On some algebraic aspects of special Kähler metrics

A thesis submitted for the degree of Ph.D. in mathematics at the university of Aarhus

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0.1 Summary

The present thesis represents my work as a Ph.D. fellow in the geometry group at the university of Aarhus. The overarching theme of my Ph.D. has been to study how certain algebraic structures and conditions can be useful for studying metric structures near the boundary or singularities of moduli spaces. In the first part, I, of the thesis, this mainly manifests itself in explicit work with the Hilbert modular cusp singularities. This part contains mostly classical material, and the most original contribution of the author will be the perspective on the possible relationship between the metric geometry and the non-Archimedean geometry found in proposition 1.2.13 and section 1.3, where we modify an ansatz by Collins and Li [CL22].

In recent years, the notion of K-stability has played an important role in forming moduli spaces of projective varieties with good properties. By a famous result of Chen, Donaldson and Sun [CDS14a],[CDS14b],[CDS14c], in the smooth fano case, K-stability is equivalent to the existence of a Kähler Einstein metric on the variety. The stability condition is algebraic and measures, roughly speaking, the limiting behaviour along algebraic paths in the space of compatible Kähler potentials. The second part, II, contains a review of some algebraic aspects of K-stability and original work undertaken with Lars Martin Sektnan from 3.2 and onwards. We consider an analogous stability criterion for families of varities, dubbed fibrational stability by Dervan and Sektnan in [DS21a]. It restricts to K-stability when the family is over a point and, in this case, there is a valuative obstruction for K-stability. We introduce a special class of valuations called horizontal and show that these provide a valuative obstruction for fibration stability. Finally we illustrate this obstruction in the case where the fibration is a projective bundle and show that exceptional divisors obtained by blowing up subbundles give a relation with slope stability of the bundle.

To the best of my ability I have named any results and definitions from the literature with references to the original article. Results which are not named are either "standard" or original. In order to make the exposition more self contained, I have included some proofs of results from the literature with more details. In accordance with GSNS rules, parts of I were also used in the progress report for the qualifying examination.

Resumé

Denne afhandling repræsenterer mit arbejde som Ph.D fellow i geometrigruppen ved Aarhus Universitet. Det overordnede tema har været at studere hvordan bestemte algebraiske strukturer og betingelser kan bruges til at studere metriske strukturer tæt på randen eller nær singulariter af moduli rum. I den første del af afhandlingen, I, kommer dette primært til udtryk ved explicit arbejde med Hilbert modular cusp singulariteterne. Denne del indeholder fortrinsvis klassiske resultater og forfatterens mest originale bidrag er således perspektivet på en mulig relation mellem den metriske geometri og den ikke-Arkimediske geometri som forefindes i proposition 1.2.13 og 1.3 hvor vi modificerer en ansatz af Collins og Li [CL22].

I de senere år, har begrebet K-stabilitet spillet en vigtig rolle i bestræbelserne på at konstruere moduli rum af projektive varieteter med særligt gode egenskaber. Et berømt resultat af Chen, Donaldson og Sun [CDS14a],[CDS14b],[CDS14c] siger at i det ikke singulære Fano tilfælde, så er K-stabilitet ækvivalent til eksistensen af en Kähler Einstein metrik på varieteten. Stabilitetsbetingelsen er algebraisk og måler groft sagt opførslen i grænsen langs stier i rummet af kompatible Kähler potentialer. Afhandlingens anden del, II, indeholder en gennemgang af nogle algebraiske aspekter af K-stabilitet og indeholder originalt arbejde foretaget med Lars Martin Sektnan fra afsnit 3.2 og frem. Vi betragter en analog stabilitetsbetingelse for familier af varieteter, kaldet fibrationsstabilietet som blev introduceret af Dervan og Sektnan i [DS21a] og reducerer til K-stabilitet når familien er over et punkt. Der er en obstruktion for K-stabilitet givet ved en invariant for værdisætninger associeret til særlige divisorer. Vi introducerer en klasse af værdisætninger kaldet horisontale og viser at de giver en analog obstruktion for fibrations stabilitet. Slutteligt illustrerer vi denne obstruktion i tilfældet hvor fibrationen er et projektivt bundt, og viser at den for eksseptionelle divisorer i et blow up af et delbundt, er relateret til hældningsstabiliteten for bundtet.

Jeg har efter bedste evne navngivet resultater of definitioner fra litteraturen med referencer til ophavsartiklen. Resultater uden navn er enten "standard" eller originale. For at gøre fremstillingen lettere at følge har jeg inkluderet beviser for resultater i litteraturen med flere detaljer. I overensstemmelse med GSNS regler er dele af I brugt i min rapport i forbindelse med min kvalifikationseksamen.

0.2 Acknowledgements

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Part I

On certain cusp singularities

1 On certain cusp singularities

Introduction

Negative Kähler Einstein metrics defined away from a singular set or divisor have been the focus of a lot of research over the years. Following the resolution of the Calabi conjecture by Yau [Yau78], a lot of attention was devoted to the study of complete metrics on non-compact spaces and their asymptotics at infinity, for instance [CY80],[Kob84],[TY87]. More recently, metrics with cone-type singularities along a divisor in projective space were studied and used in the proof of the celebrated Yau-Tian-Donaldson conjecture. A common theme in this work has been to try to find metrics $\omega = \frac{i}{2\pi}\partial\overline{\partial}\,\phi$ on a compact Kähler manifold X with $K_X + D$ ample, which is Einstein away from a simple normal crossing divisor $D = \sum d_i D_i, d_i \in [0,1]$ and with a prescribed behaviour of its local potential ϕ near the support of D. This situation arises, for example, on log smooth pairs (X, D) with ample log canonical class in the sense of the minimal model programme (see [KK13],[KM98]). If one requires the metric to extend over D as a current, then the state of the art is the results of [GW16], [BG14] who show that there is a unique such metric, which is Kähler Einstein away from D. Moreover, they show that the metric is locally holomorphically bilipschitz equivalent to a model metric which splits as a product metric with cone and cusp singularities dependent on the coefficients d_i.

In a more local direction, [DFS21], Datar, Fu and Song study the negative Einstein Dirichlet problem near certain isolated log canonical singularities, which are not log terminal (purely log canonical), and they show that any two potentials giving complete metrics must be asymptotically close as one approaches the singularity in a strong sense. Building on this, Fu, Hein and Jiang [FHJ21] gave a description of the asymptotics of the Kähler Einstein metrics obtained in [DFS21] in the case of cones over elliptic curves. The results suggest, that the asymptotics of complete Kähler Einstein metrics defined near isolated, purely log canonical singularities are in some sense canonical. It is then a natural question to ask, if the asymptotic behaviour of a complete Kähler Einstein metric can be related to some local algebraic structure associated to the log canonical singularity. To a resolution of a singularity, one can associate a dual complex, which captures how the exceptional divisors of the resolution intersect. Although dependent on the particular resolution, a first venture would be to try to establish a correspondence between the dual complex and the asymptotic behaviour of the Kähler Einstein metric. As a first step in this programme, we consider the Hilbert modular cusps in dimension 2, equipped with a special choice of negative Kähler Einstein metric. In this case, our main result is the observation that one can relate the dual complex to the non-collapsed part of the link via a geometrically induced homeomorphism 1.2.13. This is done by considering the limit of a special class of analytic geodesics, starting from points in the link arising from scaling the cone underlying the construction of the cusp singularity 1.2.3. Since this result relies on an essentially complete description of the singularity and its resolution, it appears difficult to make generalizeations. Finally, we make a remark that the Hilbert modular cusps fit in a modification of a new ansatz by Collins and Li [CL22] regarding Calabi-Yau metrics near the zero section of product of line bundles, which under a certain proportionality assumption produce negative Einstein metrics 1.3.

In this part of the thesis, we review the construction of certain family of cuspidal singularities due to Tsuchihashi, which generalizes the Hilbert-Modular cusps 1.1. Afterwards, we specialize the discussion to metric aspects of the Hilbert modular cusps 1.2, and we introduce the concept of the dual complex associated to an exceptional divisor of a good resolution. We also define a map from a neighbourhood of the exceptional divisor to the dual complex 1.2. Then, we consider the case of Hilbert modular cusps in dimension 2 1.2. Finally, we make a remark on a non-Archimedean ansatz 1.3.

1.1 Tsuchihashi cusps

In this section, we sketch the general construction of Tsuchihashi cusp singularities [Tsu83] from "permissible pairs" (C, Γ), which form a type of isolated normal singularities equipped with a natural metric, whose geometry models the cusp. The construction generalizes the Hilbert modular cusps, and their minimal toric resolution. Imperative to the construction of these cusps are certain characteristic functions $\phi\colon C \to \mathbb{R}$, which enable the contraction of the boundary at infinity to a point.

In order to fix notation, we let

- $N \cong \mathbb{Z}^r$ be a lattice and $M = Hom_{\mathbb{Z}}(N, \mathbb{Z})$ its dual.
- $Sl(N) \subset Aut_{\mathbb{Z}}(N)$ be the index 2 subgroup of elements with determinant 1.
- Set $N_{\mathbb{R}} = N \otimes_{\mathbb{R}} \mathbb{R}$ so that we can identify $Sl(N)$ with its image in $Gl(N_{\mathbb{R}})$.
- $ray(N_{\mathbb{R}}) = (N_{\mathbb{R}} \setminus \{0\}/\mathbb{R}_{>0}) \overset{homeo}{\simeq} S^{r-1}$.

Definition 1.1.1. We say a pair (C, Γ) is Tsuchihashi *cusp data* for N, if

- $C \subset N_{\mathbb{R}}$ is an open strongly convex cone, i.e., $\overline{C} \cap \{-\overline{C}\} = \{0\}$ of maximal dimension.
- $\Gamma \subset Stab(C, N_{\mathbb{R}}) = \{g \in Gl(N_{\mathbb{R}}) | gC = C\}$ is a subgroup such that the induced action on $ray(C_{\mathbb{R}})$ is effective, properly discontinuous and $ray(C_{\mathbb{R}})/\Gamma$ is compact.

Given Tsuchihashi cusp data (C, Γ) for N, then there is a dual cusp data. Indeed, if we equip M with the dual Γ- action, i.e., the action such that $(g \cdot m)(x) = m(g^{-1}x)$, then we see that the dual cone,

$$C' = \{m \in M | \langle m, c \rangle > 0, \forall c \in C \setminus \{0\}\},$$

has an induced Γ-action and that (C', Γ) is Tsuschihashi cusp data for M. In particular, when C is self dual, i.e., when $C' \cong C$ under the identification $N_{\mathbb{R}} \to M_{\mathbb{R}}$, then we have an involution on the set of Tsuchihashi cusp data. While this symmetry is certainly nice, we shall not pursue it further.

The setup

Given Tsuchihashi cusp data (C, Γ) for N, one can consider the convex hull

$$\Theta = convh(N \cap C),$$

along with its Euclidean boundary $\partial\Theta$. It is shown in [Tsu83] that the boundary $\partial\Theta$ can be decomposed into (possibly infinitely many) convex polytopes with vertices in N. Write $P(\partial\Theta)$ for this collection. There is a natural Γ action on points in $N \cap \partial\Theta$, and since Γ acts by matrices of determinant 1, no convex polytope gets dilated, and there is an induced action on $P(\partial\Theta)$ uniquely determined by the action on the vertices defining the respective polytope. Since we are given the data $\partial\Theta$ with a Γ-action on $P(\partial\Theta)$, we can form the fan

$$\Sigma = \{0\} \cup \bigcup_{\alpha \in P(\partial\Theta)} Cone_{\mathbb{R}}(\alpha),$$

where $Cone_{\mathbb{R}}(\alpha) \subset N_{\mathbb{R}}$ is the cone generated by the polytope with a Γ-action and support $|\Sigma| = C \cup \{0\}$. By standard toric geometry, there is then a toric variety $T_N\Sigma$ with an action of Γ associated to this data. The space $T_N\Sigma$ is smooth if, and only if, all $Cone_{\mathbb{R}}(\alpha)$ are regular, meaning that $Cone_{\mathbb{R}}(\alpha) \cap N$ contains a $\mathbb{Z}$ basis for N. In general, it might happen that $T_N\Sigma$ has singularities, but these can be resolved equivariantly by constructing a new fan from subdivision of the cones of Σ. Now, these cones must also be preserved by the action of Γ, so this resolution also has an induced action by Γ. Therefore, in the following, we can safely assume $T_N\Sigma$ is smooth.

Remark 1.1.2. If $\gamma \cdot Cone_{\mathbb{R}}(\alpha) = Cone_{\mathbb{R}}(\beta)$, then dually $\gamma \cdot Cone_{\mathbb{R}}(\beta)' = Cone_{\mathbb{R}}(\alpha)'$. Hence, one has the map of character rings

$$\gamma\colon \mathbb{C}[\chi^m]_{m\in Cone_{\mathbb{R}}(\beta)'\cap M} \to \mathbb{C}[\chi^m]_{m\in Cone_{\mathbb{R}}(\alpha)'\cap M}$$

given on generators by $\gamma(\chi^m) = \chi^{\gamma m}$. Thus, when the cones are regular, we can pick generators to obtain ring isomorphisms

$$F_\beta\colon \mathbb{C}[\chi^m]_{m\in Cone_{\mathbb{R}}(\beta)'\cap M} \cong \mathbb{C}[z_1, \dots, z_r]$$

$$F_\alpha\colon \mathbb{C}[\chi^m]_{m\in Cone_{\mathbb{R}}(\alpha)'\cap M} \cong \mathbb{C}[\tilde{z}_1, \dots, \tilde{z}_r]$$

giving rise to the conjugate map $F_\alpha \circ \gamma \circ F_\beta^{-1}$, which takes $z_i \mapsto F_\alpha(\chi^{\gamma m_i}) = \prod_{j=1}^r \tilde{z}_j^{c_{ij}}$, where $\chi^{\gamma m_i} = \chi^{\sum_j c_{ij} s_j}$. Therefore, when $T_N\Sigma$ is smooth, then the action of γ maps the open subsets isomorphic to $\mathbb{C}^r$ determined by a cone to each other by a transformation of the type

$$\gamma \cdot (\tilde{z}_1, \dots, \tilde{z}_r) = (\prod \tilde{z}_j^{c_{1j}}, \dots, \prod \tilde{z}_j^{c_{rj}}),$$

and the integral matrix (c_{ij}) represents the dual action of γ, i.e., is equal to the transpose of the matrix representing γ.

Recall that one has a map to a manifold with corners, which is given by the quotient map with respect to the compact torus $U(1)^n \subset \mathbb{C}^*$ acting on $T_N\Sigma$ (see e.g. [Oda88]). This map is Γ-equivariant and coincides up to a sign change with the Euclidean tropicalization map

$$trop : T_N\Sigma \to (T_N\Sigma)^{trop}$$

given on points (identified to be morphisms $\rho\colon Cone_R(\alpha) \cap M \to \mathbb{C}, \rho(0) = 1, \rho(m + m') = \rho(m)\rho(m')$) by $\rho \mapsto -\log(|\rho|)$. Now, C is naturally embedded as an open Γ-invariant subset in $(T_N\Sigma)^{trop}$, because the image of the torus $trop(N \otimes_{\mathbb{Z}} \mathbb{C}^*)^r = N_{\mathbb{R}}$ is open. Consider the set

$$\tilde{C} = C \cup ((T_N\Sigma)^{trop} \setminus N_{\mathbb{R}}),$$

which is open in $(T_N\Sigma)^{trop}$ as $U = trop^{-1}(\tilde{C})$ is a neighbourhood of the boundary divisor $D = U \setminus U \cap (N \otimes_{\mathbb{Z}} \mathbb{C}^*)$. The subset U is Γ-stable since C and the toric boundary is so. Moreover, the action is properly discontinuous and without fixed points because it is so on C. In particular, we can form the quotient

$$\begin{array}{ccc} D \subset U & \xrightarrow{trop} & \tilde{C} \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\ D/\Gamma \subset U/\Gamma & \xrightarrow{trop} & \tilde{C}/\Gamma, \end{array}$$

where D/Γ is a compact analytic subspace, because C/Γ is compact, hence there is a finite number of equivalence classes of convex polytopes in $P(\partial\Theta)$ and thus D/Γ is compact. Finally, in order to obatin the cusp, we want to contract D/Γ, which is done by the contraction criterion of Grauert [Gra62].

There is a natural plurisubharmonic function $\phi\colon C \to \mathbb{R}_{>0}$ called the characteristic function of the cone 1.1.3, invariant under the Γ-action. This admits a continuous extension $\tilde{\phi}\colon \tilde{C} \to \mathbb{R}_{\geq 0}$ by setting $\tilde{\phi}(p) = 0$ for $p \in \tilde{C} \setminus C$. The characteristic function is convex on C, hence pulling back the characteristic function $trop^*\phi$ yields a plurisubharmonic function on $U \setminus D$, which is smooth and vanishes along the boundary. Evidently the function descends to the quotient U/Γ by equivariance. By a theorem of Grauert [Gra62], the boundary divisor can then uniquely be blown down to a point in a holomorphic way. The resulting normal pointed space (V, x_0) is the Tsuchihashi cusp associated to the cusp data (C, Γ). We have the diagram

$$\begin{array}{ccccc} (U \setminus D) \cup \{x_0\} & \longleftarrow & D \subset U & \xrightarrow{trop} & \tilde{C} \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\ (V, x_0) & \longleftarrow & D/\Gamma \subset U/\Gamma & \xrightarrow{trop} & \tilde{C}/\Gamma. \end{array}$$

Unwinding the definition of the quotient map *trop*, one may describe the cusp complex analytically as the quotient of a tube domain

$$\{N_{\mathbb{R}} + iC\}/(N \rtimes \Gamma) \cup \{x_0\}$$

with distinguished open neighbourhoods of x_0 given by $W(\epsilon) = q(V(\epsilon)) \cup \{x_0\}$ where

$$V(\epsilon) = \{n + ic \in N_{\mathbb{R}} + iC | \phi(c) < \epsilon\}.$$

In this decription, the action of $N \times \Gamma$ is given by

$$(n', g) \cdot (n + ic) = g(n) + n' + ig(n).$$

Moreover, holomorphic functions on $W(\epsilon)$ can be described via its lifting to $V(\epsilon)$ as having a special Fourier expansion [Oga86]. The relation between the two descriptions is given by the diagram

$$\begin{array}{ccc} & N_{\mathbb{R}} + iC & \\ {}^{e^{2\pi i(\cdot)}}\swarrow & & \searrow^{\frac{1}{2\pi} Im} \\ U \setminus D \subset (\mathbb{C}^*)^n & \xrightarrow{\quad trop \quad} & C, \end{array}$$

where $e^{2\pi i(n+ic)} := (e^{2\pi i(n_1+ic_1)}, \ldots, e^{i(n_n+ic_n)})$ becomes biholomorphic after taking the N quotient.

The characteristic function associated to a Tsuchihashi cusp

In this section, we define the characteristic function associated to a cone as in [Oda88]. It was originally considered by Vinberg in [Vin63].

Definition 1.1.3. For an open cone $C \subset V \cong \mathbb{R}^r$ with dual C', we set

$$\phi(c) = \int_{C'} e^{-\langle w,c\rangle} dx,$$

where $\langle w, c\rangle = w(c)$, and dx is the Lebesgue measure. We call ϕ the *characteristic function* of C.

Thus, the characteristic function of C is an averaged Gaussian function. It is clearly finite away from the boundary of C, since there $\langle c, x\rangle > 0, \forall x$ and it diverges as one approaches the boundary of C. The following properties are straightforward.

Lemma 1.1.4. *The characteristic function of a cone satisfies the following properties*

- *for $L \in Stab(C, V)$, one has*

$$L^*\phi = \frac{1}{|\det(L)|}\phi \ .$$

- *ϕ is convex and $log(\phi)$ is strictly convex.*

Proof. The first property follows directly from the coordinate change formula for the Lebesgue measure. The second property is a simple calculation. If $x_1, \ldots, x_r$ are coordinates on $V \cong \mathbb{R}^r$, and $x^1, \ldots, x^r$ are coordinates on

V^*, then the pairing $\langle \cdot, \cdot \rangle$ is the standard inner product. Let $a, b \in T_x C$, then we have

$$
\begin{aligned}
\partial_i \phi(x) &= \int_{C'} \partial_i e^{-w^i x_i} dw \\
&= \int_{C'} -w^i e^{-w^i x_i} dw \\
d\phi_x(a) &= \int_{C'} -\langle w, a \rangle e^{-\langle w, x \rangle} dw \\
dlog(\phi)_x &= \frac{d\phi_x}{\phi(x)}.
\end{aligned}
$$

Taking second order derivatives gives

$$
\begin{aligned}
\partial_j \partial_i \phi(x) &= \int_{C'} w^i w^j e^{-w^i x_i} dw \\
Hess(\phi)_x(a, b) &= \int_{C} \langle w, a \rangle \langle w, b \rangle e^{-\langle w, x \rangle} dw \\
\partial_j \partial_i log(\phi)(x) &= \frac{\partial_j \partial_i \phi(x)}{\phi(x)} - \frac{\partial_j \phi(x) \partial_i \phi(x)}{\phi(x)^2} \\
Hess(log(\phi))_x(a, a) &= \frac{Hess(\phi)_x(a, a)}{\phi(x)} - (\frac{d\phi_x(a)}{\phi(x)})^2 \\
&= \frac{1}{\phi(x)} (\int_{C'} \langle w, a \rangle^2 e^{-\langle w, x \rangle} dw - \frac{1}{\phi(x)} (\int_{C'} \langle w, a \rangle e^{-\langle w, a \rangle})^2) \\
&> 0.
\end{aligned}
$$

The last inequality follows by Cauchy-Schwartz

$$
\begin{aligned}
(\int_{C'} \langle w, a \rangle e^{-\frac{1}{2}\langle w, x \rangle} e^{-\frac{1}{2}\langle w, x \rangle} dw)^2 \leq \int_{C'} \langle w, a \rangle^2 e^{-\langle w, a \rangle} dw \int_{C'} e^{-\langle w, x \rangle} dw \\
= \phi(x) \int_{C'} \langle w, a \rangle^2 e^{-\langle w, a \rangle} dw
\end{aligned}
$$

with equality iff $\langle w, a \rangle e^{-\frac{1}{2}\langle w, x \rangle} = k e^{-\frac{1}{2}\langle w, x \rangle}$ for some constant k, which cannot hold for $w \in C'$ when $a \in C$. In particular, we see that the form $(\partial_i \partial_i \phi_x)$ is positive semidefinite at any point $x \in C$, i.e., ϕ is convex, and similarly, $log(\phi)$ is strictly convex, since its Hessian is positive definite at any point. □

The characteristic function has other neat relationships with the cone, which we shall not pursure further, see for example [Oda88].

Example 1.1.5. When $C = \mathbb{R}^n_{>0}$, the characteristic function is

$$\begin{aligned}\phi(y_1, \dots, y_n) =& \int_{\mathbb{R}^n_{>0}} e^{-\sum_{k=0}^n y_k x_k} dx \\ =& \prod_{k=0}^n \int_{\mathbb{R}_{>0}} e^{-y_k x_k} dx_k \\ =& \prod_{k=0}^n \lim_{t\to\infty} \int_0^t e^{-y_k x_k} dx_k \\ =& \prod_{k=0}^n \lim_{t\to\infty} [\frac{-1}{y_k} e^{-y_k x_k}]_0^t \\ =& \frac{1}{\prod_{k=0}^n y_k}.\end{aligned}$$

In particular, $log(\phi)(y)$ is a potential for the product Kähler-Einstein metric on $(\mathbb{H}^1)^n = \mathbb{R}^n + iC \subset \mathbb{C}^n$.

From the calculation above 1.1, it follows that one can equip any open real cone C with an intrinsically defined Riemannian metric $g = (\partial_i \partial_j log(\phi))$. Moreover, as the example above indicates, when one considers the complex tube domain

$$\mathbb{R}^n + iC \subset \mathbb{C}^n,$$

then the pullback metric is automatically Kähler. This is made more concrete in the following statement whose proof is a standard calculation (see [Nee15, prop. 2.3]).

Proposition 1.1.6. *Suppose we have a manifold of the type $\mathbb{R}^n + iC \subset \mathbb{C}^n$ where $C \subset \mathbb{R}^n_{>0}$ and $M_C = (\mathbb{R}^n + iV)/\mathbb{Z}^n$, where $\mathbb{Z}^n$ act on the real part $\mathbb{R}^n$ by translations. Let $Im\colon \mathbb{R}^n + iC \to C$ and denote the induced map from M_C the same. Then,*

- *$f \in C^\infty(C)$ is (strictly) convex, if and only if, the pullback $Im^* f(z)$ is (strictly) plurisubharmonic. Moreover, one has the following identity*

$$\frac{1}{n!}(i\partial\bar{\partial} Im^* f)^n = \det(\frac{\partial^2 f}{\partial y_i \partial y_j}) dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n.$$

In particular, since the flat fiber tori have volume 1, we have an identity

$$\frac{1}{n!}\int_{M_C} (i\partial\bar{\partial} Im^* f)^n = \int_V det(\frac{\partial^2 f}{\partial y_i \partial y_j}) dy_1 \wedge \cdots \wedge dy_n.$$

In other words, we have an identity of Monge- Ampére operators

$$MA_{\mathbb{C}}(Im^* f) = n! MA_{\mathbb{R}}(f).$$

- *Suppose* ω *is a smooth,* $\mathbb{R}^n/\mathbb{Z}^n$*-invariant, exact and positive* $(1,1)$*-form on* M_C*, then there exists a smooth function* $f \in C^\infty(C)$ *with* $Hess_C(f) \geq 0$ *such that* $\omega = i\partial\bar{\partial} Im^* f$*.*

Since $d\log(\phi)_x = \frac{d\phi_x}{\phi(x)} \neq 0$ at any point $x \in C$, one can use the $N \rtimes \Gamma$ invariant characteristic function $s = \log(\phi)$ as a coordinate to obtain an $N \rtimes \Gamma$ equivariant smooth decomposition of $N_{\mathbb{R}} + iC$ into $\mathbb{R} \times \log(\phi)^{-1}(0)$. Indeed, one can consider the mapping $(n + ic) \mapsto (\log(\phi(c)), n + i\phi(c)^{1/n}c)$ with inverse $(s, n + ic) \mapsto n + ie^{-s}c$. Moreover, by equivariance, it follows that we have an identification of the respective quotients. Hence, we have an alternative C^∞ description of the distinguished neighbourhoods

$$V(\epsilon) \cong (-\infty, \log(\epsilon)) \times \log(\phi)^{-1}(0)$$

and

$$W(\epsilon) \setminus \{x_0\} \cong (-\infty, \log(\epsilon)) \times \log(\phi^{-1})(0)/N \rtimes \Gamma.$$

Definition 1.1.7. We call

$$\log(\phi)^{-1}(0)/N \rtimes \Gamma = \phi^{-1}(1)/N \rtimes \Gamma$$

the characteristic *link* of the Tsuchihashi cusp singularity.

Usually, the link of an isolated singularity $(X, x_0) \subset \mathbb{C}^n$ is given by the manifold$(X, x_0) \cap S_\epsilon^{2n-1}(x_0)$ when ϵ is sufficiently small [Mil74]. Since we think of ϕ as a "metrically relevant" radial coordinate this partly justifies the terminology. The link of the cusp associated to (C, Γ) is topologically a $U(1)^r$-bundle over $\phi|_C^{-1}(1)/\Gamma \cong ray(\phi|_C^{-1}(1))/\Gamma$, which we shall call the *base* of the characteristic link. Since *ray* is the quotient map by the scaling action, it follows that one can identify the base of the link with the rays on C/Γ obtained by scaling the cone.

1.2 Hilbert modular cusps

In this section, we specialize the construction of Tsuschihashi cusps to the case of Hilbert-Modular cusps, and we explain how the construction from the previous section fits with the resolution construction due to Hirzebruch (dimension 2) and Ehlers (dimension > 2) [Hir73],[Ehl75]. The main impetus for studying these singularities arose from my initial research question

> *Are there singularities equipped with a natural metric structure such that the metric behaviour is reflected in the dual complex of the singularity?*

The Hilbert modular cusps arise from the data

- K a totally real degree r field extension of Q. I.e., there are exactly r embeddings $a_i \colon K \to \mathbb{R}$, $1 \leq i \leq r$, giving an identification $K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^r$ by $k \otimes r \mapsto r(a_1(k), \ldots, a_r(k))$. This identification is unique up the action of the symmetric group.
- $N \subset K$ a $\mathbb{Z}$-lattice of maximal rank, i.e., $N_{\mathbb{Q}} = K$, and so one can identify $N_{\mathbb{R}} = K_{\mathbb{R}}$.

Considering the cone $\mathbb{R}^r_{>0}$ pulled back via the identification, $\mathbb{R}^r \cong N_{\mathbb{R}}$ gives a cone C_N. A totally positive $k \in K$ is an element of C_N. The group

$$\Gamma_N^+ = \{g \in K | g \text{ totally positive unit } gN = N\}$$

is a rank $r-1$ [Oda88, p. 155] commutative group. There is then an action of Γ_N^+ on $N_{\mathbb{R}}$ preserving C_N, and if $\Gamma \subset \Gamma_N^+$ is a finite index subgroup, then the action of Γ is properly discontinuous and $ray(C_N)/\Gamma \cong R^{r-1}/\mathbb{Z}^{r-1}$ is compact. Hence, (C_N, Γ) define Tsuchihashi cusp data. The associated cusp can then be described as before. More concretely we have

$$\begin{array}{ccc} & U \setminus D & \\ \swarrow^{\frac{1}{2\pi i}\log(\cdot)} & & \searrow^{trop} \\ (\mathbb{H}^1)^r/N \cong (N_R + iC_N)/N & \xrightarrow{\quad Im(\cdot) \quad} & C_N, \end{array}$$

where $\frac{1}{2\pi i}\log(\cdot) = \frac{1}{2\pi i}\sum_{i=1}^r n_i(\log(|\cdot|) + iArg(\cdot))$ now is a biholomorphism and $n_1, \ldots, n_r$is a $\mathbb{Z}$-basis for N . The cusp is then obtained as the Γ-quotient with a point added at infinity.

As discussed earlier, in the model $(\mathbb{H}^1)^r$ the function $\log(\phi)$, where ϕ is the characteristic function of $\mathbb{R}^r_{>0}$ is exactly the Kähler-Einstein product metric:

$$\omega = -i\,\partial\overline{\partial}\sum_{i=1}^r \log(y_i).$$

Splitting as before

$$(\mathbb{H}^1)^r \cong \mathbb{R}\times\log(\phi)^{-1}(0) = \mathbb{R}\times(\mathbb{R}^r\times\{(y_1,\dots,y_r)\in\mathbb{R}^r_{>0}\,|\,\prod_{i=1}^r y_i = 1\})$$

by the map $(s, x_1, \dots, x_n, y_1, \dots, y_n) \mapsto (x_i + ie^{-s}y_i)_{1\le i\le r}$, we see that the link of a Hilbert modular cusp is a $U(1)^r$ bundle over a quotient of

$$\{(y_1,\dots,y_r)|\prod_{i=1}^r y_i = 1\}$$

by a discrete group. When $r = 2$, we have in particular a $U(1)^2 = T$ torus bundle over a circle S^1. If we write the metric g associated to ω in terms of the parametrization

$$(s, x_1, \dots, x_r, \tilde{y}_1, \dots, \tilde{y}_{r-1}) \mapsto (s, x_1, \dots, x_r, e^{-s}y_1, \dots, e^{-s}y_{r-1}, \frac{e^{-s}}{\prod_{i=1}^{r-1} y_i}),$$

we see that the metric becomes

$$\begin{aligned}
g =&\sum_{i=1}^r \frac{1}{y_i^2}(dx_i^2 + dy_i^2)\\
=&\frac{1}{e^{-2s}}(\sum_{i=1}^r \frac{1}{\tilde{y}_i^2}dx_i^2 + \sum_{i=1}^{r-1}\frac{1}{\tilde{y}_i^2}(e^{-2s}\tilde{y}_i^2 ds^2 - e^{-2s}(ds\otimes d\tilde{y}_i + d\tilde{y}_i\otimes ds) + e^{-2s}d\tilde{y}_i^2)\\
&+ \frac{(\prod_{i=1}^{r-1}\tilde{y}_i)^2}{e^{-2s}}dy_r^2)\\
=&(r-1)ds^2 + \sum_{i=1}^{r-1}\frac{1}{\tilde{y}_i^2}d\tilde{y}_i^2 - \sum_{i=1}^{r-1}\frac{1}{\tilde{y}_i^2}(ds\otimes d\tilde{y}_i + d\tilde{y}_i\otimes ds) + \frac{1}{e^{-2s}}\sum_{i=1}^r\frac{1}{\tilde{y}_i^2}d\tilde{x}_i^2\\
&+ \frac{(\prod_{i=1}^{r-1}\tilde{y}_i)^2}{e^{-2s}}(\frac{e^{-2s}}{(\prod_{i=1}^{r-1}\tilde{y}_i)^2}ds^2 + \sum_{i=1}^{r-1} e^{-2s}\frac{\prod_{j\neq i}\tilde{y}_j}{(\prod_{k=1}^{r-1}\tilde{y}_k)^3}(d\tilde{y}_i\otimes ds + ds\otimes d\tilde{y}_i)\\
&+ \sum_{i,k} e^{-2s}\frac{\prod_{j\neq i}\tilde{y}_j\prod_{j\neq k}\tilde{y}_k}{(\prod_{\ell=1}^{r-1}\tilde{y}_\ell)^4}d\tilde{y}_i\otimes d\tilde{y}_k)\\
=&rds^2 + \sum_{i=1}^{r-1}\frac{1}{\tilde{y}_i^2}d\tilde{y}_i^2 + \sum_{i=1}^{r-1}(\frac{1}{\tilde{y}_i} - \frac{1}{\tilde{y}_i^2})(ds\otimes d\tilde{y}_i + d\tilde{y}_i\otimes ds)\\
&+ \sum_{i,k}\frac{1}{\tilde{y}_i\tilde{y}_k}d\tilde{y}_i\otimes d\tilde{y}_k + \frac{1}{e^{-2s}}\sum_{i=1}^r\frac{1}{\tilde{y}_i}d\tilde{x}_i^2.
\end{aligned}$$

Making the coordinate change $\tilde{y}_i = e^{\hat{y}_i}$, we can simplify a bit further

$$g = rds^2 + \sum_{i=1}^{r-1} d\hat{y}_i^2 + \sum_{i=1}^{r-1}(1 - \frac{1}{e^{\hat{y}_i}})(ds \otimes d\hat{y}_i + d\hat{y}_i \otimes ds) + \sum_{i,k} d\hat{y}_i \otimes d\hat{y}_k$$
$$+ \frac{1}{e^{-2s}} \sum_{i=1}^{r} \frac{1}{e^{\hat{y}_i}} d\tilde{x}_i^2.$$

So, when $s \to \infty$, then the fibre directions of the $U(1)^r$ fibration collapse leaving the "limit metric" towards the cusp to be

$$g \approx rds^2 + \sum_{i=1}^{r-1} d\hat{y}_i^2 + \sum_{i=1}^{r-1}(1 - \frac{1}{e^{\hat{y}_i}})(ds \otimes d\tilde{y}_i + d\tilde{y}_i \otimes ds) + \sum_{i,k} d\hat{y}_i \otimes d\hat{y}_k.$$

Hence, after a logarithmic change of coordinates on the link of the cone, one has the data of a metric on a cylinder over the quotient of

$$\{(y_1, \ldots, y_r) | \prod_{i=1}^{r} y_i = 1\}.$$

Next, we consider special geodesics with the cusp as its limit point.

Definition 1.2.1. Given an inclusion $U \subset V$ of topological spaces with $\bar{U} \subset V$, we say a ray $\gamma\colon \mathbb{R} \to U$ has limit point $p \in \bar{U} \setminus U$ if $\lim_{t\to\infty} \gamma(t) = p$.

Lemma 1.2.2. *The rays in $(\mathbb{H}^1)^r$ starting at a point $(z_1, \ldots, z_r) = (x_1 + iy_1, \ldots, x_r + iy_r)$ of the following type*

$$\gamma(t) = (x_1 + ie^t y_1, \ldots, x_r + ie^t y_r)$$

are geodesics with respect to the product Kähler-Einstein metric.

Proof. These curves project to curves on $\mathbb{H}^1$, which are all classically known to be unit speed geoedesics. Therefore, the γ are also geodesics with speed r. □

Geodesics on $(\mathbb{H}^1)^r$ descend to geodesics on the quotient by $N \rtimes \Gamma$ when taking the quotient metric. Any two of these induced geodesics γ, γ' have the same image, if, and only if, $(z_1, \ldots, z_r) \sim (z'_1, \ldots, z'_r)$ under the $N \rtimes \Gamma$ action.

Corollary 1.2.3. *The geodesics of the previous lemma descent to geodesics on a neighbourhood of the Hilbert-modular cusps with the cusp as their limit point.*

Proof. It only remains to show that these curves indeed have the cusp as limit. In general, there is a basis for the topology at the cusp $\{x_0\}$ given by $W(\epsilon) = q(V(\epsilon)) \cup \{x_0\}$, where

$$V(\epsilon) = \{n + ic \in N_{\mathbb{R}} + iC | \log(\phi(c)) < \log(\epsilon)\}.$$

In the case of Hilbert modular cusps, $\log(\phi(y_1, \ldots, y_r)) = \log(\frac{1}{\prod_{i=1}^r y_i})$, so a curve on $(\mathbb{H}^1)^r$ descends to a curve with the cusp as its limit, if any cone coordinate goes to infinity. This is indeed the case for the geodesics in question, since

$$\begin{aligned}\log(\phi(\gamma(t))) =& \log(\frac{1}{e^{rt} \prod_i y_i}) \\ =& -rt + \log(\phi(y_1, \ldots, y_r)) \underset{t\to\infty}{\to} -\infty.\end{aligned}$$

□

Following the same line of reasoning, there are many more geodesic rays on $(\mathbb{H}^1)^r/N \rtimes \Gamma$ going to the cusp. For example, the curves obtained from

$$t \mapsto (x_1 + ie^{k_1 t} y_1, \ldots, x_r + ie^{k_r t} y_r)$$

for any tuple $k = (k_1, \ldots, k_r) \in \mathbb{R}^r$ such that $\sum_i k_i > 0$. Therefore, geodesics from 1.2.2 are special in that they exactly correspond to the curves obtained as the orbit of a point under the rescaling action on the cone C.

Definition 1.2.4. We call the geodesics on $(\mathbb{H}^1)^r/N \rtimes \Gamma$ from lemma 1.2.2 *cone type* geodesics.

The dual complex

In the following, we review the construction of the dual complex associated to a resolution of a singular pair, following [dFKX]. Roughly speaking, the dual complex is a cell complex one can attach to a resolution of a complex analytic space, which contains information of how the exceptional divisors intersect. We fix a variety X. Denote for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_{>0}$ the weighted standard simplex $\Delta^{n-1}_\alpha := \{x \in \mathbb{R}^n_{\geq 0} | \sum_{i=1}^n \alpha_i x_i = 1\}$.

Definition 1.2.5. A finite Δ-complex of dimension n is the data of a topological space X and a finite collection $M = \{\sigma_i\}_{i \in I}$ of maps $\sigma_i \colon \Delta^{n(i)}_{\alpha(i)} \to X$ such that

1. σ_i is injective when restricted to the interior, and for any $x \in X$ $|\{i | x \in \sigma_i(\mathring{\Delta}_{\alpha(i)})\}| = 1$.

2. σ_i restricted to a face is another map $\sigma_j \in M$ such that

$$\begin{array}{ccc} \Delta^{n(j)}_{\alpha(j)} & \xrightarrow{\quad\sigma_j\quad} & X \\ & {\scriptstyle \iota}\searrow \quad \nearrow{\scriptstyle \sigma_i} & \\ & \Delta^{n(i)}_{\alpha(i)} & \end{array}$$

commutes, where $\alpha(j) = (\alpha(i)_1, \alpha(i)_{k-1}, \ldots, \widehat{\alpha(i)}_k, , \alpha(i)_{k+1} \ldots, \alpha(i)_{n(i)+1})$ and ι is the obvious linear inclusion map.

3. The topology is given by $A \subset X$ open, if, and only if, $\sigma_i^{-1}(A)$ is open for all $i \in I$.

A morphism between two delta complexes (X, M) and (X', M') is a map $\phi\colon X \to X'$ such that it is linear when restricted to simplexes.

It is clear that given just the data of (weighted)-simplexes satisfying compatibility as in point 2, then one can glue these to a delta complex. Suppose $D = \sum_{i=1}^{m} a_i D_i$ is an effective Weil-divisor on X. For any $I \subset \{1, 2, \ldots, m\}$, write $D_I := \bigcap_{i \in I} D_i$ and $a_I := (a_i)_{i \in I}$. Suppose D satisfies the following :

1. Each D_i is normal,

2. For any I with $D_I \neq \emptyset$, any connected component of D_I is irreducible and of codimension $|I|$ in X.

The second condition ensures connected components are the same as irreducible ones, so it implies that any irreducible component of D_J is in a unique irreducible component of D_I whenever $I \subset J$ and $D_J \neq \emptyset$.

We associate a Δ-complex $\Delta(D)$ to D, built inductively in the following way: for any irreducible component D_i, we have a vertex v_i corresponding to a 0-simplex $\Delta^0_{a_i}$. The n-cells correspond to irreducible components of the $D_I \neq \emptyset$, where $|I| = n + 1$. Each component $W_{I,j}$ of D_I corresponds to a copy of $\Delta^n_{a_I}$ attached to the $(n-1)$-skeleton according to the unique inclusions of components in $D_{I\setminus\{i\}}$ when $i \in I$ varies. That is, by the obvious linear maps $\Delta^{n-1}_{a_{I\setminus\{i\}}} \to \Delta^n_{a_I}$.

Note that one obtains a simplicial complex, i.e., nonempty intersection of two simplexes is a single face in both, if, and only if, for any I, $D_I \neq \emptyset$ implies D_I is irreducible. The above conditions $1., 2.$ clearly hold, if X is smooth and D is simple normal crossing.

Definition 1.2.6. Suppose X is smooth and D is a simple normal crossing divisor. Then, the complex constructed above is called the *weighted* dual complex associated to D, and we denote it by $\Delta_w(D)$. If we consider $\Delta(D)$ constructed without considering multiplicities, i.e., all simplexes in the construction are Δ_1^n, we simply call $\Delta(D)$ the dual complex.

Although topologically equivalent, one typically considers weighted complexes, when one wants to encode multiplicity information, for instance when dealing with degenerations and unweighted complexes when considering singularities.

Definition 1.2.7. Suppose X is a singular complex space. Given a good resolution $f\colon Y \to X$ of singularities such that the exceptional divisor D is a simple normal crossings divisor. Then, we call $\Delta(D)$ the dual complex of the resolution.

By considering the behaviour of dual complexes under blow-ups of smooth subvarieties, it is then possible to show that the homotopy type of complex $\Delta(D)$ associated to a resolution only depends on the singularity [Ste06].

The Log map associated to a resolution

Following Boucksom-Jonnson [BJ17] there is a way of attaching the Dual complex of a resolution of singular complex anlaytics to form a *Hybrid space*.

Definition 1.2.8. Let $D = \sum_{i=1}^{m} d_i D_i$ be a snc divisor on a complex manifold X, and suppose $V \subset X$ is a coordinate neighbourhood in a coordinate system $\mathbb{C}^n_{z_1,\ldots,z_n}$, where $D \cap \mathbb{C}^n = \{\prod_{i\in I} z_{k_i} = 0 | I \subset \{1,\ldots,m\}\}$ for some injection $i \mapsto k_i \in \{1,\ldots,n\}$. Then, we say V is *adapted* to D if

1. $V \subset \{(z_1,\ldots,z_n) \in \mathbb{C}^n || z_{k_i}| < 1, \forall i \in I\}$,
2. $V \cap D_I \neq \emptyset$ implies $V \cap D_I = V \cap (cc(D_I) \setminus \bigcup_{j\in\{1,\ldots,m\}\setminus I} D_j)$, where $cc(D_I)$ is some connected component of D_I.

In other words, adapted coordinates are in polydiscs of the coordinates corresponding to the equations of the snc divisor, and they are localized at a connected component of D_I such that it does not intersect a component $D_j, j \in \{1,\ldots,m\} \setminus I$. We always have adapted coordinates near any point of the exceptional divisor on a resolution of singularities. Now, we define a rescaled local "tropicalization" map.

Definition 1.2.9. Suppose $V_{z_1,\dots,z_n}$ are adapted to a snc divisor $D = \sum_{i=1}^m d_i D_i$ on X such that $D \cap V = \{\prod_{i\in I} z_{k_i} = 0 | I \subset \{1,\dots,m\}\}$. Then we define a map to the dual complex $Log_V : V \setminus D \cap V \to \Delta_1^{|I|-1} \subset \Delta(D)$ given by

$$(z_1,\dots,z_n) \mapsto (\frac{\log(|z_{k_i}|)}{\log(\prod_{i\in I}|z_{k_i}|)})_{i\in I}.$$

The $\Delta_1^{|I|-1}$ corresponds to D_I in the notation from the definition of the dual complex 1.2.

We have the following simple observations

Lemma 1.2.10. *For an adapted neighbourhood V, the image of the map Log_V defined above is dense, and if two points $p = (z_1,\dots,z_n), q = (z'_1,\dots,z'_n)$ map to the same point in the interior of the simplex then*

$$\frac{\log(|z_{k_i}|)}{\log(|z'_{k_i}|)} = \frac{\log(|z_{k_j}|)}{\log(|z'_{k_j}|)}, \text{ for any pair } i,j \in I.$$

If V' is another adapted neighbourhood of D such that $V \cap V' \neq \emptyset$, then on the intersection we have:

$$Log_V - Log_{V'} = O(\frac{-1}{\log(\prod_{i\in I}|z_i|)}).$$

Suppose $(c_{k_i})_{i\in I} \in \Delta_1^{|I|-1}$ is an interior point, then the fiber is topologically $Log_V^{-1}((c_{k_i})_{i\in I}) = \{(z_1,\dots,z_n) \in V | |z_{k_i}| = |z_{k_j}|^{\frac{c_{k_i}}{c_{k_j}}}\} \simeq \mathbb{C}^{n-|I|+1} \times (S^1)^{|I|-1}$. That is, it is generically a torus fibration. Now suppose X is compact and consider small tubular open neighbourhoods around each of the components of D. This gives a covering of D. Let U be the union of these neighbourhoods. Passing to a suitable subcover of U, we can assume U to be covered by charts adapted to D. Details of such a construction can be found in [Cle77]. So, we have an open covering $\mathcal{U}$ of D consisting of adapted neighbourhoods. Pick a partition of unity subordinate to $\mathcal{U}$ and glue the locally defined Log_V maps to a smooth map $Log_{\mathcal{U}} : U \setminus D \to \Delta(D)$.

Note that by the previous lemma, this map is uniformally approximated by any locally defined Log_V coming from an adapted chart.

Remark 1.2.11. In the case where X is a smooth toric variety and $D = X \setminus (\mathbb{C}^*)^n$ is a simple normal crossing boundary divisor, then the logarithm map is simply the part of the map *trop* relevant for the boundary divisor, rescaled by $\frac{1}{-\log(\prod|z_i|)}$. In light of this, the limit value of the log map as one approaches the boundary divisor, can be thought of as moving to the tropicalization of the Berkovich space associated to X with respect to the trivial valuation on $\mathbb{C}$. For more on this see for example [Jon15].

Two dimensional Hilbert-Modular cusps

We now shift attention to the generic two dimensional case $r = 2$ and prove that there is a geometrically realized homeomorphism between the base of the characteristic link and the dual complex of the singularity. Thus giving a class of examples where the answer to 1.2 seems to be in the affirmative. For the explicit description of the resolution, see [Hir73].

In this case, the cusp arises from a totally real degree 2 field extension $\mathbb{Q} \subset K$. Consider the maximal $\mathbb{Z}$-lattice N generated by $(1, \omega_0) \in \mathbb{R}$. Under the identification $K_{\mathbb{R}} \cong \mathbb{R}^2$, these map to $(1,1), (\omega_0^{(1)}, \omega_0^{(2)})$ respectively. Thus, in order to construct the toric resolution one needs to decompose the boundary of $convh(\mathbb{R}^2_{>0} \cap (\mathbb{Z}(1,1) + \mathbb{Z}(\omega_0^1, \omega_0^2))$ into line segments and take the cones over them in order to produce the fan Σ. The pairs of integral points giving the polyhedral decomposition of this boundary can be identified with certain totally positive bases for N. These can be obtained cyclically as follows: since the extension is of degree two, there is a periodic function $b \colon \mathbb{Z} \to \mathbb{N}_{\geq 2}$ such that one has a continued fraction expansion

$$\omega_0 = \overline{[[b(0)]]} := b(0) - \cfrac{1}{b(1) - \cfrac{1}{\ddots}}.$$

Now, let ω_k to be the numbers obtained by cyclically permuting the continued fraction expansion of ω_0 k times. Defining the numbers $A_k = (\omega_0 \dots \omega_k)^{-1}$ for $k \geq 1$, and $A_{-k} = \omega_0 \dots \omega_{-k+1}$, for $k \geq 1$ and $A_0 := 1$ gives pairs A_k, A_{k-1} which form totally positive $\mathbb{Z}$ bases for N. Indeed, A_0, A_{-1} form a basis, and one has the invertible relation

$$\begin{pmatrix} A_k \\ A_{k-1} \end{pmatrix} = \begin{pmatrix} b(k-1) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A_{k-1} \\ A_{k-2} \end{pmatrix}$$

with period equal to the period of b. Therefore, the vectors $(A_k^{(1)}, A_k^{(2)})$ and $(A_{k-1}^{(1)}, A_{k-1}^{(2)})$, for $k \in \mathbb{Z}$ generate the cones for the fan giving a covering of the associated toric variety by the local models

$$U_k = Spec(\mathbb{C}[z_1, z_2])$$

and transition morphisms $U_k \to U_{k+1}$ given by $(z_1, z_2) \mapsto (z_1^{b(k)} z_2, z_1^{-1}) = (y_1, y_2)$ in accordance with 1.1.2. One has $\Gamma = \langle A_p \rangle$, where p is the period of b. So, since $A_{k+p} = A_p A_k$, it follows that Γ takes points $U_k \to U_{k+p}$ via the identity map.

Since the boundary divisor corresponds to the one dimensional rays in the fan [Oda88, p. 66], it follows that the numbers A_k modulo the period of b parametrize the irreducible components of the boundary divisor D/Γ. Moreover, these are $\mathbb{P}^1$'s as the local equation is $\{z_1 = 0\}$ in the k'th corodinate system, which is transformed into $\{y_2 = 0\}$ in the second. Therefore, except in the case where b has period ≤ 2, the boundary divisor consists of a cycle of $\mathbb{P}^1$'s, which intersect in a single point with precisely two other such components in a simple normal crossing way. Thus, the dual complex $\Delta(D/\Gamma)$ is homeomorphic to S^1. Denote the neighbourhood of the boundary divisor D, where the cusp is defined by U and the cusp by $(V, x_0) = \mathbb{R}^2 + i\,\mathbb{R}^2_{>0}/N \rtimes \Gamma \cup \{x_0\}$. Then, the resolution morphism

$$\pi \colon U/\Gamma \to (V, x_0)$$

is given (before taking the Γ quotient) in the k'th coordinate system by

$$\pi(z_1, z_2) = \frac{1}{2\pi i}(A^{(1)}_{k-1}\log(z_1) + A^{(1)}_k \log(z_2), A^{(2)}_{k-1}\log(z_1) + A^{(2)}_k \log(z_2))$$

with inverse (before taking the Γ quotient) given by

$$(z_1, \ldots, z_2) \mapsto (e^{2\pi i \frac{A^{(2)}_k z_1 - A^{(1)}_k z-2}{\det(M_k)}}, e^{2\pi i \frac{A^{(1)}_{k-1} z_1 - A^{(2)}_{k-1} z-2}{\det(M_k)}}),$$

where

$$M_k = \begin{pmatrix} A^{(1)}_{k-1} & A^{(1)}_k \\ A^{(2)}_{k-1} & A^{(2)}_k \end{pmatrix} > 0.$$

From this description, we see that the cusp is strictly log canonical in the terminology of the minimal model programme. That is, the discrepancy between canonical divisors is given by prime divisors with multiplicity -1. Indeed, one computes it straightforwardly by pulling back a $(2,0)$-form

$$\pi^*(d\tilde{z}_1 \wedge d\tilde{z}_2) = \frac{\det(M_k)}{4\pi^2 z_1 z_2} dz_1 \wedge dz_2.$$

Pulling back the product Kähler-Einstein metric $\omega = -i\,\partial\overline{\partial}\, log(Im(\tilde{z}_1)Im(\tilde{z}_2))$ by this map gives a metric with singularities along the boundary divisor D.

$$\begin{aligned}\pi^*\omega =& -i\,\partial\overline{\partial}\log(\frac{1}{4\pi^2}(A^{(1)}_{k-1}\log(|z_1|) + A^{(1)}_k \log(|z_2|))(A^{(2)}_{k-1}\log(|z_1|) + A^{(2)}_k \log(|z_2|)))\\ =& -i\,\partial\overline{\partial}\log\big[A^{(1)}_{k-1}A^{(2)}_{k-1}\log(|z_1|)^2 + A^{(1)}_k A^{(2)}_k \log(|z_2|)^2\\ &+ (A^{(1)}_{k_1}A^{(2)}_k + A^{(1)}_k A^{(2)}_{k-1})\log(|z_1|)\log(|z_2|)\big].\end{aligned}$$

Thus, the pullback metric has worse singularities than analytic singularities in the sense of potential theory.

Map to dual complex

When considering the two dimensional Hilbert Modular cusp, the circle S^1 comes up in two ways. First, it is the base B of the $U(1)^2$-fibration obtained as the characteristic link. In this section, we give a geometrically induced homeomorphism between these two. Recall the equivariant splitting

$$(\mathbb{H}^1)^2 \cong \mathbb{R} \times \log(\phi)^{-1}(0) \cong \mathbb{R}_s \times \mathbb{R}^2_{x_1,x_2} \times \mathbb{R}_{>0}$$

given by

$$(s, x_1, x_2, y_1) \mapsto (x_1 + e^{-s} y_1, x_2 + e^{-s} \frac{1}{y_1})$$

The $N \rtimes \Gamma$ action on the coordinates (s, x_1, x_2, y_1) is given by

$$(n, A_p) \cdot (s, x_1, x_2, y_1) = (s, A_p^{(1)} x_1 + n^{(1)}, A_p^{(2)} x_2 + n^{(2)}, A_p^{(1)} y_1),$$

hence after taking the quotient, the base of the $U(1)^2$ fibration $B \simeq S^1$ can be identified with the equivalence class of $[y_1]$ under the multiplicative action by A_p.

Definition 1.2.12. Let $F\colon B \to \Delta(D/\Gamma)$ be the map defined by

$$F([y_1]) = \lim_{t\to\infty} Log_{U/\Gamma}(\pi^{-1} \circ \gamma(t))$$

where γ is a cone type geodesic starting from a point on the characteristic link projecting to $[y_1]$ on the base of the fibration and $Log_{U/\Gamma}$ is the log map associated to $\Delta(D/\Gamma)$.

We need to ensure that this is well defined. We show that $\pi^{-1} \circ \gamma(t)$ is independent of such choices, from which it follows that the limit is as well. Suppose γ is a cone type geodesic ray with the cusp as its limit starting at an equivalence class $(s, [x_1, x_2, y_1])$. Then we can pick a lift to $(\mathbb{H}^1)^2/N$ starting from any point $(s, x_1', x_2', y_1') \sim (s, x_1, x_2, y_1)$. We know that this geodesic has the form

$$\tilde{\gamma}(t) = (s - t, x_1', x_2', y_1').$$

After applying π^{-1} to this, one has the following expression in the k'th coordinate system

$$\pi^{-1} \circ \tilde{\gamma}(t) = (e^{-2\pi e^t s(\frac{A_k^{(2)} y_1' - A_k^{(1)} y_2'}{\det(M_k)})} e^{2\pi i(\frac{A_k^{(2)} x_2' - A_k^{1} x_1'}{\det(M_k)})}, e^{-2\pi e^t s(\frac{A_{k_1}^{(1)} y_2' - A_{k-1}^{(2)} y_1'}{\det(M_k)})} e^{2\pi i(\frac{A_{k-1}^{(1)} x_2' - A_{k-1}^{(2)} x_1'}{\det(M_k)})}),$$

where $y_2' = \frac{1}{y_1'}$. This does not depend on the choice of x_1', x_2'. Thus, the only ambiguity comes from the choice of y_1'. But by Γ equivariance of π, it follows by slight abuse of notation that $q \circ \pi^{-1} \circ \tilde{\gamma}(t) = \pi^{-1} \circ q \circ \tilde{\gamma}(t) = \pi^{-1} \circ \gamma(t)$. Hence, the value depends only the equivalence class of y_1. Therefore, the limit, when it exists, is also independent of such choices.

Note, that the charts for U give adapted charts for U/Γ in the sense of [BJ17]. Therefore, the locally defined maps satisfy $Log_{U/\Gamma} \circ q = q \circ Log_U$ in these coordinates. Here we think of $\Delta(D/\Gamma)$ as being a quotient of $\Delta(D)$ by the Γ-action. In particular, their limiting value in the dual complex agrees. We can therefore compute the value for $t \gg 0$ as

$$\begin{aligned} Log_{U/\Gamma}(\pi^{-1} \circ \gamma(t)) =& Log_U(\pi^{-1} \circ \tilde{\gamma}(t)) \\ =& \left(\frac{A_k^{(2)} y_1 - A_k^{(1)} y_2}{A_k^{(2)} y_1 - A_k^{(1)} y_2 + A_{k-1}^{(1)} y_2 - A_{k-1}^{(2)} y_1}, \frac{A_{k-1}^{(1)} y_2 - A_{k-1}^{(2)} y_1}{A_k^{(2)} y_1 - A_k^{(1)} y_2 + A_{k-1}^{(1)} y_2 - A_{k-1}^{(2)} y_1}\right) \end{aligned} \tag{1}$$

where, again $y_2 = \frac{1}{y_1}$. Thus the limit is well defined for any such curve.

Proposition 1.2.13. *The map F is a homeomorphism.*

Proof. It suffices to show that F is a continuous bijection. By the expression 1, the map is clearly continuous. We first show that F is surjective, and since we quotient by Γ, we will also write F for the lift to $\mathbb{R}_{>0}$ before taking the quotient by $\Gamma = \langle A_p^{(1)} \rangle$. In view of the relation $y_2 = \frac{1}{y_1}$, the inequalities $A_{k-1}^{(1)} y_2 - A_{k-1}^{(2)} y_1 \geq 0, A_k^{(2)} y_1 - A_k^{(1)} y_2 \geq 0$ imply that $\frac{A_k^{(1)}}{A_k^{(2)}} \leq y_1^2 \leq \frac{A_{k-1}^{(1)}}{A_{k-1}^{(2)}}$, which then gives a fundamental domain for the coordinate functions F_1, F_2.

$$F_1(y_1) = \frac{A_k^{(2)} y_1 - A_k^{(1)} y_2}{A_k^{(2)} y_1 - A_k^{(1)} y_2 + A_{k-1}^{(1)} y_2 - A_{k-1}^{(2)} y_1}.$$

The function F_1 is monotonely increasing when writing $y_2 = \frac{1}{y_1}$. Indeed, rewriting slightly one has

$$F_1(y_1) = \frac{A_k^{(2)} y_1^2 - A_k^{(1)}}{(A_k^{(2)} - A_{k-1}^{(2)}) y_1^2 + A_{k-1}^{(1)} - A_k^{(1)}},$$

making the monotone substitution $b = y_1^2$ one has

$$\begin{aligned}
\frac{d}{db}F_1(b) =& \frac{A_k^{(2)}((A_k^{(2)} - A_{k-1}^{(2)})b + A_{k-1}^{(1)} - A_k^{(1)}) - (A_k^{(2)} - A_{k-1}^{(2)})(A_k^{(2)}b - A_k^{(1)})}{((A_k^{(2)} - A_{k-1}^{(2)})b + A_{k-1}^{(1)} - A_k^{(1)})^2} \\
=& \frac{A_k^{(2)}A_{k-1}^{(1)} - A_{k-1}^{(2)}A_k^{(1)}}{((A_k^{(2)} - A_{k-1}^{(2)})b + A_{k-1}^{(1)} - A_k^{(1)})^2} \\
=& \frac{\det(M_k)}{((A_k^{(2)} - A_{k-1}^{(2)})b + A_{k-1}^{(1)} - A_k^{(1)})^2} \\
>& 0.
\end{aligned}$$

Since the map is to the standard 1-simplex, it follows by

$$\begin{aligned}
F_1\left(\sqrt{\frac{A_k^{(1)}}{A_k^{(2)}}}\right) =& \frac{\sqrt{A_k^{(2)}A_k^{(1)}} - \sqrt{A_k^{(1)}A_k^{(2)}}}{\sqrt{A_k^{(2)}A_k^{(1)}} - \sqrt{A_k^{(1)}A_k^{(2)}} + A_{k-1}^{(1)}\sqrt{\frac{A_k^{(2)}}{A_k^{(1))}}} - A_{k-1}^{(2)}\sqrt{\frac{A_k^{(1)}}{A_k^{(2)}}}} \\
=& 0 \\
F_1\left(\sqrt{\frac{A_{k-1}^{(1)}}{A_{k-1}^{(2)}}}\right) =& \frac{A_k^{(2)}\sqrt{\frac{A_{k-1}^{(1)}}{A_{k-1}^{(2)}}} - A_k^{(1)}\sqrt{\frac{A_{k-1}^{(2)}}{A_{k-1}^{(1)}}}}{A_k^{(2)}\sqrt{\frac{A_{k-1}^{(1)}}{A_{k-1}^{(2)}}} - A_k^{(1)}\sqrt{\frac{A_{k-1}^{(2)}}{A_{k-1}^{(1)}}} + \sqrt{A_{k-1}^{(1)}A_{k-1}^{(2)}} - \sqrt{A_{k-1}^{(2)}A_{k-1}^{(1)}}} \\
=& 1,
\end{aligned}$$

that F is onto the piece of the dual complex given by the k'th coordinate system from U. As this holds for all $k \in \mathbb{Z}$, it follows that F is surjective. Moreover, by the argument above, it follows that F restricted to fundamental domains for Γ is injective. To see that F is injective, simply observe that the action of Γ is the identity between the k'th and $k+p$'th coordinate system and that boundary values for F_1, F_2 match up with the gluing of $\Delta(D/\Gamma)$. □

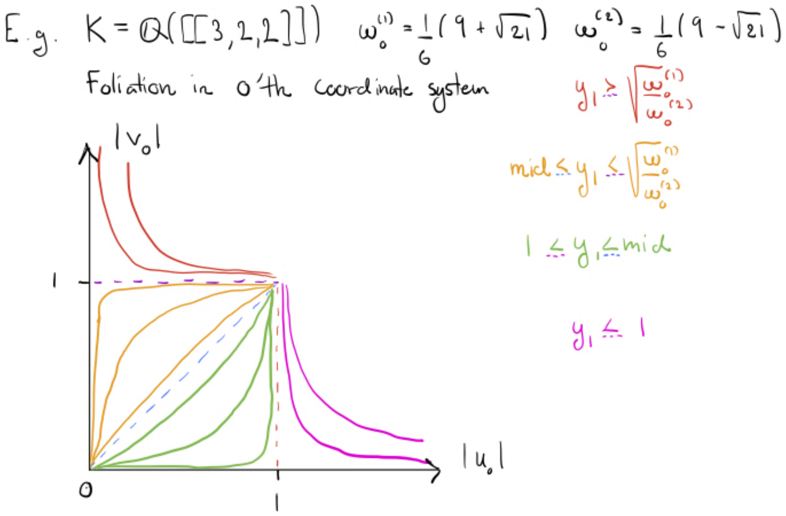


Figure 1: An illustration of the family of cone type geodesics going to the cusp.

1.3 A remark on a non-Archimedean ansatz

Recently, Li and Collins [CL22] found a generalized Calabi-ansatz [Cal79] for producing approximate Calabi-Yau metrics near the zero section of a direct sum of line bundles. These satisfy a Monge-Ampére type equation which they dubbed the non-Archimedean Monge-Ampére equation, and are supposed to model the geometry generically at the intersection of smooth prime divisors which have simple normal crossings.

The set up for the ansatz is as follows. One is given a compact (connected) Calabi-Yau manifold Y of dimension m and k positive line bundles L_i on Y. Then, one seeks a Calabi-Yau metric on an open neighbourhood of the zero section in the total space $Z = Tot(\bigoplus_{i=1}^k L_i)$. We write $\pi\colon Z \to Y$ for the projection map. Picking hermitian metrics h_i on the L_i, one obtains a splitting $Z \cong (\mathbb{R} \cup \{\infty\})^r \times U$ via the map

$$(v_{L_1}, \dots v_{L_k}) \mapsto (-\log(h_1(v_{L_1})), \dots, -\log(h_k(v_{L_k})), \frac{v_{L_1}}{h_1(v_{L_1})}, \dots \frac{v_{L_k}}{h_k(v_{L_k})}).$$

Here U denotes the total space of the direct sum of circle bundles associated to h_i over Y. Thus, one has global coordinate functions $r_i = h_{L_i}(v_{L_i})$, $y_i = -\log(r_i)$ smooth away from the divisor $\{h_i = 0\}$. Over any trivializing open $V \subset Y$ such that $Z|_{\pi^{-1}(V)} \cong V \times \mathbb{C}^k_{z_1,\dots,z_k}$, one may write $y_i = -\log(|z_i|) + \phi_i$, where ϕ_i is a smooth strictly plurisubharmonic function on V.

Now, the ansatz is to consider potentials of the form $u(y_1, \dots, y_k)$ which satisfy the equation

$$\det(D^2 u) \int_Y (\mathcal{D}u)^m = const,$$

in a region $\{\infty > y_1, \dots, y_i \gg 0\} \subset Z$ where

- $u\colon \mathbb{R}^k \to \mathbb{R}$ is smooth and (strictly) convex
- $\det(D^2 u)$ is the determinant of the Hessian in the coordinates $y_1, \dots, y_k$.
- $\mathcal{D}u = \sum_{i=1}^k \frac{du}{dy_i} c_1(L_i)$ is a function $Z \to H^2(Y, \mathbb{R})$.

As explained in [CL22], the two extreme cases $m = 0, k = 1$ are familiar. They correspond respectively to the case of the real Monge-Ampére on the cone $\mathbb{R}^k_{>0}$ in the $U(1)^k$-quotient $\mathbb{R}^k$ of $(\mathbb{C}^*)^k$ (hence by 1.1.6 correspond to torus invariant metrics on the open polydisc Δ_1^k) and the classical Calabi-ansatz on a line bundle over a compact Calabi-Yau manifold. Collins and Li, subsequently show that in the case of proportional line bundles, say

$L_i \cong L_1^{\otimes d_i}$, then the generalized Calabi ansatz, produces genuine Calabi-Yau metrics when u is strictly convex and $\sum_{i=1}^{k} \frac{du}{dy_i} d_i > 0$ [CL22, Lemma 2.1].

Following the same line of reasoning as in [CL22], it is tempting (and makes sense) to consider the same ansatz when trying to find Kähler-Einstein metrics, which are not necessarily Calabi-Yau, by modifying the non-archimedean Monge-Ampére equation accordingly to:

$$\det(D^2 u) \int_Y (\mathcal{D}u)^m = e^{-\lambda u}.$$

As in the Calabi-Yau setting, the extreme case $m = 0$ correspond to a real Monge-Ampére and the case $k = 1$ to the Calabi ansatz. Since the logarithm of the characteristic function ϕ for the cone $C = \mathbb{R}^k_{>0}$ is negative Kähler-Einstein, one has a solution for the ansatz by setting $u = \log(\phi)$ when $m = 0$, which is valid near the maximal intersection of divisors on the equivariant resolutions of the Hilbert Modular cusps.

By essentially the same argument, we show that the conclusion of [CL22, Lemma 2.1] holds in this setting as well. Assume $L_i \cong L_1^{d_i}$, and choose $h_{L_i} = h_{L_1}^{d_i}$, where h_{L_1} is picked to be the unique hermitian metric that one obtains from the solution of the Calabi Conjecture [Yau78]. That is, whose curvature form in $c_1(L_1)$ is Ricci-flat. Since Y is Calabi-Yau, there is a holmorphic volume form Ω_Y on Y such that

$$\int_Y i^{2m} \Omega_Y \wedge \bar{\Omega}_Y = 1.$$

Then, computing locally

$$\begin{aligned}
\frac{i}{2\pi} \partial\overline{\partial}\, u =& \frac{i}{2\pi}(\sum_{ij} \frac{\partial^2 u}{\partial y_i \partial y_j} \partial y_i \wedge \bar{\partial} y_j + \sum_i \frac{\partial u}{\partial y_i} \partial\overline{\partial}\, y_i) \\
=& \frac{i}{2\pi}(\sum_{ij} \frac{\partial^2 u}{\partial y_i \partial y_j} \partial y_i \wedge \bar{\partial} y_j + \sum_i \frac{\partial u}{\partial y_i} \partial\overline{\partial}\, \phi_i) \\
=& \frac{i}{2\pi}(\sum_{ij} \frac{\partial^2 u}{\partial y_i \partial y_j} \partial y_i \wedge \bar{\partial} y_j + (\sum_i \frac{\partial u}{\partial y_i} d_i)\, \partial\overline{\partial}\, \phi_1),
\end{aligned}$$

where ϕ_1 is a local potential for h_{L_1}. Since the y_i define coordinates and ϕ_1 comes from the base, we have

$$\begin{aligned}
(\frac{i}{2\pi} \sum_{ij} \frac{\partial^2 u}{\partial y_i \partial y_j} \partial y_i \wedge \bar{\partial} y_j)^{k+1} =& 0 \\
(\frac{i}{2\pi} \partial\overline{\partial}\, \phi_1)^{m+1} =& 0.
\end{aligned}$$

Then,

$$
\begin{aligned}
(\frac{i}{2\pi}\,\partial\overline{\partial}\,u)^{m+k} =&(\frac{i}{2\pi})^{m+k}(\sum_{n=1}^{m+k}\binom{m+k}{n}(\sum_{ij}\frac{\partial^2 u}{\partial y_i\partial y_j}\partial y_i\wedge\bar{\partial} y_j)^n \\
&\wedge((\sum_i \frac{\partial u}{\partial y_i}d_i)\,\partial\overline{\partial}\,\phi_1)^{m+k-n}) \\
=&(\frac{i}{2\pi})^{m+k}\frac{(m+k)!}{k!m!}(\sum_{ij}\frac{\partial^2 u}{\partial y_i\partial y_j}\partial y_i\wedge\bar{\partial} y_j)^k\wedge((\sum_i\frac{\partial u}{\partial y_i}d_i)\,\partial\overline{\partial}\,\phi_1)^m \\
=&(\frac{i}{2\pi})^{m+k}\frac{(m+k)!}{m!}\det(D^2u)(\sum_i\frac{\partial u}{\partial y_i}d_i)^m\prod_{i=1}^k\partial y_i\wedge\bar{\partial} y_i\wedge(\partial\overline{\partial}\,\phi_1)^m \\
=&(\frac{i}{2\pi})^{m+k}\frac{(m+k)!}{m!}\det(D^2u)(\sum_i\frac{\partial u}{\partial y_i}d_i)^m\prod_{i=1}^k\partial\log(|z_i|)\wedge\bar{\partial}\log(|z_i|) \\
&\wedge(\partial\overline{\partial}\,\phi_1)^m \\
=&\frac{(m+k)!}{m!}\det(D^2u)(\sum_i\frac{\partial u}{\partial y_i}d_i)^m\prod_{i=1}^k(\frac{i}{2\pi}\partial\log(|z_i|)\wedge\bar{\partial}\log(|z_i|)) \\
&\wedge(\frac{i}{2\pi}\,\partial\overline{\partial}\,\phi_1)^m.
\end{aligned}
$$

Here, we used that the only nonzero term occures when $n = k$ and the fact that $(\partial\overline{\partial}\,\phi_1)^m$ gives a volume form, so wedging with $\partial y_i = -\partial\log(|z_i|) + d_i\partial\phi_1$ is the same as wedging with $-\partial\log(|z_i|)$. Now, since $\frac{i}{2\pi}\,\partial\overline{\partial}\,\phi_1$ is Ricci flat on Y, we have

$$(\frac{i}{2\pi}\,\partial\overline{\partial}\,\phi_1)^m = c\Omega_Y\wedge\bar{\Omega}_Y$$

for some constant c. Thus, the normalization $\int_Y i^{2m}\Omega_Y\wedge\bar{\Omega}_Y = 1$ implies that

$$c = i^{2m}\int_Y c_1(L_1)^m.$$

Therefore, one has a solution to

$$\det(D^2u)\int_Y(\mathcal{D}u)^m = e^{-\lambda u},$$

if, and only if,

$$(\frac{i}{2\pi}\,\partial\overline{\partial}\,u)^{m+k} = \frac{(m+k)!}{m!}e^{-\lambda u}\prod_{i=1}^k(\frac{i}{2\pi}\partial\log(|z_i|)\wedge\bar{\partial}\log(|z_i|))\wedge(i^{2m}\Omega_Y\wedge\bar{\Omega}_Y).$$

Hence, if, and only if, $\frac{i}{2\pi}\partial\overline{\partial}\, u$ has Ricci curvature equal to λ.

It is not clear to the author, if the proportionality condition imposed on the line bundles is reasonable in general, when one considers model metrics near the simple normal crossing divisors on good resolutions of singularities, since exceptional divisors appearing on such resolutions typically are not $\mathbb{Q}$-linearly equivalent. If one restricts attention to a "maximal intersection" (say in a point) of exceptional divisors in a single chart, then the proportionality is void, and one gets back the real Monge-Ampére equation on the manifold $\mathbb{R}^k_{>0} \subset trop((\mathbb{C}^*)^k)$.

Part II

Stability of fibrations

2 The classical setting

Introduction

The search for special Kähler metrics representing cohomology classes of line bundles on compact Kähler manifolds has a long and rich history. From the problem of Calabi [Cal54] over to its eventual solution by Yau [Yau77], [Yau78] and the subsequent development of K-stability, designed to obstruct the existence of Kähler-Einstein or constant scalar curvature Kähler metrics (cscK) [Tia97],[Don02]. From the perspective of moduli theory, one of the key applications of the existence of such special metrics is to include it as part of the data for moduli problems in order to obtain a moduli space with better properties. For instance, the moduli space of Fano manifolds of dimension n with finite automorphism group, which admit Kähler-Einstein metrics is seperated [Oda12], and the moduli space of certain Fano surfaces is proper [OSS16]. In recent years, K-stability has taken on a life of its own, and has been used to prove, that the moduli space of K-polystable log fano pairs is projective [LXZ22].

In this chapter, we review some algebraic aspects of K-stability. This is done to motivate the development of fibration stability and, to give some details which will be needed later on in chapter 3. First, we describe an approach to the the cscK problem 2.1 to motivate the introduction of test configurations and the notion of K-stability 2.2 of ample test configurations. We introduce the Donaldson-Futaki invariant 2.2 and describe how to compute it intersection theoretically on the compactification of a test configuration. Afterwards, we describe a recent generalization of Fujita's β invariant 2.2 and sketch how it gives rise to a notion of stability. Finally, we sketch the link between filtrations of the section ring, test configurations and divisorial valuations, which provide the main conceptual background for the work in chapter 3 and the relation of the β invariant with the Donaldson-Futaki invariant 2.3.

2.1 CscK metrics - a motivating problem

In the following, we sketch the cscK problem and try to convey how one eventually is led to K-stability as a means for testing whether or not a given polarized manifold has a constant scalar curvature Kähler metric representing the first Chern class of the polarizing bundle. To fix a notation, let (X, ω_0) be a compact Kähler manifold of dimension n with Kähler class

$[\omega_0] \in H^2(X, \mathbb{R})$, and let

$$\mathcal{K}(\omega_0) = \{\phi \in C^\infty(X, \mathbb{R}) | \omega_\phi = \omega_0 + i\,\partial\overline{\partial}\,\phi > 0\}$$

be the space of potentials parametrizing the Kähler metrics in the class $[\omega_0]$.

Definition 2.1.1. A metric $\omega_\phi \in [\omega_0]$ is a *constant scalar curvature Kähler metric* (cscK) if

$$S(\omega_\phi) = \hat{S} = \frac{nc_1(M) \cdot [\omega_0]^{n-1}}{[\omega_0]^n},$$

where the latter is the intersection number in cohomology.

The Mabuchi functional

CscK metrics in a given cohomology class have by now several variational interpretations. They can be seen as extremal metrics with vanishing Futaki invariant. That is, they are critical points of the Calabi functional

$$Cal(\omega) = \int_X S(\omega)^2 \omega^n$$

such that, another functional, the Futaki invariant, vanishes on the space of holomorphic vector fields, admitting a holomorphy potential [Szé14].

Another variational interpretation of cscK metrics was given by Mabuchi [Mab85], who defined a functional on $\mathcal{K}(\omega_0)$, having precisely cscK metrics as its critical points. The space $\mathcal{K}(\omega_0)$ is an open cone in the vector space $C^\infty(X, \mathbb{R})$. Therefore, it has a natural differentiable structure, and one may identify $T_\phi \mathcal{K}(\omega) \cong C^\infty(X, \mathbb{R})$. In particular, one can define the 1-form

$$Fut_\phi(f) = \int_X f(\hat{S} - S(\omega_\phi))\omega_\phi^n,$$

which can be shown to be closed. Therefore, since $\mathcal{K}(\omega_0)$ is a cone containing 0, it is contractible. Thus, it follows by the infinite dimensional Poincaré Lemma [KM97] that there is a functional $\mathcal{M}\colon \mathcal{K}(\omega_0) \to \mathbb{R}$ unique up to the addition of a constant such that $d\mathcal{M}_\phi = Fut_\phi$.

Definition 2.1.2. The functional $\mathcal{M}\colon \mathcal{K}(\omega_0) \to \mathbb{R}$ is called the *Mabuchi* functional.

It is clear that $d\mathcal{M}_\phi = 0$, if, and only if, ω_ϕ is a cscK metric in the Kähler class of ω_0. One may introduce a Riemannian metric on $\mathcal{K}(\omega_0)$, which at a point ϕ is given by the formula

$$\langle f, g\rangle_\phi = \int_X fg\omega_\phi^n.$$

In turn, one obtains a notion of geodesics $t \mapsto \phi_t$ on $\mathcal{K}(\omega_0)$. One then has

Proposition 2.1.3 ([Mab87]). *The Mabuchi functional $\mathcal{M}\colon \mathcal{K}(\omega_0) \to \mathbb{R}$ is convex when restricted to any geodesic.*

In particular, an obstruction to the criticality of a point ϕ arises. If $t \mapsto \frac{\mathcal{M}(\phi_t)-\mathcal{M}(\phi)}{t}$ is asymptotically nonpositive for some geodesic path ϕ_t emmanating from ϕ, then ω_ϕ cannot be a cscK metric. This suggests that in order to have existence of a cscK metric, then for any geodesic ray $t \mapsto \phi_t$ the slope at infinity

$$\lim_{t\to\infty} \frac{\mathcal{M}(\phi_t)}{t}$$

must be nonnegative. Geodesic rays in $\mathcal{K}(\omega_0)$ give rise to trivial families $X \times \Delta_1^* \to \Delta_1^*$, where Δ_1^* is the punctured unit disc, such that the fibre X_t is equipped with the metric $\omega_{\phi_{-\log(|t|)}}$. Conversely, any S^1-invariant trivial family of fibrewise Kähler metrics represents a geodesic ray in $\mathcal{K}(\omega_0)$ when it satisfies a certain Monge-Ampére equation [Don99] on $X \times \Delta_1$.

Rephrased in this way, one might try to define a positivity notion on potential limits over 0, and then relate it to the slope at infinity for the Mabuchi functional in order to determine whether (X, ω_0) carries a cscK metric or not. This is morally the philosophy of K-stability introduced by G. Tian in [Tia97] and later refined by Donaldson in [Don02], which viewed this way is a means of probing the slopes at infinity for geodesics arising "algebraically", i.e., from $\mathbb{C}^*$-degenerations in projective space, when the Kähler class in question is the first Chern class of a line bundle.

2.2 K-stability

In the following, let (X, L) be a polarized variety. That is, L is an ample line bundle on the variety X.

Definition 2.2.1 ([Don02]). A test configuration for (X, L) is the following data $(\mathcal{X}, \mathcal{L})$ where

- $\mathcal{X}$ is a normal flat $\mathbb{C}^*$-equivariant family $\mathcal{X} \to \mathbb{C}$, where $\mathbb{C}$ is equipped with the standard action.
- The $\mathbb{C}^*$ action lifts to $\mathcal{L}$, making the bundle map $\mathcal{L} \to \mathcal{X}$, $\mathbb{C}^*$-equivariant.

- $(\mathcal{X}_1, \mathcal{L}_1) \cong (X, L^r)$ for some $r \in \mathbb{N}_{>0}$.

One calls r the exponent of the test configuration. We say the test configuration is *ample*, if $\mathcal{L}$ is relatively ample. If the central fibre is integral, we say the test configuration is *integral*.

From the definition, $\mathbb{C}^*$-equivariance of $\mathcal{L} \to \mathcal{X} \to \mathbb{C}$ implies that for all $t \neq 0$,

$$(\mathcal{X}_t, \mathcal{L}_t) \cong (\mathcal{X}_1, \mathcal{L}_1) = (X, L).$$

Hence, one has $\mathcal{L}|_{\mathcal{X}\setminus X_0} \cong p_1^* L$, as $\mathbb{C}^*$ linearized line bundles, where $p_1 \colon X \times \mathbb{C}^* \to X$ is the projection. Moreover, there is an induced $\mathbb{C}^*$ action on the central fibre $(\mathcal{X}_0, \mathcal{L}_0)$.

Definition 2.2.2. For at test configuration $(\mathcal{X}', \mathcal{L}')$ of(X, L), we call a map $f \colon (\mathcal{X}, \mathcal{L}) \dashrightarrow (\mathcal{X}', \mathcal{L}')$ a *modification* of $(\mathcal{X}', \mathcal{L}')$ if it is equivariant, proper and birational over $\mathbb{C}$, and $(\mathcal{X}, \mathcal{L})$ is also a test configuration for (X, L). We say $(\mathcal{X}, \mathcal{L})$ *dominates* $(\mathcal{X}', \mathcal{L}')$, if f is a morphism.

Example 2.2.3. Given any X with a polarization L, we always have the trivial test configuration, which is $\mathcal{X} = X \times \mathbb{C}$ equipped with the action on the second component and with the same action on $\mathcal{L} = p_1^* L = L \times \mathbb{C}$. More generally, given an effective action $\rho \colon \mathbb{C}^* \to Aut(X, L)$ one can similarly define $\mathcal{X} = X \times \mathbb{C}, \mathcal{L} = p_1^* L$ and equip the pair with the action

$$X \times \mathbb{C} \times \mathbb{C}^* \to X \times \mathbb{C}$$

given by $(x, t, t') \mapsto (\rho(t')x, t't)$. A test configuration arising in this fashion is called a product test configuration.

Any test configuration $(\mathcal{X}, \mathcal{L})$ is a modification of the trivial test configuration $(X \times \mathbb{C}, p_1^* L)$, since we by definition have an equivariant isomorphism

$$(X \times \mathbb{C}^*, p_1^* L) \cong (\mathcal{X}_{\mathbb{C}^*}, \mathcal{L}_{\mathcal{X}_{\mathbb{C}^*}})$$

over $\mathbb{C}^*$. This means that the interesting geometry of a test configuration all takes place near or at the central fibre.

A slightly less trivial, but prototypical example of a test configuration is the following:

Example 2.2.4. Given the elliptic curve $X = \{x^3 + xyz - z^3 = 0\} \subset \mathbb{P}^2_{\mathbb{C}}$ with $L = \mathcal{O}(1)_{\mathbb{P}^2_{\mathbb{C}}}|_X$ and take $\mathcal{X} = \{t^{3k_1}x^3 + t^{k_1+k_2+k_3}xyz - t^{3k_3}z^3 = 0\} \subset \mathbb{P}^2_{\mathbb{C}} \times \mathbb{C}$ with $\mathcal{L} = p_1^* L$. Now, clearly we have $\mathcal{X}_t \cong X$ for $t \neq 0$, since this is nothing but the closure of the orbit of X under the action $t \cdot [x, y, z] = [t^{k_1}x, t^{k_2}y, t^{k_3}z]$, where $k_1, k_2, k_3 \in \mathbb{Z}$. The action lifts to $\mathcal{O}(1)$ as $t \cdot l(x, y, z) = l(t^{k_1}x, t^{k_2}y, t^{k_3}z)$. Therefore, this is a test configuration of exponent $r = 1$.

In fact, the above example illustrates the principle that ample test configurations arise as flat completions of $\mathbb{C}^*$ orbits inside a projective space.

Proposition 2.2.5 ([RT06a, prop. 3.7]). *Any ample test configuration* $(\mathcal{X}, \mathcal{L})$ *of* (X, L) *is isomorphic to the Zariski closure of a* $\mathbb{C}^*$*-orbit inside a projective space.*

We shall need the following Lemma, whose proof is well known:

Lemma 2.2.6. *For any locally closed* $i\colon X \to Y$ *into a Noetherian scheme and* $f\colon Y \to \mathbb{C}$ *flat such that* $X \subset f^{-1}(\mathbb{C}^*)$ *is closed, there exists a unique subscheme* $j\colon X' \to Y$ *such that* $f \circ j$ *is flat and* $X'_{f^{-1}(\mathbb{C}^*)} \cong X$*.* X' *is scheme theoretic closure of* $i(X)$ *with its induced subscheme structure.*

Proof. Since flatness is a local property, we can prove existence by considering the local case, i.e., when $Y = Spec(B))$ is affine. Then, since X is a Zariski closed subset of the generic fibre, one has that $X = Spec((B \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm}])/I)$ for some ideal $I \subset B \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm}]$. The scheme theoretic closure is

$$X' = Supp(coker\colon \mathcal{O}_Y \to i_*(\mathcal{O}_X))$$

Hence, X' has as its defining ideal the kernel I' of the morphism

$$B \to (B \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm}])/I \to 0$$

which defines the inclusion $X \to Y$. Therefore $X' = Spec(B/I')$. Since we are over a discrete valuation ring, we have that B/I' is flat, if it is torsion free as a $\mathbb{C}[t]$-module (see e.g. [Eis95]). But since B/I' is isomorphic as a $\mathbb{C}[t]$-module to $B \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm}]/I$, the result follows if one knows that $B \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm}]/I$ is torsion free as a $\mathbb{C}[t]$-module. However, this is automatic as $B \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm}]/I$ is a flat $\mathbb{C}[t^{\pm}]$-module, and $\mathbb{C}[t^{\pm}]$ is a flat $\mathbb{C}[t]$-module. Indeed, from this it follows that $B \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm}]/I$ is a flat $\mathbb{C}[t]$-module. Therefore, the Zariski closure is flat and

$$B/I' \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm}] \cong B \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm}]/I,$$

so it satisfies the conditions of the lemma. Next we show uniqueness of this subscheme, which allows us to glue the previous construction. Suppose X', X'' are two closed subschemes of Y satisfying the conditions of the lemma. Then, the defining ideals J', J'' must satisfy

$$B/J' \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm}] \cong B/J'' \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm}].$$

Since $B/J', B/J''$ are both flat over $\mathbb{C}[t]$, it follows that the maps

$$B/J' \to B/J' \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm}]$$

given by $f \mapsto f \otimes_{\mathbb{C}[t]} 1$ are both injective (kernel is given by torsion elements). Hence, precomposing gives that the map

$$B \to B/J' \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm}]$$

has kernel exactly J'. The same goes for the map $B \to B/J'' \otimes_{\mathbb{C}[t]} \mathbb{C}[t^{\pm}]$, and so we have $J' = J''$, from which is follows that $X' = X''$ as subschemes. □

We call the X' in the previous Lemma the flat limit, or the flat completion of X.

Proof. (2.2.5) Denote the map $\pi \colon \mathcal{X} \to \mathbb{C}$. By relatively ampleness of $\mathcal{L}$ there is for some k sufficiently big an equivariant embedding

$$\mathcal{X} \to \mathbb{P}_{Spec(\mathbb{C}[t])}(\pi_*(k\mathcal{L}))$$

over $\mathbb{C} = Spec(\mathbb{C}[t])$. Note that $\mathcal{X}$ must be the flat limit of $\mathcal{X} \setminus \mathcal{X}_0$ which is isomorphic to $X \times \mathbb{C}^*$, which agrees with the orbit of $\mathcal{X}_1 \cong X$ under the $\mathbb{C}^*$-action.

Now, $\pi_*(k\mathcal{L})$ is flat over $\mathbb{C}[t]$ and finitely generated, hence it is projective, and thus free, since $\mathbb{C}[t]$ is a principal ideal domain. In other words, $\mathbb{P}_{Spec(\mathbb{C}[t])}(\pi_*(k\mathcal{L}))$ is a trivial projective bundle over $\mathbb{C}$ of the form $\mathbb{P}(V) \times \mathbb{C}$, where V is a finite dimensional $\mathbb{C}$ vector space. By embedding $\mathbb{C}$ equivariantly in $\mathbb{P}^1$, one obtains an equivariant embedding

$$\mathcal{X} \to \mathbb{P}(V) \times \mathbb{P}^1 \to \mathbb{P}^{2(\dim(V)+1)-1}$$

identifying $\mathcal{X}$ with the Zariski closure of the orbit $X \cong \mathcal{X}_1$ under a $\mathbb{C}^*$-action.

□

For a test configuration, there is in many circumstances a correspondence between scheme theoretic properties of $X, \mathcal{X}_0$ and the total space $\mathcal{X}$.

Proposition 2.2.7 ([BHJ17] prop. 2.6)**.** *Let $\mathcal{X}$ be a (not necessarily normal) test configuraiton for X, then*

- *$\mathcal{X}$ is reduced, if, and only if, X is reduced.*
- *If X is normal and $\mathcal{X}_0$ is reduced, then $\mathcal{X}$ is normal*
- *$\mathcal{X}$ is a variety if, and only if, X is a variety.*

In particular, we see that a $\mathbb{C}^*$-equivairant family $(\mathcal{X}, \mathcal{L}) \to \mathbb{C}$ extending (X, L) is normal when the central fibre is integral.

The Donaldson-Futaki Invariant

In the geometric invariant theory developed by Mumford [MFK94], the stability with respect to a group action of a point p in a projectice variety, can be characterized by the sign of the weights on a lift to affine space of the limit $\lim_{t\to 0}\rho(t)\cdot p = q$ under the action of any $\mathbb{C}^*$-subgroup. The celebrated Kempf-Ness theorem [KN79], [MFK94] gives another characterization of stability for points on an equivariantly polarized manifold (X, L) in terms of a moment-map μ of the action with respect to any Kähler metric in $c_1(L)$. Roughly speaking, the stability of a point depends on how the orbit intersects $\mu^{-1}(0)$. The moment map picture forms the basis for the notion of K-stability of a polarized variety. Indeed, a test configuration can be thought of as the closure of a $\mathbb{C}^*$-orbit in the category of polarized varieties, and the scalar curvature can be thought of as a moment map for this action whose norm squared is zero, if, and only if, the metric is cscK (see [Szé14]).

In this section, we introduce the Donaldson-Futaki invariant associated to a given test configuration. Intuitively, this should be thought of as a weight attached to the limit of the $\mathbb{C}^*$-orbit of (X, L), measuring the degeneration. Since $\mathcal{X}_0$ has a $\mathbb{C}^*$-action, there is a $\mathbb{C}^*$-action on $H^0(\mathcal{X}_0, k\mathcal{L}_0)$, and the Donaldson-Futaki invariant is defined in terms of the ratio between the asymptotic growth in dimension of $H^0(\mathcal{X}_0, k\mathcal{L}_0)$ and its total weight. By analyzing the asymptotic behaviour of Bergman kernels, Donaldson [Don05] proved that the negative of the Donaldson-Futaki invariant provides a lower bound for the norm of the moment map for any metric $\omega \in c_1(L)$. Thus, if there is a test configuration for (X, L), with negative Donaldson-Futaki invariant, then there cannot be a cscK metric in $c_1(L)$.

Definition 2.2.8. The weights of a $\mathbb{C}^*$-action, $\rho\colon \mathbb{C}^* \to Gl(V)$, on $\mathbb{C}$-vector space V is the collection of exponents appearing in the eigenspace decomposition

$$V \cong \bigoplus_{n\in\mathbb{Z}} V_{\chi_n}$$

where $V_{\chi_n} = \{v \in V | \rho(t)(v) = \chi_n(t)v\}$ and $\chi_n(t) = t^n$. The *total weight* $wt(V, \rho)$ is the weight of the action on $\det(V)$. In particular, if $n_1, \ldots, n_{\dim(V)}$ are the weights of the action, then the total weight is

$$wt(V, \rho) = \sum_{i=1}^{\dim(V)} n_i.$$

Alternatively, when writing $D\rho_1\colon T_1\,\mathbb{C}^* = \mathbb{C} \to \mathfrak{gl}(V)$, the total weight is $Tr(D\rho_1(1))$. Observe that when one has an equivariant short exact sequence,

$$0 \to (\rho_M, M) \xrightarrow{f} (\rho_N, N) \xrightarrow{g} (\rho_V, V) \to 0.$$

Then the natural equivariant isomorphism

$$\det(N) \cong \det(M) \otimes \det(V)$$

given by

$$m_1 \wedge \cdots \wedge m_{rk(M)} \otimes v_1 \wedge \cdots \wedge v_{rk(V)} \mapsto f(m_1) \wedge \cdots \wedge f(m_{rk(M)}) \wedge n_1 \wedge \cdots \wedge n_{rk(V)}$$

where n_i is arbitrary in the fibre $g^{-1}(v_i)$, shows that the total weight satisfies

$$wt(\rho_N, N) = wt(\rho_M, M) + wt(\rho_V, V).$$

If f is merely an injection such that the image is preserved by ρ_N, then we have

$$wt(\rho_N, N) = wt(\rho_N|_{Im(f)}, Im(f)) + wt(\rho_V, V).$$

Thus, the weight determines a map

$$wt\colon \{(V, \rho_V)\} \to \mathbb{Z},$$

which is is additive on equivariant short exact sequences.

Remark 2.2.9. We note here a convention. Given a linear action $\rho\colon \mathbb{C}^* \to Gl(V)$ on V, with weights $\lambda_1, \ldots, \lambda_n$ and weight vectors v_i, then we equip the dual vector space $V^\vee$ with the unique $\mathbb{C}^*$-action such that the pairing $\langle w^\vee, v \rangle = w^\vee(v)$ satisfies $\langle t \cdot w^\vee, t \cdot v \rangle = \langle w^\vee, v \rangle$ for all, $v \in V, w^\vee \in V^\vee$. Thus, the action is given by $t \cdot w^\vee(v) = w^\vee(t^{-1} \cdot v)$, and the dual basis $v_i^\vee$ become weight vectors of weight $-\lambda_i$. We call this the dual action. From an action on V, there is an induced action on $\mathbb{P}_{\mathbb{C}}(V)$ such that $\mathcal{O}(-1)$ becomes an equivariant line bundle. Indeed, the action is just given by the action on V, so $\mathcal{O}(1)$ also has the structure of an equivariant line bundle, along with the dual action.

It is well known that the global sections are $H^0(\mathbb{P}_{\mathbb{C}}(V), \mathcal{O}(1)) = V^\vee$, which has the dual action. Therefore, the weights of the global sections are $-\lambda_1, \ldots, -\lambda_n$. In particular, since $H^0(\mathbb{P}_{\mathbb{C}}(V), \mathcal{O}(k)) = Sym^k_{\mathbb{C}}(V^\vee)$, we have weights given by $(-1)^k \lambda_{i_1} \ldots \lambda_{i_k}$, where $1 \leq i_j \leq n, \forall j$. The standard action on the vector space $\mathbb{C}$ is of weight 1, hence the trivially extended action to $\mathbb{C}^2$ has weights $0, 1$. Therefore, the weights of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$ with this action is $0, -1$.

When $\mathbb{C}^*$ acts algebraically on an affine scheme $X = Spec(A)$, then this is equivalent to a $\mathbb{Z}$ grading of A say $A = \bigoplus_{\lambda \in \mathbb{Z}} A_\lambda$, where A_λ is a $\mathbb{C}$-module. Here, A_λ consists of the algebraic functions which are homogeneous of degree λ with respect to the $\mathbb{C}^*$-action. When $A = \mathbb{C}[x_1, \ldots, x_n]$ and the $\mathbb{C}^*$-action is linear, then the relation between weights of the associated vector space and this decomposition is given as follows. The associated vector space has weights $\lambda_1, \ldots, \lambda_n$,if, and only if, there are generators $z_1, \ldots, z_n$ of the $\mathbb{C}$-algebra A such that $z_i \in A_{\lambda_i}$. One has a similar result when considering $X = Proj_B(\bigoplus_{k\geq 0} A_k)$, with $A_0 = B$. If there is a graded decomposition as a B-algebra

$$\bigoplus_{k\geq 0} A_k = \bigoplus_{k\geq 0}(\bigoplus_{\lambda\in\mathbb{Z}} A_{k,\lambda}),$$

where $A_k = \bigoplus_{\lambda\in\mathbb{Z}} A_{k,\lambda}$ is a decomposition of B-modules. Then there is a $\mathbb{C}^*(B) = Spec(B[t^{\pm}])$-action on $Proj_B(\bigoplus_{k\geq 0} A_k)$. Indeed, one considers the graded comorphism

$$\bigoplus_{k\geq 0} A_k \to (\bigoplus_{k\geq 0} A_k) \otimes_B B[t^{\pm}]$$

given on homogeneous components by $a_k \mapsto \sum_\lambda a_{k,\lambda} \otimes t^\lambda$. Upon localizing to any affine given by some element $a \in A_{k,\lambda}$, this gives a morphism $D(a) \times \mathbb{C}^*(B) \to D(a)$. Since the graded ring can be generated by such elements, these then glue to a global morphism $X \times \mathbb{C}^*(B) \to X$. In the case where $A_{0,\lambda} = 0$ whenever $\lambda \neq 0$, then the morphism is over $Spec(B)$, corresponding to a $\mathbb{C}^*$-action on the fibres of $X \to Spec(B)$. In the special case where $B = \mathbb{C}[t]$, then $\mathbb{C}^*(B) = Spec(\mathbb{C}[t^{\pm}]) = \mathbb{C}^*$ gives a $\mathbb{C}^*$-action such that the map $X \to \mathbb{C}$ is equivariant, because we have commutativity of the diagram

$$\begin{array}{ccc}
(\bigoplus_{k\geq 0} A_k) \otimes_{\mathbb{C}[t']} \mathbb{C}[t^{\pm}] & \longleftarrow & \bigoplus_{k\geq 0} A_k \\
\uparrow & & \uparrow \\
\mathbb{C}[t'] \otimes \mathbb{C}[t^{\pm}] & \longleftarrow & \mathbb{C}[t'].
\end{array}$$

Continuing in this special case, assume X is complete, then one has a finite dimensional $\mathbb{C}$ vector space $H^0(X, \mathcal{O}(k)) = A_k$ with an induced $\mathbb{C}$-linear $\mathbb{C}^*$-action $A_k \times \mathbb{C}^* \to A_k$ induced by the comorphism $A_k \to A_k \otimes \mathbb{C}[t^{\pm}]$. This is constructed in such a way that the weight spaces are $A_{k,\lambda}$. Thus the total weight is

$$wt(H^0(X, \mathcal{O}(k)) = \sum_{\lambda\in\mathbb{Z}} \dim_{\mathbb{C}}(A_{k,\lambda})\lambda.$$

Lemma 2.2.10. *For any $\mathbb{C}^*$-linearized ample line bundle L on a variety X, the total weight on the global sections $wt(H^0(X, L^k))$ is a degree* $\dim(X)+1$ *polynomial in k for $k \gg 0$.*

Proof. We proceed by induction on the dimension of X. Assume $dim(X) = 0$, then a $\mathbb{C}^*$-linearized line bundle is just a 1-dimensional $\mathbb{C}$-vector space L with an equivariant identification $H^0(X, L) \cong L$. Therefore $H^0(X, kL) \cong L^{\otimes k}$ and the comments above show that

$$wt(H^0(X, kL)) = kwt(L) = kwt(H^0(X, L))$$

a polynomial of the desired type. Assume the claim holds for any pair (Y, H), where $dim(Y) < dim(X)$ and H ample. Then, since L is ample we can (by tensoring if necessary) assume that the space of invariant sections $H^0(X, L)^{\mathbb{C}^*} \neq 0$, and that $H^1(X, L^k) = 0$ for all $k > 0$. Pick $s \in H^0(X, L)^{\mathbb{C}^*}$ nonzero and consider the equivariant short exact sequence

$$0 \to L^{k-1} \stackrel{\otimes s}{\to} L^k \to \iota_* \iota^* L^k \to 0$$

where $\iota \colon V(s) \to X$ is the inclusion of the vanishing locus for s. Then, there is an equivariant short exact sequence

$$0 \to H^0(X, L^{k-1}) \to H^0(X, L^k) \to H^0(V(s), \iota^* L^k) \to 0$$

so

$$\begin{aligned} \det(H^0(X, L^k)) &= \det(H^0(X, L^{k-1})) \otimes \det(H^0(V(s), (\iota^* L)^k)) \\ &= \dots \\ &= \bigotimes_{i=0}^{k-1} \det(H^0(V(s), L^{k-i})), \end{aligned}$$

which implies that

$$wt(H^0(X, L^k)) = \sum_{i=0}^{k-1} wt(H^0(V(s), (\iota^* L)^{k-i})).$$

Since $\iota^* L$ is ample, we can use the induction hypothesis to (by potentially tensoring L further) assume that $wt(H^0(V(s), (\iota^* L)^m))$ is a polynomial

$P(m)$ for all $m \geq 0$. Thus

$$\begin{aligned}
\frac{1}{k} wt(H^0(X, L^k)) &= \frac{1}{k} \sum_{i=0}^{k-1} P(k-i) \\
&= \frac{1}{k} \sum_{i=1}^{k} P(i) \\
&= \frac{1}{k} \sum_{i=1}^{k} P(\frac{i}{k} k) \\
&\underset{k \to \infty}{\to} \int_0^1 P(tk) dt.
\end{aligned}$$

So since $wt(H^0(X, L^k))$ is integer valued we must have the equality

$$wt(H^0(X, L^k)) = k \int_0^1 P(tk) dt = \int_0^k P(t) dt$$

for $k \gg 0$. □

From the previous lemma, we see that when $dim(X) = n$, then we can expand for $k \gg 0$ as follows:

$$wt(H^0(\mathcal{X}_0, k\mathcal{L}_0)) = b_0 k^{n+1} + b_1 k^b + O(k^{n-1}).$$

Similarly, we can expand the Hilbert polynomial

$$h^0(\mathcal{X}_0, k\mathcal{L}_0) = a_0 k^n + a_1 k^{n-1} + \mathcal{O}(k^{n-2}))$$

which coincides with the Hilbert polynomial for $h^0(X, kL)$, since $\mathcal{X} \to \mathbb{C}$ is flat, and $\mathcal{L}$ is relatively ample, so $t \to h^0(\mathcal{X}_t, k\mathcal{L}|_{\mathcal{X}_t}) = \chi(\mathcal{X}_t, k\mathcal{L}|_{\mathcal{X}_t})$ is locally constant for $k \gg 0$ [Har10, III thm. 9.9]. We are now ready to define the Donaldson-Futaki invariant of a test configuration.

Definition 2.2.11. [Don02] The Donaldson-Futaki invariant of a test-configuration $(\mathcal{X}, \mathcal{L})$ of (X, L) is the integer

$$DF(\mathcal{X}, \mathcal{L}) = \frac{a_1 b_0 - a_0 b_1}{a_0}.$$

This coefficient is the $-C_1$ term in the asymptotic expansion

$$\frac{wt(H^0(\mathcal{X}_0, k\mathcal{L}_0))}{k h^0(\mathcal{X}_0, k\mathcal{L}_0)} = C_0 + C_1 k^{-1} + O(k^{-2}).$$

The invariant is a generalization of the classical Futaki invariant for polarized varieties. In fact, in the case of product test configurations the differential geometric Futaki invariant is -4π times the Donaldson-Futaki invariant [Szé14, chapter 7].

Definition 2.2.12. [Don02] A polarized variety (X, L) is

- *K-semistable* if $DF(\mathcal{X}, \mathcal{L}) \geq 0$ for any test configuration $(\mathcal{X}, \mathcal{L})$
- *K-stable* if $DF(\mathcal{X}, \mathcal{L}) > 0$ for all nontrivial test configurations.
- *K-polystable* if (X, L) is K-semistable and $DF(\mathcal{X}\mathcal{L}) = 0$, if, and only if, it is a product test configuration $(X \times \mathbb{C}, p_1^* L^r)$.

We clearly have the implications

$$\text{K-stable} \implies \text{K-polystable} \implies \text{K-semistable}.$$

As alluded to earlier, K-stability is important, because it conjectured to be an algebraic criterion for the existence of a cscK metrics in the Chern class of L.

Conjecture 1 (Yau-Tian-Donaldson (YTD)). A smooth polarized variety (X, L) has a cscK metric in $c_1(L)$, if, and only ifk (X, L) is K-polystable.

Various parts and modifications of the YTD conjecture has been established in numerous special cases, for instance the Fano case $L = -K_X$ in the breakthrough papers [CDS14a][CDS14b] [CDS14c]. It is however expected that a slightly stronger condition than K-polystability is needed for the statement above to hold in general [Szé15].

Importantly, the Donaldson-Futaki invariant of a test configuration can be computed intersection theoretically after completing it over $\mathbb{P}^1_{\mathbb{C}}$, [Wan12],[Oda13]. Concretely, the completion of a test configuration is constructed as follows: since a test configuration is equivariantly trivial away from the central fibre $(\mathcal{X} \setminus \mathcal{X}_0, \mathcal{L}|_{\mathcal{X}\setminus\mathcal{X}_0}) \cong (X \times \mathbb{C}^*, p_1^* L = L \times \mathbb{C}^*)$, one can extend this portion over infinity to a pair $(\overline{\mathcal{X}}, \overline{\mathcal{L}}) \to \mathbb{P}^1_{\mathbb{C}}$ by gluing the two pieces $(X \times \mathbb{C}, p_1^* L)$ along the portion, where one has an equivariant splitting. That is, it is the pushout of the following diagram

$$\begin{array}{ccc} (\mathcal{X} \setminus \mathcal{X}_0, \mathcal{L}|_{\mathcal{X}\setminus\mathcal{X}_0}) \cong (X \times \mathbb{C}^*, p_1^* L), & \xrightarrow{Id \times (\cdot)^{-1}} & (X \times \mathbb{C}, p_1^* L) \\ \big\downarrow{\scriptstyle i} & & \\ (\mathcal{X}, \mathcal{L}) & & \end{array}$$

where the action on $(X, \times \mathbb{C})$ is such that $Id \times (\cdot)^{-1}$ becomes equivariant. This operation gives $\overline{\mathcal{L}}$ a $\mathbb{C}^*$-linearized line bundle on $\overline{\mathcal{X}}$ such that the $\mathbb{C}^*$-action $\overline{\mathcal{L}}|_{\overline{\mathcal{X}}_\infty} \cong L$ is trivial. Observe that the extension operation

$$\mathcal{L} \mapsto \overline{\mathcal{L}}$$

is multiplicative, in the sense that it preserves tensor products when tensoring with line bundles $\mathcal{M}$ such that $\mathcal{M}|_{\mathcal{X}\setminus\mathcal{X}_0} \cong p_1^* M$, i.e.,

$$\mathcal{L} \otimes \mathcal{M} \mapsto \overline{\mathcal{L} \otimes \mathcal{M}} \cong \overline{\mathcal{L}} \otimes \overline{\mathcal{M}}.$$

From the multiplicative property and the fact that the trivial bundle extends trivially, it follows that the assignment $\mathcal{L} \mapsto \overline{\mathcal{L}}$ is injective for all such line bundles.

There is a notion of a norm of a test configuration first introduced by Dervan [Der16]

Definition 2.2.13. The (minimum)-*norm* of a completed test configuration is the number

$$||(\overline{\mathcal{X}}, \overline{\mathcal{L}})||_{min} = \frac{\overline{\mathcal{L}}^{n+1}}{n+1} - \overline{\mathcal{L}}^{n}.(\overline{\mathcal{L}} - \phi^* p_1^* L),$$

where $\phi \colon \mathcal{Y} \to X \times \mathbb{P}^1_{\mathbb{C}}$ is a resolution of indeterminacy of $\mathcal{X} \dashrightarrow X \times \mathbb{P}^1_{\mathbb{C}}$ and we have surpressed the pullback of $\overline{\mathcal{L}}$.

It is then a theorem [Der16], [BHJ17] that the vanishing of this quantity exactly detects when a completed test configuration comes from a product test configuration. The fact that there is an intersection theoretic criterion detecting product test configurations is an instance of the general principle, that the invariants arising from test configurations can be interpreted and calculated as intersection numbers on its completion.

Lemma 2.2.14. *Denote the completed test configuration by* $p \colon (\overline{\mathcal{X}}, \overline{\mathcal{L}}) \to \mathbb{P}^1_{\mathbb{C}}$*. If the test configuration* $(\mathcal{X}, \mathcal{L})$ *is ample, then one can compute the weight polynomial* $wt(H^0(\mathcal{X}_0, k\mathcal{L}_0))$ *as a Hilbert polynomial*

$$wt(H^0(\mathcal{X}_0, k\mathcal{L}_0)) = h^0(\overline{\mathcal{X}}, \overline{\mathcal{L}}^k \otimes p^* \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)).$$

Proof. The space $\bar{\mathcal{X}}$ is clearly normal when $\mathcal{X}$ is so, hence, there is a well defined theory of Weil divisors on $\bar{\mathcal{X}}$. Moreover, it is irreducible since it is glued together from two irreducibles over an open affine subset. Note that the bundle $\bar{\mathcal{L}}$ is p-ample, since the restriction to any fibre is ample. Moreover, for any coherent sheaf $\mathcal{F}$ on $\bar{\mathcal{X}}$, the derived sheaves satisfy $R^i p_*(\mathcal{F} \otimes \bar{\mathcal{L}}^k) = 0, \forall i > 0$, when $k \gg 0$ (see [Laz04, pp 94-97]) .

By definition, the two fibres $\overline{\mathcal{X}}_0 = \mathcal{X}_0, \overline{\mathcal{X}}_\infty \cong X$ are rationally equivalent effective divisors on $\overline{\mathcal{X}}$. Hence, their associated line bundles are isomorphic. Indeed, they are both vanishing loci of $\sigma_0, \sigma_\infty \in H^0(\overline{\mathcal{X}}, p^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1))$ of weight $-1, 0$ respectively.

Multiplication by σ_0, σ_∞ gives two families of short exact sequences of sheaves

$$0 \to \overline{\mathcal{L}}^k \otimes p^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1) \overset{\otimes\sigma_0}{\to} \overline{\mathcal{L}}^k \to \overline{\mathcal{L}}^k|_{\mathcal{X}_0} \to 0$$

$$0 \to \overline{\mathcal{L}}^k \otimes p^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1) \overset{\otimes\sigma_\infty}{\to} \overline{\mathcal{L}}^k \to \overline{\mathcal{L}}^k|_{\overline{\mathcal{X}}_\infty} \to 0.$$

Taking the pushforward of these short exact sequences by p, one obtains by the vanishing $R^i(p_*(\bar{L} \otimes p^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1))) = 0, i > 0$ another pair of short exact sequences of sheaves on $\mathbb{P}^1_{\mathbb{C}}$. Taking global sections of these, one gets long exact sequences in cohomology. But by the vanishing $R^i(p_*(\bar{L} \otimes p^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1))) = 0$ for $i > 0$, the Leray spectral sequence degenerates on page 2, so

$$H^i(\bar{\mathcal{X}}, \bar{\mathcal{L}}^k \otimes p^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)) = H^i(\mathbb{P}^1_{\mathbb{C}}, p_*(\mathcal{L}^k) \otimes \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)) = 0, \forall i > 0.$$

One reaches similar conclusions for the other terms in the sequence, so when $k \gg 0$, one has short exact sequences of global sections given by

$$0 \to H^0(\overline{\mathcal{X}}, \overline{\mathcal{L}}^k \otimes p^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)) \overset{\otimes\sigma_0}{\to} H^0(\overline{\mathcal{X}}, \overline{\mathcal{L}}^k) \to H^0(\mathcal{X}_0, \mathcal{L}_0) \to 0$$

$$0 \to H^0(\overline{\mathcal{X}}, \overline{\mathcal{L}}^k \otimes p^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)) \overset{\otimes\sigma_\infty}{\to} H^0(\overline{\mathcal{X}}, \overline{\mathcal{L}}^k) \to H^0(\overline{\mathcal{X}}_\infty, \overline{\mathcal{L}}^k|_{\overline{\mathcal{X}}_\infty}) \to 0.$$

Observe, that the second short exact sequence is equivariant because σ_∞ is invariant, so from the additive property of the weight, it follows that we have

$$wt(H^0(\overline{\mathcal{X}}, \overline{\mathcal{L}}^k)) = wt(Im(\otimes\sigma_\infty)) + wt(H^0(\overline{\mathcal{X}}_\infty, \overline{\mathcal{L}^k_\infty})). \tag{2}$$

The first short exact sequence is not equivariant, however, it comes from multiplication by a section σ_0 of weight -1, hence the image is an invariant subspace and thus

$$wt(H^0(\overline{\mathcal{X}}, \overline{\mathcal{L}}^k)) = wt(Im(\otimes\sigma_0)) + wt(H^0(\mathcal{X}_0, \mathcal{L}_0^k)). \tag{3}$$

Since $\overline{\mathcal{L}}^k$ is generated by global sections, and $\overline{\mathcal{X}}$ is irreducible, we can with no essential loss of information consider the $\mathbb{C}^*$-invariant Zariski dense open subset U where $\frac{\sigma_\infty}{\sigma_0}$ is defined. Then, any $s \in Im(\otimes\sigma_0)$ satisfies

$$s|_U = \frac{\sigma_0}{\sigma_\infty}\tilde{s}|_U$$

for some uniquely determined $\tilde{s} \in Im(\otimes\sigma_\infty)$. The assignment $s \mapsto \tilde{s}$ is a linear isomorphism $Im(\otimes\sigma_0) \to Im(\otimes\sigma_\infty)$. Moreover, it shifts the weight spaces by 1. Indeed suppose $t \cdot s = t^\lambda s$, then

$$\begin{aligned} t \cdot \tilde{s}|_U &= t \cdot (\frac{\sigma_\infty}{\sigma_0} s|_U) \\ &= t\frac{\sigma_\infty}{\sigma_0} t \cdot s|_U \\ &= t^{\lambda+1}\frac{\sigma_\infty}{\sigma_0} s|_U \\ &= t^{\lambda+1}\tilde{s}|_U \end{aligned}$$

Hence $t \cdot \tilde{s} = t^{\lambda+1}\tilde{s}$. Therefore,

$$wt(Im(\otimes\sigma_\infty)) = wt(Im(\otimes\sigma_0)) + h^0(\overline{\mathcal{X}}, \overline{\mathcal{L}}^k \otimes p^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)).$$

Combining the two formulas 2,3 with this relation we get

$$wt(H^0(\mathcal{X}_0, \mathcal{L}_0^k)) = h^0(\overline{\mathcal{X}}, \overline{\mathcal{L}}^k \otimes p^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)) + wt(H^0(\overline{\mathcal{X}_\infty}, \overline{\mathcal{L}^k_\infty})),$$

which is the desired equation once we note that the action on $(\overline{\mathcal{X}}_\infty, \overline{\mathcal{L}}_\infty) \cong (X, L)$ is trivial, hence all global sections must be invariant and so the total weight is 0. □

Theorem 2.2.15 ([Wan12],[Oda13])**.** *If $\bar{\pi}\colon \bar{\mathcal{X}} \to \mathbb{P}^1_{\mathbb{C}}$ is a completed test configuration associated to an ample test configuration $(\mathcal{X}, \mathcal{L})$ and $n = dim(X)$, then one has*

$$2(n!)DF(\mathcal{X}, \mathcal{L}) = \frac{n}{n+1}\mu(X, L)\bar{\mathcal{L}}^{n+1} + \bar{\mathcal{L}}^n \cdot K_{\bar{\mathcal{X}}/\mathbb{P}^1_{\mathbb{C}}}$$

where $\mu(X, L) = \frac{-K_X \cdot L^{n-1}}{L^n}$ is the slope of L with respect to X and $K_{\bar{\mathcal{X}}/\mathbb{P}^1_{\mathbb{C}}}$ is the relative canonical class given by $K_{\bar{\mathcal{X}}} - \bar{\pi}^*K_{\mathbb{P}^1_{\mathbb{C}}}$.

Proof. The Hirzebruch-Riemann-Roch theorem holds for normal quasi-projective varieties [Ful14, chapter 18], so we have

$$\begin{aligned} h^0(\mathcal{X}_0, k\mathcal{L}_0) =& h^0(X, kL) \\ =& \frac{k^n}{n!}L^n + \frac{k^{n-1}}{(n-1)!}L^{n-1} \cdot (-\frac{K_X}{2}) + O(k^{n-2}), \end{aligned}$$

when $k \gg 0$. By the previous lemma, we can calculate by Riemann-Roch

$$
\begin{aligned}
wt(H^0(\mathcal{X}_0, k\mathcal{L}_0)) =& h^0(\bar{\mathcal{X}}, k\bar{\mathcal{L}} + \bar{\pi}^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)) \\
=& \int_X ch(\bar{\mathcal{L}}^k \otimes \bar{\pi}^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)) td(\bar{\mathcal{X}}) \\
=& (\frac{k^{n+1}}{(n+1)!}\bar{\mathcal{L}}^{n+1} + \frac{k^n(n+1)}{(n+1)!}\bar{\mathcal{L}}^n \cdot \bar{\pi}^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)) td_0(\bar{\mathcal{X}}) \\
&+ \frac{k^n}{n!}\bar{\mathcal{L}}^n td_1(\bar{\mathcal{X}}) + O(k^{n-1}) \\
=& \frac{k^{n+1}}{(n+1)!}\bar{\mathcal{L}}^{n+1} + \frac{k^n}{2(n!)}\bar{\mathcal{L}}^n \cdot (-K_{\bar{\mathcal{X}}} + 2\bar{\pi}^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)) + O(k^{n-1}) \\
=& \frac{k^{n+1}}{(n+1)!}\bar{\mathcal{L}}^{n+1} - \frac{k^n}{2(n!)}\bar{\mathcal{L}}^n \cdot K_{\bar{\mathcal{X}}/\mathbb{P}^1_{\mathbb{C}}} + O(k^{n-1}),
\end{aligned}
$$

where we used that $td_0(\bar{\mathcal{X}}) = 1, td_1(\bar{\mathcal{X}}) = \frac{c_1(\bar{\mathcal{X}})}{2} = \frac{-K_{\bar{\mathcal{X}}}}{2}, K_X = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-2)$. From this we obtain

$$
\begin{aligned}
DF(\mathcal{X}, \mathcal{L}) =& \frac{a_1}{a_0}b_0 - b_1 \\
=& \frac{n!(-K_X) \cdot L^{n-1}}{(n-1)!2L^n}\frac{1}{(n+1)!}\bar{\mathcal{L}}^{n+1} + \frac{1}{2(n!)}\bar{\mathcal{L}}^n \cdot K_{\bar{\mathcal{X}}/\mathbb{P}^1_{\mathbb{C}}} \\
=& \frac{1}{2(n!)}(\frac{n}{n+1}\mu(X, L)\bar{\mathcal{L}}^{n+1} + \bar{\mathcal{L}}^n \cdot K_{\bar{\mathcal{X}}/\mathbb{P}^1_{\mathbb{C}}}).
\end{aligned}
$$

□

Fujita's β invariant and its generalization

In this section, we define and describe an invariant associated to prime divisors defined over a polarized variety (X, L) originally defined by Fujita [Fuj19] and recently generalized by Dervan and Legendre [DL22]. We fix a polarized variety (X, L). First we need some definitions:

Definition 2.2.16. The *slope* of (X, L) is the number

$$\mu(X, L) = \frac{-K_X.L^{n-1}}{L^n}.$$

Definition 2.2.17. A prime divisor $D \subset Y$ is *over* X if there is a birational morphism $f: Y \to X$.

Definition 2.2.18. The log discrepancy function

$$A_X\colon \{D \text{ divisor over X}\} \to \mathbb{Q}$$

is given by

$$A_X(D) = 1 + ord_D(K_Y - f^*K_X),$$

where $f\colon Y \to X$ is the given birational morphism with $D \subset Y$.

The log discrepancy function is a measure of the multiplicity of the local equations for D, which one has when pulling back $(n,0)$-forms over the regular part of X along f.

Volume of divisors

Definition 2.2.19. The *volume* of a Cartier divisor $D \subset Y$ over X is

$$Vol(D) = \limsup_{k\to\infty} \frac{h^0(Y, kD)}{k^{\dim(X)}},$$

if $Vol(D) > 0$ we say D is big. The big divisors form an open cone inside $Pic(X)$.

The volume is a birational invariant of normal varieties in the sense that if $g\colon Y' \to Y$ is a birational morphism of normal varieties such that $H = g^*D$, then $Vol(H) = Vol(D)$. For any Cartier divisor D over X we shall set

$$Vol(L - tD) := Vol(f^*L - tD).$$

The following are a list of properties of the volume function:

Proposition 2.2.20 ([Laz04])**.** *The volume function satisfies*

- $Vol(kD) = k^{\dim(X)}Vol(D)$
- $Vol(D)$ *depends only on the numerical equivalence class of* D.
- *Choosing a norm on the finite dimensional vector space* $N^1_{\mathbb{R}}(X)$*, then there is a constant* $C(||\cdot||) > 0$ *such that*

  $$|Vol(D) - Vol(E)| \leq C \cdot (\max(||D||, ||E||))^{\dim(X)-1} \cdot ||D - E||$$

  for any two $D, E \in N^1_{\mathbb{Q}}(X)$.

In particular, we can extend the initial definition from $Pic(X)$ to $Pic_{\mathbb{Q}}(X)$ by using the homogeneity property and from $Pic_{\mathbb{Q}}(X)$ to $Pic_{\mathbb{R}}(X)$ by utilizing the continuity estimate. By doing so, the set of Big $\mathbb{R}$-divisors

$$Big(X) \subset Pic_{\mathbb{R}}(X)$$

form an open cone (closed under addition by effective divisors), and one can prove that the volume actually has slightly more regularity on $Big(X)$.

Theorem 2.2.21 ([BFJ08]). *Vol is a $\mathcal{C}^1$ function on the Big cone, moreover one has that for any D big and E arbitrary, then*

$$\frac{d}{dt} Vol(D + tE)|_{t=0} = n\langle D^{n-1}\rangle \cdot E,$$

where $\langle\rangle$ denotes the positive intersection product [BFJ08], which is also homogeneous and superadditive in all entries and coincides with the standard intersection product when all entries are nef.

In this setup, the slope of (X, L) can be thought of quite literally as the slope of the volume in the direction of the anticanonical divisor, since we have

$$n\mu(X, L)Vol(L) = \frac{d}{dt} Vol(L - tK_X)|_{t=0}.$$

Definition 2.2.22. Let L be a big divisor. Then, the pseudoeffective threshold of a divisor D with respect to a divisor L is the quantity

$$\tau_L(D) = \sup\{t > 0 | Vol(L - tD) > 0\}.$$

Alternatively, one can decribe the pseudoeffective threshold to be

$$\tau_L(D) = \sup\{t > 0 | L - tD \geq 0\}.$$

Indeed, if $Vol(L - iD) > 0$, then $L - iD$ is in particular effective so $\tau_L(D) \leq \sup\{t > 0 | L - tD \geq 0\}$. But since the big cone is the interior of the pseudoeffective cone ([Laz04] p. 147) (i.e., closure of the effective cone), it follows that for $i > \tau_L(D)$, we must have $H^0(Y, f^*L - iD) = 0$. Therefore, we have equality. Another property to remark is the scaling properties

$$\tau_{kL}(D) = k\tau_L(D)$$

$$\tau_L(kD) = \frac{1}{k}\tau_L(D)$$

for any $k > 0$, which follow directly from the scaling property of the volume 2.2.20.

Definition 2.2.23 ([DL22]). For any prime divisor D over (X, L), the β invariant with respect to L is

$$\begin{aligned}\beta_L(D) =& A_X(D)Vol(L) + n\mu(X,L)\int_0^{\tau_L(D)} Vol(L - tD)dt \\ &+ \int_0^{\tau_L(D)} \frac{d}{ds} Vol(L - tD + sK_X)|_{s=0}dt.\end{aligned}$$

It is shown in [DL22] that this gives back Fujita's original invariant from [Fuj19] in the Fano case, i.e., when the polarization $L = -K_X$ the formula for the β-invariant reduces to

$$\beta_{-K_X}(D) = A_X(D)Vol(-K_X) - \int_0^{\tau_L(D)} Vol(-K_X - tD)dt.$$

Note that the β invariant also scales as

$$\beta_{kL}(D) = k^{\dim(X)}\beta_L(D).$$

Very recently, Boucksom and Jonsson [BJ22] extended the β invariant to positive convex linear combinations of divisorial valuations, which they interpret as certain probability measures supported finite subsets of divisorial points in an associated Berkovich space. From this, Boucksom and Jonsson obtain a valuative stability condition implying uniform K-stability, which is the requirement that the Donaldson Futaki invariant is bounded from below by a constant times the minimum norm of the test configuration.

Having defined the β invariant, we are now in a position to define a stability condition for (X, L) coming from divisors.

Definition 2.2.24. We say (X, L) is *valuatively*

- *semistable* if $\beta_L(D) \geq 0$ for all non-trivial prime divisors D over X.
- *stable* if $\beta_L(D) > 0$ for all nontrivial prime divisors D over X
- *uniformly stable* if there is an $\epsilon > 0$ such that

$$\beta_L(D) \geq \epsilon \int_0^{\tau_L(D)} Vol(L) - Vol(L - tD)dt$$

for all nontrivial prime divisors over D over X.

Note that this differs slightly from the convention in [DL22], where they only define valuative stability with respect to dreamy divisors. Uniformly valuatively stable means that when

$$\beta_L(D) = 0,$$

we are forced to have $Vol(L) = Vol(L - tD), \forall 0 \leq t < \tau_L(D)$. But this cannot be when D is non-trivial as we will now explain. The volume satisfies $Vol(L-tD) \leq Vol(L), t \in (0, \tau_L(D))$ with equality for all such t if and only if $D \equiv_{num} 0$, since in that case

$$0 = \frac{d}{dt} Vol(L - tD)|_{t=0} = -n(f^*(L)^{n-1}) \cdot D = -n(L^{n-1}) \cdot f_*(D),$$

which contradicts ampleness of L. Now, by Kleiman's characterization of numerically trivial bundles [Laz04, p. 19], one has that $nD \sim 0$ i.e. $D = 0$ in $Pic_{\mathbb{Q}}(Y)$. Therefore we have the implications

$$\text{uniformly stable} \implies \text{stable} \implies \text{semistable}.$$

2.3 A link between valuative stability and K-stability

We describe how there is a relation between the notion of valuative stability for a special class of divisors over X and K-stability of integral test configurations. First, we sketch a construction arising from filtrations of modules and graded algebras, due to Rees [Eis95, chapter 5], which is implicitly used in much of what follows. Fix a given s-module M and a seperated descending filtration $F \colon \mathbb{Z} \to s - subm(M)$, i.e.,

- $F_{j+1} \hookrightarrow F_j, \forall j$.
- $F_j \cong M$, when j is sufficiently small.
- $\bigcap_{i \in I} F_i M = 0$ in M.

Then, one can form the $s[t]$-submodule

$$\tilde{M} := \bigoplus_{j \in Z} t^{-j} F_j \subset M[t^{\pm 1}].$$

Lemma 2.3.1. *One has for any M that*

- $\tilde{M}$ *is flat over* $\mathbb{C}[t]$
- $(\tilde{M})_{(t)} \cong M[t^{\pm 1}]$
- $\tilde{M}/(t-c)\tilde{M} \cong M$
- $\tilde{M}/t\tilde{M} \cong Gr(M,F) = \bigoplus_{j\in\mathbb{Z}} F_j/F_{j+1}$.

Proof. The statements are shown in the same order.

- A module is flat over a discrete valuation ring R, if it torsion free as an R-module. Therefore, it suffices to check that multiplication by t is injective. We have

  $$t(\sum_j t^{-j}s_j) = \sum_k t^{-(j-1)}s_j = 0 \iff \forall j, 0 = s_j \in F_{j-1} \iff \forall j, s_j = 0 \in F_j$$

  since the inclusions $F_{j+1} \hookrightarrow F_j$ are all injective. Hence, $\tilde{M}$ is flat over $\mathbb{C}[t]$.

- By exactness of localization, one has

  $$(\tilde{M})_{(t)} \subset M[t^{\pm 1}]_{(t)} = M[t^{\pm 1}],$$

  but this inclusion is surjective since any element mt^i $i \geq 0$ is in the image by definition and mt^i $i < 0$ is in the image of $m\frac{1}{t^{-i}}$.

- Consider the s-linear map $\phi_c\colon \tilde{M} \to M$ given by $\phi_c(\sum_j t^{-j}s_j) = \sum c^{-j}s_j$. It is surjective, and one has $(t-c)\tilde{M} \subset \ker(\phi_c)$. To show the converse containment, it is enough to find a factorization

  $$\begin{aligned}\sum_{j\in\mathbb{Z}} t^{-j}s_j &= (t-c)\sum_{j\in\mathbb{Z}} t^{-1}k_j \\ &= \sum_{j\in\mathbb{Z}} t^{-j}(k_{j-1} - ck_j)\end{aligned}$$

  whenever $\sum_{j\in\mathbb{Z}} t^{-j}s_j \in \ker(\phi_c)$. Thus, one solves the equation $s_j = k_{j-1} - ck_j$. Since $\tilde{M}$ is direct sum, there is a minimal j_0 and a maximal j_1 such that $s_{j_0}, s_{j_1} \neq 0$. So one can set $k_{j_0} = c^{-1}s_{j_0}, k_{j_0-p} = 0 \forall p \in \mathbb{N}$. Now, we solve recursively to obtain

  $$k_j = -c^{-j}\sum_{k=j_0}^{j} c^{k-1}s_k.$$

This terminates for $k_{j_1+1} = 0$, if, and only if, $\sum_{k=j_0}^{j_1} t^{-k} s_k \in \ker(\phi_c)$, because

$$k_{j_1+1} = c^{-1} k_{j_1} = -c^{j_1} \sum_{k=0}^{j_1} c^k s_k = -c^{j_1} \phi_c(\sum_{j=j_0}^{j_1} t^{-k} s_k).$$

Thus $\ker(\phi_c) \subset (t-c)\tilde{M}$.

- Consider the map $\phi_0 \colon \tilde{M} \to Gr(M, F)$ given by $\phi_0(\sum_{j\in\mathbb{Z}} t^{-j} s_j) = ([s_j])_{j\in\mathbb{Z}}$. Then $\sum_{j\in\mathbb{Z}} t^{-j} s_j \in \ker(\phi_0)$, if, and only if, $s_j \in F_{j+1}$ for all $j \in \mathbb{Z}$. But this is equivalent to the factorization $\tilde{M}$ as

$$\sum_{j\in\mathbb{Z}} t^{-j} s_j = t(\sum_{j\in\mathbb{Z}} t^{-(j+1)} s_j).$$

□

Next, we extend the same reasoning to graded systems of filtrations, which enables us to define projective degenerations of schemes by applying the proj or relative proj functors. Suppose one has a graded system of filtrations for a $\mathbb{N}$-graded ring $R = \bigoplus_{k\geq 0} R_k$ of s-modules with $R_0 = s$. I.e., for all $k \in \mathbb{Z}_{>0}$ one has a filtration

$$F_k \colon \mathbb{Z} \to s - subm(R_k)$$

such that it is compatible with the graded multiplication:

$$F_{k,j} \cdot F_{k',j'} \subset F_{k+k',j+j'}.$$

Then one can form the graded ring of $s(t)$-modules:

$$\hat{R} = \bigoplus_{k\geq 0} \tilde{R}_k = \bigoplus_{k\geq 0} (\bigoplus_{j\in\mathbb{Z}} t^{-j} F_{k,j}) \subset R[t^{\pm 1}].$$

Lemma 2.3.2. *The ring $\hat{R}$ has the following properties*

- $\hat{R}_{(t)} \cong R[t^{\pm 1}]$
- $\hat{R}/(t-c)\hat{R} \cong R$
- $\hat{R}/(t)\hat{R} = \bigoplus_{k\geq 0} Gr(R_k, F_k)$

Proof.

- Localization commutes with arbitrary direct sums, i.e,

$$\begin{aligned}\hat{R}_{(t)} &= (\bigoplus_{k\geq 0}\tilde{R}_k)_{(t)} \\ &\cong \bigoplus_{k\geq 0}(\tilde{R}_k\ )_{(t)} \\ &\cong \bigoplus_{k\geq 0} R_k[t^{\pm 1}] \\ &\cong (\bigoplus_{k\geq 0} R_k)[t^{\pm 1}] \\ &= R[t^{\pm 1}].\end{aligned}$$

- The ideal $(t-c)\hat{R} = \bigoplus_{k\geq 0}(t-c)\tilde{R}_k$ is homogenous so

$$\begin{aligned}\hat{R}/(t-c)\hat{R} &\cong \bigoplus_{k\geq 0}\tilde{R}_k/(t-c)\tilde{R}_k \\ &\cong \bigoplus_{k\geq 0} R_k \\ &= R.\end{aligned}$$

- The ideal $t\hat{R}$ is again homogeneous, so

$$\begin{aligned}\hat{R}/(t)\hat{R} &\cong \bigoplus_{k\geq 0}\tilde{R}_k/(t)\tilde{R}_k \\ &\cong \bigoplus_{k\geq 0} Gr(R_k, F_k).\end{aligned}$$

□

A natural source of filtrations arise as seen in the following example.

Example 2.3.3. Given D an effective divisor over (X, L), then we always have an injection

$$0 \to f^*L - D \to f^*L,$$

so we have

$$H^0(Y, kf^*L - \lambda D) \subset H^0(Y, kf^*L) \cong H^0(X, kL).$$

In particular, one has a $\mathbb{Z}$-filtration of the vector space $H^0(X, kL)$ with a natural multiplication in the section ring $R(L) = \bigoplus_{k\geq 0} H^0(X, kL)$ given by the tensor product.

From divisors to test configurations

We give a class of divisors over X that always give rise to an integral test configuration. Moreover, we show that the weight function $k \to wt_k$ has a nice description.

Definition 2.3.4 ([Fuj19])**.** An effective prime divisor $D \subset Y \xrightarrow{f} X$ over X is L-dreamy if there is an integer $r > 0$ such that the graded ring

$$R(rL, D) = \bigoplus_{k \geq 0} (\bigoplus_{\lambda \in \mathbb{Z}} t^{-\lambda} H^0(Y, krf^*L - \lambda D))$$

is finitely generated as a $\mathbb{C}[t]$-algebra. Write $R(rL, D)_k$ for the k graded piece, and write $R(L, D)_{\geq j}$ for the truncated subring

$$R(L, D)_{j \geq k} = \bigoplus_{k \geq j} R(L, D)_k$$

with its $R(L, D)$-module structure.

As in the Rees construction, we think of $\bigoplus_{\lambda \in \mathbb{Z}} t^{-\lambda} H^0(Y, kf^*L - \lambda D) \subset H^0(Y, kf^*L) \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$ with the inherited $\mathbb{C}[t]$-module structure, and the multiplication is defined by the multiplication of sections

$$0 \to H^0(Y, kf^*L - \lambda D) \otimes H^0(Y, k'f^*L - \lambda' D) \to H^0(Y, (k+k')f^*L - (\lambda + \lambda')D).$$

Definition 2.3.5. Given D an L-dreamy divisor, then we call the scheme

$$(Proj_{Spec(\mathbb{C}[t])}(R(L, D)), \mathcal{O}(1))$$

the test configuration for (X, L) associated to D.

Lemma 2.3.6. *The test configuration associated to an L-dreamy divisor is an ample integral test configuration*

Proof. The scheme has a natural morphism

$$\pi \colon Proj_{Spec(\mathbb{C}[t])}(R(L, D)) \to \mathbb{C}$$

given by the identification $\mathbb{C}[t] \cong R(L, D)_0$. The morphism π is flat, because all $R(L, D)_k$ are flat $\mathbb{C}[t]$-modules and thus all homogeneous localization is flat. Moreover, it is naturally $\mathbb{C}^*$-equivariant, since the $\mathbb{C}^*$-action on $R(L, D)$ is defined by the $\mathbb{Z}$-grading as in 2.2.9.

By compatibility or relative proj with base change and the Rees construction, the fibre over 1 is

$$\begin{aligned}Proj_{Spec(\mathbb{C}[t])}(R(L,D)) \times_{\mathbb{C}} Spec(\mathbb{C}[t]/(t-1)) &\cong Proj_{Spec(\mathbb{C}[t]/(t-1))}(R(L,D)/(t-1)R(L,D))\\ &\cong Proj_{Spec(\mathbb{C})}(\bigoplus_{k\geq 0} H^0(X,kL))\\ &\cong X,\end{aligned}$$

and the fibre over 0 is

$$\begin{aligned}Proj_{Spec(\mathbb{C}[t])}(R(L,D)) \times_{\mathbb{C}} Spec(\mathbb{C}[t]/(t)) &\cong Proj_{Spec(\mathbb{C}[t]/(t))}(R(L,D)/tR(L,C))\\ &\cong Proj_{Spec(\mathbb{C})}(Gr(R(L,D))),\end{aligned}$$

where

$$Gr(R(L,D)) = \bigoplus_{k\geq 0}(\bigoplus_{\lambda\in\mathbb{Z}} H^0(X,kf^*L-\lambda D)/H^0(X,kf^*L-(\lambda+1)D)).$$

This ring is integral since any two sections $s_1 \in H^0(X,kf^*L-\lambda D), s_2 \in H^0(X,k'f^*L-\lambda' D)$ has nonzero product $s_1s_2 \in H^0(X,(k+k')f^*L-(\lambda+\lambda')D)/H^0(X,(k+k')f^*L-(\lambda+\lambda'+1)D)$, when s_1 vanish exactly to order λ and vanish exactly to order λ' along D. Thus, the central fibre is integral as well. It only remains to be seen that $\mathcal{O}(1)$ is relatively ample. By finite generation of $R(L,D)$, there is a j sufficiently big such that $\mathcal{O}(1)^{\otimes j} \cong \mathcal{O}(j)$ is the line bundle associated to the graded $R(L,D)$-module $R(L,D)_{\geq j}$, which is generated in degree 1. Therefore, one has a graded surjection

$$S((R(L,D)_{\geq j})) \to R(L,D)_{\geq j} \to 0.$$

Here, S denotes the symmetric algebra. This surjection induces a closed immersion

$$i\colon Proj_{Spec(\mathbb{C}[t])}(R(L,D)) \to \mathbb{P}_{Spec(\mathbb{C}[t])}(R(L,D)_{\geq j})$$

such that $\mathcal{O}(j) = i^*\mathcal{O}(1)$. □

From the explicit description, one readily obtains the weight.

Lemma 2.3.7. *The weight of the induced test configuration is given by*

$$wt_k = (\lambda_{min}(k)-1)h^0(X,kf^*L) + \sum_{\lambda=\lambda_{min}(k)}^{\lambda_{max}} h^0(X,kf^*L-\lambda D)),$$

where

$$\begin{aligned}\lambda_{min}(k) =& \inf\{\lambda | H^0(X, kf^*L - \lambda D) \neq H^0(X, kf^*L)\}\\ \lambda_{max}(k) =& \sup\{\lambda | H^0(X, kf^*L - \lambda D) \neq 0\}.\end{aligned}$$

Proof. Since the central fibre is

$$\mathcal{X}_0 \cong Proj_{Spec\mathbb{C}}(Gr(R(L, D))),$$

and the bundle is $\mathcal{O}(1)|_{\mathcal{X}_0}$, one knows by the remark 2.2.9 that the weight is

$$\begin{aligned}wt_k =& \sum_{\lambda\in\mathbb{Z}} \dim_{\mathbb{C}}(Gr(R(L, D)_{k,\lambda}))\lambda\\ =& \sum_{\lambda\in\mathbb{Z}} \left[h^0(X, kf^*L - \lambda D) - h^0(X, kf^*L - (\lambda + 1)D)\right]\lambda\\ =& (\lambda_{\min}(k) - 1)h^0(X, kf^*L) + \sum_{\lambda=\lambda_{min}(k)}^{\lambda_{max}(k)} h^0(X, kf^*L - \lambda D).\end{aligned}$$

□

From test configurations to filtrations

From a test configuration $(\mathcal{X}, \mathcal{L})$ of (X, L), there is a natural filtration of the section ring $R(L) = \bigoplus_{k\geq 0} H^0(X, kL)$ originally investigated by David Nyström in [Wit12].

Definition 2.3.8. A filtration of a graded ring $R = \bigoplus_{k\geq 0} R_k$ is an assignment of R_0-submodules $(\lambda, k) \mapsto F^\lambda R_k \subset R_k$ for $(\lambda, k) \in \mathbb{Z}\times\mathbb{N}$ such that the multiplication respects the grading

- $F^\lambda R_k \cdot F^{\lambda'} R_{k'} \subset F^{\lambda+\lambda'} R_{k+k'}$.

We say it is *decreasing* if for each k fixed, $\lambda \to F^\lambda R_k$ is decreasing, i.e., $F^{\lambda+1} R_k \subset F^\lambda R_k$. We say the filtration is *permissible* if it has these bounds:

- There is a λ_0 such that $F^{\lambda_0} R_k = R_k$ for all k.
- There is a constant C such that $F^{Ck} R_k = 0$ for all k.

Example 2.3.9. The filtration induced by an effective divisor over (X, L) is permissible when D is L-dreamy.

Lemma 2.3.10 ([Wit12]). *Any ample test configuration induces a permissible filtration.*

Proof. (Sketch) We set

$$F^\lambda H^0(X, kL) = \{s \in H^0(X, kL) | t^{-\lambda}\bar{s} \in H^0(X, k\mathcal{L})\},$$

where $\bar{s}$ is the invariant extension $\bar{s}(u,t) = t \cdot s(u, 1)$ obtained by identifying equivariantly $\mathcal{L}_{\mathcal{X}\setminus\mathcal{X}_0} \cong p_1^* L$ where $p_1 \colon X \times \mathbb{C} \to \mathbb{C}$. This is clearly a decreasing filtration of the section ring $R(L)$ in the sense above. It is permissible by the argument in [Wit12, proposition 6.4].

□

Lemma 2.3.11 ([Wit12]). *The weight polynomial of a test configuration can be described in terms of the filtration as*

$$\begin{aligned} wt_k &= \sum_j j(\dim(F^\lambda H^0(X, kL)) - \dim(F^\lambda H^0(X, kL))) \\ &= \sum_{j=\lambda^k_{min}}^{\lambda^k_{max}} \dim(F^\lambda H^0(X, kL) + (\lambda^k_{min} - 1)h^0(X, kL), \end{aligned}$$

where again

$$\begin{aligned} \lambda^k_{min} &= \inf\{\lambda | F^\lambda H^0(X, kL) \neq H^0(X, kL)\} \\ \lambda^k_{max} &= \sup\{\lambda | F^\lambda H^0(X, kL) \neq 0\}. \end{aligned}$$

So, this is analogous with the test configurations arising from dreamy divisors. From the Rees construction 2.3.2, it follows that the permissible filtration associated to a test configuration gives rise to another test configuration, which has the same total weight as the original test configuration, hence these two test configurations have the same Donaldson-Futaki invariants as observed in [Szé15].

Filtrations of test configurations described by divisors

In this section, we follow [BHJ17] to see how filtrations associated to certain test configurations can be described by divisors over X.

Definition 2.3.12. Given a field $\mathbb{C} \subset K$, then a valuation v on K is a map $v\colon K \to \bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ such that for all $f, g \in K$ one has

- $v(f) = 0 \iff f = 0$,
- $v(fg) = v(f) + v(g)$,
- $v(f + g) \geq \min\{v(f), v(g)\}$,
- $v(z) = 0, \quad \forall z \in \mathbb{C}^*$.

By the correspondence $v \mapsto e^{-v}$, this notion of a valuation is equivalent to a non-Archimedean norm on K which extends the trivial norm on $\mathbb{C}$. Associated to a valuation v are

- The valuation ring $v^{-1}(\bar{\mathbb{R}}_{\geq 0})$, which is a local subring of K with maximal ideal $v^{-1}(\bar{\mathbb{R}}_{\geq 0})$,
- the value group $(v(\mathbb{R}), +)$,
- the residue field $v(\bar{\mathbb{R}}_{\geq 0})/v(\bar{\mathbb{R}}_{>0})$.

Given a dominant morphism $f\colon X \to Y$ of varieties, then the map on function fields $f^*\colon K(Y) \to K(X)$ is an injection, and one can consider the pushforward f_*v of a valuation v on $K(X)$ given by $f_*v(h) = v(f^*h)$. Abhyankar [Abh56] proved the following general inequality for valuations v on a field K regarding restriction to smaller field extensions $\mathbb{C} \subset F \subset K$.

$$tr.deg(v) + rk_{\mathbb{Q}}(v) \leq tr.deg(v|_F) + rk_{\mathbb{Q}}(v|_F) + tr.deg(K/F)$$

here $tr.deg(v)$ denotes the transcendence degree of the residue field over $\mathbb{C}$, and $rk_{\mathbb{Q}}(v)$ is the rank over $\mathbb{Q}$ of the value group. Taking $K = F = K(Y)$ for some variety, then one has in particular

$$tr.deg(v) + rk_{\mathbb{Q}}(v) \leq dim(Y). \tag{4}$$

From this inequality, one identifies classes of valuations on $K(Y)$ which are special:

Definition 2.3.13. A valuation v on $K(Y)$ is

- *Abhyankar* if there is equality in 4.
- *Divisorial* if $rk_{\mathbb{Q}}(v) = 1$ and $tr.deg(v) = dim(Y) - 1$.

The general Abhyankar inequality then gives some control of the types of valuations one obtains when considering for example dominating morphisms $Y \to X$ and corresponding injection $K(X) \to K(Y)$. In fact, we shall consider only the case

$$p_1 \colon X \times \mathbb{C} \to X$$

with corresponding injection $p_1^* \colon K(X) \to K(X \times \mathbb{C}) \cong K(X)(t)$. We denote $(p_1)_* v = v|_{K(X)}$.

From the Abhyankhar inequalities, one has

Lemma 2.3.14 ([BHJ17]). *If v is Abhyankhar on $K(X \times \mathbb{C}) = K(X)(t)$, then so is $v|_{K(X)}$. Moreover, if v is divisorial, then $v|_{K(X)}$ is either divisorial or trivial.*

Proof. If v is Abhyankhar, then

$$tr.deg(v) + rk_{\mathbb{Q}}(v) = \dim(X) + 1.$$

Thus, from the Abhyankhar inequality applied twice one has

$$\dim(X) \leq tr.deg(v|_{K(X)}) + rk_{\mathbb{Q}}(v|_{K(X)}),$$

but, $tr.deg(v|_{K(X)}) + rk_{\mathbb{Q}}(v|_{K(X)}) \leq dim(X)$ always holds, hence the result. If v is divisorial, then it is in particular Abhyankhar so

$$tr.deg(v|_{K(X)}) + rk_{\mathbb{Q}}(v|_{K(X)}) = \dim(X).$$

Since the value group of $v|_{K(X)}$ is contained in the one for v, it follows that $rk_{\mathbb{Q}}(v|_{K(X)}) \leq 1$. In the case $rk_{\mathbb{Q}}(v|_{K(X)}) = 0$, one has $v|_{K(X)} = v_{triv}$, and in the case $rk_{\mathbb{Q}}(v|_{K(X)}) = 1$, one has that $v|_{K(X)}$ is divisorial. □

We shall consider $K(X)(t)$ with the dual action of the standard $\mathbb{C}^*$ action on $\mathbb{C}$, that is $z \cdot f(x,t) = f(x, z^{-1}t)$ for $f \in K(X)(t)$. There is yet another action on valautions v given by dualizing the dual action, i.e.,$(z \cdot v)(f) = v(z^{-1} \cdot fx)$.

Lemma 2.3.15 ([BHJ17]). *A valuation v on $K(X)(t)$ is $\mathbb{C}^*$ invariant, if, and only if, it has the normal form*

$$v(f) = \min_{\mu \in \mathbb{Z}} (v|_{K(X)}(f_\mu) + \mu v(t)),$$

where $f = \sum_{\mu \in \mathbb{Z}} f_\mu t^\mu$, $f_\mu \in K(X)$. In particular, a $\mathbb{C}^*$ invariant valuation restricts as $v|_{K(X)} = v_{triv}$, if, and only if, $v = c\,ord_t$ for some $c > 0$.

Proof. Observe that

$$z \cdot \sum_{\mu \in \mathbb{Z}} f_\mu t^\mu = \sum_{\mu \in \mathbb{Z}} (f_\mu z^{-\mu}) t^\mu :$$

Therefore the expression

$$h_v(f) := \min_{\mu \in \mathbb{Z}} (v|_{K(X)}(f_\mu) + \mu v(t))$$

must be invariant, since $v|_{K(X)}(z^{-\mu} f_\mu) = v|_{K(X)}(f_\mu)$.

In general one has

$$\begin{aligned} v(f) =& v(\sum_{\mu \in \mathbb{Z}} f_\mu t^\mu) \\ \geq& \min_{\mu \in \mathbb{Z}} (v(f_\mu t^\mu) \\ =& \min_{\mu \in \mathbb{Z}} (v(f_\mu) + \mu v(t)) \\ =& \min_{\mu \in \mathbb{Z}} (v|_{K(X)}(f_\mu) + \mu v(t)) \\ =& h_v(f). \end{aligned}$$

To see the converse inequality when v is invariant, consider the set $\Gamma = \{\mu | f_\mu \neq 0\}$, and consider the finite dimensional vector space $span_{\mathbb{C}}(f_\mu t^\mu | \mu \in \Gamma) \cong \mathbb{C}^{|\Gamma|}$ with its induced $\mathbb{C}^*$-action. Picking distinct $z_\mu \in \mathbb{C}^*$ for each $\mu \in \Gamma$, one has a $\mathbb{C}$-basis $z_\mu \cdot f$. Indeed,

$$\begin{aligned} 0 =& \sum_{\mu \in \Gamma} c_\mu z_\mu \cdot f \\ =& \sum_\lambda f_\lambda (\sum_\mu c_\mu z_\mu^{-\lambda}) t^\lambda \\ 0 =& \sum_\mu c_\mu z_\mu^{-\lambda}, \quad \forall \lambda \in \Gamma \\ 0 =& Zcm \end{aligned}$$

where $Z = (z_\mu^{-\lambda})_{\mu, \lambda \in \Gamma}$ and $c = (c_\mu)_{\mu \in \Gamma}$. But Z is invertible, since it is a multiple of a Vandermonde matrix, hence $c = 0$. Thus, we can write uniquely

$$f_\mu t^\mu = \sum_\lambda c_\lambda z_\lambda \cdot f.$$

So, it follows that

$$\begin{aligned} v|_{K(X)}(f_\mu) + \mu v(t) =& v(f_\mu t^\mu) \\ =& v(\sum_\lambda c_\lambda z_\lambda \cdot f) \\ \geq& \min_{\lambda\in\Gamma}(v(c_\lambda) + v(z_\lambda \cdot f)) \\ =& \min_{\lambda\in\Gamma} v(z_\lambda \cdot f) \\ =& min_{\lambda\in\Gamma} v(f) \\ =& v(f). \end{aligned}$$

□

From this characterization, one sees that the central fibre $X \times \{0\}$ of the trivial test configuration is, up to strict transforms, the only irreducible component of a central fibre of a test configuration for X which induces the valuation ord_t on $K(X \times \mathbb{C})$. In particular, for nontrivial integral test configurations it follows that the induced valuation on $K(X)$ is divisorial.

Proposition 2.3.16 ([BHJ17])**.** *A divisorial valuation v on $K(X)(t)$ satisfies $v(t) > 0$ and is $\mathbb{C}^*$-invariant, if, and only if, $v = cord_D$, $c > 0$ and D is an irreducible component of the central fibre of a test configuration $\mathcal{X}$ for X.*

Proof. (Sketch) Since any test configuration is equivariantly birational to the trivial one $\rho\colon \mathcal{X} \dashrightarrow X \times \mathbb{C}$, there is an isomorphism $K(X \times \mathbb{C}) \cong K(\mathcal{X})$. It is clear that $\rho_* ord_D(t) = ord_D(\rho^* t) > 0$ since $\mathcal{X}_0 = \{\rho^* t = 0\}$ is effective. The valuation is $\mathbb{C}^*$-invariant, because D is $\mathbb{C}^*$-invariant. For the converse, suppose v is a $\mathbb{C}^*$-invariant divisorial valuation on $K(X \times \mathbb{C})$. Then, upon compactifying the trivial test configuration and blowing up the centres of v iteratively, it follows by a theorem of Zariski 3.2.10, [KM98, lemma 2.45] that there is a divisor D on one of these dominating models $\rho\colon Y \to X \times \mathbb{P}^1_{\mathbb{C}}$, such that $c\rho_* ord_D = v$ for some $c > 0$. Since the blown up centres are $\mathbb{C}^*$-invariant and contained in the central fibre at each step, the resulting composition of maps ρ will be $\mathbb{C}^*$-equivariant and define a test configuration for X. □

Proposition 2.3.17 ([BHJ17]). *Let $(\mathcal{X}, \mathcal{L})$ be a test configuration dominating the trivial test configuration $((X \times \mathbb{C}), p_1^* L)$ via a map g, such that $\mathcal{L} = (p_1 \circ g)^* L + D$. Then, the filtration of the section ring $R(L)$, induced by a test configuration satisfies for $m \gg 0$ and all λ that*

$$F^\lambda H^0(X, mL) = \bigcap_E \{s \in H^0(X, mL) | v_E(s) + m \cdot ord_E(\mathcal{X}_0)^{-1} ord_E(D) \geq \lambda\},$$

where $v_E = \frac{g_* ord_E|_X}{g_* ord_E(t)}$ and E are all irreducible components of $\mathcal{X}_0$.

Proof. From the definition of the filtration associated to a test configuration, one has that $s \in F^\lambda H^0(X, mL)$, if, and only if, $t^{-\lambda}\bar{s} \in H^0(\mathcal{X}, m\mathcal{L})$, with $\bar{s}$ being the invariant rational extension over $\mathcal{X} \setminus \mathcal{X}_0$. Thus for any E irreducible component of $\mathcal{X}_0$ one has $ord_E(t^{-\lambda}\bar{s}) \geq 0$ as it extends holomorphically. This is equivalent to $ord_E(\bar{s}) \geq \lambda g_* ord_E(t) = \lambda ord_E(\mathcal{X}_0)$. The result follows, since $m\mathcal{L} = m(p_1 \circ g)^*(mL) + mD$ gives locally $\bar{s} = (p_1 \circ g)^* s f^m$, where f is the equation for D. Hence,

$$\begin{aligned} ord_E(\bar{s}) =& ord_E(p_1^* g^* s) + m \cdot ord_E(D) \\ =& g_* ord_E(s)|_X + m \cdot ord_E(D) \\ =& ord_E(\mathcal{X}_0) v_E(s) + m \cdot ord_E(D). \end{aligned}$$

□

When the central fibre of a test donfiguration dominating the trivial one is irreducible, then the proposition says that the filtration associated to a test configuration comes from a divisor. Since it is always possible to put oneself in the situation of the previous proposition, by considering, for example an equivariant resolution of the graph of $\rho \colon \mathcal{X} \dashrightarrow X \times \mathbb{C}$, one should be able to characterize K-stability with respect to integral test configurations in terms of valuative stability. In fact, this is exactly what is done first by Fujita [Fuj19] in the Fano case and later in [DL22] by relating the β invariant with the Donaldson Futaki invariant.

Integral K-stability and valuative stability

In this section, we reference the main result in [DL22], and give an idea of how one might prove the result.

Theorem 2.3.18 ([DL22]). *K-stability of the pair (X, L) with respect to integral test configurations coincides with the valuative stability with respect to L-dreamy divisors.*

The argument in the proof is structured as follows. To any dreamy divisor D, there is an associated integral test configuration $(\mathcal{X}, \mathcal{L})$. Since this test configuration is relatively concrete, one can obtain an asymptotic expression for its Donaldson-Futaki invariant. Then, by using techniques from the minimal model programme to calculate the integrals involved in the computation of β_L on models, where the volume can be computed polynomially, they eventually obtain an equality

$$\beta_L(D) = DF(\mathcal{X}, \mathcal{L})2(n-1)!. \tag{5}$$

Therefore, K-stability implies dreamy valuative stability. To show the converse, they observe that any integral test configuration gives rise to an L-dreamy divisor such that the filtration associated to the dreamy divisor coincides with the filtration of the test configuration. Then, finally, one can show that 5 is satisfied with the Donaldson-Futaki invariant of the given test configuration. This uses the description of the weight of the test configuration in terms of the filtration 2.3.11.

From equation 5, one then has

Theorem 2.3.19 ([DL22]). *K-semistability of (X, L) with respect to integral test configurations is equivalent to valuative semistability with respect to L-dreamy divisors.*

3 Fibration degenerations

Introduction

In this chapter, we start by describing fibration degenerations and an associated notion of stability as defined by Dervan-Sektnan in [DS21a]. It is a stability condition on a family of varieties, defined from an asymptotic expansion of the Donaldson-Futaki invariant, which arises by degenerating the family inside a projective bundle and then tensoring with ample line bundles pulled back from the base 3.1. The original contributions in this chapter (from 3.2 and onwards) come from forthcoming joint work with Lars Martin Sektnan. Motivated by the relation between the Donaldson-Futaki invariant with the beta invariant, we define a special class of divisors/divisorial valuations 3.2 for which there is an asymptotic expansion of the β invariant 3.3. In order to conclude that there is an asymptotic expansion for the β invariant for these divisors, we use equation 5 and the fact that there is an aymptotic expansion for the Donaldson-Futaki invariant of a fibration degeneration. Therefore we need to show that our special class of divisors give rise to fibration degenerations 3.2.20, and that the divisors become dreamy after twisting with sufficiently high tensor powers of any ample bundle from the base 3.2.21. We also show that our class of divisors arise naturally from the data of a fibration degeneration 3.2.31. The β invariant makes sense for any divisorial valuation, so we also make steps towards showing that one has an asymptotic expansion of the β invariant when the divisor in question satisfies milder conditions 3.3. In the case of projective bundles, we calculate to second order the asymptotic expansion of the beta invariant of the exceptional divisor blown up in a subbundle, and we show that nonnegativity of the subleading order term corresponds to an inequality of slopes 3.4. This gives a partial proof from the β-invariant side of a fact established in [DS21a], which is motivated by a similar result in the work of [RT06b] concerning slope stability with respect to subvarieties.

Canonical families of cscK metrics

Let $f\colon X \to B$ be a holomorphic submersion with a relatively ample line bundle H on X, suppose $\omega_{X/B} \in c_1(H)$ is a family of cscK metrics. That is, $\omega_{X/B}|_{X_b}$ is cscK for each $b \in B$. Such a $\omega_{X/B}$ depends on a fibrewise choice, since cscK metrics in a given Kähler class are only unique up to automorphisms preserving this class. Fixing a polarization L of B and a metric $\omega_B \in c_1(L)$, a natural question arises: can one determine a canonical choice for $\omega_{X/B}$ from the data of the family f and ω_B?

Recently, Dervan and Sektnan [DS21b] found an equation for $\omega_{X/B}$, namely

$$p(\Delta_{\mathcal{V}}(\Lambda_{\omega_B}\mu^*(F_{\mathcal{H}})) + \Lambda_{\omega_B}\rho_{\mathcal{H}}) = 0,$$

whose solutions they dubbed *optimal symplectic connections*. Shortly after conjecturing that optimal symplectic connections give a canonical choice of fibrewise cscK metrics Dervan-Sektnan [DS21c] confirmed this as they showed that any two optimal symplectic connections $\omega_{X/B}, \omega'_{X/B}$ differ by

$$\omega'_{X/B} = g^*\omega_{X/B} + f^*(i\,\partial\overline{\partial}\,\phi_B),$$

where $g \in Aut(X/B)$ and $\phi_B \in C^\infty(B,\mathbb{R})$. Hence, the family of cscK metrics $\omega_{X/B}|_{X_b}$ is indeed uniquely determined. A compelling property of optimal symplectic connections shown in [DS21a] is the following: when ω_B can be chosen to be cscK, then for any $j \gg 0$ there is a cscK metric in the class $c_1(H) + jf^*c_1(L)$. Having K-stability in mind, this suggests that one should be able to at least find an obstruction for the existence of an optimal symplectic connection in terms of suitably modified asymptotic version K-stability of the pair $(X, H + jf^*L)$ for all j big enough. Since the family structure $f\colon X \to B$ is relevant for the optimal symplectic connection equation, the test configurations to consider should then also have some sort of morphism to the base. Considerations like these led Dervan-Sektnan [DS21a] to introduce the notion of a fibration degeneration [DS21a] and a notion of stability for fibrations, which they conjecture in some form to be sufficient for the existence of an optimal symplectic connection for $f\colon (X,H) \to (B,L)$.

3.1 Fibration degenerations

In this section, we introduce the notion of a fibration degeneration as defined in [DS21a]. We shall fix the following data: (X, B, f), where $f\colon X \to B$ is a proper flat morphism between normal projective varieties such that $f_*\mathcal{O}_X \cong \mathcal{O}_B$ and $dim(X) > dim(B)$.

Definition 3.1.1. An f-degeneration of an f-ample divisor H, is a $\mathbb{C}^*$-equivariant coherent sheaf $\mathcal{E}$ on $B \times \mathbb{C}$ flat over $\mathbb{C}$ (i.e., when $p_1(b,t') = t'$, then $\mathcal{E}_{b,t'}$ is a flat $\mathbb{C}[t]_{t'}$-module), so that equivariantly $\mathcal{E}|_{B\times\mathbb{C}^*} \cong p_1^*f_*(kH)$ for some k such kH is f-very ample and $p_1^*f_*(kH)$ has the trivial $\mathbb{C}^*$-action. We call k the exponent of the f-degeneration.

Remark 3.1.2. If B is a point, then f-ample just means ample, and thus a degeneration means a coherent sheaf $\mathcal{E}$ on $\mathbb{C}$, which is flat, and hence free and equipped with a $\mathbb{C}^*$-action making the map

$$\mathbb{P}_{\mathbb{C}[t]}(\mathcal{E}) \to \mathbb{C}$$

equivariant. Thus, taking a subvariety $X \subset \mathbb{P}_{\mathbb{C}[t]}(\mathcal{E})|_1 \cong \mathbb{P}_{\mathbb{C}}(H^0(X, kH))$ and considering the closure of its $\mathbb{C}^*$-orbit recovers the notion of an ample test configuration.

For an f-degeneration $\mathcal{E}$ of H, one can consider the flat limit $\mathcal{X}$ of the $\mathbb{C}^*$-orbit of the Kodaira Embedding induced by H,

$$X \to \mathbb{P}_{B\times\mathbb{C}}(\mathcal{E})_1,$$

equipped with its induced $\mathbb{C}^*$-equivariant line bundle $\mathcal{H} = \mathcal{O}_{\mathcal{X}}(1) = \mathcal{O}(1)|_{\mathcal{X}}$. By composing

$$\mathcal{X} \to \mathbb{P}_{B\times\mathbb{C}}(\mathcal{E}) \to B \times \mathbb{C},$$

one obtains maps $\pi\colon \mathcal{X} \to B \times \mathbb{C}$ such that $p_2 \circ \pi$ is flat by definition. Since $\mathcal{X} \setminus \mathcal{X}_0 \cong_\psi X \times \mathbb{C}^*$ over $B \times \mathbb{C}^*$, one sees that $\pi|_{\mathcal{X}\setminus\mathcal{X}_0} = (f \times Id) \circ \psi$ via this isomorphism.

Definition 3.1.3 ([DS21a])**.** A fibration degeneration of (X, H, B, f) is given by $(\mathcal{X}, \mathcal{H}) \to B \times \mathbb{C}$ arising from an f-degeneration of H. We say the fibration degeneration has exponent k, if the f-degeneration of H giving rise to the fibration degeneration has exponent k.

Thus, fibration degenerations are completely analogous to 3.1.2 and can be viewed as a special relative version of test configurations. Indeed, in [DS21a], they prove among other things the following

Lemma 3.1.4. *Let $(\mathcal{X}, \mathcal{H})$ be a fibration degeneration of exponent k for (X, H, B, f) and L ample on B. Then $p_2 \circ \pi\colon (\mathcal{X}, \mathcal{H} + kj\pi^*p_1^*L) \to \mathbb{C}$ is a test configuration for $(X, H + jf^*L)$ of exponent k for all $j \gg 0$.*

This suggests that one should be able to come up with a notion of stability for fibration degenerations of (X, H, B, f) by studying the asymptotics of the invariants of the test configurations for $(X, H + jf^*L)$. The intuition that fibration degenerations correspond to families of test configurations can be made more precise. Indeed, this is true generically on the base.

Proposition 3.1.5 ([DS21a],[Hat22, lemma 4.8])**.** *For a fibration degeneration $(\mathcal{X}, \mathcal{H})$ of (X, H, B, L, f), there is a dense Zariski open set $U \subset B$, such that for any $b \in U$ the fibre $\mathcal{X}_b \to b \times \mathbb{C} \cong \mathbb{C}$ is flat, and $(\mathcal{X}_b, \mathcal{H}_b)$ is a test configuration for (X_b, H_b) of exponent k.*

Completions of fibration degenerations

Given an f-degeneration $\mathcal{E}$ of H, we can "complete" $\mathcal{E}$ trivially over $B\times\{\infty\}$ to a $\mathbb{C}^*$-equivariant coherent sheaf on $B \times \mathbb{P}^1_{\mathbb{C}}$ restricting to $\mathcal{E}$ by gluing the two pieces

- $p_1^* f_*(kH)$ over $B \times \mathbb{C}$ equipped with the inverse trivial action.
- $\mathcal{E}$ with its given action over $B \times \mathbb{C}$,

which are defined on different open affines of $B \times \mathbb{P}^1_{\mathbb{C}}$ and glued along the equivariant isomorphism $\mathcal{E}|_{B\times\mathbb{C}^*} \cong p_1^* f_*(kH)$ from the definition.

Definition 3.1.6. The $\mathbb{C}^*$-equivariant coherent sheaf $\overline{\mathcal{E}}$ on $B\times\mathbb{P}^1_{\mathbb{C}}$ obtained as above we call the completion of $\mathcal{E}$.

Note that the action on $\overline{\mathcal{E}}|_{B\times\{\infty\}} \cong f_*(kH)$ is then trivial by construction.

Suppose $\overline{\mathcal{E}}$ is the completion of $\mathcal{E}$. Then since $\overline{\mathcal{E}}|_{B\times\mathbb{C}} = \mathcal{E}$ one has

$$\begin{aligned}\mathbb{P}_{B\times\mathbb{C}}(\mathcal{E}) &= \mathbb{P}_{B\times\mathbb{C}}(\overline{\mathcal{E}}|_{B\times\mathbb{C}})\\ &= \mathbb{P}_{B\times\mathbb{P}^1_{\mathbb{C}}}(\overline{\mathcal{E}}) \times_{B\times\mathbb{P}^1_{\mathbb{C}}} (B\times\mathbb{C})\end{aligned}$$

Thus if $\mathcal{E}$ gives rise to a fibration degeneration $\mathcal{X}$, then we can consider the closed subscheme $\overline{\mathcal{X}} \subset \mathbb{P}_{B\times\mathbb{P}^1_{\mathbb{C}}}(\overline{\mathcal{E}})$ defined to be the subscheme obtained by gluing together the flat limits over 0 in either affine chart. Note that we have $\overline{\mathcal{X}}|_{B\times\mathbb{C}} = \mathcal{X}$ and $\overline{\mathcal{X}} \setminus \mathcal{X}_0 \cong X \times \mathbb{C}$ and a natural $\mathbb{C}^*$-equivariant line bundle $\overline{\mathcal{H}} = \mathcal{O}(1)|_{\overline{\mathcal{X}}}$, which agrees with $\mathcal{H}$ when restricted to $\mathcal{X}$.

Definition 3.1.7. We call $\overline{\mathcal{X}}$ the completed fibration degeneration.

Given a line bundle L on B, then we can pull back to a line bundle p_1^*L on $B \times \mathbb{P}^1_{\mathbb{C}}$ and further via the morphism $\hat{\pi}\colon \mathbb{P}_{B\times\mathbb{P}^1_{\mathbb{C}}}(\overline{\mathcal{E}}) \to B \times \mathbb{P}^1_{\mathbb{C}}$ to a line bundle. Now, by naturality of the construction, this bundle must restrict to $\overline{\mathcal{X}}$ as $(p_1 \circ \overline{\pi})^*L$, where $\overline{\pi}\colon \overline{\mathcal{X}} \to B \times \mathbb{P}^1_{\mathbb{C}}$ and similarly to $\mathcal{X}$ as $(p_1 \circ \pi)^*L$.

Lemma 3.1.8. *When $j \gg 0$ is such that $(\mathcal{X}, \mathcal{H}+jk\pi^*p_1^*L)$ is a test configuration, then the notion of completed fibration degeneration and completed test configuration coincide. Moreover, the induced line bundle on the completed test configuration $\overline{\mathcal{H}+jk\pi^*p_1^*L}$ coincides with $\overline{\mathcal{H}}+jk(p_1\circ\overline{\pi})^*L$.*

Proof. Since the $\mathbb{C}^*$ action is trivial on fibres over the affine where $\overline{\mathcal{E}}|_{B\times\mathbb{C}} = p_1^* f_*(kH)$, the flat limit of X is simply $X \times \mathbb{C}$. Hence the underlying spaces of the two completions are obtained in the same way.

The claim for the line bundles is analogous, since both $\overline{\mathcal{H}}$ and $(p_1 \circ \overline{\pi})^* L$ are the trivial extensions of $\mathcal{H}$ and $(p_1 \circ \pi)^* L$ respectively, hence the trivial extension $\overline{\mathcal{H} + jk\pi^* p_1^* L} = \overline{\mathcal{H}} + jk(p_1 \circ \overline{\pi})^* L$, by multiplicativity of the extension operation.

□

Fibrational stability

Lemma 3.1.9 ([DS21a])**.** *For a fibration degeneration $(\mathcal{X}, \mathcal{H})$ of (X, H, B, f) the asymptotic Donaldson-Futaki invariant $DF(\mathcal{X}, \mathcal{H}+kj\pi^* p_1^* L)$ has an expansion as a Laurent polynomial in j of leading order* $\dim(B)$*.*

$$DF(\mathcal{X}, \mathcal{H} + kj\pi^* p_1^* L) = W_0(\mathcal{X}, \mathcal{H}) j^{\dim(B)} + W_1(\mathcal{X}, \mathcal{H}) j^{\dim(B)-1} + \dots$$

Here, the $W_i(\mathcal{X}, \mathcal{H})$ have a dependancy on the exponent k.

Proof. Consider the completion $(\overline{\mathcal{X}}, \overline{\mathcal{H}}+jk(p_1\circ\overline{\pi})^* L)$. Suppose $n = \dim(X)$. By the formula for the Donaldson-Futaki invariant of a completed test configuration, we have, up to multiplication by a positive dimensional constant

$$\begin{aligned} DF(\mathcal{X}, \mathcal{H} + jk(p_1 \circ \pi)^* L) = & \frac{n}{n+1} \mu(X, H + jkf^* L)(\overline{\mathcal{H}} + jk(p_1 \circ \overline{\pi})^* L)^{n+1} \\ & + (\overline{\mathcal{H}} + jk(p_1 \circ \overline{\pi})^* L)^n . K_{\overline{\mathcal{X}}/\mathbb{P}^1_{\mathbb{C}}}. \end{aligned}$$

Analyzing the terms, we see that the slope is a quotient of two polynomials in j of degree at most $\dim(B)$ in the enumerator and degree at most $\dim(B)$ in the denominator, because any intersection involving terms of the type $A^{\dim(X)-\dim(B)+k}.(f^* L)^{\dim(B)+k}$ for $k > 0$ and A arbitrary is zero by the push-pull formula. Hence, the slope admits a Laurent expansion in j with leading term of order 0. Since the extensions of $(p_1 \circ \pi)^* L$ are also pullbacks, the intersection numbers $(\mathcal{H}+jk(p_1\circ\overline{\pi})^* L)^{n+1}, (\overline{\mathcal{H}}+jk(p_1\circ\overline{\pi})^* L)^n . K_{\overline{\mathcal{X}}/\mathbb{P}^1_{\mathbb{C}}}$ must also be polynomials in j of degree at most $\dim(B)$, hence the result. □

Performing the expansion in j and letting $n = dim(X), m = dim(B)$ such that $n - m = \ell$ is the fibre dimension, then one has the following formulas in terms of intersection numbers [DS21a]:

$$
\begin{aligned}
W_0(\mathcal{X},\mathcal{H}) = &\binom{n}{m} k^m(\frac{\ell}{\ell+1}(\frac{-K_{X/B}\cdot f^*L^m\cdot H^{\ell-1}}{f^*L^m\cdot H^\ell}(p_1\circ\bar{\pi})^*L^m\cdot\bar{\mathcal{H}}^{\ell+1})\\
&+(p_1\circ\bar{\pi})^*L^m\cdot\bar{\mathcal{H}}^\ell\cdot K_{\bar{\mathcal{X}}/\mathbb{P}^1_{\mathbb{C}}}) \qquad (6)\\
W_1(\mathcal{X},\mathcal{H}) = &\binom{n}{m-1} k^{m-1}(C_1(\mathcal{X},\mathcal{H})+C_2(\mathcal{X},\mathcal{H})+C_3(\mathcal{X},\mathcal{H})+C_4(\mathcal{X},\mathcal{H}))\\
C_1(\mathcal{X},\mathcal{H}) = &\frac{\ell}{\ell+2}(\frac{-K_{X/B}\cdot f^*L^m\cdot H^{\ell-1}}{f^*L^m\cdot H^\ell}(p_1\circ\bar{\pi})^*L^{m-1}\cdot\bar{\mathcal{H}}^{\ell+2})\\
C_2(\mathcal{X},\mathcal{H}) = &-\frac{\ell}{\ell+1}\frac{(-K_{X/B}\cdot f^*L^{m-1}\cdot H^{\ell-1})(f^*L^{m-1}\cdot H^{\ell+1})}{(f^*L^m\cdot H^\ell)^2}((p_1\circ\bar{\pi})^*L^m\cdot\bar{\mathcal{H}}^{\ell+1})\\
C_3(\mathcal{X},\mathcal{H}) = &\frac{-K_X\cdot f^*L^{m-1}\cdot H^\ell}{f^*L^m\cdot H^\ell}(p_1\circ\bar{\pi})^*L^m\cdot\bar{\mathcal{H}}^{\ell+1}\\
C_4(\mathcal{X},\mathcal{H}) = &(p_1\circ\bar{\pi})^*L^{m-1}\cdot\bar{\mathcal{H}}^{\ell+1}\cdot K_{\bar{\mathcal{X}}/\mathbb{P}^1_{\mathbb{C}}},
\end{aligned}
$$

where $K_X = K_B + K_{X/B}$. Similarly, by considering a resolution of indeterminacy of the canonical map from the compactified test configuration to $X\times\mathbb{P}^1_{\mathbb{C}}$, Dervan-Sektnan [DS21a] obtain an expansion in j for the norm of the family of test configurations $DF(\mathcal{X},\mathcal{H}+kj\pi^*p_1^*L)$. The leading term then gives a notion of norm for a fibration degeneration.

Definition 3.1.10 ([DS21a])**.** The norm of a fibration degeneration $(\mathcal{X},\mathcal{H})$ is

$$||(\mathcal{X},\mathcal{H})|| = \frac{L^m\cdot\mathcal{H}^\ell}{\ell+1} + L^m\cdot\mathcal{H}^\ell\cdot(\mathcal{H}-H)$$

Here we have supressed the various pullbacks and completions involved in the expression.

Now, we can give a notion of a "K-stability for families" from the asymptotic expension of the Donaldson-Futaki invariant and the notion of a norm.

Definition 3.1.11 ([DS21a])**.** Given a fibration $f\colon (X,H)\to(B,L)$ such that the fibres (X_b,H_b) are K-semistable, then the fibration is

- *semistable* if $W_0(\mathcal{X},\mathcal{H})\geq 0$ and when there is equality one has $W_1(\mathcal{X},\mathcal{H})\geq 0$
- *stable* if it is semistable and whenever $W_0(\mathcal{X},\mathcal{H})=0, W_1(\mathcal{X},\mathcal{H})=0$ then the norm is 0
- *polystable* if it is semistable and when $W_0(\mathcal{X},\mathcal{H})=0, W_1(\mathcal{X},\mathcal{H})=0$, then either the normalization of $(\mathcal{X},\mathcal{H}+kj\pi^*p_1^*L)$ is a product test configuration for $j\gg 0$ or the norm is 0

for any fibration degeneration of (X, H, B, f). We say the fibration is *integrally* semistable (resp. stable, polystable), if it holds for all fibration degenerations with irreducible central fibre.

In the original definition of fibration stability [DS21a], they require the fibres to be K-polystable. This comes from the accompanying differential geometric construction in [DS21b][DS20], where they consider cscK fibrations, i.e., fibrations where all fibres admit a cscK metric. Recent work by Ortu [Ort22] shows that fibrations with analytically K-semistable fibres also give a cscK metric in asymptotic classes when one can solve the optimal symplectic connection equation, hence we have chosen to allow K-semistable fibres as well. To get a feeling of the nature of the stability conditions, we display an explicit interpretation of the first coefficient

Proposition 3.1.12 ([DS21a])**.** *For any fibration degeneration $(\mathcal{X}, \mathcal{H})$ of (X, H, B, L, f) one has*

$$W_0(\mathcal{X}, \mathcal{H}) = \binom{n}{m} f^* L^m DF(\mathcal{X}_b, \mathcal{H}_b),$$

where $(\mathcal{X}_b, \mathcal{H}_b) \to b \times \mathbb{C} \cong \mathbb{C}$ is an induced test configuration for a generic $b \in B$.

Proof. Let $U \subset B$ be the open subset, such that $b \in U$ implies that $(\mathcal{X}_b, \mathcal{H}_b)$ is a test configuration for $(X_b.H_b)$. Then $\pi|_{\pi^{-1}(U \times \mathbb{C})}$ is flat, hence, for $b \in U$ one has that $(p_1 \circ \pi)^*[b] = [\mathcal{X}_b]$ is a well defined element in the chow ring of $\bar{\mathcal{X}}$. In particular, for any class A representing the intersection of line bundles

$$(p_1 \circ \bar{\pi})^*[b] \cdot A = [\mathcal{X}_b] \cdot A = A|_{\mathcal{X}_b}.$$

Similarly, since f is flat,$f^*[b] = [X_b]$, and one arrives at a similar conclusion when intersecting with this class. Since L is ample on B, it follows that $L^m \sim c[b]$ for some point $b \in U, c > 0$. Hence, one can write

$$\begin{aligned} (p_1 \circ \bar{\pi})^* L^m \cdot \bar{\mathcal{H}}^{\ell+1} &= c\bar{\mathcal{H}}^{\ell+1}|_{\mathcal{X}_b} \\ f^* L^m \cdot H^\ell &= cH^\ell|_{X_b}. \end{aligned}$$

Now applying the formula 6

$$\begin{aligned} W_0(\mathcal{X}, \mathcal{H}) &= \binom{n}{m} ck^m \Big(\frac{\ell}{\ell+1}\Big(\frac{(-K_{X/B} \cdot H^{\ell-1})|_{X_b}}{H^\ell|_{X_b}} \bar{\mathcal{H}}^{\ell+1}|_{\mathcal{X}_b} + (\bar{\mathcal{H}}^\ell \cdot K_{\bar{\mathcal{X}}/\mathbb{P}^1_{\mathbb{C}}})|_{\mathcal{X}_b}\Big) \\ &= \binom{n}{m} ck^m \Big(\frac{\ell}{\ell+1} \frac{-K_{X_b} \cdot H|_b^{\ell-1}}{H_b^\ell} \bar{\mathcal{H}}_b^{\ell+1} + \bar{\mathcal{H}}_b^\ell \cdot K_{\bar{\mathcal{X}}_b}\Big) \\ &= \binom{n}{m} ck^m DF(\mathcal{X}_b, \mathcal{H}_b). \end{aligned}$$

where we used the equation 2.2.15 and the compatibility of the relative canonical class under base change [Har10, II, prop. 8.10]. The result then follows, since the intersection number of $f^* L^m$ is c.

□

Recently M. Hattori [Hat22] defined a stronger notion of stability for fibrations called $\mathfrak{f}$-stability involving all the terms in the asymptotic expansion of the Donaldson-Futaki invariant. Hattori shows that $\mathfrak{f}$-stability of a fibration implies some control on the singularities of a pair (X, D), where X is the total space of the family. The difference between the two stability notions for families seem formally similar to the difference between slope stability and Gieseker stability of sheaves [HL10, chapter 1].

3.2 A valuative obstruction for fibrational stability

Horizontal valuations

In this section we define a class of valuations appearing later, where we give a valuative interpretation of the asymptotic Donaldson-Futaki invariant. Let $f\colon X \to B$ be a dominant morphism of varieties and denote the comorphism by $f^*\colon \mathcal{O}_B \to f_*\mathcal{O}_X$. Recall that one can push forward any valuation $v\colon K(X) \to \overline{\mathbb{R}}$ in this situation as follows

$$f_*v(s) = v(f^*s).$$

Definition 3.2.1. A valuation $v\colon K(X) \to \overline{\mathbb{R}}$ is f-horizontal, if

$$f_*v = v_{triv},$$

where the trivial valuation is $v_{triv}(f) = \begin{cases} 0 & f \neq 0 \\ \infty & f = 0. \end{cases}$

Remark 3.2.2. Geometrically this means that v does not provide a measure of multiplicity for any nontrivial algebraic function (or divisor) defined on an open subset of B.

Recall the notion of a center for a valuation:

Definition 3.2.3. A valuation v on $K(X)$, with value ring R has a center on a variety X, if there is a scheme point $p \in X$ such that the canonical map

$$\mathcal{O}_{X,p} \to K(X)$$

factors through R as a local morphism and R dominates $\mathcal{O}_{X,p}$, that is, $\mathfrak{m}_R \cap \mathcal{O}_{X,p} = \mathfrak{m}_p$.

Intuitively, having a center means that there is a subvariety such that v measures multiplicity along that subvariety. By the valuative critetion of properness [Har10, II, thm. 4.7], it follows that a valuation on the function field of a complete variety has a unique center on that variety.

Suppose one is given a valuation v on $K(X)$ with value ring $R = v^{-1}(\mathbb{R}_{\geq 0})$. Then, pushing v forward by f to $w = f_*v$ with value ring

$S = w^{-1}(\mathbb{R}_{\geq 0})$ yields a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & R & \xrightarrow{i} & K(X) \\ & & \uparrow & & \uparrow{\scriptstyle f^*} \\ 0 & \longrightarrow & S & \xrightarrow{i} & K(B) \\ & & \uparrow & & \uparrow \\ & & 0 & & 0. \end{array}$$

If v has a center on X, there is a prime $p \in X$ such that $\mathcal{O}_{X,p} \subset R$ and $\mathfrak{m}_R \cap \mathcal{O}_{X,p} = \mathfrak{m}_p$, in particular by taking any affine neighbourhood U of p and considering $\mathcal{O}_X|_U \to \mathcal{O}_{X,p} \to R$ there is a map

$$Spec(R) \to X,$$

whose unique closed point corresponding to $\mathfrak{m}_R$ maps to p. It is similar if w has a center on B. Now, since $S \to R$ is local we get a commutative diagram

$$\begin{array}{ccc} Spec(R) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ Spec(S) & \longrightarrow & B. \end{array}$$

Thus, in this case, the centers satisfy $c_B(w) = f(c_X(v))$. Note that the trivial valuation v_{triv} has the generic point of B as its center. Therefore the center of horizontal valuations must map to the generic point of B.

Example 3.2.4. Consider trivially (although prototypical for what we want), the coordinate projection map $p_1 \colon X = Spec(\mathbb{C}[t,w]) \to B = Spec(\mathbb{C}[t])$. Then, a valuation v is p_1 horizontal, if, and only if $v(t-c) = 0$, for all $c \in \mathbb{C}$. Indeed, it suffices to check on polynomials hence by algebraic closedness of $\mathbb{C}$, we can check on$f = \prod_{i=1}^n (t - c_i)$. Thus,

$$v(f) = \sum_{i=1}^{n} v(t - c_i).$$

In this case, we can also easily classify the horizontal valuations v. On a w-monomial $g = f(t)w^i \in \mathbb{C}(t)(w) \cong K(X)$ we have $v(g) = v(w^i)$ and so in general for $g = \sum_i f_i(t)w^i$, we have

$$v(g) \geq \min_i iv(w)$$

with equality whenever $v(w) \neq 0$, in which case $v = c \cdot ord_w$ for $c \neq 0$ is a multiple of the order of vanishing along w, and when $v(w) = 0$, then $v(g) \geq 0$ for all $g \neq 0$, and so v must be trivial. Therefore, we see that the center of a horizontal v must be the generic point of the zero section $\{w = 0\}$, or else the center must be on some compactification of $Spec(\mathbb{C}[t, w])$.

A basic property of f-horizontal valuations is that the notion is invariant under birational morphisms over B in the following sense:

Lemma 3.2.5. *Suppose $g\colon Y \dashrightarrow X$ is a birational morphism of varieties over B, and v is an f-horizontal divisor. Then $w = g_*^{-1}v$ is a h-horizontal divisor, where $h = f \circ g$.*

Proof. g induces an identification of function fields $K(X) \cong K(Y)$. Hence,

$$\begin{aligned} h_* w &= (f \circ g)_* g_*^{-1} v \\ &= f_* v \\ &= v_{triv}. \end{aligned}$$

□

Next, we relate the notion of an f-horizontal valuation to a geometric property of Cartier divisors appearing on X.

Definition 3.2.6. A divisor $D \subset X$ is f-horizontal if $f(Supp(D))$ is Zariski dense.

In particular, when f is proper, then $f(Supp(D)) = B$ for a horizontal divisor D. Any divisor $D = \sum a_i D_i$ is then f-horizontal, if, and only if, each D_i is so. Therefore they form a subspace of the Weil divisors $Div(X)$. Consider the morphism $\mathbb{P}^2_{\mathbb{C}} \setminus \{x_1 = x_2 = 0\} \to \mathbb{P}^1_{\mathbb{C}}$ given by projection to $\{x_3 = 0\}$. Then, the global sections. $x_1, x_2, x_3 \in H^0(\mathbb{P}^2_{\mathbb{C}} \setminus \{x_1 = x_2 = 0\}, \mathcal{O}(1))$ give linearly equivalent divisors with $\{x_3 = 0\}$ horizontal but $\{x_1 = 0\}, \{x_2 = 0\}$ are not horizontal. This shows by example that the notion is not preserved by linear equivalence. It is clear that an f-horizontal divisor cannot be the pullback of a divisor on B (or the flat pullback of a Weil divisor), however, they are in general not all the divisors, which are not pullbacks, since f might contract some divisors onto subsets of $codim \geq 2$, e.g. if the divisor in question is the exceptional set of a blowup along a generic smooth subvariety. Therefore, f-horizontal divisors are special in the sense that they are the divisors that are contracted the least by f in terms of dimension. A way to produce such divisors is, if one is given a map $g\colon Y \to B$, and one then blows up a subvariety $A \subset Y$ such that $g|_A$ is dominant.

Proposition 3.2.7. *For a proper dominant morphism $f\colon X \to B$, then a prime Cartier divisor D is f-horizontal, if, and only if, the associated valuation ord_D is f-horizontal.*

Proof. Suppose first that ord_D is f-horizontal. The center of ord_D is the generic point $\eta \in D$. So $f(\eta)$ is the generic point of of B. Therefore $\{f(\eta)\} \subset f(D)$, but $\{f(\eta)\}$ is Zariski dense, so $B \subset f(D)$ since $f(D)$ is closed. For the converse, suppose $f(D) = B$. We argue that any $s \in \mathcal{O}_B(V)$ such that

$$ord_D(f^*s) \neq 0$$

must satisfy $s = 0$. It suffices to do this in any stalk $b \in V$. The local comorphisms $f_b^*\colon \mathcal{O}_{B,b} \to \mathcal{O}_{X,d}$ where $d \in D$ is such that $f(d) = b$ are all injections. Thus, $s_b \in \mathfrak{m}_b$, if it maps to $\mathfrak{m}_d$. But this has to be the case because near D, we can write $f^*s = t^n g$ for $n \in \mathbb{N}$ and g invertible. Therefore, $s_b \in \mathfrak{m}_b$ for all $b \in V$, and so when V is affine, s sits in every prime ideal, which means it must be in the nilradical. But this is 0 as B is reduced. □

Corollary 3.2.8. *If $f\colon X \to B$ is a dominant proper morphism of varieties, then any prime Cartier divisor D on Y over X is h-horizontal, if, and only if, its associated valuation on X is f-horizontal.*

Proof. We have $g\colon Y \to X$ proper birational and $h\colon Y \to B$ proper and dominant such that $f \circ g = h$. Thus, ord_D h-horizontal, if, and only if, g_*ord_D is f-horizontal by 3.2.5, so the result follows from 3.2.7, because h is proper and dominant.

□

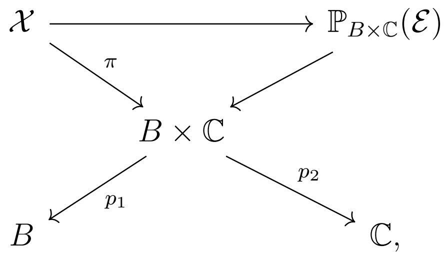
This shows that the notion of being a horizontal prime divisor on the birational models dominating $f\colon X \to B$ is independent of such a model. In fact, the theorem 3.2.10 shows that horizontal divisorial valuations are precisely characterized by the corresponding geometric property on some model dominating $f\colon X \to B$. Therefore, we shall introduce the definition

Definition 3.2.9. A divisor D on a model $g\colon Y \to X$ over $f\colon X \to B$ is horizontal, if it is $h = g \circ f$ horizontal.

Theorem 3.2.10 ([ZS97, section 14, thm. 31])**.** *For any divisorial valuation $v \in K(X)$ with center $c_X(v)$ on X, there is a birational model $g\colon Y \to X$ and a divisor $D \subset Y$, such that $cv = g_*ord_D$ on X for some $c \neq 0$.*

Since $cv_{triv} = v_{triv}, \forall c \neq 0$, it follows directly that any such divisor D from the theorem must be horizontal, when v is f-horizontal. Next, we want to give a partial justification for the introduction of horizontal divisors when studying fibration degenerations.

A fibration degeneration $(\mathcal{X}, \mathcal{H})$ of $f\colon (X,H) \to B$ gives rise to the following commutative diagram



where $\mathcal{X}$ is equal to the flat limit over $\mathbb{C}$ of $orb_{\mathbb{C}^*}(X) \cong X \times \mathbb{C}^*$ inside $\mathbb{P}_{B\times\mathbb{C}}(\mathcal{E})$. The central fibre $\mathcal{X}_0$ surjects onto B, because $\mathcal{X} \setminus \mathcal{X}_0 \cong_{B\times\mathbb{C}} X \times \mathbb{C}^*$, so

$$\pi(\chi \setminus \chi_0) = B \times \mathbb{C}^*,$$

and therefore

$$\pi(\chi) \supset \overline{\pi(\chi \setminus \chi_0)} = \overline{B \times \mathbb{C}^*} = B \times \mathbb{C},$$

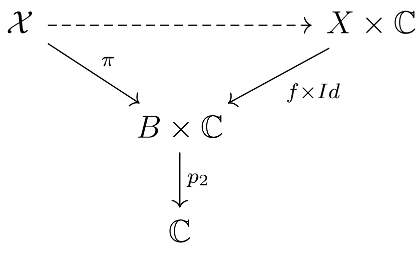
which means

$$\pi(\mathcal{X}_0) \supset B \times \{0\}.,$$

i.e., $\mathcal{X}_0 = \pi^{-1}(B \times \{0\})$.

Lemma 3.2.11. *Any non-trivial integral fibration degeneration of $f\colon (X,H) \to B$ gives rise to a horizontal divisor on a birational model $Y \to X$ over B.*

Proof. We have a $\mathbb{C}^*$-equivariant birational diagram over $B \times \mathbb{C}$



The central fibre of $p_2 \circ \pi$ is then a prime divisor, which by surjectivity onto B gives rise to a $pr_1 \circ \pi$-horizontal valuation $v_{\mathcal{X}_0}$ on $K(\mathcal{X})$. By commutativity of the diagram, it follows that the induced valuation on $K(X \times \mathbb{C})$ is $pr_1 \circ (f \times Id)$-horizontal. Let us, by abuse of notation also denote this by $v_{\mathcal{X}_0}$.

Finally, since $K(X \times \mathbb{C}) = K(X)(t)$, this is equivalent to $v_{\mathcal{X}}|_{K(X)}$ being f-horizontal. Now, by the general theory from [BHJ17], $v_{\mathcal{X}_0}|_{K(X)}$ is divisorial when the test configuration is nontrivial. In any case, there is a divisor giving rise to $v_{\mathcal{X}_0}$ realized on some birational model of X over B by 3.2.10. The fact that this divisor is horizontal follows by the birational invariance of the notion. □

The relative Rees construction

We show that horizontal divisors are precisely the divisors such that one may carry out a relative version of the classical construction by Rees, giving a geometric meaning to $\mathbb{Z}$ filtrations of the global sections of a bundle.

Definition 3.2.12. Let $f\colon X \to B$ be a proper surjective morphism of varieties. Let D be an f-horizontal divisor on X. Then, for any line bundle L on X, we define the value of any $s \in \Gamma(U, f_*(L)) = \Gamma(f^{-1}(U), L)$ to be

$$ord_D(s) = ord_D(s_i),$$

where $s_i = \phi_i s|_{U_i}$ in some local trivialization $L|_{U_i} \stackrel{\phi_i}{\to} \mathcal{O}_X|_{U_i}$, such that $U_i \cap D \neq \emptyset$ and $U_i \subset f^{-1}(V)$.

Firstly, we note that there always exists such a trivialization U_i as in the definition: one can pick any trivialization, since the divisor must map surjectively onto B. In fact, one could extend this definition to any horizontal valuation, because there is a point in every fibre that is a specialization (in the Zariski closure) of the centre.

The value $v(s)$ does then not depend on the chosen trivialization, since any two such are $\mathcal{O}_X$-linearly isomorphic over their intersection by $\phi_{ij} = \phi_i \circ \phi_j$ which is determined as multiplication by the unit $\phi_{ij}(1)$. But this does not change the order along D. This is also true for any horizontal valuation in the adapted definition.

A second benefit of the horizontal condition is that $ord_D(s)$ is well behaved when multiplying by functions from B. Indeed, let $b \in \mathcal{O}_B(V)$ and consider $f^*b \in \mathcal{O}_X(f^{-1}(V))$, then, by definition,

$$\begin{aligned} ord_D(f^*bs) &= ord_D(f^*b|_{U_i} s_i) \\ &= ord_D(f^*b|_{U_i}) + ord_D(s_i) \\ &= ord_D(f^*b) + ord_D(s_i) \\ &= 0 + ord_D(s_i) \end{aligned}$$

because $f^*b|_{U_i} \sim f^*b$ in $K(X)$. The sets

$$\mathcal{F}^{\lambda}_{ord_D} f_*(L)(V) = \{s \in f_*(L)(V) | ord_D(s) \geq \lambda\}$$

are then $\mathcal{O}_B(V)$-submodules, and we obtain a filtration of the sheaf $f_*(L)$ of $\mathcal{O}_B$-modules by considering the subsheaves

$$\lambda \mapsto \mathcal{F}^{\lambda}_{ord_D} f_*(L).$$

Moreover, when considering $f_*(kL)$ for any $k \geq 0$, one obtains a decreasing multiplicative filtration by considering

$$(\lambda, k) \to \mathcal{F}^{\lambda}_{ord_D} f_*(kH).$$

Remark 3.2.13. When $f\colon X \to Spec(\mathbb{C})$ is the canonical map, then being f-horizontal is void, so any divisor D satisfies this condition. In this case our definition of the value of a section 3.2.12 is the same as the one used in [BHJ17].

Definition 3.2.14. Let $f\colon X \to B$ be a proper surjective morphism and v an f-horizontal valuation with value group $\Gamma \cong \mathbb{Z}^n$. Then, we call the graded $\mathcal{O}_B$-algebra

$$\mathcal{R}_v(L) = \bigoplus_{k \geq 0} (\bigoplus_{\lambda \in \Gamma} t^{-\lambda} \mathcal{F}^{\lambda}_v f_*(kL))$$

the associated relative degeneration algebra or relative Rees algebra. We write $\mathcal{R}_{v,k}(L)$ for the k'th piece.

We shall only consider the case of horizontal divisorial valuations $\Gamma = \mathbb{Z}$.

Proposition 3.2.15. *When $f\colon X \to B$ is proper, surjective, and $f_*\mathcal{O}_X \cong \mathcal{O}_B$, then, the relative Rees algebra for an f-horizontal divisor v induces a family over $B \times \mathbb{C}$ with irreducible central fibre, and a $\mathbb{C}^*$-action such that the map to $B \times \mathbb{C}$ is equivariant.*

Proof. In this case the degree $k = 0$ piece is

$$\begin{aligned}\bigoplus_{\lambda \in \mathbb{Z}} t^{-\lambda} \mathcal{F}^{\lambda}_v f_*(\mathcal{O}_X) &\cong \bigoplus_{\lambda \in \mathbb{Z}} t^{-\lambda} \mathcal{F}^{\lambda}_v \mathcal{O}_B \\ &\cong \bigoplus_{-\lambda \in \mathbb{N}} t^{-\lambda} \mathcal{O}_B \\ &= \mathcal{O}_B \otimes_{\mathbb{C}} \mathbb{C}[t],\end{aligned}$$

where we used that

$$\mathcal{F}_v^\lambda \mathcal{O}_B = \mathcal{F}_{f_*v}^\lambda \mathcal{O}_B = \mathcal{F}_{v_{triv}}^\lambda \mathcal{O}_B = \begin{cases} \mathcal{O}_B & \lambda \leq 0 \\ 0 & \lambda > 0. \end{cases}$$

In particular, the $\mathcal{R}_{v,k}$ pieces have an $\mathcal{O}_B \otimes_{\mathbb{C}} \mathbb{C}[t]$ module structure, and we can form the relative proj

$$Proj_{B\times\mathbb{C}}(\mathcal{R}_v(L)) \to B \times \mathbb{C}$$

with its canonical map. There is by 2.2.9 a natural action of the group scheme $\mathbb{C}^*(B) = B \times \mathbb{C}^*$ on this space such that the canonical map commutes with the $\mathbb{C}^*(B)$-action on $B \times \mathbb{C}^*$. But this is equivalent to a $\mathbb{C}^*$-action on $Proj_{B\times\mathbb{C}}(\mathcal{R}_v(L))$ such that the canonical map to $B \times \mathbb{C}^*$ is equivariant since the group sheme acts trivially on the B component. The central fibre is the proj

$$Proj_B(Gr(\mathcal{R}_v(L))),$$

where

$$Gr(\mathcal{R}_v(L)) = \bigoplus_{k\geq 0}(\bigoplus_{\lambda\in\mathbb{Z}} \mathcal{F}_v^\lambda f_*(kL)/\mathcal{F}_v^{\lambda+1} f_*(kL)).$$

But this sheaf certainly forms an integral ring over open affines $U \subset B$. Hence the central fibre is integral. □

Now, imposing certain local finiteness conditions for $\mathcal{F}_v^\lambda f_*(kL)$ on B one can ensure that the space has desirable properties, i.e., such that it defines a fibration degeneration. This is in fact exactly what we shall do to obtain fibration degenerations from divisors.

Relatively Dreamy divisors

Suppose we are given the data (X, B, f) as in section 3.1.

Definition 3.2.16. Let H be an f-ample divisor on X and D a prime divisor over X. Then we say D is relatively H-dreamy, if there is an $r > 0$ such that rH is Cartier and

$$\mathcal{R}(rH, D) = \bigoplus_{k\geq 0}(\bigoplus_{\lambda\in\mathbb{Z}} t^{-\lambda} h_*(g^*krH - \lambda D))$$

is locally finitely generated as an $\mathcal{O}_B[t]$-algebra. Here, we think of $h_*(g^*krH - \lambda D) \subset f_*(krH)$ as a subsheaf.

When D is horizontal, then the algebra from the relatively dreamy condition agrees with

$$\mathcal{R}_{ord_D}(H)$$

defined in 3.2.14. Indeed, if $s \in f_*(kH)(V) = kH(f^{-1}(V))$, then $s \in \mathcal{F}^{\lambda}_{ord_D} f_*(kH)(V)$, if, and only if,

$$\begin{aligned}\lambda \leq g_* ord_D(s) &= g_* ord_D(s_i) \quad \forall i \\ &= ord_D(g^* s_i) \quad \forall i\end{aligned}$$

for all s_i representing s in a trivialization over $U_i \subset f^{-1}(V)$. By resolution of singularities [Hir64], we can by considering the strict transform assume the model $g\colon Y \to X$, where D appears is a smooth variety. Hence that D is Cartier. Let t be the local equation for D on an open affine subset $W_{ij} \subset g^{-1}(U_i)$. Then, $\lambda \leq ord_D(g^* s_i) \iff g^* s_i = t^{\lambda} w_{ij}$, where w_{ij} is some function on W_{ij}. The decomposition glues so we also have $g^* s = t^{\lambda} w$ for some section $w \in g^* kH(g^{-1}(f^{-1}(V))$. But this is by definition what it means to be in the image of $g^* s \in h_*(g^* kH - \lambda D)$ under the map $h_*(g^* kH - \lambda D) \to h_* g^*(kH) = f_*(kH)$. Let us summarize:

Lemma 3.2.17. *If D is a horizontal divisor, then*

$$\mathcal{R}(H, D) = \mathcal{R}_{ord_D}(H).$$

If D is relatively H-dreamy then $\mathcal{R}_{ord_D}(H)$ is locally finitely generated as an $\mathcal{O}_B[t]$-algebra.

Remark 3.2.18. Until the day of writing this document, it is unknown to the author if the conditions of D being relatively H-dreamy has a relation with the property that D is horizontal. One might wonder, if D relatively H-dreamy implies that D is $\mathbb{Q}$-linearly equivalent to a horizontal divisor.

Lemma 3.2.19. *If D is relatively H-dreamy, then for some k_0 big enough*

$$\mathcal{R}(k_0 H, D)$$

is locally generated in degree 1 as an $\mathcal{O}_B[t]$-algebra.

Proof. This follows from the fact that the base B has a finite affine cover $U_i, i \in \{1, \ldots, n\}$ such that $\mathcal{R}(rH, D)|_U$ is finitely generated. If $\mathcal{R}(rH, D)|_U$ is generated in degree d_i, we know that the subalgebra $\mathcal{R}(d_i rH, D)|_{U_i}$ is generated in degree 1. The same is true upon replacing d_i by md_i, $m \in \mathbb{N}$. Now, set $k_0 = lcm(r \cup \{d_i | i \in \{1, \ldots, n\}\})$, then

$$\mathcal{R}(k_0 H, D)|_{U_i}$$

is generated in degree 1 for all U_i. □

Proposition 3.2.20. *A horizontal and relatively H-dreamy divisor D induces a fibration degeneration $(\mathcal{X}, \mathcal{H})$ isomorphic to the relative projective spectrum*

$$\mathcal{X} \cong Proj_{B\times\mathbb{C}}(\mathcal{R}(k_0H, D))$$

with k_0 some sufficiently large constant and $\mathcal{H} = \mathcal{O}_{\mathcal{X}}(1) = \phi^*\mathcal{O}(1)$, where

$$\phi\colon \mathcal{X} \to \mathbb{P}_{B\times\mathbb{C}}(\bigoplus_{\ell\in\mathbb{Z}} t^{-\ell} h_*(g^*k_0H - \ell D)).$$

Proof. By construction of a fibration degeneration, one picks $k_0 \gg 0$ such that k_0H is f-very ample. Since this is also true for even larger k, we can ensure that $\mathcal{R}(k_0H, D)$ is locally generated in degree 1.

Then one has by the Rees construction 2.3.2 a projective bundle degeneration induced by the coherent sheaf

$$D(k_0H, D) = \bigoplus_{\ell\in\mathbb{Z}} t^{-\ell}(h)_*(g^*k_0H - \ell D),$$

That is, given by

$$\mathbb{P}_{B\times\mathbb{C}}(D(k_0H, D)) = Proj_{B\times\mathbb{C}}(Sym(D(k_0H, D))).$$

By general considerations of the Rees construction 2.3.2 the fibre over any $t \neq 0$ is

$$\mathbb{P}_B((h)_*(g^*k_0H)) = \mathbb{P}_B(f_*(k_0H)),$$

since $h_*(g^*k_0H) \cong f_*(k_0H)$. Thus, X embeds in this fibre as

$$X \cong Proj_B(\bigoplus_{k\geq 0} f_*(k_0kH)),$$

and the $\mathbb{C}^*$-orbit is isomorphic to

$$orb_{\mathbb{C}^*}(X) \cong Proj_{B\times\mathbb{C}^*}(\bigoplus_{k\geq 0} f_*(k_0kH)[t^{\pm 1}]).$$

$\mathcal{X}$ is obtained as the flat limit of the orbit inside $\mathbb{P}_{B\times\mathbb{C}}(D(k_0H, D)) \to \mathbb{C}$. Note that $\mathcal{X}$ is uniquely determined as a closed subscheme by the diagram

$$\begin{array}{ccccc} orb_{\mathbb{C}^*}(X) & \xrightarrow{\iota_2} & \mathbb{P}_{B\times\mathbb{C}}(D(k_0H, D)) & \xrightarrow{p_2\circ\pi} & \mathbb{C} \\ \downarrow{\scriptstyle \iota_1} & \nearrow{\scriptstyle \phi} & & & \\ \mathcal{X} & & & & \end{array},$$

where ϕ is a closed immersion such that $(p_2 \circ \pi) \circ \phi$ is flat. But the scheme

$$Proj_{B\times\mathbb{C}}(\mathcal{R}(k_0H, D))$$

also extends $orb_{\mathbb{C}^*}(X)$ and local generation in degree 1 ensures that it embeds into $\mathbb{P}_{B\times\mathbb{C}}(D(k_0H, D))$, because of the surjections

$$S^k(\mathcal{R}_1(k_0H, D)) \to \mathcal{R}_k(k_0H, D) \to 0$$

from $\mathcal{R}_1(k_0H, D) = D(k_0H, D)$. Moreover,

$$Proj_{B\times\mathbb{C}}(\mathcal{R}(k_0H, D))$$

also restricts to $orb_{\mathbb{C}^*}(X)$ over $\mathbb{C}^*$, and it is flat over $\mathbb{C}[t]$ by the Rees construction 2.3.2. The $\mathbb{C}^*$-equivariance follows from 2.2.9.

□

The remainder of this section will be dedicated to the proof of the following:

Theorem 3.2.21. *If D is a horizontal and relatively H-dreamy divisor, then for any ample bundle L on B, D is a dreamy divisor for $H + jf^*L$, when j is sufficiently big.*

We shall need a couple of lemmata along the way. First we recall the setup:

$$\begin{array}{ccc} D \subset Y & \xrightarrow{g} & (X, H) \\ & \searrow^{h} & \downarrow f \\ & & (B, L) \end{array}$$

g, f are proper, g birational and f is flat with connected fibres. That is $f_*\mathcal{O}_X = \mathcal{O}_B$. H is moreover f-ample, and L is ample on B. In particular, $H + jf^*L$ is ample for j sufficiently big ([Laz04] prop. 1.7.10).

Setting $r = 1$ in the relative dreamy condition, the statement of the theorem is:

If

$$\mathcal{R}(H, D) = \bigoplus_{k\geq 0}(\bigoplus_{\ell\geq 0} t^{-\ell} h_*(kg^*H - \ell D))$$

is locally finitely generated over $\mathcal{O}_B[t]$, then for $j \gg 0$,

$$R(H + jf^*L, D) = \bigoplus_{k\geq 0}(\bigoplus_{\ell\geq 0} t^{-\ell} H^0(Y, (kg^*(H + jf^*L) - \ell D))$$

is finitely generated over $\mathbb{C}[t]$. In the following, we denote

$$\mathcal{R}_k(H,D) = \bigoplus_{\ell\geq 0} t^{-\ell} h_*(kg^*H - \ell D),$$

so that $\mathcal{R}(H,D) = \bigoplus_{k\geq 0} \mathcal{R}_k(H,D)$.

Lemma 3.2.22. *Let $\mathcal{X}$ be the induced test configuration from a relatively H-dreamy divisor D, and denote the canonical map $\pi\colon \mathcal{X} \to B \times \mathbb{C}$, then for all j, we have*

$$(Proj_{B\times\mathbb{C}}(\mathcal{R}(H,D)), \mathcal{O}(1) \otimes \pi^* p_1^* L^j) \cong (Proj_{B\times\mathbb{C}}(\mathcal{R}(H + jf^*L, D)), \mathcal{O}(1)).$$

Proof. Then this is a consequence of the general theory [Har10, II, Lemma 7.9], which says that

$$(Proj_{B\times\mathbb{C}}(\mathcal{R}(H,D)), \mathcal{O}(1)\otimes\pi^* p_1^* L^j) \cong (Proj_{B\times\mathbb{C}}(\bigoplus_{k\geq 0} \mathcal{R}_k(H,D)\otimes p_1^* L^{kj}), \mathcal{O}(1))$$

and the following computation, where we write $L = \mathcal{O}_B(F)$ for some divisor.

$$\begin{aligned}
\mathcal{R}_k(H,D) \otimes_{\mathcal{O}_B[t]} p_1^* L^{kj} &= (\bigoplus_{\ell\geq 0} t^{-\ell} h_*(g^*\mathcal{O}_X(kH) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-\ell D))) \otimes_{\mathcal{O}_B[t]} p_1^* L^{kj} \\
&\cong \bigoplus_{\ell\geq 0} t^{-\ell} h_*(g^*\mathcal{O}_X(kH) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-\ell D)) \otimes_{\mathcal{O}_B} L^{kj} \\
&\cong \bigoplus_{\ell\geq 0} t^{-\ell} h_*(g^*\mathcal{O}_X(kH) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-\ell D) \otimes_{\mathcal{O}_Y} (f_Y)^* L^{kj}) \\
&\cong \bigoplus_{\ell\geq 0} t^{-\ell} h_*(g^*(\mathcal{O}_X(kH) \otimes_{\mathcal{O}_X} f^* L^{kj}) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-\ell D)) \\
&= \bigoplus_{\ell\geq 0} t^{-\ell} h_*(g^*(\mathcal{O}_X(k(H + jf^*F))) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(-\ell D)) \\
&= \mathcal{R}_k(H + jf^*L, D).
\end{aligned}$$

□

Lemma 3.2.23. *For $j \gg 0$, there are embeddings making the following diagram commute*

$$\begin{array}{ccc}
Proj_{B\times\mathbb{C}}(\mathcal{R}(H + jf^*L, D)) & \longrightarrow & \mathbb{P}_{B\times\mathbb{C}}(\mathcal{R}_1(H + jf^*L, D)) \\
\downarrow & & \downarrow \\
Proj_{B\times\mathbb{C}}(p_2^* p_{2*}\mathcal{R}(H + jf^*L, D)) & \longrightarrow & \mathbb{P}_{B\times\mathbb{C}}(p_2^* p_{2*}\mathcal{R}_1(H + jf^*L, D))
\end{array}$$

Proof. It suffices to show that the commutative diagram

$$\begin{array}{ccccc}
0 & & 0 & & \\
\uparrow & & \uparrow & & \\
p_2^* p_{2*} \mathcal{R}_k(H + jf^*L, D) & \longrightarrow & \mathcal{R}_k(H + jf^*L, D) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \\
S^k_{\mathcal{O}_{B\times\mathbb{C}}}(p_2^* p_{2*} \mathcal{R}_1(H + jf^*L, D)) & \longrightarrow & S^k_{\mathcal{O}_{B\times\mathbb{C}}}(\mathcal{R}_1(H + jf^*L, D)) & \longrightarrow & 0
\end{array}$$

obtained by canonical maps and the sheaf morphisms yielding ϕ from Lemma 3.2.22 is exact, when $j \gg 0$.

Exactness of the second column follows by local generation in degree 1 of $\mathcal{R}(H + jf^*L, D)$. Since L is ample on B, the sheaf $\pi_*\mathcal{O}(1) = \mathcal{R}_1(H + jf^*L, D)$, is generated by global sections when $j \gg 0$. Therefore, we have

$$\begin{array}{ccccc}
 & & pt^* pt_* \mathcal{R}_1(H + jf^*L, D), & & \\
 & \swarrow & & \searrow & \\
p_2^* p_{2*} \mathcal{R}_1(H + jf^*L, D) & & \longrightarrow & & \mathcal{R}_1(H + jf^*L, D) \\
 & & & & \searrow \\
 & & & & \quad 0
\end{array}$$

So, we have exactness of the bottom row:

$$S^k_{\mathcal{O}_{B\times\mathbb{C}}}(p_2^* p_{2*} \mathcal{R}_1(H + jf^*L, D)) \to S^k_{\mathcal{O}_{B\times\mathbb{C}}}(\mathcal{R}_1(H + jf^*L, D)) \to 0.$$

The first column is trivially exact in the case $k = 1$. The general case follows since $p_2^* p_{2*} \mathcal{R}(H + jf^*L, D)$ is locally generated in degree 1, hence one obtains a surjection from the symmetric powers. The top row is then automatically exact.

□

Lemma 3.2.24. *One has $p_{2*}\mathcal{R}(H + jf^*L, D) = R(H \widetilde{+ jf^*} L, D)$, i.e., it is the sheaf associated to the module $R(H + jf^*L, D)$.*

Proof. We have

$$\begin{aligned}
p_{2*}\mathcal{R}(H + jf^*L, D) &= p_{2*}\Big(\bigoplus_{k\geq 0} \mathcal{R}_k(H + jf^*L, D)\Big) \\
&= \bigoplus_{k\geq 0} p_{2*}\mathcal{R}_k(H + jf^*L, D).
\end{aligned}$$

Thus it suffices to consider these $\mathbb{C}[t]$-modules.

$$\begin{aligned} p_{2*}\mathcal{R}_k(H+jf^*L,D) &= p_{2*}\bigoplus_{\ell\geq 0} t^{-\ell}(f_Y)_*(kg^*(H+jf^*L)-\ell D) \\ &\cong \bigoplus_{\ell\geq 0} t^{-\ell}H^0(Y, kg^*(\widetilde{H+jf^*L})-\ell D) \\ &= R_k(\widetilde{H+jf^*L},D). \end{aligned}$$

The first isomorphism follows, since any quasi-coherent sheaf on an affine variety is the sheaf associated to its global sections, and the global sections [Har10, III, prop. 8.5] are

$$H^0(B\times\mathbb{C},\bigoplus_{\ell\geq 0} t^{-\ell}h_*(kg^*(H+jf^*L)-\ell D)) = \bigoplus_{\ell\geq 0} t^{-\ell}H^0(Y,kg^*(H+jf^*L)-\ell D),$$

from which the claim follows, since the sheaf associated to a module is compatible with the direct sum. □

Proof. (Theorem 3.2.21) From these lemmata, we see by the base change property of pullbacks, that

$$Proj_{\mathbb{C}}(R(H+jf^*L,D))$$

is finite dimensional, hence $R(H+jf^*L,D)$ is finitely generated and D is dreamy with respect to $H+jf^*L$ for $j\gg 0$. Indeed, we have

$$\begin{aligned} Proj_{B\times\mathbb{C}}(p_2^*p_{2*}(\mathcal{R}(H+jf^*L,D))) &= Proj_{\mathbb{C}}(p_{2*}(\mathcal{R}(H+jf^*L,D))\times_{\mathbb{C}}(B\times\mathbb{C}) \\ &\cong Proj_{\mathbb{C}}(R(H+jf^*L,D))\times_{\mathbb{C}}(B\times\mathbb{C}) \\ &\cong Proj_{\mathbb{C}}((R(H+jf^*L,D))\times B, \end{aligned}$$

where $\mathbb{C}=Spec(\mathbb{C}[t])$, which by the same operations and the previous lemma 3.2.24 then embeds into

$$\mathbb{P}_{\mathbb{C}}(R_1(H+jf^*L,D))\times B.$$

This space has to be finite dimensional as

$$R_1(H+f^*L,D) = \bigoplus_{\ell\in\mathbb{Z}} t^{-\ell}H^0(Y,g^*(H+jf^*L)-\ell D)$$

is a finitely generated $\mathbb{C}[t]$-module. □

Filtrations of fibration degenerations

The aim of this seciton is to prove that when $p_2 \circ \pi \colon (\mathcal{X}, \mathcal{H}) \to \mathbb{C}$ is a fibration degeneration associated to the data $f \colon (X, H) \to (B, L)$, then we have an natural filtration of sheaves $(\lambda, k) \mapsto \mathcal{F}^{\lambda}_{\mathcal{X}} f_*(kH)$, which is locally finitely generated. Moreover, we want to prove that this filtration is induced by a valuation, when $(\mathcal{X}, \mathcal{H}) \to \mathbb{C}$ has irreducible central fibre.

Definition 3.2.25. A multiplicative filtration of sheaves $(\lambda, k) \mapsto \mathcal{F}^{\lambda}\mathcal{E}_k$ of $\mathcal{O}_B$-modules is locally finitely generated, if the relative Rees algebra is locally finitely generated.

We want to associate a filtration

$$(\lambda, k) \mapsto \mathcal{F}^{\lambda}_{\mathcal{X}} f_*(kH) \subset f_*(kH)$$

of $\mathcal{O}_B$-submodules for any H f-ample with an associated fibration degeneration. For this, we aim to mimick the case over a point first used in [Wit12]. We have $X \cong \mathcal{X}_1$ via some morphism ϕ and $\sigma \colon X \times \mathbb{C}^* \to \mathcal{X} \setminus \mathcal{X}_0$ over $B \times \mathbb{C}^*$ given by $\sigma(x,t) = \rho(t) \cdot \phi(x)$, where ρ is the induced $\mathbb{C}^*$-action on $\mathcal{X}$. Note σ also induces $\mathbb{C}^*$-equivariant identification $\sigma_* p_1^* kH \cong k\mathcal{H}|_{\mathcal{X}\setminus\mathcal{X}_0}$. In particular, for any $\mathbb{C}^*$-invariant open subset of $\mathcal{X} \setminus \mathcal{X}_0$, say $\pi^{-1}(U \times \mathbb{C}^*) = \sigma(U \times \mathbb{C}^*)$ where U is affine, there is an equivariant isomorphism of sections

$$\begin{aligned} p_1^* kH(f^{-1}(U) \times \mathbb{C}^*) \to \sigma_* p_1^* kH((f^{-1}(U) \times \mathbb{C}^*)) &= p_1^* kH((\pi^{-1}(U \times \mathbb{C}^*)) \\ &= k\mathcal{H}|_{\mathcal{X}\setminus\mathcal{X}_0}(\pi^{-1}(U \times \mathbb{C}^*)) \\ &= \pi_*(k\mathcal{H}|_{\mathcal{X}\setminus\mathcal{X}_0})(U \times \mathbb{C}^*). \end{aligned}$$

Let us by abuse of notation call this isomorphism σ as well. There is a natural morphism

$$\tilde{s} \colon f_*(kH)(U) \to p_1^* kH(f^{-1}(U) \times \mathbb{C}^*)$$

given by $\tilde{s}(u,t) = s(u)$. Finally, we set $\overline{s} = \sigma(\tilde{s})$. This gives a map to the invariant sections of $\pi_*(k\mathcal{H}|_{\mathcal{X}\setminus\mathcal{X}_0})(U \times \mathbb{C}^*)$. Indeed, denoting all actions by ρ, we get

$$\begin{aligned} \rho(t) \cdot \overline{s} &= \sigma(\rho(t) \cdot \tilde{s}) \\ &= \sigma(\tilde{s}) \\ &= \overline{s}, \end{aligned}$$

because the action on $p_1^* kH$ is given by multiplication on the $\mathbb{C}^*$-factor. We are now ready to define the filtration.

Definition 3.2.26. Set

$$(\lambda, k) \mapsto \mathcal{F}^{\lambda}_{\mathcal{X}}(f_*(kH))_{\ b} = \{s \in f_*(kH)_b | t^{-\lambda}\bar{s} \in \pi_*(\mathcal{O}_{\chi}(k))_{b\times\mathbb{C}}\}$$

and then define subsheaves by setting

$$U \mapsto \mathcal{F}^{\lambda}_{\mathcal{X}} f_*(kH)(U) = \{s \in f_*(kH)(U) | s_b \in \mathcal{F}^{\lambda} f_*(kH)_b \forall b \in U\}.$$

Here, t acts by multiplication, which comes from the module structure dictated by the morphism $p_1 \circ \pi \colon \mathcal{X} \to \mathbb{C}$.

Remark 3.2.27. We might as well have defined it over affines $U \subset B$, and then glue the locally defined subsheaves. $\mathcal{F}^{\lambda}_{\mathcal{X}} f_*(kH)$ is indeed a sheaf of $\mathcal{O}_B$-submodule, since if $m \in \mathcal{O}_B(U)$, then for $s \in \mathcal{F}^{\lambda} f_*(kH)(U)$

$$\begin{aligned} t^{-\lambda}(\overline{m \cdot s}) &= t^{-\lambda}(\overline{f^* m \cdot s}) \\ &= t^{-\lambda} f^* m \bar{s} \\ &= f^* m t^{-\lambda} \bar{s} \in \pi_*(\mathcal{O}_{\mathcal{X}}(k))(U \times \mathbb{C}), \end{aligned}$$

because $f = p_1 \circ \pi$ over $U \times \mathbb{C}^*$ and $p_1^* m$ naturally extends.

Lemma 3.2.28. *If $s_1, \dots, s_n$ are local generators for $f_*(kH)$ around $b \in B$ such that $s_i \in \mathcal{F}^{\lambda_i}_{\mathcal{X}} f_*(kH)_b$, then there is a neighbourhood $b \in U$ such that*

- $\mathcal{F}^{min_i \lambda_i}_{\mathcal{X}} f_*(kH)|_U = f_*(kH)|_U$
- $\mathcal{F}^{max_i \lambda_i + 1}_{\mathcal{X}} f_*(kH)|_U = 0.$

Proof. Pick local generators $s_1, \dots, s_n$ for $f_*(kH)_b$. We can assume there is an open neighbourhood U, where $s_i \in \mathcal{F}^{\lambda_i} f_*(kH)|_U$, so there is some $h_i|_{U\times\mathbb{C}^*} = t^{-\lambda_i}\bar{s}_i$. Let $\mu = min_i \lambda_i$, then any $s = \sum g_i s_i$ satisfies

$$\begin{aligned} t^{-\mu}\bar{s} &= t^{-\mu} \sum g_i \bar{s}_i \\ &= \sum g_i t^{-\mu} \bar{s}_i \\ &= (\sum g_i (t^{-\mu+\lambda_i} h_i))|_{U\times\mathbb{C}^*} \end{aligned}$$

Therefore, $s \in \mathcal{F}^{\mu} f_*(kH)|_U$. Conversely, if $s \in \mathcal{F}^{\lambda} f_*(kH)|_U$, then the extension h of $t^{-\lambda}\bar{s}$ satisfies

$$\begin{aligned} t^{\lambda-\mu} h|_{U\times\mathbb{C}^*} &= t^{-\mu}\bar{s} \\ &= \sum g_i t^{-\mu} \bar{s}_i \\ &= (\sum g_i t^{-\mu+\lambda_i} h_i)|_{U\times\mathbb{C}^*}. \end{aligned}$$

So, since extensions are uniquely determined, we have

$$t^{\lambda-\mu}h = \sum g_i t^{-\mu+\lambda_i} h_i,$$

hence

$$h = \sum g_i t^{\lambda_i-\lambda} h_i,$$

which is only defined over $t = 0$, when $\lambda \leq \min_i \lambda_i$, where the minimum is taken over all i where $g_i \neq 0$. Thus, taking $\lambda = \max_i \lambda_i + 1$, one has a contradiction. □

Proposition 3.2.29. *For any fibration degeneration, the filtration of subsheaves*

$$(\lambda, k) \mapsto \mathcal{F}^{\lambda}_{\mathcal{X}} f_*(kH)$$

is decreasing and locally right and linearly left bounded.

Proof. We first show that we can bring ourselves in a position where 3.2.28 holds. Pick any ample line bundle L on the base B. Then, by Dervan-Sektnan 3.1.4, we know that changing the polarization to $(\mathcal{X}, \mathcal{H}+j\pi^*L)$ for $j \gg 0$ sufficiently big, gives an ample test configuration for $(X, H+jf^*L)$. It follows that there is a filtration

$$\begin{aligned}(\lambda, k) \mapsto & F^{\lambda}H^0(X, k(H+jf^*L)) \\ & = \{s \in H^0(X, k(H+jf^*L)) | t^{-\lambda}\bar{s} \in H^0(\mathcal{X}, k(\mathcal{O}_{\mathcal{X}}(1)+j\pi^*L))\},\end{aligned}$$

which is linearly left and right bounded, say by a constant C and λ_0 respectively 2.3.10. Fixing j big enough, we can assume that $k(\mathcal{H}+j\pi^*L), k(H+jf^*L)$ is globally generated for all k. Hence, we have a surjection of sheaves

$$H^0(X, k(H+jf^*L)) \to k(H+jf^*L) \to 0.$$

Restricting this morphism to an affine $U \subset B$ trivializing L, we have that the targets of the morphism is

$$k(H+jf^*L)(f^{-1}(U)) \cong f_*(kH)(U).$$

In particular, any finite set of local generators $s_i \in f_*(kH)(U)$ are restrictions of sections $\hat{s}_i \in H^0(B, f_*(kH)+jkf^*L)$. Since we have a filtration of these vector spaces, there are λ_i such that $\hat{s}_i \in F^{\lambda_i}H^0(X, k(H+jf^*L))$. It follows by definition of the filtration that $\hat{s}_i|_U = s_i$ must be in $\mathcal{F}^{\lambda_i}_{\mathcal{X}} f_*(kH)(U)$. Therefore, by 3.2.28, the bounds of $(\lambda, k) \mapsto F^{\lambda}_{\mathcal{X}}H^0(X, k(H+jf^*L)$ are inherited by $(\lambda, k) \mapsto \mathcal{F}^{\lambda}_{\mathcal{X}} f_*(kH)$ near b. □

The proof tells us that the optimal local bounds on the filtration are smaller than the global bounds on the filtrations

$$(\lambda, k) \mapsto F^\lambda H^0(X, k(H + jf^*L)),$$

where L can vary over the entire ample cone of B, and j is picked so that $H + jf^*L$ is ample on X.

Lemma 3.2.30. *The filtration associated to an integral fibration degeneration comes from a horizontal divisor, i.e., there is a horizontal divisor $D \subset Y$ over X such that for all (λ, k) we have*

$$\mathcal{F}^\lambda_{\mathcal{X}} f_*(kH) = \mathcal{F}^\lambda_{ord_D} f_*(kH).$$

Proof. If $(\mathcal{X}, \mathcal{H})$ is a fibration degeneration with irreducible central fibre, then the induced horizontal valuation $v_{\mathcal{X}_0}$ from 3.2.11 can be used to describe the filtration as

$$\begin{aligned} \mathcal{F}^\lambda_{\mathcal{X}} f_*(kH)(U) &= \{s \in f_*(kH)(U) | t^{-\lambda}\bar{s} \in \pi_*(k\mathcal{H})(U \times \mathbb{C})\} \\ &= \{s \in f_*(kH)(U) | v_{\mathcal{X}_0}(\bar{s}) \geq \lambda\}. \end{aligned}$$

Indeed, $t = 0$ gives an equation for $\mathcal{X}_0$ when $\mathcal{X}_0$ is irreducible. Therefore, we have $v_{\mathcal{X}_0}(\bar{s}) \geq \lambda$, if, and only if, we can write $\bar{s}|_{W_i} = t^\lambda h$ and $h \in k\mathcal{H}(\pi^{-1}(U \times \mathbb{C})) = \pi_*(k\mathcal{H})(U \times \mathbb{C})$. Recall the equivariant map $\sigma \colon X \times \mathbb{C}^* \to \mathcal{X} \setminus \mathcal{X}_0 \subset \mathcal{X}$, which gives us an isomorphism of function fields $K(\mathcal{X}) \cong K(X \times \mathbb{C}) \cong K(X)(t)$. In particular, since $\mathcal{X}_0$ is a $\mathbb{C}^*$-invariant divisor on $\mathcal{X}$, it follows that $v_{\mathcal{X}_0}$ is also $\mathbb{C}^*$-invariant when considered on $K(X)(t)$. Therefore, by 2.3.15, we can write it as

$$v_{\mathcal{X}_0}(f) = \min_\mu (v_{\mathcal{X}_0}|_X(f_\mu) + \mu v_{\mathcal{X}_0}(t)),$$

where $f = \sum_{\mu \in \mathbb{Z}} f_\mu t^\mu$, $f_\mu \in K(X)$. Now

$$\begin{aligned} v_{\mathcal{X}_0}(\bar{s}) &= v_{\mathcal{X}_0}(\sigma(\tilde{s})) \\ &= \sigma_*^{-1} v_{\mathcal{X}_0}(\tilde{s}) \\ &= \sigma_*^{-1} v_{\mathcal{X}_0}|_X(\tilde{s}_0) \\ &= \sigma_*^{-1} v_{\mathcal{X}_0}|_X(s). \end{aligned}$$

Here we used that $s \mapsto \tilde{s}$ is the invariant extension, hence $\tilde{s}$ has weight 0 in $K(X)(t)$ and one has $\tilde{s}_0 = s$. If the test configuration is not trivial, then the valuation $\sigma_*^{-1} v_{\mathcal{X}_0}|_X$ is divisorial, so by theorem 3.2.10 there is a divisor $D \subset Y \xrightarrow{g} X$ such that $g_* ord_D = \sigma_*^{-1} v_{\mathcal{X}_0}|_X$. This divisor D has to be horizontal by 3.2.5 since $\sigma_*^{-1} v_{\mathcal{X}_0}|_X$ is horizontal by 3.2.11 and 3.2.8.

Thus we can write

$$\begin{aligned} \mathcal{F}_{\mathcal{X}}^{\lambda} f_*(kH)(U) &= \{s \in f_*(kH) | \sigma_*^{-1} v_{\mathcal{X}_0}|_X(s) \geq \lambda\} \\ &= \mathcal{F}_{ord_D}^{\lambda} f_*(kH). \end{aligned}$$

□

Now, combining the results one can conclude:

Theorem 3.2.31. *An integral fibration degeneration* $(\mathcal{X}, \mathcal{H})$*, for* (X, H, B, f) *gives rise to a horizontal relatively* H*-dreamy divisor.*

Proof. In view of the previous lemma, the only thing that remains to be proven is that a locally linearly right and left bounded filtration of $f_*(kH)$ is locally finitely generated. Indeed, then the induced horizontal divisor D is relatively H-dreamy by definition.

If H is a relatively f-ample, then the algebra

$$\bigoplus_{k \geq} f_*(kH)$$

is locally finitely generated. Then, considering the local bounds any element in

$$\bigoplus_{k \geq 0} (\bigoplus_{\lambda \in \mathbb{Z}} t^{-\lambda} \mathcal{F}^{\lambda} f_*(kH))$$

can be written locally as

$$\sum_{k=0}^{k'} (t^{-\lambda} \sum_{Ck \geq \lambda \geq \lambda_0} s_{k,\lambda} + \sum_{\lambda < \lambda_0} t^{-\lambda} g_{k,\lambda}),$$

which in turn can be written in terms of local generators $m_1, \ldots, m_n$ for $\bigoplus_{k \geq 0} f_*(kH)$ as

$$\sum_{k=0}^{k'} (\sum_{Ck \geq \lambda \leq \lambda_0} t^{-\lambda} P_{k,\lambda}(m_1, \ldots, m_n) + \sum_{\lambda < \lambda_0} t^{-\lambda} Q_{k,\lambda}(m_1, \ldots, m_n))$$

for some polynomials with $\mathcal{O}_B$-coefficients. This sum then consists of finitely many terms, so indeed, one has locally finite generation. □

This completes the programme of characterizing the relevant subclass of divisors/divisorial valuations that needs to be considered in relation to fibration degenerations.

3.3 An asymptotic valuative invariant for fibration degenerations

In analogy with the notion of stability for fibration degenerations obtained in [DS21a], we shall introduce an asymptotic expansion of the β invariant, which gives a notion of stability for fibration degenerations. It follows essentially by definition that this notion of valuative stability gives an obstruction to the stability of a fibration. Finally, we compute the invariant for a class of examples.

Remark 3.3.1. Suppose one has a fibration degeneration $\pi\colon (\mathcal{X}, \mathcal{H}) \to B \times \mathbb{C}$ of (X, H, B, f) obtained by a relatively H-dreamy divisor D. Then, D is $H + jf^*L$ dreamy for $j \gg 0$ for any ample L on B and $(\mathcal{X}, \mathcal{H} + j\pi^* p_1^* L)$ is a test configuration for $H + jf^*L$ by lemma 3.1.4 of [DS21a]. By the work of Dervan-Legendre [DL22], the Donaldson-Futaki invariants of these test configurations can be computed as

$$\beta_{H+jf^*L}(D) = DF(\mathcal{X}, \mathcal{H} + j\pi^* p_1^* L) 2(\dim(X) - 1)!, \tag{7}$$

but the Donaldson-Futaki invariant $DF(\mathcal{X}, \mathcal{H} + j\pi^* p_1^* L)$ admits an expansion in j by lemma 3.1.9, when $j \gg 0$, hence, $\beta_{H+jf^*L}(D)$ admits such an expansion as well of the same degree.

In the following, we set

$$\beta_{(H,L)}(D)(j) := \beta_{H+jf^*L}(D),$$

so that when we think of it as a function or polynomial of j, we simply write $\beta_{(H,L)}$. Hence, we can write

$$\beta_{(H,L)}(D) = \beta_0(D) j^{\dim(B)} + \beta_1(D) j^{\dim(B)-1} + \dots$$

Definition 3.3.2. We say $f\colon (X, H) \to (B, L)$ is *valuatively*

- *semistable*, if for any relatively H-dreamy horizontal divisor D, we have $\beta_0(D) \geq 0$ and when $\beta_0(D) = 0$ then $\beta_1(D) \geq 0$

- *stable*, if for any relatively H-dreamy horizontal divisor D, we have $\beta_0(D) \geq 0$ and whenever it is zero, then $\beta_1(D) > 0$.

Proposition 3.3.3. *If $f\colon (X,H) \to (B,L)$ is with K-semistable fibres, then it is integrally fibrationally semistable, if, and only if, it is valuatively semistable.*

Proof. On one hand, a horizontal relatively H-dreamy divisor D gives rise to a fibration degeneration $(\mathcal{X},\mathcal{H})$ with integral central fibre. On the other hand, any $(\mathcal{X},\mathcal{H})$ gives rise to a horizontal relatively H-dreamy divisor. In both cases, the formula 7 holds by ([DL22], prop. 3.9 and 3.15). Therefore, up to multiplication by a positive constant the coefficients agree, so one has

$$\beta_0(D) = 2(\dim(X)-1)!W_0(\mathcal{X},\mathcal{H}),$$

and

$$\beta_1(D) = 2(\dim(X)-1)!W_1(\mathcal{X},\mathcal{H}),$$

from which the claim follows. □

Remark 3.3.4. The same argument also immediately gives that valuative stability of the fibration implies integral fibration stability. The converse then depends on whether $\beta_1(D) = 0$ implies that the norm is 0. Since valuative stability characterizes K-stability in the Fano case [Fuj19, main thm. 1.4], there is hope to have a similar result in the case of Fano fibrations. In general our valuative interpretation is only an obstruction to fibration stability, so it seems important to develop more tools to characterize it. It would be interesting to extend the obstruction to a more general setting as in [BJ22], [BHJ17]. This would require a non-Archimedean interpretation of stability in the setting of fibrations.

Towards a stronger stability condition

In this section, we aim to make progress towards showing that the family of β invariants $\beta_{H+jf^*L}(D)$ is polynomial in j for any D horizontal, but omitting the dreamy hypothesis. This enables us to extend the same valuative stability condition to a larger class of prime divisors. The terms in the β_{H+jf^*L} invariant, which are not a priori polynomial in j, are the terms involving the integral of the volume and its derivative. In particular, the interval over which one integrates $(0, \tau_{H+jf^*L})$ depends on j. We show here that in good circumstances when j is sufficiently large, then the interval integrated over is fixed.

Definition 3.3.5 ([dFEM14, prop.1.6.33])**.** A Cartier divisor $H \in Pic(X)$ is f-big for $f\colon X \to B$ projective, if, and only if, there are Cartier divisors A, E on X and a constant $d > 0$ such that

- A is f-ample,
- E is f-effective, i.e. $f_*\mathcal{O}_X(E) \neq 0$
- $dH = A + E$.

In particular, f-ample divisors are f-big. When B is projective itself, then the condition that E is f-effective can be restated as follows.

There exists an ample divisor C on B and an effective divisor E' on X such that

$$E = E' - f^*C.$$

Where C can be chosen to be a sufficiently big multiple of any ample bundle on B. Thus, one can refine the decomposition in the definition of an f-big divisor to be

$$H = A - f^*C + E',$$

where now all divisors are $\mathbb{Q}$-divisors. Observe f-big divisors are f-effective: in fact, the pushforward is supported fully on B. Conversely, if a divisor is f-effective, then adding any small multiple of an f-ample divisor gives an f-big divisor. Therefore, the cone of f-big divisors are the interior of the cone of f-effective divisors.

Definition 3.3.6. Let $f\colon X \to B$ be a projective morphism. Then, the relative pseudoeffective threshold of a horizontal prime divisor D with respect to an f-ample divisor H is the number

$$\begin{aligned}\tau_H^f(D) &= \sup\{t > 0 | H - tD \text{ is } f - \text{big}\}\\ &= \sup\{t > 0 | H - tD \text{ is } f - \text{effective}\}\\ &= \inf\{t > 0 | f_*(\mathcal{O}_X(k(H - tD))) = 0, \forall k \in \mathbb{N}\}.\end{aligned}$$

The fact that these descriptions are equivalent follows by the fact that the f-big divisors are the interior of the f-effective divisors [dFEM14].

Remark 3.3.7. The relative pseudoeffective threshold is equal to the supremum the pseudoeffective divisors restricted to the fibres. Indeed, if $H|_{X_b} = H_b, D|_{X_b} = D_b$ then

$$\tau_{H_b}(D_b) \leq \tau_H^f(D)$$

for all $b \in B$. Since

$$\begin{aligned}\{t > 0 | f_*(H - tD) = 0\} =& \bigcap_{b \in B} \{t > 0 | f_*(H - tD)_b \neq 0\} \\ \subset& \{t > 0 | H^0(X_b, H_b - tD_b) \neq 0\}\end{aligned}$$

if $\sup_{b \in B} \tau_{H_b}(D_b) < \tau_H^f(D)$, then one clearly has a contradiction, because for any s such that $\sup_{b \in B} \tau_{H_b}(D_b) < s < \tau_H^f(D)$, there can be no sections restricted to any fibres, yet $f_*(H - sD) \neq 0$. Thus,

$$\tau_H^f(D) = \sup_{b \in B} \tau_{H_b}(D_b).$$

Next, we show how the relative pseudoeffective threshold relates to the asymptotic pseudoeffective threshold $\tau_{H+jf^*L}(D)$.

Lemma 3.3.8. *For a horizontal prime divisor D over $f \colon (X, H) \to (B, L)$, we have*

$$\tau_{H+jf^*L}(D) \leq \tau_H^f(D).$$

Proof. Let $s \in H^0(X, H + jf^*L - tD)$ be nonzero. Then, for $U \subset B$ affine $s|_{f^{-1}(U)}$ cannot vanish everywhere. Therefore $s|_{f^{-1}(U)} \in f_*(H + jf^*L - tD)(U)$ is nontrivial. By the push-pull formula

$$\begin{aligned}f_*(H + jf^*L - tD)(U) \cong& (f_*(H - tD) + jL)(U) \\ \cong& f_*(H - tD)(U),\end{aligned}$$

because $jL(U) \cong \mathcal{O}_B(U)$. Hence, $f_*(H - tD) \neq 0$ and so $\tau_{H+jf^*L}(D) \leq \tau_H^f(D)$. □

Next, we show that in fortunate circumstances, the bound from the previous lemma is in fact achieved for j sufficiently big.

Proposition 3.3.9. *If D is a horizontal prime divisor over $f \colon (X, H) \to (B, L)$ such that $f_*(H - \tau_H^f(D)D) \neq 0$, then there exists a $j_0(D)$ such that for all $j > j_0$ one has*

$$\tau_{H+jf^*L}(D) = \tau_H^f(D).$$

Proof. By definition one knows that for each $t < \tau_H^f(D)$ there are A_t, C_t, E_t such that $H - tD = A_t - f^*C_t + E_t$ and

- A_t is f-ample
- C_t is ample on B

- E_t is effective on X

Since A_t is f-ample, there is a $j_1(t)$ such that $A_t + jf^*L$, is ample when $j > j_1(t)$ and since L is ample on B, there is a $j_2(t)$ such that $-C_t + jL$ is ample on B when $j > j_2(t)$. Therefore, when $j > j_1(t) + j_2(t)$ one has that $A_t - f^*C_t + jf^*L$ is ample. For such j

$$H + jf^*L - tD = A_t - f^*C_t + E_t + jf^*L$$

is the sum of an ample divisor and an effective divisor. In particular, $H + jf^*L - tD$ is big. It follows that we have $\lim_{j\to\infty} \tau_{H+jf^*L}(D) = \tau_H^f(D)$. In order to achieve our result, we need to show that we can bound the $j_1(t), j_2(t)$ needed uniformly in t.

Write $M_t = H - tD$. We supposed that $f_*(M_{\tau_H^f}) \neq 0$, which might not be true in general. By the remark 3.3.7, there is $C_{\tau_H^f}$ ample on B and $E_{\tau_H^f}$ effective on X such that

$$M_{\tau_H^f} = -f^*C_{\tau_H^f} + E_{\tau_H^f}.$$

Then, for all $s \in [0,1]$, we have

$$\begin{aligned} M_{s\tau_H^f} =& (1-s)M_0 + sM_{\tau_H^f} \\ =& (1-s)H - sf^*C_{\tau_H^f} + sE_{\tau_H^f}. \end{aligned}$$

H is f-ample, so there is a j_1 such that for all $j > j_1$ $H + jf^*L$ is ample. In particular,

$$(1-s)H + jf^*L = (1-s)(H + jf^*L) + sjf^*L$$

is ample too for $s \in [0,1)$, since it is the sum of an ample and nef divisor. There is a j_2 such that $jL - sC_{\tau_H^f} \geq 0$ is effective when $s \in [0,1]$ for $j > j_2$. So, the pullback is effective too. Therefore, for $j > j_1 + j_2$, one has that

$$H - s\tau_H^f D + jf^*L = M_{s\tau_H^f} + jf^*L$$

is the sum of an ample and effective divisor whenever $s \in [0,1)$, i.e., it is big for all such s. Therefore, $\tau_{H+jf^*L}(D) \geq \tau_H^f(D)$, when $j > j_1 + j_2$.

□

This shows that in good circumstances (i.e., $f_*(H - \tau_H^f(D)D) \neq 0$), the existence of a polynomial expansion for $\beta_{H+jf^*L}(D)$ only depends on whether the volume $Vol(H + jf^*L - tD)$ can be described piecewise polynomially in j for $t \in (0, \tau_H^f(D))$.

3.4 The case of projective bundles

In the following section, we compute the asymptotic β invariant up to subleading order in the case when the fibration $f\colon (X, H) \to (B, L)$ is a projective bundle with $n = \dim(B)$. Write $X = \mathbb{P}_B(\mathcal{E})$ for some locally free sheaf of $\mathcal{O}_B$-modules $\mathcal{E}$ of rank r, f for the canonical map, and $H = \mathcal{O}_X(1)$ for the hyperplane bundle. We shall also fix an injection of $0 \to \mathcal{F} \to \mathcal{E}$ of locally free sheaves or equivalently a surjection $\mathcal{E}^* \to \mathcal{F}^* \to 0$ of locally free sheaves such that $\mathcal{F}$ has rank $r - s$, giving rise to an embedding $\mathcal{P}_B(\mathcal{F}) \to \mathcal{P}_B(\mathcal{E})$ of projective bundles. Note, that unlike the previous sections we shall use the convention

$$\mathbb{P}_B(\mathcal{E}) = Proj_B(S(\mathcal{E}^*)),$$

so that projectivization of vector bundles becomes a covariant functor. Moreover, we shall often supress the base B, as it is understood to be fixed. This convention is standard when considering for example slope stability.

The main result of this section is the following, and the calculations involved in its proof will take up substantial space.

Theorem 3.4.1. *Let E be the exceptional divisor in the blow-up of X along the subbundle $\mathbb{P}_B(\mathcal{F})$. Then, the β invariant of E with respect to $H + j\pi^*L$ admits an expansion as*

$$\beta_{H,L}(E) = \frac{2r(r-s)}{r+1}\binom{n+r-1}{n-1}(\mu_L(\mathcal{E}) - \mu_L(\mathcal{F}))j^{n-1} + O(j^{n-2}).$$

It is known that one can relate slope stability with fibrational stability in this case (see [DS21a] thm. 2.3.6 , which relies on [RT06b]). Here, we pick an explicit divisor appearing on a dominating model for the projective bundle, which realizes this correspondence. Moreover, it provides a first example of asymptotic β invariant calculations. Recall the notion of slope stability of coherent sheaves (see [HL10]): Given a polarized variety (Y, L) and a coherent sheaf $\mathcal{G}$ on Y. Then, $\mathcal{G}$ has slope

$$\mu_L(G) = \frac{c_1(\mathcal{G}) \cdot L^{\dim(Y)-1}}{rk(\mathcal{G})}.$$

By Riemann-Roch, this number corresponds to the subleading order term in the monic expansion of the Euler characteristic $\mathcal{X}(Y, \mathcal{G} \otimes L^k)$ as $k \gg 0$.

Definition 3.4.2. Given an ample line bundle L on B, then a coherent sheaf $\mathcal{E}$ is *slope semistable* with respect to L, if for each proper subsheaf $0 \to \mathcal{F} \to \mathcal{E}$ one has $\mu_L(\mathcal{F}) \leq \mu_L(\mathcal{E})$, it is *slope stable*, if the inequality is always strict.

Then, one has directly from the theorem above that

Corollary 3.4.3. *Slope stability of $\mathcal{E}$ with respect to L implies valuative stability of the fibration $\pi\colon (\mathbb{P}_B(\mathcal{E}), \mathcal{O}(1)) \to (B, L)$ with respect to all divisors appearing as blow-ups along subbundles. Conversely, valuative stability of the fibration implies slope stability with respect to L of smooth subbundles.*

The Chow ring of the Blow-up

In this section, we recall a description of the Chow-ring of the blowup of $\mathbb{P}(\mathcal{F})$ in $\mathbb{P}(\mathcal{E})$ as described in [EH16] chapter 13. We write

- $X = \mathbb{P}(\mathcal{E})$
- $Z = \mathbb{P}(\mathcal{F})$
- $\pi\colon Y = Bl_Z X \to X$ and $\hat{\pi} = f \circ \pi$
- $D = \pi^{-1}(Z) \cong \mathbb{P}_Z(N)$ where $N = \mathcal{O}_Z(1) \otimes f_Z^*(\mathcal{E}/\mathcal{F})$ is the normal bundle of Z in X.

Then, we have a diagram

$$\begin{array}{ccc} D & \xrightarrow{\;j\;} & Y \\ \downarrow{\scriptstyle \pi_D} & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{\;i\;} & X, \end{array}$$

where π is proper birational and π_D is projection to the zero section. This gives maps between Chow groups

$$\begin{aligned} \pi^*\colon A^k(X) &\to A^k(Y) \\ \pi_D^*\colon A^k(Z) &\to A^k(D) \\ i_*\colon A^k(Z) &\to A^{k+s}(X) \\ j_*\colon A^k(D) &\to A^{k+1}(Y), \end{aligned}$$

where A^k denote codimension k varieties modulo rational equivalence. The intersection ring $A(Y)$ is generated by $\pi^*A(X)$ and $j_*A(D)$ with multiplication given by

$$\begin{aligned} \pi^*a \cdot \pi^*b &= \pi^*(ab) \\ \pi^*a \cdot j_*b &= j_*(b \cdot \pi_D^* i^* a) \\ j_*a \cdot j_*b &= -j_*(a \cdot b \cdot h_D). \end{aligned}$$

Here, h_D is the first Chern class of the hyperplane bundle of D. Moreover, one has an exact sequence which determines the relations among these generators

$$0 \to A(Z) \overset{-i_* \times q}{\to} A(X) \oplus A(D) \overset{\pi^* + j_*}{\to} A(Y) \to 0. \tag{8}$$

Here, q is the map $q(a) = -c_{s-1}(Q) \cdot \pi_D^* a$, where $Q = \pi_D^* N / \mathcal{O}_{\mathbb{P}_Z(N)}(-1)$ is the universal quotient bundle for D.

In the calculations to follow, we shall need

Lemma 3.4.4. *Let $\eta = \pi^* h_X, \epsilon = j_*(1) \in A^1(Y)$, where h represents is the hyperplane class $\mathcal{O}_X(1)$. Then, the subring of $A(Y)$ generated by codimension 1-subvarieties is generated by classes*

$$\hat{\pi}^* A(B)[\eta, \epsilon].$$

Moreover, one has the following relations between the generators

- $\sum_{k=0}^{s} c_{s-k}(\mathcal{E}/\mathcal{F})(\eta - \epsilon)^k = 0$
- $\sum_{i=0}^{r} c_i(\mathcal{E})\eta^{r-i} = 0$
- $\epsilon \cdot (\sum_{i=0}^{r-s} c_i(\mathcal{F})\eta^{r-s-i}) = 0,$

where we have supressed pullbacks in the notation.

Proof. We know that $A^1(Y) = \pi^* A^1(X) \oplus \langle j_*(1) \rangle = \hat{\pi} A^1(B) \oplus \langle \eta \rangle \oplus \langle \epsilon \rangle$, so these classes definitely generate the subring. The two last relations follow from the well known descriptions [EH16, chapter 9]

$$A(X) = A(B)[h_X]/(\sum_{i=0}^{r} c_i(\mathcal{E}) \cdot h_X^{r-i})$$

$$A(Z) = A(B)/(\sum_{i=0}^{r-s} c_i(\mathcal{F}) h_Z^{r-s-i})$$

by pulling back the relations to Y. Indeed,

$$
\begin{aligned}
0 &= \pi^*(\sum_{i=0}^{r} c_i(\mathcal{E})h_X^{r-i}) \\
&= \sum_{i=0}^{r} c_i(\mathcal{E})\eta^{r-i} \\
\epsilon \cdot (\sum_{i=0}^{r-s} c_i(\mathcal{F})\eta^{r-s-i}) &= j_*(1) \cdot \pi^*(\sum_{i=0}^{r-s} c_i(\mathcal{F})h_X^{r-s-i}) \\
&= j_*\pi_D^*(\sum_{i=0}^{r-s} c_i(\mathcal{F})h_Z^{r-s-i}) \\
&= 0,
\end{aligned}
$$

since $h_Z = i^*h_X$. For the last relation, note that we have

$$c_i(N) = \sum_{j=0}^{i} \binom{s-i+j}{j} c_{i-j}(\mathcal{E}/\mathcal{F})h_Z^j$$

on Z. Now, by the projectivized normal bundle description, one has

$$A(D) = A(Z)[h_D]/(\sum_{i=0}^{s} c_i(N)h_D^{s-i}).$$

so

$$
\begin{aligned}
0 &= \sum_{i=0}^{s} j_*(c_i(N) \cdot h_D^{s-i}) \\
&= \sum_{i=0}^{s} j_*(\pi_D^* i^*(\sum_{j=0}^{i} \binom{s-i+j}{j} c_{i-j}(\mathcal{E}/\mathcal{F})h_X^j) \cdot h_D^{s-i}) \\
&= \sum_{i=0}^{s}(\pi^*(\sum_{j=0}^{i} \binom{s-i+j}{j} c_{i-j}(\mathcal{E}/\mathcal{F})h_X^j) \cdot j_*(h_D^{s-i})) \\
&= \sum_{i=0}^{s}\sum_{j=0}^{i}(-1)^{s-i} \binom{s-i+j}{j} c_{i-j}(\mathcal{E}/\mathcal{F})\eta^j \cdot \epsilon^{s-i} \\
&= \sum_{k=0}^{s} c_{s-k}(\mathcal{E}/\mathcal{F})(\eta-\epsilon)^k,
\end{aligned}
$$

where we used that

$$j_*(h_D^k) = j_*(h_D^k \cdot 1) = -j_*(h_D^{k-1}) \cdot j_*(1) = \cdots = (-1)^k j_*(1)^k = (-1)^k \epsilon^k.$$

□

In the following, we shall make the change of generators

$$\alpha = \eta - \epsilon,$$
$$\beta = \eta,$$

and the corresponding relations from the previous lemma

- $\sum_{k=0}^{s} c_{s-k}(\mathcal{E}/\mathcal{F})\alpha^k = 0$
- $\sum_{i=0}^{r} c_i(\mathcal{E})\beta^{r-i} = 0$
- $(\beta - \alpha) \cdot (\sum_{i=0}^{r-s} c_i(\mathcal{F})\beta^{r-s-i}) = 0.$

Calculations

We want to find explicitly the two leading terms in the asymptotic expansion of

$$\begin{aligned}\beta_{H+jf^*L}(D) =& A_X(D)Vol(H + jf^*L) \\ &+ (n + r - 1)\mu(X, H + jf^*L) \int_0^{\tau_j(D)} Vol(H + f^*L - tD)dt \\ &+ \int_0^{\tau_j(D)} \frac{d}{ds} Vol(H + f^*L - tD + sK_X)dt,\end{aligned}$$

for j very large. We do this simply by computing the terms individually for j big. First, we note the following well known facts:

Lemma 3.4.5. *When X is a smooth variety, $Y = Bl_Z(X)$ and Z is a smooth subvariety of codimension s, then the canonical divisors are related by*

$$K_Y = \pi^* K_X + (s - 1)D$$

where D is the exceptional divisor.

Proof. This is a local compuation, so it follows by considering the blow-up of $\mathbb{C}^n$ along the ideal $I = \langle z_1, \ldots, z_s \rangle$. By definition, this is the subvariety

$$Bl_I(\mathbb{C}^n) = \{(z_1, \ldots, z_n), [x_1 : \ldots x_s] | x_i z_j = x_j z_i\} \subset \mathbb{C}^n \times \mathbb{P}^{s-1}_{\mathbb{C}}$$

and the blow-up map is $p_1 \colon Bl_I(\mathbb{C}^n) \to \mathbb{C}^n$, so the exceptional divisor is $p_1^{-1}(V(I))$. In any chart $\mathbb{C}^n \times \{x_i \neq 0\}$ we have

$$Bl_I(\mathbb{C}^n) \cap \mathbb{C}^n \times \{x_i \neq 0\} = \{(z_1, \ldots, z_n), (\tilde{x}_{1i}, \ldots \tilde{x}_{(s-1)i}) | z_j = \tilde{x}_{ji} z_i\}$$

where $\tilde{x}_{ji} = \frac{x_j}{x_i}$. Thus the exceptional divisor has equation $z_i = 0$ here. Pulling back a top degree volume form we have

$$\begin{aligned} p_1^*(dz_1 \wedge \cdots \wedge dz_n) &= \pm d(\tilde{x}_{1i} z_i) \wedge \cdots \wedge d(\tilde{x}_{(s-1)i} z_i) \wedge dz_i \wedge dz_{s+1} \wedge \ldots dz_n \\ &= \pm z_i^{s-1} d\tilde{x}_{1i} \wedge \cdots \wedge d\tilde{x}_{(s-1)i} \wedge dz_i \wedge dz_{s+1} \wedge \cdots \wedge dz_n \end{aligned}$$

Here the sign only depends on the chosen chart. In either case, it implies that $\pi^* K_X = K_Y - (s-1)D$ hence the result.

□

From this it follows directly that the log discrepancy is

$$A_X(D) = ord_D(K_Y - \pi^* K_X) + 1 = s. \quad (9)$$

Lemma 3.4.6. *The canonical bundle of $X = \mathbb{P}_B(\mathcal{E})$ is given by*

$$K_X = -r\mathcal{O}(1) + f^*(K_B - \det(\mathcal{E}))$$

Proof. We have the short exact sequence

$$0 \to f^*\Omega_B \to \Omega_X \to \Omega_{X/B} \to 0$$

where Ω_X denotes the sheaf of Kähler differentials. From this it follows that

$$K_X = f^* K_B + K_{X/B}$$

Recall the relative Euler sequence [Har10, II, thm 8.13]

$$0 \to \Omega_{X/B} \to (\bigoplus_{i=1}^{r} \mathcal{O}_X(-1)) \otimes f^*\mathcal{E} \to \mathcal{O}_X \to 0$$

from which it follows that

$$r\mathcal{O}(-1) + \det(f^*\mathcal{E}) = K_{X/B}$$

thus

$$K_X = -r\mathcal{O}(1) + f^*(K_B + \det(\mathcal{E})).$$

□

In particular since $c_1(\det(\mathcal{E})) = c_1(E)$, one has for purposes of intersection number calculations the identity

$$K_X \equiv -rH + f^*(K_B + c_1(\mathcal{E})).$$

Lemma 3.4.7. *There is a j_0 such that for all $j \geq j_0$, the pseudoeffective threshold $\tau_{H+jf^*L}(D) = 1$, in fact for all those j and $x \in [0,1)$, the bundle $H + jf^*L - xD$ is ample, so the Seshadri constant $\epsilon_{H+jf^*L}(D) = 1$ as well.*

Proof. By 3.3.9 the pseudoeffective threshold

$$\tau_{H+jf^*L}(D) = \tau_H^f(D), \quad j \gg 0.$$

Therefore, it suffices to analyze the condition on the fibres. By functoriality of the blow-up, the fibre X_b is blown up in Z_b. I.e. $X_b \cong \mathbb{P}_{\mathbb{C}}^{r-1}$ and $Z_b \cong \mathbb{P}_{\mathbb{C}}^{r-s-1}$. But

$$Bl_{Z_b}(X_b) \subset X_b \times \mathbb{P}_{\mathbb{C}}^s$$

shows that the ample cone of $Bl_{Z_b}(X_b)$ contains positive linear combinations of pullback of the hyperplane sections h_{X_b} and $h_{\mathbb{P}_{\mathbb{C}}^s}$. Moreover, in $A^1(Bl_{Z_b}(X_b))$ we have that the exceptional divisor

$$D_b = h_{X_b} - h_{\mathbb{P}_{\mathbb{C}}^s}$$

[EH16, cor. 9.12] where we have supressed the pullback. Since $H|_{X_b} = h_{X_b}$ it follows that for $t \in [0,1)$ that

$$H|_{X_b} - tD_b = (1-t)h_{X_b} + th_{\mathbb{P}_{\mathbb{C}}^s}$$

is ample. Therefore $\tau_{H+jf^*L} \geq 1$ and we see that it is less than 1 by considering the volume for $t = 1$.

$$\begin{aligned} Vol(H|_{X_b} - D_b) =& Vol(h_{\mathbb{P}_{\mathbb{C}}^s}) \\ =& h_{\mathbb{P}_{\mathbb{C}}^s}^{r-1} \\ =& 0. \end{aligned}$$

□

Thus the integrals involved in the β invariant are for j sufficiently big over the unit interval, which we shall assume from now on.

The first term of β then expands polynomially in j as

$$\begin{aligned} A_X(D)Vol(H + jf^*L) &= s(H + jf^*L)^{n+r-1} \\ &= s(\binom{n+r-1}{n} j^n f^*L^n \cdot H^{r-1} + \binom{n+r-1}{n-1} j^{n-1} L^{n-1} \cdot H^r) \\ &+ O(j^{n-2}). \end{aligned}$$

Next we calculate the slope

$$\begin{aligned}\mu(X, H + jf^*L) &= \frac{-K_X \cdot (H + jf^*L)^{n+r-2}}{(H + jf^*L)^{n+r-1}} \\ &= \frac{(rH - f^*(c_1(\det(\mathcal{E})) + K_B)) \cdot (H + f^*L)^{n+r-2}}{(H + jf^*L)^{n+r-1}}.\end{aligned}$$

Expanding this asymptotically in j gives

$$\begin{aligned}\mu(X, H + f^*L) = &\frac{r(r-1)}{n+r-1} \\ &+ j^{-1}\frac{n}{n+r-1}\left(\frac{f^*L^{n-1} \cdot H^r}{f^*L^n \cdot H^{r-1}} - \frac{f^*(c_1(\det(\mathcal{E})) + K_B) \cdot f^*L^{n-1} \cdot H^{r-1})}{f^*L^n \cdot H^{r-1}}\right) \\ &+ O(j^{-2}).\end{aligned}$$

Next we simplify the volume in the integrand. As we know by the lemma 3.4.7, we can assume $H + jf^*L - tD$ is ample as long as its big, hence we can use intersection numbers to calculate the volume as a polynomial

$$\begin{aligned}Vol(H + jf^*L - tD) =&(\beta + j\hat{\pi}^*L - t\epsilon)^{n+r-1} \\ =&(j\hat{\pi}^*L + (1-t)\beta + t\alpha)^{n+r-1} \\ =&\binom{n+r-1}{n} j^n\hat{\pi}^*L^n \cdot ((1-t)\beta + t\alpha)^{r-1} \\ &+ \binom{n+r-1}{n-1} j^{n-1}\hat{\pi}^*L^{n-1} \cdot ((1-t)\beta + t\alpha)^r + O(j^{n-2}) \\ =&\binom{n+r-1}{n} j^n\hat{\pi}^*L^n \cdot (\sum_{k=0}^{r-1}\binom{r-1}{k}(1-t)^k t^{r-1-k}\beta^k\alpha^{r-1-k}) \\ &+ \binom{n+r-1}{n-1} j^{n-1}\hat{\pi}^*L^{n-1} \cdot (\sum_{k=0}^{r}\binom{r}{k}(1-t)^k t^{r-k}\beta^k\alpha^{r-k}) \\ &+ O(j^{n-2})\end{aligned}$$

where we used that $\epsilon = j_*(1) = D$. The next goal is to write the terms above in the form $A \cdot \beta^{r-1}$, where $A \in A^n(B)$ since these give intersection numbers computed on B.

Intersecting the relations 3.4 with $\hat{\pi}^*L^n$ give

$$0 =\hat{\pi}^*L^n \cdot \alpha^s \tag{10}$$
$$0 =\hat{\pi}^*L^n \cdot \beta^r \tag{11}$$
$$\hat{\pi}^*L^n \cdot \alpha \cdot \beta^{r-s} =\hat{\pi}^*L^n \cdot \beta^{r-s+1}. \tag{12}$$

These equations also hold if we replace $\hat{\pi}^*L^n$ by any other element pulled back from $A^n(B)$. In particular if V is a bundle on B then

$$0 = \hat{\pi}^*(L^{n-1} \cdot c_1(V)) \cdot \alpha^s \tag{13}$$
$$0 = \hat{\pi}^*(L^{n-1} \cdot c_1(V)) \cdot \beta^r \tag{14}$$
$$\hat{\pi}^*(L^{n-1} \cdot c_1(V)) \cdot \alpha \cdot \beta^{r-s} = \hat{\pi}^*(L^{n-1} \cdot c_1(V)) \cdot \beta^{r-s+1}. \tag{15}$$

Similarly, if we intersect the relations 3.4 with $\hat{\pi}^*L^{n-1}$, then we obtain

$$\hat{\pi}^*L^{n-1} \cdot \alpha^s = -L^{n-1} \cdot c_1(\mathcal{E}/\mathcal{F}) \cdot \alpha^{s-1} \tag{16}$$
$$\hat{\pi}^*L^{n-1} \cdot \beta^r = -\hat{\pi}^*L^{n-1} \cdot c_1(\mathcal{E}) \cdot \beta^{r-1} \tag{17}$$
$$\hat{\pi}^*L^{n-1} \cdot \alpha \cdot \beta^{r-s} = \hat{\pi}^*L^{n-1} \cdot \beta^{r-s+1} + \hat{\pi}^*L^{n-1} \cdot c_1(\mathcal{F}) \cdot (\beta^{r-s} - \alpha \cdot \beta^{r-s-1}). \tag{18}$$

Now, the leading order term in $Vol(H + jf^*L - tD)$ reduces to

$$\begin{aligned}
&\hat{\pi}^*L^n \cdot (\sum_{k=0}^{r-1} \binom{r-1}{k} (1-t)^k t^{r-1-k} \beta^k \alpha^{r-1-k}) \\
&= \sum_{k=0}^{r-1} \binom{r-1}{k} (1-t)^{r-1-k} t^k \hat{\pi}^*L^n \cdot \beta^{r-1-k} \cdot \alpha^k \\
&= \sum_{k=0}^{s-1} \binom{r-1}{k} (1-t)^{r-1-k} t^k \hat{\pi}^*L^n \cdot \beta^{r-1-k} \cdot \alpha^k \\
&= \sum_{k=0}^{s-1} \binom{r-1}{k} (1-t)^{r-1-k} t^k \hat{\pi}^*L^n \cdot \beta^{r-1} \\
&= \hat{\pi}^*L^n \cdot \beta^{r-1} (\sum_{k=0}^{s-1} \binom{r-1}{k} (1-t)^{r-1-k} t^k).
\end{aligned}$$

Here, we first used equation 10 and then equation 12, which applies exactly when $k \leq s-1$, since then $r-1-k \geq r-s$. Computing the subleading order term, we first note that equation 16 applied twice yields

$$\hat{\pi}^*L^{n-1} \cdot (\sum_{k=0}^{r} \binom{r}{k} (1-t)^{r-k} x^k \beta^{r-k} \alpha^k) = \hat{\pi}^*L^{n-1} \cdot (\sum_{k=0}^{s} \binom{r}{k} (1-t)^{r-k} t^k \beta^{r-k} \alpha^k),$$

Indeed, when $k \geq s+1$, then

$$\begin{aligned}
\hat{\pi}^*L^{n-1} \cdot \alpha^k \cdot \beta^{r-k} &= \hat{\pi}^*(L^{n-1} \cdot c_1(\mathcal{E}/\mathcal{F})) \cdot \alpha^{k-1} \cdot \beta^{r-k} \\
&= \hat{\pi}^*(L^{n-1} \cdot c_1(\mathcal{E}/\mathcal{F})^2) \cdot \alpha^{k-2} \cdot \beta^{r-k} \\
&= 0,
\end{aligned}$$

because $\hat{\pi}^*(L^{n-1} \cdot c_1(\mathcal{E}/\mathcal{F})^2) \in \hat{\pi}^* A^{n+1}(B) = 0$. When $k < s$, then one obtains by equation 18 and 12 that

$$
\begin{aligned}
\hat{\pi^*} L^{n-1} \cdot \beta^{r-k} \cdot \alpha^k =&(\hat{\pi}^* L^{n-1} \cdot \alpha \cdot \beta^{r-s}) \cdot \alpha^{k-1} \cdot \beta^{s-k} \\
=&(\hat{\pi}^* L^{n-1} \cdot \beta^{r-s+1} + \hat{\pi}^* L^{n-1} \cdot c_1(\mathcal{F}) \cdot (\beta^{r-s} - \alpha\beta^{r-s-1})) \cdot \alpha^{k-1} \cdot \beta^{s-k} \\
=&\hat{\pi}^* L^{n-1} \cdot \alpha^{k-1} \cdot \beta^{r-k+1} + \hat{\pi}^* L^{n-1} \cdot c_1(\mathcal{F}) \cdot \alpha^{k-1} \cdot \beta^{r-k} \\
&- \hat{\pi}^* L^{n-1} \cdot c_1(\mathcal{F}) \cdot \alpha^k \cdot \beta^{r-k-1} \\
=&\hat{\pi}^* L^{n-1} \cdot \alpha^{k-1} \cdot \beta^{r-k+1} + \hat{\pi}^* L^{n-1} \cdot c_1(\mathcal{F}) \cdot \alpha^{k-1} \cdot \beta^{r-k} \\
&- \hat{\pi}^* L^{n-1} \cdot c_1(\mathcal{F}) \cdot \alpha^{k-1} \cdot \beta^{r-k} \\
=&\hat{\pi}^* L^{n-1} \cdot \alpha^{k-1} \cdot \beta^{r-k+1} \\
=&\hat{\pi}^* L^{n-1} \cdot \beta^r \\
=&- \hat{\pi}^* L^{n-1} \cdot c_1(\mathcal{E}) \cdot \beta^{r-1}.
\end{aligned}
$$

The last two equation follow by 12 and 17 respectively. The final term $k = s$ gives by 16 and repeated use of 15

$$
\begin{aligned}
\hat{\pi}^* L^{n-1} \cdot \beta^{r-s} \cdot \alpha^s =& - \hat{\pi}^* L^{n-1} \cdot c_1(\mathcal{E}/\mathcal{F}) \cdot \alpha^{s-1} \cdot \beta^{r-s} \\
=& - \hat{\pi}^* L^{n-1} \cdot c_1(\mathcal{E}/\mathcal{F}) \cdot \alpha^{s-2} \cdot \beta^{r-s+1} \\
=& - -\hat{\pi}^* L^{n-1} \cdot c_1(\mathcal{E}/\mathcal{F}) \cdot \beta^{r-1}.
\end{aligned}
$$

In conclusion, the subleading order term of the volume is

$$
\begin{aligned}
\hat{\pi}^* L^{n-1} \cdot (\sum_{k=0}^{r} \binom{r}{k} (1-t)^{r-k} t^k \beta^{r-k} \alpha^k) =& - \binom{r}{s} (1-t)^{r-s} t^s \hat{\pi}^*(L^{n-1} \cdot c_1(\mathcal{E}/\mathcal{F}) \cdot \beta^{r-1}) \\
& - \sum_{k=0}^{s-1} \binom{r}{k} (1-t)^{r-k} t^k \hat{\pi}^*(L^{n-1} \cdot c_1(\mathcal{E})) \cdot \beta^{r-1} \\
=& \binom{r}{s} (1-t)^{r-s} t^s \hat{\pi}^*(L^{n-1} \cdot c_1(\mathcal{F}) \cdot \beta^{r-1}) \\
& - \sum_{k=0}^{s} \binom{r}{k} (1-t)^{r-k} t^k \hat{\pi}^*(L^{n-1} \cdot c_1(\mathcal{E})) \cdot \beta^{r-1}.
\end{aligned}
$$

Thus, expanding the integral of the volume as

$$
\int_0^\infty Vol(H + jf^*L - tD) = v_0 j^n + v_1 j^{n-1} + O(j^{n-2}),
$$

one has

$$\begin{aligned}
v_0 &= \binom{n+r-1}{n} \int_0^1 \hat{\pi}^* L^n \cdot ((1-t)\beta + t\alpha)^{r-1} dt \\
&= \binom{n+r-1}{n} \int_0^1 \hat{\pi}^* L^n \cdot \beta^{r-1} (\sum_{k=0}^{s-1} \binom{r-1}{k} (1-t)^{r-1-k} t^k) dt \\
&= \binom{n+r-1}{n} \hat{\pi}^* L^n \cdot \beta^{r-1} \sum_{k=0}^{s-1} \binom{r-1}{k} \int_0^1 (1-t)^{r-1-k} t^k dt \\
&= \binom{n+r-1}{n} \hat{\pi}^* L^n \cdot \beta^{r-1} \hat{\pi}^* L^n \cdot \beta^{r-1} (\sum_{k=0}^{s-1} \binom{r-1}{k} \frac{(r-1-k)!k!}{r!}) \\
&= \binom{n+r-1}{n} \hat{\pi}^* L^n \cdot \beta^{r-1} \sum_{k=0}^{s-1} \frac{1}{r} \\
&= \frac{s}{r} \binom{n+r-1}{n} \hat{\pi}^* L^n \cdot \beta^{r-1},
\end{aligned}$$

and

$$\begin{aligned}
v_1 =& \binom{n+r-1}{n-1} \int_0^1 \hat{\pi}^* L^{n-1} \cdot ((1-t)\beta + t\alpha)^r \, dt \\
=& \binom{n+r-1}{n-1} \cdot \int_0^1 \binom{r}{s} (1-t)^{r-s} t^s \hat{\pi}^* L^{n-1} \cdot c_1(\mathcal{F}) \cdot \beta^{r-1} dt \\
&- \binom{n+r-1}{n-1} \cdot \int_0^1 \sum_{k=0}^{s} \binom{r}{k} (1-t)^{r-k} t^k \hat{\pi}^* (L^{n-1} \cdot c_1(\mathcal{E})) \cdot \beta^{r-1} dt \\
=& \binom{n+r-1}{n-1} \left(\binom{r}{s} \frac{(r-s)!s!}{(r+1)!} \hat{\pi}^* (L^{n-1} \cdot c_1(\mathcal{F})) \cdot \beta^{r-1} \right) \\
&- \binom{n+r-1}{n-1} \left(\sum_{k=0}^{s} \binom{r}{k} \frac{(r-k)!k!}{(r+1)!} \hat{\pi}^* (L^{n-1} \cdot c_1(\mathcal{E})) \cdot \beta^{r-1} \right) \\
=& \binom{n+r-1}{n-1} \left(\frac{1}{r+1} \hat{\pi}^* (L^{n-1} \cdot c_1(\mathcal{F})) \cdot \beta^{r-1} - \sum_{k=0}^{s} \frac{1}{r+1} \hat{\pi}^* (L^{n-1} \cdot c_1(\mathcal{E})) \cdot \beta^{r-1} \right) \\
=& \binom{n+r-1}{n-1} \left(\frac{1}{r+1} \hat{\pi}^* (L^{n-1} \cdot c_1(\mathcal{F})) \cdot \beta^{r-1} - \frac{s+1}{r+1} \hat{\pi}^* (L^{n-1} \cdot c_1(\mathcal{E})) \cdot \beta^{r-1} \right).
\end{aligned}$$

In the above, we have used the following identities

$$\int_0^1 (1-t)^a t^b dt = \frac{a!b!}{(a+b+1)!}$$

$$\binom{r}{k}\frac{(r-k)!k!}{(r+1)!} = \frac{1}{r+1}.$$

Next we aim to perform a similar reduction of the derivative of the volume in the direction K_X. Recall that one has the formula

$$\frac{d}{dt}Vol(A+tB)|_{t=0} = n\langle A^{n-1}\rangle B$$

and that the positive intersection product is the standard intersection product when A is nef. Therefore, we have

$$\begin{aligned}
\frac{d}{ds}Vol(H + jf^*L - tD + sK_X)|_{s=0}& \\
=&(n+r-1)(j\hat{\pi}^*L + (1-t)\beta + t\alpha)^{n+r-2} \cdot \pi^*K_X \\
=&(n+r-1)\binom{n+r-2}{n} j^n\hat{\pi}^*L^n((1-t)\beta + t\alpha)^{r-2} \cdot \pi^*K_X \\
&+ (n+r-1)\binom{n+r-2}{n-1} j^{n-1}\hat{\pi}^*L^{n-1}((1-t)\beta + t\alpha)^{r-1}\pi^*K_X \\
&+ O(j^{n-2}).
\end{aligned}$$

Recall that $\pi^*K_X = -r\beta + \hat{\pi}^*(K_B + c_1(\mathcal{E})$ by lemma 3.4.6, so expanding the leading term yields

$$\begin{aligned}
\hat{\pi}^*L^n((1-t)\beta + t\alpha)^{r-2}\pi^*K_X =& - r\hat{\pi}^*L^n((1-t)\beta + t\alpha)^{r-2}\beta \\
=& - r\sum_{k=0}^{r-2}\binom{r-2}{k}(1-t)^{r-2-k}t^k\beta^{r-k-1}\alpha^k\hat{\pi}^*L^n \\
=& - r\sum_{k=0}^{s-1}\binom{r-2}{k}(1-t)^{r-2-k}t^k\beta^{r-k-1}\alpha^k\hat{\pi}^*L^n \\
=& - r\beta^{r-1}\hat{\pi}^*L^n\sum_{k=0}^{s-1}\binom{r-2}{k}(1-t)^{r-k-2}t^k.
\end{aligned}$$

Here, we used that the base has dimension n, that the terms $k \geq s$ vanish by 10, and then applying 12 repeatedly for $k < s$. For the second term, one has

$$\begin{aligned}\hat{\pi}^*L^{n-1}((1-t)\beta+t\alpha)^{r-1}\pi^*K_X =& -r\sum_{k=0}^{r-1}\binom{r-1}{k}(1-t)^{r-1-k}t^k\beta^{r-k}\alpha^k\hat{\pi}^*L^{n-1}\\ &+\sum_{k=0}^{r-1}\binom{r-1}{k}(1-t)^{r-1-k}t^k\beta^{r-1-k}\alpha^k\hat{\pi}^*(L^{n-1}\cdot K_B)\\ &+\sum_{k=0}^{r-1}\binom{r-1}{k}(1-t)^{r-1-k}t^k\beta^{r-1-k}\alpha^k\hat{\pi}^*(L^{n-1}\cdot c_1(\mathcal{E})).\end{aligned}$$

The three terms in the sum are handled by using the following relations, which all follow from the equations 13,15,16, 17 and 18.

$$\hat{\pi}^*(L^{n-1}\cdot K_B)\beta^{r-1-k}\alpha^k = \begin{cases}0 & k\geq s\\ \hat{\pi}^*(L^{n-1}\cdot K_B)\beta^{r-1} & k<s\end{cases}$$

$$\hat{\pi}^*(L^{n-1}\cdot c_1(\mathcal{E}))\beta^{r-1-k}\alpha^k = \begin{cases}0 & k\geq s\\ \hat{\pi}^*(L^{n-1}\cdot c_1(\mathcal{E}))\beta^{r-1} & k<s\end{cases}$$

$$\hat{\pi}^*L^{n-1}\alpha^k\beta^{r-k} = \begin{cases}0 & k>s\\ -\hat{\pi}^*(L^{n-1}\cdot c_1(\mathcal{E}/\mathcal{F}))\beta^{r-1} & k=s\\ -\hat{\pi}^*(L^{n-1}\cdot c_1(\mathcal{E}))\beta^{r-1} & k<s.\end{cases}$$

Using these, we get

$$\begin{aligned}\hat{\pi}^*L^{n-1}((1-t)\beta+t\alpha)^{r-1}\pi^*K_X =& r\binom{r-1}{s}(1-t)^{r-1-s}t^s c_1(\mathcal{E}/\mathcal{F})\hat{\pi}^*L^{n-1}\beta^{r-1}\\ &+r\sum_{k=0}^{s-1}\binom{r-1}{k}(1-t)^{r-1-k}t^k\hat{\pi}^*(L^{n-1s}\cdot c_1(\mathcal{E}))\beta^{r-1}\\ &+\sum_{k=0}^{s-1}\binom{r-1}{k}(1-t)^{r-1-k}t^k\hat{\pi}^*(L^{n-1}\cdot K_B)\beta^{r-1-k}\\ &-\sum_{k=0}^{s-1}\binom{r-1}{k}(1-t)^{r-1-k}t^k\hat{\pi}^*(L^{n-1}\cdot c_1(\mathcal{E})\beta^{r-1}\\ =& r\binom{r-1}{s}(1-t)^{r-1-s}t^s\hat{\pi}^*(c_1(\mathcal{E}/\mathcal{F})L^{n-1})\beta^{r-1}\\ &+(r-1)\sum_{k=0}^{s-1}\binom{r-1}{k}(1-t)^{r-1-k}t^k\hat{\pi}^*(L^{n-1}\cdot c_1(\mathcal{E}))\beta^{r-1}\\ &+\hat{\pi}^*(L^{n-1}\cdot K_B)\beta^{r-1}\sum_{k=0}^{s-1}\binom{r-1}{k}(1-t)^{r-1-k}t^k\end{aligned}$$

So, we have the first two coefficients in the expansion

$$\frac{d}{ds}Vol(H + f^*L - tD + sK_X)|_{s=0} = j^n w_0 + j^{n-1} w_1 + O(j^{n-2}).$$

These are

$$\begin{aligned} w_0 =& - r(n+r-1)\binom{n+r-2}{n}\beta^{r-1}\hat{\pi}^* L^n \sum_{k=0}^{s-1}\binom{r-2}{k}(1-t)^{r-k-2}t^k \\ =& - r(r-1)\binom{n+r-1}{n}\beta^{r-1}\hat{\pi}^* L^n \sum_{k=0}^{s-1}\binom{r-2}{k}(1-t)^{r-k-2}t^k, \end{aligned}$$

and

$$\begin{aligned} w_1 =& (n+r-1)\binom{n+r-2}{n-1}\Big[r\binom{r-1}{s}(1-t)^{r-1-s}t^s\hat{\pi}^*(c_1(\mathcal{E}/\mathcal{F})L^{n-1})\beta^{r-1} \\ &+ (r-1)\sum_{k=0}^{s-1}\binom{r-1}{k}(1-t)^{r-1-k}t^k\hat{\pi}^*(L^{n-1}\cdot c_1(\mathcal{E}))\beta^{r-1} \\ &+ \hat{\pi}^*(L^{n-1}\cdot K_B)\beta^{r-1}\sum_{k=0}^{s-1}\binom{r-1}{k}(1-t)^{r-1-k}t^k\Big] \\ =& r^2\binom{n+r-1}{n-1}\binom{r-1}{s}\hat{\pi}^*(c_1(\mathcal{E}/\mathcal{F})L^{n-1})\beta^{r-1}(1-t)^{r-1-s}t^s \\ &+ r(r+1)\binom{n+r-1}{n-1}\hat{\pi}^*(c_1(\mathcal{E})L^{n-1})\beta^{r-1}\sum_{k=0}^{s-1}\binom{r-1}{k}(1-t)^{r-1-k}t^k \\ &+ r\binom{n+r-1}{n-1}\hat{\pi}^* L^{n-1}K_B\beta^{r-1}\sum_{k=0}^{s-1}\binom{r-1}{k}(1-t)^{r-1-k}t^k. \end{aligned}$$

Here, we used the idenitities

$$\begin{aligned} (n+r-1)\binom{n+r-2}{n} &= (r-1)\binom{n+r-1}{n} \\ (n+r-1)\binom{n+r-2}{n-1} &= r\binom{n+r-1}{n-1}. \end{aligned}$$

We want to integrate the asymptotic expansion, so by using 3.4 we get

$$
\begin{aligned}
&\int_0^\infty \frac{d}{ds} Vol(H + jf^*L - tD + sK_X)|_{s=0} dt \\
=& - j^n r(r-1) \binom{n+r-1}{n} \hat{\pi}^* L^n \beta^{r-1} \sum_{k=0}^{s-1} \binom{r-2}{k} \frac{(r-k-2)!k!}{(r-1)!} \\
&+ j^{n-1} r^2 \binom{n+r-1}{n-1} \binom{r-1}{s} \hat{\pi}^*(L^{n-1} c_1(\mathcal{E}/\mathcal{F})) \frac{(r-1-s)!s!}{r!} \\
&+ j^{n-1} r(r-1) \binom{n+r-1}{n-1} \hat{\pi}^*(c_1(\mathcal{E}) L^{n-1}) \beta^{r-1} \sum_{k=0}^{s-1} \binom{r-1}{k} \frac{(r-1-k)!k!}{r!} \\
&+ j^{n-1} r \binom{n+r-1}{n-1} \hat{\pi}^*(L^{n-1} K_B) \beta^{r-1} \sum_{k=0}^{s-1} \binom{r-1}{k} \frac{(r-1-k)!k!}{r!} \\
&+ O(j^{n-2}) \\
=& - j^n r(r-1) \binom{n+r-1}{n} \hat{\pi}^* L^n \beta^{r-1} \sum_{k=0}^{s-1} \frac{1}{r-1} \\
&+ j^{n-1} r^2 \binom{n+r-1}{n-1} \hat{\pi}^*(L^{n-1} c_1(\mathcal{E}/\mathcal{F})) \beta^{r-1} \frac{1}{r} \\
&+ j^{n-1} r(r-1) \binom{n+r-1}{n-1} \hat{\pi}^*(L^{n-1} c_1(\mathcal{E})) \beta^{r-1} \sum_{k=0}^{s-1} \frac{1}{r} \\
&+ j^{n-1} r \binom{n+r-1}{n-1} \hat{\pi}^*(L^{n-1} K_B) \beta^{r-1} \sum_{k=0}^{s-1} \frac{1}{r} \\
=& - j^n rs \binom{n+r-1}{n} \hat{\pi}^* L^n \beta^{r-1} + j^{n-1} r \binom{n+r-1}{n-1} \hat{\pi}^*(L^{n-1} c_1(\mathcal{E}/\mathcal{F})) \beta^{r-1} \\
&+ j^{n-1} (r-1) s \binom{n+r-1}{n-1} \hat{\pi}^*(L^{n-1} c_1(\mathcal{E})) \beta^{r-1} + j^{n-1} s \binom{n+r-1}{n-1} \hat{\pi}^*(L^{n-1} K_B) \beta^{r-1} \\
&+ O(j^{n-2}).
\end{aligned}
$$

Using the fact that $\hat{\pi}^*(A) \cdot \beta^{r-1} = A$ as numbers when A is a top intersection number on B, we can summarize the calculations above as follows:

Proposition 3.4.8.

$$\int_0^1 Vol(H + jf^*L - tD)dt = V_0 j^n + V_1 j^{n-1} + O(j^{n-2})$$
$$\int_0^1 Vol(H + jf^*L - tD)dt = W_0 j^n + W_1 j^{n-1} + O(j^{n-2}),$$

where

$$V_0 = \binom{n+r-1}{n} \frac{s}{r} L^n$$
$$V_1 = \binom{n+r-1}{n-1} (\frac{1}{r+1} L^{n-1} c_1(\mathcal{F}) - \frac{s+1}{r+1} L^{n-1} c_1(\mathcal{E}))$$
$$W_0 = -\binom{n+r-1}{n} sr L^n$$
$$W_1 = \binom{n+r-1}{n-1} (rc_1(\mathcal{E}/\mathcal{F}) L^{n-1} + s(r-1)c_1(\mathcal{E})L^{n-1} + sK_B L^{n-1}).$$

To finish up the computations needed for theorem 3.4.1 we simply need to put all these expressions together. Recall that

$$\mu(X, H + jf^*L) = M_0 + j^{-1} M_1 + O(j^{-2}),$$

where

$$\begin{aligned} M_0 =& \frac{r(r-1)}{n+r-1} \\ M_1 =& \frac{n}{n+r-1} (\frac{f^*L^{n-1} \cdot H^r}{L^n} - \frac{(c_1(\det(\mathcal{E})) + K_B) \cdot L^{n-1}}{L^n}) \\ =& \frac{n}{n+r-1} (\frac{f^*L^{n-1} c_1(\mathcal{E})}{L^n} - \frac{(c_1(\det(\mathcal{E})) + K_B) \cdot L^{n-1}}{L^n}) \\ =& -\frac{n}{n+r-1} \frac{K_B \cdot L^{n-1}}{L^n} \\ =& \frac{n}{n+r-1} \mu(B, K_B). \end{aligned}$$

Here we used that we have equation 17 giving $\pi^*(f^*L^{n-1} \cdot H^r) = \hat{\pi}^*(L^{n-1} c_1(\mathcal{E}))\beta^{r-1}$, which gives $f^*L^{n-1}H^r = f^*(L^{n-1}c_1(\mathcal{E}))H^{r-1}$. Writing

$$\begin{aligned} Vol(H + f^*L) &= N_0 j^n + N_1 j^{n-1} + O(j^{n-2}), \\ N_0 &= \binom{n+r-1}{n} f^*L^n \\ N_1 &= \binom{n+r-1}{n-1} H^r f^* L^{n-1} \\ &= -\binom{n+r-1}{n-1} c_1(\mathcal{E}) f^* L^{n-1}, \end{aligned}$$

we know that the asymptotic expansion of the β invariant is

$$\beta_{H,L} = \beta_0 j^n + \beta_1 j^{n-1} + O(j^{n-2})$$

with

$$\begin{aligned} \beta_0 =& A_X(D)N_0 + (n+r-1)M_0V_0 + W_0 \\ \beta_1 =& A_X(D)N_1 + (n+r-1)(M_0V_1 + M_1V_0) + W_1. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \beta_0 =& s\binom{n+r-1}{n} L^n + r(r-1)\binom{n+r-1}{n}\frac{s}{r}L^n - \binom{n+r-1}{n} srL^n \\ =& (s + s(r-1) - sr)\binom{n+r-1}{n} L^n \\ =& 0. \end{aligned}$$

Performing the same calculations with $n = \dim(B) = 0$, one may prove that the exceptional divisor D of a blow up of projective space in a linear subspace gives beta invariant 0, corresponding to a product test configuration. Hence, the fibrewise statement holds and the fact that the leading coefficient is 0 is thus given by proposition 3.1.12.

Similarly, we compute

$$\begin{aligned}
\beta_1 =& - s\binom{n+r-1}{n-1} c_1(\mathcal{E})L^{n-1} + r(r-1)\binom{n+r-1}{n-1}(\frac{1}{r+1}L^{n-1}c_1(\mathcal{F}) \\
&- \frac{s+1}{r+1}L^{n-1}c_1(\mathcal{E})) + n\mu(B,K_B)\binom{n+r-1}{n}\frac{s}{r}L^n \\
&+ \binom{n+r-1}{n-1}(rc_1(\mathcal{E}/\mathcal{F})L^{n-1} + s(r-1)c_1(\mathcal{E})L^{n-1} + sK_BL^{n-1}) \\
=& \binom{n+r-1}{n-1}\big[(-s - \frac{r(r-1)(s+1)}{r+1}) + s(r-1) + r)c_1(\mathcal{E})L^{n-1} \\
&+ (\frac{r(r-1)}{r+1} - r)c_1(\mathcal{F})L^{n-1}\big] \\
&= \binom{n+r-1}{n-1}[\frac{2(r-s)}{r+1}c_1(\mathcal{E})L^{n-1} - \frac{2r}{r+1}c_1(\mathcal{F})L^{n-1}] \\
=& \frac{2r(r-s)}{r+1}\binom{n+r-1}{n-1}[\frac{c_1(\mathcal{E})L^{n-1}}{r} - \frac{c_1(\mathcal{F})L^{n-1}}{r-s}] \\
=& \frac{2r(r-s)}{r+1}\binom{n+r-1}{n-1}(\mu_L(\mathcal{E}) - \mu_L(\mathcal{F})).
\end{aligned}$$

In the calculation, we used that the terms involving K_B cancel, since

$$\frac{n}{r}\binom{n+r-1}{n} = \binom{n+r-1}{n-1}.$$

This finishes the computation and the proof of theorem 3.4.1.

References

[Abh56] Shreeram Abhyankar. "On the Valuations Centered in a Local Domain". In: *American Journal of Mathematics* 78.2 (Apr. 1956), p. 321. ISSN: 00029327. DOI: 10.2307/2372519. JSTOR: 2372519.

[BFJ08] Sébastien Boucksom, Charles Favre, and Mattias Jonsson. "Differentiability of Volumes of Divisors and a Problem of Teissier". In: *Journal of Algebraic Geometry* 18.2 (Apr. 23, 2008), pp. 279–308. ISSN: 1056-3911, 1534-7486. DOI: 10.1090/S1056-3911-08-00490-6. URL: https://www.ams.org/jag/2009-18-02/S1056-3911-08-00490-6/.

[BG14] Robert J. Berman and Henri Guenancia. "Kähler–Einstein Metrics on Stable Varieties and Log Canonical Pairs". In: *Geometric and Functional Analysis* 24.6 (Dec. 2014), pp. 1683–1730. ISSN: 1016-443X, 1420-8970. DOI: 10.1007/s00039-014-0301-8. URL: http://link.springer.com/10.1007/s00039-014-0301-8.

[BHJ17] Sébastien Boucksom, Tomoyuki Hisamoto, and Mattias Jonsson. "Uniform K-stability, Duistermaat–Heckman Measures and Singularities of Pairs". In: *Annales de l'institut Fourier* 67.2 (2017), pp. 743–841. ISSN: 0373-0956, 1777-5310. DOI: 10.5802/aif.3096. URL: https://aif.centre-mersenne.org/item/AIF_2017__67_2_743_0/.

[BJ17] Sébastien Boucksom and Mattias Jonsson. "Tropical and Non-Archimedean Limits of Degenerating Families of Volume Forms". In: *Journal de l'École polytechnique — Mathématiques* 4 (2017), pp. 87–139. ISSN: 2270-518X. DOI: 10.5802/jep.39. URL: https://jep.centre-mersenne.org/item/JEP_2017__4__87_0.

[BJ22] Sebastien Boucksom and Mattias Jonsson. "A Non-Archimedean Approach to K-stability, II: Divisorial Stability and Openness". Version 1. In: (2022). DOI: 10.48550/ARXIV.2206.09492. URL: https://arxiv.org/abs/2206.09492.

[Cal54] Eugenio Calabi. "The Space of K\"ahler Metrics". In: *Proceedings of the International Congress of Mathematicans, Amsterdam* 2 (1954), pp. 206–207.

[Cal79] E. Calabi. "Métriques Kählériennes et Fibrés Holomorphes". In: *Annales scientifiques de l'École normale supérieure* 12.2 (1979), pp. 269–294. ISSN: 0012-9593, 1873-2151. DOI: 10.24033/asens.1367. URL: http://www.numdam.org/item?id=ASENS_1979_4_12_2_269_0.

[CDS14a] Xiuxiong Chen, Simon Donaldson, and Song Sun. "Kähler-Einstein Metrics on Fano Manifolds. I: Approximation of Metrics with Cone Singularities". In: *Journal of the American Mathematical Society* 28.1 (Mar. 28, 2014), pp. 183–197. ISSN: 0894-0347, 1088-6834. DOI: 10.1090/S0894-0347-2014-00799-2. URL: https://www.ams.org/jams/2015-28-01/S0894-0347-2014-00799-2/.

[CDS14b] Xiuxiong Chen, Simon Donaldson, and Song Sun. "Kähler-Einstein Metrics on Fano Manifolds. II: Limits with Cone Angle Less than \boldmath2". In: *Journal of the American Mathematical Society* 28.1 (Mar. 28, 2014), pp. 199–234. ISSN: 0894-0347, 1088-6834. DOI: 10.1090/S0894-0347-2014-00800-6. URL: https://www.ams.org/jams/2015-28-01/S0894-0347-2014-00800-6/.

[CDS14c] Xiuxiong Chen, Simon Donaldson, and Song Sun. "Kähler-Einstein Metrics on Fano Manifolds. III: Limits as Cone Angle Approaches \boldmath2 and Completion of the Main Proof". In: *Journal of the American Mathematical Society* 28.1 (Mar. 28, 2014), pp. 235–278. ISSN: 0894-0347, 1088-6834. DOI: 10.1090/S0894-0347-2014-00801-8. URL: https://www.ams.org/jams/2015-28-01/S0894-0347-2014-00801-8/.

[CL22] Tristan C. Collins and Yang Li. "Complete Calabi-Yau Metrics in the Complement of Two Divisors". Version 1. In: (2022). DOI: 10.48550/ARXIV.2203.10656. URL: https://arxiv.org/abs/2203.10656.

[Cle77] C. H. Clemens. "Degeneration of Kähler Manifolds". In: *Duke Mathematical Journal* 44.2 (June 1977), pp. 215–290. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-77-04410-6. URL: http://projecteuclid.org/euclid.dmj/1077312231.

[CY80] Shiu-Yuen Cheng and Shing-Tung Yau. "On the Existence of a Complete Kähler Metric on Non-Compact Complex Manifolds and the Regularity of Fefferman's Equation". In: *Communications on Pure and Applied Mathematics* 33.4 (July 1980), pp. 507–544. ISSN: 00103640, 10970312. DOI: 10.1002/cpa.

3160330404. URL: https://onlinelibrary.wiley.com/doi/10.1002/cpa.3160330404.

[Der16] Ruadhaí Dervan. "Uniform Stability of Twisted Constant Scalar Curvature Kähler Metrics". In: *International Mathematics Research Notices* 2016.15 (2016), pp. 4728–4783. ISSN: 1073-7928, 1687-0247. DOI: 10.1093/imrn/rnv291. URL: https://academic.oup.com/imrn/article-lookup/doi/10.1093/imrn/rnv291.

[dFEM14] Tommaso de Fernex, Lawrence Ein, and Mircea Mustata. "Vanishing Theorems and Singularities in Birational Geometry". 2014. URL: http://homepages.math.uic.edu/~ein/DFEM.pdf.

[dFKX] Tommaso de Fernex, János Kollár, and Chenyang Xu. "The Dual Complex of Singularities". In: Higher Dimensional Algebraic Geometry in Honour of Professor Yujiro Kawamata's Sixtieth Birthday. University of Tokyo, Japan, pp. 103–129. DOI: 10.2969/aspm/07410103. URL: https://projecteuclid.org/euclid.aspm/1540319484.

[DFS21] Ved Datar, Xin Fu, and Jian Song. "Kahler-Einstein Metric near an Isolated Log Canonical Singularity". Version 1. In: (2021). DOI: 10.48550/ARXIV.2106.05486. URL: https://arxiv.org/abs/2106.05486.

[DL22] Ruadhaí Dervan and Eveline Legendre. "Valuative Stability of Polarised Varieties". In: *Mathematische Annalen* (Jan. 6, 2022). ISSN: 0025-5831, 1432-1807. DOI: 10.1007/s00208-021-02313-4. URL: https://link.springer.com/10.1007/s00208-021-02313-4.

[Don02] S.K. Donaldson. "Scalar Curvature and Stability of Toric Varieties". In: *Journal of Differential Geometry* 62.2 (Oct. 1, 2002). ISSN: 0022-040X. DOI: 10.4310/jdg/1090950195. URL: https://projecteuclid.org/journals/journal-of-differential-geometry/volume-62/issue-2/Scalar-Curvature-and-Stability-of-Toric-Varieties/10.4310/jdg/1090950195.full.

[Don05] S.K. Donaldson. "Lower Bounds on the Calabi Functional". In: *Journal of Differential Geometry* 70.3 (July 1, 2005). ISSN: 0022-040X. DOI: 10.4310/jdg/1143642909. URL: https://projecteuclid.org/journals/journal-of-differential-geometry/volume-70/issue-3/Lower-bounds-on-the-Calabi-functional/10.4310/jdg/1143642909.full.

[Don99] S.K. Donaldson. "Symmetric Spaces, K\"ahler Geometry and Hamiltonian Dynamics". In: *Northern California Symplectic Geometry Seminar*. Vol. 196. American Mathematical Society Translations: Series 2. American Mathematical Society, 1999.

[DS20] Ruadhaí Dervan and Lars Martin Sektnan. "Extremal Metrics on Fibrations". In: *Proceedings of the London Mathematical Society* 120.4 (Apr. 2020), pp. 587–616. ISSN: 0024-6115, 1460-244X. DOI: 10.1112/plms.12297. URL: https://onlinelibrary.wiley.com/doi/abs/10.1112/plms.12297.

[DS21a] Ruadhaí Dervan and Lars Martin Sektnan. "Moduli Theory, Stability of Fibrations and Optimal Symplectic Connections". In: *Geometry & Topology* 25.5 (Sept. 3, 2021), pp. 2643–2697. ISSN: 1364-0380, 1465-3060. DOI: 10.2140/gt.2021.25.2643. URL: https://msp.org/gt/2021/25-5/p10.xhtml.

[DS21b] Ruadhaí Dervan and Lars Martin Sektnan. "Optimal Symplectic Connections on Holomorphic Submersions". In: *Communications on Pure and Applied Mathematics* 74.10 (Oct. 2021), pp. 2132–2184. ISSN: 0010-3640, 1097-0312. DOI: 10.1002/cpa.21930. URL: https://onlinelibrary.wiley.com/doi/10.1002/cpa.21930.

[DS21c] Ruadhaí Dervan and Lars Martin Sektnan. "Uniqueness of Optimal Symplectic Connections". In: *Forum of Mathematics, Sigma* 9 (2021), e18. ISSN: 2050-5094. DOI: 10.1017/fms.2021.15. URL: https://www.cambridge.org/core/product/identifier/S2050509421000153/type/journal_article.

[EH16] David Eisenbud and Joe Harris. *3264 and All That: A Second Course in Algebraic Geometry*. 1st ed. Cambridge University Press, Apr. 1, 2016. ISBN: 978-1-107-01708-5 978-1-139-06204-6 978-1-107-60272-4. DOI: 10.1017/CBO9781139062046. URL: https://www.cambridge.org/core/product/identifier/9781139062046/type/book.

[Ehl75] Fritz Ehlers. "Eine Klasse komplexer Mannigfaltigkeiten und die Auflösung einiger isolierter Singularitäten". In: *Mathematische Annalen* 218.2 (June 1975), pp. 127–156. ISSN: 0025-5831, 1432-1807. DOI: 10.1007/BF01370816. URL: http://link.springer.com/10.1007/BF01370816.

[Eis95] David Eisenbud. *Commutative Algebra*. Vol. 150. Graduate Texts in Mathematics. New York, NY: Springer New York, 1995. ISBN: 978-3-540-78122-6 978-1-4612-5350-1. DOI: 10.1007/978-1-4612-5350-1. URL: http://link.springer.com/10.1007/978-1-4612-5350-1.

[FHJ21] Xin Fu, Hans-Joachim Hein, and Xumin Jiang. "Asymptotics of Kähler-Einstein Metrics on Complex Hyperbolic Cusps". Version 2. In: (2021). DOI: 10.48550/ARXIV.2108.13390. URL: https://arxiv.org/abs/2108.13390.

[Fuj19] Kento Fujita. "A Valuative Criterion for Uniform K-stability of -Fano Varieties". In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2019.751 (June 1, 2019), pp. 309–338. ISSN: 1435-5345, 0075-4102. DOI: 10.1515/crelle-2016-0055. URL: https://www.degruyter.com/document/doi/10.1515/crelle-2016-0055/html.

[Ful14] William Fulton. *Intersection Theory*. Springer, 2014. ISBN: 978-1-4612-1700-8.

[Gra62] Hans Grauert. "Über die Modifikationen und exzeptionelle analytische Mengen". In: *Mathematische Annalen* 146 (1962), pp. 331–368.

[GW16] Henri Guenancia and Damin Wu. "On the Boundary Behavior of Kähler–Einstein Metrics on Log Canonical Pairs". In: *Mathematische Annalen* 366.1-2 (Oct. 2016), pp. 101–120. ISSN: 0025-5831, 1432-1807. DOI: 10.1007/s00208-015-1306-9. URL: http://link.springer.com/10.1007/s00208-015-1306-9.

[Har10] Robin Hartshorne. *Algebraic Geometry*. 2010. ISBN: 978-1-4757-3849-0. URL: https://doi.org/10.1007/978-1-4757-3849-0.

[Hat22] Masafumi Hattori. "On Fibration Stability after Dervan-Sektnan and Singularities". Version 1. In: (2022). DOI: 10.48550/ARXIV.2202.09992. URL: https://arxiv.org/abs/2202.09992.

[Hir64] Heisuke Hironaka. "Resolution of Singularities of an Algebraic Variety Over a Field of Characteristic Zero: I". In: *The Annals of Mathematics* 79.1 (Jan. 1964), p. 109. ISSN: 0003486X. DOI: 10.2307/1970486. JSTOR: 1970486.

[Hir73] Friedrich Hirzebruch E.P. "Hilbert Modular Surfaces". In: *L'Enseignement Mathématique* 19 (1973), pp. 183–281.

[HL10] Daniel Huybrechts and Manfred Lehn. *The Geometry of Moduli Spaces of Sheaves*. 2nd ed. Cambridge University Press, May 27, 2010. ISBN: 978-0-521-13420-0 978-0-511-71198-5. DOI: 10.1017/CBO9780511711985. URL: https://www.cambridge.org/core/product/identifier/9780511711985/type/book.

[Jon15] Mattias Jonsson. "Degenerations of Amoebae and Berkovich Spaces". Comment: To appear in Math. Ann. Apr. 7, 2015. URL: http://arxiv.org/abs/1406.1430.

[KK13] János Kollár and Sándor Kovács. *Singularities of the Minimal Model Program*. 1st ed. Cambridge University Press, Feb. 21, 2013. ISBN: 978-1-107-03534-8 978-1-139-54789-5 978-1-107-47125-2. DOI: 10.1017/CBO9781139547895. URL: https://www.cambridge.org/core/product/identifier/9781139547895/type/book.

[KM97] Andreas Kriegl and Peter Michor. *The Convenient Setting of Global Analysis*. Vol. 53. Mathematical Surveys and Monographs. Providence, Rhode Island: American Mathematical Society, Sept. 30, 1997. ISBN: 978-0-8218-0780-4 978-0-8218-3396-4. DOI: 10.1090/surv/053. URL: http://www.ams.org/surv/053.

[KM98] Janos Kollár and Shigefumi Mori. *Birational Geometry of Algebraic Varieties*. 1st ed. Cambridge University Press, Sept. 17, 1998. ISBN: 978-0-521-63277-5 978-0-521-06022-6 978-0-511-66256-0. DOI: 10.1017/CBO9780511662560. URL: https://www.cambridge.org/core/product/identifier/9780511662560/type/book.

[KN79] George Kempf and Linda Ness. "The Length of Vectors in Representation Spaces". In: *Algebraic Geometry*. Ed. by Knud Lønsted. Vol. 732. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer Berlin Heidelberg, 1979, pp. 233–243. ISBN: 978-3-540-09527-9 978-3-540-35049-1. DOI: 10.1007/BFb0066647. URL: http://link.springer.com/10.1007/BFb0066647.

[Kob84] Ryoichi Kobayashi. *Kähler-Einstein Metric on an Open Algebraic Manifold*. 1984. DOI: 10.18910/6442. URL: https://doi.org/10.18910/6442.

[Laz04] Robert Lazarsfeld. *Positivity in Algebraic Geometry 1, 1,* 2004. ISBN: 978-3-642-18808-4.

[LXZ22] Yuchen Liu, Chenyang Xu, and Ziquan Zhuang. “Finite Generation for Valuations Computing Stability Thresholds and Applications to K-stability”. In: *Annals of Mathematics* 196.2 (Sept. 1, 2022). ISSN: 0003-486X. DOI: 10.4007/annals.2022.196.2.2. URL: https://projecteuclid.org/journals/annals-of-mathematics/volume-196/issue-2/Finite-generation-for-valuations-computing-stability-thresholds-and-applications-to/10.4007/annals.2022.196.2.2.full (visited on 07/21/2022).

[Mab85] Toshiki Mabuchi. “A Functional Integrating Futaki’s Invariant”. In: *Proceedings of the Japan Academy, Series A, Mathematical Sciences* 61.4 (Jan. 1, 1985). ISSN: 0386-2194. DOI: 10.3792/pjaa.61.119. URL: https://projecteuclid.org/journals/proceedings-of-the-japan-academy-series-a-mathematical-sciences/volume-61/issue-4/A-functional-integrating-Futakis-invariant/10.3792/pjaa.61.119.full.

[Mab87] Toshiki Mabuchi. “Some Symplectic Geometry on Compact Kähler Manifolds. I”. In: *Osaka Journal of Mathematics* 24.2 (Jan. 1, 1987), pp. 227–252. URL: https://doi.org/.

[MFK94] David Mumford, John Fogarty, and Frances Clare Kirwan. *Geometric Invariant Theory*. 3rd enl. ed. Ergebnisse Der Mathematik Und Ihrer Grenzgebiete 34. Berlin ; New York: Springer-Verlag, 1994. 292 pp. ISBN: 978-3-540-56963-3 978-0-387-56963-5.

[Mil74] John Milnor. *Singular Points of Complex Hypersurfaces*. 2. print. Annals of Mathematics Studies 61. Princeton, NJ: Princeton Univ. Press, 1974. 122 pp. ISBN: 978-0-691-08065-9.

[Nee15] Karl-Hermann Neeb. “Kaehler Geometry, Momentum Maps and Convex Sets”. Version 1. In: (2015). DOI: 10.48550/ARXIV.1510.03289. URL: https://arxiv.org/abs/1510.03289.

[Oda12] Yuji Odaka. “On the Moduli of Kahler-Einstein Fano Manifolds”. Version 4. In: *Proceedings of Kinosaki algebraic geometry symposium 2013* (2012). DOI: 10.48550/ARXIV.1211.4833. URL: https://arxiv.org/abs/1211.4833.

[Oda13] Yuji Odaka. “A Generalization of the Ross–Thomas Slope Theory”. In: *Osaka Journal of Mathematics* 50.1 (Mar. 1, 2013), pp. 171–185. URL: https://doi.org/.

[Oda88] Tadao Oda. *Convex Bodies and Algebraic Geometry*. Springer(Berlin [u.a.]), 1988. URL: https://eudml.org/doc/203658.

[Oga86] Shōetsu Ogata. "Infinitesimal Deformations of Tsuchihashi's Cusp Singularities". In: *Tohoku Mathematical Journal* 38.2 (Jan. 1, 1986). ISSN: 0040-8735. DOI: 10.2748/tmj/1178228493. URL: https://projecteuclid.org/journals/tohoku-mathematical-journal/volume-38/issue-2/Infinitesimal-deformations-of-Tsuchihashis-cusp-singularities/10.2748/tmj/1178228493.full.

[Ort22] Annamaria Ortu. "Optimal Symplectic Connections and Deformations of Holomorphic Submersions". Version 1. In: (2022). DOI: 10.48550/ARXIV.2201.12562. URL: https://arxiv.org/abs/2201.12562.

[OSS16] Yuji Odaka, Cristiano Spotti, and Song Sun. "Compact Moduli Spaces of Del Pezzo Surfaces and Kähler–Einstein Metrics". In: *Journal of Differential Geometry* 102.1 (Jan. 1, 2016). ISSN: 0022-040X. DOI: 10.4310/jdg/1452002879. URL: https://projecteuclid.org/journals/journal-of-differential-geometry/volume-102/issue-1/Compact-moduli-spaces-of-Del-Pezzo-surfaces-and-K%c3%a4hlerEinstein-metrics/10.4310/jdg/1452002879.full.

[RT06a] Julius Ross and Richard Thomas. "A Study of the Hilbert-Mumford Criterion for the Stability of Projective Varieties". In: *Journal of Algebraic Geometry* 16.2 (Nov. 28, 2006), pp. 201–255. ISSN: 1056-3911, 1534-7486. DOI: 10.1090/S1056-3911-06-00461-9. URL: https://www.ams.org/jag/2007-16-02/S1056-3911-06-00461-9/ (visited on 07/21/2022).

[RT06b] Julius Ross and Richard Thomas. "An Obstruction to the Existence of Constant Scalar Curvature Kähler Metrics". In: *Journal of Differential Geometry* 72.3 (Mar. 1, 2006). ISSN: 0022-040X. DOI: 10.4310/jdg/1143593746. URL: https://projecteuclid.org/journals/journal-of-differential-geometry/volume-72/issue-3/An-obstruction-to-the-existence-of-constant-scalar-curvature-K%c3%a4hler/10.4310/jdg/1143593746.full.

[Ste06] D. A. Stepanov. "A Note on the Dual Complex Associated to a Resolution of Singularities". In: *Russian Mathematical Surveys* 61.1 (Feb. 28, 2006). Comment: 5 pages, pp. 181–183. ISSN: 0036-0279, 1468-4829. DOI: 10.1070/RM2006v061n01ABEH004309.

arXiv: math/0509588. URL: http://arxiv.org/abs/math/0509588.

[Szé14] Gábor Székelyhidi. *An Introduction to Extremal Kähler Metrics*. Graduate Studies in Mathematics volume 152. Providence, Rhode Island: American Mathematical Society, 2014. 192 pp. ISBN: 978-1-4704-1047-6.

[Szé15] Gábor Székelyhidi. "Filtrations and Test-Configurations: With an Appendix by Sebastien Boucksom". In: *Mathematische Annalen* 362.1-2 (June 2015), pp. 451–484. ISSN: 0025-5831, 1432-1807. DOI: 10.1007/s00208-014-1126-3. URL: http://link.springer.com/10.1007/s00208-014-1126-3.

[Tia97] Gang Tian. "Kähler-Einstein Metrics with Positive Scalar Curvature". In: *Inventiones Mathematicae* 130.1 (Sept. 3, 1997), pp. 1–37. ISSN: 0020-9910, 1432-1297. DOI: 10.1007/s002220050176. URL: http://link.springer.com/10.1007/s002220050176.

[Tsu83] Hiroyasu Tsuchihashi. "Higher-Dimensional Analogues of Periodic Continued Fractions and Cusp Singularities". In: *Tohoku Mathematical Journal* 35.4 (Jan. 1, 1983). ISSN: 0040-8735. DOI: 10.2748/tmj/1178228955. URL: https://projecteuclid.org/journals/tohoku-mathematical-journal/volume-35/issue-4/Higher-dimensional-analogues-of-periodic-continued-fractions-and-cusp-singularities/10.2748/tmj/1178228955.full.

[TY87] G. Tian and S. T. Yau. "Existence of Kähler-Einstein Metrics on Complete Kähler Manifolds and Their Applications to Algebraic Geometry". In: *Mathematical Aspects of String Theory*. Proceedings of the Conference on Mathematical Aspects of String Theory. California, USA: WORLD SCIENTIFIC, Sept. 1987, pp. 574–628. ISBN: 978-9971-5-0273-7 978-981-279-841-1. DOI: 10.1142/9789812798411_0028. URL: http://www.worldscientific.com/doi/abs/10.1142/9789812798411_0028.

[Vin63] E.B. Vinberg. "The Theory of Homogeneous Convex Cones". In: *Tr. Mosk. Mat. Obs.* 12 (1963), pp. 303–358.

[Wan12] Xiaowei Wang. "Height and GIT Weight". In: *Mathematical Research Letters* 19.4 (2012), pp. 909–926. ISSN: 10732780, 1945001X. DOI: 10.4310/MRL.2012.v19.n4.a14. URL: http://www.intlpress.com/site/pub/pages/journals/items/mrl/content/vols/0019/0004/a014/ (visited on 07/11/2022).

[Wit12] David Witt Nyström. "Test Configurations and Okounkov Bodies". In: *Compositio Mathematica* 148.6 (Nov. 2012), pp. 1736–1756. ISSN: 0010-437X, 1570-5846. DOI: 10.1112/S0010437X12000358. URL: https://www.cambridge.org/core/product/identifier/S0010437X12000358/type/journal_article (visited on 07/11/2022).

[Yau77] Shing-Tung Yau. "Calabi's Conjecture and Some New Results in Algebraic Geometry". In: *Proceedings of the National Academy of Sciences* 74.5 (May 1977), pp. 1798–1799. ISSN: 0027-8424, 1091-6490. DOI: 10.1073/pnas.74.5.1798. URL: https://pnas.org/doi/full/10.1073/pnas.74.5.1798 (visited on 07/10/2022).

[Yau78] Shing-Tung Yau. "On the Ricci Curvature of a Compact Kähler Manifold and the Complex Monge-Ampére Equation, I". In: *Communications on Pure and Applied Mathematics* 31.3 (May 1978), pp. 339–411. ISSN: 00103640, 10970312. DOI: 10.1002/cpa.3160310304. URL: https://onlinelibrary.wiley.com/doi/10.1002/cpa.3160310304 (visited on 07/10/2022).

[ZS97] Oscar Zariski and Pierre Samuel. *Commutative Algebra. 2*. Repr. of the 1958 - 1960 ed., 4. [print.] - [1997]. Graduate Texts in Mathematics 29. New York Berlin Heidelberg: Springer, 1997. ISBN: 978-0-387-90171-8 978-3-540-90171-6.