'\mapsto \langle \alpha, e'\rangle$. This is a continuous operator: If $e'_\lambda \to 0$ then we find that for every $\varphi\in\mathcal{S}_0(\mathbb{R})$,

$$\varphi\langle \alpha, e'_\lambda\rangle^{(n)} = \langle \varphi\alpha^{(n)}, e'_\lambda\rangle \to 0$$

uniformly since $(\varphi\alpha^{(n)})(\mathbb{R})$ is bounded. The association is continuous: If $H\subseteq E'$ is an equicontinuous subset and $\alpha_\lambda\to 0$ then whenever $\varphi\in\mathcal{S}_0(\mathbb{R})$ we have

$$\varphi\langle \alpha_\lambda, e'\rangle^{(n)} = \langle \varphi\alpha_\lambda^{(n)}, e'\rangle \to 0$$

uniformly as $e'\in H$ because $\varphi\alpha_\lambda^{(n)}\to 0$ uniformly. Clearly, the association is injective and we have just proven that the topology on $\mathcal{O}_M(\mathbb{R}\setminus 0, E)$ is finer than the topology induced from $\mathcal{L}_\varepsilon(E', \mathcal{O}_M(\mathbb{R}\setminus 0))$.

On the other hand, suppose that $\alpha_\lambda\to 0$ in $\mathcal{L}_\varepsilon(E', \mathcal{O}(\mathbb{R}\setminus 0))$. Then if $\varphi\in\mathcal{S}_0(\mathbb{R})$ we can go backwards and conclude that

$$\langle \varphi\alpha_\lambda^{(n)}, e'\rangle \to 0$$

uniformly as $e'\in H$ some equicontinuous subset. But the topology on a locally convex space E is equal to the topology of uniform convergence on equicontinuous subsets of E' ([39, Proposition 36.1]) so we must conclude that

$$\varphi\alpha_\lambda^{(n)}\to 0$$

uniformly.

Employing the same proposition one sees that the induced topology on

$$\mathcal{O}_M(\mathbb{R}\setminus 0)\otimes E$$

is the ε-topology which is identical to the π-topology since $\mathcal{O}_M(\mathbb{R}\setminus 0)$ is nuclear. In order to get the tensor product representation all there is left to do is show that the simple tensors are dense in $\mathcal{O}_M(\mathbb{R}\setminus 0, E)$. But this should be clear: An element $\alpha\in\mathcal{O}_M(\mathbb{R}\setminus 0, E)$ can be written as $\alpha=\alpha_0+\alpha_\infty$ where α_0 has support around 0 and α_∞ has support around ∞. Then since $\mathcal{O}_M(\mathbb{R}, E)\cong\mathcal{O}_M(\mathbb{R})\widehat{\otimes}E$ we can approximate $\iota\alpha_0$ and α_∞ by simple tensors and we can even arrange that the simple tensors respect the supports by multiplying by an appropriate function. Now, ι preserves $\mathcal{O}_M(\mathbb{R}\setminus 0)\otimes E$ and it is continuous so we get an approximation of α. □

One already knows that when E is locally convex, $\mathcal{S}(\mathbb{R}, E)$ is identical to the set of functions $\varphi:\mathbb{R}\to E$ such that $\langle\varphi, e'\rangle\in\mathcal{S}(\mathbb{R})$ for all $e'\in E$. By considering that in this case E' separates points we find

Lemma 4.3.8. *Suppose that E is locally convex. $\mathcal{S}_0(\mathbb{R}, E)$ is exactly the set of functions $\varphi : \mathbb{R} \to E$ for which $\langle \varphi, e' \rangle \in \mathcal{S}_0(\mathbb{R})$.*

Proposition 4.3.9. *Suppose that E is locally convex. If $\alpha \in \mathcal{O}_M(\mathbb{R} \setminus 0, E)$ then for all $\varphi \in \mathcal{S}_0(\mathbb{R})$, $\alpha \cdot \varphi$ extends to a function in $\mathcal{S}_0(\mathbb{R}, E)$. The resulting bilinear map*

$$\mathcal{O}_M(\mathbb{R} \setminus 0, E) \times \mathcal{S}_0(\mathbb{R}) \to \mathcal{S}_0(\mathbb{R}, E)$$

is hypocontinuous.

Proof. We have already a continuous map

$$\mathcal{O}_M(\mathbb{R} \setminus 0) \to \mathcal{L}_b(\mathcal{S}_0(\mathbb{R}), \mathcal{S}_0(\mathbb{R})).$$

By [39, Prop. 50.7] we have

$$\mathcal{L}_b(\mathcal{S}_0(\mathbb{R}), \mathcal{S}_0(\mathbb{R})) \cong \mathcal{S}_0(\mathbb{R}) \widehat{\otimes} \mathcal{S}_0'(\mathbb{R})$$

since $\mathcal{S}_0(\mathbb{R})$ is a nuclear Fréchet space being a closed subspace of the nuclear Fréchet space $\mathcal{S}(\mathbb{R})$.

Taking the tensor product with E we obtain a continuous map

$$\mathcal{O}_M(\mathbb{R} \setminus 0, E) \cong \mathcal{O}_M(\mathbb{R} \setminus 0) \widehat{\otimes} E \to (\mathcal{S}_0(\mathbb{R}) \widehat{\otimes} \mathcal{S}_0'(\mathbb{R})) \widehat{\otimes} E \cong \mathcal{L}_b(\mathcal{S}_0(\mathbb{R}), \mathcal{S}_0(\mathbb{R}, E))$$

here using [39, Prop 50.5].

This proves the the map is hypocontinuous with regards to the bounded subsets of $\mathcal{S}_0(\mathbb{R})$. Since $\mathcal{S}_0(\mathbb{R})$ is barreled we also have hypocontinuity with regards to the bounded subsets of $\mathcal{O}_M(\mathbb{R} \setminus 0, E)$.

One should verify that when $\alpha \in \mathcal{O}_M(\mathbb{R} \setminus 0, E)$ and $\varphi \in \mathcal{S}_0(\mathbb{R})$ the image of (α, φ) is actually given by pointwise multiplication. But note that any evaluation map $\mathcal{S}_0(\mathbb{R}, E) \to E$ is continuous so it is enough to see this on simple tensors. This is obvious. □

Finally, we have the following analogue of Theorem 4.0.2:

Theorem 4.3.10. *Suppose that E, F, G are complete locally convex spaces and that $\Lambda : E \times F \to G$ is hypocontinuous. Suppose that F is an $\mathcal{F}$-space. Then Λ induces a hypocontinuous bilinear map*

$$\mathcal{O}_0(\mathbb{R} \setminus 0, E) \times \mathcal{S}_0(\mathbb{R}, F) \to \mathcal{S}_0(\mathbb{R}, G)$$

given by

$$\Lambda(\varphi, \psi)(x) = \Lambda(\varphi(x), \psi(x))$$

for $x \neq 0$ and

$$\Lambda(\varphi, \psi)^{(n)}(0) = 0$$

for all n.

Proof. We will see in Theorem 4.1.8 that there is a hypocontinuous map

$$\mathcal{O}_M(\mathbb{R} \setminus 0) \mathbin{\widehat{\otimes}} E \times \mathcal{S}_0(\mathbb{R}) \mathbin{\widehat{\otimes}} F \to \mathcal{S}_0(\mathbb{R}) \mathbin{\widehat{\otimes}} G$$

given as the tensor product of the multiplication map

$$\mathcal{O}_M(\mathbb{R} \setminus 0) \times \mathcal{S}_0(\mathbb{R}) \to \mathcal{S}_0(\mathbb{R})$$

and Λ. If we compose with the evaluation map we see that the induced map has the given property.

□

Chapter 5

Fourier Theory on the Heisenberg Group

In this chapter we consider the Fourier transform on the Heisenberg group. The goal is to classify the positive definite tempered distributions in $\mathcal{S}'(H)$ via the Fourier transform. It is not completely clear how to do this: The unitary dual of the Heisenberg group comes in two families. There are the infinite dimensional representations parametrised by $\mathbb{R}^* = \mathbb{R} \setminus 0$ and there are the characters parametrised by $\mathbb{R}^d \times \mathbb{R}^d$. The classical Fourier inversion for functions take into account only the infinite dimensional representations, but it becomes clear that a corresponding Fourier inversion for distributions must take into account the characters as well.

We show that the Fourier transform is an isomorphism of the $*$-ideal of Lizorkin functions in $\mathcal{S}(H)$ onto $\mathcal{S}_0(\mathbb{R}, \mathcal{L}_0)$ so that we can at least obtain necessary conditions for positive definiteness via the Fourier transform, and conversely relate positivity on the Fourier side to positivity on a $*$-ideal in $\mathcal{S}(H)$. We then apply this to the Knapp-Stein kernel to get necessary conditions for the existence of a new G-invariant inner product. The Knapp-Stein kernel and it analytic extension has been considered in [19] but without analysis of its positive definiteness.

Definition. Consider the Heisenberg group $H = \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ with composition

$$(p, q, t)(p', q', t') = (p + p', q + q', t + t' + pq')$$

Here (p, q, t) corresponds to

$$\begin{pmatrix} 1 & p & t \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix}.$$

Eventually, the analysis here must be employed on $\overline{N}$ which is also isomorphic to H. The isomorphism takes (p, q, t) to

$$\begin{pmatrix} 1 & 0 & 0 \\ -p & 1 & 0 \\ pq - t & -q & 1 \end{pmatrix}.$$

Everything that was said in Section 2.1 applies here.

Unitary Dual of H. The characters of H are given by

$$(p, q, t) \mapsto e^{ipx} e^{iqy}.$$

For each $h \in \mathbb{R}^*$ we have the Schrödinger representation ρ_h acting on $L^2(\mathbb{R}^d)$ by

$$[\rho_h(p, q, t)\varphi](x) = e^{iht} e^{iqx} \varphi(x + hp).$$

These are all the unitary representations of H up to unitary equivalence.

Fourier Transform of L^1 Functions. For each $f \in L^1(H)$ the integral

$$\widehat{f}(h) = \rho_h(f) = \int_H f(g)\rho_h(g)\, dg$$

converges. The map $f \mapsto \widehat{f}$ is the Fourier transform. It has certain well-known algebraic properties, namely it is linear,

$$\widehat{f * g} = \widehat{f} \cdot \widehat{g}.$$

and

$$\widehat{f^*}(h) = \widehat{f}(h)^*.$$

Fourier Transform of Schwartz Functions. It will be convenient to take the Haar measure dx on $\mathbb{R}^d$ to be $(2\pi)^{-d/2} d\lambda(x)$ where λ is the Lebesgue measure. We will take the Fourier transform on $\mathbb{R}$ to be

$$\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d),$$
$$[\mathcal{F}\,\varphi](\xi) = \int \varphi(x) e^{ix\xi}\, dx.$$

This makes $\mathcal{F}$ unitary with inverse

$$[\mathcal{F}^{-1}\,\varphi](x) = \int \varphi(\xi) e^{-ix\xi}\, d\xi.$$

Likewise, it is convenient to take the Haar measure dx on H to be the tensor product of the Haar measures on $\mathbb{R}^d$, $\mathbb{R}^d$ and $\mathbb{R}$ so that dx is $(2\pi)^{-d-1/2} d\lambda(x)$ where λ is the Lebesgue measure on $\mathbb{R}^{2d+1}$.

Theorem 5.0.1. *For each $f \in \mathcal{S}(H)$, the Fourier transform $\widehat{f}(h)$ is a kernel operator with kernel in $\mathcal{L}_0 = \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ given by*

$$K_h^f(x, y) = |h|^{-d} [\mathcal{F}_{2,3}\, f](h^{-1}(y - x), x, h). \tag{5.1}$$

Furthermore, the dependence on h is smooth and we obtain a continuous linear map

$$\mathcal{S}(H) \to \mathcal{E}(\mathbb{R}^*, \mathcal{L}_0)$$

which is the Fourier transform in the context of Schwartz functions. This map also preserves the $*$-algebra structures.

Proof. Letting $\varphi \in \mathcal{S}(\mathbb{R}^d)$ we get

$$\begin{aligned}[\widehat{f}(h)\varphi](x) &= \int_H f(p,q,t)[\rho_h(p,q,t)\varphi](x)\,d(p,q,t)\\ &= \int_H f(p,q,t)e^{iht+iqx}\varphi(x+hp)\,d(p,q,t)\\ &= \int_{\mathbb{R}} [\mathcal{F}_{2,3}\,f](p,x,h)\varphi(x+hp)\,dp.\end{aligned}$$

Here $\mathcal{F}_{2,3}$ denotes Fourier transformation in the second and third variable. Define a new variable $y = x + hp$ then $dy = |h|^d\,dp$ so

$$[\widehat{f}(h)\varphi](x) = |h|^{-d}\int_{\mathbb{R}} [\mathcal{F}_{2,3}\,f](h^{-1}(y-x),x,h)\varphi(y)\,dy.$$

Since the Fourier transform is an automorphism of $\mathcal{S}(\mathbb{R}^d)$ and because of Theorem 3.3.4, it is obvious that $K_h^f \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ for all $h \in \mathbb{R}^*$.

That the map is well-defined and continuous is easy: First, the Fourier transformation is continuous, then the restriction $\mathcal{S}(H) \to \mathcal{E}(\mathbb{R}^*, \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d))$ is continuous, and then we use Theorem 4.0.2 with Theorem A.0.4 to make the linear substitution continuously, noting that we have an element of $\mathcal{E}(\mathbb{R}^*, \mathcal{L}(\mathcal{S}_{2d}))$ defined by

$$\mathbb{R}^* \ni h \mapsto \begin{pmatrix} -h^{-1} & h^{-1} \\ h^{-1} & 0 \end{pmatrix} \in \mathrm{GL}(\mathbb{R}^d)$$

as a consequence of Theorem 3.3.4. □

We also have the well-known

Theorem 5.0.2 (Inversion Theorem)**.** *For all $\varphi \in \mathcal{S}(H)$, we have*

$$\varphi(x) = \int |h|^d\,\mathrm{tr}[\widehat{\varphi}(h)\rho_h(x)^*]\,dh.$$

This of course implies that the Fourier transform $\mathcal{S}(H) \to \mathcal{E}(\mathbb{R}^*, \mathcal{L}_0)$ is injective.

The Fourier Transform of a Tempered Distribution. In the above, the Fourier transform takes a function to a family of operators. It is natural then to expect that a generalised function should be taken to a generalised family of operators; the space

$$\mathcal{E}(\mathbb{R}^*, \mathcal{L}_0) \cong \mathcal{E}(\mathbb{R}^*)\,\widehat{\otimes}\,\mathcal{L}_0$$

is replaced by

$$\mathcal{D}'(\mathbb{R}^*)\,\widehat{\otimes}\,\mathcal{L} \cong \mathcal{D}'(\mathbb{R}^*, \mathcal{L}).$$

One can also take the view that a linear functional on $\mathcal{S}(H)$ should be taken to a linear functional on the Fourier transformed object, in this case $\mathcal{E}(\mathbb{R}^*, \mathcal{L}_0)$. At first this is problematic because $\mathcal{E}(\mathbb{R}^*, \mathcal{L}_0)$ is much bigger than the image of

$\mathcal{S}(H)$. This can be salvaged by replacing $\mathcal{E}(\mathbb{R}^*, \mathcal{L}_0)$ by its subalgebra $\mathcal{D}(\mathbb{R}^*, \mathcal{L}_0)$. We will go through this approach later as well.

Now, the Fourier transform above is defined as

$$\mathcal{S}(H) \overset{\mathcal{F}_{2,3}}{\to} \mathcal{S}(H) \to \mathcal{E}(\mathbb{R}^*, \mathcal{L}_0) \overset{g}{\to} \mathcal{E}(\mathbb{R}^*, \mathcal{L}_0) \overset{|h|^{-d}}{\to} \mathcal{E}(\mathbb{R}^*, \mathcal{L}_0)$$

where $g \in \mathcal{E}(\mathbb{R}^*, \mathcal{L}(\mathcal{S}_d))$ is the linear substitution defined by

$$g_h = \begin{pmatrix} -h^{-1} & h^{-1} \\ -h^{-1} & 0 \end{pmatrix}.$$

Using the theory of tensor products of bilinear maps, it is possible to do the same thing for tempered distributions. Indeed, note first that the action of $\mathrm{GL}(d, \mathbb{R})$ on $\mathcal{S}_d$ extends uniquely to $\mathcal{S}'_d$ by

$$\langle g \cdot f, \varphi \rangle := |\det g| \langle f, g^{-1} \cdot \varphi \rangle.$$

Then obviously the smooth map $\mathrm{GL}(d, \mathbb{R}) \to \mathcal{L}_b(\mathcal{S}_d)$ extends uniquely to a smooth map $\mathrm{GL}(d, \mathbb{R}) \to \mathcal{L}_b(\mathcal{S}'_d)$.

An element $\Phi \in \mathcal{E}(\mathbb{R}^*, \mathcal{L}_0)$ is uniquely associated with a generalised function $\Phi \in \mathcal{D}'(\mathbb{R}^*, \mathcal{L})$ given by

$$\Phi(\chi) = \int |h|^d \chi(h) \Phi(h)\, dh \in \mathcal{L}_0 \subseteq \mathcal{L}\,.$$

So then it is possible to talk of extending the Fourier transform:

Theorem 5.0.3. *The Fourier transform extends uniquely from $\mathcal{S}(H)$ to give a continuous linear map*

$$\mathcal{S}'(H) \to \mathcal{D}'(\mathbb{R}^*, \mathcal{L}).$$

Properly understood, it is still the case that

$$\widehat{f * \varphi} = \widehat{f} \cdot \widehat{\varphi}$$
$$\widehat{f^*} = \widehat{f}^*.$$

for $f \in \mathcal{S}'(H)$ and $\varphi \in \mathcal{S}(H)$.

Proof. Pointwise application gives a separately continuous bilinear map

$$\mathcal{E}(\mathbb{R}^*, \mathcal{L}(\mathcal{S}_{2d})) \times \mathcal{E}(\mathbb{R}^*, \mathcal{S}_{2d}) \to \mathcal{E}(\mathbb{R}^*, \mathcal{S}_{2d}).$$

This is the tensor product of multiplication $\mathcal{E} \times \mathcal{E} \to \mathcal{E}$ and the application map $\mathcal{L}(\mathcal{S}_{2d}) \times \mathcal{S}_{2d} \to \mathcal{S}_{2d}$. We would like to instead use the tensor product of multiplication $\mathcal{E} \times \mathcal{D}' \to \mathcal{D}'$ and the application map $\mathcal{L}(\mathcal{S}'_{2d}) \times \mathcal{S}'_{2d} \to \mathcal{S}'_{2d}$ to give a separately continuous bilinear map

$$\mathcal{E}(\mathbb{R}^*, \mathcal{L}(\mathcal{S}'_{2d})) \times \mathcal{D}'(\mathbb{R}^*, \mathcal{S}'_{2d}) \to \mathcal{D}'(\mathbb{R}^*, \mathcal{S}'_{2d}).$$

Unfortunately, we do not have a theorem to facilitate this. However, the Fourier transform should be defined on $\mathcal{S}'(H)$ which is a $\mathcal{DF}$-space so in our case we can actually replace $\mathcal{E} \times \mathcal{D}' \to \mathcal{D}'$ by $\mathcal{E} \times \mathcal{S}' \to \mathcal{D}'$. Thus, Theorem 4.1.8 applies in the $\mathcal{DF}$-case. The algebraic properties hold by density of $\mathcal{S}(H)$ in $\mathcal{S}'(H)$. □

Linear Functional Approach. Here we take the second approach to the Fourier transform, namely that a linear functional should be taken to a linear functional. Note that

$$\mathcal{D}'(\mathbb{R}^*, \mathcal{L}_0) \cong \mathcal{D}(\mathbb{R}^*, \mathcal{L})'$$

according to [34, Proposition 22, Corollaire 3 (p. 104)]. The isomorphism is given on simple tensors as

$$\langle F, \chi \otimes T \rangle = \operatorname{tr}[F(\chi) T]$$

for $F \in \mathcal{D}'(\mathbb{R}^*, \mathcal{L})$.

For functions $\Phi, \Psi : \mathbb{R}^* \to \mathcal{L}_0$ we denote by

$$\langle \Phi, \Psi \rangle = \int_{-\infty}^{\infty} |h|^d \operatorname{tr}[\Phi(h)\Psi(h)]\, dh.$$

This becomes a hypocontinuous bilinear pairing

$$\mathcal{E}(\mathbb{R}^*, \mathcal{L}_0) \times \mathcal{D}(\mathbb{R}^*, \mathcal{L}_0) \to \mathbb{C}$$

which then induces an injective continuous linear map $\mathcal{E}(\mathbb{R}^*, \mathcal{L}_0) \to \mathcal{D}(\mathbb{R}^*, \mathcal{L}_0)'$. Note that the inclusions

$$\mathcal{E}(\mathbb{R}^*, \mathcal{L}_0) \to \mathcal{D}'(\mathbb{R}^*, \mathcal{L})$$
$$\mathcal{E}(\mathbb{R}^*, \mathcal{L}_0) \to \mathcal{D}(\mathbb{R}^*, \mathcal{L}_0)'$$

are chosen so that they are compatible with the isomorphism

$$\mathcal{D}'(\mathbb{R}^*, \mathcal{L}) \cong \mathcal{D}(\mathbb{R}^*, \mathcal{L}_0)'.$$

Proposition 5.0.4. *The image of $\mathcal{S}(\mathbb{R}^d)$ under the Fourier transform contains wholly $\mathcal{D}(\mathbb{R}^*, \mathcal{L}_0)$, and in fact the Fourier transform admits a continuous linear inverse*

$$\mathcal{F}^{-1} : \mathcal{D}(\mathbb{R}^*, \mathcal{L}_0) \to \mathcal{S}(H)$$

given as usual by

$$\mathcal{F}^{-1}\Phi(x) = \int |h|^d \operatorname{tr}[\Phi(h)\rho_h(x)^*]\, dh.$$

Subject to the pairings $\langle \cdot, \cdot \rangle$, the Fourier transform admits a continuous linear transpose $\mathcal{F}^t : \mathcal{D}(\mathbb{R}^*, \mathcal{L}_0) \to \mathcal{S}(H)$ given by

$$\mathcal{F}^t \Phi(x) = \mathcal{F}^{-1}\Phi(x^{-1}).$$

This means that

$$\langle \mathcal{F}\varphi, \Psi \rangle = \langle \varphi, \mathcal{F}^t \Psi \rangle$$

for all $\varphi \in \mathcal{S}(H)$, $\Psi \in \mathcal{D}(\mathbb{R}^*, \mathcal{L}_0)$.

The transpose satisfies the algebraic properties

$$\mathcal{F}^t(\Phi \cdot \Psi) = \mathcal{F}^t \Phi * \mathcal{F}^t \Psi$$
$$(\mathcal{F}^t \Phi)^* = \mathcal{F}^t \Phi^*.$$

Proof. The Fourier Inversion Theorem says that $\mathcal{F}^{-1}\,\mathcal{F}\,\varphi = \varphi$ for all $\varphi \in \mathcal{S}(H)$, but a simple calculation shows that $\mathcal{F}\,\mathcal{F}^{-1}\,\Phi = \Phi$ for all $\Phi \in \mathcal{D}(\mathbb{R}^*, \mathcal{L}_0)$ as well.

One simply applies Fubini-Tonelli to see that

$$\begin{aligned}\langle \mathcal{F}\,\varphi, \Psi\rangle &= \int |h|^d \operatorname{tr} \widehat{\varphi}(h)\Psi(h)\, dh \\ &= \int |h|^d \varphi(x) \operatorname{tr} \rho_h(x)\Psi(h)\, dxdh = \langle \varphi, \mathcal{F}^t\,\Psi\rangle\end{aligned}$$

That $\Phi \mapsto \mathcal{F}^t\,\Phi$ is a continuous linear map $\mathcal{D}(\mathbb{R}^*, \mathcal{L}_0) \to \mathcal{S}(H)$ requires an extra argument: Doing the calculations, $\widehat{\Phi}$ is explicitly given by

$$\widehat{\Phi}(p, q, t) = \int |h|^d \Phi(y + hp, y, h) e^{iht} e^{iqy}\, dydh.$$

This is again a linear substitution along h and after that a Fourier transform both of which are continuous. So then also the inverse, being simply the transpose composed with the substitution $x \mapsto x^{-1}$ is also continuous. □

So we can define the Fourier transform $\mathcal{S}'(H) \to \mathcal{D}(\mathbb{R}^*, \mathcal{L}_0)'$ by

$$\langle \widehat{f}, \Phi\rangle = \langle f, \widehat{\Phi}\rangle.$$

There is (also) a natural sesquilinear form on $\mathcal{S}(H)$ given by

$$(\varphi|\psi) = \langle \varphi, \overline{\psi}\rangle = \int f(x)\overline{g(x)}\, dx.$$

On the Fourier side of things, for functions $\Phi, \Psi : \mathbb{R}^* \to \mathcal{L}_0$, the corresponding device is

$$(\Phi|\Psi) = \langle \Phi, \Psi^*\rangle = \int |h|^d \operatorname{tr}[\Phi(h)\Psi(h)^*]\, dh.$$

Then we have the traditional Plancherel Theorem

Theorem 5.0.5. *The Fourier transform is unitary, i.e.,*

$$(\widehat{\varphi}|\widehat{\psi}) = (\varphi|\psi)$$

for all $\varphi, \psi \in \mathcal{S}(H)$.

Properties of the Fourier Transform. Recall the definition of $\mathrm{Pol}_c(H)$ from page 32.

Proposition 5.0.6. *The Fourier transform* $\mathcal{S}'(H) \to \mathcal{D}(\mathbb{R}^*, \mathcal{L}_0)'$ *is continuous with dense image and with kernel* $\mathrm{Pol}_c(H)$.

Proof. The image contains $\mathcal{D}(\mathbb{R}^*, \mathcal{L}_0)$ which is dense in $\mathcal{D}'(\mathbb{R}^*, \mathcal{L})$. Suppose that $f \in \mathcal{S}'(H)$ such that $\widehat{f} = 0$. Then the generalised family

$$|h|^{-d}[\mathcal{F}_{2,3}\, f](h^{-1}(y - x), x, h) \in \mathcal{D}'(\mathbb{R}^*, \mathcal{L})$$

is 0 and we can apply the family of operators in $\mathcal{E}(\mathbb{R}^*, \mathcal{L})$ as above and find

$$|h|^{-d}[\mathcal{F}_{2,3}\, f](x, y, h) \in \mathcal{D}'(\mathbb{R}^*, \mathcal{L})$$

is 0. We can multiply by $|h|^d$ in $\mathcal{E}(\mathbb{R}^*)$ and even apply a Fourier transform to find

$$[\mathcal{F}_3\, f](x, y, h) \in \mathcal{D}'(\mathbb{R}^*, \mathcal{L})$$

is 0. This means that $\mathcal{F}_3\, f \in \mathcal{D}'(\mathbb{R}, \mathcal{L})$ is 0 when restricted to $\mathbb{R}^*$ so Theorem 3.1.7 implies that

$$\mathcal{F}_3\, f = \sum_n \delta_0^{(n)} \otimes f_n$$

for some $f_n \in \mathcal{L}$. Since $\mathcal{L}$ is a $\mathcal{DF}$-space we have $\mathcal{F}_3\, f$ must have locally finite order so the sum is actually finite. All in all we find

$$f(x, y, h) = \sum_{n=0}^{N} f_n(x, y) h^n$$

for some $f_n \in \mathcal{L}$. □

Remark 5.0.7. Instead of using Theorem 3.1.7, since we are dealing with a distribution in $\mathcal{D}'(\mathbb{R}, E)$ where E is a distribution space we could have used the classical theorems on distributions supported on a subspace, cf. [35, Ch. III, §10].

With regards to positive definiteness we have

Proposition 5.0.8. *If $f \in \mathcal{S}'(H)$ is positive then $\widehat{f} \in \mathcal{D}(\mathbb{R}^*, \mathcal{L}_0)'$ is positive.*

Proof. In this case,

$$\langle \widehat{f}, \Phi\Phi^* \rangle = \langle f, \mathcal{F}^t\, \Phi * (\mathcal{F}^t\, \Phi)^* \rangle \geq 0.$$

□

Unfortunately, we do not get a full classification of positive linear functionals because the transpose $\mathcal{F}^t$ is not surjective. For instance, if f is a character

$$f(p, q, t) = e^{ipx} e^{iqy}$$

then the Fourier transform above is 0, but f is not a positive functional. Indeed, using the characters in the group Fourier transform we get a map $\widetilde{\mathcal{F}} : \mathcal{S}(H) \to \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ where

$$\widetilde{\mathcal{F}}\varphi(x, y) = \int_H \varphi(h) \pi_{x,y}(h)\, dh$$

for $\pi_{x,y}(p, q, t) = e^{ipx} e^{iqy}$. Explicitly,

$$\widetilde{\mathcal{F}}\varphi(x, y) = \mathcal{F}_{\mathrm{Ab}}\, \varphi(x, y, 0)$$

where $\mathcal{F}_{\text{Ab}}$ is the Abelian Fourier transform on $\mathbb{R}^d \times \mathbb{R}^d$. So $\widetilde{\mathcal{F}}$ is continuous, linear, and

$$\widetilde{\mathcal{F}}(\varphi * \psi) = \widetilde{\mathcal{F}}\varphi \cdot \widetilde{\mathcal{F}}\psi$$
$$\widetilde{\mathcal{F}}(\varphi^*) = \overline{\widetilde{\mathcal{F}}\varphi}.$$

There is no transpose of the map $\widetilde{\mathcal{F}}$ so there is no corresponding Fourier transform $\mathcal{S}'(H) \to \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$. We can use $\widetilde{\mathcal{F}}$ to pull back a tempered distribution, however. The distributions $f \circ \widetilde{\mathcal{F}}$ for $f \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ are exactly the distributions of the form

$$f(p, q, t) = f_0(p, q) \tag{5.2}$$

for a tempered distribution $f_0 \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$.

Proposition 5.0.9. *Suppose that $f \in \mathcal{S}'(H)$ is a tempered distribution that is constant along the center, i.e., given by Eq. (5.2). Then f is positive if and only if f_0 is positive in the Abelian sense, i.e, if and only if the Abelian Fourier transform of f_0 is a positive measure.*

Proof. The Fourier transform $\widetilde{\mathcal{F}}$ pulls back a positive measure on $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ to a positive definite tempered distribution on $\mathcal{S}(H)$. This gives us sufficiency. As for necessity, let $A\varphi \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ be the average along the center of any $\varphi \in \mathcal{S}(H)$, i.e.,

$$A\varphi(p, q) = \int_{-\infty}^{\infty} \varphi(p, q, t)\, dt.$$

Then a calculation shows that

$$\langle f, \varphi * \psi \rangle = \langle f_0, A\varphi * A\psi \rangle$$

where the latter convolution is over $\mathbb{R}^d \times \mathbb{R}^d$, i.e., over an Abelian group. Evidently, A is surjective so it follows that it is necessary that f_0 is positive definite as a distribution on $\mathbb{R}^d \times \mathbb{R}^d$. □

Combining this with Theorem 2.1.15 we find

Corollary 5.0.10. *The only positive definite distributions in $\mathrm{Pol}_c(H)$ are the distributions of the form*

$$(p, q, t) \mapsto f(p, q)$$

where $f \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ is positive definite in the Abelian sense.

Fourier Transform on the Lizorkin Space. We discuss again the Fourier transform associated with the regular orbits. Above we have not achieved a full Fourier inversion theorem. Indeed, the Fourier transform as outlined above is 0 on the polynomials so the Fourier transform does not give any information about them. This suggests that a modification of the spaces is required. What modification? According to [13, Prop. 1.1.3] we have

$$(\mathcal{S}^\infty)'(\mathbb{R}) \cong \mathcal{S}'(\mathbb{R}) / \operatorname{Pol}(\mathbb{R})$$

which suggests that we should focus on the Lizorkin space $\mathcal{S}^\infty(H)$ as defined on page 31.

Now applying Abelian Fourier transform in the second and third variables we have

$$\mathcal{S}^\infty(H) \to \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d) \mathbin{\widehat{\otimes}} \mathcal{S}_0(\mathbb{R}) \cong \mathcal{S}_0(\mathbb{R}, \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)).$$

This is where we would like to apply the variable shift induced by

$$h \mapsto \begin{pmatrix} 0 & 1 \\ h & 1 \end{pmatrix} = \begin{pmatrix} -h^{-1} & h^{-1} \\ 1 & 0 \end{pmatrix}^{-1}$$

which a priori is an element of $\mathcal{E}(\mathbb{R}^*, \mathcal{L}(\mathcal{S}_{2d}))$. We find

Lemma 5.0.11. *The element*

$$h \mapsto \begin{pmatrix} 0 & 1 \\ h & 1 \end{pmatrix}$$

induces an element in $\mathcal{O}_M(\mathbb{R}^*, \mathcal{L}(\mathcal{S}_{2d}))$.

Proof. Note that since $\mathcal{S}_{2d}$ is barreled, bounded in $\mathcal{L}(\mathcal{S}_{2d})$ is the same as pointwise bounded. So we need to prove that every time $\chi \in \mathcal{S}_0(\mathbb{R})$ and $\varphi \in \mathcal{S}_{2d}$ and γ is a polynomial and $m \in \mathbb{N}_0 \times \mathbb{N}_0$ we have

$$\chi(h)\gamma(x,y)\partial_x^{m_1}\partial_y^{m_2}|h|^{-d}\varphi(h^{-1}(y-x),x)$$

is uniformly bounded as $(x,y,h) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$. Doing the differentiations it is enough to see that

$$\chi(h)\gamma(x,hz+x)|h|^l\varphi^{(j,m_1)}(z,x)$$

is bounded for any l, j, m_1.

Taking $\gamma(x,y) = x^{n_1}y^{n_2}$ and doing the multiplications it is enough to see that

$$\chi(h)|h|^l x^{n_1}y^{n_2}\varphi^{(m_2,m_1)}(y,x)$$

is bounded. This is the case when $\chi \in \mathcal{S}_0(\mathbb{R})$ (by L'Hôpital's rule, e.g.) and $\varphi \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$. □

Now applying Theorem 4.3.10 we get

Theorem 5.0.12. *The Fourier transform restricts to a topological $*$-algebra isomorphism*

$$\mathcal{S}^\infty(H) \to \mathcal{S}_0(\mathbb{R}, \mathcal{L}_0).$$

The transpose gives rise to a linear isomorphism

$$(\mathcal{S}^\infty)'(H) \to \mathcal{S}_0'(\mathbb{R}, \mathcal{L}_0).$$

Comments on Synthesis. In this paragraph we discuss what is missing from a full Bochner's Theorem on the Heisenberg group. Note first what is known:

1. A distribution in $\mathcal{S}'(H)$ is given by its restriction to $\mathcal{S}^\infty(H)$ up to an element of $\mathrm{Pol}_c(H)$.
2. The positive definiteness of a distribution in $(\mathcal{S}^\infty)'(H)$ is classified by Fourier analysis.
3. A distribution in $\mathrm{Pol}_c(H)$ is positive definite if and only if it is of the form
$$(p,q,t) \mapsto f(p,q)1^0$$
where $f \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ is positive definite in the Abelian sense.

These facts when taken together suggests that

Conjecture 5.0.13. *The positive definite distributions in $\mathcal{S}'(H)$ are in bijective correspondence with pairs of positive definite distributions in $\mathrm{Pol}_c(H) \times \mathcal{S}'_\infty(H)$.*

What is missing is some way of extending a positive definite element of $\mathcal{S}'_\infty(H)$ to a positive definite element in $\mathcal{S}'(H)$, taking into account the kernel of the restriction $\mathrm{Pol}_c(H)$. An approach to this could be through complementary subspaces. Recall that a closed subspace E of some space X is complemented if there is some closed subspace $F \subseteq X$ such that $X = E + F$ and $E \cap F = 0$. Note first that

Lemma 5.0.14. *The $*$-ideal $\mathcal{S}^\infty(H)$ cannot be complemented by an ideal $I \subseteq \mathcal{S}(H)$.*

Proof. In this case we would have $\varphi * \psi = 0$ for all $\varphi \in I$, $\psi \in \mathcal{S}^\infty(H)$ so that $\widehat{\varphi} \cdot \Psi = 0$ for all $\Psi \in \mathcal{S}_0(\mathbb{R}, \mathcal{L}_0)$ so that $\widehat{\varphi} = 0$ so that $\varphi = 0$. □

Suppose that $\mathcal{S}^\infty(H)$ is complemented by a space E. Let π be the projection onto $\mathcal{S}^\infty(H)$ and p be the projection onto E. Then any distribution f decomposes $f = f \circ \pi + f \circ p$. If we want $f \circ \pi$ to be positive when f is, we need to assume something about π. It would be sufficient that π was $*$-invariant and $\mathcal{S}(H)$-invariant (i.e., $\pi(\varphi * \psi) = \varphi * \pi(\psi)$) but this would mean that E was a $*$-ideal in contradiction with the lemma. It would also be sufficient that π was a $*$-algebra homomorphism which implies that E is a $*$-subalgebra. For the positivity of f to descend to $f \circ p$ we would want p to be a $*$-algebra homomorphism as well but this is only possible if $E * \mathcal{S}^\infty(H) = 0$ which is a contradiction by the argument in the proof of the lemma.

According to [25], the space $\mathcal{S}^\infty(\mathbb{R})$ is complemented in $\mathcal{S}(\mathbb{R})$ and it follows by [18, Prop. 15.2.3] that $\mathcal{S}^\infty(H)$ is complemented in $\mathcal{S}(H)$. However, it does not seem to be the case that the complement is a subalgebra. Indeed, the complement of $\mathcal{S}^\infty(\mathbb{R})$ in $\mathcal{S}(\mathbb{R})$ as constructed in [25] is not a subalgebra. In the notation of the article one only has
$$\theta_j^a * \theta_l^a = \theta_{j+l}^a \mod \mathcal{S}^\infty(\mathbb{R})$$

which is not sufficient.

Remark 5.0.15. I have come to doubt even the result of the article: Is the constructed complement even a subspace? One is supposed to take union over a of the spaces spanned by $(\theta_j^a)_j$. But if $a \neq a'$ then both θ_j^a and $\theta_j^{a'}$ are in the complement but

$$\theta_j^a - \theta_j^{a'} \in \mathcal{S}^\infty(\mathbb{R}).$$

Perhaps the result is salvageable if one is a little bit more careful about the sequences a that the union is taken over. And perhaps it is salvageable in a way that would produce a subalgebra! Note that the fact that $\mathcal{S}^\infty(\mathbb{R})$ is complemented by a subalgebra does not imply directly that $\mathcal{S}^\infty(H)$ is complemented by a subalgebra since the convolution is not the tensor product of convolutions in $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R})$.

Dilations of the Heisenberg Group. We will make a few comments on the homogeneous distributions in $\mathcal{S}'(H)$. This will not have a great impact on the analysis of the Knapp-Stein operator, but it furnishes us with an application of Section 3.2. We take $d = 1$ for these two paragraphs.

We shall say that a family of automorphisms $\delta_s : H \to H$, $s > 0$ is a family of dilations if $s \mapsto \delta_s$ is continuous (say, strongly) and if they satisfy

$$\delta_{ss'} = \delta_s \delta_{s'}$$

and if δ_1 is the identity. For example, δ_s could be given by

$$\delta_s(p, q, t) = (s^x p, s^y q, s^{x+y} t)$$

for $x, y \in \mathbb{R}$. The case $x = y = 1$ is the case typically considered where we also have a corresponding normlike map $(p, q, t) \mapsto (p^2 + q^2)^2 + t^2$ homogeneous with regards to these dilations. We will instead be focused on $x = 1$, $y = 0$ which is the case that gives rise homogeneity after Fourier transform.

Homogeneous Functions. Let $\delta_s : H \to H$ be the transformation $\delta_s(p, q, t) = (sp, q, st)$. Then we say that a function or distribution f on H is *homogeneous* of degree $\lambda \in \mathbb{C}$ if

$$f(\delta_s(h)) = s^\lambda f(h).$$

At the same time we understand what it means to be homogeneous in

$$\mathcal{D}'(\mathbb{R}^*, \mathcal{S}'(\mathbb{R} \times \mathbb{R}))$$

and in $\mathcal{E}(\mathbb{R}^*, \mathcal{S}(\mathbb{R} \times \mathbb{R}))$, namely we must have

$$T(sx) = s^\lambda T(x)$$

for $s > 0$, $x \in \mathbb{R}^*$. It is then the case that if $f \in \mathcal{S}'(H)$ is homogeneous of degree λ then $\widehat{f}$ is homogeneous of degree $-\lambda - 2$. Indeed,

$$K_h^f(x, y) = |h|^{-1} \mathcal{F}_{2,3} f(h^{-1}(y - x), x, h)$$

so that

$$\begin{aligned} K^f_{sh}(x,y) &= |sh|^{-1}\,\mathcal{F}_{2,3}\,f((sh)^{-1}(y-x),xsh) \\ &= |sh|^{-1}\int e^{isht}\,\mathcal{F}_2\,f((sh)^{-1}(y-x),x,t)\,dt \\ &= s^{-2}|h|^{-1}\int e^{iht}\,\mathcal{F}_2\,f((sh)^{-1}(y-x),x,t/s)\,dt \\ &= s^{-\lambda-2}|h|^{-1}\int e^{iht}\,\mathcal{F}_2\,f(h^{-1}(y-x),x,t)\,dt = s^{-\lambda-2}K^f_h(x,y). \end{aligned}$$

It follows then from Theorem 3.2.8 that

$$\widehat{f}(h) = h_+^{-\lambda-2}\otimes F_+ + h_-^{-\lambda-2}\otimes F_-$$

for $F_+, F_- \in \mathcal{L}^d$. It is the case that the Knapp-Stein kernel F^λ_ε below is homogeneous of degree $\lambda_1+\lambda_2-2$ so that its Fourier transform is homogeneous of degree $-\lambda_1-\lambda_2$. This is consistent with the calculations in the next section.

5.1 The Knapp-Stein Kernel

According to [11, Eqns. (12) and (13) p. 173] we have

$$\mathcal{F}\left(\frac{|x|^\lambda_\varepsilon}{\Gamma(\lambda+1)}\right) = c^{\lambda+1}_\varepsilon |x|^{-1-\lambda}_\varepsilon$$

where

$$c^\lambda_\varepsilon = \frac{\exp(i\pi\frac{\lambda}{2})+\varepsilon\exp(-i\pi\frac{\lambda}{2})}{\sqrt{2\pi}} = \begin{cases} \sqrt{\dfrac{2}{\pi}}\cos\left(\pi\dfrac{\lambda}{2}\right), & \varepsilon=+, \\ \sqrt{\dfrac{2}{\pi}}i\sin\left(\pi\dfrac{\lambda}{2}\right), & \varepsilon=-. \end{cases}$$

We want to use the Fourier transformation to analyse the Knapp-Stein kernel $F^\lambda_\varepsilon \in \mathcal{S}'(H)$ given by

$$F^\lambda_\varepsilon(x,y,z) = \frac{|z-xy|^{\lambda_1-1}_{\varepsilon_1}|z|^{\lambda_2-1}_{\varepsilon_2}}{\Gamma(\lambda_1)\Gamma(\lambda_1+\lambda_2)\Gamma(\lambda_2)}.$$

First a useful lemma:

Lemma 5.1.1. *Consider $f\in\mathcal{S}'(\mathbb{R}^d)$ given by*

$$\widehat{f}(x) = \frac{|a\cdot x+b|^\lambda_\varepsilon}{\Gamma(\lambda+1)}$$

for $a\in\mathbb{R}^d$, $b\in\mathbb{R}$. Then $\widehat{f}$ is given by

$$\widehat{f}(\xi) = c^{\lambda+1}_\varepsilon |a|^{-1} e^{-ib\frac{a\xi}{|a|^2}}\delta_0(\mathrm{pr}_{(a)^\perp}\xi)\left|\frac{a\xi}{|a|^2}\right|^{-1-\lambda}_\varepsilon$$

which means that

$$\langle \widehat{f}, \varphi\rangle = c_\varepsilon^{\lambda+1} \int_{-\infty}^{\infty} e^{-isb} |s|_\varepsilon^{-1-\lambda} \varphi(sa)\, ds.$$

Remark 5.1.2. For any $a \in \mathbb{R}^d$ we use the shorthand $(a) = \operatorname{span}(a) \subseteq \mathbb{R}^d$, and pr_U denotes the orthogonal projection onto any subspace $U \subseteq \mathbb{R}^d$.

Proof. In the integral

$$\Gamma(\lambda+1)\widehat{f}(\xi) = \int |ax+b|_\varepsilon^\lambda e^{ix\xi}\, dx$$

we split the integration $x = x_0 + x^\perp$ subject to the linear decomposition $\mathbb{R}^d = (a) \oplus (a)^\perp$. Thus,

$$\Gamma(\lambda+1)\widehat{f}(\xi) = \int_{(a)^\perp} e^{ix^\perp \xi}\, dx^\perp \cdot \int_{(a)} |ax_0+b|_\varepsilon^\lambda e^{ix\xi}.$$

We find

$$\int_{(a)^\perp} e^{ix^\perp\xi}\, dx^\perp = \delta_0(\mathrm{pr}_{(a)}\, \xi)$$

and

$$\begin{aligned}\int_{(a)} |ax_0+b|_\varepsilon^\lambda e^{ix\xi} &= \int_{-\infty}^{\infty} \left| t\frac{a}{|a|} + b\right|_\varepsilon^\lambda e^{it\frac{a}{|a|}\xi}\, dt = \int_{-\infty}^{\infty} |s|_\varepsilon^\lambda e^{i\frac{s-b}{|a|}\frac{a}{|a|}\xi} \frac{ds}{|a|} \\ &= \Gamma(\lambda+1) c_\varepsilon^{\lambda+1} |a|^{-1} e^{-ib\frac{a\xi}{|a|^2}} \left| \frac{a\xi}{|a|^2}\right|_\varepsilon^{-1-\lambda}\end{aligned}$$

which gives us the first claim. As for the second, when $\xi = t\frac{a}{|a|}$, $\frac{a\xi}{|a|^2} = \frac{t}{|a|}$ so

$$\begin{aligned}\langle \widehat{f}, \varphi\rangle &= c_\varepsilon^{\lambda+1}|a|^{-1} \int_{-\infty}^{\infty} e^{-ibt/|a|} \left|\frac{t}{|a|}\right|_\varepsilon^{-1-\lambda} \varphi\left(t\frac{a}{|a|}\right) dt \\ &= c_\varepsilon^{\lambda+1} \int_{-\infty}^{\infty} e^{-isb}|s|_\varepsilon^{-1-\lambda}\varphi(sa)\, ds.\end{aligned}$$

□

Theorem 5.1.3. *The Fourier transform of F_ε^λ is*

$$\widehat{F_\varepsilon^\lambda}(h) = \frac{c_{\varepsilon_1}^{\lambda_1} c_{\varepsilon_2}^{\lambda_2}}{\Gamma(\lambda_1+\lambda_2)} |h|_{\varepsilon_1\varepsilon_2}^{-\lambda_1-\lambda_2-d+1} T_\varepsilon^\lambda$$

where T_ε^λ is given by

$$T_\varepsilon^\lambda \varphi(xv) = \int_{-\infty}^{\infty} |x|_{\varepsilon_1}^{-\lambda_1} |y-x|_{\varepsilon_1\varepsilon_2}^{\lambda_1+\lambda_2-1} |y|_{\varepsilon_2}^{-\lambda_2} \varphi(yv)\, dy$$

for any $x \in \mathbb{R}$ and unit vector v or

$$T_\varepsilon^\lambda \varphi(p) = \int_{-\infty}^{\infty} |t-1|_{\varepsilon_1\varepsilon_2}^{\lambda_1+\lambda_2-1} |t|_{\varepsilon_2}^{-\lambda_2} \varphi(tp)\, dt$$

for any $p \in \mathbb{R}^d$.

Proof. Let $F = F_\varepsilon^\lambda$. The lemma tells us that

$$\Gamma(\lambda_1+\lambda_2)\Gamma(\lambda_2)\,\mathcal{F}_2\,F(x,\eta,z) = c_{\varepsilon_1}^{\lambda_1}|x|^{-1}e^{-iz\frac{x\eta}{|x|^2}}\delta_0(\mathrm{pr}_{(x)^\perp}\,\xi)\left|\frac{x\eta}{|x|^2}\right|_{\varepsilon_1}^{-\lambda_1}|z|_{\varepsilon_2}^{\lambda_2-1}$$

So it follows that $\mathcal{F}_2\,F(p,q,t)$ as a distribution in p,q is supported in the submanifold

$$M = \left\{(p,q)\in\mathbb{R}^d \,\middle|\, p,q \text{ are linearly dependent.}\right\}.$$

This means we might as well compute $\mathcal{F}_2, F(p,q,t)$ for $p = xv$, $q = yv$ for some unit vector $v\in\mathbb{R}^d$. Then

$$\Gamma(\lambda_1+\lambda_2)\Gamma(\lambda_2)\,\mathcal{F}_2\,F(xv,yv,z) = c_{\varepsilon_1}^{\lambda_1}|x|^{-1}e^{-iz\frac{y}{x}}\left|\frac{y}{x}\right|_{\varepsilon_1}^{-\lambda_1}|z|_{\varepsilon_2}^{\lambda_2-1}.$$

Taking the Fourier transform in z as well,

$$\begin{aligned}\Gamma(\lambda_1+\lambda_2)\,\mathcal{F}_{2,3}\,F(xv,yv,h) &= c_{\varepsilon_1}^{\lambda_1}c_{\varepsilon_2}^{\lambda_2}|x|_{\varepsilon_1}^{\lambda_1-1}|y|_{\varepsilon_1}^{-\lambda_1}\left|h-\frac{y}{x}\right|_{\varepsilon_2}^{-\lambda_2}.\\ &= c_{\varepsilon_1}^{\lambda_1}c_{\varepsilon_2}^{\lambda_2}|x|_{\varepsilon_1\varepsilon_2}^{\lambda_1+\lambda_2-1}|y|_{\varepsilon_1}^{-\lambda_1}|xh-y|_{\varepsilon_2}^{-\lambda_2}.\end{aligned}$$

Now, note that $h^{-1}(y-x), x$ are linearly dependent if and only if x,y are linearly dependent so again it suffices with a calculation on M:

$$\Gamma(\lambda_1+\lambda_2)\,\mathcal{F}_{2,3}\,F(h^{-1}(y-x)v,xv,h) = c_{\varepsilon_1}^{\lambda_1}c_{\varepsilon_2}^{\lambda_2}|h|_{\varepsilon_1\varepsilon_2}^{-\lambda_1-\lambda_2+1}|y-x|_{\varepsilon_1\varepsilon_2}^{\lambda_1+\lambda_2-1}|x|_{\varepsilon_1}^{-\lambda_1}|y|_{\varepsilon_2}^{-\lambda_2}.$$

This is the result. □

Now, an analysis of the operator T_ε^λ is in order.

Proposition 5.1.4. *For $d=1$, $T_\varepsilon^\lambda\in\mathcal{L}(\mathcal{S}_1,\mathcal{S}_1')$ is given by the kernel*

$$T_\varepsilon^\lambda(x,y) = |x|_{\varepsilon_1}^{-\lambda_1}|y-x|_{\varepsilon_1\varepsilon_2}^{-\lambda_1-\lambda_2+1}|y|_{\varepsilon_2}^{-\lambda_2}$$

for all $x,y\in\mathbb{R}$. For $\varepsilon_1=\varepsilon_2=\varepsilon$, $\lambda_1=\overline{\lambda_2}=\lambda$, T_ε^λ is a positive operator only if $\operatorname{Re}\lambda\le\frac{1}{2}$.

Proof. The first assertion follows directly from the previous theorem. As for the other one, we find

$$(T_\varepsilon^\lambda\varphi|\varphi) = \int |x|_\varepsilon^{-\lambda}|y-x|^{2\operatorname{Re}\lambda-1}|y|_\varepsilon^{-\overline{\lambda}}\varphi(y)\overline{\varphi(x)}\,dxdy = (|x|^{2\operatorname{Re}\lambda-1}*S\varphi|S\varphi)$$

where

$$S\varphi(x) = |x|_\varepsilon^{-\overline{\lambda}}\varphi(x).$$

When φ has compact support away from 0, $S\varphi$ is also a smooth function with compact support so that for T_ε^λ to be positive, it is necessary that $|x|^{2\operatorname{Re}\lambda-1}$ is positive definite. By the classical Bochner's Theorem, this implies that $-2\operatorname{Re}\lambda\ge -1$, i.e., $\operatorname{Re}\lambda\le\frac{1}{2}$. □

Theorem 5.1.5. *For $d = 1$, $\lambda_1 = \overline{\lambda_2} = \lambda$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon$, $w = w_{13}$, the Knapp-Stein intertwiner $P(\varepsilon, \lambda) \to P(w\varepsilon, w\lambda)$ can be normalised to give a nonzero positive $P(\varepsilon, \lambda)$-invariant sesquilinear form on $C^\infty P(\varepsilon, \lambda)$ only if*

$$-\frac{1}{2} \leq \operatorname{Re}\lambda \leq \frac{1}{2}.$$

Proof. We have

$$\widehat{F_\varepsilon^\lambda}(h) = \frac{c_\varepsilon^\lambda c_\varepsilon^{\overline{\lambda}}}{\Gamma(2\operatorname{Re}\lambda)} |h|^{-2\operatorname{Re}\lambda+1} T_\varepsilon^\lambda.$$

In the case where $T_\varepsilon^\lambda \neq 0$, in order to have positivity at least up to normalisation we must insist on $-2\operatorname{Re}\lambda + 1 \geq -1$ or $\operatorname{Re}\lambda \leq 1$ so that $|h|^{-2\operatorname{Re}\lambda+1}$ is positive. Then we must have T_ε^λ is positive so that $\operatorname{Re}\lambda \leq \frac{1}{2}$.

By switching what roots we consider positive we can realise the representation on N instead of on $\overline{N}$. Then we are looking at the $NMA\overline{N}$ decomposition instead of the $\overline{N}MAN$ one. By transpose inverse it is clear that

$$g^{-t} = n(g)^{-t} m(g)^{-1} a(g)^{-1} \overline{n}(g)^{-t}$$

so that the $NMA\overline{N}$-decomposition is

$$g = n(g^{-t})^{-t} m(g^{-t})^{-1} a(g^{-t})^{-1} \overline{n}(g^{-t})^{-t}.$$

Actually, since the MA part only involves determinants of the principal submatrices, they are invariant under ^{t}. Therefore, the decomposition is actually

$$g = n(g^{-t})^{-t} m(g^{-1})^{-1} a(g^{-1})^{-1} \overline{n}(g^{-t})^{-t}.$$

The Knapp-Stein kernel on N is then

$$(a((w^{-1}n^{-1})^{-1})^{-1})^{\lambda+\rho} \varepsilon(m((w^{-1}n^{-1})^{-1})^{-1}) = a(nw)^{-\lambda-\rho} \varepsilon(m(nw)).$$

Note that

$$nw = \begin{pmatrix} -z & y & 1 \\ -x & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

so the Knapp-Stein kernel on N is

$$|xy - z|_{\varepsilon_1}^{-\lambda_1-1} |z|_{\varepsilon_2}^{-\lambda_2-1}.$$

This means that in order for the Knapp-Stein intertwiner to be positive definite it is necessary that

$$-\frac{1}{2} \leq \operatorname{Re}\lambda \leq \frac{1}{2}. \qquad \square$$

Comments for the $d > 1$-Case. One approach to the analysis of T_ε^λ in this case uses polar coordinates for the integral. Recall

$$\int_{\mathbb{R}^d} f(x)\,dx = \int_{S^{d-1}} \int_0^\infty f(rv) r^{d-1}\,drdv$$

or at least choose a normalisation of the integral on the sphere so this formula holds. The formula holds just as well for $x \mapsto f(-x)$ so we can write

$$\int_{\mathbb{R}^d} f(x)\,dx = \int_{S^{d-1}} \int_{-\infty}^\infty f(tv)|t|^{d-1}\,drdv$$

up to a proper normalisation on the sphere. But this is where trouble for T_ε^λ begins because now

$$\begin{aligned}(T_\varepsilon^\lambda\varphi|\varphi) &= \int_{S^{d-1}} \int_{-\infty}^\infty T_\varepsilon^\lambda \varphi(xv)\overline{\varphi(yv)}\,dxdydv \\ &= \int_{-\infty}^\infty \int_{-\infty}^\infty |x|_\varepsilon^{-\lambda+d-1}|y-x|^{2\operatorname{Re}\lambda-1}|y|_\varepsilon^{-\overline{\lambda}} \int_{S^{d-1}} \varphi(xv)\overline{\varphi(yv)}\,dvdxdy.\end{aligned}$$

Employing the same transformation trick as before we are basically asking the question of whether or not

$$\int_{-\infty}^\infty \int_{-\infty}^\infty |x|^{d-1}|y-x|^{2\operatorname{Re}\lambda-1}\varphi(x)\overline{\varphi(y)}\,dxdy \geq 0 \tag{5.3}$$

for all φ. It is not at all clear how to proceed in general when $d > 1$.

For $\operatorname{Re}\lambda = \frac{1}{2}$ the analysis simplifies significantly, however. In this case we are considering the operator $S_\varepsilon^\lambda : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ given by

$$S_\varepsilon^\lambda\varphi(x) = \int_{-\infty}^\infty |t|_\varepsilon^\lambda \varphi(tx)\,dt.$$

Proposition 5.1.6. *When $-1 \leq \lambda \leq d-1$ we have*

$$(S_\varepsilon^\lambda\varphi|\varphi) \geq 0$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ given by

$$\varphi(x) = P(x)e^{-|x|^2}$$

where P is a homogeneous polynomial.

Proof. Then

$$\begin{aligned}S_\varepsilon^\lambda\varphi(x) &= (1+\varepsilon(-1)^n)\int_0^\infty |t|_\varepsilon^{\lambda+n} e^{-t^2|x|^2}\,dt \cdot P(x) \\ &= \frac{1+\varepsilon(-1)^n}{2}\Gamma\Big(\frac{\lambda+n+1}{2}\Big)|x|^{-\lambda-n-1}P(x).\end{aligned}$$

Now, if Q is a polynomial homogeneous of degree m and $\psi(x) = Q(x)e^{-|x|^2}$ then using polynomial coordinates

$$\begin{aligned}(S_\varepsilon^\lambda\varphi|\psi) &= \frac{1+\varepsilon(-1)^n}{2}\Gamma\left(\frac{\lambda+n+1}{2}\right)\int_{\mathbb{R}^d}|x|^{-\lambda-n-1}P(x)\overline{Q(x)}e^{-|x|^2}\,dx \\ &= \frac{1+\varepsilon(-1)^n}{2}\Gamma\left(\frac{\lambda+n+1}{2}\right)(P|Q)\int_0^\infty r^{m+d-\lambda-2}e^{-r^2}\,dr \\ &= \frac{1+\varepsilon(-1)^n}{2}\Gamma\left(\frac{\lambda+n+1}{2}\right)\frac{(P|Q)}{2}\Gamma\left(\frac{m+d-\lambda-1}{2}\right)\end{aligned}$$

where $(P|Q)$ refers to the L^2 inner product on S^{d-1}. So for $\psi = \varphi$ we have

$$(S_\varepsilon^\lambda\varphi|\varphi) = \frac{1+\varepsilon(-1)^n}{2}\Gamma\left(\frac{\lambda+n+1}{2}\right)\frac{(P|P)}{2}\Gamma\left(\frac{n+d-\lambda-1}{2}\right) \geq 0 \qquad \square$$

It should be possible to prove

Conjecture 5.1.7. *For all d,*

$$T_{\varepsilon,\varepsilon}^{\frac{1}{2},\frac{1}{2}} = S_\varepsilon^{-\frac{1}{2}}$$

is positive.

Chapter 6

Compact Picture

In this chapter we consider the Fourier analysis over K of the Knapp-Stein kernel in two cases: First the linear case $K = \mathrm{SO}(3)$ where we induce from $\varepsilon \in \{-1, 1\}^2 \cong \widehat{M}$ and then in the nonlinear case $K = \mathrm{SU}(2)$ where we induce from the 2-dimensional representation of $M \subseteq \mathrm{SU}(2)$. This then achieves necessary and sufficient conditions for the existence of new representations constructed via the G-invariant inner product given by the Knapp-Stein intertwiner. The latter case was also considered in [38] but through a very different method.

6.1 Analysis of Certain Operators

In this section we analyse certain operators that are relevant for both the SO(3) and SU(2)-case.

Unitary Dual of SU(2)**.** Let $\mathcal{H}^n$ consist of all polynomials in two complex variables z_1, z_2 homogeneous of degree n. Then SU(2) has a representation on $\mathcal{H}^n$ given by

$$[\rho_n(k)P](z) = P(k^{-1}z).$$

Now, the simplest elements of $\mathcal{H}^n$ are given by

$$P_j(z) = z_1^j z_2^{n-j}.$$

There is a G-invariant inner product on $\mathcal{H}^n$ for which the P_j's are mutually orthogonal and for which

$$\|P_j\|^2 = \binom{n}{j}^{-1}.$$

The ρ_n make up the entire unitary dual of SU(2).

In the analysis below it will be important to understand the operators

$$A^n_{jl}(\lambda) = \frac{1}{\Gamma(\lambda)} \frac{1}{\pi} \int_0^{\pi} \sin^{\lambda-1}\theta \rho_n\left(\frac{\theta}{2}\right) d\theta.$$

This makes sense as an integral for $\mathrm{Re}\,\lambda > 0$ and can be extended analytically in λ to the entire complex plane.

Theorem 6.1.1. *We have*

$$A^n_{jl}(\lambda)^* = \rho_n(m_{jl})^{-1} A^n_{jl}(\overline{\lambda})$$

and $A^n_{jl}(\lambda)$ *has eigenvalues* $\{a(\lambda, \frac{n}{2} - m)\}_{0 \le m \le n}$ *where*

$$a(\lambda, j) = \frac{i^j}{2^{\lambda-1}\Gamma(\frac{\lambda+1+j}{2})\Gamma(\frac{\lambda+1-j}{2})}. \tag{6.1}$$

The symbol $a(\lambda, j)$ *satisfies*

$$(j + 1 + \lambda)a(\lambda, j + 2) = (j + 1 - \lambda)a(\lambda, j) \tag{6.2}$$

$$a(\lambda, -j) = (-1)^j a(\lambda, j). \tag{6.3}$$

Also, $A^0_{jl}(\lambda)$ *is invertible if and only if* $\lambda \notin -1 - 2\mathbb{N}_0$ *and for* $n \ge 1$, $A^n_{jl}(\lambda)$ *is invertible if and only if* $\lambda \notin \frac{n}{2} - 1 - \mathbb{N}_0$.

Furthermore,

$$\rho_n(m_{12})^{-1} A^n_{23}(\lambda)\rho_n(m_{12}) = A^n_{23}(\overline{\lambda})^*.$$

Proof. Making the substitution $\varphi = \pi - \theta$ we have

$$\begin{aligned}\pi A^n_{jl}(\lambda)^* &= \overline{\Gamma(\lambda)}^{-1} \int_0^\pi \sin^{\overline{\lambda-1}} \theta \rho_n\left(-\frac{\theta}{2}\right) d\theta \\ &= \Gamma(\overline{\lambda})^{-1} \int_0^\pi \sin^{\overline{\lambda-1}} \varphi \rho_n\left(\frac{\varphi - \pi}{2}\right) d\varphi = \pi \rho_n(m_{jl})^{-1} A^n_{jl}(\overline{\lambda}).\end{aligned}$$

There is an orthonormal basis $(Q_m)_{0 \le m \le n}$ for $\mathcal{H}^n$ where $\rho_n k_{jl}\theta Q_m = e^{i(n-2m)\theta} Q_m$. In this basis $A^n_{jl}(\lambda)$ is diagonal with eigenvalues $a(\lambda, \frac{n}{2} - m)$ where

$$a(\lambda, j) = \frac{1}{\Gamma(\lambda)} \frac{1}{\pi} \int_0^\pi \sin^{\lambda-1} \theta e^{ij\theta}\, d\theta.$$

We arrive at Eq. (6.1) by combining equations 8 and 1 of [12, 3.631].

We have that $A^0_{jl}(\lambda)$ has sole eigenvalue

$$a(\lambda, 0) = \frac{1}{2^{\lambda-1}\Gamma(\frac{\lambda+1}{2})}$$

which is nonzero if and only if $\lambda \notin -1 - 2\mathbb{N}_0$. Meanwhile, if $n \ge 1$ we have $A^n_{jl}(\lambda)$ is invertible if and only if $a(\lambda, \frac{n}{2} - m) \ne 0$ for $0 \le m \le n$ which is the case if and only if

$$\frac{\lambda + 1 \pm (\frac{n}{2} - m)}{2} \notin -\mathbb{N}_0$$

which is equivalent to

$$\lambda \notin \pm(\frac{n}{2} - m) - 1 - 2\mathbb{N}_0.$$

If we take $m = 0$ and $m = 1$ we find that this is the same as

$$\lambda \notin \frac{n}{2} - 1 - \mathbb{N}_0.$$

The last equation is a consequence of

$$\rho_n(m_{12})^{-1}k_{23}(\theta)\rho_n(m_{12}) = k_{23}(-\theta).$$

□

For $\varepsilon \in \{-1, 1\}$ we will use

$$E^n_{jl}(\varepsilon) := \frac{\mathrm{Id} + \varepsilon\rho_n(m_{jl})}{2}$$

Lemma 6.1.2. *We have*

$$\mathrm{Id} = E^n_{jl}(+) + E^n_{jl}(-)$$

and for even $2n$*,*

$$(E^{2n}_{jl}(\varepsilon))^2 = E^{2n}_{jl}(\varepsilon)$$

Introduce for $\lambda \in \mathbb{C}$ and $\varepsilon \in \{-1, 1\}$,

$$B^n(\lambda, \varepsilon) := E^n_{12}(\varepsilon)A^n_{23}(\lambda)E^n_{12}(\varepsilon).$$

Proposition 6.1.3. *For every* λ *and* ε*,*

$$B^{2n}(\lambda, \varepsilon) = E^{2n}_{12}(\varepsilon)\frac{A^{2n}_{23}(\lambda) + A^{2n}_{23}(\overline{\lambda})^*}{2} = \frac{A^{2n}_{23}(\lambda) + A^{2n}_{23}(\overline{\lambda})^*}{2}E^{2n}_{12}(\varepsilon).$$

For $\lambda \in -1 - 2\mathbb{N}_0$*,*

$$B^{2n}(\lambda, \varepsilon) = 0$$

for all n*.*

For $n = 0, 1$*,* $B^{2n}(\lambda, \varepsilon)$ *is positive for all* $\lambda \in \mathbb{R}$*. For* $n \geq 2$ *we have*

$$B^{2n}(\lambda, \varepsilon) \geq 0$$

with $\lambda \in \mathbb{R}$ *if and only if* $\lambda \in 1 - 2\mathbb{N}_0$ *or* $\lambda \in (-1, 1)$*.*

Proof. Indeed,

$$2A^{2n}_{23}(\lambda)E^{2n}_{12}(\varepsilon) = A^{2n}_{23}(\lambda) + \varepsilon\rho_{2n}(m_{12})A^{2n}_{23}(\overline{\lambda})^*$$

so that we have the first equation since $E^{2n}_{12}(\varepsilon)\varepsilon\rho_{2n}(m_{12}) = E^{2n}_{12}(\varepsilon)$. The second is achieved the same way.

Consider the eigenvectors Q_m of $A^{2n}_{23}(\lambda)$ with eigenvalue $a(\lambda, n - m)$ for $0 \leq m \leq n$. We change the parametrisation so that Q_m has eigenvalue $a(\lambda, m)$ for $-n \leq m \leq n$. Then the above tells us that

$$B^{2n}(\lambda, \varepsilon)E^{2n}_{12}(-\varepsilon)Q_m = 0$$
$$B^{2n}(\lambda, \varepsilon)E^{2n}_{12}(\varepsilon)Q_m = \frac{1 + (-1)^m}{2}a(\lambda, m)E^{2n}_{12}(\varepsilon)Q_m$$

since $\overline{a(\overline{\lambda}, m)} = a(\lambda, -m)(-1)^m a(\lambda, m)$. Since $\mathrm{Id} = E^{2n}_{12}(\varepsilon) + E^{2n}_{12}(-\varepsilon)$ we have found all possible eigenvalues. In order to find out whether or not they are all eigenvalues we need to figure out whether or not $E^{2n}_{12}(\varepsilon)Q_m \neq 0$.

If $P_j(z) = z_1^j z_2^{2n-j}$ then we have

$$\rho_{2n}(k_{12}\theta)P_j = e^{2i(n-j)}P_j,$$
$$\rho_{2n}(m_{23})P_j = (-1)^j P_{2n-j}.$$

We note that

$$k_{12}(\theta) = w_{13}^{-1}k_{23}(\theta)w_{13}$$

so that $\{\rho_{2n}(w_{13})P_j\}_j$ are the eigenvectors of $k_{23}(\theta)$. Since $w_{13}^{-1}m_{12}w_{13} = m_{23}^{-1}$ we have

$$\rho_{2n}(m_{12})\rho_{2n}(w_{13})P_j = \rho_{2n}(w_{13})\rho_{2n}(m_{23}^{-1})P_j = (-1)^j\rho_{2n}(w_{13})(P_{2n-j}).$$

It follows that

$$2E^{2n}_{12}(\varepsilon)\rho_{2n}(w_{13})P_j = \rho_{2n}(w_{13})(P_j + \varepsilon(-1)^j P_{2n-j})$$

which can never be 0 unless $j = 2n - j$, i.e., $j = n$. But for this j we have

$$E^{2n}_{12}(\varepsilon)\rho_{2n}(w_{13})P_n = (1 + \varepsilon(-1)^n)\rho_{2n}(w_{13})P_n$$

which is 0 if and only if $(-1)^n = -\varepsilon$. So in the case where $(-1)^n = -\varepsilon$, 0 takes the place of $a(\lambda, 0)$ as an eigenvalue. Positivity is preserved to $a(\lambda, m)$ for $m > 0$ in the same manner as above. □

6.2 Knapp-Stein on SO(3)

We want to determine the kernel $a(w^{-1}k)^{\lambda-\rho}\varepsilon(m(w^{-1}k))$. First it is necessary to find a good parametrisation of SO(3) that is congenial to the Haar measure.

We have three copies of SO(2) inside SO(3). Let $k_{jl}(t) = \iota_{jl}(k(t))$. We use the following parametrisation of the Haar measure:

$$\int_{\mathrm{SO}(3)} f = \frac{1}{8\pi^2}\int_0^{2\pi} d\psi \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi\, f(k_{12}(\psi)k_{23}(\theta)k_{12}(\varphi)).$$

Let us shorten $k(\psi, \theta, \varphi) := k_{12}(\psi)k_{23}(\theta)k_{12}(\varphi)$.

There is a double covering $\Phi : \mathrm{SU}(2) \to \mathrm{SO}(3)$. For all n, $\rho_{2n}(\sigma) = \mathrm{Id}$ and so ρ_{2n} descends to a representation $\widetilde{\rho}_{2n}$ of SO(3). Then $\{\widetilde{\rho}_{2n}\}_{n\in\mathbb{N}_0}$ exhausts the unitary dual of SO(3).

We find with $k = k(\psi, \theta, \varphi)$:

$$a(w^{-1}k)^{\lambda-\rho}\varepsilon(m(w^{-1}k)) = |\sin\psi\sin\theta|_{\varepsilon_1}^{\lambda_1-1}|\sin\theta\sin\varphi|_{\varepsilon_2}^{\lambda_2-1}.$$

We consider the Fourier transform of

$$\Phi^\lambda_\varepsilon(k) = \frac{1}{\Gamma(\lambda_1)\Gamma(\lambda_1+\lambda_2)\Gamma(\lambda_2)}a(w^{-1}k)^{\lambda-\rho}\varepsilon(m(w^{-1}k))$$

We introduce the notation for $\varepsilon \in \{-1, 1\}$,

$$E_{jl}^{2n}(\varepsilon) = \frac{\operatorname{Id} + \rho_{2n}(m_{jl})}{2}.$$

Theorem 6.2.1. *The Fourier transform of* Φ_ε^λ *is given by*

$$\widehat{\Phi_\varepsilon^\lambda}(2n) = \frac{\pi}{2} E_{12}^{2n}(\varepsilon_1) A_{12}^{2n}(\lambda_1) A_{23}^{2n}(\lambda_1 + \lambda_2) A_{12}^{2n}(\lambda_2) E_{12}^{2n}(\varepsilon_2).$$

Proof.

$$\begin{aligned}\int_{SO(3)} \Phi_\varepsilon^\lambda(k) \widetilde{\rho_{2n}}(k)\, dk = \frac{1}{8\pi^2} \int_0^{2\pi} &|\sin\psi|_{\varepsilon_1}^{\lambda_1 - 1} \rho_n(k_{12}\psi)\, d\psi \\ \cdot \int_0^\pi \sin^{\lambda_1+\lambda_2-1}\theta \rho_n(k_{23}\theta)\, d\theta & \int_0^{2\pi} |\sin\varphi|_{\varepsilon_2}^{\lambda_2-1} \rho_n(k_{12}\varphi)\, d\varphi.\end{aligned}$$

This clearly implies the result. □

Positive Definiteness of the Knapp-Stein Kernel. For the kernel Φ_ε^λ to be positive definite it must be self-adjoint. We have

$$(\Phi_{\varepsilon_1,\varepsilon_2}^{\lambda_1,\lambda_2})^* = \Phi_{\varepsilon_2,\varepsilon_1}^{\overline{\lambda_2},\overline{\lambda_1}}$$

so Φ_ε^λ is self-adjoint if and only if $\lambda_1 = \overline{\lambda_2} = \lambda$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon$. We find then

Theorem 6.2.2. *For* $(\varepsilon_1, \varepsilon_2) = (\varepsilon, \varepsilon)$ *and* $(\lambda_1, \lambda_2) = (\lambda, \overline{\lambda})$, *we have*

$$\widehat{\Phi_\varepsilon^\lambda}(2n) = 0$$

for all n *if* $\operatorname{Re}\lambda \in -\frac{1}{2} - \mathbb{N}_0$. *For* $\operatorname{Re}\lambda \notin -\frac{1}{2} - \mathbb{N}_0$,

$$\varepsilon\widehat{\Phi_\varepsilon^\lambda}(2n) \geq 0$$

for all n *if and only if* $-\frac{1}{2} < \operatorname{Re}\lambda \leq \frac{1}{2}$.

Proof. We have

$$\begin{aligned}\widehat{\Phi_\varepsilon^\lambda}(2n) &= A_{12}^{2n}(\lambda) B(2\operatorname{Re}\lambda, \varepsilon) A_{12}^{2n}(\overline{\lambda}) = A_{12}^{2n}(\lambda) B(2\operatorname{Re}\lambda, \varepsilon) \rho_{2n}(m_{12}) A_{12}^{2n}(\lambda)^* \\ &= \varepsilon A_{12}^{2n}(\lambda) B(2\operatorname{Re}\lambda, \varepsilon) A_{12}^{2n}(\lambda)^*\end{aligned}$$

since $E_{12}^{2n}(\varepsilon)\rho_{2n}(m_{12}) = \varepsilon E_{12}^{2n}(\varepsilon)$. This is 0 for all n if $2\operatorname{Re}\lambda \in -1 - 2\mathbb{N}_0$.

The case $n = 2$ provides us with the necessary condition $-1 < 2\operatorname{Re}\lambda < 1$ when $\lambda \notin 1 - \mathbb{N}_0$ because then $A_{12}^4(\lambda)$ is invertible. It is clear from the previous proposition that this is also sufficient and furthermore we find that $\operatorname{Re}\lambda = \frac{1}{2}$ is also sufficient. □

6.3 Knapp-Stein on SU(2)

In this section, the notations $k, m, a, n, \overline{n}$ refers to the elements inside $\widetilde{\mathrm{SL}(3,\mathbb{R})}$. Let $q : \widetilde{\mathrm{SL}(3,\mathbb{R})} \to \mathrm{SL}(3,\mathbb{R})$ be the projection. The nontrivial element of its kernel will be denoted by σ. We take as a representation ε of M the canonical 2-dimensional one. Again we have the Knapp-Stein kernel

$$\Phi_\varepsilon^\lambda(k) = \frac{a(w^{-1}k)^{\lambda-\rho}\varepsilon(m(w^{-1}k))}{\Gamma(\lambda_1)\Gamma(\lambda_1+\lambda_2)\Gamma(\lambda_2)}$$

The normalised Haar measure on SU(2) is given by

$$\int_{\mathrm{SU}(2)} f = \frac{1}{2\pi^2}\int_0^{2\pi} d\psi \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \int_0^{\pi} d\varphi f(k(\psi,\theta,\varphi))$$

where $k(\psi,\theta,\varphi) = k_{12}(\psi)k_{23}(\theta)k_{12}(\varphi)$.

Analogously (but not quite systematically) with the SO(3)-case we introduce

$$E^n = \frac{\mathrm{Id} + m_{12}\otimes\rho_n(m_{12})}{2}$$
$$F^n = \frac{\mathrm{Id} + m_{23}^{-1}\otimes\rho_n(m_{12})}{2}.$$

Consider also

$$W := \frac{1}{\sqrt{2}} w_{13}\otimes I$$

and note that since $m_{23}^{-1} = w_{13}^{-1}m_{12}w_{13}$,

$$F = W^{-1}EW.$$

Proposition 6.3.1. *For all odd n,*

$$E^nF^n = E^nW = WF^n$$
$$F^nE^n = W^{-1}E^n = F^nW^{-1}.$$

Proof. One finds

$$\begin{aligned} 4EF &= I + m_{12}\otimes\rho m_{12} + m_{23}^{-1}\otimes\rho m_{12} + m_{13}^{-1}\otimes\rho\sigma \\ &= \begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\otimes I + \begin{pmatrix} i & -1\\ 1 & -i\end{pmatrix}\otimes\rho m_{12} \\ &= \sqrt{2}(w_{13}\otimes I)(I + m_{23}^{-1}\otimes\rho m_{12}) = 4WF. \end{aligned}$$

And we know that $WF = EW$. The second equation follows by conjugating the first:

$$FE = F^*E^* = W^*E^* = W^{-1}E = FW^{-1}.$$

□

Proposition 6.3.2. *For every* $\lambda \in \mathbb{C}^2$, $\widehat{\Phi_\varepsilon^\lambda}(n) = 0$ *if* n *is even, and if* n *is odd,*

$$\widehat{\Phi_\varepsilon^\lambda}(n) = 4\pi(e \otimes A_{12}^n(\lambda_1))F^n(m_{12} \otimes A_{23}^n(\lambda_1 + \lambda_2))E^n(e \otimes A_{12}^n(\lambda_2))$$

Proof. We need to compute

$$a(w^{-1}k(\psi,\theta,\varphi))^{\lambda-\rho}\varepsilon(m(w^{-1}k(\psi,\theta,\varphi)))$$

for the parameters relevant for the Haar integral. The A-part is already known since q gives us an isomorphism of A and the A-part of $\mathrm{SL}(3,\mathbb{R})$. Consequently,

$$a(w^{-1}k(\psi,\theta,\varphi))^{\lambda-\rho} = |\sin 2\psi \sin 2\theta|^{\lambda_1 - 1}|\sin 2\theta \sin 2\varphi|^{\lambda_2 - 1}.$$

For the M-part, we reduce to $0 \le \psi,\theta,\varphi \le \frac{\pi}{2}$ by way of:

$$\begin{aligned} m(w^{-1}k(\psi+\pi,\theta,\varphi)) &= \sigma m(w^{-1}k(\psi,\theta,\varphi)) \\ m(w^{-1}k(\psi+\frac{\pi}{2},\theta,\varphi)) &= m_{23}^{-1}m(w^{-1}k(\psi,\theta,\varphi)) \\ m(w^{-1}k(\psi,\theta,\varphi+\frac{\pi}{2})) &= m(w^{-1}k(\psi,\theta,\varphi))m_{12}. \end{aligned}$$

Here we have used that

$$w_{13}^{-1}m_{12}w_{13} = m_{23}^{-1}.$$

Note finally that $m(w^{-1}k(\psi,\theta,\varphi))$ is constant for $0 < \psi,\theta,\varphi < \frac{\pi}{2}$. This can be seen by first computing the $\overline{N}MAN$-decomposition inside $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ to get a decomposition for every $k_{jl}(\theta)$ and then using commutation rules to get the $\overline{N}MAN$-decomposition for $k(\psi,\theta,\varphi)$ (cf. Appendix B.3).

So for these parameters,

$$m(w^{-1}k(\psi,\theta,\varphi)) = m(w^{-1}(\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{4})) = m(w_{13}^{-1}w_{12}w_{23}w_{12}) = m(m_{12}) = m_{12}.$$

The Fourier transformation is then given by

$$\begin{aligned} \widehat{\Phi_\varepsilon^\lambda}(n) &= \frac{1}{2\pi^2}\int_0^{2\pi} d\psi \int_0^{\frac{\pi}{2}} \sin 2\theta\, d\theta \int_0^{\pi} d\varphi\, \Phi_\varepsilon^\lambda(k(\psi,\theta,\varphi) \otimes \rho_n(k(\psi,\theta,\varphi)) \\ &= 4\pi\frac{1+\sigma\otimes\rho_n(\sigma)}{2}\frac{1+m_{23}^{-1}\otimes\rho_n(m_{12})}{2} \\ &\quad (m_{12}\otimes A_{12}^n(\lambda_1)A_{23}^n(\lambda_1+\lambda_2)A_{12}^n(\lambda_2))\frac{1+m_{12}\otimes\rho_n(m_{12})}{2}. \end{aligned}$$

For even n, $\sigma \otimes \rho_n(\sigma) = 0$ □

Just as before we explore the case $\lambda_1 = \overline{\lambda_2} = \lambda$. Then for every odd n,

$$\widehat{\Phi_\varepsilon^\lambda}(n) = 4\pi(e \otimes A_{12}^n(\lambda))F^n(e \otimes A_{23}^n(2\,\mathrm{Re}\,\lambda))E^n(e \otimes A_{12}^n(\lambda))^*$$

since $m_{12} \otimes \rho_n(m_{12})E^n = E^n$. Introduce

$$B(\lambda) := F^n(e \otimes A_{23}^n(\lambda))E^n.$$

Proposition 6.3.3. *For odd n,*

$$B(\lambda) = \frac{e \otimes A_{23}^n(\lambda) + m_{13}^{-1} \otimes A_{23}^n(\overline{\lambda})^*}{2} E^n = F^n \frac{e \otimes A_{23}^n(\lambda) + m_{13}^{-1} \otimes A_{23}^n(\overline{\lambda})^*}{2}.$$

Write $n = 2l+1$, $l \in \mathbb{N}_0$. In addition to 0, the eigenvalues of $B^{2l+1}(\lambda)$ are

$$\frac{(-1)^k}{2^{\lambda-1}\Gamma(\frac{\lambda+3/2}{2}+k)\Gamma(\frac{\lambda+1/2}{2}-k)}$$

for k integer, $-\lfloor \frac{l+1}{2} \rfloor \leq k \leq \lfloor \frac{l}{2} \rfloor$.

For $\lambda \in [-\frac{1}{2}, \frac{1}{2}]$, $B(\lambda)$ is positive. If $\lambda \notin \frac{1}{2} - \mathbb{N}_0$, $B(\lambda)$ can only be positive if $\lambda \in [-\frac{1}{2}, \frac{1}{2}]$.

Proof. Since $m_{23}^{-1} = m_{13}^{-1} m_{12}$,

$$2F^n(e \otimes A_{23}^n(\lambda)) = e \otimes A_{23}^n(\lambda) + (m_{13}^{-1} \otimes A_{23}^n(\overline{\lambda})^*)(m_{12} \otimes \rho_n(m_{12}))$$

so it is clear if we multiply with E^n we have the first equation. For the second, note $m_{12} = m_{23}^{-1} m_{13}^{-1}$ so

$$2(e \otimes A_{23}^n(\lambda))E^n = e \otimes A_{23}^n(\lambda) + (m_{23}^{-1} \otimes \rho_n(m_{12}))(m_{13}^{-1} \otimes A_{23}^n(\overline{\lambda})^*)$$

so if we multiply with F^n we have the second equation.

Now, suppose that Q_m, $0 \leq m \leq n$, is an eigenvector of $\rho_n k_{23}(\theta)$ such that

$$\rho_n(k_{23}\theta)Q_m = e^{i(\frac{n}{2}-m)\theta} Q_m.$$

Suppose that v_ε, $\varepsilon \in \{-1, 1\}$, is an eigenvector of w_{13} with eigenvalue $e^{\varepsilon i\pi/4}$. Then we have

$$\begin{aligned} B(\lambda)F(v_\varepsilon \otimes Q_m) &= (e \otimes A_{23}^n(\lambda) + m_{13}^{-1} \otimes A_{23}^n(\overline{\lambda})^*)EF(v_\varepsilon \otimes Q_m) \\ &= (e \otimes A_{23}^n(\lambda) + m_{13}^{-1} \otimes A_{23}^n(\overline{\lambda})^*)EW(v_\varepsilon \otimes Q_m) \\ &= \frac{1}{\sqrt{2}}(e^{\varepsilon i\pi/4} a(\lambda, \frac{n}{2} - m) + e^{-\varepsilon i\pi/4} a(\lambda, m - \frac{n}{2}))F(v_\varepsilon \otimes Q_m). \end{aligned}$$

In order to make sure that the values are actual eigenvalues we need to make sure that $F(v_\varepsilon \otimes Q_m) \neq 0$. For this, note that we can take

$$v_\varepsilon = \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix}$$

so that

$$m_{23}^{-1} v_\varepsilon = \begin{pmatrix} -\varepsilon \\ 1 \end{pmatrix} = -\varepsilon v_{-\varepsilon}.$$

Also, we know that $Q_m = \rho_n w_{13} P_m$ where $P_m(z) = z_1^m z_2^{n-m}$ so that $\rho_n(m_{23})P_m = (-1)^{n-m} P_{n-m}$ turns into $\rho_n(m_{12})Q_m = (-1)^m Q_{n-m}$. Thus,

$$2F(v_\varepsilon \otimes Q_m) = v_\varepsilon \otimes Q_m - \varepsilon(-1)^m v_{-\varepsilon} \otimes Q_{n-m}$$

which can never be 0.

We have found all the eigenvalues if we add 0. Indeed, with

$$\widetilde{F} = \frac{I - m_{23}^{-1} \otimes \rho_n m_{12}}{2}$$

we have $I = F + \widetilde{F}$ so that $F(v_\varepsilon \otimes Q_m)$, $\widetilde{F}(v_\varepsilon \otimes Q_m)$ form a basis and again, $\widetilde{F}(v_\varepsilon \otimes Q_m)$ cannot be 0. Also, $F\widetilde{F} = 0$ so that $\widetilde{F}(v_\varepsilon \otimes Q_m)$ is an eigenvector for $B(\lambda)$ with eigenvalue 0.

The possibly nonzero eigenvalues are

$$\frac{1}{2\sqrt{2}} \frac{e^{\varepsilon i\pi/4} e^{i(\frac{n}{2}-m)\pi/2} + e^{-\varepsilon i\pi/4} e^{i(m-\frac{n}{2})}}{2^{\lambda-1}\Gamma(\frac{\lambda+1+\frac{n}{2}-m}{2})\Gamma(\frac{\lambda+1+m-\frac{n}{2}}{2})}.$$

Set $n = 2l + 1$, $l \in \mathbb{N}_0$. The numerator is

$$e^{\varepsilon i\pi/4} e^{i(l+\frac{1}{2}-m)\pi/2} (1 + e^{-\varepsilon i\pi/2} e^{-i(l+\frac{1}{2}-m)} \pi)$$

where

$$1 + e^{-\varepsilon i\pi/2} e^{-i(l+\frac{1}{2}-m)\pi} = 1 + i^{-\varepsilon}(-1)^{l-m} i^{-1} = 1 - \varepsilon(-1)^{l-m}.$$

It follows that the possibly nonzero eigenvalues are

$$\frac{1-\varepsilon(-1)^{l-m}}{2} \frac{1}{\sqrt{2}} \frac{e^{(\varepsilon+1)i\pi/4} i^{l-m}}{2^{\lambda-1}\Gamma(\frac{\lambda+3/2+l-m}{2})\Gamma(\frac{\lambda+1/2+m-l}{2})}.$$

We split the analysis according to the parity of $l-m$. Note that as $0 \leq m \leq 2l+1$ we have $-(l+1) \leq m \leq l$. Also,

$$e^{i(\varepsilon+1)\pi/4} = \begin{cases} i & \varepsilon = 1, \\ 1 & \varepsilon = -1. \end{cases}$$

In order to have a nonzero eigenvalue when $l - m$ is even we must have $\varepsilon = -1$ so that the corresponding eigenvalues are

$$\frac{1}{\sqrt{2}} \frac{i^m}{2^{\lambda-1}\Gamma(\frac{\lambda+3/2+m}{2})\Gamma(\frac{\lambda+1/2-m}{2})}$$

for $-(l+1) \leq m \leq l$, m even. For $l - m$ odd we must have $\varepsilon = 1$ so that the corresponding eigenvalues are

$$\frac{1}{\sqrt{2}} \frac{i^{m+1}}{2^{\lambda-1}\Gamma(\frac{\lambda+3/2+m}{2})\Gamma(\frac{\lambda+1/2-m}{2})}$$

for $-(l+1) \leq m \leq l$, m odd. In the even case we can take $m = 2k$ and in the odd case we can take $m = 2k - 1$. Both add up to the same thing (one replaces k by $-k$), namely

$$\frac{1}{\sqrt{2}} \frac{(-1)^k}{2^{\lambda-1}\Gamma(\frac{\lambda+3/2}{2} + k)\Gamma(\frac{\lambda+1/2}{2} - k)}$$

for k integer, $-\lfloor\frac{l+1}{2}\rfloor \le k \le \lfloor\frac{l}{2}\rfloor$. Let $b(k)$ be this number. One finds that

$$(2k+3/2+\lambda)b(k+1) = (2k+3/2-\lambda)b(k)$$

for all k. By inspection we see that if $\lambda \notin \frac{1}{2}-\mathbb{N}_0$ then $b(0)$ and $b(-1)$ are nonzero so that if they both are positive then since

$$b(-1) = \frac{-1/2+\lambda}{-1/2-\lambda}b(0)$$

we need $-1/2 \le \lambda \le 1/2$.

One sees that by the Legendre duplication formula,

$$b(0) = \sqrt{\frac{2}{\pi}}\frac{1}{\Gamma(\lambda+\frac{1}{2})}$$

is positive when $\lambda \in [-\frac{1}{2}, \frac{1}{2}]$. The recursive formula above then implies that $b(k) \ge 0$ for all k when λ is in this interval. □

Theorem 6.3.4. *For $\lambda \notin \frac{1}{2}-\mathbb{N}_0$ and $2\operatorname{Re}\lambda \notin -\frac{1}{2}-\mathbb{N}_0$, $\widehat{\Phi_\varepsilon^{\lambda,\bar{\lambda}}}(3)$ can only be positive when $\operatorname{Re}\lambda \in [-\frac{1}{4}, \frac{1}{4}]$. And $\widehat{\Phi_\varepsilon^{\lambda}}(n)$ is positive for all n if $\operatorname{Re}\lambda \in [-\frac{1}{4}, \frac{1}{4}]$.*

Proof. This follows from the previous propositions since for odd n,

$$\widehat{\Phi_\varepsilon^{\lambda,\bar{\lambda}}}(n) = 4\pi(e\otimes A_{12}^n(\lambda))B(2\operatorname{Re}\lambda)(e\otimes A_{12}^n(\lambda))^*.$$

When $\lambda \notin \frac{1}{2}-\mathbb{N}_0$, $A_{12}^n(\lambda)$ is invertible so the positivity of $\widehat{\Phi_\varepsilon^{\lambda}}(3)$ reduces to the positivity of $B(3)$. □

Remark 6.3.5. We will see later that $\widehat{\Phi_\varepsilon^{\lambda,\bar{\lambda}}}(n)$ is positive when $\lambda = \frac{1}{2}$. This proposition tells us that the unitary representation at $\lambda = \frac{1}{2}$ is isolated from the other complementary series representations at $Re\lambda \in [-\frac{1}{4}, \frac{1}{4}]$.

The Case When $\lambda_1 = \lambda_2 = \frac{1}{2}$. Let

$$A^n(\lambda_1,\lambda_2) = A_{12}^n(\lambda_1)A_{23}^n(\lambda_1+\lambda_2)A_{12}^n(\lambda_2).$$

We want to set $\lambda_1 = \lambda_2 = \frac{1}{2}$. Let P_j, $j = 0,1,\ldots,n$ be the vectors that diagonalise $\rho_n(k_{12}\theta)$. Concretely, we can take $P_j(z) = z_1^j z_2^{n-j}$. Then

$$A_{12}^n(\frac{1}{2})P_j = a(\frac{1}{2},\frac{n}{2}-j)P_j.$$

Shorten $a_j^n := a(\frac{1}{2},\frac{n}{2}-j)$. Then

Proposition 6.3.6. *We have*

$$a_m^{4a+1} = 0$$

for m odd, $1 \le m \le 2a-1$, and for m even, $2a+2 \le m \le 4a$. Also,

$$a_m^{4a+3} = 0$$

for m even, $0 \le m \le 2a$, and for m odd, $2a+3 \le m \le 4a+3$.

Proof. This follows by analysing

$$a_m^n = \frac{\sqrt{2} i^{\frac{n}{2}-m}}{\Gamma(\frac{\frac{1}{2}+1+\frac{n}{2}-m}{2})\Gamma(\frac{\frac{1}{2}+1+m-\frac{n}{2}}{2})}.$$ □

The following will be convenient:

Lemma 6.3.7. *We have*

$$A^n(\frac{1}{2},\frac{1}{2})^* = \rho_n(\sigma)A^n(\frac{1}{2},\frac{1}{2})$$

and

$$(A^n(\frac{1}{2},\frac{1}{2})P_{n-j}|P_l) = i^n(-1)^{j+l}(A^n(\frac{1}{2},\frac{1}{2})P_j|P_l)$$
$$(A^n(\frac{1}{2},\frac{1}{2})P_j|P_{n-l}) = (-i)^n(-1)^{j+l}(A^n(\frac{1}{2},\frac{1}{2})P_j|P_l)$$

Proof. First,

$$A^n(\frac{1}{2},\frac{1}{2})^* = \rho_n(\sigma)A_{12}(\frac{1}{2})\rho_n(m_{12})A_{23}(1)^*\rho_n(m_{12})^{-1}A_{12}(\frac{1}{2}) = \rho_n(\sigma)A^n(\frac{1}{2},\frac{1}{2}).$$

The second set of equations is based on the fact that

$$\rho_n(m_{13})P_j = (-i)^n P_{n-j}$$

so that

$$\begin{aligned}(A^n(\frac{1}{2},\frac{1}{2})P_{n-j}|P_l) &= i^n(A^n(\frac{1}{2},\frac{1}{2})\rho_n(m_{13})P_j|P_l)\\ &= i^n(A_{12}^n(\frac{1}{2})\rho_n(m_{13}m_{23}^{-1})A_{12}^n(\frac{1}{2})\rho_n(m_{12})P_j|P_l)\\ &= i^n i^{n-2j}(A^n(\frac{1}{2},\frac{1}{2})P_j|\rho_n(m_{12})^{-1}P_l)\\ &= i^n(-1)^{j+l}(A^n(\frac{1}{2},\frac{1}{2})P_j|P_l).\end{aligned}$$

Conjugating this equation gives us

$$(A^n(\frac{1}{2},\frac{1}{2})P_l|P_{n-j}) = (-i)^n(-1)^{j+l}(A^n(\frac{1}{2},\frac{1}{2})P_l|P_j)$$ □

Theorem 6.3.8. *For all $a \in \mathbb{N}_0$,*

$$A^{4a+3}(\frac{1}{2},\frac{1}{2}) = 0$$

and in the $(P_m)_m$-basis, in blocks of size $2a+1$,

$$A^{4a+1}(\frac{1}{2},\frac{1}{2}) = (A^{4a+1}(\frac{1}{2},\frac{1}{2})P_0|P_0)\begin{pmatrix} xx^t & ixy^t \\ -iyx^t & -yy^t\end{pmatrix}$$

where $x \in \mathbb{C}^{2a+1}$ is given by $x_m = 0$ for m odd, $0 \le m \le 2a$ and

$$x_{2m} = (-1)^m \frac{(\frac{1}{2})^{\overline{m}}(-a)^{\overline{m}}}{(\frac{1}{2}-a)^{\overline{m}}(-2a)^{\overline{m}}}$$

for $0 \le m \le a$ and where $y_m = x_{2a-m}$.

Proof. We start with the facts that

$$\begin{aligned} d\rho_n(W_{23})P_j &= -jP_{j-1} + (n-j)P_{j+1} \\ A^n_{23}(1)d\rho_n W_{23} &= \rho_n m_{23} - I \\ \rho_n(m_{23})P_j &= (-1)^j P_{n-j} \end{aligned}$$

Note that the first equation does make sense for $j = 0$ or $j = n$ with the right interpretation. It follows that

$$(-1)^j P_{n-j} - P_j = -jA^n_{23}(1)P_{j-1} + (n-j)A^n_{23}(1)P_{j+1}.$$

We multiply by $(n-2j-1)a_{j+1}$, using the fact that according to Eq. (6.2),

$$(n-2j-1)a_{j+1} = (n-2j+1)a_{j-1}$$

so get

$$\begin{aligned} (n-2j-1)a_{j+1}\big((-1)^j P_{n-j} - P_j\big) &= -j(n-2j+1)A^n_{23}(1)A^n_{12}(\frac{1}{2})P_{j-1} \\ &\quad + (n-j)(n-2j-1)A^n_{23}(1)A^n_{12}(\frac{1}{2})P_{j+1}. \end{aligned}$$

Lastly, we can apply $A^n_{12}(\frac{1}{2})$, using the fact that $a_{n-j} = (-1)^j a_j$ to obtain

$$\begin{aligned} (n-2j-1)a_{j+1}a_j(P_{n-j} - P_j) &= -j(n-2j+1)A(\frac{1}{2},\frac{1}{2})P_{j-1} \\ &\quad + (n-j)(n-2j-1)A(\frac{1}{2},\frac{1}{2})P_{j+1} \end{aligned}$$

For every j, a_j or a_{j+1} is 0 unless $j = \frac{n-1}{2}$ for which $n-2j-1 = 0$ so one obtains for all j,

$$j(n-2j+1)A(\frac{1}{2},\frac{1}{2})P_{j-1} = (n-j)(n-2j-1)A(\frac{1}{2},\frac{1}{2})P_{j+1}.$$

In particular,

$$A^n(\frac{1}{2},\frac{1}{2})P_1 = A^n(\frac{1}{2},\frac{1}{2})P_{n-1} = 0.$$

So when $n = 4a+3$ we have

$$A^n(\frac{1}{2},\frac{1}{2})P_j = 0 \tag{6.4}$$

when j is odd, $j \le \frac{n-1}{2} = 2a+2$ so combined with the previous proposition we have Eq. (6.4) for all $j \le 2a+1$. Also, the above implies that Eq. (6.4) holds for j even, $j \ge \frac{n-1}{2} = 2a+2$ while the previous proposition tells us that it is true for j odd, $j \ge 2a+3$. Consequently we conclude that $A^{4a+3}(\frac{1}{2},\frac{1}{2}) = 0$.

Now for $A^{4a+1}(\frac{1}{2},\frac{1}{2})$. In the following we use A as a shorthand for this operator. We want to compute the matrix representation

$$M_{jl} = (AP_l|P_j).$$

Because of the recursive relationship between the AP_m's we must have a scalar $x_m \in \mathbb{C}$, $0 \leq m \leq 2a$ such that

$$AP_m = x_m AP_0$$

for every such m. It is clear that $x_0 = 1$ and that $x_m = 0$ if m is odd. Furthermore, we get

$$x_{2m+2} = -\frac{(\frac{1}{2}+m)(-a+m)}{(\frac{1}{2}-a+m)(-2a+m)} x_{2m}$$

which implies that $(x_m)_m$ is given as claimed.

Likewise, there are scalars $y_m \in \mathbb{C}$, $0 \leq m \leq 2a$ such that

$$AP_{2a+1+m} = y_m P_{4a+1}.$$

Then $y_{2a} = 1$ and $y_m = 0$ if m is odd. Furthermore,

$$y_{2m-2} = -\frac{m(-a-\frac{1}{2}+m)}{(a+m)(-\frac{1}{2}+m)} y_{2m}.$$

Introducing $a_m = y_{2a-2m}$ an solving its recurrence relation we find $a_m = x_{2m}$, i.e., $y_{2m} = x_{2a-2m}$.

Now we find that the matrix representation of $A^{4a+1}(\frac{1}{2},\frac{1}{2})$ is

$$M = \begin{pmatrix} (AP_0|P_0)xx^t & (AP_{4a+1}|P_0)xy^t \\ (AP_0|P_{4a+1})yx^t & (AP_{4a+1}|P_{4a+1}yy^t \end{pmatrix}.$$

For example, when $0 \leq j, l \leq 2a$ we have

$$M_{jl} = (AP_l|P_j) = -x_l(P_l|AP_j) = x_j x_l (AP_0|P_0)$$

since $A^* = -A$. *Mutatis mutandis* for the other blocks.

The lemma above informs us that

$$\begin{aligned} (AP_{4a+1}|P_0) &= i(AP_0|P_0) \\ (AP_0|P_{4a+1}) &= -i(AP_0|P_0) \\ (AP_{4a+1}|P_{4a+1}) &= -i(AP_0|P_{4a+1}) = -(AP_0|P_0) \end{aligned}$$

so we are done. □

Proposition 6.3.9. *For all a,*

$$(A^{4a+1}(\frac{1}{2},\frac{1}{2}P_0|P_0) = \frac{4}{\sqrt{\pi}} i \frac{\Gamma(2a+1)}{\Gamma(2a+\frac{3}{2})} \frac{1}{\Gamma(a+1)^2\Gamma(-a+\frac{1}{2})^2}$$

Proof. Indeed,

$$\rho_n(k_{23}\theta)P_0 = \sum_{j=0}^{n} \sin^j\theta \cos^{n-j}\theta P_j$$

so that

$$(A_{23}^n(1)P_0|P_0) = \frac{1}{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos^n\theta\, d\theta = \frac{2}{\pi}B(\frac{n+1}{2},\frac{1}{2}) = \frac{2}{\sqrt{\pi}}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})}.$$

It follows that

$$(A^n(\frac{1}{2},\frac{1}{2})P_0|P_0) = a(\frac{1}{2},\frac{n}{2})(A_{23}^n(1)P_0|P_0) = \frac{4}{\sqrt{\pi}}\frac{i^n}{\Gamma(\frac{3+n}{4})^2\Gamma(\frac{3-n}{4})^2}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})}$$

so plugging in $n = 4a+1$ we have the result. □

Proposition 6.3.10. *Explicitly,*

$$F^{4a+1}(m_{12}\otimes A^{4a+1}(\frac{1}{2},\frac{1}{2}))E^{4a+1} = -\frac{(A^{4a+1}(\frac{1}{2},\frac{1}{2})P_0|P_0)}{2}\begin{pmatrix}x\\iy\\ix\\y\end{pmatrix}\begin{pmatrix}0 & y^t & x^t & 0\end{pmatrix}.$$

Proof. Note first that since $\rho_{4a+1}(m_{12})P_j = (-1)^j iP_j$ and since that $x_m, y_m = 0$ for odd m, we must have that $\rho(m_{12})$ acts on $A = A^{4a+1}(\frac{1}{2},\frac{1}{2})$ on both the left and the right as the block matrix

$$\begin{pmatrix}iI & 0\\0 & -iI\end{pmatrix}$$

Consequently, in computing $4F(m_{12}\otimes A)E$ we can replace $2F$ by

$$I + m_{23}^{-1}\otimes\begin{pmatrix}i & 0\\0 & -i\end{pmatrix} = \begin{pmatrix}1 & 0 & -i & 0\\0 & 1 & 0 & i\\i & 0 & 1 & 0\\0 & -i & 0 & 1\end{pmatrix}$$

and we can replace $2E$ by

$$1 + m_{12}\otimes\begin{pmatrix}i & 0\\0 & -i\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 0\\0 & 2 & 0 & 0\\0 & 0 & 2 & 0\\0 & 0 & 0 & 0\end{pmatrix}$$

Then simply doing the matrix multiplication gives us the result. □

Proposition 6.3.11. *We have*

$$\widehat{\Phi_{\varepsilon}^{\frac{1}{2},\frac{1}{2}}}(4a+1) = -\pi(A^{4a+1}(\frac{1}{2},\frac{1}{2})P_0|P_0)\begin{pmatrix}x\\iy\\ix\\y\end{pmatrix}\begin{pmatrix}0 & y^t & x^t & 0\end{pmatrix}$$

has rank 1. The single nonzero eigenvalue is

$$-2\pi i|x|^2(A^{4a+1}(\frac{1}{2},\frac{1}{2}P_0|P_0) = 8\sqrt{\pi}|x|^2\frac{\Gamma(2a+1)}{\Gamma(2a+\frac{3}{2})}\frac{1}{\Gamma(a+1)^2\Gamma(-a+\frac{1}{2})^2}$$

which is strictly positive for all a.

Proof. Since $|x|^2 = |y|^2$,

$$\widehat{\Phi_\varepsilon^{\frac{1}{2},\frac{1}{2}}}(4a+1)\begin{pmatrix} x \\ iy \\ ix \\ y \end{pmatrix} = -2\pi i|x|^2(A^{4a+1}(\frac{1}{2},\frac{1}{2})P_0|P_0)\begin{pmatrix} x \\ iy \\ ix \\ y \end{pmatrix} \qquad \square$$

In summary,

Theorem 6.3.12. *We have*

$$\widehat{\Phi_\varepsilon^{\frac{1}{2},\frac{1}{2}}}(n) = 0$$

for $n \not\equiv 1 \mod 4$, and for $n \equiv 1 \mod 4$, $\widehat{\Phi_\varepsilon^{\frac{1}{2},\frac{1}{2}}}(n)$ is a rank-one positive matrix. So there is a unitary quotient representation of $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ with Langlands parameters given by the characteristic representation of $\mathrm{SU}(2)$ and $\frac{1}{2}\delta_1 + \frac{1}{2}\delta_2$. The K-types of this representation are $\mathcal{H}^{4a+1}$ with multiplicity 1 for every a.

6.4 Outlook

Bochner's Theorem in General. It is an interesting problem in general to find Bochner's Theorem for (operator-valued) distributions on an arbitrary Lie group. As we have seen in Theorems 2.3.4 and 2.3.7 the problem is solved for compact Lie groups and it seems that it is not a very great challenge to generalise from the scalar case to the operator-valued one. On the Heisenberg group we only achieved Bochner's Theorem for the Lizorkin distributions and we realised via the Abelian Fourier transform a sort of Bochner's Theorem for the distributions that are polynomial along the center. What we do not have is any way of relating these two facts with each other.

I am confident that we could achieve analogous results on the Fourier transform on more general nilpotent groups. For example one could let N be the subgroup of all unipotent upper triangular matrices in $\mathrm{SL}(d,\mathbb{R})$ which corresponds with inducing via a minimal parabolic. The change one would have to content with is that $\widehat{N}$ would split into more families corresponding to the stratification of the coadjoint orbits by their dimensions. I think it would be a good idea to finish the Heisenberg case before taking that challenge on, however.

Operators from Section 5.1. Theorem 5.1.7 made the claim that the operator $S_\varepsilon^\lambda : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ defined by

$$S_\varepsilon^\lambda \varphi(x) = \int_{-\infty}^{\infty} |t|_\varepsilon^\lambda \varphi(tx)\,dt$$

is positive when $\lambda = -\frac{1}{2}$. Our calculations show that one might even expect it to be positive for $-1 \leq \lambda \leq d-1$. Taking a step back from the case where the middle term vanishes in $\widehat{T^\lambda_\varepsilon}$, we become interested in analysis of the operator $R^\lambda_\varepsilon : \mathcal{S}_d \to \mathcal{S}'_d$ defined by

$$R^\lambda_\varepsilon \varphi(x) = \int_{-\infty}^{\infty} |t-1|^{2\operatorname{Re}\lambda - 1} |t|_\varepsilon^{-\overline{\lambda}} \varphi(tx)\, dt.$$

We have made it likely that positivity of this operator is related to the operator $\widetilde{R}^\lambda_\varepsilon : \mathcal{S}_1 \to \mathcal{S}'_1$ defined by

$$\widetilde{R}^\lambda_\varepsilon \varphi(t) = \int_{-\infty}^{\infty} |s|^{d-1} |t-s|^{2\operatorname{Re}\lambda - 1} \varphi(s)\, ds.$$

Analysis in the case $d = 1$ is easy because then $\widetilde{R}^\lambda_\varepsilon$ is a convolution operator but in general it does not seem like there is an easy way forward.

Computations for $\widetilde{\mathrm{SL}}(3, \mathbb{R})$ on $\overline{N}$. It is not impossible, merely very labour intensive, to compute the Fourier transform in the noncompact picture using the calculations of Appendix B.2. I have made some preliminary calculations of the Fourier transform and one is left with many terms. In order to carry out the analysis in this case one would need to somehow simplify the expression. At first glance, it did not seem very easy.

Appendix A

Functional Theoretic Generalities

We say that a *topological vector space* is a Hausdorff topological space X that is also a vector space such that the addition $X \times X \to X$ and scalar multiplication $\mathbb{C} \times X \to X$ are continuous. The easiest case is of course when X is a metric space. It is then natural to assume that the metric d is *translation-invariant*, i.e.,

$$d(x+t, y+t) = d(x, y).$$

In fact, the translation-invariance of d together with continuity of the scalar multiplication is enough to ensure continuity of the addition, cf. [2, Ch. III, §1]. Anyway, we say that X is an $\mathcal{F}$*-space* if it is topological vector space whose topology is defined by a translation-invariant metric with regards to which it is complete as a metric space (i.e., Cauchy sequences are convergent sequences).

Locally Convex Spaces. The simplest way of getting a translation-invariant (pseudo-)metric d is from a (semi-)norm p:

$$d(x, y) = p(x - y) \tag{A.1}$$

A $\mathcal{F}$-space whose translation-invariant metric comes from a norm like this is a *Banach space*. In general we might consider the topology on a space X generated by a collection of seminorms P, i.e., the topology is generated by the *open balls*

$$B_p(x, r) := \{y \in X \mid p(x - y) < r\} \tag{A.2}$$

for $p \in P$, $x \in X$, $r > 0$. In order to have a Hausdorff space we need to impose on P the condition that for every $x \in X$ there is some $p \in P$ with $p(x) > 0$.

One would expect p to be in some ways determined by $A = B_p(0, 1)$. Indeed one finds $p = p_A$ where

$$p_A(x) := \inf\{t > 0 \mid x/t \in A\}.$$

In general, in order for p_A to be a seminorm for arbitrary $A \subseteq X$, assumptions must be made on A: In order for the infimum to make sense for all x we must have $\bigcup_{r>0} rA = X$; in this case we say A is *absorbing*. In order to have subadditivity of p_A it is sufficient that A is *convex*, i.e., $ta + (1-t)a' \in A$ for all $a, a' \in A$, $t \in [0,1]$. In order to have homogeneity of p_A, it is sufficient that A is *balanced*, i.e., $\lambda A \subseteq A$ for $|\lambda| \leq 1$. The ball $B_p(0,1)$ has all these properties. We will say that a subset $A \subseteq X$ is *absolutely convex* if it is both balanced and convex. We have from [30, Thm. 1.35]:

Lemma A.0.1. *Suppose that A is an absolutely convex absorbing subset of a topological vector space X. Then p_A is a seminorm and*

$$B_{p_A}(0,r) \subseteq A \subseteq \overline{B_{p_A}}(0,r).$$

Here

$$\overline{B_p}(x,r) = \{y \in X \mid p(y-x) \leq r\}.$$

Suppose X is a topological vector space. A subset $U \subseteq X$ is a neighbourhood of $x \in X$ if there is some open $V \subseteq X$ such that $x \in V \subseteq U$. A *local base* for X is a collection γ of neighbourhoods of 0 such that if U is a neighbourhood of 0 then U contains some element of γ. We say that γ is (absolutely) convex if if every element in γ is (absolutely) convex. A topological vector space X is said to be *locally convex* if it has a convex local base. One can show [30, Thm 1.14] that any convex neighbourhood of 0 contains an absolutely convex neighbourhood of 0 so any locally convex space has an absolutely convex local base. Each absolutely convex neighbourhood of 0 is associated with a continuous seminorm and we find that a locally convex space is completely described by the collection of continuous seminorms.

All the spaces we will look at are locally convex. A locally convex $\mathcal{F}$-space is called a *Fréchet* space. Such a space is first countable so the absolutely convex local base can be chosen to be countable. In this way it turns out that to specify a Fréchet space it is enough to specify some countable family of seminorms on a vector space.

Completeness. In order to prove continuity it will be convenient to consider nets. A *net* in a topological vector space X is a subset $(x_\lambda)_{\lambda \in I}$ indexed by some directed set I. A nonempty set I is *directed* if it is partially ordered by some order $\leq$ and if for every $\lambda, \lambda' \in I$ there is some $\Lambda \in I$ with $\lambda, \lambda' \leq \Lambda$. A net $(x_\lambda)_\lambda$ is said to be *convergent* if there is some $x \in X$ such that for every neighbourhood U of x there is some Λ such that for all $\lambda \geq \Lambda$ we have $x_\lambda \in U$. In this case x is said to be the *limit* of the net (the Hausdorff property of X ensures uniqueness). A net (x_λ) is said to be *Cauchy* if to each neighbourhood U of 0 there is a Λ such that for $\lambda, \lambda' \geq \Lambda$ we have $x_\lambda - x_{\lambda'} \in U$. A topological vector space X is said to be *complete* if every Cauchy net in X is convergent. A space X is said to be *quasi-complete* if its closed and bounded subsets are complete.

Barreledness. The Banach-Steinhaus theorem is an important tool in functional analysis. The theorem says

Theorem A.0.2. *Suppose that X is a Banach space and Y is a normed vector space. Suppose that $T_\lambda : X \to Y$, $\lambda \in I$, is a family of continuous linear maps such that $\sup_\lambda \|T_\lambda x\| < \infty$ for all $x \in X$. Then $\sup_\lambda \|T_\lambda\| < \infty$.*

Here $\|T\|$ is the operator norm. In generalizing this to a non-normed context one needs more concepts. A subset B of a topological vector space X is said to be *bounded* if, to each neighbourhood U of 0 there is some $t \geq 0$ such that $B \subseteq tU$. In a locally convex space X it is easy to see that a subset $B \subseteq X$ is bounded if and only if $p(B) \subseteq \mathbb{R}$ is bounded for every continuous seminorm p on X. The condition $\sup_\lambda \|T_\lambda x\| < \infty$ is then naturally replaced by the condition that $\{T_\lambda x\}_\lambda$ is bounded for every $x \in X$. The condition that $C = \sup_\lambda \|T_\lambda\| < \infty$ implies that when $\|x\| < r$ we have $\|T_\lambda x\| < Cr$. This condition is naturally replace by equicontinuity. A family of maps $T_\lambda : X \to Y$, $\lambda \in I$, between topological vector spaces is said to *equicontinuous* if, to every neighbourhood $V \subseteq Y$ of 0 there is some neighbourhood $U \subseteq X$ of 0 such that $T_\lambda U \subseteq V$ holds for every λ.

We will merely assume that Y is locally convex. What condition is required on X? Suppose that $T_\lambda : X \to Y$, $\lambda \in I$, is a family of operators that is pointwise bounded, i.e., for which $\{T_\lambda x\}_\lambda$ is bounded. Let $V \subseteq Y$ be a closed absolutely convex neighbourhood of 0. Then

$$\bigcap_\lambda T_\lambda^{-1} V$$

is closed, absolutely convex, and the pointwise boundedness implies that it is also absorbing. Such a subset is said to be a *barrel*. In order to get equicontinuity one requires that it is a neighbourhood of 0. We say that a topological vector space X is *barreled* if all its barrels are neighbourhoods of 0. Then one has the uniform boundedness principle:

Theorem A.0.3. *Suppose that X is barreled and Y is locally convex. Then a family of pointwise bounded continuous operators $X \to Y$ is equicontinuous.*

$\mathcal{LF}$-spaces. Suppose that E is a vector space and that we have a family of linear subspaces E_j, $j \in I$, such that $\bigcup_j E_j = E$. Suppose that each E_j is a locally convex space. Then there is a finest topology among the topologies on E making E into a locally convex space. With this topology, E is the *inductive limit* of the E_j's. In this case, a family of maps $T_\lambda : E \to F$, $\lambda \in J$ into a locally convex space F is equicontinuous if and only if $(T_\lambda|_{E_j})_\lambda$ is equicontinuous for each j (cf. [28, Ch. V, §2, Proposition 5]).

When $I = \mathbb{N}$, $E_n \subseteq E_{n+1}$ and each E_n is a Fréchet space we will say that E is an $\mathcal{LF}$-space.

Spaces of Linear Maps. Let E, F be two locally convex spaces. Let $\mathcal{L}(E, F)$ be the set of continuous linear maps $E \to F$. We will typically give $\mathcal{L}(E, F)$ the topology of uniform convergence on the bounded subsets of E. The neighbourhood filter for this topology is given by

$$U(B,V) = \{T \in \mathcal{L}(E,F) \,|\, T(B) \subseteq V\} \tag{A.3}$$

for bounded $B \subseteq E$ and neighbourhood $V \subseteq F$ of 0. When we wish to highlight this choice of topology we will write $\mathcal{L}_b(E,F)$.

A special case that we will work with extensively is the case where $F = \mathbb{C}$. This gives us $E' = \mathcal{L}(E,\mathbb{C})$ the *dual space* of E. If one wishes to emphasize the topology given to E' one can say that this is the *strong dual* of E.

When we are looking at $\mathcal{L}(E',F)$ we will have occasion to consider another topology, namely the topology of uniform convergence on equicontinuous subsets. This is the topology on $\mathcal{L}(E',F)$ with the neighbourhood filters $U(B,V)$ in Eq. (A.3) where $B = H \subseteq E'$ is taking to be equicontinuous. The space equipped with this topology is denoted by $\mathcal{L}_\varepsilon(E',F)$.

Lemma A.0.4. *For all barreled E, the application map*

$$\mathcal{L}(E,F) \times E \to F$$

is hypocontinuous.

Proof. [5, Ch. III, §5, Exercise 12]. □

$\mathcal{DF}$-spaces. Morally, the $\mathcal{DF}$-spaces are the dual spaces of the $\mathcal{F}$-spaces. It will not be important but let us review (some of) the technicalities for completeness anyway. Let E be an $\mathcal{F}$-space and let $U_1 \supseteq U_2 \supseteq \cdots$ be a countable base for the topology. We have the corresponding *polars*

$$B_n = \{f \in E' \,|\, \forall x \in U_n,\ |\langle f,x\rangle| \leq 1\}$$

and $(B_n)_n$ turns out to give a *fundamental sequence of bounded sets* for E', i.e., a sequence of bounded sets such that every bounded subset of E' is contained in some B_n. An $\mathcal{F}$-space is barreled so a bounded subset B of E' will be equicontinuous which implies that there is some neighbourhood U of 0 in E such that $|\langle f,x\rangle| \leq 1$ for all $x \in U$, $f \in B$. We can find n such that $U_n \subseteq U$ so $B \subseteq B_n$.

A $\mathcal{DF}$-space E is a locally convex space with a fundamental sequence of bounded sets that also satisfies that every strongly bounded subset of E' which is the union of countably many equicontinuous sets is also equicontinuous. The other property here is rather technical so we will not review it in detail. What is important is that the strong dual of any $\mathcal{F}$-space is a complete $\mathcal{DF}$ and that the strong dual of any $\mathcal{DF}$-space is a $\mathcal{F}$-space.

Classes of Bilinear Maps. Suppose E, F, G are topological vector spaces. A map

$$M : E \times F \to G$$

is *bilinear* if $M(e, \cdot)$ and $M(\cdot, f)$ are linear for every e, f. M is said to be *jointly continuous* or just *continuous* if it is continuous when $E \times F$ is equipped with the product topology. B is said to be *separately continuous* if $M(e, \cdot)$ and $M(\cdot, f)$ are continuous for every e, f. In general, these are distinct properties but we do have [39, Thm 34.1]

Theorem A.0.5. *Suppose that E, F are $\mathcal{F}$-spaces and that G is locally convex. Suppose also that E is barreled. Then every separately continuous bilinear map $E \times F \to G$ is jointly continuous.*

M is said to be *hypocontinuous with regards to the bounded subsets of E* if $M(B, \cdot) = \{M(b, \cdot)\}_{b \in B}$ is equicontinuous for every bounded subset $B \subseteq E$. Likewise with regards to the bounded subsets of F. M is said to be *hypocontinuous* if it is hypocontinuous with regards to the bounded subsets of both E and F. Obviously, hypocontinuity is a property weaker than continuity but stronger than separate continuity, and

Proposition A.0.6. *Suppose that E, F, G are locally convex. If E is barreled, any separately continuous bilinear map $E \times F \to G$ is hypocontinuous with regards to the bounded subsets of F. If both E, F are barreled, any separately continuous bilinear map $E \times F \to G$ is hypocontinuous.*

Finally, for $\mathcal{DF}$-spaces there is an analogue of Theorem A.0.5, cf. [23, §40, 2. (10)]

Proposition A.0.7. *Let E, F be $\mathcal{DF}$-spaces and suppose that G is locally convex. Any hypocontinuous bilinear map $E \times F \to G$ is continuous.*

A.1 Differential Calculus

Differentiable Maps $\mathbb{R}^d \to E$. We will collect some generalities on differentiable maps here. Suppose that $U \subseteq \mathbb{R}^d$ is open and that E is a topological vector space. A map $\varphi : U \to E$ is said to be *differentiable at $x \in U$* if there is a linear map $D\varphi(x) : \mathbb{R}^d \to E$ such that

$$\lim_{h \to 0} \frac{\varphi(x+h) - \varphi(x) - D\varphi(x)h}{|h|} = 0.$$

We say that φ is *differentiable* if φ is differentiable at all $x \in U$. In this case we have a map $D\varphi : U \to \mathcal{L}(\mathbb{R}^d, E)$. As usual we make $\mathcal{L}(\mathbb{R}^d, E)$ into a topological vector space by giving it the topology of uniform convergence on bounded subsets. Since $\mathbb{R}^d$ is finite dimensional, this topology actually coincides with the topology of pointwise convergence and with the topology of uniform convergence on compact, precompact or absolutely convex compact sets, i.e., the topology on $\mathcal{L}(\mathbb{R}^d, E)$ is very natural. In general, when replacing $\mathbb{R}^d$ by another topological vector space these topologies will not coincide and a choice will have to be made, cf. [20].

Anyway, φ is *continuously differentiable* or C^1 if φ is differentiable and $D\varphi$ is continuous. Inductively, stating with the case $k = 1$ and $D^0\varphi = \varphi$ and in the case where $D^{k-1}\varphi$ is differentiable we define $D^k\varphi := D(D^{k-1}\varphi)$. Canonically, the image of this map is contained in $\mathcal{L}^k(\mathbb{R}^d, E)$ — the space of k-linear maps $(\mathbb{R}^d)^k \to E$. Again, the topology on this space is very natural and can be taken to be the topology of pointwise convergence. We then say that φ is *k-times continuously differentiable* or C^k when $D^k\varphi$ exists and is continuous.

Remark A.1.1. This definitions coincides with the definition given in [21, p. 52] in the context for Hilbert spaces. However, it is not quite the same as the definition given in [20, Def. 2.1.0] nor as the one given in [32, p. 93] nor in [39, Def. 40.1]. These definitions are all given in terms of some directional or partial derivatives, in [20] out of necessity because $\mathbb{R}^d$ is replaced by some non-normable locally convex space, in [32] the space E is locally convex and we will see that the definitions will coincide, and in [39] the space E is kept arbitrary but his aim is primarily on locally convex spaces.

Partial Derivatives. When φ is p-times continuously differentiable it is natural, for any directions $v_1, \ldots, v_p \in \mathbb{R}^d$ to consider the *partial derivatives*

$$D^p\varphi_x(v_1, \ldots, v_p).$$

When using the canonical basis vectors $e_j \in \mathbb{R}^d$ we get the canonical partial derivatives

$$\partial^p\varphi(x) = D^{|p|}\varphi_x(e_1^{p_1}, \ldots, e_d^{p_d})$$

for any tuple $p \in \mathbb{N}_0^d$. Here we understand $e_j^k = (e_j, \ldots, e_j)$, k times and for $p \in \mathbb{N}_0^d$, $|p| = \sum_j p_j$. Note that [20, Thm. 2.4.0] says that $D^p\varphi$ is symmetric so it does not matter what order we use above. We also use $\varphi^{(p)} = \partial^p\varphi$.

In particular we have the partial derivatives

$$D^1\varphi_x(v) = \lim_{h\to 0} \frac{\varphi(x + hv) - \varphi(x)}{h}. \tag{A.4}$$

The definition of differentiability in [20, Def. 1.0.0] is centered around these directional derivatives. We will say that $\varphi : U \to E$, $U \subseteq \mathbb{R}^d$ open, is *Gâteaux differentiable* if for every $x \in U$ there is a linear map $D^1\varphi(x) : \mathbb{R}^d \to E$ (necessarily continuous) such that Eq. (A.4) is satisfied for all $v \in \mathbb{R}^d$. Note this definition admits generalisation to the case where $\mathbb{R}^d$ is an arbitrary topological vector space. We say that φ is *continuously Gâteaux differentiable* if $D^1\varphi : U \to \mathcal{L}(\mathbb{R}^d, E)$ is continuous. A priori it is clear that any (continuously) differentiable map is (continuously) Gâteaux differentiable but the converse seems elusive in general.

Even more specific is the definition of differentiability given in [32, p. 93]. This is build entirely in terms of the canonical partial derivatives

$$\partial_j\varphi(x) = D^1\varphi_x(e_j) = \lim_{h\to 0} \frac{\varphi(x + he_j) - \varphi(x)}{h}.$$

We will work shortly towards proving

Proposition A.1.2. *Suppose that E is a locally convex space and that $U \subseteq \mathbb{R}^d$ is open. A function $\varphi : U \to E$ is continuously differentiable if and only if $\partial_j \varphi$ exists and is continuous for every j.*

Really, looking over the proof of the corresponding finite-dimensional case [29, Thm. 9.21] we see that what is needed is a Mean Value Theorem. Luckily we have one available:

Proposition A.1.3. *Suppose that E is locally convex and suppose that $a, b \in \mathbb{R}$, $a \leq b$. Let $\varphi : [a, b] \to E$ be continuous with $\varphi|_{(a,b)}$ differentiable. Then*

$$\frac{\varphi(b) - \varphi(a)}{b - a} \in \overline{\mathrm{cx}}\{\varphi'(t) \mid t \in [a, b]\}.$$

Here $\overline{cx}$ denotes taking the closed convex hull in the weak topology.

Remark A.1.4. In [1, Example 1.23] it is remarked local convexity is essential for this theorem and a counterexample is provided with E consisting of all measurable functions on $[0, 1]$ equipped with the topology of convergence in measure. In this example, $E' = 0$, cf. [18, 6.10 J]. In the case where E' separates points we can equip E with the weak topology which will make it a locally convex space E_s and the identity $E \to E_s$ is continuous. So the Mean Value Theorem actually holds in this case if we take the weak closure instead of the closure.

Proof. Set

$$C := \overline{\mathrm{cx}}\{\varphi'(t) \mid t \in [a, b]\}.$$

If the theorem is false, Hahn-Banach [30, Thm. 3.4] gives us a linear functional $e' \in E'$ such that

$$\mathrm{Re}\left\langle \frac{\varphi(b) - \varphi(a)}{b - a}, e' \right\rangle < \mathrm{Re}\langle c, e' \rangle$$

for all $c \in C$. But the scalar Mean Value Theorem gives us a $t \in [a, b]$ such that

$$\langle \varphi'(t), e' \rangle = \left\langle \frac{\varphi(b) - \varphi(a)}{b - a}, e' \right\rangle$$

which leads us to the absurd

$$\mathrm{Re}\langle \varphi'(t), e' \rangle < \mathrm{Re}\langle \varphi'(t), e' \rangle$$

by taking $c = \varphi'(t)$. □

Theorem A.1.5. *Suppose that E is locally convex and $U \subseteq \mathbb{R}^d$ is open. Suppose $\varphi : U \to E$ such that $\partial_j \varphi$ exists and is continuous for all j. Then φ is continuously differentiable.*

This is the argument from [29, Thm 9.21]:

Proof. Let $x \in U$ and write $h \in \sum_{j=1}^{n} h_j e_j$. Let $\delta > 0$ be so small that $|h_j| < \delta$ for all j implies that $v_k = \sum_{j=1}^{k} h_j e_j \in U$. We claim that as $h \to 0$,

$$\frac{1}{|h|}\left(\varphi(x+h) - \varphi(x) - \sum_{j=1}^{n} h_j \partial_j \varphi(x)\right) \to 0.$$

For $|h_j| < \delta$ we can write

$$\begin{aligned}\varphi(x+h) - \varphi(x) - &\sum_{j=1}^{n} h_j \partial_j \varphi(x) = \\ &\sum_{j=1}^{n}\big(\varphi(x+v_{j-1}+h_j e_j) - \varphi(x+v_{j-1}) - h_j \partial_j \varphi(x)\big).\end{aligned}$$

Note that $[0,1] \ni t \mapsto \varphi(x+v_{j-1}+th_j e_j) - th_j \partial_j \varphi(x)$ is differentiable so the Mean Value Theorem gives

$$\varphi(x+v_{j-1}+h_j e_j) - \varphi(x+v_{j-1}) - h_j \partial_j \varphi(x) \in \overline{cx}\{h_j(\partial_j \varphi(x+v_j) - \partial_j \varphi(x))\}.$$

Let $V \subseteq E$ be some neighbourhood of 0. Since E is locally convex we may assume that V is closed and absolutely convex so that if we take δ so small that $|h_j| < \delta$ implies that

$$\partial_j \varphi(x+v_j) - \partial_j \varphi(x) \in V$$

then we get

$$\frac{\varphi(x+h) - \varphi(x) - \sum_j h_j \partial_j \varphi(x)}{|h|} \in \sum_j \frac{h_j}{|h|} V \subseteq \sum_j V.$$

which concludes the proof. □

Corollary A.1.6. *Suppose that E is locally convex and $U \subseteq \mathbb{R}^d$ open. Then $\varphi \in C^m(U,E)$ if and only if $\partial^p \varphi$ exists and is continuous for all $|p| = m$.*

The norm $|\cdot|$ is a multiindex $p \in \mathbb{N}_0^d$ is defined as

$$|p| = \sum_j p_j.$$

Pairing with the Dual. Suppose that $U \subseteq \mathbb{R}^d$ is open and E is a topological vector space. For every $\varphi : U \to \mathbb{R}^d$ and every $e' \in E'$ we have a corresponding map $\langle \varphi, e' \rangle : U \to \mathbb{C}$ given by

$$\langle \varphi, e' \rangle(x) := \langle \varphi(x), e' \rangle.$$

If $\langle \varphi, e' \rangle$ is m-times continuously differentiable for all e' we say that φ is *scalarly* m-times continuously differentiable. It is clear that if φ is m-times continuously differentiable then φ is scalarly m-times continuously differentiable. In fact, there is a converse due to Grothendieck [14, Ch. 3, §8, Prop. 15, Corollary 1]:

Proposition A.1.7. *Suppose that E is a quasi-complete locally convex space and that $U \subseteq \mathbb{R}^d$ is open. If $\varphi : U \to E$ is scalarly m-times continuously differentiable then the partial derivatives of φ up to degree $m-1$ exist and are continuous.*

Remark A.1.8. Grothendieck only works with partial derivatives, hence the somewhat cumbersome formulation.

If we combine the results up to know we find

Theorem A.1.9. *Suppose that E is a quasi-complete locally convex space and $U \subseteq \mathbb{R}^d$ open. If $\varphi : U \to E$ satisfies that the degree m partial derivatives of $\langle \varphi, e' \rangle$ exist and are continuous for all $e' \in E'$ then φ is $(m-1)$-times continuously differentiable. Consequently, φ is smooth if and only if all the partial derivatives of $\langle \varphi, e' \rangle$ exist for all $e' \in E'$.*

Leibniz' Rule. Suppose that $\varphi : U \to E$ and $\psi : U \to F$ are both m-times continuously differentiable and suppose that $\Lambda : E \times F \to G$ is hypocontinuous. Then we define $\Lambda(\varphi, \psi) : U \to G$ by $\Lambda(\varphi, \psi)(x) = \Lambda(\varphi(x), \psi(x))$. We find

Theorem A.1.10. *The function $\Lambda(\varphi, \psi)$ is m-times continuously differentiable and we have Leibniz' formula*

$$\Lambda(\varphi, \psi)^{(n)}(x) = \sum_{k \leq n} \binom{n}{k} \varphi^{(k)}(x) \psi^{(n-k)}(x)$$

for every multiindex n with $|n| \leq m$.

Here the relation $k \leq n$ for $k, n \in \mathbb{N}_0^d$ means that $k_j \leq k_j$ for all j. Also, $n - k$ is to be understood pointwise; $(n-k)_j = n_j - k_j \in \mathbb{N}_0$ if $k \leq n$.

Remark A.1.11. This generalises for example [20, Prop. A.1.7].

Proof. There are linear maps $D\varphi_x : \mathbb{R}^d \to E$, $D\psi_x : \mathbb{R}^d \to F$ such that

$$\frac{\varphi(x+h) - \varphi(x) - D\varphi_x h}{|h|} \to 0$$
$$\frac{\psi(x+h) - \psi(x) - D\psi_x h}{|h|} \to 0$$

as $h \to 0$. Note that $h \mapsto \Lambda(D\varphi_x h, \psi(x)) + \Lambda(\varphi(x), D\psi_x h)$ is a linear map and

$$\frac{\Lambda(\varphi(x+h), \psi(x+h)) - \Lambda(\varphi(x), \psi(x)) - \Lambda(D\varphi_x h, \psi(x)) + \Lambda(\varphi(x), D\psi_x h)}{|h|}$$
$$= \Lambda\left(\frac{\varphi(x+h) - \varphi(x) - D\varphi_x h}{|h|}, \psi(x+h) \right) + \Lambda\left(\frac{D\varphi_x h}{|h|}, \psi(x+h) - \psi(x) \right)$$
$$+ \Lambda\left(\varphi(x), \frac{\psi(x+h) - \psi(x) - D\psi_x h}{|h|} \right)$$

which goes to 0 as $h \to 0$ by hypocontinuity: As $h \to 0$ it will in particular be bounded so $\{\psi(x+h)\}_h$ and $\{|h|^{-1}D\varphi_x h\}_h$ will be bounded.

Thus one concludes that $\Lambda(\varphi,\psi)$ is differentiable and that its differential is $h \mapsto \Lambda(D\varphi_x h, \psi(x)) + \Lambda(\varphi(x), D\psi_x h)$. Note here that Λ induces a hypocontinuous linear map $\mathcal{L}(\mathbb{R}^d, E) \times F \to \mathcal{L}(\mathbb{R}^d, G)$ so we can apply the above reasoning if $D\varphi$ is differentiable to see that $\Lambda(D\varphi,\psi)$ is differentiable, too. Continuing this way we find, if φ, ψ are n-times differentiable that $\Lambda(\varphi,\psi)$ is n-times differentiable and that the n'th derivative is

$$D^n\Lambda(\varphi,\psi)_x(h_1,\dots,h_n) = \sum_{j=0}^{n}\binom{n}{j}\Lambda(D^j\varphi_x(h_1,\dots,h_j), D^{n-j}\psi_x(h_{j+1},\dots,h_n))$$

which the understanding that $D^0\varphi_x = \varphi(x)$.

Expressed in partial derivatives this is exactly the formula that we want. □

Taylor Expansion. We will need to use Taylor expansion of some vector-valued functions.

Proposition A.1.12. *Suppose that E is a quasi-complete locally convex space and $U \subseteq \mathbb{R}^d$ is open. Suppose that $\varphi : U \to E$ is m-times continuously differentiable. Then when $x + [0,1]h \subseteq U$ we have*

$$\varphi(x+h) = \sum_{|\alpha|\le m-1}\frac{\partial^\alpha\varphi(x)}{\alpha!}h^\alpha + \sum_{|\alpha|=m}\frac{m}{\alpha!}h^\alpha\int_0^1(1-t)^{m-1}\partial^\alpha\varphi(x+th)\,dt.$$

Remark A.1.13. The integral makes sense according to [6, III, §3, no. 3, Prop. 7, Cor. 2].

Proof. When paired with any continuous linear functional this will hold. Since E' separates points we have the theorem. □

A.2 Examples of Locally Convex Spaces

Suppose that X is a set. When $f : X \to \mathbb{C}$ we use

$$\|f\|_\infty := \sup_{x\in X}|f(x)|.$$

Suppose that E is a topological space. When $f : X \to E$ is a map and p is a seminorm on E we use

$$\|f\|_p := \|p \circ f\|_\infty = \sup_{x\in X} p(f(x)).$$

The Vector-Valued Smooth Functions $\mathcal{E}(U,E)$. Let E be some locally convex space. We let $\mathcal{E}(U,E)$ be the vector space of smooth functions $U \to E$. We give $\mathcal{E}(U,E)$ the topology of uniform convergence of all derivatives on all compacts in U. So then $\varphi_\lambda \to \varphi$ if and only if $\partial^p \varphi_\lambda \to \partial^p \varphi$ uniformly on every compact $K \subseteq U$ and for every tuple p.

Suppose that $K \subseteq U$ is compact, $n \in \mathbb{N}_0$ and q is a continuous seminorm on E. Then we get a seminorm

$$p_{q,n,K}(\varphi) := \max_{|k|\le n} \|\varphi^{(k)}|_K\|_q = \max_{|k|\le n} \sup_{x\in K} q(\varphi^{(k)}(x))$$

on $\mathcal{E}(U,E)$ which defines its structure as a locally convex space. When $E = \mathbb{C}$ we let q be the usual norm and we use $p_{n,K} = p_{q,n,K}$.

The Vector-Valued Schwartz Functions. Let E be some locally convex space. We take $\mathcal{S}(\mathbb{R}^d, E) \subseteq \mathcal{E}(\mathbb{R}^d, E)$ to be the set of smooth functions φ such that whenever p is some continuous seminorm on E,

$$\|\varphi\|_{N,p} = \max_{|n|\le N} \sup_{x\in\mathbb{R}^d} p((1+|x|^2)^N \partial^n \varphi(x)) < \infty.$$

Then $\mathcal{S}(\mathbb{R}^d, E)$ is equipped with the topology induced by the norms $\|\cdot\|_{N,p}$. When $E = \mathbb{C}$ we let p be the usual norm and use $\|\cdot\|_N = \|\cdot\|_{N,p}$. For $E = \mathbb{C}$ we just write $\mathcal{S}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d, \mathbb{C})$. Also we will occasionally use the shorthand $\mathcal{S}_d$ for $\mathcal{S}(\mathbb{R}^d)$.

The Vector-Valued Test Functions $\mathcal{D}^N(U,E)$. Let E be some locally convex space and let $N \in \mathbb{N}_0 \cup \{\infty\}$. We let $\mathcal{D}^N(U,E)$ be the vector space of all functions that are N times continuously differentiable $U \to E$ with compact support (when $N = \infty$ this means smooth). For each compact $K \subseteq U$ let $\mathcal{D}_K^N(U,E)$ be the subspace of functions φ with $\operatorname{supp}\varphi \subseteq K$. The $\mathcal{D}_K(U,E)$ is given the topology of uniform convergence of all derivatives. We can equip it with the norms

$$\|\varphi\|_{p,n} = \max_{|m|\le n} \|\partial^m \varphi\|_p = \max_{|m|\le n} \sup_{x\in U} p(\partial^m \varphi(x)).$$

We then give $\mathcal{D}^N(U,E)$ the finest locally convex topology such that all the inclusions $\mathcal{D}_K^N(U,E) \to \mathcal{D}^N(U,E)$ are continuous. When $N = \infty$ we use the notation $\mathcal{D}(U,E)$ instead. For $E = \mathbb{C}$ we use the notation $\|\cdot\|_n$ instead.

The Vector-Valued Distributions $\mathcal{D}'(U,E)$. Let E be some locally convex space. We define $\mathcal{D}'(U,E) := \mathcal{L}(\mathcal{D}(U), E)$ to be the set of distributions with values in E. When E is complete there is a continuous linear injection $\mathcal{E}(U,E) \to \mathcal{D}'(U,E)$ defined by

$$\Phi(\psi) = \int \psi(x)\varphi(x)\,dx \in E$$

for $\Phi \in \mathcal{E}(U, E)$, $\psi \in \mathcal{D}(U)$. This integral is to be understood in the weak sense so that

$$\langle \Phi(\psi), e' \rangle = \int \psi(x) \langle \varphi(x), e' \rangle \, dx.$$

for all $e' \in E'$. Indeed, according to [6, III.39, No. 4, Prop. 8] the integral is continuous $\mathcal{D}(U, E) \to E$ and the multiplication

$$\mathcal{D}(U) \times \mathcal{E}(U, E) \to \mathcal{E}(U, E)$$

is hypocontinuous.

The Multiplier Space $\mathcal{O}_M(\mathbb{R}^d, E)$. Let $\mathcal{O}_M(\mathbb{R}^d, E) \subseteq \mathcal{E}(\mathbb{R}^d, E)$ be the subset of functions Φ such that $\varphi \cdot \Phi^{(n)}$ is bounded for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and all n. This space is given the topology such that $\Phi_\lambda \to 0$ when $\varphi \cdot \Phi_\lambda^{(n)} \to 0$ uniformly. For $E = \mathbb{C}$ we write $\mathcal{O}_M(\mathbb{R}^d) = \mathcal{O}_M(\mathbb{R}^d, \mathbb{C})$. In [17, 4, §11, Proposition 5] it is proven that $\mathcal{O}_M(\mathbb{R}^d)$ is exactly the subset of smooth functions Φ such that for every n there is some $m \in \mathbb{N}_0$ and some $c > 0$ such that

$$|\Phi^{(n)}(x)| \leq c|x|^m.$$

A.3 Tensor Products

Suppose that E, F are locally convex spaces. We shall work with three topologies on the tensor product $E \otimes F$. The *inductive* resp. *projective* topology is the finest locally convex topology such that the natural bilinear map $E \times F \to E \otimes F$ is separately continuous resp. jointly continuous. Let $E \otimes_i F$ resp. $E \otimes_\pi F$ be the space $E \otimes F$ with the inductive resp. the projective topology. Let $E \mathbin{\overline{\otimes}} F$ resp. $E \mathbin{\widehat{\otimes}} F$ be their completions. They are both equipped with a universal property: Any separately continuous resp. jointly continuous bilinear map on $E \times F$ descends to give a continuous linear map on $E \mathbin{\overline{\otimes}} F$ resp. $E \mathbin{\widehat{\otimes}} F$. The third topology will be identical to the projective one for most of our examples so we will not go much into detail. The space $E \otimes F$ can be identified with $B(E'_s, F'_s)$ — the space of *continuous* bilinear maps on $E'_s \times F'_s$ where E'_s, F'_s are the dual spaces given the topology of *pointwise* convergence, cf. [39, Proposition 42.5]. This space can then be given the topology of uniform convergence on products of equicontinuous subsets of E'_s and F'_s. This topology on $E \otimes F$ is called the *injective* topology. Given this topology the space is written $E \otimes_\varepsilon F$.

Nuclear Spaces. A locally convex space E is called *nuclear* if for every locally convex space F we have $E \otimes_\pi F = E \otimes_\varepsilon F$.

Since the projection $E \times F \to E \mathbin{\widehat{\otimes}} F$ is continuous, it is in particular separately continuous so there is a continuous linear map

$$\Gamma : E \mathbin{\overline{\otimes}} F \to E \mathbin{\widehat{\otimes}} F.$$

It will be convenient to know the following ([15, Ch. 1, §1, no. 3, Proposition 5 and Corollaire 1]):

Proposition A.3.1. *If both E and F are $\mathcal{F}$-spaces then $E \otimes_\pi F$ is barreled. If both E and F are barreled $\mathcal{DF}$-spaces then $E \otimes_\pi F$ is barreled.*

There is a map $E' \times F \to \mathcal{L}(E, F)$, $e' \times f \mapsto e' \otimes f$ given by $e' \otimes f(e) = \langle e', e \rangle f$. This map is always continuous so by the universal property we get a continuous linear map

$$E' \otimes_\pi F \to \mathcal{L}(E, F)$$

the image of which consist of all finite-rank operators in $\mathcal{L}(E, F)$. There are problems with extending this map to $E' \widehat{\otimes} F$ in general because $\mathcal{L}(E, F)$ is not necessarily complete. However, we have ([39, Proposition 50.5]):

Proposition A.3.2. *Suppose that E, F are complete locally convex spaces. Suppose additionally that E is barreled, and that E' is nuclear and complete. Then $\mathcal{L}(E, F)$ is complete, and we have*

$$E' \widehat{\otimes} F \cong \mathcal{L}(E, F).$$

Examples of Tensor Products. Many of the spaces introduced above have a representation as a tensor product.

Vector-Valued Smooth Functions. Suppose $U \subseteq \mathbb{R}^d$ is open, and E is a complete locally convex space. For $\varphi \in \mathcal{E}(U)$ and $\psi \in E$ we have $\varphi \otimes \psi \in \mathcal{E}(U, E)$ defined by

$$\varphi \otimes \psi(x) = \varphi(x)\psi.$$

This gives us a map $\mathcal{E}(U) \otimes E \to \mathcal{E}(U, E)$.

We have, cf. [39, p. 533]:

$$\mathcal{E}(U, E) \cong \mathcal{E}(U) \widehat{\otimes} E.$$

Vector-Valued Test Functions. Again we have a map as above $\mathcal{D}(U) \otimes E \to \mathcal{D}(U, E)$. Grothendieck shows in [15, Chapitre II, §3, no. 3, Théorème 13, Examples 4 (Ch. II, p. 84)] that if E is an $\mathcal{F}$-space then

$$\mathcal{D}(U, E) \cong \mathcal{D}(U) \overline{\otimes} E.$$

Vector-Valued Schwartz Functions. [39, p. 533] tells us that

$$\mathcal{S}(\mathbb{R}^d, E) \cong \mathcal{S}(\mathbb{R}^d) \widehat{\otimes} E$$

whenever E is a complete locally convex space.

Vector-Valued Distributions. It follows from Theorem A.3.2 that for any complete locally convex E,

$$\mathcal{D}'(U, E) \cong \mathcal{D}'(U) \widehat{\otimes} E.$$

Schwartz Kernel Theorem. Again, Theorem A.3.2 and the fact that $\mathcal{S}_n \mathbin{\widehat{\otimes}} \mathcal{S}_m \cong \mathcal{S}_{n+m}$ gives us

$$\mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^m)) \cong \mathcal{S}(\mathbb{R}^m) \mathbin{\widehat{\otimes}} \mathcal{S}(\mathbb{R}^n) \cong \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n).$$

This is known as the Schwartz Kernel Theorem. We will introduce some notation for this theorem. Suppose that $K \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n)$ is a kernel. Then the corresponding operator $T_K : \mathcal{S}_n \to \mathcal{S}_m$ is defined by

$$\langle T_K\varphi, \psi\rangle = \langle K, \psi \otimes \varphi\rangle$$

for $\varphi \in \mathcal{S}_n$, $\psi \in \mathcal{S}_m$.

Multiplier Space. According to [32, Prop. 10], if E is a complete locally convex space,

$$\mathcal{O}_M(\mathbb{R}^d, E) \cong \mathcal{O}_M(\mathbb{R}^d) \mathbin{\widehat{\otimes}} E.$$

Appendix B

An $\mathrm{SL}(2,\mathbb{R})$-Trick in $\widetilde{\mathrm{SL}}(3,\mathbb{R})$

In this chapter we compute the $\overline{N}MAN$-decomposition for some elements in $\widetilde{\mathrm{SL}}(3,\mathbb{R})$ that are relevant to the Knapp-Stein intertwiner. These computations have not been polished and might appear rough.

B.1 The $\overline{N}MAN$-Decomposition in $\widetilde{\mathrm{SL}}(2,\mathbb{R})$

First we consider the group $\mathrm{SL}(2,\mathbb{R})$. We have the important subgroup $M = \{e,\sigma\}$ where $e \in G$ is the identity element and $\sigma = -e$. For $\varepsilon \in \{-1,1\}$ we let $m(\varepsilon) = \varepsilon e$. Now, we get

Proposition B.1.1. *For all $g \in \mathrm{SL}(2,\mathbb{R})$,*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1,$$

for which $a \neq 0$ we have

$$g = \overline{n}(c/a)m(\mathrm{sgn}(a))a(|a|)n(b/a).$$

We use the explicit construction of the double covering $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ from [38, I, 2] to obtain the $\overline{N}MAN$-decomposition for this group. Note that $\mathrm{SL}(2,\mathbb{R})$ acts on the upper half plane ($\Im z > 0$) by

$$gz = \frac{az+b}{cz+d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The double cover can be realised as the set of pairs (g,Φ) where $g \in \mathrm{SL}(2,\mathbb{R})$ and Φ is a holomorphic function on the upper half plane such that

$$\Phi(z)^2 = cz + d, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The composition in $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is given by

$$(g_1,\Phi_1)(g_2,\Phi_2) = (g_3,\Phi_3)$$

where $g_3 = g_1 g_2$ and

$$\Phi_3(z) = \Phi_1(g_2 z)\Phi_2(z).$$

We use the definitions from [38] with

$$\begin{aligned}
\overline{n}(t) &= y(t) \\
a(s) = h(s) &= \left(\begin{pmatrix} s & 0 \\ 0 & 1/s \end{pmatrix}, s^{-1/2} \right) \\
n(t) &= x(t)
\end{aligned}$$

for $t \in \mathbb{R}$, $s > 0$, and additionally we define

$$m(u) := \left(\begin{pmatrix} u^2 & 0 \\ 0 & u^2 \end{pmatrix}, u \right)$$

for $u \in \{1, i, -1, -i\}$. We also use the convention that, for $z \in \mathbb{C}$, $z^{1/2}$ is the square root defined by

$$z^{1/2} = r^{1/2} e^{i\theta/2}$$

when $z = re^{i\theta}$, $r > 0$, $-\pi < \theta \leq \pi$.

Proposition B.1.2. *For* $(g, \Phi) \in \widetilde{\mathrm{SL}}(2,\mathbb{R})$*,*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1$$

with $a \neq 0$ *we have*

$$g = \overline{n}(c/a) m(u) a(|a|) n(b/a)$$

where $u \in \{1, i, -1, -i\}$ *is chosen such that* $u^2 = \mathrm{sgn}(a)$ *and*

$$\Phi(z) = u \cdot u(a, c, d)(cz + d)^{1/2}$$

where

$$u(a, c, d) = \begin{cases} +1 & a > 0 \\ -i & a < 0, c > 0 \\ -i & a < 0, c = 0, d < 0 \\ +i & a < 0, c < 0 \\ +i & a < 0, c = 0, d > 0. \end{cases}$$

B.2 $\overline{N}MAN$-Decomposition of $w_{13}^{-1}\overline{n}(x, y, z)$

Using the above we find

$$\begin{aligned}
w_{13}^{-1}\overline{n}(x, y, z) &= w_{13}^{-1}\overline{n}_{13}(z)\overline{n}_{12}(x)\overline{n}_{23}(y) \\
&= \overline{n}_{13}(-z^{-1}) m(u_1) a_{13}|z| n_{13}(z^{-1}) \overline{n}_{12}(x) \overline{n}_{23}(y)
\end{aligned}$$

where $u_1^2 = \operatorname{sgn}(-z) = -\operatorname{sgn}(z)$ and

$$u_1 = \begin{cases} +1 & z < 0 \\ +i & z > 0. \end{cases}$$

The idea is to use commutation rules arising from Lie algebra identities to move around in the equation until we have an $\overline{N}MAN$-decomposition.

Commutators. For a, b in any group we let

$$[a, b] := aba^{-1}b^{-1}.$$

There is another possible convention but this corresponds nicely with the ring commutator $[A, B] = AB - BA$. The commutator gives us the cost of making elements commute. Indeed,

$$ba[a^{-1}, b^{-1}] = ab = [a, b]ba.$$

We will need some commutators in $\widetilde{G}$, among others the commutators of $\overline{n}_{ij}$ and n_{kl} for certain i, j, k, l. Let us start with $(i, j) = (k, l)$:

Lemma B.2.1. *For all $s, t \in \mathbb{R}$,*

$$n(s)\overline{n}(t) = \overline{n}\left(\frac{t}{1+ts}\right)m(u)a|1+ts|n\left(\frac{s}{1+ts}\right)$$

where

$$u = \begin{cases} +1 & 1+ts > 0 \\ i & 1+ts < 0, t > 0 \\ -i & 1+ts < 0, t \leq 0. \end{cases}$$

In order to rapidly generate more commutators we shall note some results about certain automorphisms of $\widetilde{G}$. First there is the Cartan involution Θ uniquely given by its differential

$$\theta(X) = -X^t.$$

This implies that Θ preserves each of the embedded copies of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ given by the images of j_{kl}, and Θ restricts to the Cartan involution of the subgroup.

In particular we have

$$\begin{aligned} \Theta(n_{ij}(t)) &= \overline{n}_{ij}(-t) \\ \Theta(a_{ij}(t)) &= a_{ij}(t^{-1}) \\ \Theta(k_{ij}(t)) &= k_{ij}(t) \\ \Theta(m_{ij}(u)) &= m_{ij}(u). \end{aligned}$$

We consider

$$\widetilde{w} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R})$$

and for $n \geq 3$, $1 \leq i < j \leq n$, $\widetilde{w}_{ij} = j_{ij}(\widetilde{w})$. Note that $\mathrm{GL}(n,\mathbb{R})$ acts by conjugation on $\mathrm{SL}(n,\mathbb{R})$ so this action ascends to the universal covers which is equal to the double cover for $n \geq 3$. For $n = 2$ we note that conjugation by $\widetilde{w}$ sends the generator of π_1 into its inverse. Indeed,

$$\widetilde{w}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\widetilde{w}^{-1} = \begin{pmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta). \end{pmatrix}$$

so the action of $\widetilde{w}$ ascends to any m-fold cover of $\mathrm{SL}(2,\mathbb{R})$. In particular, it ascends to the double cover. Concretely, it seems the action is realised as

$$(g,\Phi) \mapsto (c_{\widetilde{w}}g, \left(\frac{bz+a}{dz+c}\right)^{1/2}\Phi(z^{-1})z^{1/2}), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The action of $\widetilde{w}$ on $\widetilde{\mathrm{SL}}(n,\mathbb{R})$ is given uniquely by its differential which is equal to the differential of the action on $\mathrm{SL}(n,\mathbb{R})$, i.e., it is equal to $\mathrm{Ad}(\widetilde{w})$. From this fact we can make conclusions about $c_{\widetilde{w}}$ on $\widetilde{\mathrm{SL}}(3,\mathbb{R})$. It is not difficult to see that

$$\begin{aligned} \mathrm{Ad}(\widetilde{w}_{13})dj_{13} &= dj_{13}\,\mathrm{Ad}(\widetilde{w}) \\ \mathrm{Ad}(\widetilde{w}_{13})dj_{12} &= dj_{23}\,\mathrm{Ad}(\widetilde{w}) \end{aligned}$$

the first equation being obvious and the second found by computation. This implies that

$$\begin{aligned} c_{\widetilde{w}_{13}}n_{13}(s) &= \overline{n}_{13}(s) \\ c_{\widetilde{w}_{13}}n_{12}(t) &= \overline{n}_{23}(t) \\ c_{\widetilde{w}_{13}}n_{23}(x) &= \overline{n}_{12}(x). \end{aligned}$$

Likewise we have

$$\begin{aligned} \mathrm{Ad}(\widetilde{w}_{12})dj_{12} &= dj_{12}\,\mathrm{Ad}(\widetilde{w}) \\ \mathrm{Ad}(\widetilde{w}_{12})dj_{13} &= dj_{23} \end{aligned}$$

giving

$$\begin{aligned} c_{\widetilde{w}_{12}}n_{13}(s) &= n_{23}(s) \\ c_{\widetilde{w}_{12}}n_{12}(t) &= \overline{n}_{12}(t) \\ c_{\widetilde{w}_{12}}n_{23}(x) &= n_{13}(x). \end{aligned}$$

Lastly, we have

$$\begin{aligned} \mathrm{Ad}(\widetilde{w}_{23})dj_{23} &= dj_{23}\,\mathrm{Ad}(\widetilde{w}) \\ \mathrm{Ad}(\widetilde{w}_{23})dj_{13} &= dj_{12} \end{aligned}$$

so

$$\begin{aligned} c_{\widetilde{w}_{23}}n_{13}(s) &= n_{12}(s) \\ c_{\widetilde{w}_{23}}n_{12}(t) &= n_{13}(t) \\ c_{\widetilde{w}_{23}}n_{23}(x) &= \overline{n}_{23}(x). \end{aligned}$$

Lemma B.2.2. *For all* $s, t \in \mathbb{R}$,

$$[\overline{n}_{12}(x), n_{13}(z)] = n_{23}(xz)$$
$$[\overline{n}_{12}(x), n_{23}(y)] = e$$
$$[\overline{n}_{23}(y), n_{13}(z)] = n_{12}(-yz)$$

Proof. In the Heisenberg group of course we have

$$[n_{12}(x), n_{23}(y)] = n_{13}(xy)$$

Applying $\widetilde{w}_{12}$ and $\widetilde{w}_{23}$ gives, in order,

$$[\overline{n}_{12}(x), n_{13}(y)] = n_{23}(xy)$$
$$[n_{13}(x), \overline{n}_{23}(y)] = n_{12}(xy).$$

Also in the Heisenberg group we have

$$[n_{12}(x), n_{13}(z)] = e$$

so applying $\widetilde{w}_{12}$ gives

$$[\overline{n}_{12}(x), n_{23}(z)] = e.$$

□

Proof. The one-parameter subgroup $s \mapsto c_{\overline{n}_{12}(t)} n_{13}(s)$ has differential at 0

$$\mathrm{Ad}(\overline{n}_{12}(t))X_{13} = e^{t\,\mathrm{ad}\,Y_{12}} X_{13} = X_{13} + tX_{23}$$

because $[Y_{12}, X_{13}] = X_{23}$ and $[Y_{12}, X_{23}] = 0$. Since $[X_{13}, X_{23}] = 0$ we conclude

$$\overline{n}_{12}(t) n_{13}(s) \overline{n}_{12}(-t) = n_{13}(s) n_{23}(st)$$

which is what we want after replacing s by $-s$.

Using this on the found commutator relation we have

$$[\overline{n}_{13}(s), n_{23}(t)] = \overline{n}_{12}(-st).$$

Applying the Cartan involution gives us

$$[n_{13}(-s), \overline{n}_{23}(-t)] = n_{12}(st).$$

We can replace s, t by $-s, -t$ to get the result. □

So we have

$$\overline{n}_{13}(-z^{-1}) m(u) a_{13}|z| \overline{n}_{12}(x) n_{13}(z^{-1}) n_{23}(-z^{-1}x) \overline{n}_{23}(y)$$

Lemma B.2.3. *For all* $t > 0$, $s \in \mathbb{R}$,

$$a_{13}(t)\overline{n}_{12}(s) = \overline{n}_{12}(st^{-1}) a_{13}(t)$$
$$a_{13}(t)\overline{n}_{23}(s) = \overline{n}_{23}(st^{-1}) a_{13}(t)$$
$$a_{23}(t) n_{12}(s) = n_{12}(st^{-1}) a_{23}(t)$$
$$a_{23}(t) n_{13}(s) = n_{13}(st) a_{23}(t)$$

Proof. Note first that $[H_{13}, Y_{12}] = -Y_{12}$ so

$$e^{t\,\mathrm{ad}\,H_{13}}Y_{12} = e^{-t}Y_{12}.$$

This means that the one-parameter subgroup $s \mapsto c_{a_{13}(t)}\overline{n}_{12}(s)$ has differential at 0

$$\mathrm{Ad}(a_{13}(t))Y_{12} = e^{\log t\,\mathrm{ad}\,H_{13}}Y_{12} = t^{-1}Y_{12}$$

so that

$$a_{13}(t)\overline{n}_{12}(s)a_{13}(t)^{-1} = \overline{n}_{12}(st^{-1}).$$

Applying $c_{\widetilde{w}_{13}}$ we get

$$a_{13}(t)^{-1}n_{23}(s)a_{13}(t) = n_{23}(st^{-1})$$

which after Cartan involution is

$$a_{13}(t)\overline{n}_{23}(-s)a_{13}(t)^{-1} = \overline{n}_{23}(-st^{-1}).$$

Note that

$$\begin{aligned}\mathrm{Ad}(\widetilde{w}_{12})dj_{12} &= dj_{12}\,\mathrm{Ad}(\widetilde{w})\\ \mathrm{Ad}(\widetilde{w}_{12})dj_{13} &= dj_{23}\end{aligned}$$

so applying $c_{\widetilde{w}_{12}}$ instead we have

$$a_{23}(t)n_{12}(-s)a_{23}(t)^{-1} = n_{12}(-st^{-1}).$$

Lastly,

$$\begin{aligned}\mathrm{Ad}(\widetilde{w}_{23})dj_{23} &= dj_{23}\,\mathrm{Ad}(\widetilde{w})\\ \mathrm{Ad}(\widetilde{w}_{23})dj_{13} &= dj_{12}\end{aligned}$$

so applying $c_{\widetilde{w}_{23}}$ to this last equation we have

$$a_{23}(t)^{-1}n_{13}(-s)a_{23}(t) = n_{12}(-st^{-1})$$

□

$$\overline{n}_{13}(-z^{-1})m(u)\overline{n}_{12}(x|z|^{-1})a_{13}|z|n_{13}(z^{-1})n_{23}(-z^{-1}x)\overline{n}_{23}(y)$$

Moving $\overline{n}$ past $m(u)$ will depend on u, i.e., on the sign of z so let us move the other $\overline{n}$'s first. The above decomposition lemma tells us that

$$n(-z^{-1}x)\overline{n}(y) = \overline{n}\left(\frac{zy}{z-xy}\right)m(u)a|1-xy/z|n\left(\frac{x}{xy-z}\right)$$

where $u^2 = \mathrm{sgn}(1-xy/z)$ and $(1+tz)^{1/2} = u(u^2tz+u^2)^{1/2}$. so we get

Lemma B.2.4. *For all $s \in \mathbb{R}$, $u \in \{1, i, -1, -i\}$,*

$$\begin{aligned} m_{13}(u)\overline{n}_{12}(s) &= \overline{n}_{12}(u^2 s)m_{13}(u) \\ m_{13}(u)\overline{n}_{23}(s) &= \overline{n}_{23}(u^2 s)m_{13}(u) \\ m_{23}(u)n_{12}(s) &= n_{12}(u^2 s)m_{23}(u) \\ m_{23}(u)n_{13}(s) &= n_{13}(u^2 s)m_{23}(u). \end{aligned}$$

Proof. The space spanned by (Y_{12}, X_{23}) is invariant under $\mathrm{ad}(X_{13} - Y_{13})$ and this map in matrix form is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The (time-dependent) matrix exponential of this is

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

It follows that

$$\mathrm{Ad}(k_{13}(t))Y_{12} = e^{t\,\mathrm{ad}(X_{13}-Y_{13})}Y_{12} = \cos(t)Y_{12} - \sin(t)X_{23}.$$

This is the differential at 0 of the one-parameter subgroup $s \mapsto c_{k_{13}(t)}\overline{n}_{12}(s)$. Since $[Y_{12}, X_{23}] = 0$ we conclude

$$k_{13}(t)\overline{n}_{12}(s)k_{13}(-t) = \overline{n}_{12}(\cos t)n_{23}(-\sin t).$$

Now, putting in $t = 0, \pi, -\pi, 2\pi$ gives the result.

Since $\mathrm{Ad}(\widetilde{w})(X - Y) = Y - X$ we have $c_{\widetilde{w}}k(t) = k(-t)$ so $c_{\widetilde{w}}m(\pm 1) = m(\pm 1)$, $c_{\widetilde{w}}m(\pm i) = m(\mp i)$. In other words, $c_{\widetilde{w}}m(u) = m(u^2 u)$. Applying Cartan involution to $k(t)$ gives $k(t)$ so $m(u)$ is mapped to $m(u)$.

Applying $c_{\widetilde{w}_{13}}$ we get

$$m_{13}(u^2 u)n_{23}(s) = n_{23}(u^2 s)m_{13}(u^2 u).$$

and applying Cartan involution to this we get

$$m_{13}(u^2 u)\overline{n}_{23}(-s) = \overline{n}_{23}(-u^2 s)m_{13}(u^2 u).$$

Applying $c_{\widetilde{w}_{12}}$ gets us the last two equations. □

Using these results, one finds that

Theorem B.2.5. *Let*

$$\overline{n}(x, y, z) = \exp(xY_{12} + yY_{23} + zY_{13}).$$

The $\overline{N}MAN$-decomposition of $w_{13}^{-1}\overline{n}(x, y, z)$ in $\widetilde{\mathrm{SL}}(3, \mathbb{R})$ is

$$\begin{aligned} w_{13}^{-1}\overline{n}(x, y, z) = \overline{n}\left(\frac{-x}{z}, \frac{-y}{z - xy}, \frac{-1}{z}\right) & m_{13}(u_1)m_{23}(u_2)a_{13}|z|a_{23}\left|1 - \frac{xy}{z}\right| \\ \cdot\, n\left(\frac{y}{z}, \frac{-x}{z - xy}, \frac{1}{z}\right) & \end{aligned}$$

where

$$u_1 = \begin{cases} +1 & z > 0 \\ +i & z < 0, \end{cases}$$

and

$$u_2 = \begin{cases} +1 & 1 - \frac{xy}{z} > 0 \\ +i & 1 - \frac{xy}{z} < 0, y > 0 \\ -i & 1 - \frac{xy}{z} < 0, y \le 0. \end{cases}$$

Calculating the M-part we split into six cases:

$$m_{13}(u_1)m_{23}(u_2) = \begin{cases} e, & z < 0, z - xy < 0, \\ m_{13}, & z < 0, z - xy > 0, y > 0, \\ \sigma m_{13}, & z < 0, z - xy > 0, y \le 0, \\ m_{23}, & z > 0, z - xy > 0, \\ m_{12}, & z > 0, z - xy < 0, y > 0, \\ \sigma m_{12}, & z > 0, z - xy < 0, y \le 0. \end{cases}$$

Succinctly,

$$m_{13}(u_1)m_{23}(u_2) = w_{13} w_{13}^{-\operatorname{sgn} z} (w_{23} w_{23}^{-\operatorname{sgn}(z-xy)})^{\operatorname{sgn} y}.$$

Note that $\operatorname{sgn} y$ was absent in the linear case. That it is present complicated the Fourier transform considerably. In doing the Fourier transform we cannot use the formulas that relate the Fourier transform of $|x|_\varepsilon^\lambda$ to $|x|_\varepsilon^{-1-\lambda}$. Instead, we are looking at a sum of Fourier transforms of x_ε^λ which is related to another family, namely $(x + \varepsilon i0)^{-1-\lambda}$. Concretely, the $(1,1)$-coordinate in $M_2(\mathbb{C})$ is

$$z_-^{\lambda_1 - 1}(z - xy)_-^{\lambda_2 - 1} + z_+^{\lambda_1 - 1}(z - xy)_-^{\lambda_2 - 1} \operatorname{sgn}(y).$$

Computing $\mathcal{F}_{2,3}$ of the second term is going to be difficult; that $\operatorname{sgn} y$ is present means that one cannot simply replace $z - xy$ by something else in the integral.

B.3 $\overline{N}MAN$-Decomposition of $w_{13}^{-1}k(\psi,\theta,\varphi)$

In this section we compute the $\overline{N}MAN$-decomposition of

$$k_{12}(\psi)k_{13}(\theta)k_{12}(\varphi).$$

Note first that in $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ we have

$$k(\theta) = \overline{n}(-\tan\theta)m(u_\theta)a|\cos\theta|n(\tan\theta)$$

for $\cos\theta, \sin\theta \ne 0$ where

$$u_\theta = \begin{cases} 1 & \cos\theta > 0 \\ -\operatorname{sgn}(\theta)i & \cos\theta < 0 \end{cases}$$

The $\overline{N}MAN$-decomposition of $k_{12}(\psi)k_{13}(\theta)$. The above gives us

$$\begin{aligned} k_{12}(\psi)k_{13}(\theta) = \overline{n}_{12}(-\tan\psi)m_{12}(u_\psi)a_{12}|\cos\psi|n_{12}(\tan\psi) \\ \cdot\, \overline{n}_{13}(-\tan\theta)m_{13}(u_\theta)a_{13}|\cos\theta|n_{13}(\tan\theta) \end{aligned}$$

Here we see that

$$[n_{12}(x), \overline{n}_{13}(y)] = \Theta[\overline{n}_{12}(-x), n_{13}(-y)] = \Theta n_{23}(xy) = \overline{n}_{23}(-xy)$$

so letting $x = \tan\psi$ and $y = -\tan\theta$ we get

$$n_{12}(\tan\psi)\overline{n}_{13}(-\tan\theta) = \overline{n}_{23}(\tan\psi\tan\theta)\overline{n}_{13}(-\tan\theta)n_{12}(\tan\psi).$$

This means

$$\begin{aligned} k_{12}(\psi)\overline{n}_{13}(-\tan\theta) = \overline{n}_{12}(-\tan\psi)m_{12}(u_\psi)a_{12}|\cos\psi| \\ \cdot\, \overline{n}_{23}(\tan\psi\tan\theta)\overline{n}_{13}(-\tan\theta)n_{12}(\tan\psi) \end{aligned}$$

Note that

$$\begin{aligned} C_{a_{12}(t)}\overline{n}(x,y,z) &= \overline{n}(t^{-2}x, ty, t^{-1}z) \\ C_{m_{ij}(u)}\overline{n}_{kl}(t) &= \overline{n}_{kl}(u^2 t) \end{aligned}$$

(we use the notation $C_g(x) = gxg^{-1}$) if $(i,j) \neq (k,l)$ so

$$\begin{aligned} k_{12}(\psi)\overline{n}_{13}(-\tan\theta) &= \overline{n}_{12}(-\tan\psi)\overline{n}_{23}(\sin\psi\tan\theta) \\ &\quad\cdot\, \overline{n}_{13}(-\tan\theta/\cos\psi)m_{12}(u_\psi)a_{12}|\cos\psi|n_{12}(\tan\psi) \\ &= \overline{n}(-\tan\psi, \sin\psi\tan\theta, -\frac{\tan\theta}{\cos\psi})m_{12}(u_\psi)a_{12}|\cos\psi|n_{12}(\tan\psi). \end{aligned}$$

We note that

$$C_{a_{13}}(t)n(x,y,z) = n(tx, ty, t^2 z)$$

so

$$n_{12}(\tan\psi)m_{13}(u_\theta)a_{13}|\cos\theta|n_{13}(\tan\theta) = m_{13}(u_\theta)a_{13}|\cos\theta|n(\frac{\tan\psi}{\cos\theta}, 0, \tan\theta).$$

So we conclude

$$\begin{aligned} k_{12}(\psi)k_{13}(\theta) = \overline{n}(-\tan\psi, \sin\psi\tan\theta, -\frac{\tan\theta}{\cos\psi})m_{12}(u_\psi)a_{12}|\cos\psi| \\ \cdot\, m_{13}(u_\theta)a_{13}|\cos\theta|n(\frac{\tan\psi}{\cos\theta}, 0, \tan\theta) \end{aligned}$$

One can check by projecting to $\mathrm{SL}(3,\mathbb{R})$ that this gives the correct decomposition in that group. It follows that our calculation is at least correct up to σ.

The $\overline{N}MAN$-decomposition of $k_{12}(\psi)k_{13}(\theta)k_{12}(\varphi)$. First note that

$$n_{13}(z)\overline{n}_{12}(x) = \overline{n}_{12}(x)n_{13}(z)[n_{13}(-z),\overline{n}_{12}(-x)] = \overline{n}_{12}(x)n_{13}(z)n_{23}(-xz)$$

and

$$n_{12}(x')\overline{n}_{12}(x) = \overline{n}_{12}(\frac{x}{1+xx'})m_{12}(u)a_{12}|1+xx'|n_{12}(\frac{x'}{1+xx'}).$$

This means

$$n(x',0,z)\overline{n}_{12}(x) = \overline{n}_{12}(\frac{x}{1+xx'})m_{12}(u)a_{12}|1+xx'|n_{12}(\frac{x'}{1+xx'})n_{13}(z)n_{23}(-xz).$$

Here we have

$$\begin{aligned} n_{12}(\frac{x'}{1+xx'})n_{13}(z)n_{23}(-xz) &= n(\frac{x'}{1+xx'},-xz,z-\frac{xx'z}{1+xx'}) \\ &= n(\frac{x'}{1+xx'},-xz,\frac{z}{1+xx'}) \end{aligned}$$

so setting $z = \tan\theta$, $x = -\tan\varphi$, $x' = \frac{\tan\psi}{\cos\theta}$ we conclude

$$\begin{aligned} n(\frac{\tan\psi}{\cos\theta},0,\tan\theta)\overline{n}_{12}(-\tan\varphi) &= \overline{n}_{12}(\frac{-\cos\theta\tan\varphi}{\cos\theta-\tan\psi\tan\varphi}) \\ &\cdot m_{12}(u)a_{12}|1-\frac{\tan\psi\tan\varphi}{\cos\theta}| \\ &\cdot n(\frac{\tan\psi}{\cos\theta-\tan\psi\tan\varphi},\tan\theta\tan\varphi,\frac{\sin\theta}{\cos\theta-\tan\psi\tan\varphi}). \end{aligned}$$

Next we look at

$$m_{12}(u_\psi)a_{12}|\cos\psi|m_{13}(u_\theta)a_{13}|\cos\theta|\overline{n}_{12}(\frac{-\cos\theta\tan\varphi}{\cos\theta-\tan\psi\tan\varphi}).$$

Note that

$$C_{a_{13}(t)}\overline{n}(x,y,z) = \overline{n}(t^{-1}x,t^{-1}y,t^{-2}z)$$

so combined with a result above we get that this is

$$\overline{n}_{12}(\frac{-\tan\varphi}{\cos\theta\cos^2\psi-\sin\psi\cos\psi\tan\varphi})m_{12}(u_\psi)a_{12}|\cos\psi|m_{13}(u_\theta)a_{13}|\cos\theta|.$$

So we have

$$k_{12}(\psi)k_{13}(\theta)\overline{n}_{12}(-\tan\varphi) = \overline{n}(-\tan\psi, \sin\psi\tan\theta, -\frac{\tan\theta}{\cos\psi})m_{12}(u_\psi)a_{12}|\cos\psi|$$
$$\cdot\, m_{13}(u_\theta)a_{13}|\cos\theta|n(\frac{\tan\psi}{\cos\theta}, 0, \tan\theta)\overline{n}_{12}(-\tan\varphi)$$
$$= \overline{n}(-\tan\psi, \sin\psi\tan\theta, -\frac{\tan\theta}{\cos\psi})m_{12}(u_\psi)a_{12}|\cos\psi|$$
$$\cdot\, m_{13}(u_\theta)a_{13}|\cos\theta|\overline{n}_{12}(\frac{-\cos\theta\tan\varphi}{\cos\theta-\tan\psi\tan\varphi})m_{12}(u)a_{12}|1-\frac{\tan\psi\tan\varphi}{\cos\theta}|$$
$$\cdot\, n(\frac{\tan\psi}{\cos\theta-\tan\psi\tan\varphi}, \tan\theta\tan\varphi, \frac{\sin\theta}{\cos\theta-\tan\psi\tan\varphi})$$
$$= \overline{n}(-\tan\psi, \sin\psi\tan\theta, -\frac{\tan\theta}{\cos\psi})\overline{n}_{12}(\frac{-\tan\varphi}{\cos\theta\cos^2\psi-\sin\psi\cos\psi\tan\varphi})$$
$$\cdot\, m_{12}(u_\psi)a_{12}|\cos\psi|m_{13}(u_\theta)a_{13}|\cos\theta|m_{12}(u)a_{12}|1-\frac{\tan\psi\tan\varphi}{\cos\theta}|$$
$$\cdot\, n(\frac{\tan\psi}{\cos\theta-\tan\psi\tan\varphi}, \tan\theta\tan\varphi, \frac{\sin\theta}{\cos\theta-\tan\psi\tan\varphi})$$
$$= \overline{n}(\frac{-\sin\psi\cos\theta\cos\varphi-\cos\psi\sin\varphi}{\cos\psi\cos\theta\cos\varphi-\sin\psi\sin\varphi}, \sin\psi\tan\theta, \frac{-\sin\theta\cos\varphi}{\cos\psi\cos\theta\cos\varphi-\sin\psi\sin\varphi})$$
$$\cdot\, m_{12}(u)m_{12}(u_\psi)a_{12}|\cos\psi-\frac{\sin\psi\tan\varphi}{\cos\theta}|m_{13}(u_\theta)a_{13}|\cos\theta|$$
$$\cdot\, n(\frac{\tan\psi}{\cos\theta-\tan\psi\tan\varphi}, \tan\theta\tan\varphi, \frac{\sin\theta}{\cos\theta-\tan\psi\tan\varphi}).$$

We only need to multiply with $m_{12}(u_\varphi)a_{12}|\cos\varphi|n_{12}(\tan\varphi)$ and we are done. We have

$$n(\frac{\tan\psi}{\cos\theta-\tan\psi\tan\varphi}, \tan\theta\tan\varphi, \frac{\sin\theta}{\cos\theta-\tan\psi\tan\varphi})m_{12}(u_\varphi)a_{12}|\cos\varphi|$$
$$= m_{12}(u_\varphi)a_{12}|\cos\varphi|$$
$$n(\frac{\tan\psi}{\cos\theta\cos^2\varphi-\tan\psi\cos\varphi\sin\varphi}, \tan\theta\sin\varphi, \frac{\sin\theta}{\cos\theta\cos\varphi-\tan\psi\sin\varphi}).$$

This leads to the final decomposition

Proposition B.3.1. *The $\overline{N}MAN$-decomposition of $k_{12}(\psi)k_{13}(\theta)k_{12}(\varphi)$ is*

$$k_{12}(\psi)k_{13}(\theta)k_{12}(\varphi)$$
$$= \overline{n}(\frac{-\sin\psi\cos\theta\cos\varphi-\cos\psi\sin\varphi}{\cos\psi\cos\theta\cos\varphi-\sin\psi\sin\varphi}, \sin\psi\tan\theta, \frac{-\sin\theta\cos\varphi}{\cos\psi\cos\theta\cos\varphi-\sin\psi\sin\varphi})$$
$$\cdot\, m_{12}(u)m_{12}(u_\psi)m_{12}(u_\varphi)a_{12}|\cos\psi\cos\varphi-\frac{\sin\psi\sin\varphi}{\cos\theta}|m_{13}(u_\theta)a_{13}|\cos\theta|$$
$$\cdot\, n(\frac{\cos\psi\cos\theta\sin\varphi+\sin\psi\cos\varphi}{\cos\varphi\cos\theta\cos\varphi}, \tan\theta\tan\varphi, \frac{\cos\psi\sin\theta}{\cos\psi\cos\theta\cos\varphi-\sin\psi\sin\varphi})$$

This is not quite what we are looking for. We would actually like to have a decomposition of $w_{13}^{-1}k_{12}(\psi)k_{23}(\theta)k_{12}(\varphi)$. But this could be computed in the same way.

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