Torus symmetry and nearly Kähler metrics

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To my grandmother Francesca,
who left me when all of this began

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Introduction

Let (M, g, J) be an almost Hermitian $2n$-dimensional manifold with Riemannian metric g and almost complex structure J compatible with g. Lowering the upper index of J yields a two-form $\sigma := g(J\,\cdot\,,\,\cdot\,)$. When the almost complex structure is integrable (i.e. the Nijenhuis tensor of J vanishes) and σ is a closed form, the triple (M, g, J) is called *Kähler* manifold. This condition is well known to be equivalent to saying that J is parallel with respect to the Levi-Civita connection on M, namely $\nabla J = 0$. By the General Holonomy Principle this implies that the holonomy group of ∇ reduces to a subgroup of $\mathrm{U}(n)$. The class of Kähler manifolds is widely studied and the literature contains plenty of examples.

This is a rather special set-up, but one may consider other geometries weakening the conditions on the structure, e.g. assuming that J satisfies some symmetries without being parallel, or only that σ is closed. When no particular assumptions are given, a number of interesting classes of almost Hermitian geometries arise and one can see the general picture. A classification may be carried out at the linear algebra level: the idea is that one considers the vector space of the type $(3, 0)$-tensors satisfying the same symmetries as $\nabla\sigma$. This space, which we call $\mathcal{W}$, splits under the action of the unitary group $\mathrm{U}(n)$ into the orthogonal direct sum of four irreducible submodules:

$$\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4.$$

Each combination of the $\mathcal{W}_i$s gives rise to a specific geometry, and we clearly have a total of sixteen classes. The class of Kähler manifolds corresponds to the trivial module $\{0\}$. An explicit description of these geometries was given by Gray and Hervella and published in 1980 [GH80].

In the present work we concentrate on $\mathcal{W}_1$, the class of *nearly Kähler* manifolds. They are characterised by the skew-symmetry of ∇J, which is assumed not to vanish identically. The first author who attempted an abstract, general analysis of their structure is Gray, [Gra69], [Gra70], [Gra76]. What is more, we focus on dimension six, where the first examples of nearly Kähler manifolds that are not Kähler arise: in fact it turns out that the whole theory of $\mathcal{W}_1$ relies on this case [Nag02], which is also particularly relevant for its connections with spin [FG85] and G_2-geometry [Bry87]. These links make nearly Kähler six-manifolds attractive in superstring theory and M-theory as well (see e.g. [Str86], [Agr06], [Agr08]).

One of the main problems we address is the lack of concrete examples. For the sake of completeness we quickly present the known ones here, for more details the reader may skip this paragraph and read Section 1.1 directly. The first explicit homogeneous nearly Kähler structure was found on the six-sphere in the 1950's, but a complete classification of the homogeneous, compact examples was achieved only in 2005 by Butruille [But05]. The list contains four spaces: the six-sphere $\mathbb{S}^6$, the flag manifold of $\mathbb{C}^3$, the complex projective space $\mathbb{CP}^3$, and the product of three-spheres $\mathbb{S}^3 \times \mathbb{S}^3$. Finally, in 2017, Foscolo and Haskins [FH17] proved the existence of the first two complete and non-homogeneous nearly Kähler structures on $\mathbb{S}^6$ and $\mathbb{S}^3 \times \mathbb{S}^3$. In a time interval of almost seventy years this is the complete set of compact examples found. The problem of constructing new ones reduces to solving a system of partial differential equations in terms of an $\mathrm{SU}(3)$-structure on the manifold: we shall show how the condition ∇J skew-symmetric be equivalent to the existence of a complex $(3,0)$-form $\psi_{\mathbb{C}} = \psi_+ + i\psi_-$ such that

$$d\sigma = 3\psi_+, \qquad d\psi_- = -2\sigma \wedge \sigma,$$

where $\sigma = g(J\,\cdot\,,\,\cdot\,)$. The obvious critical point is that solving these equations in complete generality is a highly non-trivial problem, so one may assume to have some kind of symmetry to simplify the analysis. This is why we will consider and look for nearly Kähler six-manifolds admitting a *two-torus symmetry*, which is a mild and natural assumption satisfied by all examples mentioned above.

In practice we assume a two-torus T^2 act on $(M, \sigma, \psi_{\mathbb{C}})$ effectively preserving the $\mathrm{SU}(3)$-structure $(\sigma, \psi_{\mathbb{C}})$. In this set-up one can construct a special T^2-invariant real-valued function on the manifold which is called *multi-moment map* [MS12], a generalisation of moment maps in symplectic geometry. This is the essential tool we use, and here is how. The T^2-symmetry yields a multi-moment map ν_M, concretely defined by $\nu_M := \sigma(U, V)$, where U, V are the infinitesimal generators of the T^2-action. Regular values s of ν_M give rise to five-dimensional invariant submanifolds $\nu_M^{-1}(s) \hookrightarrow M$, and it turns out that the action of T^2 on them is free. Therefore, the three-dimensional quotients $\nu_M^{-1}(s)/T^2$ have the structure of smooth manifolds. We will study the geometry of these spaces in detail. This structural reduction is called *T^2-reduction*. Now the goal is to reverse this construction hoping to be able to generate nearly Kähler six-manifolds from some three-dimensional manifold Q. We will then work out the conditions under which it is possible to construct a principal T^2-bundle over Q and then see how to evolve the structure of the total space of this bundle to get a nearly Kähler one. A new example of nearly Kähler six-manifold is obtained in this way starting with the three-dimensional Heisenberg group as base manifold, and potentially more new examples may be obtained.

The T^2-action is free on the level sets of the multi-moment map corresponding to regular values, but as it turns out it is not free on the whole manifold in general. A crucial point is then to analyse non-trivial stabilisers and study singular aspects corresponding to lack of smoothness. We will see how the set of fixed-points and one-dimensional orbits of the action define a trivalent graph. This encodes the geometric structure of points where the infinitesimal generators of the action are linearly dependent over the reals. As we will see, at these points the multi-moment maps and their differential vanish. We will explain the link between these two aspects in detail.

Non-degeneracy of the Hessian of the multi-moment map helps recover the topological structure of the whole six-manifold: in this context the multi-moment map should be thought of as a Morse function. This further motivates why we need to understand the structure of critical sets. We perform explicit, algebraic calculations on the homogeneous examples, the graphs will integrate this information geometrically.

The thesis is organised as follows. Chapter 1 starts off with basic definitions and a detailed motivation supporting our work. An overview on the story of nearly Kähler geometry enhances and completes this introduction, and some notable results in dimension six are recalled. In the subsequent sections we go through a series of lemmas to show equivalent characterisations of nearly Kähler six-manifolds and prove that these spaces are Einstein. Technicalities are taken steadily. The reader should find precise explanations of the basic ones, so as to be able to reproduce them autonomously. As we go along we then provide fewer details, trying to be as clear as possible anyway. The material covered in this first chapter is now classical, but we claim to have solved some issues found in various references. In particular, Sections 1.3 and 1.4 are devoted to these delicate results. In Chapter 2 we recall the structure of the homogeneous, compact nearly Kähler six-manifolds, whereas in Chapter 3 we start introducing multi-moment maps. We recall their general definition and construct explicit examples on the spaces studied in Chapter 2. Afterwards, in Chapter 4, we explain the T^2-reduction and the inverse process mentioned above, constructing a new example of nearly Kähler six-manifold with a two-torus symmetry. This material is part of a joint paper with Andrew Swann published in Journal of Geometry and Physics [RS19]. A link to it may be found in the Copyright statement. Chapter 5 deals with the analysis of the trivalent graphs corresponding to the critical loci where the multi-moment maps vanish. The final chapter includes properties of the Hessian and remarks on the topology of nearly Kähler six-manifolds with a two-torus symmetry. To come to the end, we sum up our conclusions and explain further developments in the last two sections. In particular, the reader may want to have a look at Section 6.4 for an overview of our results.

Abstract

This is a thesis in differential geometry studying a certain class of Einstein six-dimensional manifolds. Nearly Kähler manifolds were introduced by Gray in the 1960's, but until recently only a few examples were known. Our main focus is to use multi-moment maps to study nearly Kähler six-manifolds that admit an action of a two-dimensional torus.

We begin by reviewing the geometry of nearly Kähler six-manifolds. A detailed proof of Gray's result that they are Einstein manifolds of positive scalar curvature is given. We also provide details of how Gray's definition is related to the modern definition of nearly Kähler manifolds as $\mathrm{SU}(3)$-structures satisfying a certain pair of partial differential equations.

Following the classification result of Butruille, we know there are only four compact, homogeneous nearly Kähler manifolds in dimension six. All of them admit a two-torus symmetry generating multi-moment maps. Their expressions and critical sets are computed explicitly in each example.

We then switch to the general framework of nearly Kähler six-manifolds with effective two-torus symmetry. The multi-moment map for the torus action becomes an eigenfunction of the Laplace operator. At regular values, we prove the action is necessarily free on the level sets and determines the geometry of three-dimensional quotients. An inverse construction is given locally producing nearly Kähler six-manifolds from three-dimensional data. This is illustrated for structures on the Heisenberg group.

On the homogeneous cases we study the sets of points admitting non-trivial stabilisers and show how their structure may be encoded in trivalent graphs. Based on these examples, we prove a result on the configuration of such sets in the general case of an $\mathrm{SU}(3)$-structure with an effective torus symmetry. Viewing the multi-moment map as a Morse function, we work out the models of non-degenerate level sets close to orbits of local maximum and minimum.

Dansk résumé

I denne afhandling indenfor differentialgeometri studeres en særlig klasse af seks-dimensionale Einstein mangfoldigheder. Næsten-Kähler mangfoldigheder blev introduceret af Gray i 1960'erne, men indtil for nyligt var der kun få kendte eksempler. Vores primære fokus er at bruge multi-moment afbildninger til at studere næsten-Kähler seks-mangfoldigheder, der tillader en virkning af en todimensional torus.

Vi begynder med en gennemgang af geometrien på næsten-Kähler seks-mangfoldigheder. Vi giver et detaljeret bevis for Grays resultat: næsten-Kähler seks-mangfoldigheder er altid Einstein med positiv skalarkrumning. Derudover viser vi, hvordan Grays oprindelige definition er relateret til en moderne definition af næsten-Kähler mangfoldigheder, som $SU(3)$-strukturer der opfylder et par partielle differentialligninger.

Fra et klassifikationsresultat af Butruille ved vi, at der kun er fire kompakte, homogene næsten-Kähler mangfoldigheder af dimension seks. De tillader alle to-torus symmetri, hvilket giver anledning til en multi-momentafbildning. Vi beregner eksplicitte udtryk for multi-momentafbildningen og mængden af kritiske punkter i alle eksempler.

Dernæst skifter vi til den generelle teori for næsten-Kähler seksmangfoldigheder med effektiv to-torus symmetri. Multi-momentafbildningen for torusvirkningen er en egenfunktion for Laplace operatoren og vi viser, at virkningen på urbilleder af regulære værdier nødvendigvis er fri og bestemmer en geometrisk struktur på de tredimensionelle kvotienter. Vi giver også en invers konstruktion, der lokalt producerer næsten-Kähler seks-mangfoldigheder fra en tredimensional mangfoldighed med den nødvendige geometriske struktur. Konstruktionen illustreres med strukturer på Heisenberggruppen.

I de homogene tilfælde studerer vi punkterne med ikke trivielle isotropigruppe og viser, hvordan deres struktur kan indkodes i en trivalent graf. Fra disse eksempler viser vi et mere generelt resultat om mulige konfigurationer af sådanne mængder for $SU(3)$-strukturer med en effektiv to-torus symmetri. Idet vi fortolker multi-momentafbildningen som en Morse funktion, finder vi modeller for ikke degenererede niveaumængder tæt på lokale maksima og minima.

Notations and conventions

1. We write bases of vector spaces with upper case letters and lower indices, e.g. $E_1, E_2, \ldots$, dual bases with lower case letters and upper indices, e.g. $e^1, e^2, \ldots$. Correspondingly, coordinates of vectors have upper indices, coordinates of covectors have lower indices. We usually use the letters E and e for real bases and the letters F and f for complex bases.

2. We use standard notations for classical Lie groups $\mathrm{O}(n), \mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n), \mathrm{Sp}(n)$, etc. We denote n-dimensional tori by T^n. The standard Fraktur alphabet is used for Lie algebras, e.g. $\mathfrak{so}(n), \mathfrak{su}(n), \mathfrak{sp}(n)$, etc.

3. If α is a tensor, we denote $\alpha \otimes \ldots \otimes \alpha$ (n-times) by $\alpha^{\otimes n}$ and not by α^n. The latter notation may be used instead for wedging n times.

4. We adopt the convention that p-covariant tensors have type $(p, 0)$, q-contravariant tensors have type $(0, q)$. Thus vectors are $(0, 1)$ tensors, one-forms are $(1, 0)$ tensors, and so on.

5. We use the symbol $\mathfrak{S}_{X,Y,\ldots,Z}$ to denote cyclic sums over $X, Y, \ldots, Z$. When the sum is over $E_i, E_j, \ldots, E_k$ we may write $\mathfrak{S}_{i,j,\ldots,k}$ instead of $\mathfrak{S}_{E_i,E_j,\ldots,E_k}$.

6. We sometimes use the standard symbol $\mathfrak{X}(M)$ for the Lie algebra of vector fields on M.

7. In the formula for the differential of a k-form α we drop the numerical factor $k+1$. E.g. if α is a one-form we have $d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$, instead of $2d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$.

8. If A is a (p, q)-tensor then we write $\nabla A(X, Y_1, \ldots, Y_p)$ or simply $\nabla_X A(Y_1, \ldots, Y_p)$ for $(\nabla_X A)(Y_1, \ldots, Y_p)$. Analogously, $\nabla^2 A(W, X, Y_1, \ldots, Y_p)$ or $\nabla^2_{W,X} A(Y_1, \ldots, Y_p)$ for $(\nabla^2_{W,X} A)(Y_1, \ldots, Y_p)$. If A is any $(1, q)$-tensor, we write $\nabla_X AY$ for any covariant derivative of AY with respect to X. We write instead $(\nabla_X A)Y$ when the operator $\nabla_X A$ is applied to Y. We use brackets in potentially ambiguous expressions.

9. To denote group actions we usually use a dot or nothing at all: if an element g of a group acts on an object x in some space, then we write the action of g on x as $g \cdot x$ or as gx.

10. We denote the Lie derivative with respect to a vector field X by $\mathcal{L}_X$, and contractions by the symbol $\lrcorner$.

Chapter 1

The structure of nearly Kähler six-manifolds

Let us start our mathematical journey with an overview of general facts on nearly Kähler manifolds in dimension six. We first give some motivation behind this project and then let the mathematics talk. We explain the essential tools and arguments one needs to get into the geometry we will be concerned with, supplying all the necessary details in favour of a fluent reading. In particular, since the first chapter contains several technical results, we try to be as clear as possible in the structure of the exposition and show all the relevant steps. In our special six-dimensional set-up each problem is solved with adapted techniques. We need the right language rather than the general theory. However, the material supporting the whole manuscript is classical and can be safely found in the literature. The general references we used most are [KN96], [Sal89], [Hat02], [FH04], [Mor07], [Bes08], and [Sil08]. Further and more specific ones are provided along the way.

1.1 A historical overview

As disclosed in the introduction, nearly Kähler geometry places itself in the wider framework of almost Hermitian geometry. To our knowledge, the first author giving a formal definition of nearly Kähler manifolds was Gray (see e.g. [Gra65] or [Gra70]), although traces of their appearance can be found in previous works: for example, Tachibana [Tac59] and Kotō [Kot60] call them "K-spaces". We shall stick for a while to the following definition by Gray. Later on we will see equivalent characterisations.

Definition 1.1.1. Let (M, g, J) be an almost Hermitian manifold with Riemannian metric g, almost complex structure J compatible with g (i.e. J is g-orthogonal), and let ∇ be the Levi-Civita connection. Then M is called *nearly Kähler* if $(\nabla_X J)X = 0$ for all vector fields X on M.

Recall that the presence of the almost complex structure J forces the dimension of M to be even. We say that M is *strict* nearly Kähler if ∇J does not vanish identically—when it does, M is *Kähler*. From now on when we say nearly Kähler we actually mean strict nearly Kähler. In a note of 1969 [Gra69], Gray uses the terminology of the definition above to generalise the skew-symmetry of ∇J found in a specific six-dimensional case, that of the six-sphere S^6. From the chronological point of view this is the first example of strict nearly Kähler manifold noticed: an almost complex structure J on S^6 was defined by

Frölicher [Frö55] making use of the algebra of octonions, and subsequently Fukami and Ishihara [FI55] proved that ∇J is skew-symmetric. We will discuss this example in more detail in Section 2.2.

In fact it turns out that the six-dimensional case is in a sense the first significant one: Gray [Gra69] gave a short and explicit argument proving that nearly Kähler manifolds in dimension four are automatically Kähler, namely $\nabla J = 0$ (see Section 1.2, Proposition 1.2.2). This result can be easily adapted to the two-dimensional case, thus strict nearly Kähler manifolds may occur only in dimension greater than four.

It is then clear that the example of $\mathbb{S}^6$ mentioned above already makes dimension six appealing. For the moment we just recall that the Lie group G of automorphisms of the algebra of octonions acts transitively on $\mathbb{S}^6$ and any isotropy group is conjugate to the special unitary group $\mathrm{SU}(3)$, meaning that $\mathbb{S}^6 = G/\mathrm{SU}(3)$. Therefore, the six-sphere has a homogeneous nearly Kähler structure. This observation can be found for instance in [FI55]. A remarkable piece of information here is that G is isomorphic to the exceptional Lie group G_2. We shall dedicate Section 2.1 to the G_2 geometry we will be interested in.

The six-sphere does not come along alone. Gray and Wolf [GW68] found other complete and simply connected homogeneous examples in dimension six: the flag manifold $F_{1,2}(\mathbb{C}^3) = \mathrm{SU}(3)/T^2$, the complex projective space $\mathbb{CP}^3 = \mathrm{Sp}(2)/\mathrm{Sp}(1)\mathrm{U}(1)$, and the product of three-spheres $\mathbb{S}^3 \times \mathbb{S}^3 = \mathrm{SU}(2)^3/\mathrm{SU}(2)_\Delta$—here $\mathrm{SU}(2)_\Delta$ denotes the diagonal subgroup in $\mathrm{SU}(2)^3$, namely that subgroup whose elements are triples (g, g, g), with g in $\mathrm{SU}(2)$. Much later, in 2005, Butruille [But05] proved that these four examples give a complete classification of the homogeneous, compact nearly Kähler six-manifolds. Recently, in 2017, Foscolo and Haskins [FH17] proved the existence of the first complete inhomogeneous examples by showing there is at least one co-homogeneity one nearly Kähler structure on $\mathbb{S}^6$ and one on $\mathbb{S}^3 \times \mathbb{S}^3$. In these two cases the Lie group acting is $\mathrm{SU}(2) \times \mathrm{SU}(2)$. For the time being, the spaces listed so far are the only compact, six-dimensional nearly Kähler manifolds known.

The outstanding difficulty in finding more examples partly motivates our work. A first observation is that the symmetry groups of the four homogeneous spaces above and the two co-homogeneity one cases studied by Foscolo and Haskins have rank at least two (the case $\mathrm{SU}(2)^3$ has in fact rank three). This is why we will work with nearly Kähler six-manifolds admitting a two-torus symmetry. The theory will be developed in Chapter 4 and represents the core of the manuscript. The aim is to lay down the foundations of a path to find new examples with weaker symmetry.

In order to emphasise why we focus on dimension six we shall give more geometric insights. In 2002, Nagy [Nag02] provided a structure theory for general complete nearly Kähler manifolds, showing they are built from homogeneous examples (classified in [DC12]), twistor spaces of positive quaternionic Kähler manifolds, and six-dimensional strict nearly Kähler manifolds. Thus, in principle dimension six is essential for a better understanding of the whole theory of nearly Kähler spaces.

There is also a link with spin geometry: Friedrich and Grunewald [FG85] showed that a six-dimensional Riemannian manifold admits a Riemannian Killing spinor if and only if it is nearly Kähler.

Moreover, in dimension six nearly Kähler manifolds are Einstein: Gray made a massive use of curvature identities to give a proof of this fact [Gra76, Theorem 5.2]. There are however more recent arguments based on connections with G_2 geometry [Bry87] and representation theory [CS04] with a more geometric and less analytic flavour. In the former work, Bryant mentions that a common feature of the Riemannian manifolds with holonomy G_2 (and Spin(7)) he found is that they are *cones* on lower dimensional manifolds, whence one can see the link with nearly Kähler geometry: if we consider the Riemannian cone over a six-dimensional nearly Kähler manifold M, i.e. $C(M) = (\mathbb{R}_+ \times M, g_C = dt^{\otimes 2} + t^2 g)$, and assume the holonomy of the cone be in G_2, then $C(M)$ is Ricci-flat and the manifold (M, g) is Einstein, with $\mathrm{Ric}_g = 5g$ (up to renormalising the metric). It follows that the scalar curvature of M is positive. Bryant describes how to construct a metric with holonomy G_2 on the cone over the flag manifold $SU(3)/T^2$: he showed that there exist a two-form σ and a complex three-form $\psi_{\mathbb{C}}$ satisfying

$$d\sigma = 3\mathrm{Re}\psi_{\mathbb{C}}, \qquad d\psi_{\mathbb{C}} = -2i\sigma \wedge \sigma, \tag{1.1}$$

and that on the cone we can construct a parallel G_2-structure defined by

$$\varphi := t^2 dt \wedge \sigma + t^3 \mathrm{Re}\psi_{\mathbb{C}}, \qquad \psi := \tfrac{1}{2} t^4 \sigma \wedge \sigma - t^3 dt \wedge \mathrm{Im}\psi_{\mathbb{C}}.$$

By a direct calculation one proves that ψ is $*\varphi$, where $*$ denotes the Hodge star operator acting on differential forms defined on the seven-dimensional cone. Fernández and Gray [FG82] proved that the condition φ be parallel, namely that the holonomy of g_C reduces to G_2, is equivalent to the closedness of φ and $*\varphi$, hence φ is harmonic. We then say that the G_2-structure is *torsion-free*. It is readily checked that $d\varphi = 0 = d{*\varphi}$ give equations (1.1). But as we will see in Section 1.5, an SU(3)-structure $(\sigma, \psi_{\mathbb{C}})$ on M is nearly Kähler exactly when equations (1.1) hold, so M is nearly Kähler if and only if the Riemannian cone over it has holonomy in G_2.

Indeed, a special feature of dimension six is an equivalence between Definition 1.1.1 and a pair of partial differential equations as in (1.1), which appears for example in [Car93, Theorem 4.9], and will be discussed in Section 1.5: the main statement is that a six-dimensional almost Hermitian manifold (M, g, J) is nearly Kähler if and only if there exists a complex three-form $\psi_{\mathbb{C}} = \psi_+ + i\psi_-$ of type $(3, 0)$ such that

$$d\sigma = 3\psi_+, \qquad d\psi_- = -2\sigma \wedge \sigma, \tag{1.2}$$

where σ is the fundamental two-form $g(J\,\cdot\,, \cdot\,)$, and the identities hold up to homothety. Hence, a nearly Kähler six-manifold will be a Riemannian manifold with an SU(3)-structure $(\sigma, \psi_{\mathbb{C}})$ satisfying equations (1.2). A digression on how these equations encode information about the torsion of a connection adapted to the SU(3)-structure in question will follow at the end of Section 1.5. The advantage of this formulation of the original definition is that it is often easier to check, we will see some examples in Chapter 2.

As regards the topology, here are some useful properties. Myers' theorem implies that a complete, connected nearly Kähler six-manifold is compact with finite fundamental group. Its universal cover is also a complete nearly Kähler manifold. This is why we focus on the connected, simply connected case. The importance of these facts will be particularly evident in Section 6.3, when we discuss topological aspects.

We shall now start investigating first, simple consequences of the skew-symmetry of ∇J on an almost Hermitian manifold, heading quickly to the six dimensional case.

1.2 Symmetries

Let us consider an almost complex manifold (M, J), namely J is an endomorphism of each tangent space such that $J^2 = -\,\mathrm{Id}$ at each point. If M admits a Riemannian metric g and J

is an orthogonal transformation with respect to g, then (M, g, J) is called *almost Hermitian* manifold. From these data one can construct a two-form $\sigma := g(J\,\cdot\,,\,\cdot\,)$, commonly called *fundamental* two-form, which results from lowering the upper index of J. Denote by ∇ the Levi-Civita connection on (M, g, J). We assume M to be nearly Kähler as in Definition 1.1.1 throughout.

Lemma 1.2.1. *For each triple U, V, Z of vector fields on M we have*

$$\nabla\sigma(U, V, Z) = g((\nabla_U J)V, Z). \tag{1.3}$$

Further, we can move J across all the entries of $\nabla\sigma$:

$$\nabla\sigma(JU, V, Z) = \nabla\sigma(U, JV, Z) = \nabla\sigma(U, V, JZ). \tag{1.4}$$

Proof. Recall that J is g-orthogonal and the connection is metric, namely $\nabla g = 0$. For each triple U, V, Z of vector fields one has

$$\begin{aligned}\nabla\sigma(U, V, Z) &= U(g(JV, Z)) - g(J\nabla_U V, Z) - g(JV, \nabla_U Z)\\ &= g(\nabla_U JV, Z) - g(J\nabla_U V, Z)\\ &= g((\nabla_U J)V, Z).\end{aligned}$$

The second statement is then readily checked: since $0 = (\nabla J^2) = (\nabla J)J + J(\nabla J)$, J and ∇J anti-commute, so $(\nabla_{JX} J)Y = -(\nabla_Y J)JX = J(\nabla_Y J)X = -J(\nabla_X J)Y$. Therefore

$$\nabla\sigma(JX, Y, Z) = g((\nabla_{JX} J)Y, Z) = -g(J(\nabla_X J)Y, Z).$$

But J is orthogonal, thus the latter equals $\nabla\sigma(X, Y, JZ)$. On the other hand J and ∇J anti-commute, so $-g(J(\nabla_X J)Y, Z)$ coincides with $\nabla\sigma(X, JY, Z)$ as well. □

Equation (1.3) tells us that $\nabla\sigma$ is skew-symmetric and that $\nabla_U J$ is skew-adjoint. We now prove a well-known fact recalled in the previous section to motivate our interest in dimension six.

Proposition 1.2.2 ([Gra69])**.** *Nearly Kähler manifolds in dimension two and four are Kähler.*

Proof. If M is nearly Kähler and has dimension four, on each open subset of M we can consider an orthonormal frame $\{X, JX, Y, JY\}$. Our claim is that $\nabla J = 0$. Since $\nabla\sigma$ is skew-symmetric

$$g((\nabla_X J)Y, X) = g((\nabla_X J)Y, Y) = 0.$$

Moreover, ∇J anti-commutes with J. Projecting $(\nabla_X J)Y$ onto JX, JY we obtain

$$\begin{aligned} g((\nabla_X J)Y, JX) &= -g((\nabla_X J)JX, Y) = g(J(\nabla_X J)X, Y) = 0,\\ g((\nabla_X J)Y, JY) &= g(J(\nabla_X J)Y, Y) = -g((\nabla_X J)Y, JY) = 0.\end{aligned}$$

This proves $(\nabla_X J)Y = 0$ for all X, Y, which means $\nabla J = 0$, and M is Kähler.

In the two-dimensional case, our orthonormal local frame is given by $\{X, JX\}$. But then $(\nabla_X J)JX = -J(\nabla_X J)X = 0$, which proves the claim. □

Remark 1.2.3. We have thus seen that $\nabla\sigma$ is a three-form and that $(\nabla_X J)Y$ is orthogonal to X, Y, JX, JY. Conversely, if we assume $\nabla\sigma$ to be skew-symmetric, then $\nabla\sigma(X, X, Y) = g((\nabla_X J)X, Y) = 0$ for every Y, and by non-degeneracy of the metric M is nearly Kähler.

Remark 1.2.4. When M is six-dimensional, the Hodge star operator $*$ gives an automorphism of $\Lambda^3 T^* M$ and yields a three-form $*\nabla\sigma$. Then $\nabla\sigma + i*\nabla\sigma$ is a complex three-form on M. We will come back to this point in the end of Section 1.5 (cf. Remark 1.5.6).

Remark 1.2.5. From a more abstract point of view, equation (1.3) tells us $\nabla\sigma$ measures the failure of M to be Kähler in every dimension. More precisely, the symmetries of $\nabla\sigma$ on any almost Hermitian manifold determine sixteen classes of almost Hermitian geometries, as was shown by Gray and Hervella [GH80, Theorem 2.1].

For the rest of this section we assume M has dimension six. Let us introduce some ideas and notations from [Sal89]. The main intention here is to provide a unifying language to describe the symmetries of useful tensors, in view of Remark 1.2.5.

Recall the identity of Lie groups

$$\mathrm{U}(n) = \mathrm{SO}(2n) \cap \mathrm{GL}(n, \mathbb{C}). \tag{1.5}$$

In real dimension six this tells us $\mathrm{U}(3)$ is the stabiliser in $\mathrm{GL}(6, \mathbb{R})$ of an inner product g_0 and an (almost) complex structure J_0 on a copy of $\mathbb{R}^6$. At the level of Lie algebras, this identity implies in particular that elements of $\mathfrak{u}(3)$ commute with J_0. We shall always think of $\mathrm{U}(3)$ as a subgroup of $\mathrm{SO}(6)$.

At each point of M there is a representation of $\mathrm{U}(3)$ on the tangent space inducing the structure of $\mathrm{U}(3)$-module on the complexified vector space of k-forms, which we denote simply by $\Lambda^k \otimes \mathbb{C}$. Note that every orthogonal matrix coincides with the transpose of its inverse, so the representations $\Lambda^k T_p^* M$ and $\Lambda^k T_p M$ of $\mathrm{U}(3) \subset \mathrm{SO}(6)$ are equivalent and one loses no information in identifying k-forms and k-vectors. This explains our choice of the symbol Λ^k for the space of real k-forms and will allow us to identify $\mathrm{U}(3)$-modules and their duals in other circumstances. There is an isomorphism of vector spaces

$$\Lambda^k \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^p(\Lambda^{1,0}) \otimes \Lambda^q(\Lambda^{0,1}),$$

and by definition $\Lambda^{p,q} := \Lambda^p(\Lambda^{1,0}) \otimes \Lambda^q(\Lambda^{0,1})$ is the space of complex differential forms of type (p,q). Each $\Lambda^{p,q}$ is a $\mathrm{U}(3)$-invariant complex module because unitary matrices commute with the almost complex structure J.

For $p \neq q$ we denote by $[\![\Lambda^{p,q}]\!]$ the *real* vector space underlying $\Lambda^{p,q}$, whose complexification is $[\![\Lambda^{p,q}]\!] \otimes \mathbb{C} = \Lambda^{p,q} \oplus \overline{\Lambda^{p,q}} = \Lambda^{p,q} \oplus \Lambda^{q,p}$. For $p = q$, $[\Lambda^{p,p}]$ is the space of type (p,p)-forms α such that $\overline{\alpha} = \alpha$, hence $[\Lambda^{p,p}] \otimes \mathbb{C} = \Lambda^{p,p}$. Therefore, we have isomorphisms of $\mathrm{U}(3)$-modules such as

$$\Lambda^1 = [\![\Lambda^{1,0}]\!], \quad \Lambda^2 = [\![\Lambda^{2,0}]\!] \oplus [\Lambda^{1,1}], \quad \Lambda^3 = [\![\Lambda^{3,0}]\!] \oplus [\![\Lambda^{2,1}]\!], \text{ etc.}$$

Each real form of type $(p,q) + (q,p)$ satisfies a specific relation with J. To show this, we specialise for a moment to the case $k = 2$, taking the opportunity to describe a well-known decomposition of Λ^2. The same idea is exploited in the case $k = 3$ for a better understanding of the symmetries of $\nabla\sigma$.

At every point of M, the metric g yields a canonical isomorphism $\mathfrak{so}(6) = \Lambda^2$, which is obtained by mapping each A in $\mathfrak{so}(6)$ to the two-form $g(A\,\cdot\,, \cdot\,)$. Viewing $\mathfrak{so}(6)$ as the adjoint representation of $\mathrm{SO}(6) \supset \mathrm{U}(3)$, we have actually got an isomorphism of $\mathrm{U}(3)$-modules: for $A \in \mathfrak{so}(6)$ and $B \in \mathrm{U}(3)$, the action of B on two-forms gives

$$Bg(A\,\cdot\,, \cdot\,) = g(AB^{-1}\,\cdot\,, B^{-1}\,\cdot\,) = g(BAB^{-1}\,\cdot\,, \cdot\,),$$

so the map $A \mapsto g(A\,\cdot\,,\,\cdot\,)$ is $\mathrm{U}(3)$-equivariant and our claim follows.

The correspondence just found conceals interesting relations with J. Consider the splitting $\mathfrak{so}(6) = \mathfrak{u}(3) \oplus \mathfrak{u}(3)^\perp$, where $\mathfrak{u}(3)^\perp$ is the orthogonal complement of $\mathfrak{u}(3)$ as a subspace of $\mathfrak{so}(6)$. Bearing in mind the isomorphisms $\mathfrak{so}(6) = \Lambda^2 = [\![\Lambda^{2,0}]\!] \oplus [\Lambda^{1,1}]$, any endomorphism A in $\mathfrak{u}(3)$ corresponds to a two-form α such that $\alpha(JX, JY) = \alpha(X, Y)$: in fact, since A commutes with J and J is g-orthogonal

$$\alpha(JX, JY) = g(AJX, JY) = g(JAX, JY) = g(AX, Y) = \alpha(X, Y). \tag{1.6}$$

On the other hand, a two-form β in $[\Lambda^{1,1}]$ is defined so as to vanish on pairs of complex vectors of the same type, namely $\beta(X - iJX, Y - iJY) = 0$. Thus $\beta(JX, JY) = \beta(X, Y)$, and by counting dimensions the following splittings are equivalent:

$$\mathfrak{so}(6) = \mathfrak{u}(3) \oplus \mathfrak{u}(3)^\perp, \qquad \Lambda^2 = [\Lambda^{1,1}] \oplus [\![\Lambda^{2,0}]\!]. \tag{1.7}$$

We have already encountered a two-form enjoying the property of elements in $[\Lambda^{1,1}]$, that is the fundamental two-form σ. Identity (1.6) is readily checked:

$$\sigma(JX, JY) = -g(X, JY) = g(JX, Y) = \sigma(X, Y).$$

Since σ is $\mathrm{U}(3)$-invariant by definition, there is a decomposition $[\Lambda^{1,1}] = [\Lambda^{1,1}_0] \oplus \mathbb{R}$, where $[\Lambda^{1,1}_0]$ is defined as the orthogonal complement of the image of $\wedge\sigma\colon \mathbb{R} \to [\Lambda^{1,1}]$ mapping a real number r to $r \wedge \sigma = r\sigma$. We thus have a first decomposition of Λ^2 in $\mathrm{U}(3)$-submodules

$$\Lambda^2 = [\![\Lambda^{2,0}]\!] \oplus [\Lambda^{1,1}_0] \oplus \mathbb{R},$$

and $[\Lambda^{1,1}_0]$ cannot be decomposed further (cf. [FFS94]).

In this way elements of $[\Lambda^{1,1}]$ can be seen as eigenvectors of J with eigenvalue $+1$. Likewise, elements of $[\![\Lambda^{2,0}]\!]$ are eigenvectors of J with eigenvalue -1: β in $[\![\Lambda^{2,0}]\!]$ satisfies $\beta(X - iJX, Y + iJY) = 0$, which gives $\beta(X, Y) + \beta(JX, JY) = 0$.

Identity (1.4) implies

$$\nabla\sigma(U, JV, JZ) = \nabla\sigma(U, V, J^2 Z) = -\nabla\sigma(U, V, Z). \tag{1.8}$$

which we may then rephrase by saying $\nabla\sigma$ sits inside $\Lambda^1 \otimes [\![\Lambda^{2,0}]\!]$. Furthermore, since M is nearly Kähler, $\nabla\sigma$ actually takes values in $[\![\Lambda^{3,0}]\!]$: recall that $\nabla\sigma$ is skew-symmetric, so (1.8) implies

$$\nabla\sigma(U, V, Z) = \nabla\sigma(JU, JV, Z) - \nabla\sigma(U, JV, JZ) - \nabla\sigma(JU, V, JZ).$$

On the other hand this is the property characterising elements of $[\![\Lambda^{3,0}]\!]$: if $\beta \in [\![\Lambda^{3,0}]\!]$ then $\beta(X - iJX, Y - iJY, Z + iJZ) = 0$, that is

$$\beta(X, Y, Z) = \beta(JX, JY, Z) - \beta(X, JY, JZ) - \beta(JX, Y, JZ).$$

Hence $\nabla\sigma \in [\![\Lambda^{3,0}]\!]$.

A last observation is motivated by Remark 1.2.3. Since $\nabla\sigma$ is a three-form, it is then natural to ask whether there is a relation between $d\sigma$ and $\nabla\sigma$. This is readily worked out, as $d\sigma = \mathcal{A}\nabla\sigma$, where $(\mathcal{A}\nabla\sigma)(X, Y, Z) = \mathfrak{S}_{X,Y,Z}\, \nabla\sigma(X, Y, Z)$, and $\nabla\sigma$ skew-symmetric implies $d\sigma = 3\nabla\sigma$. Conversely, if $d\sigma = 3\nabla\sigma$, then $0 = d\sigma(X, X, Y) = 3\nabla\sigma(X, X, Y) = 3g((\nabla_X J)X, Y)$, so M is nearly Kähler by non-degeneracy of g. We summarise the observations done so far in

Proposition 1.2.6. *Assume (M, g, J) is an almost Hermitian six-manifold and let $\sigma = g(J\,\cdot\,,\,\cdot\,)$ be the fundamental two-form. Then the following are equivalent:*

1. *M is nearly Kähler.*
2. *$\nabla\sigma \in [\![\Lambda^{3,0}]\!]$.*
3. *$d\sigma = 3\nabla\sigma$.*

1.3 Curvature

As recalled in Remark 1.2.3, $(\nabla_X J)Y$ is orthogonal to X, JX, Y, JY. On the other hand, ∇J does not vanish identically, otherwise M would be Kähler. At each point the tangent space then splits as the orthogonal direct sum of three J-invariant planes

$$\langle X, JX\rangle \oplus \langle Y, JY\rangle \oplus \langle(\nabla_X J)Y, J(\nabla_X J)Y\rangle,$$

where angular brackets denote the real vector space spanned by a pair of vectors and Y is orthogonal to the span of X and JX. We now prove a formula giving an explicit way to calculate the norm of $(\nabla_X J)Y$.

Lemma 1.3.1. *There exists a non-negative function μ on M such that*

$$\|(\nabla_X J)Y\|^2 = \mu^2\big(\|X\|^2\|Y\|^2 - g(X,Y)^2 - \sigma(X,Y)^2\big) \tag{1.9}$$

for every pair of vector fields X, Y on M.

Proof. We define μ in terms of a local frame, then we extend it to a global function declaring it has to satisfy (1.9).

Given X and Y in a neighbourhood of a point there exists an orthonormal frame $\{E_i, JE_i\}, i = 1, 2, 3$, such that $X = aE_1$ and $Y = bE_1 + cJE_1 + dE_2$ for local functions a, b, c, d. We assume $d \neq 0$ in order to avoid the trivial case $(\nabla_X J)Y = 0$.

Define μ so that $(\nabla_{E_1} J)E_2 =: \mu E_3$. We may assume μ non-negative up to changing the orientation of the basis. Then $(\nabla_X J)Y = a(\nabla_{E_1} J)dE_2 = ad\mu E_3$, which implies $\|(\nabla_X J)Y\|^2 = a^2 d^2 \mu^2$. On the other hand, $\|X\|^2\|Y\|^2 = a^2(b^2 + c^2 + d^2)$, $g(X,Y)^2 = a^2 b^2$, and $\sigma(X,Y)^2 = a^2c^2$. So $\mu^2\big(\|X\|^2\|Y\|^2 - g(X,Y)^2 - \sigma(X,Y)^2\big) = a^2 d^2 \mu^2$, and the formula is proved locally.

We can finally extend μ to a global function imposing that (1.9) be satisfied for all pairs of vector fields X, Y on M. $\square$

The function μ cannot vanish identically because M is nearly Kähler. We shall now study how μ is related to the Riemannian and the Ricci curvature tensors on M, and finally prove that μ is locally constant—hence constant because M is connected. A first step in this direction is to consider second order covariant derivatives of σ and study their symmetries. We follow [Gra76] and [Mor14] for this part, working in general even dimension $2n$ when there is no need to restrict to the six-dimensional case.

Lemma 1.3.2. *Let $R \in \Lambda^2 \otimes \mathfrak{so}(2n)$ be the type $(3,1)$ Riemannian curvature tensor of the Levi-Civita connection on M, given by $R(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$. The following identities hold for every quadruple of vector fields W, X, Y, Z in $\mathfrak{X}(M)$:*

1. $\nabla^2\sigma(W,X,Y,Z) - \nabla^2\sigma(X,W,Y,Z) = \sigma(R(X,W)Y,Z) + \sigma(Y,R(X,W)Z)$.
2. $\nabla^2\sigma(X,X,JY,Y) = \|(\nabla_X J)Y\|^2$.

Proof. To prove the first formula we start by expanding the first term:

$$\begin{aligned}
\nabla^2\sigma(W,X,Y,Z) &= W(\nabla\sigma(X,Y,Z)) - \nabla\sigma(\nabla_W X,Y,Z) \\
&\quad - \nabla\sigma(X,\nabla_W Y,Z) - \nabla\sigma(X,Y,\nabla_W Z) \\
&= g(\nabla_W((\nabla_X J)Y),Z) + g((\nabla_X J)Y,\nabla_W Z) - g((\nabla_{\nabla_W X}J)Y,Z) \\
&\quad - g((\nabla_X J)\nabla_W Y,Z) - g((\nabla_X J)Y,\nabla_W Z) \\
&= g((\nabla_W(\nabla_X J))Y,Z) - g((\nabla_{\nabla_W X}J)Y,Z).
\end{aligned}$$

As an element of $\mathfrak{so}(2n)$, $R(W,X)$ is a skew-adjoint derivation. We can then rewrite the difference $\nabla^2\sigma(W,X,Y,Z) - \nabla^2\sigma(X,W,Y,Z)$ as

$$\begin{aligned}
g(([\nabla_W,\nabla_X]J - \nabla_{[W,X]}J)Y,Z) &= g((R(W,X)J)Y,Z) \\
&= g(R(W,X)JY,Z) - g(JR(W,X)Y,Z) \\
&= -g(JY,R(W,X)Z) - g(JR(W,X)Y,Z) \\
&= \sigma(Y,R(X,W)Z) + \sigma(R(X,W)Y,Z).
\end{aligned}$$

In order to prove the second formula we make use of (1.4):

$$\begin{aligned}
\nabla^2\sigma(X,X,JY,Y) &= X(\nabla\sigma(X,JY,Y)) - \nabla\sigma(\nabla_X X,JY,Y) \\
&\quad - \nabla\sigma(X,\nabla_X JY,Y) - \nabla\sigma(X,JY,\nabla_X Y) \\
&= X(\nabla\sigma(JX,Y,Y)) - \nabla\sigma(J\nabla_X X,Y,Y) \\
&\quad - g((\nabla_X J)\nabla_X JY,Y) - g((\nabla_X J)JY,\nabla_X Y) \\
&= g(\nabla_X JY,(\nabla_X J)Y) - g((\nabla_X J)JY,\nabla_X Y) \\
&= g((\nabla_X J)Y,(\nabla_X J)Y) + g(J\nabla_X Y,(\nabla_X J)Y) - g((\nabla_X J)JY,\nabla_X Y) \\
&= \|(\nabla_X J)Y\|^2,
\end{aligned}$$

and the statement is proved. □

Remark 1.3.3. In the next corollary we denote the Riemannian curvature tensors of type $(3,1)$ and $(4,0)$ by the same letter. We shall keep doing that in the rest of this work, the type will be clear from the context.

Corollary 1.3.4. *Let $R \in \mathrm{Sym}^2(\Lambda^2)$ be the type $(4,0)$ Riemannian curvature tensor obtained by contraction with the metric: $R(W,X,Y,Z) := g(R(W,X)Y,Z)$. Then*

$$\|(\nabla_X J)Y\|^2 = R(X,Y,JX,JY) - R(X,Y,X,Y), \quad X,Y \in \mathfrak{X}(M). \tag{1.10}$$

Proof. Since $\nabla\sigma$ is a three-form, $\nabla^2\sigma(A,B,B,C) = 0$. We can then combine the results found in Lemma 1.3.2 to get

$$\begin{aligned}
\|(\nabla_X J)Y\|^2 = \|(\nabla_X J)JY\|^2 &= -\nabla^2\sigma(X,X,Y,JY) \\
&= \nabla^2\sigma(X,Y,X,JY) - \nabla^2\sigma(Y,X,X,JY) \\
&= \sigma(R(Y,X)X,JY) + \sigma(X,R(Y,X)JY) \\
&= g(R(Y,X)X,Y) - g(R(Y,X)JX,JY) \\
&= R(X,Y,JX,JY) - R(X,Y,X,Y),
\end{aligned}$$

which was our claim. □

Formula (1.10) gives a way to calculate the norm of $(\nabla_X J)Y$—hence the function μ in (1.9)—in terms of the curvature tensor. A consequence of it is that R is invariant under the action of J. To see this, define the tensor $S(W,X,Y,Z) := R(JW,JX,JY,JZ)$. Of course S inherites the properties of algebraic curvature tensors, namely $S \in \Lambda^2 \otimes \Lambda^2$ and satisfies the first Bianchi identity

$$S(W,X,Y,Z) + S(X,Y,W,Z) + S(Y,W,X,Z) = 0.$$

To show $R = S$ we can check $R(X,Y,Y,X) = S(X,Y,Y,X)$. A straightforward calculation proves the claim:

$$\begin{aligned} R(JX,JY,JY,JX) - R(X,Y,Y,X) &= R(JX,JY,JY,JX) - R(X,Y,JY,JX) \\ &\quad + R(X,Y,JY,JX) - R(X,Y,Y,X) \\ &= \|(\nabla_{JX}J)JY\|^2 - \|(\nabla_X J)Y\|^2 = 0. \end{aligned}$$

The identity just obtained allows us to carry out a polarisation process giving a way to measure inner products of vectors of the form $(\nabla_X J)Y$ in terms of the curvature.

Lemma 1.3.5. *For every quadruple of vector fields W,X,Y,Z we have the formula*

$$g((\nabla_W J)X, (\nabla_Y J)Z) = R(W,X,JY,JZ) - R(W,X,Y,Z). \tag{1.11}$$

Proof. Mapping $X \mapsto A+B$ in formula (1.10) one has

$$\begin{aligned} \|(\nabla_{A+B}J)Y\|^2 &= R(A+B,Y,JA+JB,JY) - R(A+B,Y,A+B,Y) \\ &= R(A,Y,JA,JY) - R(A,Y,A,Y) \\ &\quad + R(B,Y,JB,JY) - R(B,Y,B,Y) \\ &\quad + R(A,Y,JB,JY) - R(A,Y,B,Y) \\ &\quad + R(B,Y,JA,JY) - R(B,Y,A,Y). \end{aligned}$$

The left hand side is

$$\|(\nabla_{A+B}J)Y\|^2 = \|(\nabla_A J)Y\|^2 + \|(\nabla_B J)Y\|^2 + 2g((\nabla_A J)Y, (\nabla_B J)Y),$$

so applying once again (1.10) we find

$$\begin{aligned} 2g((\nabla_A J)Y, (\nabla_B J)Y) &= R(A,Y,JB,JY) - R(A,Y,B,Y) \\ &\quad + R(B,Y,JA,JY) - R(B,Y,A,Y). \end{aligned}$$

Putting now $Y \mapsto C+D$, we expand $2g((\nabla_A J)(C+D), (\nabla_B J)(C+D))$ and obtain the expression

$$\begin{aligned} &R(A,C,JB,JC) - R(A,C,B,C) + R(A,C,JB,JD) - R(A,C,B,D) \\ &\quad + R(A,D,JB,JC) - R(A,D,B,C) + R(A,D,JB,JD) - R(A,D,B,D) \\ &\quad + R(B,C,JA,JC) - R(B,C,A,C) + R(B,C,JA,JD) - R(B,C,A,D) \\ &\quad + R(B,D,JA,JC) - R(B,D,A,C) + R(B,D,JA,JD) - R(B,D,A,D). \end{aligned}$$

Linearity in the various arguments implies

$$\begin{aligned} &2g((\nabla_A J)(C+D), (\nabla_B J)(C+D)) \\ &\quad = 2\big(g((\nabla_A J)C, (\nabla_B J)C) + g((\nabla_A J)C, (\nabla_B J)D) \\ &\qquad + g((\nabla_A J)D, (\nabla_B J)C) + g((\nabla_A J)D, (\nabla_B J)D)\big). \end{aligned}$$

Simplifying we are left with

$$\begin{aligned} g((\nabla_A J)C, (\nabla_B J)D) + g((\nabla_A J)D, (\nabla_B J)C) \\ = R(A,C,JB,JD) - R(A,C,B,D) \\ + R(A,D,JB,JC) - R(A,D,B,C). \end{aligned} \tag{1.12}$$

Set $L(A,B,C,D) := R(A,B,C,D) + R(A,D,C,B)$. The first Bianchi identity, together with (1.12), gives

$$\begin{aligned} 0 &= R(A,B,C,D) + R(B,C,A,D) + R(C,A,B,D) \\ &= R(A,B,C,D) - R(C,B,A,D) + \big(L(C,A,B,D) - R(C,D,B,A)\big) \\ &= R(A,B,C,D) + \big(L(C,A,B,D) + R(A,B,C,D)\big) \\ &\quad - \big(L(C,B,A,D) - R(C,D,A,B)\big) \\ &= 3R(A,B,C,D) + L(C,A,B,D) - L(C,B,A,D) \\ &= 3R(A,B,C,D) + \big(R(C,A,B,D) + R(C,D,B,A)\big) \\ &\quad - \big(R(C,B,A,D) + R(C,D,A,B)\big) \\ &= 3R(A,B,C,D) - 2R(C,D,JA,JB) \\ &\quad + R(C,A,JB,JD) - R(C,B,JA,JD) \\ &\quad + 2g((\nabla_C J)D, (\nabla_A J)B) - g((\nabla_C J)A, (\nabla_B J)D) + g((\nabla_C J)B, (\nabla_A J)D). \end{aligned} \tag{1.13}$$

Now we set $C \mapsto JC, D \mapsto JD$:

$$\begin{aligned} 0 &= 3R(A,B,JC,JD) - 2R(JC,JD,JA,JB) - R(JC,A,JB,D) + R(JC,B,JA,D) \\ &\quad + 2g((\nabla_{JC} J)JD, (\nabla_A J)B) - g((\nabla_{JC} J)A, (\nabla_B J)JD) + g((\nabla_{JC} J)B, (\nabla_A J)JD). \end{aligned}$$

Using that R is J-invariant, the difference between the latter and (1.13) becomes

$$\begin{aligned} 0 &= 3R(A,B,JC,JD) - 2R(JC,JD,JA,JB) - R(JC,A,JB,D) + R(JC,B,JA,D) \\ &\quad + 2g((\nabla_{JC} J)JD, (\nabla_A J)B) - g((\nabla_{JC} J)A, (\nabla_B J)JD) + g((\nabla_{JC} J)B, (\nabla_A J)JD) \\ &\quad - 3R(A,B,C,D) + 2R(C,D,JA,JB) - R(C,A,JB,JD) + R(C,B,JA,JD) \\ &\quad - 2g((\nabla_C J)D, (\nabla_A J)B) + g((\nabla_C J)A, (\nabla_B J)D) - g((\nabla_C J)B, (\nabla_A J)D) \\ &= 5R(A,B,JC,JD) - 5R(A,B,C,D) - R(A,C,JD,JB) - R(A,JC,D,JB) \\ &\quad - R(A,D,JB,JC) - R(A,JD,JB,C) - 4g((\nabla_A J)B, (\nabla_C J)D). \end{aligned}$$

Applying the first Bianchi identity once again we have

$$\begin{aligned} 4g((\nabla_A J)B, (\nabla_C J)D) = 5R(A,B,JC,JD) - 5R(A,B,C,D) \\ + R(A,JB,C,JD) + R(A,JB,JC,D). \end{aligned} \tag{1.14}$$

Now map $B \mapsto JB, C \mapsto JC$ and add a fifth of the result to (1.14):

$$\begin{aligned} \tfrac{24}{5} g((\nabla_A J)JB, (\nabla_{JC} J)D) &= -R(A,JB,C,JD) - R(A,JB,JC,D) \\ &\quad - \tfrac{1}{5}R(A,B,JC,JD) + \tfrac{1}{5}R(A,B,C,D) \\ &\quad + 5R(A,B,JC,JD) - 5R(A,B,C,D) \\ &\quad + R(A,JB,C,JD) + R(A,JB,JC,D) \\ &= \tfrac{24}{5}R(A,B,JC,JD) - \tfrac{24}{5}R(A,B,C,D). \end{aligned}$$

Since $g((\nabla_A J)JB, (\nabla_{JC} J)D) = g((\nabla_A J)B, (\nabla_C J)D)$ we are done. □

Lemma 1.3.6. *Let $W, X, Y, Z \in \mathfrak{X}(M)$. The following formula holds:*

$$2\nabla^2\sigma(W, X, Y, Z) = -\underset{X,Y,Z}{\mathfrak{S}} g((\nabla_W J)X, (\nabla_Y J)JZ). \tag{1.15}$$

Proof. Combine the first formula in Lemma 1.3.2 and identity (1.11):

$$\begin{aligned}
\nabla^2\sigma(W, X, Y, Z) - \nabla^2\sigma(X, W, Y, Z) &= g(JR(X, W)Y, Z) + g(JY, R(X, W)Z)\\
&= g(JR(X, W)Y, Z) - g(R(X, W)JY, Z)\\
&= g((R(X, W)J)Y, J^2 Z)\\
&= R(X, W, JY, J^2 Z) - R(X, W, Y, JZ)\\
&= g((\nabla_X J)W, (\nabla_Y J)JZ).
\end{aligned} \tag{1.16}$$

On the other hand

$$\begin{aligned}
\nabla^2\sigma(W, W, Y, Z) &= -\nabla^2\sigma(W, Y, W, Z)\\
&= \nabla^2\sigma(Y, W, W, Z) - \nabla^2\sigma(W, Y, W, Z)\\
&= g((\nabla_W J)Y, (\nabla_W J)JZ).
\end{aligned}$$

Polarising the latter, one obtains

$$\begin{aligned}
\nabla^2\sigma(W + X, W + X, Y, Z) &= \nabla^2\sigma(W, W, Y, Z) + \nabla^2\sigma(W, X, Y, Z)\\
&\quad + \nabla^2\sigma(X, W, Y, Z) + \nabla^2\sigma(X, X, Y, Z)\\
&= g((\nabla_W J)Y, (\nabla_W J)JZ) + g((\nabla_X J)Y, (\nabla_X J)JZ)\\
&\quad + \nabla^2\sigma(W, X, Y, Z) + \nabla^2\sigma(X, W, Y, Z),
\end{aligned}$$

whence

$$\begin{aligned}
&\nabla^2\sigma(W, X, Y, Z) + \nabla^2\sigma(X, W, Y, Z)\\
&\quad = -g((\nabla_W J)Y, (\nabla_W J)JZ) - g((\nabla_X J)Y, (\nabla_X J)JZ)\\
&\qquad + g((\nabla_{W+X} J)Y, (\nabla_{W+X} J)JZ)\\
&\quad = g((\nabla_W J)Y, (\nabla_X J)JZ) + g((\nabla_X J)Y, (\nabla_W J)JZ).
\end{aligned} \tag{1.17}$$

Adding (1.16) to (1.17) and using usual symmetries of ∇J the claim follows. □

The results got so far are technical, but rather explicit. Our need to be so concrete and maybe pedantic partly stems from a lack of details in the literature. For instance, the proof of Lemma 1.3.5 by Gray (cf. [Gra70, Proposition 2.1]) glosses over a number of intermediate steps, making partial results not completely clear at a first reading. A second issue, which we claim to solve here, is proving explicitly that μ is constant: this was done in the first place by Gray, showing that nearly Kähler six-manifolds are Einstein [Gra76, Theorem 5.2]. However, proving directly the constancy of μ is perhaps easier, so we pursue this point of view here. In this last part we follow and fill in the results of Morris [Mor14, Section 4.2]. What is more, a note on the fact that μ is constant is given by Carrión in a remark right after Lemma 4.8 [Car93], but there seems to be no direct proof. One of the purposes of these first sections is in effect to provide a comprehensive and detailed summary of well-known results appearing in the literature.

It is high time now to define the Ricci and the Ricci-$*$ endomorphisms. Hereafter we still assume the dimension of M be even, not necessarily six. We go back to the six-dimensional case in Proposition 1.3.9, where we prove that μ is constant on M.

Definition 1.3.7. Given any local, orthonormal frame $E_1, \dots, E_{2n}$ we define the Ricci and the Ricci-$*$ endomorphisms $\mathrm{Ric}, \mathrm{Ric}^* \in TM \otimes TM$ by

$$g(\mathrm{Ric}X, Y) := \sum_{i=1}^{2n} R(X, E_i, E_i, Y), \quad g(\mathrm{Ric}^* X, Y) := \sum_{i=1}^{2n} R(X, E_i, JE_i, JY).$$

Because of (1.11) we can write their difference as

$$g((\mathrm{Ric} - \mathrm{Ric}^*)X, Y) = \sum_{i=1}^{2n} g((\nabla_X J)E_i, (\nabla_Y J)E_i). \tag{1.18}$$

Obviously $\mathrm{Ric} - \mathrm{Ric}^*$ is self-adjoint, and so is its covariant derivative: for any self-adjoint operator A we have in fact

$$\begin{aligned} g((\nabla_Z A)X, Y) &= g(\nabla_Z AX, Y) - g(A\nabla_Z X, Y) \\ &= Z(g(AX, Y)) - g(\nabla_Z X, AY) - g(AX, \nabla_Z Y) \\ &= Z(g(X, AY)) - g(\nabla_Z X, AY) - g(X, A\nabla_Z Y) \\ &= g(X, \nabla_Z AY) - g(X, A\nabla_Z Y) = g(X, (\nabla_Z A)Y). \end{aligned}$$

Moreover, $\mathrm{Ric} - \mathrm{Ric}^*$ and J commute: put now $A := \mathrm{Ric} - \mathrm{Ric}^*$, so that

$$\begin{aligned} g(JAX, Y) &= -g(AX, JY) = -\sum_i g((\nabla_X J)E_i, (\nabla_{JY} J)E_i) \\ &= -\sum_i g(J(\nabla_X J)E_i, (\nabla_Y J)E_i) = \sum_i g((\nabla_{JX} J)E_i, (\nabla_Y J)E_i) = g(AJX, Y). \end{aligned}$$

We can then prove a last useful result.

Lemma 1.3.8. *For $X, Y, Z \in \mathfrak{X}(M)$ we have the following formula:*

$$\begin{aligned} 2g((\nabla_Z(\mathrm{Ric} - \mathrm{Ric}^*))X, Y) = g((\mathrm{Ric} - \mathrm{Ric}^*)JX, (\nabla_Z J)Y) \\ + g((\mathrm{Ric} - \mathrm{Ric}^*)JY, (\nabla_Z J)X). \end{aligned} \tag{1.19}$$

Proof. Start differentiating (1.18) with $X = Y$, still with $A := \mathrm{Ric} - \mathrm{Ric}^*$:

$$\begin{aligned} g((\nabla_Z A)X, X) + 2g(AX, \nabla_Z X) &= Z(g(AX, X)) \\ &= 2\sum_{i=1}^{2n} g(\nabla_Z((\nabla_X J)E_i), (\nabla_X J)E_i). \end{aligned}$$

Rearranging the terms

$$g((\nabla_Z A)X, X) = 2\sum_{i=1}^{2n} g(\nabla_Z((\nabla_X J)E_i), (\nabla_X J)E_i) - g((\nabla_X J)E_i, (\nabla_{\nabla_Z X} J)E_i). \tag{1.20}$$

Note that $\sum_{i=1}^{2n} g((\nabla_X J)\nabla_Z E_i, (\nabla_X J)E_i) = 0$: setting $\nabla_Z E_i = \sum_{j=1}^{2n} B_i^j E_j$ we have $0 = Z(g(E_i, E_j)) = g(\nabla_Z E_i, E_j) + g(E_i, \nabla_Z E_j) = \sum_k B_i^k \delta_{kj} + \sum_r B_j^r \delta_{ir} = B_i^j + B_j^i$. Thus

$$\begin{aligned} \sum_{i=1}^{2n} g((\nabla_X J)\nabla_Z E_i, (\nabla_X J)E_i) &= \sum_{i,j=1}^{2n} g((\nabla_X J)B_i^j E_j, (\nabla_X J)E_i) \\ &= -\sum_{i,j=1}^{2n} g((\nabla_X J)E_j, (\nabla_X J)B_j^i E_i) \\ &= -\sum_{j=1}^{2n} g((\nabla_X J)E_j, (\nabla_X J)\nabla_Z E_j) = 0. \end{aligned}$$

This last term appears in the expansion of $\nabla^2\sigma(Z, X, (\nabla_X J)E_i, E_i)$ as well. Simplifying we get

$$\begin{aligned}\nabla^2\sigma(Z, X, (\nabla_X J)E_i, E_i) &= -Z(g((\nabla_X J)E_i, (\nabla_X J)E_i)) - g((\nabla_{\nabla_Z X} J)(\nabla_X J)E_i, E_i)\\ &\quad - g((\nabla_X J)\nabla_Z((\nabla_X J)E_i), E_i) - g((\nabla_X J)(\nabla_X J)E_i, \nabla_Z E_i)\\ &= -2g(\nabla_Z((\nabla_X J)E_i), (\nabla_X J)E_i) + g((\nabla_X J)E_i, (\nabla_{\nabla_Z X} J)E_i)\\ &\quad + g(\nabla_Z((\nabla_X J)E_i), (\nabla_X J)E_i) - g((\nabla_X J)(\nabla_X J)E_i, \nabla_Z E_i)\\ &= g((\nabla_X J)E_i, (\nabla_{\nabla_Z X} J)E_i) - g((\nabla_Z(\nabla_X J))E_i, (\nabla_X J)E_i)\\ &\quad + g((\nabla_X J)E_i, (\nabla_X J)\nabla_Z E_i).\end{aligned}$$

Therefore, by formula (1.15), identity (1.20) becomes

$$\begin{aligned}&g((\nabla_Z(\mathrm{Ric} - \mathrm{Ric}^*))X, X)\\ &\quad = 2\sum_{i=1}^{2n} g(\nabla_Z((\nabla_X J)E_i), (\nabla_X J)E_i) - g((\nabla_X J)E_i, (\nabla_{\nabla_Z X} J)E_i)\\ &\quad = -2\sum_{i=1}^{2n} \nabla^2\sigma(Z, X, (\nabla_X J)E_i, E_i)\\ &\quad = \sum_{i=1}^{2n} g((\nabla_Z J)X, (\nabla_{(\nabla_X J)E_i} J)JE_i) + g((\nabla_Z J)(\nabla_X J)E_i, (\nabla_{E_i} J)JX)\\ &\qquad + g((\nabla_Z J)E_i, (\nabla_X J)J(\nabla_X J)E_i)\\ &\quad = \sum_{i=1}^{2n} g((\nabla_{E_i} J)(\nabla_Z J)X, J(\nabla_X J)E_i) + g((\nabla_Z J)(\nabla_X J)E_i, J(\nabla_X J)E_i)\\ &\qquad + g((\nabla_X J)(\nabla_Z J)E_i, (\nabla_X J)JE_i).\end{aligned}$$

The second term in the latter sum is readily seen to vanish, because of (1.4). The sum $\sum_i g((\nabla_X J)(\nabla_Z J)E_i, (\nabla_X J)JE_i)$ vanishes as well. To see this, we set $A := J(\nabla_Z J)$. In the first place A lies in $\mathfrak{so}(2n)$, because

$$g(J(\nabla_Z J)E_i, E_j) = g((\nabla_Z J)JE_j, E_i) = -g(J(\nabla_Z J)E_j, E_i).$$

Consequently, the following chain of identities leads to our claim:

$$\begin{aligned}&\sum_{i=1}^{2n} g((\nabla_X J)(\nabla_Z J)E_i, (\nabla_X J)JE_i)\\ &\quad = -\sum_{i=1}^{2n} g((\nabla_X J)J(\nabla_Z J)E_i, (\nabla_X J)E_i)\\ &\quad = -\sum_{i,j=1}^{2n} g((\nabla_X J)A_i^j E_j, (\nabla_X J)E_i)\\ &\quad = \sum_{i,j=1}^{2n} g((\nabla_X J)E_j, (\nabla_X J)A_j^i E_i) = \sum_{j=1}^{2n} g((\nabla_X J)J(\nabla_Z J)E_j, (\nabla_X J)E_j) = 0.\end{aligned}$$

We then go back to our first expansion recalling that $\mathrm{Ric} - \mathrm{Ric}^*$ commutes with J:

$$\begin{aligned}
-2\sum_{i=1}^{2n} \nabla^2\sigma(Z, X, (\nabla_X J)E_i, E_i) &= \sum_{i=1}^{2n} g((\nabla_{E_i} J)(\nabla_Z J)X, J(\nabla_X J)E_i) \\
&= -\sum_{i=1}^{2n} g((\nabla_{J(\nabla_Z J)X} J)E_i, (\nabla_X J)E_i) \\
&= -g((\mathrm{Ric} - \mathrm{Ric}^*)J(\nabla_Z J)X, X) \\
&= g((\mathrm{Ric} - \mathrm{Ric}^*)JX, (\nabla_Z J)X).
\end{aligned}$$

Thus $g((\nabla_Z(\mathrm{Ric} - \mathrm{Ric}^*))X, X) = g((\mathrm{Ric} - \mathrm{Ric}^*)JX, (\nabla_Z J)X)$. By polarisation and the symmetry of $\nabla_Z(\mathrm{Ric} - \mathrm{Ric}^*)$ the result follows. □

Let us restrict ourselves to the six-dimensional case now, so $n = 3$. Recall that in Lemma 1.3.1 we proved the existence of a special function μ on M satisfying (1.9).

Proposition 1.3.9. *If M is a connected nearly Kähler six-manifold, the function μ is constant.*

Proof. We only prove μ is locally constant, then the claim follows from the connectedness of M. Mapping X into $A + B$ in (1.9) one has

$$g((\nabla_{A+B} J)Y, (\nabla_{A+B} J)Y) = \mu^2\big(\|A + B\|^2\|Y\|^2 - g(A + B, Y)^2 - \sigma(A + B, Y)^2\big),$$

which can be simplified as

$$g((\nabla_A J)Y, (\nabla_B J)Y) = \mu^2\big(g(A, B)\|Y\|^2 - g(A, Y)g(B, Y) - g(JA, Y)g(JB, Y)\big).$$

On the other hand

$$\begin{aligned}
g((\mathrm{Ric} - \mathrm{Ric}^*)A, B) &= \sum_{i=1}^{3} g((\nabla_A J)E_i, (\nabla_B J)E_i) + g((\nabla_A J)JE_i, (\nabla_B J)JE_i) \\
&= \mu^2\big(6g(A, B) - g(A, B) - g(JA, JB)\big) = 4\mu^2 g(A, B).
\end{aligned}$$

Thus $\mathrm{Ric} - \mathrm{Ric}^* = 4\mu^2\,\mathrm{Id}$, but now formula (1.19) implies

$$\begin{aligned}
2g((\nabla_Z(\mathrm{Ric} - \mathrm{Ric}^*))X, Y) &= g((\mathrm{Ric} - \mathrm{Ric}^*)JX, (\nabla_Z J)Y) + g((\mathrm{Ric} - \mathrm{Ric}^*)JY, (\nabla_Z J)X) \\
&= 4\mu^2\big(g(JX, (\nabla_Z J)Y) + g(JY, (\nabla_Z J)X)\big) = 0.
\end{aligned}$$

This proves $\nabla_Z(\mathrm{Ric} - \mathrm{Ric}^*) = 0 = 4Z(\mu^2)\,\mathrm{Id}$ for every Z, hence μ is non-zero and locally constant. □

We have thus proved that on a connected nearly Kähler six-manifold there exists a constant μ such that

$$\|(\nabla_X J)Y\|^2 = \mu^2\big(\|X\|^2\|Y\|^2 - g(X, Y)^2 - \sigma(X, Y)^2\big), \quad X, Y \in \mathfrak{X}(M).$$

Using the terminology introduced by Gray [Gra70, Proposition 3.5] we say that connected nearly Kähler six-manifolds have *global constant type*.

1.4 The Einstein condition

The aim of this section is to push our calculations further in order to prove that nearly Kähler six-manifolds are Einstein. We follow [Gra76] to do this. We first introduce a connection adapted to the $\mathrm{U}(3)$-structure (g, J). A quick computation of the torsion of J will help us go smoothly towards it. We then work out some relevant symmetries satisfied by the curvature tensor of the new connection. We conclude proving that $\mathrm{Ric}_g = 5\mu^2 g$, where Ric_g is the Ricci curvature $(2,0)$-tensor of the Levi-Civita connection and μ is the function defined in (1.9). Since connected nearly Kähler six-manifolds have global constant type, the Einstein condition will follow.

Let us now compute the Nijenhuis tensor of J, i.e. the type $(2,1)$-tensor field N on M defined by

$$4N(X,Y) := [X,Y] - [JX,JY] + J[JX,Y] + J[X,JY], \quad X,Y \in \mathfrak{X}(M).$$

Proposition 1.4.1. *If M is nearly Kähler then $N(X,Y) = J(\nabla_X J)Y$, where $X, Y \in \mathfrak{X}(M)$.*

Proof. The key property we use here is that the Levi-Civita connection ∇ is torsion-free. Since ∇J is skew-symmetric and anti-commutes with J, the following formulas for the commutators hold:

$$\begin{aligned} [X,Y] &= \nabla_X Y - \nabla_Y X, & [JX,JY] &= -2J(\nabla_X J)Y + J\nabla_{JX}Y - J\nabla_{JY}X, \\ J[JX,Y] &= J(\nabla_{JX}Y - \nabla_Y JX), & J[X,JY] &= J(\nabla_X JY - \nabla_{JY}X). \end{aligned}$$

A straightforward calculation gives

$$\begin{aligned} 4N(X,Y) &= \nabla_X Y - \nabla_Y X + 2J(\nabla_X J)Y + J\nabla_X JY - J\nabla_Y JX \\ &= 2J(\nabla_X J)Y + J(\nabla_X J)Y - J(\nabla_Y J)X \\ &= 4J(\nabla_X J)Y, \end{aligned}$$

and we are done. □

Note that Proposition 1.4.1 is valid for every $2n$-dimensional nearly Kähler manifold. We can then use it to construct a Hermitian connection, i.e. a connection preserving the Riemannian metric and the almost complex structure: the difference $\nabla - \frac{1}{2}N$ defines a covariant derivative $\widehat{\nabla}$:

$$\widehat{\nabla}_X Y := \nabla_X Y - \tfrac{1}{2}J(\nabla_X J)Y, \quad X,Y \in \mathfrak{X}(M).$$

Identity (1.5) implies that $\widehat{\nabla}$ is a $\mathrm{U}(n)$-connection—i.e. a connection on a subbundle of the canonical frame bundle having structure group $\mathrm{U}(n)$—if and only if g and J are parallel with respect to it, namely $\widehat{\nabla} g = 0$ and $\widehat{\nabla} J = 0$. In fact, we have the following

Proposition 1.4.2. *$\widehat{\nabla}$ is a $\mathrm{U}(n)$-connection.*

Remark 1.4.3. In Proposition 1.5.7 below we prove that on nearly Kähler six-manifolds $\widehat{\nabla}$ is actually an $\mathrm{SU}(3)$-connection, first exhibiting a complex volume form $\psi_{\mathbb{C}}$ on M and then proving it is $\widehat{\nabla}$-parallel.

Proof. We show $\widehat{\nabla} g = 0$ and $\widehat{\nabla}\sigma = 0$. These two identities imply $\widehat{\nabla} J = 0$ because for every $X, Y \in \mathfrak{X}(M)$ one has

$$0 = \widehat{\nabla}\sigma(X, Y, \cdot\,) = g((\widehat{\nabla}_X J)Y, \cdot\,),$$

which gives the claim. Since $\nabla g = 0$, ∇J anti-commutes with J and is skew-adjoint

$$\begin{aligned}\widehat{\nabla} g(X,Y,Z) &= X(g(Y,Z)) - g(\widehat{\nabla}_X Y, Z) - g(Y, \widehat{\nabla}_X Z)\\ &= X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)\\ &\quad + \tfrac{1}{2}\big(g(J(\nabla_X J)Y, Z) + g(Y, J(\nabla_X J)Z)\big)\\ &= \tfrac{1}{2}\big(g(Y, (\nabla_X J)JZ)) - g(Y, (\nabla_X J)JZ)\big) = 0.\end{aligned}$$

Finally we check $\widehat{\nabla}\sigma = 0$:

$$\begin{aligned}\widehat{\nabla}\sigma(X,Y,Z) &= X(\sigma(Y,Z)) - \sigma(\widehat{\nabla}_X Y, Z) - \sigma(Y, \widehat{\nabla}_X Z)\\ &= X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)\\ &\quad + \tfrac{1}{2}\big(\sigma(J(\nabla_X J)Y, Z) + \sigma(Y, J(\nabla_X J)Z)\big)\\ &= \nabla\sigma(X,Y,Z) - \tfrac{1}{2}\big(g((\nabla_X J)Y, Z) - g(Y, (\nabla_X J)Z)\big)\\ &= \nabla\sigma(X,Y,Z) - \tfrac{1}{2}\big(\nabla\sigma(X,Y,Z) + \nabla\sigma(X,Y,Z)\big) = 0,\end{aligned}$$

so the result follows. □

Let us call $\widehat{R}$ the curvature tensor of $\widehat{\nabla}$: $\widehat{R}(W,X)Y := \widehat{\nabla}_W\widehat{\nabla}_X Y - \widehat{\nabla}_X\widehat{\nabla}_W Y - \widehat{\nabla}_{[W,X]}Y$. More explicitly:

$$\begin{aligned}\widehat{\nabla}_W\widehat{\nabla}_X Y &= \nabla_W\big(\nabla_X Y - \tfrac{1}{2}J(\nabla_X J)Y\big) - \tfrac{1}{2}J(\nabla_W J)\big(\nabla_X Y - \tfrac{1}{2}J(\nabla_X J)Y\big)\\ &= \nabla_W\nabla_X Y - \tfrac{1}{2}(\nabla_W J)(\nabla_X J)Y - \tfrac{1}{2}J(\nabla_W(\nabla_X J))Y\\ &\quad - \tfrac{1}{2}J(\nabla_X J)\nabla_W Y - \tfrac{1}{2}J(\nabla_W J)\nabla_X Y + \tfrac{1}{4}(\nabla_W J)(\nabla_X J)Y\\ &= \nabla_W\nabla_X Y - \tfrac{1}{4}(\nabla_W J)(\nabla_X J)Y - \tfrac{1}{2}J(\nabla_W(\nabla_X J))Y\\ &\quad - \tfrac{1}{2}J(\nabla_X J)\nabla_W Y - \tfrac{1}{2}J(\nabla_W J)\nabla_X Y.\end{aligned}$$

Switching the roles of W and X one obviously has

$$\begin{aligned}\widehat{\nabla}_X\widehat{\nabla}_W Y &= \nabla_X\nabla_W Y - \tfrac{1}{4}(\nabla_X J)(\nabla_W J)Y - \tfrac{1}{2}J(\nabla_X(\nabla_W J))Y\\ &\quad - \tfrac{1}{2}J(\nabla_W J)\nabla_X Y - \tfrac{1}{2}J(\nabla_X J)\nabla_W Y,\end{aligned}$$

and then the whole curvature tensor is

$$\begin{aligned}\widehat{R}(W,X)Y &= \nabla_W\nabla_X Y - \tfrac{1}{4}(\nabla_W J)(\nabla_X J)Y - \tfrac{1}{2}J(\nabla_W(\nabla_X J))Y\\ &\quad - \tfrac{1}{2}J(\nabla_X J)\nabla_W Y - \tfrac{1}{2}J(\nabla_W J)\nabla_X Y\\ &\quad - \nabla_X\nabla_W Y + \tfrac{1}{4}(\nabla_X J)(\nabla_W J)Y + \tfrac{1}{2}J(\nabla_X(\nabla_W J))Y\\ &\quad + \tfrac{1}{2}J(\nabla_W J)\nabla_X Y + \tfrac{1}{2}J(\nabla_X J)\nabla_W Y\\ &\quad - \nabla_{[W,X]}Y + \tfrac{1}{2}J(\nabla_{[W,X]}J)Y\\ &= R(W,X)Y + \tfrac{1}{4}\big((\nabla_X J)(\nabla_W J)Y - (\nabla_W J)(\nabla_X J)Y\big)\\ &\quad - \tfrac{1}{2}J(R(W,X)JY - JR(W,X)Y).\end{aligned}$$

A contraction with the metric and identity (1.11) applied to the last term yield a type $(4,0)$-tensor field, which we still denote by $\widehat{R}$. Its expression is

$$\begin{aligned}\widehat{R}(W,X,Y,Z) = R(W,X,Y,Z) &+ \tfrac{1}{2}g((\nabla_W J)X,(\nabla_Y J)Z)\\ &+ \tfrac{1}{4}\big(g((\nabla_X J)Y,(\nabla_W J)Z) - g((\nabla_W J)Y,(\nabla_X J)Z)\big).\end{aligned} \tag{1.21}$$

We can go a bit further rewriting every summand in terms of the curvature tensor R: by formula (1.11) and the first Bianchi identity, equation (1.21) becomes

$$\begin{aligned}\widehat{R}(W,X,Y,Z) &= R(W,X,Y,Z) + \tfrac{1}{2}\big(R(W,X,JY,JZ) - R(W,X,Y,Z)\big)\\ &\quad + \tfrac{1}{4}\big(R(X,Y,JW,JZ) - R(X,Y,W,Z)\big)\\ &\quad\quad - R(W,Y,JX,JZ) + R(W,Y,X,Z)\big)\\ &= \tfrac{1}{4}\big(3R(W,X,Y,Z) + 2R(W,X,JY,JZ)\\ &\quad + R(X,Y,JW,JZ) - R(W,Y,JX,JZ)\big).\end{aligned}$$

Recalling that R is J-invariant and lies in $\mathrm{Sym}^2(\Lambda^2)$ we obtain the final expression

$$\begin{aligned}\widehat{R}(W,X,Y,Z) = \tfrac{1}{4}\big(&3R(W,X,Y,Z) + 2R(W,X,JY,JZ)\\ &+ R(W,Z,JX,JY) + R(W,Y,JZ,JX)\big).\end{aligned} \tag{1.22}$$

One may use the latter equation to check whether $\widehat{R}$ satisfies the usual symmetries of algebraic curvature tensors, but it turns out that the first Bianchi identity does not hold: since R is J-invariant we calculate

$$\begin{aligned}&\widehat{R}(W,X,Y,Z) + \widehat{R}(X,Y,W,Z) + \widehat{R}(Y,W,X,Z)\\ &\quad = \tfrac{1}{4}\big(3R(W,X,Y,Z) + 3R(X,Y,W,Z) + 3R(Y,W,X,Z)\\ &\quad\quad + 2R(W,X,JY,JZ) + 2R(X,Y,JW,JZ) + 2R(Y,W,JX,JZ)\\ &\quad\quad + R(W,Z,JX,JY) + R(X,Z,JY,JW) + R(Y,Z,JW,JX)\\ &\quad\quad + R(W,Y,JZ,JX) + R(X,W,JZ,JY) + R(Y,X,JZ,JW)\big)\\ &\quad = \tfrac{1}{4}\big(3R(W,X,JY,JZ) + 3R(X,Y,JW,JZ) + 3R(Y,W,JX,JZ)\\ &\quad\quad + R(W,X,JY,JZ) + R(X,Y,JW,JZ) + R(Y,W,JX,JZ)\big)\\ &\quad = \underset{W,X,Y}{\mathfrak{S}} R(W,X,Y,Z) + g((\nabla_W J)X,(\nabla_Y J)Z) = \underset{W,X,Y}{\mathfrak{S}} g((\nabla_W J)X,(\nabla_Y J)Z).\end{aligned}$$

Nevertheless the remaining properties still hold true.

Lemma 1.4.4. *The tensor $\widehat{R}$ lies in $\Lambda^2 \otimes [\Lambda^{1,1}]$.*

Proof. Skew-symmetry in the first two arguments is straightforward by definition of $\widehat{R}$. That $\widehat{R}(W,X) \in [\Lambda^{1,1}]$ is a consequence of Proposition 1.4.2:

$$\begin{aligned}\widehat{R}(W,X,JY,JZ) &= g(\widehat{\nabla}_W\widehat{\nabla}_X JY,JZ) - g(\widehat{\nabla}_X\widehat{\nabla}_W JY,JZ) - g(\widehat{\nabla}_{[W,X]}JY,JZ)\\ &= g(J\widehat{\nabla}_W\widehat{\nabla}_X Y,JZ) - g(J\widehat{\nabla}_X\widehat{\nabla}_W Y,JZ) - g(J\widehat{\nabla}_{[W,X]}Y,JZ)\\ &= \widehat{R}(W,X,Y,Z).\end{aligned}$$

In particular $\widehat{R} \in \Lambda^2 \otimes \Lambda^2$. □

Lemma 1.4.5. *The tensor $\widehat{R}$ sits inside* $\mathrm{Sym}^2(\Lambda^2)$*, and thus $\nabla\widehat{R}$ lies in* $\Lambda^1 \otimes \mathrm{Sym}^2(\Lambda^2)$.

Proof. Lemma 1.4.4 implies that we only need to check $\widehat{R}(W,X,Y,Z) = \widehat{R}(Y,Z,W,X)$. We do this using (1.22) and applying J-invariance of R:

$$\begin{aligned}\widehat{R}(W,X,Y,Z) &= \tfrac{1}{4}\big(3R(W,X,Y,Z) + 2R(W,X,JY,JZ)\\ &\qquad + R(W,Z,JX,JY) + R(W,Y,JZ,JX)\big)\\ &= \tfrac{1}{4}\big(3R(Y,Z,W,X) + 2R(Y,Z,JW,JX)\\ &\qquad + R(Y,X,JZ,JW) + R(Y,W,JX,JZ)\big) = \widehat{R}(Y,Z,W,X).\end{aligned}$$

A straightforward calculation then yields

$$\begin{aligned}\nabla_V\widehat{R}(W,X,Y,Z) &= V(\widehat{R}(W,X,Y,Z)) - \widehat{R}(\nabla_V W,X,Y,Z)\\ &\qquad - \widehat{R}(W,\nabla_V X,Y,Z) - \widehat{R}(W,X,\nabla_V Y,Z) - \widehat{R}(W,X,Y,\nabla_V Z)\\ &= V(\widehat{R}(Y,Z,W,X)) - \widehat{R}(\nabla_V Y,Z,W,X)\\ &\qquad - \widehat{R}(Y,\nabla_V Z,W,X) - \widehat{R}(Y,Z,\nabla_V W,X) - \widehat{R}(Y,Z,W,\nabla_V X)\\ &= \nabla_V\widehat{R}(Y,Z,W,X),\end{aligned}$$

which is the second part of the claim. □

We now want more information about the exact expression of $\nabla\widehat{R}$. Since the next formulas will be rather cumbersome we sometimes avoid to write simple intermediate steps. Also, we keep working on a nearly Kähler manifold of generic dimension $2n$, focussing on the six-dimensional case only at the end, after proving Proposition 1.4.7. Incidentally, in the course of the proof of that result we will need an explicit formula for the cyclic sum $\nabla_V\widehat{R}(W,X,Y,Z) + \nabla_W\widehat{R}(X,V,Y,Z) + \nabla_X\widehat{R}(V,W,Y,Z)$, specifically the case where V,W,X are elements of some local unitary frame. The goal now is to work out this expression.

Let us start computing $\nabla_V\widehat{R}(W,X,Y,Z)$. Differentiating (1.21) one gets

$$\begin{aligned}V(\widehat{R}(W,X,Y,Z)) &= V(R(W,X,Y,Z)) + \tfrac{1}{4}g(\nabla_V((\nabla_X J)Y),(\nabla_W J)Z)\\ &\qquad + \tfrac{1}{4}g((\nabla_X J)Y,\nabla_V((\nabla_W J)Z)) - \tfrac{1}{4}g(\nabla_V((\nabla_W J)Y),(\nabla_X J)Z)\\ &\qquad - \tfrac{1}{4}g((\nabla_W J)Y,\nabla_V((\nabla_X J)Z)) + \tfrac{1}{2}g(\nabla_V((\nabla_W J)X),(\nabla_Y J)Z)\\ &\qquad + \tfrac{1}{2}g((\nabla_W J)X,\nabla_V((\nabla_Y J)Z)).\end{aligned}$$

Expanding both sides and isolating $\nabla_V\widehat{R}(W,X,Y,Z)$ on the left we have

$$\begin{aligned}&\nabla_V\widehat{R}(W,X,Y,Z)\\ &\quad= -\widehat{R}(\nabla_V W,X,Y,Z) - \widehat{R}(W,\nabla_V X,Y,Z) - \widehat{R}(W,X,\nabla_V Y,Z) - \widehat{R}(W,X,Y,\nabla_V Z)\\ &\qquad + R(\nabla_V W,X,Y,Z) + R(W,\nabla_V X,Y,Z) + R(W,X,\nabla_V Y,Z) + R(W,X,Y,\nabla_V Z)\\ &\qquad + \tfrac{1}{4}\big(g((\nabla_V(\nabla_X J))Y + (\nabla_X J)\nabla_V Y,(\nabla_W J)Z)\\ &\qquad\qquad + g((\nabla_X J)Y,(\nabla_V(\nabla_W J))Z + (\nabla_W J)\nabla_V Z)\big)\\ &\qquad - \tfrac{1}{4}\big(g((\nabla_V(\nabla_W J))Y + (\nabla_W J)\nabla_V Y,(\nabla_X J)Z)\\ &\qquad\qquad + g((\nabla_W J)Y,(\nabla_V(\nabla_X J))Z + (\nabla_X J)\nabla_V Z)\big)\\ &\qquad + \tfrac{1}{2}\big(g((\nabla_V(\nabla_W J))X + (\nabla_W J)\nabla_V X,(\nabla_Y J)Z)\\ &\qquad\qquad + g((\nabla_W J)X,(\nabla_V(\nabla_Y J))Z + (\nabla_Y J)\nabla_V Z)\big) + \nabla_V R(W,X,Y,Z).\end{aligned}$$

One can expand the first four summands on the right hand side making use of (1.21). Recall that $(\nabla^2_{A,B}J)C = (\nabla_A(\nabla_B J))C - (\nabla_{\nabla_A B}J)C$, then simplifying we are left with

$$\begin{aligned}
&\nabla_V \widehat{R}(W,X,Y,Z) \\
&\quad = \nabla_V R(W,X,Y,Z) \\
&\qquad + \tfrac{1}{2}\big(g((\nabla^2_{V,W}J)X,(\nabla_Y J)Z) + g((\nabla^2_{V,Y}J)Z,(\nabla_W J)X)\big) \\
&\qquad + \tfrac{1}{4}\big(g((\nabla^2_{V,W}J)Z,(\nabla_X J)Y) + g((\nabla^2_{V,X}J)Y,(\nabla_W J)Z)\big) \\
&\qquad - \tfrac{1}{4}\big(g((\nabla^2_{V,W}J)Y,(\nabla_X J)Z) + g((\nabla^2_{V,X}J)Z,(\nabla_W J)Y)\big).
\end{aligned}$$

Therefore, the second Bianchi identity implies

$$\begin{aligned}
&\nabla_V \widehat{R}(W,X,Y,Z) + \nabla_W \widehat{R}(X,V,Y,Z) + \nabla_X \widehat{R}(V,W,Y,Z) \\
&\quad = \underset{V,W,X}{\mathfrak{S}} \Big(\tfrac{1}{2}g((\nabla^2_{V,Y}J)Z,(\nabla_W J)X) + \tfrac{1}{2}g((\nabla^2_{V,W}J)X,(\nabla_Y J)Z) \\
&\qquad + \tfrac{1}{4}g((\nabla^2_{V,W}J)Z - (\nabla^2_{W,V}J)Z,(\nabla_X J)Y) \\
&\qquad + \tfrac{1}{4}g((\nabla^2_{V,X}J)Y - (\nabla^2_{X,V}J)Y,(\nabla_W J)Z)\Big).
\end{aligned} \tag{1.23}$$

Besides formula (1.23), in the proof of Proposition 1.4.7 we will need a last technical result. Consider an orthonormal frame $E_1,\dots,E_{2n}$ around each point, where $JE_i = E_{n+i}$, for $i = 1,\dots,n$.

Lemma 1.4.6. *Let Y be a vector field on M and $\{E_i, JE_i\}, i = 1,\dots,n$, be a local orthonormal frame. Then the following formula holds:*

$$\sum_{j=1}^{2n} (\nabla^2_{E_j,E_j}J)Y = -(\mathrm{Ric} - \mathrm{Ric}^*)JY. \tag{1.24}$$

Proof. This is a consequence of formula (1.15):

$$\begin{aligned}
g((\nabla^2_{E_j,E_j}J)Y,X) &= \nabla^2\sigma(E_j,E_j,Y,X) \\
&= -\tfrac{1}{2}\big(g((\nabla_{E_j}J)Y,(\nabla_X J)JE_j) + g((\nabla_{E_j}J)X,(\nabla_{E_j}J)JY)\big) \\
&= \tfrac{1}{2}\big(g((\nabla_{E_j}J)Y,J(\nabla_X J)E_j) - g(J(\nabla_{E_j}J)X,(\nabla_{E_j}J)Y)\big) \\
&= \tfrac{1}{2}\big(g((\nabla_{E_j}J)Y,J(\nabla_X J)E_j) + g(J(\nabla_X J)E_j,(\nabla_{E_j}J)Y)\big) \\
&= g((\nabla_{E_j}J)Y,(\nabla_{E_j}J)JX).
\end{aligned}$$

Then summing over j and identity (1.18) give

$$\begin{aligned}
\sum_{j=1}^{2n} g\big((\nabla^2_{E_j,E_j}J)Y,X\big) &= \sum_{j=1}^{2n} g((\nabla_{E_j}J)Y,(\nabla_{E_j}J)JX) \\
&= g((\mathrm{Ric} - \mathrm{Ric}^*)Y,JX) = -g((\mathrm{Ric} - \mathrm{Ric}^*)JY,X),
\end{aligned}$$

because $\mathrm{Ric} - \mathrm{Ric}^*$ commutes with J. □

We are thus ready to prove our final result.

Proposition 1.4.7. *Let W, X be two vector fields on M and $\{E_i, JE_i\}, i = 1, \dots, n$, be a local orthonormal frame. Then*

$$\sum_{i,j=1}^{2n} g((\mathrm{Ric} - \mathrm{Ric}^*)E_i, E_j)\big(R(W, E_i, E_j, X) - 5R(W, E_i, JE_j, JX)\big) = 0. \tag{1.25}$$

Proof. Since $\widehat{R} \in \Lambda^2 \otimes [\Lambda^{1,1}]$ by Lemma 1.4.4 and $JE_i = E_{n+i}$ for $i = 1, \dots, n$, we have

$$\begin{aligned}
\sum_{i=1}^{2n} \widehat{R}(W, X, E_i, (\nabla_V J)E_i) &= \tfrac{1}{2} \sum_{i=1}^{2n} \widehat{R}(W, X, E_i, (\nabla_V J)E_i) + \widehat{R}(W, X, JE_i, J(\nabla_V J)E_i) \\
&= \tfrac{1}{2} \sum_{i=1}^{2n} \widehat{R}(W, X, E_i, (\nabla_V J)E_i) - \widehat{R}(W, X, JE_i, (\nabla_V J)JE_i) \\
&= \tfrac{1}{2} \sum_{i=1}^{n} \widehat{R}(W, X, E_i, (\nabla_V J)E_i) - \widehat{R}(W, X, JE_i, (\nabla_V J)JE_i) \\
&\quad + \tfrac{1}{2} \sum_{i=1}^{n} \widehat{R}(W, X, JE_i, (\nabla_V J)JE_i) - \widehat{R}(W, X, E_i, (\nabla_V J)E_i) = 0.
\end{aligned}$$

We can thus differentiate the identity obtained with respect to a vector field U viewing each summand on the left hand side as a function $p \in M \mapsto \widehat{R}_p(\,\cdot\,, \cdot\,, \cdot\,, (\nabla_V J)_p\, \cdot\,)$:

$$\sum_{i=1}^{2n} \nabla_U \widehat{R}(W, X, E_i, (\nabla_V J)E_i) + \widehat{R}(W, X, E_i, (\nabla^2_{U,V} J)E_i) = 0. \tag{1.26}$$

Set $U = V = E_j$ and sum over j. The second term in the latter sum becomes

$$\sum_{i,j=1}^{2n} \widehat{R}(W, X, E_i, (\nabla^2_{E_j,E_j} J)E_i). \tag{1.27}$$

By (1.24), the sum (1.27) becomes

$$\begin{aligned}
\sum_{i,j=1}^{2n} \widehat{R}(W, X, E_i, (\nabla^2_{E_j,E_j} J)E_i) &= -\sum_{i=1}^{2n} \widehat{R}(W, X, E_i, (\mathrm{Ric} - \mathrm{Ric}^*)JE_i) \\
&= -\sum_{i,j=1}^{2n} \widehat{R}(W, X, E_i, g((\mathrm{Ric} - \mathrm{Ric}^*)JE_i, JE_j)JE_j) \\
&= -\sum_{i,j=1}^{2n} g((\mathrm{Ric} - \mathrm{Ric}^*)E_i, E_j)\widehat{R}(W, X, E_i, JE_j).
\end{aligned}$$

Set $X = JW$. Then J-invariance of R and the first Bianchi identity give

$$\begin{aligned}
\widehat{R}(W, JW, E_i, JE_j) &= \tfrac{1}{4}\big(3R(W, JW, E_i, JE_j) - 2R(W, JW, JE_i, E_j) \\
&\qquad - R(JW, E_i, JW, E_j) - R(W, E_i, W, E_j)\big) \\
&= \tfrac{1}{4}\big(5R(W, JW, E_i, JE_j) - R(W, E_i, W, E_j) - R(W, JE_i, W, JE_j)\big) \\
&= \tfrac{1}{4}\big(5R(W, E_i, JW, JE_j) - 5R(W, JE_j, JW, E_i) \\
&\qquad - R(W, E_i, W, E_j) - R(W, JE_i, W, JE_j)\big).
\end{aligned}$$

Using (1.24) and (1.22) we have

$$\begin{aligned}
&\sum_{i,j=1}^{2n} \widehat{R}(W, JW, E_i, (\nabla^2_{E_j,E_j} J)E_i) \\
&\qquad = -\sum_{i,j=1}^{2n} g((\mathrm{Ric} - \mathrm{Ric}^*)E_i, E_j)\widehat{R}(W, JW, E_i, JE_j) \\
&\qquad = \tfrac{1}{4}\sum_{i,j=1}^{2n} g((\mathrm{Ric} - \mathrm{Ric}^*)E_i, E_j)\big(-5R(W, E_i, JW, JE_j) + 5R(W, JE_j, JW, E_i) \\
&\qquad\qquad\qquad\qquad + R(W, E_i, W, E_j) + R(W, JE_i, W, JE_j)\big).
\end{aligned}$$

We now split this expression in four different sums where the indices i, j always run from 1 to n, just to make them easier to handle. Set $A := \mathrm{Ric} - \mathrm{Ric}^*$ and

$$\begin{aligned}
L(E_i, E_j) &:= -5R(W, E_i, JW, JE_j) + 5R(W, JE_j, JW, E_i) \\
H(E_i, E_j) &:= R(W, E_i, W, E_j) + R(W, JE_i, W, JE_j),
\end{aligned}$$

so as to write $\sum_{i,j=1}^{2n} \widehat{R}(W, JW, E_i, (\nabla^2_{E_j,E_j} J)E_i)$ as

$$\begin{aligned}
&\tfrac{1}{4}\sum_{i,j=1}^{n} \big(g(AE_i, E_j)(L+H)(E_i, E_j) + g(AE_i, JE_j)(L+H)(E_i, JE_j) \\
&\qquad + g(AJE_i, E_j)(L+H)(JE_i, E_j) + g(AJE_i, JE_j)(L+H)(JE_i, JE_j)\big).
\end{aligned}$$

The symmetries of R, its J-invariance and the identity $AJ = JA$ yield

$$\begin{aligned}
&\sum_{i,j=1}^{2n} \widehat{R}(W, JW, E_i, (\nabla^2_{E_j,E_j} J)E_i) \\
&\quad = \tfrac{1}{2}\sum_{i,j=1}^{n} \Big(g(AE_i, E_j)\big(L(E_j, E_i) + H(E_i, E_j)\big) + g(AE_i, JE_j)\big(L(E_i, JE_j) + H(E_i, JE_j)\big)\Big).
\end{aligned}$$

Going back to our usual notation we find

$$\begin{aligned}
&\sum_{i,j=1}^{2n} \widehat{R}(W, JW, E_i, (\nabla^2_{E_j,E_j} J)E_i) \\
&\qquad = \tfrac{1}{2}\sum_{i,j=1}^{2n} g(AE_i, E_j)\big(R(W, E_i, W, E_j) - 5R(W, E_i, JW, JE_j)\big) \\
&\qquad\quad + \tfrac{1}{2}\sum_{i,j=1}^{n} g(AE_i, JE_j)\big(R(W, E_i, W, JE_j) + 5R(W, E_i, JW, E_j)\big) \\
&\qquad\quad + \tfrac{1}{2}\sum_{i,j=1}^{n} g(AJE_i, E_j)\big(R(W, JE_i, W, E_j) - 5R(W, JE_i, JW, JE_j)\big) \\
&\qquad\quad + \tfrac{1}{2}\sum_{i,j=1}^{n} g(AJE_i, JE_j)\big(R(W, JE_i, W, JE_j) + 5R(W, JE_i, JW, E_j)\big) \\
&\qquad = \tfrac{1}{2}\sum_{i,j=1}^{2n} g(AE_i, E_j)\big(R(W, E_i, W, E_j) - 5R(W, E_i, JW, JE_j)\big).
\end{aligned}$$

Let us go back to (1.26) and focus on the first term now. Setting again $U = V = E_j$, $X = JW$, applying Lemma 1.4.5, and summing over j we have:

$$\begin{aligned}\sum_{i,j=1}^{2n} \nabla_{E_j}\widehat{R}(W, JW, E_i, (\nabla_{E_j}J)E_i) &= \sum_{i,j,k=1}^{2n} \nabla_{E_j}\widehat{R}(W, JW, E_i, g((\nabla_{E_j}J)E_i, E_k)E_k) \\ &= \sum_{i,j,k=1}^{2n} \nabla\sigma(E_j, E_i, E_k)\nabla_{E_j}\widehat{R}(E_i, E_k, W, JW) \\ &= \tfrac{1}{2}\sum_{i<j<k} \nabla\sigma(E_i, E_j, E_k) \underset{i,j,k}{\mathfrak{S}} \nabla_{E_i}\widehat{R}(E_j, E_k, W, JW).\end{aligned}$$

The sum $\mathfrak{S}_{i,j,k} \nabla_{E_i}\widehat{R}(E_j, E_k, W, JW)$ is actually zero: by formula (1.23)

$$\begin{aligned}&\nabla_{E_i}\widehat{R}(E_j, E_k, W, JW) + \nabla_{E_j}\widehat{R}(E_k, E_i, W, JW) + \nabla_{E_k}\widehat{R}(E_i, E_j, W, JW) \\ &\quad = \tfrac{1}{2}\underset{i,j,k}{\mathfrak{S}}(g((\nabla^2_{E_k,W}J)JW, (\nabla_{E_i}J)E_j) + g((\nabla^2_{E_i,E_j}J)E_k, (\nabla_W J)JW)) \\ &\qquad + \tfrac{1}{4}\underset{i,j,k}{\mathfrak{S}} g((\nabla^2_{E_i,E_j}J)JW - (\nabla^2_{E_j,E_i}J)JW, (\nabla_{E_k}J)W) \\ &\qquad + \tfrac{1}{4}\underset{i,j,k}{\mathfrak{S}} g((\nabla^2_{E_j,E_i}J)W - (\nabla^2_{E_i,E_j}J)W, (\nabla_{E_k}J)JW).\end{aligned}$$

Recall that $\nabla^2\sigma(W, X, Y, Z) = g((\nabla^2_{W,X}J)Y, Z)$. Applying (1.15) and simplifying we have

$$\begin{aligned}&\nabla_{E_i}\widehat{R}(E_j, E_k, W, JW) + \nabla_{E_j}\widehat{R}(E_k, E_i, W, JW) + \nabla_{E_k}\widehat{R}(E_i, E_j, W, JW) \\ &\quad = \tfrac{1}{2}\underset{i,j,k}{\mathfrak{S}} \nabla^2\sigma(E_k, W, JW, (\nabla_{E_i}J)E_j) \\ &\qquad + \tfrac{1}{4}\underset{i,j,k}{\mathfrak{S}}(\nabla^2\sigma(E_i, E_j, JW, (\nabla_{E_k}J)W) - \tfrac{1}{4}\nabla^2\sigma(E_j, E_i, JW, (\nabla_{E_k}J)W)) \\ &\qquad - \tfrac{1}{4}\underset{i,j,k}{\mathfrak{S}}(\nabla^2\sigma(E_i, E_j, W, (\nabla_{E_k}J)JW) + \tfrac{1}{4}\nabla^2\sigma(E_j, E_i, W, (\nabla_{E_k}J)JW)) \\ &\quad = \tfrac{1}{2}g((\nabla_W J)(\nabla_{E_i}J)E_j, (\nabla_{E_k}J)W) + \tfrac{1}{2}g((\nabla_{E_k}J)E_i, (\nabla_W J)(\nabla_{E_j}J)W) \\ &\qquad + \tfrac{1}{2}g((\nabla_W J)(\nabla_{E_j}J)E_k, (\nabla_{E_i}J)W) + \tfrac{1}{2}g((\nabla_{E_i}J)E_j, (\nabla_W J)(\nabla_{E_k}J)W) \\ &\qquad + \tfrac{1}{2}g((\nabla_{E_j}J)E_k, (\nabla_W J)(\nabla_{E_i}J)W) + \tfrac{1}{2}g((\nabla_W J)(\nabla_{E_k}J)E_i, (\nabla_{E_j}J)W) = 0.\end{aligned}$$

Then $\sum_{i,j=1}^{2n} \nabla_{E_j}\widehat{R}(W, JW, E_i, (\nabla_{E_i}J)E_j) = 0$. Polarisation of (1.26) with $X = JW$ concludes the proof. □

Let now Ric_g be the Ricci curvature $(2, 0)$ tensor field of g. We conclude this section with a final, fundamental result.

Theorem 1.4.8. *Nearly Kähler six-manifolds are Einstein with positive scalar curvature.*

Proof. Consider the six-dimensional case in Proposition 1.4.7, i.e. $n = 3$. In the course of the proof of Proposition 1.3.9 we got $\mathrm{Ric} - \mathrm{Ric}^* = 4\mu^2\,\mathrm{Id}$, with μ a non-zero constant. Thus, since $g(E_i, E_j) = \delta_{ij}$, formula (1.25) reduces to

$$\sum_{i=1}^{6} R(W, E_i, E_i, X) - 5R(W, E_i, JE_i, JX) = 0,$$

which is equivalent to saying $\mathrm{Ric} = 5\mathrm{Ric}^*$. Therefore, $\mathrm{Ric} - \mathrm{Ric}^* = \mathrm{Ric} - \tfrac{1}{5}\mathrm{Ric} = 4\mu^2\,\mathrm{Id}$, namely $\mathrm{Ric}_g = 5\mu^2 g$, and M is Einstein with positive scalar curvature. □

1.5 Formulation in terms of PDEs

In the introduction we explained roughly how Definition 1.1.1 is linked to a system of partial differential equations specifying the properties of an $SU(3)$-structure on M. We go through the details of the whole story in this section, thus concluding the presentation of the equivalent characterisations of nearly Kähler six-manifolds. We follow [Car93, Section 4.3] for this last part.

It will be convenient to work on the complexified tangent bundle $T \otimes \mathbb{C}$ of M. We use the standard notations $T^{1,0}$ and $T^{0,1}$ for the eigenspaces of J corresponding to the eigenvalues i and $-i$ respectively, so that $T \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$. All linear operations are extended by $\mathbb{C}$-linearity. Certainly the nearly Kähler condition translates onto the complex field: let $X = X_1 + X_2$, where $JX_1 = iX_1$ and $JX_2 = -iX_2$, and assume M is nearly Kähler. Then

$$\begin{aligned}(\nabla_X J)X &= (\nabla_{X_1+X_2} J)(X_1 + X_2) \\ &= (\nabla_{X_1} J)X_1 + (\nabla_{X_2} J)X_2 + (\nabla_{X_1} J)X_2 + (\nabla_{X_2} J)X_1 = 0.\end{aligned}$$

A first step in the direction we want to take was Proposition 1.2.6, where we proved that having a nearly Kähler structure on (M, g, J) is equivalent to saying $\nabla\sigma$ is a type $(3,0) + (0,3)$ form or that $d\sigma = 3\nabla\sigma$, for $\sigma = g(J\,\cdot\,,\,\cdot\,)$. We now give further characterisations.

Lemma 1.5.1. *The following assertions hold:*

1. *M is nearly Kähler if and only if $\nabla_X Y + \nabla_Y X \in T^{1,0}$ for $X, Y \in T^{1,0}$.*
2. *If M is nearly Kähler then $\nabla_{\overline{X}} Y \in T^{1,0}$, for $X, Y \in T^{1,0}$.*

Proof. For $X, Y \in T^{1,0}$ we have

$$\begin{aligned}J(\nabla_X Y + \nabla_Y X) &= \nabla_X JY + \nabla_Y JX - (\nabla_X J)Y - (\nabla_Y J)X \\ &= i(\nabla_X Y + \nabla_Y X) - (\nabla_X J)Y - (\nabla_Y J)X,\end{aligned}$$

from which our first claim follows. The second is a plain check that $(\nabla_{\overline{X}} J)Y = 0$, for $J\nabla_{\overline{X}} Y = \nabla_{\overline{X}} JY - (\nabla_{\overline{X}} J)Y = i\nabla_{\overline{X}} Y - (\nabla_{\overline{X}} J)Y$. □

Lemma 1.5.2. *Let us consider $\{F_i\}_{i=1,2,3}$, a local orthonormal basis of $T^{1,0}$ on M. Denote by $\{f^i\}_{i=1,2,3}$ its dual in $\Lambda^{1,0}$. The following facts are equivalent:*

1. *$\nabla_X Y + \nabla_Y X \in T^{1,0}$ for $X, Y \in T^{1,0}$.*
2. *There exists a constant, complex-valued function λ such that $[F_i, F_j]^{0,1} = -\overline{\lambda F_k}$, where (i, j, k) is a cyclic permutation of $(1, 2, 3)$.*
3. *There exists a constant, complex-valued function λ such that $(df^i)^{0,2} = \lambda \overline{f^j} \wedge \overline{f^k}$, where (i, j, k) is a cyclic permutation of $(1, 2, 3)$.*

Proof. Let us prove that 2 and 3 are equivalent first. Suppose $(df^i)^{0,2} = \lambda \overline{f^j} \wedge \overline{f^k}$ for some constant $\lambda \in \mathbb{C}$. Since type $(0,1)$ forms vanish on $(1,0)$ vectors and $(d\overline{f^k})^{2,0} = \overline{(df^k)^{0,2}} = \overline{\lambda} f^i \wedge f^j$, we get

$$\begin{aligned}[F_i, F_j]^{0,1} &= \sum_{k=1}^{3} \overline{f^k}([F_i, F_j])\overline{F}_k = \sum_{k=1}^{3} (F_i(\overline{f^k}(F_j)) - F_j(\overline{f^k}(F_i)) - d\overline{f^k}(F_i, F_j))\overline{F}_k \\ &= -\sum_{k=1}^{3} (d\overline{f^k})^{2,0}(F_i, F_j)\overline{F}_k = -\overline{\lambda F_k}.\end{aligned}$$

Conversely, assume $[F_i,F_j]^{0,1}=-\overline{\lambda F_k}$ holds for some complex constant λ. If we set $\overline{X}=\sum_{\ell=1}^3 a_\ell \overline{F}_\ell$, $\overline{Y}=\sum_{\ell=1}^3 b_\ell \overline{F}_\ell$, using that $[\overline{F}_j,\overline{F}_k]^{1,0}=\overline{[F_j,F_k]^{0,1}}=-\lambda F_i$ we have

$$\begin{aligned}
\lambda \overline{f}^j\wedge\overline{f}^k(\overline{X},\overline{Y}) &= \lambda(a_jb_k-b_ja_k) = -f^i\Big(\mathfrak{S}_{i,j,k}(a_jb_k-b_ja_k)(-\lambda F_i)\Big)\\
&= -f^i\Big(\sum_{j<k}(a_jb_k-b_ja_k)([\overline{F}_j,\overline{F}_k]^{1,0})\Big) = -f^i([\overline{X},\overline{Y}]^{1,0})\\
&= -\overline{X}(f^i(\overline{Y}))+\overline{Y}(f^i(\overline{X}))+df^i(\overline{X},\overline{Y})\\
&= df^i(\overline{X},\overline{Y}).
\end{aligned}$$

This yields our first equivalence.

Let us assume now that $[F_i,F_j]^{0,1}=-\overline{\lambda F_k}$ for $\lambda\in\mathbb{C}$. We use that $g(\nabla_{F_j}F_k,F_k)=0$ to compute $g(\nabla_{F_1}F_2+\nabla_{F_2}F_1,F_i)$ for all $i=1,2,3$. We have

$$\begin{aligned}
g(\nabla_{F_1}F_2+\nabla_{F_2}F_1,F_1) &= g(\nabla_{F_1}F_2-\nabla_{F_2}F_1,F_1)\\
&= g([F_1,F_2],F_1) = -\overline{\lambda}g(\overline{F}_3,F_1)=0.\\
g(\nabla_{F_1}F_2+\nabla_{F_2}F_1,F_2) &= g(-\nabla_{F_1}F_2+\nabla_{F_2}F_1,F_2)\\
&= g([F_2,F_1],F_2) = \overline{\lambda}g(\overline{F}_3,F_2)=0.\\
g(\nabla_{F_1}F_2+\nabla_{F_2}F_1,F_3) &= g(\nabla_{F_1}F_2,F_3)+g(\nabla_{F_2}F_1,F_3)\\
&= -g(F_2,\nabla_{F_1}F_3)-g(F_1,\nabla_{F_2}F_3).
\end{aligned}$$

Now note that $g(\nabla_{F_2}F_3-\nabla_{F_3}F_2,F_1)=g(\nabla_{F_3}F_1-\nabla_{F_1}F_3,F_2)=-\overline{\lambda}$. This yields

$$\begin{aligned}
-g(F_2,\nabla_{F_1}F_3)-g(F_1,\nabla_{F_2}F_3) &= -g(F_2,\nabla_{F_1}F_3)+\overline{\lambda}-g(F_1,\nabla_{F_3}F_2)\\
&= g(F_2,\nabla_{F_3}F_1-\nabla_{F_1}F_3)+\overline{\lambda}\\
&= -\overline{\lambda}+\overline{\lambda}=0.
\end{aligned}$$

The other cases are analogous and 1 follows.

Finally, we prove that 1 implies 2. Assuming $\nabla_XY+\nabla_YX\in T^{1,0}$ with $X,Y\in T^{1,0}$, we have

$$g([F_i,F_j]^{0,1},F_k)=g([F_i,F_j],F_k)=g(\nabla_{F_i}F_j,F_k)-g(\nabla_{F_j}F_i,F_k)=2g(\nabla_{F_i}F_j,F_k),$$

as the metric is of type $(1,1)$ and $\nabla_{F_i}F_j=-\nabla_{F_j}F_i+W$, $W\in T^{1,0}$ by assumption. The basis has type $(1,0)$, so

$$\begin{aligned}
g(\nabla_{F_i}F_j,F_k) &= g(J\nabla_{F_i}F_j,JF_k)\\
&= g(\nabla_{F_i}JF_j-(\nabla_{F_i}J)F_j,JF_k)\\
&= -g(\nabla_{F_i}F_j,F_k)-g((\nabla_{F_i}J)F_j,iF_k),
\end{aligned}$$

which implies $2g(\nabla_{F_i}F_j,F_k)=-i\nabla\sigma(F_i,F_j,F_k)$. By Lemma 1.5.1, M is nearly Kähler, hence $2g(\nabla_{F_i}F_j,F_k)$ is totally skew-symmetric in i,j,k. So we can write it as

$$2g(\nabla_{F_i}F_j,F_k)=-\varepsilon_{ijk}\overline{\lambda}$$

for some complex valued function λ on M, where ε_{ijk} is the sign of the permutation (i,j,k) and takes value 0 when any two indices coincide. Note that λ is a global function because its definition does not depend on the coordinate system, in the same fashion as for μ in Section 1.3. There remains to prove that λ is constant. To this aim, take any real, local orthonormal set $\{E_1, JE_1, E_2, JE_2\}$. We put $(\nabla_{E_1} J)E_2 := \mu E_3$, where E_3 is a unit vector and μ a non-negative real function satisfying (1.9). Then set $F_k := (1/\sqrt{2})(E_k - iJE_k)$ in $T^{1,0}, k = 1, 2, 3$, and recall that $g(\overline{X}, \overline{Y}) = \overline{g(X, Y)}$ and $\overline{\nabla_X Y} = \nabla_{\overline{X}} \overline{Y}$ for every $X, Y \in T \otimes \mathbb{C}$. Hence $-\lambda = 2g(\nabla_{\overline{F}_1} \overline{F}_2, \overline{F}_3)$. Here below we find the relationship between λ and μ:

$$\begin{aligned} -\lambda &= 2g(\nabla_{\overline{F}_1} \overline{F}_2, \overline{F}_3) \\ &= g(\nabla_{E_1} E_2 - \nabla_{JE_1} JE_2 + i(\nabla_{E_1} JE_2 + \nabla_{JE_1} E_2), \overline{F}_3) \\ &= g(\nabla_{E_1} E_2 + iJ\nabla_{E_1} E_2, \overline{F}_3) + ig(\nabla_{JE_1} E_2 + iJ\nabla_{JE_1} E_2, \overline{F}_3) \\ &\quad + g(J(\nabla_{E_1} J)E_2 + i(\nabla_{E_1} J)E_2, \overline{F}_3). \end{aligned}$$

Observe that $\nabla_{E_1} E_2 + iJ\nabla_{E_1} E_2$ and $\nabla_{JE_1} E_2 + iJ\nabla_{JE_1} E_2$ are of type $(0, 1)$, so the first two terms vanish, and expanding the last term we find

$$g(J(\nabla_{E_1} J)E_2 + i(\nabla_{E_1} J)E_2, \overline{F}_3) = i\mu g(E_3 - iJE_3, \overline{F}_3) = i\sqrt{2}\mu.$$

In Proposition 1.3.9 we proved that μ is constant, so λ is constant as well, and this completes the proof. □

Theorem 1.5.3. *Let (M, g, J) be an almost Hermitian six-manifold. Then M is nearly Kähler if and only if there exist a complex three-form $\psi_{\mathbb{C}} = \psi_+ + i\psi_-$ and a constant μ such that*

$$d\sigma = 3\mu\psi_+, \qquad d\psi_- = -2\mu\sigma \wedge \sigma. \tag{1.28}$$

Proof. Assume M is nearly Kähler. Using the local orthonormal basis as in Lemma 1.5.2, we can write locally $\sigma = i \sum_{k=1}^{3} f^k \wedge \overline{f}^k$, where $f^k = (1/\sqrt{2})(e^1 + iJe^1)$. Let us define

$$\psi_{\mathbb{C}} = \psi_+ + i\psi_- := 2\sqrt{2} f^1 \wedge f^2 \wedge f^3.$$

We know by Proposition 1.2.6 that M nearly Kähler implies $d\sigma \in [\![\Lambda^{3,0}]\!]$. We thus calculate its $(3, 0) + (0, 3)$ part.

$$(id\sigma)^{3,0} = -\Big(\sum_{k=1}^{3} df^k \wedge \overline{f}^k - f^k \wedge d\overline{f}^k\Big)^{3,0} = \sum_{k=1}^{3} (f^k \wedge d\overline{f}^k)^{3,0} = \sum_{k=1}^{3} f^k \wedge (d\overline{f}^k)^{2,0}.$$

Lemma 1.5.2 implies

$$(id\sigma)^{3,0} = \sum_{k=1}^{3} f^k \wedge (d\overline{f}^k)^{2,0} = \overline{\lambda} \underset{1,2,3}{\mathfrak{S}} f^1 \wedge f^2 \wedge f^3 = \tfrac{3}{2\sqrt{2}} \overline{\lambda} \psi_{\mathbb{C}}.$$

Similarly, $(id\sigma)^{0,3} = -\frac{3}{2\sqrt{2}} \lambda \overline{\psi_{\mathbb{C}}}$. We found $\lambda = -i\sqrt{2}\mu$, so

$$id\sigma = (id\sigma)^{3,0} + (id\sigma)^{0,3} = 3i\mu\psi_+.$$

This implies $0 = d\psi_+$, hence $d\psi_{\mathbb{C}} = -d\overline{\psi}_{\mathbb{C}}$. Differentiating $\psi_{\mathbb{C}}$ we find

$$\begin{aligned} d\psi_{\mathbb{C}} &= 2\sqrt{2}\big(df^1 \wedge f^2 \wedge f^3 - f^1 \wedge df^2 \wedge f^3 + f^1 \wedge f^2 \wedge df^3\big) \\ &= 2\sqrt{2}\big((df^1)^{1,1} \wedge f^2 \wedge f^3 + (df^1)^{0,2} \wedge f^2 \wedge f^3 \\ &\quad - f^1 \wedge (df^2)^{1,1} \wedge f^3 - f^1 \wedge (df^2)^{0,2} \wedge f^3 \\ &\quad + f^1 \wedge f^2 \wedge (df^3)^{1,1} + f^1 \wedge f^2 \wedge (df^3)^{0,2}\big) \in \Lambda^{3,1} + \Lambda^{2,2}. \end{aligned}$$

With similar computations one can see that $d\overline{\psi}_{\mathbb{C}} \in \Lambda^{1,3} + \Lambda^{2,2}$. We proved that $d\psi_{\mathbb{C}} = -d\overline{\psi}_{\mathbb{C}}$, so the $(1,3)$ part of $d\psi_{\mathbb{C}}$ vanishes. We then have

$$id\psi_- = 2\sqrt{2}\lambda \sum_{j<k} \overline{f}^j \wedge \overline{f}^k \wedge f^j \wedge f^k = -2i\mu\sigma \wedge \sigma$$

and the first implication is done.

Conversely, given $d\sigma = 3\mu\psi_+$ and $d\psi_- = -2\mu\sigma \wedge \sigma$, it is enough to prove that $(df^i)^{0,2} = \lambda \overline{f}^j \wedge \overline{f}^k$ for (i,j,k) cyclic permutation of $(1,2,3)$ and some constant $\lambda \in \mathbb{C}$. To get it, we first see that

$$\psi_{\mathbb{C}} \wedge (df^i)^{0,2} = \psi_{\mathbb{C}} \wedge df^i = d\psi_{\mathbb{C}} \wedge f^i = id\psi_- \wedge f^i = \psi_{\mathbb{C}} \wedge \lambda(\overline{f}^j \wedge \overline{f}^k).$$

Now observe that the map $\Lambda^{0,2} \to \Lambda^{3,2}$ given by the wedge product with $\psi_{\mathbb{C}}$ is injective. This implies $(df^i)^{0,2} = \lambda \overline{f}^j \wedge \overline{f}^k$. $\square$

Remark 1.5.4. We can rescale our basis so that $\sigma \mapsto \widetilde{\sigma} := \mu^2\sigma$ and $\psi_\pm \mapsto \widetilde{\psi}_\pm := \mu^3\psi_\pm$. Then

$$d\widetilde{\sigma} = 3\widetilde{\psi}_+, \qquad d\widetilde{\psi}_- = -2\widetilde{\sigma} \wedge \widetilde{\sigma}.$$

Thus Theorem 1.5.3 provides us with a characterisation of nearly Kähler six-manifolds in terms of an SU(3)-structure. We can then give a final definition we will use throughout the rest of this work.

Definition 1.5.5. Let (M, g, J) be an almost Hermitian six-dimensional manifold with an SU(3)-structure $(\sigma = g(J\,\cdot\,,\,\cdot\,), \psi_{\mathbb{C}} = \psi_+ + i\psi_-)$. We say that M is *nearly Kähler* if and only if

$$d\sigma = 3\psi_+, \qquad d\psi_- = -2\sigma \wedge \sigma. \tag{1.29}$$

Observe that locally σ and $\psi_{\mathbb{C}}$ were expressed in terms of type $(1,0)$ vectors f^i as $\sigma = i\sum_{k=1}^3 f^k \wedge \overline{f}^k$ and $\psi_{\mathbb{C}} = 2\sqrt{2}f^1 \wedge f^2 \wedge f^3$, thus giving the real models

$$\sigma = e^1 \wedge Je^1 + e^2 \wedge Je^2 + e^3 \wedge Je^3, \tag{1.30}$$
$$\psi_+ = e^1 \wedge e^2 \wedge e^3 - Je^1 \wedge Je^2 \wedge e^3 - e^1 \wedge Je^2 \wedge Je^3 - Je^1 \wedge e^2 \wedge Je^3, \tag{1.31}$$
$$\psi_- = e^1 \wedge e^2 \wedge Je^3 - Je^1 \wedge Je^2 \wedge Je^3 + e^1 \wedge Je^2 \wedge e^3 + Je^1 \wedge e^2 \wedge e^3, \tag{1.32}$$

which are obtained by the definition $f^k := (1/\sqrt{2})(e^k + iJe^k), k = 1,2,3$. By $Je^i = -e^i \circ J$, expressions (1.31) and (1.32), the relation $\psi_- = -\psi_+(\,\cdot\,,\,\cdot\,,J\,\cdot\,)$ readily follows. On the other hand by equations (1.28) and Proposition 1.2.6 we have $d\sigma = 3\mu\psi_+ = 3\nabla\sigma$, so $\mu\psi_+ = \nabla\sigma$, but since $\nabla\sigma \in \Lambda^1 \otimes [\![\Lambda^{2,0}]\!]$ we find

$$\begin{aligned}\psi_-(X,Y,Z) &= -\psi_+(X,Y,JZ) = -\mu^{-1}\nabla\sigma(X,Y,JZ) = -\mu^{-1}\nabla\sigma(JZ,X,Y)\\ &= \mu^{-1}\nabla\sigma(JZ,JX,JY) = \mu^{-1}\nabla\sigma(JX,JY,JZ) = -J\psi_+(X,Y,Z).\end{aligned}$$

Therefore $\psi_- = -J\psi_+$.

Remark 1.5.6. Let us set vol $:= e^1 \wedge Je^1 \wedge e^2 \wedge Je^2 \wedge e^3 \wedge Je^3$. A straightforward calculation of $\psi_+ \wedge \psi_-$ and $\sigma \wedge \psi_\pm$ gives

$$\psi_+ \wedge \psi_- = 4\text{vol} = \tfrac{2}{3}\sigma^3, \tag{1.33}$$
$$\sigma \wedge \psi_\pm = 0. \tag{1.34}$$

Since $g(\psi_+, \psi_+) = 4$ the first equation tells us that $\psi_+ \wedge \psi_- = 4\mathrm{vol} = g(\psi_+, \psi_+)\mathrm{vol} = \psi_+ \wedge *\psi_+$, so by uniqueness of $*\psi_+$ we deduce $*\psi_+ = \psi_-$. This final digression completes the argument we touched upon in Remark 1.2.4. Using the terminology in [CS02, Definition 4.1] we say that nearly Kähler six-manifolds are *half-flat*, because $d\psi_+ = \frac{1}{3}d^2\sigma = 0$ and (1.34) implies $\sigma \wedge d\sigma = 0$.

Recall that $\widehat{\nabla} := \nabla - \frac{1}{2}J(\nabla J)$ is a $\mathrm{U}(3)$-connection by Proposition 1.4.2. Now we can make this statement more precise.

Proposition 1.5.7. *$\widehat{\nabla}$ is an* $\mathrm{SU}(3)$*-connection.*

Proof. We calculate $\widehat{\nabla}(\nabla\sigma)$. By Lemma 1.15 we have

$$\begin{aligned}\widehat{\nabla}(\nabla\sigma)(W, X, Y, Z) &= W(\nabla\sigma(X, Y, Z)) - \nabla\sigma(\widehat{\nabla}_W X, Y, Z)\\ &\qquad - \nabla\sigma(X, \widehat{\nabla}_W Y, Z) - \nabla\sigma(X, Y, \widehat{\nabla}_W Z)\\ &= \nabla^2\sigma(W, X, Y, Z) + \tfrac{1}{2} \underset{X,Y,Z}{\mathfrak{S}} g((\nabla_W J)X, (\nabla_Y J)JZ) = 0,\end{aligned}$$

which proves $\widehat{\nabla}(\nabla\sigma) = 0 = \widehat{\nabla}\psi_+$, thus ψ_+ is parallel. Further, by $\psi_- = -J\psi_+$ we have at once $\widehat{\nabla}\psi_- = 0$, namely $\widehat{\nabla}\psi_{\mathbb{C}} = 0$, which proves $\widehat{\nabla}$ is actually an $\mathrm{SU}(3)$-connection. □

The difference $\nabla - \widehat{\nabla}$ takes then values in $\Lambda^1 \otimes \mathfrak{su}(3)^\perp$ (where $\mathfrak{su}(3)^\perp$ is the orthogonal complement of $\mathfrak{su}(3)$ in $\mathfrak{so}(6)$) and is called *intrinsic torsion* of the $\mathrm{SU}(3)$-structure. It measures the failure of the holonomy group of the Levi-Civita connection to reduce to $\mathrm{SU}(3)$.

Remark 1.5.8. We mentioned already in Proposition 1.2.6 that $\nabla\sigma$ lies in $[\![\Lambda^{3,0}]\!]$, so obviously it is only the $(3,0) + (0,3)$ part of $\nabla\sigma$ that measures the failure of M to be Kähler. This clarifies Remark 1.2.5. Therefore, we can say that it is exactly the type of $\nabla\sigma$ that determines the class of nearly Kähler manifolds in the classification completed by Gray and Hervella. On the other hand, equation (1.3) tells us $\nabla\sigma$ may be identified with ∇J, which may in turn be identified with the Nijenhuis tensor N of J by Proposition 1.4.1. The latter is the intrinsic torsion of the $\mathrm{SU}(3)$-structure $(\sigma, \psi_\pm)$ by Proposition 1.5.7. A detailed study of this object for $\mathrm{SU}(3)$- and G_2-structures was pursued by Chiossi and Salamon (see [CS02], in particular Theorem 1.1 for what regards our set-up).

Chapter 2

Homogeneous nearly Kähler structures

In the introduction we mentioned in dimension six there are only four compact, homogeneous spaces with a nearly Kähler structure: the six-sphere $\mathbb{S}^6 = \mathrm{G}_2/\mathrm{SU}(3)$, the flag manifold $F_{1,2}(\mathbb{C}^3) = \mathrm{SU}(3)/T^2$, the complex projective space $\mathbb{CP}^3 = \mathrm{Sp}(2)/\mathrm{Sp}(1)\mathrm{U}(1)$, and the product of three spheres $\mathbb{S}^3 \times \mathbb{S}^3 = \mathrm{SU}(2)^3/\mathrm{SU}(2)_\Delta$. We now recall their nearly Kähler structures, which will be essential for constructing multi-moment maps and computing their critical sets in the next chapter. In doing this in the case of the six-sphere, we will use a few basic concepts of G_2 geometry, which we shall dedicate the first section to. For the spaces $F_{1,2}(\mathbb{C}^3)$ and $\mathbb{CP}^3$ we follow Gray [Gra72], whereas the material on $\mathbb{S}^3 \times \mathbb{S}^3$ may be found in different references, e.g. [But05], [Bol+15], [Dix18].

2.1 On G_2 geometry

Definition (1.5.5) gives an alternative way of checking that an almost Hermitian six-manifold is nearly Kähler, provided that it is equipped with an $\mathrm{SU}(3)$-structure $(\sigma, \psi_\pm)$ subject to the constraints (1.33) and (1.34). What we shall do in Section 2.2 is to define an explicit $\mathrm{SU}(3)$-structure $(\sigma, \psi_\pm)$ on the six-sphere $\mathbb{S}^6$ and check it satisfies equations (1.29). Here below we collect elementary facts about G_2 geometry that will help us go towards this construction. To do this we need to introduce the algebra of octonions first.

Let us consider $\mathbb{R}^8$ with basis $(E_0, E_1, \dots, E_7)$ and $V \cong \mathbb{R}^7$ the subspace spanned by $E_1, \dots, E_7$. We denote by 1 the vector E_0 and by $\{e^1, \dots, e^7\}$ the dual basis of V. We can define a multiplication on $\mathbb{R}^8$ using the following rules: 1 is the identity, and $E_i \cdot E_j := -\delta_{ij}1 + \varepsilon_{ijk}E_k$, where ε_{ijk} is a totally skew-symmetric symbol with value $+1$ when $(ijk) = (123), (145), (167), (246), (275), (374), (365)$, and 0 otherwise. This gives $\mathbb{R}^8$ the structure of a (non-commutative and non-associative) algebra whose elements are called *octonions*. The space V contains the *imaginary* octonions.

We now introduce a three-form on V encoding the multiplication table for the basis elements of V. We use the notation e^{ijk} as a shorthand for $e^i \wedge e^j \wedge e^k$, and similarly for differential forms of higher or lower degree. Consider the three-form

$$\varphi_0 := e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}. \tag{2.1}$$

One can check that G_2 is a subgroup of $\mathrm{SO}(7)$, so φ_0 yields an inner product g_0 and an orientation: in fact φ_0 induces the map $b\colon V \times V \to \Lambda^7 V^*$ given by

$$b(X,Y) = \tfrac{1}{6}(X \lrcorner\, \varphi_0) \wedge (Y \lrcorner\, \varphi_0) \wedge \varphi_0,$$

which turns out to be $b(X,Y) = g_0(X,Y)e^{1234567}$, where $g_0(X,Y) = \sum_{k=1}^{6} X^k Y^k$. The basis $E_1, \dots, E_7$ is orthonormal with respect to g_0. All of this is part of the proof of the proposition below. The Hodge star operator $*$ then yields a four-form

$$*\varphi_0 = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}.$$

Define the group

$$\mathrm{G}_2 := \{A \in \mathrm{GL}(V) : A\varphi_0 = \varphi_0\},$$

namely the stabilizer of φ_0 in $\mathrm{GL}(V)$.

Proposition 2.1.1. *The group $\mathrm{G}_2 \subseteq \mathrm{GL}(V)$ is compact, connected, simple, simply connected and of dimension 14. Moreover, G_2 acts irreducibly on V and transitively on the space of lines in V and two-planes in V. Finally, G_2 is isomorphic to the group of algebra automorphisms of the octonions.*

A detailed proof may be found in [Bry87, Section 2], so we omit it. We can raise one index of φ_0 defining the so-called G_2-cross product, a G_2-equivariant map $P_0\colon V \times V \to V$ given by

$$g_0(P_0(X,Y),Z) = \varphi_0(X,Y,Z). \tag{2.2}$$

This new map allows one to rephrase the multiplication of imaginary octonions, for

$$X \cdot Y = -g_0(X,Y)1 + P_0(X,Y), \quad X,Y \in V.$$

Also, P_0 is certainly skew-symmetric, $P_0(X,Y)$ is orthogonal to X and Y, and

$$\|P_0(X,Y)\|^2 = \|X\|^2\|Y\|^2 - g_0(X,Y)^2, \tag{2.3}$$

as can be checked by a direct calculation in terms of the basis. More material on the construction of generic two-fold vector cross products may be found in [FG82]. By the non-degeneracy of g_0, φ_0 is a *fully non-degenerate* form, namely $\varphi_0(X,Y,\,\cdot\,)$ is non-zero if X, Y are linearly independent (cf. [MS13], in particular Definition 2.1 and Theorem 2.1). Finally, φ_0 is *stable* in the sense of Hitchin: its orbit under the standard action by pullback of the general linear group $\mathrm{GL}(V)$ is open in $\Lambda^3 V^*$. For more details on this point we refer to [Hit01].

The choice of a unit vector N in V yields what we claim to be a complex structure J_0 on $\langle N\rangle^\perp \cong \mathbb{R}^6 \subset V$, obtained by the contraction $N \lrcorner\, P_0$. It is readily seen that J_0 maps $\langle N\rangle^\perp$ into itself, because $g_0(J_0X,N) = g_0(P_0(N,X),N) = \varphi_0(N,X,N) = 0$. Now we observe that the condition $J_0^2 = -\mathrm{Id}$ is equivalent to the compatibility between g_0 and J_0: a simple calculation shows

$$\begin{aligned}
g_0(J_0^2 Y, Z) &= g_0(P_0(N, J_0 Y), Z) \\
&= g_0(P_0(N, P_0(N,Y)), Z) \\
&= \varphi_0(N, P_0(N,Y), Z) \\
&= -\varphi_0(N, Z, P_0(N,Y)) \\
&= -g_0(P_0(N,Y), P_0(N,Z)) = -g_0(J_0Y, J_0Z).
\end{aligned}$$

On the other hand, mapping $Y \mapsto Y + Z$ in (2.3), we find

$$\begin{aligned} g_0(P_0(X, Y+Z), P_0(X, Y+Z)) &= \|X\|^2 g_0(Y+Z, Y+Z) - g_0(X, Y+Z)^2 \\ &= \|X\|^2\|Y\|^2 + \|X\|^2\|Z\|^2 + 2\|X\|^2 g_0(Y, Z) \\ &\quad - g_0(X, Y)^2 - g_0(X, Z)^2 - 2g_0(X, Y)g_0(X, Z). \end{aligned}$$

The left hand side is $\|P_0(X, Y)\|^2 + \|P_0(X, Z)\|^2 + 2g_0(P_0(X, Y), P_0(X, Z))$, so (2.3) implies

$$g_0(P_0(X, Y), P_0(X, Z)) = \|X\|^2 g_0(Y, Z) - g_0(X, Y)g_0(X, Z).$$

Therefore, when Y, Z are orthogonal to N we have

$$\begin{aligned} g_0(J_0Y, J_0Z) &= g_0(P_0(N, Y), P_0(N, Z)) \\ &= \|N\|^2 g_0(Y, Z) - g_0(N, Y)g_0(N, Z) = g_0(Y, Z), \end{aligned}$$

so J_0 does define a complex structure.

2.2 The six-sphere

Consider $\mathbb{S}^6 \subset \mathbb{R}^7$. Let $x^k, k = 1, \dots, 7$, be global coordinates on $\mathbb{R}^7$, ∂_k the associated coordinate vector fields, and $dx^k, k = 1, \dots, 7$, their duals. In analogy with (2.1), we define the three-form

$$\varphi = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356}, \tag{2.4}$$

which induces an inner product g_0 and an orientation on $\mathbb{R}^7$. A metric on the six-sphere is defined by the pullback $g = i^* g_0$, where $i \colon \mathbb{S}^6 \hookrightarrow \mathbb{R}^7$ is the natural immersion. As in (2.2), we construct a G_2-cross product $P \colon \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$ such that $g(P(X, Y), Z) = \varphi(X, Y, Z)$. Recall that the Hodge star operator yields a four-form

$$*\varphi = dx^{4567} + dx^{2367} + dx^{2345} + dx^{1357} - dx^{1346} - dx^{1256} - dx^{1247}.$$

We write N for the unit normal vector field to the six-sphere. Its expression is given by the sum $\sum_{k=1}^{7} x^k \partial_k$, where of course $\sum_{k=1}^{7} (x^k)^2 = 1$. The contraction

$$J_p := N_p \lrcorner P_p \colon T_p\mathbb{S}^6 \to T_p\mathbb{S}^6$$

defines an almost complex structure on $\mathbb{S}^6$ as shown in the previous section.

We may define an $\mathrm{SU}(3)$-structure $(\sigma, \psi_+ + i\psi_-)$ by setting

$$\sigma := g(J\,\cdot\,, \cdot\,), \qquad \psi_+ := i^*\varphi, \qquad \psi_- := -i^*(N \lrcorner *\varphi).$$

To show that this structure is nearly Kähler we perform the computations on $\mathbb{R}^7$ and then restrict to $\mathbb{S}^6$. Firstly, by the definition of the G_2-cross product above we find

$$\begin{aligned} \sigma = {} & x^3 dx^{12} - x^2 dx^{13} + x^5 dx^{14} - x^4 dx^{15} + x^7 dx^{16} - x^6 dx^{17} \\ & + x^1 dx^{23} + x^6 dx^{24} - x^7 dx^{25} - x^4 dx^{26} + x^5 dx^{27} \\ & - x^7 dx^{34} - x^6 dx^{35} + x^5 dx^{36} + x^4 dx^{37} \\ & + x^1 dx^{45} + x^2 dx^{46} - x^3 dx^{47} \\ & - x^3 dx^{56} - x^2 dx^{57} \\ & + x^1 dx^{67}. \end{aligned} \tag{2.5}$$

The differential is then

$$
\begin{aligned}
d\sigma = {} & dx^{123} + dx^{123} + dx^{145} + dx^{145} + dx^{167} + dx^{167} \\
& + dx^{123} + dx^{246} - dx^{257} + dx^{246} - dx^{257} \\
& - dx^{347} - dx^{356} - dx^{356} - dx^{347} \\
& + dx^{145} + dx^{246} - dx^{347} \\
& - dx^{356} - dx^{257} \\
& + dx^{167} \\
= {} & 3(dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356}) = 3\varphi,
\end{aligned}
$$

so their restrictions on $\mathbb{S}^6$ coincide. Secondly, $d\psi_- = -i^* d(N \lrcorner *\varphi)$. We have

$$
*\varphi = dx^{4567} + dx^{2367} + dx^{2345} + dx^{1357} - dx^{1346} - dx^{1256} - dx^{1247}.
$$

so $d(N \lrcorner *\varphi) = 4*\varphi$ and again the restrictions on $\mathbb{S}^6$ are equal. Thus the claim is $4i^**\varphi = 2\sigma \wedge \sigma$, but this can be checked at every point p. Up to a rotation in G_2 mapping p to E_7, and so N to ∂_7—such a rotation exists by Proposition 2.1.1—we have $\sigma = N \lrcorner \varphi = \partial_7 \lrcorner \varphi = dx^{16} - dx^{25} - dx^{34}$, so $\sigma \wedge \sigma = -2(dx^{1256} + dx^{1346} - dx^{2345})$ and $i^**\varphi = dx^{2345} - dx^{1346} - dx^{1256}$ is unchanged. Hence $d\psi_- = -2\sigma \wedge \sigma$. We have just checked that $(\sigma, \psi_\pm)$ satisfy (1.29), therefore $\mathbb{S}^6$ with the above structure is a nearly Kähler manifold.

2.3 The flag manifold of $\mathbb{C}^3$

Let us now switch to $F_{1,2}(\mathbb{C}^3)$, the set of pairs (L, U) of subspaces in $\mathbb{C}^3$, where L is a complex line contained in the complex plane U. We first show that $F_{1,2}(\mathbb{C}^3)$ is homogeneous for the group $\mathrm{SU}(3)$ and that the isotropy group of a particular point is isomorphic to the two-torus T^2. This proves that $F_{1,2}(\mathbb{C}^3)$ is a smooth manifold diffeomorphic to $\mathrm{SU}(3)/T^2$, which is called *flag manifold* of $\mathbb{C}^3$.

Given the standard basis $\{F_1, F_2, F_3\}$ of $\mathbb{C}^3$ and the point $(\langle F_1\rangle, \langle F_1, F_2\rangle)$ in $F_{1,2}(\mathbb{C}^3)$, an element B in $\mathrm{SU}(3)$ maps $(\langle F_1\rangle, \langle F_1, F_2\rangle)$ into $(\langle BF_1\rangle, \langle BF_1, BF_2\rangle)$. This map is seen to be surjective, thus defines a transitive action of $\mathrm{SU}(3)$ on $F_{1,2}(\mathbb{C}^3)$. Now $A \in \mathrm{SU}(3)$ fixes the point $(\langle F_1\rangle, \langle F_1, F_2\rangle)$ when F_1 is an eigenvector of A and AF_2 is in the span of F_1, F_2. Hence there are constants $\lambda, \mu, \rho \in \mathbb{C}$ such that $AF_1 = \lambda F_1$ and $AF_2 = \mu F_1 + \rho F_2$. But A preserves the Hermitian product, thus $\lambda = e^{i\alpha}$, $\mu = 0$ and $\rho = e^{i\beta}$. Since A is unitary with determinant 1, it must be of the form $\mathrm{diag}(e^{i\alpha}, e^{i\beta}, e^{-i(\alpha+\beta)})$. Therefore, the stabiliser of the point chosen is a maximal torus T^2 in $\mathrm{SU}(3)$, and $F_{1,2}(\mathbb{C}^3)$ is then diffeomorphic to the quotient $\mathrm{SU}(3)/T^2$.

We now equip $\mathrm{SU}(3)/T^2$ with a Riemannian metric and an almost complex structure. A matrix $p \in \mathrm{SU}(3)$ acts on $\mathrm{SU}(3)$ by left translation and induces a pullback map $(p_{\mathrm{Id}}^{-1})^* \colon \mathfrak{su}^*(3)^{\otimes 2} \to T_p^*\mathrm{SU}(3)^{\otimes 2}$. We can thus define an inner product g_p at the point $p \in \mathrm{SU}(3)$ as

$$
g_p := \mathrm{Re}((p_{\mathrm{Id}}^{-1})^* g_0), \tag{2.6}
$$

where g_0 denotes the Killing form on $\mathfrak{su}(3)$ and is normalised as follows:

$$
g_0(X, Y) := \tfrac{1}{2}\mathrm{Tr}({}^t\overline{X}Y), \quad X, Y \in \mathfrak{su}(3).
$$

The metric g is bi-invariant because g_0 is. In particular g is invariant under the action of the maximal torus in $\mathrm{SU}(3)$ above, so it descends to a metric on the flag, which we still denote by g. To construct an almost complex structure J we follow the process described by Gray [Gra72, Section 3]. Let us take the matrix $A := \operatorname{diag}(e^{2\pi i/3}, e^{4\pi i/3}, 1)$ and define the conjugation map $\tilde{\vartheta}\colon \mathrm{SU}(3) \to \mathrm{SU}(3)$ so that $\tilde{\vartheta}(B) = ABA^{-1}$. It is clear by this definition that $\tilde{\vartheta} \circ \tilde{\vartheta} \circ \tilde{\vartheta} = \mathrm{Id}$ and that $\tilde{\vartheta}$ fixes the maximal torus T^2 in $\mathrm{SU}(3)$ above. So $\tilde{\vartheta}$ induces a map on the quotient $\vartheta\colon \mathrm{SU}(3)/T^2 \to \mathrm{SU}(3)/T^2$ that fixes the coset T^2 and satisfies $\vartheta \circ \vartheta \circ \vartheta = \mathrm{Id}$. From now on we write ϑ^3 instead of $\vartheta \circ \vartheta \circ \vartheta$. We define J_0 at the identity as follows: for $X \in \mathfrak{su}(3)/\mathfrak{t}^2$ write $d\vartheta(X) = AXA^{-1} =: -(1/2)X + (\sqrt{3}/2)J_0X$, so that

$$J_0X = \tfrac{2}{\sqrt{3}}\left(AXA^{-1} + \tfrac{1}{2}X\right). \tag{2.7}$$

The map $J_0\colon \mathfrak{su}(3)/\mathfrak{t}^2 \to \mathfrak{su}(3)/\mathfrak{t}^2$ is well defined as A commutes with diagonal matrices. We now check that $J_0^2 = -\mathrm{Id}$. Firstly, observe that $d\vartheta - \mathrm{Id}$ is injective: if $AXA^{-1} - X = 0$, then $AX = XA$, and since X is diagonalisable then X is diagonal. Thus $X = 0$ in $\mathfrak{su}(3)/\mathfrak{t}^2$ and $\mathrm{Id} - d\vartheta$ is left-invertible. We denote the left inverse by $(\mathrm{Id} - d\vartheta)^{-1}$. This amounts to say that $0 = (\mathrm{Id} - d\vartheta)^{-1}(\mathrm{Id} - d\vartheta^3) = \mathrm{Id} + d\vartheta + d\vartheta^2$, or more explicitly that $X + AXA^{-1} + A^2XA^{-2} = 0$. Therefore

$$\begin{aligned} J_0^2X = J_0(J_0X) &= \tfrac{2}{\sqrt{3}}(AJ_0XA^{-1} + \tfrac{1}{2}J_0X) \\ &= \tfrac{4}{3}(A(AXA^{-1} + \tfrac{1}{2}X)A^{-1} + \tfrac{1}{2}(AXA^{-1} + \tfrac{1}{2}X)) \\ &= \tfrac{4}{3}(A^2XA^{-2} + AXA^{-1} + X - \tfrac{3}{4}X) = \tfrac{4}{3}(-\tfrac{3}{4}X) = -X, \end{aligned}$$

as we wanted. We can move the operator J_0 to every point $p \in \mathrm{SU}(3)$ so that for each $Y \in T_p(\mathrm{SU}(3)/T^2)$ one has

$$J_p(Y) = pJ_0(p^{-1}Y).$$

Further, J_0 is an isometry:

$$\begin{aligned} g_0(J_0X, J_0Y) &= -\tfrac{1}{2}\mathrm{Tr}(J_0X \cdot J_0Y) \\ &= -\tfrac{2}{3}\left(\tfrac{5}{4}\mathrm{Tr}(XY) + \tfrac{1}{2}(\mathrm{Tr}((d\vartheta(X))Y) + \mathrm{Tr}((d\vartheta^2(X))Y))\right) \\ &= -\tfrac{2}{3}\left(\tfrac{3}{4}\mathrm{Tr}(XY) + \tfrac{1}{2}\left(\mathrm{Tr}(XY) + \mathrm{Tr}((d\vartheta(X))Y) + \mathrm{Tr}((d\vartheta^2(X))Y)\right)\right) \\ &= -\tfrac{1}{2}\mathrm{Tr}(XY) = g_0(X, Y). \end{aligned}$$

From the invariance of g and J it follows that (g, J) is an almost Hermitian structure on the flag manifold.

Now the goal is to go further and show that g and J may be used to construct a nearly Kähler structure. To do this we work at the identity of $\mathrm{SU}(3)$ and define an explicit $\mathrm{SU}(3)$-structure satisfying equations (1.29). We finally extend this structure to the whole manifold as we did for J_0.

Matrices p in $\mathfrak{su}(3)$ satisfy ${}^t\overline{p} + p = 0$ and are traceless, so the following vectors are a basis of $\mathfrak{su}(3)$:

$$E_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$E_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad E_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad E_7 = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix}, \quad E_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}.$$

Clearly $\mathfrak{su}(3)/\mathfrak{t}^2$ is generated by $E_1, \ldots, E_6$. Denote by e^k the dual of E_k. Using (2.6) and (2.7) one can check that

$$
\begin{aligned}
g_0 &= e^1 \otimes e^1 + \ldots + e^6 \otimes e^6, \\
J_0 &= E_2 \otimes e^1 - E_1 \otimes e^2 + E_4 \otimes e^3 - E_3 \otimes e^4 + E_6 \otimes e^5 - E_5 \otimes e^6.
\end{aligned}
$$

Recall e^{ij} stands for $e^i \wedge e^j$, and similarly for higher degree forms. By a direct computation it follows that $[E_7, E_8] = 0$ and

$$
\begin{array}{llll}
[E_1, E_7] = E_2, & [E_4, E_7] = 2E_3, & [E_1, E_8] = -E_2, & [E_4, E_8] = E_3, \\
[E_2, E_7] = -E_1, & [E_5, E_7] = E_6, & [E_2, E_8] = E_1, & [E_5, E_8] = 2E_6, \\
[E_3, E_7] = -2E_4, & [E_6, E_7] = -E_5, & [E_3, E_8] = -E_4, & [E_6, E_8] = -2E_5.
\end{array}
$$

The differentials of $e^1, \ldots, e^6$ follow on from the expressions of the commutators:

$$
\begin{aligned}
de^1 &= e^{46} - e^{35} + e^{27} - e^{28}, & de^2 &= e^{36} + e^{45} - e^{17} + e^{18}, \\
de^3 &= e^{15} - e^{26} - 2e^{47} - e^{48}, & de^4 &= e^{52} + e^{61} + 2e^{37} + e^{38}, \\
de^5 &= e^{24} - e^{13} + e^{67} + 2e^{68}, & de^6 &= e^{23} + e^{14} - e^{57} - 2e^{58}.
\end{aligned}
$$

Moreover $\mathcal{L}_{E_7} g_0 = \mathcal{L}_{E_8} g_0 = 0$ and $\mathcal{L}_{E_7} J = \mathcal{L}_{E_8} J = 0$, so g_0 and J_0 descend to the quotient $\mathfrak{su}(3)/\mathfrak{t}^2$. Define now the following differential forms on $\mathfrak{su}(3)$:

$$
\begin{aligned}
\sigma_0 &:= g_0(J_0 \cdot\,, \cdot\,) = e^{12} + e^{34} + e^{56}, \\
\varphi_0 &:= -e^{136} + e^{246} - e^{235} - e^{145}, \\
\psi_0 &:= e^{135} - e^{245} - e^{146} - e^{236}.
\end{aligned}
$$

They descend to the quotient as well because $\mathcal{L}_{E_k} \sigma_0 = 0 = \mathcal{L}_{E_k} \varphi_0 = \mathcal{L}_{E_k} \psi_0$, for $k = 7, 8$, and their contractions with E_7, E_8 vanish. We want to check that $(\sigma_0, \varphi_0, \psi_0)$ satisfy the nearly Kähler structure equations. The results for the differentials $de^k, k = 1, \ldots, 6$, obtained above yield

$$
\begin{aligned}
d\sigma_0 = e^{462} &- e^{352} - e^{136} - e^{145} + e^{154} - e^{264} \\
&- e^{352} - e^{361} + e^{246} - e^{136} - e^{523} - e^{514} = 3\varphi_0,
\end{aligned}
$$

and

$$
\begin{aligned}
d\psi_0 = e^{4635} &+ e^{2735} - e^{2835} + e^{1265} + 2e^{1475} + e^{1485} + e^{1324} + e^{1367} + 2e^{1368} \\
&- e^{3645} + e^{1745} - e^{1845} + e^{2615} + 2e^{2375} + e^{2385} + e^{2413} - e^{2467} - 2e^{2468} \\
&+ e^{3546} - e^{2746} + e^{2846} + e^{1526} + 2e^{1376} + e^{1386} - e^{1423} + e^{1457} + 2e^{1458} \\
&- e^{4536} + e^{1736} - e^{1836} + e^{2156} - 2e^{2476} - e^{2486} - e^{2314} + e^{2357} + 2e^{2358} \\
= -4(e^{1234} &- e^{1256} - e^{3456}) = -2\sigma_0 \wedge \sigma_0.
\end{aligned}
$$

Note that the expressions of $d\sigma_0$ and $d\psi_0$ are T^2-invariant because φ_0 and $\sigma_0 \wedge \sigma_0$ are, so these equations still hold on $\mathfrak{su}(3)/\mathfrak{t}^2$. Finally, we extend the differential forms $\sigma_0, \varphi_0, \psi_0$ to the whole space: using the notations as in Section 1.5 we define

$$
\begin{aligned}
\sigma_p(X_p, Y_p) &:= g_p(J_p X_p, Y_p), \\
\psi_{+|p}(X_p, Y_p, Z_p) &:= \varphi_0(p^{-1}X_p, p^{-1}Y_p, p^{-1}Z_p), \\
\psi_{-|p}(X_p, Y_p, Z_p) &:= \psi_0(p^{-1}X_p, p^{-1}Y_p, p^{-1}Z_p).
\end{aligned}
$$

The equations $d\sigma = 3\psi_+$ and $d\psi_- = -2\sigma \wedge \sigma$ follow automatically and we then get an explicit nearly Kähler structure $(\sigma, \psi_\pm)$.

2.4 The complex projective space $\mathbb{CP}^3$

We proceed in a similar fashion as in the previous section, thus skip some technicalities. The compact symplectic group $\mathrm{Sp}(2)$ acts by isometries on $\mathbb{C}^4 \cong \mathbb{H}^2$ and transitively on $\mathbb{S}^7$, so transitively on the projective space $\mathbb{CP}^3 := (\mathbb{C}^4 \setminus \{0\})/\mathbb{C}^*$ as well. An element in $\mathrm{Sp}(2)$ is $p = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$ such that ${}^t\overline{p}p = \mathrm{Id}$. Consequently $x, w \in \mathbb{S}^3 \cong \mathrm{Sp}(1)$ and $\overline{x}y + \overline{z}w = 0$, where conjugation denotes quaternionic conjugation. The matrix p fixes $[1:0:0:0] \in \mathbb{CP}^3$ if and only if x is a combination of $1, i$ and the quaternions z and y vanish. Since x has unit length it must lie in a circle $\mathrm{U}(1)$. The isotropy group of $[1:0:0:0]$ is then isomorphic to $\mathrm{Sp}(1)\mathrm{U}(1)$, therefore $\mathbb{CP}^3$ is diffeomorphic to $\mathrm{Sp}(2)/\mathrm{Sp}(1)\mathrm{U}(1)$. Write $H := \mathrm{Sp}(1)\mathrm{U}(1)$ and $G := \mathrm{Sp}(2)$. We then identify H with a subgroup of G containing elements of the form $\mathrm{diag}(e^{i\vartheta}, \alpha)$, where α is a unit quaternion and ϑ an angle. In the following we denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively.

On the Lie algebra $\mathfrak{g}$ we define the Killing form as $g_0(X,Y) := \mathrm{Tr}({}^t\overline{X}Y) = -\mathrm{Tr}(XY)$. This can be translated to any point p yielding an inner product

$$g_p := \mathrm{Re}((p_{\mathrm{Id}}^{-1})^* g_0)$$

on every tangent space T_pG, and descends to the quotient modulo H because it is bi-invariant. We then get a Riemannian metric on $\mathbb{CP}^3$.

The construction of the almost complex structure J follows from the existence of a diffeomorphism of order three as in the case of the flag: consider $A := \mathrm{diag}(e^{2\pi i/3}, 1)$ in G and define $\tilde{\vartheta}(B) = ABA^{-1}$. Since $A^3 = \mathrm{Id}$ one has $\tilde{\vartheta}^3 = \mathrm{Id}$ and $\tilde{\vartheta}$ fixes H, thus $\tilde{\vartheta}$ descends to $\vartheta\colon \mathbb{CP}^3 \to \mathbb{CP}^3$, which is a cubic diffeomorphism fixing the coset H. The Lie algebra $\mathfrak{g}$ splits as the direct sum $\mathfrak{h} \oplus \mathfrak{m}$. We then define an almost complex structure $J_0\colon \mathfrak{m} \to \mathfrak{m}$ as

$$J_0X := \tfrac{2}{\sqrt{3}}(AXA^{-1} + \tfrac{1}{2}X).$$

In order to construct an explicit nearly Kähler structure we fix a basis of $\mathfrak{g}$ and go through the same steps of the previous section again. The Lie algebra $\mathfrak{g}$ contains 2×2 quaternionic matrices p such that ${}^t\overline{p} + p = 0$. This forces p to be spanned by the following elements:

$$\begin{aligned}
&E_0 = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, \quad E_1 = \begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E_3 = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\
&E_4 = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \quad E_5 = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}, \quad E_6 = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \quad E_7 = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} \\
&E_8 = \begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix}, \quad E_9 = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}.
\end{aligned}$$

Note that the indices range from 0 to 9, and not from 1 to 10. This will help keep the notation shorter in the calculations later. The quotient $\mathfrak{g}/\mathfrak{h}$ is obviously spanned by $E_0, \ldots, E_5$. One can check $E_0, \ldots, E_5$ are orthonormal, so the metric and the almost complex structure have the familiar form

$$\begin{aligned}
g_0 &= e^0 \otimes e^0 + \ldots + e^5 \otimes e^5, \\
J_0 &= E_1 \otimes e^0 - E_0 \otimes e^1 + E_3 \otimes e^2 - E_2 \otimes e^3 + E_5 \otimes e^4 - E_4 \otimes e^5.
\end{aligned}$$

Now define

$$\begin{aligned}\sigma_0 &:= g_0(J_0\,\cdot\,,\,\cdot\,) = e^{01}+e^{23}+e^{45},\\ \varphi_0 &:= e^{024}-e^{134}-e^{035}-e^{125},\\ \psi_0 &:= e^{025}-e^{135}+e^{034}+e^{124}.\end{aligned}$$

To check that this structure is nearly Kähler we may observe that it is identical to the one defined in (1.30), (1.31), (1.32). For completeness we check it explicitly in our special case. We first need to compute the differentials of $e^k, k=0,\dots,5$. The results are

$$\begin{aligned}de^0 &= 2e^{16}-e^{25}-e^{34}, & de^1 &= -2e^{06}-e^{24}+e^{35},\\ de^2 &= e^{05}+e^{14}-e^{36}+e^{37}+e^{48}+e^{59}, & de^3 &= e^{04}-e^{15}+e^{26}-e^{27}-e^{49}+e^{58},\\ de^4 &= -e^{03}-e^{12}-e^{28}+e^{39}-e^{56}-e^{57}, & de^5 &= -e^{02}+e^{13}-e^{38}+e^{46}+e^{47}-e^{29}.\end{aligned}$$

The nearly Kähler structure equations follow by simplifying a routine computation:

$$\begin{aligned}d\sigma = {}& -e^{251}-e^{341}+e^{024}-e^{035}+e^{053}+e^{143}+e^{483}+e^{593}\\ & -e^{204}+e^{215}+e^{249}-e^{258}-e^{035}-e^{125}-e^{285}+e^{395}\\ & +e^{402}-e^{413}+e^{438}+e^{429} = 3(e^{024}-e^{035}-e^{125}-e^{134}) = 3\varphi_0,\end{aligned}$$

$$\begin{aligned}d\psi_0 = {}& 2e^{1625}-e^{3425}-e^{0145}+e^{0365}-e^{0375}-e^{0485}+e^{0213}-e^{0238}\\ & +e^{0246}+e^{0247}+2e^{0635}+e^{2435}+e^{1045}+e^{1265}-e^{1275}-e^{1495}\\ & +e^{1302}-e^{1346}-e^{1347}+e^{1329}+2e^{1634}-e^{2534}+e^{0154}-e^{0264}\\ & +e^{0274}-e^{0584}-e^{0312}-e^{0328}-e^{0356}-e^{0357}-2e^{0624}+e^{3524}\\ & -e^{1054}+e^{1364}-e^{1374}-e^{1594}-e^{1203}+e^{1239}-e^{1256}-e^{1257}\\ = {}& -4(e^{0123}+e^{0145}+e^{2345}) = -2\sigma_0\wedge\sigma_0.\end{aligned}$$

Note these equations and the structure used to check them descend to the quotient $\mathfrak{g}/\mathfrak{h}$. By left-translation of $(\sigma_0,\varphi_0,\psi_0)$ we get an $\mathrm{SU}(3)$-structure $(\sigma,\psi_\pm)$ all over $\mathbb{CP}^3$ that is nearly Kähler.

2.5 The product of three-spheres $\mathbb{S}^3\times\mathbb{S}^3$

It is convenient to view $\mathbb{S}^3\times\mathbb{S}^3$ as $\mathrm{Sp}(1)\times\mathrm{Sp}(1)\subset\mathbb{H}\times\mathbb{H}$. Recall that $\mathrm{Sp}(1)$ is the group of unit quaternions and is isomorphic to $\mathrm{SU}(2)$. A triple $(h,k,l)\in\mathrm{SU}(2)^3$ acts on $\mathbb{S}^3\times\mathbb{S}^3$ as

$$((h,k,l),(p,q))\mapsto(hpl^{-1},kql^{-1}).$$

This action is obviously transitive and the stabiliser of the point $(1,1)$ is given by the triples $(h,h,h)\in\mathrm{SU}(2)^3$. We denote this isotropy group by $\mathrm{SU}(2)_\Delta$. Therefore $\mathbb{S}^3\times\mathbb{S}^3$ has the structure of smooth manifold and is diffeomorphic to $\mathrm{SU}(2)^3/\mathrm{SU}(2)_\Delta$.

We define an almost complex structure at the identity $(1,1)\in\mathbb{S}^3\times\mathbb{S}^3$ as follows:

$$J_0(X,Y) := \tfrac{1}{\sqrt{3}}(X-2Y,2X-Y),\quad (X,Y)\in\mathfrak{sp}(1)\oplus\mathfrak{sp}(1).$$

It is trivial to check that $J^2 = -\operatorname{Id}$. Now we translate J_0 to any pair (p, q) by quaternionic multiplication: if $(X, Y) \in T_p\mathbb{S}^3 \oplus T_q\mathbb{S}^3$ then $(p^{-1}X, q^{-1}Y) \in \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$, so

$$J_0(p^{-1}X, q^{-1}Y) = \tfrac{1}{\sqrt{3}}(p^{-1}X - 2q^{-1}Y, 2p^{-1}X - q^{-1}Y).$$

Translating the result back to the point (p, q) we then find the vector

$$\tfrac{1}{\sqrt{3}}(X - 2pq^{-1}Y, 2qp^{-1}X - Y).$$

Therefore, we define an almost complex structure J on $\mathbb{S}^3 \times \mathbb{S}^3$ as

$$J_{(p,q)}(X, Y) := \tfrac{1}{\sqrt{3}}(X - 2pq^{-1}Y, 2qp^{-1}X - Y). \tag{2.8}$$

The standard product metric $\langle \cdot , \cdot \rangle$ on $\mathbb{S}^3 \times \mathbb{S}^3$ is not invariant under J, so we define a metric g as the average of $\langle \cdot , \cdot \rangle$ and $\langle J \cdot , J \cdot \rangle$, and normalise it by a factor $1/3$. At the point (p, q) its expression is

$$g(X, Y) := \tfrac{1}{6}(\langle X, Y\rangle + \langle JX, JY\rangle), \quad X, Y \in T_{(p,q)}(\mathbb{S}^3 \times \mathbb{S}^3). \tag{2.9}$$

Trivially, J is g-orthogonal and (g, J) is then an almost Hermitian structure.

Since $\mathbb{S}^3 \times \mathbb{S}^3$ is homogeneous for $\mathrm{SU}(2)^3$ we can work again at the identity $(1, 1) \in \mathbb{S}^3 \times \mathbb{S}^3$ to construct a nearly Kähler structure. A matrix $p \in \mathfrak{su}(2)$ is traceless and such that $\overline{p} + p = 0$, so may be identified with an imaginary quaternion in $\mathfrak{sp}(1) \cong \mathbb{R}^3$. A basis for $\mathfrak{su}(2)^2$ is then given by the vectors

$$\begin{aligned} E_1 &= (i, 0), & E_2 &= (j, 0), & E_3 &= (-k, 0), \\ E_4 &= (0, i), & E_5 &= (0, j), & E_6 &= (0, -k). \end{aligned}$$

Hence, a basis of each tangent space $T_{(p,q)}(\mathbb{S}^3 \times \mathbb{S}^3)$ is obtained by quaternionic multiplication by the point $(p, q) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ on the left:

$$\begin{aligned} E_1(p, q) &= (pi, 0), & E_2(p, q) &= (pj, 0), & E_3(p, q) &= (-pk, 0), \\ E_4(p, q) &= (0, qi), & E_5(p, q) &= (0, qj), & E_6(p, q) &= (0, -qk). \end{aligned}$$

The differentials of the duals e^k of E_k satisfy $de^i = 2e^{jk}$ for (ijk) cyclic permutation of (123) and (456). We define the differential forms

$$\begin{aligned} \sigma_0 &:= g_0(J_0 \cdot , \cdot) = \tfrac{2}{3\sqrt{3}}\big(e^{14} + e^{25} + e^{36}\big), \\ \varphi_0 &:= \tfrac{4}{9\sqrt{3}}\big(e^{126} - e^{135} - e^{156} + e^{234} + e^{246} - e^{345}\big), \\ \psi_0 &:= -J\varphi_0 = -\tfrac{4}{27}\big(2e^{123} + 2e^{456} + e^{135} - e^{156} - e^{234} - e^{126} + e^{246} - e^{345}\big). \end{aligned}$$

Since $de^i = 2e^{jk}$ for cyclic permutations $(ijk) = (123), (456)$, then

$$d\sigma_0 = \tfrac{2}{3\sqrt{3}}\big(2e^{234} - 2e^{156} + 2e^{315} - 2e^{264} + 2e^{126} - 2e^{345}\big) = 3\varphi_0,$$

$$\begin{aligned} d\psi_0 &= -\tfrac{4}{27}\big(2e^{1364} - 2e^{2356} - 2e^{2356} - 2e^{1245} + 2e^{3146} - 2e^{1245}\big) \\ &= -\tfrac{8}{27}\big(2e^{1436} + 2e^{2536} + e^{1425}\big) \\ &= -2\sigma_0 \wedge \sigma_0. \end{aligned}$$

The nearly Kähler structure equations follow. By extending $\sigma_0, \varphi_0, \phi_0$ we get a nearly Kähler structure $(\sigma, \psi_\pm)$ on $\mathbb{S}^3 \times \mathbb{S}^3$.

Chapter 3

Multi-moment maps

In this chapter we introduce multi-moment maps, the essential ingredient in our theory of nearly Kähler six-manifolds with two-torus symmetry. They generalise moment maps in symplectic geometry, we are going to see how.

The set-up we are interested in is characterised by a smooth manifold M, a closed three-form α on M, and a Lie group G acting on M preserving α. One can also consider geometries defined by forms of higher degree [MS13]. In general there are topological obstructions to the existence of such maps, but we do not touch upon this question. We refer to [MS12] for a general theory, here we focus on concrete examples. Unlike in the case of symplectic geometry, the dimension of the manifold is not relevant.

The goal is to head to the case where the Lie group acting is a two-torus T^2. What we aim for in the first section, following [MS12, Section 2], is a general definition. The main construction we use throughout the rest of this work will be part of a final example. In the remaining sections of this chapter we specialise Example 3.1.5 to the homogeneous nearly Kähler six-manifolds. We write the explicit expression of the multi-moment maps in terms of the structures defined in Chapter 2, find their range and critical points. As already mentioned, we consider actions of a two-torus, which in the first three cases will be a maximal torus contained in the symmetry groups of the various examples. We will have to discuss more carefully the case of $\mathbb{S}^3 \times \mathbb{S}^3$, where there is no preferred choice of the T^2-action.

An important piece of information is contained in the critical sets of the multi-moment maps, which is why we shall investigate their structure. In the next chapter we will see that regular values for our maps exist, and that if s is such a value for a multi-moment map ν, then we can take the quotient of $\nu^{-1}(s)$ by T^2 and find a three-dimensional smooth manifold. The study of critical sets in the homogeneous cases partly integrates this picture and will play a more general role in topological arguments we go through in Chapter 6. Also, at critical points the infinitesimal generators of the action are linearly dependent over $\mathbb{C}$, (cf. (4.2) below). In particular, when the dependence is over the reals then the torus action cannot be free, and non-trivial stabilisers arise. We will study the structure of the critical sets computing fixed points and one-dimensional orbits, and see how the information obtained may be encoded in trivalent graphs (see Chapter 5).

Let us now start describing the general framework. As mentioned already, we introduce one example we will use in the applications. More examples related to other geometries may be found in Section 6.5.

3.1 The general set-up

Let M be a smooth manifold and α a closed three-form. We call the pair (M, α) a *strong geometry*. Now let G be a group of symmetries for (M, α), i.e. a Lie group acting smoothly on M—say on the left—preserving α. An element X in the Lie algebra $\mathfrak{g}$ of G generates a one-parameter group via the exponential map: if $p \in M$ and $\cdot$ denotes the action of G on M, then

$$\phi_t(p) := \exp(tX) \cdot p$$

is a curve on M for any real value of t and $\phi_0 = \mathrm{Id}_M$, $\phi_{t+s} = \phi_t \circ \phi_s$. Hence X generates a vector field on M, which we still denote by X. Its expression at p is

$$X_p = \frac{d}{dt}\Big(\exp(tX) \cdot p\Big)\Big|_{t=0}.$$

The correspondence $\mathfrak{g} \to \mathfrak{X}(M)$ is a Lie algebra homomorphism. In particular, if X, Y in $\mathfrak{g}$ commute, then the induced vector fields on M commute as well. Since the action of G preserves α, the Lie derivative of α with respect to X vanishes, so Cartan's formula and $d\alpha = 0$ yield

$$0 = \mathcal{L}_X \alpha = d(X \lrcorner \alpha) + X \lrcorner d\alpha = d(X \lrcorner \alpha). \tag{3.1}$$

Now consider $Y \in \mathfrak{g}$ commuting with X. The induced vector fields on M commute, so

$$0 = X \lrcorner \mathcal{L}_Y \alpha = \mathcal{L}_Y (X \lrcorner \alpha) = d(Y \lrcorner X \lrcorner \alpha). \tag{3.2}$$

This shows the one-form $\alpha(X, Y, \cdot\,)$ is closed when X and Y commute.

We look at pairs of commuting vector fields as inside the kernel of the linear map $L\colon \Lambda^2\mathfrak{g} \to \mathfrak{g}$ induced by the Lie bracket. We can check that L is $\mathfrak{g}$-linear, beucase $\mathfrak{g}$ acts on itself by $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, $\mathrm{ad}_X := [X, \cdot\,]$, and on $\Lambda^2\mathfrak{g}$ by $\rho\colon \mathfrak{g} \to \mathfrak{gl}(\Lambda^2\mathfrak{g})$, defined as $\rho(X)(Y \wedge Z) := \mathrm{ad}_X(Y) \wedge Z + Y \wedge \mathrm{ad}_X(Z)$. The Jacobi identity implies

$$\begin{aligned} L(\rho(X)(Y \wedge Z)) &= L([X, Y] \wedge Z + Y \wedge [X, Z]) \\ &= [[X, Y], Z] + [[Z, X], Y] = \mathrm{ad}_X(L(Y \wedge Z)). \end{aligned}$$

Consequently, $\ker L$ is a $\mathfrak{g}$-submodule of $\Lambda^2\mathfrak{g}$. We can then give the following

Definition 3.1.1. The *Lie kernel* $\mathcal{P}_\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is the $\mathfrak{g}$-module

$$\mathcal{P}_\mathfrak{g} := \ker\left(L\colon \Lambda^2\mathfrak{g} \to \mathfrak{g}\right),$$

where L is the $\mathfrak{g}$-linear map induced by the Lie bracket.

Remark 3.1.2. Note that if G is Abelian, then all pairs in $\mathfrak{g}$ commute, thus $\mathcal{P}_\mathfrak{g} = \Lambda^2\mathfrak{g}$.

We extend here the calculations in (3.1) and (3.2) to elements of the Lie kernel. Let us fix the notation first: for a bivector $p = \sum_{i=1}^k X_i \wedge Y_i$ we write $p \lrcorner \alpha := \sum_{i=1}^k \alpha(X_i, Y_i, \cdot\,)$.

Lemma 3.1.3 ([MS12]). *Let G be a group of symmetries for a strong geometry (M, α). Let $p = \sum_{i=1}^k X_i \wedge Y_i$ be an element of the Lie kernel $\mathcal{P}_\mathfrak{g}$ and $p = \sum_{i=1}^k X_i \wedge Y_i$ be the corresponding bivector on M. Then*

$$d(p \lrcorner \alpha) = 0.$$

Proof. We have $0 = L(p) = \sum_{i=1}^{k}[X_i, Y_i]$ because p sits in $\mathcal{P}_{\mathfrak{g}}$. This together with (3.1) and $d\alpha = 0$ gives

$$\begin{aligned} 0 &= \sum_{i=1}^{k}[Y_i, X_i] \lrcorner \alpha = \sum_{i=1}^{k}(\mathcal{L}_{Y_i}X_i) \lrcorner \alpha \\ &= \sum_{i=1}^{k}\mathcal{L}_{Y_i}(X_i \lrcorner \alpha) - X_i \lrcorner \mathcal{L}_{Y_i}\alpha = \sum_{i=1}^{k} d(Y_i \lrcorner X_i \lrcorner \alpha) = d(p \lrcorner \alpha). \end{aligned}$$

□

Thus, if for example the first Betti number $b_1(M) = 0$, there is a smooth function $\nu_p \colon M \to \mathbb{R}$ with $d\nu_p = p \lrcorner \alpha$ for each $p \in \mathcal{P}_{\mathfrak{g}}$. We then give the following

Definition 3.1.4 ([MS12]). Let (M, α) be a strong geometry with a symmetry group G. A *multi-moment map* is an equivariant map $\nu \colon M \to \mathcal{P}_{\mathfrak{g}}^*$ satisfying

$$d\langle \nu, p\rangle = p \lrcorner \alpha, \quad p \in \mathcal{P}_{\mathfrak{g}}. \tag{3.3}$$

Example 3.1.5. On a nearly Kähler six-manifold we have an SU(3)-structure $(\sigma, \psi_+ + i\psi_-)$ satisfying in particular $d\sigma = 3\psi_+$. Thus $(M, 3\psi_+)$ is a strong geometry in the above sense. Assume a two-torus T^2 acts on M preserving σ and ψ_+. Since T^2 is Abelian, by Remark 3.1.2 we have that $\mathcal{P}_{\mathfrak{t}^2} = \Lambda^2\mathfrak{t}^2 \cong \mathbb{R}$. Let U, V be the infinitesimal generators of the action, so $\mathcal{L}_U\sigma = 0 = \mathcal{L}_V\sigma$. Further, since U and V commute, $\mathcal{L}_V(U \lrcorner \sigma) = U \lrcorner \mathcal{L}_V\sigma = 0$. Now, define the real-valued function

$$\nu_M := \sigma(U, V).$$

We claim this is a multi-moment map on M: first of all it is T^2-invariant by construction. Further, by Cartan's formula one has

$$\begin{aligned} d\nu_M = d(V \lrcorner U \lrcorner \sigma) &= \mathcal{L}_V(U \lrcorner \sigma) - V \lrcorner d(U \lrcorner \sigma) \\ &= -V \lrcorner \mathcal{L}_U\sigma + V \lrcorner U \lrcorner d\sigma = 3\psi_+(U, V, \cdot\,), \end{aligned}$$

which is exactly (3.3) with $\alpha = 3\psi_+$ and $p = U \wedge V$.

3.2 On the six-sphere

Let us start applying the construction in Example 3.1.5 to the six-sphere $S^6 \subset \mathbb{R}^7$. Concretely, the torus action on $\mathbb{R}^7 = \mathbb{C}^3 \oplus \mathbb{R}$ is the following: consider the maximal torus T^2 inside SU(3) given by matrices of the form $A_{\vartheta,\phi} := \operatorname{diag}(e^{i\vartheta}, e^{i\phi}, e^{-i(\vartheta+\phi)})$. If $(z, t) = (z^1, z^2, z^3, t) \in \mathbb{C}^3 \oplus \mathbb{R}$, then a matrix $A_{\vartheta,\phi}$ acts on it on the left as

$$A_{\vartheta,\phi}(z^1, z^2, z^3, t) := (e^{i\vartheta}z^1, e^{i\phi}z^2, e^{-i(\vartheta+\phi)}z^3, t).$$

Remark 3.2.1. Note that this action is effective: assume $A_{\vartheta,\phi}(z, t) = (z, t)$ for all $(z, t) \in \mathbb{C}^3 \oplus \mathbb{R}$. Then in particular $e^{i\vartheta}z^1 = z^1$ and $e^{i\phi}z^2 = z^2$ for all z^1, z^2, so $e^{i\vartheta} = e^{i\phi} = 1$, namely $A_{\vartheta,\phi} = \mathrm{Id}$.

Due to the particular convention chosen for the three-form (2.4) it is convenient to set $z^1 = x^1 + ix^6, z^2 = x^5 + ix^2, z^3 = x^4 + ix^3$, and $t = x^7$. Let $p = (z^1, z^2, z^3, t) \in \mathbb{C}^3 \oplus \mathbb{R}$, then we have the following fundamental vector fields corresponding to $(1,0), (0,1) \in \mathfrak{t}^2$:

$$U_p = \frac{d}{dt}\Big(\exp((t,0)) \cdot p\Big)\Big|_{t=0} = -x^6\partial_1 - x^4\partial_3 + x^3\partial_4 + x^1\partial_6,$$
$$V_p = \frac{d}{dt}\Big(\exp((0,t)) \cdot p\Big)\Big|_{t=0} = x^5\partial_2 - x^4\partial_3 + x^3\partial_4 - x^2\partial_5.$$

Plugging the two vectors in the expression of σ found in (2.5), one can compute

$$\nu_{\mathbb{R}^7}(p) = 3\big(x^1(x^4x^5 - x^2x^3) - x^6(x^3x^5 + x^2x^4)\big). \tag{3.4}$$

On the other hand $z^2z^3 = x^4x^5 - x^2x^3 + i(x^3x^5 + x^2x^4)$, so our map can also be written as

$$\nu_{\mathbb{R}^7}(p) = 3(\operatorname{Re}(z^1)\operatorname{Re}(z^2z^3) - \operatorname{Im}(z^1)\operatorname{Im}(z^2z^3)) = 3\operatorname{Re}(z^1z^2z^3).$$

Note that this expression is invariant under the action of T^2 as expected, because mapping $z^1 \mapsto e^{i\vartheta}z^1, z^2 \mapsto e^{i\phi}z^2, z^3 \mapsto e^{-i(\vartheta+\phi)}z^3$ leaves $\nu_{\mathbb{R}^7}$ unchanged.

What we need now is the restriction of this map to the six-sphere, which we call ν_{S^6}. Since $\psi_+ = i^*\varphi$, we have

$$d\nu_{S^6} = 3\psi_+(U, V, \,\cdot\,) = 3(i^*\varphi)(U, V, \,\cdot\,) = 3g(P(U,V), \,\cdot\,)_{TS^6},$$

which vanishes if and only if $P(U,V)$ is parallel to N, by non-degeneracy of the metric. We work out when this happens on the sphere by computing $P(U,V)$ and imposing it is parallel to N. One finds

$$\begin{aligned}
g(P(U,V),\partial_1) &= \phi(\partial_1, -x^6\partial_1 - x^4\partial_3 + x^3\partial_4 + x^1\partial_6, x^5\partial_2 - x^4\partial_3 + x^3\partial_4 - x^2\partial_5)\\
&= \phi(\partial_1, -x^4\partial_3, +x^5\partial_2) + \phi(\partial_1, x^3\partial_4, -x^2\partial_5)\\
&= x^4x^5 - x^2x^3,\\
g(P(U,V),\partial_2) &= \phi(\partial_2, -x^6\partial_1 - x^4\partial_3 + x^3\partial_4 + x^1\partial_6, x^5\partial_2 - x^4\partial_3 + x^3\partial_4 - x^2\partial_5)\\
&= \phi(\partial_2, -x^6\partial_1, -x^4\partial_3) + \phi(\partial_2, x^1\partial_6, x^3\partial_4)\\
&= -x^4x^6 - x^1x^3,\\
g(P(U,V),\partial_3) &= \phi(\partial_3, -x^6\partial_1 - x^4\partial_3 + x^3\partial_4 + x^1\partial_6, x^5\partial_2 - x^4\partial_3 + x^3\partial_4 - x^2\partial_5)\\
&= \phi(\partial_3, -x^6\partial_1, x^5\partial_2) + \phi(\partial_3, x^1\partial_6, -x^2\partial_5)\\
&= -x^5x^6 - x^1x^2,\\
g(P(U,V),\partial_4) &= \phi(\partial_4, -x^6\partial_1 - x^4\partial_3 + x^3\partial_4 + x^1\partial_6, x^5\partial_2 - x^4\partial_3 + x^3\partial_4 - x^2\partial_5)\\
&= \phi(\partial_4, -x^6\partial_1, -x^2\partial_5) + \phi(\partial_4, x^1\partial_6, x^5\partial_2)\\
&= -x^2x^6 + x^1x^5,\\
g(P(U,V),\partial_5) &= \phi(\partial_5, -x^6\partial_1 - x^4\partial_3 + x^3\partial_4 + x^1\partial_6, x^5\partial_2 - x^4\partial_3 + x^3\partial_4 - x^2\partial_5)\\
&= \phi(\partial_5, -x^6\partial_1, x^3\partial_4) + \phi(\partial_5, x^1\partial_6, -x^4\partial_3)\\
&= -x^3x^6 + x^1x^4,
\end{aligned}$$

$$\begin{aligned}
g(P(U,V),\partial_6) &= \phi(\partial_6, -x^6\partial_1 - x^4\partial_3 + x^3\partial_4 + x^1\partial_6, x^5\partial_2 - x^4\partial_3 + x^3\partial_4 - x^2\partial_5) \\
&= \phi(\partial_6, -x^4\partial_3, -x^2\partial_5) + \phi(\partial_6, x^3\partial_4, x^5\partial_2) \\
&= -x^2x^4 - x^3x^5, \\
g(P(U,V),\partial_7) &= \phi(\partial_7, -x^6\partial_1 - x^4\partial_3 + x^3\partial_4 + x^1\partial_6, x^5\partial_2 - x^4\partial_3 + x^3\partial_4 - x^2\partial_5) \\
&= \phi(\partial_7, -x^4\partial_3, x^3\partial_4) + \phi(\partial_7, x^3\partial_4, -x^4\partial_3) \\
&= 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
P(U,V) = (x^4x^5 - x^2x^3)\partial_1 &- (x^4x^6 + x^1x^3)\partial_2 \\
&- (x^5x^6 + x^1x^2)\partial_3 + (x^1x^5 - x^2x^6)\partial_4 \\
&+ (x^1x^4 - x^3x^6)\partial_5 - (x^2x^4 + x^3x^5)\partial_6.
\end{aligned}$$

To see where $P(U,V)$ is proportional to $N = x^1\partial_{x^1} + \ldots + x^7\partial_{x^7}$ we need to find the points $(x^1,\ldots,x^7)$ such that $\lambda x^7 = 0$ and

$$\begin{cases} x^4x^5 - x^2x^3 = \lambda x^1 \\ -x^4x^6 - x^1x^3 = \lambda x^2 \\ -x^5x^6 - x^1x^2 = \lambda x^3, \end{cases} \qquad \begin{cases} x^1x^5 - x^2x^6 = \lambda x^4 \\ x^1x^4 - x^3x^6 = \lambda x^5 \\ -x^2x^4 - x^3x^5 = \lambda x^6, \end{cases}$$

for some real number λ. We start with the case $\lambda = 0$, which gives points p where $P(U,V) = 0$, i.e. U_p and V_p are linearly dependent over $\mathbb{R}$, namely the multi-moment map vanishes.

Assume $x^1 = 0$, then one gets a trivial case if $x^6 \neq 0$, namely $x^2 = x^3 = x^4 = x^5 = 0$, whereas when $x^6 = 0$ we get only two equations: $x^4x^5 - x^2x^3 = 0 = x^2x^4 + x^3x^5$. Now if $x^2 = x^5 = 0$ then x^3, x^4, x^7 are subject to no constraint and we are done. Otherwise, the system tells us the vector (x^3, x^4) is parallel and orthogonal to (x^2, x^5) with respect to the standard scalar product in $\mathbb{R}^2$, so $x^3 = x^4 = 0$. The results may be summarised as follows:

1. $x^1 = x^2 = x^3 = x^4 = x^5 = 0$, $(x^6)^2 + (x^7)^2 = 1$.
2. $x^1 = x^2 = x^5 = x^6 = 0$, $(x^3)^2 + (x^4)^2 + (x^7)^2 = 1$.
3. $x^1 = x^3 = x^4 = x^6 = 0$, $(x^2)^2 + (x^5)^2 + (x^7)^2 = 1$.

When $x^1 \neq 0$ we end up with $x^4x^5 - x^2x^3 = 0 = x^2x^4 + x^3x^5$ and the relations

$$x^2 = -x^5x^6/x^1, \quad x^3 = -x^4x^6/x^1, \quad x^4 = x^3x^6/x^1, \quad x^5 = x^2x^6/x^1. \tag{3.5}$$

Then $x^4x^5 - x^2x^3 = 2x^2x^4x^6/x^1 = 0$, thus one of the following cases is necessary: $x^2 = 0$, $x^4 = 0$, or $x^6 = 0$.

If $x^2 = 0$ then $x^5 = 0$ as well by (3.5). Combining the relations giving x^3 and x^4 one finds that $x^3 = x^4 = 0$. Thus $x^1 \neq 0$ and $x^2, \ldots, x^5$ vanish. This result together with solution 1 above implies there is a third two-sphere of critical points, that is $(x^1)^2 + (x^6)^2 + (x^7)^2 = 1$. In the case $x^4 = 0$ we get the same result. When $x^6 = 0$ then $x^2, \ldots, x^5 = 0$, with $x^1 \neq 0$.

Therefore, we obtain three two-spheres of critical points where the multi-moment map vanishes:

1. $(x^1)^2+(x^6)^2+(x^7)^2=1$ and $x^2=x^3=x^4=x^5=0$.
2. $(x^2)^2+(x^5)^2+(x^7)^2=1$ and $x^1=x^3=x^4=x^6=0$.
3. $(x^3)^2+(x^4)^2+(x^7)^2=1$ and $x^1=x^2=x^5=x^6=0$.

Note that they intersect only at the poles $(x^7)^2=1, x^i=0, i\neq 7$. We will come back to this fact later. Consider now the case $\lambda\neq 0$. Assume we are at a critica point p, so in particular $x^7=0$. If $x^1\neq 0$, then we can move p in the direction of U—i.e. multiplying p by $\mathrm{diag}(e^{i\vartheta},1,e^{-i\vartheta})$ on the left—getting a point of the form

$$(x^1\cos\vartheta - x^6\sin\vartheta,\ \cdot\,,\ \cdot\,,\ \cdot\,,\ \cdot\,, x^6\cos\vartheta + x^1\sin\vartheta, 0),$$

and this is still a critical point by the T^2-invariance of the differential of the multi-moment map. We can always choose ϑ such that $x^1\cos\vartheta - x^6\sin\vartheta = 0$. So we assume $x^1=0$ and $x^6\geq 0$ up to the action of U. Now one can move this new point in the direction of V multiplying by $\mathrm{diag}(1,e^{i\phi},e^{-i\phi})$. The action fixes the plane x^1+ix^6, so the first component remains the same. We get

$$(0, x^2\cos\phi - x^5\sin\phi,\ \cdot\,,\ \cdot\,,\ \cdot\,,\ \cdot\,, 0).$$

As above, we can choose ϕ such that $x^2\cos\phi - x^5\sin\phi = 0$. So, up to the action of U and V, we can assume $x^1=x^2=0$ and $x^5,x^6\geq 0$. The system of equations characterising critical points yields $x^4=0$ and $x^5x^6=-\lambda x^3, x^3x^6=-\lambda x^5, x^3x^5=-\lambda x^6$. If one among x^3,x^5,x^6 vanishes, so do all the others, and we get a contradiction because we need solutions on the sphere. Therefore we can assume without loss of generality all of them non-zero, which gives $(x^i)^2=\lambda^2$ for $i=3,5,6$, namely $x^5=x^6=1/\sqrt{3}$ and $(x^3)^2=1/3$. We thus obtain two stationary T^2-orbits where $\nu_{\mathbb{S}^6}$ attains its maximum and minimum, and $\nu_{\mathbb{S}^6}(\mathbb{S}^6)=[-1/\sqrt{3},1/\sqrt{3}]$.

3.3 On the flag manifold

Consider the maximal torus T^2 in $\mathrm{SU}(3)$ given by the matrices $\mathrm{diag}(e^{i\alpha},e^{i\beta},e^{-i(\alpha+\beta)})$. Two linearly independent generators of its Lie algebra are then

$$U=\mathrm{diag}(i,0,-i),\quad V=\mathrm{diag}(0,i,-i),$$

and the infinitesimal generators of the action at $p\in\mathrm{SU}(3)$ are given by

$$U_p := Up = \frac{d}{dt}\Big(\exp(tU)\cdot p\Big)\Big|_{t=0},\quad V_p := Vp = \frac{d}{dt}\Big(\exp(tV)\cdot p\Big)\Big|_{t=0}.$$

Thus $p^{-1}Up, p^{-1}Vp$ are vectors in the Lie algebra $\mathfrak{su}(3)$, which splits as $\mathfrak{t}^2\oplus\mathfrak{m}$, $\mathfrak{m}$ containing matrices with zeros on the diagonal. So when we work on the quotient $\mathrm{SU}(3)/T^2$ we need to consider the projections $(p^{-1}Up)_{\mathfrak{m}}$ and $(p^{-1}Vp)_{\mathfrak{m}}$.

A matrix $p=(p_{ij})\in\mathrm{SU}(3)$ satisfies the conditions $\det p=1$ and

$$\begin{cases}|p_{11}|^2+|p_{21}|^2+|p_{31}|^2=1\\ |p_{12}|^2+|p_{22}|^2+|p_{32}|^2=1\\ |p_{13}|^2+|p_{23}|^2+|p_{33}|^2=1,\end{cases}\qquad \begin{cases}\overline{p}_{11}p_{12}+\overline{p}_{21}p_{22}+\overline{p}_{31}p_{32}=0\\ \overline{p}_{11}p_{13}+\overline{p}_{21}p_{23}+\overline{p}_{31}p_{33}=0\\ \overline{p}_{12}p_{13}+\overline{p}_{22}p_{23}+\overline{p}_{32}p_{33}=0.\end{cases}$$

One can compute explicitly $p^{-1}Up, p^{-1}Vp$ and project them onto $\mathfrak{m}$:

$$(p^{-1}Up)_{\mathfrak{m}} = \begin{pmatrix} & iz^1 & iz^2 \\ i\overline{z}^1 & & iz^3 \\ i\overline{z}^2 & i\overline{z}^3 & \end{pmatrix}, \qquad \begin{cases} z^1 = \overline{p}_{11}p_{12} - \overline{p}_{31}p_{32} \\ z^2 = \overline{p}_{11}p_{13} - \overline{p}_{31}p_{33} \\ z^3 = \overline{p}_{12}p_{13} - \overline{p}_{32}p_{33}, \end{cases}$$

$$(p^{-1}Vp)_{\mathfrak{m}} = \begin{pmatrix} & iw^1 & iw^2 \\ i\overline{w}^1 & & iw^3 \\ i\overline{w}^2 & i\overline{w}^3 & \end{pmatrix}, \qquad \begin{cases} w^1 = \overline{p}_{21}p_{22} - \overline{p}_{31}p_{32} \\ w^2 = \overline{p}_{21}p_{23} - \overline{p}_{31}p_{33} \\ w^3 = \overline{p}_{22}p_{23} - \overline{p}_{32}p_{33}. \end{cases}$$

Note that the coefficients z^i and w^k are all T^2-invariant. Using the set-up as in Section 2.3 we write $(p^{-1}Up)_{\mathfrak{m}}, (p^{-1}Vp)_{\mathfrak{m}}$ in terms of the basis of $\mathfrak{su}(3)/\mathfrak{t}^2$, the vectors $J(p^{-1}Up)_{\mathfrak{m}}$ and $J(p^{-1}Vp)_{\mathfrak{m}}$ follow trivially by (2.7):

$$\begin{aligned} (p^{-1}Up)_{\mathfrak{m}} &= \operatorname{Re} z^1 E_1 - \operatorname{Im} z^1 JE_1 - \operatorname{Im} z^2 E_3 \\ &\quad + \operatorname{Re} z^2 JE_3 + \operatorname{Re} z^3 E_5 - \operatorname{Im} z^3 JE_5, \end{aligned} \tag{3.6}$$

$$\begin{aligned} (p^{-1}Vp)_{\mathfrak{m}} &= \operatorname{Re} w^1 E_1 - \operatorname{Im} w^1 JE_1 - \operatorname{Im} w^2 E_3 \\ &\quad + \operatorname{Re} w^2 JE_3 + \operatorname{Re} w^3 E_5 - \operatorname{Im} w^3 JE_5. \end{aligned} \tag{3.7}$$

The multi-moment map is then $\nu_{F_{1,2}(\mathbb{C}^3)}(p) = \sigma_p(U_p, V_p) = \sigma_0((p^{-1}Up)_{\mathfrak{m}}, (p^{-1}Vp)_{\mathfrak{m}})$, namely

$$\begin{aligned} \nu_{F_{1,2}(\mathbb{C}^3)}(p) &= -\operatorname{Re} z^1 \operatorname{Im} w^1 + \operatorname{Re} w^1 \operatorname{Im} z^1 - \operatorname{Im} z^2 \operatorname{Re} w^2 \\ &\quad + \operatorname{Re} z^2 \operatorname{Im} w^2 - \operatorname{Re} z^3 \operatorname{Im} w^3 + \operatorname{Re} w^3 \operatorname{Im} z^3 \\ &= -\operatorname{Im}(\overline{z}^1 w^1 - \overline{z}^2 w^2 + \overline{z}^3 w^3). \end{aligned}$$

Since z^i, w^k are T^2-invariant it is clear that $\nu_{F_{1,2}(\mathbb{C}^3)}$ is T^2-invariant as well.

We go ahead and compute $d\nu_{F_{1,2}(\mathbb{C}^3)} = 3\psi_+(U, V, \cdot\,)$, where U, V are shorthands for the vectors in (3.6), (3.7). Recall that

$$\psi_+ = -e^1 \wedge e^3 \wedge Je^5 + Je^1 \wedge Je^3 \wedge Je^5 - Je^1 \wedge e^3 \wedge e^5 - e^1 \wedge Je^3 \wedge e^5.$$

Therefore

$$\begin{aligned} -e^1 \wedge e^3 \wedge Je^5(U, V, \cdot\,) &= -e^1(U)e^3(V)Je^5 + e^1(U)Je^5(V)e^3 + e^3(U)e^1(V)Je^5 \\ &\quad - e^3(U)Je^5(V)e^1 - Je^5(U)e^1(V)e^3 + Je^5(U)e^3(V)e^1 \\ &= (\operatorname{Im} z^3 \operatorname{Im} w^2 - \operatorname{Im} z^2 \operatorname{Im} w^3)e^1 \\ &\quad + (\operatorname{Im} z^3 \operatorname{Re} w^1 - \operatorname{Re} z^1 \operatorname{Im} w^3)e^3 \\ &\quad + (\operatorname{Re} z^1 \operatorname{Im} w^2 - \operatorname{Im} z^2 \operatorname{Re} w^1)Je^5, \\ Je^1 \wedge Je^3 \wedge Je^5(U, V, \cdot\,) &= Je^1(U)Je^3(V)Je^5 - Je^1(U)Je^5(V)Je^3 - Je^3(U)Je^1(V)Je^5 \\ &\quad + Je^3(U)Je^5(V)Je^1 + Je^5(U)Je^1(V)Je^3 - Je^5(U)Je^3(V)Je^1 \\ &= (\operatorname{Im} z^3 \operatorname{Re} w^2 - \operatorname{Re} z^2 \operatorname{Im} w^3)Je^1 \\ &\quad + (\operatorname{Im} z^3 \operatorname{Im} w^1 - \operatorname{Im} z^1 \operatorname{Im} w^3)Je^3 \\ &\quad + (\operatorname{Re} z^2 \operatorname{Im} w^1 - \operatorname{Im} z^1 \operatorname{Re} w^2)Je^5, \end{aligned}$$

$$\begin{aligned}
-Je^1 \wedge e^3 \wedge e^5(U,V,\,\cdot\,) &= -Je^1(U)e^3(V)e^5 + Je^1(U)e^5(V)e^3 + e^3(U)Je^1(V)e^5 \\
&\qquad - e^3(U)e^5(V)Je^1 - e^5(U)Je^1(V)e^3 + e^5(U)e^3(V)Je^1 \\
&= (\operatorname{Im} z^2 \operatorname{Re} w^3 - \operatorname{Re} z^3 \operatorname{Im} w^2)Je^1 \\
&\qquad + (\operatorname{Re} z^3 \operatorname{Im} w^1 - \operatorname{Im} z^1 \operatorname{Re} w^3)e^3 \\
&\qquad + (\operatorname{Im} z^2 \operatorname{Im} w^1 - \operatorname{Im} z^1 \operatorname{Im} w^2)e^5, \\
-e^1 \wedge Je^3 \wedge e^5(U,V,\,\cdot\,) &= -e^1(U)Je^3(V)e^5 + e^1(U)e^5(V)Je^3 + Je^3(U)e^1(V)e^5 \\
&\qquad - Je^3(U)e^5(V)e^1 - e^5(U)e^1(V)Je^3 + e^5(U)Je^3(V)e^1 \\
&= (\operatorname{Re} z^3 \operatorname{Re} w^2 - \operatorname{Re} z^2 \operatorname{Re} w^3)e^1 \\
&\qquad + (\operatorname{Re} z^1 \operatorname{Re} w^3 - \operatorname{Re} z^3 \operatorname{Re} w^1)Je^3 \\
&\qquad + (\operatorname{Re} z^2 \operatorname{Re} w^1 - \operatorname{Re} z^1 \operatorname{Re} w^2)e^5.
\end{aligned}$$

Thus summing all terms

$$\begin{aligned}
\psi_+(U,V,\,\cdot\,)_{|p} &= (\operatorname{Im} z^3 \operatorname{Im} w^2 - \operatorname{Im} z^2 \operatorname{Im} w^3 + \operatorname{Re} z^3 \operatorname{Re} w^2 - \operatorname{Re} z^2 \operatorname{Re} w^3)e^1 \\
&\quad + (\operatorname{Im} z^3 \operatorname{Re} w^2 - \operatorname{Re} z^2 \operatorname{Im} w^3 + \operatorname{Im} z^2 \operatorname{Re} w^3 - \operatorname{Re} z^3 \operatorname{Im} w^2)Je^1 \\
&\quad + (\operatorname{Im} z^3 \operatorname{Re} w^1 - \operatorname{Re} z^1 \operatorname{Im} w^3 + \operatorname{Re} z^3 \operatorname{Im} w^1 - \operatorname{Im} z^1 \operatorname{Re} w^3)e^3 \\
&\quad + (\operatorname{Im} z^3 \operatorname{Im} w^1 - \operatorname{Im} z^1 \operatorname{Im} w^3 + \operatorname{Re} z^1 \operatorname{Re} w^3 - \operatorname{Re} z^3 \operatorname{Re} w^1)Je^3 \\
&\quad + (\operatorname{Im} z^2 \operatorname{Im} w^1 - \operatorname{Im} z^1 \operatorname{Im} w^2 + \operatorname{Re} z^2 \operatorname{Re} w^1 - \operatorname{Re} z^1 \operatorname{Re} w^2)e^5 \\
&\quad + (\operatorname{Re} z^1 \operatorname{Im} w^2 - \operatorname{Im} z^2 \operatorname{Re} w^1 + \operatorname{Re} z^2 \operatorname{Im} w^1 - \operatorname{Im} z^1 \operatorname{Re} w^2)Je^5 \\
&= \operatorname{Re}(z^3\overline{w}^2 - z^2\overline{w}^3)e^1 + \operatorname{Im}(z^3\overline{w}^2 + z^2\overline{w}^3)Je^1 + \operatorname{Im}(z^3w^1 - z^1w^3)e^3 \\
&\quad + \operatorname{Re}(z^1w^3 - z^3w^1)Je^3 + \operatorname{Re}(\overline{z}^2w^1 - \overline{z}^1w^2)e^5 + \operatorname{Im}(\overline{z}^2w^1 + \overline{z}^1w^2)Je^5.
\end{aligned}$$

This implies that a point $p \in \mathrm{SU}(3)/T^2$ is critical if and only if

$$z^2\overline{w}^3 = \overline{z}^3w^2, \qquad z^1w^3 = z^3w^1, \qquad \overline{z}^1w^2 = z^2\overline{w}^1.$$

Since p lies in $\mathrm{SU}(3)$, using the relations $z^1 + w^1 = -3\overline{p}_{31}p_{32}, z^2 + w^2 = -3\overline{p}_{31}p_{33}, z^3 + w^3 = -3\overline{p}_{32}p_{33}$ we find that p is critical precisely when

$$\begin{cases} p_{22}\overline{p}_{23}\overline{p}_{31}p_{33} = \overline{p}_{21}p_{23}p_{32}\overline{p}_{33} \\ \overline{p}_{22}p_{23}\overline{p}_{31}p_{32} = \overline{p}_{21}p_{22}\overline{p}_{32}p_{33} \\ \overline{p}_{21}p_{23}p_{31}\overline{p}_{32} = p_{21}\overline{p}_{22}\overline{p}_{31}p_{33}. \end{cases} \tag{3.8}$$

We may use the T^2-actions on the left and on the right to simplify the system: the matrix p becomes

$$\begin{pmatrix} e^{i(\alpha+\sigma)}p_{11} & e^{i(\alpha+\rho)}p_{12} & e^{i(\alpha-\sigma-\rho)}p_{13} \\ e^{i(\beta+\sigma)}p_{21} & e^{i(\beta+\rho)}p_{22} & e^{i(\beta-\sigma-\rho)}p_{23} \\ e^{i(\gamma+\sigma)}p_{31} & e^{i(\gamma+\rho)}p_{32} & e^{i(\gamma-\sigma-\rho)}p_{33} \end{pmatrix},$$

so we can make $p_{21}, p_{22}, p_{23}, p_{32}$ real and non-negative. This is possible because there are unique values of $\beta, \gamma, \sigma, \rho$ such that

$$\begin{cases} \beta + \sigma = -c_{21} \\ \beta + \rho = -c_{22} \\ \beta - \sigma - \rho = -c_{23} \\ \gamma + \rho = -c_{32}, \end{cases}$$

where c_{ij} is the argument of p_{ij}. Let us write $a = p_{21}, b = p_{22}, c = p_{23}, d = p_{32}$. Then (3.8) becomes

$$\begin{cases} bc\overline{p}_{31}p_{33} = acd\overline{p}_{33} \\ bcd\overline{p}_{31} = abdp_{33} \\ acdp_{31} = ab\overline{p}_{31}p_{33}. \end{cases}$$

Our set-up is invariant under cyclic permutations of columns or rows of p up to a sign of $\nu_{F_{1,2}(\mathbb{C}^3)}$, so in order to work out maxima and minima we can distinguish three cases:

1. $a = b = 0, c \neq 0,$
2. $a = 0, b \neq 0, c \neq 0,$
3. $a \neq 0, b \neq 0, c \neq 0.$

In the first one the criticality conditions are automatically satisfied. Since rows and columns are unitary, one necessarily has $p_{13} = p_{33} = 0$ and $c = 1$. The conditions characterising $p \in \mathrm{SU}(3)$ imply

$$\begin{cases} |p_{11}|^2 + |p_{31}|^2 = 1 \\ |p_{12}|^2 + d^2 = 1 \\ \overline{p}_{11}p_{12} + \overline{p}_{31}d = 0 \\ p_{12}p_{31} - p_{11}d = 1. \end{cases}$$

When $d = 0$ then $p_{31} = e^{i\vartheta}$, which implies $p_{12} = e^{-i\vartheta}$ and $p_{11} = 0$. If $d \neq 0$ then the last two equations yield $p_{11} = (p_{12}p_{31} - 1)/d$ and $\overline{p}_{31} = -\overline{p}_{11}p_{12}/d$. Using that $d^2 + |p_{12}|^2 = 1$ and $|p_{11}|^2 + |p_{31}|^2 = 1$ we find $\overline{p}_{31} = p_{12}$ and $p_{11} = -|p_{11}|^2/d$. In particular, p_{11} is real and non-positive. If it is zero then we get $p_{31} = e^{i\vartheta}$ and $p_{12} = e^{-i\vartheta}$, whereas all the other entries except for c vanish. If $p_{11} \neq 0$, then $p_{11} = -d$. In all cases p has the following form:

$$\begin{pmatrix} -d & p_{12} & 0 \\ 0 & 0 & 1 \\ \overline{p}_{12} & d & 0 \end{pmatrix}, \quad \text{with } |p_{12}|^2 + d^2 = 1.$$

In the second case the critical conditions become $\overline{p}_{31}p_{33} = 0$ and $\overline{p}_{31}d = 0$. Assume first $p_{31} = 0$. Then $p_{11} = e^{i\vartheta}, p_{12} = p_{13} = 0$ and the conditions on p yield

$$\begin{cases} b^2 + d^2 = 1 \\ c^2 + |p_{33}|^2 = 1, \end{cases} \qquad \begin{cases} bc + dp_{33} = 0 \\ bp_{33}e^{i\vartheta} - cde^{i\vartheta} = 1. \end{cases}$$

If $d = 0$ then $c = 0$ as well, but this is a contradiction, so $d \neq 0$. We then find $p_{33} = -bc/d$, so in particular p_{33} is real. Then $bp_{33}e^{i\vartheta} - cde^{i\vartheta} = 1$ forces $e^{i\vartheta} = \pm 1$, so $d = -c$ when $e^{i\vartheta} = 1$ and $c = d$ when $e^{i\vartheta} = -1$. This yields the solutions

$$p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & c \\ 0 & -c & b \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & b & c \\ 0 & c & -b \end{pmatrix}, \quad \text{with } b^2 + c^2 = 1.$$

If $p_{31} \neq 0$ then necessarily $p_{33} = 0$ and $d = 0$ by criticality, which implies $p_{31} = e^{i\vartheta}$ and $p_{11} = 0$. Imposing that p lies in $\mathrm{SU}(3)$ we find the conditions

$$\begin{cases} e^{i\vartheta}(p_{12}c - p_{13}b) = 1 \\ |p_{12}|^2 + b^2 = 1 \\ |p_{13}|^2 + c^2 = 1 \\ \overline{p}_{12}p_{13} + bc = 0. \end{cases}$$

Now $p_{13} = 0$ would imply $b = 0$, so $p_{13} \neq 0$ and then $p_{12} = -(c/b)p_{13}$. The first equation yields $p_{13} = -be^{-i\vartheta}$, so $p_{12} = ce^{-i\vartheta}$. Then p has the form

$$\begin{pmatrix} 0 & ce^{-i\vartheta} & -be^{-i\vartheta} \\ 0 & b & c \\ e^{i\vartheta} & 0 & 0 \end{pmatrix}, \quad \text{with } b^2 + c^2 = 1.$$

The multi-moment map vanishes at all the critical points found.

In the third case $a, b, c \neq 0$ then d cannot be 0, otherwise the criticality conditions would imply either $p_{31} = 0$ or $p_{33} = 0$, thus either $a = 0$ or $c = 0$. Then our equations are

$$\begin{cases} b\overline{p}_{31}p_{33} = a\overline{p}_{33}d \\ cp_{31} = a\overline{p}_{33} \\ cp_{31}d = b\overline{p}_{31}p_{33}. \end{cases}$$

Set $p_{31} := \rho e^{i\vartheta}, p_{33} := \sigma e^{i\varphi}$, so that the system becomes

$$\begin{cases} b\rho\sigma e^{i(\varphi-\vartheta)} = a\sigma e^{-i\varphi}d \\ c\rho e^{i\vartheta} = a\sigma e^{-i\varphi} \\ c\rho e^{i\vartheta}d = b\rho\sigma e^{i(\varphi-\vartheta)}. \end{cases}$$

Observe that if $\rho = 0$ then $\sigma = 0$, so $d = 1$ and $b = 0$, which is a contradiction. Analogously, if $\sigma = 0$ then $\rho = 0$, contradiction. Then $p_{31}, p_{33} \neq 0$ and a comparison of the arguments in the system shows that $3\varphi \equiv 0 \pmod{2\pi}, \vartheta \equiv -\varphi \pmod{2\pi}$. Comparing the radii we obtain $ad = b\rho, c\rho = a\sigma, cd = b\sigma$, so $\rho = ad/b$ and $\sigma = cd/b$ and $p_{31} = (ad/b)e^{-i\varphi}, p_{33} = (cd/b)e^{i\varphi}$. Now second and third row have unit length, whence $a^2d^2/b^2 + d^2 + c^2d^2/b^2 = 1$, which implies $d = b$. Our matrix has then the form

$$\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ a & b & c \\ ae^{-i\varphi} & b & ce^{i\varphi} \end{pmatrix},$$

and the constraint $p \in \mathrm{SU}(3)$ gives

$$\begin{cases} |p_{11}|^2 + 2a^2 = 1 \\ |p_{12}|^2 + 2b^2 = 1 \\ |p_{13}|^2 + 2c^2 = 1, \end{cases} \qquad \begin{cases} \overline{p}_{11} p_{12} = -ab(1+e^{i\varphi}) \\ p_{11}\overline{p}_{13} = -ac(1+e^{i\varphi}) \\ \overline{p}_{12} p_{13} = -bc(1+e^{i\varphi}). \end{cases}$$

The second system in particular implies the chain of equations

$$\frac{\overline{p}_{11} p_{12}}{ab} = \frac{p_{11}\overline{p}_{13}}{ac} = \frac{\overline{p}_{12} p_{13}}{bc} = -1 - e^{i\varphi},$$

whereas the first allows us to write $p_{11} = \sqrt{1-2a^2}e^{i\alpha}, p_{12} = \sqrt{1-2b^2}e^{i\beta}, p_{13} = \sqrt{1-2c^2}e^{i\gamma}$. Now we have three possibilities: $\varphi = 0, \varphi = 2\pi/3, \varphi = 4\pi/3$. In the first one two rows in the matrix p are the same, so the determinant vanishes, which is a contradiction. We can then assume $\varphi = 2\pi/3$, so that

$$\frac{\overline{p}_{11} p_{12}}{ab} = \frac{p_{11}\overline{p}_{13}}{ac} = \frac{\overline{p}_{12} p_{13}}{bc} = e^{4\pi i/3}.$$

Comparing the arguments we find

$$\begin{cases} \beta - \alpha \equiv 4\pi/3 \pmod{2\pi} \\ \alpha - \gamma \equiv 4\pi/3 \pmod{2\pi} \\ \gamma - \beta \equiv 4\pi/3 \pmod{2\pi}, \end{cases}$$

which implies $\beta \equiv \alpha + 4\pi/3 \pmod{2\pi}$ and $\gamma \equiv \alpha + 2\pi/3 \pmod{2\pi}$. Comparing the radii instead, we get

$$\begin{cases} c\sqrt{1-2a^2}\sqrt{1-2b^2} = b\sqrt{1-2a^2}\sqrt{1-2c^2} \\ c\sqrt{1-2a^2}\sqrt{1-2b^2} = a\sqrt{1-2b^2}\sqrt{1-2c^2} \\ b\sqrt{1-2a^2}\sqrt{1-2c^2} = a\sqrt{1-2b^2}\sqrt{1-2c^2}. \end{cases}$$

We can assume $a, b, c \neq 1/\sqrt{2}$, otherwise we swap first and second row and end up with one of the cases we started with. Hence $a = b = c = 1/\sqrt{3}$. We have an expression for all the entries of our matrix p, which then has the form

$$\frac{1}{\sqrt{3}} \begin{pmatrix} e^{i\alpha} & e^{i(\alpha+4\pi/3)} & e^{i(\alpha+2\pi/3)} \\ 1 & 1 & 1 \\ e^{-2\pi i/3} & 1 & e^{2\pi i/3} \end{pmatrix},$$

Imposing the condition $\det p = 1$ we find $\alpha \equiv 7\pi/6 \pmod{2\pi}$, so

$$p = \frac{1}{\sqrt{3}} \begin{pmatrix} i\omega & i & i\omega^2 \\ 1 & 1 & 1 \\ \omega^2 & 1 & \omega \end{pmatrix}, \quad \text{with } \omega = e^{2\pi i/3}. \tag{3.9}$$

This is a point of minimum, the value of the multi-moment map at p is $-\sqrt{3}/2$.

The last case $\varphi = 4\pi/3$ is similar: we have $\overline{p}_{11}p_{12}/ab = p_{11}\overline{p}_{13}/ac = \overline{p}_{12}p_{13}/bc = e^{2\pi i/3}$, whence

$$\begin{cases} \beta - \alpha \equiv 2\pi/3 \pmod{2\pi} \\ \alpha - \gamma \equiv 2\pi/3 \pmod{2\pi} \\ \gamma - \beta \equiv 2\pi/3 \pmod{2\pi}, \end{cases}$$

so $\beta \equiv \alpha + 2\pi/3 \pmod{2\pi}, \gamma \equiv \alpha + 4\pi/3 \pmod{2\pi}$. The comparison of the radii still yields $a = b = c = 1/\sqrt{3}$. The condition $\det p = 1$ implies $\alpha = 5\pi/6$. So

$$p = \frac{1}{\sqrt{3}} \begin{pmatrix} -i\omega^2 & -i & -i\omega \\ 1 & 1 & 1 \\ \omega & 1 & \omega^2 \end{pmatrix}, \quad \omega = e^{2\pi i/3},$$

and since $\nu_{F_{1,2}(\mathbb{C}^3)}(\overline{p}) = -\nu_{F_{1,2}(\mathbb{C}^3)}(p)$, by (3.9) the value of the multi-moment map is $\sqrt{3}/2$. Summing up, we got two stationary orbits corresponding to maximum and minimum, and $\operatorname{Im} \nu_{F_{1,2}(\mathbb{C}^3)} = [-\sqrt{3}/2, \sqrt{3}/2]$.

3.4 On the complex projective space

Let T^2 be the maximal torus in $G = \mathrm{Sp}(2)$ given by matrices of the form $\mathrm{diag}(e^{i\vartheta}, e^{i\varphi})$. Then the generators of $\mathfrak{sp}(2)$ are $U = \mathrm{diag}(i, 0)$ and $V = \mathrm{diag}(0, i)$. We want to compute the vectors $p^{-1}Up$ and $p^{-1}Vp$ in terms of the coefficients of p as an element of $\mathrm{Sp}(2)$. We split p as $p_{ik} = p^1_{ik} + p^2_{ik}j$, where p^1, p^2 are now 2×2 matrices of complex numbers, and recall that $p^{-1} = {}^t\overline{p}$, which is equivalent to

$$\begin{cases} {}^t\overline{p}^1 p^1 + {}^t p^2 \overline{p}^2 = \mathrm{Id} \\ {}^t\overline{p}^1 p^2 - {}^t p^2 \overline{p}^1 = 0. \end{cases}$$

Expanding this system we get the conditions

$$\begin{cases} |p^1_{11}|^2 + |p^1_{21}|^2 + |p^2_{11}|^2 + |p^2_{21}|^2 = 1 \\ |p^1_{12}|^2 + |p^1_{22}|^2 + |p^2_{12}|^2 + |p^2_{22}|^2 = 1 \\ \overline{p}^1_{11} p^1_{12} + \overline{p}^1_{21} p^1_{22} + \overline{p}^2_{12} p^2_{11} + \overline{p}^2_{22} p^2_{21} = 0 \\ \overline{p}^1_{11} p^2_{12} + \overline{p}^1_{21} p^2_{22} - \overline{p}^1_{12} p^2_{11} - \overline{p}^1_{22} p^2_{21} = 0. \end{cases}$$

We can thus calculate the vectors generating the action:

$$\begin{aligned} p^{-1}Up &= \left(\begin{pmatrix} \overline{p}^1_{11} & \overline{p}^1_{21} \\ \overline{p}^1_{12} & \overline{p}^1_{22} \end{pmatrix} - \begin{pmatrix} p^2_{11} & p^2_{21} \\ p^2_{12} & p^2_{22} \end{pmatrix} j \right) \left(\begin{pmatrix} ip^1_{11} & ip^1_{12} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} ip^2_{11} & ip^2_{12} \\ 0 & 0 \end{pmatrix} j \right) \\ &= \begin{pmatrix} i(|p^1_{11}|^2 - |p^2_{11}|^2) & i(\overline{p}^1_{11} p^1_{12} - p^2_{11} \overline{p}^2_{12}) \\ i(\overline{p}^1_{12} p^1_{11} - p^2_{12} \overline{p}^2_{11}) & i(|p^1_{12}|^2 - |p^2_{12}|^2) \end{pmatrix} \\ &\quad + \begin{pmatrix} 2i\overline{p}^1_{11} p^2_{11} & i(\overline{p}^1_{11} p^2_{12} + p^2_{11} \overline{p}^1_{12}) \\ i(\overline{p}^1_{12} p^2_{11} + p^2_{12} \overline{p}^1_{11}) & 2ip^2_{12} \overline{p}^1_{12} \end{pmatrix} j, \end{aligned}$$

$$\begin{aligned} p^{-1}Vp &= \left(\begin{pmatrix} \overline{p}^1_{11} & \overline{p}^1_{21} \\ \overline{p}^1_{12} & \overline{p}^1_{22} \end{pmatrix} - \begin{pmatrix} p^2_{11} & p^2_{21} \\ p^2_{12} & p^2_{22} \end{pmatrix} j \right) \left(\begin{pmatrix} 0 & 0 \\ ip^1_{21} & ip^1_{22} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ ip^2_{21} & ip^2_{22} \end{pmatrix} j \right) \\ &= \begin{pmatrix} i(|p^1_{21}|^2 - |p^2_{21}|^2) & i(\overline{p}^1_{21} p^1_{22} - p^2_{21} \overline{p}^2_{22}) \\ i(\overline{p}^1_{22} p^1_{21} - p^2_{22} \overline{p}^2_{21}) & i(|p^1_{22}|^2 - |p^2_{22}|^2) \end{pmatrix} \\ &\quad + \begin{pmatrix} 2i\overline{p}^1_{21} p^2_{21} & i(\overline{p}^1_{21} p^2_{22} + p^2_{21} \overline{p}^1_{22}) \\ i(\overline{p}^1_{22} p^2_{21} + p^2_{22} \overline{p}^1_{21}) & 2i\overline{p}^1_{22} p^2_{22} \end{pmatrix} j. \end{aligned}$$

The Lie algebra $\mathfrak{h}$ contains elements of the form $\left(\begin{smallmatrix} ia & 0 \\ 0 & ib \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 0 & c \end{smallmatrix}\right) j$, with a, b real and c complex. The Lie algebra $\mathfrak{g}$ splits as $\mathfrak{h} \oplus \mathfrak{m}$. The projections $(p^{-1}Up)_{\mathfrak{m}}, (p^{-1}Vp)_{\mathfrak{m}}$ must be of the form $\left(\begin{smallmatrix} xj & \rho \\ -\overline{\rho} & 0 \end{smallmatrix}\right)$, for x complex and ρ quaternion, so

$$(p^{-1}Up)_{\mathfrak{m}} = \begin{pmatrix} & \alpha \\ -\overline{\alpha} & \end{pmatrix} + \begin{pmatrix} \gamma j & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{cases} \alpha = i(\overline{p}^1_{11}p^1_{12} - p^2_{11}\overline{p}^2_{12}) + i(\overline{p}^1_{11}p^2_{12} + p^2_{11}\overline{p}^1_{12})j \\ \gamma = 2i\overline{p}^1_{11}p^2_{11}, \end{cases}$$
$$(p^{-1}Vp)_{\mathfrak{m}} = \begin{pmatrix} & \beta \\ -\overline{\beta} & \end{pmatrix} + \begin{pmatrix} \delta j & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{cases} \beta = i(\overline{p}^1_{21}p^1_{22} - p^2_{21}\overline{p}^2_{22}) + i(\overline{p}^1_{21}p^2_{22} + p^2_{21}\overline{p}^1_{22})j \\ \delta = 2i\overline{p}^1_{21}p^2_{21}. \end{cases}$$

We now write $(p^{-1}Up)_{\mathfrak{m}}, (p^{-1}Vp)_{\mathfrak{m}}$ in terms of the basis introduced in Section 2.4: note that we shift the indices so that $E_0 \mapsto E_1, \dots, E_5 \mapsto E_6$, the first convention we used is no longer needed. Then

$$\begin{aligned}
(p^{-1}Up)_{\mathfrak{m}} &= 2\operatorname{Re}(\overline{p}^1_{11}p^2_{11})E_1 - 2\operatorname{Im}(\overline{p}^1_{11}p^2_{11})E_2 \\
&\qquad + \sqrt{2}\Big(-\operatorname{Im}(\overline{p}^1_{11}p^1_{12} - p^2_{11}\overline{p}^2_{12})E_3 + \operatorname{Re}(\overline{p}^1_{11}p^1_{12} - p^2_{11}\overline{p}^2_{12})E_4 \\
&\qquad\qquad - \operatorname{Im}(\overline{p}^1_{11}p^2_{12} + p^2_{11}\overline{p}^1_{12})E_5 + \operatorname{Re}(\overline{p}^1_{11}p^2_{12} + p^2_{11}\overline{p}^1_{12})E_6\Big), \\
(p^{-1}Vp)_{\mathfrak{m}} &= 2\operatorname{Re}(\overline{p}^1_{21}p^2_{21})E_1 - 2\operatorname{Im}(\overline{p}^1_{21}p^2_{21})E_2 \\
&\qquad + \sqrt{2}\Big(-\operatorname{Im}(\overline{p}^1_{21}p^1_{22} - p^2_{21}\overline{p}^2_{22})E_3 + \operatorname{Re}(\overline{p}^1_{21}p^1_{22} - p^2_{21}\overline{p}^2_{22})E_4 \\
&\qquad\qquad - \operatorname{Im}(\overline{p}^1_{21}p^2_{22} + p^2_{21}\overline{p}^1_{22})E_5 + \operatorname{Re}(\overline{p}^1_{21}p^2_{22} + p^2_{21}\overline{p}^1_{22})E_6\Big).
\end{aligned}$$

It is convenient to write

$$\begin{aligned}
(p^{-1}Up)_{\mathfrak{m}} &= f'E_1 + e'E_2 + \sqrt{2}(a'E_3 + b'E_4 + c'E_5 + d'E_6), \\
(p^{-1}Vp)_{\mathfrak{m}} &= f''E_1 + e''E_2 + \sqrt{2}(a''E_3 + b''E_4 + c''E_5 + d''E_6),
\end{aligned}$$

where $\alpha = a' + b'i + c'j + d'k, \beta = a'' + b''i + c''j + d''k, \gamma = e' + f'i, \delta = e'' + f''i$. The multi-moment map $\nu_{\mathbb{C}P^3}(p) = \sigma((p^{-1}Up)_{\mathfrak{m}}, (p^{-1}Vp)_{\mathfrak{m}})$, with σ as in Section 2.4, has the form

$$\begin{aligned}
\nu_{\mathbb{C}P^3}(p) &= (f'e'' - f''e') + 2(a'b'' - a''b' + c'd'' - c''d') \\
&= \operatorname{Im}\gamma\overline{\delta} + 2\operatorname{Re}(i\alpha\overline{\beta}),
\end{aligned} \tag{3.10}$$

which is indeed invariant under the torus action, because $\alpha, \beta, \gamma, \delta$ are invariant.

We now want to find the points where $\psi_+(U, V, \cdot\,) = 0$. In this case at the identity

$$\psi_+ = e^1 \wedge e^3 \wedge e^5 - Je^1 \wedge Je^3 \wedge e^5 - e^1 \wedge Je^3 \wedge Je^5 - Je^1 \wedge e^3 \wedge Je^5.$$

Again, we use U and V as shorthands for $(p^{-1}Up)_{\mathfrak{m}}, (p^{-1}Vp)_{\mathfrak{m}}$. Therefore

$$\begin{aligned}
V \lrcorner U \lrcorner (e^1 \wedge e^3 \wedge e^5) &= 2(a'c'' - a''c')e^1 + \sqrt{2}(c'f'' - f'c'')e^3 + \sqrt{2}(f'a'' - a'f'')e^5, \\
-V \lrcorner U \lrcorner (Je^1 \wedge Je^3 \wedge e^5) &= 2(c'b'' - b'c'')Je^1 + \sqrt{2}(e'c'' - c'e'')Je^3 + \sqrt{2}(b'e'' - b''e')e^5, \\
-V \lrcorner U \lrcorner (e^1 \wedge Je^3 \wedge Je^5) &= 2(b''d' - b'd'')e^1 + \sqrt{2}(f'd'' - d'f'')Je^3 + \sqrt{2}(b'f'' - f'b'')Je^5, \\
-V \lrcorner U \lrcorner (Je^1 \wedge e^3 \wedge Je^5) &= 2(a''d' - a'd'')Je^1 + \sqrt{2}(e'd'' - d'e'')e^3 + \sqrt{2}(a'e'' - e'a'')Je^5.
\end{aligned}$$

The equation $\psi_+(U,V,\,\cdot\,)_{|p}=0$ is then equivalent to the system

$$\begin{cases}(a'c''-a''c')+(b''d'-b'd'')=0\\(b''c'-b'c'')+(a''d'-a'd'')=0\\(c'f''-f'c'')+(e'd''-d'e'')=0\\(e'c''-c'e'')+(f'd''-d'f'')=0\\(f'a''-a'f'')+(b'e''-b''e')=0\\(b'f''-f'b'')+(a'e''-e'a'')=0.\end{cases}$$

A direct calculation shows that these conditions may be rephrased in terms of $\alpha,\beta,\gamma,\delta$ as

$$\alpha\overline{\beta}\in\mathrm{Span}\{1,i\},\quad \alpha\delta-\beta\gamma\in\mathrm{Span}\{1,i\},\quad \alpha\overline{\delta}-\beta\overline{\gamma}\in\mathrm{Span}\{j,k\}.$$

In terms of the p^k_{ij}, the latter three conditions are respectively

$$\begin{cases}\overline{p}^1_{11}\overline{p}^1_{21}(p^1_{12}p^2_{22}-p^2_{12}p^1_{22})+p^2_{11}p^2_{21}(\overline{p}^1_{12}\overline{p}^2_{22}-\overline{p}^2_{12}\overline{p}^1_{22})\\\qquad+\overline{p}^1_{11}p^2_{21}(p^2_{12}\overline{p}^2_{22}+p^1_{12}\overline{p}^1_{22})-p^2_{11}\overline{p}^1_{21}(\overline{p}^1_{12}p^1_{22}+\overline{p}^2_{12}p^2_{22})=0\\\overline{p}^1_{11}p^2_{11}(\overline{p}^1_{21}p^2_{22}+p^2_{21}\overline{p}^1_{22})-\overline{p}^1_{21}p^2_{21}(\overline{p}^1_{11}p^2_{12}+p^2_{11}\overline{p}^1_{12})=0\\p^1_{11}\overline{p}^2_{11}(\overline{p}^1_{21}p^1_{22}-p^2_{21}\overline{p}^2_{22})-p^1_{21}\overline{p}^2_{21}(\overline{p}^1_{11}p^1_{12}-p^2_{11}\overline{p}^2_{12})=0.\end{cases}$$

As in the previous case, we combine the left action of T^2 and the right action of $\mathrm{Sp}(1)\mathrm{U}(1)$:

$$\begin{pmatrix}e^{i\vartheta}&\\&e^{i\varphi}\end{pmatrix}\begin{pmatrix}p_{11}&p_{12}\\p_{21}&p_{22}\end{pmatrix}\begin{pmatrix}e^{i\tau}&\\&\omega\end{pmatrix}=\begin{pmatrix}e^{i\vartheta}p_{11}e^{i\tau}&e^{i\vartheta}p_{12}\omega\\e^{i\varphi}p_{21}e^{i\tau}&e^{i\varphi}p_{22}\omega\end{pmatrix}.$$

Thus $\omega\in\mathrm{Sp}(1)$ can be fixed so that $p_{12}=c$ is a non-negative real number, ϑ,φ,τ can be chosen so that $p_{11}=a+bj$, where a,b are non-negative real numbers, and $p_{21}=d+\rho j$, where d is a non-negative real and ρ is complex. Therefore we can assume without loss of generality that p has the form

$$p=\begin{pmatrix}a+bj&c\\d+\rho j&\sigma+\tau j\end{pmatrix}.$$

The system giving critical points then reduces to

$$\begin{cases}c\big(a(\tau d+\rho\overline{\sigma})-b(\sigma d-\rho\overline{\tau})\big)=0\\b\big(a(\tau d+\rho\overline{\sigma})-c\overline{\rho}d\big)=0\\a\big(b(\sigma d-\rho\overline{\tau})-c\overline{\rho}d\big)=0,\end{cases}$$

and the conditions defining $\mathrm{Sp}(2)$ are

$$\begin{cases}a^2+b^2+d^2+|\rho|^2=1\\ac+\sigma d+\rho\overline{\tau}=0\\bc-\tau d+\rho\overline{\sigma}=0\\c^2+|\sigma|^2+|\tau|^2=1.\end{cases}$$

In order to solve these systems we first distinguish two main cases: $c=0$ and $c>0$.

When $c = 0$ then the critical conditions become $ab(\tau d + \rho\overline{\sigma}) = 0$ and $ab(\sigma d - \rho\overline{\tau}) = 0$. Then either $ab = 0$ or $ab \neq 0$. However, in the case $c = 0$ we cannot have $a = 0 = b$, so the first subcase implies $a = 0$ and $b \neq 0$ or $a \neq 0$ and $b = 0$. In both, the conditions giving criticality are satisfied, moreover either $b = 1$ or $a = 1$ respectively. This implies $d = 0 = \rho$, so the matrices we get are simply

$$\begin{pmatrix} j & 0 \\ 0 & \sigma + \tau j \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & \sigma + \tau j \end{pmatrix}.$$

When both a and b are non-zero, then $\tau d + \rho\overline{\sigma} = 0 = \sigma d - \rho\overline{\tau}$ by criticality and $\sigma d + \rho\overline{\tau} = 0 = \rho\overline{\sigma} - \tau d$ by the conditions on $\mathrm{Sp}(2)$. Thus $\rho\overline{\sigma} = 0 = \sigma d$. So if $\sigma = 0$ then we are at a critical point, and τ has length one. This implies $d = 0 = \rho$. If $\sigma \neq 0$ instead, then $d = 0 = \rho$. So in the respective cases we find the critical points

$$\begin{pmatrix} a + bj & 0 \\ 0 & \tau j \end{pmatrix}, \qquad \begin{pmatrix} a + bj & 0 \\ 0 & \sigma + \tau j \end{pmatrix}.$$

All the points found are zeros of the multi-moment map.

To find non-zero critical points we can then safely assume $c > 0$. The system giving criticality simplifies as

$$\begin{cases} a(d\tau + \rho\overline{\sigma}) - b(d\sigma + \rho\overline{\tau}) = 0 \\ b(a(d\tau + \rho\overline{\sigma}) - cd\overline{\rho}) = 0 \\ a(b(d\sigma - \rho\overline{\tau}) - cd\overline{\rho}) = 0. \end{cases}$$

We distinguish four cases:

1. $a = b = 0$.
2. $a = 0, b \neq 0$.
3. $a \neq 0, b = 0$.
4. $a \neq 0, b \neq 0$.

The first one yields critical points automatically and we can then split the system defining $\mathrm{Sp}(2)$ according to the subcases $d = 0$ or $d \neq 0$. If $d = 0$ then $\rho = e^{i\vartheta}$, thus $\sigma = 0 = \tau$. When $d \neq 0$ then $\sigma = -\rho\overline{\tau}/d$ and $\tau = \rho\overline{\sigma}/d$. Therefore $\tau = -(|\rho|^2/d^2)\tau$, which implies $\tau = 0$, hence $\sigma = 0$. So $c = 1$, and we have found the critical points

$$\begin{pmatrix} 0 & c \\ d + \rho j & \sigma + \tau j \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ d + \rho j & 0 \end{pmatrix}.$$

Again, the multi-moment map vanishes at them.

In the second case $a = 0$ and $b \neq 0$. Critical points are thus characterised by the two equations $-\sigma d + \rho\overline{\tau} = 0$ and $\overline{\rho}d = 0$. But $\sigma d + \rho\overline{\tau} = 0$, so the two conditions are in fact $\rho\overline{\tau} = 0 = \rho d$. There are two subcases: $\rho = 0$ or $\rho \neq 0$ and $d = 0 = \tau$. In the first one $d \neq 0$, otherwise $c = 0$, and we are assuming c positive. Then $\sigma = 0$ and we are done. In the case $\rho \neq 0$ and $d = 0 = \tau$ then we get a critical point with $bc + \rho\overline{\sigma} = 0$, which specifies σ. But the conditions on $\mathrm{Sp}(2)$ imply that either $\rho = 0$ or $c = 0$ or $b = 0$, so a contradiction. All in all we find the critical points

$$\begin{pmatrix} bj & c \\ d & \tau j \end{pmatrix},$$

and the multi-moment map vanishes at them.

The third case $a \neq 0, b = 0$ is similar. The constraints yielding critical points are $\rho\overline{\sigma} = 0 = d\rho$, so either $\rho = 0$ or $\rho \neq 0$ and $\sigma = 0 = d$. In both cases we find critical points where $\nu_{\mathbb{CP}^3}$ vanishes.

Lastly, we have the fourth case $a, b \neq 0$. One can easily see that in the system giving critical points the equation $a(\tau d + \rho\overline{\sigma}) - b(\sigma d + \rho\overline{\tau}) = 0$ is redundant. The orthogonality relations for $\mathrm{Sp}(2)$ yield the condition $b\sigma d = a\rho\overline{\sigma}$. So we have the subcases $\sigma = 0$ and $\sigma \neq 0$. In the first one we find the critical point

$$\begin{pmatrix} a + bj & c \\ d + \rho j & \tau j \end{pmatrix},$$

where now ρ and τ are reals. The multi-moment map vanishes at these points.

We have finally arrived to the last case $a, b, c, \sigma \neq 0$, where we get the first non-zero critical points. The critical conditions are

$$\begin{cases} a(\tau d + \rho\overline{\sigma}) = c\overline{\rho}d \\ b(\sigma d - \rho\overline{\tau}) = c\overline{\rho}d, \end{cases}$$

and orthogonality of the columns of matrices in $\mathrm{Sp}(2)$ yields

$$\begin{cases} \tau d - \rho\overline{\sigma} = bc \\ \sigma d + \rho\overline{\tau} = -ac. \end{cases}$$

Plugging the last two equations in the critical conditions above we find four equations giving information on σ and τ:

$$\begin{aligned} a\rho\overline{\sigma} &= b\sigma d, \\ a\tau d &= -b\rho\overline{\tau}, \\ abc &= a\tau d - b\sigma d, \\ abc &= -a\rho\overline{\sigma} - b\rho\overline{\tau}. \end{aligned}$$

The first two force the values of ρ, σ, τ in the following way: write $\rho = Re^{ir}, \sigma = Se^{is}$, and $\tau = Te^{it}$. Comparing the angles in the first two equations we find the congruences

$$\begin{aligned} s &\equiv r - s \pmod{2\pi}, \\ t &\equiv \pi + r - t \pmod{2\pi}, \end{aligned}$$

which imply $t = \pi/2 + s \pmod{\pi}$. This gives two subcases: $e^{it} = ie^{is}$ and $e^{it} = -ie^{is}$, which we solve in the same fashion. Observe that $e^{ir} = e^{2is}$, so plugging these results in our starting equations $a\rho\overline{\sigma} = b\sigma d$ and $a\tau d = -b\rho\overline{\tau}$ we find $R = ad/b = bd/a$, hence $a = b$. Therefore, the equations $a\rho\overline{\sigma} = b\sigma d$ and $a\tau d = -b\rho\overline{\tau}$ simplify as $\rho\overline{\sigma} = \sigma d, \tau d = -\rho\overline{\tau}$. They imply $\sigma = \rho\overline{\sigma}/d$ and $\tau = -\rho\overline{\tau}/d$, so

$$|\sigma|^2 + |\tau|^2 = \tfrac{|\rho|^2}{d^2}(|\sigma|^2 + |\tau|^2),$$

whence $d = |\rho|$, namely $\rho = de^{ir}$. But $2a^2 + c^2 = a^2 + b^2 + c^2 = 1$ and $a^2 + b^2 + d^2 + |\rho|^2 = 1 = 2a^2 + 2d^2$, so $c^2 = 2d^2$, that is $c = \sqrt{2}d$. Observe now that by the conditions $abc = a\tau d - b\sigma d$ and $abc = -a\rho\overline{\sigma} - b\rho\overline{\tau}$ we have $\tau - \sigma = \sqrt{2}a$ and $\sigma + \tau = -\sqrt{2}ae^{ir}$, so

$$\begin{aligned} 2\tau &= \sqrt{2}a(1 - e^{ir}), \\ 2\sigma &= -\sqrt{2}a(1 + e^{ir}). \end{aligned}$$

Then the critical condition $bd\sigma - b\rho\overline{\tau} = cd\overline{\rho}$ becomes

$$-\tfrac{\sqrt{2}}{2}a^2 d(1+e^{ir}) - \tfrac{\sqrt{2}}{2}a^2 de^{ir}(1-e^{-ir}) = \sqrt{2}d^3 e^{-ir},$$

and simplifying $-a^2e^{2ir} = d^2$. Then e^{2ir} must be real and negative, thus $2r \equiv \pi \pmod{2\pi}$, or equivalently $r = \pm\pi/2$, and $a = d$. But since the first column has unit length $4a^2 = 1$, so $a = 1/2 = b = d = c/\sqrt{2} = |\rho|$. We then obtain the critical points

$$\begin{pmatrix} \frac{1}{2}+\frac{1}{2}j & \frac{1}{\sqrt{2}} \\ \frac{1}{2}+\frac{1}{2}ij & -\frac{1}{2\sqrt{2}}(1+i)+\frac{1}{2\sqrt{2}}(1-i)j \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2}+\frac{1}{2}j & \frac{1}{\sqrt{2}} \\ \frac{1}{2}-\frac{1}{2}ij & -\frac{1}{2\sqrt{2}}(1-i)+\frac{1}{2\sqrt{2}}(1+i)j \end{pmatrix}.$$

The values of the multi-moment map at these two points are respectively $-3/4$ and $3/4$, and we end up with two critical T^2-orbits giving maximum and minimum.

3.5 On the product of three-spheres

Finally we consider $\mathrm{SU}(2)^3/\mathrm{SU}(2)_\Delta \cong \mathbb{S}^3 \times \mathbb{S}^3$. An element $(t_1, t_2, t_3), t_k = e^{i\vartheta_k}$, of a maximal three-torus $T^3 \subset \mathrm{SU}(2)^3$ acts on $(g_1, g_2, g_3)\mathrm{SU}(2)_\Delta$ as

$$(t_1,t_2,t_3)(g_1,g_2,g_3)\mathrm{SU}(2)_\Delta := (t_1g_1, t_2g_2, t_3g_3)\mathrm{SU}(2)_\Delta$$

and then on $(p,q) \in \mathbb{S}^3 \times \mathbb{S}^3$ as $(t_1pt_3^{-1}, t_2qt_3^{-1})$. Each pair of linearly independent vectors $a = (a_1, a_2)$ in $\mathbb{Z}^3$ yields a two-torus $T_a^2 = \mathbb{R}a_1 \oplus \mathbb{R}a_2/\mathbb{Z}a_1 \oplus \mathbb{Z}a_2$ in our three-torus T^3, so in this case we end up with infinitely many possible choices for the two-torus acting. Recall that in Section 2.5 we introduced a basis of the tangent space $T_{(p,q)}(\mathbb{S}^3 \times \mathbb{S}^3) \cong T_p\mathbb{S}^3 \times T_q\mathbb{S}^3 \cong \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ given by

$$\begin{aligned} &E_1(p,q) = (pi, 0), \quad E_2(p,q) = (pj, 0), \quad E_3(p,q) = (-pk, 0), \\ &E_4(p,q) = (0, qi), \quad E_5(p,q) = (0, qj), \quad E_6(p,q) = (0, -qk). \end{aligned}$$

The T^3-action defined above yields the following infinitesimal generators at the point $(p,q) \in \mathbb{S}^3 \times \mathbb{S}^3$:

$$U_1(p,q) = (ip, 0), \quad U_2(p,q) = (0, iq), \quad U_3(p,q) = (-pi, -qi).$$

Note that $\overline{\overline{p}ip} + \overline{p}ip = -\overline{p}ip + \overline{p}ip = 0$, so $\overline{p}ip = \langle \overline{p}ip, i\rangle i + \langle \overline{p}jp, j\rangle j + \langle \overline{p}kp, k\rangle k$ in $\mathrm{Sp}(1)$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the standard inner product on the quaternions. This is equivalent to saying $ip = \langle \overline{p}ip, i\rangle pi + \langle \overline{p}jp, j\rangle pj + \langle \overline{p}kp, k\rangle pk$, because p has unit length. But then $(ip, 0) = \langle \overline{p}ip, i\rangle (pi, 0) + \langle \overline{p}jp, j\rangle (pj, 0) - \langle \overline{p}kp, k\rangle (-pk, 0)$ in $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$, which implies

$$U_1(p,q) = \langle \overline{p}ip, i\rangle E_1(p,q) + \langle \overline{p}ip, j\rangle E_2(p,q) - \langle \overline{p}ip, k\rangle E_3(p,q).$$

A similar expression is obtained for U_2, whereas the one for U_3 is trivial: we have

$$\begin{aligned} U_1(p,q) &= \langle \overline{p}ip, i\rangle E_1(p,q) + \langle \overline{p}ip, j\rangle E_2(p,q) - \langle \overline{p}ip, k\rangle E_3(p,q), \\ U_2(p,q) &= \langle \overline{q}iq, i\rangle E_4(p,q) + \langle \overline{q}iq, j\rangle E_5(p,q) - \langle \overline{q}iq, k\rangle E_6(p,q), \\ U_3(p,q) &= -E_1(p,q) - E_4(p,q). \end{aligned}$$

A multi-moment map in this case is an equivariant map $\nu\colon \mathbb{S}^3\times\mathbb{S}^3 \to \Lambda^2\mathbb{R}^3 \cong \mathbb{R}^3$. Its three real-valued components correspond to $\nu_i := \sigma(U_j, U_k)$, with σ as in Section 2.5 and for (ijk) cyclic permutation:

$$\begin{aligned}\nu(p,q) &= \big(\sigma(U_2,U_3), \sigma(U_3,U_1), \sigma(U_1,U_2)\big)\\ &= \tfrac{2}{3\sqrt{3}}\big(\langle \overline{q}iq, i\rangle, \langle \overline{p}ip, i\rangle, \langle \overline{p}ip, \overline{q}iq\rangle\big).\end{aligned}$$

Pointwise, the generators U, V of the T^2-action we are interested in are then linear combinations of the U_is:

$$U = a_{11}U_1 + a_{12}U_2 + a_{13}U_3, \quad V = a_{21}U_1 + a_{22}U_2 + a_{23}U_3.$$

The multi-moment map for the T^2-action is $\nu_{\mathbb{S}^3\times\mathbb{S}^3} := \sigma(U,V)$:

$$\nu_{\mathbb{S}^3\times\mathbb{S}^3}(p,q) = (a_{12}a_{23} - a_{22}a_{13})\nu_1 - (a_{11}a_{23} - a_{21}a_{13})\nu_2 + (a_{11}a_{22} - a_{12}a_{21})\nu_3.$$

Now let us focus on the critical points of this map. As usual, these are the points $(p,q) \in \mathbb{S}^3\times\mathbb{S}^3$ where $\psi_+(U,V,\,\cdot\,) = 0$. The generators U, V in terms of $E_1,\dots,E_6$ are

$$\begin{aligned}U &= (a_{11}\langle \overline{p}ip, i\rangle - a_{13})E_1 + a_{11}\langle \overline{p}ip, j\rangle E_2 - a_{11}\langle \overline{p}ip, k\rangle E_3\\ &\quad + (a_{12}\langle \overline{q}iq, i\rangle - a_{13})E_4 + a_{12}\langle \overline{q}iq, j\rangle E_5 - a_{12}\langle \overline{q}iq, k\rangle E_6,\\ V &= (a_{21}\langle \overline{p}ip, i\rangle - a_{23})E_1 + a_{21}\langle \overline{p}ip, j\rangle E_2 - a_{21}\langle \overline{p}ip, k\rangle E_3\\ &\quad + (a_{22}\langle \overline{q}iq, i\rangle - a_{23})E_4 + a_{22}\langle \overline{q}iq, j\rangle E_5 - a_{22}\langle \overline{q}iq, k\rangle E_6.\end{aligned}$$

Set $b = a_1 \times a_2 = (a_{12}a_{23} - a_{13}a_{22}, -a_{11}a_{23} + a_{13}a_{21}, a_{11}a_{22} - a_{12}a_{21})$ and $x = (x^1,x^2,x^3) := (\langle \overline{p}ip, i\rangle, \langle \overline{p}ip, j\rangle, \langle \overline{p}ip, k\rangle)$, $y = (y^1,y^2,y^3) := (\langle \overline{q}iq, i\rangle, \langle \overline{q}iq, j\rangle, \langle \overline{q}iq, k\rangle)$. The multi-moment map has then the form

$$\begin{aligned}\nu_{\mathbb{S}^3\times\mathbb{S}^3}(p,q) &= b_1\nu_1 + b_2\nu_2 + b_3\nu_3\\ &= \tfrac{2}{3\sqrt{3}}(b_1y_1 + b_2x_1 + b_3\langle x, y\rangle).\end{aligned}$$

A computation of $\psi_+(U,V,\,\cdot\,)_{|(p,q)}$ gives

$$\begin{aligned}V \lrcorner U \lrcorner e^{126} &= -b_3x^2y^3e^1 + \big(b_3x^1y^3 + b_1y^3\big)e^2 - b_2x^2e^6,\\ V \lrcorner U \lrcorner e^{315} &= b_3x^3y^2e^1 + \big(b_3x^1y^2 + b_1y^2\big)e^3 - b_2x^3e^5,\\ V \lrcorner U \lrcorner e^{156} &= \big(b_3x^1y^3 + b_1y^3\big)e^5 + \big(b_3x^1y^2 + b_1y^2\big)e^6,\\ V \lrcorner U \lrcorner e^{234} &= -\big(b_3x^3y^1 + b_2x^3\big)e^2 - \big(b_3x^2y^1 + b_2x^2\big)e^3,\\ V \lrcorner U \lrcorner e^{264} &= b_1y^3e^2 - b_3x^2y^3e^4 - \big(b_3x^2y^1 + b_2x^2\big)e^6,\\ V \lrcorner U \lrcorner e^{345} &= b_1y^2e^3 + b_3x^3y^2e^4 - \big(b_3x^3y^1 + b_2x^3\big)e^5.\end{aligned}$$

Therefore, the equation $\psi_+(U,V,\,\cdot\,) = 0$ is equivalent to the following system:

$$\begin{cases} b_3(x^3y^2 - x^2y^3) = 0\\ b_3(x^1y^3 - x^3y^1) - b_2x^3 = 0\\ b_3(x^1y^2 - x^2y^1) - b_2x^2 = 0\\ b_3(x^3y^1 - x^1y^3) - b_1y^3 = 0\\ b_3(x^2y^1 - x^1y^2) - b_1y^2 = 0.\end{cases} \tag{3.11}$$

This set of equations yields a variety of peculiar situations: as it turns out $\mathbb{S}^3 \times \mathbb{S}^3$ is the only homogeneous example where saddle points appear and where maximum and minimum of the multi-moment map are not symmetric with respect to 0. We assign explicit values to our parameters b_1, b_2, b_3 so as to see which critical sets arise.

Recall that the vectors x and y lie in $\mathbb{S}^2 \subset \operatorname{Im}\mathbb{H}$, because $\operatorname{Re}(\overline{p}ip) = 0$ and $|\overline{p}ip|^2 = 1$, similarly for y. We distinguish the cases $b_3 = 0$ and $b_3 \neq 0$. Since not all the b_is vanish, when $b_3 = 0$ we have three subcases:

1. If $b_1 \neq 0$ and $b_2 = 0$ then $y = \pm i$.
2. If $b_1 = 0$ and $b_2 \neq 0$ then $x = \pm i$.
3. If $b_1 \neq 0 \neq b_2$ then $x = \pm i, y = \pm i$.

If $b_3 \neq 0$ then the first equation yields $x^3y^2 - x^2y^3 = 0$. Summing respectively second and fourth, third and fifth equation in (3.11) we get

$$b_1y^3 + b_2x^3 = 0 = b_1y^2 + b_2x^2. \tag{3.12}$$

We end up with three more cases:

1. If $b_1 = b_2 = 0$ then we obtain at once that x is parallel to y, thus $y = \pm x$.
2. If $b_1 \neq 0$ and $b_2 = 0$ or $b_1 = 0$ and $b_2 \neq 0$ then x is parallel to y and $y = \pm i$.
3. If $b_1 \neq 0$ and $b_2 \neq 0$ then by (3.12) we get $x^2 = -(b_1/b_2)y^2, x^3 = -(b_1/b_2)y^3$, so plugging these solutions in the system one obtains

   $$\begin{cases} y^2(b_2b_3x^1 + b_1b_3y^1 + b_1b_2) = 0 \\ y^3(b_2b_3x^1 + b_1b_3y^1 + b_1b_2) = 0, \end{cases}$$

   so either $y = \pm i$ (and then $x = \pm i$) or $b_2b_3x^1 + b_1b_3y^1 + b_1b_2 = 0$.

In the latter case the point $(x^1, y^1) \in \mathbb{R}^2$ lies on the line

$$r\colon\ b_2b_3x^1 + b_1b_3y^1 + b_1b_2 = 0.$$

On the other hand, since $|x|^2 = 1$ and $x^2 = -(b_1/b_2)y^2, x^3 = -(b_1/b_2)y^3$, we have $(x^1)^2 + (b_1^2/b_2^2)\big((y^2)^2 + (y^3)^2\big) = 1$. But $|y|^2 = 1$ as well, so $(y^2)^2 + (y^3)^2 = 1 - (y^1)^2$, and replacing this in the former identity we find a curve

$$h\colon\ b_2^2(x^1)^2 - b_1^2(y^1)^2 = b_2^2 - b_1^2.$$

Therefore (x^1, y^1) lies in the intersection between h and r.

Note that when $b_1 = b_2$ the curve h is the union of the two lines $x^1 = \pm y^1$. The slope of r is $-b_2/b_1$ in general, so it is -1 when $b_1 = b_2$. Since $b_2 \neq 0$ the only non-trivial intersection is between $y^1 = x^1$ and $y^1 = -x^1 - b_2/b_3$, which gives $x^1 = y^1 = -b_2/2b_3$. Similarly, when $b_2 = -b_1$ the only non-trivial intersection is between $x^1 = -y^1$ and $y^1 = x^1 - b_2/b_3$, namely $x^1 = -y^1 = b_2/2b_3$.

When $b_2 \neq \pm b_1$, the curve h is a hyperbola and its asymptotes have equations $y^1 = \pm(b_2/b_1)x^1$. Since $b_2 \neq 0$ there is a unique intersection between h and r with $x^1 = (b_1^2b_3^2 - b_1^2b_2^2 - b_2^2b_3^2)/2b_1b_2^2b_3$ and $y^1 = (b_2^2b_3^2 - b_1^2b_2^2 - b_1^2b_3^2)/2b_1^2b_2b_3$.

Summing up, in every case we have a uniquely determined solution that may be written as

$$x^1 = (b_1^2 b_3^2 - b_1^2 b_2^2 - b_2^2 b_3^2)/2b_1 b_2^2 b_3, \quad y^1 = (b_2^2 b_3^2 - b_1^2 b_2^2 - b_1^2 b_3^2)/2b_1^2 b_2 b_3,$$
$$x^2 = -(b_1/b_2)y^2, \quad x^3 = -(b_1/b_2)y^3.$$

Note that even in the special cases $b_2 = \pm b_1$ these expressions reduce to the ones found above. Hereafter we list the possible values of the multi-moment map:

1. If $b_3 = 0$ we may summarise all the subcases saying $\nu_{\mathbb{S}^3\times\mathbb{S}^3}(p,q) = \frac{2}{3\sqrt{3}}(y_1 b_1 + x_1 b_2)$.
2. If $b_3 \neq 0$ and at least one between b_1 and b_2 is zero, then
$$\nu_{\mathbb{S}^3\times\mathbb{S}^3}(p,q) = \tfrac{2}{3\sqrt{3}}(y_1 b_1 + x_1 b_2 \pm b_3).$$
3. If $b_1, b_2, b_3 \neq 0$, first observe that
$$b_1 y^1 + b_2 x^1 = \frac{b_2^2 b_3^2 - b_1^2 b_2^2 - b_1^2 b_3^2}{2b_1 b_2 b_3} + \frac{b_1^2 b_3^2 - b_1^2 b_2^2 - b_2^2 b_3^2}{2b_1 b_2 b_3} = -\frac{b_1 b_2}{b_3}.$$
Secondly, it is convenient to write $\langle x, y\rangle$ as $\cos\vartheta$, where ϑ is the angle between the vectors x and y. The system giving critical points and the conditions $x^2 = -(b_1/b_2)y^2, x^3 = -(b_1/b_2)y^3$ are saying that
$$1 - \cos^2\vartheta = \sin^2\vartheta = \|x \times y\|^2 = (b_1^2/b_3^2)\big(1 - (y^1)^2\big),$$
namely
$$\cos^2\vartheta = 1 - \frac{b_1^2}{b_3^2}\left(1 - \frac{(b_2^2 b_3^2 - b_1^2 b_2^2 - b_1^2 b_3^2)^2}{4b_1^4 b_2^2 b_3^2}\right) = \left(\frac{b_2^2 b_3^2 - b_1^2 b_2^2 + b_1^2 b_3^2}{2b_1 b_2 b_3^2}\right)^2.$$
Consequently $\langle x, y\rangle = \cos\vartheta = \pm(b_2^2 b_3^2 - b_1^2 b_2^2 + b_1^2 b_3^2)/2b_1 b_2 b_3^2$, and
$$\begin{aligned}\nu_{\mathbb{S}^3\times\mathbb{S}^3}(p,q) &= \frac{2}{3\sqrt{3}}(b_1 y^1 + b_2 x^1 + b_3\langle x, y\rangle)\\ &= \frac{2}{3\sqrt{3}}\left(-\frac{b_1 b_2}{b_3} \pm \frac{b_2^2 b_3^2 - b_1^2 b_2^2 + b_1^2 b_3^2}{2b_1 b_2 b_3}\right),\end{aligned}$$
the signs $\pm$ giving two stationary orbits.

As regards the set of critical points, the situation is significantly different compared to the other homogeneous examples. First, consider the case $b_3 = 0$ with $b_1, b_2 \neq 0$, so that $x = \pm i, y = \pm i$. We write $x = \varepsilon_1 i, y = \varepsilon_2 i$, with $\varepsilon_k \in \{\pm 1\}$. If e.g. $a_1 = (2,3,1)$ and $a_2 = (2,3,5)$, then $b = (12, -8, 0)$ and

$$12\varepsilon_2 - 8\varepsilon_1 \in \{-20, -4, 4, 20\},$$

so we have four different non-zero critical values. The points corresponding to the value 4 are obtained when $x = (1,0,0) = y$ and are actually saddle points. To check this recall that the T^2-symmetry allows one to evaluate the multi-moment map on points in $\mathbb{S}^2 \times \mathbb{S}^2$

rather than in $\mathbb{S}^3 \times \mathbb{S}^3$. So considering particular points around $(x, y) = ((1,0,0),(1,0,0))$ we can prove our claim. For example, for $\alpha \neq 0$ and small

$$\begin{aligned}\nu_{\mathbb{S}^3\times\mathbb{S}^3}((\cos\alpha, \sin\alpha, 0),(1,0,0)) &= \frac{2}{3\sqrt{3}}(12 - 8\cos\alpha)\\ &> \frac{8}{3\sqrt{3}} = \nu_{\mathbb{S}^3\times\mathbb{S}^3}((1,0,0),(1,0,0)).\end{aligned}$$

Likewise

$$\begin{aligned}\nu_{\mathbb{S}^3\times\mathbb{S}^3}((1,0,0),(\cos\alpha, \sin\alpha, 0)) &= \frac{2}{3\sqrt{3}}(12\cos\alpha - 8)\\ &< \frac{8}{3\sqrt{3}} = \nu_{\mathbb{S}^3\times\mathbb{S}^3}((1,0,0),(1,0,0)),\end{aligned}$$

hence $((1,0,0),(0,0,1))$ in $\mathbb{S}^2 \times \mathbb{S}^2$ corresponds to an orbit of saddle points in $\mathbb{S}^3 \times \mathbb{S}^3$. The same steps can be repeated for the value -4.

Choosing $a_1 = (1,-1,0)$ and $a_2 = (1,1,-1)$ we have $b = (1,1,2)$, so all the b_is are non-zero and the multi-moment map takes all values between $-3/2\sqrt{3}$ and $5/6\sqrt{3}$. This shows that maximum and minimum may occur without being symmetric with respect to the origin.

Finally, if $a_1 = (1,0,0), a_2 = (0,1,0)$, then $b = (0,0,1)$ and $\nu_{\mathbb{S}^3\times\mathbb{S}^3}(p,q) = \pm 2/3\sqrt{3}$. The corresponding critical orbits in $\mathbb{S}^3 \times \mathbb{S}^3$ are four-dimensional.

Chapter 4

Torus symmetry

In Chapter 1, specifically in Section 1.5, we proved that a nearly Kähler six-manifold is an almost Hermitian manifold (M, g, J) equipped with an SU(3)-structure $(\sigma, \psi_{\pm})$ such that

$$d\sigma = 3\psi_+, \qquad d\psi_- = -2\sigma \wedge \sigma.$$

This statement resulted in Definition 1.5.5. Assume a two-torus T^2 acts *effectively* on M preserving g, J, and the complex form $\psi_{\mathbb{C}} = \psi_+ + i\psi_-$. Let U, V denote the infinitesimal generators of the action. Since the two-torus acts effectively on M, the vector fields U and V are linearly independent over M. This follows from [KN96, Proposition 4.1].

In Example 3.1.5 we introduced a multi-moment map $\nu_M := \sigma(U, V)$. We now study its general properties in this set-up and use it to perform the so-called T^2-*reduction*. We first show the existence of regular values for ν_M, hence prove that if $s \in \mathbb{R}$ is such, then T^2 acts freely on the level set $\nu_M^{-1}(s)$. The quotients $\nu_M^{-1}(s)/T^2$ corresponding to regular values s are then smooth three-dimensional manifolds. In a second stage we reverse this construction. We study under which conditions this is possible and then apply the result to the three-dimensional Heisenberg group, getting a new example of nearly Kähler six-manifold.

4.1 The infinitesimal generators

In what follows we denote $g(U, U), g(U, V), g(V, V)$ by g_{UU}, g_{UV}, g_{VV}. We call h the non-negative real-valued function on M satisfying

$$h^2 := g_{UU}g_{VV} - g_{UV}^2. \tag{4.1}$$

We use the same notations even when working pointwise.

Assume at the point p the vectors U_p and V_p are in general position and consider an SU(3)-basis $\{E_k, JE_k\}, k = 1, 2, 3$, with dual basis $\{e^k, Je^k\}$. Up to applying a special unitary transformation we have

$$U_p = g_{UU}^{1/2}E_1, \qquad V_p = x^1E_1 + y^1JE_1 + x^2E_2,$$

for some real numbers x^1, x^2, y^1. By the definition of ν_M and the conditions $(x^1)^2 + (y^1)^2 + (x^2)^2 = g_{VV}$, $g_{UV} = g_{UU}^{1/2}x^1$, we find

$$U_p = g_{UU}^{1/2}E_1, \qquad g_{UU}^{1/2}V_p = \Big(g_{UV}E_1 + \nu_M JE_1 + \big(h^2 - \nu_M^2\big)^{1/2}E_2\Big). \tag{4.2}$$

Thus, (1.31) implies $d\nu_M = 3\psi_+(U, V, \cdot\,) = 3\big(h^2 - \nu_M^2\big)^{1/2} e^3$ pointwise. Further, $h^2 - \nu_M^2$ is non-negative because of (1.9), so it makes sense to take its square root.

We now prove a simple characterisation of critical points where the multi-moment map vanishes.

Proposition 4.1.1. *The vectors U_p and V_p are linearly dependent over $\mathbb{R}$ if and only if the multi-moment map ν_M and its differential $d\nu_M$ vanish at p.*

Proof. If U_p and V_p are linearly dependent over $\mathbb{R}$ at p then both $\nu_M = \sigma(U, V)$ and its differential $d\nu_M = 3\psi_+(U, V, \cdot\,)$ vanish at p.

Conversely, since $d\nu_M = 3\psi_+(U, V, \cdot\,)$, the expressions in (4.2) imply that a point p is critical if and only if U_p and V_p are linearly dependent over $\mathbb{C}$, or equivalently that V_p is a linear combination of U_p and JU_p. Therefore

$$g_{UU} V_p = g_{UV} U_p + \nu_M J U_p,$$

so since ν_M vanishes the result follows. □

This is what we need for the moment. More properties of U and V will be worked out in Section 6.1.

4.2 The multi-moment map and its properties

Expanding $V \lrcorner\, U \lrcorner\, (\sigma \wedge \sigma)$ we obtain a useful formula we are going to use in the next lemma:

$$V \lrcorner\, U \lrcorner\, (\sigma \wedge \sigma) = 2\big(\nu_M \sigma - (U \lrcorner\, \sigma) \wedge (V \lrcorner\, \sigma)\big). \tag{4.3}$$

Lemma 4.2.1. *Let Δ be the Laplace operator on $C^\infty(M)$, defined as $\Delta = d^* d := -{*}d{*}d$. The multi-moment map is an eigenfunction of Δ:*

$$\Delta \nu_M = 24 \nu_M. \tag{4.4}$$

Proof. Firstly, we show that ${*}d\nu_M = {*}\big(3\psi_+(U, V, \cdot\,)\big) = \tfrac{3}{2}\sigma \wedge \sigma \wedge \alpha_0$, where α_0 is defined as $V \lrcorner\, U \lrcorner\, \psi_-$. From the expressions of σ and ψ_- in (1.30), (1.32) we get pointwise

$$\begin{aligned} \sigma \wedge \sigma &= 2(e^1 \wedge Je^1 \wedge e^2 \wedge Je^2 + e^1 \wedge Je^1 \wedge e^3 \wedge Je^3 + e^2 \wedge Je^2 \wedge e^3 \wedge Je^3), \\ \alpha_0 &= \big(h^2 - \nu_M^2\big)^{1/2} Je^3. \end{aligned}$$

Hence $\sigma \wedge \sigma \wedge \alpha_0 = 2\big(h^2 - \nu_M^2\big)^{1/2} e^1 \wedge Je^1 \wedge e^2 \wedge Je^2 \wedge Je^3$, and

$$3\psi_+(U, V, \cdot\,) \wedge (\sigma \wedge \sigma \wedge \alpha_0) = \tfrac{2}{3}\big\|3\psi_+(U, V, \cdot\,)\big\|^2 \mathrm{vol}_M,$$

from which we obtain ${*}d\nu_M = {*}\big(3\psi_+(U, V, \cdot\,)\big) = \tfrac{3}{2}\sigma \wedge \sigma \wedge \alpha_0$. Thus $d{*}d\nu_M = \tfrac{3}{2}\sigma \wedge \sigma \wedge d\alpha_0$, because $d(\sigma \wedge \sigma) = 0$. By Cartan's formula $d\alpha_0 = d(V \lrcorner\, U \lrcorner\, \psi_-) = -2V \lrcorner\, U \lrcorner\, (\sigma \wedge \sigma)$. Then, identity (4.3) yields

$$\begin{aligned} d{*}d\nu_M &= -6\nu_M \sigma \wedge \sigma \wedge \sigma + 6\sigma \wedge \sigma \wedge (U \lrcorner\, \sigma) \wedge (V \lrcorner\, \sigma) \\ &= -36\nu_M \mathrm{vol}_M + 12\nu_M \mathrm{vol}_M \\ &= -24\nu_M \mathrm{vol}_M, \end{aligned}$$

and we are done. □

Proposition 4.2.2. *The average value of the multi-moment map ν_M is 0. Moreover, the range of ν_M is a compact interval containing 0 in its interior.*

Proof. We can apply Stokes' theorem and Lemma 4.2.1: since M has no boundary we find

$$24 \int_M \nu_M \operatorname{vol}_M = \int_M \Delta \nu_M \operatorname{vol}_M = -\int_M *d*d\nu_M \wedge *1$$
$$= \int_M d(*d\nu_M) = \int_{\partial M} *d\nu_M = 0.$$

We then have our first claim

$$\int_M \nu_M \operatorname{vol}_M = 0.$$

However, ν_M cannot be constantly zero: if it was, by Proposition 4.1.1 we would have that U and V are linearly dependent vector fields. Thus the action of T^2 would not be effective on M, which is a contradiction. This allows us to write $\nu_M \colon M \to [a,b]$, where $a < 0 < b$, for M is compact and connected. □

Now assume s is a regular value for ν_M. In the next proposition we show that the T^2-action on $\nu_M^{-1}(s)$ is *free*, so $\nu_M^{-1}(s)/T^2$ is a smooth three-dimensional manifold by the compactness of the two-torus, and $\nu_M^{-1}(s) \to \nu_M^{-1}(s)/T^2$ is a principal T^2-bundle. We call its base space the *T^2-reduction of M at level s*. We will study the geometry of the quotients $\nu_M^{-1}(s)/T^2$ in Section 4.3. Now we recall a useful definition and a result from topology, see e.g. [Bre72], in particular Section 2 in Chapter VI.

Let H be a compact subgroup of a group G acting on a vector space V on the left and on G by right translation. Then $h \in H$ acts on the left on $(g, X) \in G \times V$ by $h(g, X) := (gh^{-1}, h_* X)$.

Definition 4.2.3. The orbit space of this action is denoted by $G \times_H V$ and is called the *twisted product* of G and V with respect to H.

Theorem 4.2.4 (Equivariant Tubular Neighbourhood Theorem). *Let G be a compact Lie group acting smoothly on M on the left. Let p be a point in M and H be the stabiliser of p in G. Then there exists an open neighbourhood of p equivariantly diffeomorphic to the twisted product $G \times_H V$, where V is the normal space $T_pM/T_p(G \cdot p)$.*

Remark 4.2.5. When H fixes all of T_pM, then the action of H on V is trivial and $G \times_H V = (G/H) \times V$.

Let us go back to our multi-moment map. A consequence of Theorem 4.2.4 is

Proposition 4.2.6. *The multi-moment map ν_M has non-zero regular values. For any regular value s the T^2-action on $\nu_M^{-1}(s)$ is free. Thus $\nu_M^{-1}(s)/T^2$ are smooth three-dimensional manifolds.*

Proof. By Sard's Theorem the set of critical values has Lebesgue measure 0 in $[a,b]$, so we can assert there exist infinitely many regular values for ν_M in $(a,b) \neq \varnothing$. Let s be any of them. Then $d\nu_{M|p}$ has rank one at each $p \in \nu_M^{-1}(s)$, so $3\psi_+(U, V, \cdot\,)_p = d\nu_{M|p} \neq 0$. Thus U, V are linearly independent over the complex numbers on $\nu_M^{-1}(s)$, and this yields a discrete stabiliser of p.

On the other hand, if H is the stabiliser in T^2 of some $p \in \nu_M^{-1}(s)$, it preserves g, J and $\psi_\pm$ as well as U and V. Hence, H fixes $U, JU, V, JV, (\nabla_U J)V$, and $J(\nabla_U J)V$—because $((\nabla_U J)V)^\flat = \psi_+(U, V, \cdot\,)$—so all of T_pM. Then, by Theorem 4.2.4 and Remark 4.2.5, the set $B := \{q \in M : hq = q, h_{*|q} = \mathrm{Id}_{T_qM}$ for each $h \in H\}$ is open in M: this is because every point in B admits an open neighbourhood $A = (T^2/H) \times V$ in M, with $V = T_pM/T_p(T^2 \cdot p)$, and every $h \in H$ acts trivially on it, so $A \subseteq B$ by the equivariance. Obviously B is also closed and not empty because it contains p, so $B = M$ for M is connected. If H is not trivial, then the action is not effective, and we are done. $\square$

The next step is then to study T^2-reductions at levels corresponding to regular values.

4.3 Reduction to three-manifolds

In order to study the geometric structure of the quotients $Q_s^3 := \nu_M^{-1}(s)/T^2$, with $s \neq 0$ a regular value for ν_M, we need to determine which forms on $\nu_M^{-1}(s)$ descend to them. A k-form β on $\nu_M^{-1}(s)$ descends to Q_s^3 if and only if it is *basic*, that is $\mathcal{L}_U\beta = \mathcal{L}_V\beta = 0$ and $U \lrcorner \beta = V \lrcorner \beta = 0$. We consider only regular values away from zero: as we will see, the behaviour of this case is critical. Before starting, let us define the dual forms of U and V:

$$\vartheta_1 := h^{-2}(g_{VV}U^\flat - g_{UV}V^\flat), \qquad \vartheta_2 := h^{-2}(g_{UU}V^\flat - g_{UV}U^\flat),$$

where h satisfies (4.1). The pair $(\vartheta_1, \vartheta_2)$ is a connection one-form for the T^2-bundle $\nu_M^{-1}(s) \to \nu_M^{-1}(s)/T^2$. The invariant functions we find are g_{UU}, g_{UV}, g_{VV}. Then we have the basic one-forms

$$\alpha_0 := V \lrcorner U \lrcorner \psi_-, \qquad \alpha_1 := s\vartheta_1 + V \lrcorner \sigma, \qquad \alpha_2 := s\vartheta_2 - U \lrcorner \sigma, \tag{4.5}$$

and lastly the two-forms $U \lrcorner \psi_+, V \lrcorner \psi_+, U \lrcorner V \lrcorner \sigma^2$ are basic.

The next step is to specify $g, \sigma, \psi_\pm$ on M in terms of the forms $d\nu_M, \vartheta_1, \vartheta_2, \alpha_0, \alpha_1, \alpha_2$. We work on M, pointing out what holds in particular on the level sets $\nu_M^{-1}(s)$. We already found the pointwise expressions $d\nu_M = 3(h^2 - \nu_M^2)^{1/2}e^3$ and $\alpha_0 = (h^2 - \nu_M^2)^{1/2}Je^3$ (cf. the proof of Lemma 4.2.1). The connection one-form $(\vartheta_1, \vartheta_2)$ can be written as

$$\begin{aligned}
\vartheta_1 &= h^{-2}g_{UU}^{-1/2}\Big(h^2e^1 - g_{UV}\nu_MJe^1 - g_{UV}\left(h^2 - \nu_M^2\right)^{1/2}e^2\Big),\\
\vartheta_2 &= h^{-2}g_{UU}^{1/2}\Big(\nu_MJe^1 + \left(h^2 - \nu_M^2\right)^{1/2}e^2\Big),
\end{aligned}$$

whereas the three one-forms $\alpha_0, \alpha_1, \alpha_2$ are

$$\begin{aligned}
\alpha_0 &= \left(h^2 - \nu_M^2\right)^{1/2}Je^3,\\
\alpha_1 &= h^{-2}g_{UU}^{-1/2}(h^2 - \nu_M^2)^{1/2}\Big(g_{UV}\left(h^2 - \nu_M^2\right)^{1/2}Je^1 - g_{UV}\nu_Me^2 + h^2Je^2\Big),\\
\alpha_2 &= h^{-2}g_{UU}^{1/2}(h^2 - \nu_M^2)^{1/2}\Big(\nu_Me^2 - \left(h^2 - \nu_M^2\right)^{1/2}Je^1\Big).
\end{aligned}$$

Now we invert this system and write all the terms needed in the expressions of

$g, \sigma, \psi_+, \psi_-$. The metric g is $\sum_{k=1}^3 e^k \otimes e^k + Je^k \otimes Je^k$, where

$$
\begin{aligned}
e^1 \otimes e^1 &= g_{UU}\vartheta_1 \otimes \vartheta_1 + \frac{g_{UV}^2}{g_{UU}}\vartheta_2 \otimes \vartheta_2 + g_{UV}(\vartheta_1 \otimes \vartheta_2 + \vartheta_2 \otimes \vartheta_1),\\
Je^1 \otimes Je^1 &= \frac{\nu_M^2}{g_{UU}}\vartheta_2 \otimes \vartheta_2 - \frac{\nu_M}{g_{UU}}(\vartheta_2 \otimes \alpha_2 + \alpha_2 \otimes \vartheta_2) + \frac{1}{g_{UU}}\alpha_2 \otimes \alpha_2,\\
e^2 \otimes e^2 &= \frac{\nu_M^2}{g_{UU}(h^2-\nu_M^2)}\alpha_2 \otimes \alpha_2 + \frac{\nu_M}{g_{UU}}(\alpha_2 \otimes \vartheta_2 + \vartheta_2 \otimes \alpha_2) + \frac{h^2-\nu_M^2}{g_{UU}}\vartheta_2 \otimes \vartheta_2,\\
Je^2 \otimes Je^2 &= \frac{g_{UU}}{h^2-\nu_M^2}\alpha_1 \otimes \alpha_1 + \frac{g_{UV}^2}{g_{UU}(h^2-\nu_M^2)}\alpha_2 \otimes \alpha_2 + \frac{g_{UV}}{h^2-\nu_M^2}(\alpha_1 \otimes \alpha_2 + \alpha_2 \otimes \alpha_1),\\
e^3 \otimes e^3 &= \frac{1}{9(h^2-\nu_M^2)}d\nu_M \otimes d\nu_M,\\
Je^3 \otimes Je^3 &= \frac{1}{h^2-\nu_M^2}\alpha_0 \otimes \alpha_0.
\end{aligned}
$$

The two-form σ is $\sum_{k=1}^3 e^k \wedge Je^k$, wose summands are

$$
\begin{aligned}
e^1 \wedge Je^1 &= \nu_M\vartheta_1 \wedge \vartheta_2 - \vartheta_1 \wedge \alpha_2 - \frac{g_{UV}}{g_{UU}}\vartheta_2 \wedge \alpha_2,\\
e^2 \wedge Je^2 &= \vartheta_2 \wedge \alpha_1 + \frac{g_{UV}}{g_{UU}}\vartheta_2 \wedge \alpha_2 - \frac{\nu_M}{h^2-\nu_M^2}\alpha_1 \wedge \alpha_2,\\
e^3 \wedge Je^3 &= (3(h^2-\nu_M^2))^{-1}d\nu_M \wedge \alpha_0.
\end{aligned}
$$

The terms for ψ_+ are:

$$
\begin{aligned}
e^1 \wedge e^2 \wedge e^3 &= \frac{\nu_M}{3(h^2-\nu_M^2)}\vartheta_1 \wedge \alpha_2 \wedge d\nu_M + \frac{1}{3}\vartheta_1 \wedge \vartheta_2 \wedge d\nu_M\\
&\quad + \frac{\nu_M g_{UV}}{3g_{UU}(h^2-\nu_M^2)}\vartheta_2 \wedge \alpha_2 \wedge d\nu_M,\\
Je^1 \wedge Je^2 \wedge e^3 &= \frac{\nu_M}{3(h^2-\nu_M^2)}\vartheta_2 \wedge \alpha_1 \wedge d\nu_M + \frac{\nu_M g_{UV}}{3g_{UU}(h^2-\nu_M^2)}\vartheta_2 \wedge \alpha_2 \wedge d\nu_M\\
&\quad - \frac{1}{3(h^2-\nu_M^2)}\alpha_2 \wedge \alpha_1 \wedge d\nu_M,\\
e^1 \wedge Je^2 \wedge Je^3 &= \frac{g_{UU}}{h^2-\nu_M^2}\vartheta_1 \wedge \alpha_1 \wedge \alpha_0 + \frac{g_{UV}}{h^2-\nu_M^2}\vartheta_1 \wedge \alpha_2 \wedge \alpha_0\\
&\quad + \frac{g_{UV}}{h^2-\nu_M^2}\vartheta_2 \wedge \alpha_1 \wedge \alpha_0 + \frac{g_{UV}^2}{g_{UU}(h^2-\nu_M^2)}\vartheta_2 \wedge \alpha_2 \wedge \alpha_0,\\
Je^1 \wedge e^2 \wedge Je^3 &= \frac{h^2}{g_{UU}(h^2-\nu_M^2)}\vartheta_2 \wedge \alpha_2 \wedge \alpha_0.
\end{aligned}
$$

In order to get the terms for ψ_- it is enough to replace e^3 with Je^3 and change the signs of the last two terms in ψ_+.

Now let $s \neq 0$. The five one-forms ϑ_i, α_k are linearly independent on $\nu_M^{-1}(s)$ if and only if $h^2 \neq s^2$. Observe that $h^2 - \nu_M^2 = \|(h^2-\nu_M^2)^{1/2}e^3\|^2 = \frac{1}{9}\|d\nu_M\|^2 = \|\psi_+(U, V, \cdot\,)\|^2$, and this quantity is non-zero under our assumptions. Therefore, we have the following general expressions of $g, \sigma, \psi_\pm$ on M:

$$\begin{aligned} g = {} & \frac{1}{9\big(h^2-\nu_M^2\big)} d\nu_M^{\otimes 2} + g_{UU}\vartheta_1^{\otimes 2} + g_{VV}\vartheta_2^{\otimes 2} + g_{UV}\big(\vartheta_1\otimes\vartheta_2+\vartheta_2\otimes\vartheta_1\big) \\ & + \frac{1}{h^2-\nu_M^2}\Big(\alpha_0^{\otimes 2} + g_{UU}\alpha_1^{\otimes 2} + g_{VV}\alpha_2^{\otimes 2} + g_{UV}\big(\alpha_1\otimes\alpha_2+\alpha_2\otimes\alpha_1\big)\Big), \end{aligned} \tag{4.6}$$

$$\sigma = \frac{1}{3\big(h^2-\nu_M^2\big)} d\nu_M\wedge\alpha_0 + \nu_M\vartheta_1\wedge\vartheta_2 - \vartheta_1\wedge\alpha_2 + \vartheta_2\wedge\alpha_1 - \frac{\nu_M}{h^2-\nu_M^2}\alpha_1\wedge\alpha_2, \tag{4.7}$$

$$\begin{aligned} \psi_+ = {} & \frac{1}{3\big(h^2-\nu_M^2\big)} d\nu_M\wedge\Big(\big(h^2-\nu_M^2\big)\vartheta_1\wedge\vartheta_2 + \nu_M\big(\vartheta_1\wedge\alpha_2-\vartheta_2\wedge\alpha_1\big) - \alpha_1\wedge\alpha_2\Big) \\ & - \frac{1}{h^2-\nu_M^2}\Big(\vartheta_1\wedge\big(g_{UU}\alpha_1+g_{UV}\alpha_2\big) + \vartheta_2\wedge\big(g_{UV}\alpha_1+g_{VV}\alpha_2\big)\Big)\wedge\alpha_0, \end{aligned} \tag{4.8}$$

$$\begin{aligned} \psi_- = {} & \frac{1}{3\big(h^2-\nu_M^2\big)} d\nu_M\wedge\Big(\vartheta_1\wedge\big(g_{UU}\alpha_1+g_{UV}\alpha_2\big) + \vartheta_2\wedge\big(g_{UV}\alpha_1+g_{VV}\alpha_2\big)\Big) \\ & + \frac{1}{h^2-\nu_M^2}\Big(\big(h^2-\nu_M^2\big)\vartheta_1\wedge\vartheta_2 + \nu_M\big(\vartheta_1\wedge\alpha_2-\vartheta_2\wedge\alpha_1\big) - \alpha_1\wedge\alpha_2\Big)\wedge\alpha_0. \end{aligned} \tag{4.9}$$

If we use the nearly Kähler structure equations we get further relationships. The cotangent space of M splits as the direct sum of vertical and horizontal spaces $V\oplus H$, where $\vartheta_i\in V, i=1,2$, and H contains $d\nu_M, \alpha_k, k=0,1,2$.

Comparing coefficients in $d\sigma = 3\psi_+$ we obtain

$$\nu_M d\vartheta_2 = d\alpha_2 + \frac{1}{h^2-\nu_M^2}\big(3g_{UU}\alpha_1\wedge\alpha_0 + 3g_{UV}\alpha_2\wedge\alpha_0 + \nu_M d\nu_M\wedge\alpha_2\big), \tag{4.10}$$

$$\nu_M d\vartheta_1 = d\alpha_1 - \frac{1}{h^2-\nu_M^2}\big(3g_{UV}\alpha_1\wedge\alpha_0 + 3g_{VV}\alpha_2\wedge\alpha_0 - \nu_M d\nu_M\wedge\alpha_1\big), \tag{4.11}$$

$$\begin{aligned} d\vartheta_1\wedge\alpha_2 - d\vartheta_2\wedge\alpha_1 = {} & \left(\frac{2\nu_M^2}{\big(h^2-\nu_M^2\big)^2} d\nu_M - \frac{\nu_M}{\big(h^2-\nu_M^2\big)^2} d(h^2)\right)\wedge\alpha_2\wedge\alpha_1 \\ & + d\nu_M\wedge\left(\frac{1}{3\big(h^2-\nu_M^2\big)^2} d(h^2)\wedge\alpha_0 - \frac{1}{3\big(h^2-\nu_M^2\big)} d\alpha_0\right) \\ & + \frac{\nu_M}{h^2-\nu_M^2} d(\alpha_2\wedge\alpha_1). \end{aligned} \tag{4.12}$$

The equation $d\psi_- = -2\sigma\wedge\sigma$ gives

$$d\alpha_0 = -\frac{4\nu_M}{3\big(h^2-\nu_M^2\big)} d\nu_M\wedge\alpha_0 + \frac{4h^2}{h^2-\nu_M^2}\alpha_1\wedge\alpha_2, \tag{4.13}$$

$$
\begin{aligned}
d\vartheta_2 \wedge \alpha_0 = & \frac{-1}{3(h^2 - \nu_M^2)^2} \Big(d\nu_M \wedge (g_{UU}\alpha_1 + g_{UV}\alpha_2) + 3\nu_M \alpha_0 \wedge \alpha_2 \Big) \wedge d(h^2) \\
& - \frac{1}{3(h^2 - \nu_M^2)} \Big(d\nu_M \wedge d(g_{UU}\alpha_1 + g_{UV}\alpha_2) + 3\nu_M d\alpha_2 \wedge \alpha_0 \Big) \\
& - \frac{h^2 - 3\nu_M^2}{3(h^2 - \nu_M^2)^2} d\nu_M \wedge \alpha_0 \wedge \alpha_2, \qquad (4.14) \\
d\vartheta_1 \wedge \alpha_0 = & \frac{1}{3(h^2 - \nu_M^2)^2} \Big(d\nu_M \wedge (g_{UV}\alpha_1 + g_{VV}\alpha_2) - 3\nu_M \alpha_0 \wedge \alpha_1 \Big) \wedge d(h^2) \\
& + \frac{1}{3(h^2 - \nu_M^2)} \Big(d\nu_M \wedge d(g_{UV}\alpha_1 + g_{VV}\alpha_2) - 3\nu_M d\alpha_1 \wedge \alpha_0 \Big) \\
& - \frac{h^2 - 3\nu_M^2}{3(h^2 - \nu_M^2)^2} d\nu_M \wedge \alpha_0 \wedge \alpha_1. \qquad (4.15)
\end{aligned}
$$

The relations among $\alpha_0, \alpha_1, \alpha_2$ on the T^2-reduction at level s are then

$$
d\alpha_0 = \frac{4h^2}{h^2 - s^2} \alpha_1 \wedge \alpha_2, \qquad (4.16)
$$

$$
d\alpha_1 \wedge \alpha_0 = \frac{s^2}{h^2(h^2 - s^2)} d(h^2) \wedge \alpha_1 \wedge \alpha_0, \qquad (4.17)
$$

$$
d\alpha_2 \wedge \alpha_0 = \frac{s^2}{h^2(h^2 - s^2)} d(h^2) \wedge \alpha_2 \wedge \alpha_0. \qquad (4.18)
$$

Define $f := 4h^2/(h^2 - s^2)$. Trivially $f > 4$, which will be relevant later, and

$$
\frac{df}{f} = -\frac{s^2}{h^2(h^2 - s^2)} d(h^2).
$$

Hence we can summarise our results as follows.

Proposition 4.3.1. *On the level sets $\nu_M^{-1}(s)$, with $s \neq 0$ regular value for ν_M, the curvature two-form is given by*

$$
\begin{aligned}
s d\vartheta_1 &= d\alpha_1 - \frac{1}{h^2 - s^2}(3g_{UV}\alpha_1 + 3g_{VV}\alpha_2) \wedge \alpha_0, \\
s d\vartheta_2 &= d\alpha_2 + \frac{1}{h^2 - s^2}(3g_{UU}\alpha_1 + 3g_{UV}\alpha_2) \wedge \alpha_0.
\end{aligned}
$$

Proposition 4.3.2. *Define $f := 4h^2/(h^2 - s^2)$. The relations among $\alpha_0, \alpha_1, \alpha_2$ on the T^2-reduction at level $s \neq 0$ are given by*

$$
d\alpha_0 = f\alpha_1 \wedge \alpha_2, \quad d\alpha_1 \wedge \alpha_0 = -\frac{df}{f} \wedge \alpha_1 \wedge \alpha_0, \quad d\alpha_2 \wedge \alpha_0 = -\frac{df}{f} \wedge \alpha_2 \wedge \alpha_0. \qquad (4.19)
$$

Remark 4.3.3. Observe that at points where ν_M vanishes we get no information on the curvature two-form. Define $\beta_0 := \alpha_0$ and $\beta_i := f\alpha_i, i = 1, 2$. Equations (4.19) are then equivalent to

$$
d\beta_0 = \frac{1}{f}\beta_1 \wedge \beta_2, \quad d\beta_1 \wedge \beta_0 = 0, \quad d\beta_2 \wedge \beta_0 = 0.
$$

4.4 Inverse construction

Now we wish to invert the construction described above. Assume we are given a three-dimensional smooth manifold Q^3, and let g_{UU}, g_{UV}, g_{VV} be three functions on Q^3 such that $g_{UU} > 0$ and $g_{UU}g_{VV} - g_{UV}^2 > 0$. We define the latter quantity as $h^2 := g_{UU}g_{VV} - g_{UV}^2$. Let $f > 4$ be a real function and $\alpha_0, \alpha_1, \alpha_2$ be a basis of one-forms satisfying (4.19). Our first goal is to construct a principal T^2-bundle over Q^3.

Let $s \neq 0$ be a real number. Given

$$\Theta_1 := \frac{1}{s}\left(d\alpha_1 - \frac{3}{h^2 - s^2}(g_{UV}\alpha_1 + g_{VV}\alpha_2) \wedge \alpha_0\right), \tag{4.20}$$

$$\Theta_2 := \frac{1}{s}\left(d\alpha_2 + \frac{3}{h^2 - s^2}(g_{UU}\alpha_1 + g_{UV}\alpha_2) \wedge \alpha_0\right), \tag{4.21}$$

we find the conditions for which they are closed and with integral period, namely $[\Theta_i] \in H^2(Q^3, \mathbb{Z})$. We follow [Swa10, Section 2.1], for this last part. If

$$d\Theta_1 = 0 = d\left(\frac{3}{h^2 - s^2}(g_{UV}\alpha_1 + g_{VV}\alpha_2) \wedge \alpha_0\right)$$

we get

$$\frac{g_{UV}}{h^2} d(h^2) \wedge \alpha_1 \wedge \alpha_0 + \frac{g_{VV}}{h^2} d(h^2) \wedge \alpha_2 \wedge \alpha_0 = dg_{UV} \wedge \alpha_1 \wedge \alpha_0 + dg_{VV} \wedge \alpha_2 \wedge \alpha_0. \tag{4.22}$$

Similarly $d\Theta_2 = 0$ yields

$$\frac{g_{UU}}{h^2} d(h^2) \wedge \alpha_1 \wedge \alpha_0 + \frac{g_{UV}}{h^2} d(h^2) \wedge \alpha_2 \wedge \alpha_0 = dg_{UU} \wedge \alpha_1 \wedge \alpha_0 + dg_{UV} \wedge \alpha_2 \wedge \alpha_0. \tag{4.23}$$

Under the conditions (4.22) and (4.23) one can apply Chern–Weil theory and find a principal T^2-bundle $E^5 \to Q^3$ with connection one-form $(\vartheta_1, \vartheta_2)$ such that $d\vartheta_k = \Theta_k$, $k = 1, 2$. The space E^5 must be thought of as the level set $\nu_M^{-1}(s)$ of the previous section. So as s varies we have a five-dimensional foliation of a six-dimensional manifold $M = E \times (c, d)$, for some real numbers $c < d$. To construct a nearly Kähler structure on M starting from E^5, we flow the ϑ_ks and the α_is along the normal vector field to E^5, given by $\partial/\partial s = \frac{1}{9}(h^2 - s^2)^{-1} ds^\sharp$. Note that $\mathcal{L}_{\partial/\partial s}\nu_M = 1$, so $\partial/\partial s$ maps level sets to level sets.

In order to establish which equations must be satisfied, we first define $\sigma, \psi_\pm$ as in (4.7)–(4.9), then impose the nearly Kähler conditions as we have done above, getting (4.10)–(4.15). Note that on M the differential d can be split as the sum of the differential on E and the one in the remaining direction: we write $d_6 = d_5 + d_s$, where d_5 is the differential on E and $d_s\gamma = \gamma' \wedge ds$, the prime denoting the derivative with respect to s of the form γ. We use this on (4.10)–(4.15) and then contract with $\partial/\partial s$. The equations found will tell us how our forms evolve in the direction defined by $\partial/\partial s$.

We have seen that $d\sigma = 3\psi_+$ implies (4.11) in particular. We rewrite this equation as

$$\begin{aligned} sd_5\vartheta_1 + sd_s\vartheta_1 = {} & d_3\alpha_1 + d_s\alpha_1 \\ & - \frac{1}{h^2 - s^2}(3g_{UV}\alpha_1 \wedge \alpha_0 + 3g_{VV}\alpha_2 \wedge \alpha_0 + s\alpha_1 \wedge d\nu_M). \end{aligned}$$

By assumption, on E we have

$$sd_5\vartheta_1 = d_3\alpha_1 - \frac{1}{h^2 - s^2}(3g_{UV}\alpha_1 + 3g_{VV}\alpha_2) \wedge \alpha_0,$$

so we can simplify our equation getting

$$\vartheta_1' \wedge ds = \frac{1}{s}\alpha_1' \wedge ds - \frac{1}{h^2 - s^2}\alpha_1 \wedge ds.$$

Contracting with $\partial/\partial s$, we obtain

$$\vartheta_1' = \frac{1}{s}\alpha_1' - \frac{1}{h^2 - s^2}\alpha_1. \tag{4.24}$$

Similarly, from (4.10) we have

$$\vartheta_2' = \frac{1}{s}\alpha_2' - \frac{1}{h^2 - s^2}\alpha_2. \tag{4.25}$$

From (4.12) and (4.13) we get

$$\begin{aligned}\vartheta_2' \wedge \alpha_1 - \vartheta_1' \wedge \alpha_2 = {} & \frac{-4h^2 - 6s^2 + 3s(h^2)'}{3\big(h^2 - s^2\big)^2}\alpha_1 \wedge \alpha_2 \\ & + \frac{s}{h^2 - s^2}(\alpha_2 \wedge \alpha_1)' + \frac{1}{3\big(h^2 - s^2\big)^2} d_3(h^2) \wedge \alpha_0,\end{aligned} \tag{4.26}$$

and by (4.13) itself we find

$$\alpha_0' = \frac{4s}{3\big(h^2 - s^2\big)}\alpha_0.$$

The remaining equations yield

$$\begin{aligned}\alpha_0 \wedge \vartheta_2' = {} & \frac{s}{\big(h^2 - s^2\big)^2}(h^2)'\alpha_2 \wedge \alpha_0 + \frac{1}{3\big(h^2 - s^2\big)^2} d_3(h^2) \wedge \big(g_{UU}\alpha_1 + g_{UV}\alpha_2\big) \\ & + \frac{s}{h^2 - s^2}\alpha_2' \wedge \alpha_0 + \frac{h^2 - 3s^2}{3\big(h^2 - s^2\big)^2}\alpha_2 \wedge \alpha_0 - \frac{1}{3\big(h^2 - s^2\big)} d_3\big(g_{UU}\alpha_1 + g_{UV}\alpha_2\big),\end{aligned} \tag{4.27}$$

$$\begin{aligned}\alpha_0 \wedge \vartheta_1' = {} & \frac{s}{\big(h^2 - s^2\big)^2}(h^2)'\alpha_1 \wedge \alpha_0 - \frac{1}{3\big(h^2 - s^2\big)^2} d_3(h^2) \wedge \big(g_{UV}\alpha_1 + g_{VV}\alpha_2\big) \\ & + \frac{s}{h^2 - s^2}\alpha_1' \wedge \alpha_0 + \frac{h^2 - 3s^2}{3\big(h^2 - s^2\big)^2}\alpha_1 \wedge \alpha_0 + \frac{1}{3\big(h^2 - s^2\big)} d_3\big(g_{UV}\alpha_1 + g_{VV}\alpha_2\big).\end{aligned} \tag{4.28}$$

If we set $\alpha_1' = \sum_i a_{1i}\alpha_i$, $\alpha_2' = \sum_j a_{2j}\alpha_j$, and

$$d_3\alpha_1 = \sum_{i<j} b_{ij}\alpha_i \wedge \alpha_j, \qquad d_3\alpha_2 = \sum_{i<j} c_{ij}\alpha_i \wedge \alpha_j,$$

we can find equations giving $a_{10}, a_{11}, a_{12}, a_{20}, a_{21}, a_{22}, g'_{UU}, g'_{UV}, g'_{VV}, (h^2)'$. Denote by X_i the dual of α_i. Using (4.24) and (4.25) in (4.26), (4.27) and (4.28), we get

$$\begin{aligned}
(\alpha_2 \wedge \alpha_1)' &= -\frac{s^2}{h^2(h^2-s^2)}(h^2)'\alpha_1 \wedge \alpha_2 + \frac{10s}{3(h^2-s^2)}\alpha_1 \wedge \alpha_2 - \frac{s}{3h^2(h^2-s^2)} d_3(h^2) \wedge \alpha_0, \\
\alpha_0 \wedge \alpha_2' &= -\frac{s^2}{h^2(h^2-s^2)}(h^2)'\alpha_0 \wedge \alpha_2 + \frac{2s}{3(h^2-s^2)}\alpha_0 \wedge \alpha_2 \\
&\quad + \frac{s}{3h^2(h^2-s^2)} d_3(h^2) \wedge (g_{UU}\alpha_1 + g_{UV}\alpha_2) - \frac{s}{3h^2} d_3(g_{UU}\alpha_1 + g_{UV}\alpha_2), \\
\alpha_0 \wedge \alpha_1' &= -\frac{s^2}{h^2(h^2-s^2)}(h^2)'\alpha_0 \wedge \alpha_1 + \frac{2s}{3(h^2-s^2)}\alpha_0 \wedge \alpha_1 \\
&\quad - \frac{s}{3h^2(h^2-s^2)} d_3(h^2) \wedge (g_{UV}\alpha_1 + g_{VV}\alpha_2) + \frac{s}{3h^2} d_3(g_{UV}\alpha_1 + g_{VV}\alpha_2).
\end{aligned}$$

From these equations, comparing the coefficients of $\alpha_0 \wedge \alpha_1, \alpha_0 \wedge \alpha_2$ and $\alpha_1 \wedge \alpha_2$, one finds

$$a_{10} = \frac{s}{3h^2(h^2-s^2)} X_2(h^2), \qquad a_{20} = \frac{-s}{3h^2(h^2-s^2)} X_1(h^2), \tag{4.29}$$

$$a_{21} = \frac{g_{UU}s}{3h^2(h^2-s^2)} X_0(h^2) - \frac{s}{3h^2}\left(X_0(g_{UU}) + g_{UU}b_{01} + g_{UV}c_{01}\right), \tag{4.30}$$

$$a_{12} = \frac{-g_{VV}s}{3h^2(h^2-s^2)} X_0(h^2) + \frac{s}{3h^2}\left(X_0(g_{VV}) + g_{UV}b_{02} + g_{VV}c_{02}\right), \tag{4.31}$$

$$(h^2)' = -\frac{2h^2}{s} + \frac{h^2-s^2}{3s}\left((b_{01}-c_{02})g_{UV} + c_{01}g_{VV} - b_{02}g_{UU}\right), \tag{4.32}$$

$$a_{11} = \frac{8s}{3(h^2-s^2)} + \frac{s}{3h^2}\left(X_0(g_{UV}) + b_{02}g_{UU} + c_{02}g_{UV}\right) - \frac{s g_{UV}}{3h^2(h^2-s^2)} X_0(h^2), \tag{4.33}$$

$$a_{22} = \frac{8s}{3(h^2-s^2)} - \frac{s}{3h^2}\left(X_0(g_{UV}) + b_{01}g_{UV} + c_{01}g_{VV}\right) + \frac{s g_{UV}}{3h^2(h^2-s^2)} X_0(h^2). \tag{4.34}$$

Further, differentiating (4.10), (4.11), and repeating the same process, we get

$$\begin{aligned}
g'_{UU} &= g_{UU}\left(\frac{(h^2)'}{h^2-s^2} + \frac{1}{s} + a_{11} - \frac{2s}{3(h^2-s^2)}\right) - \frac{h^2}{3s}c_{01} + g_{UV}a_{21}, \\
g'_{UV} &= g_{UV}\left(\frac{(h^2)'}{h^2-s^2} + \frac{1}{s} + a_{22} - \frac{2s}{3(h^2-s^2)}\right) - \frac{h^2}{3s}c_{02} + g_{UU}a_{12} + \frac{sX_0(h^2)}{3(h^2-s^2)}, \\
g'_{UV} &= g_{UV}\left(\frac{(h^2)'}{h^2-s^2} + \frac{1}{s} + a_{11} - \frac{2s}{3(h^2-s^2)}\right) + \frac{h^2}{3s}b_{01} + g_{VV}a_{21} - \frac{sX_0(h^2)}{3(h^2-s^2)}, \\
g'_{VV} &= g_{VV}\left(\frac{(h^2)'}{h^2-s^2} + \frac{1}{s} + a_{22} - \frac{2s}{3(h^2-s^2)}\right) + \frac{h^2}{3s}b_{02} + g_{UV}a_{12}.
\end{aligned}$$

These results imply that $\alpha_0, \alpha_1, \alpha_2, g_{UU}, g_{UV}, g_{VV}$ can be found from a system of first order ordinary differential equations, so by Cauchy theorem we find a unique local solution on

$E \times (s_0 - \varepsilon, s_0 + \varepsilon)$, for some $\varepsilon > 0$, where $s_0 \neq 0$ is an initial data. Observe that the value of s_0 is specified by f and h through the equation $f = 4h^2/(h^2 - s_0^2)$. Finally, (4.24) and (4.25), together with the expressions of α_1' and α_2' found, give differential equations for ϑ_1, ϑ_2, which we can apply the same theorem to. Thus we have the final result:

Theorem 4.4.1. *Let Q^3 be a smooth three-manifold, $f > 4$ a smooth real function on Q^3, and $\{\alpha_i\}_{i=0,1,2}$ a basis of one-forms on Q^3 satisfying (4.19). Suppose there exists a smooth positive definite $G = \left(\begin{smallmatrix} g_{UU} & g_{UV} \\ g_{UV} & g_{VV} \end{smallmatrix}\right)$ on Q^3 such that (4.22) and (4.23) are fulfilled, and that $s = s_0 = (1 - 4/f)^{1/2}h$ is constant. Put $h^2 = \det G$, and define ϑ_1, ϑ_2 by (4.20) and (4.21) for $s = s_0$.*

Then, if ϑ_ks have integral periods, there exist a T^2-bundle $E^5 \to Q^3$ with connection one-form $(\vartheta_1, \vartheta_2)$, such that $d\vartheta_k = \Theta_k$, and an $\varepsilon > 0$ such that $E^5 \times (s_0 - \varepsilon, s_0 + \varepsilon)$ has a unique nearly Kähler structure of the form (4.7)–(4.9).

Proof. Let $'$ denote differentiation with respect to s and assume the functions a_{ij}s are those listed in (4.29)–(4.34). Then our forms satisfy the equations

$$\alpha_0' = \frac{4s}{3(h^2 - s^2)}\alpha_0, \qquad \alpha_1' = \sum_{i=0}^{2} a_{1i}\alpha_i, \qquad \alpha_2' = \sum_{j=0}^{2} a_{2j}\alpha_j,$$
$$\vartheta_1' = \frac{1}{s}\alpha_1' - \frac{1}{h^2 - s^2}\alpha_1, \qquad \vartheta_2' = \frac{1}{s}\alpha_2' - \frac{1}{h^2 - s^2}\alpha_2.$$

For the initial data $s = s_0 = (1 - 4/f)^{1/2}h$ they have a unique solution, which corresponds to a nearly Kähler structure on $E^5 \times (s_0 - \varepsilon, s_0 + \varepsilon)$, for some $\varepsilon > 0$. □

4.5 Invariant structures on the Heisenberg group

In this section we are going to study the construction described above in the particular case where Q^3 is the three-dimensional Heisenberg group H_3. Making specific choices of the forms involved and assuming (4.16)–(4.18), we write the equations in Theorem 4.4.1 and solve them getting explicit solutions. Finally, Proposition 4.5.2 proves our solution is general.

Let us consider the Heisenberg group H_3, i.e. the unipotent Lie group given by the upper triangular, real, 3×3 matrices of the form $(a_{ij}), a_{ii} = 1$, and $a_{ij} = 0, i > j$. Its Lie algebra is generated by

$$E_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

They satisfy the commutation relations $[E_1, E_2] = -E_0, [E_0, E_1] = [E_0, E_2] = 0$. If σ_i is the dual of E_i, then we have

$$d\sigma_0 = \sigma_1 \wedge \sigma_2, \qquad d\sigma_1 = 0, \qquad d\sigma_2 = 0.$$

Define $\alpha_k := f_k(s)\sigma_k$, and set $g_{UU}(s) = g_{VV}(s) =: h(s)$, and $g_{UV}(s_0) = 0$ for some initial data $s_0 \neq 0$. With this choice, equations (4.22) and (4.23) are automatically fulfilled. Then, according to Theorem 4.4.1, there exists a T^2-bundle $E^5 \to H_3$ with connection one-form $(\vartheta_1, \vartheta_2)$ satisfying

$$sd_5\vartheta_1 = -\frac{3h}{h^2 - s^2}\alpha_2 \wedge \alpha_0, \qquad sd_5\vartheta_2 = \frac{3h}{h^2 - s^2}\alpha_1 \wedge \alpha_0.$$

Equations $d_3\alpha_k = 0, k = 1,2,$ imply that all the coefficients b_{ij}, c_{ij} vanish. Furthermore, we have an algebraic relation among the f_ks given by

$$f_0/f_1f_2 = 4h^2/(h^2 - s^2).$$

Then we can compute: $a_{10} = a_{20} = a_{12} = a_{21} = 0, a_{11} = a_{22} = 8s/3(h^2 - s^2)$, and $h' = g'_{UU} = g'_{VV} = -h/s, g'_{UV} = 0$. So the following differential equations for $\alpha_0, \alpha_1, \alpha_2, \vartheta_1, \vartheta_2$ hold:

$$\alpha_0' = \frac{4s}{3\left(h^2 - s^2\right)}\alpha_0, \quad \alpha_k' = \frac{8s}{3\left(h^2 - s^2\right)}\alpha_k, \quad \vartheta_k' = \frac{5}{3\left(h^2 - s^2\right)}\alpha_k, \quad k = 1,2.$$

Since $h = g_{UU} > 0$, we obtain the expression $h(s) = |s_0 h(s_0)|/|s| =: C/s$, and the following differential equations for f_0, f_1, f_2:

$$f_0' = -f_0\left(\frac{4s}{3\left(h^2 - s^2\right)}\right), \quad f_k' = -f_k\left(\frac{8s}{3\left(h^2 - s^2\right)}\right), \quad k = 1,2.$$

Hence one can solve them getting

$$f_0(s) = f_0(s_0)\left(\frac{C^2 - s^4}{C^2 - s_0^4}\right)^{1/3}, \quad f_k(s) = f_k(s_0)\left(\frac{C^2 - s^4}{C^2 - s_0^4}\right)^{2/3}, \quad k = 1,2.$$

Let us set $f_0(s_0) = f_1(s_0) = f_2(s_0) = f(s_0)^{-1}$. In this case, having the expressions of h, f_0, f_1, f_2, we can write equations (4.6)–(4.9) explicitly: for $0 \neq s^2 < |C|$ the nearly Kähler metric we obtain is

$$\begin{aligned} g = {} & \frac{s^2}{9\left(C^2 - s^4\right)} ds^{\otimes 2} + \frac{C}{s}\left(\vartheta_1^{\otimes 2} + \vartheta_2^{\otimes 2}\right) \\ & + \frac{s^2\left(C^2 - s_0^4\right)}{16C^4}\left(\left(\frac{C^2 - s_0^4}{C^2 - s^4}\right)^{1/3} \sigma_0^{\otimes 2} + \frac{C}{s}\left(\frac{C^2 - s^4}{C^2 - s_0^4}\right)^{1/3}\left(\sigma_1^{\otimes 2} + \sigma_2^{\otimes 2}\right)\right). \end{aligned} \tag{4.35}$$

The fundamental two-form σ and the volume form $\psi_{\mathbb{C}}$ are given by

$$\begin{aligned} \sigma = {} & \frac{s^2}{12C^2}\left(\frac{C^2 - s_0^4}{C^2 - s^4}\right)^{2/3} ds \wedge \sigma_0 + s\vartheta_1 \wedge \vartheta_2 \\ & + \frac{C^2 - s_0^4}{4C^2}\left(\frac{C^2 - s^4}{C^2 - s_0^4}\right)^{2/3}\left(\vartheta_2 \wedge \sigma_1 - \vartheta_1 \wedge \sigma_2\right) - \frac{s^3\left(C^2 - s_0^4\right)}{16C^4}\left(\frac{C^2 - s^4}{C^2 - s_0^4}\right)^{1/3} \sigma_1 \wedge \sigma_2, \end{aligned} \tag{4.36}$$

$$\begin{aligned} \psi_+ = {} & \frac{1}{3} ds \wedge \left(\vartheta_1 \wedge \vartheta_2 + \frac{s^3}{4C^2}\left(\frac{C^2 - s_0^4}{C^2 - s^4}\right)^{1/3}\left(\vartheta_1 \wedge \sigma_2 - \vartheta_2 \wedge \sigma_1\right)\right) \\ & - \frac{C^2 - s_0^4}{16C^4}\left(\frac{s^2}{3}\left(\frac{C^2 - s^4}{C^2 - s_0^4}\right)^{1/3} ds \wedge \sigma_1 \wedge \sigma_2 + Cs\left(\vartheta_1 \wedge \sigma_1 + \vartheta_2 \wedge \sigma_2\right) \wedge \sigma_0\right), \end{aligned} \tag{4.37}$$

$$\begin{aligned} \psi_- = {} & \frac{s}{12C}\left(\frac{C^2 - s_0^4}{C^2 - s^4}\right)^{1/3} ds \wedge \left(\vartheta_1 \wedge \sigma_1 + \vartheta_2 \wedge \sigma_2\right) + \frac{C^2 - s_0^4}{4C^2}\left(\frac{C^2 - s^4}{C^2 - s_0^4}\right)^{1/3} \vartheta_1 \wedge \vartheta_2 \wedge \sigma_0 \\ & + \frac{s^2\left(C^2 - s_0^4\right)}{16C^4}\left(s\left(\vartheta_1 \wedge \sigma_2 - \vartheta_2 \wedge \sigma_1\right) - \frac{\left(C^2 - s_0^4\right)^{1/3}\left(C^2 - s^4\right)^{2/3}}{4C^2} \sigma_1 \wedge \sigma_2\right) \wedge \sigma_0. \end{aligned} \tag{4.38}$$

One can check explicitly that $d\sigma = 3\psi_+$ and $d\psi_- = -2\sigma \wedge \sigma$ by comparing the coefficients of the various bits. Here are the results for those that are less trivial to compute:

$$\begin{aligned} d\sigma = &-\frac{s^2}{12C^2}\left(\frac{C^2-s_0^4}{C^2-s^4}\right)^{2/3} ds\wedge\sigma_1\wedge\sigma_2 + ds\wedge\vartheta_1\wedge\vartheta_2 \\ &-\frac{3s(C^2-s_0^4)}{16C^3}\sigma_2\wedge\sigma_0\wedge\vartheta_2 - \frac{3s(C^2-s_0^4)}{16C^3}\vartheta_1\wedge\sigma_1\wedge\sigma_0 \\ &-\frac{2s^3}{3C^2}\left(\frac{C^2-s_0^4}{C^2-s^4}\right)^{1/3} ds\wedge\vartheta_2\wedge\sigma_1 - \frac{5s^3}{12C^2}\left(\frac{C^2-s_0^4}{C^2-s^4}\right)^{1/3}\vartheta_1\wedge\sigma_2\wedge ds \\ &+\frac{2s^3}{3C^2}\left(\frac{C^2-s_0^4}{C^2-s^4}\right)^{1/3} ds\wedge\vartheta_1\wedge\sigma_2 + \frac{5s^2(C^2-s_0^4)}{24C^4}\left(\frac{C^2-s^4}{C^2-s_0^4}\right)^{1/3} ds\wedge\sigma_1\wedge\sigma_2 \\ &-\frac{9s^2C^2-13s^6}{48C^4}\left(\frac{C^2-s_0^4}{C^2-s^4}\right)^{2/3} ds\wedge\sigma_1\wedge\sigma_2 + \frac{5s^3}{12C^2}\left(\frac{C^2-s_0^4}{C^2-s^4}\right)^{1/3}\sigma_1\wedge ds\wedge\vartheta_2, \end{aligned}$$

$$\begin{aligned} -2\sigma\wedge\sigma = &-\frac{s^3}{3C^2}\left(\frac{C^2-s_0^4}{C^2-s^4}\right)^{2/3} ds\wedge\sigma_0\wedge\vartheta_1\wedge\vartheta_2 - \frac{s^2(C^2-s_0^4)}{12C^4} ds\wedge\sigma_0\wedge\vartheta_2\wedge\sigma_1 \\ &+\frac{s^2}{12C^4}(C^2-s_0^4)ds\wedge\sigma_0\wedge\vartheta_1\wedge\sigma_2 \\ &+\frac{s^5}{48C^6}\left(\frac{C^2-s_0^4}{C^2-s^4}\right)^{1/3}(C^2-s_0^4)ds\wedge\sigma_0\wedge\sigma_1\wedge\sigma_2 \\ &+\frac{s^4(C^2-s_0^4)^{2/3}(C^2-s^4)^{1/3}}{4C^4}\vartheta_1\wedge\vartheta_2\wedge\sigma_1\wedge\sigma_2 \\ &+\frac{(C^2-s_0^4)^{2/3}(C^2-s^4)^{4/3}}{4C^4}\vartheta_2\wedge\sigma_1\wedge\vartheta_2\wedge\sigma_2, \end{aligned}$$

$$\begin{aligned} d\psi_- = &\frac{s(C^2-s_0^4)}{32C^4}\left(\frac{C^2-s_0^4}{C^2-s^4}\right)^{1/3} ds\wedge\sigma_2\wedge\sigma_0\wedge\sigma_1 - \frac{s^3}{3C^2}\left(\frac{C^2-s_0^4}{C^2-s^4}\right)^{2/3} ds\wedge\vartheta_1\wedge\vartheta_2\wedge\sigma_0 \\ &+\frac{5s^2(C^2-s_0^4)}{48C^4}\sigma_1\wedge ds\wedge\vartheta_2\wedge\sigma_0 - \frac{5s^2(C^2-s_0^4)}{48C^4}\vartheta_1\wedge\sigma_2\wedge ds\wedge\sigma_0 \\ &+\frac{(C^2-s_0^4)^{2/3}(C^2-s^4)^{1/3}}{4C^2}\vartheta_1\wedge\vartheta_2\wedge\sigma_1\wedge\sigma_2 + \frac{3s^2(C^2-s_0^4)}{16C^4} ds\wedge\vartheta_1\wedge\sigma_2\wedge\sigma_0 \\ &+\frac{5s^5(C^2-s_0^4)^{4/3}}{192C^6(C^2-s^4)^{1/3}}\sigma_1\wedge ds\wedge\sigma_2\wedge\sigma_0 - \frac{3s^2(C^2-s_0^4)}{16C^4} ds\wedge\vartheta_2\wedge\sigma_1\wedge\sigma_0 \\ &-\frac{5s^5(C^2-s_0^4)^{4/3}}{192C^6(C^2-s^4)^{1/3}}\sigma_2\wedge ds\wedge\sigma_1\wedge\sigma_0 \\ &-\frac{s(C^2-s_0^4)^{4/3}(6C^2-14s^4)}{192C^6(C^2-s^4)^{1/3}} ds\wedge\sigma_1\wedge\sigma_2\wedge\sigma_0. \end{aligned}$$

Remark 4.5.1. By the expression of the metric g in (4.35) we can observe that the fiber blows up when $s \to 0$, whereas the remaining four-dimensional subspace collapses to a point. If $s^2 \to |C|$, the fiber stabilises, a two-dimensional subspace of the base space collapses to a point, and the rest blows up.

The following proposition states this is indeed a general solution.

Proposition 4.5.2. *The nearly Kähler structure in* (4.35)–(4.38) *gives a general left-invariant structure for the case of the Heisenberg group H_3.*

Proof. As seen in Remark 4.3.3, the geometry of Q^3 can be described by three one-forms $\beta_0, \beta_1, \beta_2$ satisfying

$$d\beta_0 = \frac{1}{f}\beta_1 \wedge \beta_2, \qquad d\beta_1 \wedge \beta_0 = 0, \qquad d\beta_2 \wedge \beta_0 = 0.$$

Denote by τ_0, τ_1, τ_2 a dual basis of the Lie algebra of H_3 satisfying $d\tau_0 = \tau_1 \wedge \tau_2$ and $d\tau_1 = d\tau_2 = 0$. In our particular case we can observe that $d\beta_0 \in \mathrm{Span}\{\tau_1 \wedge \tau_2\}$, so $\beta_1 \wedge \beta_2 = c_1\tau_1 \wedge \tau_2$ for some real number $c_1 \neq 0$. This happens only when $\beta_1, \beta_2 \in \mathrm{Span}\{\tau_1, \tau_2\}$, so $d\beta_1 = d\beta_2 = 0$, and then $d\beta_1 \wedge \beta_0 = d\beta_2 \wedge \beta_0 = 0$, as we wanted.

Therefore $\beta_0 = (c_1/f)\tau_0 + a\tau_1 + b\tau_2$ for some real numbers a, b. Now define $\sigma_0 := f\beta_0$, and choose $\sigma_1, \sigma_2 \in \mathrm{Span}\{\tau_1, \tau_2\}$ so that σ_1 is $\tilde{g}$-orthogonal to σ_2, $\|\sigma_1\|_{\tilde{g}} = \|\sigma_2\|_{\tilde{g}}$, $\sigma_1 \wedge \sigma_2 = \beta_1 \wedge \beta_2$, and $\sigma_1 \wedge \sigma_2 = c_2\tau_1 \wedge \tau_2$, for some positive constant c_2. Define $\tilde{\beta}_0 := \beta_0$, $\tilde{\beta}_1 := \sigma_1$, and $\tilde{\beta}_2 := \sigma_2$. This new dual frame satisfies

$$d\tilde{\beta}_0 = \frac{1}{f}\tilde{\beta}_1 \wedge \tilde{\beta}_2, \qquad d\tilde{\beta}_1 = d\tilde{\beta}_2 = 0, \qquad \tilde{g} = h\,\mathrm{Id},$$

As in the end of Section 4.3, we can define three one-forms $\tilde{\alpha}_i, i = 0, 1, 2$ such that $\tilde{\beta}_0 =: \tilde{\alpha}_0$ and $\tilde{\beta}_i =: f\tilde{\alpha}_i, i = 1, 2$. Hence

$$\begin{aligned}
\tilde{\alpha}_0(s_0) &= \tilde{\beta}_0(s_0) = \frac{1}{f(s_0)}\sigma_0 \\
\tilde{\alpha}_k(s_0) &= \frac{1}{f(s_0)}\tilde{\beta}_k(s_0) = \frac{1}{f(s_0)}\sigma_k, \quad k = 1, 2,
\end{aligned}$$

so if f_0, f_1, f_2 are such that $\tilde{\alpha}_k = f_k(s)\sigma_k$, we get $f_0(s_0) = f_1(s_0) = f_2(s_0) = f(s_0)^{-1}$. Thus it is always possible to restrict ourselves to the case studied above. $\square$

Chapter 5

Critical sets and graphs

In Chapter 3 we computed explicitly some multi-moment maps and their critical sets. As Proposition 4.1.1 tells us, there is a simple characterisation of critical points where the multi-moment map vanishes in terms of the infinitesimal generators of the action: the multi-moment map ν_M and its differential vanish at p if and only if the generators U_p and V_p are linearly dependent over the reals. But if U and V are linearly dependent at some point p, the torus-action cannot be free (cf. [KN96, Proposition 4.1]). Therefore, Proposition 4.1.1 tells us the stabiliser of those points where the multi-moment map and its differential vanish is non-trivial.

Our task now is to compute these stabilisers in the homogeneous cases. We look for those points with non-trivial stabiliser first, then we draw a graph whose vertices correspond to points fixed by all of the two-torus, and whose edges correspond to points with one-dimensional stabiliser. Discrete and non-trivial stabilisers exist when the torus-action is not effective. This construction will be formalised in Theorem 5.5.1 and subsequent remarks. We stress that just the existence of such a graph in a specific case means that the torus action cannot be free. We already computed critical sets where the multi-moment maps vanish and got algebraic solutions, so we expect the graphs to encode and clarify their structure geometrically.

5.1 The six-sphere

Let us start off with $\mathbb{S}^6 \subset \mathbb{R}^7 \cong \mathbb{C}^3 \oplus \mathbb{R}$. We use the same notations as in Section 3.2. Recall that an element $A_{\vartheta,\phi} \in T^2$ acts on $\mathbb{C}^3 \oplus \mathbb{R}$ in the following way:

$$A_{\vartheta,\phi}(z^1, z^2, z^3, t) := (e^{i\vartheta}z^1, e^{i\phi}z^2, e^{-i(\vartheta+\phi)}z^3, t).$$

Our goal is to solve $A_{\vartheta,\phi}(z^1, z^2, z^3, t) = (z^1, z^2, z^3, t)$, namely the system $e^{i\vartheta}z^1 = z^1, e^{i\phi}z^2 = z^2, e^{-i(\vartheta+\phi)}z^3 = z^3$. The first equation implies either $e^{i\vartheta} = 1$ or $z^1 = 0$, so we have the following four possibilities:

$$\begin{cases} z^1 = 0, \vartheta \text{ free} \\ z^2 = 0, \phi \text{ free} \\ e^{-i(\vartheta+\phi)}z^3 = z^3, \end{cases} \quad \begin{cases} z^1 = 0, \vartheta \text{ free} \\ e^{i\phi} = 1, z^2 \text{ free} \\ e^{-i\vartheta}z^3 = z^3, \end{cases} \quad \begin{cases} e^{i\vartheta} = 1, z^1 \text{ free} \\ e^{i\phi} = 1, z^2 \text{ free} \\ z^3 \text{ free}, \end{cases} \quad \begin{cases} e^{i\vartheta} = 1, z^1 \text{ free} \\ z^2 = 0, \phi \text{ free} \\ e^{-i\phi}z^3 = z^3. \end{cases}$$

It is readily seen that the third system gives a trivial stabiliser, so we can ignore it. From the first system we get either $z^3 = 0$ or $\vartheta + \phi \equiv 0 \pmod{2\pi}$. In the former case we get the

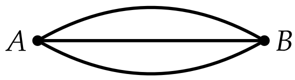

Figure 5.1: Graph of the fixed subspaces of $\mathbb{S}^6$

solution $(0,0,0,t)$, which corresponds to the points $(0,0,0,\pm 1)$ on the six-sphere. Their stabiliser is a two-torus because ϑ, ϕ are free, so we have two distinct points in our graph. If $\vartheta + \phi \equiv 0 \pmod{2\pi}$ and $z^3 \neq 0$, we get a set of fixed points on $\mathbb{S}^6$ of the form $(0,0,z^3,t)$ which satisfy $t^2 + |z^3|^2 = 1$. The equation $\vartheta + \phi \equiv 0 \pmod{2\pi}$ implies that the stabilisers of these points are copies of $\mathbb{S}^1$. Note that $t^2 + |z^3|^2 = 1$ corresponds to the equation of the two-sphere $(x^3)^2 + (x^4)^2 + (x^7)^2 = 1$ already found in Section 3.2.

The second system gives $z^3 = 0$ as ϑ is free. We obtain a fixed subspace of $\mathbb{S}^6$ given by the points of the form $(0, z^2, 0, t)$ such that $t^2 + |z^2|^2 = 1$, namely the two-sphere $(x^2)^2 + (x^5)^2 + (x^7)^2 = 1$.

The last system yields $z^3 = 0$ as ϕ is free, so finally we have the third two-sphere $t^2 + |z^1|^2 = 1$, namely $(x^1)^2 + (x^6)^2 + (x^7)^2 = 1$.

One can see we have just recovered the three two-spheres of critical points where the multi-moment map vanishes. The poles are fixed by the whole two-torus, so they play a distinguished role here:

$$\begin{cases} (0,0,0,\pm 1), & \text{poles fixed by all of } T^2, \\ t^2 + |z^i|^2 = 1, i = 1,2,3, & \text{two-spheres of points fixed by } \mathbb{S}^1. \end{cases}$$

Our graph will be then given by two points and three edges. As $|z^i| \to 0$ the two-spheres collapse to the common poles. Moreover, the spheres do not intersect each other at any point but the poles, so the edges of the graph do not intersect (see Figure 5.1). Observe that the graph we find is *trivalent*, which is to say there are three edges departing from each vertex.

5.2 The flag manifold

Recall the flag is defined as $F_{1,2}(\mathbb{C}^3) := \{(L, U) : L \leq U \leq \mathbb{C}^3, \dim L = 1, \dim U = 2\}$. The T^2-action on $\mathbb{C}^3$ is given by $A_{\vartheta,\phi} = \operatorname{diag}(e^{i\vartheta}, e^{i\phi}, e^{-i(\vartheta+\phi)})$ as in the previous case. Our aim now is to find which pairs of subspaces (L, U) are fixed by the $A_{\vartheta,\phi}$s. As we are going to see, it turns out that this action is not really effective: a copy of $\mathbb{Z}_3$ in T^2 fixes all the flags. However, there is an isomorphism between T^2 and $T^2/\mathbb{Z}_3$: the map $(e^{i\vartheta}, e^{i\phi}) \mapsto (e^{3i\vartheta}, e^{i(\vartheta-\phi)})$ is surjective onto T^2, and its kernel is a subgroup of T^2 isomorphic to $\mathbb{Z}_3$, so it yields an isomorphism $T^2/\mathbb{Z}_3 \cong T^2$. In the case below where $\mathbb{Z}_3$ appears as a discrete stabilizer of all the flags, we can use this trick to argue that the action of $T^2 \cong T^2/\mathbb{Z}_3$ is effective and the discrete stabilizers are all trivial.

Let us consider a non-zero $z = (z^1, z^2, z^3) \in \mathbb{C}^3$ and assume that $L := \operatorname{Span}(z)$ is a T^2-invariant one-dimensional subspace of $\mathbb{C}^3$. The equation we want to solve is $A_{\vartheta,\phi} z = \lambda(\vartheta, \phi) z$, where λ is some complex-valued function of ϑ, ϕ. Explicitly $e^{i\vartheta} z^1 = \lambda z^1, e^{i\phi} z^2 =$

$\lambda z^2, e^{-i(\vartheta+\phi)}z^3 = \lambda z^3$. As before, we have the following cases:

$$\begin{cases} e^{i\vartheta} = \lambda, z^1 \text{ free} \\ e^{i\phi} = e^{i\vartheta}, z^2 \text{ free} \\ e^{-i3\vartheta}z^3 = z^3 \end{cases} \quad \begin{cases} e^{i\vartheta} = \lambda, z^1 \text{ free} \\ z^2 = 0, \phi \text{ free} \\ e^{-i(2\vartheta+\phi)}z^3 = z^3 \end{cases} \quad \begin{cases} z^1 = 0, \vartheta \text{ free} \\ e^{i\phi} = \lambda, z^2 \text{ free} \\ e^{-i(\vartheta+2\phi)}z^3 = z^3. \end{cases} \quad \begin{cases} z^1 = 0, \vartheta \text{ free} \\ z^2 = 0, \phi \text{ free} \\ e^{-i(\vartheta+\phi)}z^3 = \lambda z^3. \end{cases}$$

The first system gives two subcases: $3\vartheta \equiv 0 \pmod{2\pi}$ or $z^3 = 0$. In the former we have

$$\vartheta \in \left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\} \pmod{2\pi}$$

and $z^i \neq 0$ for any $i = 1, 2, 3$, which gives a discrete stabilizer of L since $\vartheta \equiv \phi \pmod{2\pi}$. This is a copy of $\mathbb{Z}_3$ and we can argue as above to conclude that the stabilizer is trivial. In the latter case we have $(z^1, z^2, 0)$ which is fixed by an $\mathbb{S}^1$ given by $A_{\vartheta,\phi} = \operatorname{diag}(\lambda, \lambda, \lambda^{-2})$, because $\vartheta \equiv \phi \pmod{2\pi}$ and ϑ is free.

The second system gives either $2\vartheta + \phi \equiv 0 \pmod{2\pi}$ or $z^3 = 0$. From the first we see that $(z^1, 0, z^3)$ is fixed by a copy of $\mathbb{S}^1$, that is $A_{\vartheta,\phi} = \operatorname{diag}(\lambda, \lambda^{-2}, \lambda)$. The second one gives $(z^1, 0, 0)$ fixed by all of T^2 as we have no restrictions on ϑ and ϕ.

The third system gives either $\vartheta + 2\phi \equiv 0 \pmod{2\pi}$ or $z^3 = 0$. The first one yields $(0, z^2, z^3)$ fixed by an $\mathbb{S}^1$, which is given by matrices of the form $A_{\vartheta,\phi} = \operatorname{diag}(\lambda^{-2}, \lambda, \lambda)$, whereas the second one gives $(0, z^2, 0)$ fixed by all of T^2.

Finally, the last system gives only $(0, 0, z^3)$ fixed by all of T^2 as $z \neq 0$. Denote by F_1, F_2, F_3 the vectors $(1,0,0), (0,1,0), (0,0,1) \in \mathbb{C}^3$ respectively. We write the solutions as

$$\begin{cases} \mathbb{C}F_1, \mathbb{C}F_2, \mathbb{C}F_3, & \text{fixed by all of } T^2, \\ \mathbb{C}F_1 \oplus \mathbb{C}F_2, \mathbb{C}F_1 \oplus \mathbb{C}F_3, \mathbb{C}F_2 \oplus \mathbb{C}F_3, & \text{fixed by } \mathbb{S}^1. \end{cases}$$

Since the T^2-action preserves $\mathbb{C}F_i$ and the angles between two vectors, we have that $\mathbb{C}F_j \oplus \mathbb{C}F_k$ is preserved by T^2 as well, for different i, j, k that range in $\{1, 2, 3\}$. On the other hand, since $\mathbb{S}^1$ preserves $\mathbb{C}z$, where $z \in \operatorname{Span}\{F_i, F_j\}, i \neq j$, then the pairs $(\mathbb{C}z, \mathbb{C}F_i \oplus \mathbb{C}F_j)$ and $(\mathbb{C}F_i, \mathbb{C}F_j \oplus \operatorname{Span}\{z\})$, are fixed by $\mathbb{S}^1$. Therefore, we have six points in $F_{1,2}(\mathbb{C}^3)$ fixed by all of T^2 and nine edges corresponding to two-dimensional subspaces of points fixed by $\mathbb{S}^1$. The six points are represented by the following $A_{\alpha,\beta\gamma}$, which are obviously symmetric in β and γ:

$$\begin{cases} A_{1,12} = (\mathbb{C}F_1, \mathbb{C}F_1 \oplus \mathbb{C}F_2) \\ A_{1,13} = (\mathbb{C}F_1, \mathbb{C}F_1 \oplus \mathbb{C}F_3) \\ A_{2,12} = (\mathbb{C}F_2, \mathbb{C}F_1 \oplus \mathbb{C}F_2), \end{cases} \quad \begin{cases} A_{2,23} = (\mathbb{C}F_2, \mathbb{C}F_2 \oplus \mathbb{C}F_3) \\ A_{3,13} = (\mathbb{C}F_3, \mathbb{C}F_1 \oplus \mathbb{C}F_3) \\ A_{3,23} = (\mathbb{C}F_3, \mathbb{C}F_2 \oplus \mathbb{C}F_3), \end{cases}$$

and the edges $a_i, i = 1 \ldots, 9$ are

$$\begin{cases} \text{If } z \in \operatorname{Span}\{F_1, F_2\} \\ a_1 = (\mathbb{C}z, \mathbb{C}F_1 \oplus \mathbb{C}F_2) \\ a_2 = (\mathbb{C}z, \mathbb{C}F_3 \oplus \mathbb{C}z) \\ a_3 = (\mathbb{C}F_3, \mathbb{C}z \oplus \mathbb{C}F_3), \end{cases} \quad \begin{cases} \text{If } z \in \operatorname{Span}\{F_1, F_3\} \\ a_4 = (\mathbb{C}z, \mathbb{C}F_1 \oplus \mathbb{C}F_3) \\ a_5 = (\mathbb{C}F_2, \mathbb{C}z \oplus \mathbb{C}F_2) \\ a_6 = (\mathbb{C}z, \mathbb{C}F_2 \oplus \mathbb{C}z), \end{cases} \quad \begin{cases} \text{If } z \in \operatorname{Span}\{F_2, F_3\} \\ a_7 = (\mathbb{C}F_1, \mathbb{C}F_1 \oplus \mathbb{C}z) \\ a_8 = (\mathbb{C}z, \mathbb{C}F_2 \oplus \mathbb{C}F_3) \\ a_9 = (\mathbb{C}z, \mathbb{C}F_1 \oplus \mathbb{C}z). \end{cases}$$

In order to figure out what the vertices of, say, a_1 are, one can take the limit $z \to F_1$ (resp. $z \to F_2$) and see that $a_1 \to A_{1,12}$ (resp. $a_1 \to A_{2,12}$). The same can be applied to the other edges. The resulting trivalent graph is shown in Figure 5.2.

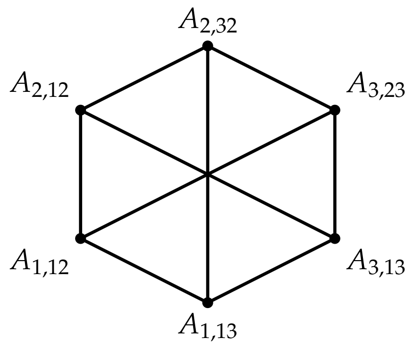

Figure 5.2: Graph of the fixed subspaces of $F_{1,2}(\mathbb{C}^3)$

5.3 The complex projective space

Let us now consider the action of T^2 in $\mathrm{SU}(4)$ on $\mathbb{C}^4 \setminus \{0\}$ given by the injection

$$\mathrm{diag}(e^{i\vartheta}, e^{i\phi}) \hookrightarrow A_{\vartheta,\phi} = \mathrm{diag}(e^{i\vartheta}, e^{i\phi}, e^{-i\vartheta}, e^{-i\phi}) \in \mathrm{SU}(4).$$

Explicitly on $\mathbb{CP}^3 = (\mathbb{C}^4 \setminus \{0\})_{/\mathbb{C}^*}$ we have

$$A_{\vartheta,\phi}([z^1 \colon z^2 \colon z^3 \colon z^4]) = [e^{i\vartheta} z^1 \colon e^{i\phi} z^2 \colon e^{-i\vartheta} z^3 \colon e^{-i\phi} z^4].$$

Observe that this action is not effective, because $A_{0,0}$, $A_{\pi,\pi}$ fix all points of $\mathbb{CP}^3$, and these are the only elements of the torus doing that. The morphism $(e^{i\vartheta}, e^{i\phi}) \mapsto (e^{i2\vartheta}, e^{i(\phi-\vartheta)})$ from the torus to itself induces an isomorphism $T^2 \cong T^2/\mathbb{Z}_2$, so the action of $T^2/\mathbb{Z}_2 \cong T^2$ on $\mathbb{CP}^3$ is effective.

We want the solutions of $A_{\vartheta,\phi}([z^1 \colon z^2 \colon z^3 \colon z^4]) = [z^1 \colon z^2 \colon z^3 \colon z^4]$. The homogeneous coordinates allow us to simplify the equations, because

$$[e^{i\vartheta} z^1 \colon e^{i\phi} z^2 \colon e^{-i\vartheta} z^3 \colon e^{-i\phi} z^4] = [z^1 \colon e^{i(\phi-\vartheta)} z^2 \colon e^{-i2\vartheta} z^3 \colon e^{-i(\vartheta+\phi)} z^4].$$

It is in fact enough to study this case: if we divide by $e^{i\phi}$ instead of $e^{i\vartheta}$ then we can permute the indices so as to swap (z^1, z^3) and (z^2, z^4), ϑ and ϕ. We then get the same system of the previous case. Similarly in the other two cases: if we divide by $e^{-i\vartheta}$, we need to map first $(\vartheta, \phi) \mapsto (-\vartheta, -\phi)$ and then swap (z^1, z^2) and (z^3, z^4), whereas if we divide by $e^{-i\phi}$ we map $(\vartheta, \phi) \mapsto (-\vartheta, -\phi)$ and finally swap (z^1, z^2) and (z^4, z^3). Once we find the solutions of the first case it will suffice to perform these steps to solve the others.

If we divide by $e^{i\vartheta}$ we get $z^1 = \lambda z^1, e^{i(\phi-\vartheta)} z^2 = \lambda z^2, e^{-i2\vartheta} z^3 = \lambda z^3, e^{-i(\vartheta+\phi)} z^4 = \lambda z^4$, where $\lambda = \lambda(\vartheta, \phi)$ is a complex-valued function. We distinguish the cases $\lambda = 1$, z^1 free and λ free, $z^1 = 0$. In the former case we then have to find the solutions of

$$\begin{cases} e^{i(\phi-\vartheta)} z^2 = z^2 \\ e^{-i2\vartheta} z^3 = z^3 \\ e^{-i(\vartheta+\phi)} z^4 = z^4. \end{cases}$$

This yields the following cases

$$\begin{cases} z^2 = 0 \\ e^{i2\vartheta} = 1, z^3 \text{ free} \\ e^{-i(\vartheta+\phi)} z^4 = z^4, \end{cases} \quad \begin{cases} z^2 = 0 \\ z^3 = 0, \vartheta \text{ free} \\ e^{-i(\vartheta+\phi)} z^4 = z^4, \end{cases} \quad \begin{cases} e^{i(\phi-\vartheta)} = 1, z^2 \text{ free} \\ e^{i2\vartheta} = 1, z^3 \text{ free} \\ e^{-i2\phi} z^4 = z^4, \end{cases} \quad \begin{cases} e^{i(\phi-\vartheta)} = 1, z^2 \text{ free} \\ z^3 = 0, \vartheta \text{ free} \\ e^{-i2\phi} z^4 = z^4. \end{cases}$$

From the first system we have that ϕ is free and so is $z^4 = 0$, otherwise we get a trivial stabilizer. Note that z^3 is free, so we get a two-dimensional subspace given by points of the form $[z^1 : 0 : z^3 : 0]$ fixed by an $\mathbb{S}^1$.

In the second system ϑ is free. If $z^4 = 0$ we get the point $[1 : 0 : 0 : 0]$ fixed by all of T^2. On the other hand, if $\vartheta + \phi \equiv 0 \pmod{2\pi}$ then z^4 is free and we get a two-dimensional subspace given by the points $[z^1 : 0 : 0 : z^4]$ fixed by $\mathbb{S}^1$.

The solutions of the third system correspond to a trivial stabilizer because $\phi \equiv \vartheta \pmod{2\pi}$ and $2\vartheta \equiv 0 \pmod{2\pi}$.

The fourth system yields the two-dimensional subspace given by $[z^1 : z^2 : 0 : 0]$ fixed by a copy of $\mathbb{S}^1$. The solutions are then the following:

$$\begin{cases} [1 : 0 : 0 : 0], & \text{fixed by all of } T^2, \\ [z^1 : z^2 : 0 : 0], [z^1 : 0 : z^3 : 0], [z^1 : 0 : 0 : z^4], & \text{fixed by } \mathbb{S}^1. \end{cases}$$

Now we discuss the case λ free and $z^1 = 0$. We have $e^{i(\phi-\vartheta)}z^2 = \lambda z^2, e^{-2i\vartheta}z^3 = \lambda z^3, e^{-i(\vartheta+\phi)}z^4 = \lambda z^4$. This implies

$$\begin{cases} e^{i(\phi-\vartheta)} = \lambda, z^2 \text{ free} \\ e^{-i(\phi+\vartheta)} = 1, z^3 \text{ free} \\ e^{-i2\phi}z^4 = z^4, \end{cases} \qquad \begin{cases} e^{i(\phi-\vartheta)} = \lambda, z^2 \text{ free} \\ z^3 = 0 \\ e^{-i2\phi}z^4 = z^4, \end{cases}$$

$$\begin{cases} z^2 = 0 \\ e^{-i2\vartheta} = \lambda, z^3 \text{ free} \\ e^{-i(\vartheta+\phi)}z^4 = \lambda z^4, \end{cases} \qquad \begin{cases} z^2 = 0 \\ z^3 = 0 \\ e^{-i(\vartheta+\phi)}z^4 = \lambda z^4. \end{cases}$$

The system on the top left gives points of the form $[0 : z^2 : z^3 : 0]$ fixed by $\mathbb{S}^1$ if $z^4 = 0$, otherwise its solutions correspond to a trivial stabilizer.

The system on the top right gives $[0 : 1 : 0 : 0]$ fixed by all of T^2 if $z^4 = 0$, otherwise the complex line $[0 : z^2 : 0 : z^4]$ fixed by an $\mathbb{S}^1$.

The system on the bottom left gives $[0 : 0 : 1 : 0]$ fixed by all of T^2 if $z^4 = 0$, otherwise the complex line $[0 : 0 : z^3 : z^4]$ fixed by $\mathbb{S}^1$.

The last system gives necessarily $[0 : 0 : 0 : 1]$ fixed by all of T^2. Thus, we have got four points fixed by all of T^2, namely

$$\begin{array}{ll} [1 : 0 : 0 : 0], & [0 : 1 : 0 : 0], \\ [0 : 0 : 1 : 0], & [0 : 0 : 0 : 1], \end{array}$$

and six points fixed by $\mathbb{S}^1$, that are

$$\begin{array}{lll} [z^1 : z^2 : 0 : 0], & [z^1 : 0 : z^3 : 0], & [z^1 : 0 : 0 : z^4], \\ [0 : z^2 : z^3 : 0], & [0 : z^2 : 0 : z^4], & [0 : 0 : z^3 : z^4]. \end{array}$$

Observe that every change of parameters and variables described above will lead us to the same solutions, because the edges are given by all the possible combinations of two elements out of four. For any point fixed by $\mathbb{S}^1$ we see that if one of the coordinates approaches 0 then it collapses to one of the points fixed by all of T^2. This yields the trivalent graph shown in Figure 5.3.

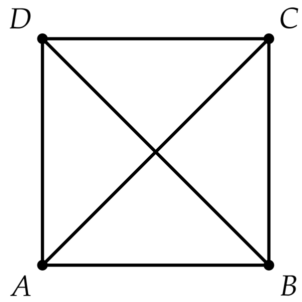

Figure 5.3: Graph of the fixed subspaces of $\mathbb{CP}^3$

5.4 The product of three-spheres

We conclude by studying $\mathbb{S}^3 \times \mathbb{S}^3$, which we recall to be diffeomorphic to $\mathrm{SU}(2)^3/\mathrm{SU}(2)_\Delta$. In this case there is a T^3-action: we consider $(t_1, t_2, t_3) \in T^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, with $t_k = e^{i\vartheta_k}$ for some $\vartheta_k \in \mathbb{R}$ and map each of them into SU(2) so that $t_i \mapsto \mathrm{diag}(t_i, t_i^{-1}) \in \mathrm{SU}(2)$. The action is then given by $(t_1, t_2, t_3)(g_1, g_2, g_3)\mathrm{SU}(2)_\Delta := (t_1g_1, t_2g_2, t_3g_3)\mathrm{SU}(2)_\Delta$. If $(g_1, g_2, g_3)\mathrm{SU}(2)_\Delta$ is fixed by (t_1, t_2, t_3) then we have $(t_1g_1, t_2g_2, t_3g_3) = (g_1g, g_2g, g_3g)$ for some $g \in \mathrm{SU}(2)$, which yields the system of equations $t_1g_1 = g_1g, t_2g_2 = g_2g, t_3g_3 = g_3g$. Isolating g on one side we find $g_1{}^{-1}t_1g_1 = g_2{}^{-1}t_2g_2$ and $g_1{}^{-1}t_1g_1 = g_3{}^{-1}t_3g_3$, so

$$\begin{cases} t_1 = (g_1g_2{}^{-1})t_2(g_1g_2{}^{-1})^{-1} \\ t_1 = (g_1g_3{}^{-1})t_3(g_1g_3{}^{-1})^{-1}. \end{cases}$$

This shows that t_1 and t_2 are conjugate, as well as t_1 and t_3. Thus each pair has to have the same eigenvalues. Since t_1, t_2, t_3 are diagonal matrices we see that if $t_1 = \mathrm{diag}(e^{i\vartheta}, e^{-i\vartheta})$ then $t_k = \mathrm{diag}(e^{i\vartheta}, e^{-i\vartheta})$ or $t_k = \mathrm{diag}(e^{-i\vartheta}, e^{i\vartheta})$ for $k = 2, 3$. This leads us to consider four cases: write $t_1 = t$, then (t_1, t_2, t_3) can be written as either (t, t, t), (t, t^{-1}, t), (t, t, t^{-1}), or (t, t^{-1}, t^{-1}). Note there is a discrete stabiliser given by $t = \pm\,\mathrm{Id}$, meaning that the action is not effective. By the usual argument as in the two cases above we can thus ignore it.

In the first case we get either $g_2g_1{}^{-1} = \mathrm{diag}(\lambda, \overline{\lambda})$ with $|\lambda| = 1$, as $g_2g_1{}^{-1} \in \mathrm{SU}(2)$ commutes with t. The same holds for $g_3g_1{}^{-1}$, so we can write $g_3g_1{}^{-1} = \mathrm{diag}(\lambda', \overline{\lambda}')$, with $|\lambda'| = 1$. Hence

$$\begin{aligned} (g_1, g_2, g_3)\mathrm{SU}(2)_\Delta &= (\mathrm{Id}, g_2g_1{}^{-1}, g_3g_1{}^{-1})\mathrm{SU}(2)_\Delta \\ &= (\mathrm{Id}, \mathrm{diag}(\lambda, \overline{\lambda}), \mathrm{diag}(\lambda', \overline{\lambda'}))\mathrm{SU}(2)_\Delta \\ &\cong \mathbb{S}^1 \times \mathbb{S}^1 = T^2. \end{aligned}$$

This shows we have a two-torus whose points are fixed by $\mathbb{S}^1$.

The second case is similar: $g_2g_1{}^{-1} = \left(\begin{smallmatrix} & -\lambda \\ \overline{\lambda} & \end{smallmatrix}\right)$, and $g_3g_1{}^{-1} = \left(\begin{smallmatrix} \lambda' & \\ & \overline{\lambda}' \end{smallmatrix}\right)$, with $|\lambda| = |\lambda'| = 1$, as before. Then

$$\begin{aligned} (g_1, g_2, g_3)\mathrm{SU}(2)_\Delta &= (\mathrm{Id}, g_2g_1{}^{-1}, g_3g_1{}^{-1})\mathrm{SU}(2)_\Delta \\ &= \left(\mathrm{Id}, \left(\begin{smallmatrix} & -\lambda \\ \overline{\lambda} & \end{smallmatrix}\right), \left(\begin{smallmatrix} \lambda' & \\ & \overline{\lambda}' \end{smallmatrix}\right)\right)\mathrm{SU}(2)_\Delta \\ &\cong \mathbb{S}^1 \times \mathbb{S}^1 = T^2. \end{aligned}$$

We obtain a second two-torus of points fixed by a copy of $\mathbb{S}^1$.

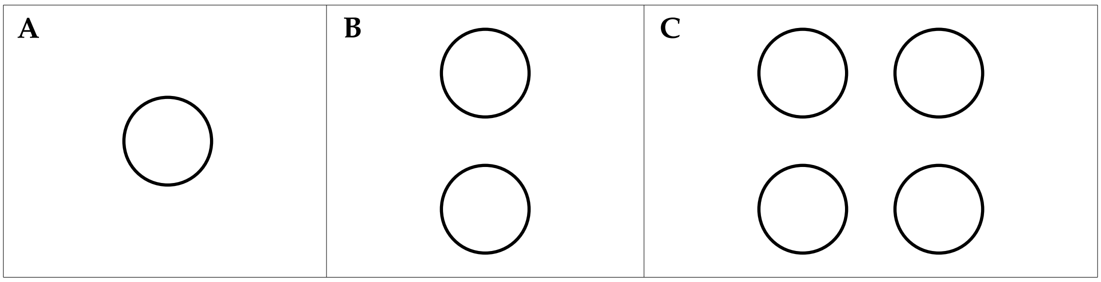

Figure 5.4: In **A** and **B** the two fixed-point sets for the T^2-action when this is not free. In **C** the four circles corresponding to the T^3-action.

The third case can be discussed analogously switching the roles of $g_2{g_1}^{-1}$ and $g_3{g_1}^{-1}$ in the second case. We get

$$\begin{aligned}(g_1, g_2, g_3)\mathrm{SU}(2)_\Delta &= (\mathrm{Id}, g_2{g_1}^{-1}, g_3{g_1}^{-1})\mathrm{SU}(2)_\Delta \\ &= \left(\mathrm{Id}, \begin{pmatrix} \lambda & \\ & \bar\lambda \end{pmatrix}, \begin{pmatrix} & \lambda' \\ -\bar\lambda' & \end{pmatrix}\right)\mathrm{SU}(2)_\Delta \\ &\cong \mathbb{S}^1 \times \mathbb{S}^1 = T^2,\end{aligned}$$

obtaining a third two-torus with points fixed by $\mathbb{S}^1$.

Finally, in the last case one has

$$\begin{aligned}(g_1, g_2, g_3)\mathrm{SU}(2)_\Delta &= (\mathrm{Id}, g_2{g_1}^{-1}, g_3{g_1}^{-1})\mathrm{SU}(2)_\Delta \\ &= \left(\mathrm{Id}, \begin{pmatrix} & \lambda \\ -\bar\lambda & \end{pmatrix}, \begin{pmatrix} & \lambda' \\ -\bar\lambda' & \end{pmatrix}\right)\mathrm{SU}(2)_\Delta \\ &\cong \mathbb{S}^1 \times \mathbb{S}^1 = T^2.\end{aligned}$$

so there is a fourth two-torus of points fixed by $\mathbb{S}^1$. For every T^2 in T^3, the stabilizers are still zero- or one-dimensional as $\mathrm{Stab}_{T^2}(p) \subset \mathrm{Stab}_{T^3}(p)$. Thus there are no vertices in our graph, we get only disjoint circles. Further, a $T^2 \subset T^3$ cannot contain all the circles $(t, t, t), (t, t^{-1}, t), (t, t, t^{-1}), (t, t^{-1}, t^{-1})$. There are three cases: the two-torus may contain none, one or two of the circles above. For example, the first case happens when T^2 is of the form (r, s, Id), $r, s \in \mathbb{S}^1$, so in this case we get an empty graph and the T^2-action is free. If it contains triples (r, rs, rs^2), $r, s \in \mathbb{S}^1$, then it contains the circle (r, r, r), so the graph is a single circle. Thirdly, if it is of the form (r, s, r), then it contains the circles (t, t, t) and (t, t^{-1}, t), but does not include (t, t, t^{-1}) and (t, t^{-1}, t^{-1}), so we get two circles in our graph (in Figure 5.4 the non-trivial cases are shown).

5.5 A general result

In the first three cases above each graph contains points fixed by all of T^2 and is trivalent. The fourth one exhibits no such points, although it may be considered as a trivalent graph with an empty set of vertices. On the other hand, $\mathbb{S}^3 \times \mathbb{S}^3$ is certainly the only homogeneous case with a disconnected graph. Inspired by these concrete examples, we can prove a general statement on the configuration of fixed-points and one-dimensional orbits: the next result holds for $\mathrm{SU}(3)$-structures with a T^2-symmetry and not necessarily nearly Kähler.

Theorem 5.5.1. *Let $(M, \sigma, \psi_\pm)$ be a six-dimensional manifold with an* $\mathrm{SU}(3)$*-structure admitting a two-torus symmetry. Assume the T^2-action is effective on M. Let p be a point in M and H_p its stabiliser in T^2.*

1. *If $\dim H_p = 2$ then $H_p = T^2$ and there is a neighbourhood W of p in M with the following properties: the stabiliser of each point of W is either trivial or a one-dimensional circle $\mathbb{S}^1 < T^2$, and the set of points in W with one-dimensional stabilisers is a disjoint union of three totally geodesic two-dimensional submanifolds which are complex with respect to J and whose closures only meet at p.*

2. *If $\dim H_p = 1$ then $H_p = \mathbb{S}^1 < T^2$ and there is a neighbourhood W of p in M with the following properties: the stabiliser of each point of W is either trivial or H_p and the set of points $\{q \in W : Stab_{T^2}(q) = H_p\}$ is a smooth totally geodesic submanifold of dimension two which is complex with respect to J.*

3. *If $\dim H_p = 0$ and H_p is non-trivial, then $H_p \cong \mathbb{Z}_k$ for some $k > 1$. The T^2-orbit E through p is a totally geodesic two-dimensional submanifold, complex with respect to J, and there is a neighbourhood W of this orbit where T^2 acts freely on $W \setminus E$.*

Proof. Let $g \in T^2$ and denote by $\vartheta_g \colon M \to M$, the diffeomorphism of M mapping q to gq. Its differential $T_p\vartheta_g$ is in general an isomorphism between T_pM and $T_{gp}M$. Since T^2 preserves the $\mathrm{SU}(3)$-structure, $T_p\vartheta_g$ preserves the metric g, the almost complex structure J and the volume form $\psi_{\mathbb{C}} = \psi_+ + i\psi_-$. Assume that p is fixed by $g \in T^2$. Then $T_p\vartheta_g$ is an automorphism of T_pM, which is isomorphic to $\mathbb{C}^3$ with its standard $\mathrm{SU}(3)$-structure

$$\sigma_0 = \tfrac{i}{2}\sum_{k=1}^{3} dz^k \wedge d\bar{z}^k, \qquad \psi_0 = dz^1 \wedge dz^2 \wedge dz^3,$$

so $T_p\vartheta_g \in \mathrm{SU}(3)$. Up to conjugation, $T_p\vartheta_g$ is an element of a maximal torus in $\mathrm{SU}(3)$, so for concreteness we assume $T_p\vartheta_g = \mathrm{diag}(e^{i\vartheta}, e^{i\varphi}, e^{-i(\vartheta+\varphi)})$ with respect to the standard basis of $\mathbb{C}^3$.

When $\dim H_p = 2$ then H_p is exactly T^2 by the Closed Subgroup Theorem, and by Theorem 4.2.4 there is an open neighbourhood of p equivariantly diffeomorphic to

$$T^2 \times_{T^2} (T_pM / T_p(T^2 \cdot p)) \cong T_pM \cong \mathbb{C}^3.$$

We now look for points with non-trivial stabiliser in this neighbourhood of p. A point $q \neq p$ in the neighbourhood coincides then with a vector X in $\mathbb{C}^3$, and by equivariance the requirement $gq = q$ in M translates to $T_q\vartheta_g X = X$ in $\mathbb{C}^3$. Denote X by $(z^1, z^2, z^3) \in \mathbb{C}^3$ with respect to the standard basis. Then we can write the action explicitly as

$$\mathrm{diag}(e^{i\vartheta}, e^{i\zeta}, e^{-i(\vartheta+\zeta)}) \cdot (z^1, z^2, z^3) = (z^1, z^2, z^3),$$

getting the non-trivial cases

$$\begin{cases} e^{i\vartheta} = 1, z_1 \in \mathbb{C} \\ z_2 = 0, \zeta \in \mathbb{R} \\ e^{-i\zeta} z_3 = z_3, \end{cases} \quad \begin{cases} z_1 = 0, \vartheta \in \mathbb{R} \\ e^{i\zeta} = 1, z_2 \in \mathbb{C} \\ e^{-i\vartheta} z_3 = z_3, \end{cases} \quad \begin{cases} z_1 = 0, \vartheta \in \mathbb{R} \\ z_2 = 0, \zeta \in \mathbb{R} \\ e^{-i(\vartheta+\zeta)} z_3 = z_3. \end{cases}$$

One can solve the systems and find there are three $\mathbb{S}^1$-invariant directions F_1, F_2, F_3 corresponding to the standard basis of $\mathbb{C}^3$. Thus the lines zF_1, zF_2, zF_3 correspond to three two-dimensional invariant subspaces in $\mathbb{C}^3$ whose points have one-dimensional stabiliser. This proves points p with stabiliser T^2 are isolated when exist, and there are three two-dimensional, disjoint submanifolds in a neighbourhood of p in M, intersecting at p, and whose points are fixed by a one-dimensional stabiliser. The fact that they are totally geodesic follows from e.g. [Kob72, Theorem 5.1].

Assume now p has one-dimensional stabiliser H_p. Choosing U in the Lie algebra of H_p and V such that $\mathrm{Span}\{U, V\} = \mathfrak{t}^2$, we have $U_p = 0$ and $V_p \neq 0$ in T_pM. So $T_pM \cong \mathrm{Span}\{V_p, JV_p\} \oplus \mathbb{R}^4 \cong \mathbb{C} \oplus \mathbb{C}^2$. Since V_p and JV_p are H_p-invariant, $T_p\vartheta_g \in \mathrm{SU}(2)$ for $g \in H_p$. We then claim $H_p \cong \mathbb{S}^1$: the connected component of the identity in H_p is conjugate to $\mathbb{S}^1$, so up to a change of basis its elements are diagonal matrices of the form $\mathrm{diag}(e^{i\alpha}, e^{-i\alpha})$. But $T_p\vartheta_g$ and $\mathrm{diag}(e^{i\alpha}, e^{-i\alpha})$ commute because H_p is Abelian, thus $T_p\vartheta_g$ must be diagonal, hence in $\mathbb{S}^1$, and the claim is proved. Therefore, p has a neighbourhood diffeomorphic to

$$T^2 \times_{\mathbb{S}^1} (T_pM/T_p(T^2 \cdot p)) \cong \mathbb{S}^1 \times \mathbb{R}^5 \cong \mathbb{S}^1 \times (\mathbb{R} \oplus \mathbb{C}^2).$$

Call $\mathbb{S}^1_-$ the stabiliser H_p, so that $T^2 = \mathbb{S}^1_+ \times \mathbb{S}^1_-$. The torus-action on $\mathbb{S}^1 \times (\mathbb{R} \oplus \mathbb{C}^2)$ can be chosen as follows: $\mathbb{S}^1_+$ acts on $\mathbb{S}^1$, and $\mathbb{S}^1_-$ acts on $\mathbb{R} \oplus \mathbb{C}^2$ trivially on $\mathbb{R}$ and as the usual maximal torus in $\mathrm{SU}(2)$ on $\mathbb{C}^2$. But an element in $\mathbb{S}^1_-$ preserves JV_p, so a point q in the neighbourhood $\mathbb{S}^1 \times (\mathbb{R} \oplus \mathbb{C}^2)$ is fixed by an element ℓ in the two-torus when the corresponding component in $\mathbb{R} \oplus \mathbb{C}^2$ is fixed, namely $T_q\vartheta_\ell X = X$ in $\mathbb{R} \oplus \mathbb{C}^2$. Since the action of H_p on $\mathbb{R}$ is trivial, this condition translates to a condition on $\mathbb{C}^2 \subset \mathbb{C}^3 \cong T_pM$. The same calculation as above shows there is only one invariant direction. Thus there is only one invariant two-dimensional totally geodesic submanifold containing p.

Finally, when p has zero-dimensional stabiliser H_p, then there are two invariant independent directions $U_p, V_p \neq 0$. Two cases may occur: either $V_p \in \mathrm{Span}\{U_p, JU_p\}$ or $V_p \notin \mathrm{Span}\{U_p, JU_p\}$.

In the former case, $T_pM = \langle U_p, JU_p \rangle \oplus \mathbb{C}^2$, so $H_p \leq \mathrm{SU}(2)$ is a discrete subgroup of $\mathbb{S}^1$. But H_p is compact and Abelian, so it is finite in $\mathrm{SU}(2)$ and is then conjugate to $\mathbb{Z}_k$ for some integer k. In this case p has a neighbourhood diffeomorphic to

$$\begin{aligned} T^2 \times_{\mathbb{Z}_k} \mathbb{C}^2 &= (T^2 \times_{\mathbb{Z}_k} \{0\}) \cup (T^2 \times_{\mathbb{Z}_k} (\mathbb{C}^2 \setminus \{0\})) \\ &= (T^2/\mathbb{Z}_k) \cup (T^2 \times_{\mathbb{Z}_k} (\mathbb{C}^2 \setminus \{0\})). \end{aligned}$$

Now, assume a point q in this neighbourhood be fixed by $\mathbb{Z}_k$. Since the action of $\mathbb{Z}_k$ is trivial on $T^2/\mathbb{Z}_k$ and is free on $\mathbb{C}^2 \setminus \{0\}$, q must lie in $T^2/\mathbb{Z}_k \cong T^2$, so it belongs to the orbit of p.

In the case $V_p \notin \mathrm{Span}\{U_p, JU_p\}$ then H_p fixes all of T_pM, so it is a subgroup of $\mathrm{SU}(1) = \{1\}$, and is then trivial. □

Remark 5.5.2. When H_p has positive dimension, the generators of the action are linearly dependent over the reals, whereas when H_p is zero-dimensional and non-trivial they are linearly dependent over the complex numbers. This implies that in the first two cases listed in the theorem above, our multi-moment map ν_M vanishes at p, and when H_p is discrete and not trivial $\nu_M(p) \neq 0$.

Remark 5.5.3. Consider the projection $\pi \colon M \to M/T^2$. The graphs are obtained by mapping fixed-points and two-submanifolds of points with one-dimensional stabiliser to M/T^2. In the first two cases W/T^2 is homeomorphic to $\mathbb{R}^4$. That $\mathbb{C}^3/T^2$ is homeomorphic to $\mathbb{R}^4$ follows from the homeomorphism between $\mathbb{S}^5/T^2$ and $\mathbb{S}^3$ [Mar16] and by taking the cones on the respective spaces. For the second case the homeomorphism is obtained by looking at $\mathbb{C}^2$ as a cone over $\mathbb{S}^3$ and at the sphere $\mathbb{S}^3$ as a principal $\mathbb{S}^1$-bundle over $\mathbb{S}^2$:

$$\begin{aligned} \mathbb{S}^1 \times (\mathbb{R} \oplus \mathbb{C}^2)/T^2 &\cong (\mathbb{R} \oplus \mathbb{C}^2)/\mathbb{S}^1_- \cong \mathbb{R} \times (\mathbb{C}^2/\mathbb{S}^1) \\ &\cong \mathbb{R} \times (C(\mathbb{S}^3)/\mathbb{S}^1) \cong \mathbb{R} \times C(\mathbb{S}^3/\mathbb{S}^1) \\ &\cong \mathbb{R} \times C(\mathbb{S}^2) \cong \mathbb{R}^4. \end{aligned}$$

In the third case the image of the exceptional orbit is an orbifold point in M/T^2. Lastly, we observe that the shape of the graphs for the examples constructed by Foscolo and Haskins are the same as for the homogeneous cases, although the general critical sets may be different.

Chapter 6

Topological aspects

In this final chapter we start by expanding Section 4.1. We describe properties and symmetries of the generators of a two-torus action on any nearly Kähler six-manifold, then use part of the data obtained to work out a formula on the Hessian of the multi-moment map. The idea is to collect material to study nearly Kähler six-manifolds with a two-torus symmetry from a topological point of view. In this regard, the multi-moment map must be thought of as a Morse function, so non-degenerate critical points play a distinguished role. This is why we need information on the Hessian and will show its explicit expression on one of the homogeneous examples. The results presented can be elaborated further, so we conclude explaining possible directions and potential applications.

6.1 Further symmetries

The usual, general set-up consists of a six-manifolds M equipped with a nearly Kähler structure $(g, J, \psi_\pm)$ and admitting a two-torus symmetry. Explicitly, this amounts to say that the infinitesimal generators of the T^2-action U and V satisfy the following properties:

1. $[U, V] = 0 = \mathcal{L}_U V$.
2. For $X = U, V$ we have $\mathcal{L}_X g = 0, \mathcal{L}_X J = 0$, and $\mathcal{L}_X \psi_\pm = 0$.

The first property implies $\nabla_U V = \nabla_V U$, whereas the second gives $\mathcal{L}_U \sigma = 0 = \mathcal{L}_V \sigma$. We have of course made use of all these properties already. A multi-moment map is defined as $\nu_M = \sigma(U, V)$ and its differential is given by $d\nu_M = 3\psi_+(U, V, \cdot\,) = 3\nabla\sigma(U, V, \cdot\,) = 3g((\nabla_U J)V, \cdot\,)$. At critical points p then V_p is in the span of U_p, JU_p, or equivalently $(\nabla_U J)V_{|p} = 0$. Recall that $\widehat{\nabla} = \nabla - \frac{1}{2}J(\nabla J)$ is a Hermitian connection on M (cf. Proposition 1.4.2).

We now want general information about $\nabla U, \nabla V, \widehat{\nabla} U$, and $\widehat{\nabla} V$. We work only with $\nabla U, \widehat{\nabla} U$, the same respective conclusions hold for the remaining fields. Let us start working with ∇, then switch to $\widehat{\nabla}$.

Our first observation is that the field ∇U preserves the metric, namely $\nabla U \in \mathfrak{so}(6)$. The identities $\mathcal{L}_U g = 0$ and $\nabla g = 0$ imply

$$\begin{aligned} U(g(X, Y)) &= g([U, X], Y) + g(X, [U, Y]), \\ U(g(X, Y)) &= g(\nabla_U X, Y) + g(X, \nabla_U Y). \end{aligned}$$

Comparing the two right hand sides we find

$$g(\nabla_U X - \nabla_X U, Y) + g(X, \nabla_U Y - \nabla_Y U) = g(\nabla_U X, Y) + g(X, \nabla_U Y),$$

thus $g((\nabla U)X, Y) + g(X, (\nabla U)Y) = 0$, namely $\nabla U \in \mathfrak{so}(6)$.

We can repeat the same steps with σ, but since it is not parallel with respect to ∇ we find that ∇U does not preserve σ. This implies that the $(2,0)$ component of ∇U in the decomposition of the Lie algebra of skew-symmetric endomorphisms $\mathfrak{so}(6) = \mathfrak{u}(3) \oplus \mathfrak{u}(3)^\perp$ is non-zero (cf. identities (1.7)). We write $\nabla U = (\nabla U)^{1,1} + (\nabla U)^{2,0}$, with $(\nabla U)^{2,0} \neq 0$. The behaviour of ∇U with respect to J is then non-trivial and interesting, we will compute $[\nabla U, J]$ later. Since $\mathcal{L}_U \sigma = 0$ we have

$$\begin{aligned} U(\sigma(X,Y)) &= \sigma([U,X],Y) + \sigma(X,[U,Y]), \\ U(\sigma(X,Y)) &= \nabla\sigma(U,X,Y) + \sigma(\nabla_U X, Y) + \sigma(X, \nabla_U Y). \end{aligned}$$

Therefore

$$\nabla\sigma(U,X,Y) + \sigma((\nabla U)X, Y) + \sigma(X, (\nabla U)Y) = 0. \tag{6.1}$$

Let us turn to the relation between ∇U and ψ_+. Since $\psi_+ = \nabla\sigma$, we expect to get an expression in terms of $\nabla^2\sigma = \nabla\psi_+$. In fact we have

$$\begin{aligned} U(\psi_+(X,Y,Z)) &= \psi_+([U,X],Y,Z) + \psi_+(X,[U,Y],Z) + \psi_+(X,Y,[U,Z]), \\ U(\psi_+(X,Y,Z)) &= \nabla\psi_+(U,X,Y,Z) + \psi_+(\nabla_U X, Y, Z) \\ &\quad + \psi_+(X, \nabla_U Y, Z) + \psi_+(X, Y, \nabla_U Z). \end{aligned}$$

Simplifying we find

$$\begin{aligned} &\nabla\psi_+(U,X,Y,Z) + \psi_+((\nabla U)X, Y, Z) \\ &\quad + \psi_+(X, (\nabla U)Y, Z) + \psi_+(X, Y, (\nabla U)Z) = 0. \end{aligned}$$

Doing the same for ψ_- and recalling $\psi_- = -J\psi_+$ we get

$$\begin{aligned} &\nabla\psi_-(U,X,Y,Z) \\ &\quad = -\psi_+((\nabla U)JX, Y, Z) - \psi_+(X, (\nabla U)JY, Z) - \psi_+(X, Y, (\nabla U)Z) \\ &\qquad + \psi_+((\nabla_U J)X, Y, Z) + \psi_+(X, (\nabla_U J)Y, Z) + \psi_+(X, Y, (\nabla_U J)Z). \end{aligned}$$

We now observe a simple and useful fact about the interplay between U and J. For every vector field X on M we have $[U, JX] = J[U, X]$. This is an easy consequence of $\mathcal{L}_U J = 0$, which implies

$$0 = \mathcal{L}_U JX - J\mathcal{L}_U X = [U, JX] - J[U, X], \tag{6.2}$$

whence the result.

Motivated by the observations above, we finish computing $[\nabla U, J]$: using (6.2), the commutator of the operators ∇U and J acts on any vector field X as

$$\begin{aligned} [\nabla U, J]X &= \nabla_{JX} U - J\nabla_X U = [JX, U] + \nabla_U JX - J\nabla_X U \\ &= J[X, U] + (\nabla_U J)X + J[U, X] = (\nabla_U J)X. \end{aligned}$$

Since the left hand side is actually $[(\nabla U)^{2,0}, J]X$ the identity found is

$$[(\nabla U)^{2,0}, J] = (\nabla_U J). \tag{6.3}$$

We now switch to the same information for $\widehat{\nabla} U$. First of all it is true that $\widehat{\nabla} U$ preserves the metric, because $\mathcal{L}_U g = 0$ and $\widehat{\nabla}_U g = 0$. Since $\widehat{\nabla}$ is not torsion-free, $\widehat{\nabla} U$ has a non-zero component in $\mathfrak{u}(3)^\perp \subset \mathfrak{so}(6)$. In general $\widehat{\nabla}_A B - \widehat{\nabla}_B A - [A, B] = J(\nabla_B J)A$. An explicit calculation using $\mathcal{L}_U \sigma = 0$ and $\widehat{\nabla}_U \sigma = 0$ yields

$$\begin{aligned} U(\sigma(X,Y)) &= \sigma([U,X],Y) + \sigma(X,[U,Y]) \\ &= \sigma(\widehat{\nabla}_U X - \widehat{\nabla}_X U - J(\nabla_X J)U, Y) + \sigma(X, \widehat{\nabla}_U Y - \widehat{\nabla}_Y U - J(\nabla_Y J)U), \\ U(\sigma(X,Y)) &= \sigma(\widehat{\nabla}_U X, Y) + \sigma(X, \widehat{\nabla}_U Y). \end{aligned}$$

Therefore

$$2\nabla\sigma(U,X,Y) + \sigma((\widehat{\nabla} U)X, Y) + \sigma(X, (\widehat{\nabla} U)Y) = 0. \tag{6.4}$$

Note that (6.1) and (6.4) imply

$$\sigma((\widehat{\nabla} U - 2\nabla U)X, Y) + \sigma(X, (\widehat{\nabla} U - 2\nabla U)Y) = 0,$$

so $\widehat{\nabla} U - 2\nabla U \in \mathfrak{u}(3)$. Note that $\widehat{\nabla} U - 2\nabla U \in \mathfrak{u}(3)$ implies $(\widehat{\nabla} U)^{2,0} - 2(\nabla U)^{2,0} \in \mathfrak{u}(3)$, namely $(\widehat{\nabla} U)^{2,0} = 2(\nabla U)^{2,0}$. We will come back to this point in a moment. The identities $\mathcal{L}_U \psi_+ = 0$ and $\widehat{\nabla} \psi_+ = 0$ imply

$$\begin{aligned} &U(\psi_+(X,Y,Z)) = \\ &\quad = \psi_+(\widehat{\nabla}_U X - \widehat{\nabla}_X U + J(\nabla_U J)X, Y, Z) + \psi_+(X, \widehat{\nabla}_U Y - \widehat{\nabla}_Y U + J(\nabla_U J)Y, Z) \\ &\qquad + \psi_+(X, Y, \widehat{\nabla}_U Z - \widehat{\nabla}_Z U + J(\nabla_U J)Z), \\ &U(\psi_+(X,Y,Z)) = \psi_+(\widehat{\nabla}_U X, Y, Z) + \psi_+(X, \widehat{\nabla}_U Y, Z) + \psi_+(X, Y, \widehat{\nabla}_U Z). \end{aligned}$$

It follows

$$\begin{aligned} &2\nabla^2\sigma(U,X,Y,Z) + \psi_+((\widehat{\nabla} U)X, Y, Z) \\ &\qquad + \psi_+(X, (\widehat{\nabla} U)Y, Z) + \psi_+(X, Y, (\widehat{\nabla} U)Z) = 0. \end{aligned}$$

Lastly, we can extend (6.3) to $\widehat{\nabla}$:

$$\begin{aligned} [\widehat{\nabla} U, J]X &= \widehat{\nabla}_{JX} U - J\widehat{\nabla}_X U \\ &= \nabla_{JX} U - J\nabla_X U - \tfrac{1}{2}J(\nabla_{JX} J)U + \tfrac{1}{2}J^2(\nabla_X J)U \\ &= (\nabla_U J)X - 2\cdot\tfrac{1}{2}(\nabla_X J)U = 2(\nabla_U J)X, \end{aligned}$$

which is to say $[(\widehat{\nabla} U)^{2,0}, J] = 2(\nabla_U J)$. This identity together with (6.3) implies $(\widehat{\nabla} U)^{2,0} - 2(\nabla U)^{2,0} \in \mathfrak{u}(3) \cap \mathfrak{u}(3)^\perp = \{0\}$, thus $(\widehat{\nabla} U)^{2,0} = 2(\nabla U)^{2,0}$. But $\widehat{\nabla} U = \nabla U - \frac{1}{2}J(\nabla J)U$, so since $J(\nabla J)U \in \mathfrak{u}(3)^\perp$ one has

$$\begin{aligned} \widehat{\nabla} U &= (\widehat{\nabla} U)^{1,1} + (\widehat{\nabla} U)^{2,0} = (\widehat{\nabla} U)^{1,1} + 2(\nabla U)^{2,0} \\ &= (\nabla U)^{1,1} + (\nabla U)^{2,0} - \tfrac{1}{2}J(\nabla J)U, \end{aligned}$$

that is

$$\begin{cases} (\widehat{\nabla} U)^{1,1} = (\nabla U)^{1,1} \\ (\widehat{\nabla} U)^{2,0} = 2(\nabla U)^{2,0} \\ (\nabla U)^{2,0} = -\tfrac{1}{2}J(\nabla J)U. \end{cases} \tag{6.5}$$

Summarising, we obtain the following results.

Proposition 6.1.1. *The vector field U satisfies the following properties:*

1. *The vector fields $\nabla U, \widehat{\nabla} U$ preserve the metric, namely $\nabla U, \widehat{\nabla} U \in \mathfrak{so}(6)$, and for every vector field X we have $[U, JX] = J[U, X]$.*

2. *Let X be in $\mathfrak{so}(6)$ and denote by $X^{1,1}$ and $X^{2,0}$ its $(1,1)$- and $(2,0)$-part in the decomposition $\mathfrak{so}(6) = \mathfrak{u}(3) \oplus \mathfrak{u}(3)^{\perp}$. Then*
$$(\widehat{\nabla} U)^{1,1} = (\nabla U)^{1,1}, \quad (\widehat{\nabla} U)^{2,0} = 2(\nabla U)^{2,0}, \quad (\nabla U)^{2,0} = -\tfrac{1}{2} J(\nabla J)U.$$

3. *The interplay between ∇U and $\sigma, \psi_{\pm}$ is expressed by the formulas*
$$\begin{aligned} 0 &= \nabla\sigma(U, X, Y) + \sigma((\nabla U)X, Y) + \sigma(X, (\nabla U)Y), \\ 0 &= \nabla^2\sigma(U, X, Y, Z) + \underset{X,Y,Z}{\mathfrak{S}} \nabla\sigma((\nabla U)X, Y, Z). \end{aligned}$$

4. *The interplay between $\widehat{\nabla} U$ and $\sigma, \psi_{\pm}$ is expressed by the formulas*
$$\begin{aligned} 0 &= 2\nabla\sigma(U, X, Y) + \sigma((\widehat{\nabla} U)X, Y) + \sigma(X, (\widehat{\nabla} U)Y), \\ 0 &= 2\nabla^2\sigma(U, X, Y, Z) + \underset{X,Y,Z}{\mathfrak{S}} \nabla\sigma((\widehat{\nabla} U)X, Y, Z) = 0. \end{aligned}$$

The same properties hold for V.

6.2 The Hessian

Let us call $\widehat{H}$ the Hessian of ν_M. The associated $(2,0)$ tensor is $\widehat{H}(X, Y) := \widehat{\nabla} d\nu_M$:
$$\widehat{H}(X, Y) = X(d\nu_M(Y)) - d\nu_M(\widehat{\nabla}_X Y). \tag{6.6}$$

Remark 6.2.1. When we compute the Hessian at critical points the last term on the right hand side of (6.6) vanishes, so the choice of the connection does not matter. Since $\widehat{\nabla}$ satisfies more symmetries than ∇, we choose to work with it.

The main point of this section is to study the behaviour of $\widehat{H}$ with respect to J. In Lemma 4.2.1 we proved $24\nu_M = \Delta\nu_M = -\mathrm{Tr}\widehat{H}$, so one cannot expect $\widehat{H}$ to be of type $(2,0)$: if this was the case, then $\widehat{H}J = -J\widehat{H}$, so taking some $\mathrm{U}(3)$-basis $\{E_i, JE_i\}, i = 1, 2, 3$, of the tangent space at each point we would have
$$\mathrm{Tr}(\widehat{H}) = \sum_{i=1}^{3} g(\widehat{H}E_i, E_i) + g(\widehat{H}JE_i, JE_i) = \sum_{i=1}^{3} g(\widehat{H}E_i, E_i) - g(J\widehat{H}E_i, JE_i) = 0,$$
which is a contradiction. On the other hand, if $\widehat{H}$ had type $(1,1)$, then its eigenvectors would come in pairs: in fact if λ was an eigenvalue of $\widehat{H}$ with eigenvector X one would have
$$\widehat{H}JX = J\widehat{H}X = J(\lambda X) = \lambda JX,$$
and JX would be a second eigenvector with eigenvalue λ. An explicit computation will tell us the answer is more complicated than this.

Recall that $dv_M = 3\psi_+(U, V, \cdot\,)$. Then $\widehat{\nabla}\psi_+ = 0$ implies

$$\begin{aligned} X(dv_M(Y)) - dv_M(\widehat{\nabla}_X Y) &= 3\big(X(\psi_+(U,V,Y)) - \psi_+(U,V,\widehat{\nabla}_X Y)\big) \\ &= 3\big(\psi_+(\widehat{\nabla}_X U, V, Y) + \psi_+(U, \widehat{\nabla}_X V, Y)\big). \end{aligned}$$

Since $\widehat{\nabla}_A B - \widehat{\nabla}_B A - [A,B] = J(\nabla_B J)A$, $[JX,U] = J[X,U]$ by (6.2), and $\widehat{\nabla}$ preserves J

$$\begin{aligned} \widehat{H}(JX,JY) &= 3\big(\psi_+(\widehat{\nabla}_{JX} U, V, JY) + \psi_+(U, \widehat{\nabla}_{JX} V, JY)\big) \\ &= 3\big(\psi_+(\widehat{\nabla}_U JX, V, JY) + \psi_+([JX,U], V, JY) + \psi_+(J(\nabla_U J)JX, V, JY) \\ &\qquad + \psi_+(U, \widehat{\nabla}_V JX, JY) + \psi_+(U, [JX,V], JY) + \psi_+(U, J(\nabla_V J)JX, JY)\big) \\ &= 3\big(\psi_+(J\widehat{\nabla}_U X, V, JY) + \psi_+(J[X,U], V, JY) + \psi_+(J(\nabla_U J)JX, V, JY) \\ &\qquad + \psi_+(U, J\widehat{\nabla}_V X, JY) + \psi_+(U, J[X,V], JY) + \psi_+(U, J(\nabla_V J)JX, JY)\big) \\ &= 3\big(-\psi_+(\widehat{\nabla}_U X, V, Y) - \psi_+([X,U], V, Y) + \psi_+((\nabla_U J)X, V, JY) \\ &\qquad - \psi_+(U, \widehat{\nabla}_V X, Y) - \psi_+(U, [X,V], Y) + \psi_+(U, (\nabla_V J)X, JY)\big). \end{aligned}$$

Applying once again $\widehat{\nabla}_A B - \widehat{\nabla}_B A - [A,B] = J(\nabla_B J)A$:

$$\begin{aligned} \widehat{H}(JX,JY) &= 3\big(-\psi_+(\widehat{\nabla}_X U, V, Y) - \psi_+([U,X], V, Y) - \psi_+(J(\nabla_X J)U, V, Y) \\ &\qquad - \psi_+(U, \widehat{\nabla}_X V, Y) - \psi_+(U, [V,X], Y) - \psi_+(U, J(\nabla_X J)V, Y) \\ &\qquad - \psi_+([X,U], V, Y) - \psi_+(U, [X,V], Y) + \psi_+((\nabla_U J)X, V, JY) \\ &\qquad + \psi_+(U, (\nabla_V J)X, JY)\big) \\ &= -\widehat{H}(X,Y) + 6\big(\psi_+((\nabla_U J)X, V, JY) + \psi_+(U, (\nabla_V J)X, JY)\big). \end{aligned}$$

Assume we are at a critical point p, so that V_p is in the span of U_p, JU_p. We then choose a basis $\{E_i\}, i = 1, \ldots, 6$ of T_pM in such a way that $U_p = g_{UU}^{1/2} E_1$, $V_p = g_{UU}^{-1/2}(g_{UV}E_1 + v_M E_2)$, and $E_2 = JE_1, E_4 = JE_3, E_6 = JE_5$. Recall that the expression of ψ_+ is given pointwise by $\psi_+ = e^{135} - e^{245} - e^{146} - e^{236}$, in analogy with (1.31). We can compute the term $\psi_+((\nabla_U J)X, V, JY) + \psi_+(U, (\nabla_V J)X, JY)$ appearing in the expression found at p:

$$\begin{aligned} &\psi_+((\nabla_U J)X, V, JY) + \psi_+(U, (\nabla_V J)X, JY) \\ &\quad = g_{UU}^{-1}\big(\psi_+((\nabla_U J)X, g_{UV}U + v_M JU, JY) + \psi_+(U, (\nabla_{g_{UV}U + v_M JU} J)X, JY)\big) \\ &\quad = g_{UU}^{-1}\big(g_{UV}\psi_+((\nabla_U J)X, U, JY) - v_M\psi_+((\nabla_U J)X, U, Y) \\ &\qquad\quad + g_{UV}\psi_+(U, (\nabla_U J)X, JY) + v_M\psi_+(U, (\nabla_U J)X, Y)\big) \\ &\quad = 2g_{UU}^{-1} v_M \psi_+(U, (\nabla_U J)X, Y). \end{aligned}$$

Write $(\nabla_U J)X$ as a combination of the vectors E_i so that

$$\begin{aligned} 2g_{UU}^{-1} v_M \psi_+(U, (\nabla_U J)X, Y) &= 2v_M \sum_{i=1}^{6} \psi_+(E_1, g((\nabla_{E_1} J)X, E_i)E_i, Y) \\ &= -2v_M \sum_{i=1}^{6} \psi_+(E_1, E_i, X)\psi_+(E_1, E_i, Y) \\ &= -2v_M(e^3 \otimes e^3 + \ldots + e^6 \otimes e^6)(X,Y) \\ &= -2v_M g^{\perp}(X,Y), \end{aligned}$$

where $g^{\perp}$ is the metric g restricted to the orthogonal complement of the span of E_1 and E_2 in the tangent space. We can formulate the result as

Proposition 6.2.2. *At critical points the relation between J and $\widehat{H}$ is given by the formula*

$$\widehat{H}(JX, JY) = -\widehat{H}(X, Y) - 12\nu_M g^{\perp}(X, Y),$$

where $g^{\perp}$ is the metric g restricted to the orthogonal complement of the span of E_1, E_2.

Example 6.2.3. To give an example of explicit Hessian, we restrict our attention to the six-sphere $\mathbb{S}^6 \subset \mathbb{R}^7 \cong \mathbb{C}^3 \oplus \mathbb{R}$. Recall that in this case the multi-moment map was obtained in (3.4) and at the point $p = (x^1, \dots, x^7) \in \mathbb{S}^6$ has the form

$$\nu_{\mathbb{S}^6}(p) = 3\big(x^1(x^4x^5 - x^2x^3) - x^6(x^3x^5 + x^2x^4)\big).$$

In the general set-up described in Section 2.2 we introduced the unit normal N, which in terms of the coordinate vector fields is $\sum_k x^k\partial_k$, and $\sum_k (x^k)^2 = 1$. One may easily compute $N\nu_M$ and find

$$\begin{aligned} N\nu_M &= 3\sum_k x^k\partial_k\big(x^1x^4x^5 - x^1x^2x^3 - x^6x^3x^5 - x^6x^2x^4\big) \\ &= 9\big(x^1x^4x^5 - x^1x^2x^3 - x^6x^3x^5 - x^6x^2x^4\big) = 3\nu_M. \end{aligned}$$

Let us denote by $\overline{\nabla}$ the flat connection on $\mathbb{R}^7$ and by ∇ the Levi-Civita connection on $\mathbb{S}^6$. Consider two non-zero tangent vectors $X_i = \partial_i - x^iN, X_j = \partial_j - x^jN$ at a point. Then $H(X_i, X_j)$ is nothing but $X_i(X_j\nu_M) - (\nabla_{X_i}X_j)\nu_M$. To our purposes there is no need to use the Hermitian connection for this computation. In the following we denote by δ_{ij} the Kronecker delta and by π the pushforward of the projection from $\mathbb{R}^7 \setminus \{0\}$ to $\mathbb{S}^6$. Recall that since $\overline{\nabla}$ is flat, then $\overline{\nabla}_{\partial_i}\partial_j = 0$. We then find

$$\begin{aligned} X_i(X_j\nu_M) &= X_i(\partial_j\nu_M - x_jN\nu_M) \\ &= X_i(\partial_j\nu_M - 3x_j\nu_M) \\ &= \partial_i\partial_j\nu_M - 3\delta_{ij}\nu_M - 3x^j\partial_i\nu_M - x^i\sum_k x^k\partial_k\partial_j\nu_M + 12x^ix^j\nu_M, \end{aligned}$$

whereas the second bit can be computed in general as

$$\begin{aligned} \nabla_{X_i}X_j &= \pi(\overline{\nabla}_{X_i}X_j) \\ &= \pi(\overline{\nabla}_{\partial_i - x^iN}(\partial_j - x^jN)) \\ &= \pi(-x^i\overline{\nabla}_N\partial_j - \overline{\nabla}_{\partial_i}x^jN + x^i\overline{\nabla}_N x^jN) \\ &= \pi(-\delta_{ij}N - x^j\overline{\nabla}_{\partial_i}N + x^iN(x^j)N + x^ix^j\overline{\nabla}_N N) \\ &= \pi(-x^j\partial_i + x^ix^jN + x^ix^jN) \\ &= -\pi(x^j\partial_i) = -x^jX_i. \end{aligned}$$

Observe that $3\partial_j\nu_M = \partial_j(N\nu_M) = \sum_k \partial_j(x^k\partial_k\nu_M) = \sum_k \delta_j^k\partial_k\nu_M + \sum_k x^k\partial_j\partial_k\nu_M$, whence $-x^i\sum_k x^k\partial_j\partial_k\nu_M = -2x^i\partial_j\nu_M$. Therefore the expression of the Hessian is given by

$$H(X_i, X_j) = \partial_i\partial_j\nu_M - 3\delta_{ij}\nu_M - 2(x^j\partial_i + x^i\partial_j)\nu_M + 9x^ix^j\nu_M.$$

6.3 The multi-moment map as Morse function

The idea of this section is to describe the structure of critical sets of ν_M from a topological point of view, in the same spirit as in the introduction of [Mil69]. We restrict our attention to the *equivariantly non-degenerate* ones, i.e. those orbits where the Hessian is non-degenerate. This is an essential assumption: in fact our results follow from Morse Lemma—recalled below—which gives information on non-degenerate critical points of a smooth function. In our case, the latter will be the multi-moment map.

The following definition is just about simple terminology.

Definition 6.3.1. Let H be a symmetric bilinear form over a vector space V. The *index* of H is defined to be the maximal dimension of a subspace of V on which H is negative definite.

In our set-up one should think of the bilinear form H as the Hessian of the multi-moment map. The result we apply is the following theorem from [Mil69].

Theorem 6.3.2 (Morse Lemma)**.** *Let p be a non-degenerate critical point for a smooth function f. Then there is a local coordinate system $(x^1, \dots, x^n)$ in a neighbourhood U of p, centered at p, and such that the identity*

$$f = f(p) - (x^1)^2 - \dots - (x^\ell)^2 + (x^{\ell+1})^2 + \dots + (x^n)^2$$

holds throughout U, where ℓ is the index of f at p.

Theorem 6.3.3. *Let $p \in \nu_M^{-1}(s), s \neq 0$ be a non-degenerate critical point that is a local maximum or minimum. Assume the stabiliser of p in T^2 is a finite group H. Then for t in a neighbourhood of s there is a diffeomorphism*

$$\nu_M^{-1}(t) \cong T^2 \times_H S^3.$$

Proof. By Theorem 4.2.4 a critical point $p \in \nu_M^{-1}(s)$ has an open neighbourhood equivariantly diffeomorphic to $U = T^2 \times_H V$, where V is the normal bundle at p, namely a copy of $\mathbb{R}^4$. The multi-moment map ν_M is T^2-invariant, thus its restriction to V is nothing but a smooth function with non-degenerate critical point at 0. By Morse Lemma, Theorem 6.3.2, there is a local coordinate system (x^1, x^2, x^3, x^4) in V such that

$$\nu_{M|V} = s + \varepsilon_1(x^1)^2 + \varepsilon_2(x^2)^2 + \varepsilon_3(x^3)^2 + \varepsilon_4(x^4)^2,$$

where $\varepsilon_k \in \{\pm 1\}$. So $\nu_M^{-1}(t) \cong T^2 \times \nu_{M|V}^{-1}(t)$. In the particular case where s is a maximum, then the index of the Hessian of ν_M at p is 4 and $t = s - (x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2$, or, which is the same,

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = s - t > 0.$$

Thus $\nu_{M|V}^{-1}(t)$ is homeomorphic to a three-sphere S^3. When s is a minimum, then the index of the Hessian at p is 0, so

$$\nu_{M|V} = s + (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2.$$

On the level set corresponding to $t > s$ then

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = t - s > 0,$$

and again we obtain $\nu_{M|V}^{-1}(t) \cong S^3$. $\square$

Corollary 6.3.4. *Let $(M, \sigma, \psi_\pm)$ be nearly Kähler with a T^2-symmetry. If the T^2-action is free on M then the multi-moment map ν_M cannot have only two non-degenerate critical sets.*

Proof. Suppose the multi-moment map $\nu_M \colon M \to [a,b]$ has only two non-degenerate critical sets. Then they must correspond to two copies of T^2 where ν_M attains its maximum and minimum, so in particular they are connected and non-degenerate. Consider $U_{\min}$ and $U_{\max}$ defined as

$$U_{\min} := \nu_M^{-1}([a, a+\varepsilon)), \quad U_{\max} := \nu_M^{-1}((b-\varepsilon, b]),$$

where ε is positive and such that $a + \varepsilon > b - \varepsilon$. Therefore $M = U_{\min} \cup U_{\max}$. Since the action is free both $U_{\min}$ and $U_{\max}$ are equivariantly diffeomorphic to $T^2 \times \mathbb{R}^4$ by Theorem 4.2.4. Also, $U_{\min} \cap U_{\max} = \nu_M^{-1}((b-\varepsilon, a+\varepsilon))$, and is diffeomorphic to $(b-\varepsilon, a+\varepsilon) \times T^2 \times \mathbb{S}^3$ by Theorem 6.3.3. So we have the long exact Mayer–Vietoris sequence in de Rham cohomology

$$\begin{aligned} H^0(M) \longrightarrow H^0(U_{\min}) \oplus H^0(U_{\max}) \longrightarrow H^0(U_{\min} \cap U_{\max}) \longrightarrow \\ \longrightarrow H^1(M) \longrightarrow H^1(U_{\min}) \oplus H^1(U_{\max}) \longrightarrow H^1(U_{\min} \cap U_{\max}) \longrightarrow \dots \end{aligned}$$

Now, M is connected as well as $U_{\min}, U_{\max}$, and $U_{\max} \cap U_{\min}$, so $H^0(M) = H^0(U_{\min}) = H^0(U_{\max}) = H^0(U_{\min} \cap U_{\max}) = \mathbb{R}$. Further, since M is connected and with finite fundamental group, $H^1(M) = 0$, and the homotopic equivalences $T^2 \sim T^2 \times \mathbb{R}^4$ and $(b-\varepsilon, a+\varepsilon) \times T^2 \times \mathbb{S}^3 \sim T^2 \times \mathbb{S}^3$ generate the isomorphisms $H^1(U_{\max}) \cong \mathbb{R}^2 \cong H^1(U_{\min})$ and $H^1(U_{\min} \cap U_{\max}) \cong \mathbb{R}^2$—the latter holds by Künneth formula. Therefore our sequence has the form

$$\mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \longrightarrow 0 \longrightarrow \mathbb{R}^2 \oplus \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \longrightarrow \dots$$

and there is an injective homomorphism $\mathbb{R}^4 \to \mathbb{R}^2$, which is a contradiction. □

6.4 Conclusions

Let us summarise our results. We described properties of multi-moment maps on nearly Kähler six-manifolds with a torus symmetry and illustrated how these maps can be a powerful tool in constructing explicit nearly Kähler metrics. In three of the homogeneous cases, namely $\mathbb{S}^6, F_{1,2}(\mathbb{C}^3), \mathbb{CP}^3$, we found out that our multi-moment maps only have two critical orbits, which necessarily are the maximum and the minimum. The corresponding values are symmetric with respect to the origin (Sections 3.2–3.4). In the case of $\mathbb{S}^3 \times \mathbb{S}^3$ different choices of a two-torus inside the three-torus acting yield different outcomes: orbits of saddle points arise, critical sets need not be two-dimensional, and maximum and minimum need not be symmetric (Section 3.5). Incidentally, the latter tells us we cannot assert that the range of ν_M has the general form $[-a, a]$ in Proposition 4.2.2.

The information obtained on critical sets is partly recovered and clarified when we look for points fixed by some subgroup of the action. The nature of multi-moment maps links these two aspects of the story. In the particular case of the six-sphere this correspondence is evident (see Section 5.1): the graph representing points with non-trivial stabiliser is given by two points and three edges, which correspond respectively to the two poles and the three two-spheres found after computing critical points in Section 3.2. In the other cases purely algebraic calculations of critical points perhaps hide the geometric structure of these two-dimensional submanifolds. Nonetheless, an application of the Equivariant Tubular Neighbourhood Theorem gives general information on the configuration of points with non-trivial stabiliser, depending on the dimension of the isotropy groups (Theorem 5.5.1). When looking at these points in the quotient M/T^2 we find graphs as those computed in the homogeneous cases (cf. Remark 5.5.3).

Just the non-emptiness of the graph tells us the two-torus action cannot be free on the whole manifold: this is in fact the case for $\mathbb{S}^6, F_{1,2}(\mathbb{C}^3), \mathbb{CP}^3$. A particular choice of a two-torus in the three-torus acting on $\mathbb{S}^3 \times \mathbb{S}^3$ yields an empty graph, so this is the only homogeneous example where the action can be free (see the remarks at the end of Section 5.4). Regardless of these special cases, the two-torus action is always free on the level sets of the multi-moment map corresponding to regular values, as it is stated in Proposition 4.2.6. This is what allows us to perform the T^2-reduction and then construct nearly Kähler six-manifolds from three-dimensional spaces (Chapter 4, in particular Section 4.5). Part of the future projects will be to apply this theory on the homogeneous examples, find their T^2-reductions and try to construct explicit nearly Kähler (necessarily non-homogeneous) structures through the inverse process.

The material contained in this last chapter is yet to be completed, and is an attempt to collect information on topology of nearly Kähler six-manifolds with torus symmetry. In particular, one possible direction could be to get closer to an answer to the open conjecture stating that $\mathbb{S}^3 \times \mathbb{S}^3$ is the only nearly Kähler six-manifold with a T^3-symmetry. Being far from a conclusive statement, we work out the topology of non-degenerate critical sets of points of local maximum and minimum on nearly Kähler six-manifolds with two-torus symmetry. The answer obtained implies that if the multi-moment map has only two non-degenerate critical sets then the action cannot be free, which is in line with the study of the special cases summarised above.

6.5 Further developments

Besides the points highlighted in the previous section, there are several possible directions to extend the work contained in this thesis. The set-up of two of them is sketched here below, without too many technical details. I take the opportunity to thank Lorenzo Foscolo, Andrei Moroianu, and Andrew Swann, who pointed me to these projects at different stages.

The first one concerns nearly Kähler structures on six-dimensional sine-cones over Sasaki–Einstein five-manifolds. Let Σ be a five-dimensional smooth manifold equipped with an SU(2)-structure given by $(\eta, \omega_1, \omega_2, \omega_3)$, where

1. η is a nowhere vanishing one-form (dual of a vector field N) splitting each tangent space as $T_x\Sigma = \mathbb{R} \oplus \ker \eta_x$,
2. ω_1 is a non-degenerate two-form on $\ker \eta$ such that $\eta \wedge \omega_1^2 \neq 0$,
3. ω_2, ω_3 are two-forms such that $\omega_i \wedge \omega_j = \delta_{ij}\omega_1^2$, where δ_{ij} is the Kronecker delta.

On $(\Sigma, \eta, \omega_1, \omega_2, \omega_3)$ there exist a unique metric g_Σ and an orientation compatible with the SU(2)-structure.

On the Riemannian cone $C(\Sigma) := (\Sigma \times \mathbb{R}_{>0}, g_{C(\Sigma)} := dr^{\otimes 2} + r^2 g_\Sigma)$ there is an SU(3)-structure (ω, Ω) given by

$$\omega = rdr \wedge \eta + r^2\omega_1, \qquad \Omega = (dr + ir\eta) \wedge r^2(\omega_2 + i\omega_3). \tag{6.7}$$

The volume form Ω splits into real and imaginary parts

$$\operatorname{Re}\Omega = r^2 dr \wedge \omega_2 - r^3 \eta \wedge \omega_3, \qquad \operatorname{Im}\Omega = r^2 dr \wedge \omega_3 + r^3 \eta \wedge \omega_2,$$

and the identity $\omega_i \wedge \omega_j = \delta_{ij}\omega_1^2$ implies

$$\omega \wedge \Omega = 0, \qquad \operatorname{Re}\Omega \wedge \operatorname{Im}\Omega = \tfrac{2}{3}\omega^3.$$

Therefore, (ω, Ω) defines an $\mathrm{SU}(3)$-structure (cf. (1.33) and (1.34)).

Definition 6.5.1. The $\mathrm{SU}(2)$-structure $(\eta, \omega_1, \omega_2, \omega_3)$ on Σ is called *Sasaki–Einstein* if

$$d\eta = 2\omega_1, \qquad d\omega_2 = -3\eta \wedge \omega_3, \qquad d\omega_3 = 3\eta \wedge \omega_2.$$

Definition 6.5.2. The $\mathrm{SU}(3)$-structure (ω, Ω) on $C(\Sigma)$ is called *Calabi–Yau* if

$$d\omega = 0 = d\Omega.$$

Differentiating (6.7) we obtain

$$\begin{aligned} d\omega &= rdr \wedge (-d\eta + 2\omega_1) + r^2 d\omega_1, \\ d\Omega &= r^2 dr \wedge (-d\omega_2 - 3\eta \wedge \omega_3) - r^3 d(\eta \wedge \omega_3) \\ &\quad + i(r^2 dr \wedge (-d\omega_3 + 3\eta \wedge \omega_2) + r^3 d(\eta \wedge \omega_2)), \end{aligned}$$

from which it is clear that $C(\Sigma)$ is Calabi–Yau if and only if Σ is Sasaki–Einstein.

We now introduce the *sine-cone* over Σ: consider the Riemannian product

$$SC(\Sigma) := ((0, \pi) \times \Sigma, g_{SC} = ds^{\otimes 2} + \sin^2 s\, g_\Sigma),$$

where s is a parameter in $(0, \pi)$. The Riemannian cone over it is

$$C(SC(\Sigma)) := (\mathbb{R}_{>0} \times (0, \pi) \times \Sigma, g = d\rho^{\otimes 2} + \rho^2 (ds^{\otimes 2} + \sin^2 s\, g_\Sigma)).$$

The cylinder over the Calabi–Yau cone $C(\Sigma)$ defined by

$$(\mathbb{R} \times \mathbb{R}_{>0} \times \Sigma, g = dt^{\otimes 2} + dr^{\otimes 2} + r^2 g_\Sigma)$$

is diffeomorphic and isometric to $C(SC(\Sigma))$.

We know that a six-dimensional manifold is nearly Kähler if and only if the cone constructed over it has holonomy contained in G_2. Hence, the sine-cone $SC(\Sigma)$ is nearly Kähler if and only if the cylinder $(\mathbb{R} \times C(\Sigma), g = dt^{\otimes 2} + g_{C(\Sigma)})$ has a parallel G_2-structure. This is in fact given by the two closed three-forms

$$\varphi = dt \wedge \omega + \operatorname{Re}\Omega, \qquad *\varphi = -dt \wedge \operatorname{Im}\Omega + \tfrac{1}{2}\omega^2,$$

and the nearly Kähler structure $(\sigma, \psi_\pm)$ on $SC(\Sigma)$ turns out to be

$$\begin{aligned} \sigma &:= \sin s\, ds \wedge \eta + \sin^2 s \cos s\, \omega_1 + \sin^3 s\, \omega_2, \\ \psi_+ &:= -\sin^3 s\, ds \wedge \omega_1 + \sin^2 s \cos s\, ds \wedge \omega_2 - \sin^3 s\, \eta \wedge \omega_3, \\ \psi_- &:= \sin^2 s\, ds \wedge \omega_3 + \sin^3 s \cos s\, \eta \wedge \omega_2 - \sin^4 s\, \eta \wedge \omega_1. \end{aligned}$$

One may now assume that a two-torus T^2 acts effectively on Σ preserving the Sasaki–Einstein structure $(\eta, \omega_1, \omega_2, \omega_3)$, and this yields a two-torus symmetry on the sine-cone over Σ. Let U, V be the Killing vector fields generating the action. Note that $\omega_1(U, V)$ is proportional to $d\eta(U, V) = U(\eta(V)) - V(\eta(U)) = \mathcal{L}_U(V \lrcorner \eta) - \mathcal{L}_V(U \lrcorner \eta) = 0$ since U and V commute, hence vanishes itself. So one gets a multi-moment map on the sine-cone given by

$$\nu_M := \sigma(U, V) = \omega_2(U, V) \sin^3 s.$$

The goal will be to apply the construction as in Chapter 4 to obtain a structure theory and new concrete examples.

Another possible direction regards G_2 geometry. In Section 2.1 we saw how the group G_2 may be defined as the stabilizer of a three-form φ_0 on $\mathbb{R}^7$ in the special orthogonal group $SO(7)$. Equivalently, one may also define $G_2 \hookrightarrow \mathrm{Spin}(7)$ as the stabilizer of a non-zero spinor ψ_0 in an eight-dimensional, real representation, the *spin representation*. Incidentally, this close relation with spin geometry explains the importance of G_2 in the world of particles and supersymmetry (see e.g. [Str86; Agr08]).

Definition 6.5.3. A G_2-*structure* on a seven-dimensional Riemannian manifold (M, g) is a three-form φ pointwise equivalent to φ_0. The manifold (M, φ) is then called G_2-*manifold*.

Every G_2-manifold carries a spinor field ψ canonically induced by φ_0 and ψ_0. Among the various classes of compact spaces equipped with a G_2-structure, we focus on *nearly parallel* G_2-manifolds.

Definition 6.5.4. A G_2-manifold (M, φ) is *nearly parallel* if and only if $\nabla\varphi = \lambda{*}\varphi$ for some non-zero constant λ, and $*$ the Hodge star operator.

The condition $\nabla\varphi = \lambda{*}\varphi$ is equivalent to saying that the corresponding spinor ψ is Killing, i.e. $\nabla_X\psi = \mu X \cdot \psi$, with X a vector field and μ a real constant, hence M is Einstein. Examples of nearly parallel G_2-manifolds exist: remarkable ones are e.g. the Aloff–Wallach spaces, i.e. compact, homogeneous, seven-manifolds of the form $SU(3)/U(1)$. Nonetheless, constructing explicit G_2-metrics remains a non-trivial challenge.

The main point of this project is to study geometry and topology of non-homogeneous nearly parallel G_2-manifolds (M, φ) having a three-torus symmetry, and find new, explicit examples. We assume a three-torus T^3 acts effectively on our space preserving the G_2-structure. A T^3-symmetry induces Killing vector fields U, V, W, and the three-form φ can be used to generate a T^3-invariant real valued function

$$\nu_M := \varphi(U, V, W),$$

which in out set-up is a multi-moment map. Again, this is the starting point to apply the machinery developed in Chapter 4, study the structure theory and find new examples.

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