Branching laws for tensor products of unitary irreducible representations of $\mathrm{PGL}(2, \mathbb{R})$

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PhD Dissertation

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Another roof, another proof.
- Paul Erdős

Abstract

In Paper A we consider the real reductive Lie group $\mathrm{SL}(2,\mathbb{R})$ and its subgroup H consisting of diagonal matrices. The homogeneous space G/H can be identified with the one-sheeted hyperboloid. For χ a "trivial" unitary character of H we decompose the induced representations $\mathrm{Ind}_H^G(\chi)$, by using the known spectral theory for the Casimir operator Δ. We obtain an explicit Plancherel formula by studying intertwining operators between $\mathrm{Ind}_H^G(\chi)$ and principal series representations of $\mathrm{SL}(2,\mathbb{R})$. We then generalize the result to all unitary characters χ of H, by constructing an explicit isomorphism. From here we obtain the general Plancherel formula and we derive the corresponding direct integral decomposition of $\mathrm{Ind}_H^G(\chi)$.

In Paper B we study tensor products of unitary irreducible representations of $G = \mathrm{PGL}(2,\mathbb{R})$ and their restriction the the diagonal subgroup $\Delta(G)$. We study the corresponding branching problem by studying symmetry breaking operators and give a detailed description of their meromorphic nature. We derive Bernstein-Sato relations which allows for a holomorphic extension of the symmetry breaking operators, in the sense of distribution theory. We investigate the resulting families of zeroes and functional equations, by evaluating on lowest K-types for $G\times G$. Using the detailed description of symmetry breaking operators, we reduce the problem of decomposing the restriction tensor products of unitarily induced principal series representations of $G \times G$, to $\Delta(G)$, to the study of the Plancherel formula on the open dense orbit $\mathcal{O} \cong G/\,\mathrm{GL}(1,\mathbb{R})$. By applying the results in Paper A, we derive the Plancherel formula on $L^2(\mathcal{O},\chi)$, for unitary characters χ of $\mathrm{GL}(1,\mathbb{R})$, and in turn achieve the decomposition of tensor products of unitarily induced principal series representations of G. From the holomorphic dependence of the family of bilinear pairings on principal series representations, of $G \times G$, we extend this result to non-unitarily induced principal series representations, by means of an analytic continuation process. With some minor technical assumptions we solve the full branching problem for the strongly spherical pair $(G \times G, \Delta(G))$.

Lastly, in Paper C, we investigate the invariant inner product on discrete series representations of the group $SO_0(4,1)$ by studying the Fourier-transform on principal series representations, in the non-compact picture. We derive a formula for the invariant inner-product by studying the action of the nilradical N in the Fourier transformed picture. The corresponding invariant inner-product is described in terms of a multiplication operator acting on the eigenspaces of an operator, appearing from the action of Lie algebra of $M \cong SO(3)$.

Resumé

Afhandlingen består af tre manuskripter Paper A, Paper B og Paper C.

I Paper A betragter vi den reelle reduktive Lie-gruppe $\mathrm{SL}(2,\mathbb{R})$ og dens undergruppe H, der består af diagonalmatricer. Det homogene rum G/H kan identificeres med hyperboloiden med en enkelt flade. For χ en "triviel" unitær karakter af H dekomponerer vi rummet af inducerede repræsentationer $\mathrm{Ind}_H^G(\chi)$ ved hjælp af den kendte spektralteori for Casimir-operatoren Δ. Vi opnår en eksplicit Plancherel-formel ved at studere intertwining operatorer mellem $\mathrm{Ind}_H^G(\chi)$ og principale række-repræsentationer af $\mathrm{SL}(2,\mathbb{R})$. Derefter generaliserer vi resultatet til alle unitære karakterer χ af H ved at konstruere en eksplicit isomorfi. Herfra opnår vi den generelle Plancherel-formel, og vi udleder den tilsvarende direkte integral dekomposition af rummet $\mathrm{Ind}_H^G(\chi)$.

I Paper B studerer vi tensorprodukter af unitære irreducible repræsentationer af $G = \mathrm{PGL}(2,\mathbb{R})$ og deres restriktion til diagonal-undergruppen $\Delta(G)$. Vi studerer det tilhørende branching problem gennem symmetry breaking operatorer og giver en detaljeret beskrivelse af deres meromorfe natur. Vi udleder Bernstein-Sato-relationer, som tillader en holomorf udvidelse af symmetry breaking operatorene i distributions-teoriens perspektiv. Vi undersøger familien af nulpunkter for disse og udleder funktionalligninger ved at evaluere på laveste K-typer for $G \times G$. Fra den detaljerede beskrivelse af familien af symmetry breaking operatorer, reducerer vi problemet at dekomponere restriktionen af tensorprodukter af unitært inducerede principale række-repræsentationer af $G \times G$ til $\Delta(G)$, til studiet af Plancherel-formlen på den åbne tætte bane $\mathcal{O} \cong G/\mathrm{GL}(1,\mathbb{R})$. Ved hjælp af resultaterne i Paper A udleder vi Plancherel-formlen på $L^2(\mathcal{O},\chi)$ for unitære karakterer af $\mathrm{GL}(1,\mathbb{R})$ og opnår dermed dekomponeringen af tensorprodukter af unitært inducerede principale serie-repræsentationer af G. Fra den holomorfe natur af familien af bilineære paringer på principal række-representationer, udvider vi dette resultat til principale række-repræsentationer, som ikke er unitært inducerede, ved hjælp af en analytisk fortsættelsesproces. Med nogle mindre tekniske antagelser løser vi derved det fulde branching problem for det strongly spherical par $(G \times G, \Delta(G))$.

I det afsluttende manuskript Paper C undersøger vi det invariante indre produkt på diskrete række-repræsentationer af gruppen $SO_0(4,1)$, ved at studere Fourier-transformen på principale række-repræsentationer, i det ikke-kompakte billede. Vi udleder en formel for det invariante indre produkt ved at studere virkningen af nilradikalen N, i det Fourier-transformerede billede. Det tilhørende invariante indre produkt beskrives i form af en multiplikationsoperator, der virker på egenrummene af en bestemt operator, som opstår fra virkningen af Lie-algebraen for $M \cong SO(3)$.

Contents

Preface

This thesis concludes my 3 years as a PhD student at the Department of Mathematics at Aarhus University. While starting my PhD studies during the Covid pandemic, and resulting shutdowns, did not pose a great start for conducting research, the project eventually picked up pace and ultimately I am happy with the results obtained.

The thesis consists of an introduction and 3 manuscripts

- Paper A: An explicit Plancherel formula for line bundles over the one-sheeted hyperboloid.
- Paper B: Tensor products of unitary irreducible representations of $\mathrm{PGL}(2, \mathbb{R})$.
- Paper C: A L^2-model for discrete series representations of $SO_0(4,1)$.

Paper A has been co-authored with Jonathan Ditlevsen and is published in the Journal of Lie Theory. Up to typesetting the paper appears as in the published version. I played a vital role in obtaining the results of the paper and in the writing of the paper itself.

Both Paper B and Paper C are written in the style of an article, but perhaps with a few additional details than typically presented in published papers in mathematics. A small part of sections 2 and 9 of Paper B has been taken from an unpublished manuscript written with Jonathan Ditlevsen. In accordance with GSNS rules, parts of Paper B were also used in the progress report for the qualifying examination. Furthermore parts of section 1.1 in Paper C was used in a project that was part of some coursework completed during my PhD studies.

Paper B was originally meant to study tensor products of the group $\mathrm{SL}(2, \mathbb{R})$ instead of $\mathrm{PGL}(2, \mathbb{R})$. But due to some unforeseen technical difficulty, we pivoted to the group $\mathrm{PGL}(2, \mathbb{R})$ instead. The paper thus additionally contain some results for statements about tensor products of representations of $\mathrm{SL}(2, \mathbb{R})$, and the general methods used in the paper are influenced by this "detour".

The ideas presented in Paper C were developed during my stay at Chalmers university of technology, where I visited Professor Genkai Zhang and Postdoctoral researcher Clemens Weiske.

All the manuscripts are written to be self-contained and include both their own introduction and references.

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Introduction

Let G be a group and H a subgroup. For a representation π of G its restriction to H naturally defines a representation of H. If G is a compact Lie group and π is a irreducible representation of G then π is finite dimensional and its restriction to H need not be irreducible anymore. In this case the restriction decomposes into irreducible representations of H

$$\pi|_H \cong \bigoplus_{\tau \in \widehat{H}} m(\pi, \tau) \cdot \tau, \qquad m(\pi, \tau) \in \mathbb{N}_0.$$

Here $\widehat{H}$ denotes the unitary dual of H, i.e the irreducible unitary representations of H up to equivalence and $m(\pi, \tau)$ denotes the multiplicity of τ in $\pi|_H$. The question of how such irreducible representations of G, when restricted to H, decomposes as irreducible representations of H is called a *branching problem* or *branching law* and solving it requires explicitly determining the multiplicities $m(\pi, \tau)$. Branching laws for the classical compact groups $O(n), U(n)$ and $Sp(n)$, and the respective subgroups $O(n-1), U(n-1)$ and $Sp(n-1)$, was studied by Weyl, Murnaghan and Zhelobenko in the early 1930's and early 1960's. Later Kostant proved the Kostant Multiplicity Formula, which provided a uniform method of proof for branching problems in the compact setting. However if G is a real reductive non-compact Lie group, then the irreducible representations of G need not be finite dimensional and such branching problems may be ill-defined. But when π is unitary then its restriction to H instead admits a direct integral decomposition

$$\pi|_H \cong \int_{\widehat{H}}^{\oplus} m(\pi, \tau) \cdot \tau \, d_\pi(\tau)$$

with multiplicities $m(\pi, \tau) \in \mathbb{N}_0 \cup \{\infty\}$ and $d_\pi(\tau)$ some measure on $\widehat{H}$. In this case one can study the branching problems. Understanding the corresponding branching problem boils down to determining both the multiplicity function $m(\pi, \tau)$ and the measure $d_\pi(\tau)$ explicitly. The support of the measure $d_\pi(\tau)$ may both have a discrete and continuous part.

Kobayashi proposed a program for studying branching problems in [Kob15] for smooth admissable representation of real reductive groups, consisting of three parts

Stage A Abstract features of $\pi|_H$.
Stage B Branching laws.
Stage C Construction of symmetry breaking operators.

With a symmetry breaking operator being a continuous H-homomorphism from the representation $\pi|_H$ to an irreducible representation τ of H. In recent times **Stage C** has received growing amounts of attention (See e.g [KS15], [KS18], [Fra23], [Cle16], [Cle17] and [FW20]). However most results that have been produced in this setting are for groups are of rank 1, and not until recently has a systematical approach been applied to rank one groups (G, H), where the measure $d_\pi(\tau)$ admits both discrete and continuous spectrum (See [Wei20] and [Wei21]).

This thesis takes some steps toward applying the result outside the rank 1 cases, namely to the case $G = \mathrm{PGL}(2, \mathbb{R}) \times \mathrm{PGL}(2, \mathbb{R})$ and H is the diagonal subgroup of G. The thesis is split into three Papers, Paper A, Paper A and Paper C.

Paper A concerns itself with the study of the Plancherel formula for the one-sheeted hyperboloid $\mathrm{SL}(2, \mathbb{R})/MA$, with MA the subgroup of diagonal matrices in $\mathrm{SL}(2, \mathbb{R})$, and the corresponding direct integral decomposition.

Paper B concerns itself with the branching problem for tensor products of irreducible unitary representations of $\mathrm{PGL}(2, \mathbb{R})$ and the surrounding theory.

Lastly, Paper C concerns discrete series representations for the group $SO_0(4, 1)$ and their realizations as L^2-spaces in the Fourier picture.

We now give a brief overview of some topics relevant to the papers.

Strongly spherical pairs of real reductive Lie groups

Let G be a real reductive Lie group and H be a reductive subgroup of G. Unlike the case for compact Lie groups, the multiplicities $m(\pi, \tau)$ need no longer be finite. A suitable condition for the pair (G, H), in which the multiplicities are finite, was singled out by Kobayashi-Oshima in [KO13]. Assuming that both groups are defined algebraically over $\mathbb{R}$, then the corresponding multiplicities $m(\pi, \tau)$ are finite for all smooth irreducible admissible representations π of G and τ of H if and only if the pair (G, H) is strongly spherical, i.e if a minimal parabolic subgroup $P_G \times P_H$ of $G \times H$ has an open orbit. Such reductive pairs were fully classified by Knop-Krötz-Pecher-Schlittkrull in [KKPS19]. The case where $G = \mathrm{PGL}(2, \mathbb{R}) \times \mathrm{PGL}(2, \mathbb{R})$ and H the diagonal subgroup of G is an example of a strongly spherical pair and the corresponding branching problem is equivalent to the branching problem for tensor products of representations of $\mathrm{PGL}(2, \mathbb{R})$.

Symmetry breaking operators

Let (G, H) be a strongly spherical pair and $\pi_{\xi,\lambda}$ and $\tau_{\eta,\nu}$ be principal series representations of G and H, induced from characters of minimal parabolic subgroups P_G and P_H of G and H respectively. The space of symmetry breaking operators

$$\mathrm{Hom}_H(\pi_{\xi,\lambda}|_H, \tau_{\eta,\nu})$$

can be realized as distribution sections on G/P_G with specific P_H equivariances. Under this setup the symmetry breaking operators can be identified by their corresponding distribution kernels, determined by their values on the P_H orbits in G/P_G. The kernels are often given as meromorphic families of distributions, in terms of the induction parameters, and can under suitable conditions often be extended to a holomorphic family by means of normalization. For the case where $G = \mathrm{PGL}(2, \mathbb{R}) \times \mathrm{PGL}(2, \mathbb{R})$ and $H = \Delta(\mathrm{PGL}(2, \mathbb{R}))$, the space of symmetry breaking operators

$$\mathrm{Hom}_H(\pi_{\xi,\lambda} \otimes \pi_{\eta,\nu}|_H, \pi_{\zeta,\nu})$$

is generically spanned by a single family of symmetry breaking operators

$$\mathrm{Hom}_H(\pi_{\xi,\lambda} \otimes \pi_{\eta,\mu}|_H, \pi_{\zeta,\nu}) = \mathbb{C}A^{\zeta}_{\lambda,\mu,\nu},$$

depending meromorphically on the induction parameters. This family of symmetry breaking operators can then be normalized to a family depending holomorphically on the induction parameters.

Knapp–Stein intertwining operators and functional equations for symmetry breaking operators

For a principal series representation $\pi_{\xi,\lambda}$ of a real reductive Lie group G the maps

$$T^w_{\xi,\lambda} : \pi_{\xi,\lambda} \to \pi_{w\xi,w\lambda},$$
$$T^w_{\xi,\lambda} f(g) = \int_{\overline{N} \cap w^{-1}Nw} f(gw\overline{n}) d\overline{n}$$

defines a meromorphic family of G intertwining operators, which can be extended to a holomorphic family by normalization. Here w denotes a representative of an element of the Weyl group of G. For a principal series representation $\tau_{\eta,\nu}$ of a real reductive subgroup H of G and a symmetry breaking operator

$$A \in \mathrm{Hom}_H(\pi_{w\xi,w\lambda}|H, \tau_{\eta,\nu})$$

the composition $A \circ T^w_{\xi,\lambda}$ again defines a symmetry breaking operator

$$A \circ T^w_{\xi,\lambda} \in \mathrm{Hom}_H(\pi_{\xi,\lambda}|_H, \tau_{\eta,\nu}).$$

If A depends meromorphically on the induction parameters λ and ν, the composition again defines a meromorphic family of symmetry breaking operators. When the space of symmetry breaking operators is generically one dimensional, a functional equation necessarily exists. Determining such functional identities can be difficult by means of direct computation, but may instead be understood through evaluation on a sufficiently nice choice of vector $\psi \in \pi_{\xi,\lambda}$. Such functional identities can also be used to investigate the holomorphic extension of symmetry breaking operators, for cases of potential "over normalization", i.e determine for which parameters ξ, η, λ and ν the corresponding symmetry breaking operator is zero, for the chosen normalization. The Knapp-Stein intertwiners also play an essential role in the unitarization of non-unitarily induced principal series representations and unitarizable composition factors. Such representations are typically unitarized by equipping them with their canonical K-pairing, and then composing with a Knapp–Stein intertwiner in a single argument. In this picture one gets a family, depending holomorphically on the induction parameters, of bilinear pairings that unitarize the unitarizable principal series representations, at the corresponding induction parameters.

Plancherel formulas and symmetry breaking

For a strongly spherical pair (G, H) for which P_H acts with a unique open orbit on G/P_G, the subgroup H also acts with an open orbit $\mathcal{O}$ on G/P_G. Picking a basepoint $x_0 \in \mathcal{O}$ one can consider the restriction of a principal series representation $\pi_{\xi,\lambda}$ to the open orbit $\mathcal{O}$, by letting H act on the basepoint x_0. This induces a natural H-intertwining map

$$\Theta_\lambda : \pi_{\xi,\lambda}|_H \to C^\infty(H \times_{H_{x_0}} \chi)$$

to the smooth section of some homogeneous vector bundle

$$H \times_{H_{x_0}} V_\chi|_{H_{x_0}} \to H/H_{x_0}.$$

This map can be understood to depend holomorphically on the parameter λ. If $\pi_{\xi,\lambda}$ is a unitarily induced principal series representation and $C^\infty(H \times_{H_{x_0}} \chi)$ has unitary closure $L^2(H \times_{H_{x_0}} \chi)$, then Θ_λ extends to a unitary isomorphism between Hilbert spaces. In this case the decomposition of the unitary principal series representation reduces to the study of the Plancherel formula on the corresponding L^2-sections, for the associated homogeneous vector bundle. In the case where $G = \mathrm{PGL}(2,\mathbb{R}) \times \mathrm{PGL}(2,\mathbb{R})$ and $H = \Delta(\mathrm{PGL}(2,\mathbb{R}))$ the corresponding open orbit is $\mathcal{O} \cong \mathrm{PGL}(2,\mathbb{R})/\mathrm{GL}(1,\mathbb{R})$ and the homogeneous H space to study becomes the one–sheeted hyperboloid. We study this in detail in Paper A and give the explicit Plancherel formula and the corresponding direct integral decomposition. In Paper B we study how the Plancherel formula extends to principal series representations which are not unitarily induced, by means of analytic continuation.

A brief summary of results

In Paper A we study the Plancherel formula on the one–sheeted hyperboloid $\mathrm{SL}(2,\mathbb{R})/MA$ with MA being the subgroup of $\mathrm{SL}(2,\mathbb{R})$ consisting of diagonal matrices. We do so by studying the action of the Casimir operator Δ_0, on the space $\mathrm{Ind}_{MA}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon \otimes \mathbf{1})$. Using the known spectral theory for the Casimir operator, in some well chosen coordinates on G/H, we derive the Plancherel formula for $\mathrm{Ind}_{MA}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon \otimes \mathbf{1})$, written in terms of $\mathrm{SL}(2,\mathbb{R})$ intertwining operators

$$A_{0,\mu}^{\varepsilon,\xi} : \mathrm{Ind}_{MA}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon \otimes \mathbf{1}) \to \mathrm{Ind}_{MAN}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon \otimes e^\mu \otimes 1),$$

between the induced representations $\mathrm{Ind}_{MA}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon \otimes \mathbf{1})$ and the principal series representations $\mathrm{Ind}_{MAN}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon \otimes e^\mu \otimes 1)$, of $\mathrm{SL}(2,\mathbb{R})$. We then show that $\mathrm{Ind}_{MA}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon \otimes \mathbf{1}) \cong \mathrm{Ind}_{MA}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon \otimes e^\lambda)$ for all unitary characters $\varepsilon \otimes e^\lambda$ of H, by constructing an explicit isometry. Using this we obtain the general Plancherel formula on $\mathrm{Ind}_{MA}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon \otimes e^\lambda)$ and derive the corresponding direct integral decomposition of $\mathrm{Ind}_{MA}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon \otimes e^\lambda)$, in terms of unitary irreducible representations of $\mathrm{SL}(2,\mathbb{R})$.

Paper B is about decomposing tensor products of unitary irreducible representations of $G = \mathrm{PGL}(2,\mathbb{R})$. We study symmetry breaking operators $A_{\lambda,\mu,\nu}^{\zeta}$ for the strongly spherical pair $(G \times G, \Delta(G))$ via their corresponding distributional kernels, investigating their meromorphic nature, Bernstein–Sato identities and resulting holomorphic extensions. We derive functional equations for their composition with Knapp–Stein intertwining operators by studying the associated invariant trilinear forms on $S^1 \times S^1 \times S^1$, of the symmetry breaking operators $A_{\lambda,\mu,\nu}^{\sigma}$. This allows us to give a detailed description of most zeroes of the holomorphic family of symmetry breaking operators. We obtain direct integral decompositions of the restriction of tensor products of unitary irreducible representations of $G \times G$. Excluding some minor technical details, we thus describe the full branching law for the pair $(G \times G, \Delta(G))$.

In Paper C we investigate the invariant inner product on discrete series representations of $G = SO_0(4,1)$, realized as quotients inside principal series representations of G, by studying the Fourier-transform of the principal series representations in the non-compact picture. We investigate the action of the nilradical N in this framework and derive an explicit formula for the G invariant sesquilinear form on the discrete series. We thus obtain a L^2-model for all discrete series representations of $G = SO_0(4,1)$.

Outlook

In Paper B we examine the theory of symmetry breaking operators for tensor products of principal series representations for $\mathrm{PGL}(2,\mathbb{R})$ by first studying the problem for $\mathrm{SL}(2,\mathbb{R})$. Hence it would be natural to use this theory to give a uniform proof for the branching laws found by Repka in [Rep78]. This is however a tricky endeavour, as the framework used for the proof is not necessarily well suited to handle the holomorphic and anti-holomorphic discrete series representations of $\mathrm{SL}(2,\mathbb{R})$. We also make some minor technical assumptions in order to achieve the full branching laws for tensor products of $\mathrm{PGL}(2,\mathbb{R})$. Absolving these assumptions is of course a natural place to start, for further research. The methods used in Paper B are in principal also suited for vector valued principal series representations. Hence it would be interesting to attempt to apply the methods used in Paper B to study branching laws for certain representations of e.g $SO(4,1)$. This may prove to be too technical in practice though, since the analysis required in Paper B is already quite challenging.

The results of Paper C potentially allow for the study of branching laws for tensor products of discrete series representations of $SO_0(4,1)$. L^2-models for complementary series representations of rank 1 groups was used by Zhang in [Zha17] to study discrete components appearing in the tensor product of complementary series representations, for $SO_0(n,1)$.

Similarly Möllers and Oshima studied branching laws for all unitary representations contained in spherical principal series representations of $O(1,n+1)$, restricted to the subgroup $O(1,m+1) \times O(n-m)$, in [MO15]. If one can find a suitable generalization of the methods used in Paper C to all rank 1 indefinite orthogonal groups $SO(n,1)$, one could in principle find L^2-models for all discrete series representations of $SO(n,1)$ and in turn study the corresponding branching laws by using similar methods to those used in either [Zha17] or [MO15].

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Paper A

An explicit Plancherel formula for line bundles over the one-sheeted hyperboloid

Frederik Bang–Jensen and Jonathan Ditlevsen

Abstract

In this paper we consider $G = \mathrm{SL}(2,\mathbb{R})$ and H the subgroup of diagonal matrices. Then $X = G/H$ is a unimodular homogeneous space which can be identified with the one-sheeted hyperboloid. For each unitary character χ of H we decompose the induced representations $\mathrm{Ind}_H^G(\chi)$ into irreducible unitary representations, known as a Plancherel formula. This is done by studying explicit intertwining operators between $\mathrm{Ind}_H^G(\chi)$ and principal series representations of G. These operators depends holomorphically on the induction parameters.

Introduction

The Plancherel formula for a unimodular homogeneous space $X = G/H$ of a Lie group G describes the decomposition of the left-regular representation of G on $L^2(X)$ into irreducible unitary representations. More generally, one can ask for the decomposition of $L^2(G \times_H V_\chi)$, the L^2-sections of a homogeneous vector bundle associated with a unitary representation (χ, V_χ) of H. In representation theoretic language, this corresponds to the induced representation $\mathrm{Ind}_H^G(\chi)$ of G, and for the trivial representation $\chi = \mathbf{1}$ we recover $L^2(G/H)$.

By abstract theory, the unitary representation $\mathrm{Ind}_H^G(\chi)$ decomposes into a direct integral of irreducible unitary representations of G, i.e. there exists a measure μ on the unitary dual $\widehat{G}$ of G and a multiplicity function $m : \widehat{G} \to \mathbb{N} \cup \{\infty\}$ such that

$$\mathrm{Ind}_H^G(\chi) \simeq \int_{\widehat{G}}^{\oplus} m(\pi) \cdot \pi \, d\mu(\pi).$$

An *abstract* Plancherel formula describes the support of the Plancherel measure μ as well as the multiplicity function m. Such abstract Plancherel formulas have been established for certain classes of homogeneous spaces such as semisimple symmetric spaces (see e.g. [B05]).

However, for some applications an abstract Plancherel formula is not sufficient, and a more *explicit* version is needed (see e.g. [FW, W21]). By this, we mean an explicit formula for the measure μ as well as explicit linearly independent intertwining operators $A_{\pi,j} : \mathrm{Ind}_H^G(\chi)^\infty \to \pi^\infty$, $j = 1, \ldots, m(\pi)$, for μ-almost every $\pi \in \widehat{G}$ such that

$$\|f\|^2_{\mathrm{Ind}_H^G(\chi)} = \int_{\widehat{G}} \sum_{j=1}^{m(\pi)} \|A_{\pi,j} f\|^2_\pi \, d\mu(\pi) \qquad (f \in \mathrm{Ind}_H^G(\chi)^\infty).$$

Such an explicit Plancherel formula is for instance known for Riemannian symmetric spaces $X = G/K$, where the Plancherel measure μ is explicitly given in terms of Harish-Chandra's c-function and the intertwining operators $A_{\pi,j}$ can be described in terms of spherical functions (see e.g. [H08], and also [S94] for the case of line bundles over Hermitian symmetric spaces). This explicit Plancherel formula has recently been applied in the context of branching problems for unitary representations where the explicit Plancherel measure and in particular its singularities play a crucial role (see e.g. [FW, W21]). In order to apply the same strategy to other branching problems, explicit Plancherel formulas are needed for more general homogeneous spaces.

In this paper, we determine the explicit Plancherel formula for line bundles over the one-sheeted hyperboloid $X = G/H$, where $G = \mathrm{SL}(2,\mathbb{R})$ and H the subgroup of diagonal matrices. This specific Plancherel formula has direct applications to branching problems for the pairs $(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}), \mathrm{diag}(\mathrm{SL}(2,\mathbb{R}))$ and $(\mathrm{GL}(3,\mathbb{R}), \mathrm{GL}(2,\mathbb{R}))$. The homogeneous Hermitian line bundles over X are parameterized by $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ and $\lambda \in i\mathbb{R}$, the corresponding unitary character of H being

$$\chi_{\varepsilon,\lambda}\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \mathrm{sgn}(t)^{\varepsilon}|t|^{\lambda} \qquad (t \in \mathbb{R}^{\times}).$$

We find intertwining operators $A^{\xi}_{\lambda,\mu} : \mathrm{Ind}_H^G(\chi_{\lambda,\varepsilon}) \to \mathrm{Ind}_P^G(\varepsilon \otimes e^{\mu} \otimes 1)$, $\xi = 0, 1$ between the line bundles over X and the principal series representation (see Proposition 5.1).

Theorem 0.1 (See Corollary 7.6)**.** *For $f \in \mathrm{Ind}_H^G(\chi_{\lambda,\varepsilon})$, $\lambda \in i\mathbb{R}$ and $\varepsilon \in \{0,1\}$ we have*

$$\|f\|^2 = \int_{i\mathbb{R}} \sum_{\xi=0}^{1} \|\mathbf{A}^{\xi}_{\lambda,\mu} f\|^2 \frac{d\mu}{|a(\mu,\varepsilon)|^2} + \sum_{\mu \in 1-\varepsilon-2\mathbb{N}} c(\mu,\varepsilon)\|\mathbb{A}^{\varepsilon}_{\lambda,\mu} f\|^2, \tag{0.1}$$

where $\mathbf{A}_{\lambda,\mu}$ and $\mathbb{A}_{\lambda,\mu}$ are some combinations of $A^0_{\lambda,\mu}$ and $A^1_{\lambda,\mu}$.

The proof of (0.1) consists of two steps. First, we prove (0.1) in the case $\lambda = 0$ separately for each K-isotypic component. On a fixed K-isotypic component, the intertwining operators $A^{\xi}_{\lambda,\mu}$ are essentially Fourier–Jacobi transforms, and the Plancherel formula follows from the spectral decomposition of the corresponding ordinary second order differential operator by Sturm–Liouville theory. The main difficulty is that the continuous spectrum occurs with multiplicity two, while the discrete part occurs with multiplicity-one, and it is non-trivial to find the right linear combination of A^0_{μ} and A^1_{μ} that corresponds to a direct summand. In fact, this linear combination is very different for the cases $\varepsilon = 0$ and $\varepsilon = 1$. In the second step, we show that, as a representation of G, $L^2(G/H, \mathcal{L}_{\varepsilon,\lambda})$ is independent of λ, and by finding an explicit unitary isomorphism $L^2(G/H, \mathcal{L}_{\varepsilon,\lambda}) \to L^2(G/H, \mathcal{L}_{\varepsilon,0})$ we deduce the claimed formula.

We remark that for $\varepsilon = 0$ and general $\lambda \in i\mathbb{R}$ the Plancherel formula was recently obtained by Zhu [Z18]. Moreover, for $\varepsilon = 0$ and $\lambda = 0$ our Plancherel formula can be viewed as a special case of the one for pseudo-Riemannian real hyperbolic spaces $\mathrm{O}(p,q)/\mathrm{O}(p,q-1)$ with $p = 1$ and $q = 2$ which was obtained by Faraut [F79], Rossmann [R78] and Strichartz [S73]. Note also that the corresponding *abstract* Plancherel formula, i.e. the description of the representations occurring in the direct integral decomposition, also follows from the general theory (see e.g. [B05]).

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Notation: $\mathbb{N} = \{1,2,3\ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $A \subseteq \mathbb{R}$ and $b, c \in \mathbb{R}$ we denote by $b + cA = \{b + ca \,|\, a \in A\}$. The Pochhammer symbol is $(x)_n = x(x+1)\cdots(x+n-1)$. We denote Lie groups by Roman capitals and their corresponding Lie algebras by the corresponding Fraktur lower cases. For $m \in \mathbb{Z}$ we let $[m]_2 \in \{0,1\}$ be the remainder of m after division by 2.

1 The principal series of SL$(2,\mathbb{R})$

In this section we recall some results about the representation theory of SL$(2,\mathbb{R})$ following [C20]. Let $G = \mathrm{SL}(2,\mathbb{R})$ and consider the following subgroups

$$M = \{\pm I\}, \quad A = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{R}_{>0} \right\}, \quad N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\},$$

then $P = MAN$ is a minimal parabolic subgroup of G. Identify $\widehat{M} \cong \mathbb{Z}/2\mathbb{Z}$ by mapping $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ to the character

$$M \to \{\pm 1\}, \qquad \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \mapsto (\pm 1)^{\varepsilon}.$$

Further, we identify $\mathfrak{a}_{\mathbb{C}}^* \cong \mathbb{C}$ by mapping $\lambda \mapsto \lambda\big(\operatorname{diag}(1,-1)\big)$. We can then observe that any character of $H := MA$ is of the form $\chi_{\varepsilon,\lambda} = \varepsilon \otimes e^{\lambda}$ where

$$\chi_{\varepsilon,\lambda} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = |t|_{\varepsilon}^{\lambda} := \operatorname{sgn}(t)^{\varepsilon} |t|^{\lambda}, \qquad (t \in \mathbb{R}^{\times}).$$

As the commutator subgroup of P is N the characters of P is of the form $\varepsilon \otimes e^{\mu} \otimes 1$ and these characters are unitary exactly when $\lambda \in i\mathbb{R}$.

Let $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ and $\mu \in \mathbb{C}$. For any character $\varepsilon \otimes e^{\mu} \otimes 1$ of P define the principal series representation $\pi_{\varepsilon,\mu}$induced by it to be the left regular representation of G on

$$\operatorname{Ind}_P^G(\varepsilon \otimes e^{\mu} \otimes 1) = \left\{ f \in C^{\infty}(G) \;\middle|\; f(gman) = |t|_{\varepsilon}^{-\mu-1} f(g),\, m \in M,\, a \in A,\, n \in N \right\},$$

where $ma = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \in MA$. We introduce the notation

$$k_{\theta} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

and $\zeta_m(k_\theta) = e^{im\theta}$. According to the theory of Fourier series we have the K-type decomposition

$$\operatorname{Ind}_P^G(\varepsilon \otimes e^{\mu} \otimes 1) \cong \widehat{\bigoplus_{m \in 2\mathbb{Z}+\varepsilon}} \mathbb{C}\zeta_m.$$

We let $\operatorname{Ind}_P^G(\varepsilon \otimes e^{\mu} \otimes 1)_m$ denote the set of functions contained in the K-type given by $m \in \mathbb{Z}$, that is $\operatorname{Ind}_P^G(\varepsilon \otimes e^{\mu} \otimes 1)_m = \mathbb{C}\zeta_m$.

A basis of $\mathfrak{g}$ is given by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Consider the Casimir operator

$$\Delta_{\mu} = d\pi(H)^2 + d\pi(E+F)^2 - d\pi(E-F)^2,$$

where $\pi = \pi_{\varepsilon,\mu}$.

Proposition 1.1 (See e.g. [C20, Prop. 10.7])**.** *For $f \in \operatorname{Ind}_P^G(\varepsilon \otimes e^\mu \otimes 1)$ we have*

$$\Delta_\mu f = (\mu^2 - 1) f.$$

Proposition 1.2 (See [C20, Prop. 10.8])**.** *The representation $\operatorname{Ind}_P^G(\varepsilon \otimes e^\mu \otimes 1)$ is irreducible except when $\mu \in 1 - \varepsilon - 2\mathbb{Z}$. If $\mu \in 1 - \varepsilon - 2\mathbb{N}$ then $\operatorname{Ind}_P^G(\varepsilon \otimes e^\mu \otimes 1)$ decomposes as $V_0 \oplus V_1 \oplus V_2$ where V_0 is an irreducible representation containing exactly the K-types with $|m| \leq -\mu$. The quotient $\pi^{ds}_{\varepsilon,\mu}$ is a direct sum of two infinite dimensional representations $\pi^{\mathrm{hol}}_{\varepsilon,\mu}$ and $\pi^{\mathrm{ahol}}_{\varepsilon,\mu}$.*

Let $w_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, a representative of the longest Weyl group element of G. Recall the definition of the Knapp–Stein intertwining operator

$$T^\varepsilon_\mu : \operatorname{Ind}_P^G(\varepsilon \otimes e^\mu \otimes 1) \to \operatorname{Ind}_P^G(\varepsilon \otimes e^{-\mu} \otimes 1), \quad T^\varepsilon_\mu f(g) = \frac{1}{\Gamma(\frac{\mu+\varepsilon}{2})} \int_{\overline{N}} f(g w_0 \overline{n}) d\overline{n},$$

for $\operatorname{Re}(\mu) > 0$. The normalization is chosen such that T^ε_μ extends holomorphically to $\mu \in \mathbb{C}$.

Proposition 1.3. *For $f \in \operatorname{Ind}_P^G(\varepsilon \otimes e^\mu \otimes 1)_m$ we have*

$$T^\varepsilon_\mu f = b^\varepsilon_m(\mu) f,$$

where

$$b^\varepsilon_m(\mu) = \sqrt{\pi} i^{[\varepsilon]_2} (-1)^{\frac{m+|m|}{2} - [\varepsilon]_2} \frac{\left(\frac{1+\varepsilon-\mu}{2}\right)_{\frac{|m|-\varepsilon}{2}}}{\Gamma(\frac{\mu+1+|m|}{2})}.$$

For $\varepsilon = 0$ and $\mu \in 1 - 2\mathbb{N}$ we have $b^0_m(\mu) \geq 0$ for all $m \in 2\mathbb{Z}$. Whereas for $\varepsilon = 1$, m odd and $\mu \in -2\mathbb{N}$ we have $-i b^1_m(\mu) \geq 0$ for $m > 0$ and $i b^1_m(\mu) \geq 0$ for $m < 0$.

Proof. As T^ε_μ maps K-types to K-types we have $T^\varepsilon_\mu f = T^\varepsilon_\mu f(e) f$. Now decompose

$$w_0 \overline{n}_x = kan = \frac{1}{\sqrt{1+x^2}} \begin{pmatrix} x & 1 \\ -1 & x \end{pmatrix} \begin{pmatrix} \sqrt{x^2+1} & 0 \\ 0 & \frac{1}{\sqrt{1+x^2}} \end{pmatrix} \begin{pmatrix} 1 & \frac{x}{x^2+1} \\ 0 & 1 \end{pmatrix},$$

then applying f's equivariance properties, we arrive at

$$\int_{\mathbb{R}} f(w_0 \overline{n}_x)\, dx = \int_{\mathbb{R}} (x+i)^{\frac{m-\mu-1}{2}} (x-i)^{\frac{-m-\mu-1}{2}}\, dx = \frac{2^{1-\mu} \pi i^m \Gamma(\mu)}{\Gamma(\frac{\mu-|m|+1}{2}) \Gamma(\frac{\mu+|m|+1}{2})},$$

where in the last equality we used Lemma A.1. Now dividing by $\Gamma(\frac{\mu+\varepsilon}{2})$ and shuffling around Gamma-factors we arrive at the result. ☐

For $\mu \in i\mathbb{R}$ we equip the space $\operatorname{Ind}_P^G(\varepsilon \otimes e^\mu \otimes 1)$ with the usual L^2-norm. Using Proposition 2 we can for $\varepsilon = 0$ and $\mu \in 1 - 2\mathbb{N}$ equip $\operatorname{Ind}_P^G(0 \otimes e^\mu \otimes 1)$ with the norm

$$\|f\|^2 = \int_K f(k) \overline{T^0_\mu f(k)}\, dk.$$

Similarly for $\varepsilon = 1$ and $\mu \in -2\mathbb{N}$ we can equip $\operatorname{Ind}_P^G(1 \otimes e^\mu \otimes 1)$ with the norm

$$\|f\|^2 = \int_K f(k) \overline{\hat{T}^1_\mu f(k)}\, dk$$

where

$$\hat{T}^1_\mu f = \begin{cases} i T^1_\mu f, & \text{for } m > 0, \\ -i T^1_\mu f, & \text{for } m < 0 \end{cases}$$

for $f \in \operatorname{Ind}_P^G(1 \otimes e^\mu \otimes 1)_m$. The operator $\hat{T}^1_\mu$ is still an intertwining operator as it vanishes on V_0 per Proposition 2 and thus we just altered it by a scalar on each of the summands in Proposition 1.

2 The homogeneous space G/H

For a unitary character $\chi_{\varepsilon,\lambda} = \varepsilon \otimes e^{\lambda}$ with $\lambda \in i\mathbb{R}$ the left-regular action $\tau_{\varepsilon,\lambda}$ of G on the space of L^2-sections associated to the line bundle $G \times_H \mathbb{C}_{\varepsilon,\lambda} \to G/H$, given by

$$\operatorname{Ind}_H^G(\varepsilon \otimes e^{\lambda}) = \left\{ f : G \to \mathbb{C},\ \textit{measurable} \mid f(gh) = \chi_{\varepsilon,\lambda}(h)^{-1} f(g),\ \int_{G/H} |f(g)|^2 d(gH) < \infty \right\},$$

defines a unitary representation of G. The goal of this paper is to decompose this space. Furthermore we consider the subspace of compactly supported smooth functions

$$C_c^{\infty}\text{-}\operatorname{Ind}_H^G(\varepsilon \otimes e^{\lambda}) = \left\{ f \in C^{\infty}(G) \cap \operatorname{Ind}_H^G(\varepsilon \otimes e^{\lambda}) \mid \operatorname{supp}(f) \subseteq \Omega,\ \Omega H \text{ is compact in } G/H \right\}.$$

We will denote the smooth vectors in $\operatorname{Ind}_H^G(\varepsilon \otimes e^{\lambda})$ by $\operatorname{Ind}_H^G(\varepsilon \otimes e^{\lambda})^{\infty}$. We introduce the notation

$$b_u = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix}, \quad \overline{n}_x = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}.$$

Using the decomposition $G = KBA$, where $B = \{b_u \mid u \in \mathbb{R}\}$, we consider G/H in the global coordinates $(\theta, u) \in [0, \pi) \times \mathbb{R}$ where $xH = k_\theta b_u H$ and the invariant measure is $d(xH) = \cosh(2u) du d\theta$, see e.g. [M84]. Now in terms of these coordinates we have the K-type decomposition

$$C_c^{\infty}\text{-}\operatorname{Ind}_H^G(\varepsilon \otimes e^{\lambda}) = \widehat{\bigoplus_{m \in 2\mathbb{Z}+\varepsilon}} \mathbb{C}\zeta_m \otimes C_c^{\infty}(\mathbb{R}) \tag{2.1}$$

with $\zeta_m(k_\theta) = e^{im\theta}$. We let $\operatorname{Ind}_H^G(\varepsilon \otimes e^{\lambda})_m$ denote the set of functions contained in the K-type given by $m \in 2\mathbb{Z} + \varepsilon$, that is $\operatorname{Ind}_H^G(\varepsilon \otimes e^{\lambda})_m = \mathbb{C}\zeta_m \otimes C_c^{\infty}(\mathbb{R})$.

We denote by Δ_λ the Casimir operator for the representation $\operatorname{Ind}_H^G(\varepsilon \otimes e^{\lambda})$ defined in a similar fashion as for the principal series.

Proposition 2.1. *Written in the coordinates (θ, u) the Casimir operator Δ_λ is given by*

$$\Delta_\lambda = \frac{\lambda^2}{\cosh^2(2u)} + 2\lambda \frac{\tanh(2u)}{\cosh(2u)} \partial_\theta + 2\tanh(2u)\partial_u - \frac{1}{\cosh^2(2u)} \partial_\theta^2 + \partial_u^2.$$

Proof. This is a standard computation. □

Another set of coordinates can be obtained by using the Iwasawa decomposition $G = KA\overline{N}$ with $(\theta, y) \in [0, \pi) \times \mathbb{R}$ where $xH = k_\theta \overline{n}_y H$. The invariant measure is given by $d(xH) = \frac{1}{2} dy d\theta$ see [K16, Chap. 5, §6].

3 Constructing an isomorphism

The goal of this section is to construct the explicit isomorphism in the following theorem, of which the proof was in large presented to us by Jan Frahm.

Theorem 3.1. *For $\nu, \lambda \in i\mathbb{R}$ the map*

$$☥_\lambda^\nu : \operatorname{Ind}_H^G(\varepsilon \otimes e^{\lambda}) \to \operatorname{Ind}_H^G(\varepsilon \otimes e^{\nu})$$

given by

$$\text{☥}^{\nu}_{\lambda} f(g) = \frac{1}{\sqrt{\pi}2^{\frac{\lambda-\nu}{4}}} \frac{\Gamma\left(\frac{2+\nu-\lambda}{4}\right)}{\Gamma\left(\frac{\lambda-\nu}{4}\right)} \int_{\mathbb{R}} |x|^{\frac{\lambda-\nu}{2}-1} f(g\overline{n}_x)\, dx$$

defines a unitary isomorphism intertwining $\mathrm{Ind}_H^G(\varepsilon \otimes e^{\lambda})$ *and* $\mathrm{Ind}_H^G(\varepsilon \otimes e^{\nu})$*.*

To this extent we consider the minimal parabolic subgroup $\overline{P} = \overline{N}AM \subset G$ and let

$$\mathrm{Ind}_{MA}^{P}(\varepsilon \otimes e^{\lambda}) = \left\{ f : \overline{N}AM \to \mathbb{C} \,|\, f(gma) = \mathrm{sgn}(m)^{\varepsilon} a^{-\lambda-1} f(g),\ \&\ \int_{P/MA} |f(g)|^2\, dg < \infty \right\},$$

where $a^{\lambda} := e^{\lambda(X)}$ for $a = e^X$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^* \cong \mathbb{C}$.

Lemma 3.2 (Induction in stages)**.**

$$\mathrm{Ind}_{MA}^{G}(\varepsilon \otimes e^{\lambda}) \simeq \mathrm{Ind}_{MA\overline{N}}^{G}(\mathrm{Ind}_{MA}^{MA\overline{N}}(\varepsilon \otimes e^{\lambda})),$$

where the map is given by $f \mapsto F$ *where* $F(g)(\overline{p}) = f(g\overline{p})$ *and thus the inverse is given by* $f(g) = F(g)(1)$*.*

Proof. See e.g. [G12, Chapter VI, section 9] □

Proof of Theorem 3.1. We first show the isomorphism claim of Theorem 3.1. By Lemma 3.2 it suffices to show that $\mathrm{Ind}_{MA}^{\overline{P}}(\varepsilon \otimes e^{\lambda}) \simeq \mathrm{Ind}_{MA}^{\overline{P}}(\varepsilon \otimes e^{\nu})$. Let $\mathrm{rest}_{\overline{N}} : \mathrm{Ind}_{MA}^{\overline{P}}(\varepsilon \otimes e^{\lambda}) \to L^2(\overline{N})$ be the restriction from $\overline{P}$ to $\overline{N}$. We let Φ be the inverse map which is given by $\Phi F(\overline{n}am) = \mathrm{sgn}(m)^{\varepsilon} a^{-\lambda-1} F(\overline{n})$. Let $\pi_{\varepsilon,\lambda}$ be the left regular representation on $\mathrm{Ind}_{MA}^{\overline{P}}(\varepsilon \otimes e^{\lambda})$ and define $\tilde{\pi}_{\varepsilon,\lambda}(g) = \mathrm{rest}_{\overline{N}} \circ \pi_{\varepsilon,\lambda}(g) \circ \Phi$. Then $\overline{P}$ acts on $L^2(\overline{N})$ via $\tilde{\pi}_{\varepsilon,\lambda}$, and the above statement reduces to showing that $\tilde{\pi}_{\varepsilon,\lambda} \cong \tilde{\pi}_{\varepsilon,\nu}$ for $\lambda, \nu \in i\mathbb{R}$.

To construct an isomorphism $H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ intertwining $\tilde{\pi}_{\varepsilon,\lambda}$ and $\tilde{\pi}_{\varepsilon,\nu}$ we note that the action of $P = \overline{N}AM$ on $f \in L^2(\overline{N})$ is given by

$$\begin{aligned} \tilde{\pi}_{\varepsilon,\lambda}(\overline{n}) f(\overline{n}') &= f(\overline{n}^{-1}\overline{n}') && \overline{n}, \overline{n}' \in \overline{N}, \\ \tilde{\pi}_{\varepsilon,\lambda}(ma) f(\overline{n}) &= \mathrm{sgn}(m)^{\varepsilon} a^{\lambda+1} f((ma)^{-1}\overline{n}(ma)), && m \in M,\ a \in A,\ \overline{n} \in \overline{N}. \end{aligned}$$

Identifying $\overline{N} \simeq \mathbb{R}$, $M \simeq \{\pm 1\}$ and $A \simeq \mathbb{R}_{>0}$, the above becomes

$$\begin{aligned} \tilde{\pi}_{\varepsilon,\lambda}(y) f(x) &= f(x-y), && x, y \in \mathbb{R}, \\ \tilde{\pi}_{\varepsilon,\lambda}(t) f(x) &= t^{\lambda+1} f(t^2 x), && x \in \mathbb{R},\ t \in \mathbb{R}_{>0}. \end{aligned}$$

Since $\overline{N}$ acts by translation any such intertwining operator H must be a translation invariant operator on $L^2(\mathbb{R})$, hence there exists some tempered distribution $u \in \mathcal{S}'(\mathbb{R})$ such that H is given by convolution with u, that is $HF(x) = \langle u, \tau_x \check{F} \rangle$, where $\tau_x F(y) = F(y-x)$ and $\check{F}(x) = F(-x)$. Furthermore let $d_t f$ denote the dilation of f by $t \in \mathbb{R} \setminus \{0\}$, i.e. $d_t f(x) = f(tx)$, then

$$H \circ \tilde{\pi}_{\varepsilon,\lambda}(t) F(x) = \tilde{\pi}_{\varepsilon,\nu}(t) \circ HF(x).$$

Evaluating at $x = 0$ then yields $\langle d_{t^{-2}} u, \check{F} \rangle = t^{\nu-\lambda+2} \langle u, \check{F} \rangle$. Hence u is a homogeneous distribution of degree $\frac{\lambda-\nu-2}{2}$ and we conclude that

$$Hf = f * |x|_{\delta}^{\frac{\lambda-\nu-2}{2}} \qquad \text{for some } \delta \in \{0,1\}.$$

Comparing with Lemma B.2 we see that these are both necessary and sufficient conditions for H to establish an isomorphism between $\mathrm{Ind}_{MA}^{P}(\varepsilon \otimes e^{\lambda})$ and $\mathrm{Ind}_{MA}^{P}(\varepsilon \otimes e^{\nu})$. Putting $\delta = 0$, composing with the map from Lemma 3.2 then yields the desired isomorphism. To see that the normalization indeed makes $☥_{\lambda}^{\nu}$ unitary, it suffices to note that for $\lambda, \nu \in i\mathbb{R}$ Lemma B.2 gives

$$\left| \frac{1}{\sqrt{\pi} 2^{\frac{\lambda-\nu}{4}}} \frac{\Gamma\left(\frac{2+\nu-\lambda}{4}\right)}{\Gamma\left(\frac{\lambda-\nu}{4}\right)} \mathcal{F}(|x|^{\frac{\lambda-\nu}{2}-1}) \right| = 1.$$

□

4 Eigenfunctions for the Casimir operator

By Theorem 3.1 a Plancherel formula on $\mathrm{Ind}_{H}^{G}(\varepsilon \otimes e^{\lambda})$ for some fixed $\lambda \in i\mathbb{R}$ can be extended to all $\nu \in i\mathbb{R}$ by compositon with the unitary isomorphism $☥_{\lambda}^{\nu}$ from Theorem 3.1. Following this we will therefore mostly consider the cases of which $\lambda = 0$, which often simplifies matters considerably.

For $f \in \mathrm{Ind}_{H}^{G}(\varepsilon \otimes e^{0})_m$ with $f(k_\theta b_u) = e^{im\theta} \cdot h(u),\ h \in C_c^\infty(\mathbb{R})$ we have

$$\Delta_0 f = e^{im\theta} \tilde{\Delta}_m h(u),$$

for some differential operator $\tilde{\Delta}_m$.

Lemma 4.1. *Let $h \in C_c^\infty(\mathbb{R})$ and $m \in \mathbb{Z}$. Then we have*

$$\tilde{\Delta}_m \cosh^{\frac{m}{2}}(2u) h\big(-\sinh^2(2u)\big) = \cosh^{\frac{m}{2}}(2u) \square_m h\big(-\sinh^2(2u)\big).$$

For $t = -\sinh^2(2u)$ the operator $\square_m$ is given by

$$\square_m = m(m+2) + 8(-1 + (3+m)\,t)\frac{d}{dt} - 16t \cdot (1-t)\frac{d^2}{dt^2}.$$

Proof. Follows directly from Proposition 2.1. □

Recall that a hypergeometric differential equation has the form

$$t(1-t)\frac{d^2}{dt^2} + [c - (a+b+1)t]\frac{d}{dt} - ab = 0.$$

If c is not a non-positive integer there are two independent solutions (around $t = 0$)

$${}_2F_1(a, b; c; t) \quad \text{and} \quad t^{1-c}\ {}_2F_1(1 + a - c, 1 + b - c; 2 - c; t),$$

expressed in terms of the hypergeometric function ${}_2F_1(a, b; c; t)$.

We note that the eigenvalue problem $\square_m f = (\mu^2 - 1)f$ is a hypergeometric differential equation thus giving us two linearly independent solutions φ_μ^m and ψ_μ^m. Using the notation from appendix C we can express these solutions as

$$\varphi_\mu^m(u) = \phi_{\frac{-\mu}{2}}^{-\frac{1}{2}, \frac{m}{2}}(u), \quad \psi_\mu^m(u) = i \sinh(u) \cdot \phi_{\frac{-\mu}{2}}^{\frac{1}{2}, \frac{m}{2}}(u),$$

where $u \in [0, \infty)$. Note that these functions allow for natural extensions from $[0, \infty)$ to $\mathbb{R}$. We now restate the results from Appendix C in terms of φ_μ^m and ψ_μ^m.

Proposition 4.2. *For $f \in C_c^\infty([0,\infty))$, let $(J_j f)(\mu)$, $j=0,1$, denote the Fourier–Jacobi transforms of f given by*

$$J_0 f(\mu) = \int_0^\infty f(t)\varphi_\mu^m(t)\cosh^{m+1}(t)\,dt,$$
$$J_1 f(\mu) = \int_0^\infty f(t)\psi_\mu^m(t)\sinh(t)\cosh^{m+1}(t)\,dt.$$

Then we have the following inversion formulas

$$f(t) = \frac{1}{8\pi^2}\int_{i\mathbb{R}} J_0 f(\mu)\varphi_\mu^m(t)\frac{d\mu}{|\ell_0(\mu)|^2} - \frac{1}{2\pi}\sum_{\mu\in D_0} J_0 f(\mu)\varphi_\mu^m(t)\operatorname*{Res}_{\nu=\mu}(\ell_0(\nu)\ell_0(-\nu))^{-1}$$
$$\sinh(t)f(t) = \frac{-1}{2\pi^2}\int_{i\mathbb{R}} J_1 f(\mu)\psi_\mu^m(t)\frac{d\mu}{|\ell_1(\mu)|^2} + \frac{2}{\pi}\sum_{\mu\in D_1} J_1 f(\mu)\psi_\mu^m(t)\operatorname*{Res}_{\nu=\mu}(\ell_1(\nu)\ell_1(-\nu))^{-1}.$$

with $D_j = \{\eta\in\mathbb{R} \mid \eta = 4k+1+2j-|m| < 0,\ k\in\mathbb{N}_0\}$ and

$$\ell_j(\mu) = \frac{\Gamma(\frac{\mu}{2})}{\Gamma(\frac{\mu+1+2j+|m|}{4})\Gamma(\frac{\mu+1+2j-|m|}{4})}.$$

Remark 4.3. *As φ_μ^m and ψ_μ^m are given in terms of hypergeometric functions we get*

$$\varphi_\mu^m = \varphi_{-\mu}^m, \quad \text{and} \quad \psi_\mu^m = \psi_{-\mu}^m,$$

as ${}_2F_1(a,b;c;t) = {}_2F_1(b,a;c;t)$. The Euler transformation ${}_2F_1(a,b;c;t) = (1-t)^{c-a-b}\ {}_2F_1(c-a,c-b;c;t)$ amounts to

$$\varphi_\mu^m(u) = \cosh^{-m}(u)\varphi_\mu^{-m}(u), \quad \text{and} \quad \psi_\mu^m(u) = \cosh^{-m}(u)\psi_\mu^{-m}(u).$$

5 Intertwining operators

To obtain an explicit Plancherel formula for representation theoretic purposes, we require expressions for intertwining operators between the representation spaces introduced in earlier sections. More explicitly we consider intertwining operators

$$P : \operatorname{Ind}_P^G(\delta\otimes e^\mu\otimes 1) \to \operatorname{Ind}_H^G(\varepsilon\otimes e^\lambda)^\infty,$$
$$A : C_c^\infty\text{–}\operatorname{Ind}_H^G(\varepsilon\otimes e^\lambda) \to \operatorname{Ind}_P^G(\delta\otimes e^\mu\otimes 1)$$

and their realizations in terms of the coordinates introduced in earlier sections. Such operators only exist when $\varepsilon=\delta$ as M lies in the center of G.

We fix $\varepsilon\in\mathbb{Z}/2\mathbb{Z}$ and supress it in the notation for the rest of this section. When ε appear in formulas we will consider it as number in $\{0,1\}$ where we will use the notation $[\,\cdot\,]_2$ when confusions can occur.

For $\xi\in\mathbb{Z}/2\mathbb{Z}$ and $\lambda\in i\mathbb{R}$ consider the kernel

$$K_{\lambda,\mu}^\xi(g) = |g_{11}|_{\xi+\varepsilon}^{\frac{\lambda+\mu-1}{2}}|g_{21}|_\xi^{\frac{\mu-\lambda-1}{2}}, \quad g\in G,$$

where g_{ij} is the (i,j)'th entry in G. As g_{11} and g_{21} does not simultaneously vanish in G this kernel enjoys many of the similar properties as a Riesz distribution (see Appendix B). $\Gamma(\frac{\mu+1}{2})^{-1}K_{\lambda,\mu}^\xi$ is locally integrable for $\operatorname{Re}(\mu) > -1$ and admits a holomorphic continuation as a distribution to $\mu\in\mathbb{C}$.

Proposition 5.1. *The map given by*

$$P^{\xi}_{\lambda,\mu} f(g) = \int_K K^{\xi}_{\lambda,\mu}(g^{-1}k) f(k)\, dk,$$

defines an intertwining operator $\mathrm{Ind}_P^G(\varepsilon \otimes e^{\mu} \otimes 1) \to \mathrm{Ind}_H^G(\varepsilon \otimes e^{\lambda})^{\infty}$. *Similarly the map given by*

$$A^{\xi}_{\lambda,\mu} f(g) = \int_{G/H} K^{\xi}_{-\lambda,-\mu}(x^{-1}g) f(x)\, d(xH),$$

defines an intertwining operator C_c^{∞}*–*$\mathrm{Ind}_H^G(\varepsilon \otimes e^{\lambda}) \to \mathrm{Ind}_P^G(\varepsilon \otimes e^{\mu} \otimes 1)$. *Both integrals should be understood in the distributional sense.*

Proof. The equivariance properties follows by direct verification. □

Proposition 5.2. *For* $\xi, \varepsilon \in \mathbb{Z}/2\mathbb{Z}$ *we have the following relation*

$$T^{\varepsilon}_{\mu} \circ A^{\xi}_{\lambda,\mu} = d^{\xi}_{\lambda,\mu} A^{\xi+\varepsilon}_{\lambda,-\mu},$$

where

$$d^{\xi}_{\lambda,\mu} = (-1)^{\lfloor \frac{\varepsilon+\xi}{2} \rfloor} \sqrt{\pi} \frac{\Gamma(\frac{1-\lambda-\mu+2[\xi+\varepsilon]_2}{4})\Gamma(\frac{1+\lambda-\mu+2\xi}{4})}{\Gamma(\frac{1+\lambda+\mu+2[\xi+\varepsilon]_2}{4})\Gamma(\frac{1-\lambda+\mu+2\xi}{4})\Gamma(\frac{1-\mu+\varepsilon}{2})}.$$

Proof. Fix $g \in G$ and put $z = x^{-1}g$. Then the set $\{xH \in G/H \mid z_{11}z_{21} = 0\}$ is a $d(xH)$-null set. Using Lemma B.2 for $0 < \mathrm{Re}\,(\mu) < 1$ we get

$$\begin{aligned}\int_N K^{\xi}_{-\lambda,-\mu}(zw_0\overline{n}) d\overline{n} &= |z_{11}|_{\xi+\varepsilon}^{\frac{-\mu-\lambda-1}{2}} |z_{21}|_{\xi}^{\frac{\lambda-\mu-1}{2}} \int_{\mathbb{R}} |x|_{\xi+\varepsilon}^{\frac{-\lambda-\mu-1}{2}} |x - \frac{1}{z_{11}z_{21}}|_{\xi}^{\frac{\lambda-\mu-1}{2}} dx \\ &= \Gamma(\frac{\mu+\varepsilon}{2}) d^{\xi}_{\lambda,\mu} K^{\xi+\varepsilon}_{-\lambda,\mu}(z), \quad a.e.\end{aligned}$$

The claim then follows by analytic continuation. □

We introduce the notation

$$\omega^{\xi}_m = (-1)^{\xi} + (-1)^m i^m.$$

Note that $\omega^{\xi}_{-m} = \overline{\omega}^{\xi}_m$ and as $\varepsilon \equiv m \mod 2$ we have

$$\omega^{\xi}_m = \begin{cases} (-1)^{\xi} + (-1)^{\frac{m}{2}}, & \varepsilon = 0, \\ (-1)^{\xi} + i(-1)^{\frac{m+1}{2}}, & \varepsilon = 1, \end{cases} \quad \text{and} \quad \omega^0_m \overline{\omega}^1_m = \begin{cases} 0, & \varepsilon = 0, \\ 2i(-1)^{\frac{m-1}{2}}, & \varepsilon = 1. \end{cases}$$

Proposition 5.3. *For* $\mu \in \mathbb{C}$ *and* $m \in \mathbb{Z}$ *we have*

$$\frac{1}{\Gamma(\frac{\mu+1}{2})} P^{\xi}_{\mu} \zeta_m(k_{\theta} b_u) = \zeta_m(k_{\theta}) \cosh^{\frac{m}{2}}(2u) \left(\omega^{\xi}_m c_m(\mu) \varphi^m_{\mu}(2u) + \frac{i}{2} \omega^{\xi+1}_m c_m(\mu-2) \psi^m_{\mu}(2u) \right),$$

where

$$c_m(\mu) = \frac{2^{1-\mu} \pi e^{i\frac{m\pi}{4}}}{\Gamma(\frac{\mu+3+|m|}{4})\Gamma(\frac{\mu+3-|m|}{4})}.$$

Proof. As P^ξ_μ intertwines $\pi_{\varepsilon,\mu}$ and $\tau_{\varepsilon,0}$ it also intertwines the derived representations $d\pi_{\varepsilon,\mu}$ and $d\pi_{\varepsilon,0}$. Hence P^ξ_μ intertwines Δ_0 and Δ_μ and therefore the image of P^ξ_μ is contained in the eigenspace of Δ_0 to the eigenvalue μ^2-1 by Proposition 1.1. Fix μ with $\mathrm{Re}(\mu)>1$. From Lemma 4.1 it follows for generic μ that

$$P^\xi_\mu\zeta_m(k_\theta b_u)=\cosh^{\frac{m}{2}}(2u)\zeta_m(k_\theta)(a^\xi_m(\mu)\cdot\varphi^m_\mu(2u)+b^\xi_m(\mu)\cdot\psi^m_\mu(2u))$$

for some $a^\xi_m(\mu),b^\xi_m(\mu)\in\mathbb{C}$. Hence, it only remains to compute $a^\xi_m(\mu)$ and $b^\xi_m(\mu)$. Note that $\varphi^m_\mu(0)=1$ and $\psi^m_\mu(0)=0$ and hence $P^\xi_\mu\zeta_m(k_\theta)=\zeta_m(k_\theta)a^\xi_m(\mu)$ so

$$\begin{aligned}a^\xi_m(\mu)=P^\xi_\mu\zeta_m(e)&=\int_K K^\xi_\mu(k_\theta)\zeta_m(k_\theta)dk_\theta=2\int_0^\pi|\cos\theta|^{\frac{\mu-1}{2}}_{\xi+\varepsilon}|-\sin\theta|^{\frac{\mu-1}{2}}_{\xi}e^{im\theta}\,d\theta\\&=2^{\frac{1-\mu}{2}}\omega^\xi_m\int_0^\pi(\sin\theta)^{\frac{\mu-1}{2}}e^{i\frac{m}{2}\theta}d\theta=\frac{2^{1-\mu}\pi\omega^\xi_m e^{i\frac{m\pi}{4}}\Gamma(\frac{\mu+1}{2})}{\Gamma(\frac{\mu+3+|m|}{4})\Gamma(\frac{\mu+3-|m|}{4})},\end{aligned}$$

by Lemma 4.

To compute $b^\xi_m(\mu)$ it suffices to note that $\frac{d}{du}\varphi^m_\mu(2u)\mid_{u=0}=0$ and $\frac{d}{du}\psi^m_\mu(2u)\mid_{u=0}=2i$, hence

$$\begin{aligned}2i\cdot b^\xi_m(\mu)&=\frac{d}{du}P^\xi_\mu\zeta_m(b_u)\big|_{u=0}=\int_K\frac{d}{du}K^\xi_\mu(b_{-u}k_\theta)\big|_{u=0}\zeta_m(k_\theta)dk_\theta\\&=\frac{1-\mu}{2}\int_K K^{\xi+1}_{\mu-2}(k_\theta)\zeta_m(k_\theta)dk_\theta\\&=\frac{1-\mu}{2}a^{\xi+1}_m(\mu-2),\end{aligned}$$

from which the result follows by analytic continuation. □

Let $f\in\mathrm{Ind}^G_H(\varepsilon\otimes e^0)_m$ and write $f(k_\theta b_u)=\zeta_m(k_\theta)h(u)$ for some $h\in L^2(\mathbb{R},\cosh(2u)du)$. Now let $h_e(u)$ be the even part of h and $h_o(u)$ the odd part. We introduce the following notation

$$\begin{aligned}J_0f(\mu,\theta)&=\zeta_m(k_\theta)J_0(\cosh^{-\frac{m}{2}}(x)h_e(b_{\frac{x}{2}}))(\mu),\\J_1f(\mu,\theta)&=\zeta_m(k_\theta)J_1(\sinh^{-1}(x)\cosh^{-\frac{m}{2}}(x)h_o(b_{\frac{x}{2}}))(\mu)\end{aligned}$$

where the x denotes the variable the Fourier–Jacobi transform is done with respect to.

Proposition 5.4. *Let $f\in\mathrm{Ind}^G_H(\varepsilon\otimes e^0)_m$ then*

$$\frac{1}{\Gamma(\frac{1-\mu}{2})}A^\xi_\mu f(k_\theta)=\frac{\overline{\omega}^\xi_m}{2}c_{-m}(-\mu)J_0f(\mu,\theta)+i\frac{\overline{\omega}^{\xi+1}_m}{4}c_{-m}(-\mu-2)J_1f(\mu,\theta).$$

Proof. As A^ξ_μ is an intertwining operator, it maps K-types to K-types, thus $A^\xi_\mu f(k_\theta)=A^\xi_\mu f(e)\times\zeta_m(k_\theta)$. Now

$$A^\xi_\mu f(e)=\int_0^\pi\int_{\mathbb{R}}K^\xi_{-\mu}(b_u^{-1}k_\theta^{-1})f(k_\theta b_u)\cosh(2u)dud\theta=\frac{1}{2}\int_{\mathbb{R}}\cosh(2u)h(u)P^\xi_{-\mu}\zeta_{-m}(b_u)du.$$

From Proposition 5.3 and Remark 4.3 we have

$$\begin{aligned}\frac{A^{\xi}_{\mu}f(e)}{\Gamma(\frac{1-\mu}{2})} &= \frac{1}{2}\int_{\mathbb{R}} h(u)\cosh^{-\frac{m}{2}+1}(2u)\Big(\omega^{\xi}_{-m}c_{-m}(-\mu)\varphi^{-m}_{-\mu}(2u) + \frac{i}{2}\omega^{\xi+1}_{-m}c_{-m}(-\mu-2)\psi^{-m}_{-\mu}(2u)\Big)du \\ &= \int_0^\infty h_e(u)\cosh^{\frac{m}{2}+1}(2u)\overline{\omega}^{\xi}_{m}c_{-m}(-\mu)\varphi^{m}_{\mu}(2u)du \\ &\quad + \frac{i}{2}\int_0^\infty h_o(u)\cosh^{\frac{m}{2}+1}(2u)\overline{\omega}^{\xi+1}_{m}c_{-m}(-\mu-2)\psi^{m}_{\mu}(2u)du \\ &= \frac{1}{2}\overline{\omega}^{\xi}_{m}c_{-m}(-\mu)(J_0\cosh^{-\frac{m}{2}}(u)h_e(\tfrac{x}{2}))(\mu) \\ &\quad + \frac{i}{4}\overline{\omega}^{\xi+1}_{m}c_{-m}(-\mu-2)(J_1\sinh^{-1}(x)\cosh^{-\frac{m}{2}}(x)h_o(\tfrac{x}{2}))(\mu). \qquad \square\end{aligned}$$

Combining Proposition 5.3 and Proposition 5.4 yields an explicit intertwining operator

$$\frac{P^{\xi}_{\mu}A^{\xi'}_{\mu}}{\Gamma(\frac{1+\mu}{2})\Gamma(\frac{1-\mu}{2})} : C^{\infty}_{c}\text{--}\mathrm{Ind}^{G}_{H}(\varepsilon\otimes e^0)_m \to \mathrm{Ind}^{G}_{H}(\varepsilon\otimes e^0)^{\infty}_m.$$

By Propositions 5.3 and 5.4, the above intertwining operator is holomorphic in μ, i.e. the above defines a holomorphic family of intertwining operators, intertwining $\mathrm{Ind}^{G}_{H}(\varepsilon\otimes e^0)_m$ with itself.

6 Combining intertwining operators

In this section we consider a function $f \in C^{\infty}_{c}\text{--}\mathrm{Ind}^{G}_{H}(\varepsilon\otimes e^0)_m$ and then write

$$f(k_\theta b_{\frac{u}{2}}) = \cosh^{\frac{m}{2}}(u)\Big[\big(\cosh^{-\frac{m}{2}}(u)f_e(k_\theta b_{\frac{u}{2}})\big) + \sinh(u)\big(\cosh^{-\frac{m}{2}}(u)\sinh^{-1}(u)f_o(k_\theta b_{\frac{u}{2}})\big)\Big],$$

then apply the two inversion formulas from Proposition 4.2 to each of the two terms giving

$$\begin{aligned} f(k_\theta b_{\frac{u}{2}}) = \cosh^{\frac{m}{2}}(u)\Big[&\frac{1}{\pi^2}\int_{i\mathbb{R}} J_0f(\mu,\theta)\varphi^{m}_{\mu}(u)\frac{d\mu}{8|\ell_0(\mu)|^2} - \frac{1}{\pi^2}\int_{i\mathbb{R}} J_1f(\mu,\theta)\psi^{m}_{\mu}(u)\frac{d\mu}{2|\ell_1(\mu)|^2} \\ - \frac{1}{2\pi}\sum_{\mu\in D_0} J_0f(\mu,\theta)\varphi^{m}_{\mu}(u)&\operatorname*{Res}_{\nu=\mu}\big(\ell_0(\nu)\ell_0(-\nu)\big)^{-1} + \frac{2}{\pi}\sum_{\mu\in D_1} J_1f(\mu,\theta)\psi^{m}_{\mu}(u)\operatorname*{Res}_{\nu=\mu}\big(\ell_1(\nu)\ell_1(-\nu)\big)^{-1}\Big].\end{aligned}$$

The goal is then to express this decomposition in terms of some combination of the operators $P^{\xi}_{\mu}A^{\xi'}_{\mu}f(k_\theta b_{\frac{u}{2}})$ which by a quick glance at Propositions 5.3 and 5.4 appears plausible. The following identity will be used multiple times in the following subsections

$$\begin{aligned}2^4 c_m(\nu)c_{-m}(-\nu)\ell_0(\nu)\ell_0(-\nu) &= c_m(\nu-2)c_{-m}(-\nu-2)\ell_1(\nu)\ell_1(-\nu) \\ &= \frac{2^5(-1)^{1+\varepsilon}}{\pi}\frac{\cos^2\big(\frac{\pi(\nu+\varepsilon)}{2}\big)}{\nu\sin\big(\frac{\pi\nu}{2}\big)}, \qquad (6.1)\end{aligned}$$

which follows from Gamma-function identities and recalling that $m \equiv \varepsilon \mod 2$.

6.1 The continuous part

For $\mu \in \mathbb{C}$ we introduce the following maps

$$\mathbf{P}^{\xi}_{\mu} = \frac{P^{\xi}_{\mu}}{\Gamma(\frac{\mu+1}{2})}, \quad \mathbf{A}^{\xi}_{\mu} = \frac{A^{\xi}_{\mu}}{\Gamma(\frac{1-\mu}{2})},$$

which are holomorphic in μ.

Proposition 6.1. *We have*

$$\sum_{\xi=0}^{1}\mathbf{P}_\mu^\xi\mathbf{A}_\mu^\xi f(k_\theta b_u)$$
$$=\cosh^{\frac{m}{2}}(2u)\Big[2c_m(\mu)c_{-m}(-\mu)J_0f(\mu,\theta)\varphi_\mu^m(2u)-\frac{1}{2}c_m(\mu-2)c_{-m}(-\mu-2)J_1f(\mu,\theta)\psi_\mu^m(2u)\Big].$$

Combining this with (6.1) we get

$$\int_{i\mathbb{R}}\sum_{\xi=0}^{1}\mathbf{P}_\mu^\xi\mathbf{A}_\mu^\xi f(k_\theta b_{\frac{u}{2}})\frac{d\mu}{|a(\mu)|^2}$$
$$=\cosh^{\frac{m}{2}}(u)\Big[\frac{1}{\pi^2}\int_{i\mathbb{R}}J_0f(\mu,\theta)\varphi_\mu^m(u)\frac{d\mu}{8|\ell_0(\mu)|^2}-\frac{1}{\pi^2}\int_{i\mathbb{R}}J_1f(\mu,\theta)\psi_\mu^m(u)\frac{d\mu}{2|\ell_1(\mu)|^2}\Big]$$

where

$$a(\mu)=4\pi\frac{\Gamma(\frac{\mu}{2})}{\Gamma(\frac{1+\mu+\varepsilon}{2})\Gamma(\frac{1+\mu-\varepsilon}{2})}.$$

Proof. When computing $\sum_{\xi=0}^{1}\mathbf{P}_\mu^\xi\mathbf{A}_\mu^\xi f(k_\theta b_{\frac{u}{2}})$ we apply Proposition 5.3 and 5.4. We obtain some cross-terms, containing factors like $J_0f(\mu,\theta)\psi_\mu^m(u)$, but since

$$\sum_{\xi=0}^{1}\omega_m^\xi\overline{\omega}_m^{\xi+1}=0 \text{ and } \sum_{\xi=0}^{1}\omega_m^\xi\overline{\omega}_m^\xi=4,$$

no cross-terms survive and the assertion follows. □

To express the discrete part in terms of $P_\mu^\xi A_\mu^{\xi'}f$ is a bit more delicate as we cannot simply take a sum to make the cross terms disappear thus we need to make a suitable choice of normalization. The cases for $\varepsilon=0$ and $\varepsilon=1$ will be treated differently and the main culprit as to why is the factor ω_m^ξ which for $\varepsilon=0$ vanishes depending on the parity of $\frac{m}{2}$ and for $\varepsilon=1$ never vanishes.

6.2 The discrete part for $\varepsilon=0$

In this subsection we fix $\varepsilon=0$. Consider the following normalizations

$$\widehat{P}_\mu^\xi=\frac{\Gamma(\frac{\mu+3-2\xi}{4})}{\Gamma(\frac{\mu+1}{2})\Gamma(\frac{\mu+1+2\xi}{4})}P_\mu^\xi,\quad \widehat{A}_\mu^\xi=\frac{\Gamma(\frac{-\mu+3-2\xi}{4})}{\Gamma(\frac{-\mu+1}{2})\Gamma(\frac{-\mu+1+2\xi}{4})}A_\mu^\xi,$$

which, by the duplication formula for the Gamma-function, does not introduce any poles. Now introduce the operators

$$\mathbb{P}_\mu:=\widehat{P}_\mu^0+\widehat{P}_\mu^1 \quad\text{and}\quad \mathbb{A}_\mu:=\widehat{A}_\mu^0+\widehat{A}_\mu^1.$$

Lemma 6.2. *For a fixed $m\in2\mathbb{Z}$ we have*

$$\mathbb{P}_\mu\zeta_m(k_\theta b_u)=\zeta_m(k_\theta)\cosh^{\frac{m}{2}}(2u)\big(\alpha_m(\mu)\varphi_\mu^m(2u)+\beta_m(\mu)\psi_\mu^m(2u)\big),$$

where

$$\alpha_m(\mu) = c_m(\mu)\left(\omega_m^0\frac{\Gamma(\frac{\mu+3}{4})}{\Gamma(\frac{\mu+1}{4})} + \omega_m^1\frac{\Gamma(\frac{\mu+1}{4})}{\Gamma(\frac{\mu+3}{4})}\right),$$

and

$$\beta_m(\mu) = \frac{i}{2}c_m(\mu-2)\left(\omega_m^1\frac{\Gamma(\frac{\mu+3}{4})}{\Gamma(\frac{\mu+1}{4})} + \omega_m^0\frac{\Gamma(\frac{\mu+1}{4})}{\Gamma(\frac{\mu+3}{4})}\right).$$

Furthermore if $\mu \in 1-2\mathbb{N}$ then $\alpha_m(\mu)$ is only non-zero when $\mu \in D_0$ and $\beta_m(\mu)$ is only non-zero when $\mu \in D_1$.

Proof. The first identity is a direct consequence of Proposition 5.3. To see the second assertion rewrite

$$\alpha_m(\mu) = \frac{2^{1-\mu}\pi e^{\frac{\pi i m}{4}}}{\Gamma(\frac{\mu+3+|m|}{4})}\left(\omega_m^0\frac{(\frac{\mu+3-|m|}{4})_{\frac{|m|}{4}}}{\Gamma(\frac{\mu+1}{4})} + \omega_m^1\frac{(\frac{\mu+3-|m|}{4})_{\frac{|m|-2}{4}}}{\Gamma(\frac{\mu+3}{4})}\right).$$

As either ω_m^0 or ω_m^1 is vanishing this makes sense term by term. When μ is of the form $\mu = 4k+3-|m|$ for $k \in \mathbb{Z}$ then term by term $\Gamma(\frac{\mu+1}{4})^{-1}$ and $\Gamma(\frac{\mu+3}{4})^{-1}$ vanishes. When μ has the form $\mu = 4k+1-|m|$ for $k \in -\mathbb{N}$ then $\Gamma(\frac{\mu+3+|m|}{4})^{-1}$ vanishes. A similarly argument applies to $\beta_m(\mu)$. □

Lemma 6.3. *We have*

$$\mathbb{A}_\mu f(k_\theta) = \tilde{\alpha}_m(\mu)J_0 f(\mu,\theta) + \tilde{\beta}_m(\mu)J_1 f(\mu,\theta),$$

where

$$\tilde{\alpha}_m(\mu) = \frac{1}{2}c_{-m}(-\mu)\left(\overline{\omega}_m^0\frac{\Gamma(\frac{3-\mu}{4})}{\Gamma(\frac{1-\mu}{4})} + \overline{\omega}_m^1\frac{\Gamma(\frac{1-\mu}{4})}{\Gamma(\frac{3-\mu}{4})}\right),$$

$$\tilde{\beta}_m(\mu) = \frac{i}{4}c_{-m}(-\mu-2)\left(\overline{\omega}_m^1\frac{\Gamma(\frac{-\mu+3}{4})}{\Gamma(\frac{-\mu+1}{4})} + \overline{\omega}_m^0\frac{\Gamma(\frac{-\mu+1}{4})}{\Gamma(\frac{-\mu+3}{4})}\right).$$

Furthermore, if $\mu \in 1-2\mathbb{N}$ of the form $\mu = 4k+1-|m|$ we have $\tilde{\beta}(\mu) = 0$ and similarly for μ of the form $\mu = 4k+3-|m|$ we have $\tilde{\alpha}_m(\mu) = 0$.

Proof. This follows from Proposition 5.4 and considerations similar to those in the proof of Lemma 6.2. □

Lemma 6.4. *For $\mu \in D_0$:*

$$\mathbb{P}_\mu\mathbb{A}_\mu f(k_\theta b_u) = \cosh^{\frac{m}{2}}(2u)\alpha_m(\mu)\tilde{\alpha}_m(\mu)J_0 f(\mu,\theta)\varphi_\mu^m(2u),$$

and for $\mu \in D_1$:

$$\mathbb{P}_\mu\mathbb{A}_\mu f(k_\theta b_u) = \cosh^{\frac{m}{2}}(2u)\beta_m(\mu)\tilde{\beta}_m(\mu)J_1 f(\mu,\theta)\psi_\mu^m(2u).$$

Furthermore, if $\mu \in (1-2\mathbb{N}) \setminus (D_0 \cup D_1)$ then $\mathbb{P}_\mu\mathbb{A}_\mu f(k_\theta b_u) = 0$.

Proof. This is a direct consequence of the two preceding lemmas. □

Consider the non-vanishing entire analytic function

$$\mho(\mu)=\frac{1}{\Gamma(\frac{1+\mu}{4})^2\Gamma(\frac{1-\mu}{4})^2}+\frac{1}{\Gamma(\frac{3+\mu}{4})^2\Gamma(\frac{3-\mu}{4})^2}.$$

Lemma 6.5. *For $\mu\in D_0$*

$$(-2\pi)\alpha_m(\mu)\widetilde{\alpha}_m(\mu)\ell_0(\mu)\ell_0(-\mu)=16\pi^2\frac{\cot\left(\frac{\pi\mu}{2}\right)\mho(\mu)}{\mu}.$$

For $\mu\in D_1$

$$\frac{\pi}{2}\beta_m(\mu)\widetilde{\beta}_m(\mu)\ell_1(\mu)\ell_1(-\mu)=16\pi^2\frac{\cot\left(\frac{\pi\mu}{2}\right)\mho(\mu)}{\mu}.$$

Proof. This follows from (6.1). One trick is used which arises when a term like

$$\frac{|\omega_m^0|^2}{\Gamma(\frac{\mu+1}{2})^2},$$

is obtained. As ω_m^0 is either 0 or 2 we can set $\omega_m^0=2$ as $\Gamma(\frac{\mu+1}{2})^{-1}$ vanishes in the same cases as ω_m^0. □

Proposition 6.6. *We have*

$$\frac{1}{(2\pi)^3}\sum_{\mu\in 1-2\mathbb{N}}\frac{-\mu}{\mho(\mu)}\mathbb{P}_\mu\mathbb{A}_\mu f(k_\theta b_u)=\cosh^{\frac{m}{2}}(2u)\Bigg[\frac{-1}{2\pi}\sum_{\mu\in D_0}J_0f(\mu,\theta)\varphi_\mu^m(2u)\operatorname*{Res}_{\nu=\mu}(\ell_0(\nu)\ell_0(-\nu))^{-1}$$
$$+\frac{2}{\pi}\sum_{\mu\in D_1}J_1f(\mu,\theta)\psi_\mu^m(2u)\operatorname*{Res}_{\nu=\mu}(\ell_1(\nu)\ell_1(-\nu))^{-1}\Bigg].$$

Proof. Apply Lemma 6.4 to the right hand side. Now note that $c_m(\mu)c_{-m}(-\mu)$ is regular for $\mu\in D_0$ and $c_m(\mu-2)c_{-m}(-\mu-2)$ is regular for $\mu\in D_1$, thus they can be moved inside the residues. Then everything follows from Lemma 6.5 after recalling that $\mathrm{Res}_{\nu=\mu}\tan(\frac{\pi\nu}{2})=-\frac{2}{\pi}$. □

6.3 The discrete part for $\varepsilon=1$

In this subsection we fix $\varepsilon=1$. The proof will proceed using the same ideas as for $\varepsilon=0$. For $\mu\in-2\mathbb{N}$ let

$$\mathcal{A}_\mu=\frac{1}{\Gamma(\frac{1-\mu}{2})}A_\mu^0,\quad\text{and}\quad\mathcal{P}_\mu\zeta_m=\frac{(-1)^{\frac{m+|m|-2}{2}}}{\Gamma(\frac{1+\mu}{2})}P_\mu^1\zeta_m,$$

that is we define $\mathcal{P}_\mu$ by its eigenvalues on K-types. By Proposition 5.3 we get $\mathcal{P}_\mu$ is intertwining by the same argument we used for $\hat{T}_\mu^1$ in section 1.

Lemma 6.7. *For $\mu\in D_0$ we have*

$$\mathcal{P}_\mu\mathcal{A}_\mu f(k_\theta b_u)=\alpha_m(\mu)\cosh^{\frac{m}{2}}(2u)\varphi_\mu^m(2u)J_0f(\mu,\theta),$$

where

$$\alpha_m(\mu)=i(-1)^{\frac{|m|+1}{2}}c_m(\mu)c_{-m}(-\mu)$$

For $\mu\in D_1$ we have

$$\mathcal{P}_\mu\mathcal{A}_\mu f(k_\theta b_u)=\beta_m(\mu)\cosh^{\frac{m}{2}}(2u)\psi_\mu^m(2u)J_1f(\mu,\theta),$$

where

$$\beta_m(\mu) = \frac{1}{4} i(-1)^{\frac{|m|+1}{2}} c_m(\mu-2)c_{-m}(-\mu-2).$$

Furthermore if $\mu \in -2\mathbb{N}$ then $\alpha_m(\mu)$ is only non-zero if $\mu \in D_0$ and $\beta_m(\mu)$ is only non-zero if $\mu \in D_1$.

Proof. The proof is an application of Propositions 5.3 and 5.4. □

Lemma 6.8. *For $\mu \in D_0$*

$$(-2\pi)\alpha_m(\mu)\ell_0(\mu)\ell_0(-\mu) = 4i(-1)^{\frac{|m|-1}{2}} \frac{\sin\left(\frac{\pi\mu}{2}\right)}{\mu},$$

and for $\mu \in D_1$

$$\frac{\pi}{2}\beta_m(\mu)\ell_1(\mu)\ell_1(-\mu) = 4i(-1)^{\frac{|m|+1}{2}} \frac{\sin\left(\frac{\pi\mu}{2}\right)}{\mu}.$$

Proof. This is a direct consequence of (6.1). □

Proposition 6.9. *We have*

$$\begin{aligned}\frac{1}{2\pi i}\sum_{\mu\in -2\mathbb{N}} \mu\mathcal{P}_\mu\mathcal{A}_\mu f(k_\theta b_u) = \cosh^{\frac{m}{2}}(2u)\Big[&-\frac{1}{2\pi}\sum_{\mu\in D_0} J_0 f(\mu,\theta)\varphi_\mu^m(2u)\operatorname*{Res}_{\nu=\mu}\big(\ell_0(\nu)\ell_0(-\nu)\big)^{-1} \\ &+\frac{2}{\pi}\sum_{\mu\in D_1} J_1 f(\mu,\theta)\psi_\mu^m(2u)\operatorname*{Res}_{\nu=\mu}\big(\ell_1(\nu)\ell_1(-\nu)\big)^{-1}\Big].\end{aligned}$$

Proof. This follows in the same manner as the proof for Proposition 6.6, where we here note for $\mu = 4k+1-|m| \in D_0$ that $\operatorname{Res}_{\nu=\mu}\sin\left(\frac{\pi\nu}{2}\right)^{-1} = \frac{2}{\pi}(-1)^{\frac{|m|-1}{2}}$, and for $\mu \in D_1$ we have $\operatorname{Res}_{\nu=\mu}\sin\left(\frac{\pi\nu}{2}\right)^{-1} = \frac{2}{\pi}(-1)^{\frac{|m|+1}{2}}$. □

7 The Plancherel formula

The intertwining operators P_μ^ξ and A_μ^ξ are continuous maps and hence the intertwining operators introduced in the previous section are also continuous. This allows for an extension of the results obtained for K-types, described by the first theorem of this section. We then extend this theorem to arbitrary $\lambda \in i\mathbb{R}$ by virtue of Theorem 3.1

Recall that

$$a(\mu) = 4\pi\frac{\Gamma(\frac{\mu}{2})}{\Gamma(\frac{1+\mu+\varepsilon}{2})\Gamma(\frac{1+\mu-\varepsilon}{2})}, \quad \text{and} \quad \mathfrak{F}(\mu) = \frac{1}{\Gamma(\frac{1+\mu}{4})^2\Gamma(\frac{1-\mu}{4})^2} + \frac{1}{\Gamma(\frac{3+\mu}{4})^2\Gamma(\frac{3-\mu}{4})^2}. \tag{7.1}$$

Theorem 7.1 (Plancherel formula for $\lambda = 0$)**.** *For $\varepsilon = 0$ and $f \in C_c^\infty\text{–}\operatorname{Ind}_H^G(\varepsilon\otimes e^0)$ we have the following inversion formula*

$$f(k_\theta b_u) = \int_{i\mathbb{R}}\sum_{\xi=0}^{1}\boldsymbol{P}_\mu^\xi\boldsymbol{A}_\mu^\xi f(k_\theta b_u)\frac{d\mu}{|a(\mu)|^2} + \frac{1}{(2\pi)^3}\sum_{\mu\in 1-2\mathbb{N}}\frac{-\mu}{\mathfrak{F}(\mu)}\mathbb{P}_\mu\mathbb{A}_\mu f(k_\theta b_u),$$

and the corresponding Plancherel formula

$$\|f\|^2 = \int_{i\mathbb{R}}\sum_{\xi=0}^{1}\|\boldsymbol{A}_\mu^\xi f\|^2\frac{d\mu}{|a(\mu)|^2} + \frac{1}{(2\pi)^3}\sum_{\mu\in 1-2\mathbb{N}}\frac{2\Gamma(1-\mu)}{\Gamma(\frac{-\mu}{2})\,\mathfrak{F}(\mu)}\|\mathbb{A}_\mu f\|^2.$$

For $\varepsilon = 1$ and $f \in C_c^\infty\text{–}\mathrm{Ind}_H^G(\varepsilon \otimes e^0)$ we have the following inversion formula

$$f(k_\theta b_u) = \int_{i\mathbb{R}} \sum_{\xi=0}^{1} \boldsymbol{P}_\mu^\xi \boldsymbol{A}_\mu^\xi f(k_\theta b_u) \frac{d\mu}{|a(\mu)|^2} + \frac{1}{2\pi i} \sum_{\mu \in -2\mathbb{N}} \mu \mathcal{P}_\mu \mathcal{A}_\mu f(k_\theta b_u),$$

and the corresponding Plancherel formula

$$\|f\|^2 = \int_{i\mathbb{R}} \sum_{\xi=0}^{1} \|\boldsymbol{A}_\mu^\xi f\|^2 \frac{d\mu}{|a(\mu)|^2} + \frac{1}{2\pi} \sum_{\mu \in -2\mathbb{N}} \frac{\Gamma(-\mu)\mu^2}{\Gamma(\frac{1-\mu}{2})} \|\mathcal{A}_\mu f\|^2.$$

Proof. The inversion formulas follow directly from the introduction and results of Section 6. To get the Plancherel formula write

$$\|f\|^2 = \int_0^\pi \int_{\mathbb{R}} f(k_\theta b_u) \overline{f(k_\theta b_u)} \cosh(2u)\, du d\theta,$$

and use the inversion formula on $f(k_\theta b_u)$ and apply that

$$\int_{G/H} P_\mu^\xi f(xH) \overline{g(xH)}\, d(xH) = \int_K f(k) \overline{A_{-\overline{\mu}}^\xi g(k)}\, dk,$$

for $f \in \mathrm{Ind}_P^G(\varepsilon \otimes e^\mu)$ and $g \in \mathrm{Ind}_H^G(\varepsilon \otimes e^0)$. Lastly for the discrete part, we apply Proposition 5.2 to get

$$T_\mu^0 \mathbb{A}_\mu = \frac{\sqrt{\pi} 2^\mu}{\Gamma(\frac{1-\mu}{2})} \mathbb{A}_{-\mu} \quad \text{and} \quad T_\mu^1 \mathcal{A}_\mu = \frac{\sqrt{\pi} 2^\mu}{\Gamma(\frac{2-\mu}{2}) \Gamma(\frac{1+\mu}{2})} A_{-\mu}^1,$$

giving the final result. □

We now extend the previous result from $\lambda = 0$ to $\lambda \in i\mathbb{R}$ using Theorem 3.1. We want to compose A_μ and $☥_\lambda^0$ but as we cannot ensure the regularity of the functions in the image of $☥_\lambda^0$ we end up doing this in an L^2-sense using direct integrals. Consider the following operators

$$\mathbb{A}_{\lambda,\mu} := \frac{2^{\frac{1+\mu}{2}} \sqrt{\pi} \Gamma(\frac{1+\mu}{4} + \frac{\lambda}{4})}{\Gamma(\frac{1-\mu}{4}) \Gamma(\frac{1+\mu}{4}) \Gamma(\frac{1-\mu}{4} - \frac{\lambda}{4})} A_{\lambda,\mu}^0 + \frac{2^{\frac{1+\mu}{2}} \sqrt{\pi} \Gamma(\frac{3+\mu}{4} + \frac{\lambda}{4})}{\Gamma(\frac{3-\mu}{4}) \Gamma(\frac{3+\mu}{4}) \Gamma(\frac{3-\mu}{4} - \frac{\lambda}{4})} A_{\lambda,\mu}^1,$$

$$\mathbf{A}_{\lambda,\mu}^\xi := \frac{A_{\lambda,\mu}^\xi}{\Gamma(\frac{1-\mu}{2})}, \quad \text{and} \quad \mathcal{A}_{\lambda,\mu} := \frac{2^{\frac{1+\mu}{2}} \sqrt{\pi} \Gamma(\frac{1+\mu}{4} + \frac{\lambda}{4})}{\Gamma(\frac{1-\mu}{4}) \Gamma(\frac{3+\mu}{4}) \Gamma(\frac{1-\mu}{4} - \frac{\lambda}{4})} A_{\lambda,\mu}^0,$$

which are extensions of $\mathbb{A}_\mu$, $\mathbf{A}_\mu^\xi$ and $\mathcal{A}_\mu$ e.g. $\mathbb{A}_{0,\mu} = \mathbb{A}_\mu$. Furthermore let

$$\mathcal{H}_\varepsilon = \int_{i\mathbb{R}}^\oplus \pi_{\varepsilon,\mu} \otimes \mathbb{C}^2\, d\mu \oplus \bigoplus_{\mu \in 1-\varepsilon-2\mathbb{N}} \pi_{\varepsilon,\mu}^{\mathrm{ds}}.$$

where $\int_U^\oplus H_\mu d\mu$ denotes a direct integral of Hilbert spaces, see e.g [F22] for a short exposition. The inner-product on $\mathcal{H}_\varepsilon$ is given by Theorem 7.1, i.e. for $\varepsilon = 0$ and $f, h \in \mathcal{H}_0$

$$\langle g, h \rangle_{\mathcal{H}} = \int_{i\mathbb{R}} \sum_{\xi=0}^{1} \langle g_\mu^\xi, h_\mu^\xi \rangle_{L^2(K)} \frac{d\mu}{|a(\mu)|^2} + \frac{1}{16\pi} \sum_{\mu \in 1-2\mathbb{N}} \frac{\Gamma(1-\mu)}{\Gamma(\frac{-\mu}{2}) \boldsymbol{\mathfrak{F}}(\mu)} \langle T_\mu^0 g_\mu, h \rangle_{L^2(K)}.$$

For simplicity we shall assume that $\varepsilon = 0$ for the remainder of the section. All arguments made can be done for $\varepsilon = 1$ as well using the corresponding results from the previous section.

Abusing notation, Theorem 7.1 defines an isometry $A_0 : C_c^\infty(G/H) \to \mathcal{H}_0$ which extends to an isometry

$$A_0 : \mathrm{Ind}_H^G(\varepsilon \otimes 1) \to \mathcal{H}_0.$$

For $f \in \mathcal{H}_0$ with $f = (f^0, f^1, f^d)$ we introduce the following map

$$P_0 : \mathcal{H}_0 \to L^2(G/H)$$
$$f = (f^0, f^1, f^d) \mapsto \int_{i\mathbb{R}} \sum_{\xi=0}^{1} \mathbf{P}_\mu^\xi f_\mu^\xi \frac{d\mu}{|a(\mu)|^2} + \frac{1}{(2\pi)^3} \sum_{\mu \in 1-2\mathbb{N}} \frac{-\mu}{\boldsymbol{\sigma}(\mu)} \mathbb{P}_\mu f_\mu^d.$$

Lemma 7.2. *For $f \in \mathrm{Ind}_H^G(\varepsilon \otimes e^0)$ and $h \in \mathcal{H}_0$ we have the following relation:*

$$\langle A_0 f, h\rangle_{\mathcal{H}} = \langle f, P_0 h\rangle_{L^2(G/H)}.$$

Proof. Let $f \in \mathrm{Ind}_H^G(\varepsilon \otimes e^\lambda)$ and $h \in C_c^\infty(\mathcal{H}_0)$. We then find

$$\begin{aligned}
\langle A_0 f, h\rangle_{\mathcal{H}} &= \sum_{\xi=0}^{1} \int_{i\mathbb{R}} \int_K \mathbf{A}_\mu^\xi f(k) \overline{h_\mu^\xi(k)} dk \frac{d\mu}{|a(\mu)|^2} + \frac{1}{(2\pi)^3} \sum_{\mu \in 1-2\mathbb{N}} \frac{2\Gamma(1-\mu)}{\Gamma(\frac{-\mu}{2}) \boldsymbol{\sigma}(\mu)} \langle T_\mu^0 \circ \mathbb{A}_\mu f, h_\mu^\xi\rangle_{L^2(K)} \\
&= \sum_{\xi=0}^{1} \int_{i\mathbb{R}} \int_{G/H} f(x) \overline{\mathbf{P}_\mu^\xi h_\mu^\xi(x)} d(xH) \frac{d\mu}{|a(\mu)|^2} + \frac{1}{(2\pi)^3} \sum_{\mu \in 1-2\mathbb{N}} \frac{2\Gamma(1-\mu)}{\Gamma(\frac{-\mu}{2}) \boldsymbol{\sigma}(\mu)} \frac{\sqrt{\pi} 2^\mu}{\Gamma(\frac{1-\mu}{2})} \langle \mathbb{A}_{-\mu} f, h_\mu^d\rangle_{L^2(K)} \\
&= \sum_{\xi=0}^{1} \int_{i\mathbb{R}} \langle f, \mathbf{P}_\mu^\xi h_\mu^\xi\rangle_{L^2(G/H)} \frac{d\mu}{|a(\mu)|^2} + \frac{1}{(2\pi)^3} \sum_{\mu \in 1-2\mathbb{N}} \frac{-\mu}{\boldsymbol{\sigma}(\mu)} \langle f, \mathbb{P}_\mu h_\mu^d\rangle_{L^2(G/H)} \\
&= \langle f, P_0 h\rangle_{L^2(G/H)}.
\end{aligned}$$

□

Lemma 7.3 (See [P76, Theorem 1])**.** *Suppose $S : \mathcal{H}^\infty \to \mathcal{H}^\infty$ is a continuous intertwining operator for $\mathcal{H}^\infty$. Then for a.e $\mu \in i\mathbb{R} \cup (1-\varepsilon-2\mathbb{N})$ there exists unique $\mathcal{H}^\infty$ intertwining operators S_μ for $\pi_{\varepsilon,\mu}^\infty \otimes \mathbb{C}^2$ if $\mu \in i\mathbb{R}$ and for $\pi_{\varepsilon,\mu}^{ds}$ if $\mu \in 1-\varepsilon-2\mathbb{N}$ such that*

$$(Sf)_\mu = S_\mu f_\mu \quad a.e\ \mu \in i\mathbb{R} \cup (\mu \in 1-\varepsilon-2\mathbb{N}),\ f \in \mathcal{H}^\infty.$$

Proposition 7.4. *The map $A_0^\varepsilon : \mathrm{Ind}_H^G(\varepsilon \otimes e^0) \to \mathcal{H}_0$ is surjective. In particular A_0 is an isometric isomorphism.*

Proof. Since the discrete and continuous part of $\mathcal{H}_0$ consist of pairwise inequivalent representations of G it suffices to show that the projection of $W = \overline{\mathrm{Image}(A_0)}$ onto the continuous part and the discrete part respectively, is surjective. For the projection to the discrete part we can consider the projection of W onto each summand $\pi_{0,\mu}^{\mathrm{ds}}$. Proposition 1 gives $\pi_{0,\mu}^{\mathrm{ds}} = \pi_{0,\mu}^{\mathrm{hol}} \oplus \pi_{0,\mu}^{\mathrm{ahol}}$ and since these representations are inequivalent it again suffices to show that the projection on each of them are onto. Lemma 6.3 then shows that $\mathrm{proj}_{\pi_{0,\mu}^{\mathrm{hol}}}(W) \neq 0$ and $\mathrm{proj}_{\pi_{0,\mu}^{\mathrm{ahol}}}(W) \neq 0$. But since the projection is G-equivariant the image is a subrepresentation and it follows that both projections must be onto.

Since the projection onto an integrand of the continuous part of $\mathcal{H}_0$ is in general not pointwise defined, the proof differs to that of the discrete part. Abusing notation slightly we shall write $f_\mu = (f_\mu^0, f_\mu^1)$ when $\mu \in i\mathbb{R}$ and $f \in \mathcal{H}_0$, omitting the discrete part.

By Lemma 7.2 $(A_0)^* = P_0$ and since the adjoint is G-equivariant we have

$$P_0(\mathcal{H}_0^\infty) \subseteq L^2(G/H)^\infty \quad \text{and} \quad A_0(L^2(G/H)^\infty) \subseteq \mathcal{H}_0^\infty.$$

Let $A_0^\infty = A_0|_{L^2(G/H)^\infty}$ and $P_0^\infty = P_0|_{\mathcal{H}_0^\infty}$. Then

$$S = A_0^\infty \circ P_0^\infty : \mathcal{H}_0^\infty \to \mathcal{H}_0^\infty$$

is a $\mathcal{H}_0^\infty$ intertwining map and by Lemma 7.3 there exists a family of $\mathcal{H}_0^\infty$ intertwining operators (S_μ) such that $((A_0^\infty \circ P_0^\infty)f)_\mu = S_\mu f_\mu$ for a.e $\mu \in i\mathbb{R}$ and all $f \in \mathcal{H}_0^\infty$. By Schur's lemma this implies $S_\mu = (\mathrm{id} \otimes B_\mu)$ for a.e $\mu \in i\mathbb{R}$, with $B_\mu : \mathbb{C}^2 \to \mathbb{C}^2$ a linear map. Let N denote the corresponding null-set, we then show that for $\mu \in i\mathbb{R} \setminus N$ we have $S_\mu = \mathrm{id}$.

To this extent let $\mu \in i\mathbb{R} \setminus N$ and let f be a K-finite vector in $\mathcal{H}_0^\infty$ and note that this implies f_μ must be a K-finite vector in π_μ^∞. Assume therefore without loss of generality that $f_\mu = (c_1\zeta_m, c_2\zeta_n)$ for some $m, n \in 2\mathbb{Z}$ and pick by Proposition 5.4 a K-finite vector in $L^2(G/H)^\infty$ such that $A_0(w)_\mu = f_\mu$. One can e.g pick w of the form $w = \zeta_m f_1 + \zeta_n f_2$ with f_1 an even function and f_2 an odd function of correct regularity and growth. Then we have

$$(A_0^\infty \circ P_0^\infty f)_\mu = A_0^\infty \circ P_0^\infty \circ A_0^\infty(w)_\mu = A_0^\infty(w)_\mu = (A_\mu^0 w, A_\mu^1 w),$$

where the second equality follows from the inversion formula given by Theorem 7.1. On the other hand we have

$$(A_0^\infty \circ P_0^\infty f)_\mu = S_\mu f_\mu = (\mathrm{id} \otimes B_\mu) A_0^\infty(w)_\mu = ((b_{11})_\mu A_\mu^0 w + (b_{12})_\mu A_\mu^1 w, (b_{21})_\mu A_\mu^0 w + (b_{22})_\mu A_\mu^1 w)$$

hence

$$(A_\mu^0 w, A_\mu^1 w) = (a_\mu A_\mu^0 w + b_\mu A_\mu^1 w, c_\mu A_\mu^0 w + d_\mu A_\mu^1 w).$$

Since A_μ^0 and A_μ^1 are linearly independent for $\mu \in i\mathbb{R}$ it follows that $B_\mu = \mathrm{id}$ and hence $S_\mu = \mathrm{id}$ on the K-finite vectors of $\mathcal{H}_0$, for a.e $\mu \in i\mathbb{R}$ and since the K-finite vectors form a dense subset the result follows. □

Theorem 7.5. *For $\lambda \in i\mathbb{R}$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ we have*

$$\mathrm{Ind}_H^G(\varepsilon \otimes e^\lambda) \cong \int_{i\mathbb{R}}^{\oplus} \pi_{\varepsilon,\mu} \oplus \pi_{\varepsilon,\mu} \frac{d\mu}{|a(\mu)|^2} \oplus \bigoplus_{\mu \in 1-\varepsilon-2\mathbb{N}} \pi_{\varepsilon,\mu}^{\mathrm{hol}} \oplus \pi_{\varepsilon,\mu}^{\mathrm{ahol}},$$

where the map is given by

$$\begin{aligned} f &\mapsto (p_{\lambda,\mu}^0 \boldsymbol{A}_{\lambda,\mu}^0 f, p_{\lambda,\mu}^1 \boldsymbol{A}_{\lambda,\mu}^1 f, \mathbb{A}_{\lambda,\mu} f), && \text{for } f \in \mathrm{Ind}_H^G(0 \otimes e^\lambda) \\ f &\mapsto (p_{\lambda,\mu}^0 \boldsymbol{A}_{\lambda,\mu}^0 f, p_{\lambda,\mu}^1 \boldsymbol{A}_{\lambda,\mu}^1 f, \mathcal{A}_{\lambda,\mu} f), && \text{for } f \in \mathrm{Ind}_H^G(1 \otimes e^\lambda) \end{aligned}$$

with

$$p_{\lambda,\mu}^\xi = \frac{\Gamma(\frac{1-\mu+2\xi}{4})\Gamma(\frac{\mu-\lambda+1+2\xi}{4})}{2^{\frac{\lambda}{4}}\Gamma(\frac{1+\mu+2\xi}{4})\Gamma(\frac{\lambda-\mu+1+2\xi}{4})}.$$

Proof. For $\lambda \in i\mathbb{R}$ the map $A_0 : \mathrm{Ind}_H^G(\varepsilon \otimes 1) \to \mathcal{H}_0$ gives rise to an isometric isomorphism $\mathrm{Ind}_H^G(\varepsilon \otimes e^\lambda) \to \mathcal{H}_0$ by composition with $\dagger_\lambda^0 : \mathrm{Ind}_H^G(\varepsilon \otimes e^\lambda) \to \mathrm{Ind}_H^G(\varepsilon \otimes e^0)$ from Theorem 3.1. Let $\lambda \in i\mathbb{R}$, $f \in \mathrm{Ind}_H^G(\varepsilon \otimes e^\lambda)$ and $h \in C_c^\infty(\mathcal{H}_0)$. Then by Lemma 7.2 we have

$$\begin{aligned} &\langle A_0 \circ \dagger_\lambda^0 f, h\rangle_{\mathcal{H}} = \langle \dagger_\lambda^0 f, P_0 h\rangle_{L^2(G/H)} \\ &= \sum_{\xi=0}^{1} \int_{i\mathbb{R}} \langle \dagger_\lambda^0 f, \mathbf{P}_\mu^\xi h_\mu^\xi\rangle_{L^2(G/H)} \frac{d\mu}{|a(\mu)|^2} + \frac{1}{(2\pi)^3} \sum_{\mu \in 1-2\mathbb{N}} \frac{-\mu}{\mho(\mu)} \langle \dagger_\lambda^0 f, \mathbb{P}_\mu h_\mu^d\rangle_{L^2(G/H)} \\ &= \sum_{\xi=0}^{1} \int_{i\mathbb{R}} \int_K \int_{G/H} \dagger_\lambda^0 f(x) \frac{K_{-\mu}^\xi(x^{-1}k)}{\Gamma(\frac{1-\mu}{2})} \overline{h_\mu^\xi(k)} d(xH) dk \frac{d\mu}{|a(\mu)|^2} \\ &\quad + \frac{1}{(2\pi)^3} \sum_{\mu \in 1-2\mathbb{N}} \frac{-\mu}{\mho(\mu)} \langle \dagger_\lambda^0 f, \mathbb{P}_\mu h_\mu^d\rangle_{L^2(G/H)}. \end{aligned}$$

Using the coordinates $xH = k_\theta \overline{n}_y H$ and applying Lemma B.2 in the distributional sense, we find

$$\begin{aligned}
&\frac{1}{\Gamma(\frac{1-\mu}{2})}\int_{G/H} K^{\xi}_{-\mu}(x^{-1}k)\,\text{☥}^0_\lambda\, f(x)d(xH)\\
&= \frac{\Gamma(\frac{2-\lambda}{4})}{\sqrt{\pi}2^{\frac{\lambda}{4}}\Gamma(\frac{1-\mu}{2})\Gamma(\frac{\lambda}{4})}\int_0^\pi\int_{\mathbb{R}} |\cos\theta|_\varepsilon^{-\mu-1} f(gk_\theta\overline{n}_z)\int_{\mathbb{R}} |x|_\varepsilon^{\frac{-\lambda-2}{2}}|z-\tan\theta - x|_\xi^{\frac{-\mu-1}{2}}\frac{1}{2}dxdzd\theta\\
&= p^\xi_{\lambda,\mu}\mathbf{A}^\xi_{\lambda,\mu} f(k),
\end{aligned}$$

An analogous calculation applies to the discrete part after applying that $\mathbb{A}_{-\mu} = \frac{\Gamma\left(\frac{1-\mu}{2}\right)}{\sqrt{\pi}2^\mu} T^0_\mu \mathbb{A}_\mu$. In conclusion we find

$$\langle A_0 \circ \text{☥}^0_\lambda\, f, h\rangle_{\mathcal{H}} = \sum_{\xi=0}^{1}\int_{i\mathbb{R}} \langle p^\xi_{\lambda,\mu}\mathbf{A}^\xi_{\lambda,\mu} f, h^\xi_\mu\rangle_{L^2(K)}\frac{d\mu}{|a(\mu)|^2} + \frac{1}{(2\pi)^3}\sum_{\mu\in 1-2\mathbb{N}}\frac{2\Gamma(1-\mu)}{\Gamma(\frac{-\mu}{2})\,\boldsymbol{\mathfrak{z}}(\mu)}\langle T^0_\mu \mathbb{A}_{\lambda,\mu} f, h^d_\mu\rangle_{L^2(K)}$$

Hence $A_0 \circ \text{☥}^0_\lambda = A_\lambda$ with $A_\lambda : \operatorname{Ind}_H^G(\lambda\otimes e^\lambda) \to \mathcal{H}_0$ given by

$$\langle A_\lambda f, h\rangle_{\mathcal{H}} = \sum_{\xi=0}^{1}\int_{i\mathbb{R}} \langle p^\xi_{\lambda,\mu}\mathbf{A}^\xi_{\lambda,\mu} f, h^\xi_\mu\rangle_{L^2(K)}\frac{d\mu}{|a(\mu)|^2} + \frac{1}{(2\pi)^3}\sum_{\mu\in 1-2\mathbb{N}}\frac{2\Gamma(1-\mu)}{\Gamma(\frac{-\mu}{2})\,\boldsymbol{\mathfrak{z}}(\mu)}\langle T^0_\mu \mathbb{A}_{\lambda,\mu} f, h^d_\mu\rangle_{L^2(K)}$$

□

Corollary 7.6. *For $\varepsilon = 0$ and $\operatorname{Ind}_H^G(\varepsilon\otimes e^\lambda)$ we have the following Plancherel formula*

$$\|f\|^2 = \int_{i\mathbb{R}}\sum_{\xi=0}^{1}\|\boldsymbol{A}^\xi_{\lambda,\mu} f\|^2\frac{d\mu}{|a(\mu)|^2} + \frac{1}{(2\pi)^3}\sum_{\mu\in 1-2\mathbb{N}}\frac{2\Gamma(1-\mu)}{\Gamma(\frac{-\mu}{2})\,\boldsymbol{\mathfrak{z}}(\mu)}\|\mathbb{A}_{\lambda,\mu} f\|^2.$$

For $\varepsilon = 1$ and $f \in \operatorname{Ind}_H^G(\varepsilon\otimes e^\lambda)$ we have the following Plancherel formula

$$\|f\|^2 = \int_{i\mathbb{R}}\sum_{\xi=0}^{1}\|\boldsymbol{A}^\xi_{\lambda,\mu} f\|^2\frac{d\mu}{|a(\mu)|^2} + \frac{1}{2\pi}\sum_{\mu\in -2\mathbb{N}}\frac{\mu^2\Gamma(-\mu)}{\Gamma(\frac{1-\mu}{2})}\|\mathcal{A}_{\lambda,\mu} f\|^2.$$

with $a(\mu)$ and $\boldsymbol{\mathfrak{z}}(\mu)$ given by (7.1).

Proof. Since $|p^\xi_{\lambda,\mu}| = 1$ for $\lambda,\mu\in i\mathbb{R}$ the assertion follows. □

A Integral formulas

Lemma A.1 (See [C20, Proposition 16.8])**.** *For $\operatorname{Re}(\alpha+\beta) > 0$ we have*

$$\int_{\mathbb{R}}\frac{dx}{(x-i)^\alpha(x+i)^\beta} = \frac{2^{2-\alpha-\beta}\pi i^{\alpha-\beta}\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\Gamma(\beta)}$$

Lemma A.2 (See [GR94, Section 3.631])**.** *For $\operatorname{Re}\nu > 0$ we have*

$$\int_0^\pi \sin^{\nu-1}(x)\cos(ax)\,dx = \frac{2^{1-\nu}\pi\cos\left(\frac{a\pi}{2}\right)\Gamma(\nu)}{\Gamma(\frac{\nu+1-a}{2})\Gamma(\frac{\nu+1+a}{2})}.$$

Lemma A.3 (See [GR94, Section 3.631])**.** *For* $\operatorname{Re}\nu > 0$ *we have*

$$\int_0^\pi \sin^{\nu-1}(x)\sin(ax)\,dx = \frac{2^{1-\nu}\pi\sin\left(\frac{a\pi}{2}\right)\Gamma(\nu)}{\Gamma\left(\frac{\nu+1-a}{2}\right)\Gamma\left(\frac{\nu+1+a}{2}\right)}.$$

Lemma A.4. *For* $\operatorname{Re}\nu > 0$ *we have*

$$\int_0^\pi \sin^{\nu-1}(x)e^{iax}\,dx = \frac{2^{1-\nu}\pi e^{\frac{ai\pi}{2}}\Gamma(\nu)}{\Gamma\left(\frac{\nu+1-a}{2}\right)\Gamma\left(\frac{\nu+1+a}{2}\right)}.$$

Lemma A.5 (See [GR94, Section 3.251])**.** *For* $\operatorname{Re}\beta > -1$ *and* $\operatorname{Re}(\alpha+\beta) < -1$ *we have*

$$\int_1^\infty x^\alpha(x-1)^\beta\,dx = B\big(-\alpha-\beta-1,\beta+1\big).$$

Lemma A.6 (See [GR94, Section 3.194])**.** *For* $\operatorname{Re}\beta > -1$ *and* $\operatorname{Re}(\alpha+\beta) < -1$ *we have*

$$\int_0^\infty x^\alpha(x+1)^\beta\,dx = B\big(-\alpha-\beta-1,\alpha+1\big).$$

B The Fourier Transform and Riesz distributions

Define the Fourier transform of $\varphi \in C_c(\mathbb{R})$ as

$$\mathcal{F}[\varphi](\xi) = \int_{\mathbb{R}} \varphi(x)e^{ix\xi}\,dx$$

which makes the inversion formula $\mathcal{F}\mathcal{F}[\varphi](x) = 2\pi\varphi(-x)$. Extend this to distributions in the usual way.

For $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > -1$ and $\varepsilon \in \{0,1\}$ the function

$$u_\alpha^\varepsilon(x) = \frac{1}{2^{\frac{\alpha}{2}}\Gamma\left(\frac{\alpha+1+\varepsilon}{2}\right)}|x|_\varepsilon^\alpha,$$

is locally integrable and can thus be considered as a distribution.

Lemma B.1. *The family of distributions* u_α^ε *extends analytically to a holomorpic family in* $\alpha \in \mathbb{C}$*. For* $\alpha = 1-\varepsilon-2n \in 1-\varepsilon-2\mathbb{N}$ *we have*

$$u_{1-\varepsilon-2n}^\varepsilon(x) = \frac{(-1)^{n+\varepsilon-1}(n-1)!}{2^{\frac{1-\varepsilon}{2}-n}(2n+\varepsilon-2)!}\delta^{(2n+\varepsilon-2)}(x),$$

where $\delta(x)$ *is the Dirac* δ*–function.*

Lemma B.2. *For* $\alpha \in \mathbb{C}$ *we have*

$$\mathcal{F}[u_\alpha^\varepsilon] = \sqrt{2\pi}i^\varepsilon u_{-\alpha-1}^\varepsilon.$$

Furthermore for $\alpha,\beta \in \mathbb{C}$ *with* $\operatorname{Re}\alpha, \operatorname{Re}\beta > -1$ *and* $\operatorname{Re}(\alpha+\beta) < -1$ *we get*

$$\int_{\mathbb{R}} u_\alpha^\varepsilon(x)u_\beta^\xi(y-x)\,dx = (-1)^{\lfloor\frac{\varepsilon+\xi}{2}\rfloor}\sqrt{2\pi}\frac{\Gamma\left(\frac{-1-\alpha-\beta+[\varepsilon+\xi]_2}{2}\right)}{\Gamma\left(\frac{-\alpha+\varepsilon}{2}\right)\Gamma\left(\frac{-\beta+\xi}{2}\right)}u_{\alpha+\beta+1}^{\varepsilon+\xi}(y),$$

for $y \neq 0$.

Proof. The first assertion can be found in [GS64, p.170]. For $y \neq 0$ we have

$$\int_{\mathbb{R}} |x|_{\varepsilon}^{\alpha} |y-x|_{\xi}^{\beta}\, dx = |y|_{\xi+\varepsilon}^{\alpha+\beta+1} \int_{\mathbb{R}} |x|_{\varepsilon}^{\alpha} |1-x|_{\xi}^{\beta}\, dx,$$

by change of variables. Now writing

$$\int_{\mathbb{R}} |x|_{\varepsilon}^{\alpha} |1-x|_{\xi}^{\beta}\, dx = (-1)^{\varepsilon} \int_0^{\infty} x^{\alpha}(1+x)^{\beta}\, dx + \int_0^1 x^{\alpha}(1-x)^{\beta}\, dx + (-1)^{\xi} \int_1^{\infty} x^{\alpha}(x-1)^{\beta}\, dx,$$

we can use the integral formula for the Beta-function, apply Lemma A.5, A.6 and arrive at

$$(-1)^{\varepsilon} B(\alpha+1, -\alpha-\beta-1) + B(\alpha+1, \beta+1) + (-1)^{\xi} B(\beta+1, -\alpha-\beta-1).$$

Now rewriting the Beta-function in terms of the Gamma-function, applying Euler's reflection formula for the Gamma-function and factoring out common factors we get

$$\pi^{-1}\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(-\alpha-\beta-1)\Big[\sin\left((\alpha+\beta)\pi\right) + (-1)^{\varepsilon+1}\sin(\beta\pi) + (-1)^{\xi+1}\sin(\alpha\pi)\Big].$$

We can now apply the identity

$$\begin{aligned}\sin\left((\alpha+\beta)\pi\right) &+ (-1)^{\varepsilon+1}\sin(\beta\pi) + (-1)^{\xi+1}\sin(\alpha\pi) \\ &= 4(-1)^{\lfloor \frac{1+\varepsilon+\xi}{2} \rfloor} \sin\Big(\frac{(\alpha+\varepsilon)\pi}{2}\Big) \sin\Big(\frac{(\beta+\xi)\pi}{2}\Big) \sin\Big(\frac{(\alpha+\beta+2+[\varepsilon+\xi]_2)\pi}{2}\Big),\end{aligned}$$

which can be verified on a case by case basis depending on $\varepsilon, \xi \in \{0,1\}$. Lastly rewrite the sine-functions as Gamma-functions using Euler's reflection formula and cancel out Gamma-functions case by case for $\varepsilon, \xi \in \{0,1\}$.

□

C Fourier-Jacobi transform

This section is a condensed form of [FJ77, Appendix 1]. For $\alpha, \beta \in \mathbb{C}$ with $\alpha \notin -\mathbb{N}$ and $\operatorname{Re}\beta > -1$, define the Fourier–Jacobi transform of $f \in C_c^{\infty}(\mathbb{R}_{\geq 0})$ by

$$J_{\alpha,\beta} f(\mu) = \int_0^{\infty} f(t)\phi_{\mu}^{\alpha,\beta}(t) \sinh^{2\alpha+1}(t) \cosh^{2\beta+1}(t)\, dt,$$

where $\phi_{\mu}^{\alpha,\beta}$ are the Jacobi functions given by

$$\phi_{\mu}^{\alpha,\beta}(t) = {}_2F_1\left(\frac{\alpha+\beta+1+\mu}{2}, \frac{\alpha+\beta+1-\mu}{2}; \alpha+1; -\sinh^2(t)\right).$$

Then we have the following inversion formula:

$$f(t) = \frac{1}{4\pi}\int_{i\mathbb{R}} J_{\alpha,\beta} f(\mu)\phi_{\mu}^{\alpha,\beta}(t) \frac{d\mu}{|c_{\alpha,\beta}(\mu)|^2} - \sum_{\mu \in D_{\alpha,\beta}} J_{\alpha,\beta} f(\mu)\phi_{\mu}^{\alpha,\beta}(t) \operatorname*{Res}_{\nu=\mu}\left(c_{\alpha,\beta}(\nu)c_{\alpha,\beta}(-\nu)\right)^{-1},$$

where

$$c_{\alpha,\beta}(\mu) = \frac{\Gamma(\mu)\Gamma(\alpha+1)}{\Gamma\big(\frac{\alpha+|\beta|+1+\mu}{2}\big)\Gamma\big(\frac{\alpha-|\beta|+1+\mu}{2}\big)},$$

and

$$D_{\alpha,\beta} = \{x \in \mathbb{R} \mid k \in \mathbb{N}_0,\, x = 2k+1+\alpha-|\beta| < 0\}.$$

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Paper B

Tensor products of unitary irreducible representations of $\mathrm{PGL}(2,\mathbb{R})$

Frederik Bang–Jensen

Introduction

Let G be a group, H a subgroup of G and π a unitary irreducible representation of G. The restriction of π to H again defines a unitary representation of H, but $\pi|_H$ need not be irreducible anymore. If G is a compact Lie group, then π is necessarily finite dimensional and its restriction $\pi|_H$ decomposes into a direct sum of irreducible unitary representations of H

$$\pi|_H \cong \bigoplus_{\tau\in\widehat{H}} m(\pi,\tau)\cdot\tau,$$

with multiplicities $m(\pi,\tau)\in\mathbb{N}_0$.
However when G is non-compact the unitary irreducible representations of G need not be finite dimensional and thus one can not expect a direct sum decomposition of the restriction of such representations. Instead the restriction decomposes into a direct integral

$$\pi|_H \cong \int_{\widehat{H}}^{\oplus} m(\pi,\tau)\cdot\tau\, d_\pi(\tau),$$

with the multiplicities $m(\pi,\tau)\in\mathbb{N}_0\cup\{\infty\}$ and measure $d_\pi(\tau)$. Hence understanding the branching problem in the non-compact case requires both understanding the multiplicity function $m(\pi,\tau)$ and the measure $d_\pi(\tau)$ explicitly. The measure $d_\pi(\tau)$ may both have a discrete and continuous part for certain pairs of Lie groups and subgroups (G,H).
In the case where the measure $d_\pi(\tau)$ has both discrete and continuous parts, the branching problem appears to require detailed analysis (See e.g. [Rep78], [MO15], [Zha17] for examples). Recently a uniform method of study has been applied to the case where both G and H are real reductive Lie groups of rank 1 by Frahm and Weiske (See [Wei21] and [Wei20]).

In this paper we apply this method to the rank 2 case where $G=\mathrm{PGL}(2,\mathbb{R})\times\mathrm{PGL}(2,\mathbb{R})$ and $H=\Delta(G)$, providing a uniform method of proof for results obtained by Repka in [Rep78]. We obtain the following results:

Main results

Let $G=\mathrm{PGL}(2,\mathbb{R})$ and $P_G=M_GA_GN_G$ be a minimal parabolic subgroup of G. Then $\Delta(G)$ acts on $G\times G/P_G\times P_G$ with an open dense orbit $\mathcal{O}\cong\mathrm{PGL}(2,\mathbb{R})/\mathrm{GL}(1,\mathbb{R})$. For $\pi_{\xi,\lambda}$ and $\pi_{\eta,\mu}$

unitary principal series representations of G, the restriction of the tensor product $\pi_{\xi,\lambda} \otimes \pi_{\eta,\mu}$, to the open dense orbit $\mathcal{O}$ induces a unitary isomorphism

$$\Theta_{\lambda,\mu} : (\pi_{\xi,\lambda} \otimes \pi_{\eta,\mu})|_{\Delta(G)} \to \mathrm{Ind}_{M_G A_G}^{G}((\xi + \eta) \otimes e^{\lambda-\mu,\mu-\lambda})$$

onto a line-bundle over the one–sheeted hyperboloid $G/M_G A_G$. The Plancherel formula and corresponding direct integral decomposition of the one–sheeted hyperboloid was computed in [BJD23]. Using the restriction map $\Theta_{\lambda,\mu}$ we obtain the corresponding (explicit) direct integral decomposition for tensor products of unitarily induced principal series representations of G.

Theorem 1. *For $\lambda, \mu \in i\mathbb{R}$ the tensor product of unitary principal series representations $\pi_{\xi,\lambda}$ and $\pi_{\eta,\mu}$ of G decomposes as*

$$\pi_{\xi,\lambda} \otimes \pi_{\eta,\mu}|_G \cong \bigoplus_{\sigma=0,1} \int_{i\mathbb{R}}^{\oplus} \pi_{\sigma,\nu}\, d\nu \oplus \bigoplus_{\nu \in \frac{1}{2}-\mathbb{N}} \pi_{\nu}^{\mathrm{ds}}.$$

The Plancherel formula used to obtain the decomposition is given explicitly in terms of symmetry breaking operators $A_{\lambda,\mu,\nu}^{\zeta} \in \mathrm{Hom}_G(\pi_{\xi,\lambda} \otimes \pi_{\eta,\mu}|_G, \pi_{\zeta,\nu})$ depending meromorphically on the induction parameters $\lambda, \mu \in \mathbb{C}$. We give a detailed description of such operators, finding a holomorphic extension of the operators by constructing differential operators and studying the subsequent Bernstein–Sato identities. We also give a detailed description of the zeroes of these holomorphic families of symmetry breaking operators by considering the associated Bernstein–Reznikov integrals. Using the holomorphic extensions of the symmetry breaking operators we extend the Plancherel formula (with some slight assumptions) from the unitary setting to $\mathbb{C}^2$ by method of analytic continuation. By this process we obtain an extension to $\lambda, \mu \in \mathbb{C}$, for which the principal series representations $\pi_{\xi,\lambda}$ and $\pi_{\eta,\mu}$ are unitarizable or contain unitarizable quotients. In the process we collect residues of the symmetry breaking operators which, in the special case of tensor products of complementary series, becomes discrete components in the decomposition:

Theorem 2. *For $\lambda, \mu \in (-\frac{1}{2}, 0)$ with $\lambda + \mu < -\frac{1}{2}$ and $\xi, \eta \in \mathbb{Z}/2\mathbb{Z}$ we have*

$$\pi_{\xi,\lambda}^{c} \otimes \pi_{\xi,\mu}^{c}|_G \cong \bigoplus_{\sigma=0,1} \int_{i\mathbb{R}}^{\oplus} \pi_{\sigma,\nu}\, d\nu \oplus \bigoplus_{\nu \in \frac{1}{2}-\mathbb{N}} \pi_{\nu}^{\mathrm{ds}} \oplus \pi_{\xi+\eta,\lambda+\mu+\frac{1}{2}}^{c}.$$

Part I

Principal series representations

Let G be a real reductive Lie group and $P = MAN$ a minimal parabolic subgroup of G. For a unitary irreducible representation (ξ, V_ξ) and a character $\lambda \cong \mathfrak{a}_\mathbb{C}^*$ we construct a representation $(\xi \otimes e^\lambda \otimes \mathbf{1}, V_\xi)$ of $P = MAN$, by letting N act trivially on V_ξ. This P representation gives rise to a principal series representation of G

$$\pi_{\xi,\lambda} = \mathrm{Ind}_P^G(\xi \otimes e^\lambda \otimes \mathbf{1})$$

as the left regular representation of G on the function space

$$\{f \in C^\infty(G, V_\xi) \mid f(gman) = \xi(m)^{-1} e^{-(\lambda+\rho)\log a} f(g)\ \forall man \in P\},$$

with $\rho := \frac{1}{2}\operatorname{tr}\operatorname{ad}|_\mathfrak{n}$. Denote by $\mathcal{V}_{\xi,\lambda} := G \times_P V_{\xi,\lambda+\rho} \to G/P$ the homogeneous vector bundle associated to $V_{\xi,\lambda+\rho}$. Then $\pi_{\xi,\lambda}$ may be interpreted as the left regular action of G on the space of smooth sections $C^\infty(G, \mathcal{V}_{\xi,\lambda})$.

1 Principal series representations of $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{GL}(2,\mathbb{R})$ and $\mathrm{PGL}(2,\mathbb{R})$

Let $G = \mathrm{PGL}(2,\mathbb{R})$ and let $P_G = M_G A_G N_G \subset G$ denote the minimal parabolic subgroup of G with corresponding Langlands decomposition. We identify G with 2 by 2 matrices with at least one entry in the second row equal to 1. Under this identification we have

$$M_G = \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_G = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \quad N_G = \begin{pmatrix} 1 & x \\ 0 & 1. \end{pmatrix}, \qquad a \in \mathbb{R} \setminus \{> 0\}, x \in \mathbb{R}.$$

Furthermore we identify $\mathrm{PSL}(2,\mathbb{R})$ as a subgroup of G as the image of $\mathrm{SL}(2,\mathbb{R}) \subset \mathrm{GL}(2,\mathbb{R})$ under the quotient map. We identify $\widehat{M_G} \cong \mathbb{Z}/2\mathbb{Z}$ and $\mathfrak{a}_{G,\mathbb{C}}^* \cong \mathbb{C}$ and let $K_G \cong O(2)/\pm\mathrm{id}$ denote the maximal compact subgroup of G. The principal series representations of G can be equipped with the canonical K_G pairing on $\pi_{\xi,\lambda} \times \pi_{\xi,-\lambda}$

$$(f, f')_{K_G} = \int_{K_G/M_G} f(k) f'(k) dk,$$

and the corresponding sesquilinear form $\langle f, f' \rangle_{K_G} = (f, \overline{f'})_{K_G}$. Let $\xi \in \widehat{M_G}$ and $\lambda \in \mathfrak{a}_{G,\mathbb{C}}^*$, for $\pi_{\xi,\lambda}$ the corresponding principal series representation we can extend $\pi_{\xi,\lambda}$ to a representation of $\mathrm{GL}(2,\mathbb{R})$ by composing with the quotient $q : \mathrm{GL}(2,\mathbb{R}) \to \mathrm{PGL}(2,\mathbb{R})$, i.e letting the center of $\mathrm{GL}(2,\mathbb{R})$ act trivially. Under the above identification the corresponding representation for $\mathrm{GL}(2,\mathbb{R})$ is again a principal series representation $\tau_{(\xi_1,\xi_2),(\lambda_1,\lambda_2)}$ of $\mathrm{GL}(2,\mathbb{R})$, corresponding to the induction parameters $(\lambda_1,\lambda_2)=(\lambda,-\lambda)\in\mathfrak{a}^*_{\mathrm{GL}(2,\mathbb{R}),\mathbb{C}}\cong\mathbb{C}^2$ and $(\xi_1,\xi_2)=(\xi,\xi)\in\widehat{M}_{\mathrm{GL}(2,\mathbb{R})}\cong(\mathbb{Z}/2\mathbb{Z})^2$. Likewise, the restriction map $\mathrm{Res}:\tau_{(\xi,\xi),(\lambda,-\lambda)}\to\omega_{0,2\lambda}$ maps the principal series representation $\tau_{(\xi,\xi),(\lambda,-\lambda)}$ to the principal series representation $\omega_{\zeta,\nu}$ of $\mathrm{SL}(2,\mathbb{R})$, corresponding to the inductions parameters $\zeta=0\in\widehat{M}_{\mathrm{SL}(2,\mathbb{R})}\cong\mathbb{Z}/2\mathbb{Z}$ and $\nu\in\mathfrak{a}^*_{\mathrm{SL}(2,\mathbb{R}),\mathbb{C}}\cong\mathbb{C}$. Hence the compositions give a natural map

$$\pi_{\xi,\lambda}\xrightarrow{q}\tau_{(\xi,\xi),(\lambda,-\lambda)}\xrightarrow{\mathrm{Res}}\omega_{0,2\lambda}.$$

Naturally the restriction commutes with the action of $\mathrm{SL}(2,\mathbb{R})$ and for this specific choice of parameters it defines a bijection. To make this process a bit more clear we consider the following lemma relating principal series representations of $\mathrm{GL}(2,\mathbb{R})$ to principal series representations of $\mathrm{SL}(2,\mathbb{R})$:

Lemma 1. *Let* $f\in\mathrm{Ind}_{P_{\mathrm{SL}(2,\mathbb{R})}}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon\otimes e^{\lambda}\otimes 1)$ *and let*

$$\widetilde{L}_{\delta,s}:\mathrm{Ind}_{P_{\mathrm{SL}(2,\mathbb{R})}}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon\otimes e^{\lambda}\otimes 1)\to\mathrm{Ind}_{P_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}((\varepsilon+\delta,\delta)\otimes e^{(\lambda+s,s)}\otimes 1),\quad f\mapsto\widetilde{F}\tag{1.1}$$

with $\widetilde{F}(g)=|\det(g)|_{\delta}^{\frac{1}{2}-\lambda}f(g\,\mathrm{diag}(1,\det(g))^{-1})$. *Then we have the following:*

1. *If* $\widetilde{F}\in\mathrm{Ind}_{P_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}((\eta_1,\eta_2)\otimes e^{(\nu_1,\nu_2)}\otimes 1)$ *satisfies* $\widetilde{F}|_{\mathrm{SL}(2,\mathbb{R})}=f$ *then* $\widetilde{F}=\widetilde{L}_{\eta_2,\nu_2}f$ *and* $(\varepsilon,\lambda)=(\eta_1+\eta_2,\nu_1-\nu_2)$.
2. $\langle\widetilde{L}_{\delta,s}f,\widetilde{L}_{\delta,-\overline{s}}f'\rangle_{K_{\mathrm{GL}(2,\mathbb{R})}}=\langle f,f'\rangle_{K_{\mathrm{SL}(2,\mathbb{R})}}$, *where* $f'\in\mathrm{Ind}_{P_{\mathrm{SL}(2,\mathbb{R})}}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon\otimes e^{-\overline{\lambda}}\otimes 1)$.
3. $T_{(\varepsilon+\delta,\delta),(\lambda+s,s)}\widetilde{L}_{\delta,s}=\widetilde{L}_{\delta+\varepsilon,s+\lambda}T_{\varepsilon,\lambda}$.
4. $\langle f,T_{\varepsilon,\lambda}f\rangle=\langle\widetilde{L}_{\delta,s}f,T_{(\delta,\delta+\varepsilon),(-\overline{s},-\overline{s}-\lambda)}\widetilde{L}_{\delta+\varepsilon,-\overline{s}-\lambda}f\rangle$ *for* $\lambda\in\mathbb{R}$.

with $T_{\varepsilon,\lambda}$ *denoting the standard intertwining operator*

$$T_{\varepsilon,\lambda}:\mathrm{Ind}_{P_{\mathrm{SL}(2,\mathbb{R})}}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon\otimes e^{\lambda}\otimes 1)\to\mathrm{Ind}_{P_{\mathrm{SL}(2,\mathbb{R})}}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon\otimes e^{-\lambda}\otimes 1)$$

and $T_{(\varepsilon,\varepsilon+\delta),(\lambda+s,s)}$ *denoting the standard intertwining operator*

$$T_{(\varepsilon,\varepsilon+\delta),(\lambda+s,s)}:\mathrm{Ind}_{P_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}((\varepsilon,\varepsilon+\delta)\otimes e^{(\lambda+s,s)}\otimes 1)\to\mathrm{Ind}_{P_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}((\varepsilon+\delta,\varepsilon)\otimes e^{(s,\lambda+s)}\otimes 1),$$

both given by

$$f\mapsto Tf,\quad\textit{where}\quad Tf(g)=\int_{\overline{N_S}}f(gw_0\overline{n})\,d\overline{n},\qquad w_0=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

Proof. **1:** Assume that $\widetilde{F}\in\mathrm{Ind}_{P_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}((\eta_1,\eta_2)\otimes e^{(\nu_1,\nu_2)}\otimes 1)$ such that $\widetilde{F}|_{\mathrm{SL}(2,\mathbb{R})}=f$. Then we have

$$\widetilde{F}(g)=\widetilde{F}\left(g\begin{pmatrix}1&0\\0&\det(g)^{-1}\end{pmatrix}\begin{pmatrix}1&0\\0&\det(g)\end{pmatrix}\right)=|\det(g)|_{\eta_2}^{\frac{1}{2}-\nu_2}\widetilde{F}(g\,\mathrm{diag}(1,\det(g))^{-1})=\widetilde{L}_{\eta_2,\nu_2}f(g)$$

Further more for $a \in \mathbb{R}$ we have

$$|a|_{\eta_1+\eta_2}^{\nu_2-\nu_1-1}\widetilde{F}(1) = \widetilde{F}(\operatorname{diag}(a, a^{-1})) = f(\operatorname{diag}(a, a^{-1})) = |a|_{\varepsilon}^{-\lambda-1} f(1),$$

hence $(\varepsilon, \lambda) = (\eta_1 + \eta_2, \nu_1 - \nu_2)$.
2: Since the map

$$k \mapsto k \cdot \begin{pmatrix} 1 & 0 \\ 0 & \det(k)^{-1} \end{pmatrix} M_{\mathrm{SL}(2,\mathbb{R})}$$

has kernel $K_{\mathrm{GL}(2,\mathbb{R})} = O(2)$ it follows that $K_{\mathrm{GL}(2,\mathbb{R})}/M_{\mathrm{GL}(2,\mathbb{R})} \cong K_{\mathrm{SL}(2,\mathbb{R})}/M_{\mathrm{SL}(2,\mathbb{R})}$. We then have that

$$\int_{K_{\mathrm{GL}(2,\mathbb{R})}/M_{\mathrm{GL}(2,\mathbb{R})}} f(kM) dgM = \int_{K_{\mathrm{SL}(2,\mathbb{R})}/M_{\mathrm{SL}(2,\mathbb{R})}} f(k'M') dk'M',$$

from which the result follows.
3: Let $f \in \operatorname{Ind}_{P_{\mathrm{SL}(2,\mathbb{R})}}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon \otimes e^{\lambda} \otimes 1)$. Then we have

$$\begin{aligned} T_{(\varepsilon+\delta,\delta),(\lambda+s,s)}\widetilde{L}_{\delta,s} f(g) &= \int_{\overline{N}_{\mathrm{GL}(2,\mathbb{R})}} |\det(gw_0\overline{n})|_{\delta}^{\frac{1}{2}-s} f\left(gw_0\overline{n}\begin{pmatrix} 1 & 0 \\ 0 & \det(gw_0\overline{n})^{-1} \end{pmatrix}\right) d\overline{n} \\ &= |\det(g)|_{\delta}^{-\frac{1}{2}-s} \int_{\overline{N}_{\mathrm{SL}(2,\mathbb{R})}} f\left(g\begin{pmatrix} \det(g)^{-1} & 0 \\ 0 & 1 \end{pmatrix} w_0\overline{n}\right) d\overline{n} \\ &= |\det(g)|_{\delta}^{-\frac{1}{2}-s} T_{\varepsilon,\lambda} f\left(g\begin{pmatrix} \det(g)^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= |\det(g)|_{\delta+\varepsilon}^{\frac{1}{2}-s-\lambda} T_{\varepsilon,\lambda} f\left(g\begin{pmatrix} 1 & 0 \\ 0 & \det(g)^{-1} \end{pmatrix}\right) \\ &= \widetilde{L}_{\delta+\varepsilon,s+\lambda} T_{\varepsilon,\lambda} f(g). \end{aligned}$$

4: This follows directly from combining properties 2 and 3 □

By the discussion preceding the lemma, any principal series representation of G can be identified as a principal series representation of GL(2, ℝ), with the center acting trivially. Similarly any principal series representation of PSL(2, ℝ) can be identified with a principal series representation of SL(2, ℝ), again with the center acting trivially. For the purpose of this paper it will be convenient to restate the lemma in these terms:

Lemma 2. *Let* $f \in \operatorname{Ind}_{P_{\mathrm{PSL}(2,\mathbb{R})}}^{\mathrm{PSL}(2,\mathbb{R})}(0 \otimes e^{\lambda} \otimes 1)$ *and let*

$$L_{\xi} : \operatorname{Ind}_{P_{\mathrm{PSL}(2,\mathbb{R})}}^{\mathrm{PSL}(2,\mathbb{R})}(0 \otimes e^{\lambda} \otimes 1) \to \operatorname{Ind}_{P_{\mathrm{PGL}(2,\mathbb{R})}}^{\mathrm{PGL}(2,\mathbb{R})}(\xi \otimes e^{\frac{\lambda}{2}} \otimes 1), \quad f \mapsto F \tag{1.2}$$

with $F([g]) = |\det(g)|_{\xi}^{\frac{1+\lambda}{2}} f\left(\left[g\begin{pmatrix} 1 & 0 \\ 0 & \det(g)^{-1} \end{pmatrix}\right]\right)$. *Let*

$$q_{\mathrm{PGL}} : \operatorname{Ind}_{P_G}^{G}(\xi \otimes e^{\lambda} \otimes 1) \to \operatorname{Ind}_{P_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}((\xi, \xi) \otimes e^{(\lambda,-\lambda)} \otimes 1)$$

and

$$q_{\mathrm{PSL}} : \operatorname{Ind}_{P_{\mathrm{PSL}(2,\mathbb{R})}}^{\mathrm{PSL}(2,\mathbb{R})}(0 \otimes e^{\lambda} \otimes 1) \to \operatorname{Ind}_{P_{\mathrm{SL}(2,\mathbb{R})}}^{\mathrm{SL}(2,\mathbb{R})}(0 \otimes e^{\lambda} \otimes 1)$$

denote the map mapping a function to its composition with the respective quotient. Then L_{ξ} *is well-defined and we have the following*

1. $L_\xi = q_{\mathrm{PGL}}^{-1} \circ L_{\xi,-\frac{\lambda}{2}} \circ q_{\mathrm{PSL}}^{-1}$
2. $\langle L_\xi f, L_\xi f' \rangle_{K_{\mathrm{PGL}(2,\mathbb{R})}} = \langle f, f' \rangle_{K_{\mathrm{PSL}(2,\mathbb{R})}}$, *where* $f' \in \mathrm{Ind}_{P_{\mathrm{PSL}(2,\mathbb{R})}}^{\mathrm{PSL}(2,\mathbb{R})}(0 \otimes e^{-\overline{\lambda}} \otimes 1)$.
3. $T^G_{\xi,\frac{\lambda}{2}} L_\xi = L_\xi T^{\mathrm{PSL}}_\lambda$.
4. $\langle f, T^{\mathrm{PSL}}_\lambda f \rangle = \langle L_{\xi,\frac{-\lambda}{2}} f, T^G_{\xi,\frac{\lambda}{2}} L_{\xi,-\frac{\lambda}{2}} f \rangle$ *for* $\lambda \in \mathbb{R}$.

with $T^G_{\xi,\lambda}$ *and* T^{PSL}_λ *defined analogously to Lemma 1.*

Proof. It suffices to check that L_ξ is well-defined, since the rest follows from direct verification or directly from Lemma 1. Let $f \in \mathrm{Ind}_{P_{\mathrm{PSL}(2,\mathbb{R})}}^{\mathrm{PSL}(2,\mathbb{R})}(0 \otimes e^{\lambda} \otimes 1)$ and let $g, g' \in \mathrm{GL}(2,\mathbb{R})$ such that $[g] = [g']$, i.e there exists $a \in \mathbb{R} \setminus \{0\}$ such that $g' = g \cdot a \cdot \mathrm{id}$. Then

$$\begin{aligned} L_\xi f([g']) &= |\det(g)|_\xi^{\frac{1+\lambda}{2}} |a|^{1+\lambda} f\left(\left[g \cdot \begin{pmatrix} 1 & 0 \\ 0 & \det(g)^{-1} \end{pmatrix}\right]\left[\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right]\right) \\ &= |\det(g)|_\xi^{\frac{1+\lambda}{2}} f\left(\left[g \cdot \begin{pmatrix} 1 & 0 \\ 0 & \det(g)^{-1} \end{pmatrix}\right]\right) = L_\xi f([g]). \end{aligned}$$

□

2 The unitary dual of $\mathrm{SL}(2,\mathbb{R}), \mathrm{GL}(2,\mathbb{R})$ and $\mathrm{PGL}(2,\mathbb{R})$

Let us briefly recall the description of the unitary dual $\widehat{\mathrm{SL}(2,\mathbb{R})}$ of $\mathrm{SL}(2,\mathbb{R})$ in terms of the induction parameters for principal series representations $(\xi,\mu) \subset \widehat{M}_{\mathrm{SL}(2,\mathbb{R})} \times \mathfrak{a}^*_{\mathrm{SL}(2,\mathbb{R}),\mathbb{C}}$. To this extend we introduce some notation. Let $P_S = M_S A_S N_S \subset \mathrm{SL}(2,\mathbb{R})$ denote the usual minimal parabolic subgroup in $\mathrm{SL}(2,\mathbb{R})$, i.e

$$M_S = \{\pm I\}, \quad A_S = \left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{R}_{>0}\right\}, \quad N_S = \left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R}\right\}.$$

We furthermore let $K_S = SO(2)$ denote the maximal compact subgroup of $\mathrm{SL}(2,\mathbb{R})$. We then identify $\widehat{M_S} \cong \mathbb{Z}/2\mathbb{Z}$ and $\mathfrak{a}^*_{S,\mathbb{C}} \cong \mathbb{C}$. Under this identification $\rho_{\mathrm{SL}(2,\mathbb{R})} = 1$. Let $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ and $\mu \in \mathbb{C}$. For any character $\varepsilon \otimes e^\mu \otimes 1$ of P_S define the principal series representation $\pi_{\varepsilon,\mu}$ to be the left regular representation of $\mathrm{SL}(2,\mathbb{R})$ on

$$\mathrm{Ind}_{P_S}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon \otimes e^\mu \otimes 1) = \left\{f \in C^\infty(\mathrm{SL}(2,\mathbb{R})) \mid f(gman) = |t|_\varepsilon^{-\mu-1} f(g),\, man \in M_S A_S N_S\right\},$$

where $ma = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \in M_S A_S$. We introduce the notation

$$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

and the characters $\psi_m(k_\theta) = e^{im\theta}$ on K_S. According to the theory of Fourier series we have the K_S-type decomposition

$$\mathrm{Ind}_{P_S}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon \otimes e^\mu \otimes 1) \cong \widehat{\bigoplus_{m \in 2\mathbb{Z}+\varepsilon}} \mathbb{C}\psi_m.$$

We let $\mathrm{Ind}_{P_S}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon \otimes e^\mu \otimes 1)_m$ denote the set of functions contained in the K_S-type given by $m \in \mathbb{Z}$, that is $\mathrm{Ind}_{P_S}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon \otimes e^\mu \otimes 1)_m = \mathbb{C}\psi_m$. Then we have:

Proposition 1 (See [Cas20, Prop. 10.8])**.** *The representation* $\mathrm{Ind}_{P_S}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon\otimes e^{\mu}\otimes 1)$ *is irreducible except when* $\mu\in 1-\varepsilon-2\mathbb{Z}$ *. If* $\mu\in 1-\varepsilon-2\mathbb{N}$ *then* $\mathrm{Ind}_{P_S}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon\otimes e^{\mu}\otimes 1)$ *decomposes as* $V_0\oplus V_1\oplus V_2$ *where* V_0 *is an irreducible representation containing exactly the* K*-types with* $|m|\le -\mu$*. The quotient* $\tau^{ds}_{\varepsilon,\mu}$ *is a direct sum of two infinite dimensional representations* $\tau^{\mathrm{hol}}_{\varepsilon,\mu}$ *and* $\tau^{\mathrm{ahol}}_{\varepsilon,\mu}$*. For* $\mu\in\varepsilon-1+2\mathbb{N}$ $\mathrm{Ind}_{P_S}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon\otimes e^{\mu}\otimes 1)$ *decomposes as* $V_0\oplus V_1\oplus V_2$ *where* V_1 *and* V_2 *are irreducible representations with* V_1 *containing exactly the* K*-types with* $m>\mu$ *and* V_2 *containing the* K*-types* $m<-\mu$*. The quotient is a finite dimensional representation.*

For $\varepsilon=0$ and $\mu\in i\mathbb{R}$ the principal series representations $\mathrm{Ind}_{P_S}^{\mathrm{SL}(2,\mathbb{R})}(0\otimes e^{\mu}\otimes 1)$ can be unitarized by being equipped with the canonical $L^2(K_S/M_S)$ inner-product. Similarly for $\varepsilon=1$ and $\mu\in i\mathbb{R}\setminus\{0\}$ the principal series representation $\mathrm{Ind}_{P_S}^{\mathrm{SL}(2,\mathbb{R})}(1\otimes e^{\mu}\otimes 1)$ can be unitarized in the same way. However for $\mu\in 1-\varepsilon-2\mathbb{Z}$ the infinite dimensional representations $\tau^{\mathrm{hol}}_{\varepsilon,\mu}$ and $\tau^{\mathrm{ahol}}_{\varepsilon,\mu}$ does not come with such a simple unitarization. Instead let $w_0=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ denote a representative of the longest Weyl group element of $\mathrm{SL}(2,\mathbb{R})$. Recall the definition of the normalized Knapp–Stein intertwining operator

$$\widetilde{T}_{\varepsilon,\mu}:\mathrm{Ind}_{P_S}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon\otimes e^{\mu}\otimes 1)\to\mathrm{Ind}_{P_S}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon\otimes e^{-\mu}\otimes 1),\quad \widetilde{T}_{\varepsilon,\mu}f(g)=\frac{1}{\Gamma(\frac{\mu+\varepsilon}{2})}\int_N f(gw_0\overline{n})d\overline{n},$$

for $\mathrm{Re}\,(\mu)>0$. The normalization is chosen such that $\widetilde{T}_{\varepsilon,\mu}$ extends holomorphically to $\mu\in\mathbb{C}$.

Proposition 2 (See [BJD23, Prop. 2.3])**.** *For* $f\in\mathrm{Ind}_{P_S}^{\mathrm{SL}(2,\mathbb{R})}(\varepsilon\otimes e^{\mu}\otimes 1)_m$ *we have*

$$T_{\varepsilon,\mu}f=b^{\varepsilon}_m(\mu)f,$$

where

$$b^{\varepsilon}_m(\mu)=\sqrt{\pi}\,i^{[\varepsilon]_2}(-1)^{\frac{m+|m|}{2}-[\varepsilon]_2}\frac{\left(\frac{1+\varepsilon-\mu}{2}\right)_{\frac{|m|-\varepsilon}{2}}}{\Gamma\left(\frac{\mu+1+|m|}{2}\right)}.$$

For $\varepsilon=0$ *and* $\mu\in 1-2\mathbb{N}$ *we have* $b^0_m(\mu)\ge 0$ *for all* $m\in 2\mathbb{Z}$*. Whereas for* $\varepsilon=1$*,* m *odd and* $\mu\in-2\mathbb{N}$ *we have* $-ib^1_m(\mu)\ge 0$ *for* $m>0$ *and* $ib^1_m(\mu)\ge 0$ *for* $m<0$*.*

Using Proposition 2 we can for $\varepsilon=0$ and $\mu\in 1-2\mathbb{N}$ equip $\mathrm{Ind}_{P_S}^{\mathrm{SL}(2,\mathbb{R})}(0\otimes e^{\mu}\otimes 1)$ with the norm

$$\|f\|^2=\langle f,\widetilde{T}_{0,\mu}f\rangle=\int_{K_{\mathrm{SL}(2,\mathbb{R})}/M_S}f(k)\overline{T_{0,\mu}f(k)}\,dk.$$

Similarly when $\varepsilon=1$ and $\mu\in-2\mathbb{N}$ we equip $\mathrm{Ind}_{P_S}^{\mathrm{SL}(2,\mathbb{R})}(1\otimes e^{\mu}\otimes 1)$ with

$$\|f\|^2=\langle f,\widehat{T}_{1,\mu}f\rangle=\int_{K_{\mathrm{SL}(2,\mathbb{R})}/M_S}f(k)\overline{\widehat{T}_{1,\mu}f(k)}\,dk$$

where $\widehat{T}_{1,\mu}$ is given by

$$\widehat{T}_{1,\mu}f=\begin{cases}iT_{1,\mu}f, & \text{for } m>0,\\ -iT_{1,\mu}f, & \text{for } m<0\end{cases}$$

for $f\in\mathrm{Ind}_{P_S}^{\mathrm{SL}(2,\mathbb{R})}(1\otimes e^{\mu}\otimes 1)_m$. By Lemma 1 the restriction map

$$\mathrm{Res}_{\eta,\nu}:\mathrm{Ind}_{P_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}((\eta_1,\eta_2)\otimes e^{(\nu_1,\nu_2)}\otimes 1)\to\mathrm{Ind}_{P_S}^{\mathrm{SL}(2,\mathbb{R})}(\eta_1+\eta_2\otimes e^{\nu_1-\nu_2}\otimes 1),\quad f\to f|_S,$$

satisfies

$$\mathrm{Res}_{(\varepsilon+\delta,\delta),(s+t,s)} \circ \widetilde{L}_{\delta,s} = \mathrm{id}, \quad \text{and} \quad \widetilde{L}_{\eta_2,\nu_2} \circ \mathrm{Res}_{\eta,\nu} = \mathrm{id}\,.$$

Furthermore, since both maps are $\mathrm{SL}(2,\mathbb{R})$ intertwining the image $\widetilde{\psi}_m := \widetilde{L}_{\eta_2,\nu_2}\psi_m$ of $\psi_m \in \mathrm{Ind}_{P_S}^{\mathrm{SL}(2,\mathbb{R})}(\eta_1+\eta_2 \otimes e^{\nu_1-\nu_2} \otimes 1)_m$ is fixed under the action of $\mathrm{SL}(2,\mathbb{R})$. Since $K_{\mathrm{GL}(2,\mathbb{R})}$ can be written as the disjoint union

$$K_{\mathrm{GL}(2,\mathbb{R})} = K_S \cup K_S\kappa, \quad \text{where} \quad \kappa = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the action of $K_{\mathrm{GL}(2,\mathbb{R})}$ on $\widetilde{\psi}_m$ is determined by the action of κ. Let $\ell \in \{0,1\}$, then

$$\widetilde{\psi}_m(\kappa k_\theta \kappa^\ell) = |\det(\kappa^{1+\ell})|_{\eta_2}^{\frac{1}{2}-\nu_2}\psi_m\left(\kappa k_\theta\kappa^\ell \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{1+\ell}\end{pmatrix}\right) = (-1)^{\eta_2}\widetilde{\psi}_{-m}\left(k_\theta\kappa^\ell\right).$$

Hence the K-type decomposition for $\mathrm{Ind}_{P_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}((\eta_1,\eta_2)\otimes e^{(\nu_1,\nu_2)}\otimes 1)$ becomes

$$\bigoplus_{m\in 2\mathbb{N}+[\eta_1+\eta_2]} \mathbb{C}\widetilde{\psi}_m \oplus \mathbb{C}\widetilde{\psi}_{-m} \oplus \mathbb{C}\psi_0.$$

For $(\nu_1,\nu_2) \in (i\mathbb{R})^2$ we unitarize $\mathrm{Ind}_{P_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}((\eta_1,\eta_2)\otimes e^{(\nu_1,\nu_2)}\otimes 1)$ analogously to the case for $\mathrm{SL}(2,\mathbb{R})$. For $\eta_1+\eta_2 = 0$ and $\nu_1-\nu_2 \in 1-2\mathbb{N}$ we equip $\mathrm{Ind}_{P_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}((\eta_1,\eta_2)\otimes e^{(\nu_1,\nu_2)}\otimes 1)/\ker(\widetilde{T}_{\eta,\nu})$ with the norm

$$\|f\|^2 = \langle f, \widetilde{T}_{(\eta_1,\eta_2),(\nu_1,\nu_2)}f\rangle,$$

which is positive by (3) in Lemma 1 and Proposition 2. We then have

$$T_{(\eta,\eta),(\nu_1,\nu_2)}\widetilde{\psi}_m = T_{(\eta,\eta),(\nu_1,\nu_2)}\widetilde{L}_{\eta_2,\nu_2}\psi_m = \widetilde{L}_{0,\nu_1}T_{0,\nu_1-\nu_2}\psi_m = b_m^0(\nu_1-\nu_2)\widetilde{L}_{0,\nu_1}\psi_m = b_m^0(\nu_1-\nu_2)\widetilde{\psi}_m,$$

where the last equality follows from the second parameter, in the extension map, not mattering in the $K_{\mathrm{GL}(2,\mathbb{R})}$-picture. For $\eta_1+\eta_2 = 1$ and $\nu_1-\nu_2 \in -2\mathbb{N}$ the $\mathrm{SL}(2,\mathbb{R})$-intertwining operator

$$\widehat{T}_{(\eta_1,\eta_2),(\nu_1,\nu_2)} = \widetilde{L}_{\eta_2,\nu_1} \circ \widehat{T}_{1,\nu_1-\nu_2} \circ \mathrm{Res}_{\mathrm{SL}(2,\mathbb{R})} : \mathrm{Ind}_{P_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}((\eta_1,\eta_2)\otimes e^{(\nu_1,\nu_2)}\otimes 1) \to \mathrm{Ind}_{P_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}(\eta\otimes e^{(\nu_2,\nu_1)}\otimes 1).$$

defines a $\mathrm{GL}(2,\mathbb{R})$-intertwining operator. To see this it suffices to check that it intertwines the center of $\mathrm{GL}(2,\mathbb{R})$ and the action of κ. To see that it intertwines the action of κ we get by Proposition 2 that

$$\widehat{T}_{\eta,\nu}(c_0\widetilde{\psi}_m + c_1\widetilde{\psi}_{-m}) = \widehat{L}_{\eta_1,\nu_2}a_m(\nu_1-\nu_2)(c_0\psi_m + c_1\psi_{-m}) = a_m(\nu_1-\nu_2)(c_0\widetilde{\psi}_m + c_1\widetilde{\psi}_{-m}),$$

where $c_0, c_1 \in \mathbb{C}$, $m \in 2\mathbb{N}_0+1$ and

$$a_m(\nu_1-\nu_2) = \sqrt{\pi}\frac{(\frac{2-\nu_1+\nu_2}{2})_{\frac{m-1}{2}}}{\Gamma(\frac{\nu_1-\nu_2+1+m}{2})}.$$

To see that it also intertwines the action of the center we note that the action of $t\,\mathrm{id}$ is multiplication by $|t|^{-(\nu_1+\nu_2)}$, which is invariant under permutation of ν_1,ν_2. We remark that the above also shows that $\mathrm{Ind}_{P_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}((\eta_1,\eta_2)\otimes e^{(\nu_1,\nu_2)}\otimes 1)/\ker(\widehat{T}_{(\eta_1,\eta_2),(\nu_1,\nu_2)})$ can be equipped with the norm

$$\|f\|^2 = \langle f, \widehat{T}_{(\eta_1,\eta_2),(\nu_1,\nu_2)}f\rangle.$$

Theorem 3 ([Bum97])**.** *Let $\nu \in \mathbb{C}^2$ and $\eta \in (\mathbb{Z}/2\mathbb{Z})^2$. Let $\nu_\pm = \nu_1 \pm \nu_2$ and $\eta_* = \eta_1 + \eta_2$. The unitary dual of* $\mathrm{GL}(2,\mathbb{R})$ *can be described as follows: In all cases we $\nu_+ \in i\mathbb{R}$:*

1. *For $\nu_- \in i\mathbb{R}$ we have $\tau_{\eta,\nu}$, unitary induced principal series.*
2. *For $\eta_* = 0$ and $\nu_- \in (-1, 0)$ we have $\tau_{\eta,\nu}$, the complementary series.*
3. *For $\nu_- \in 1 - \eta_* - 2\mathbb{N}$ we have $\tau_{\eta,\nu}/\ker(\widetilde{T}^{\mathrm{GL}(2,\mathbb{R})}_{\eta,\nu})$, the discrete series representations.*
4. *For $\eta_* = 1$ and $\nu_- = 0$ then $\tau_{\eta,\nu}$ is a limit of discrete series representations*

From this we immediately obtain the unitary dual of $\mathrm{PGL}(2,\mathbb{R})$ by only considering the principal series representations for which the center acts trivially:

Corollary 1. *Let $\nu \in \mathbb{C}$ and $\eta \in (\mathbb{Z}/2\mathbb{Z})^2$. The unitary dual of* $\mathrm{PGL}(2,\mathbb{R})$ *can be described as follows:*

1. *For $\nu \in i\mathbb{R}$ we have $\pi_{\eta,\nu}$, unitary induced principal series.*
2. *For $\nu \in (-\frac{1}{2}, 0)$ we have $\pi^c_{\eta,\nu}$, the complementary series.*
3. *For $\nu \in \frac{1}{2} - \mathbb{N}$ we have $\pi^{\mathrm{ds}}_\nu = \pi_{\eta,\nu}/\ker(\widetilde{T}^G_{\eta,\nu})$, the discrete series representations.*

We have purposefully omitted the trivial representations from the lists.

Part II

Symmetry breaking operators

Let $G' := \Delta(G) \subset G \times G$. The space of symmetry breaking operators for principal series representations of $G \times G$ and G'

$$\operatorname{Hom}_{G'}\left((\pi_{\xi,\lambda} \otimes \pi_{\eta,\mu})\,|_{\Delta(G)}, \pi_{\zeta,\nu}\right)$$

identifies with distribution sections on $G \times G/(P_G \times P_G)$ with certain $P' := \Delta(P_G)$-equivariance, by the Schwartz Kernel Theorem. Such kernels are for generic induction parameters in correspondence with the double co-sets $P_{G'}\backslash G/P_{G\times G}$. Especially, for $G = \mathrm{PGL}(2,\mathbb{R})$ one has

$$\dim \operatorname{Hom}_{G'}\left((\pi_{\xi,\lambda} \otimes \pi_{\eta,\mu})\,|_{G'}, \pi_{\zeta,\nu}\right) \leq 1,$$

for generic $\lambda, \mu, \nu \in \mathfrak{a}_{\mathbb{C}}^*$ and $\xi, \eta, \zeta \in \widehat{M}$. Since principal series representations of $\mathrm{PSL}(2,\mathbb{R})$ can be identified with principal series representations of $\mathrm{SL}(2,\mathbb{R})$ and principal series representation of $\mathrm{PGL}(2,\mathbb{R})$ can be identified with principal series representations of $\mathrm{GL}(2,\mathbb{R})$, with the center acting trivially, we use Lemma 2 to extend symmetry breaking operators for the pair $(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}), \Delta(\mathrm{SL}(2,\mathbb{R})))$ to symmetry breaking operators of $(\mathrm{PGL}(2,\mathbb{R}) \times \mathrm{PGL}(2,\mathbb{R}), \Delta(\mathrm{PGL}(2,\mathbb{R})))$

3 Symmetry breaking operators for tensor products of $\mathrm{SL}(2,\mathbb{R})$

Let $\lambda, \mu, \nu \in \mathbb{C}$ and $\xi, \eta, \zeta \in \mathbb{Z}/2\mathbb{Z}$ and note that

$$\dim \operatorname{Hom}_{\Delta(\mathrm{SL}(2,\mathbb{R}))}\left((\pi_{\xi,\lambda} \otimes \pi_{\eta,\mu})\,|_{\mathrm{SL}(2,\mathbb{R})}, \pi_{\zeta,\nu}\right) = 0 \qquad \text{if } \zeta \neq \xi + \eta$$

since the center of $\mathrm{SL}(2,\mathbb{R})$ is precisely M_S. Using the compact picture, any intertwining operator $A_K : \pi_{\xi,\lambda} \otimes \pi_{\eta,\mu}|_{\mathrm{SL}(2,\mathbb{R})} \to \pi_{\xi+\eta,\nu}$ can be written as

$$A_K f(g) = \int_{K_S \times K_S} K(g_1, g_2) f(gg_1, gg_2) d(g_1, g_2),$$

with $K \in \mathcal{D}'(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))$ satisfying the equivariance properties:

$$\begin{aligned} K(mang_1, mang_2) &= \xi(m)\eta(m)e^{(\nu+\rho)\log a}K(g_1, g_2) \\ K(g_1 man, g_2) &= \xi(m)e^{(\lambda-\rho)\log a}K(g_1, g_2) \\ K(g_1, g_2 man) &= \eta(m)e^{(\mu-\rho)\log a}K(g_1, g_2) \end{aligned}$$

for all $g_1, g_2 \in \mathrm{SL}(2,\mathbb{R})$ and $man \in P_S$. Since $(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}), \mathrm{SL}(2,\mathbb{R}))$ is a real spherical pair, $\Delta(P_S)\backslash \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})/(P_S \times P_S)$ contains an open dense double co-set. Let

$$w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

be a representative of the longest Weyl-group element for $\mathrm{SL}(2,\mathbb{R})$. Then $(e, w_0) \in \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ is a representative the open dense co-set, and the possible kernels $K \in \mathcal{D}'(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))$ is generically determined by their values on $(p_0p_1, p_0w_0p_2)$ for $p_j \in P_S$. For $g_1, g_2 \in \mathrm{SL}(2,\mathbb{R})$ we put

$$\Phi_0(g_1, g_2) = (g_2)_{11}(g_1)_{21} - (g_1)_{11}(g_2)_{21}, \quad \Phi_1(g_1, g_2) = (g_1)_{21}, \quad \Phi_2(g_1, g_2) = (g_2)_{21}, \tag{3.1}$$

with $(g_k)_{ij}$ denoting the $(i,j)'th$ entry of $g_k \in \mathrm{SL}(2,\mathbb{R})$. Then we obtain the following result:

Theorem 4. *For $\lambda, \mu, \nu \in \mathbb{C}$ and $\xi, \eta \in \mathbb{Z}/2\mathbb{Z}$ any kernel $K \in \mathcal{D}'(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})) \cap L^1_{Loc}(\mathrm{SL}(2,\mathbb{R}))$ such that the corresponding distribution A_K satisfies*

$$A_K \in \mathrm{Hom}_{\Delta(\mathrm{SL}(2,\mathbb{R})}\left((\pi_{\xi,\lambda} \otimes \pi_{\eta,\mu})\,|_{\Delta(\mathrm{SL}(2,\mathbb{R})}, \pi_{\eta+\zeta,\nu}\right)$$

is up to normalization given by

$$K^{\sigma,\xi,\eta}_{\lambda,\mu,\nu}(g_1, g_2) = |\Phi_0(g_1, g_2)|^{s_1}_{\sigma} |\Phi_1(g_1, g_2)|^{s_2}_{\xi+\sigma} |\Phi_2(g_1, g_2)|^{s_3}_{\eta+\sigma} \tag{3.2}$$

with $|x|^s_\varepsilon = \mathrm{sgn}(x)^\varepsilon |x|^s$, and $s_1, s_2, s_3 \in \mathbb{C}$ being affine linear transformations of λ, μ and ν given by

$$s_1 = \frac{1}{2}(\lambda + \mu + \nu - 1), \quad s_2 = \frac{1}{2}(\lambda - \mu - \nu - 1), \quad s_3 = \frac{1}{2}(-\lambda + \mu - \nu - 1).$$

To simplify notation we will at times consider the dependence of the kernels in Theorem 4 in the parameters $s_1, s_2, s_3 \in \mathbb{C}$ instead of $\lambda, \mu, \nu \in \mathbb{C}$. Abusing notation we will often write

$$K^{\sigma,\xi,\eta}_{s_1,s_2,s_3} = |\Phi_0(g_1, g_2)|^{s_1}_{\sigma} |\Phi_1(g_1, g_2)|^{s_2}_{\xi+\sigma} |\Phi_2(g_1, g_2)|^{s_3}_{\eta+\sigma} \tag{3.3}$$

for the kernels in Theorem 4, and instead treat them in generality, as a family of distribution valued maps in the powers s_1, s_2, s_3.

The existence of the kernels from Theorem 4 is not necessarily immediately obvious from their definition. To see that there indeed exists choices of $s_1, s_2, s_3 \in \mathbb{C}$ such that the kernels define locally integrable functions on $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ we use the non-compact picture for the principal series representations, i.e we realize the kernels on $\overline{N}_S \times \overline{N}_S$ with $\overline{N}_S := N_S^{\mathrm{tr}}$. Using the integral formula

$$\int_{K_S} f(k)dk = \int_{\overline{N}_S} f(\kappa(\overline{n}))e^{-2\rho H(\overline{n})}d\overline{n},$$

with $\overline{n} = \kappa(\overline{n})\mu(\overline{n})e^{H(\overline{n})}n' \in G = K_SM_SA_SN_S$, we instead consider expressions of the form

$$A^{\sigma,\xi,\eta}_{\lambda,\mu,\nu} f(\overline{n}) = \int_{\overline{N}_S \times \overline{N}_S} K^{\sigma,\xi,\eta}_{s_1,s_2,s_3}(\overline{n}_1, \overline{n}_2) f(\overline{n}\overline{n}_1, \overline{n}\overline{n}_2) d(\overline{n}_1, \overline{n}_2), \qquad f \in \pi_{\xi,\lambda} \otimes \pi_{\eta,\mu}.$$

Identifying $\overline{N}_S \cong \mathbb{R}$ by letting $\overline{n}_a = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \quad a \in \mathbb{R}$ we find

$$\Phi_0(\overline{n}_x, \overline{n}_y) = x - y, \quad \Phi_1(\overline{n}_x, \overline{n}_y) = x, \quad \Phi_2(\overline{n}_x, \overline{n}_y) = y.$$

Hence the integral kernels $K^{\sigma,\xi,\eta}_{s_1,s_2,s_3}$ may be realized on $\mathbb{R}^2$ as

$$K^{\sigma,\xi,\eta}_{s_1,s_2,s_3}(\overline{n}_x,\overline{n}_y) = |x-y|^{s_1}_{\sigma}|x|^{s_2}_{\xi+\sigma}|y|^{s_3}_{\eta+\sigma}. \tag{3.4}$$

For the spherical case $\xi=\eta=\sigma=0$, the integrability of such kernels were classified in [OC11] and from this we obtain

Theorem 5 (Clerc-Ørsted [OC11, p. 9])**.** *The kernels $K^{\sigma,\xi,\eta}_{s_1,s_2,s_3}$ are locally integrable as functions on $\mathbb{R}^2$ in the region*

$$\Omega = \{(s_1,s_2,s_3)\in\mathbb{C}^3 \mid \mathrm{Re}(s_i)>-1,\ i=1,2,3,\ \mathrm{Re}(s_1+s_2+s_3)>-1\}.$$

In terms of the original parameters, the kernels $K^{\sigma,\xi,\eta}_{\lambda,\mu,\nu}$ are locally integrable as functions on $\mathbb{R}^2$ in the region

$$\Omega = \{(\lambda,\mu,\nu)\in\mathbb{C}^3 \mid \mathrm{Re}(\lambda),\mathrm{Re}(\mu)>-1,\ -1<\mathrm{Re}(\nu)<1\}.$$

4 Symmetry breaking operators for tensor products of $\mathrm{PGL}(2,\mathbb{R})$

Let $\xi,\eta,\zeta,\sigma\in\mathbb{Z}/2\mathbb{Z}$ and $\lambda,\mu,\nu\in\mathbb{C}$. We consider the following diagram in light of Lemma 2

$$\begin{tikzcd}
\mathrm{Ind}^G_{P_G}(\xi\otimes e^{\lambda}\otimes 1)\otimes\mathrm{Ind}^G_{P_G}(\eta\otimes e^{\mu}\otimes 1) \arrow[d, "\mathrm{Res}_{\mathrm{PSL}\times\mathrm{PSL}}"] & \mathrm{Ind}^G_{P_G}(\zeta\otimes e^{\frac{\nu}{2}}\otimes 1) \\
\mathrm{Ind}^{\mathrm{SL}(2,\mathbb{R})}_{P_S}(0\otimes e^{2\lambda}\otimes 1)\otimes\mathrm{Ind}^{\mathrm{SL}(2,\mathbb{R})}_{P_S}(0\otimes e^{2\mu}\otimes 1) \arrow[r, "A^{\sigma,0,0}_{2\lambda,2\mu,\nu}"] & \mathrm{Ind}^{\mathrm{SL}(2,\mathbb{R})}_{P_S}(0\otimes e^{\nu}\otimes 1) \arrow[u, "L_\zeta"']
\end{tikzcd}$$

which is $\mathrm{PSL}(2,\mathbb{R})$ intertwining by Lemma 2.

Lemma 3. *For $\zeta=\sigma+\xi+\eta$ the above diagram is $\mathrm{PGL}(2,\mathbb{R})$ intertwining, making the composition*

$$A^{\zeta}_{\lambda,\mu,\nu} := L_\zeta\circ A^{\sigma,0,0}_{2\lambda,2\mu,\nu}\circ\mathrm{Res}_{\mathrm{PSL}\times\mathrm{PSL}}$$

a symmetry breaking operator for the strongly spherical pair $(\mathrm{PGL}(2,\mathbb{R})\times\mathrm{PGL}(2,\mathbb{R}),\Delta(\mathrm{PGL}(2,\mathbb{R})))$

Proof. Since

$$\mathrm{PGL}(2,\mathbb{R}) = \mathrm{PSL}(2,\mathbb{R})\cup[\kappa]\,\mathrm{PSL}(2,\mathbb{R}),\qquad \kappa=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

it suffices to check that $A^{\zeta}_{\lambda,\mu,\nu}$ intertwines the action of $[\kappa]$. To this extend let $f\in\pi_{\xi,\lambda}\otimes\pi_{\eta,\mu}$ and realize $A^{\sigma,0,0}_{2\lambda,2\mu,\nu}$ in the non-compact picture. Note that by the Iwasawa decomposition $G=K_GA_GN_G$ it suffices to evaluate at $[k]\in K_G$. Note that $\kappa=\kappa^{-1}$ and $\kappa k\kappa=k^{-1}$, thus we have

$$\begin{aligned}
\pi_{\zeta,\nu}([\kappa])A^{\zeta}_{\lambda,\mu,\nu}f([k]) &= A^{\zeta}_{\lambda,\mu,\nu}f([\kappa k]) = |\det(\kappa k)|^{\frac{1+\nu}{2}}_{\zeta}A^{\zeta}_{\lambda,\mu,\nu}f\left(\left[\kappa\cdot k\cdot\begin{pmatrix}1&0\\0&\det(\kappa k)\end{pmatrix}\right]\right)\\
&= (-1)^{\zeta}\int_{\overline{N}_G\times\overline{N}_G}K^{\sigma,0,0}_{2\lambda,2\mu,\nu}\left(\left(\kappa\cdot k\cdot\begin{pmatrix}1&0\\0&\det(\kappa k)\end{pmatrix}\right)^{-1}(\overline{n}_1,\overline{n}_2)\right)f(\overline{n}_1,\overline{n}_2)d(\overline{n}_1,\overline{n}_2)\\
&= (-1)^{\zeta}\int_{\overline{N}_S\times\overline{N}_S}K^{\sigma,0,0}_{2\lambda,2\mu,\nu}\left(k(\overline{n}_1,\overline{n}_2)\right)f(\overline{n}_1,\overline{n}_2)d(\overline{n}_1,\overline{n}_2).
\end{aligned}$$

Meanwhile we have

$$\begin{aligned}
A^{\zeta}_{\lambda,\mu,\nu}(\pi_{\xi,\lambda}([\kappa])\otimes\pi_{\eta,\mu}([\kappa])f)([k]) &= \int_{\overline{N}_S\times\overline{N}_S} K^{\sigma,0,0}_{2\lambda,2\mu,\nu}\left(k^{-1}(\overline{n}_1,\overline{n}_2)\right) f(\kappa\overline{n}_1,\kappa\overline{n}_2)d(\overline{n}_1,\overline{n}_2)\\
&= (-1)^{\xi+\eta}\int_{\overline{N}_S\times\overline{N}_S} K^{\sigma,0,0}_{2\lambda,2\mu,\nu}\left(k^{-1}\kappa(\overline{n}_1,\overline{n}_2)\kappa\right) f(\overline{n}_1,\overline{n}_2)d(\overline{n}_1,\overline{n}_2)\\
&= (-1)^{\xi+\eta}\int_{\overline{N}_S\times\overline{N}_S} K^{\sigma,0,0}_{2\lambda,2\mu,\nu}\left(\kappa k(\overline{n}_1,\overline{n}_2)\kappa\right) f(\overline{n}_1,\overline{n}_2)d(\overline{n}_1,\overline{n}_2)\\
&= (-1)^{\xi+\eta+\sigma}\int_{\overline{N}_S\times\overline{N}_S} K^{\sigma,0,0}_{2\lambda,2\mu,\nu}\left(k(\overline{n}_1,\overline{n}_2)\right) f(\overline{n}_1,\overline{n}_2)d(\overline{n}_1,\overline{n}_2)
\end{aligned}$$

where the last equality follows directly from (3.2). □

Note that as a distribution

$$A^{\zeta}_{\lambda,\mu,\nu}f(g) = |\det(g)|_{\zeta}^{\frac{1}{2}-\nu}\langle K^{\zeta+\xi+\eta,0,0}_{2\lambda,2\mu,\nu}, f(g\,\cdot)\rangle.$$

Since $\det(g)\neq 0$ for any $g\in \mathrm{GL}(2,\mathbb{R})$ the analytic properties of $A^{\zeta}_{\lambda,\mu,\nu}$ depends only on the analytic properties $K^{\sigma,0,0}_{2\lambda,2\mu,\nu}$, or in other words the analytic properties of $A^{\sigma,0,0}_{\lambda,\mu,\nu}$. Hence it suffices to study such properties for the symmetry breaking operators of $(\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R}),\Delta(\mathrm{SL}(2,\mathbb{R}))$. Going forward we shall abuse notation and write $A^{\zeta}_{s_1,s_2,s_3}$ similarly to the case for $A^{\sigma,\xi,\eta}_{s_1,s_2,s_3}$. Note however that the s_i's do not coincide with those for $A^{\sigma,\xi,\eta}_{s_1,s_2,s_3}$.

5 Analytic continuation of symmetry breaking operators

The family of kernels $K^{\sigma,\xi,\eta}_{s_1,s_2,s_3}$ defines a holomorphic family of distributions on Ω, in the sense that for any $\varphi\in C^\infty(\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R}))$ the map $(s_1,s_2,s_3)\mapsto\langle K^{\sigma,\xi,\eta}_{s_1,s_2,s_3},\varphi\rangle$ is a holomorphic map on Ω. This notion of holomorphic dependence makes sense, since the compact picture for principal series representations does not depend on the induction parameters $\lambda,\mu,\nu\in\mathbb{C}$.
To extend the domain on which this family is defined, we can construct a meromorphic extension of $K^{\sigma,\xi,\eta}_{s_1,s_2,s_3}$ by deriving explicit Bernstein–Sato identities for the kernels. To do this we employ a trick introduced in [BC12]. Let (G,H) be a strongly spherical pair of real reductive groups. Fix parabolic subgroups $P_G = M_GA_GN_G$ and $P_H = M_HA_HN_H$ of G and H respectively. Let

$$\pi_{\xi,\lambda} = \mathrm{Ind}_{P_G}^G(\xi\otimes e^{\lambda}\otimes 1)\quad\text{and}\quad \tau_{\xi,\lambda} = \mathrm{Ind}_{P_G}^G(\eta\otimes e^{\nu}\otimes 1)$$

denote principal series representations of G and H induced from $(\xi,V_\xi)\in\widehat{M}_G$, $\lambda\in(\mathfrak{a}_G^*)_{\mathbb{C}}$ and $(\eta,W_\eta)\in\widehat{M}_H$, $\nu\in\mathfrak{a}^*_{H,\mathbb{C}}$ respectively. We identify

$$\mathrm{Hom}_H(\pi_{\xi,\lambda}|_H,\tau_{\eta,\nu})\cong\mathcal{D}'(G)_{(\xi,\lambda),(\eta,\nu)}$$

with $\mathcal{D}'(G)_{(\xi,\lambda),(\eta,\nu)}$, given by

$$\begin{aligned}
\mathcal{D}'(G)_{(\xi,\lambda),(\eta,\nu)} = \{K\in D'(G)\otimes\mathrm{Hom}(V_\xi,W_\eta)\mid K(&m_Ha_Hn_Hgm_Ga_Gn_G)\\
&= a_G^{\lambda-\rho_G}a_H^{\nu+\rho_H}\eta(m_H)\circ K(g)\circ\eta(m_G)\},
\end{aligned}$$

by the map mapping $K\in\mathcal{D}'(G)_{(\xi,\lambda),(\eta,\nu)}$ to the intertwining operator

$$Af(h) = \int_{K_G} K(h^{-1}g)f(g)dg,$$

with the integral being understood in the distributional sense. We quickly remark that using the integral formula [Kna16][Formula (5.25)]

$$\int_{K_G} f(k)dk = \int_{\overline{N}_G} f(\kappa(\overline{n})e^{-2\rho_G(H(\overline{n}))}d\overline{n} \tag{5.1}$$

the intertwining operator A can also be computed in the non-compact picture whenever the kernel $K \in \mathcal{D}'(G)_{(\xi,\lambda),(\eta,\nu)}$ is given by a locally integrable function;

$$Af(h) = \int_{\overline{N}_G} K(h^{-1}\overline{n})f(\overline{n})d\overline{n}.$$

We also remark that the distribution $g \mapsto K(h^{-1}g)$ can be viewed as a distribution section of the homogeneous vector bundle over G/P_G defining $\pi_{\xi^\vee,-\lambda}$, with the check denoting the contragradient representation to (ξ, V_ξ). Similarly $h \mapsto K(h^{-1}g)$ can be viewed as a distribution section of the homogeneous vector bundle over H/P_H defining $\tau_{\eta^\vee,\nu}$. We shall write $K(h^{-1}\cdot) \in \pi^{-\infty}_{\xi^\vee}$ and $K((\cdot)^{-1}g) \in \tau^{-\infty}_{\eta,\nu}$ for short. Suppose that the space $\mathcal{D}'(G)_{(\xi,\lambda),(\eta,\nu)}$ is given by some family of distributions $K^{\sigma_1,\dots,\sigma_r}_{s_1,\dots,s_r}$ as

$$K^{\sigma_1,\dots,\sigma_r}_{s_1,\dots,s_r}(g) = |\Phi_1(g)|^{s_1}_{\sigma_1}\cdots|\Phi_r(g)|^{s_r}_{\sigma_r}$$

where $\Phi_1,\dots,\Phi_r$ are analytic functions on G and $s_1,\dots,s_r$ correspond to $(\lambda,\nu) \in (\mathfrak{a}^*_G)_\mathbb{C} \times (\mathfrak{a}^*_H)_\mathbb{C}$ via an affine coordinate transformation. This is for instance the case for $(G,H) = (\mathrm{GL}(n+1,\mathbb{R}),\mathrm{GL}(n,\mathbb{R}))$, $(G,H) = (\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R}),\mathrm{SL}(2,\mathbb{R}))$ and $(G,H) = (\mathrm{PGL}(2,\mathbb{R})\times\mathrm{PGL}(2,\mathbb{R}),\mathrm{PGL}(2,\mathbb{R}))$. Multiplication with Φ_r then maps $K^{\sigma_1,\dots,\sigma_i,\dots,\sigma_r}_{s_1,\dots,s_i,\dots,s_r}$ to $K^{\sigma_1,\dots,\sigma_i+1,\dots,\sigma_r}_{s_1,\dots,s_i+1,\dots,s_r}$ and hence it maps $\mathcal{D}'(G)_{(\sigma_1,\dots,\sigma_i,\dots,\sigma_r),(s_1,\dots,s_i,\dots,s_r)}$ to $\mathcal{D}'(G)_{(\sigma_1,\dots,\sigma_i+1,\dots\sigma_r),(s_1,\dots,s_i+1,\dots,s_r)}$, assuming that $(\sigma_1,\dots,\sigma_i+1,\dots\sigma_r)$ still corresponds to some pair $(\xi',\eta') \in \widehat{M}_G \times \widehat{M}_H$. The trick used in [BC12] is to conjugate the multiplication operator by standard intertwining operators to find Bernstein–Sato identities that relate $K^{\sigma_1,\dots,\sigma_i,\dots,\sigma_r}_{s_1,\dots,s_i,\dots,s_r}$ to $K^{\sigma_1,\dots,\sigma_i-1,\dots,\sigma_r}_{s_1,\dots,s_i-1,\dots,s_r}$. We briefly explain how standard intertwining operators can be used on $\mathcal{D}'(G)_{(\xi,\eta),(\lambda,\nu)}$:
Let $W_G = N_{K_G}(A_G)/Z_{K_G}(A_G)$ denote the Weyl group of G. Then for every $w = [\tilde{w}] \in W_G$ the integral

$$T_{w,\xi,\eta}f(g) = \int_{\overline{N_G}\cap\tilde{w}^{-1}N_G\tilde{w}} f(g\tilde{w}\overline{n})$$

converges absolutely in some range of $(\mathfrak{a}^*_G)_\mathbb{C}$ and defines an intertwining operator $\pi_{\xi,\lambda} \to \pi_{w\xi,w\lambda}$. It can be meromorphically extended in $(\mathfrak{a}^*_G)_\mathbb{C}$ to a family of intertwining operators. For $K \in \mathcal{D}'(G)_{(\xi,\eta),(\lambda,\nu)}$ we have $K(h^{-1}\cdot) \in \pi^{-\infty}_{\xi^\vee,-\lambda}$ and hence $T_{w,\xi^\vee,-\lambda}K(h^{-1}\cdot) \in \pi^{-\infty}_{w\xi^\vee,-w\lambda}$. Since this does not influence the equivariance properties of the kernel with respect to h we obtain a new kernel $K_w \in \mathcal{D}'(G)_{(w\xi,w\eta),(\lambda,\nu)}$ given by

$$K_w(g) = T_{w,\xi^\vee,-\lambda}K(g) = \int_{\overline{N}_G\cap\tilde{w}^{-1}N_G\tilde{w}} K(g\tilde{w}\overline{n}).$$

Similarly we let $W_H = N_{K_H}(A_G)/Z_{K_H}(A_H)$ denote the Weyl group of H. For $w \in W_H$ we define the standard intertwining operator $S_{w,\eta,\nu} : \tau_{\eta,\nu} \to \tau_{w\eta,w\nu}$ analogously to the case for G. For every $K \in \mathcal{D}'(G)_{(\xi,\eta),(\lambda,\nu)}$ we have $K((\cdot)^{-1})g) \in \tau^{-\infty}_{\eta,\nu}$ and we obtain $S_{w,\eta,\nu}K((\cdot)^{-1})g) \in \tau^{-\infty}_{w\eta,w\nu}$. This does not change the equivariance properties in g and hence we obtain a new kernel ${}_wK \in \mathcal{D}'(G)_{(\xi,\eta),(w\lambda,w\nu)}$ given by

$${}_wK(g) = S_{w,\eta,\nu}K((\cdot)^{-1}g) = \int_{\overline{N}_H\cap\tilde{w}^{-1}N_H\tilde{w}} K(\overline{n}\tilde{w}g)d\overline{n}.$$

In this setting we can also realize this in a slightly different way, avoiding the direct application of standard intertwining operators on $\mathcal{D}'(G)_{(\xi,\eta),(\lambda,\nu)}$: For every $A \in \mathrm{Hom}(\pi_{\xi,\eta}|H, \tau_{\eta,\nu})$ we have $S_{w,\eta,\nu} \circ A \in \mathrm{Hom}(\pi_{\xi,\eta}|H, \tau_{w\eta,w\nu})$. Hence, if A is given by the kernel K then

$$\begin{aligned} S_{w,\eta,\nu} \circ Af(h) &= \int_{\overline{N}_H \cap \tilde{w}^{-1} N_H \tilde{w}} \int_{K_G} K(\overline{n}^{-1}\tilde{w}^{-1}h^{-1}g)f(g)dgd\overline{n} \\ &= \int_{K_G} \left(\int_{\overline{N}_H \cap \tilde{w}^{-1} N_H \tilde{w}} K(\overline{n}^{-1}\tilde{w}^{-1}h^{-1}g) \right) d\overline{n} f(g)dg \\ &= \int_{K_G} {}_wK(h^{-1}g)f(g)dg. \end{aligned}$$

Hence $S_{w,\eta,\nu} \circ A$ is given by the kernel $S_{w,\eta,\nu} \circ A$. In summary we have two maps

$$\begin{aligned} &w \in W_G : \qquad \mathcal{D}'(G)_{(\xi,\eta),(\lambda,\nu)} \to \mathcal{D}'(G)_{(w\xi,w\eta),(\lambda,\nu)}, \qquad K \mapsto K_w, \\ &w \in W_H : \qquad \mathcal{D}'(G)_{(\xi,\eta),(\lambda,\nu)} \to \mathcal{D}'(G)_{(\xi,\eta),(w\lambda,w\nu)}, \qquad K \mapsto {}_wK. \end{aligned}$$

Note that the two maps commute and that for when the Weyl group element is chosen to be the identity the respective map is also the identity. For $w \in W_G$ and $w' \in W_H$ we define the conjugation map $A_{w,w'}$ by

$$\begin{aligned} A_{w,w'} : \mathcal{D}'(G)_{(\xi,\eta),(\lambda,\nu)} &\to \mathcal{D}'(G)_{(w\xi,w'\eta),(w\lambda,w'\nu)} \\ K &\mapsto {}_{w'}K_w \end{aligned}$$

with the kernel ${}_{w'}K_w$ given by

$${}_{w'}K_w = \int_{\overline{N}_G \cap \tilde{w}^{-1} N_G \tilde{w}} \int_{\overline{N}_G \cap \tilde{w}' N_H \tilde{w}'^{-1}} K(\overline{n}_H \tilde{w}'^{-1} g \tilde{w} \overline{n}_G) d\overline{n}_H d\overline{n}_G.$$

Let

$$\begin{aligned} M_{\Phi_i} : \mathcal{D}'(G)_{s_1,\ldots,s_r} &\to \mathcal{D}'(G)_{s_1,\ldots,s_i+1,\ldots,s_r} \\ K &\mapsto \Phi_i \cdot K \end{aligned}$$

denote the multiplication operator by Φ_i. If $w_1, w_2 \in W_G$ and $w_1', w_2' \in W_H$ is such that $A_{w_2,w_2'} \circ M_{\Phi_i} \circ A_{w_1,w_1'}$ is a differential operator that maps $\mathcal{D}'(G)_{s_1,\ldots,s_r}$ to $\mathcal{D}'(G)_{s_1,\ldots,s_i-1,\ldots,s_r}$, then the result is a Bernstein–Sato identity which can be used to meromorphically extend $K_{s_1,\ldots,s_r}$. To see how this works for the case when $(G,H) = (\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}), \Delta(\mathrm{SL}(2,\mathbb{R}))$ let w_0 be a representative for the longest Weyl group element of $\mathrm{SL}(2,\mathbb{R})$ and put $w_0^i = w_0$ if $i = 1$ and $w_0^i = \mathrm{id}$ if $i = 0$. Then we have

$$A_{((w_0^i, w_0^j), w_0^l)} K^{\sigma,\xi,\eta}_{\lambda,\mu,\nu} = \mathrm{const} \times K^{\sigma,\xi,\eta}_{(-1)^i\lambda,(-1)^j\mu,(-1)^l\nu}$$

Meanwhile we have the multiplication maps

$$\begin{aligned} M_{\Phi_0} &: (\lambda,\mu,\nu) \to (\lambda+1,\mu+1,\nu) \\ M_{\Phi_1} &: (\lambda,\mu,\nu) \to (\lambda+1,\mu,\nu-1) \\ M_{\Phi_2} &: (\lambda,\mu,\nu) \to (\lambda,\mu+1,\nu-1) \end{aligned}$$

This suggests that the relevant composition maps are

$$\begin{aligned}
D_{1,\mathbf{s}} &= A_{(w_0,w_0),\mathrm{id}} \circ M_{\Phi_0} \circ A_{(w_0,w_0),\mathrm{id}} : (s_1,s_2,s_3) \to (s_1-1,s_2,s_3)\\
D_{2,\mathbf{s}} &= A_{(w_0,\mathrm{id}),w_0} \circ M_{\Phi_1} \circ A_{(w_0,\mathrm{id}),w_0} : (s_1,s_2,s_3) \to (s_1,s_2-1,s_3)\\
D_{3,\mathbf{s}} &= A_{(\mathrm{id},w_0),w_0} \circ M_{\Phi_1} \circ A_{(\mathrm{id},w_0),w_0} : (s_1,s_2,s_3) \to (s_1,s_2,s_3-1)
\end{aligned}$$

To see that these are indeed differential operators we note that since $T_w \circ T_{w^{-1}} = \text{const} \times \mathrm{id}$ and $S_w \circ S_{w^{-1}} = \text{const} \times \mathrm{id}$ we have $D \circ A_{w_1,w_1'} = A_{w_2,w_2'} \circ M_{\Phi_r}$. Realising $A_{(w_0,w_0),\mathrm{id}}$ in the non-compact by decomposing $w_0\overline{n}_x = \overline{n}_y a_t n$ we find $t = x$ and $y = -\frac{1}{x}$ and hence

$$\begin{aligned}
A_{(w_0,w_0),\mathrm{id}} \circ K^{\sigma,\xi,\eta}_{\lambda,\mu,\nu}(\overline{n}_z,\overline{n}_w) &= \int_{\overline{N}_S}\int_{\overline{N}_S} |t_1|_\xi^{\lambda-1}|t_2|_\eta^{\mu-1} K^{\sigma,\xi,\eta}_{-\lambda,-\mu,\nu}((\overline{n}_z\overline{n_1},\overline{n}_w\overline{n_2}))d(\overline{n_1},\overline{n_2})\\
&= (-1)^{\xi+\eta}\int_{\mathbb{R}}\int_{\mathbb{R}} |x|_\xi^{-\lambda-1}|y|_\eta^{-\mu-1}K^{\sigma,\xi,\eta}_{-\lambda,-\mu,\nu}(z+x,w+y)dxdy.
\end{aligned}$$

On the other hand we find

$$\begin{aligned}
&A_{(w_0,w_0),\mathrm{id}} \circ M_{\Phi_0} \circ K^{\sigma,\xi,\eta}_{\lambda,\mu,\nu}(\overline{n}_z,\overline{n}_w)\\
&\quad= (-1)^{\xi+\eta}\int_{\mathbb{R}}\int_{\mathbb{R}} |x|_\xi^{-\lambda-2}|y|_\eta^{\mu-2}\Phi_0(\overline{n}_{z+x},\overline{n}_{w+y})K^{\sigma,\eta,\xi}_{-\lambda,-\mu,\nu}(z+x,w+y)dxdy\\
&\quad= (-1)^{\xi+\eta}\int_{\mathbb{R}}\int_{\mathbb{R}} |x|_\xi^{-\lambda-2}|y|_\eta^{\mu-2}(x-y+z-w)K^{\sigma,\xi,\eta}_{-\lambda,-\mu,\nu}(z+x,w+y)dxdy.
\end{aligned}$$

We get similar setups for $D_{2,\mathbf{s}}$ and $D_{3,\mathbf{s}}$. The corresponding differential operator is then obtained by using partial integration to relate the expressions. This yields the following

Theorem 6. *For $\boldsymbol{s} = (s_1,s_2,s_3) \in \mathbb{C}^3$ and $\Lambda = (\sigma,\xi,\eta)$ the composition maps $D_{i,\boldsymbol{s}}$, in the non-compact picture, are given by*

$$D_{1,\boldsymbol{s}} = (x-y)\partial_x\partial_y + (s_1+s_3)\partial_x - (s_1+s_2)\partial_y \tag{5.2}$$
$$D_{2,\boldsymbol{s}} = (s_1+2s_2+s_3)\partial_x + (s_1+s_2)\partial_y - x(\partial_x\partial_x + \partial_x\partial_y) \tag{5.3}$$
$$D_{3,\boldsymbol{s}} = (s_1+s_3)\partial_x + (s_1+s_2+2s_3)\partial_y - y(\partial_y\partial_y + \partial_x\partial_y). \tag{5.4}$$

For $K^\Lambda_{\boldsymbol{s}} = K^{\sigma,\xi,\eta}_{s_1,s_2,s_3}$ we have the following Bernstein–Sato identities:

$$D_{i,\boldsymbol{s}}K^\Lambda_{\boldsymbol{s}} = b_i(\boldsymbol{s})K^{\Lambda-\boldsymbol{e}_i}_{\boldsymbol{s}-\boldsymbol{e}_i} \tag{5.5}$$

where $\boldsymbol{e}_i = (\delta_{1i},\delta_{2i},\delta_{3i})$ and

$$b_i(s_1,s_2,s_3) = s_i(1+s_1+s_2+s_3). \tag{5.6}$$

The Bernstein–Sato relations allow for a meromorphic continuation of the kernels since

$$K^\Lambda_{\mathbf{s}} = \frac{1}{b_i(\mathbf{s}+e_i)}D_{i,\mathbf{s}}K^{\Lambda+e_i}_{\mathbf{s}+e_i}.$$

If we normalize the kernels by canceling out the zeroes of the Bernstein–Sato polynomials $b_i(\mathbf{s})$, we obtain a new family of kernels $\widetilde{K}^\Lambda_{\mathbf{s}}$, which depend holomorphically on the parameters $\mathbf{s} = (s_1,s_2,s_3)$:

$$\widetilde{K}^\Lambda_{\mathbf{s}} = \frac{K^{\sigma,\xi,\eta}_{s_1,s_2,s_3}}{\Gamma(s_1+1)\Gamma(s_2+1)\Gamma(s_3+1)\Gamma(s_1+s_2+s_3+2)}. \tag{5.7}$$

Remark. The classical Riesz distribution $r_{\xi,\lambda}(x) = |x|_\xi^\lambda$ on $\mathbb{R}$, defined for $\mathrm{Re}(\lambda) > -1$, satisfies the Bernstein–Sato relation $\frac{d|x|_\xi^\lambda}{dx} = \lambda |x|_{\xi+1}^\lambda$, and the procedure we deployed above would then imply that the holomorphic extension to $\lambda \in \mathbb{C}$ should be $\widetilde{r}_\lambda = \frac{r_{\xi,\lambda}}{\Gamma(\lambda+1)}$. However it is well known that this normalization introduces unnecessary zeroes, i.e there exists an analytic extension of r_λ to $\lambda \in \mathbb{C}$ such that $r_\lambda \neq 0$ for any $\lambda \in \mathbb{C}$. The same problem happens to occur for our normalization $\widetilde{K}_{\mathbf{s}}^\Lambda$, however the problem is a bit more finicky in this setting, since we are dealing with several complex variables.

Part III

Zeroes of the Kernels $\widetilde{K}_{\mathbf{s}}^{\Lambda}$

Analogously to the case of the holomorphic extension of the Riesz distributions $\widetilde{r}_\lambda$, we can investigate the set of zeroes

$$Z(\widetilde{K}_{\mathbf{s}}^{\Lambda}) = \{(s_1, s_2, s_3) \in \mathbb{C}^3 \mid A^{\sigma,\xi,\eta}_{s_1,s_2,s_3} f(g) = 0 \ \forall f \in C^\infty(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})\},$$

by evaluating the kernel against against a specific test-function $f \in C^\infty(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))$. The kernels arise from a representation theoretic viewpoint and this suggests that for the case $\Lambda = (\sigma, 0, 0)$ a suitable candidate for f should be the spherical vector $\varphi_0 \otimes \varphi_0$ in $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. However for arbitrary choice of Λ we don't necessarily have a K-finite vector. A suitable alternative is the tensor product of lowest K-types of $\mathrm{SL}(2,\mathbb{R})$. The value of $A^{0,0,0}_{s_1,s_2,s_3}$ on $\varphi_0 \otimes \varphi_0$ is already well documented in the literature, see e.g. [OC11][p. 15], however this does not seem to be the case for lowest K-types. We employ the method used in [CKØP11], extending the result for $n = 1$ to the signed case.

6 Bernstein–Reznikov integrals

Consider the integral operators on $C^\infty(K_S)$ given by

$$R^\sigma_\mu f(y) = \int_{K_S} |\sin(x-y)|^s_\sigma f(x) dx \quad \text{and} \quad B^s_\sigma f(y) = \int_{K_S} |\sin(x-y)|^s_\sigma \cos(x-y) f(x) dx.$$

Let $\psi_m = e^{im\theta}$ and recall that the K_S-finite vectors for a principal series representation $\pi_{\xi,\lambda}$ of $\mathrm{SL}(2,\mathbb{R})$ are given by $\bigoplus_{m \in 2\mathbb{Z}+\varepsilon} \mathbb{C} \cdot \psi_m$. We realize $C^\infty(K_S)$ as $C^\infty(K_S) \cong C^\infty(S^1)$. Then we have

$$\begin{aligned}
&\langle A^{\sigma,\xi,\eta}_{s_1,s_2,s_3}(\psi_m \otimes \psi_n), \psi_{(m+n)} \rangle_{K_S} \\
&= \int_{K_S}\int_{K_S}\int_{K_S} K^{\sigma,\varepsilon,\eta}_{s_1,s_2,s_3}(k_{\theta_1} \cdot k_{\theta_3}^{-1}, k_{\theta_2} \cdot k_{\theta_3}^{-1}) \psi_m(k_{\theta_1}) \psi_n(k_{\theta_2}) \psi_{-(m+n)}(k_{\theta_3}) dk_{\theta_1} dk_{\theta_2} dk_{\theta_3} \\
&= \int_{K_S}\int_{K_S}\int_{K_S} |\sin(\theta_2 - \theta_1)|^{s_1}_\sigma |\sin(\theta_1 - \theta_3)|^{s_2}_{\sigma+\xi} |\sin(\theta_2 - \theta_3)|^{s_3}_{\sigma+\eta} e^{im(\theta_1-\theta_3)} e^{in(\theta_2-\theta_3)} d\theta_1 d\theta_2 d\theta_3.
\end{aligned}$$

Meanwhile we also have

$$\begin{aligned}
R^{s_2}_{\sigma_2} \circ R^{s_1}_{\sigma_1} \circ R^{s_3}_{\sigma_3} f(\varphi) &= \int_0^{2\pi} |\sin(\theta_1 - \varphi)|^{s_2}_{\sigma_2} R^{s_1}_{\sigma_1} \circ R^{s_3}_{\sigma_3} f(\theta_1) d\theta_1 \\
&= \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} |\sin(\theta_1 - \varphi)|^{s_2}_{\sigma_2} |\sin(\theta_2 - \theta_1)|^{s_1}_{\sigma_1} |\sin(\theta_3 - \theta_2)|^{s_3}_{\sigma_3} f(\theta_3) d\theta_1 d\theta_2 d\theta_3,
\end{aligned}$$

and hence we find that

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} |\sin(\theta_2-\theta_1)|_{\sigma_1}^{s_1}|\sin(\theta_1-\theta_3)|_{\sigma_2}^{s_2}|\sin(\theta_2-\theta_3)|_{\sigma_3}^{s_3} d\theta_1 d\theta_2 d\theta_3 = (-1)^{\sigma_3}\operatorname{tr}(R_{\sigma_2}^{s_2}\circ R_{\sigma_1}^{s_1}\circ R_{\sigma_3}^{s_3}).$$

By the same computation we also find

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} |\sin(\theta_2-\theta_1)|_{\sigma_1}^{s_1}|\sin(\theta_1-\theta_3)|_{\sigma_2}^{s_2}|\sin(\theta_2-\theta_3)|_{\sigma_3}^{s_3}\cos(\theta_1-\theta_3) d\theta_1 d\theta_2 d\theta_3 \\ = (-1)^{\sigma_3}\operatorname{tr}(B_{\sigma_2}^{s_2}\circ R_{\sigma_1}^{s_1}\circ R_{\sigma_3}^{s_3}).\end{aligned}$$

Hence for $m,n\in\{-1,0,1\}$ the evaluation of $A_{s_1,s_2,s_3}^{\sigma,\xi,\eta}$ against the lowest K_S-type can be described by computing the spectrum of $R_{\sigma_2}^{s_2}\circ R_{\sigma_1}^{s_1}\circ R_{\sigma_3}^{s_3}$ and $B_{\sigma_2}^{s_2}\circ R_{\sigma_1}^{s_1}\circ R_{\sigma_3}^{s_3}$. In principle, evaluation against any K_S-finite vector can be computed using this method in combination with the classical trigonometric identities, however the corresponding computation might be exceedingly difficult to carry out or the result might be hard to interpret in terms of poles and zeroes of $A_{s_1,s_2,s_3}^{\sigma,\xi,\eta}$. As such we shall focus on the cases when $m,n\in\{-1,0,1\}$ and the special case when $m=2$ and $n=0$. An easy calculation shows that the integral operators R_σ^s and B_σ^s have shared eigenvectors ψ_m and using the integral formula

Lemma 4. *For* $\operatorname{Re}(\nu)>0$ *we have*

$$\int_0^\pi \sin^{\nu-1}(x)e^{iax}\,dx = \frac{2^{1-\nu}\pi e^{\frac{ai\pi}{2}}\Gamma(\nu)}{\Gamma(\frac{\nu+1-a}{2})\Gamma(\frac{\nu+1+a}{2})}.$$

We then find that their corresponding eigenvalues are given by

$$\begin{aligned}R_\sigma^s\psi_m(x) &= \int_0^{2\pi}|\sin(\theta-x)|_\sigma^s e^{im\theta}d\theta \\ &= \int_{0-x}^{2\pi-x}|\sin(\theta)|_\sigma^s e^{im\theta}d\theta\cdot\psi_m(x) \\ &= (1+(-1)^{\sigma+m})\int_0^\pi \sin(\theta)e^{im\theta}d\theta\cdot\psi_m(x) \\ &= (1+(-1)^{\sigma+m})\frac{2^{-s}\pi i^m\Gamma(s+1)}{\Gamma(\frac{s+m}{2}+1)\Gamma(\frac{s-m}{2}+1)}\psi_m(x) \\ &= \omega_m^\sigma \ell_m(s)\psi_m(x),\end{aligned}$$

with $\ell_m(s)=\frac{2^{-s}\pi i^m\Gamma(s+1)}{\Gamma(\frac{s+m}{2}+1)\Gamma(\frac{s-m}{2}+1)}$. Likewise we also get

$$\begin{aligned}&B_\sigma^s\psi_m(x) = \int_0^{2\pi}|\sin(\theta-x)|_\sigma^s\cos(\theta-x)e^{im\theta}d\theta = \frac{1}{2}\int_{0-x}^{2\pi-x}|\sin(\theta)|_\sigma^s(e^{i(m+1)\theta}+e^{i(m-1)\theta})d\theta\psi_m(x) \\ &= \frac{\omega_{m+1}^\sigma}{2}(\ell_{m+1}(s)+\ell_{m-1}(s))\psi_m(x) \\ &= \omega_{m+1}^\sigma 2^{-s-1}\pi\Gamma(s+1)i^{m+1}\left(\frac{1}{\Gamma(\frac{s+m+1}{2}+1)\Gamma(\frac{s-m-1}{2}+1)}-\frac{1}{\Gamma(\frac{s+m-1}{2}+1)\Gamma(\frac{s-m+1}{2}+1)}\right)\psi_m(x) \\ &= \omega_{m+1}^\sigma 2^{-s-1}\pi\Gamma(s+1)i^{m+1}\frac{1}{\Gamma(\frac{s+m+1}{2}+1)\Gamma(\frac{s-m+1}{2}+1)}\left(\frac{s-m-1}{2}+1-\frac{s+m-1}{2}-1\right)\psi_m(x) \\ &= \omega_{m+1}^\sigma 2^{-s}\pi\Gamma(s+1)i^{m-1}\frac{m}{\Gamma(\frac{s+m+1}{2}+1)\Gamma(\frac{s-m+1}{2}+1)}\psi_m(x) \\ &= \omega_{m+1}^\sigma q_m(s)\psi_m(x),\end{aligned}$$

with $q_m = 2^{-s}\pi\Gamma(s+1)i^{m-1}\frac{m}{\Gamma(\frac{s+m+1}{2}+1)\Gamma(\frac{s-m+1}{2}+1)}$. We note the recursion relation

$$\frac{\ell_{m+2}(s)}{\ell_m(s)} = (-1)\cdot\frac{\Gamma(\frac{s+m}{2}+1)\Gamma(\frac{s-m}{2}+1)}{\Gamma(\frac{s+m}{2}+2)\Gamma(\frac{s-m}{2})} = \frac{\frac{m}{2}-\frac{s}{2}}{\frac{m}{2}+\frac{s}{2}+1}$$

which implies

$$\ell_m(s) = \begin{cases} \frac{(\frac{-s}{2})_k}{(\frac{s}{2}+1)_k}\ell_0(s), & \sigma = 0 \text{ and } m = 2k \\ \frac{(\frac{1-s}{2})_k}{(\frac{s+1}{2}+1)_k}\ell_1(s), & \sigma = 1 \text{ and } m = 2k+1, \end{cases}$$

where $(a)_k = a\cdot(a+1)\cdots(a+k-1)$ denotes the Pochhammer symbol. Analogously, for q_m we find

$$\frac{q_{m+2}(s)}{q_m(s)} = \frac{(m+2)(m-s-1)}{m(s+3+m)}.$$

Hence for $m = 2k$ we have

$$\begin{aligned} q_{2k} &= q_2(s)\prod_{j=1}^{k-1}\frac{j+1}{j}\frac{j-\frac{s+1}{2}}{j+\frac{s+3}{2}} = q_2(s)k\cdot\prod_{j=0}^{k-2}\frac{j+1-\frac{s+1}{2}}{j+1+\frac{s+3}{2}} \\ &= k\cdot\frac{(\frac{1-s}{2})_{k-1}}{(\frac{s+3}{2}+1)_{k-1}}q_2(s) = \frac{(2)_{k-1}}{(1)_{k-1}}\cdot\frac{(\frac{1-s}{2})_{k-1}}{(\frac{s+3}{2}+1)_{k-1}}q_2(s) \end{aligned}$$

and for $m = 2k+1$ we find

$$q_{2k+1} = q_1(s)\prod_{j=0}^{k-1}\frac{2j+3}{2j+1}\frac{j-\frac{s}{2}}{j+\frac{s}{2}+2} = (2k+1)q_1(s)\frac{(\frac{-s}{2})_k}{(\frac{s+2}{2}+1)_k} = q_1(s)\frac{(\frac{3}{2})_k}{(\frac{1}{2})_k}\frac{(\frac{-s}{2})_k}{(\frac{s+2}{2}+1)_k}.$$

Evaluation on lowest K_S-types

Let at first $\xi = \eta = 1$. Then $\psi_1\otimes\psi_{-1}\in\pi_{\xi,\lambda}\otimes\pi_{\eta,\mu}$ and we find that

$$\begin{aligned} &\langle A^{\sigma,1,1}_{s_1,s_2,s_3}(\psi_1\otimes\psi_{-1}),\psi_0\rangle_K \\ &= \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}|\sin(\theta_2-\theta_1)|^{s_1}_{\sigma}|\sin(\theta_1-\theta_3)|^{s_2}_{\sigma+1}|\sin(\theta_2-\theta_3)|^{s_3}_{\sigma+1}e^{i(\theta_1-\theta_2)}d\theta_1 d\theta_2 d\theta_3 \\ &= \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}|\sin(\theta_2-\theta_1)|^{s_1}_{\sigma}|\sin(\theta_1-\theta_3)|^{s_2}_{\sigma+1}|\sin(\theta_2-\theta_3)|^{s_3}_{\sigma+1}\cos(\theta_1-\theta_2)d\theta_1 d\theta_2 d\theta_3 \\ &- i\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}|\sin(\theta_2-\theta_1)|^{s_1+1}_{\sigma+1}|\sin(\theta_1-\theta_3)|^{s_2}_{\sigma+1}|\sin(\theta_2-\theta_3)|^{s_3}_{\sigma+1}d\theta_1 d\theta_2 d\theta_3. \end{aligned}$$

For $\sigma = 0$ the second integral is not invariant under the transformation $(\theta_1, \theta_2, \theta_3) \mapsto (-\theta_1, -\theta_2, -\theta_3)$ and hence vanishes. Hence we find

$$\begin{aligned}
&\langle A^{0,1,1}_{s_1,s_2,s_3}(\psi_1 \otimes \psi_{-1}), \psi_0\rangle_K = (-1)\operatorname{tr}(R_1^{s_2} \circ B_0^{s_1} \circ R_1^{s_3}) = (-1)\sum_{m\in\mathbb{Z}} (\omega_m^1)^3 q_m(s_1)\ell_m(s_2)\ell_m(s_3) \\
&= 2^4 q_1(s_1)\ell_1(s_2)\ell_1(s_3) \sum_{k=0}^{\infty} (1)_k \frac{(\frac{3}{2})_k}{(\frac{1}{2})_k} \frac{(\frac{-s_1}{2})_k}{(\frac{s_1+2}{2}+1)_k} \frac{(\frac{1-s_2}{2})_k}{(\frac{s_2+1}{2}+1)_k} \frac{(\frac{1-s_3}{2})_k}{(\frac{s_3+1}{2}+1)_k} \frac{1}{k!} \\
&= 2^4 q_1(s_1)\ell_1(s_2)\ell_1(s_3) {}_5F_4\begin{pmatrix} 1 & \frac{3}{2} & \frac{-s_1}{2} & \frac{1-s_2}{2} & \frac{1-s_2}{2} \\ & \frac{1}{2} & \frac{s_1+2}{2}+1 & \frac{s_2+1}{2}+1 & \frac{s_3+1}{2}+1 \end{pmatrix} \\
&= \frac{\pi^3 2^{4-s_1-s_2-s_3}\Gamma(s_1+1)\Gamma(s_2+1)\Gamma(s_3+1))}{\Gamma(\frac{s_1+4}{2})\Gamma(\frac{s_1+2}{2})\Gamma(\frac{s_2+3}{2})\Gamma(\frac{s_2+1}{2})\Gamma(\frac{s_3+3}{2})\Gamma(\frac{s_3+1}{2})} {}_5F_4\begin{pmatrix} 1 & \frac{3}{2} & \frac{-s_1}{2} & \frac{1-s_2}{2} & \frac{1-s_2}{2} \\ & \frac{1}{2} & \frac{s_1+2}{2}+1 & \frac{s_2+1}{2}+1 & \frac{s_3+1}{2}+1 \end{pmatrix} \\
&= \frac{\pi^{\frac{3}{2}} 2^4 \Gamma(\frac{s_1+1}{2})\Gamma(\frac{s_2}{2}+1)\Gamma(\frac{s_3}{2}+1)}{\Gamma(\frac{s_1+4}{2})\Gamma(\frac{s_2+3}{2})\Gamma(\frac{s_3+3}{2})} {}_5F_4\begin{pmatrix} 1 & \frac{3}{2} & \frac{-s_1}{2} & \frac{1-s_2}{2} & \frac{1-s_2}{2} \\ & \frac{1}{2} & \frac{s_1+2}{2}+1 & \frac{s_2+1}{2}+1 & \frac{s_3+1}{2}+1 \end{pmatrix},
\end{aligned}$$

where ${}_5F_4$ denotes the generalized hypergemeotric funciton. Using the Dougall-Ramanujan identity

$$ {}_5F_4\begin{pmatrix} m-1 & \frac{m+1}{2} & -x & -y & -z \\ & \frac{m-1}{2} & x+m & y+m & z+m \end{pmatrix} = \frac{\Gamma(x+m)\Gamma(y+m)\Gamma(z+m)\Gamma(x+y+z+m)}{\Gamma(m)\Gamma(x+y+m)\Gamma(y+z+m)\Gamma(x+z+m)}$$

then gives

$$\begin{aligned}
&\langle A^{0,1,1}_{s_1,s_2,s_3}(\psi_1 \otimes \psi_{-1}), \psi_0\rangle_K \\
&= (-1)2^4 q_1(s_1)\ell_1(s_2)\ell_1(s_3) \sum_{k=0}^{\infty} (1)_k \frac{(\frac{3}{2})_k}{(\frac{1}{2})_k} \frac{(\frac{-s_1}{2})_k}{(\frac{s_1+2}{2}+1)_k} \frac{(\frac{1-s_2}{2})_k}{(\frac{s_2+1}{2}+1)_k} \frac{(\frac{1-s_3}{2})_k}{(\frac{s_3+1}{2}+1)_k} \frac{1}{k!} \\
&= \frac{\pi^{\frac{3}{2}} 2^4 \Gamma(\frac{s_1+1}{2})\Gamma(\frac{s_2}{2}+1)\Gamma(\frac{s_3}{2}+1)}{\Gamma(\frac{s_1+4}{2})\Gamma(\frac{s_2+3}{2})\Gamma(\frac{s_3+3}{2})} \frac{\Gamma(\frac{s_1+4}{2})\Gamma(\frac{s_2+3}{2})\Gamma(\frac{s_3+3}{2})\Gamma(\frac{s_1+s_2+s_3}{2}+1)}{\Gamma(2)\Gamma(\frac{s_1+s_2+3}{2})\Gamma(\frac{s_1+s_3+3}{2})\Gamma(\frac{s_2+s_3}{2}+1)} \\
&= \frac{2^4 \pi^{\frac{3}{2}} \Gamma(\frac{s_1+1}{2})\Gamma(\frac{s_2}{2}+1)\Gamma(\frac{s_3}{2}+1)\Gamma(\frac{s_1+s_2+s_3}{2}+1)}{\Gamma(\frac{s_1+s_2+3}{2})\Gamma(\frac{s_1+s_3+3}{2})\Gamma(\frac{s_2+s_3}{2}+1)}.
\end{aligned}$$

To make the calculations to follow a bit more manageable we introduce the following meromorphic function

$$B(s_1, s_2, s_3) = -\operatorname{tr}(R_1^{s_2} \circ B_0^{s_1} \circ R_1^{s_3}).$$

By the above we have

$$B(s_1, s_2, s_3) = \frac{2^4 \pi^{\frac{3}{2}} \Gamma(\frac{s_1+1}{2})\Gamma(\frac{s_2}{2}+1)\Gamma(\frac{s_3}{2}+1)\Gamma(\frac{s_1+s_2+s_3}{2}+1)}{\Gamma(\frac{s_1+s_2+3}{2})\Gamma(\frac{s_1+s_3+3}{2})\Gamma(\frac{s_2+s_3}{2}+1)}. \tag{6.1}$$

For $\sigma = 1$ the first integral vanishes and we find

$$\langle A^{1,1,1}_{s_1,s_2,s_3}(\psi_1 \otimes \psi_{-1}), \psi_0\rangle_K = -i\operatorname{tr}(R_0^{s_2} \circ R_0^{s_1+1} \circ R_0^{s_3}).$$

However the Dougall-Ramanujan identity is not available in this case. We instead note that

$$
\begin{aligned}
&\langle A^{1,1,1}_{s_1,s_2,s_3}(\psi_1\otimes\psi_{-1}),\psi_0\rangle_K\\
&=\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}|\sin(\theta_2-\theta_1)|_1^{s_1}|\sin(\theta_1-\theta_3)|_0^{s_2}|\sin(\theta_2-\theta_3)|_0^{s_3}e^{i(\theta_1-\theta_3)}e^{-i(\theta_2-\theta_3)}d\theta_1d\theta_2d\theta_3\\
&=i\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}|\sin(\theta_2-\theta_1)|_1^{s_1}|\sin(\theta_1-\theta_3)|_1^{s_2+1}|\sin(\theta_2-\theta_3)|_0^{s_3}\cos(\theta_3-\theta_2)d\theta_1d\theta_2d\theta_3\\
&-i\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}|\sin(\theta_2-\theta_1)|_1^{s_1}|\sin(\theta_1-\theta_3)|_0^{s_2}|\sin(\theta_2-\theta_3)|_1^{s_3+1}\cos(\theta_1-\theta_3)d\theta_1d\theta_2d\theta_3\\
&=i(\operatorname{tr}(R_1^{s_2+1}\circ R_1^{s_1}\circ B_0^{s_3})+\operatorname{tr}(B_0^{s_2}\circ R_1^{s_1}\circ R_1^{s_3+1}))=-i(B(s_3,s_2+1,s_1)+B(s_2,s_1,s_3+1))\\
&=\frac{1}{i}\left(\frac{2^4\pi^{\frac32}\Gamma(\frac{s_3+1}{2})\Gamma(\frac{s_2+3}{2})\Gamma(\frac{s_1}{2}+1)\Gamma(\frac{s_1+s_2+s_3+3}{2})}{\Gamma(\frac{s_2+s_3}{2}+2)\Gamma(\frac{s_1+s_3+3}{2})\Gamma(\frac{s_1+s_2+3}{2})}+\frac{2^4\pi^{\frac32}\Gamma(\frac{s_2+1}{2})\Gamma(\frac{s_1}{2}+1)\Gamma(\frac{s_3+3}{2})\Gamma(\frac{s_1+s_2+s_3+3}{2})}{\Gamma(\frac{s_1+s_2+3}{2})\Gamma(\frac{s_2+s_3}{2}+2)\Gamma(\frac{s_1+s_3+3}{2})}\right)\\
&=-i2^4\pi^{\frac32}\frac{\Gamma(\frac{s_1}{2}+1)\Gamma(\frac{s_1+s_2+s_3+3}{2})}{\Gamma(\frac{s_1+s_2+3}{2})\Gamma(\frac{s_2+s_3}{2}+2)\Gamma(\frac{s_1+s_3+3}{2})}\left(\Gamma(\frac{s_2+3}{2})\Gamma(\frac{s_3+1}{2})+\Gamma(\frac{s_2+1}{2})\Gamma(\frac{s_3+3}{2})\right)\\
&=-i2^4\pi^{\frac32}\frac{\left(\frac{s_2+s_3}{2}+1\right)\Gamma(\frac{s_1}{2}+1)\Gamma(\frac{s_2+1}{2})\Gamma(\frac{s_3+1}{2})\Gamma(\frac{s_1+s_2+s_3+3}{2})}{\Gamma(\frac{s_1+s_2+3}{2})\Gamma(\frac{s_2+s_3}{2}+2)\Gamma(\frac{s_1+s_3+3}{2})}\\
&=-i2^4\pi^{\frac32}\frac{\Gamma(\frac{s_1}{2}+1)\Gamma(\frac{s_2+1}{2})\Gamma(\frac{s_3+1}{2})\Gamma(\frac{s_1+s_2+s_3+3}{2})}{\Gamma(\frac{s_1+s_2+3}{2})\Gamma(\frac{s_2+s_3}{2}+1)\Gamma(\frac{s_1+s_3+3}{2})}.
\end{aligned}
$$

As before we introduce the following notation:

$$R(s_1,s_2,s_3)=-i\operatorname{tr}(R_0^{s_2}\circ R_0^{s_1+1}\circ R_0^{s_3}) \tag{6.2}$$

and we note that the previous computations show that

$$R(s_1,s_2,s_3)=\langle A^{1,1,1}_{s_1,s_2,s_3}(\psi_1\otimes\psi_{-1}),\psi_0\rangle_K=\frac{2^4\pi^{\frac32}}{i}\frac{\Gamma(\frac{s_1}{2}+1)\Gamma(\frac{s_2+1}{2})\Gamma(\frac{s_3+1}{2})\Gamma(\frac{s_1+s_2+s_3+3}{2})}{\Gamma(\frac{s_1+s_2+3}{2})\Gamma(\frac{s_2+s_3}{2}+1)\Gamma(\frac{s_1+s_3+3}{2})}. \tag{6.3}$$

First spherical and second non-spherical, i.e $\xi=1$ and $\eta=0$

Let $\xi=1$ and $\eta=0$. Then we have that $\psi_1\otimes\psi_0\in\pi_{\xi,\lambda}\otimes\pi_{\eta,\mu}$ and we find that

$$
\begin{aligned}
&\langle A^{\sigma,1,0}_{s_1,s_2,s_3}(\psi_1\otimes\psi_0),\psi_1\rangle_K\\
&=\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}|\sin(\theta_2-\theta_1)|_\sigma^{s_1}|\sin(\theta_1-\theta_3)|_{\sigma+1}^{s_2}|\sin(\theta_2-\theta_3)|_\sigma^{s_3}e^{i(\theta_1-\theta_3)}d\theta_1d\theta_2d\theta_3\\
&=\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}|\sin(\theta_2-\theta_1)|_\sigma^{s_1}|\sin(\theta_1-\theta_3)|_{\sigma+1}^{s_2}|\sin(\theta_2-\theta_3)|_\sigma^{s_3}\cos(\theta_1-\theta_3)d\theta_1d\theta_2d\theta_3\\
&+i\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}|\sin(\theta_2-\theta_1)|_\sigma^{s_1}|\sin(\theta_1-\theta_3)|_\sigma^{s_2+1}|\sin(\theta_2-\theta_3)|_\sigma^{s_3}d\theta_1d\theta_2d\theta_3
\end{aligned}
$$

For $\sigma=0$ the first integral vanishes and we find

$$\langle A^{0,1,0}_{s_1,s_2,s_3}(\psi_1\otimes\psi_0),\psi_1\rangle_K=i\operatorname{tr}(R_0^{s_2+1}\circ R_0^{s_1}\circ R_0^{s_3})=i\operatorname{tr}(R_0^{s_1}\circ R_0^{s_2+1}\circ R_0^{s_3})=-R(s_2,s_1,s_3)$$

For $\sigma=1$ we find

$$\langle A^{1,1,0}_{s_1,s_2,s_3}(\psi_1\otimes\psi_0),\psi_1\rangle_K=-\operatorname{tr}(B_0^{s_2}\circ R_1^{s_1}\circ R_1^{s_3})=-\operatorname{tr}(R_1^{s_1}\circ B_0^{s_2}\circ R_1^{s_3})=B(s_2,s_1,s_3)$$

First non-spherical and second spherical, i.e $\xi = 0$ and $\eta = 1$

Let $\xi = 0$ and $\eta = 1$. Since permuting λ and μ permutes s_2 and s_3, this computation is in practice redundant, but acts as a nice sanity check. We have that $\psi_0 \otimes \psi_1 \in \pi_{\xi,\lambda} \otimes \pi_{\eta,\mu}$ and it follows from symmetry that we must have that for $\sigma = 0$

$$\langle A^{0,0,1}_{s_1,s_2,s_3}(\psi_0 \otimes \psi_1), \psi_1\rangle_K = i\,\mathrm{tr}(R_0^{s_2} \circ R_0^{s_1} \circ R_0^{s_3+1}) = i\,\mathrm{tr}(R_0^{s_1} \circ R_0^{s_3+1} \circ R_0^{s_2}) = -R(s_3, s_1, s_2).$$

For $\sigma = 1$ we find

$$\langle A^{1,0,1}_{s_1,s_2,s_3}(\psi_0 \otimes \psi_1), \psi_1\rangle_K = -\,\mathrm{tr}(R_1^{s_2} \circ R_1^{s_1} \circ B_0^{s_3}) = -\,\mathrm{tr}(R_1^{s_1} \circ B_0^{s_3} \circ R_1^{s_2}) = B(s_3, s_1, s_2).$$

The Spherical case, i.e $\xi = 0$ and $\eta = 0$

Let $\xi = 0$ and $\eta = 0$. Then we have that $\psi_0 \otimes \psi_0 \in \pi_{\xi,\lambda} \otimes \pi_{\eta,\mu}$ and we have

$$\langle A^{\sigma,0,0}_{s_1,s_2,s_3}(\psi_0 \otimes \psi_0), \psi_0\rangle_K = \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} |\sin(\theta_2-\theta_1)|_\sigma^{s_1} |\sin(\theta_1-\theta_3)|_\sigma^{s_2} |\sin(\theta_2-\theta_3)|_\sigma^{s_3} d\theta_1 d\theta_2 d\theta_3.$$

For $\sigma = 0$ we have

$$\langle A^{0,0,0}_{s_1,s_2,s_3}(\psi_0 \otimes \psi_0), \psi_0\rangle_K = \mathrm{tr}(R_0^{s_2} \circ R_0^{s_1} \circ R_0^{s_3}) = iR(s_1-1, s_2, s_3)$$

For $\sigma = 1$ then $\langle A^{0,0,0}_{s_1,s_2,s_3}(\psi_0 \otimes \psi_0), \psi_0\rangle_K = 0$. Hence we instead consider the following expression:

$$\begin{aligned}
&\langle A^{1,0,0}_{s_1,s_2,s_3}(\psi_2 \otimes \psi_0), \psi_2\rangle_K \\
&= \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} |\sin(\theta_2-\theta_1)|_1^{s_1} |\sin(\theta_1-\theta_3)|_1^{s_2} |\sin(\theta_2-\theta_3)|_1^{s_3} e^{i2(\theta_1-\theta_3)} d\theta_1 d\theta_2 d\theta_3 \\
&= i\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} |\sin(\theta_2-\theta_1)|_1^{s_1} |\sin(\theta_1-\theta_3)|_1^{s_2} |\sin(\theta_2-\theta_3)|_1^{s_3} \sin(2(\theta_1-\theta_3)) d\theta_1 d\theta_2 d\theta_3 \\
&= 2i\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} |\sin(\theta_2-\theta_1)|_1^{s_1} |\sin(\theta_1-\theta_3)|_1^{s_2} |\sin(\theta_2-\theta_3)|_1^{s_3} \sin(\theta_1-\theta_3)\cos(\theta_1-\theta_3) d\theta_1 d\theta_2 d\theta_3 \\
&= 2i\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} |\sin(\theta_2-\theta_1)|_1^{s_1} |\sin(\theta_1-\theta_3)|_0^{s_2+1} |\sin(\theta_2-\theta_3)|_1^{s_3} \cos(\theta_1-\theta_3) d\theta_1 d\theta_2 d\theta_3 \\
&= -2i\,\mathrm{tr}(B_0^{s_2+1} \circ R_1^{s_1} \circ R_1^{s_3}) = -2i\,\mathrm{tr}(R_1^{s_1} \circ B_0^{s_2+1} \circ R_1^{s_3}) = 2iB(s_2+1, s_1, s_3).
\end{aligned}$$

In conclusion we have found

$$\begin{aligned}
\langle A^{0,1,1}_{s_1,s_2,s_3}(\psi_1 \otimes \psi_{-1}), \psi_0\rangle_K &= B(s_1, s_2, s_3) \\
\langle A^{1,1,1}_{s_1,s_2,s_3}(\psi_1 \otimes \psi_{-1}), \psi_0\rangle_K &= R(s_1, s_2, s_3) \\
\langle A^{0,1,0}_{s_1,s_2,s_3}(\psi_1 \otimes \psi_0), \psi_1\rangle_K &= R(s_2, s_1, s_3) \\
\langle A^{1,1,0}_{s_1,s_2,s_3}(\psi_1 \otimes \psi_0), \psi_1\rangle_K &= B(s_2, s_1, s_3) \\
\langle A^{0,0,1}_{s_1,s_2,s_3}(\psi_0 \otimes \psi_1), \psi_1\rangle_K &= -R(s_3, s_1, s_2) \\
\langle A^{1,0,1}_{s_1,s_2,s_3}(\psi_0 \otimes \psi_1), \psi_0\rangle_K &= B(s_3, s_1, s_2) \\
\langle A^{0,0,0}_{s_1,s_2,s_3}(\psi_0 \otimes \psi_0), \psi_0\rangle_K &= iR(s_1-1, s_2, s_3) \\
\langle A^{1,1,1}_{s_1,s_2,s_3}(\psi_2 \otimes \psi_0), \psi_2\rangle_K &= 2iB(s_1, s_2, s_3).
\end{aligned}$$

with

$$R(s_1,s_2,s_3)=\frac{2^4\pi^{\frac{3}{2}}}{i}\frac{\Gamma(\frac{s_1}{2}+1)\Gamma(\frac{s_2+1}{2})\Gamma(\frac{s_3+1}{2})\Gamma(\frac{s_1+s_2+s_3+3}{2})}{\Gamma(\frac{s_1+s_2+3}{2})\Gamma(\frac{s_2+s_3}{2}+1)\Gamma(\frac{s_1+s_3+3}{2})}$$

$$B(s_1,s_2,s_3)=\frac{2^4\pi^{\frac{3}{2}}\Gamma(\frac{s_1+1}{2})\Gamma(\frac{s_2}{2}+1)\Gamma(\frac{s_3}{2}+1)\Gamma(\frac{s_1+s_2+s_3}{2}+1)}{\Gamma(\frac{s_1+s_2+3}{2})\Gamma(\frac{s_1+s_3+3}{2})\Gamma(\frac{s_2+s_3}{2}+1)}.$$

7 Functional equations and the renormalization of $A^\sigma_{s_1,s_2,s_3}$

We saw in the previous chapter that the symmetry breaking operators

$$\frac{A^\sigma_{s_1,s_2,s_3}}{\Gamma(s_1+1)\Gamma(s_2+1)\Gamma(s_3+1)\Gamma(s_1+s_2+s_3+2)}$$

extends holomorphically, as a distribution, to $\mathbb{C}^3$. It turns out that this normalization introduces some unnecessary zeroes. Hence we can normalize through different complex lines through (s_1,s_2,s_3) to obtain new non-zero kernels. To do this we show some functional identities using the Knapp-Stein intertwining operators and use these to find a suitable renormalization.

Recall from the discussion in chapter 2 that for $\pi_{\xi,\lambda},\pi_{\eta,\mu}\in\widehat{G}$ with the $G\times G$ invariant form is given by

$$\langle f,f'\rangle_{\lambda,\mu}=\langle f,T^{(ij)}_{\lambda,\mu}f'\rangle_{K_G\times K_G},\quad f,f'\in\pi_{\xi,\lambda}\otimes\pi_{\eta,\mu}$$

with $T^{(ij)}_{\lambda,\mu}:=T^i_\lambda\otimes T^j_\mu,\ i,j=0,1$ with $T^i_\lambda=\mathrm{id}$ if $i=0$ and the standard (normalized) Knapp–Stein intertwiner for G if $i=1$. By the third property in Lemma 2 the composition $A^\xi_{\lambda,\mu,\nu}\circ T^{(ij)}_{(-1)^i\lambda,(-1)^j\mu}$ can be computed by computing $A^{\sigma,\xi,\eta}_{s_1,s_2,s_3}\circ(T^S)^{(ij)}_{(-1)^i\lambda,(-1)^j\mu}$, with $(T^S)^{(ij)}_{\lambda,\mu}$ being defined analogously to $T^{(ij)}_{\lambda,\mu}$ but with respect to $\mathrm{SL}(2,\mathbb{R})$ (or $\mathrm{PSL}(2,\mathbb{R})$).

Lemma 5. *For* $(\sigma,\xi,\eta)\in(\mathbb{Z}/2\mathbb{Z})^3$ *we have the following*

$$A^{\sigma,\xi,\eta}_{s_1,s_2,s_3}\circ(T^S)^{(ij)}_{(-1)^i\lambda,(-1)^j\mu}=d^{\sigma,\eta,\xi}_{(ij)}(s_1,s_2,s_3)A^{\sigma+i\xi+j\eta,\xi,\eta}_{s'_1,s'_2,s'_3},\quad i,j=0,1$$

where $(T^S)^{(ij)}_{\lambda,\mu}(f_1\otimes f_2)=(T^S)^i_{\xi,\lambda}(f_1)\otimes(T^S)^j_{\eta,\mu}(f_2)$ *with* $(T^S)^0_{\sigma,\nu}=\mathrm{id}$, $(T^S)^1_{\sigma,\nu}=(T^S)_{\sigma,\nu}$ *and* (s'_1,s'_2,s'_3) *given by* (7.2), (7.3) *and* (7.4) *for the 3 cases* $(ij)=(10),(01),(11)$ *respectively.*

Proof. Note that $A^{\sigma,\xi,\eta}_{s_1,s_2,s_3}\circ(T^S)^{(ij)}_{(-1)^i\lambda,(-1)^j\mu}$ again defines a symmetry breaking operator and hence is on the form

$$A^{\sigma,\xi,\eta}_{s_1,s_2,s_3}\circ T^{(ij)}_{\lambda,\mu}=\sum_{\sigma=0,1}a_\sigma A^{\sigma,\xi,\eta}_{s'_1,s'_2,s'_3}. \tag{7.1}$$

Both sides of the above are completely determined by their corresponding integral kernel. The $(s'_1s'_2,s'_3)$ indices are determined by the coordinate change $(\lambda,\mu)\mapsto((-1)^i\lambda,(-1)^j\mu)$. These coordinate changes corresponds to the following in the s_1,s_2,s_3 picture:

$$(-\lambda,\mu)\Rightarrow(s_1,s_2,s_3)\mapsto(s'_1,s'_2,s'_3)=(-s_2-1,-s_1-1,s_1+s_2+s_3+1) \tag{7.2}$$

$$(\lambda,-\mu)\Rightarrow(s_1,s_2,s_3)\mapsto(s'_1,s'_2,s'_3)=(-s_3-1,s_1+s_2+s_3+1,-s_1-1) \tag{7.3}$$

$$(-\lambda,-\mu)\Rightarrow(s_1,s_2,s_3)\mapsto(s'_1,s'_2,s'_3)=(-s_1-s_2-s_3-2,s_3,s_2). \tag{7.4}$$

By computing both sides of (7.1) in the non-compact picture we can note that the kernel of the left hand side has specific equivariance property under a the coordinate change $(x,y)\mapsto(-x,-y)$, hence the kernel of the right hand side of (7.1) must have the same equivariance property. By symmetry it suffices to only check the cases $(ij)=(10)$ and $(ij)=(11)$. Here we find

$$A^{\sigma,\xi,\eta}_{s_1,s_2,s_3}(T^S_{\xi,-\lambda}f_1\otimes f_2)(z)=\int_{\mathbb{R}}\int_{\mathbb{R}}|x-y|^{s_1}_{\sigma}|x|^{s_2}_{\sigma+\xi}|y|^{s_3}_{\sigma+\eta}f_2(z+y)\int_{\mathbb{R}}f_1(\overline{n}_{z+x}w_0\overline{n}_w)dwdxdy.$$

Decomposing $w_0\overline{n}_w=\overline{n}_u a_t n$ gives $t=w$ and $u=-w^{-1}$. Hence we have

$$\begin{aligned}
A^{\sigma,\xi,\eta}_{s_1,s_2,s_3}(T^S_{\xi,-\lambda}f_1\otimes f_2)(z)&=\int_{\mathbb{R}}\int_{\mathbb{R}}|x-y|^{s_1}_{\sigma}|x|^{s_2}_{\sigma+\xi}|y|^{s_3}_{\sigma+\eta}f_2(z+y)\int_{\mathbb{R}}f_1(\overline{n}_{z+x-w^{-1}})|w|^{-\lambda-1}_{\xi}dwdxdy\\
&=(-1)^{\xi}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}|x-y|^{s_1}_{\sigma}|x|^{s_2}_{\sigma+\xi}|y|^{s_3}_{\sigma+\eta}f_2(z+y)f_1(z+x+w)|w|^{\lambda-1}_{\xi}dwdxdy\\
&=\int_{\mathbb{R}}\int_{\mathbb{R}}|y|^{s_3}_{\sigma+\eta}f_2(z+y)f_1(z+u)\left(\int_{\mathbb{R}}|x-u|^{\lambda-1}_{\xi}|x|^{s_2}_{\sigma+\xi}|x-y|^{s_1}_{\sigma}dx\right)dudy\\
&=\int_{\mathbb{R}}\int_{\mathbb{R}}K'(u,y)f_1(z+u)f_2(z+y)dudy
\end{aligned}$$

Since $K'(-x,-y)=(-1)^{\sigma+\eta}K'(x,y)$ we must have $A^{\sigma,\xi,\eta}_{s_1,s_2,s_3}\circ T^{(10)}_{-\lambda,\mu}=d^{\sigma,\xi,\eta}_{(10)}(s_1,s_2,s_3)A^{\sigma+\xi,\xi,\eta}_{s_1,s_2,s_3}$. A similar calculation for $A^{\sigma,\xi,\eta}_{s_1,s_2,s_3}\circ T^{(11)}_{-\lambda,-\mu}$ shows that $A^{\sigma,\xi,\eta}_{s_1,s_2,s_3}\circ T^{(11)}_{-\lambda,-\mu}=d^{\sigma,\xi,\eta}_{(11)}(s_1,s_2,s_3)A^{\sigma+\xi+\eta,\xi,\eta}_{s_1,s_2,s_3}$. □

By Lemma 5 it suffices compute the Knapp-Stein intertwiners $(T^S)^{(ij)}_{\lambda,\mu}$ on the K_S-types considered in chapter 6. Since $(T^S)^{(ij)}_{\lambda,\mu}$ is $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ intertwining we have

$$T^{(ij)}_{\lambda,\mu}(\psi_m\otimes\psi_n)=(T^i_{\xi,\lambda}\psi_m\otimes T^j_{\eta,\mu}\psi_n)=(c_1(m,\xi,\lambda))^i(c_2(n,\eta,\mu))^j(\psi_m\otimes\psi_n),$$

for some $c_k(m,\zeta,\nu)\in\mathbb{C}$. To compute these constants we decompose $w_0\overline{n}_x=k_\theta a_t n$

$$w_0\overline{n}_x=\begin{pmatrix}x&1\\-1&0\end{pmatrix}=\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}\begin{pmatrix}1&y\\0&1\end{pmatrix}=\begin{pmatrix}t\cos\theta&ty\cos\theta+t^{-1}\sin\theta\\-t\sin\theta&t^{-1}\cos\theta-ty\sin\theta\end{pmatrix}.$$

Hence $t=\sqrt{1+x^2}$ and $\cos\theta=\frac{x}{\sqrt{1+x^2}}$. Then, since $\psi_m\in\mathrm{Ind}^{\mathrm{SL}(2,\mathbb{R})}_{P_S}(\xi\otimes e^{\lambda}\otimes 1)$ we get

$$\psi_m(w_0\overline{n}_x)=\left(\frac{x+i}{\sqrt{1+x^2}}\right)^m\left(\frac{1}{\sqrt{1+x^2}}\right)^{\lambda+1}$$

We therefore find

$$\begin{aligned}
T^S_{\xi,\lambda}\psi_m(\overline{n}_0)&=\int_{\mathbb{R}}\psi_m(w_0\overline{n}_x)dx=\int_{\mathbb{R}}\left(\frac{x+i}{\sqrt{1+x^2}}\right)^m\left(\frac{1}{\sqrt{1+x^2}}\right)^{\lambda+1}dx\\
&=\int_{\mathbb{R}}\psi_m(w_0\overline{n}_x)dx=\int_{\mathbb{R}}(x+i)^m(1+x^2)^{\frac{-m-\lambda-1}{2}}dx\\
&=\int_{\mathbb{R}}(x+i)^m((x-i)(i+x))^{\frac{-m-\lambda-1}{2}}dx\\
&=\int_{\mathbb{R}}(x-i)^{\frac{-m-\lambda-1}{2}}(x+i)^{\frac{m-\lambda-1}{2}}dx\\
&=\frac{2^{1-\lambda}\pi i^m\Gamma(\lambda)}{\Gamma(\frac{\lambda+1-m}{2})\Gamma(\frac{\lambda+1+m}{2})}.
\end{aligned}$$

where we used the integration formula:

Lemma 6 (Casselman 16.7)**.**

$$\int_{\mathbb{R}}(x-i)^{\alpha}(x+i)^{\beta}dx = \frac{2^{\alpha+\beta+2}\pi i^{\beta-\alpha}\Gamma(-\alpha-\beta-1)}{\Gamma(-\alpha)\Gamma(-\beta)}.$$

Going forward it will be convenient to consider the normalized Knapp-Stein intertwining operators $(\widetilde{T}^S)_{\xi,\lambda}:\pi_{\xi,\lambda}\to\pi_{\xi,-\lambda}$ given by $(\widetilde{T}^S)_{\xi,\lambda}=\frac{T^S_{\xi,\lambda}}{\Gamma(\frac{\lambda+\xi}{2})}$, making $\widetilde{T}_{\xi,\lambda}$ holomorphic in λ. We define $(\widetilde{T}^S)^{(ij)}_{\lambda,\mu}$ accordingly. Denote $\widetilde{d}^{\sigma,\xi,\eta}_{(ij)}(\lambda,\mu,\nu)$ the corresponding coefficients from Lemma 5. For our intents and purposes we only require $d^{\sigma,0,0}_{(ij)}(\lambda,\mu,\nu)$ but we refer to the Appendix for the remaining cases. To this extend recall that $s_1+s_2=\lambda-1$ and $s_1+s_3=\mu-1$. Then

$$\begin{aligned}
A^{0,0,0}_{s_1,s_2,s_3}\circ(\widetilde{T}^S)^{(10)}_{-\lambda,\mu}(\psi_0\otimes\psi_0)(e) &= \frac{\sqrt{\pi}}{\Gamma(\frac{-s_1-s_2}{2})}A^{0,0,0}_{s_1,s_2,s_3}(\psi_0\otimes\psi_0)(e)\\
&= \frac{\sqrt{\pi}}{\Gamma(\frac{-s_1-s_2}{2})}\frac{iR(s_1-1,s_2,s_3)}{2\pi}\\
&= \widetilde{d}^{0,0,0}_{(10)}(s_1,s_2,s_3)A^{0,0,0}_{s_1',s_2',s_3'}(\psi_0\otimes\psi_0)(e)\\
&= \widetilde{d}^{0,0,0}_{(10)}(s_1,s_2,s_3)\frac{R(-s_2-2,-s_1-1,s_1+s_2+s_3+1)}{2\pi}.
\end{aligned}$$

Hence moving things around we find

$$\begin{aligned}
\widetilde{d}^{0,0,0}_{(10)}(s_1,s_2,s_3) &= \frac{\sqrt{\pi}}{\Gamma(\frac{-s_1-s_2}{2})}\frac{R(s_1-1,s_2,s_3)}{R(-s_2-2,-s_1-1,s_1+s_2+s_3+1)}\\
&= \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}s_2+\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2}s_1\right)\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}s_2+1\right)\Gamma\left(-\frac{1}{2}s_2\right)}.
\end{aligned}$$

Since the tensor product is symmetric we obtain

$$\begin{aligned}
d^{0,0,0}_{(01)}(s_1,s_2,s_3) &= \frac{\sqrt{\pi}}{\Gamma(\frac{-s_1-s_3}{2})}\frac{R(s_1-1,s_2,s_3)}{R(-s_3-2,s_1+s_2+s_3+1,-s_1-1)}\\
&= \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}s_3+\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2}s_1\right)\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}s_3+1\right)\Gamma\left(-\frac{1}{2}s_3\right)}.
\end{aligned}$$

Lastly, using the same method of computation, we find

$$\begin{aligned}
d^{0,0,0}_{(11)}(s_1,s_2,s_3) &= \frac{\sqrt{\pi}}{\Gamma(\frac{-s_1-s_2}{2})}\frac{\sqrt{\pi}}{\Gamma(\frac{-s_1-s_3}{2})}\frac{R(s_1-1,s_2,s_3)}{R(-s_1-s_2-s_3-3,s_3,s_2)}\\
&= \frac{\pi\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}s_2+\frac{1}{2}s_3+1\right)\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2}s_1\right)\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}s_2+1\right)\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}s_3+1\right)\Gamma\left(-\frac{1}{2}s_1-\frac{1}{2}s_2-\frac{1}{2}s_3-\frac{1}{2}\right)}.
\end{aligned}$$

When $\sigma = 0$ we instead find

$$
\begin{aligned}
A^{1,0,0}_{s_1,s_2,s_3} \circ (\widetilde{T}^S)^{(10)}_{-\lambda,\mu}(\psi_2 \otimes \psi_0) &= \frac{2^{1+\lambda}\pi i^2 \Gamma(-\lambda)}{\Gamma(\frac{-\lambda}{2})\Gamma(\frac{-\lambda-1}{2})\Gamma(\frac{-\lambda+3}{2})} \frac{2iB(s_2+1,s_1,s_3)}{2\pi} \\
&= \frac{\sqrt{\pi}}{\Gamma(\frac{1-\lambda}{2})} \frac{1+\lambda}{1-\lambda} \frac{2iB(s_2+1,s_1,s_3)}{2\pi} \\
&= \frac{-\sqrt{\pi}}{\Gamma(\frac{-s_1-s_2}{2})} \frac{s_1+s_2+2}{s_1+s_2} \frac{2iB(s_2+1,s_1,s_3)}{2\pi} \\
&= \widetilde{d}^{1,0,0}_{(10)}(s_1,s_2,s_3) \frac{B(s_2'+1,s_1',s_3')}{2\pi} \\
&= \widetilde{d}^{1,0,0}_{(10)}(s_1,s_2,s_3) \frac{2iB(-s_1,-s_2-1,s_1+s_2+s_3+1)}{2\pi}.
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
\widetilde{d}^{1,0,0}_{(10)}(s_1,s_2,s_3) &= \frac{-\sqrt{\pi}}{\Gamma(\frac{-s_1-s_2}{2})} \frac{s_1+s_2+2}{s_1+s_2} \frac{B(s_2+1,s_1,s_3)}{B(-s_1,-s_2-1,s_1+s_2+s_3+1)} \\
&= \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}s_1+1\right)\Gamma\left(\frac{1}{2}s_2+1\right)}{\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}s_2+1\right)\Gamma\left(-\frac{1}{2}s_1+\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}s_2+\frac{1}{2}\right)}.
\end{aligned}
$$

Again, using the symmetry properties, we get

$$
\begin{aligned}
\widetilde{d}^{1,0,0}_{(01)}(s_1,s_2,s_3) &= \frac{\sqrt{\pi}}{\Gamma(\frac{-s_1-s_3}{2})} \frac{B(s_2+1,s_1,s_3)}{B(s_1+s_2+s_3+2,-s_3-1,-s_1-1)} \\
&= \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}s_1+1\right)\Gamma\left(\frac{1}{2}s_3+1\right)}{\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}s_3+1\right)\Gamma\left(-\frac{1}{2}s_1+\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}s_3+\frac{1}{2}\right)}.
\end{aligned}
$$

Lastly, by direct computation, we get

$$
\begin{aligned}
\widetilde{d}^{1,0,0}_{(11)}(s_1,s_2,s_3) &= \frac{\sqrt{\pi}\,(\frac{s_1+s_2}{2}+1)}{\Gamma(\frac{-s_1-s_2}{2}+1)} \frac{\sqrt{\pi}}{\Gamma(\frac{-s_1-s_3}{2})} \frac{B(s_2+1,s_1,s_3)}{B(s_3+1,-s_1-s_2-s_3-2,s_2)} \\
&= \frac{\pi\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}s_2+\frac{1}{2}s_3+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}s_1+1\right)}{\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}s_2+1\right)\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}s_3+1\right)\Gamma\left(-\frac{1}{2}s_1-\frac{1}{2}s_2-\frac{1}{2}s_3\right)\Gamma\left(-\frac{1}{2}s_1+\frac{1}{2}\right)}.
\end{aligned}
$$

Abusing the notation slightly, we obtain from a case by case analysis the following closed formulas

$$
\widetilde{d}^{\sigma,0,0}_{(10)}(\lambda,\mu,\nu) = \frac{\sqrt{\pi}\Gamma\left(\frac{\lambda+\mu+\nu+1+2\sigma}{4}\right)\Gamma\left(\frac{\lambda-\mu-\nu+1+2\sigma}{4}\right)}{\Gamma\left(\frac{-\lambda-\mu-\nu+1+2\sigma}{4}\right)\Gamma\left(\frac{-\lambda+\mu+\nu+1+2\sigma}{4}\right)\Gamma\left(\frac{\lambda+1}{2}\right)} \tag{7.5}
$$

$$
\widetilde{d}^{\sigma,0,0}_{(01)}(\lambda,\mu,\nu) = \frac{\sqrt{\pi}\Gamma\left(\frac{\lambda+\mu+\nu+1+2\sigma}{4}\right)\Gamma\left(\frac{-\lambda+\mu-\nu+1+2\sigma}{4}\right)}{\Gamma\left(\frac{-\lambda-\mu-\nu+1+2\sigma}{4}\right)\Gamma\left(\frac{\lambda-\mu+\nu+1+2\sigma}{4}\right)\Gamma\left(\frac{\mu+1}{2}\right)} \tag{7.6}
$$

$$
\widetilde{d}^{\sigma,0,0}_{(11)}(\lambda,\mu,\nu) = \frac{\pi\Gamma\left(\frac{\lambda+\mu+\nu+1+2\sigma}{4}\right)\Gamma\left(\frac{\lambda+\mu-\nu+1+2\sigma}{4}\right)}{\Gamma\left(\frac{-\lambda-\mu-\nu+1+2\sigma}{2}\right)\Gamma\left(\frac{-\lambda-\mu+\nu+1+2\sigma}{4}\right)\Gamma\left(\frac{\lambda+1}{2}\right)\Gamma\left(\frac{\mu+1}{2}\right)}. \tag{7.7}
$$

Lemma 7. *For generic $\lambda,\mu,\nu \in \mathbb{C}$, $\sigma,\eta,\xi \in \mathbb{Z}/2\mathbb{Z}$ and $f \in \pi_{\xi,\lambda} \otimes \pi_{\eta,\mu}$ we have*

$$
A^{\sigma+\xi+\eta}_{\lambda,\mu,\nu} \circ \widetilde{T}^{(ij)}_{(-1)^i\lambda,(-1)^j\mu} = \widetilde{d}^{\sigma,0,0}_{(ij)}(2\lambda,2\mu,\nu)\, A^{\sigma+\xi+\eta}_{(-1)^i\lambda,(-1)^j\mu,\nu}.
$$

Proof. Using Lemma 5 we have

$$\begin{aligned} A^{\sigma+\xi+\eta}_{\lambda,\mu,\nu} \circ \widetilde{T}^{(ij)}_{(-1)^i\lambda,(-1)^j\mu} &= L_{\sigma+\xi+\eta} \circ A^{\sigma,0,0}_{2\lambda,2\mu,\nu} \circ \mathrm{Res}_{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})} \circ T^{(ij)}_{(-1)^i\lambda,(-1)^j\mu} \\ &= L_{\sigma+\xi+\eta} \circ A^{\sigma,0,0}_{2\lambda,2\mu,\nu} \circ \widetilde{T}^{(ij)}_{(-1)^i\lambda,(-1)^j\mu} \circ \mathrm{Res}_{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})} \\ &= \widetilde{d}^{\sigma,0,0}_{(ij)}(2\lambda,2\mu,\nu) L_{\sigma+\xi+\eta} \circ A^{\sigma,0,0}_{2(-1)^i\lambda,2(-1)^j\mu,\nu} \circ \mathrm{Res}_{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})} \\ &= \widetilde{d}^{\sigma,0,0}_{(ij)}(2\lambda,2\mu,\nu)\, A^{\sigma+\xi+\eta}_{(-1)^i\lambda,(-1)^j\mu,\nu}. \end{aligned}$$

□

Proposition 3. *The (renormalized) symmetry breaking operator*

$$\widetilde{A}^\sigma_{s_1,s_2,s_3} := \frac{A^\sigma_{s_1,s_2,s_3}}{\Gamma(\frac{s_1+1+\sigma}{2})\Gamma(\frac{s_2+1+\sigma}{2})\Gamma(\frac{s_3+1+\sigma}{2})\Gamma(\frac{s_1+s_2+s_3+2+\sigma}{2})}$$

extends holomorphically to $s_1, s_2, s_3 \in \mathbb{C}$.

Proof. Recall that the symmetry breaking operator

$$\widehat{A}^\sigma_{s_1,s_2,s_3} := \frac{A^\sigma_{s_1,s_2,s_3}}{\Gamma(s_1+1)\Gamma(s_2+1)\Gamma(s_3+1)\Gamma(s_1+s_2+s_3+2)}$$

extends holomorphically to $(s_1, s_2, s_3) \in \mathbb{C}^3$. Using the functional identity we just found, we have

$$\begin{aligned} \widehat{A}^\sigma_{-\lambda,\mu,\nu} \circ \widetilde{T}^{(10)}_{\lambda,\mu} &= \frac{A^\sigma_{-\lambda,\mu,\nu} \circ \widetilde{T}^{(10)}_{\lambda,\mu}}{\Gamma(\frac{-2\lambda+2\mu+\nu+1}{2})\Gamma(\frac{-2\lambda-2\mu-\nu+1}{2})\Gamma(\frac{2\lambda+2\mu-\nu+1}{2})\Gamma(\frac{-2\lambda+2\mu-\nu+1}{2})} \\ &= \frac{\Gamma(\frac{2\lambda+2\mu+\nu+1}{2})\Gamma(\frac{2\lambda-2\mu-\nu+1}{2})\widehat{A}^\sigma_{\lambda,\mu,\nu}}{d^{\sigma,0,0}_{(10)}(-2\lambda,2\mu,\nu)\Gamma(\frac{-2\lambda+2\mu+\nu+1}{2})\Gamma(\frac{-2\lambda-2\mu-\nu+1}{2})} \\ &= \frac{\sqrt{\pi}\Gamma\left(\frac{-2\lambda+2\mu+\nu+1+2\sigma}{4}\right)\Gamma\left(\frac{-2\lambda-2\mu-\nu+1+2\sigma}{4}\right)\Gamma(\frac{2\lambda+2\mu+\nu+1}{2})\Gamma(\frac{2\lambda-2\mu-\nu+1}{2})\widehat{A}^\sigma_{\lambda,\mu,\nu}}{\Gamma\left(\frac{2\lambda-2\mu-\nu+1+2\sigma}{4}\right)\Gamma\left(\frac{2\lambda+2\mu+\nu+1+2\sigma}{4}\right)\Gamma\left(\frac{1-2\lambda}{2}\right)\Gamma(\frac{-2\lambda+2\mu+\nu+1}{2})\Gamma(\frac{-2\lambda-2\mu-\nu+1}{2})} \\ &= \frac{\sqrt{\pi}\Gamma\left(\frac{-2\lambda+2\mu+\nu+1+2\sigma}{4}\right)\Gamma\left(\frac{-2\lambda-2\mu-\nu+1+2\sigma}{4}\right)\Gamma(s_1+1)\Gamma(s_2+1)\widehat{A}^\sigma_{s_1,s_2,s_3}}{\Gamma\left(\frac{1-2\lambda}{2}\right)\Gamma(\frac{-2\lambda+2\mu+\nu+1}{2})\Gamma(\frac{-2\lambda-2\mu-\nu+1}{2})\Gamma\left(\frac{s_2+1+\sigma}{2}\right)\Gamma\left(\frac{s_1+1+\sigma}{2}\right)}. \end{aligned}$$

Since the far left hand side is holomorphic in λ, μ, ν, the far right hand side must also be holomorphic in λ, μ, ν. Using the duplication formula for the Gamma function we then find that

$$\frac{\Gamma(s_1+1)\Gamma(s_2+1)\widehat{A}^\sigma_{s_1,s_2,s_3}}{\Gamma\left(\frac{s_2+1+\sigma}{2}\right)\Gamma\left(\frac{s_1+1+\sigma}{2}\right)} = \frac{2^{s_1+s_2}\Gamma\left(\frac{s_2+1}{2}\right)\Gamma\left(\frac{s_2+2}{2}\right)\Gamma\left(\frac{s_1+1}{2}\right)\Gamma\left(\frac{s_1+2}{2}\right)\widehat{A}^\sigma_{s_1,s_2,s_3}}{\pi\Gamma\left(\frac{s_2+1+\sigma}{2}\right)\Gamma\left(\frac{s_1+1+\sigma}{2}\right)}.$$

Since the right hand side must be holomorphic in s_1, s_2, s_3 by the previous computation, we must have that $\widehat{A}^0_{s_1,s_2,s_3} = 0$ when $s_1 + 2 \in -2\mathbb{N}_0$ and/or $s_2 + 1 \in -2\mathbb{N}_0$ and $\widehat{A}^1_{s_1,s_2,s_3} = 0$ when $s_1 + 2 \in -2\mathbb{N}_0$ and/or $s_2 + 1 \in -2\mathbb{N}_0$, showing that

$$\Gamma(\frac{s_1+2+\sigma}{2})\Gamma(\frac{s_2+2+\sigma}{2})\widehat{A}^\sigma_{s_1,s_2,s_3}$$

is holomorphic in s_1, s_2, s_3. Using the other functional identities we obtain that

$$\Gamma(\frac{s_1+2+\sigma}{2})\Gamma(\frac{s_2+2+\sigma}{2})\Gamma(\frac{s_3+2+\sigma}{2})\Gamma(\frac{s_1+s_2+s_3+3+\sigma}{2})\widehat{A}^\sigma_{s_1,s_2,s_3}$$

is holomorphic in s_1, s_2, s_3. The claim then follows from the duplication formula for the Gamma function, and the definition of $\widetilde{A}^{\sigma}_{s_1,s_2,s_3}$.

□

Considering this new family of symmetry breaking operators, we can restate the functional identities for the Knapp–Stein intertwiners in terms of $\widetilde{A}^{\sigma}_{\lambda,\mu,\nu}$ as

$$\widetilde{A}^{\sigma}_{\lambda,\mu,\nu} \circ \widetilde{T}^{(ij)}_{(-1)^i\lambda,(-1)^j\mu} = \frac{\sqrt{\pi}^{i+j}\widetilde{A}^{\sigma}_{(-1)^i\lambda,(-1)^j\mu,\nu}}{\Gamma(\frac{1+\lambda}{2})^i\Gamma(\frac{1+\mu}{2})^j}.$$

We also note that

$$\widetilde{T}^{G}_{\sigma+\xi+\eta,\nu} \circ \widetilde{A}^{\sigma}_{\lambda,\mu,\nu} f = \frac{1}{\Gamma(\frac{1-\nu}{2})}\widetilde{A}^{\sigma}_{\lambda,\mu,-\nu}$$

which can be seen using the same method we used to compute $\widetilde{A}^{\sigma}_{\lambda,\mu,\nu} \circ (\widetilde{T}^{G})^{(ij)}_{(-1)^i\lambda,(-1)^j\mu}$.

8 The zero set of $\widetilde{A}^{\sigma,0,0}_{s_1,s_2,s_3}$

As we saw previously the symmetry breaking operators $\widetilde{A}^{\sigma,\xi,\eta}_{s_1,s_2,s_3}$ can be realised as invariant trilinear forms on the sphere $K_S = SO(2) \cong S^1$ using the natural K_S-pairing to make the identification

$$\mathrm{Hom}_G(\pi_{\xi,\lambda} \otimes \pi_{\eta,\mu}, \pi_{\zeta,\nu}) \cong \mathrm{Hom}_G(\pi_{\xi,\lambda} \otimes \pi_{\eta,\mu} \otimes \pi_{\zeta,-\nu}, \mathbb{C}).$$

We also saw how to evaluate such trilinear forms on some simple K_S invariant functions, i.e functions on $S^1 \times S^1 \times S^1$ that are invariant under the diagonal action of K_S. Similarly we define K_S-invariant distributions by duality. In [Cle16] Clerc showed that the study of zeroes of K-invariant distributions reduces to evaluation on the algebra of K-invariant polynomials. Using this method Clerc was able to give a detailed description of the zeroes of trilinear invariant forms on $S^{n-1} \times S^{n-1} \times S^{n-1}$ with $n \geq 4$. In this chapter we intend to study the special case $n = 2$. For the sake of completeness we restate some key lemmas, the proof of which are rather simple and can be found in [Cle16].

Lemma 8 ([Cle16, Lem. 3.2])**.** *The space of K-invariant polynomial functions is dense in the space of K-invariant functions on $C(S^1 \times S^1 \times S^1)$.*

Lemma 9. *A K-invariant distribution A on $S^1 \times S^1 \times S^1$ is identically 0 if and only if A is zero on every K-invariant polynomial function on $S^1 \times S^1 \times S^1$.*

Lemma 10. *The algebra of K-invariant polynomial functions on $S^1 \times S^1 \times S^1$ is generated by the elements*

$$1, \quad \langle x_i, x_j \rangle, \quad \det(x_i, x_j) \qquad i \neq j = 1, 2, 3, \quad x_i, x_j \in S^1. \tag{8.1}$$

Proof. See e.g. [Wey66] Theorem 2.9.A □

Lemma 10 is what separates the $n = 2, 3$ case from $n \geq 4$, since for $n \geq 4$ the determinants are generated by the other elements. Using the natural parameterization of S^1 we find that (8.1) becomes

$$1, \quad \cos(\theta_i - \theta_j), \quad \sin(\theta_i - \theta_j) \qquad i \neq j = 1, 2, 3.$$

Lemma 11. *Let $s_1, s_2, s_3 \in \mathbb{C}$ and $\sigma \in \mathbb{Z}/2\mathbb{Z}$. Then $\widetilde{A}^{\sigma,0,0}_{s_1,s_2,s_3} = 0$ if and only if the corresponding trilinear form*

$$\widetilde{\mathcal{A}}^{\sigma,0,0}_{s_1,s_2,s_3}(f) := (-1)^\sigma \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} |\sin(\theta_1-\theta_2)|^{s_1}_\sigma |\sin(\theta_1-\theta_3)|^{s_2}_\sigma |\sin(\theta_2-\theta_3)|^{s_3}_\sigma f(\theta_1,\theta_2,\theta_3)d\theta_1 d\theta_2 d\theta_3$$

is zero on elements of the form

$$\cos(\theta_i-\theta_j)^\varepsilon \sin(\theta_1-\theta_2)^{a_1}\sin(\theta_1-\theta_3)^{a_2}\sin(\theta_2-\theta_3)^{a_3} \qquad i \neq j,\ a_1,a_2,a_3 \in \mathbb{N}_0,\ \varepsilon = 0,1$$

Proof. We have that $\widetilde{A}^{\sigma,0,0}_{s_1,s_2,s_3} = 0$ if and only if $\widetilde{\mathcal{A}}^{\sigma,0,0}_{s_1,s_2,s_3} = 0$. Furthermore by the previous lemmas we have that $\widetilde{\mathcal{A}}^{\sigma,0,0}_{s_1,s_2,s_3} = 0$ if and only if it vanishes on (8.1). But since $\cos(\theta_i-\theta_j)^2 = 1-\sin(\theta_i-\theta_j)^2$ it suffices to vanish on elements of the desired form. $\square$

Since the integral kernel of $\widetilde{\mathcal{A}}^{\sigma,0,0}_{s_1,s_2,s_3}$ transforms with $(-1)^\sigma$ under the coordinate transformation $(\theta_1,\theta_2,\theta_3) \to (-\theta_1,-\theta_2,-\theta_3)$ it follows that it vanishes on any function that does not have the same equivariance. Hence at least two of the exponents a_i must have same parity. However we also have

$$\widetilde{\mathcal{A}}^{\sigma,0,0}_{s_1,s_2,s_3}(\sin(\theta_1-\theta_2)^{a_1}\sin(\theta_1-\theta_3)^{a_2}\sin(\theta_2-\theta_3)^{a_3}) = (-1)^\sigma \operatorname{tr}(R^{s_1+a_1}_{\sigma+a_1} \circ R^{s_2+a_2}_{\sigma+a_2} \circ R^{s_3+a_3}_{\sigma+a_3})$$

which is zero if $\sigma + [a_i] \neq \sigma + [a_j]$ for $i \neq j$, where $[a]$ denotes the remainder of a by division with 2, by looking at their eigenvalues. Hence if $\varepsilon = 0$ all $a_i's$ must have same parity and since the corresponding function must also transform as $(-1)^\sigma$ under the reflection $(\theta_1,\theta_2,\theta_3) \to (-\theta_1,-\theta_2,-\theta_3)$ we conclude that $a_i = 2k_i+\sigma$ when $\varepsilon = 0$. Likewise if $\varepsilon = 1$ we can use a similarly argument, but with one $R^{s_i+a_i}_{\sigma+a_i}$ replaced with $B^{s_i+a_i}_{\sigma+a_i}$. Looking at the eigenvalues again we conclude that the a_i corresponding to cosine factor must satisfy $a_i = 2k_i+\sigma$ and the remaining a_j for $j \neq i$ must satisfy $a_j = 1-\sigma+2k_j$ with $k_i, k_j \in \mathbb{N}_0$.

Theorem 7. *Let $s_1, s_2, s_3 \in \mathbb{C}$ and $\sigma \in \mathbb{Z}/2\mathbb{Z}$. Then, up to permutation of the indices, the trilinear form $\widetilde{\mathcal{A}}^{\sigma,0,0}_{s_1,s_2,s_3}$ vanishes if*

$$s_1 + 1 + \sigma = -2k_1 \quad \text{and} \quad s_2+1+\sigma = -2k_2, \tag{8.2}$$

or if

$$s_1+s_2+s_3+2+\sigma = -2k \quad \text{and} \quad s_1+1+\sigma = 2(\ell+\sigma)+1, \qquad \ell, a \in \mathbb{N}_0. \tag{8.3}$$

Proof. By symmetry, and the discussion preceding the Theorem, it suffices to check if

$$\widetilde{\mathcal{A}}^{\sigma,0,0}_{s_1,s_2,s_3}(\sin(\theta_1-\theta_2)^{2a_1+\sigma}\sin(\theta_1-\theta_3)^{2a_2+\sigma}\sin(\theta_2-\theta_3)^{2a_3+\sigma}) = 0$$

and

$$\widetilde{\mathcal{A}}^{\sigma,0,0}_{s_1,s_2,s_3}(\cos(\theta_1-\theta_3)\sin(\theta_1-\theta_2)^{2a_1+\sigma}\sin(\theta_1-\theta_3)^{1-\sigma+2a_2}\sin(\theta_2-\theta_3)^{1-\sigma+2a_3}) = 0$$

for all $a_1, a_2, a_3 \in \mathbb{N}_0$. Since we have

$$\widetilde{\mathcal{A}}^{\sigma,0,0}_{s_1,s_2,s_3}(\sin(\theta_1-\theta_2)^{2a_1+\sigma}\sin(\theta_1-\theta_3)^{2a_2+\sigma}\sin(\theta_2-\theta_3)^{2a_3+\sigma}) = \operatorname{tr}(R^{2a_2+\sigma}_0 \circ R^{2a_1+\sigma}_0 \circ R^{2a_3+\sigma}_0)$$

and

$$\widetilde{\mathcal{A}}^{\sigma,0,0}_{s_1,s_2,s_3}(\cos(\theta_1-\theta_3)\sin(\theta_1-\theta_2)^{2a_1+\sigma}\sin(\theta_1-\theta_3)^{1-\sigma+2a_2}\sin(\theta_2-\theta_3)^{1-\sigma+2a_3})$$
$$=\operatorname{tr}(R_0^{2a_2+1-\sigma}\circ B_0^{2a_1+\sigma}\circ R_0^{2a_3+1-\sigma})$$

we find, for $s_1+1+\sigma=-2k_1$ and $s_2+1+\sigma=-2k_2$ that (up to some non-zero constant)

$$\begin{aligned}
&\widetilde{\mathcal{A}}^{\sigma,0,0}_{s_1,s_2,s_3}(\sin(\theta_1-\theta_2)^{2a_1+\sigma}\sin(\theta_1-\theta_3)^{2a_2+\sigma}\sin(\theta_2-\theta_3)^{2a_3+\sigma})\\
&=\frac{\left(\frac{s_1+1+\sigma}{2}\right)_{a_1}\left(\frac{s_2+1+\sigma}{2}\right)_{a_2}\left(\frac{s_3+1+\sigma}{2}\right)_{a_3}\left(\frac{s_1+s_2+s_3+2+\sigma}{2}\right)_{a_1+a_2+a_3+\sigma}}{\Gamma(\frac{s_1+s_2}{2}+1+\sigma+a_1+a_2)\Gamma(\frac{s_1+s_3}{2}+1+\sigma+a_1+a_3)\Gamma(\frac{s_2+s_3}{2}+1+\sigma+a_2+a_3)}\\
&=\frac{(-k_1)_{a_1}(-k_2)_{a_2}\left(\frac{s_3+1+\sigma}{2}\right)_{a_3}\left(\frac{s_1+s_2+s_3+2+\sigma}{2}\right)_{a_1+a_2+a_3+\sigma}}{\Gamma(a_1+a_2-k_1-k_2)}\times\dots
\end{aligned}$$

If $a_1+a_2\leq k_1+k_2$ then the Gamma factor in the denominator has a pole and the resulting expression is 0. Otherwise, if $a_1+a_2>k_1+k_2$ then either $a_1>k_1$ and/or $a_2>k_2$ which results in the corresponding Pochhammer symbol being 0. Likewise we have (up to some non-zero constant)

$$\begin{aligned}
&\widetilde{\mathcal{A}}^{\sigma,0,0}_{s_1,s_2,s_3}(\cos(\theta_1-\theta_3)\sin(\theta_1-\theta_2)^{2a_1+\sigma}\sin(\theta_1-\theta_3)^{1-\sigma+2a_2}\sin(\theta_2-\theta_3)^{1-\sigma+2a_3})\\
&=\frac{\left(\frac{s_1+1+\sigma}{2}\right)_{a_1}\left(\frac{s_2+1+\sigma}{2}\right)_{1-\sigma+a_2}\left(\frac{s_3+1+\sigma}{2}\right)_{1-\sigma+a_3}\left(\frac{s_1+s_2+s_3+2+\sigma}{2}\right)_{a_1+a_2+a_3+1-\sigma}}{\Gamma(\frac{s_1+s_2}{2}+2+a_1+a_2)\Gamma(\frac{s_1+s_3}{2}+2+a_1+a_3)\Gamma(\frac{s_2+s_3}{2}+2-\sigma+a_2+a_3)}\\
&=\frac{(-k_1)_{a_1}(-k_2)_{1-\sigma+a_2}}{\Gamma(1-\sigma+a_1+a_2-k_1-k_2)}\times\dots
\end{aligned}$$

If $a_1+1-\sigma+a_2\leq k_1+k_2$ the Gamma factor in the denominator has a pole and thus yields a 0. Otherwise if $a_1+1-\sigma+a_2>k_1+k_2$ then $a_1>k_1$ and/or $1-\sigma+a_2>k_2$ resulting in the Pochhammer symbol in the numerator to yield a 0. Since the above is no longer symmetric in the s_i's, we also need to check the "special case" when $s_2+1+\sigma=-2k_2$ and $s_3+1+\sigma=-2k_3$. Here we find

$$\begin{aligned}
&\widetilde{\mathcal{A}}^{\sigma,0,0}_{s_1,s_2,s_3}(\cos(\theta_1-\theta_3)\sin(\theta_1-\theta_2)^{2a_1+\sigma}\sin(\theta_1-\theta_3)^{1-\sigma+2a_2}\sin(\theta_2-\theta_3)^{1-\sigma+2a_3})\\
&=\frac{(-k_2)_{1-\sigma+a_2}(-k_3)_{1-\sigma+a_3}}{\Gamma(1-2\sigma+a_2+a_3-k_2-k_3)}\times\dots
\end{aligned}$$

If $1-2\sigma+a_2+a_3\leq k_2+k_3$ then the Gamma factor in the denominator has a pole and thus yields a 0. Otherwise if $1-2\sigma+a_2+a_3>k_2+k_3$ we especially have that $1-\sigma+a_2+1-\sigma+a_3>k_2+k_3$ and thus $1-\sigma+a_2>k_2$ and/or $1-\sigma+a_3>k_3$ and the Pochhammer symbol in the numerator yields a 0. Assume now that $s_1+s_2+s_3+2+\sigma=-2k$ and $s_1+1+\sigma=2\ell+1$. Then we have that $\frac{s_2+s_3}{2}=-k-\ell-1$ and hence

$$\begin{aligned}
&\widetilde{\mathcal{A}}^{\sigma,0,0}_{s_1,s_2,s_3}(\sin(\theta_1-\theta_2)^{2a_1+\sigma}\sin(\theta_1-\theta_3)^{2a_2+\sigma}\sin(\theta_2-\theta_3)^{2a_3+\sigma})\\
&=\frac{(-k)_{\sigma+a_1+a_2+a_3}}{\Gamma(a_2+a_3-k-\ell)}\times\dots
\end{aligned}$$

If $a_2+a_3\leq k+\ell$ then the denominator has a pole and thus yields a 0. Otherwise if $a_2+a_3>k+\ell$ then $\sigma+a_1+a_2+a_3>k$ and the Pochhammer symbol in the numerator yields a 0. Likewise we have

$$\begin{aligned}&\widetilde{\mathcal{A}}^{\sigma,0,0}_{s_1,s_2,s_3}(\cos(\theta_1-\theta_3)\sin(\theta_1-\theta_2)^{2a_1+\sigma}\sin(\theta_1-\theta_3)^{1-\sigma+2a_2}\sin(\theta_2-\theta_3)^{1-\sigma+2a_3})\\&=\frac{(-k)_{1-\sigma+a_1+a_2+a_3}}{\Gamma(1-2\sigma+a_2+a_3-k-\ell)}\times\dots\end{aligned}$$

If $1-2\sigma+a_2+a_3\leq k+\ell$ the Gamma factor in the denominator has a pole and thus yields a 0. Otherwise if $1-2\sigma+a_2+a_3>k+\ell$ then $1-\sigma+a_1+a_2+a_3>k$ and the Pochhammer symbol yields a 0. Since the ladder expression is not symmetric we also need to check the case where $s_1+s_2+s_3+2+\sigma=-2k$ and $s_2+1+\sigma=2\ell+1$. Here we find

$$\begin{aligned}&\widetilde{\mathcal{A}}^{\sigma,0,0}_{s_1,s_2,s_3}(\cos(\theta_1-\theta_3)\sin(\theta_1-\theta_2)^{2a_1+\sigma}\sin(\theta_1-\theta_3)^{1-\sigma+2a_2}\sin(\theta_2-\theta_3)^{1-\sigma+2a_3})\\&=\frac{(-k)_{1-\sigma+a_1+a_2+a_3}}{\Gamma(1+a_1+a_3-k-\ell-\sigma)}\times\dots\end{aligned}$$

If $1+a_2+a_3\leq k+\ell+\sigma$ the Gamma factor in the denominator has a pole and the corresponding expression is hence 0. Otherwise if $1-\sigma+a_1+a_2+a_3>k$ the Pochhammer symbol in the numerator yields a 0. $\square$

This method can in principle be applied to find zeroes for symmetry breaking operators for $(\mathrm{SL}(2,\mathbb{R}),\mathrm{SL}(2,\mathbb{R}),\mathrm{SL}(2,\mathbb{R}))$ as well, but requires some tedious technical work. We remark that for $\sigma=0$ the result we obtained is a weaker version of that obtained by Clerc in [Cle16], since he was able to obtain a full classification (for $n\geq4$). Due to time constraints we were not able to obtain this full classification, at the time of writing this paper, however we conjecture that the converse statement of the theorem also holds in this case, and that the same methods used by Clerc should carry over with minor modifications.

Part IV

The unitary Plancherel formula

Let $G = \mathrm{PGL}(2,\mathbb{R})$ and consider the strongly spherical pair $(G \times G, \Delta(G))$. Then (eP, w_0P) is a representative for the dense open orbit and since $\Delta(P) \subset \Delta(G)$, we must have that $\Delta(G)$ also acts with a dense orbit $\mathcal{O}$. Let $x_0 = (e, w_0)$ and denote by G_{x_0} the stabilizer of $x_0\Delta(P)$ in $\Delta(G) \cong G$. Then the stabilizer G_{x_0} is

$$G_{x_0} = \mathrm{Stab}_G(x_0) \cong P \cap w_0 P w_0^{-1} = M_G A_G.$$

For principal series representations $\pi_{\xi,\lambda}$ and $\pi_{\eta,\mu}$ of G, the completion of the tensor product $\pi_{\xi,\lambda} \otimes \pi_{\eta,\nu}$ is given by the smooth sections of the line bundle $\left(\mathcal{V}^{x_0}_{(\xi,\lambda+\rho),(\eta,\nu+\rho)}, G/P \times G/P, q\right)$ with $\mathcal{V}^{x_0}_{(\xi,\lambda+\rho),(\eta,\nu+\rho)} = (G \times G) \times_{P_G \times P_G} \mathbb{C}_{(\xi,\eta),(\lambda+\rho,\mu+\rho)}$. Hence the restriction map

$$\Theta_{\lambda,\mu} : (\pi_{\xi,\lambda} \otimes \pi_{\eta,\mu})^\infty |_G \to C^\infty(G/M_G A_G, \mathcal{V}^{x_0}_{(\xi,\lambda+\rho),(\eta,\mu+\rho)})$$

defines a continuous linear G-map to the sections of the associated homogeneous bundle

$$\left(\mathcal{V}^{x_0}_{(\xi,\lambda+\rho),(\eta,\mu+\rho)}, G/P \times G/P, q\right)$$

with respect to the representation twisted at x_0, i.e

$$g \in G_{x_0} \mapsto (\xi \otimes e^\lambda \otimes \mathbf{1}) \boxtimes (\eta \otimes e^\mu \otimes \mathbf{1})(x_0^{-1} g x_0),$$

by restricting elements of $\pi_{\xi,\lambda} \otimes \pi_{\xi,\mu}$ to $\mathcal{O}$ and letting G act on the basepoint. Hence the restriction map is given by $\Theta_{\lambda,\mu} f(g) = f(g \cdot x_0)$ for $g \in G$. Note that for $h = (ma, ma) \in G_{x_0} \cong M_G A_G$ we have

$$\begin{aligned} \Theta_{\lambda,\mu} f(gh) &= f(g \cdot (ma, maw_0)) = f(g \cdot (ma, w_0(ma)^{-1})) \\ &= a^{\mu-\lambda}(\xi+\eta)(m)\Theta_{\lambda,\mu} f(g) := \chi^{-1}_{\xi+\eta,\lambda-\mu}(ma) f(g). \end{aligned}$$

For $\lambda - \mu \in i\mathfrak{a} \cong i\mathbb{R}$ the map $\Theta_{\lambda,\mu}$ extends to a unitary map between the L^2-sections of $(\pi_{\xi,\lambda} \otimes \pi_{\eta,\mu})|_G$ and the L^2-sections of G/G_{x_0} associated to the character $\chi_{\xi+\eta,\lambda-\mu}$, i.e the L^2-sections of the line bundle over $G/M_G A_G$ associated to $\chi_{\xi+\eta,\lambda-\mu}$;

$$\mathrm{Ind}^G_{M_G A_G}(\xi + \eta \otimes e^{\lambda-\mu}) = \left\{ f : G \to \mathbb{C} \mid f(gh) = \chi^{-1}_{\xi+\eta,\lambda-\mu}(h) f(g), \int_{G/M_G A_G} |f(g)|^2 d(g M_G A_G) < \infty \right\},$$

with $\chi_{\xi+\eta,\lambda-\mu}(ma) = a^{\lambda-\mu}(\xi+\eta)(m)$, $ma \in M_G A_G$. Similar to the case of principal series representation, the space $\mathrm{Ind}^G_{M_G A_G}(\xi + \eta \otimes e^{\lambda-\mu})$ identifies with

$$\mathrm{Ind}^G_{M_G A_G}(\xi + \eta \otimes e^{\lambda-\mu}) \cong \mathrm{Ind}^{\mathrm{GL}(2,\mathbb{R})}_{M_{\mathrm{GL}(2,\mathbb{R})} A_{\mathrm{GL}(2,\mathbb{R})}}((\xi+\eta, \xi+\eta) \otimes e^{(\lambda-\mu,\mu-\lambda)}),$$

by composing the relevant quotient function.

9 The Plancherel formula for the one–sheeted hyperboloid.

In [BJD23] Bang–Jensen & Ditlevsen introduced the $\mathrm{SL}(2,\mathbb{R})$-intertwining operators

$$A^{\varepsilon,\sigma}_{\lambda,\mu} : \mathrm{Ind}^{\mathrm{SL}(2,\mathbb{R})}_{M_S A_S}(\varepsilon \otimes e^{\lambda}) \to \mathrm{Ind}^{S}_{P_S}(\varepsilon \otimes e^{\mu} \otimes 1), \quad A^{\varepsilon,\sigma}_{\lambda,\mu} f(g) = \int_{S/D_S} K^{\varepsilon,\sigma}_{\lambda,\mu}(x^{-1}g) f(x)\, d(xD_S),$$

for $\sigma = 0, 1$, where the kernels $K^{\varepsilon,\sigma}_{\lambda,\mu}$ are given by

$$K^{\varepsilon,\sigma}_{\lambda,\mu}(g) = |g_{11}|_{\varepsilon+\sigma}^{\frac{-\lambda-\mu-1}{2}} |g_{21}|_{\sigma}^{\frac{\lambda-\mu-1}{2}}. \tag{9.1}$$

They proved Plancherel formula for the one-sheeted hyperboloid $\mathrm{SL}(2,\mathbb{R})/M_S A_S$

Proposition 4 ([BJD23])**.** *For $f \in C^\infty - \mathrm{Ind}^{\mathrm{SL}(2,\mathbb{R})}_{M_S A_S}(\varepsilon \otimes e^{\mu})$ we have*

$$\|f\|^2 = \int_{i\mathbb{R}} \sum_{\sigma=0}^{1} \|\mathbf{A}^{\varepsilon,\sigma}_{\lambda,\mu} f\|^2 \frac{d\mu}{|a(\varepsilon,\mu)|^2} + \sum_{\mu \in 1-\varepsilon-2\mathbb{N}} d(\varepsilon,\mu) \|\mathbb{A}^{\varepsilon}_{\lambda,\mu} f\|^2,$$

where

$$a(\varepsilon,\mu) = 2^{3/2}\pi \frac{\Gamma(\frac{\mu}{2})}{\Gamma(\frac{1+\mu+\varepsilon}{2})\Gamma(\frac{1+\mu-\varepsilon}{2})}, \ \mathcal{S}(\mu) = \frac{1}{\Gamma(\frac{1+\mu}{4})^2\Gamma(\frac{1-\mu}{4})^2} + \frac{1}{\Gamma(\frac{3+\mu}{4})^2\Gamma(\frac{3-\mu}{4})^2},$$

and

$$d(0,\mu) = \frac{\Gamma(1-\mu)}{8\pi^3\Gamma(\frac{-\mu}{2})\mathcal{S}(\mu)}, \quad d(1,\mu) = \frac{\mu^2\Gamma(-\mu)}{2\pi\Gamma(\frac{1-\mu}{2})}.$$

The operators are given by

$$\mathbb{A}^{0}_{\lambda,\mu} = \frac{2^{\frac{1+\mu}{2}}\sqrt{\pi}\,\Gamma(\frac{1+\mu}{4}+\frac{\lambda}{4})}{\Gamma(\frac{1-\mu}{4})\Gamma(\frac{1+\mu}{4})\Gamma(\frac{1-\mu}{4}-\frac{\lambda}{4})} A^{0,0}_{\lambda,\mu} + \frac{2^{\frac{1+\mu}{2}}\sqrt{\pi}\,\Gamma(\frac{3+\mu}{4}+\frac{\lambda}{4})}{\Gamma(\frac{3-\mu}{4})\Gamma(\frac{3+\mu}{4})\Gamma(\frac{3-\mu}{4}-\frac{\lambda}{4})} A^{0,1}_{\lambda,\mu},$$

$$\mathbf{A}^{\varepsilon,\sigma}_{\lambda,\mu} = \frac{A^{\varepsilon,\sigma}_{\lambda,\mu}}{\Gamma(\frac{1-\mu}{2})}, \quad \textit{and} \quad \mathbb{A}^{1}_{\lambda,\mu} = \frac{2^{\frac{1+\mu}{2}}\sqrt{\pi}\,\Gamma(\frac{1+\mu}{4}+\frac{\lambda}{4})}{\Gamma(\frac{1-\mu}{4})\Gamma(\frac{3+\mu}{4})\Gamma(\frac{1-\mu}{4}-\frac{\lambda}{4})} A^{1,0}_{\lambda,\mu}.$$

Note that when we have $\mu \in 1-\varepsilon-2\mathbb{N}$ the kernels $K^{\varepsilon,\sigma}_{\lambda,\mu}$ are locally integrable since g_{11} and g_{21} does not vanish simultaneously and $\mathrm{Re}\,(\frac{-\lambda-\mu-1}{2}), \mathrm{Re}\,(\frac{-\lambda-\mu-1}{2}) > -1$. So in the case that $\varepsilon = 0$ we find

$$\Gamma\Big(\frac{1+\mu}{4}\Big)^{-1}\Big|_{\mu=1-2n} = \Gamma\Big(\frac{1-n}{2}\Big)^{-1} \quad \text{and} \quad \Gamma\Big(\frac{3+\mu}{4}\Big)^{-1}\Big|_{\mu=1-2n} = \Gamma\Big(1-\frac{n}{2}\Big)^{-1},$$

implying that one of the two terms in $\mathbb{A}^{0}_{\lambda,\mu}$ vanish when summing over $\mu \in 1-2\mathbb{N}$. Thus we rewrite the above as

$$\|f\|^2 = \int_{i\mathbb{R}} \sum_{\sigma=0}^{1} \|\mathbf{A}^{0,\sigma}_{\lambda,\mu} f\|^2 \frac{d\mu}{|a(0,\mu)|^2} + \sum_{\sigma=0}^{1} \sum_{\mu \in 1+2\sigma-4\mathbb{N}} d'(0,\mu) \|\mathbb{A}^{0,\sigma}_{\lambda,\mu} f\|^2,$$

with

$$\mathbb{A}^{\varepsilon,\sigma}_{\lambda,\mu} = \frac{\Gamma(\frac{1+2\sigma+\mu+\lambda}{4})}{\Gamma(\frac{1+2\sigma-\mu-\lambda}{4})} A^{\varepsilon,\sigma}_{\lambda,\mu}, \quad d'(0,\mu) = \frac{2^{\mu}\Gamma(1-\mu)}{4\pi^2\Gamma(-\frac{\mu}{2})}.$$

To extend the Plancherel formula to $\mathrm{Ind}_{M_{\mathrm{GL}(2,\mathbb{R})}A_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}\left((\varepsilon,\varepsilon)\otimes e^{(\nu_1,\nu_2)}\otimes 1\right)$ we first require an "extension" of $A_{\lambda,\mu}^{\varepsilon,\xi}$ in the following sense:

$$\begin{tikzcd}
\mathrm{Ind}_{M_{\mathrm{GL}(2,\mathbb{R})}A_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}\left((\varepsilon,\varepsilon)\otimes e^{(\lambda_1,\lambda_2)}\right) \arrow[d, "\mathrm{Res}_{\mathrm{SL}(2,\mathbb{R})}"] & \mathrm{Ind}_{P_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}((\varepsilon+\sigma,\varepsilon+\sigma)\otimes e^{(\frac{\lambda_1+\lambda_2+\mu}{2},\frac{\lambda_1+\lambda_2-\mu}{2})}\otimes 1) \\
\mathrm{Ind}_{M_SA_S}^{\mathrm{SL}(2,\mathbb{R})}(0\otimes e^{\lambda_1-\lambda_2}) \arrow[r, "A^{0,\sigma}_{\lambda_1-\lambda_2,\mu}"] & \mathrm{Ind}_{P_S}^{\mathrm{SL}(2,\mathbb{R})}(0\otimes e^{\mu}\otimes 1) \arrow[u, "\widetilde{L}_{\varepsilon+\sigma,\frac{1}{2}(\lambda_1+\lambda_2-\mu)}"']
\end{tikzcd}$$

Following the above diagram we let

$$B_{\lambda,\mu}^{\varepsilon,\sigma}:=\widetilde{L}_{\varepsilon+\sigma,\frac{1}{2}(\lambda_1+\lambda_2-\mu)}\circ A_{\lambda_1-\lambda_2,\mu}^{0,\sigma}\circ\mathrm{Res}_{\mathrm{SL}(2,\mathbb{R})}$$

and thus we have

$$\begin{aligned}
B_{\lambda,\mu}^{\varepsilon,\sigma}f(g)&=|\det(g)|_{\varepsilon+\sigma}^{\frac{1+\mu-\lambda_1-\lambda_2}{2}}\int_{\mathrm{SL}(2,\mathbb{R})/M_SA_S}K_{\lambda_1-\lambda_2,\mu}^{0,\sigma}\left(x^{-1}g\begin{pmatrix}1&0\\0&\det(g)^{-1}\end{pmatrix}\right)f(x)\,d(xM_SA_S)\\
&=|\det(g)|_{\varepsilon+\sigma}^{\frac{1+\mu-\lambda_1-\lambda_2}{2}}\int_{\mathrm{SL}(2,\mathbb{R})/M_SA_S}K_{\lambda_1-\lambda_2,\mu}^{0,\sigma}\left(x^{-1}g\right)f(x)\,d(xM_SA_S). \qquad (9.2)
\end{aligned}$$

By Lemma 1 $B_{\lambda,\mu}^{\varepsilon,\sigma}$ is already $\mathrm{SL}(2,\mathbb{R})$ intertwining.

Lemma 12. *The map*

$$B_{\lambda,\mu}^{\varepsilon,\sigma}:\mathrm{Ind}_{M_{\mathrm{GL}(2,\mathbb{R})}A_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}\left((\varepsilon,\varepsilon)\otimes e^{(\lambda_1,\lambda_2)}\right)\to\mathrm{Ind}_{P_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}((\varepsilon+\sigma,\varepsilon+\sigma)\otimes e^{(\frac{\lambda_1+\lambda_2+\mu}{2},\frac{\lambda_1+\lambda_2-\mu}{2})}\otimes 1)$$

given by (9.2) *is* $\mathrm{GL}(2,\mathbb{R})$ *intertwining.*

Proof. Since any $g\in\mathrm{GL}(2,\mathbb{R})$ can be written as

$$g=g\begin{pmatrix}1&0\\0&\det(g)^{-1}\end{pmatrix}\cdot\begin{pmatrix}1&0\\0&\det(g)\end{pmatrix}$$

it suffices to check that $B_{\lambda,\mu}^{\varepsilon,\sigma}$ intertwines the action of elements of the form $j_t=\begin{pmatrix}1&0\\0&t\end{pmatrix}$, $t\in\mathbb{R}\setminus\{0\}$. To this extend let $f\in C_c^\infty-\mathrm{Ind}_{M_{\mathrm{GL}(2,\mathbb{R})}A_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}\left((\varepsilon,\varepsilon)\otimes e^{(\nu_1,\nu_2)}\otimes 1\right)$ and let

$$\tau_{(\varepsilon+\sigma,\varepsilon+\sigma),(\frac{\lambda_1+\lambda_2+\mu}{2},\frac{\lambda_1+\lambda_2-\mu}{2})}=\mathrm{Ind}_{P_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}((\varepsilon+\sigma,\varepsilon+\sigma)\otimes e^{(\frac{\lambda_1+\lambda_2+\mu}{2},\frac{\lambda_1+\lambda_2-\mu}{2})}\otimes 1).$$

Then we have

$$
\begin{aligned}
&B^{\varepsilon,\sigma}_{\lambda,\mu}f(j_t^{-1}g)\\
&= |\det(j_t^{-1}g)|_{\varepsilon+\sigma}^{\frac{1+\mu-\lambda_1-\lambda_2}{2}} \int_{\mathrm{SL}(2,\mathbb{R})/M_SA_S} K^{0,\sigma}_{\lambda_1-\lambda_2,\mu}\left(x^{-1}j_t^{-1}g\begin{pmatrix}1 & 0\\ 0 & \det(j_t^{-1}g)^{-1}\end{pmatrix}\right) f(x)\, d(xM_SA_S)\\
&= |\det(j_t^{-1}g)|_{\varepsilon+\sigma}^{\frac{1+\mu-\lambda_1-\lambda_2}{2}} \int_{\mathrm{SL}(2,\mathbb{R})/M_SA_S} K^{0,\sigma}_{\lambda_1-\lambda_2,\mu}\left(j_t^{-1}\left(j_tx^{-1}j_t^{-1}\right)g\right) f(x)\, d(xM_SA_S)\\
&= |t|_{\varepsilon+\sigma}^{-\frac{1+\mu-\lambda_1-\lambda_2}{2}}|\det(g)|_{\varepsilon+\sigma}^{\frac{1+\mu-\lambda_1-\lambda_2}{2}} \int_{\mathrm{SL}(2,\mathbb{R})/M_SA_S} K^{0,\sigma}_{\lambda_1-\lambda_2,\mu}\left(j_t^{-1}x^{-1}g\right) f(j_t^{-1}xj_t)\, d(xM_SA_S)\\
&= |t|_{\varepsilon}^{\lambda_2}|t|_{\varepsilon+\sigma}^{-\frac{1+\mu-\lambda_1-\lambda_2}{2}}|\det(g)|_{\varepsilon+\sigma}^{\frac{1+\mu-\lambda_1-\lambda_2}{2}} \int_{\mathrm{SL}(2,\mathbb{R})/M_SA_S} K^{0,\sigma}_{\lambda_1-\lambda_2,\mu}\left(j_t^{-1}x^{-1}g\right) f(j_t^{-1}x)\, d(xM_SA_S)\\
&= |\det(g)|_{\varepsilon+\sigma}^{\frac{1+\mu-\lambda_1-\lambda_2}{2}} \int_{\mathrm{SL}(2,\mathbb{R})/M_SA_S} K^{0,\sigma}_{\lambda_1-\lambda_2,\mu}\left(x^{-1}g\right) f(j_t^{-1}x)\, d(xM_SA_S)\\
&= B^{\varepsilon,\sigma}_{\lambda,\mu}\tau_{(\varepsilon+\sigma,\varepsilon+\sigma),(\frac{\lambda_1+\lambda_2+\mu}{2},\frac{\lambda_1+\lambda_2-\mu}{2})}(j_t)f(g)
\end{aligned}
$$

Where the second equality follows from (9.1). □

Recall that for $\lambda_1,\lambda_2\in i\mathbb{R}$ the restriction map

$$\mathrm{Res}_{\mathrm{SL}(2,\mathbb{R})}:\mathrm{Ind}_{P_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}((\varepsilon,\varepsilon)\otimes e^{(\lambda_1,\lambda_2)}\otimes 1)\to \mathrm{Ind}_{P_S}^{\mathrm{SL}(2,\mathbb{R})}(0\otimes e^{\lambda_1-\lambda_2}\otimes 1),$$

is an isometry. For $\mu\in i\mathbb{R}$ and $f\in\mathrm{Ind}_{P_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}((\varepsilon,\varepsilon)\otimes e^{(\lambda_1,\lambda_2)}\otimes 1)$ we then have

$$
\begin{aligned}
\|A^{0,\sigma}_{\lambda_1-\lambda_2,\mu}\circ\mathrm{Res}_{\mathrm{SL}(2,\mathbb{R})}f\|^2_{K_S/M_S} &= \|\widetilde{L}_{\varepsilon+\sigma,\frac{1}{2}(\lambda_1+\lambda_2-\mu)}\circ A^{0,\sigma}_{\lambda_1-\lambda_2,\mu}\circ\mathrm{Res}_{\mathrm{SL}(2,\mathbb{R})}f\|^2_{K_{\mathrm{GL}(2,\mathbb{R})}/M_{\mathrm{GL}(2,\mathbb{R})}}\\
&= \|B^{\varepsilon,\sigma}_{\lambda,\mu}f\|^2_{K_{\mathrm{GL}(2,\mathbb{R})}/M_{\mathrm{GL}(2,\mathbb{R})}}
\end{aligned}
$$

by Lemma 1.

Going forward we suppress the subscript on the norms and inner-products for the sake of readability.
For $\mu\in\mathbb{R}$ the last property in Lemma 1 gives

$$
\begin{aligned}
&\langle A^{0,\sigma}_{\lambda_1-\lambda_2,\mu}\circ\mathrm{Res}_{\mathrm{SL}(2,\mathbb{R})}f, T_{0,\mu}\circ A^{0,\sigma}_{\lambda_1-\lambda_2}\circ\mathrm{Res}_{\mathrm{SL}(2,\mathbb{R})}f\rangle\\
&\qquad= \langle B^{\varepsilon,\sigma}_{\lambda,\mu}f, T_{(\varepsilon+\sigma,\varepsilon+\sigma),(\frac{1}{2}(\lambda_1+\lambda_2+\mu),\frac{1}{2}(\lambda_1+\lambda_2-\mu))}\circ B^{\varepsilon,\sigma}_{\lambda,\mu}f\rangle.
\end{aligned}
$$

Using Proposition 4 we then obtain the Plancherel formula for $\mathrm{Ind}_{P_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}((\varepsilon,\varepsilon)\otimes e^{(\lambda_1,\lambda_2)}\otimes 1)$

Theorem 8. *Let $\lambda_1-\lambda_2\in i\mathbb{R}$ and $\varepsilon\in\mathbb{Z}/2\mathbb{Z}$. For $f\in\mathrm{Ind}_{M_{\mathrm{GL}(2,\mathbb{R})}A_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}((\varepsilon,\varepsilon)\otimes e^{(\lambda_1,\lambda_2)})$ we have*

$$\|f\|^2=\int_{i\mathbb{R}}\sum_{\sigma=0}^{1}\|\boldsymbol{B}^{\varepsilon,\sigma}_{\lambda,\mu}f\|^2\frac{d\mu}{|a(\mu)|^2}+\sum_{\sigma=0}^{1}\sum_{\mu\in 1+2\sigma-4\mathbb{N}}b(\mu)\|\mathbb{B}^{\varepsilon,\sigma}_{\lambda,\mu}f\|^2,$$

where

$$\boldsymbol{B}^{\varepsilon,\sigma}_{\lambda,\mu}=\frac{B^{\varepsilon,\sigma}_{\lambda,\mu}}{\Gamma(\frac{1-\mu}{2})},\quad \mathbb{B}^{\varepsilon,\sigma}_{\lambda,\mu}=\frac{\Gamma(\frac{1+2\sigma+\mu+\lambda_1-\lambda_2}{4})}{\Gamma(\frac{1+2\sigma-\mu-\lambda_1+\lambda_2}{4})}B^{\varepsilon,\sigma}_{\lambda,\mu},$$

and

$$b(\mu)=\frac{2^{\mu-2}\Gamma(1-\mu)}{\pi^2\Gamma(\frac{-\mu}{2})},\quad a(\mu)=\frac{2^{\frac{3}{2}}\pi\Gamma(\frac{\mu}{2})}{\Gamma(\frac{1+\mu}{2})^2}.$$

10 Tensor-products of unitary principal series representation of $\mathrm{PGL}(2,\mathbb{R})$

Let $\xi, \eta \in \mathbb{Z}/2\mathbb{Z}$ and $\lambda, \mu, \in \mathbb{C}$ and recall that the restriction map

$$\Theta_{\lambda,\mu} : (\pi_{\xi,\lambda} \otimes \pi_{\eta,\nu})^{\infty}|_G \to C^{\infty}(G/M_G A_G, \mathcal{V}^{x_0}_{(\xi,\lambda+\rho),(\eta,\nu+\rho)})$$

defines a continuous G-intertwining map.

Lemma 13. *When* $\mathrm{Re}(\lambda) > -\frac{1}{4}$ *we have*

$$\mathrm{im}\,(\Theta_{\lambda,\mu}) \subseteq \mathrm{Ind}^{\mathrm{GL}(2,\mathbb{R})}_{M_{\mathrm{GL}(2,\mathbb{R})}A_{\mathrm{GL}(2,\mathbb{R})}} \left((\xi+\eta, \xi+\eta) \otimes e^{\lambda-\mu,\mu-\lambda}\right)$$

Proof. Let $f \in (\pi_{\xi,\lambda} \otimes \pi_{\eta,\mu})^{\infty}|_G$. Then by the discussion in the start of the chapter it is clear that $\Theta_{\lambda,\mu} f$ has the desired equivariance properties. It only remains to show that $\Theta_{\lambda,\mu} f$ is square integrable in the desired region:

$$\begin{aligned}
\|\Theta_{\lambda,\mu} f\|^2 &= \int_{G/M_G A_G} |\Theta_{\lambda,\mu} f(x)|^2 d\,(x M_G A_G) \\
&= \int_{K_G/M_G} \int_{\overline{N}_G} |f(k_\theta \overline{n}_y \cdot (e, w_0))|^2 d\theta dy \\
&= \int_0^{\pi} \int_{\mathbb{R}} \left|(1+y^2)^{\frac{-2\lambda-1}{2}}\right|^2 |f(k_{\varphi(\theta,y)}, k_\theta w_0)|^2 d\theta dy \\
&\le c_f \int_{\mathbb{R}} (1+y^2)^{-\mathrm{Re}(2\lambda+1)} dy.
\end{aligned}$$

The last integral converges when $\mathrm{Re}(\lambda) > -\frac{1}{4}$. ☐

Lemma 14. *For* $f \in C^{\infty}(\mathrm{SL}(2,\mathbb{R})/M_S A_S) \cap L^2(\mathrm{SL}(2,\mathbb{R})/M_S A_S)$ *we have*

$$\int_{\mathrm{SL}(2,\mathbb{R})/M_S A_S} f(x M_S A_S) d(x M_S A_S) = \int_{\mathbb{R}^2} f(\overline{n}_x n_y) dy dx.$$

Proof. Recall that $\overline{N}_S M_S A_S N_S$ exhausts $\mathrm{SL}(2,\mathbb{R})$ except on a lower dimensional set. Furthermore $\overline{N}_S M_S A_S N_S = \overline{N}_S N_S M_S A_S$ hence $\mathrm{SL}(2,\mathbb{R})/M_S A_S \cong \overline{N}_S \times N_S \cong \mathbb{R}^2$ except on a set of measure zero. The Jacobian of this transformation may be computed as follows.
We write

$$\int_{\mathrm{SL}(2,\mathbb{R})/M_S A_S} f(x M_S A_S) d(x M_S A_S) = \int_{\mathbb{R}^2} f(\overline{n}_x n_y) J(\overline{n}_x n_y) dy dx.$$

Since the measure on the left-hand side is invariant under the left action of $g \in \mathrm{SL}(2,\mathbb{R})$ we get the following equations:

$$\begin{aligned}
0 &= \frac{d}{dt}\Big|_{t=0} \int_{\mathbb{R}^2} f(A_t^{-1} \overline{n}_x n_y) J(\overline{n}_x n_y) dy dx, \quad A_t = \mathrm{diag}(e^{-t}, e^t), \\
\int_{\mathbb{R}^2} f(\overline{n}_x n_y) J(\overline{n}_x n_y) dy dx &= \int_{\mathbb{R}^2} f(\overline{n}_x n_y) J(x+h, y) dy dx.
\end{aligned}$$

Note that the second equation implies that $J(x,y) = J(y)$. Noting that $A_t \overline{n}_x = \overline{n}_{t^2 x} A_t$ and $A_t n_y = n_{t^{-2} y} A_t$ we find

$$0 = \frac{d}{dt} J(e^{2t} y)\big|_{t=0} = 2y \frac{d}{dy} J(y).$$

Hence the Jacobian $J(x,y)$ is constant. To compute its value we note $f(g)=(g_{11}^2+g_{21}^2)^{-2}(g_{12}^2+g_{22}^2)^{-2}\in L^2(\mathrm{SL}(2,\mathbb{R})/M_SA_S)$. An explicit computation then shows that

$$\int_{\mathrm{SL}(2,\mathbb{R})/M_SA_S} f(gM_SA_S)d(gM_SA_S)=\int_{\overline{N}_S}\int_{N_S} f(\overline{n}_x n_y)d\overline{n}_x dn_y.$$

□

Since the restriction map $\Theta_{\lambda,\mu}:(\pi_{\xi,\lambda}\otimes\pi_{\eta,\nu})^\infty|_G\to \mathrm{Ind}_{P_{\mathrm{GL}(2,\mathbb{R})}}^{\mathrm{GL}(2,\mathbb{R})}\Big((\xi+\eta,\xi+\eta)\otimes e^{(\lambda-\mu,\mu-\lambda)}\otimes 1\Big)$ is G equivariant the composition

$$B^{\xi+\eta,\sigma}_{2(\lambda-\mu),\nu}\circ\Theta_{\lambda,\mu}:\pi_{\xi,\lambda}\otimes\pi_{\eta,\mu}\to\pi_{\xi+\eta+\sigma,\nu}$$

is G equivariant and hence defines a symmetry breaking operator. For generic parameters this immediately implies that $B^{\xi+\eta,\sigma}_{2(\lambda-\mu),\nu}\circ\Theta_{\lambda,\mu}=\mathrm{const}\times A^{\xi+\eta+\sigma}_{\lambda,\mu,\nu}$. An explicit computation shows that:

$$\begin{aligned}
&B^{\xi+\eta,\sigma}_{2(\lambda-\mu),\nu}\circ\Theta_{\lambda,\mu}f(g)=\widetilde{L}_{\xi+\eta+\sigma,\frac{-\nu}{2}}\circ A^{0,\sigma}_{2(\lambda-\mu),\nu}\circ\mathrm{Res}_{\mathrm{SL}(2,\mathbb{R})}\circ\Theta_{\lambda,\mu}f(g)\\
&=|\det(g)|^{\frac{1+\nu}{2}}_{\xi+\eta+\sigma}\int_{\mathrm{SL}(2,\mathbb{R})/M_SA_S}K^{0,\sigma}_{2(\lambda-\mu),\nu}\left(x^{-1}g\right)\mathrm{Res}_{\mathrm{SL}(2,\mathbb{R})}\circ\Theta_{\lambda,\mu}f(x)dx(M_SA_S)\\
&=|\det(g)|^{\frac{1+\nu}{2}}_{\xi+\eta+\sigma}|\int_{\mathbb{R}}\int_{\mathbb{R}}K^{0,\sigma}_{2(\lambda-\mu),\nu}(n_{-y}\overline{n}_{-x})f\left(g\begin{pmatrix}1&0\\0&\det(g)^{-1}\end{pmatrix}(\overline{n}_x,\overline{n}_xn_yw_0)\right)dxdy\\
&=|\det(g)|^{\frac{1+\nu}{2}}_{\xi+\eta+\sigma}\int_{\mathbb{R}}\int_{\mathbb{R}}K^{0,\sigma}_{2(\lambda-\mu),\nu}(n_{-y}\overline{n}_{-x})f\left(g\begin{pmatrix}1&0\\0&\det(g)^{-1}\end{pmatrix}(\overline{n}_x,\overline{n}_{x+\frac{1}{y}})\right)|-y|^{-2\mu-1}dxdy\\
&=|\det(g)|^{\frac{1+\nu}{2}}_{\xi+\eta+\sigma}\int_{\mathbb{R}}\int_{\mathbb{R}}K^{0,\sigma}_{2(\lambda-\mu),\nu}(n_{(x-z)^{-1}}\overline{n}_{-x})f\left(g\begin{pmatrix}1&0\\0&\det(g)^{-1}\end{pmatrix}(\overline{n}_x,\overline{n}_z)\right)|x-z|^{2\mu+1}(x-z)^{-2}dxdz\\
&=|\det(g)|^{\frac{1+\nu}{2}}_{\xi+\eta+\sigma}\int_{\mathbb{R}}\int_{\mathbb{R}}\left|1-\frac{x}{x-z}\right|_\sigma^{\frac{-2\lambda+2\mu-\nu-1}{2}}|-x|_\sigma^{\frac{2\lambda-2\mu-\nu-1}{2}}|x-z|^{2\mu-1}f\left(g\begin{pmatrix}1&0\\0&\det(g)^{-1}\end{pmatrix}(\overline{n}_x,\overline{n}_z)\right)dxdz\\
&=|\det(g)|^{\frac{1+\nu}{2}}_{\xi+\eta+\sigma}\int_{\mathbb{R}}\int_{\mathbb{R}}|x-z|_\sigma^{\frac{2\lambda+2\mu+\nu-1}{2}}|z|_\sigma^{\frac{-2\lambda+2\mu-\nu-1}{2}}|x|_\sigma^{\frac{2\lambda-2\mu-\nu-1}{2}}f\left(g\begin{pmatrix}1&0\\0&\det(g)^{-1}\end{pmatrix}(\overline{n}_x,\overline{n}_z)\right)dxdz\\
&=A^{\sigma+\eta+\xi}_{\lambda,\mu,\nu}.
\end{aligned}$$

This together with Theorem 8 then gives the following Plancherel formula:

Proposition 5. *Let $f\in\pi_{\xi,\lambda}\otimes\pi_{\eta,\mu}$ with $\mathrm{Re}(\lambda)=\mathrm{Re}(\mu)\in(-\frac{1}{4},\frac{1}{4})$. Then we have that*

$$\begin{aligned}
\langle f,f\rangle&=\langle\Theta_{\lambda,\mu}f,\Theta_{\lambda,\mu}f\rangle\\
&=\int_{i\mathbb{R}}\sum_{\sigma=0}^{1}\|\mathbf{B}^{\varepsilon,\sigma}_{2(\lambda-\mu),\nu}\circ\Theta_{\lambda,\mu}f\|^2\frac{d\nu}{|a(0,\nu)|^2}+\sum_{\sigma=0}^{1}\sum_{\nu\in1+2\sigma-4\mathbb{N}}b(0,\nu)\|\mathbb{B}^{\varepsilon,\sigma}_{2(\lambda-\mu),\nu}\circ\Theta_{\lambda,\mu}f\|^2\\
&=\int_{i\mathbb{R}}\sum_{\sigma=0}^{1}\|A^\sigma_{\lambda,\mu,\nu}f\|^2\frac{d\nu}{|\widetilde{a}(\sigma,\nu)|^2}+\sum_{\sigma=0}^{1}\sum_{\nu\in1+2[\sigma]_2-4\mathbb{N}}\widetilde{b}(\sigma,\nu)\|A^\sigma_{\lambda,\mu,\nu}f\|^2,
\end{aligned}$$

with the coefficients $\widetilde{a}(\sigma,\mu)$ and $\widetilde{b}(\sigma,\mu)$ given by

$$\widetilde{b}(\sigma,\nu)=\frac{2^{\nu-2}\Gamma(1-\nu)}{\pi^2\Gamma(\frac{-\nu}{2})}\left|\frac{\Gamma(\frac{1+2[\sigma]+\nu+2(\lambda-\mu)}{4})}{\Gamma(\frac{1+2[\sigma]-\nu-2(\lambda-\mu)}{4})}\right|^2,\quad a(\sigma,\nu)=\frac{2^{\frac{3}{2}}\pi\Gamma(\frac{\nu}{2})}{\Gamma(\frac{1+\nu}{2})},\quad(\sigma\in\mathbb{Z}/2\mathbb{Z}).$$

Part V

Analytic continuation of the Plancherel formula

For the final chapter of this paper, due to time constraints, we unfortunately have to make some assumptions in order to achieve the full decomposition for tensor products of unitary irreducible representations of $\mathrm{PGL}(2,\mathbb{R})$. At the end we discuss why the assumptions are "reasonable" and we also briefly discuss possible methods of proof. The definitions of the relevant quantities below can be found in Theorem 9.

11 Assumptions

We make the following assumptions for the remainder of the paper:

1. For $\lambda,\mu,\nu\in\mathbb{C}$ and $\sigma\in\mathbb{Z}/2\mathbb{Z}$ we **assume** there *exists* $R\geq\frac{1}{2}$ and $C_R>0$ such that when $|\operatorname{Re}(\nu)|\leq R$ and $\frac{\left(\widetilde{A}^{\ell}_{\lambda,\mu,\nu}f|\widetilde{A}^{\ell}_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu,-\nu}f'\right)}{\widetilde{a}_{(ij)}(\lambda,\mu,\nu,\ell)}$ is regular we have the bound
$$\left|\frac{\left(\widetilde{A}^{\ell}_{\lambda,\mu,\nu}f\mid\widetilde{A}^{\ell}_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu,-\nu}f'\right)}{\widetilde{a}_{(ij)}(\lambda,\mu,\nu,\ell)}\right|<C_R(1+|\nu|)^{-N}\qquad\text{for all }N\in\mathbb{N}.$$
We make this assumption for two reasons. First it guarantees that we can make the contour shift $i\mathbb{R}\to i\mathbb{R}+\frac{1}{2}$, since the square contour with vertices $iP,-iP,\frac{1}{2}-iP$ and $\frac{1}{2}+iP$ will have the horizontal lines vanish as $P\to\infty$.
Secondly it guarantees that the contour along $i\mathbb{R}$ and $i\mathbb{R}+\frac{1}{2}$ converges when $\lambda,\mu\in\mathbb{C}$ such that $\frac{\left(\widetilde{A}^{\ell}_{\lambda,\mu,\nu}f|\widetilde{A}^{\ell}_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu,-\nu}f'\right)}{\widetilde{a}_{(ij)}(\lambda,\mu,\nu,\ell)}$ is regular along the contour.

2. We **assume** that the expressions
$$\operatorname{Res}_{\nu=\pm(2(\lambda+\mu)+1+2\ell)}\frac{\left(\widetilde{A}^{\ell}_{\lambda,\mu,\nu}f\mid\widetilde{A}^{\ell}_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu,-\nu}f'\right)}{\widetilde{a}_{(ij)}(\lambda,\mu,\nu,\ell)}$$
and
$$\operatorname{Res}_{\nu=\pm(2(\lambda-\mu)+1+2\ell)}\frac{\left(\widetilde{A}^{\ell}_{\lambda,\mu,\nu}f\mid\widetilde{A}^{\ell}_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu,-\nu}f'\right)}{\widetilde{a}_{(ij)}(\lambda,\mu,\nu,\ell)}$$

are *holomorphic* in λ, μ. This assumption implies that the bilinear pairing cancels any potential poles of the measure, when restricting (λ, μ, ν) to some specific hyperplanes determined by λ, μ.

3. We **assume** that

$$\widetilde{b}_{(ij)}(\lambda, \mu, \nu, \sigma) \left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu} f \mid \widetilde{T}^{G}_{\sigma+\xi+\eta,\nu} \circ \widetilde{A}^{\sigma}_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu,\nu} f' \right)$$

depends holomorphically on λ, μ when $\nu \in 1 + 2\sigma - 4\mathbb{N}$.

4. We **assume** that

$$\widetilde{b}_{(ij)}(\lambda, \mu, \nu, \sigma) \left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu} f \mid \widetilde{T}^{G}_{\sigma+\xi+\eta,\nu} \circ \widetilde{A}^{\sigma}_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu,\nu} f' \right)$$

is *non-zero* when restricted to $f \in \pi_{\xi,\lambda} \otimes \pi_{\eta,\mu}$ for $\lambda, \mu \in i\mathbb{R} \cup (-\frac{1}{2}, 0)$, $f \in \pi^{\mathrm{ds}}_{\lambda} \otimes \pi_{\eta,\mu}$ for $\mu \in i\mathbb{R} \cup (-\frac{1}{2}, 0)$ and $f \in \pi^{\mathrm{ds}}_{\lambda} \otimes \pi^{\mathrm{ds}}_{\mu}$.

12 The analytic continuation theorem

With the assumptions we are now ready to prove the main technical result:

Theorem 9. *Let $\lambda \in (-\infty, 0)$ and $\mu \in (-\infty, 0] \cup i\mathbb{R}$, $\xi, \eta \in \mathbb{Z}/2\mathbb{Z}$, $f \in \pi_{\xi,\lambda} \otimes \pi_{\eta,\mu}$ and $f' \in \pi_{\xi,(-1)^{i+1}\lambda} \otimes \pi_{\eta,(-1)^{j+1}\mu}$. For $\lambda + \mu \in (-\frac{1}{2} - k, 0)$ and $\lambda - \mu \in (-\frac{1}{2} - m, 0]$ we have*

$$\begin{aligned}
(f, \widetilde{T}^{(ij)}_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu} f') &= \sum_{\sigma=0}^{1} \int_{i\mathbb{R}} \left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu} f \mid \widetilde{A}^{\sigma}_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu,-\nu} f' \right) \frac{d\nu}{\widetilde{a}_{(ij)}(\lambda, \mu, \nu, \sigma)} \\
&+ \sum_{\sigma=0}^{1} \sum_{\nu \in 1+2[\sigma+\xi+\eta]_2 - 4\mathbb{N}} \widetilde{b}_{(11)}(\lambda, \mu, \nu, \sigma) \left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu} f \mid \widetilde{T}^{G}_{\sigma+\xi+\eta,\nu} \circ \widetilde{A}^{\sigma}_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu,\nu} f' \right) \\
&- 2\pi \sum_{\ell=0}^{k-1} \mathrm{Res}_{\nu=-2(\lambda+\mu)-1-2\ell} \frac{\left(\widetilde{A}^{\ell}_{\lambda,\mu,\nu} f \mid \widetilde{A}^{\ell}_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu,-\nu} f' \right)}{\widetilde{a}_{(ij)}(\lambda, \mu, \nu, \ell)} \\
&+ 2\pi \sum_{\ell=0}^{k-1} \mathrm{Res}_{\nu=2(\lambda+\mu)+1+2\ell} \frac{\left(\widetilde{A}^{\ell}_{\lambda,\mu,\nu} f \mid \widetilde{A}^{\ell}_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu,-\nu} f' \right)}{\widetilde{a}_{(ij)}(\lambda, \mu, \nu, \ell)} \\
&- 2\pi \sum_{\ell=0}^{m-1} \mathrm{Res}_{\nu=-2(\lambda-\mu)-1-2\ell} \frac{\left(\widetilde{A}^{\ell}_{\lambda,\mu,\nu} f \mid \widetilde{A}^{\ell}_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu,-\nu} f' \right)}{\widetilde{a}_{(ij)}(\lambda, \mu, \nu, \ell)} \\
&+ 2\pi \sum_{\ell=0}^{m-1} \mathrm{Res}_{\nu=2(\lambda-\mu)+1+2\ell} \frac{\left(\widetilde{A}^{\ell}_{\lambda,\mu,\nu} f \mid \widetilde{A}^{\ell}_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu,-\nu} f' \right)}{\widetilde{a}_{(ij)}(\lambda, \mu, \nu, \ell)}
\end{aligned}$$

with $\widetilde{a}_{(ij)}(\lambda,\mu,\nu,\ell)$ *and* $\widetilde{b}_{(ij)}(\lambda,\mu,\nu,\ell)$ *given by*

$$\begin{aligned}
&\widetilde{a}_{(ij)}(\lambda,\mu,\nu,\sigma)^{-1}\\
&\quad=\Gamma\left(\frac{2\lambda-2\mu-\nu+1+2[\sigma]}{4}\right)\Gamma\left(\frac{-2\lambda+2\mu-\nu+1+2[\sigma]}{4}\right)\Gamma\left(\frac{2\lambda+2\mu-\nu+1+2[\sigma]}{4}\right)\\
&\quad\times\Gamma\left(\frac{2\lambda+2\mu+\nu+1+2[\sigma]}{4}\right)\Gamma\left(\frac{-2\lambda-2\mu+\nu+1+2[\sigma]}{4}\right)\Gamma\left(\frac{2\lambda-2\mu+\nu+1+2[\sigma]}{4}\right)\\
&\quad\times\frac{\Gamma(\frac{1+\nu}{2})\Gamma(\frac{1-\nu}{2})}{2^3\pi^{\frac{3}{2}}\Gamma(\frac{\nu}{2})\Gamma(\frac{-\nu}{2})}\,\frac{\Gamma\left(\frac{-2\lambda-2\mu-\nu+1+2[\sigma]}{4}\right)\Gamma\left(\frac{-2\lambda+2\mu+\nu+1+2[\sigma]}{4}\right)}{\Gamma\left(\frac{-2\lambda+1}{2}\right)^i\Gamma\left(\frac{-2\mu+1}{2}\right)^j}
\end{aligned}$$

$$\begin{aligned}
&\widetilde{b}_{(ij)}(\lambda,\mu,\nu,\sigma)\\
&\quad=\frac{2^{\nu-2}\Gamma(1-\nu)\Gamma\left(\frac{2\lambda+2\mu+\nu+1+2[\sigma]}{4}\right)\Gamma\left(\frac{2\lambda-2\mu-\nu+1+2[\sigma]}{4}\right)\Gamma\left(\frac{2\lambda+2\mu-\nu+1+2[\sigma]}{4}\right)\Gamma\left(\frac{-2\lambda-2\mu-\nu+1+2[\sigma]}{4}\right)}{\pi^2\Gamma(\frac{-\nu}{2})}\\
&\quad\times\frac{\Gamma(\frac{2\lambda-2\mu+\nu+1+2[\sigma]}{4})\Gamma(\frac{-2\lambda+2\mu+\nu+1+2[\sigma]}{4})\Gamma\left(\frac{-2\lambda-2\mu+\nu+1+2[\sigma]}{4}\right)\Gamma\left(\frac{-2\lambda+2\mu-\nu+1+2[\sigma]}{4}\right)}{\Gamma\left(\frac{-2\lambda+1}{2}\right)^i\Gamma\left(\frac{-2\mu+1}{2}\right)^j}.
\end{aligned}$$

Proof. Assume first that $\lambda,\mu\in i\mathbb{R}$ and let $f\in\pi_{\xi,\lambda}\otimes\pi_{\eta,\mu}$ and $f'\in\pi_{\xi,(-1)^{i+1}\lambda}\otimes\pi_{\eta,(-1)^{j+1}\mu}$. Then we have

$$(f,\widetilde{T}^{(ij)}_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu}f')=\langle f,\overline{\widetilde{T}^{(ij)}_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu}f'}\rangle=\langle\Theta_{\lambda,\mu}f,\Theta_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu}\overline{\widetilde{T}^{(ij)}_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu}f'}\rangle$$

where $(\cdot\mid\cdot)$ denotes the usual K-pairing. Interpolating the unitary Plancherel formula then gives

$$\begin{aligned}
&(f,\widetilde{T}^{(ij)}_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu}f')\\
&\quad=\int_{i\mathbb{R}}\sum_{\sigma=0}^{1}\langle A^\sigma_{\lambda,\mu,\nu}f,A^\sigma_{\lambda,\mu,\nu}\circ\overline{\widetilde{T}^{(ij)}_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu}f'}\rangle\frac{d\nu}{|\widetilde{a}(\sigma,\nu)|^2}\\
&\quad+\sum_{\sigma=0}^{1}\sum_{\nu\in1+2[\sigma]_2-4\mathbb{N}}\widetilde{b}(\sigma,\nu)\langle A^\sigma_{\lambda,\mu,\nu}f,\widetilde{T}^G_{\sigma+\xi+\eta,\nu}\circ A^\sigma_{\lambda,\mu,\nu}\circ\overline{\widetilde{T}^{(ij)}_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu}f'}\rangle\\
&\quad=\int_{i\mathbb{R}}\sum_{\sigma=0}^{1}\widetilde{d}^{\sigma,0,0}_{(ij)}(-2\lambda,-2\mu,-\nu)\left(A^\sigma_{\lambda,\mu,\nu}f\mid A^\sigma_{(-1)^{i+1}\lambda,(-1)^{j+1},\mu,-\nu}f'\right)\frac{d\nu}{|\widetilde{a}(\sigma,\nu)|^2}\\
&\quad+\sum_{\sigma=0}^{1}\sum_{\nu\in1+2[\sigma]_2-4\mathbb{N}}\widetilde{d}^{\sigma,0,0}_{(ij)}(-2\lambda,-2\mu,\nu)\,\widetilde{b}(\sigma,\nu)\left(A^\sigma_{\lambda,\mu,\nu}f,\widetilde{T}^G_{\sigma+\xi+\eta,\nu}\circ A^\sigma_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu,\nu}f'\right)\\
&\quad=\int_{i\mathbb{R}}\sum_{\sigma=0}^{1}\widetilde{d}^{\sigma,0,0}_{(ij)}(-2\lambda,-2\mu,-\nu)\left(A^\sigma_{\lambda,\mu,\nu}f\mid A^\sigma_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu,-\nu}f'\right)\frac{d\nu}{|\widetilde{a}(\sigma,\nu)|^2}\\
&\quad+\sum_{\sigma=0}^{1}\sum_{\nu\in1+2[\sigma]_2-4\mathbb{N}}\widetilde{d}^{\sigma,0,0}_{(ij)}(-2\lambda,-2\mu,\nu)\,\widetilde{b}(\sigma,\nu)\left(A^\sigma_{\lambda,\mu,\nu}f\mid\widetilde{T}^G_{\sigma+\xi+\eta,\nu}\circ A^\sigma_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu,\nu}f'\right),
\end{aligned}$$

We let

$$\begin{aligned}
N(\lambda,\mu,\nu,\sigma)&=\Gamma\left(\frac{\lambda+\mu+\nu+1+2[\sigma]}{4}\right)\Gamma\left(\frac{\lambda-\mu-\nu+1+2[\sigma]}{4}\right)\Gamma\left(\frac{-\lambda+\mu-\nu+1+2[\sigma]}{4}\right)\\
&\quad\times\Gamma\left(\frac{\lambda+\mu-\nu+1+2[\sigma]}{4}\right).
\end{aligned}$$

Then $\widetilde{A}^{\sigma}_{\lambda,\mu,\nu} = \frac{A^{\sigma}_{\lambda,\mu,\nu}}{N(\lambda,\mu,\nu,\sigma)}$ extends holomorphically to $\mathbb{C}^3$ and we find

$$\begin{aligned}(f, \widetilde{T}^{(ij)}_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu} f') &= \int_{i\mathbb{R}} \sum_{\sigma=0}^{1} \left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu} f \mid \widetilde{A}^{\sigma}_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu,-\nu} f' \right) \frac{d\nu}{\widetilde{a}_{(ij)}(\lambda,\mu,\nu,\sigma)} \\ &\quad + \sum_{\sigma=0}^{1} \sum_{\nu \in 1+2[\sigma+\xi+\eta]_2 - 4\mathbb{N}} \widetilde{b}_{(ij)}(\lambda,\mu,\nu,\sigma) \left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu} f \mid \widetilde{T}^{G}_{\sigma+\xi+\eta,\nu} \circ \widetilde{A}^{\sigma}_{(-1)^{i+1}\lambda,(-1)^{j+1}\mu,\nu} f' \right).\end{aligned} \tag{12.1}$$

with

$$\widetilde{a}_{(ij)}(\lambda,\mu,\nu,\sigma) = \frac{\widetilde{a}(\sigma,\nu)\widetilde{a}(\sigma,-\nu)}{\widetilde{d}^{\sigma,0,0}_{(ij)}(-2\lambda,-2\mu,-\nu)\, N(2\lambda,2\mu,\nu,\sigma) N((-1)^{i+1}2\lambda,(-1)^{j+1}2\mu,-\nu,\sigma)}$$

and

$$\widetilde{b}_{(ij)}(\lambda,\mu,\nu) = \widetilde{b}(\sigma,\nu) N(2\lambda,2\mu,\nu,\sigma) N((-1)^{i+1}2\lambda,(-1)^{j+1}2\mu,\nu,\sigma) \widetilde{d}^{\sigma,0,0}_{(ij)}(-2\lambda,-2\mu,\nu).$$

Using Lemma 7 we find that $\widetilde{a}_{(ij)}(\lambda,\mu,\nu,\sigma)$ is given by

$$\begin{aligned}&\widetilde{a}_{(ij)}(\lambda,\mu,\nu,\sigma)^{-1} \\ &\quad = \Gamma\left(\frac{2\lambda-2\mu-\nu+1+2[\sigma]}{4}\right)\Gamma\left(\frac{-2\lambda+2\mu-\nu+1+2[\sigma]}{4}\right)\Gamma\left(\frac{2\lambda+2\mu-\nu+1+2[\sigma]}{4}\right) \\ &\quad \times \Gamma\left(\frac{2\lambda+2\mu+\nu+1+2[\sigma]}{4}\right)\Gamma\left(\frac{-2\lambda-2\mu+\nu+1+2[\sigma]}{4}\right)\Gamma\left(\frac{2\lambda-2\mu+\nu+1+2[\sigma]}{4}\right) \\ &\quad \times \frac{\Gamma(\frac{1+\nu}{2})\Gamma(\frac{1-\nu}{2})}{2^3\pi^{\frac{3}{2}}\Gamma(\frac{\nu}{2})\Gamma(\frac{-\nu}{2})} \frac{\Gamma\left(\frac{-2\lambda-2\mu-\nu+1+2\sigma}{4}\right)\Gamma\left(\frac{-2\lambda+2\mu+\nu+1+2\sigma}{4}\right)}{\Gamma\left(\frac{-2\lambda+1}{2}\right)^i\Gamma\left(\frac{-2\mu+1}{2}\right)^j},\end{aligned}$$

and $\widetilde{b}_{(ij)}(\lambda,\mu,\nu,\sigma)$ given by

$$\begin{aligned}&\widetilde{b}_{(ij)}(\lambda,\mu,\nu,\sigma) \\ &\quad = \frac{2^{\nu-2}\Gamma(1-\nu)\Gamma\left(\frac{2\lambda+2\mu+\nu+1+2[\sigma]}{4}\right)\Gamma\left(\frac{2\lambda-2\mu-\nu+1+2[\sigma]}{4}\right)\Gamma\left(\frac{2\lambda+2\mu-\nu+1+2[\sigma]}{4}\right)\Gamma\left(\frac{-2\lambda-2\mu-\nu+1+2[\sigma]}{4}\right)}{\pi^2\Gamma(\frac{-\nu}{2})} \\ &\quad \times \frac{\Gamma(\frac{2\lambda-2\mu+\nu+1+2[\sigma]}{4})\Gamma(\frac{-2\lambda+2\mu+\nu+1+2[\sigma]}{4})\Gamma\left(\frac{-2\lambda-2\mu+\nu+1+2[\sigma]}{4}\right)\Gamma\left(\frac{-2\lambda+2\mu-\nu+1+2[\sigma]}{4}\right)}{\Gamma\left(\frac{-2\lambda+1}{2}\right)^i\Gamma\left(\frac{-2\mu+1}{2}\right)^j}.\end{aligned}$$

Letting $x = \lambda + \mu$, $y = \lambda - \mu$ and

$$\begin{aligned}P(z,\nu,\sigma) &= \Gamma\left(\frac{2z-\nu+1+2[\sigma]}{4}\right)\Gamma\left(\frac{-2z-\nu+1+2[\sigma]}{4}\right) \\ &\quad \times \Gamma\left(\frac{2z+\nu+1+2[\sigma]}{4}\right)\Gamma\left(\frac{-2z+\nu+1+2[\sigma]}{4}\right)\end{aligned}$$

we can simplify $\widetilde{a}_{(ij)}(\lambda,\mu,\nu,\sigma)^{-1}$ and $\widetilde{b}_{(ij)}(\lambda,\mu,\nu,\sigma)$ as

$$\widetilde{a}_{(ij)}(\lambda,\mu,\nu,\sigma)^{-1} = \frac{\Gamma(\frac{1+\nu}{2})\Gamma(\frac{1-\nu}{2})}{2^3\pi^{\frac{3}{2}}\Gamma(\frac{\nu}{2})\Gamma(\frac{-\nu}{2})\Gamma\left(\frac{1-2\lambda}{2}\right)^i\Gamma\left(\frac{1-2\mu}{2}\right)^j} P(x,\nu,\sigma)P(y,\nu,\sigma)$$

$$\widetilde{b}_{(ij)}(\lambda,\mu,\nu,\sigma) = \frac{2^{\nu-2}\Gamma(1-\nu)}{\pi^2\Gamma(\frac{-\nu}{2})\Gamma\left(\frac{1-2\lambda}{2}\right)^i\Gamma\left(\frac{1-2\mu}{2}\right)^j} P(x,\nu,\sigma)P(y,\nu,\sigma).$$

Abusing notation slightly we shall write $\widetilde{a}_{(ij)}(x,y,\nu,\sigma)^{-1}$ and $\widetilde{b}_{(ij)}(x,y,\nu,\sigma)$ instead. We carry out the proof for the case $(ij)=(11)$, however the other cases follow be identical computations. The left hand side of (12.1) depends holomorphically on $\lambda,\mu\in\mathbb{C}$. Meanwhile the right hand side becomes

$$(f,\widetilde{T}^{(ij)}_{\lambda,\mu}f') = \int_{i\mathbb{R}}\sum_{\sigma=0}^{1}\left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu}f \mid \widetilde{A}^{\sigma}_{\lambda,\mu,-\nu}f'\right)\frac{d\nu}{\widetilde{a}_{(11)}(\lambda,\mu,\nu,\sigma)}$$
$$+\sum_{\sigma=0}^{1}\sum_{\nu\in 1+2[\sigma+\xi+\eta]_2-4\mathbb{N}}\widetilde{b}_{(11)}(\lambda,\mu,\nu,\sigma)\left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu}f \mid \widetilde{T}^{G}_{\sigma+\xi+\eta,\nu}\circ\widetilde{A}^{\sigma}_{\lambda,\mu,\nu}f'\right).$$

Note the integral in (12.1) is defined when $\mathrm{Re}(x),\mathrm{Re}(y)\in\left(-\frac{1}{2}-\sigma,\frac{1}{2}+\sigma\right)$. By assumption it suffices to find an extension to each of the integrals

$$\int_{i\mathbb{R}}\left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu}f \mid \widetilde{A}^{\sigma}_{\lambda,\mu,-\nu}f'\right)\frac{d\nu}{\widetilde{a}_{(11)}(\lambda,\mu,\nu,\sigma)}. \tag{12.2}$$

We do this by moving the contour of the integral to obtain an extension in x. In doing so we pick up some residues in x, which are holomorphic by the assumption at the start of the chapter. Assume that $\mathrm{Re}(x)\in\left(-\frac{1}{2}-\sigma,-\frac{1}{4}-\sigma\right)$ and $\mathrm{Re}(y)\in\left(-\frac{1}{4},\frac{1}{4}\right)$. Then $\widetilde{a}_{(11)}(x,y,\nu,\sigma)$ has exactly a single pole in the range $\mathrm{Re}(\nu)\in(0,\frac{1}{2})$ at $\nu=2x+1+2\sigma$. Hence shifting the contour of the integral from $i\mathbb{R}$ to $i\mathbb{R}+\frac{1}{2}$ we find

$$\int_{i\mathbb{R}}\left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu}f \mid \widetilde{A}^{\sigma}_{\lambda,\mu,-\nu}f'\right)\frac{d\nu}{\widetilde{a}_{(11)}(\lambda,\mu,\nu,\sigma)}$$
$$=\int_{i\mathbb{R}+\frac{1}{2}}\left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu}f \mid \widetilde{A}^{\sigma}_{\lambda,\mu,-\nu}f'\right)\frac{d\nu}{\widetilde{a}_{(11)}(\lambda,\mu,\nu,\sigma)}-2\pi\,\mathrm{Res}_{\nu=2x+1+2\sigma}\frac{\left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu}f \mid \widetilde{A}^{\sigma}_{\lambda,\mu,-\nu}f'\right)}{\widetilde{a}_{(11)}(\lambda,\mu,\nu,\sigma)}. \tag{12.3}$$

The right hand side of (12.3) is then well-defined for $\mathrm{Re}(x)\in\left(-\frac{3}{4}-\sigma,-\frac{1}{4}-\sigma\right)$ and $\mathrm{Re}(y)\in\left(-\frac{1}{4},\frac{1}{4}\right)$, hence (12.3) defines an analytic continuation of (12.2) in x. If we fix the same y as before but let $x\in\mathbb{C}$ with $\mathrm{Re}(x)\in\left(-\frac{3}{4}-\sigma,-\frac{1}{2}-\sigma\right)$ then $\widetilde{a}_{(11)}(x,y,\nu,\sigma)$ has a single pole at $\nu=-2x-1-2\sigma$ in the range $\mathrm{Re}(\nu)\in(0,\frac{1}{2})$. Shifting the contour of the integral back to $i\mathbb{R}$ we thus find

$$\int_{i\mathbb{R}+\frac{1}{2}}\left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu}f \mid \widetilde{A}^{\sigma}_{\lambda,\mu,-\nu}f'\right)\frac{d\nu}{\widetilde{a}_{(11)}(\lambda,\mu,\nu,\sigma)}-2\pi\,\mathrm{Res}_{\nu=-2x-1-2\sigma}\frac{\left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu}f \mid \widetilde{A}^{\sigma}_{\lambda,\mu,-\nu}f'\right)}{\widetilde{a}_{(11)}(\lambda,\mu,\nu,\sigma)}$$
$$=\int_{i\mathbb{R}}\left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu}f \mid \widetilde{A}^{\sigma}_{\lambda,\mu,-\nu}f'\right)\frac{d\nu}{\widetilde{a}_{(11)}(\lambda,\mu,\nu,\sigma)}$$
$$+2\pi\,\mathrm{Res}_{\nu=2x+1+2\sigma}\frac{\left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu}f \mid \widetilde{A}^{\sigma}_{\lambda,\mu,-\nu}f'\right)}{\widetilde{a}_{(11)}(\lambda,\mu,\nu,\sigma)}-2\pi\,\mathrm{Res}_{\nu=-2x-1-2\sigma}\frac{\left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu}f \mid \widetilde{A}^{\sigma}_{\lambda,\mu,-\nu}f'\right)}{\widetilde{a}_{(11)}(\lambda,\mu,\nu,\sigma)}. \tag{12.4}$$

The right hand side of (12.4) is well-defined for $\mathrm{Re}(x)\in(-\frac{5}{2}-\sigma,-\frac{1}{2}-\sigma)$. Note that if $x=-\frac{1}{2}-\sigma$ then $\widetilde{a}_{(11)}(x,y,\nu,\sigma)$ has no poles in the range $\mathrm{Re}(\nu)\in(0,\frac{1}{2})$ and $\widetilde{a}_{(11)}(x,y,\nu,\sigma)^{-1}$ is regular for $\mathrm{Re}(\nu)=0$, hence

$$\int_{i\mathbb{R}+\frac{1}{2}}\left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu}f \mid \widetilde{A}^{\sigma}_{\lambda,-\mu,-\nu}f'\right)\frac{d\nu}{\widetilde{a}_{(10)}(\lambda,\mu,\nu,\sigma)}=\int_{i\mathbb{R}}\left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu}f \mid \widetilde{A}^{\sigma}_{\lambda,-\mu,-\nu}f'\right)\frac{d\nu}{\widetilde{a}_{(10)}(\lambda,\mu,\nu,\sigma)}.$$

Thus (12.4) in combination with (12.3) defines an analytic extension of (12.2) to $x \in \mathbb{C}$ with $\mathrm{Re}(x) \in (-\frac{5}{2} - \sigma, -\frac{1}{2} - \sigma]$ and $\mathrm{Re}(y) \in (-\frac{1}{4}, \frac{1}{4})$. Assume now we have found an an analytic continuation to $x \in \mathbb{C}$ with $\mathrm{Re}(x) \in (-\frac{1}{2} - 2k - \sigma, -\frac{1}{4} - 2k - \sigma)$ and $\mathrm{Re}(y) \in (-\frac{1}{4}, \frac{1}{4})$ for $k \in \mathbb{N}_0$. Then the integral

$$\int_{i\mathbb{R}} \left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu} f \mid \widetilde{A}^{\sigma}_{\lambda,\mu,-\nu} f'\right) \frac{d\nu}{\widetilde{a}_{(11)}(\lambda,\mu,\nu,\sigma)}$$

has exactly one pole at $\nu = 2x + 1 + 2\sigma + 4k$ in the range $\mathrm{Re}(\nu) \in (0, \frac{1}{2})$. Hence shifting the contour we find

$$\begin{aligned}
&\int_{i\mathbb{R}} \left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu} f \mid \widetilde{A}^{\sigma}_{\lambda,\mu,-\nu} f'\right) \frac{d\nu}{\widetilde{a}_{(11)}(\lambda,\mu,\nu,\sigma)} \\
&\quad = \int_{i\mathbb{R}+\frac{1}{2}} \left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu} f \mid \widetilde{A}^{\sigma}_{\lambda,\mu,-\nu} f'\right) \frac{d\nu}{\widetilde{a}_{(10)}(\lambda,\mu,\nu,\sigma)} \\
&\quad - 2\pi \,\mathrm{Res}_{\nu=2x+1+2\sigma+4k} \frac{\left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu} f \mid \widetilde{A}^{\sigma}_{\lambda,\mu,-\nu} f'\right)}{\widetilde{a}_{(11)}(\lambda,\mu,\nu,\sigma)}
\end{aligned}$$

For $\mathrm{Re}(\nu) = \frac{1}{2}$ we have that $\widetilde{a}_{(11)}(x,y,\nu,\sigma)^{-1}$ is regular for $\mathrm{Re}(x) \in (-\frac{3}{4} - 2k - \sigma, -\frac{1}{4} - 2k - \sigma)$ and hence the right hand side defines an analytic continuation of (12.2) to $\mathrm{Re}(x) \in (-\frac{3}{4} - 2k - \sigma, -\frac{1}{4} - 2k - \sigma)$ and $\mathrm{Re}(y) \in (-\frac{1}{4}, \frac{1}{4})$. We now consider $x \in \mathbb{C}$ with $\mathrm{Re}(x) \in (-\frac{3}{4} - 2k - \sigma, -\frac{1}{2} - 2k - \sigma)$. Then $\widetilde{a}_{(11)}(x,y,\nu,\sigma)^{-1}$ has exactly one pole at $\nu = -2x - 1 - 2\sigma - 4k$ in the range $\nu \in (0, \frac{1}{2})$. Hence shifting the contour back to $i\mathbb{R}$ we find

$$\begin{aligned}
&\int_{i\mathbb{R}+\frac{1}{2}} \left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu} f \mid \widetilde{A}^{\sigma}_{\lambda,\mu,-\nu} f'\right) \frac{d\nu}{\widetilde{a}_{(11)}(\lambda,\mu,\nu,\sigma)} - 2\pi \,\mathrm{Res}_{\nu=-2x-1-2\sigma-4k} \frac{\left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu} f \mid \widetilde{A}^{\sigma}_{\lambda,\mu,-\nu} f'\right)}{\widetilde{a}_{(11)}(\lambda,\mu,\nu,\sigma)} \\
&\quad = \int_{i\mathbb{R}} \left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu} f \mid \widetilde{A}^{\sigma}_{\lambda,\mu,-\nu} f'\right) \frac{d\nu}{\widetilde{a}_{(11)}(\lambda,\mu,\nu,\sigma)} \\
&\quad + 2\pi \,\mathrm{Res}_{\nu=2x+1+2\sigma+4k} \frac{\left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu} f \mid \widetilde{A}^{\sigma}_{\lambda,\mu,-\nu} f'\right)}{\widetilde{a}_{(11)}(\lambda,\mu,\nu,\sigma)} - 2\pi \,\mathrm{Res}_{\nu=-2x-1-2\sigma-4k} \frac{\left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu} f \mid \widetilde{A}^{\sigma}_{\lambda,\mu,-\nu} f'\right)}{\widetilde{a}_{(11)}(\lambda,\mu,\nu,\sigma)}.
\end{aligned}$$

Since $\widetilde{a}_{(11)}(x,y,\nu,\sigma)^{-1}$ is regular for $\mathrm{Re}(x) \in (-\frac{1}{2} - 2(k+1) - \sigma, -\frac{1}{2} - 2k - \sigma)$ the right hand side together with previous extension defines an analytic continuation of (12.2) to $(-\frac{1}{2} - 2(k+1) - \sigma, 0]$. The desired extension in x then follows by induction. Note that the assumption that $\mathrm{Re}(y) \in (-\frac{1}{4}, \frac{1}{4})$ is essentially redundant. We could have assumed that $\mathrm{Re}(y) \in (-\frac{1}{2} - \sigma, \frac{1}{2} + \sigma)$, since fixing y in this region at every step causes any residue with respect to y to cancel when we move the contour back. Since $\widetilde{a}_{(11)}(x,y,\nu,\sigma) = \widetilde{a}_{(11)}(y,x,\nu,\sigma)$ and $\widetilde{a}_{(11)}(x,-y,\nu,\sigma) = \widetilde{a}_{(11)}(x,y,\nu,\sigma)$ the analytic continuation with respect to y follows immediately after the continuation from the continuation in x . $\square$

13 Decomposition of tensor products of principal series representations of $\mathrm{PGL}(2,\mathbb{R})$

Using the analytic continuation found in the previous section we obtain the following unitary branching laws:

Corollary 2. *For $\lambda, \mu \in i\mathbb{R}$ and $\xi, \eta \in \mathbb{Z}/2\mathbb{Z}$ we have*

$$\pi_{\xi,\lambda} \otimes \pi_{\eta,\mu}|_G \cong \bigoplus_{\sigma=0,1} \int_{i\mathbb{R}}^{\oplus} \pi_{\sigma,\nu}\, d\nu \oplus \bigoplus_{\nu \in \frac{1}{2}-\mathbb{N}} \pi_\nu^{\mathrm{ds}}. \tag{13.1}$$

For $\lambda \in (-\frac{1}{2}, 0)$, $\xi, \eta \in \mathbb{Z}/2\mathbb{Z}$ and $\mu \in i\mathbb{R}$ we have

$$\pi^c_{\xi,\lambda} \otimes \pi_{\eta,\mu}|_G \cong \bigoplus_{\sigma=0,1} \int_{i\mathbb{R}}^{\oplus} \pi_{\sigma,\nu}\, d\nu \oplus \bigoplus_{\nu \in \frac{1}{2}-\mathbb{N}} \pi_\nu^{\mathrm{ds}}. \tag{13.2}$$

For $\lambda, \mu \in (-\frac{1}{2}, 0)$ with $\lambda + \mu \geq -\frac{1}{2}$ and $\xi, \eta \in \mathbb{Z}/2\mathbb{Z}$ we have

$$\pi^c_{\xi,\lambda} \otimes \pi^c_{\eta,\mu}|_G \cong \bigoplus_{\sigma=0,1} \int_{i\mathbb{R}}^{\oplus} \pi_{\sigma,\nu}\, d\nu \oplus \bigoplus_{\nu \in \frac{1}{2}-\mathbb{N}} \pi_\nu^{\mathrm{ds}}. \tag{13.3}$$

For $\lambda, \mu \in (-\frac{1}{2}, 0)$ with $\lambda + \mu < -\frac{1}{2}$ and $\xi, \eta \in \mathbb{Z}/2\mathbb{Z}$ we have

$$\pi^c_{\xi,\lambda} \otimes \pi^c_{\xi,\mu}|_G \cong \bigoplus_{\sigma=0,1} \int_{i\mathbb{R}}^{\oplus} \pi_{\sigma,\nu}\, d\nu \oplus \bigoplus_{\nu \in \frac{1}{2}-\mathbb{N}} \pi_\nu^{\mathrm{ds}} \oplus \pi^c_{\xi+\eta,\lambda+\mu+\frac{1}{2}}. \tag{13.4}$$

For $\lambda \in \frac{1}{2} - \mathbb{N}$, $\eta \in \mathbb{Z}/2\mathbb{Z}$ and $\mu \in i\mathbb{R}$ we have

$$\pi_\lambda^{\mathrm{ds}} \otimes \pi_{\eta,\mu}|_G \cong \bigoplus_{\sigma=0,1} \int_{i\mathbb{R}}^{\oplus} \pi_{\sigma,\nu}\, d\nu \oplus \bigoplus_{\nu \in \frac{1}{2}-\mathbb{N}} \pi_\nu^{\mathrm{ds}}. \tag{13.5}$$

For $\lambda \in \frac{1}{2} - \mathbb{N}$ and $\mu \in (-\frac{1}{2}, 0)$ we have

$$\pi_\lambda^{\mathrm{ds}} \otimes \pi_{\eta,\mu}|_G \cong \bigoplus_{\sigma=0,1} \int_{i\mathbb{R}}^{\oplus} \pi_{\sigma,\nu}\, d\nu \oplus \bigoplus_{\nu \in \frac{1}{2}-\mathbb{N}} \pi_\nu^{\mathrm{ds}}. \tag{13.6}$$

For $\lambda \in \frac{1}{2} - \mathbb{N}$ and $\mu \in \frac{1}{2} - \mathbb{N}$ we have

$$\pi_\lambda^{\mathrm{ds}} \otimes \pi_\mu^{\mathrm{ds}}|_G \cong \bigoplus_{\sigma=0,1} \int_{i\mathbb{R}}^{\oplus} \pi_{\sigma,\nu}\, d\nu \oplus \bigoplus_{\nu \in \frac{1}{2}-\mathbb{N}} \pi_\nu^{\mathrm{ds}} \tag{13.7}$$

Proof. We start by showing how to achieve (13.1). Consider the map

$$\begin{aligned}\psi : \pi_{\xi,\lambda} \otimes \pi_{\eta,\mu}|_G \to \bigoplus_{\sigma=0,1} \int_{i\mathbb{R}}^{\oplus} \pi_{\sigma,\nu}\, d\nu \oplus \bigoplus_{\nu \in \frac{1}{2}-\mathbb{N}} \pi_\nu^{\mathrm{ds}} \\ f \mapsto \left((A^0_{\lambda,\mu,\nu} f)_{\nu \in i\mathbb{R}}, (A^1_{\lambda,\mu,\nu} f)_{\nu \in i\mathbb{R}}, (A^0_{\lambda,\mu,\nu} f)_{\nu \in 1-4\mathbb{N}}, (A^1_{\lambda,\mu,\nu} f)_{\nu \in 3-4\mathbb{N}} \right).\end{aligned}$$

By proposition 5 this map is injective. Since $A^\sigma_{\lambda,\mu,\nu} \neq 0$ when $\lambda, \mu \in i\mathbb{R}$ and $\nu \in i\mathbb{R} \cup 1 - 2\mathbb{N}$ we have that ψ is surjective since the image is a sub-representation. Hence ψ is an isomorphism and the result follows.

(13.2) follows from Theorem 9 as follows. We let $\lambda \in (-\frac{1}{2}, 0)$, $\mu \in i\mathbb{R}$ and $\eta \in \mathbb{Z}/2\mathbb{Z}$. Then we let $f' = \overline{f}$ to get

$$\begin{aligned}(f, \widetilde{T}^{(10)}_{\lambda,-\mu}\overline{f}) = \|f\|^2 &= \sum_{\sigma=0}^{1} \int_{i\mathbb{R}} \left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu} f \mid \widetilde{A}^{\sigma}_{\lambda,-\mu,-\nu}\overline{f} \right) \frac{d\nu}{\widetilde{a}_{(ij)}(\lambda,\mu,\nu,\sigma)} \\ &+ \sum_{\sigma=0}^{1} \sum_{\nu \in 1+2[\sigma+\xi+\eta]_2 - 4\mathbb{N}} \widetilde{b}_{(11)}(\lambda,\mu,\nu,\sigma) \left(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu} f \mid \widetilde{T}^{G}_{\sigma+\eta,\nu} \circ \widetilde{A}^{\sigma}_{\lambda,-\mu,\nu}\overline{f'} \right) \\ &= \sum_{\sigma=0}^{1} \int_{i\mathbb{R}} \|\widetilde{A}^{\sigma}_{\lambda,\mu,\nu} f\|^2 \frac{d\nu}{\widetilde{a}_{(ij)}(\lambda,\mu,\nu,\sigma)} + \sum_{\sigma=0}^{1} \sum_{\nu \in 1+2[\sigma+\xi+\eta]_2 - 4\mathbb{N}} \widetilde{b}_{(11)}(\lambda,\mu,\nu,\sigma) \|\widetilde{A}^{\sigma}_{\lambda,\mu,\nu} f\|^2 \end{aligned} \tag{13.8}$$

and the result then follows from Lemmas 2 and 3. Note that we can with out loss of generality assume that $\mathrm{Re}(\lambda) \leq \mathrm{Re}(\mu)$ since the tensor product is symmetric.
Then (13.3) follows from the same line of reasoning.
For (13.4) we can use the same reasoning as before for the continuous and discrete parts, but Theorem 9 yields two additional residues to consider. To investigate these we first recall that

$$\widetilde{T}^{G}_{\sigma,\nu} \circ \widetilde{A}^{\sigma}_{\lambda,\mu,\nu} f = \frac{1}{\Gamma(\frac{1-\nu}{2})} \widetilde{A}^{\sigma}_{\lambda,\mu,-\nu}.$$

Applying this we find

$$\begin{aligned} &- 2\pi \operatorname{Res}_{\nu=-2(\lambda+\mu)-1} \frac{\left(\widetilde{A}^{0}_{\lambda,\mu,\nu} f \mid \widetilde{A}^{0}_{\lambda,\mu,-\nu}\overline{f} \right)}{\widetilde{a}_{(11)}(\lambda,\mu,\nu,0)} + 2\pi \operatorname{Res}_{\nu=2(\lambda+\mu)+1} \frac{\left(\widetilde{A}^{0}_{\lambda,\mu,\nu} f \mid \widetilde{A}^{0}_{\lambda,\mu,-\nu}\overline{f} \right)}{\widetilde{a}_{(11)}(\lambda,\mu,\nu,0)} \\ &= 4\pi \operatorname{Res}_{\nu=2(\lambda+\mu)+1} \frac{\Gamma(\frac{1+\nu}{2}) \|\widetilde{A}^{0}_{\lambda,\mu,\nu} f\|^2}{\widetilde{a}_{(11)}(\lambda,\mu,\nu,0)}. \end{aligned}$$

The result then follows by the same arguments as used previously. We remark that the residues can in this case be seen to be non-zero by evaluating on the Spherical vector $\psi_0 \otimes \psi_0$.
To show (13.5) it suffices to show that any residue from Theorem 9 vanishes when $\lambda = \frac{1}{2} - m$ for $m \in \mathbb{N}$ and $\mu \in i\mathbb{R}$. The residues to consider are

$$4\pi \sum_{\ell=0}^{m-1} \operatorname{Res}_{\nu=2(\lambda+\mu)+1+2\ell} \frac{(\widetilde{A}^{\ell}_{\lambda,\mu,\nu} f, \widetilde{A}^{\ell}_{\lambda,-\mu,\nu}\overline{f})}{\widetilde{a}_{(10)}(\lambda,\mu,\nu,\ell)} + 4\pi \sum_{\ell=0}^{m-1} \operatorname{Res}_{\nu=2(\lambda-\mu)+1+2\ell} \frac{(\widetilde{A}^{\ell}_{\lambda,\mu,\nu} f, \widetilde{A}^{\ell}_{\lambda,-\mu,\nu}\overline{f})}{\widetilde{a}_{(10)}(\lambda,\mu,\nu,\ell)}$$

with $\widetilde{a}_{(10)}(\lambda,\mu,\nu,\ell)^{-1}$ at $\nu=2(\lambda+\mu)+1+2\ell$ given by

$$\begin{aligned}
&\operatorname{Res}_{\nu=2(\lambda+\mu)+1+2\ell}(\widetilde{a}_{(10)}(\lambda,\mu,\nu,\ell)^{-1})\\
&=\frac{(-1)^{\frac{[\ell]-\ell}{2}}}{\left(\frac{[\ell]-\ell}{2}\right)!}\Gamma\left(-\mu+\frac{[\ell]-\ell}{2}\right)\Gamma\left(-\lambda+\frac{[\ell]-\ell}{2}\right)\Gamma\left(\lambda+\mu+\frac{1}{2}+\frac{[\ell]+\ell}{2}\right)\Gamma\left(-\lambda-\mu+\frac{1}{2}+\frac{[\ell]+\ell}{2}\right)\\
&\times\frac{\Gamma\left(\lambda+\frac{1}{2}+\frac{[\ell]+\ell}{2}\right)\Gamma(\lambda+\mu+1+\ell)\Gamma(-\lambda-\mu-\ell)}{2^3\pi^{\frac{3}{2}}\Gamma(\lambda+\mu+\ell+\frac{1}{2})\Gamma(-\lambda-\mu-\ell-\frac{1}{2})}\frac{\Gamma\left(-\lambda-\mu+\frac{[\ell]-\ell}{2}\right)\Gamma\left(\mu+1+\frac{[\ell]+\ell}{2}\right)}{\Gamma\left(\frac{-2\lambda+1}{2}\right)}\\
&=\frac{(-1)^{\frac{[\ell]-\ell}{2}}\Gamma\left(-\mu+\frac{[\ell]-\ell}{2}\right)\Gamma\left(\frac{[\ell]-\ell-1+2m}{2}\right)\Gamma\left(1+\mu+\frac{[\ell]+\ell-2m}{2}\right)\Gamma\left(\mu+\frac{[\ell]+\ell+2m}{2}\right)}{\left(\frac{[\ell]-\ell}{2}\right)!}\\
&\times\frac{\Gamma\left(1+\frac{[\ell]+\ell-2m}{2}\right)\Gamma(\mu+\frac{3}{2}+\ell-m)\Gamma(\frac{-1}{2}-\mu-\ell+m)}{2^3\pi^{\frac{3}{2}}\Gamma(1+\mu+\ell-m)\Gamma(-1-\mu+m-\ell)}\frac{\Gamma\left(-\frac{1}{2}-\mu+\frac{[\ell]-\ell+2m}{2}\right)\Gamma\left(\mu+1+\frac{[\ell]+\ell}{2}\right)}{\Gamma\left(m\right)}
\end{aligned}$$

When $\mu\in i\mathbb{R}\setminus\{0\}$ then the above only has a single pole for each $0\le\ell\le m-1$, but since we also have

$$\frac{-2\lambda+2\mu-\nu+1+2\sigma}{4}=m+\frac{[\ell]-\ell+1}{2}\ge 0\quad\text{and}\quad\frac{2\lambda+2\mu-\nu+1+2\sigma}{4}=\frac{[\ell]-\ell-2m}{2}\le 0$$

and the second case of Theorem 7 implies that $\widetilde{A}^{\ell}_{\lambda,\pm\mu,\nu}f=0$ at $\nu=2(\lambda+\mu)+1+2\ell$ making the product vanish. If $\mu=0$ then $\Gamma\left(-\frac{1}{2}-\mu+\frac{[\ell]-\ell+2m}{2}\right)^{-1}=0$ and the result again follows by Theorem 7. The residues of the form

$$\operatorname{Res}_{\nu=2(\lambda-\mu)+1+2\ell}\frac{(\widetilde{A}^{\ell}_{\lambda,\mu,\nu}f,\widetilde{A}^{\ell}_{\lambda,-\mu,-\nu}\overline{f})}{\widetilde{a}_{(10)}(\lambda,\mu,\nu,\ell)}$$

vanish by the same reasoning, but by the first case of Theorem 7 instead.
The decompositions in (13.6) and (13.7) follows along a similar line of reasoning. □

14 Discussion of assumptions

We start by first remarking that the general method used to obtain the analytic extension in Theorem 9 has been used previously in literature, see e.g. [Wei21]. Furthermore the bilinear form on $\pi_{\xi,\lambda}\otimes\pi_{\eta,\mu}\times\pi_{\xi,-\lambda}\otimes\pi_{\eta,-\mu}$ depends holomorphically on $\lambda,\mu\in\mathbb{C}$. Hence its composition with the Knapp-Stein intertwiner $T^{(ij)}_{\lambda,\mu}$ again depends holomorphically on λ,μ. This suggests that the residues picked up in the analytic continuation process should depend holomorphically on λ,μ as well. Since most poles of $\widetilde{a}_{(ij)}(\lambda,\mu,\nu,\sigma)$ comes directly from the normalization of the symmetry breaking operators, these can in most cases be neutralized by appealing to Theorem 7. However in some cases a more detailed understanding of the bilinear pairing $(\widetilde{A}^{\sigma}_{\lambda,\mu,\nu}f,\widetilde{A}^{\sigma}_{\lambda',\mu',\nu'})$ appears to be needed. The same goes for the discrete part and $\widetilde{b}_{(ij)}(\lambda,\mu,\nu)$. Due to time restriction we were unfortunately not able to carry out the full analysis of this problem, and hence we had to make assumptions 2 and 3.

Assumption 4 is made to guarantee that the Plancherel formula actually carries the full information about the desired branching laws. It seems unlikely that any terms of the discrete part should vanish, but to check this we would require some vector (or family of vectors)

$\psi \in \pi_{\xi,\lambda} \otimes \pi_{\eta,\mu}$ such that we can explicitly compute $\widetilde{A}^{\sigma}_{\lambda,\mu,\nu}\psi$. A good candidate for ψ is the K-finite vectors of $\pi_{\xi,\lambda} \otimes \pi_{\eta,\mu}$. However since the norm in the discrete part contains a Knapp-Stein intertwiner, the terms of the discrete part will vanish on any fixed K-finite vector at some point. However if one could find an explicit formula to evaluate on any K-finite vector assumption 4 would be redundant. Note that such a formula would potentially also resolve assumptions 2 and 3. Since we already saw how to evaluate on lowest K-types in chapter III, it is perhaps possible to understand evaluation on any K-type by finding recurrence relations between evaluation on different K-types. Indeed such a method has been used previously in the literature, see e.g [FØ19] Theorem 4.1 and Proposition 4.6.

Assumption 1 is perhaps, like assumption 4, a bit redundant. The symmetry breaking operators $\widetilde{A}^{\sigma}_{\lambda,\mu,\nu}$ heuristically play the role of a Fourier-transform in e.g. (13.8). Hence the assumption made in 1 is in the nature of existence of some Paley–Wiener type theory for such operators. This is perhaps not so far fetched since the unitary Plancherel formula from Proposition 5 originates from the theory described in [BJD23], where the operators considered are Jacobi-transforms, for which a Paley–Wiener type theory does exist, see e.g. [Koo75].

We end by remarking that (13.1), (13.2), (13.3) and (13.4) does not need assumptions 2 and 3.

Appendix A

Appendix

Listed below are the results for the composition of symmetry breaking operators $A^{\sigma,\xi,\eta}_{\lambda,\mu,\nu}$ with the standard intertwining operators $T^{(ij)}_{(-1)^i\lambda,(-1)^j\mu}$, given in terms of the parameters

$$s_1 = \frac{1}{2}(\lambda+\mu+\nu-1), \quad s_2 = \frac{1}{2}(\lambda-\mu-\nu-1), \quad s_3 = \frac{1}{2}(-\lambda+\mu-\nu-1).$$

$$\widetilde{d}^{0,0,0}_{(10)}(s_1,s_2,s_3) = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}s_2+\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2}s_1\right)\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}s_2+1\right)\Gamma\left(-\frac{1}{2}s_2\right)} \tag{0.1}$$

$$\widetilde{d}^{0,0,0}_{(01)}(s_1,s_2,s_3) = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}s_3+\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2}s_1\right)\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}s_3+1\right)\Gamma\left(-\frac{1}{2}s_3\right)} \tag{0.2}$$

$$\widetilde{d}^{0,0,0}_{(11)}(s_1,s_2,s_3) = \frac{\pi\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}s_2+\frac{1}{2}s_3+1\right)\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2}s_1\right)\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}s_2+1\right)\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}s_3+1\right)\Gamma\left(-\frac{1}{2}s_1-\frac{1}{2}s_2-\frac{1}{2}s_3-\frac{1}{2}\right)} \tag{0.3}$$

$$\widetilde{d}^{1,0,0}_{(10)}(s_1,s_2,s_3) = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}s_1+1\right)\Gamma\left(\frac{1}{2}s_2+1\right)}{\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}s_2+1\right)\Gamma\left(-\frac{1}{2}s_1+\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}s_2+\frac{1}{2}\right)} \tag{0.4}$$

$$\widetilde{d}^{1,0,0}_{(01)}(s_1,s_2,s_3) = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}s_1+1\right)\Gamma\left(\frac{1}{2}s_3+1\right)}{\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}s_3+1\right)\Gamma\left(-\frac{1}{2}s_1+\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}s_3+\frac{1}{2}\right)} \tag{0.5}$$

$$\widetilde{d}^{1,0,0}_{(11)}(s_1,s_2,s_3) = \frac{\pi\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}s_2+\frac{1}{2}s_3+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}s_1+1\right)}{\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}s_2+1\right)\Gamma\left(\frac{1}{2}s_1+\frac{1}{2}s_3+1\right)\Gamma\left(-\frac{1}{2}s_1-\frac{1}{2}s_2-\frac{1}{2}s_3\right)\Gamma\left(-\frac{1}{2}s_1+\frac{1}{2}\right)} \tag{0.6}$$

$$\widetilde{d}^{0,1,1}_{(10)}(s_1,s_2,s_3) = \frac{-\sqrt{\pi}\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\,s_2+1\right)}{\Gamma\left(-\frac{1}{2}\,s_1\right)\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_2+\frac{3}{2}\right)\Gamma\left(-\frac{1}{2}\,s_2+\frac{1}{2}\right)} \tag{0.7}$$

$$\widetilde{d}^{0,1,1}_{(01)}(s_1,s_2,s_3) = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\,s_3+1\right)}{\Gamma\left(-\frac{1}{2}\,s_1\right)\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_3+\frac{3}{2}\right)\Gamma\left(-\frac{1}{2}\,s_3+\frac{1}{2}\right)} \tag{0.8}$$

$$\widetilde{d}^{0,1,1}_{(11)}(s_1,s_2,s_3) = \frac{\pi\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_2+\frac{1}{2}\,s_3+1\right)\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2}\,s_1\right)\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_2+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_3+\frac{3}{2}\right)\Gamma\left(-\frac{1}{2}\,s_1-\frac{1}{2}\,s_2-\frac{1}{2}\,s_3-\frac{1}{2}\right)} \tag{0.9}$$

$$\widetilde{d}^{1,1,1}_{(10)}(s_1,s_2,s_3) = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}\,s_1+1\right)\Gamma\left(\frac{1}{2}\,s_2+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_2+\frac{3}{2}\right)\Gamma\left(-\frac{1}{2}\,s_1+\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\,s_2\right)} \tag{0.10}$$

$$\widetilde{d}^{1,1,1}_{(01)}(s_1,s_2,s_3) = \frac{-\sqrt{\pi}\Gamma\left(\frac{1}{2}\,s_1+1\right)\Gamma\left(\frac{1}{2}\,s_3+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_3+\frac{3}{2}\right)\Gamma\left(-\frac{1}{2}\,s_1+\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\,s_3\right)} \tag{0.11}$$

$$\widetilde{d}^{1,1,1}_{(11)}(s_1,s_2,s_3) = \frac{\pi\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_2+\frac{1}{2}\,s_3+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\,s_1+1\right)}{\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_2+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_3+\frac{3}{2}\right)\Gamma\left(-\frac{1}{2}\,s_1-\frac{1}{2}\,s_2-\frac{1}{2}\,s_3\right)\Gamma\left(-\frac{1}{2}\,s_1+\frac{1}{2}\right)} \tag{0.12}$$

$$\widetilde{d}^{0,1,0}_{(10)}(s_1,s_2,s_3) = \frac{-\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\,s_2+1\right)}{\Gamma\left(-\frac{1}{2}\,s_1\right)\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_2+\frac{3}{2}\right)\Gamma\left(-\frac{1}{2}\,s_2+\frac{1}{2}\right)} \tag{0.13}$$

$$\widetilde{d}^{0,1,0}_{(01)}(s_1,s_2,s_3) = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\,s_3+\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2}\,s_1\right)\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_3+1\right)\Gamma\left(-\frac{1}{2}\,s_3\right)} \tag{0.14}$$

$$\widetilde{d}^{0,1,0}_{(11)}(s_1,s_2,s_3) = \frac{-\pi\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_2+\frac{1}{2}\,s_3+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2}\,s_1\right)\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_2+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_3+1\right)\Gamma\left(-\frac{1}{2}\,s_1-\frac{1}{2}\,s_2-\frac{1}{2}\,s_3\right)} \tag{0.15}$$

$$\widetilde{d}^{1,1,0}_{(10)}(s_1,s_2,s_3) = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}\,s_1+1\right)\Gamma\left(\frac{1}{2}\,s_2+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_2+\frac{3}{2}\right)\Gamma\left(-\frac{1}{2}\,s_1+\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\,s_2\right)} \tag{0.16}$$

$$\widetilde{d}^{1,1,0}_{(01)}(s_1,s_2,s_3) = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}\,s_1+1\right)\Gamma\left(\frac{1}{2}\,s_3+1\right)}{\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_3+1\right)\Gamma\left(-\frac{1}{2}\,s_1+\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\,s_3+\frac{1}{2}\right)} \tag{0.17}$$

$$\tag{0.18}$$

$$\widetilde{d}^{1,1,0}_{(11)}(s_1,s_2,s_3) = \frac{\pi\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_2+\frac{1}{2}\,s_3+1\right)\Gamma\left(\frac{1}{2}\,s_1+1\right)}{\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_2+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_3+1\right)\Gamma\left(-\frac{1}{2}\,s_1-\frac{1}{2}\,s_2-\frac{1}{2}\,s_3-\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\,s_1+\frac{1}{2}\right)} \tag{0.19}$$

$$\widetilde{d}^{0,0,1}_{(10)}(s_1,s_2,s_3) = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\,s_2+\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2}\,s_1\right)\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_2+1\right)\Gamma\left(-\frac{1}{2}\,s_2\right)} \tag{0.20}$$

$$\widetilde{d}^{0,0,1}_{(01)}(s_1,s_2,s_3) = \frac{-\sqrt{\pi}\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\,s_3+1\right)}{\Gamma\left(-\frac{1}{2}\,s_1\right)\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_3+\frac{3}{2}\right)\Gamma\left(-\frac{1}{2}\,s_3+\frac{1}{2}\right)} \tag{0.21}$$

$$\widetilde{d}^{0,0,1}_{(11)}(s_1,s_2,s_3) = \frac{-\pi\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_2+\frac{1}{2}\,s_3+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\right)}{\Gamma\left(-\frac{1}{2}\,s_1\right)\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_2+1\right)\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_3+\frac{3}{2}\right)\Gamma\left(-\frac{1}{2}\,s_1-\frac{1}{2}\,s_2-\frac{1}{2}\,s_3\right)} \tag{0.22}$$

$$\widetilde{d}^{1,0,1}_{(10)}(s_1,s_2,s_3) = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}\,s_1+1\right)\Gamma\left(\frac{1}{2}\,s_2+1\right)}{\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_2+1\right)\Gamma\left(-\frac{1}{2}\,s_1+\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\,s_2+\frac{1}{2}\right)} \tag{0.23}$$

$$\widetilde{d}^{1,0,1}_{(01)}(s_1,s_2,s_3) = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}\,s_1+1\right)\Gamma\left(\frac{1}{2}\,s_3+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_3+\frac{3}{2}\right)\Gamma\left(-\frac{1}{2}\,s_1+\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\,s_3\right)} \tag{0.24}$$

$$\widetilde{d}^{1,0,1}_{(11)}(s_1,s_2,s_3) = \frac{-\pi\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_2+\frac{1}{2}\,s_3+1\right)\Gamma\left(\frac{1}{2}\,s_1+1\right)}{\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_2+1\right)\Gamma\left(\frac{1}{2}\,s_1+\frac{1}{2}\,s_3+\frac{3}{2}\right)\Gamma\left(-\frac{1}{2}\,s_1-\frac{1}{2}\,s_2-\frac{1}{2}\,s_3-\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\,s_1+\frac{1}{2}\right)} \tag{0.25}$$

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Paper C

A L^2-model for discrete series representations of $SO_0(4,1)$

Frederik Bang-Jensen

Introduction

When studying the representation theory of real reductive Lie groups in an analytical setting it is often fruitful to consider the restriction of a principal series representation π_λ to the non-compact picture. The restriction induces an isomorphism and allows one to realize π_λ on a L^2 space. If π_λ is not unitarily induced and irreducible (or contains an irreducible quotient/sub-representation), the unitarization of π_λ (or the quotient/sub-representation) then typically involves the use of the Knapp-Stein intertwining operators

$$J_\lambda^w : \pi_\lambda \to \pi_{-\lambda}, \quad f \mapsto \int_{\overline{N}\cap w^{-1}Nw} f(gw\overline{n})d\overline{n}.$$

For the case of the group $SO(n,1)$ then $\overline{N} \cong \mathbb{R}^{n-1}$ acts via translations, in the non-compact picture, which implies that J_λ^w becomes a convolution operator in the Fourier picture. Thus it is often useful to study π_λ (or its quotient/sub-representations) in terms of its Fourier transform. This allows one to realize the unitarization of π_λ as an explicit L^2-space. This setup was used by Zhang in [Zha17] to study discrete components of tensor products of complementary series representations of $SO(n,1)$ and by Möllers and Oshima to study branching problems for $(O(1,n),(O(1,m)\times O(n-m))$ for spherical principal series representations.

In this paper we study the $G = SO_0(4,1)$ invariant sesquilinear form on a principal series representation $\pi_{\sigma,\lambda}$ via its Fourier-transform $\tau_{\sigma,\lambda}$, when $\pi_{\sigma,\lambda}$ contains a discrete series representation of G. The sesquilinear form is necessarily on the form

$$(f \mid g) = \int_{\overline{N}} \langle f(\overline{n}), A(\overline{n})g(\overline{n})\rangle_{V_\sigma} d\overline{n}$$

for some multiplication operator $A \in C^\infty(\overline{N}) \otimes \mathrm{End}(V_\sigma)$. We study A by studying the action of the minimal parabolic subgroup $P = MAN$ in the Fourier transformed picture. The Nilradical N acts via a differential operator similar to that in [MO15] and this differential operator plays a vital role in understanding the behavior of A when expanded in a specific Eigenbasis $V_m(\xi)$ of $d\sigma(B_\xi)$, for some specific element $B_\xi \in \mathfrak{m} \cong \mathfrak{su}(2)$. This gives the G-invariant form

Theorem 0.1.

$$(f \mid g) = \int_{\mathbb{R}^3} \langle f(\xi), \widetilde{A}(\xi)g(\xi)\rangle_{V_\sigma} \|\xi\|^{-2\lambda} d\xi, \quad f,g \in \mathcal{F}(I_\sigma(\lambda))$$

defines a G-invariant form on $\tau_{\sigma,\lambda}$, with $\widetilde{A}(\xi)$ given by

$$\widetilde{A}(\xi)f(\xi) = a_m f(\xi) \in V_m(\xi)$$

and a_m given by the recursion

$$a_m(m - 2\lambda + 1) = a_{m+2}(m + 2\lambda + 1). \tag{0.1}$$

A similar approach was used by Liu-Oshima-Yu in [LOY21] to construct such L^2-models for $Spin(m+1,1)$ and $O(m+1,1)$ when $\sigma \in \widehat{M}$ can be realized on the space of p-forms.

At the end of the paper we discuss possible applications of such models.

1 Preliminaries

Let $G = SO(4,1)_0 \subset \mathrm{GL}(5,\mathbb{R})$ denote the identity component of the Lie group of matrices $g \in \mathrm{GL}(5,\mathbb{R})$ leaving the quadratic form

$$\mathbb{R}^5 \ni x \mapsto x^t I_{4,1} x, \qquad I_{4,1} = \mathrm{diag}(1,1,1,1,-1)$$

invariant. The Lie algebra of G can be written in block form as

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ b^t & 0 \end{pmatrix} \mid a \in \mathfrak{o}(4),\ b \in \mathbb{R}^4. \right\}$$

We fix the Cartan involution on G given by $\theta(g) = (g^t)^{-1},\ g \in G$. Then $\mathfrak{g}$ decomposes into the -1 and $+1$ Eigenspaces $\mathfrak{k}$ and $\mathfrak{p}$ of θ on $\mathfrak{g}$. We fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ by letting $\mathfrak{a} := \mathbb{R}H_0$, with

$$H_0 := E_{4,5} + E_{5,4},$$

where $E_{i,j}$ denotes the 5×5 matrix whose (i,j) entry is equal to 1 and 0 everywhere else. The root system for the pair $(\mathfrak{g}, \mathfrak{a})$ consists of the roots $\pm\gamma$, with $\gamma \in \mathfrak{a}_{\mathbb{C}}^*$ such that $\gamma(H_0) = 1$. We let

$$\mathfrak{n} := \mathfrak{g}_\gamma, \qquad \overline{\mathfrak{n}} := \mathfrak{g}_{-\gamma} = \theta\mathfrak{n},$$

and we put

$$N := \exp(\mathfrak{n}), \qquad \overline{N} := \exp(\overline{\mathfrak{n}}) = \theta(N)$$

for the corresponding analytic subgroups of G. Then the half sum of the positive roots becomes $\rho = \frac{3}{2}\gamma$. We introduce the following coordinates on N and $\overline{N}$ respectively: For $1 \leq j \leq 3$ let

$$\begin{aligned} N_j &:= E_{j,4} - E_{j,5} - E_{4,j} - E_{5,j}, \\ \overline{N}_j &:= E_{j,4} + E_{j,5} - E_{4,j} + E_{5,j}. \end{aligned}$$

for $x \in \mathbb{R}^3$ we let

$$n_x := \exp\left(\sum_{j=1}^{3} x_j N_j\right), \qquad \overline{n}_x = \exp\left(\sum_{j=1}^{3} x_j \overline{N}_j\right).$$

Furthermore we put $K = \exp(\mathfrak{k}) = G^\theta \cong SO(4)$, $M = Z_K(\mathfrak{a})$ and $A := \exp(\mathfrak{a})$. Let $W := N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ be the Weyl group corresponding to $\mathfrak{a}$. Then $W = \{\mathbf{1}, [w_0]\}$ with the non-trivial Weyl group element given by

$$w_0 = \operatorname{diag}(-1,-1,-1,-1,1).$$

Then $\overline{N}P \subset G$ forms an open dense subset of G and a straight forward computation gives

Lemma 1.1. *for $x \in \mathbb{R}^3$, $x \neq 0$ we have $w_0 \overline{n}_x = \overline{n}_y m e^{tH_0} n_z \in \overline{N}P$ with*

$$y = -\frac{x}{\|x\|^2}, \quad m = 2\frac{xx^t}{\|x\|^2} - \mathrm{id}_3, \quad t = 2\log\|x\|$$

Lemma 1.2. *For $x, y \in \mathbb{R}^3$ with $\|y\|^2 x - y \neq 0$ we have $n_y^{-1} n_x = \overline{n}_v m e^{tH_0} n_z \in \overline{N}P$ with*

$$v = \frac{\|y\|^2(x - \|x\|^2 y)}{\|\|y\|^2 x - y\|^2}, \quad t = 2\log\left(\frac{\|\|y\|^2 x - y\|}{\|y\|}\right)$$
$$m = \mathrm{id}_3 - 2yx^t - 2\frac{\|y\|^2}{\|\|y\|^2 x - y\|^2}(x - \|x\|^2 y)(\|y\|^2 x - y)$$

1.1 The unitary dual of G

We shall briefly recall some aspects of the representation theory of G, using the spectrum generating method introduced in [BÒØ96]. We refer to [BÒØ96] for the general theory and details. The setup is as follows:
The (normalized) Killing form on $\mathfrak{g}$ is given by

$$\widetilde{B}(X,Y) = -\frac{1}{2}\operatorname{tr}(XY)$$

and the spectrum generating element $\mathcal{P}$ of $U(\mathfrak{k})$ is given by

$$\mathcal{P} = \mathrm{Cas}_{\mathfrak{k}} - \mathrm{Cas}_{\mathfrak{m}}.$$

The unitary dual of $K \cong SO(4)$ is paramatized via highest weights $\tau = (\tau_1, \tau_2) \in \mathbb{Z}^2$ with

$$\tau_1 \geq |\tau_2|.$$

The value of the Cassimir $\mathrm{Cas}_{\mathfrak{so}(4)}$, with respect to the Killing form $\widetilde{B}$, is

$$\tau(\mathrm{Cas}_{\mathfrak{so}(4)}) = \langle \tau + 2\rho_{\mathfrak{so}(4)}, \tau \rangle$$

where $2\rho_{\mathfrak{so}(4)} = (2,0)$, on the irreducible $SO(4)$ module with highest weight τ. Likewise we can parametrize the unitary dual of $M \cong SO(3)$ via highest weights $\tau \in \mathbb{Z}^+$. The Branching rule from K-to-M is for $\tau = (\tau_1, \tau_2)$ an irreducible representation of $SO(4)$ given as follows:

$$\dim \operatorname{Hom}_M(\tau|_M, \sigma) = \begin{cases} 1, & \tau_1 \geq \sigma \geq |\tau_2| \\ 0, & \text{else.} \end{cases} \tag{1.1}$$

Since the K-to-M branching is multiplicity free, the spectrum generating element acts by some scalar R_α on the K-type $\alpha \in \widehat{K}$. We find that

$$\begin{aligned} R_\beta - R_\alpha &= \beta(\mathrm{Cas}_{\mathfrak{so}(4)}) - \alpha(\mathrm{Cas}_{\mathfrak{so}(4)}) = \langle \beta + 2\rho_{\mathfrak{so}(4)}, \beta \rangle - \langle \alpha + 2\rho_{\mathfrak{so}(4)}, \alpha \rangle \\ &= \langle \alpha + \beta + (2,0), \beta - \alpha \rangle. \end{aligned} \tag{1.2}$$

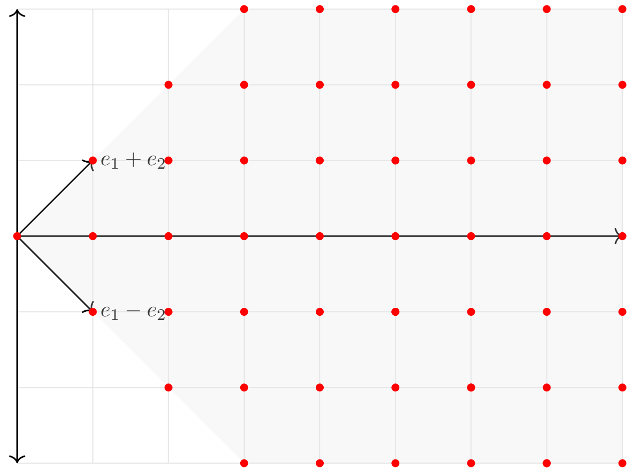

Figure 1.1: K-Isotypic components

Theorem 1.3 ([BÒØ96]). *Suppose V is a G invariant subspace of the K-finite vectors of $\pi_{\sigma,\lambda}$, and let $V(\alpha)$ denote the projection of V to the K-isotopic component $\alpha \in \widehat{K}$. Then each map*

$$\{\alpha \in \widehat{K} \mid V(\alpha) \neq 0\} \to \mathbb{C}, \quad \alpha \mapsto T_\alpha$$

satisfying

$$(R_\beta - R_\alpha + 2\lambda)T_\beta = (R_\beta - R_\alpha + 2\lambda)T_\alpha \tag{1.3}$$

for all $\alpha, \beta \in \{\alpha \in \widehat{K} \mid V(\alpha) \neq 0\}$, gives rise to an intertwiner $T : V \to \pi_{\sigma,\lambda}$.

If we consider λ as a parameter we can rewrite (1.3) as an equation of rational functions

$$(R_\beta - R_\alpha - 2\lambda)Z(\lambda; \sigma, \beta) = (R_\beta - R_\alpha - 2\lambda)Z(\lambda; \sigma, \alpha),$$

with $Z(\lambda; \sigma, \alpha)$ being the spectral function, which was computed in [BÒØ96]

$$\begin{aligned} Z(r; \sigma; \alpha) &= \frac{\Gamma(\frac{3}{2} + \alpha_1 + r)}{\Gamma(\frac{3}{2} + \sigma + r)} \frac{\Gamma(\frac{3}{2} + \sigma - r)}{\Gamma(\frac{3}{2} + \alpha_1 - r)} \frac{\Gamma(\frac{1}{2} + \alpha_2 + r)}{\Gamma(\frac{1}{2} + r)} \frac{\Gamma(\frac{1}{2} - r)}{\Gamma(\frac{1}{2} + \alpha_2 - r)} \\ &= \frac{(\frac{3}{2} + \sigma + r)_{\alpha_1 - \sigma}}{(\frac{3}{2} + \sigma - r)_{\alpha_1 - \sigma}} \frac{\Gamma(\frac{1}{2} + \alpha_2 + r)}{\Gamma(\frac{1}{2} + r)} \frac{\Gamma(\frac{1}{2} - r)}{\Gamma(\frac{1}{2} + \alpha_2 - r)}. \end{aligned} \tag{1.4}$$

The poles and zeroes of (1.4) indicate invariant subspaces. The invariant subspaces are given by

$$\begin{aligned} V_0^\lambda &= \operatorname{span}\{\alpha \mid Z(\lambda, \sigma, \alpha) = 0, \ \alpha \downarrow \sigma\} \\ V_\infty^\lambda &= \operatorname{span}\{\alpha \mid Z(\lambda, \sigma, \alpha) \neq \infty, \ \alpha \downarrow \sigma\} \end{aligned}$$

with $\alpha \downarrow \sigma$ meaning that α appears as a K-type in $\pi_{\sigma,\lambda}$. We thus obtain

Proposition 1.4. *The principal series representations $\pi_{\sigma,\lambda}$ $\sigma \in \widehat{M}$ $\lambda \in \mathbb{C}$ are irreducible, except for when $\lambda \in (\frac{1}{2} + \mathbb{Z}) \setminus \{\pm(\sigma + \frac{1}{2})\}$.*

When $\lambda \in (\frac{1}{2}+\mathbb{Z}) \setminus \{\pm(\sigma+\frac{1}{2})\}$ and $\lambda < 0$ the representation $(d\pi_{\sigma,r\lambda}, \pi_\sigma)$ contains an infinite dimensional invariant subspace V_0^λ given by

$$V_0^\lambda = \begin{cases} \operatorname{span}\{\alpha = (\alpha_1,\alpha_2) \in \widehat{K} \mid \alpha \downarrow \sigma,\ \alpha_1 \geq \frac{-1}{2} - \lambda\} & \text{if } \lambda \leq \frac{-3}{2} - \sigma \\ \operatorname{span}\{\alpha = (\alpha_1,\alpha_2) \in \widehat{K} \mid \alpha \downarrow \sigma,\ -\lambda - \frac{1}{2} < \alpha_2 \text{ or } \alpha_2 \leq \lambda - \frac{1}{2}\} & \text{else.} \end{cases} \tag{1.5}$$

When $\lambda \in \frac{1}{2}+\mathbb{Z}$ and $\lambda > 0$ the representation $(d\pi_{\sigma,\lambda}, \pi_\sigma)$ contains a (possibly finite-dimensional) invariant subspace W_∞^λ given by

$$W_\infty^\lambda = \begin{cases} \operatorname{span}\{\alpha = (\alpha_1,\alpha_2) \in \widehat{K} \mid \alpha \downarrow \sigma,\ \alpha_1 < \frac{-1}{2} + \lambda\} & \text{if } \lambda \geq \frac{3}{2} + \sigma \\ \operatorname{span}\{\alpha = (\alpha_1,\alpha_2) \in \widehat{K} \mid \alpha \downarrow \sigma,\ -\lambda - \frac{1}{2} < \alpha_2 \leq \lambda - \frac{1}{2}\} & \text{else .} \end{cases} \tag{1.6}$$

The spectrum generating function defines an intertwiner T for generic parameters $\lambda \in \mathbb{C}$, intertwining $\pi_{\sigma,\lambda}$ and $\pi_{\sigma,-\lambda}$ by $T(v_\alpha) = Z(\lambda;\sigma;\alpha)v_\alpha$. Using this intertwiner we obtain an invariant non-degenerate sesquilinear form on $\pi_{\sigma,\lambda}$, for generic $\lambda \in \mathbb{C}$, by

$$\langle \varphi, \psi \rangle_\lambda := \langle \varphi, T\psi \rangle_{L^2(K;\sigma)}, \qquad \psi, \varphi \in \pi_{\sigma,\lambda} \tag{1.7}$$

However for "non-generic" $\lambda \in \mathbb{C}$, i.e at the poles and zeroes of the spectrum generating function $Z(\lambda;\sigma;\alpha)$, (1.7) cannot yield a non-degenerate positive definite form on $\pi_{\sigma,\lambda}$. However Proposition 1.4 imply that (1.7) define a non-degenerate invariant form on subspace or sub-quotient of $\pi_{\sigma,\lambda}$ for these non generic λ. Hence the question of unitarity of these representations reduces to the positive definiteness of (1.7), which depends entirely on if the spectrum generating function has constant sign on these sub-representations.

Lemma 1.5 ([BÒØ96])**.** *Let $\sigma \in \widehat{M}$. If $r \in (-R_\sigma, R_\sigma)$ with*

$$C_\sigma = \frac{1}{2}\min\{|R_\beta - R_\alpha| \mid \alpha, \beta \downarrow \sigma,\ \alpha \leftrightarrow \beta\} \tag{1.8}$$

then the sesquilinear form given by (1.7) is positive definite.

To compute R_σ for $G = SO_0(4,1)$ we need the following Lemma, the proof of which is follows from the theory of highest weights. We write

$$\alpha \leftrightarrow \beta \Leftrightarrow \operatorname{Hom}_K(\mathfrak{p} \otimes \alpha, \beta) \neq 0,$$

then

Lemma 1.6. *Let $\sigma \in \widehat{M}$ and $\alpha, \beta \in \widehat{K}$ with $\alpha, \beta \downarrow \sigma$. Then $\alpha \leftrightarrow \beta$ if and only if $\beta = \alpha \pm e_a$, $a = 1, 2$.*

Using (1.2) we then find

$$|R_{\beta \pm e_a} - R_\beta| = |2\beta_a \pm 1 + 4 - 2a|$$

In conclusion we have

Proposition 1.7. *Let $\sigma \in \widehat{M}$. Then the complementary series for is parametrized by $\lambda \in (-\frac{1}{2}, \frac{1}{2})$ if $\sigma \neq 0$ and $\lambda \in (-\frac{3}{2}, \frac{3}{2})$ if $\sigma = 0$.*

From Proposition 1.4 we have that at certain points $\lambda \in \frac{1}{2} + \mathbb{Z}$ the representation $\pi_{\sigma,\lambda}$ contains an irreducible sub-representation and an irreducible sub-quotient, depending on the sign of λ. By the earlier discussion if the spectrum generating function $Z(\lambda, \sigma; \alpha)$ has constant sign on these sub-representations or sub-quotients, we obtain a G invariant positive definite form on the corresponding sub-representation or quotient. In conclusion we have

Theorem 1.8. *The unitary dual $\widehat{G}$ of G consists of the following representations: The **unitary principal series** $\lambda \in i\mathbb{R}$ and **the complementary series** $\lambda \in (-\frac{1}{2}, \frac{1}{2})$ if $\sigma \neq 0$ and $\lambda \in (-\frac{3}{2}, \frac{3}{2})$ if $\sigma = 0$. If $\sigma = 0$ the principal series contains a irreducible sub-quotient $V'_\infty(\lambda)$ at $\lambda \in \{\frac{3}{2}, \frac{5}{2}, \dots\}$ and if $\sigma \neq 0$ the principal series representation contains two irreducible subquotients $V^+_\infty(\lambda)$ and $V^-_\infty(\lambda)$ at $\lambda \in \{\frac{1}{2}, \dots, \sigma - \frac{1}{2}\}$. The sub-quotients $V'_\infty(\lambda), V^+_\infty(\lambda), V^-_\infty(\lambda)$ are given by*

$$V^+_\infty(\lambda) = \mathrm{span}\{\alpha = (\alpha_1, \alpha_2) \in \widehat{K} \mid \alpha \downarrow \sigma,\, \alpha_2 \geq \lambda + \frac{1}{2}\} \tag{1.9}$$

$$V^-_\infty(\lambda) = \mathrm{span}\{\alpha = (\alpha_1, \alpha_2) \in \widehat{K} \mid \alpha \downarrow \sigma,\, -\lambda - \frac{1}{2} \geq \alpha_2\} \tag{1.10}$$

$$V'_\infty(\lambda) = \mathrm{span}\{\alpha = (\alpha_1, \alpha_2) \in \widehat{K} \mid \alpha \downarrow \sigma,\, \alpha_1 \geq \frac{-1}{2} + \lambda\}, \tag{1.11}$$

with the span *understood in the sense of a quotient.*

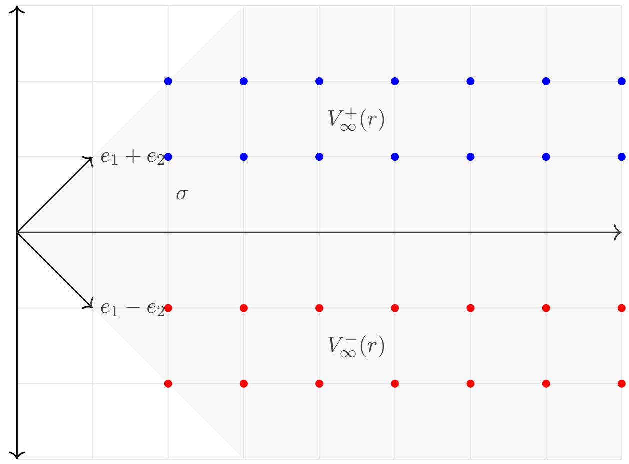

Figure 1.2: $V^{\pm}_\infty(r)$ for $\sigma = 2$ and $r = \frac{1}{2}$

By a result of Harish-Chandra G admits discrete series representations if an only if $rank(G) = rank(K)$. Since $rank(G) = 2 = rank(K)$ for $G = SO_0(1,4)$ it is natural to investigate which representations of Theorem 1.8 are discrete series representations.
Let us shortly review how to find discrete series representations for connected linear real semisimple Lie groups whose complexification is simply connected.
Assume that $rank(G) = rank(K)$. Let $\mathfrak{g}_\mathbb{C}$ denote the complexification of $\mathfrak{g}$ and let $T \subset K$ be a maximal compact Cartan subgroup of G, which exists by assumption. Denote by $\mathfrak{t}$ its Lie algebra, $\mathfrak{t}_\mathbb{C}$ its complexification and $\Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ the set of roots with respect to this choice of Cartan sub-algebra. Fix a set of positive roots $\Sigma^+_\mathfrak{g}$ of $\Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. Then $\Sigma(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C}) \subseteq \Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ and the intersection $\Sigma^+_\mathfrak{k} := \Sigma(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C}) \cap \Sigma^+_\mathfrak{g}$ yields a positive system $\Sigma^+_\mathfrak{k}$ of $\Sigma(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. Denote by $\Sigma^+_\mathfrak{s}$ the complement of $\Sigma^+_\mathfrak{k}$ in $\Sigma^+_\mathfrak{g}$. The corresponding root spaces $\mathfrak{g}_\alpha$ for $\alpha \in \Sigma^+_\mathfrak{s} \cup -\Sigma^+_\mathfrak{s}$ span

$\mathfrak{s}_{\mathbb{C}}$. Let $\alpha \in i\mathfrak{t}_{\mathbb{R}}^*$ be a $\Sigma_{\mathfrak{g}}^+$ dominant integral linear form and denote by $\rho_{\mathfrak{g}}$ the half sum of roots in $\Sigma_{\mathfrak{g}}^+$, $\rho_{\mathfrak{k}}$ the half sum of roots in $\Sigma_{\mathfrak{k}}^+$ and $\rho_{\mathfrak{s}}$ the half sum of roots in $\Sigma_{\mathfrak{s}}^+$. Then $\alpha + \rho_{\mathfrak{g}}$ is $\Sigma_{\mathfrak{g}}^+$-dominant regular integral and $\rho_{\mathfrak{s}}$ is $\Sigma_{\mathfrak{k}}^+$-dominant integral. Let $\varpi(\lambda + \rho_{\mathfrak{g}})$ be the discrete series representation of G with Harish-Chandra parameter $\lambda + \rho_{\mathfrak{g}}$. We then have the following result on discrete series representations:

Theorem 1.9 (See [Par15, Theorem 1.1.1])**.** *Let π be an irreducible unitary representation of G. Suppose that the finite-dimensional K representation $\tau_{\alpha+2\rho_{\mathfrak{s}}}$ with $\Sigma_{\mathfrak{k}}^+$ highest weight $\alpha + 2\rho_{\mathfrak{s}}$ occurs in $\pi|_K$; then one of the following cases occurs:*

1. *There is no $\tau \in \Sigma_{\mathfrak{s}}^+$ such that $\alpha + 2\rho_{\mathfrak{s}} - \tau$ is $\Sigma_{\mathfrak{k}}^+$-dominant. In this case $\pi \cong \varpi(\alpha + \rho_{\mathfrak{g}})$.*

2. *There is an $\tau \in \Sigma_{\mathfrak{s}}^+$ such that $\alpha+2\rho_{\mathfrak{s}}-\tau$ is $\Sigma_{\mathfrak{k}}^+$-dominant; but no $\Sigma_{\mathfrak{k}}^+$ irreducible constituent of $\pi|_{\mathfrak{k}}$ is of this form. In this case $\pi \cong \varpi(\alpha + \rho_{\mathfrak{g}})$.*

3. *There is an $\tau \in \Sigma_{\mathfrak{s}}^+$ such that $\alpha + 2\rho_{\mathfrak{s}} - \tau$ is $\Sigma_{\mathfrak{k}}^+$-dominant and is the highest weight of an irreducible $\mathfrak{k}$-constituent of $\pi|_{\mathfrak{k}}$. In this case $\pi \ncong \varpi(\alpha + \rho_{\mathfrak{g}})$.*

Now for $G = SO_0(4,1)$ the story goes as follows: The Lie algebra of G is $\mathfrak{g} = \mathfrak{so}(4,1)$ and its complexification is $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(5,\mathbb{C})$. The maximal compact subgroup is $K \cong SO(4)$ embedded in the top left corner, and the Cartan subgroup is $T \cong SO(2) \times SO(2) \hookrightarrow K \hookrightarrow G$. The complexification is therefore $\mathfrak{t}_{\mathbb{C}} \cong \mathfrak{so}(2,\mathbb{C}) \oplus \mathfrak{so}(2,\mathbb{C}) \hookrightarrow \mathfrak{k}_{\mathbb{C}} \cong \mathfrak{so}(4,\mathbb{C}) \hookrightarrow \mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(5,\mathbb{C})$. The roots are given by $\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) = \{\pm e_1 \pm e_2, \pm e_1, \pm e_2\}$. We will work with two notions of positivity: Fix first a notion of positivity by declaring that an element $\varphi = a_1 e_1 + a_2 e_2 \in \mathfrak{t}_{\mathbb{R}}^*$ is positive if the first non-zero coefficient is positive. Then the positive roots becomes $\Sigma_{\mathfrak{g}}^+ = \{e_1, e_2, e_1+e_2, e_1-e_2\}$ and the intersection with $\Sigma(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is $\Sigma_{\mathfrak{k}}^+ = \{e_1 \pm e_2\}$. The non-compact positive roots becomes $\Sigma_{\mathfrak{s}}^+ = \{e_1, e_2\}$. The half sums then become $\rho_{\mathfrak{g}} = (\frac{3}{2}, \frac{1}{2})$, $\rho_{\mathfrak{k}} = (1,0)$ and $\rho_{\mathfrak{s}} = (\frac{1}{2}, \frac{1}{2})$. The $\Sigma_{\mathfrak{g}}^+$ dominant integral linear forms α are parameterized by $(m,n) \in \mathbb{Z}^2$ with $m \geq n \geq 0$ and the $\Sigma_{\mathfrak{k}}^+$ dominant integral forms are those for which $m \geq |n|$.
We may also fix a second notion of positivity such that the positive roots become $\Delta_{\mathfrak{g}}^+ := \{e_1 \pm e_2, e_1, -e_2\}$ and define $\Delta_{\mathfrak{s}}^+ = \{e_1, -e_2\}$. This choice doesn't affect the compact roots, but interchanges e_2 with $-e_2$ in the non-compact roots. Note that integral dominant weights of $\Sigma_{\mathfrak{g}}^+$ and the integral dominant weights of $\Delta_{\mathfrak{g}}^+$ exhaust the set of K-types. Denote by $\rho_{\mathfrak{g}}' = (\frac{3}{2}, -\frac{1}{2})$ and $\rho_{\mathfrak{s}}' = (\frac{1}{2}, -\frac{1}{2})$ the half sum of positive roots for $\Delta_{\mathfrak{g}}^+$ and $\Delta_{\mathfrak{s}}^+$ respectively.
Let us start by showing that $V_\infty'(\lambda)$ is not a discrete series representation; Assume that $\lambda \geq \frac{3}{2}$ and $\sigma = 0$. Then $V_\infty'(\lambda)$ is a unirrep and any $\alpha \downarrow V_\infty'(\lambda)$ must have $\alpha_2 = 0$. If α is $\Sigma_{\mathfrak{g}}^+$ integral dominant then $\alpha + 2\rho_{\mathfrak{s}} \not\downarrow V_\infty'(\lambda)$ as this would imply $\alpha_2 = -1$ contradicting that α was $\Sigma_{\mathfrak{g}}^+$ dominant. The same argument works for α being $\Sigma_{\mathfrak{s}}^+$ dominant and as these together exhaust the K-isotypic components we conclude that $V_\infty^-(\lambda) \ncong \varpi(\alpha + \rho_{\mathfrak{g}})$ for any K-isotypic component α. Since these parameterize all discrete series representations the claim follows.
Assume now that $\sigma \neq 0$ and that $0 < \lambda < \frac{1}{2} + \sigma$ with $\lambda \in (\frac{1}{2} + \mathbb{Z}) \setminus \{\pm(\sigma + \frac{1}{2})\}$. Consider then $\alpha + 2\rho_{\mathfrak{s}} = (\sigma, \sigma) \downarrow V_\infty^+(\lambda)$. Then $\alpha + \rho_{\mathfrak{s}} - e_2$ is $\Sigma_{\mathfrak{k}}^+$ dominant but does not occur in $V_\infty^+(r)$ and since $\alpha = (\sigma - 1, \sigma - 1)$ is $\Sigma_{\mathfrak{g}}^+$ dominant Theorem 1.9 (2) implies that $V_\infty^+(r) \cong \varpi(\alpha + \rho_{\mathfrak{g}})$. An analogous argument with $\alpha + 2\rho_{\mathfrak{s}}' = (\sigma, -\sigma)$ shows that $V_\infty^-(\lambda)$ is also a discrete series representation. In conclusion we have

Theorem 1.10. *The discrete series representations of $\widehat{G}$ are exactly the subquotients $V_\infty^\pm$ for $\lambda \in \{\frac{1}{2}, \ldots, \sigma - \frac{1}{2}\}$ for $\sigma > 0$.*

2 Induced representations - the non-compact picture and its Fourier-transform

We identify $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathbb{C}$ by $\mathbb{C} \ni \lambda \mapsto \lambda\gamma \in \mathfrak{a}_{\mathbb{C}}^*$. Then $\rho = \frac{3}{2}$ under this identification. We define a character e^{λ} on A for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ by declaring that $e^{\lambda}(\exp(tH_0)) = e^{t\lambda}$ for $t \in \mathbb{R}$. For $\sigma \in \widehat{M}$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we induce a representation of G from the character $\sigma \otimes e^{\lambda} \otimes \mathbf{1}$ by letting

$$\begin{aligned}\widetilde{I}_{\sigma}(\lambda) : &= \mathrm{Ind}_P^G(\sigma \otimes e^{\lambda} \otimes \mathbf{1}) \\ &= \{f \in C^{\infty}(G, V_{\sigma}) \mid f(gman) = a^{-(\lambda+\rho)}\sigma(m)^{-1}f(g)\ \forall g \in G, man \in P\}.\end{aligned}$$

G acts on $I_{\sigma}(\lambda)$ by the left-regular representation, which we will denote by $\widetilde{\pi}_{\sigma,\lambda}$. Denote by $f_{\overline{N}}$ the restriction of $f \in \widetilde{I}_{\sigma}(\lambda)$ to $\overline{N}$. The restriction map $f \mapsto f_{\overline{N}}$ is one-to-one and denote by $I_{\sigma}(\lambda)$ the completion of its image. $I_{\sigma}(\lambda)$ then defines a representation of G, denoted $\pi_{\sigma,\lambda}$, by letting G act on $I_{\sigma}(\lambda)$ as $\pi_{\sigma,\lambda}(g)f_{\overline{N}} = (\widetilde{\pi}_{\sigma}(g)f)_{\overline{N}}$. Note that $N, \overline{N} \cong (\mathbb{R}^3, +)$ as groups, hence going forward we shall abbreviate $f(\overline{n}_x) = f(x), x \in \mathbb{R}^3$ and $f \in I_{\sigma}(\lambda)$. The action of $\pi_{\sigma,\lambda}$ can then be described using Lemma 1.1 and some basic computations:

$$\begin{aligned}\pi_{\sigma,\lambda}(\overline{n}_y)f(x) &= f(x-y), && y \in \mathbb{R}^3\\ \pi_{\sigma,\lambda}(m)f(x) &= \sigma(m)f(m^{-1}x), && m \in M \cong SO(3)\\ \pi_{\sigma,\lambda}\left(e^{tH_0}\right)f(x) &= e^{(\lambda+\rho)t}f(e^t x), && e^{tH_0} \in A\\ \pi_{\sigma,\lambda}(w_0)f(x) &= \|x\|^{-2(\lambda+\rho)}\sigma\left(2\frac{xx^t}{\|x\|^2} - \mathrm{id}_3\right)f\left(-\frac{x}{\|x\|^2}\right).\end{aligned}$$

This also yields the following expressions for the derived action $d\pi_{\sigma,\lambda}$

$$\begin{aligned}d\pi_{\sigma,\lambda}(\overline{N}_j)f(x) &= -\partial_j f(x), && j = 1,2,3\\ \pi_{\sigma,\lambda}(T)f(x) &= d\sigma(T) - D_{Tx}f(x), && T \in \mathfrak{m} \cong \mathfrak{o}(3)\\ \pi_{\sigma,\lambda}(H_0)f(x) &= (E+\lambda+\rho)f(x)\\ \pi_{\sigma,\lambda}(N_j)f(x) &= -\|x\|^2\partial_j + 2x_j(E+\lambda+\rho) - 2d\sigma(xe_j^t - e_jx^t), && j = 1,2,3\end{aligned}$$

Where D_a denotes the directional derivative in the direction of $a \in \mathbb{R}^3$ and $E = \sum_{j=1}^{3} x_j\partial_j$ is the Euler operator on $\mathbb{R}^3$. The last identity is obtained using Lemma 1.2.

2.1 The F picture

As remarked earlier $\overline{N} \cong (\mathbb{R}^3, +)$, which induces an injection $I_{\sigma}(\nu) \hookrightarrow L^2(\mathbb{R}^3, V_{\sigma})$ to some subspace of $L^2(\mathbb{R}^3, V_{\sigma})$. We define a Fourier-transform on (a dense subset of) $I_{\sigma}(\lambda)$ using the above identification and the Euclidean Fourier transform:

$$\mathcal{F}(f)(\xi) = (2\pi)^{-\frac{3}{2}}\int_{\mathbb{R}^3} e^{-i\langle x,\xi\rangle}f(x)dx.$$

For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $\sigma \in \widehat{M}$ we define a representation $\tau_{\sigma,\lambda}$ of G on $\mathcal{F}\left((I_{\sigma}(\nu)\right)$ by

$$\pi_{\sigma,\lambda}(g) \circ \mathcal{F} = \mathcal{F} \circ \tau_{\sigma,\lambda}(g), \quad g \in G.$$

The action of $\overline{P} = MA\overline{N}$ follows by some easy computations:

$$\begin{aligned} \tau_{\sigma,\lambda}(\overline{n}_y)f(\xi) &= e^{i\langle\xi,y\rangle}f(\xi), && \overline{n}_y \in \overline{N} \\ \tau_{\sigma,\lambda}(m)f(\xi) &= \sigma(m)f(m^{-1}\xi), && m \in M \\ \tau_{\sigma,\lambda}(e^{tH_0})f(\xi) &= e^{(\lambda-\rho)t}f(e^{-t}\xi), && e^{tH_0} \in A \end{aligned}$$

From the classical properties of the Fourier transform:

$$\begin{aligned} x_j \circ \mathcal{F} &= \mathcal{F} \circ (-i\partial_j), \\ \partial_j \circ \mathcal{F} &= \mathcal{F} \circ (-i\xi_j) \end{aligned}$$

we easily obtain the action of the derived representation $d\tau_{\sigma,\lambda}$:

$$d\tau_{\sigma,\lambda}(\overline{N}_j)f(\xi) = i\xi_j f(\xi), \qquad j = 1,2,3 \tag{2.1}$$

$$d\tau_{\sigma,\lambda}(T)f(\xi) = d\sigma(T)f(\xi) - D_{T\xi}f(\xi), \qquad T \in \mathfrak{m} \cong \mathfrak{o}(3) \tag{2.2}$$

$$d\tau_{\sigma,\lambda}(H_0)f(\xi) = -(E - \lambda + \rho)f(\xi) \tag{2.3}$$

$$d\tau_{\sigma,\lambda}(N_j)f(\xi) = i\left(\xi_j\Delta - 2(E-\lambda+\rho)\partial_j - 2d\sigma(\partial e_j^t - e_j\partial^t)\right)f(\xi), \qquad j = 1,2,3 \tag{2.4}$$

$$\tag{2.5}$$

with $\partial = \sum_{k=1}^{3} \partial_k e_k$ and e_k for $k = 1,2,3$ denoting the standard basis of $\mathbb{R}^3$. Going forward we will abbreviate

$$\mathcal{B}^{\sigma}_{\lambda,j} := \xi_j\Delta - 2(E-\lambda+\rho)\partial_j - 2d\sigma(\partial e_j^t - e_j\partial^t). \tag{2.6}$$

For $\lambda \in i\mathbb{R}$ the L^2-inner product

$$(f \mid g) = \int_{\mathbb{R}^3} \langle f(x), g(x)\rangle_{V_\sigma} dx, \qquad f,g \in I_\sigma(\nu),$$

where $\langle \cdot \mid \cdot \rangle_{V_\sigma}$ is a M-invariant inner product on V_σ, provides unitarizations for the representations $\pi_{\sigma,\lambda}$ on $L^2(\mathbb{R}^3, V_\sigma)$. However elsewhere in the unitary dual, the natural L^2-inner product does not allow for unitarizations of the forementioned representations. Such unitarizations involves considering intertwining operators

$$J_{\sigma,\lambda} : I_\sigma(\lambda) \to I_\sigma(-\lambda)$$

and then constructing a G-invariant Hermitian form

$$(f,g) \mapsto (f \mid J_{\sigma,\lambda}g)_{L^2}.$$

For the case of $G = SO(4,1)$ and $\sigma = \mathbf{1}$ this hermitian form is given by convolution with a Riesz kernel and using the Plancherel theory of the Fourier transform on $L^2(\mathbb{R}^3)$, one can obtain a explicit expression for this G-invariant Hermitian form in the Fourier picture, see e.g [MO15]. However for the non-spherical case the representations are vector-valued and the Fourier picture is more involved. Suppose

$$J_{\sigma,\lambda} : I_\sigma(\lambda) \to I_\sigma(-\lambda)$$

is such an intertwiner. Restricting to the non-compact picture we have that $J_{\sigma,\lambda}$ is an translation invariant operator on $L^2(\mathbb{R}^3)\otimes V_\sigma$. Hence $J_{\sigma,\lambda}$ is given by a convolution operator, or equivalently, in the Fourier picture

$$\widehat{J}_{\sigma,\lambda} : \mathcal{F}(I_\sigma(\lambda)) \to \mathcal{F}(I_\sigma(-\lambda))$$

is given by a multiplication operator $A \in C^\infty(\mathbb{R}^3 \setminus \{0\}) \otimes \mathrm{End}(V_\sigma)$

$$\widehat{J}_{\sigma,\lambda} f(\xi) = A(\xi) f(\xi). \tag{2.7}$$

Since $\widehat{J}_{\sigma,\lambda}$ must be MA equivariant we find that $A(e^{-t}\xi) = e^{2\lambda t} A(\xi)$ and hence

$$A(\xi) = \|\xi\|^{-2\lambda} A\left(\frac{\xi}{\|\xi\|^2}\right) := \|\xi\|^{-2\lambda} \widetilde{A}(\xi).$$

Furthermore the M equivariance implies that

$$\sigma(m^{-1}) A(\xi) \sigma(m) = A(m^{-1}\xi). \tag{2.8}$$

For convenience we parameterize $\widehat{M}$ via the highest weight theory of $M \cong SO(3)$, i.e we shall identify $\widehat{M} \cong \mathbb{N}_0$ with $\sigma = 0$ corresponding to the trivial representation. We will then show that

Theorem 2.1. *Let $\sigma \in \mathbb{N}_0$ with $\sigma \neq 0$ and let $\lambda \in \{\frac{1}{2}, \frac{3}{2}, \dots, \sigma - \frac{1}{2}\}$. Then*

$$(f \mid g)_{I_\sigma(\lambda)} := \int_{\mathbb{R}^3} \langle f(\xi), \widetilde{A}(\xi) g(\xi) \rangle_{V_\sigma} \|\xi\|^{-2\lambda} d\xi, \quad f, g \in \mathcal{F}(I_\sigma(\lambda))$$

defines a G-invariant positive definite form on $\mathcal{F}(I_\sigma(\lambda))$, with $\widetilde{A}(\xi)$ given by

$$\widetilde{A}(\xi) f(\xi) = a_m f(\xi) \in V_m(\xi)$$

with $a_0 = 1$ and a_m given by the recursion

$$a_m(m - 2\lambda + 1) = a_{m+2}(m + 2\lambda + 1). \tag{2.9}$$

3 Sub-representations of $L^2(\mathbb{R}^3, V_\sigma)$

We start by introducing some coordinates on $\mathfrak{m} \cong \mathfrak{o}(3)$ as follows: Let $X_{i,j} = 2(e_i e_j^t - e_j e_i^t)$ and let $B_\xi = \xi_2 X_{1,3} - \xi_3 X_{1,2} - \xi_1 X_{2,3}$ for $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. Then $\mathfrak{o}(3) = \mathrm{span}\{X_{1,2}, X_{1,3}, X_{2,3}\}$ with the usual commutator relations

$$[X_{1,2}, X_{1,3}] = -2X_{2,3}, \quad [X_{1,2}, X_{2,3}] = 2X_{1,3}, \quad [X_{1,3}, X_{2,3}] = -2X_{1,2}$$

If we put $B_i = \frac{\partial}{\partial \xi_i} B_\xi$ we have the following useful relations

$$[X_{j,k}, B_j] = X_{j,k} B_j + X_{i,k} B_i = -2B_k, \qquad i,j,k = 1,2,3,\ i \neq j,\ i,j \neq k \tag{3.1}$$

$$[B_j, X_{j,k}] = 2B_k, \qquad j,k = 1,2,3,\ j \neq k \tag{3.2}$$

$$2(\xi e_k^t - e_k \xi^t) = -\frac{1}{2}[B_\xi, B_k], \qquad k = 1,2,3,\ \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3. \tag{3.3}$$

For $\xi \neq 0$ B_ξ spans a Cartan-subalgebra of $\mathfrak{o}(3) \cong \mathfrak{su}(2)$ and the representation theory of $SU(2)$ implies that $d\sigma(B_\xi)$ acts with eigenvalues $\{-2im\|\xi\|, -2i(m-1)\|\xi\|, \dots, 2i(m-1)\|\xi\|, 2im\|\xi\|\}$ on V_σ, with $\dim(V_\sigma) = 2\sigma + 1$.

Lemma 3.1. *Let $\sigma \in \widehat{M}$, $f \in C^\infty(\mathbb{R}^3 \setminus \{0\}, V_\sigma)$ such that $d\sigma(B_\xi)f(\xi) = im\|\xi\|f(\xi) \forall \xi \in \mathbb{R}^3$ for some $m \in 2\mathbb{Z}$ and $k \in \{1,2,3\}$. Then we have the following identities:*

$$d\sigma(B_\xi)\partial_k f(\xi) = im\frac{\xi_k}{\|\xi\|}f(\xi) + im\|\xi\|\partial_k f(\xi) - d\sigma(B_k)f(\xi) \tag{3.4}$$

$$d\sigma(B_\xi)Ef(\xi) = im\|\xi\|Ef(\xi) \tag{3.5}$$

$$d\sigma(B_k)Ef(\xi) + d\sigma(B_\xi)\partial_k Ef(\xi) = im\frac{\xi_k}{\|\xi\|}Ef(\xi) + im\|\xi\|\partial_k Ef(\xi) \tag{3.6}$$

$$2d\sigma(B_k)\partial_k f(\xi) + d\sigma(B_\xi)\partial_k^2 = im\left(\|\xi\|^{-1}f(\xi) - \frac{\xi_k^2}{\|\xi\|^3}f(\xi) + 2\frac{\xi_k}{\|\xi\|}\partial_k f(\xi) + \|\xi\|\partial_k^2 f(\xi)\right) \tag{3.7}$$

$$\begin{aligned} 2d\sigma(B_\xi)d\sigma(\partial e_k^t - e_k\partial^t)f(\xi) &= 2im\|\xi\|d\sigma(\partial e_k^t - e_k\partial^t)f(\xi) + 2\frac{im}{\|\xi\|}d\sigma(\xi e_k^t - e_k\xi^t)f(\xi) \\ &\quad - \sum_{j\neq k} d\sigma(X_{j,k})d\sigma(B_j)f(\xi) + \sum_{j\neq k} d\sigma([B_\xi, X_{j,k}])\partial_j f(\xi) \end{aligned} \tag{3.8}$$

Proof. Let f be as in the lemma. Then (3.4) follows directly from taking partial derivatives of the eigenvalue equation $d\sigma(B_\xi)f(\xi) = im\|\xi\|f(\xi)$.
Equation (3.5) follows from (3.4) by

$$d\sigma(B_\xi)Ef(\xi) = im\|\xi\|f(\xi) + im\sum_{k=1}^{3}\|\xi\|\xi_k\partial_k f(\xi) - d\sigma(B_\xi)f(\xi) = im\|\xi\|Ef(\xi).$$

(3.6) follows directly from (3.5) by taking partial derivatives on both sides. (3.7) follows directly by taking partial derivatives of (3.4). Lastly, to obtain (3.8), we have

$$\begin{aligned} 2d\sigma(B_\xi)d\sigma(\partial e_k^t - e_k\partial^t)f(\xi) &= \sum_{j\neq k} d\sigma(B_\xi)d\sigma(X_{j,k})\partial_j f(\xi) \\ &= \sum_{j\neq k}\left(d\sigma(X_{j,k})d\sigma(B_\xi)\partial_j + d\sigma([B_\xi, X_{j,k}])\partial_j\right)f(\xi) \\ &= \sum_{j\neq k}\left(d\sigma(X_{j,k})(im\frac{\xi_j}{\|\xi\|} + im\|\xi\|\partial_j - d\sigma(B_j)) + d\sigma([B_\xi, X_{j,k}])\partial_j\right)f(\xi) \\ &= \left(2\frac{im}{\|\xi\|}d\sigma(\xi e_k^t - e_k\xi^t) + 2im\|\xi\|d\sigma(\partial e_k^t - e_k\partial^t) + \sum_{j\neq k} d\sigma([B_\xi, X_{j,k}])\partial_j - d\sigma(X_{j,k})d\sigma(B_j)\right)f(\xi), \end{aligned}$$

which proves (3.8). □

Using Lemma 3.1 we can now give a complete description of the action of $\mathfrak{n}$ on the Eigenspaces of $d\sigma(B_\xi)$. The result is contained in the following Lemmas:

Lemma 3.2. *Let $\sigma \in \widehat{M}$, $f \in C^\infty(\mathbb{R}^3, V_\sigma)$ such that $d\sigma(B_\xi)f(\xi) = im\|\xi\|\ \forall \xi \in \mathbb{R}^3$ for some $m \in 2\mathbb{Z}$. Then*

$$\left(\mathcal{B}^\sigma_{\lambda,k} - \frac{im}{2\|\xi\|}d\sigma(B_k) - 2(2+\lambda-\rho)\partial_k\right)f(\xi) \in V_m(\xi)$$

with $V_m(\xi)$ the Eigenspace corresponding to the eigenvalue $im\|\xi\|$ of $d\sigma(B_\xi)$.

Proof. Using Lemma 3.1 we have

$$d\sigma(B_\xi)\mathcal{B}^\sigma_{\lambda,k}f(\xi) = d\sigma(B_\xi)\left(x_k\Delta - 2\partial_k E + 2(1+\lambda-\rho)\partial_k - 2d\sigma(\partial e_k^t - e_k\partial^t)\right)f(\xi)$$
$$= im\xi_k\left(2\|\xi\|^{-1}f(\xi) + 2\frac{1}{\|\xi\|}Ef(\xi) + \|\xi\|\Delta f(\xi)\right) - 2\xi_k\sum_{j=1}^{3} d\sigma(B_j)\partial_j f(\xi)$$
$$- 2im\frac{\xi_k}{\|\xi\|}Ef(\xi) - 2im\|\xi\|\partial_k Ef(\xi) + 2d\sigma(B_k)Ef(\xi)$$
$$+ 2im(1+\lambda-\rho)\frac{\xi_k}{\|\xi\|}f(\xi) + 2im(1+\lambda-\rho)\|\xi\|\partial_k f(\xi) - 2(1+\lambda-\rho)d\sigma(B_k)f(\xi)$$
$$- 2im\|\xi\|d\sigma(\partial e_k^t - e_k\partial^t)f(\xi) - 2\frac{im}{\|\xi\|}d\sigma(\xi e_k^t - e_k\xi^t)f(\xi)$$
$$+ \sum_{j\neq k} d\sigma(X_{j,k})d\sigma(B_j)f(\xi) - \sum_{j\neq k} d\sigma([B_\xi, X_{j,k}])\partial_j f(\xi)$$

Grouping relevant terms and applying the identities (3.1),(3.2) and (3.3) we obtain

$$im\|\xi\|\left(\xi_k\Delta - 2\partial_k E + 2(1+\lambda-\rho)\partial_k - 2d\sigma(\partial e_k^t - e_k\partial^t)\right)f(\xi)$$
$$- 2\xi_k\sum_{j=1}^{3} d\sigma(B_j)\partial_j f(\xi) + 2d\sigma(B_k)Ef(\xi) - \sum_{j\neq k} d\sigma([B_\xi, X_{j,k}])\partial_j f(\xi)$$
$$- 2(1+\lambda-\rho)d\sigma(B_k)f(\xi) + \sum_{j\neq k} d\sigma(X_{j,k})d\sigma(B_j)f(\xi)$$
$$+ 2im(2+\lambda-\rho)\frac{\xi_k}{\|\xi\|}f(\xi) - 2\frac{im}{\|\xi\|}d\sigma(\xi e_k^t - e_k\xi^t)f(\xi)$$
$$= im\|\xi\|\mathcal{B}^\sigma_{\lambda,k}f(\xi) - 2(2+\lambda-\rho)d\sigma(B_k)f(\xi)$$
$$- \sum_{j\neq k} d\sigma([B_\xi, X_{j,k}])\partial_j f(\xi) + 2\sum_{j\neq k} d\sigma(\xi_j B_k - \xi_k B_j)\partial_j f(\xi)$$
$$+ 2im(2+\lambda-\rho)\frac{\xi_k}{\|\xi\|}f(\xi) - 2\frac{im}{\|\xi\|}d\sigma(\xi e_k^t - e_k\xi^t)f(\xi)$$
$$= im\|\xi\|\mathcal{B}^\sigma_{\lambda,k}f(\xi) + 2(2+\lambda-\rho)(im\frac{\xi_k}{\|\xi\|} - d\sigma(B_k))f(\xi)$$
$$+ \frac{im}{2\|\xi\|}\left(d\sigma(B_\xi)d\sigma(B_k)f(\xi) - im\|\xi\|d\sigma(B_k)\right)f(\xi)$$

where the last equality follows from the eigenvalue equation and from (3.2) since

$$[B_\xi, X_{j,k}] = \xi_j[B_j, X_{j,k}] - \xi_k[B_k, X_{k,j}] = 2(\xi_j B_k - \xi_k B_j).$$

Rearranging the terms and applying (3.4) we find

$$d\sigma(B_\xi)\left(\mathcal{B}^\sigma_{\lambda,k} - \frac{im}{2\|\xi\|}d\sigma(B_k) - 2(2+\lambda-\rho)\partial_k\right)f(\xi)$$
$$= im\|\xi\|\left(\mathcal{B}^\sigma_{\lambda,k} - \frac{im}{2\|\xi\|}d\sigma(B_k) - 2(2+\lambda-\rho)\partial_k\right)f(\xi)$$

as wanted. □

Lemma 3.3. *Let* $\sigma \in \widehat{M}$ *and let* $f \in C^\infty(\mathbb{R}, V_\sigma)$ *such that* $f(\xi) \in V_m(\xi)$ *for all* $\xi \in \mathbb{R}^3$ *for some* $m \in 2\mathbb{Z}$. *Then* $\partial_k f = f_{m+2} + f_m + f_{m-2}$ *with*

$$f_{m+2}(\xi) = \frac{1}{4\|\xi\|^2}\left(\frac{1}{2}d\sigma([B_\xi, B_k])f(\xi) + \xi_k m f(\xi) + i\|\xi\| d\sigma(B_k)f(\xi)\right) \in V_{m+2}(\xi) \tag{3.9}$$

$$f_{m-2}(\xi) = \frac{1}{4\|\xi\|^2}\left(\frac{1}{2}d\sigma([B_\xi, B_k])f(\xi) - \xi_k m f(\xi) - i\|\xi\| d\sigma(B_k)f(\xi)\right) \in V_{m-2}(\xi) \tag{3.10}$$

$$f_m(\xi) = \partial_k f(\xi) - \frac{1}{4\|\xi\|^2} d\sigma([B_\xi, B_k])f(\xi) \in V_m(\xi). \tag{3.11}$$

Proof. Let f be as in the lemma and denote by $f_\ell(\xi)$ the projection of $f(\xi)$ onto $V_\ell(\xi)$. Then for $\xi \in \mathbb{R}^3$ the matrix $B_\xi \in \mathfrak{o}(3)$ spans a Cartan sub-algebra $\mathfrak{h} = \operatorname{span}\{B_\xi\}$ of $\mathfrak{o}(3, \mathbb{C})$ and we have the corresponding root-space decomposition

$$\mathfrak{o}(3, \mathbb{C}) = \mathfrak{g}_+ \oplus \mathfrak{g}_- \oplus \mathfrak{h}.$$

With $\mathfrak{g}_\pm = \operatorname{span}\{B_\xi^\pm\}$ satisfying $[B_\xi, B_\xi^\pm] = \pm 2i\|\xi\| B_\xi^\pm$. By (3.4) we have

$$(d\sigma(B_\xi) - i(m-2)\|\xi\|)\partial_k f(\xi) = im\frac{\xi_k}{\|\xi\|} f(\xi) + 2i\|\xi\|\partial_k f(\xi) - d\sigma(B_k)f(\xi)$$

which implies

$$\begin{aligned}
&(d\sigma(B_\xi) - i(m+2)\|\xi\|)(d\sigma(B_\xi) - i(m-2)\|\xi\|)\partial_k f \\
&= -m^2\xi_k f(\xi) + 2i\|\xi\|\left(im\frac{\xi_k}{\|\xi\|} f(\xi) + im\|\xi\|\partial_k f(\xi) - d\sigma(B_k)f(\xi)\right) - d\sigma(B_\xi)d\sigma(B_k)f(\xi) \\
&- i(m+2)\|\xi\|\left(im\frac{\xi_k}{\|\xi\|} f(\xi) + 2i\|\xi\|\partial_k f(\xi) - d\sigma(B_k)f(\xi)\right) \\
&= 4\|\xi\|^2\partial_k f(\xi) - d\sigma([B_\xi, B_k])f(\xi).
\end{aligned}$$

We claim that $\partial_k f(\xi) - \frac{1}{4\|\xi\|^2}d\sigma([B_\xi, B_k])f(\xi) \in V_m(\xi)$ and $\frac{1}{4\|\xi\|^2}d\sigma([B_\xi, B_k])f(\xi) \in V_{m+2} \oplus V_{m-2}$. To see this note first that for $B_k = a_1 B_\xi + a_2 B_\xi^+ + a_3 B_\xi^-$ we have

$$[B_\xi, [B_\xi, B_k]] = -4\|\xi\|^2 B_k + 4\xi_k B_\xi.$$

Then (3.4) gives

$$\begin{aligned}
&d\sigma(B_\xi)\left(\partial_k f(\xi) - \frac{1}{4\|\xi\|^2}d\sigma([B_\xi, B_k])f(\xi)\right) \\
&= im\|\xi\|\partial_k f(\xi) + \xi_k d\sigma(B_\xi)f(\xi) - d\sigma(B_k)f(\xi) - \frac{1}{4\|\xi\|}(im\|\xi\| d\sigma([B_\xi, B_k]) + d\sigma([B_\xi, [B_\xi, B_k]]))f(\xi) \\
&= im\|\xi\|\left(\partial_k f(\xi) - \frac{1}{4\|\xi\|^2}d\sigma([B_\xi, B_k])f(\xi)\right).
\end{aligned}$$

The other claims follows from similar computations. This implies that $\partial_k f = f_{m+2} + f_m + f_{m-2}$ and thus

$$\begin{aligned}
4i\|\xi\| f_{m+2}(\xi) &= \frac{1}{4\|\xi\|^2}\left(d\sigma(B_\xi) - i(m-2)\right)d\sigma([B_\xi, B_k])f(\xi) \\
&= \frac{1}{4\|\xi\|^2}\left(2i\|\xi\| d\sigma([B_\xi, B_k])f(\xi) + 4\xi_k im\|\xi\| f(\xi) - 4\|\xi\|^2 d\sigma(B_k)f(\xi)\right)
\end{aligned}$$

A similar computation for f_{m-2} yields

$$f_{m+2}(\xi) = \frac{1}{4\|\xi\|^2}\left(\frac{1}{2}d\sigma([B_\xi, B_k])f(\xi) + \xi_k m f(\xi) + i\|\xi\| d\sigma(B_k)f(\xi)\right)$$
$$f_{m-2}(\xi) = \frac{1}{4\|\xi\|^2}\left(\frac{1}{2}d\sigma([B_\xi, B_k])f(\xi) - \xi_k m f(\xi) - i\|\xi\| d\sigma(B_k)f(\xi)\right)$$
$$f_m(\xi) = \partial_k f(\xi) - \frac{1}{4\|\xi\|^2} d\sigma([B_\xi, B_k])f(\xi).$$

as wanted. □

Similarly, we require the Eigenspace decomposition of the $d\sigma(B_k)f(\xi)$ term in Lemma 3.2.

Lemma 3.4. *Let $\sigma \in \widehat{M}$ and let $f \in C^\infty(\mathbb{R}, V_\sigma)$ such that $f(\xi) \in V_m(\xi)$ for all $\xi \in \mathbb{R}^3$ for some $m \in 2\mathbb{Z}$. Then $d\sigma(B_k)f(\xi) = f_{m+2} + f_m + f_{m-2}$ with*

$$f_{m+2} = \frac{1}{2\|\xi\|}\left(-\frac{i}{2}d\sigma([B_\xi, B_k])f(\xi) - i\xi_k m f(\xi) + \|\xi\| d\sigma(B_k)f(\xi)\right) \tag{3.12}$$
$$f_{m-2}(\xi) = \frac{1}{2\|\xi\|}\left(\frac{i}{2}d\sigma([B_\xi, B_k])f(\xi) - i\xi_k m f(\xi) + \|\xi\| d\sigma(B_k)f(\xi)\right) \tag{3.13}$$
$$f_m(\xi) = im\frac{\xi_k}{\|\xi\|}f(\xi). \tag{3.14}$$

Proof. Let f be as in the Lemma and write $\partial_k f(\xi) = f'_{m+2}(\xi) + f'_m(\xi) + f'_{m-2}(\xi)$ for $\xi \in \mathbb{R}^3$ with f'_{m+2}, f'_m and f'_{m-2} as in Lemma 3.3. Then using (3.4) we find

$$\begin{aligned} d\sigma(B_k)f(\xi) &= im\frac{\xi_k}{\|\xi\|}f(\xi) + im\|\xi\|\partial_k f(\xi) - d\sigma(B_\xi)\partial_k f(\xi) \\ &= im\frac{\xi_k}{\|\xi\|}f(\xi) + i\|\xi\|\left(2f'_{m-2} - 2f'_{m+2}\right). \end{aligned}$$

From which it follows that

$$f_{m+2} = \frac{1}{2\|\xi\|}\left(-\frac{i}{2}d\sigma([B_\xi, B_k])f(\xi) - i\xi_k m f(\xi) + \|\xi\| d\sigma(B_k)f(\xi)\right)$$
$$f_{m-2}(\xi) = \frac{1}{2\|\xi\|}\left(\frac{i}{2}d\sigma([B_\xi, B_k])f(\xi) - i\xi_k m f(\xi) + \|\xi\| d\sigma(B_k)f(\xi)\right)$$
$$f_m(\xi) = im\frac{\xi_k}{\|\xi\|}f(\xi),$$

as wanted. □

Using Lemmas 3.3 and 3.4 it is clear by Lemma 3.2 that the operators $\mathcal{B}^\sigma_{\lambda,k}$ maps V_m to $V_{m+2} \oplus V_m \oplus V_{m-2}$. For $f \in C^\infty(\mathbb{R}^3, V_\sigma)$ with $f(\xi) \in V_m(\xi)$ denote by $(\mathcal{B}^\sigma_{\lambda,k}f(\xi))_{m\pm2}$ the projection of $B_{\lambda,k}f(\xi)$ to $V_{m\pm2}(\xi)$. Then we have

$$(\mathcal{B}^\sigma_{\lambda,k}f(\xi))_{m\pm2} = \frac{(2+\lambda-\rho\pm\frac{m}{2})}{2\|\xi\|^2}\left(\frac{1}{2}d\sigma([B_\xi, B_k])f(\xi) \pm \xi_k m f(\xi) \pm i\|\xi\| d\sigma(B_k)f(\xi)\right). \tag{3.15}$$

Remark 3.5. *Equation (3.15) implies that for $\lambda = \frac{1}{2} \pm m$ for some $m \in 2\mathbb{Z}$ such that $im\|\xi\|$ is an eigenvalue of $d\sigma(B_\xi)$ then $\tau_{\sigma,\lambda}$ contains a $\mathfrak{n}$-stable subspace. By (3.5), $-2(E-\lambda+\rho)$ leaves the Eigenspaces of $d\sigma(B_\xi)$ invariant and since*

$$d\sigma(B_\xi)\tau_{\sigma,\lambda}(k)f(\xi) = \sigma(k)d\sigma(B_{k^{-1}\xi})f(k^{-1}\xi) = im\|\xi\|\tau_{\sigma,\lambda}(k)f(\xi), \quad k \in M \cong SO(3)$$

it follows that the $\mathfrak{n}$-stable subspace is actually $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \overline{\mathfrak{n}}$ stable.

Combining the previous lemmas we can now prove Theorem 2.1:

Proof of Theorem 2.1. Let $f, g \in \mathcal{F}(I_\sigma(\lambda))$ and $\lambda \in \mathbb{R}$. Suppose that

$$(f \mid g)_{I_\sigma(\lambda)} := \int_{\mathbb{R}^3} \langle f(\xi), \widetilde{A}(\xi)g(\xi)\rangle_{V_\sigma} \|\xi\|^{-2\lambda} d\xi, \quad f, g \in \mathcal{F}(I_\sigma(\lambda))$$

defines an G-invariant form on $\mathcal{F}(I_\sigma(\lambda))$. Since $\widetilde{A}(\xi)$ commutes with $d\sigma(B_\xi)$ we must have that $\widetilde{A}(\xi)$ acts on V_σ diagonally with respect to the basis of Eigenvectors of $d\sigma(B_\xi)$. The action of M preserves the Eigenspaces of $d\sigma(B_\xi)$ and since $d\sigma(B_\xi)$ is skew-adjoint the Eigenspaces are orthogonal. Furthermore assume that $f(\xi), g(\xi) \in V_m(\xi)$ for all $\xi \in \mathbb{R}^3$, then we have

$$(\tau_{\sigma,\lambda}(h)f \mid \tau_{\sigma,\lambda}(h)g) = \int_{\mathbb{R}^3} \langle \sigma(h)f(h^{-1}\xi), \widetilde{A}(\xi)\sigma(h)f(h^{-1}\xi)\rangle_{V_\sigma} \|\xi\|^{-2\lambda} d\xi, \quad h \in M$$

Since M preserves $V_m(\xi)$ we must have that $a_m(\xi) = a_m(h^{-1}\xi)$ by (2.8). Hence $\widetilde{A}(\xi)$ is both homogeneous of degree 0 and $SO(3)$ invariant, hence constant. Assume $f(\xi), g(\xi)$ are contained in a single Eigenspace. Then $\widetilde{A}(\xi)$ acts as a scalar $a_m \in \mathbb{C}$ on $V_m(\xi)$. To determine a_m let $f, g \in \mathcal{F}(I_\sigma(\lambda))$ such that $f(\xi) \in V_{m+2}(\xi)$ and $g(\xi) \in V_m(\xi)$ for all $\xi \in \mathbb{R}^3$. Then by (2.4) we have

$$\begin{aligned}
(f \mid \mathcal{B}^\sigma_{\lambda,j} g) &= \int_{\mathbb{R}^3} a_{m+2} \langle g(\xi), (\mathcal{B}^\sigma_{\lambda,j} g(\xi))_{m+2}\rangle_{V_\sigma} \|\xi\|^{-2\lambda} d\xi \\
&= \frac{(2+\lambda-\rho+\frac{m}{2})}{2} \int_{\mathbb{R}^3} a_{m+2} \left\langle f(\xi), \left(\frac{1}{2} d\sigma([B_\xi, B_k]) + \xi_k m + i\|\xi\| d\sigma(B_k)\right) g(\xi)\right\rangle_{V_\sigma} \|\xi\|^{-2\lambda-2} d\xi \\
&= \frac{(2+\lambda-\rho+\frac{m}{2})}{2} \int_{\mathbb{R}^3} a_{m+2} \left\langle -\left(\frac{1}{2} d\sigma([B_\xi, B_k]) - \xi_k m - i\|\xi\| d\sigma(B_k)\right) f(\xi), g(\xi)\right\rangle_{V_\sigma} \|\xi\|^{-2\lambda-2} d\xi \\
&= \frac{a_{m+2}}{a_m} \frac{(-2-\lambda+\rho-\frac{m}{2})}{(2+\lambda-\rho-\frac{m+2}{2})} \int_{\mathbb{R}^3} \left\langle (\mathcal{B}^\sigma_{\lambda,j} f(\xi))_{(m+2)-2}, a_m g(\xi)\right\rangle_{V_\sigma} \|\xi\|^{-2\lambda} d\xi \\
&= \frac{a_{m+2}}{a_m} \frac{(m+2\lambda+1)}{(m-2\lambda+1)} (f \mid \mathcal{B}^\sigma_{\lambda,j} g).
\end{aligned}$$

Hence a_m must satisfy the recurrence relation

$$a_m(m-2\lambda+1) = a_{m+2}(m+2\lambda+1)$$

as wanted. ☐

4 Outlook

We finish by discussing some potential applications of the results. In [MO15] Möllers and Oshima constructed a L^2 model for some representations of $O(1,N)$ in the spherical case. They used such L^2 models to study branching problems to the subgroup $O(1,m+1) \times O(n-m)$, reducing the branching problem to the spectral theory of a Bessel type differential operator acting on the L^2-model, with the Bessel type operator analogously to the one appearing in this paper. A similar approach might work, if one can generalize the result in this paper to $O(1,N)$, to study branching laws outside the spherical domain. The results may also prove useful in studying tensor-products of discrete series representations. Such a generalization would require a suitable substitute for B_ξ in the more general setting.

In [Zha17] Zhang successfully used the existence of an L^2 model for complementary series representations of rank one Lie groups to find discrete components in the tensor product of complementary series representations. A similar approach might be possible to apply in our setting. In fact one might make an ansatz for an intertwining operator

$$\psi : \pi_{\zeta,\nu} \to \pi_{\xi,\lambda} \otimes \pi_{\eta,\mu}$$

by taking inspiration from the approach used in [MO15]. Based on the equivariance properties with respect to the action of $\mathfrak{n}, M$ and A and the approach used by Oshima and Möllers, we attempted to make an ansatz for such an operator

$$\psi(f)(x,y) = |x+y|^{\alpha} F\left(\frac{|x-y|^2}{|x+y|^2}\right) T\left(f(x+y)\right)_{x-y}, \qquad T \in \mathrm{Hom}_M(V_\zeta, V_\xi \otimes V_\eta \otimes \mathcal{H}_k)$$

with $\mathcal{H}_k$ denoting the spherical harmonics of degree k and $F \in C^\infty(\mathbb{R})$. Möllers and Oshima made a similar ansatz and was able to determine a differential equation on F, the spectral theory of which in turn described the corresponding branching law. However after spending a considerable amount of time at this approach, it appears to too technical in this setting and another angle of approach appears to be needed. The method employed by Zhang in [Zha17] appears to be well suited for this purpose and would be the next natural step to consider. But due to time constrains we were not able to pursue this any further.

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