Restricting representations of $\mathrm{GL}(3, \mathbb{R})$ to $\mathrm{GL}(2, \mathbb{R})$

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PhD Dissertation

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August, 2023

"If you understand a handfull of good examples,
there is a 90 % chance that you understand the general theory"

Bent Ørsted

Abstract

Let (G, H) be the pair $(\mathrm{GL}(n+1, \mathbb{R}), \mathrm{GL}(n, \mathbb{R}))$ where we consider H as a subgroup of G embedded in the upper left corner. This dissertation focuses on the branching problem, also known as the restriction problem, concerning the restriction of representations of G to the subgroup H. Specifically, we delve into the study of *symmetry breaking operators*, as elements of $\mathrm{Hom}_H(\pi_{\xi,\lambda}|_H, \tau_{\eta,\nu})$ where $\pi_{\xi,\lambda}$ and $\tau_{\eta,\nu}$ are principal series representations of G and H respectively. We consider a meromorphic family of symmetry breaking operators and study it by the corresponding meromorphic family of integral kernels. For these kernels we find Bernstein–Sato identities that enable us to normalize the family such that it becomes holomorphic therefore giving us a holomorphic family of symmetry breaking operators. For this holomorphic family we find functional equations with the Knapp–Stein intertwining operators as well as some sets of codimension two, where the family of symmetry breaking operators vanish.

In the case of $n = 2$ we obtain direct integral decompositions for all unitary irreducible representations of G restricted to H. Some of these decompositions are found by first establishing a Plancherel formula for the unitary principal series $\pi_{\xi,\lambda}$of G and then by using an analytic continuation procedure in λ to move this Plancherel formula to the complementary series and then further on to points of reducibility, where the degenerate series and the generalized principal series sits as quotients in the principal series. To do this analytic continuation all the results obtained for the family of symmetry breaking operators for (G, H) play a crucial role. Alas, the analytic continuation method can not decompose all unitary irreducible representations of G. However, we also present an argument using the Whittaker Plancherel formula, that establishes the direct integral decomposition for all the unitary representations of G except degenerate series.

To obtain the Plancherel formula for the unitary principal series we restrict to the open H-orbit in G/P where P is a minimal parabolic subgroup of G. This restriction is a H-equivariant isomorphism meaning instead of obtaining a Plancherel formula directly for $\pi_{\xi,\lambda}$ we rather have to find a Plancherel formula for the space of L^2-section of homogeneous Hermitian line bundles over $X = H/MA$ where MA is the subgroup of diagonal matrices in H. This is obtained by looking at a family of H-intertwining operators between the L^2-sections and principal series representations which is holomorphic in its parameters. This family of operators is essentially the Fourier–Jacobi transform, and the Plancherel formula follows from the spectral decomposition of the corresponding ordinary second order differential operator.

Resumé

Lad (G, H) være parret $(\mathrm{GL}(n+1, \mathbb{R}), \mathrm{GL}(n, \mathbb{R}))$, hvor vi betragter H som en undergruppe af G ved at indlejre den i øverste venstre hjørne af G. Denne afhandling omhandler forgreningsproblemet, også kendt som restriktionsproblemet, der beskæftiger sig med at studere restriktionen af repræsentationer af G til undergruppen H. Mere specifikt kigger vi på såkaldte *symmetribrydendeoperatorer*, som vi betragter som elementer af $\mathrm{Hom}_H(\pi_{\xi,\lambda}|_H, \tau_{\eta,\nu})$, hvor $\pi_{\xi,\lambda}$ og $\tau_{\eta,\nu}$ er principalrække-repræsentationer for respektivt G og H. Vi betragter en meromorf familie af symmetribrydende operatorer, som vi studerer via dens meromorfe familie af integralkerner. For disse kerner finder vi såkaldte *Bernstein–Sato*-identiteter, der gør, at vi kan normaliserer familien, så den bliver holomorf og på den måde giver dette os en holomorf familie af symmetribrydende operatorer. For denne holomorfe familie finder vi funktionalligninger med Knapp–Stein-fletningsoperatorerne samt nogle mængder af kodimension to, hvor den holomorfe familie af symmetribrydendeoperatorer forsvinder.

I tilfældet hvor $n = 2$, opnår vi en direkte integraldekomposition af alle unitære irreducible repræsentationer af G restringeret til H. Nogle af disse dekompositioner bliver fundet ved at først at etablere en Plancherel-formel for den unitære principale række $\pi_{\xi,\lambda}$ for G også bruge en analytisk fortsættelses-procedure i λ til at flytte Plancherel formlen til den komplementære række og derfra videre til reducibilitetspunkter, hvor den degenererede række og den generaliserede principale række sidder som kvotienter af den principale række. For at lave denne analytiske fortsættelses-procedure er resultaterne omkring de symmetribrydende operatorer altafgørende. Desværre kan den analytiske udvidelse ikke opnå dekompositioner af enhver unitær irreducibel repræsentation. Vi præsenterer også et argument, der benytter Whittaker-Plancherel-formlen, som giver en direkte integraldekomposition for alle de unitære irreducible repræsentationer af G undtagen de degenererede række-repræsentationer.

For at opnå en Plancherel formel for den unitære principale række restringerer vi til den åbne tætte H-bane i G/P, hvor P er en minimal-parabolsk undergruppe af G. Dette er en isomorfi, så i stedet for at finde en Plancherel-formel direkte for $\pi_{\xi,\lambda}$, kan vi i stedet finde en Plancherel-formel for rummet af L^2-sektioner af homogene Hermitiske linje-bundter over $X = H/MA$, hvor MA er undergruppen af diagonale matricer i H. Dette er opnået ved at betragte en familie af fletningsoperatorer mellem L^2-sektionerne og principalrække-repræsentationer for H, som er holomorf i sine parametre. Denne familie er essentielt Fourier–Jacobi transformationen og Plancherel-formlen følger fra spektral dekomponeringen af den tilhørende ordinære anden-ordens differential operator.

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Preface

This thesis concludes my eight years of study at the department of mathematics at Aarhus university with this PhD-project being the last three of them. This dissertation consists of three papers

Paper A An explicit Plancherel formula for line bundles over the one-sheeted hyperboloid

Paper B Analytic continuation of symmetry breaking operators of the pair $(\mathrm{GL}(n+1,\mathbb{R}), \mathrm{GL}(n,\mathbb{R}))$

Paper C Unitary branching from $\mathrm{GL}(3,\mathbb{R})$ to $\mathrm{GL}(2,\mathbb{R})$

The original project for the PhD is the subject of Paper C, but along the way I needed results which yielded Paper A and B. Paper A has been published in the Journal of Lie Theory, and is co-authored with Frederik Bang-Jensen. Apart from typesetting, the version contained here is identical to the published version. The paper is joint-work and is a result of countless hours in front of blackboard computing and discussing. Paper B is not published or available as a preprint since I have a few ideas for some results in the same vicinity which would make sense to include. Paper C is missing some technical details in some proofs and is therefore not ready to be published. All the papers have been under Jan Frahm's excellent guidance and supervision.

Just as I began my PhD project, a global pandemic struck, disrupting the normal course of things and making the transition to being a PhD-student more difficult than expected. It took time to adjust working in isolation, delaying the pace of the project. However, despite the hurdles, I am pleased with the results presented in this dissertation.

I am deeply grateful to my supervisor Jan Frahm, whose contagious optimism and passion for mathematics have been a constant source of inspiration. His availability and patience in answering my questions has been incredibly supportive for my development as a mathematician. A warm thanks to Toshiyuki Kobayashi for generously hosting me during my visit to the University of Tokyo, a fantastic opportunity to learn and grow.

I am also thankful to my colleagues at the Department of Mathematics, Aarhus University, whose collegiality has has made my academic journey all the more pleasant. Among them, I wish to express special thanks to Frederik for the countless productive discussions, and for being a companion through the highs and lows of the PhD experience.

Finally, I am deeply grateful to my friends and family for their steady support, particularly during the challenging last few months of thesis writing. Their encouragement and understanding have been invaluable in bringing this work to completion.

Jonathan Ditlevsen
Aarhus, 2023

Introduction

In representation theory there are two fundamental problems, classifying the smallest objects (i.e. irreducible representations) and decomposing representations into these smallest objects. An example of the latter is a *branching problem*, by which we mean the problem of understanding how an irreducible representation of a group behaves as a representation of a subgroup.

Let G be a real reductive Lie group, H a closed subgroup of G and π an irreducible unitary representation of G. The restricted representation $\pi|_H$ is in general no longer irreducible, however it is unitary and thus has a decomposition as a direct integral

$$\pi|_H \simeq \int_{\widehat{H}}^{\oplus} m_\pi(\tau)\tau \, d\mu_\pi(\tau),$$

where $\widehat{H}$ is the unitary dual of H i.e. the irreducible unitary representations of H up to equivalence, $m_\pi(\tau) \in \mathbb{N}_0 \cup \{\infty\}$ are the multiplicities and μ_π a measure on $\widehat{H}$. Given $\widehat{H}$ is known, a complete answer to the *unitary branching problem* consists of finding the multiplicities $m_\pi(\tau)$ and the measure μ_π. We call the support of μ_π the spectrum, it can in general contain both a discrete part and a continuous part. If for example we consider the case where G is compact then

$$\pi|_H \simeq \bigoplus_{\tau \in \widehat{H}} m_\pi(\tau)\tau,$$

where each τ is finite dimensional and $m_\pi(\tau)$ are all finite. In this case the spectrum is purely discrete.

Consider the example of $G = \mathrm{SL}(2,\mathbb{R})$ and $\pi = L^2(\mathbb{R})$ with action of G given as

$$\pi(g)f(x) = |cx+d|^{-1} f\Big(\frac{ax+b}{cx+d}\Big), \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If H is the subgroup of upper triangular matrices with 1's on the diagonal we have

$$\pi|_H(n)f(x) = f(x-b), \quad n = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

Letting χ_ξ denote the character $x \mapsto e^{i\xi x}$ we can consider the intertwining map

$$A_\xi : \mathcal{S}(\mathbb{R}) \to \mathbb{C}\chi_\xi, \quad f \mapsto \hat{f}(\xi)\chi_\xi,$$

where $\hat{f}$ denotes the Fourier transform of f and $\mathcal{S}(\mathbb{R})$ is the Schwartz space. We restrict to the Schwartz space as L^2-function famously are not defined point-wise. Defining a map

$$A : \mathcal{S}(\mathbb{R}) \to \int_{\mathbb{R}}^{\oplus} \mathbb{C}\chi_\xi \, d\xi, \quad Af = (A_\xi f)_{\xi \in \mathbb{R}},$$

we see immediately from the Plancherel theorem for the Fourier transform that it is an isometry since

$$||Af||^2 = \int_{\mathbb{R}} |A_\xi f|^2 d\xi = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi = ||\hat{f}||^2 = ||f||^2.$$

As the Schwartz space is dense in $L^2(\mathbb{R})$ we can extend A to all of $L^2(\mathbb{R})$ and as $\mathbb{C}\chi_\xi$ are irreducible and each A_ξ is non-zero we also get that A is surjective and in total that

$$\pi|_H \simeq \int_{\mathbb{R}}^{\oplus} \mathbb{C}\chi_\xi \, d\xi,$$

which has purely continuous spectrum.

This example highlights multiple aspects of the unitary branching problem. The first key step was finding intertwining operators in $\mathrm{Hom}_H(\pi|_H, \tau)$. However, these maps are only defined in an L^2-sense and not point-wise, so we have to restrict ourselves to the smooth vectors of the representation. Thus we consider the space of *symmetry breaking operators*

$$\mathrm{Hom}_H(\pi^\infty|_H, \tau^\infty),$$

which makes the map

$$A : \pi^\infty|_H \to \int_{\widehat{H}}^{\oplus} m_\pi(\tau) \cdot \tau \, d\mu_\pi(\tau), \quad Av = (A_\tau^j v)_{\tau \in \widehat{H}, j=1,\ldots,m_\pi(\tau)},$$

point-wise defined for some $A_\tau^j \in \mathrm{Hom}_H(\pi^\infty|_H, \tau^\infty)$. It turns out that it is often useful to study families of symmetry breaking operators and so we stray away from our assumption on π and τ to be unitary and instead assume them to be a admissible representations, as many of the members of these families will fit into this more general setup.

We call the dimension of the space of symmetry breaking operators for the multiplicity. These multiplicities might be infinite and therefore a good selection of groups (G, H) is on order for a nice branching problem. Work by Kobayashi–Oshima in [KO13] shows that when G and H are algebraically defined over $\mathbb{R}$ then finite multiplicity is equivalent to the pair (G, H) being strongly spherical i.e. $(G \times H)/\operatorname{diag}(H)$ is real spherical. A full classification of strongly spherical pairs have been done in the case of symmetric pairs by Kobayashi–Matsuki in [KM14], and and in general by Knop–Krötz–Pecher–Schlichtkrull in [KKPS19]. Going back to the example, we could easily argue that the map to the direct integral was surjective by noting that it was a non-zero map to an irreducible representation, this along with many other aspects makes multiplicity one much easier to deal with. Fortunately Sun-Zhu found in [SZ12] that the following pairs have at most multiplicity one

$$(\mathrm{GL}(n+1, \mathbb{R}), \mathrm{GL}(n, \mathbb{R})), \quad (\mathrm{GL}(n+1, \mathbb{C}), \mathrm{GL}(n, \mathbb{C})), \quad (\mathrm{U}(p, q+1), \mathrm{U}(p, q)),$$
$$(\mathrm{SO}(n+1, \mathbb{C}), \mathrm{SO}(n, \mathbb{C})), \quad (\mathrm{SO}(p, q+1), \mathrm{SO}(p, q)).$$

Having determined the nice pairs to study, we need a model for the representations π and τ to study explicit symmetry breaking operators. Consider minimal parabolic subgroups

$$P_G = M_G A_G N_G \subseteq G, \quad P_H = M_H A_H N_H \subseteq H,$$

with $A_H \subseteq A_G$. Then letting $\xi \in \widehat{M}_G$, $\lambda \in \mathfrak{a}_{G,\mathbb{C}}^*$ and $\eta \in \widehat{M}_H$, $\nu \in \mathfrak{a}_{H,\mathbb{C}}^*$ we can consider the *principal series representations*

$$\pi_{\xi,\lambda} = C^\infty - \mathrm{Ind}_{P_G}^G(\xi \otimes e^\lambda \otimes 1), \quad \tau_{\eta,\nu} = C^\infty - \mathrm{Ind}_{P_H}^H(\eta \otimes e^\nu \otimes 1).$$

By the Casselman embedding theorem, every irreducible unitary representation is a subrepresentation of the principal series representations. When (G,H) is one of the multiplicity one pairs Frahm showed in [F21] that for generic $(\lambda,\nu) \in \mathfrak{a}_{G,\mathbb{C}}^* \times \mathfrak{a}_{H,\mathbb{C}}^*$

$$\dim \mathrm{Hom}_H(\pi_{\xi,\lambda}|_H, \tau_{\eta,\nu}) = \dim \mathrm{Hom}_M(\xi|_M, \eta|_M) \in \{0,1\}, \quad (\text{for some } M \subseteq M_G \cap M_H),$$

and in the case of $=1$ there exists an explicit family $A_{\xi,\lambda}^{\eta,\nu}$ of symmetry breaking operators that depends meromorphically on the induction parameters (λ,ν).

Methods

For $\lambda \in i\mathfrak{a}_G^*$ the principal series representation $\pi_{\xi,\lambda}$ is unitary on $L^2(G/P_G, \mathcal{V}_{\xi,\lambda})$ where $\mathcal{V}_{\xi,\lambda}$ is the homogeneous vector bundle $\mathcal{V}_{\xi,\lambda} = G \times_{P_G} (\xi \otimes e^{\lambda+\rho_G} \otimes 1) \to G/P_G$. As (G,H) is a multiplicity one pair, H acts on G/P_G by a single open dense orbit $\mathcal{O} \subseteq G/P_G$. Then the restriction to this open orbit becomes an isometric isomorphism

$$L^2(G/P_G, \mathcal{V}_{\xi,\lambda}) \xrightarrow{\sim} L^2(\mathcal{O}, \mathcal{V}_{\xi,\lambda}|_{\mathcal{O}}).$$

Given an explicit Plancherel formula for the space $L^2(\mathcal{O}, \mathcal{V}_{\xi,\lambda}|_{\mathcal{O}})$ is known, we can decompose

$$\pi_{\xi,\lambda}|_H \simeq \bigoplus_{\eta \in \widehat{M_H}} \int^{\oplus} m_{\xi,\lambda}(\eta,\nu) \tau_{\eta,\nu} \, d\mu_{\xi,\lambda,\eta}(\nu),$$

with the corresponding Plancherel formula

$$||f||_{\xi,\lambda}^2 = \sum_{\eta \in \widehat{M}_H} \int ||A_{\xi,\lambda}^{\eta,\nu} f||_{\eta,\nu}^2 \, d\mu_{\xi,\lambda,\eta}(\nu).$$

The left-hand side of this equation can be made holomorphic for $\lambda \in \mathfrak{a}_{G,\mathbb{C}}^*$ by $||f||_{\xi,\lambda}^2 = \langle f, T_{\xi,\lambda}^w \overline{f} \rangle$ where $T_{\xi,\lambda}^w$ is a Knapp–Stein intertwining operator. The idea is then to analytically extend the right hand side to $\lambda \in \mathfrak{a}_{G,\mathbb{C}}^*$, and as the two holomorphic functions coincide on $i\mathfrak{a}_G^*$ they coincide everywhere. We then get a direct integral decomposition of $\pi_{\xi,\lambda}|_H$ for the (ξ,λ) where $\pi_{\xi,\lambda}$ is unitary or contains a unitary quotient.

To make the analytic extension, the meromorphic properties of the Knapp–Stein operators, the measure $\mu_{\xi,\lambda,\eta}(\nu)$ and the family of symmetry breaking operators must be established. The Knapp–Stein operators are often well studied, and the measure depends on how explicit the Plancherel formula at hand is. The symmetry breaking operators $A_{\xi,\lambda}^{\eta,\nu}$ can via the Schwartz kernel theorem be identified to a distributional kernel $K_{\xi,\lambda}^{\eta,\nu}$ on G by

$$A_{\xi,\lambda}^{\eta,\nu} f(h) = \int_{K_G} K_{\xi,\lambda}^{\eta,\nu}(h^{-1}k) f(k) \, dk,$$

where K_G is the maximal compact subgroup for G and the integral should be understood in distributional sense. The distributional kernel will have the form

$$K^{\eta,\nu}_{\xi,\lambda}(g) = |\Phi_1(g)|^{s_1}_{\varepsilon_1} \dots |\Phi_n(g)|^{s_n}_{\varepsilon_n}, \quad (|x|^s_\varepsilon = \operatorname{sgn}(x)^\varepsilon |x|^s, \quad (x \in \mathbb{R}))$$

where Φ_i are analytic functions on G and (ε, s) is an affine transformation of $(\xi, \lambda, \eta, \nu)$. This kernel is L^1_{loc} for s in some domain and the goal is to normalize it, and analytically extend it as a distribution, obtaining a holomorphic family $A^{\eta,\nu}_{\xi,\lambda}$ of symmetry breaking operators. To do this we employ the so-called Clerc–Beckmann trick, found in [BC12, Section 3.3]. This involves "conjugating" a multiplication operator $M : K^{\eta,\nu}_{\xi,\lambda} \mapsto \Phi_i K^{\eta,\nu}_{\xi,\lambda}$ by Knapp–Stein intertwining operators. Loosely speaking, as the multiplication operator shifts (ε, s) to $(\varepsilon, s + e_i)$ the conjugation will shift (ε, s) to $(\varepsilon, s - e_i)$. If one can show that this conjugation is in fact a differential operator $D(s,t)$, then a Bernstein–Sato identity

$$D(\varepsilon, s)K_{\varepsilon,s} = b(\varepsilon, s)K_{\varepsilon,s-e_i},$$

where $b(\varepsilon, s)$ is a polynomial in s, is established. Using this identity we can normalize $K_{\varepsilon,s}$ such that $D(\varepsilon, s)$ simply shifts the parameters (ε, s) of $K_{\varepsilon,s}$ and use this to define an analytic extension of $K^{\eta,\nu}_{\xi,\lambda}$.

Results

Paper A

For $G = \mathrm{SL}(2,\mathbb{R})$ and $H = MA$ where MA is the subgroup of diagonal matrices of G we consider $L^2(G/H, \mathcal{L}_{\varepsilon,\lambda})$, the L^2-section of a homogeneous line bundle associated to a unitary character $\chi_{\varepsilon,\lambda}(h) = |t|^\lambda_\varepsilon$ of H or alternatively $\mathrm{Ind}^G_H(\chi_{\varepsilon,\lambda})$. We study holomorphic families of intertwining operators $A^\xi_{\lambda,\mu} : \mathrm{Ind}^G_H(\chi_{\varepsilon,\lambda}) \to \mathrm{Ind}^G_P(\varepsilon \otimes e^\mu \otimes 1)$ to the principal series representations of G. The main result is:

Theorem. *For $f \in \mathrm{Ind}^G_H(\chi_{\varepsilon,\lambda})$, $\lambda \in i\mathbb{R}$ and $\varepsilon \in \{0,1\}$ we have*

$$||f||^2 = \int_{i\mathbb{R}} \sum_{\xi=0}^{1} ||\boldsymbol{A}^\xi_{\lambda,\mu} f||^2 \frac{d\mu}{|a(\mu,\varepsilon)|^2} + \sum_{\mu \in 1-\varepsilon-2\mathbb{N}} c(\mu,\varepsilon)||\mathbb{A}^\varepsilon_{\lambda,\mu}||^2,$$

where $\boldsymbol{A}_{\lambda,\mu}$ and $\mathbb{A}^\varepsilon_{\lambda,\mu}$ are some combinations of $A^0_{\lambda,\mu}$ and $A^1_{\lambda,\mu}$.

The proof consists of two steps. First, we prove the theorem in the case where $\lambda = 0$ on each K-type. On a fixed K-type $A^\xi_{0,\mu}$ behaves as the Fourier–Jacobi transform and the Plancherel formula follows from a spectral decomposition. The second step we show that as a representation of G the space $\mathrm{Ind}^G_H(\chi_{\varepsilon,\lambda})$ is independent of λ, and by composing with an explicit isomorphism $\mathrm{Ind}^G_H(\chi_{\varepsilon,\lambda}) \to \mathrm{Ind}^G_H(\chi_{\varepsilon,0})$ we deduce the claimed Plancherel formula. One of the major obstacles in the proof is dealing with consequences of having multiplicity two in the continuous spectrum, but only multiplicity one in the discrete spectrum.

Paper B

In this paper we consider the pair of groups $(G, H) = (\mathrm{GL}(n+1,\mathbb{R}), \mathrm{GL}(n,\mathbb{R}))$ and intertwining operators $A^{\eta,\nu}_{\xi,\lambda} : \pi_{\xi,\lambda} \to \tau_{\eta,\nu}$ between principal series representations $\pi_{\xi,\lambda}$ and $\tau_{\eta,\nu}$ of respectively G and H. There is one meromorphic family of such intertwining operators which can be identified by a corresponding meromorphic family of integral kernels $K^{\eta,\nu}_{\xi,\lambda}$. This family of kernels is locally integrable in some domain of $(\lambda,\nu) \in \mathbb{C}^{2n+1}$ and the main result of the paper is to do analytic continuation to extend this domain.

Theorem. *Let* $\boldsymbol{K}^{\eta,\nu}_{\xi,\lambda} = K^{\eta,\nu}_{\xi,\lambda}/n(\xi,\lambda,\eta,\nu)$*, where* $n(\xi,\lambda,\eta,\nu)$ *is an explicit function given in terms of Gamma-functions.* $\boldsymbol{K}^{\eta,\nu}_{\xi,\lambda}$ *extends analytically as a distribution to all of* $(\lambda,\nu) \in \mathbb{C}^{2n+1}$.

We prove this theorem by finding Bernstein–Sato identities of $K^{\eta,\nu}_{\xi,\lambda}$ using the method outlined in the methods subsection. Using the normalized integral kernels we can get the corresponding intertwining operators $\mathbf{A}^{\eta,\nu}_{\xi,\lambda}$. We argue that the normalization $n(\xi,\lambda,\eta,\nu)$ is most likely optimal in the sense that it does not introduce any redundant zeroes, while simultaneously keeping $\mathbf{A}^{\eta,\nu}_{\xi,\lambda}$ holomorphic.

Theorem. *We have the following functional equations when composing* $\boldsymbol{A}^{\eta,\nu}_{\xi,\lambda}$ *by the Knapp–Stein intertwining operators* $\boldsymbol{T}^{w}_{\xi,\lambda}$ *and* $\boldsymbol{T}^{w'}_{\eta,\nu}$*:*

$$\boldsymbol{A}^{\eta,\nu}_{w(\xi,\lambda)} \circ \boldsymbol{T}^{w}_{\xi,\lambda} = c(\xi,\lambda,\eta)\, \boldsymbol{A}^{\eta,\nu}_{\xi,\lambda}, \quad \boldsymbol{T}^{w'}_{\eta,\nu} \circ \boldsymbol{A}^{\eta,\nu}_{\xi,\lambda} = d(\xi,\eta,\nu)\, \boldsymbol{A}^{w'(\eta,\nu)}_{\xi,\lambda},$$

where c, d *are explicit holomorphic functions given in terms of Gamma-functions.*

This Theorem follows by a direct computation. Lastly, we find a family of zeroes for $\mathbf{A}^{\eta,\nu}_{\xi,\lambda}$.

Paper C

In this unfinished paper we decompose unitary representations of $\mathrm{GL}(3,\mathbb{R})$ restricted to $\mathrm{GL}(2,\mathbb{R})$ in terms of unitary irreducible $\mathrm{GL}(2,\mathbb{R})$ representations. We use the results of paper A and B and apply the method of analytic continuation explained in the methods section.

Theorem. *The unitary principal series, complementary series and unitarily induced generalized principal series of* $\mathrm{GL}(3,\mathbb{R})$ *all decomposes as*

$$\pi|_H \simeq \bigoplus_{\eta\in(\mathbb{Z}/2\mathbb{Z})^2} \int^{\oplus}_{i\mathbb{R}^2} \tau_{\eta,\nu}\, d\nu \oplus \bigoplus_{\eta\in\mathbb{Z}/2\mathbb{Z}} \bigoplus_{\nu_-\in 1+\eta-2\mathbb{N}} \int^{\oplus}_{i\mathbb{R}} \tau^{ds}_{\eta,\nu}\, d\nu_+,$$

whereas the unitarily induced degenerate series decomposes as

$$\pi_\lambda|_H \simeq \bigoplus_{\eta\in\mathbb{Z}/2\mathbb{Z}} \int_{i\mathbb{R}} \tau_{(\eta,\eta),(\lambda,z)}\, dz, \quad (\lambda \in i\mathbb{R})$$

where $\tau_{\eta,\nu}$ *is the unitary principal series for* $\mathrm{GL}(2,\mathbb{R})$ *and* $\tau^{ds}_{\eta,\nu}$ *is the almost discrete series for* $\mathrm{GL}(2,\mathbb{R})$.

In the argumentation of this theorem there are a few gaps, but the framework for the proof is there. We also prove the first part of the theorem using a completely different method that relies on the Whittaker Plancherel formula.

Related work & Outlook

Paper A

For $\varepsilon = 0$ and $\lambda \in i\mathbb{R}$ the same Plancherel formula was obtained by Zhu in [Z18]. Moreover, for $\varepsilon = 0$ and $\lambda = 0$, corresponding to decomposing $L^2(G/H)$, our Plancherel formula is a special case of the one for pseudo-Riemannian real hyperbolic spaces $O(p,q)/O(p,q-1)$ with $p = 1$ and $q = 2$ which was obtained by Faraut, Rossmann and Strichartz in [F79], [R78], [S73]. The corresponding abstract Plancherel formula, i.e. the description of the representations occurring in the direct integral decomposition, also follows from general theory (see e.g. [B05]). The techniques in this paper are very centered around calculations of $\mathrm{SL}(2,\mathbb{R})$ and will most likely not apply in a broader setting.

Paper B

The study of symmetry breaking operators (SBOs) have gained a lot of attention in the past few years. They have found applications in analytic number theory, unitary branching and partial differential equations (see e.g. [FS18], [MZØ15], [MØ17]) but are also interesting objects in their own right. Looking at Kobayashi's ABC-program ([K15]) which describes different steps for solving branching problems, the C-part of the program can divided into subcategories as:

1. Construct SBOs explicitly,
2. Classify SBOs,
3. Find residue formlas for SBOs,
4. Study functional equations among the SBOs,
5. Determine the image of sub-quotients of SBOs.

These sub-steps was suggested by Kobayashi–Speh in their book [KS17] which contains a complete answer for the pair $(O(n+1,1), O(n,1))$. Later their work was generalized by Frahm and Weiske to strongly spherical real reductive (G,H) where both G and H are of real rank one, in [FW20].

The work of paper B fits into step 1 of this subprogram. The techniques used for obtaining Bernstein–Sato identities have good chances of working for other families of symmetry breaking operators. The computation boils down to a rank one reduction, which seems feasible for other families of symmetry breaking operators, but we have no general proof or intuition for why the conjugation of the multiplication operator by standard intertwining operators should give a differential operator.

Paper C

For real reductive groups there is no general theory available for solving branching problems. In the case of nilpotent Lie groups the orbit method provides a complete answer. In the 90s Kobayashi initiated the study of the discretely decomposable branching problem (see [K94]) where the measure is discrete and the direct integral is in fact a direct sum, in this case more algebraic methods are available see e.g. [DV10], [SØ08], [EW04]. On the other hand if the restricted representation has purely continuous spectrum more analytic methods can be applied see e.g. [R79] and [CKØP11].

In the case of mixed continuous and discrete spectrum no systematic approach has been developed. In [W21] Weiske obtained branching laws for unitary representations of $O(1, n+1)$ for the scalar principal series, using similar techniques we have used in this paper. The main difference between this paper and Weiske's is that we are considering groups of higher rank than one. Before starting the project we thought that the higher rank would complicate the analysis of the SBOs significantly but as Paper B shows this was not the case. However, the analytic continuation process is way more involved and technical then for rank one. In view of Paper B and C it might be feasible to obtain branching laws for $(\mathrm{GL}(n+1, \mathbb{R}), \mathrm{GL}(n, \mathbb{R}))$, but the limiting factors would be if there exists explicit Plancherel formulas for the open orbit and the complication of the analytic continuation process introducing even more variables.

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Paper A

An explicit Plancherel formula for line bundles over the one-sheeted hyperboloid

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Abstract

In this paper we consider $G = \mathrm{SL}(2, \mathbb{R})$ and H the subgroup of diagonal matrices. Then $X = G/H$ is a unimodular homogeneous space which can be identified with the one-sheeted hyperboloid. For each unitary character χ of H we decompose the induced representations $\mathrm{Ind}_H^G(\chi)$ into irreducible unitary representations, known as a Plancherel formula. This is done by studying explicit intertwining operators between $\mathrm{Ind}_H^G(\chi)$ and principal series representations of G. These operators depends holomorphically on the induction parameters.

Introduction

The Plancherel formula for a unimodular homogeneous space $X = G/H$ of a Lie group G describes the decomposition of the left-regular representation of G on $L^2(X)$ into irreducible unitary representations. More generally, one can ask for the decomposition of $L^2(G \times_H V_\chi)$, the L^2-sections of a homogeneous vector bundle associated with a unitary representation (χ, V_χ) of H. In representation theoretic language, this corresponds to the induced representation $\mathrm{Ind}_H^G(\chi)$ of G, and for the trivial representation $\chi = \mathbf{1}$ we recover $L^2(G/H)$.

By abstract theory, the unitary representation $\mathrm{Ind}_H^G(\chi)$ decomposes into a direct integral of irreducible unitary representations of G, i.e. there exists a measure μ on the unitary dual $\widehat{G}$ of G and a multiplicity function $m : \widehat{G} \to \mathbb{N} \cup \{\infty\}$ such that

$$\mathrm{Ind}_H^G(\chi) \simeq \int_{\widehat{G}}^{\oplus} m(\pi) \cdot \pi \, d\mu(\pi).$$

An *abstract* Plancherel formula describes the support of the Plancherel measure μ as well as the multiplicity function m. Such abstract Plancherel formulas have been established for certain classes of homogeneous spaces such as semisimple symmetric spaces (see e.g. [B05]).

However, for some applications an abstract Plancherel formula is not sufficient, and a more *explicit* version is needed (see e.g. [FW, W21]). By this, we mean an explicit formula for the measure μ as well as explicit linearly independent intertwining operators $A_{\pi,j}:\mathrm{Ind}_H^G(\chi)^\infty\to\pi^\infty$, $j=1,\dots,m(\pi)$, for μ-almost every $\pi\in\widehat{G}$ such that

$$\|f\|^2_{\mathrm{Ind}_H^G(\chi)}=\int_{\widehat{G}}\sum_{j=1}^{m(\pi)}\|A_{\pi,j}f\|_\pi^2\,d\mu(\pi)\qquad(f\in\mathrm{Ind}_H^G(\chi)^\infty).$$

Such an explicit Plancherel formula is for instance known for Riemannian symmetric spaces $X=G/K$, where the Plancherel measure μ is explicitly given in terms of Harish-Chandra's c-function, and the intertwining operators $A_{\pi,j}$ can be described in terms of spherical functions (see e.g. [H08], and also [S94] for the case of line bundles over Hermitian symmetric spaces). This explicit Plancherel formula has recently been applied in the context of branching problems for unitary representations, where the explicit Plancherel measure and in particular its singularities play a crucial role (see e.g. [FW, W21]). In order to apply the same strategy to other branching problems, explicit Plancherel formulas are needed for more general homogeneous spaces.

In this paper, we determine the explicit Plancherel formula for line bundles over the one-sheeted hyperboloid $X=G/H$, where $G=\mathrm{SL}(2,\mathbb{R})$ and H the subgroup of diagonal matrices. This specific Plancherel formula has direct applications to branching problems for the pairs $(\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R}),\mathrm{diag}(\mathrm{SL}(2,\mathbb{R}))$ and $(\mathrm{GL}(3,\mathbb{R}),\mathrm{GL}(2,\mathbb{R}))$. The homogeneous Hermitian line bundles over X are parameterized by $\varepsilon\in\mathbb{Z}/2\mathbb{Z}$ and $\lambda\in i\mathbb{R}$, the corresponding unitary character of H being

$$\chi_{\varepsilon,\lambda}\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}=\mathrm{sgn}(t)^\varepsilon|t|^\lambda\qquad(t\in\mathbb{R}^\times).$$

We find intertwining operators $A^\xi_{\lambda,\mu}:\mathrm{Ind}_H^G(\chi_{\lambda,\varepsilon})\to\mathrm{Ind}_P^G(\varepsilon\otimes e^\mu\otimes 1)$, $\xi=0,1$ between the line bundles over X and the principal series representation (see Proposition 5.1).

Theorem (See Corollary 7.6)**.** *For $f\in\mathrm{Ind}_H^G(\chi_{\lambda,\varepsilon})$, $\lambda\in i\mathbb{R}$ and $\varepsilon\in\{0,1\}$ we have*

$$\|f\|^2=\int_{i\mathbb{R}}\sum_{\xi=0}^{1}\|\mathbf{A}^\xi_{\lambda,\mu}f\|^2\frac{d\mu}{|a(\mu,\varepsilon)|^2}+\sum_{\mu\in 1-\varepsilon-2\mathbb{N}}c(\mu,\varepsilon)\|\mathbb{A}^\varepsilon_{\lambda,\mu}f\|^2,\tag{0.1}$$

where $\mathbf{A}_{\lambda,\mu}$ and $\mathbb{A}_{\lambda,\mu}$ are some combinations of $A^0_{\lambda,\mu}$ and $A^1_{\lambda,\mu}$.

The proof of (0.1) consists of two steps. First, we prove (0.1) in the case $\lambda=0$ separately for each K-isotypic component. On a fixed K-isotypic component, the intertwining operators $A^\xi_{\lambda,\mu}$ are essentially Fourier–Jacobi transforms, and the Plancherel formula follows from the spectral decomposition of the corresponding ordinary second order differential operator by Sturm–Liouville theory. The main difficulty is that the continuous spectrum occurs with multiplicity two, while the discrete part occurs with multiplicity-one, and it is non-trivial to find the right linear combination of A^0_μ and A^1_μ that corresponds to a direct summand. In fact, this linear combination is very different for the cases $\varepsilon=0$ and $\varepsilon=1$. In the second step, we show that, as a representation of G, $L^2(G/H,\mathcal{L}_{\varepsilon,\lambda})$ is independent of λ, and by finding an explicit unitary isomorphism $L^2(G/H,\mathcal{L}_{\varepsilon,\lambda})\to L^2(G/H,\mathcal{L}_{\varepsilon,0})$ we deduce the claimed formula.

We remark that for $\varepsilon = 0$ and general $\lambda \in i\mathbb{R}$ the Plancherel formula was recently obtained by Zhu [Z18]. Moreover, for $\varepsilon = 0$ and $\lambda = 0$ our Plancherel formula can be viewed as a special case of the one for pseudo-Riemannian real hyperbolic spaces $\mathrm{O}(p,q)/\mathrm{O}(p,q-1)$ with $p = 1$ and $q = 2$ which was obtained by Faraut [F79], Rossmann [R78] and Strichartz [S73]. Note also that the corresponding *abstract* Plancherel formula, i.e. the description of the representations occurring in the direct integral decomposition, also follows from the general theory (see e.g. [B05]).

Acknowledgements: We would like to thank our supervisor Jan Frahm for his help and input on the topics of this paper.

Notation: $\mathbb{N} = \{1, 2, 3 \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $A \subseteq \mathbb{R}$ and $b, c \in \mathbb{R}$ we denote by $b + cA = \{b + ca \mid a \in A\}$. The Pochhammer symbol is $(x)_n = x(x+1)\cdots(x+n-1)$. We denote Lie groups by Roman capitals and their corresponding Lie algebras by the corresponding Fraktur lower cases. For $m \in \mathbb{Z}$ we let $[m]_2 \in \{0, 1\}$ be the remainder of m after division by 2.

1 The principal series of $\mathrm{SL}(2,\mathbb{R})$

In this section we recall some results about the representation theory of $\mathrm{SL}(2,\mathbb{R})$ following [C20]. Let $G = \mathrm{SL}(2,\mathbb{R})$ and consider the following subgroups

$$M = \{\pm I\}, \quad A = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{R}_{>0} \right\}, \quad N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\},$$

then $P = MAN$ is a minimal parabolic subgroup of G. Identify $\widehat{M} \cong \mathbb{Z}/2\mathbb{Z}$ by mapping $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ to the character

$$M \to \{\pm 1\}, \qquad \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \mapsto (\pm 1)^{\varepsilon}.$$

Further, we identify $\mathfrak{a}_{\mathbb{C}}^* \cong \mathbb{C}$ by mapping $\lambda \mapsto \lambda(\operatorname{diag}(1,-1))$. We can then observe that any character of $H := MA$ is of the form $\chi_{\varepsilon,\lambda} = \varepsilon \otimes e^{\lambda}$ where

$$\chi_{\varepsilon,\lambda} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = |t|_{\varepsilon}^{\lambda} := \operatorname{sgn}(t)^{\varepsilon} |t|^{\lambda}, \qquad (t \in \mathbb{R}^{\times}).$$

As the commutator subgroup of P is N the characters of P is of the form $\varepsilon \otimes e^{\mu} \otimes 1$ and these characters are unitary exactly when $\lambda \in i\mathbb{R}$.

Let $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ and $\mu \in \mathbb{C}$. For any character $\varepsilon \otimes e^{\mu} \otimes 1$ of P define the principal series representation $\pi_{\varepsilon,\mu}$induced by it to be the left regular representation of G on

$$\operatorname{Ind}_P^G(\varepsilon \otimes e^{\mu} \otimes 1) = \left\{ f \in C^{\infty}(G) \mid f(gman) = |t|_{\varepsilon}^{-\mu-1} f(g),\, m \in M,\, a \in A,\, n \in N \right\},$$

where $ma = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \in MA$. We introduce the notation

$$k_{\theta} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

and $\zeta_m(k_\theta) = e^{im\theta}$. According to the theory of Fourier series we have the K-type decomposition

$$\mathrm{Ind}_P^G(\varepsilon \otimes e^\mu \otimes 1) \cong \widehat{\bigoplus_{m \in 2\mathbb{Z}+\varepsilon}} \mathbb{C}\zeta_m.$$

We let $\mathrm{Ind}_P^G(\varepsilon \otimes e^\mu \otimes 1)_m$ denote the set of functions contained in the K-type given by $m \in \mathbb{Z}$, that is $\mathrm{Ind}_P^G(\varepsilon \otimes e^\mu \otimes 1)_m = \mathbb{C}\zeta_m$.

A basis of $\mathfrak{g}$ is given by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Consider the Casimir operator

$$\Delta_\mu = d\pi(H)^2 + d\pi(E+F)^2 - d\pi(E-F)^2,$$

where $\pi = \pi_{\varepsilon,\mu}$.

Proposition 1.1 (See e.g. [C20, Prop. 10.7])**.** *For $f \in \mathrm{Ind}_P^G(\varepsilon \otimes e^\mu \otimes 1)$ we have*

$$\Delta_\mu f = (\mu^2 - 1)f.$$

Proposition 1.2 (See [C20, Prop. 10.8])**.** *The representation $\mathrm{Ind}_P^G(\varepsilon \otimes e^\mu \otimes 1)$ is irreducible except when $\mu \in 1 - \varepsilon - 2\mathbb{Z}$. If $\mu \in 1 - \varepsilon - 2\mathbb{N}$ then $\mathrm{Ind}_P^G(\varepsilon \otimes e^\mu \otimes 1)$ decomposes as $V_0 \oplus V_1 \oplus V_2$ where V_0 is an irreducible representation containing exactly the K-types with $|m| \leq -\mu$. The quotient $\pi_{\varepsilon,\mu}^{ds}$ is a direct sum of two infinite dimensional representations $\pi_{\varepsilon,\mu}^{\mathrm{hol}}$ and $\pi_{\varepsilon,\mu}^{\mathrm{ahol}}$.*

Let $w_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, a representative of the longest Weyl group element of G. Recall the definition of the Knapp–Stein intertwining operator

$$T_\mu^\varepsilon : \mathrm{Ind}_P^G(\varepsilon \otimes e^\mu \otimes 1) \to \mathrm{Ind}_P^G(\varepsilon \otimes e^{-\mu} \otimes 1), \quad T_\mu^\varepsilon f(g) = \frac{1}{\Gamma(\frac{\mu+\varepsilon}{2})} \int_{\overline{N}} f(gw_0\overline{n}) d\overline{n},$$

for $\mathrm{Re}\,(\mu) > 0$. The normalization is chosen such that T_μ^ε extends holomorphically to $\mu \in \mathbb{C}$.

Proposition 1.3. *For $f \in \mathrm{Ind}_P^G(\varepsilon \otimes e^\mu \otimes 1)_m$ we have*

$$T_\mu^\varepsilon f = b_m^\varepsilon(\mu) f,$$

where

$$b_m^\varepsilon(\mu) = \sqrt{\pi} i^{[\varepsilon]_2} (-1)^{\frac{m+|m|}{2} - [\varepsilon]_2} \frac{\left(\frac{1+\varepsilon-\mu}{2}\right)_{\frac{|m|-\varepsilon}{2}}}{\Gamma\left(\frac{\mu+1+|m|}{2}\right)}.$$

For $\varepsilon = 0$ and $\mu \in 1 - 2\mathbb{N}$ we have $b_m^0(\mu) \geq 0$ for all $m \in 2\mathbb{Z}$. Whereas for $\varepsilon = 1$, m odd and $\mu \in -2\mathbb{N}$ we have $-ib_m^1(\mu) \geq 0$ for $m > 0$ and $ib_m^1(\mu) \geq 0$ for $m < 0$.

Proof. As T_μ^ε maps K-types to K-types we have $T_\mu^\varepsilon f = T_\mu^\varepsilon f(e) f$. Now decompose

$$w_0 \overline{n}_x = kan = \frac{1}{\sqrt{1+x^2}} \begin{pmatrix} x & 1 \\ -1 & x \end{pmatrix} \begin{pmatrix} \sqrt{x^2+1} & 0 \\ 0 & \frac{1}{\sqrt{1+x^2}} \end{pmatrix} \begin{pmatrix} 1 & \frac{x}{x^2+1} \\ 0 & 1 \end{pmatrix},$$

then applying f's equivariance properties, we arrive at

$$\int_{\mathbb{R}} f(w_0\overline{n}_x)\,dx = \int_{\mathbb{R}} (x+i)^{\frac{m-\mu-1}{2}}(x-i)^{\frac{-m-\mu-1}{2}}\,dx = \frac{2^{1-\mu}\pi i^m\Gamma(\mu)}{\Gamma(\frac{\mu-|m|+1}{2})\Gamma(\frac{\mu+|m|+1}{2})},$$

where in the last equality we used Lemma A.1. Now dividing by $\Gamma(\frac{\mu+\varepsilon}{2})$ and shuffling around Gamma-factors we arrive at the result. □

For $\mu \in i\mathbb{R}$ we equip the space $\mathrm{Ind}_P^G(\varepsilon\otimes e^{\mu}\otimes 1)$ with the usual L^2-norm. Using Proposition 1.3 we can for $\varepsilon = 0$ and $\mu \in 1-2\mathbb{N}$ equip $\mathrm{Ind}_P^G(0\otimes e^{\mu}\otimes 1)$ with the norm

$$\|f\|^2 = \int_K f(k)\overline{T_\mu^0 f(k)}\,dk.$$

Similarly, for $\varepsilon = 1$ and $\mu \in -2\mathbb{N}$ we can equip $\mathrm{Ind}_P^G(1\otimes e^{\mu}\otimes 1)$ with the norm

$$\|f\|^2 = \int_K f(k)\overline{\hat{T}_\mu^1 f(k)}\,dk$$

where

$$\hat{T}_\mu^1 f = \begin{cases} iT_\mu^1 f, & \text{for } m>0,\\ -iT_\mu^1 f, & \text{for } m<0 \end{cases}$$

for $f \in \mathrm{Ind}_P^G(1\otimes e^{\mu}\otimes 1)_m$. The operator $\hat{T}_\mu^1$ is still an intertwining operator as it vanishes on V_0 per Proposition 1.3, and thus we just altered it by a scalar on each of the summands in Proposition 1.2.

2 The homogeneous space G/H

For a unitary character $\chi_{\varepsilon,\lambda} = \varepsilon\otimes e^{\lambda}$ with $\lambda \in i\mathbb{R}$ the left-regular action $\tau_{\varepsilon,\lambda}$ of G on the space of L^2-sections associated to the line bundle $G\times_H \mathbb{C}_{\varepsilon,\lambda} \to G/H$, given by

$$\mathrm{Ind}_H^G(\varepsilon\otimes e^{\lambda}) = \left\{ f: G\to\mathbb{C},\ \textit{measurable} \mid f(gh) = \chi_{\varepsilon,\lambda}(h)^{-1}f(g),\ \int_{G/H}|f(g)|^2 d(gH) < \infty \right\},$$

defines a unitary representation of G. The goal of this paper is to decompose this space. Furthermore we consider the subspace of compactly supported smooth functions

$$C_c^\infty\text{-}\mathrm{Ind}_H^G(\varepsilon\otimes e^{\lambda}) = \left\{ f\in C^\infty(G)\cap \mathrm{Ind}_H^G(\varepsilon\otimes e^{\lambda}) \mid \mathrm{supp}(f)\subseteq\Omega,\ \Omega H \text{ is compact in } G/H \right\}$$

We will denote the smooth vectors in $\mathrm{Ind}_H^G(\varepsilon\otimes e^{\lambda})$ by $\mathrm{Ind}_H^G(\varepsilon\otimes e^{\lambda})^\infty$. We introduce the notation

$$b_u = \begin{pmatrix}\cosh u & \sinh u\\ \sinh u & \cosh u\end{pmatrix}, \quad \overline{n}_x = \begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}.$$

Using the decomposition $G = KBA$, where $B = \{b_u \mid u\in\mathbb{R}\}$, we consider G/H in the global coordinates $(\theta,u)\in[0,\pi)\times\mathbb{R}$ where $xH = k_\theta b_u H$ and the invariant measure is

$d(xH) = \cosh(2u)dud\theta$, see e.g. [M84]. Now in terms of these coordinates we have the K-type decomposition

$$C_c^\infty\text{-}\operatorname{Ind}_H^G(\varepsilon\otimes e^\lambda) = \widehat{\bigoplus_{m\in 2\mathbb{Z}+\varepsilon}} \mathbb{C}\zeta_m \otimes C_c^\infty(\mathbb{R}) \tag{2.1}$$

with $\zeta_m(k_\theta) = e^{im\theta}$. We let $\operatorname{Ind}_H^G(\varepsilon\otimes e^\lambda)_m$ denote the set of functions contained in the K-type given by $m \in 2\mathbb{Z}+\varepsilon$, that is $\operatorname{Ind}_H^G(\varepsilon\otimes e^\lambda)_m = \mathbb{C}\zeta_m \otimes C_c^\infty(\mathbb{R})$.

We denote by Δ_λ the Casimir operator for the representation $\operatorname{Ind}_H^G(\varepsilon\otimes e^\lambda)$ defined in a similar fashion as for the principal series.

Proposition 2.1. *Written in the coordinates (θ, u) the Casimir operator Δ_λ is given by*

$$\Delta_\lambda = \frac{\lambda^2}{\cosh^2(2u)} + 2\lambda\frac{\tanh(2u)}{\cosh(2u)}\partial_\theta + 2\tanh(2u)\partial_u - \frac{1}{\cosh^2(2u)}\partial_\theta^2 + \partial_u^2.$$

Proof. This is a standard computation. □

Another set of coordinates can be obtained by using the Iwasawa decomposition $G = KA\overline{N}$ with $(\theta, y) \in [0,\pi)\times\mathbb{R}$ where $xH = k_\theta\overline{n}_yH$. The invariant measure is given by $d(xH) = \frac{1}{2}dyd\theta$ see [K16, Chap. 5, §6].

3 Constructing an isomorphism

The goal of this section is to construct the explicit isomorphism in the following theorem, of which the proof was in large presented to us by Jan Frahm.

Theorem 3.1. *For $\nu, \lambda \in i\mathbb{R}$ the map*

$$☥_\lambda^\nu : \operatorname{Ind}_H^G(\varepsilon\otimes e^\lambda) \to \operatorname{Ind}_H^G(\varepsilon\otimes e^\nu)$$

given by

$$☥_\lambda^\nu f(g) = \frac{1}{\sqrt{\pi}2^{\frac{\lambda-\nu}{4}}}\frac{\Gamma\left(\frac{2+\nu-\lambda}{4}\right)}{\Gamma\left(\frac{\lambda-\nu}{4}\right)}\int_{\mathbb{R}} |x|^{\frac{\lambda-\nu}{2}-1} f(g\overline{n}_x)\,dx$$

defines a unitary isomorphism intertwining $\operatorname{Ind}_H^G(\varepsilon\otimes e^\lambda)$ and $\operatorname{Ind}_H^G(\varepsilon\otimes e^\nu)$.

To this extent we consider the minimal parabolic subgroup $\overline{P} = \overline{N}AM \subset G$ and let

$$\operatorname{Ind}_{MA}^P(\varepsilon\otimes e^\lambda) = \left\{ f : \overline{N}AM \to \mathbb{C} \,|\, f(gma) = \operatorname{sgn}(m)^\varepsilon a^{-\lambda-1}f(g),\ \int_{P/MA} |f(g)|^2\,dg < \infty \right\},$$

where $a^\lambda := e^{\lambda(X)}$ for $a = e^X$ and $\lambda \in \mathfrak{a}_\mathbb{C}^* \cong \mathbb{C}$.

Lemma 3.2 (Induction in stages)**.**

$$\operatorname{Ind}_{MA}^G(\varepsilon\otimes e^\lambda) \simeq \operatorname{Ind}_{MA\overline{N}}^G(\operatorname{Ind}_{MA}^{MA\overline{N}}(\varepsilon\otimes e^\lambda)),$$

where the map is given by $f \mapsto F$ where $F(g)(\overline{p}) = f(g\overline{p})$ and thus the inverse is given by $f(g) = F(g)(1)$.

Proof. See e.g. [G12, Chapter VI, section 9] □

Proof of Theorem 3.1. We first show the isomorphism claim of Theorem 3.1. By Lemma 3.2 it suffices to show that $\mathrm{Ind}_{MA}^{\overline{P}}(\varepsilon\otimes e^{\lambda}) \simeq \mathrm{Ind}_{MA}^{\overline{P}}(\varepsilon\otimes e^{\nu})$. Let $\mathrm{rest}_{\overline{N}} : \mathrm{Ind}_{MA}^{\overline{P}}(\varepsilon\otimes e^{\lambda}) \to L^2(\overline{N})$ be the restriction from $\overline{P}$ to $\overline{N}$. We let Φ be the inverse map which is given by $\Phi F(\overline{n}am) = \mathrm{sgn}(m)^{\varepsilon}a^{-\lambda-1}F(\overline{n})$. Let $\pi_{\varepsilon,\lambda}$ be the left regular representation on $\mathrm{Ind}_{MA}^{\overline{P}}(\varepsilon\otimes e^{\lambda})$ and define $\tilde{\pi}_{\varepsilon,\lambda}(g) = \mathrm{rest}_{\overline{N}} \circ \pi_{\varepsilon,\lambda}(g) \circ \Phi$. Then $\overline{P}$ acts on $L^2(\overline{N})$ via $\tilde{\pi}_{\varepsilon,\lambda}$, and the above statement reduces to showing that $\tilde{\pi}_{\varepsilon,\lambda} \cong \tilde{\pi}_{\varepsilon,\nu}$ for $\lambda, \nu \in i\mathbb{R}$.

To construct an isomorphism $H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ intertwining $\tilde{\pi}_{\varepsilon,\lambda}$ and $\tilde{\pi}_{\varepsilon,\nu}$ we note that the action of $P = \overline{N}AM$ on $f \in L^2(\overline{N})$ is given by

$$\begin{aligned} &\tilde{\pi}_{\varepsilon,\lambda}(\overline{n})f(\overline{n}') = f(\overline{n}^{-1}\overline{n}') && \overline{n}, \overline{n}' \in \overline{N},\\ &\tilde{\pi}_{\varepsilon,\lambda}(ma)f(\overline{n}) = \mathrm{sgn}(m)^{\varepsilon}a^{\lambda+1}f((ma)^{-1}\overline{n}(ma)), && m \in M,\ a \in A,\ \overline{n} \in \overline{N}. \end{aligned}$$

Identifying $\overline{N} \simeq \mathbb{R}$, $M \simeq \{\pm 1\}$ and $A \simeq \mathbb{R}_{>0}$, the above becomes

$$\begin{aligned} &\tilde{\pi}_{\varepsilon,\lambda}(y)f(x) = f(x-y), && x, y \in \mathbb{R},\\ &\tilde{\pi}_{\varepsilon,\lambda}(t)f(x) = t^{\lambda+1}f(t^2x), && x \in \mathbb{R},\ t \in \mathbb{R}_{>0}. \end{aligned}$$

Since $\overline{N}$ acts by translation any such intertwining operator H must be a translation invariant operator on $L^2(\mathbb{R})$, hence there exists some tempered distribution $u \in \mathcal{S}'(\mathbb{R})$ such that H is given by convolution with u, that is $HF(x) = \langle u, \tau_x\check{F}\rangle$, where $\tau_xF(y) = F(y-x)$ and $\check{F}(x) = F(-x)$. Furthermore let d_tf denote the dilation of f by $t \in \mathbb{R}\setminus\{0\}$, i.e. $d_tf(x) = f(tx)$, then

$$H \circ \tilde{\pi}_{\varepsilon,\lambda}(t)F(x) = \tilde{\pi}_{\varepsilon,\nu}(t) \circ HF(x).$$

Evaluating at $x = 0$ then yields $\langle d_{t^{-2}}u, \check{F}\rangle = t^{\nu-\lambda+2}\langle u, \check{F}\rangle$. Hence u is a homogeneous distribution of degree $\frac{\lambda-\nu-2}{2}$ and we conclude that

$$Hf = f * |x|_{\delta}^{\frac{\lambda-\nu-2}{2}} \qquad \text{for some } \delta \in \{0,1\}.$$

Comparing with Lemma B.2 we see that these are both necessary and sufficient conditions for H to establish an isomorphism between $\mathrm{Ind}_{MA}^{P}(\varepsilon\otimes e^{\lambda})$ and $\mathrm{Ind}_{MA}^{P}(\varepsilon\otimes e^{\nu})$. Putting $\delta = 0$, composing with the map from Lemma 3.2 then yields the desired isomorphism. To see that the normalization indeed makes $\dagger_{\lambda}^{\nu}$ unitary, it suffices to note that for $\lambda, \nu \in i\mathbb{R}$ Lemma B.2 gives

$$\left|\frac{1}{\sqrt{\pi}2^{\frac{\lambda-\nu}{4}}}\frac{\Gamma\left(\frac{2+\nu-\lambda}{4}\right)}{\Gamma\left(\frac{\lambda-\nu}{4}\right)}\mathcal{F}(|x|^{\frac{\lambda-\nu}{2}-1})\right| = 1.$$

□

4 Eigenfunctions for the Casimir operator

By Theorem 3.1 a Plancherel formula on $\mathrm{Ind}_H^G(\varepsilon\otimes e^{\lambda})$ for some fixed $\lambda \in i\mathbb{R}$ can be extended to all $\nu \in i\mathbb{R}$ by compositon with the unitary isomorphism $\dagger_{\lambda}^{\nu}$ from Theorem 3.1.

Following this, we will therefore mostly consider the cases of which $\lambda = 0$, which often simplifies matters considerably.

For $f \in \mathrm{Ind}_H^G(\varepsilon \otimes e^0)_m$ with $f(k_\theta b_u) = e^{im\theta} \cdot h(u),\ h \in C_c^\infty(\mathbb{R})$ we have

$$\Delta_0 f = e^{im\theta} \tilde{\Delta}_m h(u),$$

for some differential operator $\tilde{\Delta}_m$.

Lemma 4.1. *Let $h \in C_c^\infty(\mathbb{R})$ and $m \in \mathbb{Z}$. Then we have*

$$\tilde{\Delta}_m \cosh^{\frac{m}{2}}(2u) h(-\sinh^2(2u)) = \cosh^{\frac{m}{2}}(2u) \Box_m h(-\sinh^2(2u)).$$

For $t = -\sinh^2(2u)$ the operator $\Box_m$ is given by

$$\Box_m = m(m+2) + 8(-1 + (3+m)\,t)\frac{d}{dt} - 16t \cdot (1-t)\frac{d^2}{dt^2}.$$

Proof. Follows directly from Proposition 2.1. □

Recall that a hypergeometric differential equation has the form

$$t(1-t)\frac{d^2}{dt^2} + [c - (a+b+1)t]\frac{d}{dt} - ab = 0.$$

If c is not a non-positive integer there are two independent solutions (around $t = 0$)

$${}_2F_1(a,b;c;t) \quad \text{and} \quad t^{1-c}\,{}_2F_1(1+a-c, 1+b-c; 2-c; t),$$

expressed in terms of the hypergeometric function ${}_2F_1(a,b;c;t)$.

We note that the eigenvalue problem $\Box_m f = (\mu^2 - 1)f$ is a hypergeometric differential equation, thus giving us two linearly independent solutions φ_μ^m and ψ_μ^m. Using the notation from Appendix C, we can express these solutions as

$$\varphi_\mu^m(u) = \phi_{\frac{-\mu}{2}}^{-\frac{1}{2},\frac{m}{2}}(u), \quad \psi_\mu^m(u) = i\sinh(u) \cdot \phi_{\frac{-\mu}{2}}^{\frac{1}{2},\frac{m}{2}}(u),$$

where $u \in [0,\infty)$. Note that these functions allow for natural extensions from $[0,\infty)$ to $\mathbb{R}$. We now restate the results from Appendix C in terms of φ_μ^m and ψ_μ^m.

Proposition 4.2. *For $f \in C_c^\infty([0,\infty))$, let $(J_j f)(\mu)$, $j = 0,1$, denote the Fourier–Jacobi transforms of f given by*

$$J_0 f(\mu) = \int_0^\infty f(t)\varphi_\mu^m(t)\cosh^{m+1}(t)\,dt,$$
$$J_1 f(\mu) = \int_0^\infty f(t)\psi_\mu^m(t)\sinh(t)\cosh^{m+1}(t)\,dt.$$

Then we have the following inversion formulas

$$f(t) = \frac{1}{8\pi^2}\int_{i\mathbb{R}} J_0 f(\mu)\varphi_\mu^m(t)\frac{d\mu}{|\ell_0(\mu)|^2} - \frac{1}{2\pi}\sum_{\mu\in D_0} J_0 f(\mu)\varphi_\mu^m(t)\operatorname*{Res}_{\nu=\mu}(\ell_0(\nu)\ell_0(-\nu))^{-1}$$
$$\sinh(t)f(t) = \frac{-1}{2\pi^2}\int_{i\mathbb{R}} J_1 f(\mu)\psi_\mu^m(t)\frac{d\mu}{|\ell_1(\mu)|^2} + \frac{2}{\pi}\sum_{\mu\in D_1} J_1 f(\mu)\psi_\mu^m(t)\operatorname*{Res}_{\nu=\mu}(\ell_1(\nu)\ell_1(-\nu))^{-1}.$$

with $D_j = \{\eta \in \mathbb{R} \mid \eta = 4k + 1 + 2j - |m| < 0,\ k \in \mathbb{N}_0\}$ and

$$\ell_j(\mu) = \frac{\Gamma(\frac{\mu}{2})}{\Gamma(\frac{\mu+1+2j+|m|}{4})\Gamma(\frac{\mu+1+2j-|m|}{4})}.$$

Remark 4.3. *As φ_μ^m and ψ_μ^m are given in terms of hypergeometric functions we get*

$$\varphi_\mu^m = \varphi_{-\mu}^m, \quad \text{and} \quad \psi_\mu^m = \psi_{-\mu}^m,$$

as ${}_2F_1(a,b;c;t) = {}_2F_1(b,a;c;t)$. The Euler transformation ${}_2F_1(a,b;c;t) = (1-t)^{c-a-b}\,{}_2F_1(c-a, c-b;c;t)$ amounts to

$$\varphi_\mu^m(u) = \cosh^{-m}(u)\varphi_\mu^{-m}(u), \quad \text{and} \quad \psi_\mu^m(u) = \cosh^{-m}(u)\psi_\mu^{-m}(u).$$

5 Intertwining operators

To obtain an explicit Plancherel formula for representation theoretic purposes, we require expressions for intertwining operators between the representation spaces introduced in earlier sections. More explicitly we consider intertwining operators

$$P : \mathrm{Ind}_P^G(\delta \otimes e^\mu \otimes 1) \to \mathrm{Ind}_H^G(\varepsilon \otimes e^\lambda)^\infty,$$
$$A : C_c^\infty\text{–}\mathrm{Ind}_H^G(\varepsilon \otimes e^\lambda) \to \mathrm{Ind}_P^G(\delta \otimes e^\mu \otimes 1),$$

and their realizations in terms of the coordinates introduced in earlier sections. Such operators only exist when $\varepsilon = \delta$ as M lies in the center of G.

We fix $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ and supress it in the notation for the rest of this section. When ε appear in formulas we will consider it as number in $\{0,1\}$ where we will use the notation $[\,\cdot\,]_2$ when confusions can occur.

For $\xi \in \mathbb{Z}/2\mathbb{Z}$ and $\lambda \in i\mathbb{R}$ consider the kernel

$$K_{\lambda,\mu}^\xi(g) = |g_{11}|_{\xi+\varepsilon}^{\frac{\lambda+\mu-1}{2}} |g_{21}|_\xi^{\frac{\mu-\lambda-1}{2}}, \quad g \in G,$$

where g_{ij} is the (i,j)'th entry in G. As g_{11} and g_{21} does not simultaneously vanish in G this kernel enjoys many of the similar properties as a Riesz distribution (see Appendix B). $\Gamma(\frac{\mu+1}{2})^{-1}K_{\lambda,\mu}^\xi$ is locally integrable for $\mathrm{Re}\,(\mu) > -1$ and admits a holomorphic continuation as a distribution to $\mu \in \mathbb{C}$.

Proposition 5.1. *The map given by*

$$P_{\lambda,\mu}^\xi f(g) = \int_K K_{\lambda,\mu}^\xi(g^{-1}k) f(k)\, dk,$$

defines an intertwining operator $\mathrm{Ind}_P^G(\varepsilon \otimes e^\mu \otimes 1) \to \mathrm{Ind}_H^G(\varepsilon \otimes e^\lambda)^\infty$. Similarly the map given by

$$A_{\lambda,\mu}^\xi f(g) = \int_{G/H} K_{-\lambda,-\mu}^\xi(x^{-1}g) f(x)\, d(xH),$$

defines an intertwining operator $C_c^\infty\text{–}\mathrm{Ind}_H^G(\varepsilon \otimes e^\lambda) \to \mathrm{Ind}_P^G(\varepsilon \otimes e^\mu \otimes 1)$. Both integrals should be understood in the distributional sense.

Proof. The equivariance properties follows by direct verification. □

Proposition 5.2. *For $\xi,\varepsilon\in\mathbb{Z}/2\mathbb{Z}$ we have the following relation*

$$T_\mu^\varepsilon\circ A^\xi_{\lambda,\mu}=d^\xi_{\lambda,\mu}A^{\xi+\varepsilon}_{\lambda,-\mu},$$

where

$$d^\xi_{\lambda,\mu}=(-1)^{\lfloor\frac{\varepsilon+\xi}{2}\rfloor}\sqrt{\pi}\frac{\Gamma(\frac{1-\lambda-\mu+2[\xi+\varepsilon]_2}{4})\Gamma(\frac{1+\lambda-\mu+2\xi}{4})}{\Gamma(\frac{1+\lambda+\mu+2[\xi+\varepsilon]_2}{4})\Gamma(\frac{1-\lambda+\mu+2\xi}{4})\Gamma(\frac{1-\mu+\varepsilon}{2})}.$$

Proof. Fix $g\in G$ and put $z=x^{-1}g$. Then the set $\{xH\in G/H\mid z_{11}z_{21}=0\}$ is a $d(xH)$-null set. Using Lemma B.2 for $0<\operatorname{Re}(\mu)<1$ we get

$$\begin{aligned}\int_{\overline{N}}K^\xi_{-\lambda,-\mu}(zw_0\overline{n})d\overline{n}&=|z_{11}|_{\xi+\varepsilon}^{\frac{-\mu-\lambda-1}{2}}|z_{21}|_\xi^{\frac{\lambda-\mu-1}{2}}\int_{\mathbb{R}}|x|_{\xi+\varepsilon}^{\frac{-\lambda-\mu-1}{2}}|x-\frac{1}{z_{11}z_{21}}|_\xi^{\frac{\lambda-\mu-1}{2}}dx\\&=\Gamma(\frac{\mu+\varepsilon}{2})d^\xi_{\lambda,\mu}K^{\xi+\varepsilon}_{-\lambda,\mu}(z),\quad a.e.\end{aligned}$$

The claim then follows by analytic continuation. □

We introduce the notation

$$\omega^\xi_m=(-1)^\xi+(-1)^m i^m.$$

Note that $\omega^\xi_{-m}=\overline{\omega}^\xi_m$ and as $\varepsilon\equiv m \mod 2$ we have

$$\omega^\xi_m=\begin{cases}(-1)^\xi+(-1)^{\frac{m}{2}}, & \varepsilon=0,\\(-1)^\xi+i(-1)^{\frac{m+1}{2}}, & \varepsilon=1,\end{cases}\quad\text{and}\quad \omega^0_m\overline{\omega}^1_m=\begin{cases}0, & \varepsilon=0,\\2i(-1)^{\frac{m-1}{2}}, & \varepsilon=1.\end{cases}$$

Proposition 5.3. *For $\mu\in\mathbb{C}$ and $m\in\mathbb{Z}$ we have*

$$\frac{1}{\Gamma(\frac{\mu+1}{2})}P^\xi_\mu\zeta_m(k_\theta b_u)=\zeta_m(k_\theta)\cosh^{\frac{m}{2}}(2u)\left(\omega^\xi_m c_m(\mu)\varphi^m_\mu(2u)+\frac{i}{2}\omega^{\xi+1}_m c_m(\mu-2)\psi^m_\mu(2u)\right),$$

where

$$c_m(\mu)=\frac{2^{1-\mu}\pi e^{i\frac{m\pi}{4}}}{\Gamma(\frac{\mu+3+|m|}{4})\Gamma(\frac{\mu+3-|m|}{4})}.$$

Proof. As P^ξ_μ intertwines $\pi_{\varepsilon,\mu}$ and $\tau_{\varepsilon,0}$ it also intertwines the derived representations $d\pi_{\varepsilon,\mu}$ and $d\pi_{\varepsilon,0}$. Hence P^ξ_μ intertwines Δ_0 and Δ_μ and therefore the image of P^ξ_μ is contained in the eigenspace of Δ_0 to the eigenvalue μ^2-1 by Proposition 1.1. Fix μ with $\operatorname{Re}(\mu)>1$. From Lemma 4.1 it follows for generic μ that

$$P^\xi_\mu\zeta_m(k_\theta b_u)=\cosh^{\frac{m}{2}}(2u)\zeta_m(k_\theta)(a^\xi_m(\mu)\cdot\varphi^m_\mu(2u)+b^\xi_m(\mu)\cdot\psi^m_\mu(2u))$$

for some $a^\xi_m(\mu),b^\xi_m(\mu)\in\mathbb{C}$. Hence, it only remains to compute $a^\xi_m(\mu)$ and $b^\xi_m(\mu)$. Note that $\varphi^m_\mu(0)=1$ and $\psi^m_\mu(0)=0$, and hence $P^\xi_\mu\zeta_m(k_\theta)=\zeta_m(k_\theta)a^\xi_m(\mu)$ so

$$\begin{aligned}a^\xi_m(\mu)=P^\xi_\mu\zeta_m(e)&=\int_K K^\xi_\mu(k_\theta)\zeta_m(k_\theta)dk_\theta=2\int_0^\pi|\cos\theta|_{\xi+\varepsilon}^{\frac{\mu-1}{2}}|-\sin\theta|_\xi^{\frac{\mu-1}{2}}e^{im\theta}\,d\theta\\&=2^{\frac{1-\mu}{2}}\omega^\xi_m\int_0^\pi(\sin\theta)^{\frac{\mu-1}{2}}e^{i\frac{m}{2}\theta}d\theta=\frac{2^{1-\mu}\pi\omega^\xi_m e^{i\frac{m\pi}{4}}\Gamma(\frac{\mu+1}{2})}{\Gamma(\frac{\mu+3+|m|}{4})\Gamma(\frac{\mu+3-|m|}{4})},\end{aligned}$$

by Lemma A.4.

To compute $b_m^\xi(\mu)$ it suffices to note that $\frac{d}{du}\varphi_\mu^m(2u)\mid_{u=0}=0$ and $\frac{d}{du}\psi_\mu^m(2u)\mid_{u=0}=2i$, hence

$$\begin{aligned} 2i\cdot b_m^\xi(\mu) &= \frac{d}{du}P_\mu^\xi\zeta_m(b_u)\big|_{u=0} = \int_K \frac{d}{du}K_\mu^\xi(b_{-u}k_\theta)\big|_{u=0}\zeta_m(k_\theta)dk_\theta \\ &= \frac{1-\mu}{2}\int_K K_{\mu-2}^{\xi+1}(k_\theta)\zeta_m(k_\theta)dk_\theta \\ &= \frac{1-\mu}{2}a_m^{\xi+1}(\mu-2), \end{aligned}$$

from which the result follows by analytic continuation. □

Let $f\in\mathrm{Ind}_H^G(\varepsilon\otimes e^0)_m$ and write $f(k_\theta b_u)=\zeta_m(k_\theta)h(u)$ for some $h\in L^2(\mathbb{R},\cosh(2u)du)$. Now let $h_e(u)$ be the even part of h and $h_o(u)$ the odd part. We introduce the following notation

$$\begin{aligned} J_0f(\mu,\theta) &= \zeta_m(k_\theta)J_0(\cosh^{-\frac{m}{2}}(x)h_e(b_{\frac{x}{2}}))(\mu), \\ J_1f(\mu,\theta) &= \zeta_m(k_\theta)J_1(\sinh^{-1}(x)\cosh^{-\frac{m}{2}}(x)h_o(b_{\frac{x}{2}}))(\mu) \end{aligned}$$

where the x denotes the variable the Fourier–Jacobi transform is done with respect to.

Proposition 5.4. *Let $f\in\mathrm{Ind}_H^G(\varepsilon\otimes e^0)_m$ then*

$$\frac{1}{\Gamma(\frac{1-\mu}{2})}A_\mu^\xi f(k_\theta) = \frac{\overline{\omega}_m^\xi}{2}c_{-m}(-\mu)J_0f(\mu,\theta) + i\frac{\overline{\omega}_m^{\xi+1}}{4}c_{-m}(-\mu-2)J_1f(\mu,\theta).$$

Proof. As A_μ^ξ is an intertwining operator, it maps K-types to K-types, thus $A_\mu^\xi f(k_\theta) = A_\mu^\xi f(e)\times\zeta_m(k_\theta)$. Now

$$A_\mu^\xi f(e) = \int_0^\pi\int_{\mathbb{R}} K_{-\mu}^\xi(b_u^{-1}k_\theta^{-1})f(k_\theta b_u)\cosh(2u)dud\theta = \frac{1}{2}\int_{\mathbb{R}}\cosh(2u)h(u)P_{-\mu}^\xi\zeta_{-m}(b_u)du.$$

From Proposition 5.3 and Remark 4.3 we have

$$\begin{aligned} \frac{A_\mu^\xi f(e)}{\Gamma(\frac{1-\mu}{2})} &= \frac{1}{2}\int_{\mathbb{R}} h(u)\cosh^{-\frac{m}{2}+1}(2u)\Big(\omega_{-m}^\xi c_{-m}(-\mu)\varphi_{-\mu}^{-m}(2u) \\ &\qquad \times + \frac{i}{2}\omega_{-m}^{\xi+1}c_{-m}(-\mu-2)\psi_{-\mu}^{-m}(2u)\Big)du \\ &= \int_0^\infty h_e(u)\cosh^{\frac{m}{2}+1}(2u)\overline{\omega}_m^\xi c_{-m}(-\mu)\varphi_\mu^m(2u)du \\ &\quad + \frac{i}{2}\int_0^\infty h_o(u)\cosh^{\frac{m}{2}+1}(2u)\overline{\omega}_m^{\xi+1}c_{-m}(-\mu-2)\psi_\mu^m(2u)du \\ &= \frac{1}{2}\overline{\omega}_m^\xi c_{-m}(-\mu)(J_0\cosh^{-\frac{m}{2}}(u)h_e(\tfrac{x}{2}))(\mu) \\ &\quad + \frac{i}{4}\overline{\omega}_m^{\xi+1}c_{-m}(-\mu-2)(J_1\sinh^{-1}(x)\cosh^{-\frac{m}{2}}(x)h_o(\tfrac{x}{2}))(\mu). \end{aligned}$$ □

Combining Proposition 5.3 and Proposition 5.4 yields an explicit intertwining operator

$$\frac{P_\mu^\xi A_\mu^{\xi'}}{\Gamma(\frac{1+\mu}{2})\Gamma(\frac{1-\mu}{2})} : C_c^\infty\text{–}\mathrm{Ind}_H^G(\varepsilon\otimes e^0)_m \to \mathrm{Ind}_H^G(\varepsilon\otimes e^0)_m^\infty.$$

By Propositions 5.3 and 5.4, the above intertwining operator is holomorphic in μ, i.e. the above defines a holomorphic family of intertwining operators, intertwining $\mathrm{Ind}_H^G(\varepsilon\otimes e^0)_m$ with itself.

6 Combining intertwining operators

In this section we consider a function $f \in C_c^\infty\text{–}\mathrm{Ind}_H^G(\varepsilon \otimes e^0)_m$ and then write

$$f(k_\theta b_{\frac{u}{2}}) = \cosh^{\frac{m}{2}}(u)\Big[\big(\cosh^{-\frac{m}{2}}(u) f_e(k_\theta b_{\frac{u}{2}})\big) + \sinh(u)\big(\cosh^{-\frac{m}{2}}(u)\sinh^{-1}(u) f_o(k_\theta b_{\frac{u}{2}})\big)\Big],$$

then apply the two inversion formulas from Proposition 4.2 to each of the two terms giving

$$f(k_\theta b_{\frac{u}{2}}) = \cosh^{\frac{m}{2}}(u)\Big[\frac{1}{\pi^2}\int_{i\mathbb{R}} J_0 f(\mu,\theta)\varphi_\mu^m(u)\frac{d\mu}{8|\ell_0(\mu)|^2} - \frac{1}{\pi^2}\int_{i\mathbb{R}} J_1 f(\mu,\theta)\psi_\mu^m(u)\frac{d\mu}{2|\ell_1(\mu)|^2}$$
$$-\frac{1}{2\pi}\sum_{\mu\in D_0} J_0 f(\mu,\theta)\varphi_\mu^m(u)\operatorname*{Res}_{\nu=\mu}\big(\ell_0(\nu)\ell_0(-\nu)\big)^{-1} + \frac{2}{\pi}\sum_{\mu\in D_1} J_1 f(\mu,\theta)\psi_\mu^m(u)\operatorname*{Res}_{\nu=\mu}\big(\ell_1(\nu)\ell_1(-\nu)\big)^{-1}\Big]$$

The goal is then to express this decomposition in terms of some combination of the operators $P_\mu^\xi A_\mu^{\xi'} f(k_\theta b_{\frac{u}{2}})$ which by a quick glance at Propositions 5.3 and 5.4 appears plausible. The following identity will be used multiple times in the following subsections

$$\begin{aligned}2^4 c_m(\nu)c_{-m}(-\nu)\ell_0(\nu)\ell_0(-\nu) &= c_m(\nu-2)c_{-m}(-\nu-2)\ell_1(\nu)\ell_1(-\nu)\\ &= \frac{2^5(-1)^{1+\varepsilon}}{\pi}\frac{\cos^2\left(\frac{\pi(\nu+\varepsilon)}{2}\right)}{\nu\sin\left(\frac{\pi\nu}{2}\right)},\end{aligned} \tag{6.1}$$

which follows from Gamma-function identities and recalling that $m \equiv \varepsilon \mod 2$.

6.1 The continuous part

For $\mu \in \mathbb{C}$ we introduce the following maps

$$\mathbf{P}_\mu^\xi = \frac{P_\mu^\xi}{\Gamma(\frac{\mu+1}{2})}, \quad \mathbf{A}_\mu^\xi = \frac{A_\mu^\xi}{\Gamma(\frac{1-\mu}{2})},$$

which are holomorphic in μ.

Proposition 6.1. *We have*

$$\sum_{\xi=0}^{1}\mathbf{P}_\mu^\xi\mathbf{A}_\mu^\xi f(k_\theta b_u) = \cosh^{\frac{m}{2}}(2u)\Big[2c_m(\mu)c_{-m}(-\mu)J_0 f(\mu,\theta)\varphi_\mu^m(2u)$$
$$-\frac{1}{2}c_m(\mu-2)c_{-m}(-\mu-2)J_1 f(\mu,\theta)\psi_\mu^m(2u)\Big].$$

Combining this with (6.1) we get

$$\int_{i\mathbb{R}}\sum_{\xi=0}^{1}\mathbf{P}_\mu^\xi\mathbf{A}_\mu^\xi f(k_\theta b_{\frac{u}{2}})\frac{d\mu}{|a(\mu)|^2}$$
$$= \cosh^{\frac{m}{2}}(u)\Big[\frac{1}{\pi^2}\int_{i\mathbb{R}} J_0 f(\mu,\theta)\varphi_\mu^m(u)\frac{d\mu}{8|\ell_0(\mu)|^2} - \frac{1}{\pi^2}\int_{i\mathbb{R}} J_1 f(\mu,\theta)\psi_\mu^m(u)\frac{d\mu}{2|\ell_1(\mu)|^2}\Big],$$

where

$$a(\mu) = 4\pi \frac{\Gamma(\frac{\mu}{2})}{\Gamma(\frac{1+\mu+\varepsilon}{2})\Gamma(\frac{1+\mu-\varepsilon}{2})}.$$

Proof. When computing $\sum_{\xi=0}^{1} \mathbf{P}_\mu^\xi \mathbf{A}_\mu^\xi f(k_\theta b_{\frac{u}{2}})$ we apply Proposition 5.3 and 5.4. We obtain some cross-terms, containing factors like $J_0 f(\mu,\theta)\psi_\mu^m(u)$, but since

$$\sum_{\xi=0}^{1} \omega_m^\xi \overline{\omega}_m^{\xi+1} = 0 \text{ and } \sum_{\xi=0}^{1} \omega_m^\xi \overline{\omega}_m^\xi = 4,$$

no cross-terms survive and the assertion follows. □

To express the discrete part in terms of $P_\mu^\xi A_\mu^{\xi'} f$ is a bit more delicate as we cannot simply take a sum to make the cross terms disappear, thus we need to make a suitable choice of normalization. The cases for $\varepsilon = 0$ and $\varepsilon = 1$ will be treated differently and the main culprit as to why is the factor ω_m^ξ which for $\varepsilon = 0$ vanishes depending on the parity of $\frac{m}{2}$ and for $\varepsilon = 1$ never vanishes.

6.2 The discrete part for $\varepsilon = 0$

In this subsection we fix $\varepsilon = 0$. Consider the following normalizations

$$\widehat{P}_\mu^\xi = \frac{\Gamma(\frac{\mu+3-2\xi}{4})}{\Gamma(\frac{\mu+1}{2})\Gamma(\frac{\mu+1+2\xi}{4})} P_\mu^\xi, \quad \widehat{A}_\mu^\xi = \frac{\Gamma(\frac{-\mu+3-2\xi}{4})}{\Gamma(\frac{-\mu+1}{2})\Gamma(\frac{-\mu+1+2\xi}{4})} A_\mu^\xi,$$

which, by the duplication formula for the Gamma-function, does not introduce any poles. Now introduce the operators

$$\mathbb{P}_\mu := \widehat{P}_\mu^0 + \widehat{P}_\mu^1 \quad \text{and} \quad \mathbb{A}_\mu := \widehat{A}_\mu^0 + \widehat{A}_\mu^1.$$

Lemma 6.2. *For a fixed $m \in 2\mathbb{Z}$ we have*

$$\mathbb{P}_\mu \zeta_m(k_\theta b_u) = \zeta_m(k_\theta) \cosh^{\frac{m}{2}}(2u)\big(\alpha_m(\mu)\varphi_\mu^m(2u) + \beta_m(\mu)\psi_\mu^m(2u)\big),$$

where

$$\alpha_m(\mu) = c_m(\mu) \left(\omega_m^0 \frac{\Gamma(\frac{\mu+3}{4})}{\Gamma(\frac{\mu+1}{4})} + \omega_m^1 \frac{\Gamma(\frac{\mu+1}{4})}{\Gamma(\frac{\mu+3}{4})} \right),$$

and

$$\beta_m(\mu) = \frac{i}{2} c_m(\mu-2) \left(\omega_m^1 \frac{\Gamma(\frac{\mu+3}{4})}{\Gamma(\frac{\mu+1}{4})} + \omega_m^0 \frac{\Gamma(\frac{\mu+1}{4})}{\Gamma(\frac{\mu+3}{4})} \right).$$

Furthermore if $\mu \in 1-2\mathbb{N}$ then $\alpha_m(\mu)$ is only non-zero when $\mu \in D_0$ and $\beta_m(\mu)$ is only non-zero when $\mu \in D_1$.

Proof. The first identity is a direct consequence of Proposition 5.3. To see the second assertion rewrite

$$\alpha_m(\mu) = \frac{2^{1-\mu}\pi e^{\frac{\pi i m}{4}}}{\Gamma(\frac{\mu+3+|m|}{4})}\left(\omega_m^0 \frac{(\frac{\mu+3-|m|}{4})_{\frac{|m|}{4}}}{\Gamma(\frac{\mu+1}{4})} + \omega_m^1 \frac{(\frac{\mu+3-|m|}{4})_{\frac{|m|-2}{4}}}{\Gamma(\frac{\mu+3}{4})}\right).$$

As either ω_m^0 or ω_m^1 is vanishing this makes sense term by term. When μ is of the form $\mu = 4k + 3 - |m|$ for $k \in \mathbb{Z}$ then term by term $\Gamma(\frac{\mu+1}{4})^{-1}$ and $\Gamma(\frac{\mu+3}{4})^{-1}$ vanishes. When μ has the form $\mu = 4k + 1 - |m|$ for $k \in -\mathbb{N}$ then $\Gamma(\frac{\mu+3+|m|}{4})^{-1}$ vanishes. A similarly argument applies to $\beta_m(\mu)$. □

Lemma 6.3. *We have*

$$\mathbb{A}_\mu f(k_\theta) = \tilde{\alpha}_m(\mu) J_0 f(\mu,\theta) + \tilde{\beta}_m(\mu) J_1 f(\mu,\theta),$$

where

$$\tilde{\alpha}_m(\mu) = \frac{1}{2}c_{-m}(-\mu)\left(\overline{\omega}_m^0 \frac{\Gamma(\frac{3-\mu}{4})}{\Gamma(\frac{1-\mu}{4})} + \overline{\omega}_m^1 \frac{\Gamma(\frac{1-\mu}{4})}{\Gamma(\frac{3-\mu}{4})}\right),$$

$$\tilde{\beta}_m(\mu) = \frac{i}{4}c_{-m}(-\mu-2)\left(\overline{\omega}_m^1 \frac{\Gamma(\frac{-\mu+3}{4})}{\Gamma(\frac{-\mu+1}{4})} + \overline{\omega}_m^0 \frac{\Gamma(\frac{-\mu+1}{4})}{\Gamma(\frac{-\mu+3}{4})}\right).$$

Furthermore, if $\mu \in 1 - 2\mathbb{N}$ of the form $\mu = 4k + 1 - |m|$ we have $\tilde{\beta}(\mu) = 0$ and similarly for μ of the form $\mu = 4k + 3 - |m|$ we have $\tilde{\alpha}_m(\mu) = 0$.

Proof. This follows from Proposition 5.4 and considerations similar to those in the proof of Lemma 6.2. □

Lemma 6.4. *For $\mu \in D_0$:*

$$\mathbb{P}_\mu \mathbb{A}_\mu f(k_\theta b_u) = \cosh^{\frac{m}{2}}(2u)\alpha_m(\mu)\tilde{\alpha}_m(\mu) J_0 f(\mu,\theta)\varphi_\mu^m(2u),$$

and for $\mu \in D_1$:

$$\mathbb{P}_\mu \mathbb{A}_\mu f(k_\theta b_u) = \cosh^{\frac{m}{2}}(2u)\beta_m(\mu)\tilde{\beta}_m(\mu) J_1 f(\mu,\theta)\psi_\mu^m(2u).$$

Furthermore, if $\mu \in (1 - 2\mathbb{N}) \setminus (D_0 \cup D_1)$ then $\mathbb{P}_\mu \mathbb{A}_\mu f(k_\theta b_u) = 0$.

Proof. This is a direct consequence of the two preceding lemmas. □

Consider the non-vanishing entire analytic function

$$\mathfrak{T}(\mu) = \frac{1}{\Gamma(\frac{1+\mu}{4})^2\Gamma(\frac{1-\mu}{4})^2} + \frac{1}{\Gamma(\frac{3+\mu}{4})^2\Gamma(\frac{3-\mu}{4})^2}.$$

Lemma 6.5. *For $\mu \in D_0$*

$$(-2\pi)\alpha_m(\mu)\tilde{\alpha}_m(\mu)\ell_0(\mu)\ell_0(-\mu) = 16\pi^2 \frac{\cot\left(\frac{\pi\mu}{2}\right)\mathfrak{T}(\mu)}{\mu}.$$

For $\mu \in D_1$

$$\frac{\pi}{2}\beta_m(\mu)\tilde{\beta}_m(\mu)\ell_1(\mu)\ell_1(-\mu) = 16\pi^2 \frac{\cot\left(\frac{\pi\mu}{2}\right)\mathfrak{T}(\mu)}{\mu}.$$

Proof. This follows from (6.1). One trick is used which arises when a term like

$$\frac{|\omega_m^0|^2}{\Gamma(\frac{\mu+1}{2})^2},$$

is obtained. As ω_m^0 is either 0 or 2 we can set $\omega_m^0 = 2$ as $\Gamma(\frac{\mu+1}{2})^{-1}$ vanishes in the same cases as ω_m^0. □

Proposition 6.6. *We have*

$$\frac{1}{(2\pi)^3} \sum_{\mu \in 1-2\mathbb{N}} \frac{-\mu}{\mathfrak{F}(\mu)} \mathbb{P}_\mu \mathbb{A}_\mu f(k_\theta b_u) = \cosh^{\frac{m}{2}}(2u)$$
$$\times \left[\frac{-1}{2\pi} \sum_{\mu \in D_0} J_0 f(\mu, \theta) \varphi_\mu^m(2u) \operatorname*{Res}_{\nu=\mu} (\ell_0(\nu)\ell_0(-\nu))^{-1} \right.$$
$$\left. + \frac{2}{\pi} \sum_{\mu \in D_1} J_1 f(\mu, \theta) \psi_\mu^m(2u) \operatorname*{Res}_{\nu=\mu} (\ell_1(\nu)\ell_1(-\nu))^{-1} \right].$$

Proof. Apply Lemma 6.4 to the right hand side. Now, note that $c_m(\mu)c_{-m}(-\mu)$ is regular for $\mu \in D_0$ and $c_m(\mu-2)c_{-m}(-\mu-2)$ is regular for $\mu \in D_1$, thus they can be moved inside the residues. Then everything follows from Lemma 6.5 after recalling that $\operatorname{Res}_{\nu=\mu} \tan(\frac{\pi\nu}{2}) = -\frac{2}{\pi}$. □

6.3 The discrete part for $\varepsilon = 1$

In this subsection we fix $\varepsilon = 1$. The proof will proceed using the same ideas as for $\varepsilon = 0$. For $\mu \in -2\mathbb{N}$ let

$$\mathcal{A}_\mu = \frac{1}{\Gamma(\frac{1-\mu}{2})} A_\mu^0, \quad \text{and} \quad \mathcal{P}_\mu \zeta_m = \frac{(-1)^{\frac{m+|m|-2}{2}}}{\Gamma(\frac{1+\mu}{2})} P_\mu^1 \zeta_m,$$

that is we define $\mathcal{P}_\mu$ by its eigenvalues on K-types. By Proposition 5.3 we get $\mathcal{P}_\mu$ is intertwining by the same argument we used for $\hat{T}_\mu^1$ in section 1.

Lemma 6.7. *For $\mu \in D_0$ we have*

$$\mathcal{P}_\mu \mathcal{A}_\mu f(k_\theta b_u) = \alpha_m(\mu) \cosh^{\frac{m}{2}}(2u) \varphi_\mu^m(2u) J_0 f(\mu, \theta),$$

where

$$\alpha_m(\mu) = i(-1)^{\frac{|m|+1}{2}} c_m(\mu) c_{-m}(-\mu)$$

For $\mu \in D_1$ we have

$$\mathcal{P}_\mu \mathcal{A}_\mu f(k_\theta b_u) = \beta_m(\mu) \cosh^{\frac{m}{2}}(2u) \psi_\mu^m(2u) J_1 f(\mu, \theta),$$

where

$$\beta_m(\mu) = \frac{1}{4} i(-1)^{\frac{|m|+1}{2}} c_m(\mu-2) c_{-m}(-\mu-2).$$

Furthermore if $\mu \in -2\mathbb{N}$ then $\alpha_m(\mu)$ is only non-zero if $\mu \in D_0$ and $\beta_m(\mu)$ is only non-zero if $\mu \in D_1$.

Proof. The proof is an application of Propositions 5.3 and 5.4. □

Lemma 6.8. *For $\mu \in D_0$*

$$(-2\pi)\alpha_m(\mu)\ell_0(\mu)\ell_0(-\mu) = 4i(-1)^{\frac{|m|-1}{2}} \frac{\sin\left(\frac{\pi\mu}{2}\right)}{\mu},$$

and for $\mu \in D_1$

$$\frac{\pi}{2}\beta_m(\mu)\ell_1(\mu)\ell_1(-\mu) = 4i(-1)^{\frac{|m|+1}{2}} \frac{\sin\left(\frac{\pi\mu}{2}\right)}{\mu}.$$

Proof. This is a direct consequence of (6.1). □

Proposition 6.9. *We have*

$$\frac{1}{2\pi i}\sum_{\mu\in-2\mathbb{N}} \mu\mathcal{P}_\mu\mathcal{A}_\mu f(k_\theta b_u) = \cosh^{\frac{m}{2}}(2u)\Big[-\frac{1}{2\pi}\sum_{\mu\in D_0} J_0 f(\mu,\theta)\varphi_\mu^m(2u)\operatorname*{Res}_{\nu=\mu}(\ell_0(\nu)\ell_0(-\nu))^{-1} + \frac{2}{\pi}\sum_{\mu\in D_1} J_1 f(\mu,\theta)\psi_\mu^m(2u)\operatorname*{Res}_{\nu=\mu}(\ell_1(\nu)\ell_1(-\nu))^{-1}\Big].$$

Proof. This follows in the same manner as the proof for Proposition 6.6, where we here note for $\mu = 4k+1-|m| \in D_0$ that $\mathrm{Res}_{\nu=\mu}\sin\left(\frac{\pi\nu}{2}\right)^{-1} = \frac{2}{\pi}(-1)^{\frac{|m|-1}{2}}$, and for $\mu \in D_1$ we have $\mathrm{Res}_{\nu=\mu}\sin\left(\frac{\pi\nu}{2}\right)^{-1} = \frac{2}{\pi}(-1)^{\frac{|m|+1}{2}}$. □

7 The Plancherel formula

The intertwining operators P_μ^ξ and A_μ^ξ are continuous maps and hence the intertwining operators introduced in the previous section are also continuous. This allows for an extension of the results obtained for K-types, described by the first theorem of this section. We then extend this theorem to arbitrary $\lambda \in i\mathbb{R}$ by virtue of Theorem 3.1

Recall that

$$a(\mu) = 4\pi\frac{\Gamma(\frac{\mu}{2})}{\Gamma(\frac{1+\mu+\varepsilon}{2})\Gamma(\frac{1+\mu-\varepsilon}{2})}, \quad \text{and} \quad \mathcal{b}(\mu) = \frac{1}{\Gamma(\frac{1+\mu}{4})^2\Gamma(\frac{1-\mu}{4})^2} + \frac{1}{\Gamma(\frac{3+\mu}{4})^2\Gamma(\frac{3-\mu}{4})^2}. \tag{7.1}$$

Theorem 7.1 (Plancherel formula for $\lambda = 0$)**.** *For $\varepsilon = 0$ and $f \in C_c^\infty\text{–}\mathrm{Ind}_H^G(\varepsilon\otimes e^0)$ we have the following inversion formula*

$$f(k_\theta b_u) = \int_{i\mathbb{R}}\sum_{\xi=0}^{1}\boldsymbol{P}_\mu^\xi\boldsymbol{A}_\mu^\xi f(k_\theta b_u)\frac{d\mu}{|a(\mu)|^2} + \frac{1}{(2\pi)^3}\sum_{\mu\in 1-2\mathbb{N}}\frac{-\mu}{\mathcal{b}(\mu)}\mathbb{P}_\mu\mathbb{A}_\mu f(k_\theta b_u),$$

and the corresponding Plancherel formula

$$\|f\|^2 = \int_{i\mathbb{R}}\sum_{\xi=0}^{1}\|\boldsymbol{A}_\mu^\xi f\|^2\frac{d\mu}{|a(\mu)|^2} + \frac{1}{(2\pi)^3}\sum_{\mu\in 1-2\mathbb{N}}\frac{2\Gamma(1-\mu)}{\Gamma(\frac{-\mu}{2})\,\mathcal{b}(\mu)}\|\mathbb{A}_\mu f\|^2.$$

For $\varepsilon = 1$ *and* $f \in C_c^\infty\text{--}\mathrm{Ind}_H^G(\varepsilon \otimes e^0)$ *we have the following inversion formula*

$$f(k_\theta b_u) = \int_{i\mathbb{R}} \sum_{\xi=0}^{1} \boldsymbol{P}_\mu^\xi \boldsymbol{A}_\mu^\xi f(k_\theta b_u) \frac{d\mu}{|a(\mu)|^2} + \frac{1}{2\pi i} \sum_{\mu \in -2\mathbb{N}} \mu \mathcal{P}_\mu \mathcal{A}_\mu f(k_\theta b_u),$$

and the corresponding Plancherel formula

$$\|f\|^2 = \int_{i\mathbb{R}} \sum_{\xi=0}^{1} \|\boldsymbol{A}_\mu^\xi f\|^2 \frac{d\mu}{|a(\mu)|^2} + \frac{1}{2\pi} \sum_{\mu \in -2\mathbb{N}} \frac{\Gamma(-\mu)\mu^2}{\Gamma(\frac{1-\mu}{2})} \|\mathcal{A}_\mu f\|^2.$$

Proof. The inversion formulas follow directly from the introduction and results of Section 6. To get the Plancherel formula write

$$\|f\|^2 = \int_0^\pi \int_{\mathbb{R}} f(k_\theta b_u)\overline{f(k_\theta b_u)} \cosh(2u)\, du d\theta,$$

and use the inversion formula on $f(k_\theta b_u)$ and apply that

$$\int_{G/H} P_\mu^\xi f(xH)\overline{g(xH)}\, d(xH) = \int_K f(k)\overline{A_{-\overline{\mu}}^\xi g(k)}\, dk,$$

for $f \in \mathrm{Ind}_P^G(\varepsilon \otimes e^\mu)$ and $g \in \mathrm{Ind}_H^G(\varepsilon \otimes e^0)$. Lastly for the discrete part, we apply Proposition 5.2 to get

$$T_\mu^0 \mathbb{A}_\mu = \frac{\sqrt{\pi}2^\mu}{\Gamma(\frac{1-\mu}{2})} \mathbb{A}_{-\mu} \quad \text{and} \quad T_\mu^1 \mathcal{A}_\mu = \frac{\sqrt{\pi}2^\mu}{\Gamma(\frac{2-\mu}{2})\Gamma(\frac{1+\mu}{2})} A_{-\mu}^1,$$

giving the final result. □

We now extend the previous result from $\lambda = 0$ to $\lambda \in i\mathbb{R}$ using Theorem 3.1. We want to compose A_μ and $☥_\lambda^0$ but as we cannot ensure the regularity of the functions in the image of $☥_\lambda^0$ we end up doing this in an L^2-sense using direct integrals. Consider the following operators

$$\mathbb{A}_{\lambda,\mu} := \frac{2^{\frac{1+\mu}{2}}\sqrt{\pi}\Gamma(\frac{1+\mu}{4} + \frac{\lambda}{4})}{\Gamma(\frac{1-\mu}{4})\Gamma(\frac{1+\mu}{4})\Gamma(\frac{1-\mu}{4} - \frac{\lambda}{4})} A_{\lambda,\mu}^0 + \frac{2^{\frac{1+\mu}{2}}\sqrt{\pi}\Gamma(\frac{3+\mu}{4} + \frac{\lambda}{4})}{\Gamma(\frac{3-\mu}{4})\Gamma(\frac{3+\mu}{4})\Gamma(\frac{3-\mu}{4} - \frac{\lambda}{4})} A_{\lambda,\mu}^1,$$

$$\mathbf{A}_{\lambda,\mu}^\xi := \frac{A_{\lambda,\mu}^\xi}{\Gamma(\frac{1-\mu}{2})}, \quad \text{and} \quad \mathcal{A}_{\lambda,\mu} := \frac{2^{\frac{1+\mu}{2}}\sqrt{\pi}\Gamma(\frac{1+\mu}{4} + \frac{\lambda}{4})}{\Gamma(\frac{1-\mu}{4})\Gamma(\frac{3+\mu}{4})\Gamma(\frac{1-\mu}{4} - \frac{\lambda}{4})} A_{\lambda,\mu}^0,$$

which are extensions of $\mathbb{A}_\mu$, $\mathbf{A}_\mu^\xi$ and $\mathcal{A}_\mu$ e.g. $\mathbb{A}_{0,\mu} = \mathbb{A}_\mu$. Furthermore let

$$\mathcal{H}_\varepsilon = \int_{i\mathbb{R}}^\oplus \pi_{\varepsilon,\mu} \otimes \mathbb{C}^2\, d\mu \oplus \bigoplus_{\mu \in 1-\varepsilon-2\mathbb{N}} \pi_{\varepsilon,\mu}^{\mathrm{ds}}.$$

where $\int_U^\oplus H_\mu d\mu$ denotes a direct integral of Hilbert spaces, see e.g [F22] for a short exposition. The inner-product on $\mathcal{H}_\varepsilon$ is given by Theorem 7.1, i.e. for $\varepsilon = 0$ and $f, h \in \mathcal{H}_0$

$$\langle g, h\rangle_{\mathcal{H}} = \int_{i\mathbb{R}} \sum_{\xi=0}^{1} \langle g_\mu^\xi, h_\mu^\xi\rangle_{L^2(K)} \frac{d\mu}{|a(\mu)|^2} + \frac{1}{16\pi} \sum_{\mu \in 1-2\mathbb{N}} \frac{\Gamma(1-\mu)}{\Gamma(\frac{-\mu}{2})\, \mathfrak{Z}(\mu)} \langle T_\mu^0 g_\mu, h\rangle_{L^2(K)}.$$

For simplicity we shall assume that $\varepsilon = 0$ for the remainder of the section. All arguments made can be done for $\varepsilon = 1$ as well using the corresponding results from the previous section.

Abusing notation, Theorem 7.1 defines an isometry $A_0 : C_c^\infty(G/H) \to \mathcal{H}_0$ which extends to an isometry

$$A_0 : \mathrm{Ind}_H^G(\varepsilon \otimes 1) \to \mathcal{H}_0.$$

For $f \in \mathcal{H}_0$ with $f = (f^0, f^1, f^d)$ we introduce the following map

$$P_0 : \mathcal{H}_0 \to L^2(G/H)$$
$$f = (f^0, f^1, f^d) \mapsto \int_{i\mathbb{R}} \sum_{\xi=0}^{1} \mathbf{P}_\mu^\xi f_\mu^\xi \frac{d\mu}{|a(\mu)|^2} + \frac{1}{(2\pi)^3} \sum_{\mu \in 1-2\mathbb{N}} \frac{-\mu}{\mathfrak{F}(\mu)} \mathbb{P}_\mu f_\mu^d.$$

Lemma 7.2. *For $f \in \mathrm{Ind}_H^G(\varepsilon \otimes e^0)$ and $h \in \mathcal{H}_0$ we have the following relation:*

$$\langle A_0 f, h\rangle_{\mathcal{H}} = \langle f, P_0 h\rangle_{L^2(G/H)}.$$

Proof. Let $f \in \mathrm{Ind}_H^G(\varepsilon \otimes e^\lambda)$ and $h \in C_c^\infty(\mathcal{H}_0)$. We then find

$$\begin{aligned}
\langle A_0 f, h\rangle_{\mathcal{H}} &= \sum_{\xi=0}^{1} \int_{i\mathbb{R}} \int_K \mathbf{A}_\mu^\xi f(k) \overline{h_\mu^\xi(k)} dk \frac{d\mu}{|a(\mu)|^2} + \frac{1}{(2\pi)^3} \sum_{\mu \in 1-2\mathbb{N}} \frac{2\Gamma(1-\mu)}{\Gamma(\frac{-\mu}{2}) \mathfrak{F}(\mu)} \langle T_\mu^0 \circ \mathbb{A}_\mu f, h_\mu^\xi\rangle \\
&= \sum_{\xi=0}^{1} \int_{i\mathbb{R}} \int_{G/H} f(x) \overline{\mathbf{P}_\mu^\xi h_\mu^\xi(x)} d(xH) \frac{d\mu}{|a(\mu)|^2} + \frac{1}{(2\pi)^3} \sum_{\mu \in 1-2\mathbb{N}} \frac{2\Gamma(1-\mu)}{\Gamma(\frac{-\mu}{2}) \mathfrak{F}(\mu)} \frac{\sqrt{\pi} 2^\mu}{\Gamma(\frac{1-\mu}{2})} \langle \mathbb{A}_{-\mu} f, h_\mu^d\rangle \\
&= \sum_{\xi=0}^{1} \int_{i\mathbb{R}} \langle f, \mathbf{P}_\mu^\xi h_\mu^\xi\rangle_{L^2(G/H)} \frac{d\mu}{|a(\mu)|^2} + \frac{1}{(2\pi)^3} \sum_{\mu \in 1-2\mathbb{N}} \frac{-\mu}{\mathfrak{F}(\mu)} \langle f, \mathbb{P}_\mu h_\mu^d\rangle_{L^2(G/H)} \\
&= \langle f, P_0 h\rangle_{L^2(G/H)}.
\end{aligned}$$

□

Lemma 7.3 (See [P76, Theorem 1])**.** *Suppose $S : \mathcal{H}^\infty \to \mathcal{H}^\infty$ is a continuous intertwining operator for $\mathcal{H}^\infty$. Then for a.e $\mu \in i\mathbb{R} \cup (1-\varepsilon-2\mathbb{N})$ there exists unique $\mathcal{H}^\infty$ intertwining operators S_μ for $\pi_{\varepsilon,\mu}^\infty \otimes \mathbb{C}^2$ if $\mu \in i\mathbb{R}$ and for $\pi_{\varepsilon,\mu}^{\mathrm{ds}}$ if $\mu \in 1-\varepsilon-2\mathbb{N}$ such that*

$$(Sf)_\mu = S_\mu f_\mu \quad a.e\ \mu \in i\mathbb{R} \cup (\mu \in 1-\varepsilon-2\mathbb{N}),\ f \in \mathcal{H}^\infty.$$

Proposition 7.4. *The map $A_0^\varepsilon : \mathrm{Ind}_H^G(\varepsilon \otimes e^0) \to \mathcal{H}_0$ is surjective. In particular A_0 is an isometric isomorphism.*

Proof. Since the discrete and continuous part of $\mathcal{H}_0$ consist of pairwise inequivalent representations of G, it suffices to show that the projection of $W = \overline{\mathrm{Image}(A_0)}$ onto the continuous part and the discrete part respectively, is surjective. For the projection to the discrete part, we can consider the projection of W onto each summand $\pi_{0,\mu}^{\mathrm{ds}}$. Proposition 1.2 gives $\pi_{0,\mu}^{\mathrm{ds}} = \pi_{0,\mu}^{\mathrm{hol}} \oplus \pi_{0,\mu}^{\mathrm{ahol}}$ and since these representations are inequivalent it again suffices to show that the projection on each of them are onto. Lemma 6.3 then shows that $\mathrm{proj}_{\pi_{0,\mu}^{\mathrm{hol}}}(W) \neq 0$ and $\mathrm{proj}_{\pi_{0,\mu}^{\mathrm{ahol}}}(W) \neq 0$. But since the projection is G-equivariant the image is a subrepresentation and it follows that both projections must be onto.

Since the projection onto an integrand of the continuous part of $\mathcal{H}_0$ is in general not point-wise defined, the proof differs to that of the discrete part. Abusing notation slightly we shall write $f_\mu = (f^0_\mu, f^1_\mu)$ when $\mu \in i\mathbb{R}$ and $f \in \mathcal{H}_0$, omitting the discrete part.

By Lemma 7.2 $(A_0)^* = P_0$ and since the adjoint is G-equivariant we have

$$P_0(\mathcal{H}_0^\infty) \subseteq L^2(G/H)^\infty \quad \text{and} \quad A_0(L^2(G/H)^\infty) \subseteq \mathcal{H}_0^\infty.$$

Let $A_0^\infty = A_0|_{L^2(G/H)^\infty}$ and $P_0^\infty = P_0|_{\mathcal{H}_0^\infty}$. Then

$$S = A_0^\infty \circ P_0^\infty : \mathcal{H}_0^\infty \to \mathcal{H}_0^\infty$$

is a $\mathcal{H}_0^\infty$ intertwining map and by Lemma 7.3 there exists a family of $\mathcal{H}_0^\infty$ intertwining operators (S_μ) such that $((A_0^\infty \circ P_0^\infty)f)_\mu = S_\mu f_\mu$ for a.e $\mu \in i\mathbb{R}$ and all $f \in \mathcal{H}_0^\infty$. By Schur's lemma this implies $S_\mu = (\mathrm{id} \otimes B_\mu)$ for a.e $\mu \in i\mathbb{R}$, with $B_\mu : \mathbb{C}^2 \to \mathbb{C}^2$ a linear map. Let N denote the corresponding null-set, we then show that for $\mu \in i\mathbb{R} \setminus N$ we have $S_\mu = \mathrm{id}$.

To this extent let $\mu \in i\mathbb{R} \setminus N$ and let f be a K-finite vector in $\mathcal{H}_0^\infty$ and note that this implies f_μ must be a K-finite vector in π_μ^∞. Assume therefore without loss of generality that $f_\mu = (c_1\zeta_m, c_2\zeta_n)$ for some $m, n \in 2\mathbb{Z}$ and pick by Proposition 5.4 a K-finite vector in $L^2(G/H)^\infty$ such that $A_0(w)_\mu = f_\mu$. One can e.g pick w of the form $w = \zeta_m f_1 + \zeta_n f_2$ with f_1 an even function and f_2 an odd function of correct regularity and growth. Then we have

$$(A_0^\infty \circ P_0^\infty f)_\mu = A_0^\infty \circ P_0^\infty \circ A_0^\infty(w)_\mu = A_0^\infty(w)_\mu = (A^0_\mu w, A^1_\mu w),$$

where the second equality follows from the inversion formula given by Theorem 7.1. On the other hand we have

$$\begin{aligned}(A_0^\infty \circ P_0^\infty f)_\mu = S_\mu f_\mu &= (\mathrm{id} \otimes B_\mu) A_0^\infty(w)_\mu \\ &= ((b_{11})_\mu A^0_\mu w + (b_{12})_\mu A^1_\mu w, (b_{21})_\mu A^0_\mu w + (b_{22})_\mu A^1_\mu w)\end{aligned}$$

hence

$$(A^0_\mu w, A^1_\mu w) = (a_\mu A^0_\mu w + b_\mu A^1_\mu w, c_\mu A^0_\mu w + d_\mu A^1_\mu w).$$

Since A^0_μ and A^1_μ are linearly independent for $\mu \in i\mathbb{R}$ it follows that $B_\mu = \mathrm{id}$ and hence $S_\mu = \mathrm{id}$ on the K-finite vectors of $\mathcal{H}_0$, for a.e $\mu \in i\mathbb{R}$ and since the K-finite vectors form a dense subset the result follows. □

Theorem 7.5. *For $\lambda \in i\mathbb{R}$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ we have*

$$\mathrm{Ind}_H^G(\varepsilon \otimes e^\lambda) \cong \int_{i\mathbb{R}}^{\oplus} \pi_{\varepsilon,\mu} \oplus \pi_{\varepsilon,\mu} \frac{d\mu}{|a(\mu)|^2} \oplus \bigoplus_{\mu \in 1-\varepsilon-2\mathbb{N}} \pi^{\mathrm{hol}}_{\varepsilon,\mu} \oplus \pi^{\mathrm{ahol}}_{\varepsilon,\mu},$$

where the map is given by

$$\begin{aligned} f &\mapsto (p^0_{\lambda,\mu}\boldsymbol{A}^0_{\lambda,\mu} f, p^1_{\lambda,\mu}\boldsymbol{A}^1_{\lambda,\mu} f, \mathbb{A}_{\lambda,\mu} f), &\quad &\textit{for } f \in \mathrm{Ind}_H^G(0 \otimes e^\lambda) \\ f &\mapsto (p^0_{\lambda,\mu}\boldsymbol{A}^0_{\lambda,\mu} f, p^1_{\lambda,\mu}\boldsymbol{A}^1_{\lambda,\mu} f, \mathcal{A}_{\lambda,\mu} f), &\quad &\textit{for } f \in \mathrm{Ind}_H^G(1 \otimes e^\lambda)\end{aligned}$$

with

$$p^\xi_{\lambda,\mu} = \frac{\Gamma\left(\frac{1-\mu+2\xi}{4}\right)\Gamma\left(\frac{\mu-\lambda+1+2\xi}{4}\right)}{2^{\frac{\lambda}{4}}\Gamma\left(\frac{1+\mu+2\xi}{4}\right)\Gamma\left(\frac{\lambda-\mu+1+2\xi}{4}\right)}.$$

Proof. For $\lambda \in i\mathbb{R}$ the map $A_0 : \mathrm{Ind}_H^G(\varepsilon \otimes 1) \to \mathcal{H}_0$ gives rise to an isometric isomorphism $\mathrm{Ind}_H^G(\varepsilon \otimes e^\lambda) \to \mathcal{H}_0$ by composition with $\text{☥}_\lambda^0 : \mathrm{Ind}_H^G(\varepsilon \otimes e^\lambda) \to \mathrm{Ind}_H^G(\varepsilon \otimes e^0)$ from Theorem 3.1. Let $\lambda \in i\mathbb{R}$, $f \in \mathrm{Ind}_H^G(\varepsilon \otimes e^\lambda)$ and $h \in C_c^\infty(\mathcal{H}_0)$. Then by Lemma 7.2 we have

$$\begin{aligned}
&\langle A_0 \circ \text{☥}_\lambda^0 f, h\rangle_{\mathcal{H}} = \langle \text{☥}_\lambda^0 f, P_0 h\rangle_{L^2(G/H)} \\
&= \sum_{\xi=0}^{1} \int_{i\mathbb{R}} \langle \text{☥}_\lambda^0 f, \mathbf{P}_\mu^\xi h_\mu^\xi\rangle_{L^2(G/H)} \frac{d\mu}{|a(\mu)|^2} + \frac{1}{(2\pi)^3} \sum_{\mu \in 1-2\mathbb{N}} \frac{-\mu}{\mathfrak{b}(\mu)} \langle \text{☥}_\lambda^0 f, \mathbb{P}_\mu h_\mu^d\rangle_{L^2(G/H)} \\
&= \sum_{\xi=0}^{1} \int_{i\mathbb{R}} \int_K \int_{G/H} \text{☥}_\lambda^0 f(x) \frac{K_{-\mu}^\xi(x^{-1}k)}{\Gamma(\frac{1-\mu}{2})} \overline{h_\mu^\xi(k)} d(xH) dk \frac{d\mu}{|a(\mu)|^2} \\
&\quad + \frac{1}{(2\pi)^3} \sum_{\mu \in 1-2\mathbb{N}} \frac{-\mu}{\mathfrak{b}(\mu)} \langle \text{☥}_\lambda^0 f, \mathbb{P}_\mu h_\mu^d\rangle_{L^2(G/H)}.
\end{aligned}$$

Using the coordinates $xH = k_\theta \overline{n}_y H$ and applying Lemma B.2 in the distributional sense, we find

$$\begin{aligned}
&\frac{1}{\Gamma(\frac{1-\mu}{2})} \int_{G/H} K_{-\mu}^\xi(x^{-1}k)\, \text{☥}_\lambda^0 f(x) d(xH) \\
&= \frac{\Gamma(\frac{2-\lambda}{4})}{\sqrt{\pi} 2^{\frac{\lambda}{4}} \Gamma(\frac{1-\mu}{2}) \Gamma(\frac{\lambda}{4})} \int_0^\pi \int_{\mathbb{R}} |\cos\theta|_\varepsilon^{-\mu-1} f(gk_\theta \overline{n}_z) \int_{\mathbb{R}} |x|_\varepsilon^{\frac{-\lambda-2}{2}} |z - \tan\theta - x|_\xi^{\frac{-\mu-1}{2}} \frac{1}{2} dx dz d\theta \\
&= p_{\lambda,\mu}^\xi \mathbf{A}_{\lambda,\mu}^\xi f(k),
\end{aligned}$$

An analogous calculation applies to the discrete part after applying that

$$\mathbb{A}_{-\mu} = \frac{\Gamma(\frac{1-\mu}{2})}{\sqrt{\pi} 2^\mu} T_\mu^0 \mathbb{A}_\mu.$$

In conclusion we find

$$\begin{aligned}
\langle A_0 \circ \text{☥}_\lambda^0 f, h\rangle_{\mathcal{H}} = \sum_{\xi=0}^{1} \int_{i\mathbb{R}} \langle p_{\lambda,\mu}^\xi \mathbf{A}_{\lambda,\mu}^\xi f, h_\mu^\xi\rangle \frac{d\mu}{|a(\mu)|^2} \\
+ \frac{1}{(2\pi)^3} \sum_{\mu \in 1-2\mathbb{N}} \frac{2\Gamma(1-\mu)}{\Gamma(\frac{-\mu}{2})\, \mathfrak{b}(\mu)} \langle T_\mu^0 \mathbb{A}_{\lambda,\mu} f, h_\mu^d\rangle.
\end{aligned}$$

Hence $A_0 \circ \text{☥}_\lambda^0 = A_\lambda$ with $A_\lambda : \mathrm{Ind}_H^G(\lambda \otimes e^\lambda) \to \mathcal{H}_0$ given by

$$\langle A_\lambda f, h\rangle_{\mathcal{H}} = \sum_{\xi=0}^{1} \int_{i\mathbb{R}} \langle p_{\lambda,\mu}^\xi \mathbf{A}_{\lambda,\mu}^\xi f, h_\mu^\xi\rangle \frac{d\mu}{|a(\mu)|^2} + \frac{1}{(2\pi)^3} \sum_{\mu \in 1-2\mathbb{N}} \frac{2\Gamma(1-\mu)}{\Gamma(\frac{-\mu}{2})\, \mathfrak{b}(\mu)} \langle T_\mu^0 \mathbb{A}_{\lambda,\mu} f, h_\mu^d\rangle$$

∎

Corollary 7.6. *For $\varepsilon = 0$ and $\mathrm{Ind}_H^G(\varepsilon \otimes e^\lambda)$ we have the following Plancherel formula*

$$\|f\|^2 = \int_{i\mathbb{R}} \sum_{\xi=0}^{1} \|\mathbf{A}_{\lambda,\mu}^\xi f\|^2 \frac{d\mu}{|a(\mu)|^2} + \frac{1}{(2\pi)^3} \sum_{\mu \in 1-2\mathbb{N}} \frac{2\Gamma(1-\mu)}{\Gamma(\frac{-\mu}{2})\, \mathfrak{b}(\mu)} \|\mathbb{A}_{\lambda,\mu} f\|^2.$$

For $\varepsilon = 1$ and $f \in \mathrm{Ind}_H^G(\varepsilon \otimes e^\lambda)$ we have the following Plancherel formula

$$\|f\|^2 = \int_{i\mathbb{R}} \sum_{\xi=0}^{1} \|\mathbf{A}_{\lambda,\mu}^\xi f\|^2 \frac{d\mu}{|a(\mu)|^2} + \frac{1}{2\pi} \sum_{\mu \in -2\mathbb{N}} \frac{\mu^2 \Gamma(-\mu)}{\Gamma(\frac{1-\mu}{2})} \|\mathcal{A}_{\lambda,\mu} f\|^2.$$

with $a(\mu)$ and $\mathfrak{b}(\mu)$ given by (7.1).

Proof. Since $|p^{\xi}_{\lambda,\mu}| = 1$ for $\lambda, \mu \in i\mathbb{R}$ the assertion follows. ☐

A Integral formulas

Lemma A.1 (See [C20, Proposition 16.8]). *For* $\mathrm{Re}\,(\alpha+\beta) > 0$ *we have*

$$\int_{\mathbb{R}} \frac{dx}{(x-i)^{\alpha}(x+i)^{\beta}} = \frac{2^{2-\alpha-\beta}\pi i^{\alpha-\beta}\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\Gamma(\beta)}$$

Lemma A.2 (See [GR94, Section 3.631]). *For* $\mathrm{Re}\,\nu > 0$ *we have*

$$\int_0^{\pi} \sin^{\nu-1}(x)\cos(ax)\,dx = \frac{2^{1-\nu}\pi\cos\left(\frac{a\pi}{2}\right)\Gamma(\nu)}{\Gamma\left(\frac{\nu+1-a}{2}\right)\Gamma\left(\frac{\nu+1+a}{2}\right)}.$$

Lemma A.3 (See [GR94, Section 3.631]). *For* $\mathrm{Re}\,\nu > 0$ *we have*

$$\int_0^{\pi} \sin^{\nu-1}(x)\sin(ax)\,dx = \frac{2^{1-\nu}\pi\sin\left(\frac{a\pi}{2}\right)\Gamma(\nu)}{\Gamma\left(\frac{\nu+1-a}{2}\right)\Gamma\left(\frac{\nu+1+a}{2}\right)}.$$

Lemma A.4. *For* $\mathrm{Re}\,\nu > 0$ *we have*

$$\int_0^{\pi} \sin^{\nu-1}(x)e^{iax}\,dx = \frac{2^{1-\nu}\pi e^{\frac{ai\pi}{2}}\Gamma(\nu)}{\Gamma\left(\frac{\nu+1-a}{2}\right)\Gamma\left(\frac{\nu+1+a}{2}\right)}.$$

Lemma A.5 (See [GR94, Section 3.251]). *For* $\mathrm{Re}\,\beta > -1$ *and* $\mathrm{Re}\,(\alpha+\beta) < -1$ *we have*

$$\int_1^{\infty} x^{\alpha}(x-1)^{\beta}\,dx = B\big(-\alpha-\beta-1, \beta+1\big).$$

Lemma A.6 (See [GR94, Section 3.194]). *For* $\mathrm{Re}\,\beta > -1$ *and* $\mathrm{Re}\,(\alpha+\beta) < -1$ *we have*

$$\int_0^{\infty} x^{\alpha}(x+1)^{\beta}\,dx = B\big(-\alpha-\beta-1, \alpha+1\big).$$

B The Fourier Transform and Riesz distributions

Define the Fourier transform of $\varphi \in C_c(\mathbb{R})$ as

$$\mathcal{F}[\varphi](\xi) = \int_{\mathbb{R}} \varphi(x)e^{ix\xi}\,dx$$

which makes the inversion formula $\mathcal{F}\mathcal{F}[\varphi](x) = 2\pi\varphi(-x)$. Extend this to distributions in the usual way.

For $\alpha \in \mathbb{C}$ with $\mathrm{Re}\,\alpha > -1$ and $\varepsilon \in \{0,1\}$ the function

$$u^{\varepsilon}_{\alpha}(x) = \frac{1}{2^{\frac{\alpha}{2}}\Gamma\left(\frac{\alpha+1+\varepsilon}{2}\right)}|x|^{\alpha}_{\varepsilon},$$

is locally integrable and can thus be considered as a distribution.

Lemma B.1. *The family of distributions u_α^ε extends analytically to a holomorpic family in $\alpha \in \mathbb{C}$. For $\alpha = 1-\varepsilon-2n \in 1-\varepsilon-2\mathbb{N}$ we have*

$$u_{1-\varepsilon-2n}^\varepsilon(x) = \frac{(-1)^{n+\varepsilon-1}(n-1)!}{2^{\frac{1-\varepsilon}{2}-n}(2n+\varepsilon-2)!}\delta^{(2n+\varepsilon-2)}(x),$$

where $\delta(x)$ is the Dirac δ–function.

Lemma B.2. *For $\alpha \in \mathbb{C}$ we have*

$$\mathcal{F}[u_\alpha^\varepsilon] = \sqrt{2\pi} i^\varepsilon u_{-\alpha-1}^\varepsilon.$$

Furthermore for $\alpha, \beta \in \mathbb{C}$ with $\operatorname{Re}\alpha, \operatorname{Re}\beta > -1$ and $\operatorname{Re}(\alpha+\beta) < -1$ we get

$$\int_{\mathbb{R}} u_\alpha^\varepsilon(x) u_\beta^\xi(y-x)\,dx = (-1)^{\lfloor\frac{\varepsilon+\xi}{2}\rfloor}\sqrt{2\pi}\frac{\Gamma(\frac{-1-\alpha-\beta+[\varepsilon+\xi]_2}{2})}{\Gamma(\frac{-\alpha+\varepsilon}{2})\Gamma(\frac{-\beta+\xi}{2})} u_{\alpha+\beta+1}^{\varepsilon+\xi}(y),$$

for $y \neq 0$.

Proof. The first assertion can be found in [GS64, p.170]. For $y \neq 0$ we have

$$\int_{\mathbb{R}} |x|_\varepsilon^\alpha |y-x|_\xi^\beta\,dx = |y|_{\xi+\varepsilon}^{\alpha+\beta+1}\int_{\mathbb{R}} |x|_\varepsilon^\alpha|1-x|_\xi^\beta\,dx,$$

by change of variables. Now writing

$$\int_{\mathbb{R}} |x|_\varepsilon^\alpha|1-x|_\xi^\beta\,dx = (-1)^\varepsilon\int_0^\infty x^\alpha(1+x)^\beta\,dx + \int_0^1 x^\alpha(1-x)^\beta\,dx + (-1)^\xi\int_1^\infty x^\alpha(x-1)^\beta\,dx,$$

we can use the integral formula for the Beta-function, apply Lemma A.5, A.6 and arrive at

$$(-1)^\varepsilon B(\alpha+1,-\alpha-\beta-1) + B(\alpha+1,\beta+1) + (-1)^\xi B(\beta+1,-\alpha-\beta-1).$$

Now rewriting the Beta-function in terms of the Gamma-function, applying Euler's reflection formula for the Gamma-function, and factoring out common factors we get

$$\pi^{-1}\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(-\alpha-\beta-1)\Big[\sin((\alpha+\beta)\pi) + (-1)^{\varepsilon+1}\sin(\beta\pi) + (-1)^{\xi+1}\sin(\alpha\pi)\Big].$$

We can now apply the identity

$$\begin{aligned}&\sin((\alpha+\beta)\pi) + (-1)^{\varepsilon+1}\sin(\beta\pi) + (-1)^{\xi+1}\sin(\alpha\pi)\\ &\quad = 4(-1)^{\lfloor\frac{1+\varepsilon+\xi}{2}\rfloor}\sin\Big(\frac{(\alpha+\varepsilon)\pi}{2}\Big)\sin\Big(\frac{(\beta+\xi)\pi}{2}\Big)\sin\Big(\frac{(\alpha+\beta+2+[\varepsilon+\xi]_2)\pi}{2}\Big),\end{aligned}$$

which can be verified on a case by case basis depending on $\varepsilon, \xi \in \{0,1\}$. Lastly rewrite the sine-functions as Gamma-functions using Euler's reflection formula and cancel out Gamma-functions case by case for $\varepsilon, \xi \in \{0,1\}$. □

C Fourier-Jacobi transform

This section is a condensed form of [FJ77, Appendix 1]. For $\alpha, \beta \in \mathbb{C}$ with $\alpha \notin -\mathbb{N}$ and $\operatorname{Re} \beta > -1$, define the Fourier–Jacobi transform of $f \in C_c^\infty(\mathbb{R}_{\geq 0})$ by

$$J_{\alpha,\beta} f(\mu) = \int_0^\infty f(t)\phi_\mu^{\alpha,\beta}(t) \sinh^{2\alpha+1}(t) \cosh^{2\beta+1}(t)\, dt,$$

where $\phi_\mu^{\alpha,\beta}$ are the Jacobi functions given by

$$\phi_\mu^{\alpha,\beta}(t) = {}_2F_1\left(\frac{\alpha+\beta+1+\mu}{2}, \frac{\alpha+\beta+1-\mu}{2}; \alpha+1; -\sinh^2(t)\right).$$

Then we have the following inversion formula:

$$f(t) = \frac{1}{4\pi}\int_{i\mathbb{R}} J_{\alpha,\beta} f(\mu)\phi_\mu^{\alpha,\beta}(t)\frac{d\mu}{|c_{\alpha,\beta}(\mu)|^2} - \sum_{\mu \in D_{\alpha,\beta}} J_{\alpha,\beta} f(\mu)\phi_\mu^{\alpha,\beta}(t) \operatorname*{Res}_{\nu=\mu}\big(c_{\alpha,\beta}(\nu)c_{\alpha,\beta}(-\nu)\big)^{-1}$$

where

$$c_{\alpha,\beta}(\mu) = \frac{\Gamma(\mu)\Gamma(\alpha+1)}{\Gamma\big(\frac{\alpha+|\beta|+1+\mu}{2}\big)\Gamma\big(\frac{\alpha-|\beta|+1+\mu}{2}\big)},$$

and

$$D_{\alpha,\beta} = \{x \in \mathbb{R} \mid k \in \mathbb{N}_0,\ x = 2k+1+\alpha-|\beta| < 0\}.$$

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Paper B

Analytic continuation of symmetry breaking operators of the pair $(\mathrm{GL}(n+1,\mathbb{R}),\mathrm{GL}(n,\mathbb{R}))$

Jonathan Ditlevsen

1 Introduction

A continuous representation π of a real reductive Lie group G defines a representation of a closed subgroup H by restriction. Given an irreducible representation τ of H, we can consider the so-called *symmetry breaking operators*, that is elements of $\mathrm{Hom}_H(\pi|_H,\tau)$. Following Kobayashi's ABC-program [K15] the symmetry breaking operators play an essential role for studying restrictions of representations of reductive groups.

The classification and construction of symmetry breaking operators between spherical principal series representations, has been studied in the past for certain pairs of groups (G,H), such as $(O(n+1,1),O(n,1))$ by Kobayashi and Speh [KS15] which later was extended to all strongly spherical reductive pairs (G,H) where both G and H have real rank one by Frahm and Weiske[FW20]. The real rank two case of $(O(1,n)\times O(1,n),O(1,n))$ was considered by Clerc [C16].

In this paper we consider the pair $(G,H)=(\mathrm{GL}(n+1,\mathbb{R}),\mathrm{GL}(n,\mathbb{R}))$ of real rank $n+1$ and n. We study symmetry breaking operators between principal series representations $\pi_{\xi,\lambda}$ and $\tau_{\eta,\nu}$ of G and H respectively. The main results of this paper concerns a family of symmetry breaking operators $\mathbf{A}^{\eta,\nu}_{\xi,\lambda}\in\mathrm{Hom}_H(\pi_{\xi,\lambda}|_H,\tau_{\eta,\nu})$ which is holomorphic in its paramters. These constitute "most" of the symmetry breaking operators as in [F21] it is shown that $\dim\mathrm{Hom}_H(\pi_{\xi,\lambda}|_H,\tau_{\eta,\nu})\geq 1$ and equal to one for generic parameters.

1.1 Methods and results

For $1\leq p\leq n+1$ and $1\leq q\leq n$ define the following polynomial functions for $g\in\mathrm{GL}(n+1,\mathbb{R})$:

$$\Phi_p(g)=\det((w_0g)_{1\leq i,j\leq p}),\quad \Psi_q(g)=\det((w_0g)_{2\leq i\leq q+1,1\leq j\leq q}),$$

where w_0 is a representative for the longest Weyl-group element given by

$$w_0=\begin{pmatrix} & & 1\\ & \iddots & \\ 1 & & \end{pmatrix}.$$

Consider the functions

$$K^{\eta,\nu}_{\xi,\lambda}(g) = |\Phi_{n+1}(g)|^{\lambda_{n+1}+\frac{n}{2}}_{\xi_{n+1}} \prod_{i=1}^{n} |\Phi_i(g)|^{\lambda_i-\nu_{n+1-i}-\frac{1}{2}}_{\xi_i+\eta_{n+1-i}} |\Psi_i(g)|^{\nu_{n+1-i}-\lambda_{i+1}-\frac{1}{2}}_{\eta_{n+1-i}+\xi_{i+1}},$$

which are locally integrable for (λ,ν) in some open subset of $\mathbb{C}^{n+1}\times\mathbb{C}^n$. Here we used the notation

$$|x|^{\mu}_{\xi} := \operatorname{sgn}(x)^{\xi}|x|^{\mu}, \quad x\in\mathbb{R}^{\times}, \mu\in\mathbb{C}, \xi\in\mathbb{Z}/2\mathbb{Z}.$$

For (ξ,η) in $(\mathbb{Z}/2\mathbb{Z})^{n+1}\times(\mathbb{Z}/2\mathbb{Z})^n$ we normalize $K^{\eta,\nu}_{\xi,\lambda}$ by $\mathbf{K}^{\eta,\nu}_{\xi,\lambda} = n(\xi,\lambda,\eta,\nu)^{-1}K^{\eta,\nu}_{\xi,\lambda}$ where

$$n(\xi,\lambda,\eta,\nu) = \prod_{j=1}^{n}\Big[\prod_{i=1}^{n+1-j}\Gamma\Big(\frac{\lambda_i-\nu_j+\frac{1}{2}+[\xi_i+\eta_j]}{2}\Big)\Big]\Big[\prod_{i=n+2-j}^{n+1}\Gamma\Big(\frac{\nu_j-\lambda_i+\frac{1}{2}+[\xi_i+\eta_j]}{2}\Big)\Big]. \tag{1.1}$$

Theorem 1.1. *The family of distributions $\mathbf{K}^{\eta,\nu}_{\xi,\lambda}$ can be analytically extended such that it depends holomorphically on (λ,ν) in all of $\mathbb{C}^{n+1}\times\mathbb{C}^n$.*

In [F21] it is shown that $K^{\eta,\nu}_{\xi,\lambda}$ satisfies equivariance properties such that it defines an integral kernel for a symmetry breaking operator between principal series representations of $\pi_{\xi,\lambda}$ of $\mathrm{GL}(n+1,\mathbb{R})$, and $\tau_{\eta,\nu}$ of $\mathrm{GL}(n,\mathbb{R})$. In this light Theorem 1.1 shows that we have an explicit holomorphic family of symmetry breaking operators $\mathbf{A}^{\eta,\nu}_{\xi,\lambda}\in\mathrm{Hom}_H(\pi_{\xi,\lambda}|_H,\tau_{\eta,\nu})$.

Alternatively, Theorem 1.1 can be phrased in terms of L-factors or Euler–factors. For a character $\chi_{\varepsilon,\mu}(x) = |x|^{\mu}_{\varepsilon}$ of $\mathrm{GL}(1,\mathbb{R})$ the L-factor is given by

$$L(s,\chi_{\varepsilon,\mu}) = \pi^{-\frac{1}{2}(\mu+\varepsilon+s)}\Gamma\Big(\frac{\mu+\varepsilon+s}{2}\Big).$$

By fixing $\chi_i = \chi_{\xi_i,\lambda_i}$ and $\psi_j = \chi_{\eta_j,\nu_j}$ we can write $\pi_{\xi,\lambda}$ as $\mathrm{Ind}^G_{P_G}(\chi_1\otimes\cdots\otimes\chi_{n+1})$ and $\tau_{\eta,\nu}$ as $\mathrm{Ind}^H_{P_H}(\psi_1\otimes\cdots\otimes\psi_n)$ and then the normalization of (1.1) is

$$\prod_{j=1}^{n} L(\tfrac{1}{2},\chi_1\psi_j^{-1})\dots L(\tfrac{1}{2},\chi_{n+1-j}\psi_j^{-1})L(\tfrac{1}{2},\chi^{-1}_{n+2-j}\psi_j)\dots L(\tfrac{1}{2},\chi^{-1}_{n+1}\psi_j),$$

up to some power of π.

We review how to analytically extend the Riesz distribution since the proof of Theorem 1.1 use many of the same ideas. For $\mathrm{Re}(\mu) > -1$ and $\varepsilon\in\mathbb{Z}/2\mathbb{Z}$ consider the locally integrable function

$$|x|^{\mu}_{\varepsilon} = \operatorname{sgn}(x)^{\varepsilon}|x|^{\mu}, \quad x\in\mathbb{R}^{\times}. \tag{1.2}$$

Step 1: Considering $|x|^{\mu}_{\varepsilon}$ as a distribution we have the Bernstein–Sato identity

$$\frac{d}{dx}|x|^{\mu}_{\varepsilon} = \mu|x|^{\mu-1}_{\varepsilon-1},$$

that relates $|x|^{\mu}_{\varepsilon}$ to itself with shifted parameters via a differential operator. Now for $\mathrm{Re}(\mu) > -1$ consider the distribution

$$u_{\varepsilon,\mu}(x) = \frac{|x|^{\mu}_{\varepsilon}}{\Gamma(\mu+1)}.$$

For all $\varphi \in C_c^\infty(\mathbb{R})$ the map $\mu \mapsto \langle u_{\varepsilon,\mu}, \varphi\rangle$ is holomorphic and by $\frac{d}{dx}u_{\varepsilon,\mu} = u_{\varepsilon-1,\mu-1}$ it analytically extends to all $\mu \in \mathbb{C}$.

Step 2: Notice that as $u_{0,0} = 1$, we get $\frac{d}{dx}u_{0,0} = 0$, implying that the extension of $u_{\varepsilon,\mu}$ is identically zero when μ is either a negative even integer (if $\varepsilon = 0$) or negative odd integer (if $\varepsilon = 1$). Therefore the normalization $\Gamma(\mu+1)^{-1}$ has more zeroes than is necessary for the analytical extension. As a result we get the family of distributions

$$\widehat{u}_{\varepsilon,\mu} = \frac{|x|_\varepsilon^\mu}{\Gamma(\frac{\mu+1+\varepsilon}{2})}, \quad \varepsilon \in \mathbb{Z}/2\mathbb{Z},$$

is holomorphic in $\mu \in \mathbb{C}$.

Step 3: For $\mathrm{Re}(\mu) > -1$ we have $x|x|_\varepsilon^\mu = |x|_{\varepsilon+1}^{\mu+1}$ and after normalization both sides of the equality extends analytically, and so a similar identity is obtained for $\widehat{u}_{\varepsilon,\mu}$. For $n \in \mathbb{Z}_{>0}$ we get

$$x^{2n}\widehat{u}_{\varepsilon,\mu} = \frac{\Gamma(\frac{\mu+2n+1+\varepsilon}{2})}{\Gamma(\frac{\mu+1+\varepsilon}{2})}\widehat{u}_{\varepsilon,\mu+2n} = \Big(\frac{\mu+1+\varepsilon}{2}\Big)_n \widehat{u}_{\varepsilon,\mu+2n},$$

where $(x)_n$ is the Pochhammer symbol. Thus when $\mu = 1-\varepsilon-2k$ we get that the support of $\widehat{u}_{\varepsilon,\mu}$ is contained in $\{0\}$. This implies that $\hat{u}_{\varepsilon,\mu}$ can be expressed as a linear combination of derivatives of the Dirac delta function, and by comparing even/odd-ness and homogeneity of the distributions we get

$$\widehat{u}_{\varepsilon,\mu}\Big|_{\mu=-1-\varepsilon-2n} = \alpha_{k,\varepsilon}\delta^{2k-\varepsilon}(x),$$

where $\alpha_{k,\varepsilon}$ is a constant which can be found by testing $\widehat{u}_{\varepsilon,\mu}|_{\mu=-1-\varepsilon-2n}$ against a test-function.

Analogously to Step 1, a collection of Bernstein–Sato identities are established. The method for finding these is inspired by [BC12], whereby conjugating multiplication operators of Φ_p or Ψ_q by Knapp–Stein intertwining operators we obtain explicit differential operators that relate $K_{\xi,\lambda}^{\eta,\nu}$ to itself with parameters shifted (See Theorem 6.5). Using these Bernstein–Sato identities we find a renormalization $\mathbb{K}_{\xi,\lambda}^{\eta,\nu}$ of $K_{\xi,\lambda}^{\eta,\nu}$ such that it can be analytically continued as a family of distributions. We find the following functional identities:

Theorem 1.2 (See Section 5)**.** *Composing $\mathbb{K}_{\xi,\lambda}^{\eta,\nu}$ by normalized Knapp–Stein intertwining operators $\boldsymbol{T}_w$ (normalized to be holomorphic) gives functional equations of the type*

$$\mathbb{K}_{\xi,\lambda}^{\eta,\nu} = \beta_w(\xi,\lambda,\eta,\nu)\boldsymbol{T}_w \circ \mathbb{K}_{w(\xi,\lambda)}^{\eta,\nu},$$

where $\beta_w(\xi,\lambda,\eta,\nu)$ is found explicitly as a ratio of Gamma-functions or L-factors.

Considering these functional identities we see that $\mathbf{T}_w \circ \mathbb{K}_{w(\xi,\lambda)}^{\eta,\nu}$ is holomorphic so the zeroes of $\beta_w(\xi,\lambda,\eta,\nu)$ is also zeroes for $\mathbb{K}_{\xi,\lambda}^{\eta,\nu}$. This gives a collection of zeroes just like in Step 2, so we can re-normalize $\mathbb{K}_{\xi,\lambda}^{\eta,\nu}$ to $\mathbf{K}_{\xi,\lambda}^{\eta,\nu}$, giving Theorem 1.1.

Lastly, we are able to extract some information about the support of $\mathbf{K}_{\xi,\lambda}^{\eta,\nu}$ for the parameters where the normalization have zeroes by the same arguments as Step 3. Unfortunately a similar argument to reduce to a sum of Dirac functions does not apply as the support does not reduce to a single point.

1.2 Notation

Let $\mathbb{1}$ be the vector having 1's in all entries. $e_i \in \mathbb{R}^k$ is the i'th standard basis vector. $E_{i,j}$ is the matrix with zeroes in all entries except the $(i,j)'th$ entry which contains a one. $|x|^{\mu}_{\varepsilon} := \operatorname{sgn}(x)^{\varepsilon}|x|^{\mu}$. For $\xi, \nu \in \mathbb{Z}/2\mathbb{Z}$ and $z \in \mathbb{C}$ we write $z + [\xi + \nu]$ to mean the addition $\xi + \nu$ should be done in $\mathbb{Z}/2\mathbb{Z}$ first and then either a 0 or 1 should be added to z.

2 Principal Series representations and symmetry breaking operators

Let G be a real reductive Lie group, and $\mathfrak{g}$ its Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let K be the maximal compact subgroup given by the elements invariant to the Cartan involution. Fix $\mathfrak{a} \subseteq \mathfrak{p}$ a maximal abelian subspace and put $A = \exp(\mathfrak{a})$. We consider the restricted roots of $(\mathfrak{g}, \mathfrak{a})$ and introduce an ordering allowing us to define $\mathfrak{n}$ as the sum of the positive root spaces and $N = \exp(\mathfrak{n})$. Furthermore we put M to be the centralizer of A in K.

Now $P = MAN \subseteq G$ is a minimal parabolic subgroup defined by its Langlands decomposition. Let (ξ, V_ξ) be a finite-dimensional representation of M and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ a character. Then $(\xi \otimes e^{\lambda} \otimes 1, V_\xi)$ is a finite-dimensional representation of $P = MAN$ where 1 is the trivial representation of N. Using smooth parabolic induction we obtain the principal series representation $\pi_{\xi,\lambda} = \operatorname{Ind}_P^G(\xi \otimes e^{\lambda} \otimes 1)$ as the left-regular representation of G on

$$\{f \in C^\infty(G, V_\xi) \mid f(gman) = \xi(m)^{-1} a^{-\lambda-\rho} f(g)\ \forall man \in MAN\},$$

where ρ is the half sum of the positive roots.

Let $W := N_K(A)/Z_K(A)$ be the Weyl-group of G. The Weyl group carries an action to $\widehat{M}$ by $[w\xi](m) = \xi(\tilde{w}^{-1} m \tilde{w})$ where $w = [\tilde{w}] \in W$ and similarly to $\mathfrak{a}^*_{\mathbb{C}}$ by $[w\lambda](a) = \lambda(\tilde{w}^{-1} a \tilde{w})$. Let $\overline{N}$ be the nilradical of the parabolic opposite to P . For every $w = [\tilde{w}] \in W$ the integral

$$T^w_{\xi,\lambda} f(g) = \int_{\overline{N} \cap \tilde{w}^{-1} N \tilde{w}} f(g \tilde{w} \overline{n})\, d\overline{n},$$

converges absolutely in some range of $\mathfrak{a}^*_{\mathbb{C}}$ and defines an intertwining operator $\pi_{\xi,\lambda} \to \pi_{w(\xi,\lambda)}$ known as the Knapp–Stein intertwining operator. These operators satisfies that for $w, w' \in W$ we have

$$T^{w'w}_{\xi,\lambda} = T^{w'}_{w(\xi,\lambda)} \circ T^w_{\xi,\lambda} \tag{2.1}$$

when $\ell(w'w) = \ell(w') + \ell(w)$, where ℓ denotes the length of an element in W see e.g. [K16].

Let (G, H) be a strongly spherical pair of real reductive groups, e.g. $(G, H) = (\mathrm{GL}(n+1, \mathbb{R}), \mathrm{GL}(n, \mathbb{R}))$. Fix parabolic subgroups $P_G = M_G A_G N_G$ of G and $P_H = M_H A_H N_H$ of H as in [F21] and let

$$\pi_{\xi,\lambda} = \operatorname{Ind}_{P_G}^G(\xi \otimes e^{\lambda} \otimes 1) \qquad \text{and} \qquad \tau_{\eta,\nu} = \operatorname{Ind}_{P_H}^H(\eta \otimes e^{\nu} \otimes 1),$$

denote the principal series representations induced from $(\xi, V_\xi) \in \widehat{M}_G$, $\lambda \in \mathfrak{a}^*_{G,\mathbb{C}}$ and $(\eta, W_\eta) \in \widehat{M}_H$, $\nu \in \mathfrak{a}^*_{H,\mathbb{C}}$. A *symmetry breaking operator* is an continuous H-intertwining map between $\pi_{\xi,\lambda}|_H$ and $\tau_{\eta,\nu}$. Following [F21], identify

$$\operatorname{Hom}_H(\pi_{\xi,\lambda}|_H, \tau_{\eta,\nu}) \simeq \mathcal{D}'(G)^{\eta,\nu}_{\xi,\lambda},$$

where

$$\mathcal{D}'(G)^{\eta,\nu}_{\xi,\lambda} = \{K \in \mathcal{D}'(G) \otimes \mathrm{Hom}(V_\xi, W_\eta) : K(m_H a_H n_H g m_G a_G n_G) \\ = a_G^{\lambda-\rho_G} a_H^{\nu+\rho_H} \eta(m_H) \circ K(g) \circ \xi(m_G)\},$$

in the sense that $K \in \mathcal{D}'(G)^{\eta,\nu}_{\xi,\lambda}$ defines a symmetry breaking operator $A \in \mathrm{Hom}_H(\pi_{\xi,\lambda}|_H, \tau_{\eta,\nu})$ by

$$Af(h) = \int_{K_G} K(h^{-1}g) f(g)\, dg,$$

where the integral has to be understood in the distribution sense. Note that using [K16, formula (5.25)] one can show that whenever K is given by a locally integrable function, the operator A can also be computed in the non-compact picture:

$$Af(h) = \int_{\overline{N}_G} K(h^{-1}g) f(g)\, dg.$$

3 Principal Series representations and the Knapp-Stein Intertwiners for GL$(k,\mathbb{R})$

We now set $G = \mathrm{GL}(k,\mathbb{R})$ and make some choices to make the notions of Section 2 more explicit for calculations. Let $P \subseteq G$ be the group of upper triangular matrices with Langlands decomposition $P = MAN$. Here M is the group of diagonal matrices with entires from $\{\pm 1\}$, A is the group of diagonal matrices with positive real entries and N is the group of upper triangular matrices with 1′s on the diagonal.

Identify $\widehat{M} \simeq (\mathbb{Z}/2\mathbb{Z})^k$ by taking $\xi = (\xi_1, \dots, \xi_k) \in \{0,1\}^k$ to the character

$$M \to \{-1, 1\}$$
$$\mathrm{diag}(\varepsilon_1, \dots, \varepsilon_k) \mapsto \mathrm{sgn}(\varepsilon_1)^{\xi_1} \cdots \mathrm{sgn}(\varepsilon_k)^{\xi_k}.$$

Furthermore make the identification $\mathfrak{a}^*_\mathbb{C} \simeq \mathbb{C}^k$ by mapping $\lambda \mapsto (\lambda(E_{1,1}), \dots, \lambda(E_{k,k}))$. We can then consider the principal series representations as functions $f \in C^\infty(G)$ where

$$f(gman) = |x_1|_{\xi_1}^{-\lambda_1 - \frac{k-1}{2}} \dots |x_k|_{\xi_k}^{-\lambda_k - \frac{1-k}{2}} f(g), \quad ma = \mathrm{diag}(x_1, \dots, x_n) \in MA,\, n \in N,$$

since $\rho = \frac{1}{2}(k-1, k-3, \dots, 3-k, 1-k)$.

As alluded to in the introduction it can simplify the notation quite a bit if we denote by characters $\chi_i(x) = |x|_{\xi_i}^{\lambda_i}$, so that we can describe the L-function as

$$L(s, \chi_i) = \pi^{-\frac{s+\xi_i+\lambda_i}{2}} \Gamma\Big(\frac{s+\xi_i+\lambda_i}{2}\Big).$$

The Weyl group W of G can be identified with the symmetric group $\mathbb{S}_k$ and the action of W on $\widehat{M}$ and $\mathfrak{a}^*_\mathbb{C}$ then corresponds to permuting the entries of vectors in $(\mathbb{Z}/2\mathbb{Z})^k$ and $\mathbb{C}^k$. Abusing notation, we do not distinguish between elements of W, $\mathbb{S}_k$ and the permutation matrices corresponding to elements of $\mathbb{S}_k$.

Now for $i = 1, \dots, k-1$ let w_i denote the simple transposition swapping i and $i+1$ considered as the matrix having 1's in the $(i, i+1)$ and $(i+1, i)$ entries and zero everywhere else. Then $\overline{N} \cap w_i^{-1} N w_i = e^{\mathbb{R}E_{i+1,i}}$. Let $\overline{n}_i(x) = e^{xE_{i+1,i}}$. Using the $\mathrm{GL}(2,\mathbb{R})$ computation

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{x} & 1 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & \frac{-1}{x} \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{x} \\ 0 & 1 \end{pmatrix}, \quad (x \neq 0),$$

we can decompose $w_i\overline{n}_i(x)$ into$\overline{N}MAN$, in the sense that $w_i\overline{n}_i(x)$ is contained in a copy of $\mathrm{GL}(2,\mathbb{R})$ embedded into G as block matrices

$$\left(\begin{array}{c|c|c} I_{i-1} & & \\ \hline & GL(2,\mathbb{R}) & \\ \hline & & I_{k-i-1} \end{array}\right),$$

where I_j is the $j \times j$ identity matrix. This allows us to get a more explicit formula for the Knapp–Stein intertwiner in the case where w is a simple transposition:

$$\begin{aligned} T^{w_i}_{\xi,\lambda} f(g) &= \int_{\overline{N}\cap w_i N w_i} f(g w_i \overline{n}_i(x))\, dx \\ &= (-1)^{\xi_{i+1}} \int_{\mathbb{R}} |x|^{\lambda_i - \lambda_{i+1} - 1}_{\xi_i + \xi_{i+1}} f(g\overline{n}_i(x))\, dx, \\ &= (-1)^{\xi_{i+1}} \int_{\mathbb{R}} \chi_i(x)\chi_{i+1}^{-1}(x) f(g\overline{n}_i(x))\, \frac{dx}{|x|}. \end{aligned}$$

by change of variables $x \to x^{-1}$. This operator has poles and we therefore consider the normalized version

$$\mathbf{T}^{w_i}_{\xi,\lambda} = \frac{1}{L(0, \chi_i\chi_{i+1}^{-1})} T^{w_i}_{\xi,\lambda},$$

which has a holomorphic extention to all of $\mathbb{C}^k$ see e.g. [K16]. This normalization makes $\mathbf{T}^{w_i}_{\xi,\lambda}$ holomorphic and nowhere vanishing, see e.g. [BD22, Prop 1.3]. For a general $w \in W$ we normalize $T^{w}_{\xi,\lambda}$ by writing w in terms of simple transpositions w_i and then using the normalization for each $T^{w_i}_{\xi,\lambda}$ which is justified by (2.1).

Similarly, we can decompose $\overline{n}_i(x) = k(x)a(x)n(x)$ to the compact picture KAN by the $\mathrm{GL}(2,\mathbb{R})$-computation

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \left[\frac{1}{\sqrt{1+x^2}}\begin{pmatrix} 1 & -x \\ x & 1 \end{pmatrix}\right]\begin{pmatrix} \sqrt{1+x^2} & 0 \\ 0 & \frac{1}{\sqrt{1+x^2}} \end{pmatrix}\begin{pmatrix} 1 & \frac{x}{1+x^2} \\ 0 & 1 \end{pmatrix}. \tag{3.1}$$

Which allows for another form for the Knapp–Stein intertwiner

$$T^{w_i}_{\xi,\lambda} f(g) = \int_{\mathbb{R}} (1+x^2)^{\frac{\lambda_{i+1}-\lambda_i-1}{2}} f(g w_i k(x))\, dx.$$

4 Symmetry breaking operators between principal series representations

Now fix $(G, H) = (\mathrm{GL}(n+1,\mathbb{R}), \mathrm{GL}(n,\mathbb{R}))$ and the corresponding principal series representations

$$\pi_{\xi,\lambda} = \mathrm{Ind}^G_{P_G}(\xi \otimes e^{\lambda} \otimes 1) = \mathrm{Ind}^G_{P_G}(\chi_1 \otimes \cdots \otimes \chi_{n+1}),$$

and

$$\tau_{\eta,\nu} = \mathrm{Ind}_{P_H}^H(\eta \otimes e^{\nu} \otimes 1) = \mathrm{Ind}_{P_H}^H(\psi_1 \otimes \cdots \otimes \psi_n).$$

Following [F21] consider the functions

$$\widetilde{\Phi_p}(g) = \det\left((g_{ij})_{1\le i,j\le p}\right) \quad \text{and} \quad \widetilde{\Psi}_q(g) = \det\left((g_{ij})_{2\le i\le q+1, 1\le j\le q}\right),$$

where $g \in G$. Let

$$w_0 = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix},$$

be a representative of the longest Weyl group element and set $\Phi_p(g) = \widetilde{\Phi}_p(w_0 g)$ and $\Psi_q(g) = \widetilde{\Psi}_q(w_0 g)$. Now consider the kernel

$$K_{\xi,\lambda}^{\eta,\nu}(g) = |\Phi_1(g)|_{\delta_1}^{s_1} \cdots |\Phi_{n+1}(g)|_{\delta_{n+1}}^{s_{n+1}} |\Psi_1(g)|_{\varepsilon_1}^{t_1} \cdots |\Psi_n(g)|_{\varepsilon_n}^{t_n},$$

for $g \in G$, where $s_i = \lambda_i - \nu_{n+1-i} - \frac{1}{2}$, $t_i = \nu_{n+1-i} - \lambda_{i+1} - \frac{1}{2}$ for $i = 1, \ldots, n$ and $s_{n+1} = \lambda_{n+1} + \frac{n}{2}$. Likewise $\delta_i = \xi_i - \eta_{n+1-i}$, $\varepsilon_i = \eta_{n+1-i} - \xi_{i+1}$ for $i = 1, \ldots, n$ and $\delta_{n+1} = \xi_{n+1}$. The exponents (s,t) are related to $(\lambda, \nu) \in \mathbb{C}^{n+1} \times \mathbb{C}^n$ by an invertible affine linear coordinate transformation given by

$$\begin{pmatrix} s_1 \\ t_1 \\ \vdots \\ t_n \\ s_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \nu_n \\ \lambda_2 \\ \vdots \\ \nu_1 \\ \lambda_{n+1} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ -n \end{pmatrix},$$

with inverse

$$\begin{pmatrix} \lambda_1 \\ \nu_n \\ \lambda_2 \\ \vdots \\ \lambda_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} s_1 \\ t_1 \\ \vdots \\ t_n \\ s_{n+1} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} n \\ n-1 \\ \vdots \\ -(n-1) \\ -n \end{pmatrix}.$$

The same coordinate transformation holds between (ξ, η) and (δ, ε) if we disregard the affine part. Then $K_{\xi,\lambda}^{\eta,\nu}$ belongs to $\mathcal{D}'(G)_{\xi,\lambda}^{\eta,\nu}$. Hence we get a meromorphic family of intertwining operators $A_{\xi,\lambda}^{\eta,\nu} : \pi_{\xi,\lambda}|_H \to \tau_{\eta,\nu}$ defined by

$$A_{\xi,\lambda}^{\eta,\nu} f(h) = \int_{K_G} K_{\xi,\lambda}^{\eta,\nu}(h^{-1}k) f(k) dk = \int_{\overline{N}_G} K_{\xi,\lambda}^{\eta,\nu}(h^{-1}\overline{n}) f(\overline{n}) d\overline{n}.$$

The distribution kernel $K_{\xi,\lambda}^{\eta,\nu}$ is $L^1_{\mathrm{loc}}(G)$ when $\mathrm{Re}(\lambda_1) \gg \mathrm{Re}(\nu_n) \gg \mathrm{Re}(\lambda_2) \gg \cdots \gg \mathrm{Re}(\nu_1) \gg \mathrm{Re}(\lambda_{n+1})$, or even better when $\mathrm{Re}(s_i), \mathrm{Re}(t_i) \ge 0$ for $i = 1, \ldots, n$. This is most likely not the largest domain where $K_{\xi,\lambda}^{\eta,\nu}$ is $L^1_{\mathrm{loc}}(G)$ but it will suffice for our purposes.

5 Functional identity between Knapp–Stein intertwiners and the distribution kernel

By [F21] the space $\mathcal{D}'(G)^{\eta,\nu}_{\xi,\lambda}$ is one-dimensional for generic (λ,ν). Moreover we note that for a fixed $h\in H$, the distribution $g\mapsto K^{\eta,\nu}_{\xi,\lambda}(h^{-1}g)$ can we viewed as a distribution section of the homogeneous vector bundle over G/P_G.

By duality we can apply the Knapp–Stein intertwiners to get maps $\mathcal{D}'(G)^{\eta,\nu}_{\xi,\lambda}\to\mathcal{D}'(G)^{\eta,\nu}_{w(\xi,\lambda)}$ meaning we are aiming for functional identities of the type

$$T^w_{\xi,-\lambda}K^{\eta,\nu}_{\xi,\lambda}=c_w(\xi,\lambda,\eta,\nu)K^{\eta,\nu}_{w(\xi,\lambda)},$$

as the spaces $\mathcal{D}'(G)^{\eta,\nu}_{\xi,\lambda}$ are generically of dimension one. The first thing we notice in our quest is

$$\Phi_j(gw_i\overline{n}_i(x))=\begin{cases}\Phi_j(g), & j<i,\\ \Phi_j(gw_i)+x\Phi_j(g), & j=i,\\ -\Phi_j(g), & j>i.\end{cases}$$

$$\Psi_j(gw_i\overline{n}_i(x))=\begin{cases}\Psi_j(g), & j<i,\\ \Psi_j(gw_i)+x\Psi_j(g), & j=i,\\ -\Psi_j(g), & j>i.\end{cases}$$

$$\Phi_j(\overline{n}_i(x)w_ig)=\begin{cases}\Phi_j(g), & j<n+1-i,\\ \Phi_j(w_ig)+x\Phi_j(g), & j=n+1-i,\\ -\Phi_j(g), & j>n+1-i.\end{cases}$$

$$\Psi_j(\overline{n}_i(x)w_ig)=\begin{cases}\Psi_j(g), & j<n-i,\\ \Psi_j(w_ig)+x\Psi_j(g), & j=n-i,\\ -\Psi_j(g), & j>n-i.\end{cases}$$

All of these identities follow from basic properties of the determinant. Multiplying g by $\overline{n}_i(x)$ from the right corresponds to adding x times the $(i+1)$'th column of g to the i'th column of g. So if the determinant contains none of the columns or both of them nothing happens, but in the case it only contains the i'th column and not the $(i+1)$'th, multilinearity of the determinant can be applied. Multiplying by w_i from the right correspond to swapping the i'th and $(i+1)$'th column, so if a determinant contains none of the two columns nothing happens, if it contains both we get a sign. Similar reasoning can be applied to multiplying from the left and considering rows instead of columns.

Lemma 5.1. *For $g\in\mathrm{GL}(n+1,\mathbb{R})$ we have the following two identities:*

$$\Phi_i(g)\Psi_i(gw_i)-\Phi_i(gw_i)\Psi_i(g)=\Psi_{i-1}(g)\Phi_{i+1}(g),$$

$$\Phi_{n+1-i}(w_ig)\Psi_{n-i}(g)-\Phi_{n+1-i}(g)\Psi_{n-i}(w_ig)=\Psi_{n-i-1}(g)\Phi_{n+2-i}(g),$$

where we define $\Psi_0=1$.

Proof. We show the first identity and the second one follows by analogous arguments. We notice that both sides of the identity have the same equivariance properties from the left by P_H. Similarly both sides have the same equivariance properties from the right by P_G e.g. for $n \in N_G$ we have $w_i n w_i = n' \overline{n}_i(x)$ for some $n' \in N_G$ and $x = n_{i,i+1}$ thus

$$\begin{aligned}\Phi_i(gn)\Psi_i(gnw_i) - \Phi_i(gnw_i)\Psi_i(gn) &= \Phi_i(g)\big[\Psi_i(gw_i) + x\Psi_i(g)\big] - \Psi_i(g)\big[\Phi_i(gw_i) + x\Phi_i(g)\big] \\ &= \Phi_i(g)\Psi_i(gw_i) - \Phi_i(gw_i)\Psi_i(g).\end{aligned}$$

As we are proving an equality of continuous functions it suffices to check that they coincide on a dense subset. There is an open dense double $P_H g_0 P_G$ in G where g_0 given as w_0 multiplied with the matrix with only $1's$ on the diagonal and subdiagonal and zero everywhere else, see [F21, Lemma 6.3 & Lemma 3.6]. The two sides of the equation coincide on g_0 and hence on ω by the equvariance properties. □

Remark 5.2. To avoid technical considerations about restrictions of distributions to submanifolds we use the convention that $T^{w_i}_{\eta,\nu} K^{\eta,\nu}_{\xi,\lambda}$ means the distributional kernel of the intertwining operator $T^{w_i}_{\eta,\nu} \circ A^{\eta,\nu}_{\xi,\lambda}$.

Theorem 5.3. *For fixed i where $\operatorname{Re}(\lambda_i - \nu_{n+1-i}), \operatorname{Re}(\nu_{n+1-i} - \lambda_{i+1}) > -\frac{1}{2}$ and $\operatorname{Re}(\lambda_i - \lambda_{i+1}) < 0$ we have*

$$\boldsymbol{T}^{w_i}_{\xi,-\lambda} K^{\eta,\nu}_{\xi,\lambda} = c_i(\xi,\lambda,\eta,\nu) K^{\eta,\nu}_{w_i(\xi,\lambda)},$$

where

$$c_i(\xi,\lambda,\eta,\nu) = \frac{(-1)^{(\xi_i+\xi_{i+1})(\eta_{n+1-i}+1)+\xi_i\xi_{i+1}} L(\frac{1}{2}, \chi_i \psi^{-1}_{n+1-i}) L(\frac{1}{2}, \chi^{-1}_{i+1} \psi_{n+1-i})}{\sqrt{\pi} L(1, \chi_i \chi^{-1}_{i+1}) L(\frac{1}{2}, \chi_i^{-1} \psi_{n+1-i}) L(\frac{1}{2}, \chi_{i+1} \psi^{-1}_{n+1-i})}.$$

Similarly, for $\operatorname{Re}(\lambda_{n+1-i} - \nu_i), \operatorname{Re}(\nu_{i+1} - \lambda_{n+1-i}) > -\frac{1}{2}$ and $\operatorname{Re}(\nu_{i+1} - \nu_i) < 0$ we have

$$\boldsymbol{T}^{w_i}_{\eta,\nu} K^{\eta,\nu}_{\xi,\lambda} = d_i(\xi,\lambda,\eta,\nu) K^{w_i(\eta,\nu)}_{\xi,\lambda}.$$

where

$$d_i(\xi,\lambda,\eta,\nu) = \frac{(-1)^{(\eta_i+\eta_{i+1})(\xi_{n+1-i}+1)+\eta_i\eta_{i+1}} L(\frac{1}{2}, \chi_{n+1-i} \psi^{-1}_i) L(\frac{1}{2}, \chi^{-1}_{n+1-i} \psi_{i+1})}{\sqrt{\pi} L(1, \psi_{i+1} \psi^{-1}_i) L(\frac{1}{2}, \chi^{-1}_{n+1-i} \psi_i) L(\frac{1}{2}, \chi_{n+1-i} \psi^{-1}_{i+1})}.$$

Proof. To ease notation we put $|\Psi_{n+1}|^{t_{n+1}}_{\varepsilon_{n+1}} = 1$. For g in the open dense set where $\Phi_i(g)$, $\Phi_i(gw_i), \Psi_i(g), \Psi_i(gw_i) \neq 0$ we have

$$\begin{aligned}T^{w_i}_{\xi,-\lambda} K^{\eta,\nu}_{\xi,\lambda}(g) &= \int_{\mathbb{R}} K^{\eta,\nu}_{\xi,\lambda}(g w_i \overline{n}_i(x)) dx \\ &= (-1)^{\delta_{n+1} + \sum_{j=i+1}^{n} (\delta_j + \varepsilon_j)} \Bigg(\prod_{\substack{j=1 \\ j \neq i}}^{n+1} |\Phi_j(g)|^{s_j}_{\delta_j} |\Psi_j(g)|^{t_j}_{\varepsilon_j} \Bigg) \\ &\quad \times \int_{\mathbb{R}} |\Phi_i(gw_i) + x\Phi_i(g)|^{s_i}_{\delta_i} |\Psi_i(gw_i) + x\Psi_i(g)|^{t_i}_{\varepsilon_j} dx.\end{aligned}$$

The latter integral can be evaluated by Corollary A.2 giving

$$(-1)^{\varepsilon_i} t(s_i,t_i,\delta_i,\varepsilon_i) |\Phi_i(g)|^{-t_i-1}_{\varepsilon_i} |\Psi_i(g)|^{-s_i-1}_{\delta_i} |\Phi_i(g)\Psi_i(gw_i) - \Phi_i(gw_i)\Psi_i(g)|^{s_i+t_i+1}_{\varepsilon_i+\delta_i}.$$

Lastly, applying Lemma 5.1, we arrive at the first result. For the second assertion we recall Remark 5.2 and by switching the integrals we get

$$T^{w_i}_{\eta,\nu} A^{\eta,\nu}_{\xi,\lambda} f(h) = \int_{\overline{N}} f(\overline{n}) \int_{\mathbb{R}} K^{\eta,\nu}_{\xi,\lambda}(n_i(x) w_i h^{-1} \overline{n}) dx d\overline{n}.$$

This allows us to do the same type calculation as for the first assertion but inside the integrals, in turn the constant is given by $(-1)^{\eta_i+\eta_{i+1}+\xi_{n+1-i}} t(s_{n+1-i}, t_{n-i}, \delta_{n+1-i}, \varepsilon_{n-i})$. □

Remark 5.4. By replacing (ξ, λ) by $w_i(\xi, \lambda)$ we can phrase the result in a slightly different way for $\operatorname{Re}(\lambda_{i+1} - \nu_{n+1-i}), \operatorname{Re}(\nu_{n+1-i} - \lambda_i) > -\frac{1}{2}$ and $\operatorname{Re}(\lambda_{i+1} - \lambda_i) < 0$

$$K^{\eta,\nu}_{\xi,\lambda} = \widetilde{c}_i(\xi, \lambda, \eta, \nu) \mathbf{T}^{w_i}_{w_i(\xi,-\lambda)} K^{\eta,\nu}_{w_i(\xi,\lambda)},$$

where

$$\widetilde{c}_i(\xi, \lambda, \eta, \nu) = \frac{\sqrt{\pi} L(1, \chi_i^{-1}\chi_{i+1}) L(\frac{1}{2}, \chi_{i+1}^{-1}\psi_{n+1-i}) L(\frac{1}{2}, \chi_i \psi_{n+1-i}^{-1})}{(-1)^{(\xi_i+\xi_{i+1})(\eta_{n+1-i}+1)+\xi_i\xi_{i+1}} L(\frac{1}{2}, \chi_{i+1}\psi_{n+1-i}^{-1}) L(\frac{1}{2}, \chi_i^{-1}\psi_{n+1-i})}.$$

Corollary 5.5. *Let*

$$A^+ = \bigcap_{j=i}^{n} \Big\{ (\lambda, \nu) | \operatorname{Re}(\lambda_{j+1} - \nu_{n+1-j}), \operatorname{Re}(\nu_{n+1-j} - \lambda_i) > -\frac{1}{2}, \operatorname{Re}(\lambda_{j+1} - \lambda_i) < 0 \Big\} \neq \varnothing,$$

and

$$A^- = \bigcap_{j=i}^{n} \Big\{ (\lambda, \nu) | \operatorname{Re}(\lambda_i - \nu_{n+1-j}), \operatorname{Re}(\nu_{n+1-j} - \lambda_j) > -\frac{1}{2}, \operatorname{Re}(\lambda_i - \lambda_j) < 0 \Big\} \neq \varnothing.$$

For $i = 1, \dots, n$ let $w_+ = w_i w_{i+1} \dots w_n$ and $w_- = w_{i-1} w_{i-2} \dots w_1$. For $(\lambda, \nu) \in A^{\pm}$ we have

$$K^{\eta,\nu}_{\xi,\lambda} = b_i^{\pm}(\xi, \lambda, \eta.\nu) \boldsymbol{T}^{w_\pm}_{w_\pm^{-1}(\xi,-\lambda)} K^{\eta,\nu}_{w_\pm^{-1}(\xi,\lambda)},$$

where

$$b_i^+(\xi, \lambda, \eta, \nu) = \alpha^+ \prod_{j=i}^{n} \frac{L(1, \chi_{j+1}\chi_i^{-1}) L(\frac{1}{2}, \chi_{j+1}^{-1}\psi_{n+1-j}) L(\frac{1}{2}, \chi_i \psi_{n+1-j}^{-1})}{L(\frac{1}{2}, \chi_{j+1}\psi_{n+1-j}^{-1}) L(\frac{1}{2}, \chi_i^{-1}\psi_{n+1-j})},$$

and

$$b_i^-(\xi, \lambda, \eta, \nu) = \alpha^- \prod_{j=1}^{i-1} \frac{L(1, \chi_i \chi_j^{-1}) L(\frac{1}{2}, \chi_i^{-1}\psi_{n+1-j}) L(\frac{1}{2}, \chi_j \psi_{n+1-j}^{-1})}{L(\frac{1}{2}, \chi_i \psi_{n+1-j}^{-1}) L(\frac{1}{2}, \chi_j^{-1}\psi_{n+1-j})},$$

and $\alpha^{\pm}$ are powers of π and (-1) that depend on ξ and η.

Proof. Using (2.1) we get that the right-hand side of the $(+)$–equation is

$$b_i(\xi, \lambda, \eta, \nu) \mathbf{T}^{w_i}_{w_i(\xi,-\lambda)} \circ \dots \circ \mathbf{T}^{w^n}_{w_n \dots w_i(\xi,-\lambda)} K^{\eta,\nu}_{w_n \dots w_i(\xi,\lambda)}$$

thus by successive use of Remark 5.4, we get

$$b^+(\xi.\lambda, \eta, \nu) = \widetilde{c}_i(\xi, \lambda, \eta, \nu) \prod_{j=i+1}^{n} \widetilde{c}_j(w_{j-1} \dots w_i(\xi, \lambda), \eta, \nu).$$ □

6 Bernstein–Sato identities for the distribution kernel

For $w \in W_G$ and $f : G \to \mathbb{C}$ we consider the left/right regular action $\ell(w)f = f(w^{-1}\,\cdot)$ and $r(w)f = f(\,\cdot\, w)$. Define

$$w_{i,j} = \begin{cases} w_i w_{i+1} \cdots w_j, & \text{for } i < j, \\ w_i w_{i-1} \cdots w_j, & \text{for } i > j, \\ w_i, & \text{for } i = j. \end{cases}$$

We let $\lambda_{i,j} := \lambda_i - \lambda_j - 1$, $\nu_{i,j} := \nu_i - \nu_j - 1$, $\varepsilon^{i,j} := \sum_{k=1}^{n+1} g_{k,i}\partial_{g_{k,j}}$, $\varepsilon_{i,j} := \sum_{k=1}^{n+1} g_{i,k}\partial_{g_{j,k}}$ and $\widetilde{\varepsilon}^{i,j} = (-1)^{i+j+1}\varepsilon^{i,j}$. Now, consider the differential operators

$$\mathcal{D}_i(\lambda) = (-1)^{i+1} \begin{vmatrix} \Phi_1 & \lambda_{1,i+1} & 0 & \cdots & 0 \\ r(w_{1,1})\Phi_1 & \varepsilon^{2,1} & \lambda_{2,i+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r(w_{i-1},1)\Phi_1 & \varepsilon^{i,1} & \varepsilon^{i,2} & \cdots & \lambda_{i,i+1} \\ r(w_{i,1})\Phi_1 & \varepsilon^{i+1,1} & \varepsilon^{i+1,2} & \cdots & \varepsilon^{i+1,i} \end{vmatrix}, \quad i = 1, 2, \ldots, n,$$

$$\mathcal{C}_i(\nu) = (-1)^{n-i} \begin{vmatrix} \Psi_1 & \nu_{n,i} & 0 & \cdots & 0 \\ \ell(w_{n-1,n-1})\Psi_1 & \varepsilon_{n-1,n} & \nu_{n-1,i} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \ell(w_{i+1,n-1})\Psi_1 & \varepsilon_{i+1,n} & \varepsilon_{i+1,n-1} & \cdots & \nu_{i+1,i} \\ \ell(w_{i,n-1})\Psi_1 & \varepsilon_{i,n} & \varepsilon_{i,n-1} & \cdots & \varepsilon_{i,i+1} \end{vmatrix}, \quad i = 1, 2, \ldots, n-1,$$

$$\mathcal{F}_i(\lambda) = (-1)^{n+1-i} \begin{vmatrix} \Psi_n & \lambda_{i,n+1} & 0 & \cdots & 0 \\ r(w_{n,n})\Psi_n & \widetilde{\varepsilon}^{n+1,n} & \lambda_{i,n} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r(w_{i+1,n})\Psi_n & \widetilde{\varepsilon}^{n+1,i+1} & \widetilde{\varepsilon}^{n,i+1} & \cdots & \lambda_{i,i+1} \\ r(w_{i,n})\Psi_n & \widetilde{\varepsilon}^{n+1,i} & \widetilde{\varepsilon}^{n,i} & \cdots & \widetilde{\varepsilon}^{i+1,i} \end{vmatrix}, \quad i = 1, 2 \ldots, n.$$

where $|\cdot|$ denotes the determinant. As the entries of this determinant are non-commuting, we specify that this determinant should be considered as

$$|A| = \sum_{\sigma \in \mathbb{S}_n} \operatorname{sgn}(\sigma) a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n}.$$

In the case of $\mathcal{D}_i(\lambda)$ this implies each term of the determinant has the form $r(w_{1,j})\Phi_1\varepsilon^{j_1,1}\varepsilon^{j_2,2}\cdots\varepsilon^{j_i,i}$ if we put $\varepsilon^{k,k} = \lambda_{k,i+1}$. With this settled, we can notice that the differential operator $\mathcal{D}_i(\lambda)$ has order i, $\mathcal{C}_i(\nu)$ has order $n-i$ and $\mathcal{F}_i(\lambda)$ has order $n+1-i$.

Lemma 6.1. *Fix $i = 1, 2, \ldots, n$. For $\operatorname{Re}(\lambda_i - \lambda_{i+1}) \gg 1$ and $f \in \pi_{w_i(\xi,-\lambda)}$ we have*

$$\begin{aligned} \int_{\mathbb{R}} \partial_x\Big(|x|_{\xi_i+\xi_{i+1}}^{\lambda_i-\lambda_{i+1}-1}\Big) f(g\overline{n}_i(x))\,dx &= -\int_{\mathbb{R}} |x|_{\xi_i+\xi_{i+1}}^{\lambda_i-\lambda_{i+1}-1}\partial_x f(g\overline{n}_i(x))\,dx \\ &= -\varepsilon^{i+1,i}\int_{\mathbb{R}} |x|_{\xi_i+\xi_{i+1}}^{\lambda_i-\lambda_{i+1}-1} f(g\overline{n}_i(x))\,dx. \end{aligned}$$

In fact, if $\mathcal{D}$,$\mathcal{D}'$ are differential operators on G satisfying $(\mathcal{D}f)(g\overline{n}_i(x)) = \mathcal{D}'[f(g\overline{n}_i(x))]$ then

$$\int_{\mathbb{R}} |x|^{\lambda_i-\lambda_{i+1}-1}_{\xi_i+\xi_{i+1}}[\mathcal{D}f](g\overline{n}_i(x))\,dx = \mathcal{D}'\int_{\mathbb{R}} |x|^{\lambda_i-\lambda_{i+1}-1}_{\xi_i+\xi_{i+1}} f(g\overline{n}_i(x))\,dx,$$

where on the right hand sides, $\mathcal{D}'$ should be considered as differentiating in g.

Proof. For the first equality, we argue there is no boundary term in partial integration. Considering $\overline{n}_i(x)$ as an $\mathrm{SL}(2,\mathbb{R})$-element, we can use (3.1) and find

$$\begin{aligned}|x|^{\lambda_i-\lambda_{i+1}-1}_{\xi_i+\xi_{i+1}} f(g\overline{n}_i(x)) &= |x|^{\lambda_i-\lambda_{i+1}-1}_{\xi_i+\xi_{i+1}}(1+x^2)^{\frac{\lambda_{i+1}-\lambda_i-1}{2}} f(gk(x)) \\ &= |x|^{-1}_{\xi_i+\xi_{i+1}}(1+x^2)^{-\frac{1}{2}}\Big(\frac{|x|}{\sqrt{1+x^2}}\Big)^{\lambda_i-\lambda_{i+1}} f(gk(x)),\end{aligned}$$

which vanishes at $\pm\infty$. As $g\overline{n}_i(x)$ is the same as g, but where in the i'th column x times the $i+1$'th column of g is added, we get that $\partial_x[f(g\overline{n}_i(x))] = (\varepsilon^{i+1,i}f)(g\overline{n}_i(x))$. Furthermore, as differentiating with respect to the i'th column and multiplying by elements of the $i+1$'th column is not affected by multiplying by $\overline{n}_i(x)$ from the right, we get $(\varepsilon^{i+1,i}f)(g\overline{n}_i(x)) = \varepsilon^{i+1,i}[f(g\overline{n}_i(x))]$. Thus it suffices to show the last assertion. For g in a compact subset of G we get, using the Iwasawa decomposition, that

$$\begin{aligned}\Big||x|^{\lambda_i-\lambda_{i+1}-1}_{\xi_i+\xi_{i+1}}\mathcal{D}'[f(g\overline{n}_i(x))]\Big| &= \Big||x|^{\lambda_i-\lambda_{i+1}-1}_{\xi_i+\xi_{i+1}}(1+x^2)^{\frac{\lambda_{i+1}-\lambda_i-1}{2}}\mathcal{D}'[f(gk(x))]\Big| \\ &\leq c(1+x^2)^{\frac{\mathrm{Re}(\lambda_{i+1}-\lambda_i-1)}{2}}|x|^{\mathrm{Re}(\lambda_i-\lambda_{i+1}-1)},\end{aligned}$$

where the c is a constant depending on f,g and $\mathcal{D}'$. The right hand side is integrable, allowing us to pull the differential operator out of the integral. □

Remark 6.2. Consider r the right regular representation of $e^{\mathbb{R}E_{i+1,i}}$ on $C^\infty(G)$. Then

$$\partial_x r(e^{xE_{i+1,i}}) = r(e^{xE_{i+1,i}})dr(E_{i+1,i}) = dr(E_{i+1,i})r(e^{xE_{i+1,i}}),$$

corresponds to

$$\partial_x[f(g\overline{n}_i(x))] = (\varepsilon^{i+1,i}f)(g\overline{n}_i(x)) = \varepsilon^{i+1,i}[f(g\overline{n}_i(x))]$$

as argued in the proof. That is $\varepsilon^{i+1,i} = dr(E_{i+1,i})$ which could be useful for generalizations to other groups.

For a $\mathbb{C}$-valued function f let M_f be the multiplication operator given by $M_f\varphi = f\varphi$.

Proposition 6.3. *For (generic) (λ,ν) such that $\mathrm{Re}(\lambda_1) \gg \cdots \gg \mathrm{Re}(\lambda_{n+1})$ and $\mathrm{Re}(\nu_n) \gg \cdots \gg \mathrm{Re}(\nu_1)$ the differential operators can be expressed in the following way*

$$\begin{aligned}\mathcal{D}_1(\lambda) &= \lambda_{1,2}T^{w_1}_{-w_1(\xi,\lambda)-(e_1,e_1)} \circ M_{\Phi_1} \circ \Big(T^{w_1}_{-w_1(\xi,\lambda)}\Big)^{-1}, \\ \mathcal{D}_i(\lambda) &= \lambda_{i,i+1}T^{w_i}_{-w_i(\xi,\lambda)-(e_i,e_i)} \circ \mathcal{D}_{i-1}(w_i\lambda) \circ \Big(T^{w_i}_{-w_i(\xi,\lambda)}\Big)^{-1}, \\ \mathcal{C}_{n-1}(\nu) &= \nu_{n,n-1}T^{w_{n-1}}_{w_{n-1}(\eta,\nu)+(e_n,e_n)} \circ M_{\Psi_1} \circ \Big(T^{w_{n-1}}_{w_{n-1}(\eta,\nu)}\Big)^{-1}, \\ \mathcal{C}_i(\nu) &= \nu_{i+1,i}T^{w_i}_{w_i(\eta,\nu+(e_{i+1},e_{i+1}))} \circ \mathcal{C}_{i+1}(w_i\nu) \circ \Big(T^{w_i}_{w_i(\eta,\nu)}\Big)^{-1}, \\ \mathcal{F}_n(\lambda) &= \lambda_{n,n+1}T^{w_n}_{-w_n(\xi,\lambda)-(\mathbb{1}-e_{n+1},\mathbb{1}-e_{n+1})} \circ M_{\Psi_n} \circ \Big(T^{w_n}_{-w_n(\xi,\lambda)}\Big)^{-1}, \\ \mathcal{F}_i(\lambda) &= \lambda_{i,i+1}T^{w_i}_{-w_i(\xi,\lambda)-(\mathbb{1}-e_{i+1},\mathbb{1}-e_{i+1})} \circ \mathcal{F}_{i+1}(w_i\lambda) \circ \Big(T^{w_i}_{-w_i(\xi,\lambda)}\Big)^{-1}.\end{aligned}$$

Proof. We start off by showing that this indeed holds for the family $\mathcal{D}_i(\lambda)$ using induction in i. Let $f \in \pi_{-w_1(\xi,\lambda)}$ then using Lemma 6.1 we get

$$\begin{aligned}\lambda_{1,2}T^{w_1}_{-w_1(\xi,\lambda)-(e_1,e_1)}(\Phi_1 f)(g) &= \lambda_{1,2}(-1)^{\xi_1}\int_{\mathbb{R}}|x|^{\lambda_1-\lambda_2-2}_{\xi_1+\xi_2+1}(\Phi_1 f)(g\overline{n}_1(x))\,dx \\ &= \lambda_{1,2}(-1)^{\xi_1}\int_{\mathbb{R}}|x|^{\lambda_1-\lambda_2-2}_{\xi_1+\xi_2+1}[\Phi_1(g)+xr(w_1)\Phi_1(g)]f(g\overline{n}_1(x))\,dx \\ &= \Phi_1(g)(-1)^{\xi_1}\int_{\mathbb{R}}\partial_x\Big(|x|^{\lambda_1-\lambda_2-1}_{\xi_1+\xi_2}\Big)f(g\overline{n}_1(x))\,dx \\ &\quad + \lambda_{1,2}r(w_1)\Phi_1(g)T^{w_1}_{-w_1(\xi,\lambda)}f(g) \\ &= -[\Phi_1(g)\varepsilon^{2,1} - r(w_1)\Phi_1(g)\lambda_{1,2}]T^{w_1}_{-w_1(\xi,\lambda)}f(g).\end{aligned}$$

This shows the base case for induction as $\mathcal{D}_1(\lambda) = -[\Phi_1(g)\varepsilon^{2,1} - \lambda_{1,2}r(w_1)\Phi_1(g)]$.

Assume the assertion holds for $i-1$ and let $f \in \pi_{-w_i(\xi,\lambda)}$. For $j \leq i-1$, we have $[r(w_{j,1})\Phi_1](g\overline{n}_i(x)) = [r(w_{j,1})\Phi_1](g) + \delta_{i-1,j}x[r(w_i)r(w_{j,1})\Phi_1](g)$ and for $k < j \leq i$, we have $(\varepsilon^{j,k}f)(g\overline{n}_i(x)) = (\varepsilon^{j,k} + \delta_{i,j}x\varepsilon^{j+1,k})[f(g\overline{n}_i(x))]$ where $\delta_{i,j}$ is the Kronecker delta. Thus, only the last row of the matrix is affected, and by multilinearity of the determinant we get

$$(\mathcal{D}_{i-1}(w_i\lambda)f)(g\overline{n}_i(x)) = \mathcal{D}_{i-1}(w_i\lambda)[f(g\overline{n}_i(x))] + x\widetilde{\mathcal{D}}_{i-1}(w_i\lambda)[f(g\overline{n}_i(x))],$$

where $\widetilde{\mathcal{D}}_{i-1}(w_i\lambda)$ is the determinant of the same matrix as $\mathcal{D}_{i-1}(w_i\lambda)$, but where $r(w_{1,i-1})$ is replaced with $r(w_{1,i})$, and $\varepsilon^{i,k}$ is replaced by $\varepsilon^{i+1,k}$ for $1 \leq k \leq i-1$. Now using the same steps as for $i = 1$ where $\mathcal{D}_{i-1}(w_i\lambda) + x\widetilde{\mathcal{D}}_{i-1}(w_i\lambda)$ plays the role of $\Phi_1(g) + xr(w_1)\Phi_1(g)$, we get

$$\begin{aligned}&\lambda_{i,i+1}T^{\eta,\nu}_{-w_i(\xi,\lambda)-(e_i,e_i)}(\mathcal{D}_{i-1}(w_i\lambda)f)(g) \\ &\qquad\qquad = -\Big[\mathcal{D}_{i-1}(w_i\lambda)\varepsilon^{i+1,i} - \widetilde{\mathcal{D}}_{i-1}(w_i\lambda)\lambda_{i,i+1}\Big]\,T^{\eta,\nu}_{-w_i(\xi,\lambda)}f(g).\end{aligned}$$

By multilinearity of the determinant, we can write

$$-\mathcal{D}_{i-1}(w_i\lambda)\varepsilon^{i+1,i} + \widetilde{\mathcal{D}}_{i-1}(w_i\lambda)\lambda_{i,i+1} = \mathcal{D}_i(\lambda),$$

showing the assertion for i.

The proof for $\mathcal{C}_i(\nu)$ is identical, but where everything is acting from the left. For $\mathcal{F}_i(\lambda)$ it is again similar, but with a little twist. As for $k \geq i$, we have $[r(w_{k,n})\Psi_n](g\overline{n}_{i-1}(x)) = [r(w_{k,n})\Psi_n](g) + \delta_{i,j}x[r(w_{k-1}r(w_{k,n})\Psi_n](g)$ and $(\varepsilon^{k+1,i}f)(g\overline{n}_{i-1}(x)) = [\varepsilon^{k+1,i} - x\varepsilon^{k+1,i-1}]$ $f(g\overline{n}_{i-1}(x))$, meaning we do not get addition with same signs in the last row. But by changing to the $\sim$-notation, we get $(\widetilde{\varepsilon}^{k+1,i}f)(g\overline{n}_{i-1}(x)) = [\widetilde{\varepsilon}^{k+1,i} + x\widetilde{\varepsilon}^{k+1,i-1}]f(g\overline{n}_{i-1}(x))$ and the proof follows in the same manner as for $\mathcal{D}_i(\lambda)$. □

Remark 6.4. For generic (λ,ν) the principal series representations $\pi_{(\xi,\lambda)}, \tau_{(\eta,\nu)}$ are irreducible, thus by Schur's lemma we get $T^{w^{-1}}_{w(\xi,\lambda)} \circ T^{w}_{\xi,\lambda}$ is non-zero constant times the identity allowing us to talk about the inverse of $T^{w}_{\xi,\lambda}$.

Theorem 6.5. *We have the following Bernstein–Sato identities*

$$\mathcal{D}_i(\lambda)K^{\eta,\nu}_{\xi,\lambda} = p_{\mathcal{D}_i}(\lambda,\nu)K^{\eta,\nu}_{(\xi+e_{i+1},\lambda+e_{i+1})},$$
$$\mathcal{C}_i(\nu)K^{\eta,\nu}_{\xi,\lambda} = p_{\mathcal{C}_i}(\lambda,\nu)K^{(\eta+e_i,\nu+e_i)}_{(\xi+e_1,\lambda+e_1)},$$
$$\mathcal{F}_i(\lambda)K^{\eta,\nu}_{\xi,\lambda} = p_{\mathcal{F}_i}(\lambda,\nu)K^{(\eta+\mathbb{1},\nu+\mathbb{1})}_{(\xi+\mathbb{1}-e_i,\lambda+\mathbb{1}-e_i)},$$

where

$$p_{\mathcal{D}_i}(\lambda,\nu) = (-1)^{i+1}\prod_{j=n+1-i}^{n}\Big(\nu_j-\lambda_{i+1}-\frac{1}{2}\Big),$$
$$p_{\mathcal{C}_i}(\lambda,\nu) = (-1)^{n-i}\prod_{j=2}^{n+1-i}\Big(\lambda_j-\nu_i-\frac{1}{2}\Big),$$
$$p_{\mathcal{F}_i}(\lambda,\nu) = \prod_{j=1}^{n+1-i}\Big(\lambda_i-\nu_j-\frac{1}{2}\Big).$$

Proof. As the identities of Theorem 5.3 and Proposition 6.3 are identities of distributions that can be meromorphically extended, we can regard them as formal distributional equalities and disregard the domain of parameters (λ,ν) for which they hold. Let $c'_i(\xi,\lambda,\nu,\eta) = L(0,\chi_i^{-1}\chi_{i+1})c_i(\xi,\lambda,\nu,\eta)$ that is $c_i(\xi,\lambda,\eta,\nu)$ multiplied by the normalization of $T^{w_i}_{\xi,-\lambda}$. Now by definition of $\mathcal{D}_1(\lambda)$ and Theorem 5.3 we get

$$\begin{aligned}\lambda_{1,2}c'_1(w_1(\xi,\lambda)+(e_1,e_1),\eta,\nu)K^{\eta,\nu}_{(\xi,\lambda)+(e_2,e_2)} &= \lambda_{1,2}T^{w_1}_{-w_1(\xi,\lambda)-(e_1,e_1)}(\Phi_1K^{\eta,\nu}_{w_1(\xi,\lambda)})\\ &= \mathcal{D}_1(\lambda)T^{w_1}_{-w_1(\xi,\lambda)}K^{\eta,\nu}_{w_1(\xi,\lambda)} = c'_1(w_1(\xi,\lambda),\eta,\nu)\mathcal{D}_1(\lambda)K^{\eta,\nu}_{\xi,\lambda},\end{aligned}$$

so we get $p_{\mathcal{D}_1}(\lambda,\nu) = \frac{\lambda_{1,2}c'_1(w_1(\xi,\lambda)+(e_1,e_1),\eta,\nu)}{c'_1(w_1(\xi,\lambda),\eta,\nu)}$. Arguing like this, we arrive at

$$p_{\mathcal{D}_i}(\lambda,\nu) = \frac{\lambda_{i,i+1}c'_i(w_i(\xi,\lambda)+(e_i,e_i),\eta,\nu)}{c'_i(\xi,\lambda,\eta,\nu)}p_{\mathcal{D}_{i-1}}(w_i\lambda,\nu).$$

As $\Gamma(z+1) = z\Gamma(z)$ we can, by checking cases for $\varepsilon\in\{0,1\}$, confirm the following identity:

$$\frac{\Gamma(\frac{z-1+[\varepsilon+1]}{2})\Gamma(\frac{1-z+[\varepsilon]}{2})}{\Gamma(\frac{z+[\varepsilon]}{2})\Gamma(\frac{2-z+[\varepsilon+1]}{2})} = (-1)^{\varepsilon+1}\frac{2}{z-1}.$$

Using this we get

$$\frac{\lambda_{i,i+1}c'_i(w_i(\xi,\lambda)+(e_i,e_i),\eta,\nu)}{c'_i(\xi,\lambda,\eta,\nu)} = (-1)\Big(\nu_{n+1-i}-\lambda_{i+1}-\frac{1}{2}\Big).$$

Similarly setting $d'_i(\xi,\lambda,\eta,\nu) = L(0,\psi_i\psi_{i+1}^{-1})d_i(\xi,\lambda,\eta,\nu)$ we get

$$\frac{\nu_{i+1,i}d'_i(\xi+e_1,\lambda+e_1,w_i(\eta,\nu)+(e_{i+1},e_{i+1}))}{d'_i(\xi,\lambda,w_i(\eta,\nu))} = (-1)\Big(\lambda_{n+1-i}-\nu_i-\frac{1}{2}\Big),$$

and

$$\frac{\lambda_{i,i+1}c'_i(w_i(\xi,\lambda)+(\mathbb{1}-e_{i+1},\mathbb{1}-e_{i+1}),(\eta,\nu)+(\mathbb{1},\mathbb{1}))}{c'_i(w_i(\xi,\lambda),\eta,\nu)} = \lambda_i-\nu_{n+1-i}-\frac{1}{2}. \qquad \square$$

Remark 6.6. The differential operators $\mathcal{D}_i(\lambda)$, $\mathcal{C}_i(\nu)$, and $\mathcal{F}_i(\lambda)$ are somewhat arbitrary as there are many of choices in the process. As we will see below the choices are made so we get $2n$ differential operators which can be used to analytically extend $K^{\eta,\nu}_{\xi,\lambda}$. Many different choices of multiplication operators of Φ_i or Ψ_j and permutations w, can also give a collection of differential operators that can extend $K^{\eta,\nu}_{\xi,\lambda}$, but there are a few immediate limitations.

For a choice of Φ_i or Ψ_j, say Φ_i, there is only one choice of simple transposition where one picks up any differential operator since $T^{w_j}_{-w_j(\xi,\lambda)-(e_i,e_i)}$ and M_{Φ_i} commute when $j \neq i$.

We can also mix Knapp–Stein intertwiners from $\pi_{\xi,-\lambda}$ and $\tau_{\eta,\nu}$ in one differential operator, but as these intertwiners commute, it ends up being the composition of two differential operators. If we let $n = 2$ and consider every outcome of conjugating Φ_i or Ψ_j with only intertwiners $T^w_{\xi-\lambda}$ for $\pi_{\xi,-\lambda}$, then a collection of differential operators extending $K^{\eta,\nu}_{\xi,\lambda}$ can not be obtained, which suggest that at some point the intertwiners $T^w_{\eta,\nu}$ for $\tau_{\eta,\nu}$ must be used.

We could hope to find a collection of differential operators where the amount of times we need to conjugate by intertwiners from simple transpositions are minimal, thus making the order of the differential operators smaller. But as analytic extensions are unique the Bernstein–Sato polynomials must have the same roots no matter the collection. If we arrange (λ,ν) as $(\lambda_1,\nu_n,\lambda_2,\dots,\nu_1,\lambda_{n+1})$ then each conjugation by an intertwiner from a simple transposition grants a linear factor as a difference between two adjacent element in this vector. So to obtain a factor like $\lambda_1-\nu_1-\frac{1}{2}$, a minimum of n simple transpositions must be done.

7 Analytic extension of the integral kernel

Consider the normalized kernel

$$\mathbb{K}^{\eta,\nu}_{\xi,\lambda} = \frac{K^{\eta,\nu}_{\xi,\lambda}}{\prod_{j=1}^{n}\Big[\prod_{i=1}^{n+1-j}\Gamma\Big(\lambda_i-\nu_j+\frac{1}{2}\Big)\Big]\times\Big[\prod_{i=n+2-j}^{n+1}\Gamma\Big(\nu_j-\lambda_i+\frac{1}{2}\Big)\Big]}.$$

Rewriting the Bernstein–Sato identities of Theorem 6.5 in terms of $\mathbb{K}^{\eta,\nu}_{\xi,\lambda}$, we have

$$\begin{aligned}
\mathcal{D}_i(\lambda)\mathbb{K}^{\eta,\nu}_{\xi,\lambda} &= \mathbb{K}^{\eta,\nu}_{(\xi+e_{i+1},\lambda+e_{i+1})},\\
\mathcal{C}_i(\nu)\mathbb{K}^{\eta,\nu}_{\xi,\lambda} &= \mathbb{K}^{(\eta+e_i,\nu+e_i)}_{(\xi+e_1,\lambda+e_1)},\\
\mathcal{F}_i(\lambda)\mathbb{K}^{\eta,\nu}_{\xi,\lambda} &= \mathbb{K}^{(\eta+\mathbb{1},\nu+\mathbb{1})}_{(\xi+\mathbb{1}-e_i,\lambda+\mathbb{1}-e_i)}.
\end{aligned}$$

Proposition 7.1. *$\mathbb{K}^{\eta,\nu}_{\xi,\lambda}$ extends analytically to a family of distributions that depends holomorphically on $(\lambda,\nu)\in\mathbb{C}^{n+1}\times\mathbb{C}^n$ and $\mathbb{K}^{\eta,\nu}_{\xi,\lambda}\in\mathcal{D}'(G)^{\eta,\nu}_{\xi,\lambda}$ for all (λ,ν).*

Proof. Rewriting the Bernstein–Sato identities in terms of (δ,s,ε,t), we get

$$\begin{aligned}
\mathcal{D}_i(s,t)\mathbb{K}^{\varepsilon,t}_{\delta,s} &= \mathbb{K}^{(\varepsilon-e_i,t-e_i)}_{(\delta+e_{i+1},s+e_{i+1})},\\
\mathcal{C}_i(s,t)\mathbb{K}^{\varepsilon,t}_{\delta,s} &= \mathbb{K}^{(\varepsilon+e_{n+1-i},t+e_{n+1-i})}_{(\delta+e_1-e_{n+1-i},s+e_1-e_{n+1-i})},\\
\mathcal{F}_1(s,t)\mathbb{K}^{\varepsilon,t}_{\delta,s} &= \mathbb{K}^{\varepsilon,t}_{(\delta-e_1+e_{n+1},s-e_1+e_{n+1})}.
\end{aligned}$$

As $\mathbb{K}^{\varepsilon,t}_{\delta,s}$ is already defined for all $s_{n+1} \in \mathbb{C}$ we can use $\mathcal{F}_1(s,t)$ to extend $\mathbb{K}^{\varepsilon,t}_{\delta,s}$ to all $s_1 \in \mathbb{C}$. Then using $\mathcal{D}_n(s,t)$ we can extend $\mathbb{K}^{\varepsilon,t}_{\delta,s}$ to all $t_n \in \mathbb{C}$. Now using $\mathcal{C}_1(s,t)$ we can extend to all $s_n \in \mathbb{C}$ and so forth, switching between $\mathcal{D}_i(s,t)$ and $\mathcal{C}_j(s,t)$ we can extend to all (s,t) and therefore (λ,ν). □

Using the duplication formula write

$$\Gamma\Big(\lambda_i - \nu_j + \frac{1}{2}\Big) = \frac{2^{\lambda_i-\nu_j-\frac{1}{2}}}{\sqrt{\pi}}\Gamma\Big(\frac{\lambda_i-\nu_j+\frac{1}{2}+[\xi_i+\eta_j]}{2}\Big)\Gamma\Big(\frac{\lambda_i-\nu_j+\frac{3}{2}-[\xi_i+\eta_j]}{2}\Big).$$

For $\xi,\eta \in \mathbb{Z}/2\mathbb{Z}$ and $z \in \mathbb{C}$ we write $z+[\xi+\eta]$ to mean the addition $\xi+\eta$ should be done in $\mathbb{Z}/2\mathbb{Z}$ first and then either a 0 or 1 should be added to z. Comparing this with the analytically extended Remark 5.4, we get

$$\mathbb{K}^{\eta,\nu}_{\xi,\lambda} = \beta_i(\xi,\lambda,\eta,\nu)\mathbf{T}^{w_i}_{w_i(\xi,-\lambda)}\mathbb{K}^{\eta,\nu}_{w_i(\xi,\lambda)},$$

where $\beta_i(\xi,\lambda,\eta,\nu)$ is given as

$$\frac{\Gamma(\frac{\lambda_{i+1}-\lambda_i+1+[\xi_i+\xi_{i+1}]}{2})\Gamma(\frac{\lambda_{i+1}-\nu_{n+1-i}+\frac{3}{2}-[\xi_{i+1}+\eta_{n+1-i}]}{2})\Gamma(\frac{\nu_{n+1-i}-\lambda_i+\frac{3}{2}-[\xi_i+\eta_{n+1-i}]}{2})}{c\Gamma(\frac{\nu_{n+1-i}-\lambda_{i+1}+\frac{3}{2}-[\xi_{i+1}+\eta_{n+1-i}]}{2})\Gamma(\frac{\lambda_i-\nu_{n+1-i}+\frac{3}{2}-[\xi_i+\eta_{n+1-i}]}{2})}$$

and c is a product of powers of π, 2 and -1. As $\mathbb{K}^{\eta,\nu}_{\xi,\lambda}$ and $\mathbf{T}^{w_i}_{w_i(\xi,-\lambda)}\mathbb{K}^{\eta,\nu}_{w_i(\xi,\lambda)}$ are holomorphic in (λ,ν), we see that $\mathbb{K}^{\eta,\nu}_{\xi,\lambda}$ has zeroes given by the zeroes of $\beta_i(\xi,\lambda,\eta,\nu)$. This shows that $\mathbb{K}^{\eta,\nu}_{\xi,\lambda}$ is over-normalized and the factors $\Gamma(\lambda_i-\nu_{n+1-i}+\frac{1}{2})$, $\Gamma(\nu_{n+1-i}-\lambda_{i+1}+\frac{1}{2})$ can be replaced by $\Gamma(\frac{\lambda_i-\nu_{n+1-i}+\frac{1}{2}+[\xi_i+\eta_{n+1-i}]}{2})$, $\Gamma(\frac{\nu_{n+1-i}-\lambda_{i+1}+\frac{1}{2}+[\xi_{i+1}+\eta_{n+1-i}]}{2})$ in the normalization of $K^{\eta,\nu}_{\xi,\lambda}$.

Consider the re-normalized kernel

$$\mathbf{K}^{\eta,\nu}_{\xi,\lambda} = \frac{K^{\eta,\nu}_{\xi,\lambda}}{\prod_{j=1}^{n} L(\frac{1}{2},\chi_1\psi_j^{-1})\dots L(\frac{1}{2},\chi_{n+1-j}\psi_j^{-1})L(\frac{1}{2},\chi_{n+2-j}^{-1}\psi_j)\dots L(\frac{1}{2},\chi_{n+1}^{-1}\psi_j)},$$

which, by the duplication formula, can be seen as a product between $\mathbb{K}^{\eta,\nu}_{\xi,\lambda}$,

$$\prod_{j=1}^{n}\Big[\prod_{i=1}^{n+1-j}\frac{2^{\lambda_i-\nu_j-\frac{1}{2}}}{\sqrt{\pi}}\Gamma\Big(\frac{\lambda_i-\nu_j+\frac{3}{2}-[\xi_i+\eta_j]}{2}\Big)\Big] \times\Big[\prod_{i=n+2-j}^{n+1}\frac{2^{\nu_j-\lambda_i-\frac{1}{2}}}{\sqrt{\pi}}\Gamma\Big(\frac{\nu_j-\lambda_i+\frac{3}{2}-[\xi_i+\eta_j]}{2}\Big)\Big] \tag{7.1}$$

and some powers of π.

Theorem 7.2. *The normalized kernel $\mathbf{K}^{\eta,\nu}_{\xi,\lambda}$ extends analytically to a family of distributions that depends holomorphically on $(\lambda,\nu)\in\mathbb{C}^{n+1}\times\mathbb{C}^n$ and $\mathbf{K}^{\eta,\nu}_{\xi,\lambda}\in\mathcal{D}'(G)^{\eta,\nu}_{\xi,\lambda}$ for all $(\lambda,\nu)\in\mathbb{C}^{n+1}\times\mathbb{C}^n$.*

Proof. To prove this, it suffices to show $\mathbb{K}^{\eta,\nu}_{\xi,\lambda}$ vanishes at all the poles from (7.1). Arguing like above instead of using Remark 5.4 we can use Corollary 5.5. Considering all $i = 1,2,\dots,n$ the functions $b_i^{\pm}(\xi,\lambda,\eta,\nu)$ contains all the different Gamma-factors appearing in the new normalization. □

Remark 7.3. With this new normalization Theorem 5.3 can be stated in the slightly more crisp way

$$\mathbf{T}^{w_i}_{\xi,-\lambda}\mathbf{K}^{\eta,\nu}_{\xi,\lambda} = \frac{\sqrt{\pi}(-1)^{(\xi_i+\xi_{i+1})(\eta_{n+1-i}+1)+\xi_i\xi_{i+1}}}{L(1,\chi_i\chi_{i+1}^{-1})}\mathbf{K}^{\eta,\nu}_{w_i(\xi,\lambda)},$$

and

$$\mathbf{T}^{w_i}_{\eta,\nu}\mathbf{K}^{\eta,\nu}_{\xi,\lambda} = \frac{\sqrt{\pi}(-1)^{(\eta_i+\eta_{i+1})(\xi_{n+1-i}+1)+\eta_i\eta_{i+1}}}{L(1,\psi_i^{-1}\psi_{i+1})}\mathbf{K}^{w_i(\eta,\nu)}_{\xi,\lambda}.$$

8 On the zeroes for the symmetry breaking operators

As explained in Section 2 from the integral kernel $\mathbf{K}^{\eta,\nu}_{\xi,\lambda}$, we obtain a symmetry breaking operator $\mathbf{A}^{\eta,\nu}_{\xi,\lambda} : \pi_{\xi,\lambda}|_H \to \tau_{\eta,\nu}$ by

$$\mathbf{A}^{\eta,\nu}_{\xi,\lambda}f(h) = \int_{\overline{N}_G} \mathbf{K}^{\eta,\nu}_{\xi,\lambda}(h^{-1}\overline{n})f(\overline{n})\,d\overline{n},$$

where the integral has to be considered in the distributional sense and Theorem 7.2 allows us to do this for every $(\lambda,\nu) \in \mathbb{C}^{n+1}\times\mathbb{C}^n$. The functional equations with the Knapp–Stein intertwining operators from Remark 7.3, can then be rephrased as

$$\mathbf{A}^{\eta,\nu}_{\xi,\lambda}\circ\mathbf{T}^{w_i}_{w_i(\xi,\lambda)} = \frac{\sqrt{\pi}(-1)^{(\xi_i+\xi_{i+1})(\eta_{n+1-i}+1)+\xi_i\xi_{i+1}}}{L(1,\chi_i\chi_{i+1}^{-1})}\mathbf{A}^{\eta,\nu}_{w_i(\xi,\lambda)},$$

and

$$\mathbf{T}^{w_i}_{\eta,\nu}\circ\mathbf{A}^{\eta,\nu}_{\xi,\lambda} = \frac{\sqrt{\pi}(-1)^{(\eta_i+\eta_{i+1})(\xi_{n+1-i}+1)+\eta_i\eta_{i+1}}}{L(1,\psi_i^{-1}\psi_{i+1})}\mathbf{A}^{w_i(\eta,\nu)}_{\xi,\lambda}.$$

To investigate if the normalization found in Section 7 is optimal, a good place to start is to see if we can evaluate the symmetry breaking operator on the K-invariant vector $\mathbb{1}_\lambda$, that is the vector that is constant on K with normalization $\mathbb{1}_\lambda(e) = 1$. Consider the analytic function $\beta(\lambda,\nu)$ defined by

$$\mathbf{A}^{\nu}_{\lambda}\mathbb{1}_\lambda = \beta(\lambda,\nu)\mathbb{1}_\nu,$$

where we suppress the trivial M-representations in the notation. Now by Proposition A.3

$$\mathbf{T}^{w_i}_{\lambda}\mathbb{1}_\lambda(e) = \frac{1}{L(0,\chi_i\chi_{i+1}^{-1})}\int_{\mathbb{R}}(1+x^2)^{\frac{\lambda_{i+1}-\lambda_i-1}{2}}dx = \frac{1}{L(1,\chi_i\chi_{i+1}^{-1})}\mathbb{1}_{w_i\lambda}(e). \tag{8.1}$$

Using this and the functional equations above we get that

$$\beta(w_i\lambda,\nu) = \frac{L(1,\chi_i\chi_{i+1}^{-1})}{L(1,\chi_i^{-1}\chi_{i+1})}\beta(\lambda,\nu) \quad \& \quad \beta(\lambda,w_i\nu) = \frac{L(1,\psi_i^{-1}\psi_{i+1})}{L(1,\psi_i\psi_{i+1}^{-1})}\beta(\lambda,\nu). \tag{8.2}$$

As $\beta(\lambda,\nu)$ is analytic this implies that $\beta(\lambda,\nu) = 0$ when $\lambda_i - \lambda_j \in -2\mathbb{N}_0 - 1$ or $\nu_j - \nu_i \in -2\mathbb{N}_0 - 1$ and $i < j$, which corresponds to points of reducibility for π_λ and τ_ν see [SV80]. In [FS22] Frahm and Su calculated $\beta(\lambda,\nu)$ in the case where $n = 2$, that is for the pair $(\mathrm{GL}(3,\mathbb{R}),\mathrm{GL}(2,\mathbb{R}))$, and found that

$$\beta(\lambda,\nu) = \frac{\pi^{\frac{3}{2}}}{L(1,\chi_1\chi_2^{-1})L(1,\chi_2\chi_3^{-1})L(1,\chi_1\chi_3^{-1})L(1,\psi_1^{-1}\psi_2)}.$$

The L-factors that appear are related to the Harish–Chandra c-function $c(\lambda)$, which is given by the equation

$$T_\lambda^{w_0}\mathbb{1}_\lambda(e) = c(\lambda),$$

where w_0 is the longest Weyl-group element. A formula for $c(\lambda)$ was found by Gindikin & Karpelevič in [GK62]. We consider normalized Harish–Chandra c-functions (also known as the e-function) defined by

$$\mathbf{T}_\lambda^{w_0}\mathbb{1}_\lambda(e) = \mathbf{c}_G(\lambda), \quad \mathbf{T}_\nu^{w_0'}\mathbb{1}_\nu(e) = \mathbf{c}_H(\nu),$$

where w_0' is the longest Weyl-group element of H. In the case $n = 2$, we can calculate $\mathbf{c}_G(\lambda)$ directly by decomposing $w_0 = w_1 w_2 w_1$ and using (8.1) and (2.1) to get

$$\mathbf{c}_G(\lambda) = \mathbf{T}_\lambda^{w_0}\mathbb{1}_\lambda(e) = \mathbf{T}^{w_1}_{w_2 w_1 \lambda}\mathbf{T}^{w_2}_{w_1\lambda}\mathbf{T}^{w_1}_\lambda\mathbb{1}_\lambda = \frac{1}{L(1,\chi_1\chi_2^{-1})L(1,\chi_2\chi_3^{-1})L(1,\chi_1\chi_3^{-1})}.$$

In [FS21] Frahm and Su conjectures:

Conjecture 8.1 (Frahm & Su 2021)**.**

$$\beta(\lambda,\nu) = \mathbf{c}_G(\lambda)\mathbf{c}_H(-\nu).$$

The identities of (8.2) supports this conjecture. In the case where this conjecture is true then the following Theorem also holds for $n > 2$ with the same proof.

Theorem 8.2. *For $n = 2$ the normalization used for $\mathbf{A}_\lambda^\nu$ is optimal, in the sense that the zeroes of $\mathbf{A}_\lambda^\nu$ are of codimension two in $(\lambda,\nu) \in \mathbb{C}^{n+1} \times \mathbb{C}^n$.*

Proof. The zeroes of $\mathbf{A}_\lambda^\nu$ can only occur at places where the normalization multiplies by a zero. At a zero either $\lambda_i - \nu_j + \frac{1}{2} \in -2\mathbb{N}_0$ or $\nu_j - \lambda_i + \frac{1}{2} \in -2\mathbb{N}_0$ must be satisfied. The zeroes of $\beta(\lambda,\nu)$ are of the form $\lambda_i - \lambda_j + 1 \in -2\mathbb{N}_0$ and $\nu_j - \nu_i + 1 \in -2\mathbb{N}_0$. Thus at least one more equation must be satisfied in order to have a zero for $\mathbf{A}_\lambda^\nu$. □

Proposition 8.3. *For $1 \le j \le n+1-i \le n$ and $m \in \mathbb{N}_0$ if $\lambda_i - \nu_j + \frac{1}{2} + [\xi_i + \eta_j] = -2m$, the support of $\mathbf{K}_{\xi,\lambda}^{\eta,\nu}$ is contained in*

$$\{\Phi_i = 0\} \cap \{\Psi_i = 0\} \cap \cdots \cap \{\Phi_{n+1-j} = 0\}.$$

For $2 \le n+2-j \le i \le n+1$ and $m \in \mathbb{N}_0$ if $\nu_j - \lambda_i + \frac{1}{2} + [\eta_j + \xi_i] = -2m$, the support of $\mathbf{K}_{\xi,\lambda}^{\eta,\nu}$ is contained in

$$\{\Psi_{n+1-j} = 0\} \cap \{\Phi_{n+2-j} = 0\} \cdots \cap \{\Psi_{i-1} = 0\}.$$

Proof. The proof follows in the same way as for the Riesz distribution. As functions, for $\mathrm{Re}(s), \mathrm{Re}(t) > 0$ and $m \in \mathbb{N}$, we have

$$\Phi_i^{2m}\mathbf{K}_{\delta,s}^{\varepsilon,t} = \Big(\prod_{k=1}^{i} \alpha_{k,i,m}(\delta,s,\varepsilon,t)\Big)\Big(\prod_{k=1}^{i-1} \gamma_{k,i,m}(\delta,s,\varepsilon,t)\Big)\mathbf{K}_{\delta,s+2me_i}^{\varepsilon,t},$$

$$\Psi_i^{2m}\mathbf{K}_{\delta,s}^{\varepsilon,t} = \Big(\prod_{k=1}^{i} \alpha_{k,i+1,m}(\delta,s,\varepsilon,t)\Big)\Big(\prod_{k=1}^{i} \gamma_{k,i,m}(\delta,s,\varepsilon,t)\Big)\mathbf{K}_{\delta,s}^{\varepsilon,t+2me_i},$$

where for $i \leq k$

$$\alpha_{k,i,m}(\delta, s, \varepsilon, t) = \prod_{j=k}^{n} \Big(\frac{s_i + t_i + \cdots + t_{j-1} + s_j + j + 1 - k + [\delta_i + \varepsilon_i + \cdots + \varepsilon_{j-1} + \delta_j]}{2}\Big)_m$$

$$\gamma_{k,i,m}(\delta, s, \varepsilon, t) = \prod_{j=k}^{n} \Big(\frac{t_i + s_{i+1} + \cdots + s_j + t_j + j + 1 - k + [\varepsilon_i + \delta_{i+1} + \cdots + \delta_j + \varepsilon_j]}{2}\Big)_m$$

Both sides of these equations extends analytically as distributions to all (s, t). If $\lambda_i - \nu_j + \frac{1}{2} + [\xi_i + \eta_j] = -2\ell$ then $s_i + t_i + \cdots + s_{n+1-j} + n + 1 - j - i + [\delta_i + \varepsilon_i + \cdots + \delta_{n+1-j}] = -2\ell$. Thus by keeping track of which $\alpha_{k,i,m}(\delta, s, \varepsilon, t)$ and $\gamma_{k,i,m}(\delta, s, \varepsilon, t)$ vanishes, we get the result. □

Theorem 8.4. *The operator $A^{\eta,\nu}_{\xi,\lambda}$ vanishes on the the sets*

$$\mathcal{N}_{i,j,k} = \{\lambda_i - \nu_k + \tfrac{1}{2} + [\xi_i + \eta_k] \in -2\mathbb{N}_0\} \cap \{\nu_k - \lambda_j + \tfrac{1}{2} + [\eta_k + \xi_j] \in -2\mathbb{N}_0\},$$

where $i < j$ and $k \in \{1, \ldots, n\}$ and

$$\mathcal{M}_{i,j,k} = \{\nu_j - \lambda_k + \tfrac{1}{2} + [\eta_j + \xi_k] \in -2\mathbb{N}_0\} \cap \{\lambda_k - \nu_i + \tfrac{1}{2} + [\xi_k + \eta_i] \in -2\mathbb{N}_0\},$$

where $i < j$ and $k \in \{1, \ldots, n+1\}$.

We prove this theorem in a series of lemmas.

Lemma 8.5. *Let $t_1 + 1 = 0$. If $\mathrm{Re}(s_i) \geq 0$, and $\mathrm{Re}(t_j) \geq 0$ for $i = 1, \ldots, n$, and $j = 2, \ldots, n$ then*

$$\boldsymbol{K}^t_s(g) = \frac{\delta(g_{n,1})|g_{n+1,1}|^{s_1+s_2}|g_{n,2}|^{s_2+t_2}|g_{n-1,1}|^{t_2}\prod_{i=3}^{n+1}|\Phi_i(g)|^{s_i}\prod_{j=3}^{n}|\Psi_j(g)|^{t_j}}{n'(s,t)},$$

where $n'(s,t) = (n(s,t)/\Gamma(\frac{t_1+1}{2}))|_{t_1=-1}$. Furthermore, $\boldsymbol{A}^t_s$ vanishes when $s_1 + 1 \in -2\mathbb{N}_0$ and $s_2 + 1 \in -2\mathbb{N}_0$.

Proof. For the (s, t) considered, we have that $|\Phi_i|^{s_i}$ and $|\Psi_i|^{t_j}$ are continuous functions. The Riesz distribution $|\Psi_1(g)|^{t_1}/\Gamma(\frac{t_1+1}{2}) = \delta(g_{n,1})$ is a distribution on continuous functions with compact support. Thus, the product above is defined and as Diracs delta function satisfies $f(x)\delta(x) = f(0)\delta(x)$, we get the form for $\mathbf{K}^t_s$ as claimed. Strictly speaking, $\mathbf{K}^t_s$ is defined by its analytic continuation which comes from the Bernstein–Sato identities, but as it is continuous we can consider setting $t_1 = -1$ as taking a limit from the right.

The normalization $n(s,t)$ contains the Gamma-factors $\Gamma(\frac{s_1+1}{2})$ and $\Gamma(\frac{s_2+1}{2})$. Considering the case for $s_1 + 1 \in -2\mathbb{N}_0$, if we let $s_1 = -2n - 1$ and $\mathrm{Re}(s_2) \geq 2n + 1$ along with the rest of the exponents having a real-part ≥ 0, then we still have a continuous function multiplied on a Dirac delta function, but $n'(s,t)^{-1} = 0$ so the whole thing vanishes. As $\mathbf{K}^t_s$ is entire holomorphic in s_i, t_i for $i \geq 2$ and vanishes on an open set it is identically zero. The case with $s_2 + 1 \in -2\mathbb{N}_0$ follows in a similar way. □

We now use the Bernstein–Sato identities to shift the condition on $t_1 + 1$ to even negative integers. Let $\widetilde{\mathcal{D}}_1(s,t) = \mathcal{D}_1(s + e_2, t - e_1)\mathcal{D}_1(s,t)$ and note that

$$\widetilde{\mathcal{D}}_1(s,t)\mathbf{K}^t_s = t_1(t_1 - 1)\frac{n(s + 2e_2, t_1 - 2e_1)}{n(s,t)}\mathbf{K}^{t-2e_1}_{s+2e_2} = t_1(s_2 + 1)\mathbf{K}^{t-2e_1}_{s+2e_2}. \tag{8.3}$$

Lemma 8.6. *Let $t_1+1 \in -2\mathbb{N}_0$ then $\mathbf{A}_s^t$ vanishes for $s_1+1 \in -2\mathbb{N}_0$ and for $s_2+1 \in -2\mathbb{N}_0$.*

Proof. We do this by induction on $t_1+1 = -2k$ in k, where the base case have been done in Lemma 8.5. Assume that for $t_1+1=-2k$ then $\mathbf{A}_s^t$ vanishes for $s_1+1 \in -2\mathbb{N}_0$ and for $s_2+1 \in -2\mathbb{N}_0$. By (8.3), we have

$$\widetilde{\mathcal{D}}_1(s-2e_2, t+2e_1)\mathbf{K}_{s-2e_2}^{t+2e_1} = (t_1+2)(s_2-1)\mathbf{K}_s^t.$$

By the induction hypothesis the left hand side vanishes for $t_1+3=-2k$, and either $s_1+1 \in -2\mathbb{N}_0$ or $s_2-1 \in -2\mathbb{N}_0$. If $t_1+1=-2(k+1)$, $s_1+1 \in -2\mathbb{N}_0$, and $s_2 \neq 1$ we get that $\mathbf{K}_s^t = 0$ and by continuity of $\mathbf{K}_s^t$, we can remove the condition $s_2 \neq 0$. For $n \geq 0$ if $t_1+1=-2(k+1)$, $s_2-1=-2n$ and $s_2 \neq 1$ then $\mathbf{K}_s^t = 0$, but the two conditions on s_2 corresponds to $s_2+1 \in -2\mathbb{N}_0$. ☐

We now extend the zeroes to the non-spherical case.

Lemma 8.7. *Let $t_1+\varepsilon_1+1 = -2k$ then $\mathbf{A}_{\delta,s}^{\varepsilon,t}$ vanishes for $s_1+\delta_1+1 \in -2\mathbb{N}_0$, and for $s_2+\delta_2+1 \in -2\mathbb{N}_0$.*

Proof. Consider the identity,

$$\mathbf{K}_{\delta,s}^{\varepsilon,t} = \frac{n(0,s-\delta,0,t-\varepsilon)}{n(\delta,s,\varepsilon,t)}\Big(\prod_{i=1}^{n+1}\Phi_i^{\delta_i}\prod_{j=1}^{n}\Psi_j^{\varepsilon_j}\Big)\mathbf{K}_{0,s-\delta}^{0,t-\varepsilon}.$$

As $\Phi_i^{\delta_i}$ and $\Psi_j^{\varepsilon_j}$ are just polynomials if we can argue that generically $n(0,s-\delta,0,t-\varepsilon)/n(\delta,s,\varepsilon,t)$ does not have poles at the zeroes we found for $\mathbf{K}_{s-\delta}^{t-\varepsilon}$ in Lemma 8.6 then we get the claimed result. The first kind of factors that appears in $n(0,s-\delta,0,t-\varepsilon)/n(\delta,s,\varepsilon,t)$ is

$$\frac{\Gamma(\frac{s_1+1-\delta_1}{2})}{\Gamma(\frac{s_1+1+\delta_1}{2})} = \begin{cases} 1, & \delta_1 = 0, \\ \frac{2}{s_1}, & \delta_1 = 1, \end{cases}$$

which are regular for the points considered. The next kind of factor consists of sums of s_i and t_j with an odd number of terms e.g.

$$\frac{\Gamma(\frac{s_1+t_1+s_2+2-\delta_1-\varepsilon_1-\delta_2}{2})}{\Gamma(\frac{s_1+t_1+s_2+2+[\delta_1+\varepsilon_1+\delta_2]}{2})} = \begin{cases} 1, & \delta_1+\varepsilon_1+\delta_2 = 0, \\ \frac{2}{s_1+t_1+s_2+1}, & \delta_1+\varepsilon_1+\delta_2 = 1, \\ \frac{2}{s_1+t_1+s_2+2}, & \delta_1+\varepsilon_1+\delta_2 = 2, \\ \frac{4}{(s_1+t_1+s_2+1)(s_1+t_1+s_2-1)}, & \delta_1+\varepsilon_1+\delta_2 = 3, \end{cases}$$

in either case we are at most restricting two variables so generically this type of factor is regular. ☐

We are now ready to prove Theorem 8.4.

Proof. The zeroes we found in Lemma 8.7 in (ξ,λ,η,ν)-coordinates are

$$\mathcal{N}_{1,2,n} = \{\lambda_1-\nu_n+\tfrac{1}{2}+[\xi_1+\eta_n] \in -2\mathbb{N}_0\} \cap \{\nu_n-\lambda_2+\tfrac{1}{2}+[\eta_n+\xi_2] \in -2\mathbb{N}_0\}$$
$$\mathcal{M}_{n-1,n,2} = \{\nu_n-\lambda_2+\tfrac{1}{2}+[\eta_n+\xi_2] \in -2\mathbb{N}_0\} \cap \{\lambda_2-\nu_{n-1}+\tfrac{1}{2}+[\xi_2+\eta_{n-1}] \in -2\mathbb{N}_0\}.$$

Consider the two functional identities

$$\mathbf{A}^{\eta,\nu}_{\xi,\lambda} \circ \mathbf{T}^{w_i}_{w_i(\xi,\lambda)} = \frac{c}{\Gamma(\frac{\lambda_i - \lambda_{i+1} + 1 + [\xi_i + \xi_{i+1}]}{2})} \mathbf{A}^{\eta,\nu}_{w_i(\xi,\lambda)},$$
$$\mathbf{T}^{w_i}_{\eta,\nu} \circ \mathbf{A}^{\eta,\nu}_{\xi,\lambda} = \frac{c'}{\Gamma(\frac{\nu_{i+1} - \nu_i + 1 + [\eta_{i+1} + \eta_i]}{2})} \mathbf{A}^{w_i(\eta,\nu)}_{\xi,\lambda},$$

for some constants c, c' that depends on (ξ, η). Using the first one for $i = 2$ we can for $\lambda_2 - \lambda_3 + 1 + [\xi_2 + \xi_3] \notin -2\mathbb{N}_0$ conclude that the zeroes of $\mathbf{A}^{\eta,\nu}_{\xi,\lambda}$ from $\mathcal{N}_{1,2,n}$ are also zeroes for $\mathbf{A}^{\eta,\nu}_{w_2(\xi,\lambda)}$. This implies that $\mathcal{N}_{1,3,n} \cap \{\lambda_3 - \lambda_2 + 1 + [\xi_2 + \xi_3] \notin -2\mathbb{N}_0\}$ are zeroes for $\mathbf{A}^{\eta,\nu}_{\xi,\lambda}$. If $(\xi, \lambda, \eta, \nu) \in \mathcal{N}_{1,3,n} \cap \{\lambda_3 - \lambda_2 + 1 + [\xi_3 + \xi_2] \in -2\mathbb{N}_0\}$, then $\mathbf{A}^{\eta,\nu}_{\xi,\lambda}$ vanishes on the sequence $(\xi, \lambda + \frac{1}{n+1} e_2, \eta, \nu)$, and by continuity we get $\mathbf{A}^{\eta,\nu}_{\xi,\lambda}$ vanishes on all of $\mathcal{N}_{1,3,n}$. Applying this procedure for both the functional identities and all permutations w_i we get all the vanishing sets from Theorem 8.4. We note that using the first identity for $i = 1$ would not provide any new zeroes, as the Gamma-factor vanishes on all of $\mathcal{N}_{1,2,n}$ which means we are not allowed to swap the order $i < j$. □

A Integral Formulas

Proposition A.1 (See [BD22][Prop. B.2]). *For* $Re(\alpha), Re(\beta) > -1$ *with* $0 > Re(\alpha + \beta + 1)$ *we have*

$$\int_{\mathbb{R}} |y|^{\alpha}_{\varepsilon} |x - y|^{\beta}_{\xi} dy = t(\alpha, \beta, \varepsilon, \xi) |x|^{\alpha+\beta+1}_{\varepsilon+\xi},$$

where

$$t(\alpha, \beta, \varepsilon, \xi) = (-1)^{\varepsilon\xi} \sqrt{\pi} \frac{\Gamma(\frac{\alpha+1+\varepsilon}{2}) \Gamma(\frac{\beta+1+\xi}{2}) \Gamma(\frac{-\alpha-\beta-1+[\varepsilon+\xi]_2}{2})}{\Gamma(\frac{-\alpha+\varepsilon}{2}) \Gamma(\frac{-\beta+\xi}{2}) \Gamma(\frac{\alpha+\beta+2+[\xi+\varepsilon]_2}{2})}.$$

Corollary A.2. *Let* $a, c \in \mathbb{R}^{\times}$ *and either* b *or* d *non-zero in* $\mathbb{R}$. *For* $Re(\alpha), Re(\beta) > -1$ *with* $0 > Re(\alpha + \beta + 1)$ *we have*

$$\int_{\mathbb{R}} |ax + b|^{\alpha}_{\varepsilon} |cx + d|^{\beta}_{\xi} dx = (-1)^{\varepsilon} t(\alpha, \beta, \varepsilon, \xi) |a|^{-\beta-1}_{\xi} |c|^{-\alpha-1}_{\varepsilon} |ad - bc|^{\alpha+\beta+1}_{\varepsilon+\xi}.$$

Proposition A.3. *For* $\operatorname{Re}(\mu) > 0$ *and* $\operatorname{Re}(\nu + \frac{\mu}{2}) < 1$ *we have*

$$\int_0^{\infty} x^{\mu-1} (1 + x^2)^{\nu-1} \, dx = \frac{1}{2} B\Big(\frac{\mu}{2}, 1 - \nu - \frac{\mu}{2}\Big).$$

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Paper C

Unitary branching from $\mathrm{GL}(3,\mathbb{R})$ to $\mathrm{GL}(2,\mathbb{R})$

Jonathan Ditlevsen

1 Introduction

This is a unfinished paper on the branching problem of restricting unitary representations of $\mathrm{GL}(3,\mathbb{R})$ to $\mathrm{GL}(2,\mathbb{R})$. By using analytic methods we show how to decompose any unitary representation of $\mathrm{GL}(3,\mathbb{R})$ into a direct integral of unitary representations of $\mathrm{GL}(2,\mathbb{R})$, with a few arguments missing which we highlight in one of the sections.

1.1 Results and methods

The unitary dual of $\mathrm{GL}(3,\mathbb{R})$ consists of characters, unitary principal series, complementary series, unitarily induced generalized principal series and unitarily induced degenerate series.

Theorem 1.1. *The unitary principal series, complementary series and unitarily induced generalized principal series all decomposes as*

$$\pi|_H \simeq \bigoplus_{\eta\in(\mathbb{Z}/2\mathbb{Z})^2} \int_{i\mathbb{R}^2}^{\oplus} \tau_{\eta,\nu}\, d\nu \oplus \bigoplus_{\eta\in\mathbb{Z}/2\mathbb{Z}} \bigoplus_{\nu_-\in 1+\eta-2\mathbb{N}} \int_{i\mathbb{R}}^{\oplus} \tau^{ds}_{\eta,\nu}\, d\nu_+,$$

whereas the unitarily induced degenerate series decomposes as

$$\pi_\lambda|_H \simeq \bigoplus_{\eta\in\mathbb{Z}/2\mathbb{Z}} \int_{i\mathbb{R}} \tau_{(\eta,\eta),(\lambda,z)}\, dz, \quad (\lambda\in i\mathbb{R})$$

where $\tau_{\eta,\nu}$ is the unitary principal series for $\mathrm{GL}(2,\mathbb{R})$ and $\tau^{ds}_{\eta,\nu}$ is the almost discrete series for $\mathrm{GL}(2,\mathbb{R})$.

We essentially obtain the first part of this result in two distinct ways. The first method comes from restricting principal series representations $\pi_{\xi,\lambda}$ for $\lambda\in i\mathbb{R}^3$ to the open $\mathrm{GL}(2,\mathbb{R})$-orbit in $\mathrm{GL}(3,\mathbb{R})/P$ where P is a minimal parabolic of $\mathrm{GL}(3,\mathbb{R})$. Using a Plancherel formula we can decompose vectors in $f\in\pi_{\xi,\lambda}$ by

$$||f||^2 = \sum_\eta \int_{i\mathbb{R}^2} ||\,\mathbf{A}^{\eta,\nu}_{\xi,\lambda} f||^2 \frac{d\nu_1 d\nu_2}{|a(\lambda,\nu)|^2} + \sum_{\eta,\nu_1-\nu_2} \int_{i\mathbb{R}} ||\,\mathbf{A}^{\eta,\nu}_{\xi,\lambda} f||^2 \frac{d(\nu_1+\nu_2)}{|b(\lambda,\nu)|^2},$$

where $\mathbf{A}_{\xi,\lambda}^{\eta,\nu}$ is a holomorphic family of intertwinining operators. The way to obtain the direct integral decomposition comes from analytically extending the right hand side in terms of λ from $i\mathbb{R}^3$ to parameters λ, where $\pi_{\xi,\lambda}$ is unitary or contains a unitary quotient. To do so we need to know the meromorphic nature of $a(\lambda,\nu)$, $b(\lambda,\nu)$ and $\mathbf{A}_{\xi,\lambda}^{\eta,\nu}$. The Plancherel formula used is from [BJD23], and is very explicit giving us the functions $a(\lambda,\nu)$, $b(\lambda,\nu)$ in terms of Gamma-functions. The operators $\mathbf{A}_{\xi,\lambda}^{\eta,\nu}$ have been studied in Paper B, where all the necessary zeroes, functional equations and a normalization which makes $\mathbf{A}_{\xi,\lambda}^{\eta,\nu}$ holomorphic in (λ,ν) have been established.

The second method comes from considering generalized principal series representations and again restricting them to an open $\mathrm{GL}(2,\mathbb{R})$-orbit of $\mathrm{GL}(3,\mathbb{R})/P'$, where P' is a maximal parabolic subgroup of $\mathrm{GL}(3,\mathbb{R})$. This restriction is in most cases the space $\mathrm{Ind}_S^{\mathrm{GL}(2,\mathbb{R})}(\omega)$, where S is the $ax+b$-group, and ω is its only unitary irreducible infinite dimensional representation. Picking a non-trivial character χ of N, the group of upper triangular matrices in $\mathrm{GL}(2,\mathbb{R})$ with 1's on the diagonal, we can express $\omega = \mathrm{Ind}_N^S(\chi)$ and by induction in stages, the restriction is a Whittaker model. The decomposition then follows from the Whittaker Plancherel formula.

Notation: We denote by $\mathbb{N} = \{1,2,3,\dots\}$ and by $\mathbb{N}_0 = \{0,1,2,\dots\}$. For operators we use normal letters for the unnormalized meromorphic version and bold letters for the normalized holomorphic versions.

2 Structure of $\mathrm{GL}(3,\mathbb{R})$ and $\mathrm{GL}(2,\mathbb{R})$

Let $(G,H) = (\mathrm{GL}(3,\mathbb{R}),\mathrm{GL}(2,\mathbb{R}))$, where we consider $\mathrm{GL}(2,\mathbb{R})$ as a subgroup of $\mathrm{GL}(3,\mathbb{R})$ by embedding it into the upper left corner. We introduce the notation $d(s,t,u)$ for the diagonal matrix in G with entries s, t, and u starting in the upper left corner. Abusing notation, we use similar notation for diagonal matrices in H, namely $d(s,t,1) = d(s,t)$.

In G, we fix the minimal parabolic subgroup P_G as the upper triangular matrices with Langlands decomposition $P_G = M_G A_G N_G$ where

$$M_G = \{d(m_1,m_2,m_3) \,|\, m_i \in \{\pm 1\}\}, \quad A_G = \{d(a_1,a_2,a_3) \,|\, a_i \in \mathbb{R}_{>0}\},$$

and

$$N_G = \{n(x,y,z) \,|\, x,y,z \in \mathbb{R}\}, \quad \text{where} \quad n(x,y,z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly we define $\overline{N}_G$, the nilradical of the opposite parabolic, as

$$\overline{N}_G = \{\overline{n}(x,y,z) \,:\, x,y,z \in \mathbb{R}\}, \quad \text{where} \quad \overline{n}(x,y,z) = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix}.$$

Let W_G be the Weyl group of G which is isomorphic to the symmetric group of 3 letters. We can consider W_G as a subgroup of G as the subgroup of permutation matrices. Fixing the maximal compact $K_G = O(3)$ of G allows us to write any element in terms of the

Iwasawa decomposition $G = K_G A_G N_G$ (or more rudimentary, in terms of the Gram–Schimidt process). As an example we can decompose $\overline{n}(x,y,z) = kan$ where

$$k = \begin{pmatrix} \frac{1}{\alpha} & \frac{-(x+zy)}{\alpha\beta} & \frac{xy-z}{\beta} \\ \frac{x}{\alpha} & \frac{1+z(z-xy)}{\alpha\beta} & \frac{-y}{\beta} \\ \frac{z}{\alpha} & \frac{y(1+x^2)-xz}{\alpha\beta} & \frac{1}{\beta} \end{pmatrix}, \quad \alpha = \sqrt{1+x^2+z^2}, \quad \beta = \sqrt{1+y^2+(z-xy)^2}. \tag{2.1}$$

In H we fix the minimal parabolic subgroup P_H as the group of upper triangular matrices with Langlands-decomposition $P_H = M_H A_H N_H$ where

$$M_H = \{d(m_1,m_2) \,|\, m_i \in \{\pm 1\}\}, \quad A_H = \{d(a_1,a_2) \,|\, a_i \in \mathbb{R}_{>0}\},$$

and

$$N_H = \{n(x) \,|\, x \in \mathbb{R}\}, \quad \text{where} \quad n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Similarly we define $\overline{N}_H$ as

$$\overline{N}_H = \{\overline{n}(x) \,|\, x \in \mathbb{R}\}, \quad \text{where} \quad \overline{n}(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}.$$

The Weyl group W_H of H is isomorphic to the symmetric group of 2 letters. We can consider it as a subgroup of H as the identity matrix and the permutation matrix corresponding to the simple transposition $(1,2)$. Lastly, we set the maximal compact $K_H = O(2)$. All the subgroups for H have been selected such that they are embedded into their G-counterparts, allowing the decomposition of elements in H in terms of G-decompositions to be done in H. As an example, we get the k of the Iwasawa decomposition of $\overline{n}(x)$ simply by setting $y = z = 0$ in (2.1) as

$$k = \begin{pmatrix} \frac{1}{\sqrt{1+x^2}} & -\frac{x}{\sqrt{1+x^2}} & 0 \\ \frac{x}{\sqrt{1+x^2}} & \frac{1}{\sqrt{1+x^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1+x^2}} & -\frac{x}{\sqrt{1+x^2}} \\ \frac{x}{\sqrt{1+x^2}} & \frac{1}{\sqrt{1+x^2}} \end{pmatrix}.$$

In G, we also have two maximal parabolic subgroups containing P_G, namely

$$P_0 = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}, \quad \text{and} \quad P_1 = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix},$$

which have the Langlands decompositions $P_i = M_i A_i N_i$ where

$$M_0 = \begin{pmatrix} \mathrm{SL}^{\pm}(2,\mathbb{R}) & \\ & \pm 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} \pm 1 & \\ & \mathrm{SL}^{\pm}(2,\mathbb{R}) \end{pmatrix},$$

$$A_0 = \{d(a,a,b) \,:\, a,b \in \mathbb{R}_{>0}\}, \quad A_1 = \{d(a,b,b) \,:\, a,b \in \mathbb{R}_{>0}\},$$

$$N_0 = \left\{ \begin{pmatrix} 1 & & \alpha \\ & 1 & \beta \\ & & 1 \end{pmatrix} \,:\, \alpha,\beta \in \mathbb{R} \right\}, \quad N_1 = \left\{ \begin{pmatrix} 1 & \beta & \alpha \\ & 1 & \\ & & 1 \end{pmatrix} \,:\, \alpha,\beta \in \mathbb{R} \right\}.$$

3 Unitary principal series representations

3.1 Unitary dual of $\mathrm{SL}(2,\mathbb{R})$ in terms of principal series representations

Let $\mathrm{SL}^{\pm}(2,\mathbb{R})$ be the group of 2×2 matrices with determinant ± 1 and consider the parabolic subgroup $P_S = M_S A_S N_S$ of upper triangular matrices in $\mathrm{SL}^{\pm}(2,\mathbb{R})$ where $M_S = M_H$, $A_S = \{d(t,t^{-1})\colon t>0\}$ and $N_S = N_H$.

For $\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^2 \simeq \{0,1\}^2$ and $\mu \in \mathbb{C}$ the *principal series representation* $\rho_{\varepsilon,\mu} = \mathrm{Ind}_{P_S}^{\mathrm{SL}^{\pm}(2,\mathbb{R})}(\varepsilon \otimes e^{\mu} \otimes 1)$ of $\mathrm{SL}^{\pm}(2,\mathbb{R})$ is the left regular action of $\mathrm{SL}^{\pm}(2,\mathbb{R})$ on $f \in L^2(O(2)/M_S)$ satisfying the equivariance properties

$$\begin{aligned} f(gm) &= m_1^{\varepsilon_1} m_2^{\varepsilon_2} f(g), && (d(m_1,m_2)\in M_S) \\ f(ga) &= t^{-\mu-1} f(g), && (a = d(t,t^{-1}) \in A_S) \\ f(gn) &= f(g), && (n \in N_S). \end{aligned}$$

Proposition 3.1 ([B98])**.** *Let $\varepsilon \in \{0,1\}^2$ and set $\varepsilon_+ = \varepsilon_1 + \varepsilon_2$. The unitary irreducible representations of $\mathrm{SL}^{\pm}(2,\mathbb{R})$ can be listed in terms of principal series representations as follows:*

1. *The unitary principal series, $\rho_{\varepsilon,\mu}$ where $\mu \in i\mathbb{R}_{\geq 0}$.*
2. *The complementary series, $\rho_{\varepsilon,\mu}$ where $\varepsilon_+ = 0$ and $\mu \in (-1,0)$.*
3. *The discrete series representations, $\rho^{ds}_{\varepsilon,\mu}$ which appears as a quotient inside $\rho_{\varepsilon,\mu}$ where $\mu \in 1-\varepsilon_+ - 2\mathbb{N}$.*
4. *The trivial representation and the sign character which appears as a subrepresentation of $\rho_{\varepsilon,\mu}$ where $\mu = -1$.*

The above are non-isomorphic except for the discrete series representations which are isomorphic under $\varepsilon + (1,1)$.

3.2 Unitary dual of $\mathrm{GL}(2,\mathbb{R})$ in terms of principal series representations

For $\eta \in (\mathbb{Z}/2\mathbb{Z})^2$ and $\nu \in \mathbb{C}^2$ the *principal series representations* $\tau_{\eta,\nu} = \mathrm{Ind}_{P_H}^{H}(\eta \otimes e^{\nu} \otimes 1)$ of H is the left regular action of H on smooth functions $f\colon H \to \mathbb{C}$ satisfying the equvariance properties

$$\begin{aligned} f(gm) &= m_1^{\eta_1} m_2^{\eta_2} f(g), && (d(m_1,m_2)\in M_H) \\ f(ga) &= a_1^{-\nu-1-\frac{1}{2}} a_2^{-\nu_2+\frac{1}{2}} f(g), && (a = d(a_1,a_2) \in A_H) \\ f(gn) &= f(g), && (n \in N_H). \end{aligned}$$

Denote by w_H a representative of the longest (and only non-trivial) Weyl group element in W_H, that is the 2×2 matrix with $1'$s on the off-diagonal. We define the normalized the Knapp–Stein intertwining operator $\mathbf{T}^{w_H}_{\eta,\nu} : \tau_{\eta,\nu} \to \tau_{w_H(\eta,\nu)}$ as

$$\mathbf{T}^{w_H}_{\eta,\nu} f(h) = \frac{1}{\Gamma(\frac{\nu_1-\nu_2+[\eta_1+\eta_2]}{2})} \int_{\mathbb{R}} f(h w_H \overline{n}(x))\, dx,$$

where w_H acting on η and ν simply swaps their first and second entry. Here $z + [\eta + \eta']$ mean adding $\eta + \eta'$ in $\mathbb{Z}/2\mathbb{Z}$ first and then adding either a 0 or 1 to z.

The integral only converges for ν in some domain, but with the chosen normalization it can be analytically extended to all $\nu \in \mathbb{C}^2$. In order for the principal series to be unitary we need the parameter $\nu_1 + \nu_2$ coming from the action of the center to be pure imaginary, and the rest $\nu_1 - \nu_2$ to essentially be a unitary representation of $\mathrm{SL}^{\pm}(2, \mathbb{R})$. Thus, we can consider the discrete series representations for $\nu_1 + \nu_2 \in i\mathbb{R}$ and $\nu_1 - \nu_2 \in 1 - [\eta_1 + \eta_2] - 2\mathbb{N}$

$$\tau^{\mathrm{ds}}_{\eta,\nu} = \tau_{\eta,\nu} / \ker(\mathbf{T}^{w_H}_{\eta,\nu}).$$

These are equivalent when adding $(1, 1)$ to η, so we parameterize them by a single element from $\mathbb{Z}/2\mathbb{Z}$ corresponding to the sum $\eta_1 + \eta_2$. When $\nu \in i\mathbb{R}^2$, the space $\tau_{\eta,\nu}$ comes with the usual L^2-norm from $L^2(K_H/M_H)$ whereas when $\nu \in (0, 1)$ or $\nu_1 - \nu_2 \in 1 - [\eta_1 + \eta_2] - 2\mathbb{N}$, we use the norm

$$\|f\|^2 = \int_{K_H/M_H} f(k)\overline{\mathbf{T}^{w_H}_{\eta,\nu} f(k)}d(kM_H).$$

These norms makes $\tau_{\eta,\nu}$ and $\tau^{\mathrm{ds}}_{\eta,\nu}$ a pre-Hilbert space but we will not explicitly state when we are talking about $\tau_{\eta,\nu}$ or its completion, as this should be clear from context.

Theorem 3.2 ([B98])**.** *Let $\nu \in \mathbb{C}^2$ and $\eta \in \mathbb{Z}/2\mathbb{Z}$ with $\nu_{\pm} = \nu_1 \pm \nu_2$ and $\eta_+ = \eta_1 + \eta_2$. All the unitary irreducible representations of $\mathrm{GL}(2, \mathbb{R})$ can be described as follows: In all cases $\nu_+ \in i\mathbb{R}$*

1. *The characters $h \mapsto |\det(h)|_{\eta_+}^{-\frac{\nu_{\pm}}{2}}$, which sits inside $\tau_{\eta,\nu}$ as a subrepresentation for $\nu_- = -1$.*
2. *The unitary principal series $\tau_{\eta,\nu}$ for $\nu \in i\mathbb{R}^2$.*
3. *The complementary series $\tau_{\eta,\nu}$ for $\eta_+ = 0$ and $\nu_- \in (0, 1)$.*
4. *The discrete series $\tau^{ds}_{\eta_+,\nu}$ which sits inside $\tau_{\eta,\nu}$ as a quotient for $\nu_- \in 1 - \eta_+ - 2\mathbb{N}$.*

3.3 Unitary dual for $\mathrm{GL}(3, \mathbb{R})$ in terms of principal series representations and generalized principal series representations

For $\xi \in (\mathbb{Z}/2\mathbb{Z})^3 \simeq \{0, 1\}^3$ and $\lambda \in \mathbb{C}^3$ the *principal series representations* $\pi_{\xi,\lambda} = \mathrm{Ind}^G_{P_G}(\xi \otimes e^{\lambda} \otimes 1)$ of G, is the left regular action of G on smooth functions $f : G \to \mathbb{C}$ satisfying the equivariance properties from P_G as following

$$\begin{aligned} f(gm) &= m_1^{\xi_1} m_2^{\xi_2} m_3^{\xi_3} f(g), && (d(m_1, m_2, m_3) \in M_G) \\ f(ga) &= a_1^{-\lambda_1 - 1} a_2^{-\lambda_2} a_3^{-\lambda_3 + 1} f(g), && (a = d(a_1, a_2, a_3) \in A_G) \\ f(gn) &= f(g), && (n \in N_G). \end{aligned}$$

Between the principal series representations, we have the Knapp–Stein intertwining operators $T^w_{\xi,\lambda} : \pi_{\xi,\lambda} \to \pi_{w(\xi,\lambda)}$ which are defined by

$$T^w_{\xi,\lambda} f(g) = \int_{\overline{N}_G \cap w^{-1} N_G w} f(gw\overline{n})\, d\overline{n}$$

where w is a representative of an element in W_G, and w acts on ξ and λ by permuting the entries. This integral converges only for λ in some domain, but can be meromorpically extended to all $\lambda \in \mathbb{C}^3$. The Knapp–Stein intertwining operator satisfies that

$$T^{w_1 w_2}_{\xi,\lambda} = T^{w_1}_{w_2(\xi,\lambda)} \circ T^{w_2}_{\xi,\lambda}. \tag{3.1}$$

By considering the case where $w = (i, i+1)$ is a simple reflection then inspired by the $\mathrm{GL}(2,\mathbb{R})$-case, we can use the normalization

$$\mathbf{T}^{w}_{\xi,\lambda} = \frac{1}{\Gamma(\frac{\lambda_i - \lambda_{i+1} + [\xi_i + \xi_{i+1}]}{2})} T^{w}_{\xi,\lambda},$$

to analytically extend this to all $\lambda \in \mathbb{C}^3$ and then use (3.1) to define $\mathbf{T}^{w}_{\xi,\lambda}$ for any w by writing w as a product of simple reflections.

Let ρ be an irreducible representation of $\mathrm{SL}^{\pm}(2,\mathbb{R})$, ε a character of $\{0,1\}$ and $\lambda \in \mathbb{C}^2$ the *generalized principal series representation* $\sigma_{\rho,\varepsilon,\lambda} = \mathrm{Ind}^{G}_{P_0}((\rho \otimes \varepsilon) \otimes e^{\lambda} \otimes 1)$ of G is the left regular action on the smooth functions $f : G \to \rho$ with the equivariance properties

$$\begin{aligned} f(gm) &= \rho^{-1}(m) f(g), & (m \in M_0)\\ f(ga) &= a_1^{-\lambda_1 - 1} a_2^{-\lambda_2 + 1} f(g), \quad & (a = d(a_1, a_1, a_2) \in A_0)\\ f(gn) &= f(g), & (n \in N_0). \end{aligned}$$

By induction in stages, the generalized principal series representations often coincide with the principal series representations, so we give names to the instances where this is not the case.

If ρ is a discrete series representation $\rho^{ds}_{\varepsilon,\mu}$ of $\mathrm{SL}^{\pm}(2,\mathbb{R})$ $\mu = -m$ and $\lambda \in i\mathbb{R}^2$ then we call $\sigma_{\rho,\varepsilon,\lambda}$ the *unitarily induced generalized principal series representation*, and denote it by $\pi^{\mathrm{gen}}_{m,\varepsilon,\lambda}$. As we are mostly going to be working with unitary representation we may use *generalized series representation* for short.

The discrete series representation $\rho^{ds}_{\varepsilon,\mu}$ sits inside a principal series representation $\rho_{\varepsilon,\mu}$ for $\mathrm{SL}^{\pm}(2,\mathbb{R})$ as quotient for $\mu = -m$ and $\delta_1 + \delta_2$, the same parity as m. By induction in stages, (see [K16, (7.5)]) we get that $\pi^{\mathrm{gen}}_{m,\varepsilon,\lambda}$ sits inside $\pi_{\xi,\lambda'}$ as quotient since parabolic induction is functorial where

$$\xi = (\delta_1, \delta_2, \varepsilon), \quad \lambda' = (\frac{\lambda_1 - m}{2}, \frac{\lambda_1 + m}{2}, \lambda_2).$$

By [SV80, Corrolary 2.8] the composition factors are independent of the parabolic used, so we can instead consider $\pi^{\mathrm{gen}}_{m,\varepsilon,\lambda}$ as a quotient inside $\pi_{\xi',\lambda''}$, where

$$\xi' = (\delta_1, \varepsilon, \delta_2), \quad \lambda'' = (\frac{\lambda_1 - m}{2}, \lambda_2, \frac{\lambda_1 + m}{2}).$$

For $\delta = 0, 1$, let $\chi_\delta(h) = \mathrm{sgn}(\det(h))^{\delta}$ be a unitary character of $\mathrm{SL}^{\pm}(2,\mathbb{R})$, and $\lambda \in i\mathbb{R}^2$ then we call $\sigma_{\chi_\delta,\varepsilon,\lambda}$ the *unitarily induced degenerate series representation*, and denote it by $\pi^{\deg}_{\delta,\varepsilon,\lambda}$. We may use *degenerate series representation* for short. The character χ_δ sits inside the principal series representation $\rho_{\delta',\mu}$ for $\mathrm{SL}^{\pm}(2,\mathbb{R})$ as a quotient for $\mu = 1$, where $\delta'_1 + \delta'_2 = \delta$. By induction in stages, we get that $\pi^{\deg}_{\delta,\varepsilon,\lambda}$ sits inside $\pi_{\xi,\lambda'}$ as a quotient where

$$\xi = (\delta'_1, \delta'_2, \varepsilon), \quad \lambda = (\frac{\lambda_1 + 1}{2}, \frac{\lambda_1 - 1}{2}, \lambda_2),$$

and as the composition factors are independent of the parabolic used, we can instead consider $\pi^{\deg}_{\delta,\varepsilon,\lambda}$ as a quotient inside $\pi_{\xi',\lambda''}$, where

$$\xi' = (\delta_1', \varepsilon, \delta_2'), \quad \lambda'' = (\frac{\lambda_1+1}{2}, \lambda_2, \frac{\lambda_1-1}{2}).$$

Theorem 3.3 ([S80]). *The unitary irreducible representations of* $\mathrm{GL}(3,\mathbb{R})$ *are*

1. *The characters* $g \mapsto |\det(g)|^s_\varepsilon$, *where* $s \in i\mathbb{R}$ *and* $\varepsilon \in \{0,1\}$
2. *The unitary principal series* $\pi_{\xi,\lambda}$, *where* $\lambda \in i\mathbb{R}^3$.
3. *The complementary series* $\pi_{\xi,\lambda}$, *where* $\xi_1 = \xi_3$, $\lambda_1 + \overline{\lambda_3} = 0$, $\lambda_2 \in i\mathbb{R}$ *and* $\mathrm{Re}(\lambda_1) \in (-\frac{1}{2}, \frac{1}{2})$.
4. *The unitarily induced generalized principal series* $\pi^{\mathrm{gen}}_{m,\varepsilon,\lambda}$, *where* $\lambda \in i\mathbb{R}^2$.
5. *The unitarily induced degenerate series* $\pi^{\deg}_{\delta,\varepsilon,\lambda}$, *where* $\lambda \in i\mathbb{R}^2$.

With the embedding into the principal series as described above, we have that at each end of the complementary series, sits $\pi^{\mathrm{gen}}_{1,\xi_2,\lambda}$ and $\pi^{\deg}_{0,\xi_2,\lambda}$.

4 Symmetry breaking operators between principal series representations

In this section we introduce a family of symmetry breaking operator and recall some properties about it which shown in Paper B. Consider the polynomials

$$\Phi_1(g) = g_{31}, \quad \Phi_2(g) = g_{31}g_{22} - g_{32}g_{21}, \quad \Phi_3(g) = -\det(g)$$

$$\Psi_1(g) = g_{21}, \quad \Psi_2(g) = g_{21}g_{12} - g_{22}g_{11},$$

and the integral kernel

$$K^{\eta,\nu}_{\xi,\lambda}(g) = |\Phi_1(g)|^{s_1}_{\delta_1} |\Psi_1(g)|^{t_1}_{\varepsilon_1} |\Phi_2(g)|^{s_2}_{\delta_2} |\Psi_2(g)|^{t_2}_{\varepsilon_2} |\Phi_3(g)|^{s_3}_{\delta_3},$$

where

$$s_1 = \lambda_1 - \nu_2 - \tfrac{1}{2}, \quad t_1 = \nu_2 - \lambda_2 - \tfrac{1}{2}, \quad s_2 = \lambda_2 - \nu_1 - \tfrac{1}{2}, \quad t_2 = \nu_1 - \lambda_3 - \tfrac{1}{2}, \quad s_3 = \lambda_3 + 1,$$

and

$$\delta_1 = \xi_1 + \eta_2, \quad \varepsilon_1 = \eta_2 + \xi_2, \quad \delta_2 = \xi_2 + \eta_1, \quad \varepsilon_2 = \eta_1 + \xi_3, \quad \delta_3 = \xi_3.$$

We will use the parameters $(\delta, s, \varepsilon, t)$ and $(\xi, \lambda, \eta, \nu)$ interchangeably as they are related by an invertible affine transformation. This kernel satisfy the equivariance properties

$$K^{\eta,\nu}_{\xi,\lambda}(gm_G a_G n_G) = |\ell_1|^{\lambda_1-1}_{\xi_1} |\ell_2|^{\lambda_2}_{\xi_2} |\ell_3|^{\lambda_3+1}_{\xi_3} K^{\eta,\nu}_{\xi,\lambda}(g), \quad (d(\ell_1,\ell_2,\ell_3) \in M_G A_G, n_G \in N_G),$$

$$K^{\eta,\nu}_{\xi,\lambda}(m_H a_H n_H g) = |\ell_1|^{\nu_1+\frac{1}{2}}_{\eta_1} |\ell_2|^{\nu_2-\frac{1}{2}} K^{\eta,\nu}_{\xi,\lambda}(g), \quad (d(\ell_1,\ell_2) \in M_H A_H, n_H \in N_H),$$

and is locally integrable when $\mathrm{Re}(s_i), \mathrm{Re}(t_i) \geq 0$ for $i \in \{1,2\}$ and $s_3 \in \mathbb{C}$. For those parameters it defines an intertwining operator $A^{\eta,\nu}_{\xi,\lambda} : \pi_{\xi,\lambda}|_H \to \tau_{\eta,\nu}$ by

$$A^{\eta,\nu}_{\xi,\lambda} f(h) = \int_{\overline{N}_G} K^{\eta,\nu}_{\xi,\lambda}(h^{-1}\overline{n}) f(\overline{n})\, d\overline{n} = \int_{K_G} K^{\eta,\nu}_{\xi,\lambda}(h^{-1}k) f(k)\, dk,$$

where the last equality follows from Knapp [K16, (5.25)]. By [F21] the space $\mathrm{Hom}_H(\pi_{\xi,\lambda}|_H, \tau_{\eta,\nu})$ is for generic parameters (λ, ν) of dimension 1. The kernel $K^{\eta,\nu}_{\xi,\lambda}$ in the non-compact picture $\overline{N}_G$ is given by

$$K^{\eta,\nu}_{\xi,\lambda}(\overline{n}(x,y,z)) = |z|^{\lambda_1-\nu_2-\frac{1}{2}}_{\xi_1+\eta_2}|x|^{\nu_2-\lambda_2-\frac{1}{2}}_{\eta_2+\xi_2}|z-xy|^{\lambda_2-\nu_1-\frac{1}{2}}_{\xi_2+\eta_1}. \tag{4.1}$$

We now state some Bernstein–Sato identitites given in the non-compact picture coordinates. The result in global coordinates can be found in Paper B. Consider the following differential operators

$$\begin{aligned}
\mathcal{D}_1(\lambda) &= z(\partial_x + y\partial_z) - (\lambda_1 - \lambda_2 - 1)y,\\
\mathcal{C}_1(\nu) &= x\partial_x - (\nu_2 - \nu_1 - 1),\\
\mathcal{F}_1(\lambda) &= \partial_y(y\partial_z + \partial_x) + (\lambda_1 - \lambda_2 - 1)\partial_z\\
\mathcal{D}_2(\lambda) &= z\partial_x\partial_y + (y\partial_y - (\lambda_2 - \lambda_3 - 1))\partial_z\\
&\quad - (\lambda_1 - \lambda_3 - 1)\partial_y + (\lambda_1 - \lambda_2 - 1)(\lambda_1 - \lambda_3 - 1).
\end{aligned}$$

Theorem 4.1. *We have the following Bernstein–Sato identities*

$$\begin{aligned}
\mathcal{D}_1(\lambda)K^{\varepsilon,t}_{\delta,s} &= (\nu_2 - \lambda_2 - \tfrac{1}{2})K^{\varepsilon-e_1,t-e_1}_{\delta+e_2,s+e_2}\\
\mathcal{D}_2(\lambda)K^{\varepsilon,t}_{\delta,s} &= (\nu_1 - \lambda_3 - \tfrac{1}{2})(\nu_2 - \lambda_3 - \tfrac{1}{2})K^{\varepsilon-e_2,t-e_2}_{\delta+e_3,s+e_3},\\
\mathcal{C}_1(\nu)K^{\varepsilon,t}_{\delta,s} &= (\lambda_2 - \nu_1 - \tfrac{1}{2})K^{\varepsilon+e_2,t+e_2}_{\delta+e_1-e_2,s+e_1-e_2},\\
\mathcal{F}_1(\lambda)K^{\varepsilon,t}_{\delta,s} &= (\lambda_1 - \nu_1 - \tfrac{1}{2})(\lambda_1 - \nu_2 - \tfrac{1}{2})K^{\varepsilon,t}_{\delta+e_3-e_1,s+e_3-e_1}.
\end{aligned}$$

Let

$$\begin{aligned}
n_0(\lambda,\nu) &= \Gamma(\lambda_1 - \nu_2 + \tfrac{1}{2})\Gamma(\nu_2 - \lambda_2 + \tfrac{1}{2})\Gamma(\lambda_2 - \nu_1 + \tfrac{1}{2})\\
&\quad\times \Gamma(\nu_1 - \lambda_3 + \tfrac{1}{2})\Gamma(\lambda_1 - \nu_1 + \tfrac{1}{2})\Gamma(\nu_2 - \lambda_3 + \tfrac{1}{2}),
\end{aligned}$$

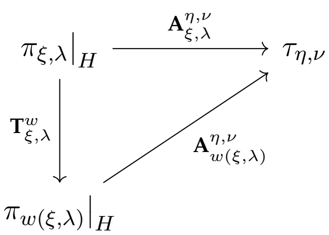
and set $\mathbb{K}^{\eta,\nu}_{\xi,\lambda} = n_0(\lambda,\nu)^{-1}K^{\eta,\nu}_{\xi,\lambda}$. Then using Theorem 4.1, we can analytically extend $\mathbb{K}^{\eta,\nu}_{\xi,\lambda}$, as a distribution to $(\lambda,\nu) \in \mathbb{C}^3 \times \mathbb{C}^2$. This normalization is however not optimal, so instead we use

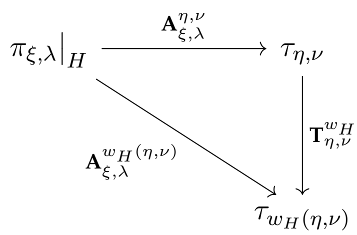
$$\begin{aligned}
n(\xi,\lambda,\eta,\nu) &= \Gamma\Big(\tfrac{\lambda_1-\nu_2+\frac{1}{2}+[\xi_1+\eta_2]_2}{2}\Big)\Gamma\Big(\tfrac{\lambda_1-\nu_1+\frac{1}{2}+[\xi_1+\eta_1]_2}{2}\Big)\Gamma\Big(\tfrac{\lambda_2-\nu_1+\frac{1}{2}+[\xi_2+\eta_1]_2}{2}\Big)\\
&\quad\times \Gamma\Big(\tfrac{\nu_2-\lambda_2+\frac{1}{2}+[\xi_2+\eta_2]_2}{2}\Big)\Gamma\Big(\tfrac{\nu_2-\lambda_3+\frac{1}{2}+[\xi_3+\eta_2]_2}{2}\Big)\Gamma\Big(\tfrac{\nu_1-\lambda_3+\frac{1}{2}+[\xi_3+\eta_1]_2}{2}\Big).
\end{aligned}$$

Let

$$\mathbf{K}^{\eta,\nu}_{\xi,\lambda} = n(\xi,\lambda,\eta,\nu)^{-1}K^{\eta,\nu}_{\xi,\lambda}, \quad \text{and} \quad \mathbf{A}^{\eta,\nu}_{\xi,\lambda} = n(\xi,\lambda,\eta,\nu)^{-1}A^{\eta,\nu}_{\xi,\lambda}.$$

The family $\mathbf{A}^{\eta,\nu}_{\xi,\lambda}$ is defined for $(\lambda,\nu) \in \mathbb{C}^3 \times \mathbb{C}^2$ and is holomorphic in these parameters. Composing $\mathbf{A}^{\eta,\nu}_{\xi,\lambda}$ by the Knapp–Stein intertwining operators as described in the following diagram, we get the following result.





Theorem 4.2. *For $w_i = (i, i+1)$, a simple transposition, we have*

$$\boldsymbol{A}^{\eta,\nu}_{\xi,\lambda}\boldsymbol{T}^{w_i}_{\xi,\lambda} = \frac{\sqrt{\pi}(-1)^{(\xi_i+\xi_{i+1})(\eta_{3-i}+1)+\xi_i\xi_{i+1}}}{\Gamma(\frac{\lambda_i-\lambda_{i+1}+1+[\xi_i+\xi_{i+1}]}{2})}\boldsymbol{A}^{\eta,\nu}_{w_i(\xi,\lambda)}.$$

Similarly, we have

$$\boldsymbol{T}^{w_H}_{\eta,\nu}\boldsymbol{A}^{\eta,\nu}_{\xi,\lambda} = \frac{\sqrt{\pi}(-1)^{(\eta_1+\eta_2)(\xi_2+1)+\eta_1\eta_2}}{\Gamma(\frac{\nu_2-\nu_1+1+[\eta_1+\eta_2]}{2})}\boldsymbol{A}^{w_H(\eta,\nu)}_{\xi,\lambda}.$$

By writing the longest Weyl group element $w_0 = (1,3)$ as $w_1 w_2 w_1$ and applying this theorem thrice, we get

Corollary 4.3.

$$\boldsymbol{A}^{\eta,\nu}_{\xi,\lambda}\boldsymbol{T}^{w_0}_{\xi,\lambda} = \pi^{\frac{3}{2}}(-1)^{(\xi_1+\xi_3)(\eta_1+\eta_2+\xi_2)+\xi_1\xi_3}\alpha(\xi,\lambda)\boldsymbol{A}^{\eta,\nu}_{w_0(\xi,\lambda)},$$

where α is the holomorphic function given by

$$\alpha(\xi,\lambda) = \frac{1}{\Gamma(\frac{\lambda_1-\lambda_2+1+[\xi_1+\xi_2]}{2})\Gamma(\frac{\lambda_2-\lambda_3+1+[\xi_2+\xi_3]}{2})\Gamma(\frac{\lambda_1-\lambda_3+1+[\xi_1+\xi_3]}{2})}.$$

We denote by $\mathcal{N}$ the vanishing set for $\mathbf{A}^{\eta,\nu}_{\xi,\lambda}$ i.e.

$$\mathcal{N} = \{(\xi,\lambda,\eta,\nu) |\, \mathbf{A}^{\eta,\nu}_{\xi,\lambda} = 0\}.$$

Theorem 4.4. *For $i, j \in \{1,2,3\}$ where $i < j$ and $k \in \{1,2\}$ then*

$$\mathcal{N}_{i,j,k} = \{\lambda_i - \nu_k + \tfrac{1}{2} + [\xi_i + \eta_k] \in -2\mathbb{N}_0\} \cap \{\nu_k - \lambda_j + \tfrac{1}{2} + [\eta_k + \xi_j] \in -2\mathbb{N}_0\},$$

and

$$\mathcal{M}_i = \{\nu_2 - \lambda_i + \tfrac{1}{2} + [\eta_2 + \xi_i] \in -2\mathbb{N}_0\} \cap \{\lambda_i - \nu_1 + \tfrac{1}{2} + [\xi_i + \eta_1] \in -2\mathbb{N}_0\},$$

are subsets of $\mathcal{N}$.

This shows when restricting to certain hyperplanes, we only need some of the factors from $n(\xi,\lambda,\eta,\nu)$ and on those hyperplanes we can renormalize the symmetry breaking operators and obtain potentially non-zero symmetry breaking operators. Restricting to the hyperplane $\lambda_i - \nu_k + \frac{1}{2} + [\xi_i + \eta_k] = -2n$, we can consider the operator

$$\mathbf{B}^{\eta,\nu}_{\xi,\lambda} = \Big(\prod_{k<\ell\leq 2}\Gamma(\tfrac{\nu_\ell-\lambda_i+\frac{1}{2}+[\xi_i+\eta_\ell]}{2})\prod_{i<j\leq 3}\Gamma(\tfrac{\nu_k-\lambda_j+\frac{1}{2}+[\xi_j+\eta_k]}{2})\,\mathbf{A}^{\eta,\nu}_{\xi,\lambda}\Big)\Big|_{\lambda_i-\nu_k+\frac{1}{2}+[\eta_k+\xi_i]=-2n},$$

and similarly for $\nu_k - \lambda_j + \frac{1}{2} + [\eta_k + \xi_j] = -2n$ the operator

$$\mathbf{B}^{\eta,\nu}_{\xi,\lambda} = \Big(\prod_{1\leq\ell<k}\Gamma(\tfrac{\lambda_j-\nu_\ell+\frac{1}{2}+[\xi_j+\eta_\ell]}{2})\prod_{1\leq i<j}\Gamma(\tfrac{\lambda_i-\nu_k+\frac{1}{2}+[\xi_i+\eta_k]}{2})\,\mathbf{A}^{\eta,\nu}_{\xi,\lambda}\Big)\Big|_{\nu_k-\lambda_j+\frac{1}{2}+[\eta_k+\xi_j]=-2n},$$

where we consider the empty product to be equal to one. These operators are still holomorphic in the parameters (λ,ν) on their respective hyperplanes. We have hidden which hyperplane we are restricting to in the notation, as it would be too cumbersome, but we will be very explicit about which hyperplane and thus which $\mathbf{B}^{\eta,\nu}_{\xi,\lambda}$ we are considering.

5 Restricting to an open orbit

Consider the base point

$$g_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

for the open dense H-orbit in G/P_G. Then

$$hg_0 = \begin{pmatrix} h_{12} & h_{11} & 0 \\ h_{22} & h_{21} & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad g_0^{-1}hg_0 = \begin{pmatrix} h_{22} & h_{21} & 0 \\ h_{12} & h_{11} & 0 \\ 1-h_{22} & -h_{21} & 1 \end{pmatrix},$$

which shows that the open dense H-orbit of g_0 in G/P_G has stabilizer $D_0 := \{d(s,1) : s \in \mathbb{R}^\times\}$. In terms of $\overline{N}_G M_G A_G N_G$, we can decompose hg_0 as

$$hg_0 = \begin{pmatrix} 1 & & \\ \frac{h_{22}}{h_{12}} & 1 & \\ \frac{1}{h_{12}} & \frac{h_{11}}{\det(h)} & 1 \end{pmatrix} \begin{pmatrix} h_{12} & & \\ & -\frac{\det(h)}{h_{12}} & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{h_{11}}{h_{12}} & \\ & 1 & \\ & & 1 \end{pmatrix}. \tag{5.1}$$

For computing the Iwasawa decomposition we can use the classical trick of writing $g = kan$ then $g^\intercal g = n^\intercal a^2 n$. So to decompose elements hg_0 in the orbit in terms of the Iwasawa decomposition the following suffices

$$g_0^\intercal hg_0 = \begin{pmatrix} 1 & & \\ \frac{h_{12}}{h_{22}+1} & 1 & \\ \frac{-1}{h_{22}+1} & \frac{h_{21}}{\det(h)+h_{11}} & 1 \end{pmatrix} \begin{pmatrix} h_{22}+1 & & \\ & \frac{\det(h)+h_{11}}{h_{22}+1} & \\ & & \frac{\det(h)}{\det(h)+h_{11}} \end{pmatrix} \begin{pmatrix} 1 & \frac{h_{21}}{h_{11}+1} & \frac{-1}{h_{22}+1} \\ & 1 & \frac{h_{12}}{\det(h)+h_{11}} \\ & & 1 \end{pmatrix} \tag{5.2}$$

Consider the H-intertwining map

$$\begin{aligned} \mathcal{O} : \pi_{\xi,\lambda}|_H &\to \operatorname{Ind}_{D_0}^H(\xi_2 \otimes e^{\lambda_2}), \\ f &\mapsto F, \end{aligned}$$

where $F(h) = f(hg_0)$ where the equivariance property follows from

$$\mathcal{O}f(gd(t,1)) = f(gd(t,1)g_0) = f(gg_0 d(1,t)) = |t|_{\xi_2}^{-\lambda_2} \mathcal{O}f(g).$$

Proposition 5.1. *The image of $\mathcal{O}$ is contained in $L^2(H/D_0)$ when $-1 < \mathrm{Re}\,(\lambda_1)$, $\mathrm{Re}(\lambda_1 - \lambda_2) > -\frac{1}{2}$, $-\frac{1}{2} < \mathrm{Re}(\lambda_2)\ \mathrm{Re}\,(\lambda_3) < \frac{1}{2}$. Furthermore, it is unitary when $\lambda \in i\mathbb{R}^3$.*

Proof. Let $D = \{d(s,t) : s,t \in \mathbb{R}^\times\}$ then decomposing $H = \overline{N}_H M_H A_H N_H$ we get the coordinates $\overline{n}(x)n(y)D$ on H/D with the invariant measure as $dydx$ see [K16, Chap. 5]. Now note that $H/D_0 \simeq H/D \times D_0$, so we get

$$\begin{aligned} \int_{H/D_0} |\mathcal{O}f(g)|^2 \, d(gD_0) &= \int_{H/D} \int_{\mathbb{R}^\times} |\mathcal{O}f(gd(1,t))|^2 \frac{dt}{|t|} d(gD) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}^\times} |f(\overline{n}(x)n(y)d(1,t)g_0)|^2 \frac{dt}{|t|} dydx \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}^\times} |f(\overline{n}(x)n(yt^{-1})d(1,t)g_0)|^2 \frac{dt}{t^2} dydx. \end{aligned}$$

Using (5.2) we decompose $\overline{n}(x)n(yt^{-1})d(1,t)g_0 = kan$, and the integral becomes

$$\int_{\mathbb{R}^3}(1+y^2+(t-xy)^2)^{\mathrm{Re}(\lambda_2-\lambda_1-1)}(1+x^2+t^2)^{\mathrm{Re}(\lambda_3-\lambda_2-1)}|t|^{-2\,\mathrm{Re}(\lambda_3)}|f(k)|^2dxdydt. \tag{5.3}$$

We can bound $|f(k)|$ by some some non-negative constant c_f, giving us

$$\le c_f\int_{\mathbb{R}^3}(1+x^2)^{-2\,\mathrm{Re}(\lambda_2)-1}(1+y^2)^{\mathrm{Re}(\lambda_2-\lambda_1-1)}(1+z^2)^{\mathrm{Re}(\lambda_3-\lambda_1-\frac{3}{2})}|z|^{-2\,\mathrm{Re}(\lambda_3)}\,dxdydz$$

which converges for

$$\mathrm{Re}(\lambda_2) > -\frac{1}{2},\quad \mathrm{Re}(\lambda_1-\lambda_2) > -\frac{1}{2},\quad \mathrm{Re}(\lambda_3) < \frac{1}{2},\quad \mathrm{Re}(\lambda_1) > -1.$$

In the case where $\lambda \in (i\mathbb{R})^3$ then

$$e^{-2\rho H(\overline{n}(x,y,z))} = (1+x^2+z^2)^{-1}(1+y^2+(z-xy)^2)^{-1},$$

so transforming (5.3) by [K16, (5.25)] we see $\mathcal{O}$ is unitary. □

5.1 The Mellin fibration

Consider the Mellin transform given by

$$\mathcal{M}'f(s) = \frac{1}{\sqrt{2\pi}}\int_0^\infty t^{is}f(t)\frac{dt}{t}.$$

This can be regarded as a unitary map between $L^2(\mathbb{R}_+,\frac{dt}{t})$ and $L^2(\mathbb{R},dt)$, since if $g(u) = f(e^u)$ then

$$\|\mathcal{M}'f\|^2 = \frac{1}{2\pi}\int_{\mathbb{R}}\Big|\int_0^\infty t^{is}f(t)\frac{dt}{t}\Big|^2ds = \int_{\mathbb{R}}\Big|\int_0^\infty g(u)e^{ius}du\Big|^2ds = \|\mathcal{F}g\|^2 = \|g\|^2 = \|f\|^2,$$

by the Plancherel formula for the Fourier transform. Let $D = \{d(s,t)\ :\ s,t\in\mathbb{R}^\times\}$, and consider the map

$$\mathcal{M}_{\delta,s} : \mathrm{Ind}_{D_0}^H(\xi_2\otimes e^{\lambda_2}) \to \mathrm{Ind}_D^H((\xi_2,\delta)\otimes e^{(\lambda_2,s)})$$

given by

$$\mathcal{M}_{\delta,s}f(h) = \frac{1}{\sqrt{2\pi}}\int_0^\infty t^s[f(hd(1,t)) + (-1)^\delta f(hd(1,-t))]\frac{dt}{t}$$
$$= \frac{1}{2\sqrt{\pi}}\int_{\mathbb{R}^\times}|t|_\delta^s f(hd(1,t))\frac{dt}{|t|},$$

where it is easily checked that $\mathcal{M}_{\delta,s}f$ has the desired equivariance properties.

Proposition 5.2. *The map*

$$\mathcal{M} : \mathrm{Ind}_{D_0}^H(\xi_2\otimes e^{\lambda_2}) \to \bigoplus_{\delta=0}^{1}\int_{i\mathbb{R}}^{\oplus}\mathrm{Ind}_D^H((\xi_2,\delta)\otimes e^{(\lambda_2,s)})ds,$$
$$f \mapsto (\mathcal{M}_{\delta,s}f)_{\delta=0,1,\,s\in i\mathbb{R}},$$

is a unitary isomorphism.

Proof. For a fixed $h \in H/D$ we consider the δ-part (that is even or odd part) of f given by $\tilde{f}^h_\delta(t) = \frac{1}{2}f(hd(1,t)) + (-1)^\delta \frac{1}{2}f(hd(1,-t))$ then

$$\begin{aligned}\|\mathcal{M}f\|^2 &= \sum_{\delta=0}^{1}\int_{i\mathbb{R}} \|\mathcal{M}^{\delta,s}_{\xi,\lambda}f\|^2 ds = \sum_{\delta=0}^{1}\int_{H/D}\int_{i\mathbb{R}} |\sqrt{2}[\mathcal{M}'\tilde{f}^h_\delta](-is)|^2 ds\, d(hD) \\ &= 2\sum_{\delta=0}^{1}\int_{H/D}\int_0^\infty |\tilde{f}^h_\delta(t)|^2\frac{dt}{t}dh = \int_{H/D}\int_{\mathbb{R}^\times}|f(hd(1,t))|^2\frac{dt}{|t|}d(hD) = \|f\|^2_{H/D_0},\end{aligned}$$

where we used the Plancherel formula for the Mellin transform, and that the even and odd part of a function are orthogonal to one another. Being an isometry it is an injection and being a surjection then follows from the inversion formula for the Mellin transformation. □

5.2 A Plancherel formula for H/D

Consider the operator

$$B^{\eta,\nu}_{\varepsilon,\mu} : \mathrm{Ind}_D^H(\varepsilon\otimes e^\mu) \to \mathrm{Ind}_{P_H}^H(\eta\otimes e^\nu\otimes 1), \quad B^{\eta,\nu}_{\varepsilon,\mu}f(g) = \int_{H/D} K^{\eta,\nu}_{\varepsilon,\mu}(x^{-1}g)f(x)\,d(xD),$$

where

$$K^{\eta,\nu}_{\varepsilon,\mu}(g) = |g_{11}|_{\eta_1+\varepsilon_2}^{\frac{\nu_2-\nu_1-(\mu_1-\mu_2)-1}{2}}|g_{21}|_{\eta_2+\varepsilon_2}^{\frac{\nu_2-\nu_1+(\mu_1-\mu_2)-1}{2}}|\det(g)|_{\eta_2}^{\frac{1}{2}-\nu_2}.$$

By acting by the center of H, we see that

$$\varepsilon_1+\varepsilon_2 = \eta_1+\eta_2, \quad \mu_1+\mu_2 = \nu_1+\nu_2.$$

Theorem 5.3 ([BJD23])**.** *Let $\mu \in i\mathbb{R}^2$ and put $\mu_\pm = \mu_1 \pm \mu_2$ and $\varepsilon_* = \varepsilon_1+\varepsilon_2$. For $f \in \mathrm{Ind}_D^H(\varepsilon\otimes e^\mu)$*

$$\begin{aligned}\|f\|^2 = \int_{i\mathbb{R}}\sum_{\sigma=0}^{1}\|B^{\eta(\sigma),\nu(t)}_{\varepsilon,\mu}f\|^2\frac{dt}{|a(\varepsilon_*,t)|^2} \\ + \sum_{\sigma=0}^{1}\sum_{t\in 1-\varepsilon_*+2\sigma-4\mathbb{N}} b(\varepsilon_*,\sigma(1-\varepsilon_*),t)\|B^{\eta(\sigma(1-\varepsilon_*)),\nu(t)}_{\varepsilon,\mu}f\|^2,\end{aligned}$$

where

$$(\eta(\sigma),\nu(t)) = \Big(\varepsilon+\sigma(1,1),(\tfrac{\mu_++t}{2}, t\frac{\mu_+-t}{2})\Big),$$

and

$$b(\delta,\sigma,\mu_-,t) = \frac{2^{t-2+3\delta}(-t)^\delta\Gamma(1-t)|\Gamma(\frac{1+2\sigma+t+\mu_-}{4})|^2}{\pi^2\Gamma(\frac{\delta-t}{2})|\Gamma(\frac{1+2\sigma-t-\mu_-}{4})|^2}, \quad a(\delta,t) = \frac{2^{\frac{3}{2}}\pi\Gamma(\frac{t}{2})\Gamma(\frac{1-t}{2})}{\Gamma(\frac{1+t+\delta}{2})\Gamma(\frac{1+t-\delta}{2})}.$$

Composing the maps above we get an H-intertwining map

$$B^{\eta,\nu}_{(\xi_2,\delta),(\lambda_2,s)}\mathcal{M}_{\delta,s}\mathcal{O} : \pi_{\xi,\lambda}|_H \to \tau_{\eta,\nu},$$

and for generic parameters $\mathrm{Hom}_H(\pi_{\xi,\lambda}|_H,\tau_{\eta,\nu})$ has dimension at most one so this composition should give us multiple of $A^{\eta,\nu}_{\xi,\lambda}$.

Proposition 5.4. *Let $t \in \mathbb{C}$ and $\sigma \in \mathbb{Z}/2\mathbb{Z}$ and put $\eta = (\xi_2 + \sigma, \delta + \sigma)$ and $\nu = \frac{1}{2}(\lambda_2 + s + t, \lambda_2 + s - t)$. We have the following relation*

$$B^{\eta,\nu}_{(\xi_2,\delta),(\lambda_2,s)} \mathcal{M}_{\delta,s}\mathcal{O} = \frac{(-1)^{\xi_2}}{2\sqrt{\pi}} A^{\eta,\nu}_{\xi,\lambda}. \tag{5.4}$$

Proof. Using (5.1) we get that

$$\begin{aligned}\mathcal{M}_{\delta,s}\mathcal{O}f(\overline{n}(x)n(y)) &= \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}^\times} |t|^s_\delta f(\overline{n}(x)n(y)d(1,t)g_0) \frac{dt}{|t|} \\ &= \frac{|y|^{\lambda_2-\lambda_1-1}_{\xi_1+\xi_2}(-1)^{\xi_2}}{2\sqrt{\pi}} \int_{\mathbb{R}^\times} |t|^{\lambda_1-s}_{\xi_1+\delta} f(\overline{n}(x+y^{-1}, t, (ty)^{-1}))\, dt.\end{aligned}$$

As both maps are H-intertwining it suffices to show the relation at the identity element. The left hand side of (5.4) at the identity element is

$$\begin{aligned}\frac{(-1)^{\xi_2+\eta_2+\delta}}{2\sqrt{\pi}} \int_{\mathbb{R}^3} |y|^{\lambda_2-\lambda_1-1}_{\xi_1+\xi_2} |t|^{\lambda_1-s}_{\delta+\xi_1} |1+xy|^{\frac{t+s-\lambda_2-1}{2}}_{\xi_2+\sigma+\delta} \\ \times |x|^{\frac{t+\lambda_2-s-1}{2}}_\sigma f(\overline{n}(x+y^{-1}, t, (ty)^{-1}))dxdtdy\end{aligned}$$

Next changing variables $(x+y^{-1}, t, (ty)^{-1}) \to (x, y, z)$, we get

$$\frac{(-1)^{\xi_2}}{2\sqrt{\pi}} \int_{\mathbb{R}^3} |z-xy|^{\lambda_2-\frac{s+\lambda_2+t}{2}-\frac{1}{2}}_\sigma |z|^{\lambda_1-\frac{\lambda_2+t+s}{2}-\frac{1}{2}}_{\xi_1+\sigma+\delta} |x|^{\frac{\lambda_2+s-t}{2}-\lambda_2-\frac{1}{2}}_{\xi_2+\sigma+\delta} f(\overline{n}(x,y,z))\, dxdydz.$$

Comparing with (4.1) shows (5.4). □

6 The Plancherel formula on the unitary axis

On $\pi_{\xi,\lambda} \times \pi_{\xi,-\lambda}$, we have the bilinear pairing

$$\langle f, f' \rangle = \int_{K_G} f(k) f'(k)\, dk,$$

and the corresponding sesquilinear pairing on $\pi_{\xi,\lambda} \times \pi_{\xi,-\overline{\lambda}}$ given by $(f \,|\, f') = \langle f, \overline{f'} \rangle$. For $\lambda \in i\mathbb{R}^3$ and $(f, f') \in \pi_{\xi,\lambda} \times \pi_{\xi,-\lambda}$, we have that

$$\langle f, f' \rangle = (f \,|\, \overline{f'}) = (\mathcal{M}\mathcal{O}f \,|\, \mathcal{M}\mathcal{O}\overline{f'}) = \sum_{\delta=0}^{1} \int_{i\mathbb{R}} (\mathcal{M}_{\delta,s}\mathcal{O}f \,|\, \mathcal{M}_{\delta,s}\mathcal{O}\overline{f'}) ds.$$

Let $\varepsilon(\delta) = \xi_2 + \delta$ and $(f, f') \in \pi_{\xi,\lambda} \times \pi_{w_0(\xi,-\lambda)}$. For $\lambda \in i\mathbb{R}^3$ and $\xi_1 = \xi_3$ we get, using Theorem 5.3, that

$$\begin{aligned}\langle f, \mathbf{T}^{w_0}_{w_0(\xi,-\lambda)} f' \rangle &= \sum_{\delta=0}^{1} \int_{i\mathbb{R}} (\mathcal{M}_{\delta,s}\mathcal{O}f \,|\, \mathcal{M}_{\delta,s}\mathcal{O}\mathbf{T}^{w_0}_{w_0(\xi,\lambda)} \overline{f'}) ds \\ &= \sum_{\delta=0}^{1} \int_{i\mathbb{R}} \Bigg[\int_{i\mathbb{R}} \sum_{\sigma=0}^{1} \Big(B^{\eta(\sigma,\delta),\nu(s,t)}_{(\xi_2,\delta),(\lambda_2,s)} \mathcal{M}_{\delta,s}\mathcal{O}f \,\Big|\, B^{\eta(\sigma,\delta),\nu(s,t)}_{(\xi_2,\delta),(\lambda_2,s)} \mathcal{M}_{\delta,s}\mathcal{O}\mathbf{T}^{w_0}_{w_0(\xi,\lambda)} \overline{f'} \Big) \frac{dt}{|a(\varepsilon(\delta),t)|^2} \\ &\quad + \sum_{\sigma=0}^{1} \sum_{t \in 1-[\varepsilon(\delta)]+2\sigma-4\mathbb{N}} b(\varepsilon(\delta), \sigma(1-\varepsilon(\delta)), s+\lambda_2, t) \\ &\qquad \times \Big(B^{\eta(\sigma(1-\varepsilon(\delta),\delta),\nu(s,t)}_{(\xi_2,\delta),(\lambda_2,s)} \mathcal{M}_{\delta,s}\mathcal{O}f \,\Big|\, B^{\eta(\sigma(1-\varepsilon(\delta),\delta),\nu(s,t)}_{(\xi_2,\delta),(\lambda_2,s)} \mathcal{M}_{\delta,s}\mathcal{O}\mathbf{T}^{w_0}_{w_0(\xi,\lambda)} \overline{f'} \Big) \Bigg] ds,\end{aligned}$$

where $\nu(s,t) = \frac{1}{2}(\lambda_2 + s + t, \lambda_2 + s - t)$ and $\eta(\sigma,\delta) = (\xi_2 + \sigma, \delta + \sigma)$. Applying Proposition 5.4, we get

$$\sum_{\sigma,\delta=0}^{1} \int_{i\mathbb{R}^2} \left(A_{\xi,\lambda}^{\eta(\sigma,\eta),\nu(s,t)} f \middle| A_{\xi,\lambda}^{\eta(\sigma,\delta),\nu(s,t)} \mathbf{T}_{w_0(\xi,\lambda)}^{w_0} \overline{f'} \right) \frac{dtds}{4\pi |a(\varepsilon(\delta),t)|^2}$$
$$+ \sum_{\sigma,\delta=0}^{1} \int_{i\mathbb{R}} \sum_{t \in 1-\varepsilon(\delta)+2\sigma-4\mathbb{N}} \frac{b(\varepsilon(\delta), \sigma(1-\varepsilon(\delta)), s+\lambda_2, t)}{4\pi}$$
$$\times \left(A_{\xi,\lambda}^{\eta(\sigma(1-\varepsilon(\delta)),\delta),\nu(s,t)} f \middle| A_{\xi,\lambda}^{\eta(\sigma(1-\varepsilon(\delta)),\delta),\nu(s,t)} \mathbf{T}_{w_0(\xi,\lambda)}^{w_0} \overline{f'} \right) ds \quad (6.1)$$

We will refer to this equation often and we will call the first line of this expression for the *continuous part* and the remaining for the *discrete part* even though strictly speaking it is not discrete.

The primary objective is now to consider the right hand side as a holomorphic function in λ, and analytically extend it to λ, where $\pi_{\xi,\lambda}$ is unitary or contains a unitary quotient. We know that for $\lambda \in i\mathbb{R}^3$, this expression coincides with $\langle f, \mathbf{T}_{w_0(\xi,-\lambda)}^{w_0} f' \rangle$ which is an entire holomorphic function, so by the identity theorem for holomorphic functions they coincide for all λ. When we then restrict (ξ,λ) to the parameters for which $\pi_{\xi,\lambda}$ is unitary (or contains a unitary quotient), and set $f' = \overline{f}$ we get something along the lines of

$$||f||^2 = \langle f, \mathbf{T}_{\xi,\overline{\lambda}}^{w_0} \overline{f} \rangle = \sum_{\eta} \int_{i\mathbb{R}^2} || \mathbf{A}_{\xi,\lambda}^{\eta,\nu} f ||^2 \frac{d\nu_1 d\nu_2}{|a(\lambda,\nu)|^2} + \sum_{\eta,\nu_1-\nu_2} \int_{i\mathbb{R}} || \mathbf{A}_{\xi,\lambda}^{\eta,\nu} f ||^2 \frac{d(\nu_1+\nu_2)}{|b(\lambda,\nu)|^2},$$

this is then a Plancherel formula for

$$\pi_{\xi,\lambda}|_H \simeq \bigoplus_{\eta} \int_{i\mathbb{R}^2}^{\oplus} \tau_{\eta,\nu}\, d\nu \oplus \bigoplus_{\eta,\nu_1-\nu_2} \int_{i\mathbb{R}}^{\oplus} \tau_{\eta,\nu}\, d(\nu_1+\nu_2),$$

giving a unitary branching law.

But before we start cheering in pure ecstasy let us talk about some limitations about this approach. Looking at the expression $\langle f, \mathbf{T}_{\xi,\overline{\lambda}}^{w_0} \overline{f} \rangle$ we see that it can only be an inner product if $\xi_1 = \xi_3$, as w_0 swap ξ_1 and ξ_3. This implies that outside of the unitary axis we can at best hope to decompose representations with $\xi_1 = \xi_3$. However, this contains the complementary series, the generalized series induced from an odd discrete series of $\mathrm{SL}^{\pm}(2,\mathbb{R})$, and the degenerate series induced from the trivial representation of $\mathrm{SL}^{\pm}(2,\mathbb{R})$, which is still a big portion of the unitary representations.

We introduce the following meromorphic function which will show up in both the continuous part and the discrete part:

$$p(\xi,\lambda,\eta,\nu) := n(\xi,\lambda,\eta,\nu) n(\xi,-\lambda,\eta,-\nu) = \prod_{i=1}^{3} p_i(\xi,\lambda,\eta,\nu),$$

where

$$p_i(\xi,\lambda,\eta,\nu) = \Gamma(\tfrac{\lambda_i - \nu_1 + \frac{1}{2} + [\xi_i + \eta_1]}{2}) \Gamma(\tfrac{\nu_1 - \lambda_i + \frac{1}{2} + [\xi_i + \eta_1]}{2}) \Gamma(\tfrac{\lambda_i - \nu_2 + \frac{1}{2} + [\xi_i + \eta_2]}{2}) \Gamma(\tfrac{\nu_2 - \lambda_i + \frac{1}{2} + [\eta_2 + \xi_i]}{2}).$$

6.1 The continuous part

Recall that we are still assuming that $\lambda \in i\mathbb{R}^3$, $\xi_1 = \xi_3$ and $(f, f') \in \pi_{\xi,\lambda} \times \pi_{w_0(\xi,-\lambda)}$. Only considering the continuous part of (6.1). We make the change of variables in the sums $\eta_1 = \xi_2 + \sigma$ and $\eta_2 = \delta + \sigma$ and in the integrals $\mu_1 = \frac{1}{2}(\lambda_2 + s)$ and $\mu_2 = \frac{1}{2}t$ and abusing notation we set $\eta = (\eta_1, \eta_2)$ and $\nu(\mu) = (\mu_1 + \mu_2, \mu_1 - \mu_2)$, so we get

$$\sum_{\eta \in (\mathbb{Z}/2\mathbb{Z})^2} \int_{i\mathbb{R}^2} \left\langle A_{\xi,\lambda}^{\eta,\nu(\mu)} f, A_{\xi,-\lambda}^{\eta,-\nu(\mu)} \mathbf{T}_{w_0(\xi,-\lambda)}^{w_0} f' \right\rangle \frac{d\mu_1 d\mu_2}{\pi |a(\eta_1 + \eta_2, 2\mu_2)|^2}.$$

By artificially introducing $n(\xi, \lambda, \eta, \nu)$ and using Corollary 4.3, we get

$$(-1)^{\xi_1} \sqrt{\pi} \sum_{\eta \in (\mathbb{Z}/2\mathbb{Z})^2} \alpha(\xi, -\lambda) \int_{i\mathbb{R}^2} \left\langle \mathbf{A}_{\xi,\lambda}^{\eta,\nu(\mu)} f,\, \mathbf{A}_{w_0(\xi,-\lambda)}^{\eta,-\nu(\mu)} f' \right\rangle \frac{p(\xi,\lambda,\eta,\nu(\mu))}{|a(\eta_1+\eta_2, 2\mu_2)|^2} d\mu_1 d\mu_2. \tag{6.2}$$

As the symmetry breaking operators have been normalized to be holomorphic, the poles are given by the functions in the measure. Abusing notation, we write $p(\xi, \lambda, \eta, \mu)$ for $p(\xi, \lambda, \eta, \nu(\mu))$. The poles of $p(\xi, \lambda, \eta, \mu)$ are

$$\begin{aligned} \mu_1 &= \lambda_i - \mu_2 + \frac{1}{2} + [\xi_i + \eta_1] + 2n, \quad & \mu_1 &= \lambda_i - \mu_2 - \frac{1}{2} - [\xi_i + \eta_1] - 2n, \\ \mu_1 &= \lambda_i + \mu_2 + \frac{1}{2} + [\xi_i + \eta_2] + 2n, \quad & \mu_1 &= \lambda_i + \mu_2 - \frac{1}{2} - [\xi_i + \eta_2] - 2n, \end{aligned} \tag{6.3}$$

where $n \in \mathbb{N}_0$. Let q be the meromorphic function defined by

$$q(\eta, \mu_2) = a(\eta_1 + \eta_2, 2\mu_2)^{-1} a(\eta_1 + \eta_2, -2\mu_2)^{-1} = \frac{\Gamma\left(\frac{1+[\eta_1+\eta_2]}{2} + \mu_2\right)\Gamma\left(\frac{1+[\eta_1+\eta_2]}{2} - \mu_2\right)}{2^3\pi^2\Gamma\left(\frac{[\eta_1+\eta_2]}{2} + \mu_2\right)\Gamma\left(\frac{[\eta_1+\eta_2]}{2} - \mu_2\right)},$$

which has poles at

$$\mu_2 = -\frac{1 + [\eta_1 + \eta_2]}{2} - n, \quad \mu_2 = \frac{1 + [\eta_1 + \eta_2]}{2} + n,$$

and vanishes at

$$\mu_2 = -\frac{[\eta_1 + \eta_2]}{2} - n, \quad \mu_2 = \frac{[\eta_1 + \eta_2]}{2} + n,$$

where $n \in \mathbb{N}_0$. Lastly, we set

$$Q(\xi, \lambda, \eta, \mu) = \left\langle \mathbf{A}_{\xi,\lambda}^{\eta,\nu(\mu)} f,\, \mathbf{A}_{w_0(\xi,-\lambda)}^{\eta,-\nu(\mu)} f' \right\rangle,$$

which is holomorphic as a function in (λ, μ). Up to constants we can then express each summand of the continuous part as

$$I(\xi, \lambda, \eta) := \alpha(\xi, -\lambda) \int_{i\mathbb{R}} \int_{i\mathbb{R}} Q(\xi, \lambda, \eta, \mu) p(\xi, \lambda, \eta, \mu) q(\eta, \mu_2)\, d\mu_1 d\mu_2. \tag{6.4}$$

Now that we have repacked everything into a complex analysis problem, we can start to view this as a function of λ and in doing so we see that none of the functions have poles as long as $\mathrm{Re}(\lambda) \in (-\frac{1}{2}, \frac{1}{2})^3$.

As a service to the reader, we will briefly explain some notational choices before the forthcoming technical analytic continuation part begins. We denote by lower case Latin letters p, q, r, q' meromorphic functions which has poles we need to be concerned about. By upper case Latin letters Q, Q', R, R' we denote holomorphic functions which we need to be less concerned about. At times, we restrict our parameters to certain hyperplanes where factors of the meromorphic functions have first order poles at the same time as the holomorphic function has zeroes, we then move the corresponding meromorphic factor into the holomorphic function and update our notation which is the reason for so many different functions introduced.

6.2 The discrete part

Recall that $\lambda \in i\mathbb{R}^3$ with $f \in \pi_{\xi,\lambda}$ and $f' \in \pi_{w_0(\xi,-\lambda)}$ and consider the discrete part of (6.1)

$$\sum_{\sigma,\delta=0}^{1} \int_{i\mathbb{R}} \sum_{t \in 1-\varepsilon(\delta)+2\sigma-4\mathbb{N}} \frac{b(\varepsilon(\delta), \sigma(1-\varepsilon(\delta)), s, t)}{4\pi} \times \left(A_{\xi,\lambda}^{\eta(\sigma(1-\varepsilon(\delta)),\delta),\nu(s,t)} f \middle| A_{\xi,-\lambda}^{\eta(\sigma(1-\varepsilon(\delta)),\delta),\nu(s,t)} \mathbf{T}_{w_0(\xi,\lambda)}^{w_0} \overline{f'} \right) ds. \quad (6.5)$$

Separating the analysis into the cases where $\varepsilon(\delta) = 0$ and $\varepsilon(\delta) = 1$, that is the term where $\delta = \xi_2$ and $\delta \neq \xi_2$, turns out to be a bit cleaner. In both cases we will do the substitutions $\mu_1 = \frac{1}{2}(s+\lambda_2)$ and $\mu_2 = \frac{t}{2}$ and set $\nu(\mu) = (\mu_1+\mu_2, \mu_1-\mu_2)$. As the sum puts the parameters in certain hyperplanes, we can renormalize $\mathbf{A}_{\xi,\lambda}^{\eta,\nu}$ on these hyperplanes as follows.

Lemma 6.1. *Let $\nu_1 - \nu_2 + 1 + [\eta_1 + \eta_2] = -2n$. Then for $f \in \pi_{\xi,\lambda}$ and $f' \in \pi_{w_0(\xi,-\lambda)}$ the function*

$$R(\xi, \lambda, \eta, \nu) = \frac{\langle \boldsymbol{A}_{\xi,\lambda}^{\eta,\nu} f, \boldsymbol{A}_{w_0(\xi,-\lambda)}^{\eta,-\nu} f' \rangle}{\prod_{i=1}^{3} (\frac{\lambda_i - \nu_2 + \frac{1}{2} + [\xi_i + \eta_2]}{2})_{n+1-m_i} (\frac{\nu_1 - \lambda_i + \frac{1}{2} + [\xi_i + \eta_1]}{2})_{n+1-m_i}},$$

where $m_i = m_i(\xi, \eta) = (1 - [\eta_1 + \eta_2])[\xi_i + \eta_1]$, is holomorphic. That is, the pairing vanishes exactly at the zeroes of the Pochhammer symbols in the denominator.

Proof. Note that for $k = 0, 1, \ldots, n - m_i$ the set

$$\{\nu_2 - \nu_1 + 1 + [\eta_1 + \eta_2] = -2n\} \cap \{\lambda_i - \nu_1 + \tfrac{1}{2} + [\xi_i + \eta_1] = -2k\},$$

is contained in $\mathcal{M}_i$, that is $\mathbf{A}_{\xi,\lambda}^{\eta,\nu}$ vanishes on it. Thus, we get that

$$\mathbf{C}_{w_0(\xi,-\lambda)}^{\eta,-\nu} := \frac{\mathbf{A}_{w_0(\xi,-\lambda)}^{\eta,-\nu}}{\prod_{i=1}^{3} (\frac{\nu_1 - \lambda_i + \frac{1}{2} + [\xi_i + \eta_1]}{2})_{n+1-m_i}},$$

is regular. Furthermore, we see that $\mathrm{im}(\mathbf{C}_{w_0(\xi,-\lambda)}^{\eta,-\nu}) \subseteq \ker(\mathbf{T}_{\eta,-\nu}^{w_H})$ as by Theorem 3.1

$$\mathbf{T}_{\eta,-\nu}^{w_H} \mathbf{C}_{w_0(\xi,-\lambda)}^{\eta,-\nu} = \frac{c\Gamma(\frac{\nu_1 - \nu_2 + 1 + [\eta_1 + \eta_2]}{2})^{-1}}{\prod_{i=1}^{3} (\frac{\nu_1 - \lambda_i + \frac{1}{2} + [\xi_i + \eta_2]}{2})_{n+1-m_i}} \mathbf{A}_{w_0(\xi,-\lambda)}^{w_H(\eta,-\nu)} = 0.$$

On the other hand, we have

$$\mathbf{T}^{w_H}_{\eta,\nu}\,\mathbf{A}^{\eta,\nu}_{\xi,\lambda} = c'\Gamma(\tfrac{\nu_2-\nu_1+1+[\eta_1+\eta_2]}{2})^{-1}\,\mathbf{A}^{w_H(\eta,\nu)}_{\xi,\lambda},$$

which vanishes for $\lambda_i - \nu_2 + \frac{1}{2} + [\xi_i + \eta_2] = -2k$ for $k = 0, 1, \ldots, n - m_i$ i.e. $\operatorname{im}(\mathbf{A}^{\eta,\nu}_{\xi,\lambda}) \subseteq \ker(\mathbf{T}^{w_H}_{\eta,\nu})$ for $\lambda_i - \nu_2 + \frac{1}{2} + [\xi_i + \eta_2] = -2k$ where $k = 0, 1, \ldots, n - m_i$. At these parameters we conclude that one entry of the K_H-pairing is in the finite dimensional subrepresentation of $\tau_{\eta,\nu}$, and the other entry is in the infinite dimensional subrepresentation of $\tau_{\eta,-\nu}$, giving us that the K_H-pairing is zero. □

Proposition 6.2. *For $\delta = \xi_2$ the expression the discrete part in (6.1) reduces to*

$$\pi^{\frac{5}{2}}\alpha(\xi,-\lambda)\sum_{\sigma=0}^{1}(-1)^{\xi_1+\xi_2+\sigma}\sum_{n=1}^{\infty}16^{\sigma-2n}(4n-2\sigma-1)$$
$$\times\int_{i\mathbb{R}}\frac{\Gamma(\lambda_1-\mu_1-\sigma+2n)\Gamma(\mu_1-\lambda_1-\sigma+2n)\Gamma(\lambda_3-\mu_1-\sigma+2n)\Gamma(\mu_1-\lambda_3-\sigma+2n)}{\cos^2\left(\frac{\pi}{2}(\mu_1-\lambda_2)\right)}R(\xi,\lambda,\eta,\nu)\,d\mu_1,$$

and for $\delta \neq \xi_2$ it reduces to

$$\pi^{\frac{1}{2}}\alpha(\xi,-\lambda)(-1)^{\xi_1}\sum_{n=1}^{\infty}16^{1-2n}n$$
$$\times\int_{i\mathbb{R}}\frac{\Gamma(\lambda_1-\mu_1+\frac{1}{2}+n)\Gamma(\mu_1-\lambda_1+\frac{1}{2}+n)\Gamma(\lambda_3-\mu_1+\frac{1}{2}+n)\Gamma(\mu_1-\lambda_3+\frac{1}{2}+n)}{\cos(\pi(\lambda_2-\mu_1))}R(\xi,\lambda,\eta,\mu)d\mu_1.$$

In both cases this is holomorphic for $\operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_3) \in (-1,1)$ and $\operatorname{Re}(\lambda_2) \in (-\frac{1}{2},\frac{1}{2})$.

Proof. Using Lemma 6.1, we can for $\nu_1 - \nu_2 + 1 + [\eta_1 + \eta_2] = -2n$ define

$$\begin{aligned}
p_i'(\xi,\lambda,\eta,\nu) &= p(\xi_i,\lambda_i,\eta,\nu)(\tfrac{\lambda_i-\nu_2+\frac{1}{2}+[\xi_i+\eta_2]}{2})_{n+1-m_i}(\tfrac{\nu_1-\lambda_i+\frac{1}{2}+[\xi_i+\eta_1]}{2})_{n+1-m_i}\\
&= \Gamma(\tfrac{\lambda_i-\nu_1+\frac{1}{2}+[\xi_i+\eta_1]}{2})\Gamma(\tfrac{\nu_1-\lambda_i+\frac{1}{2}+[\xi_i+\eta_2]}{2}+n+1-m_i)\\
&\quad\times\Gamma(\tfrac{\lambda_i-\nu_2+\frac{1}{2}+[\xi_i+\eta_2]}{2}+n+1-m_i)\Gamma(\tfrac{\nu_2-\lambda_i+\frac{1}{2}+[\eta_2+\xi_i]}{2}),
\end{aligned}$$

and set $p' = \prod_{i=1}^{3} p_i'$. Like before we will abuse notation and write $p_i'(\xi,\lambda,\eta,\mu)$ for $p_i'(\xi,\lambda,\eta,\nu(\mu))$.

If $\delta = \xi_2$ then we have $\eta(\sigma) = (\xi_2+\sigma,\xi_2+\sigma)$ and the discrete part becomes

$$\frac{1}{2\pi}\sum_{\sigma=0}^{1}\sum_{\mu_2\in\frac{1}{2}+\sigma-2\mathbb{N}}\int_{i\mathbb{R}}p(\xi,\lambda,\eta(\sigma),\nu(\mu))b(0,\sigma,2\mu_1-\lambda_2,2\mu_2)$$
$$\times\left\langle\mathbf{A}^{\eta(\sigma),\nu(\mu)}_{\xi,\lambda}f,\overline{\mathbf{T}^{w_H}_{\eta(\sigma),\nu(\mu)}\,\mathbf{A}^{\eta(\sigma),\nu(\mu)}_{\xi,\lambda}\,\mathbf{T}^{w_0}_{w_0(\xi,\lambda)}\,f'}\right\rangle d\mu_1.$$

By using Corollary 4.3 we have

$$\frac{\pi\alpha(\xi,-\lambda)}{2}\sum_{\sigma=0}^{1}(-1)^{\xi_1+\xi_2+\sigma}\sum_{\mu_2\in\frac{1}{2}+\sigma-2\mathbb{N}}\int_{i\mathbb{R}}\frac{p(\xi,\lambda,\eta(\sigma),\nu(\mu))b(0,\sigma,2\mu_1-\lambda_2,2\mu_2)}{\Gamma(\frac{1}{2}-\mu_2)}$$
$$\times\left\langle\mathbf{A}^{\eta(\sigma),\nu(\mu)}_{\xi,\lambda}f,\mathbf{A}^{\eta(\sigma),-\nu(\mu)}_{w_0(\xi,-\lambda)}f'\right\rangle d\mu_1. \quad (6.6)$$

We let $\mu_2 = \frac{1}{2} + \sigma - 2n$ then $\nu(\mu)_1 - \nu(\mu)_2 + 1 = -2(2n - 1 - \sigma)$ and $m_i = [\xi_i + \xi_2 + \sigma]$ so that

$$\begin{aligned} p_i'(\xi,\lambda,\eta,\mu) &= \Gamma(\tfrac{\lambda_i-\mu_1+[\xi_i+\xi_2+\sigma]-\sigma}{2}+n)\Gamma(\tfrac{\mu_1-\lambda_i+1-\sigma-[\xi_i+\xi_2+\sigma]}{2}+n) \\ &\quad \times \Gamma(\tfrac{\lambda_i-\mu_1+1-\sigma-[\xi_i+\xi_2+\sigma]}{2}+n)\Gamma(\tfrac{\mu_1-\lambda_i+[\xi_i+\xi_2+\sigma]-\sigma}{2}+n) \\ &= \pi 2^{2+2\sigma-4n}\Gamma(\lambda_i-\mu_1-\sigma+2n)\Gamma(\mu_1-\lambda_i-\sigma+2n), \end{aligned}$$

by the duplication formula for the Gamma-function. Similarly we get that

$$p_2(\xi_2,\lambda_2,\nu,\mu)b(0,\sigma,2\mu_1-\lambda_2,2\mu_2) = \frac{2^{-1+2\sigma-4n}\Gamma(4n-2\sigma)}{\Gamma(2n-\frac{1}{2}-\sigma)\cos^2\left(\frac{\pi}{2}(\mu_1-\lambda_2)\right)},$$

by the reflection formula for the Gamma-function. Collecting all of the above we get the desired form for $\delta = \xi_2$.

If $\delta \neq \xi_2$ then $\eta = (\xi_2, \xi_2 + 1)$ where we set $\eta' = (\xi_2 + 1, \xi_2)$ and the discrete part becomes

$$\frac{1}{2\pi}\int_{i\mathbb{R}} \sum_{\mu_2\in-\mathbb{N}} b(1,0,2\mu_1-\lambda_2,2\mu_2)\Big(A_{\xi,\lambda}^{\eta,\nu(\mu)}f\Big|A_{\xi,\lambda}^{\eta,\nu(\mu)}\mathbf{T}_{w_0(\xi,\lambda)}^{w_0}\overline{f'}\Big)d\mu_1,$$

and applying Corollary 4.3, we get

$$\tfrac{\pi\alpha(\xi,-\lambda)(-1)^{\xi_1}}{2}\sum_{\mu_2\in-\mathbb{N}}\int_{i\mathbb{R}} \tfrac{p(\xi,\lambda,\eta,\nu(\mu))b(1,0,2\mu_1-\lambda_2,2\mu_2)}{\Gamma(1-\mu_2)}\Big\langle \mathbf{A}_{\xi,\lambda}^{\eta,\nu(\mu)}f, \mathbf{A}_{w_0(\xi,-\lambda)}^{\eta',-\nu(\mu)}f'\Big\rangle d\mu_1, \qquad (6.7)$$

We let $\mu_2 = -n$ then $\nu(\mu)_1 - \nu(\mu)_2 + 2 = -2(n-1)$, and $m_i = 0$, so that

$$\begin{aligned} p_i'(\xi,\lambda,\eta,\mu) &= \Gamma(\tfrac{\lambda_i-\mu_1+\frac{1}{2}+n}{2})\Gamma(\tfrac{\lambda_i-\mu_1+\frac{3}{2}+n}{2})\Gamma(\tfrac{\mu_1-\lambda_i+\frac{1}{2}+n}{2})\Gamma(\tfrac{\mu_1-\lambda_i+\frac{3}{2}+n}{2}) \\ &= \pi 2^{1-2n}\Gamma(\lambda_i-\mu_1+\tfrac{1}{2}+n)\Gamma(\mu_1-\lambda_i+\tfrac{1}{2}+n), \end{aligned}$$

by the duplication formula for the Gamma-function. Similarly

$$b(1,0,2\mu_1-\lambda_2,2\mu_2)p_2(\xi,\lambda,\eta,\mu) = \frac{2^{3-2n}n\Gamma(1+2n)}{\Gamma(\frac{1}{2}+n)\cos(\pi(\lambda_2-\mu_1))},$$

by the reflection formula. Collecting all the above, we get the desired form for $\delta \neq \xi_2$.

□

We are now ready to prove our first direct integral decomposition. We established the Plancherel formula for $\lambda \in i\mathbb{R}^3$, but both the continuous and discrete parts are defined for $\mathrm{Re}(\lambda) \in (-\frac{1}{2}, \frac{1}{2})$ and so the Plancherel formula also holds for the complementary series.

Theorem 6.3. *The unitary principal series decomposes as*

$$\pi_{\xi,\lambda}|_H \simeq \bigoplus_{\eta\in(\mathbb{Z}/2\mathbb{Z})^2}\int_{i\mathbb{R}^2}^{\oplus}\tau_{\eta,\nu}\,d\nu \oplus \bigoplus_{\eta\in\mathbb{Z}/2\mathbb{Z}}\bigoplus_{\nu_-\in 1+\eta-2\mathbb{N}}\int_{i\mathbb{R}}^{\oplus}\tau_{\eta,\nu}^{ds}\,d\nu_+.$$

The complementary series decomposes as

$$\pi_{\xi,\lambda}|_H \simeq \bigoplus_{\eta\in(\mathbb{Z}/2\mathbb{Z})^2}\int_{i\mathbb{R}^2}^{\oplus}\tau_{\eta,\nu}\,d\nu \oplus \bigoplus_{\eta\in\mathbb{Z}/2\mathbb{Z}}\bigoplus_{\nu_-\in 1+\eta-2\mathbb{N}}\int_{i\mathbb{R}}^{\oplus}\tau_{\eta,\nu}^{ds}\,d\nu_+.$$

Proof. Restricting to the parameters $\lambda \in i\mathbb{R}^3$ with $\lambda_1 = \lambda_3$ and $\xi_1 = \xi_3$ then by (6.2) and Proposition 6.2 with $f' = \overline{f}$ we get

$$\langle f, \mathbf{T}^{w_0}_{\xi,-\lambda}\overline{f}\rangle = \frac{(-1)^{\xi_1}\pi^{\frac{1}{2}}}{|\Gamma(\frac{\lambda_1-\lambda_2+1+[\xi_1+\xi_2]}{2})|^2}\Bigg(\int_{i\mathbb{R}^2}\frac{|n(\xi,\lambda,\eta,\nu(\mu))|^2}{|a(\eta,\nu(\mu))|^2}||\,\mathbf{A}^{\eta,\nu(\mu)}_{\xi,\lambda}\,f||^2\,d\mu$$
$$+\sum_{\sigma=0}^{1}\sum_{n=1}^{\infty}16^{\sigma-2n}(4n-2\sigma-1)\Gamma(2n-2\sigma-1)\pi^2$$
$$\times\int_{i\mathbb{R}}\frac{|\Gamma(\lambda_1-\mu_1-\sigma+2n)|^2}{\cos^2(\frac{\pi(\mu_1-\lambda_2)}{2})}||\,\mathbf{B}^{\eta(\sigma),\nu(\mu_1,\frac{1}{2}+\sigma-2n)}_{\xi,\lambda}\,f||^2 d\mu_1$$
$$+\sum_{n=1}^{\infty}16^{1-2n}\Gamma(n+1)n\int_{i\mathbb{R}}\frac{|\Gamma(\lambda_1-\mu_1+\frac{1}{2}+2n)|^4}{\cos(\pi(\mu_1-\lambda_2))}||\,\mathbf{B}^{(\xi_2,\xi_2+1),\nu(\mu_1,-n)}_{\xi,\lambda}\,f||^2 d\mu_1\Bigg),$$

we see that everything is definite so the map

$$f\mapsto(\mathbf{A}^{\eta,\nu}_{\xi,\lambda}\,f)_{\eta\in(\mathbb{Z}/2\mathbb{Z})^2,\,\nu\in i\mathbb{R}^2}\cup(\mathbf{B}^{\eta,\nu}_{\xi,\lambda}\,f)_{\eta\in\mathbb{Z}/2\mathbb{Z},\,\nu_1-\nu_2+\eta_+\in 1-2\mathbb{N},\,\nu_1+\nu_2\in i\mathbb{R}},$$

mapping from $\pi_{\xi,\lambda}|_H$ to the direct integral is injective as it preserves norm. On the other hand, we see that it is surjective as each $\mathbf{A}^{\eta,\nu}_{\xi,\lambda}$ and $\mathbf{B}^{\eta,\nu}_{\xi,\lambda}$ are non-vanishing and each $\tau_{\eta,\nu}$ is irreducible. The other direct integral decomposition follows by similar observations. □

7 Decomposing the unitarily induced generalized principal series

In this section we decompose the "first" unitarily induced generalized principal series. As we noted in Section 3, the generalized principal series $\pi^{\text{gen}}_{1,\xi_2,\mu}$ sits inside $\pi_{\xi,\lambda}$ as a quotient when $\xi_1 = \xi_3$, $\lambda_1 + \overline{\lambda_3} = 0$, $\lambda_2 \in i\mathbb{R}$ and $\mathrm{Re}(\lambda_1) = -\frac{1}{2}$ where $\mu = (i\,\mathrm{Im}(\lambda_1), \lambda_2)$. However, our initial Plancherel formula is only holomorphic for $\mathrm{Re}(\lambda) \in (-\frac{1}{2}, \frac{1}{2})^3$, but the discrete part is holomorphic for $\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_3) \in (-1, 1)$ and $\mathrm{Re}(\lambda_2) \in (-\frac{1}{2}, \frac{1}{2})$, so we need to analytically extend the continuous part. We start by stating the theorem and then spend the subsection proving it.

Theorem 7.1. *Let $\nu_\pm = \nu_1 \pm \nu_2$. The unitarily induced generalized principal series decomposes as*

$$\pi^{\text{gen}}_{1,\xi_2,\mu}|_H \simeq \bigoplus_{\eta\in(\mathbb{Z}/2\mathbb{Z})^2}\int^{\oplus}_{i\mathbb{R}^2}\tau_{\eta,\nu}\,d\nu\oplus\bigoplus_{\eta\in\mathbb{Z}/2\mathbb{Z}}\bigoplus_{\nu_-\in 1+\eta-2\mathbb{N}}\int^{\oplus}_{i\mathbb{R}}\tau^{ds}_{\eta,\nu}\,d\nu_+.$$

In example B.3 we showcase a toy example of the analytic continuation procedure about to be employed.

Lemma 7.2. *Let $\eta_1 = \xi_1$. Consider the holomorphic function*

$$R'(\xi,\lambda,\eta,\mu_2) = \Gamma(\tfrac{[\eta_1+\eta_2]}{2}-\mu_2)^{-1}\langle\,\boldsymbol{A}^{\eta,\nu'(\mu_2)}_{\xi,\lambda}\,f, \boldsymbol{A}^{\eta,-\nu'(\mu_2)}_{w_0(\xi,-\lambda)}\,f'\rangle,$$

where $\nu'(\mu_2)$ is $\nu(\mu)$ for $\mu_1 \in \{\lambda_1 - \mu_2 + \frac{1}{2}, \lambda_3 - \mu_2 + \frac{1}{2}\}$. For $i = 1, 2$ let

$$r_i(\sigma,\delta,\mu_2) = \Gamma(\tfrac{\sigma_i+\delta_i}{2}+\mu_2)\Gamma(\tfrac{1+\delta_i-\sigma_i}{2}-\mu_2), \quad q'(\delta_1,\mu_2) = \frac{\Gamma(\frac{1+\delta_1}{2}+\mu_2)\Gamma(\frac{1+\delta_1}{2}-\mu_2)}{\Gamma(\frac{\delta_1}{2}-\mu_2)},$$

where $((\xi_1,\delta),(\lambda_1,\sigma))$ *is an invertible linear transformation of* (ξ,λ) *where* $\delta \in (\mathbb{Z}/2\mathbb{Z})^2$ *and* $\sigma \in \mathbb{C}^2$. *The function*

$$J(\delta,\sigma) = \int_{i\mathbb{R}} r_1(\sigma,\delta,\mu_2) r_2(\sigma,\delta,\mu_2) q'(\delta_1,\mu_2) R'((\xi_1,\delta),(\lambda_1,\sigma),\eta,\mu_2)\, d\mu_2,$$

is originally defined for $\mathrm{Re}(\sigma) \in (0,1)^2$ *but analytically extends to* $\mathrm{Re}(\sigma_1-\sigma_2) \in (0,1)$ *and* $\mathrm{Re}(\sigma_1) \in (-2,2)$, *where it is given as*

$$\begin{aligned}&\int_{i\mathbb{R}+\frac{1-\delta_1}{4}} r_1(\sigma,\delta,\mu_2) r_2(\sigma,\delta,\mu_2) q'(\delta,\mu_2) R'((\xi_1,\delta),(\lambda_1,\sigma),\eta,\mu_2)\, d\mu_2 \\ &+ 2(1-\delta_1)\pi^{\frac{3}{2}} \frac{\Gamma(\frac{1+\sigma_2-\sigma_1+\delta_2}{2})\Gamma(\frac{\sigma_1-\sigma_2+\delta_2}{2})\Gamma(1+\frac{\sigma_1}{2})\Gamma(1-\frac{\sigma_1}{2})}{\Gamma(\frac{\sigma_1-1}{2})} R'((\xi_1,\delta),(\lambda_1,\sigma),\eta,\tfrac{1-\sigma_1}{2}).\end{aligned} \tag{7.1}$$

Proof. In the case where $\delta_1 = 1$, then (7.1) is the same as the initial expression for $J(\delta,\sigma)$, so we let $\delta_1 = 0$. The poles of r_i are at

$$\mu_2 = -\frac{\sigma_i+\delta_i}{2} - n, \quad \mu_2 = \frac{1+\delta_i-\sigma_i}{2} + n,$$

for $n \in \mathbb{N}_0$. Restricting $J(\delta,\sigma)$ to $\mathrm{Re}(\sigma_1) \in (\frac{1}{2},1)$ and $\mathrm{Re}(\sigma_2) \in (0,\frac{1}{2})$ we see that the poles for r_i are at

$$\begin{aligned}\mathrm{Re}(\mu_2) &\in (-\tfrac{1}{2},-\tfrac{1}{4}) - n, \quad \mathrm{Re}(\mu_2) \in (0,\tfrac{1}{4}) + n, \\ \mathrm{Re}(\mu_2) &\in (-\tfrac{1}{4},0) - \tfrac{\delta_2}{2} - n, \quad \mathrm{Re}(\mu_2) \in (\tfrac{1}{2},1) + \tfrac{\delta_2}{2} + n.\end{aligned}$$

So, if we shift the contour of integration we get

$$\int_{i\mathbb{R}+\frac{1}{4}} r_1 r_2 q' R' d\mu_2 - 2\pi\, \mathrm{Res}(r_1 r_2 q' R', \mu_2 = \tfrac{1-\sigma_1}{2}),$$

and by calculating the residue we get the result. □

Proposition 7.3. *The integral* $I(\xi,\lambda,\eta)$ *from* (6.4) *analytically extends to*

$$U = \{\lambda \in \mathbb{C}^3 : \mathrm{Re}(\lambda) \in (-1,0)\times(0,1)^2, \mathrm{Re}(\lambda_3-\lambda_1) \in (0,2), \mathrm{Re}(\lambda_3-\lambda_2) \in (0,1)\}.$$

Restricting $I(\xi,\lambda,\eta)$ *to* $\lambda_1 - \lambda_3 + 1 = 0$ *it is holomorphic for* $\mathrm{Re}(\lambda_1) \in (-\frac{3}{2},\frac{1}{2})$ *and* $\mathrm{Re}(\lambda_2) \in (-\frac{1}{2},\frac{1}{2})$ *and is given as*

$$\begin{aligned}I(\xi,\lambda,\eta) = \alpha(\xi,-\lambda)\pi \int_{i\mathbb{R}^2} & p_2(\xi,\lambda,\eta,\mu) Q'(\xi,\lambda,\eta,\mu) q(\eta,\mu_2) \\ &\times \prod_{k=1}^{2} \Gamma(\lambda_1 - \mu_1 + (-1)^k\mu_2 + \tfrac{3}{2})\Gamma(\mu_1 + (-1)^k\mu_2 - \lambda_1 + \tfrac{1}{2})\, d\mu_1 d\mu_2,\end{aligned}$$

where

$$Q'(\xi,\lambda,\eta,\mu) = \frac{Q(\xi,\lambda,\eta,\mu)}{\prod_{k=1}^{2}(\frac{\lambda_1-\nu_k+\frac{1}{2}+[\xi_1+\eta_k]}{2})_{1-[\xi_1+\eta_k]}(\frac{\nu_k-\lambda_3+\frac{1}{2}+[\eta_k+\xi_3]}{2})_{1-[\xi_1+\eta_k]}}.$$

Proof. Restrict $I(\xi,\lambda,\eta)$ to $\mathrm{Re}(\lambda_1)\in(-\frac{1}{2},0)$ and $\mathrm{Re}(\lambda_2),\mathrm{Re}(\lambda_3)\in(0,\frac{1}{2})$, then by a shift in contour we have

$$\alpha\int_{i\mathbb{R}^2} qpQd\mu=\alpha\int_{i\mathbb{R}}\Bigg(q\int_{i\mathbb{R}+\frac{1}{2}} pQd\mu_1-2(1-[\xi_1+\eta_1])\pi\,\mathrm{Res}(pQ,\mu_1=\lambda_1-\mu_2+\tfrac{1}{2})$$
$$-2(1-[\xi_1+\eta_2])\pi\,\mathrm{Res}(pQ,\mu_1=\lambda_1+\mu_2+\tfrac{1}{2})\Bigg)d\mu_2. \quad (7.2)$$

We now argue that the right hand side is defined on U. The first term

$$\alpha(\xi,-\lambda)\int_{i\mathbb{R}}\int_{i\mathbb{R}+\frac{1}{2}} q(\eta,\mu_2)p(\xi,\lambda,\eta,\mu)Q(\xi,\lambda,\eta,\mu)d\mu_1d\mu_2,$$

is holomorphic for $\mathrm{Re}(\lambda_1)\in(-1,0)$ and $\mathrm{Re}(\lambda_2),\mathrm{Re}(\lambda_3)\in(0,1)$. For $\xi_1=\eta_1$ we calculate the residue as

$$\begin{aligned}\mathrm{Res}(p,\mu_1=\lambda_1-\mu_2+\tfrac{1}{2})&=-2\sqrt{\pi}\Gamma(\tfrac{[\xi_1+\eta_2]}{2}+\mu_2)\Gamma(\tfrac{1+[\xi_1+\eta_2]}{2}-\mu_2)\Gamma(\tfrac{\lambda_2-\lambda_1+[\xi_1+\xi_2]}{2})\\&\times\Gamma(\tfrac{\lambda_1-\lambda_2+1+[\xi_1+\xi_2]}{2})\Gamma(\tfrac{\lambda_2-\lambda_1+[\eta_2+\xi_2]}{2}+\mu_2)\Gamma(\tfrac{\lambda_1-\lambda_2+[\eta_2+\xi_2]}{2}-\mu_2)\\&\times\Gamma(\tfrac{\lambda_3-\lambda_1}{2})\Gamma(\tfrac{\lambda_1-\lambda_3+1}{2})\Gamma(\tfrac{\lambda_3-\lambda_1+[\eta_2+\xi_3]}{2}+\mu_2)\\&\times\Gamma(\tfrac{\lambda_1-\lambda_3+1+[\xi_3+\eta_2]}{2}-\mu_2).\end{aligned}$$

When restricting Q to $\mu_1=\lambda_1-\mu_2+\frac{1}{2}$, we are restricting $\mathbf{A}^{\eta,\nu}_{\xi,\lambda}$ and $\mathbf{A}^{\eta,\nu}_{w_0(\xi,-\lambda)}$ to $\lambda_1-\nu_1+\frac{1}{2}+[\eta_1+\xi_1]=0$ which means we can renormalize as follows

$$\tilde{Q}(\xi,\lambda,\eta,\mu_2)=\Bigg\langle\frac{\mathbf{A}^{(\xi_1,\eta_2),(\lambda_1+\frac{1}{2},\lambda_1+\frac{1}{2}-2\mu_2)}_{\xi,\lambda}f}{\Gamma(\frac{\lambda_1-\lambda_2+\frac{1}{2}+[\xi_1+\xi_2]}{2})^{-1}\Gamma(\frac{\lambda_1-\lambda_3+1}{2})^{-1}\Gamma(\frac{1+[\xi_1+\eta_2]}{2}-\mu_2)^{-1}},$$
$$\frac{\mathbf{A}^{(\xi_1,\eta_2),-(\lambda_1+\frac{1}{2},\lambda_1+\frac{1}{2}-2\mu_2)}_{w_0(\xi,-\lambda)}f'}{\Gamma(\frac{\lambda_1-\lambda_2+\frac{1}{2}+[\xi_1+\xi_2]}{2})^{-1}\Gamma(\frac{\lambda_1-\lambda_3+1}{2})^{-1}}\Bigg\rangle,$$

and $Q'(\xi,\lambda,\eta,\mu_2)$ is still holomorphic. Collecting all of this, we have that the integral of the residue becomes

$$\begin{aligned}&-\sqrt{\pi}\alpha(\xi,-\lambda)\frac{\Gamma(\frac{\lambda_2-\lambda_1+[\xi_1+\xi_2]}{2})\Gamma(\frac{\lambda_3-\lambda_1}{2})}{\Gamma(\frac{\lambda_1-\lambda_2+1+[\xi_1+\xi_2]}{2})\Gamma(\frac{\lambda_1-\lambda_3+1}{2})}\\&\quad\times\int_{i\mathbb{R}}\Gamma(\tfrac{\lambda_2-\lambda_1+[\eta_2+\xi_2]}{2}+\mu_2)\Gamma(\tfrac{\lambda_1-\lambda_2+1+[\eta_2+\xi_2]}{2}-\mu_2)\Gamma(\tfrac{\lambda_3-\lambda_1+[\eta_2+\xi_3]}{2}+\mu_2)\\&\quad\times\Gamma(\tfrac{\lambda_1-\lambda_3+1+[\xi_1+\eta_2]}{2}-\mu_2)\frac{\Gamma(\frac{1+[\xi_1+\eta_2]}{2}+\mu_2)\Gamma(\frac{1+[\xi_1+\eta_2]}{2}-\mu_2)}{\Gamma(\frac{[\xi_1+\eta_2]}{2}-\mu_2)}\tilde{Q}(\xi,\lambda,\eta,\mu_2)dz.\end{aligned} \quad (7.3)$$

Notice that if we had looked at the case where $\xi_1=\eta_2$, we would get the same integral but with (μ_2,η_2) changed to $(-\mu_2,\eta_1)$, and the entries of ν in $\mathbf{A}^{\eta,\nu}_{\xi,\lambda}$ switched. We can rewrite the factor in front of the integral as

$$-\frac{\cos(\frac{\pi(\lambda_3-\lambda_1)}{2})\cos(\frac{\pi(\lambda_2-\lambda_1)+\pi[\xi_2+\xi_3]}{2})\Gamma(\frac{\lambda_3-\lambda_1}{2})\Gamma(\frac{\lambda_2-\lambda_1+[\xi_2+\xi_1]}{2})}{\pi^{\frac{3}{2}}\Gamma(\frac{\lambda_3-\lambda_2+1+[\xi_2+\xi_1]}{2})},$$

using the reflection formula for the Gamma-function. By setting $\sigma_1 = \lambda_3 - \lambda_1$, $\sigma_2 = \lambda_2 - \lambda_1$, $\delta_1 = \xi_1 + \eta_2$, and $\delta_2 = \eta_2 + \xi_2$, we see that the integral is in fact $J(\delta, \sigma)$ from Lemma 7.2, which gives us analytic continuation of the residue to $\mathrm{Re}(\lambda_3 - \lambda_1) \in (0, 2)$ and $\mathrm{Re}(\lambda_3 - \lambda_2) \in (0, 1)$. In total the right hand side of (7.2) is thus defined for U defining an analytic extension, giving us the first part of the result.

We notice that this new region contains an open subset of the hyperplane $\lambda_3 - \lambda_1 = 1$ and that from the factor in front of the J integral, we have $\cos(\frac{\pi(\lambda_3 - \lambda_1)}{2})$ which vanishes, so we are left with

$$\alpha(\xi, -\lambda) \int_{i\mathbb{R}} \int_{i\mathbb{R}+\frac{1}{2}} q(\eta, \mu_2) p(\xi, \lambda, \eta, \mu) Q(\xi, \lambda, \eta, \mu) d\mu_1 d\mu_2,$$

which is holomorphic for $\mathrm{Re}(\lambda_1) \in (-1, 0)$ and $\mathrm{Re}(\lambda_2), \mathrm{Re}(\lambda_3) \in (0, 1)$. When restricted to $\lambda_1 - \lambda_3 + 1 = 0$, then Q vanishes at $\lambda_1 - \nu_k + \frac{1}{2} + [\xi_1 + \eta_k] = 0$, and at $\nu_k - \lambda_3 + \frac{1}{2} + [\xi_1 + \eta_k] = 0$ when $\xi_1 = \eta_k$, making Q' regular. Then by the duplication formula for the Gamma-function we get

$$\begin{aligned} p_1(\xi, \lambda, \eta, \nu) p_3(\xi, \lambda, \eta, \nu) \prod_{k=1}^{2} (\tfrac{\lambda_1 - \nu_k + [\xi_1 + \eta_k]}{2})_{1-[\xi_1+\eta_k]} (\tfrac{\nu_k - \lambda_3 + [\xi_1 + \eta_k]}{2})_{1-[\xi_1+\eta_k]} \\ = \pi \prod_{k=1}^{2} \Gamma(\lambda_1 - \nu_k + \tfrac{3}{2}) \Gamma(\nu_k - \lambda_1 + \tfrac{1}{2}), \end{aligned}$$

which has poles at $\lambda_1 = \nu_k - \frac{3}{2} - k$ and $\lambda_1 = \nu_k + \frac{1}{2} + k$. Restrict $\mathrm{Re}(\lambda_1) \in (-1, 0)$ and $\mathrm{Re}(\lambda_2) \in (0, \frac{1}{2})$, then shifting the contour of integration we get

$$\begin{aligned} \alpha(\xi, -\lambda) \pi \int_{i\mathbb{R}^2} q(\eta, \mu_2) p_2(\xi, \lambda, \eta, \mu) Q'(\xi, \lambda, \eta, \mu) \\ \times \prod_{k=1}^{2} \Gamma(\lambda_1 - \mu_1 + (-1)^k \mu_2 + \tfrac{3}{2}) \Gamma(\mu_1 + (-1)^k \mu_2 - \lambda_1 + \tfrac{1}{2}) d\mu_1 d\mu_2, \end{aligned}$$

which is holomorphic for $\mathrm{Re}(\lambda_1) \in (-\frac{3}{2}, \frac{1}{2})$ and $\mathrm{Re}(\lambda_2) \in (-\frac{1}{2}, \frac{1}{2})$. □

We are now ready to begin the proof of Theorem 7.1

Proof. In Proposition 7.3 we have essentially argued for why we can just let $f' = \overline{f}$, $\lambda_1 + \overline{\lambda}_3 = 0$, $\lambda_2 \in i\mathbb{R}$ and $\mathrm{Re}(\lambda_1) = -\frac{1}{2}$ and plug it into $I(\xi, \lambda, \eta)$. If we call $a = i\,\mathrm{Im}(\lambda_1)$, then the continuous part of (6.1) looks like

$$\begin{aligned} &\frac{(-1)^{\xi_1}}{2^4 \sqrt{\pi}} \sum_{\eta \in (\mathbb{Z}/2\mathbb{Z})^2} \int_{i\mathbb{R}^2} \left\| \frac{\mathbf{A}^{\eta,\nu}_{\xi,\lambda} f}{\prod_{k=1}^{2} (\frac{\lambda_1 - \nu_k + \frac{1}{2} + [\xi_1 + \eta_k]}{2})_{1-[\xi_1+\eta_k]}} \right\|^2 \\ &\times \frac{|\Gamma(a - \nu_1 + 1)|^2 |\Gamma(a - \nu_2 + 1)|^2 |\Gamma(\frac{\lambda_2 - \nu_1 + \frac{1}{2} + [\eta_1 + \xi_2]}{2})|^2 |\Gamma(\frac{\lambda_2 - \nu_2 + \frac{1}{2} + [\eta_2 + \xi_2]}{2})|^2 |\Gamma(\frac{1 + [\eta_1 + \eta_2] + \nu_1 - \nu_2}{2})|^2}{|\Gamma(\frac{\lambda_2 - a + \frac{3}{2} + [\xi_1 + \xi_2]}{2})|^2 |\Gamma(\frac{[\eta_1 + \eta_2] + \nu_1 - \nu_2}{2})|^2} d\nu, \end{aligned}$$

which is beautifully definite and gives the Plancherel formula in combination with the discrete part. □

8 Decomposing the unitarily induced degenerate series representations

In this section, we will decompose the unitarily induced degenerate series using similar arguments to those of Section 7

Theorem 8.1. *The unitarily induced degenerate series representation decomposes as*

$$\pi^{deg}_{\delta,\varepsilon,\lambda}|_H \simeq \bigoplus_{\eta\in\mathbb{Z}/2\mathbb{Z}} \int_{i\mathbb{R}}^{\oplus} \tau_{(\eta,\eta),(\lambda_1,z)}\, dz.$$

Proof. As the degenerate generalized principal series for $\xi_1 = \xi_3$, $\lambda_1 + \overline{\lambda}_3 = 0$ and $\lambda_2 \in i\mathbb{R}$ is at the point $\mathrm{Re}(\lambda_1) = \frac{1}{2}$. It turns out, it gives a much clearer picture if we first analytically extend the formula and then restrict it to $\lambda_1 - \lambda_3 = 1$.

The first thing we notice is that if $\lambda_1 - \lambda_3 = 1$, then $\alpha(\xi, -\lambda) = 0$ so as the discrete part of (6.1) is regular for $\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_3) \in (-1, 1)$ and $\mathrm{Re}(\lambda_2) \in (-\frac{1}{2}, \frac{1}{2})$ the whole thing vanishes. So we turn our focus towards the continuous part of (6.1).

Let $\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2) \in (0, \frac{1}{2})$, and $\mathrm{Re}(\lambda_3) \in (-\frac{1}{2}, 0)$, then the shifting the contour of the integral

$$\begin{aligned} I(\xi,\lambda,\eta) = \alpha \int_{i\mathbb{R}} \int_{i\mathbb{R}+\frac{1}{2}} Qpq\, d\mu_1 d\mu_2 - 2\pi(1-[\xi_1+\eta_1])\alpha \int_{i\mathbb{R}} q \,\mathrm{Res}(Qp, \mu_1 = \lambda_3 - \mu_2 + \tfrac{1}{2}) d\mu_2 \\ - 2\pi(1-[\xi_1+\eta_2])\alpha \int_{i\mathbb{R}} q \,\mathrm{Res}(Qp, \mu_1 = \lambda_3 + \mu_2 + \tfrac{1}{2}) d\mu_2. \quad (8.1) \end{aligned}$$

As $\int_{i\mathbb{R}} \int_{i\mathbb{R}+\frac{1}{2}} Qpq\, d\mu_1 d\mu_2$ is regular for $\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2) \in (0, 1)$, $\mathrm{Re}(\lambda_3) \in (-1, 0)$ we move our attention to the residues and as Q is holomorphic, we look at the residue of p first. Assume that $\xi_1 = \eta_1$, then

$$\begin{aligned} \mathrm{Res}(p, \mu_1 = \lambda_3 - \mu_2 + \tfrac{1}{2}) = -2\sqrt{\pi}\Gamma(\tfrac{\lambda_1-\lambda_3}{2})\Gamma(\tfrac{\lambda_3-\lambda_1+1}{2})\Gamma(\tfrac{\lambda_2-\lambda_3+[\xi_2+\xi_1]}{2})\Gamma(\tfrac{\lambda_3-\lambda_2+1+[\xi_2+\xi_1]}{2}) \\ \times \Gamma(\tfrac{[\eta_1+\eta_2]}{2} + \mu_2)\Gamma(\tfrac{\lambda_1-\lambda_3+[\xi_1+\eta_2]}{2} + \mu_2)\Gamma(\tfrac{\lambda_3-\lambda_1+1+[\xi_1+\eta_2]}{2} - \mu_2) \\ \times \Gamma(\tfrac{1+[\eta_1+\eta_2]}{2} - \mu_2)\Gamma(\tfrac{\lambda_2-\lambda_3+[\xi_2+\eta_2]}{2})\Gamma(\tfrac{\lambda_3-\lambda_2+1+[\xi_2+\eta_2]}{2} - \mu_2). \end{aligned}$$

When we restrict Q to $\mu_1 = \lambda_3 - \mu_2 + \frac{1}{2}$, it corresponds to restricting to $\lambda_3 - \nu_1 + \frac{1}{2} = 0$ on which one of the symmetry breaking operators have a zero, so we renormalize

$$Q'(\xi,\lambda,\eta,\mu_2) = \left\langle \frac{\mathbf{A}^{\eta,(\lambda_3+\frac{1}{2},\lambda_3+\frac{1}{2}-2\mu_2)}_{\xi,\lambda} f}{\Gamma(\frac{1+[\xi_1+\eta_2]}{2} - \mu_2)^{-1}}, \mathbf{A}^{\eta,-(\lambda_3+\frac{1}{2},\lambda_3+\frac{1}{2}-2\mu_2)}_{w_0(\xi,-\lambda)} f' \right\rangle$$

For $\xi_3 = \eta_2$ the other residue is given in the same way, but with (η_2, μ_2) swapped to $(\eta_1, -\mu_2)$, and $\mathbf{A}^{\eta,\nu}_{\xi,\lambda}$ is the restriction to $\lambda_3 - \nu_2 + \frac{1}{2} = 0$. In total the residue is then given by

$$\begin{aligned} \frac{\pi^{\frac{3}{2}}\Gamma(\frac{\lambda_1-\lambda_3}{2})\Gamma(\frac{\lambda_2-\lambda_3+[\xi_1+\xi_2]}{2})}{\Gamma(\frac{\lambda_2-\lambda_1+1+[\xi_1+\xi_2]}{2})} \int_{i\mathbb{R}} \Gamma(\tfrac{\lambda_1-\lambda_3+[\xi_1+\eta_2]}{2} + \mu_2)\Gamma(\tfrac{\lambda_3-\lambda_1+1+[\xi_1+\eta_2]}{2} - \mu_2) \\ \times \Gamma(\tfrac{\lambda_2-\lambda_3+[\xi_2+\eta_2]}{2} + \mu_2)\Gamma(\tfrac{\lambda_3-\lambda_2+1+[\xi_2+\eta_2])}{2} - \mu_2) \\ \times \frac{\Gamma(\frac{1+[\xi_1+\eta_2]}{2}+\mu_2)\Gamma(\frac{1+[\xi_1+\eta_2]}{2}-\mu_2)}{\Gamma(\frac{[\xi_1+\eta_2]}{2}-\mu_2)} Q'(\xi,\lambda,\eta,\mu_2) d\mu_2. \end{aligned}$$

Letting $\sigma_1 = \lambda_1 - \lambda_3$, $\sigma_2 = \lambda_2 - \lambda_3$, $\delta_1 = \xi_1 + \eta_2$ and $\delta_2 = \xi_2 + \eta_2$, we see that the integral is our good friend from Lemma 7.2, so the integral can be analytically extended to $\mathrm{Re}(\lambda_1 - \lambda_3) \in (-2, 2)$ and $\mathrm{Re}(\lambda_1 - \lambda_2) \in (0, 1)$. In total the right hand side of (8.1) is defined for

$$\{\lambda \in \mathbb{C}^3 : \ \mathrm{Re}(\lambda) \in (0,1)^2 \times (-1,0), \mathrm{Re}(\lambda_1 - \lambda_2) \in (0,1), \mathrm{Re}(\lambda_2 - \lambda_3) > 0\}.$$

Restricting to the hyperplane $\lambda_1 - \lambda_3 = 1$, we have that $\alpha(\xi, -\lambda) = 0$ so $\alpha \int_{i\mathbb{R}} \int_{i\mathbb{R}+\frac{1}{2}} Qpq' d\mu_1 d\mu_2$ vanishes and similarly the residue from Lemma 7.2 vanishes as we set $\sigma_1 = 1$.

The only remaining term is the integral J, which after shifting the contour back to the imaginary axis for $\xi_1 = \eta_1$, has been reduced to

$$\pi^2 \int_{i\mathbb{R}} |\Gamma(\tfrac{1+[\xi_1+\eta_2]}{2} + \mu_2)|^2 \Gamma(\tfrac{\lambda_1-\lambda_2+[\xi_2+\eta_2]}{2} + \mu_2)\Gamma(\tfrac{\lambda_2-\lambda_1+1+[\xi_2+\eta_2]}{2} - \mu_2) \\ \times \Big\langle \frac{\mathbf{A}_{\xi,\lambda}^{(\xi_1,\eta_2),(\lambda_1-\frac{1}{2},\lambda_1-\frac{1}{2}-2\mu_2)} f}{\Gamma(\frac{1+[\xi_1+\eta_2]}{2} - \mu_2)^{-1}}, \frac{\mathbf{A}_{\xi,-w_0\lambda}^{(\xi_1,\eta_2),-(\lambda_1-\frac{1}{2},\lambda_1-\frac{1}{2}-2\mu_2)} f'}{\Gamma(\frac{1+[\xi_1+\eta_2]}{2} + \mu_2)^{-1}} \Big\rangle dz$$

Summing up the different $I(\xi, \lambda, \eta)$ and setting $\lambda_1 + \overline{\lambda_3} = 0$ with $i\,\mathrm{Im}(\lambda_1) = a$ and $\lambda_2 \in i\mathbb{R}$, we get some terms with norms containing $\mathbf{A}_{\xi,\lambda}^{\eta,(a-2\mu_2,a)}$ and some containing $\mathbf{A}_{\xi,\lambda}^{\eta,(a,a-2\mu_2)}$, but as $(a - 2\mu_2, a) \in i\mathbb{R}^2$, we are on the unitary axis and the Knapp–Stein intertwining operator is an isometry, so we can collect those terms and in total (6.1) becomes

$$(-1)^{\xi_1} \sum_{\eta=0}^{1} \int_{i\mathbb{R}} |\Gamma(\tfrac{\lambda_2+\frac{1}{2}+[\xi_2+\eta]-z}{2})|^2 |\Gamma(\tfrac{1+\eta+a-z}{2})|^2 (1 + |\Gamma(\tfrac{1+\eta+a-z}{2})|^{-2}) ||\, \mathbf{A}_{\xi,\lambda}^{(\eta,\eta),(a,z)}\, ||^2 \, dz,$$

which gives us the Plancherel formula for

$$\pi_{0,\varepsilon,\lambda}^{\mathrm{deg}}|_H \simeq \bigoplus_{\eta \in \mathbb{Z}/2\mathbb{Z}} \int_{i\mathbb{R}}^{\oplus} \tau_{(\eta,\eta),(\lambda_1,z)} \, dz.$$

To get the decomposition for $\pi_{1,\varepsilon,\lambda}^{\mathrm{deg}}|_H$, notice that

$$\pi_{1,\varepsilon,\lambda}^{\mathrm{deg}} = \mathrm{Ind}_{P_0}^G((\chi_1 \otimes \varepsilon) \otimes e^\lambda \otimes 1) = \mathrm{Ind}_{P_0}^G((\chi_0 \otimes (\varepsilon + 1)) \otimes e^\lambda \otimes 1) \otimes \mathrm{sgn}(\det),$$

so by decomposing the first factor of the right hand side and then tensoring with $\mathrm{sgn}(\det)|_H$ on each term of the decomposition, we get the full result. □

9 Gaps in the argument

In the line of argumentation above there are a few missing links that we highlight in this section.

Firstly, when doing the shift of contour from $\int_{i\mathbb{R}} f(\lambda, z)dz$ to $\int_{i\mathbb{R}+\frac{1}{2}} f(\lambda, z)dz$, there needs to be some conditions on f to ensure that the top and bottom edges of the contour actually vanishes in the limit. Furthermore, we argue since $\int_{i\mathbb{R}+\frac{1}{2}} f(\lambda, z)dz$ does not have any poles for λ in some region, then it defines a continuation of the original function, but strictly speaking, we need to ensure that the integral actually converges in this region as well. Both of these issues boils down to finding a growth condition in the parameters of the bilinear pairing of the symmetry breaking operators.

Secondly, when making the Plancherel formulas into direct integral decompositions, we do not argue that the symmetry breaking operators does not vanish, which is an essential step for establishing the surjectivity of the direct integral map. In most cases, this can be done by arguing that the integral kernel is non-zero on some Bruhat cell where its more explicit. However, in the case where the representation we are decomposing is a quotient inside a principal series representation, it is more difficult to argue that even if the symmetry breaking operator is non-zero, that it does in fact not vanish on the quotient. This would require a more detailed analysis maybe by looking at the quotient as a collection of K_G-types and evaluate the symmetry breaking operator on these.

10 Branching laws using the Whittaker Plancherel formula

10.1 Unitary branching of $\mathrm{GL}(3,\mathbb{R})$

The proof in this section was presented to me by Jan Frahm. Recall we use $\sigma_{\pi,\varepsilon,\lambda}$ for the generalized principal series, given by

$$\sigma_{\pi,\delta,\lambda} = \mathrm{Ind}_{P_0}^{G}\left((\pi\otimes\delta)\otimes e^{\lambda}\otimes 1\right).$$

By induction in stages we have that

- $\sigma_{\pi,\delta,\lambda}$ is the unitary principal series for G when $\pi = \rho_{\varepsilon,\mu}$ where $\mu \in i\mathbb{R}$
- $\sigma_{\pi,\delta,\lambda}$ is the complementary series for G when $\phi = \rho_{\varepsilon,\mu}$ where $\mu \in (0,1)$ and $\varepsilon_+ = 0$.
- $\sigma_{\pi,\delta,\lambda}$ is the unitarily induced generalized principal series for G when $\pi = \rho^{ds}_{\varepsilon,\mu}$.
- $\sigma_{\pi,\delta,\lambda}$ is the unitarily induced degenerate series for G when π is a unitary character.

Recall that $H = \mathrm{GL}(2,\mathbb{R})$ is the subgroup of G embedded into the upper left corner. The subgroup H acts by multiplication on $G/P \simeq \mathbb{R}P^2 = \{\mathbb{R}v \,:\, v \in \mathbb{R}^3\backslash\{0\}\}$ with the orbits

$$\begin{aligned}
\mathcal{O}_1 &= H\cdot\mathbb{R}e_1 = \{\mathbb{R}v \,:\, v_3 = 0\},\\
\mathcal{O}_2 &= H\cdot\mathbb{R}e_3 = \{\mathbb{R}e_3\},\\
\mathcal{O}_3 &= \{\mathbb{R}(v_1,v_2,1) \,:\, (v_1,v_2)\neq 0\}.
\end{aligned}$$

The orbit $\mathcal{O}_3$ is open and dense in G/P, and we pick the base-point

$$x_0 = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & 1\end{pmatrix},$$

which has stabilizer

$$S = \left\{\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix} \,:\, a\in\mathbb{R}^\times, b\in\mathbb{R}\right\},$$

also known as the $ax+b$*-group*. We denote the elements of S by $s(a,b)$. S has one infinite dimensional unitary representation which we call ω see more in Appendix A. Now consider the orbit restriction map

$$\mathcal{R}:\tau_{\pi,\delta,\lambda}\to \mathrm{Ind}_S^H\left(((\pi\otimes\delta)\otimes e^{\lambda})|_S\right),\quad f\mapsto F,$$

where $F(h)=f(hx_0)$. That F has the desired equivariances follows from

$$F(hs)=f(hsx_0)=f(hx_0s)=((\pi\otimes\varepsilon)\otimes e^{\lambda})(s)^{-1}F(h),$$

where $s\in S$. Furthermore we see that

Lemma 10.1. *If* π *is an infinite dimensional irreducible unitary representation of* $\mathrm{SL}^{\pm}(2,\mathbb{R})$ *then*

$$\mathrm{Ind}_{P_0}^G\left((\pi\otimes\delta)\otimes e^{\lambda}\otimes 1\right)\big|_H\simeq \mathrm{Ind}_S^H(\omega).$$

Proof. Note that

$$((\pi\otimes\delta)\otimes e^{\lambda})(s(a,b))=\pi\begin{pmatrix}\mathrm{sgn}(a)|a|^{\frac{1}{2}} & |a|^{-\frac{1}{2}}b\\ & |a|^{-\frac{1}{2}}\end{pmatrix}e^{\lambda}\begin{pmatrix}|a|^{\frac{1}{2}} & \\ & |a|^{\frac{1}{2}}\end{pmatrix}=\pi(\phi(s(a,b)))|a|^{\frac{\lambda_1}{2}},$$

where ϕ is the isomorpism

$$\begin{pmatrix}a & b\\ & 1\end{pmatrix}\mapsto |a|^{-\frac{1}{2}}\begin{pmatrix}a & b\\ & 1\end{pmatrix}=\begin{pmatrix}\mathrm{sgn}(a)|a|^{\frac{1}{2}} & |a|^{-\frac{1}{2}}b\\ & |a|^{-\frac{1}{2}}\end{pmatrix},$$

from S to the $ax+b$-group embedded into $\mathrm{SL}^{\pm}(2,\mathbb{R})$. Thus showing that $((\pi\otimes\delta)\otimes e^{\lambda})|_S=\pi|_S\otimes\chi_{0,\frac{\lambda_1}{2}}$, and if π is infinite dimensional then by Proposition A.2 we get $\pi|_S\simeq\omega$ and as $\omega\otimes\chi_{0,\frac{\lambda_1}{2}}$ is still a infinite dimensional irreducible unitary representation it must be equivalent to ω by Proposition A.1. □

Theorem 10.2. *The unitary principal series, complementary series and unitarily induced generalized principal series all decomposes as*

$$\pi|_H\simeq\bigoplus_{\eta\in(\mathbb{Z}/2\mathbb{Z})^2}\int_{i\mathbb{R}^2}^{\oplus}\tau_{\eta,\nu}\,d\nu\oplus\bigoplus_{\eta\in\mathbb{Z}/2\mathbb{Z}}\bigoplus_{\nu_-\in 1+\eta-2\mathbb{N}}\int_{i\mathbb{R}}^{\oplus}\tau_{\eta,\nu}^{ds}\,d\nu_+.$$

Proof. The representation ω can be realized as $\mathrm{Ind}_{N_H}^S(\psi_\alpha)$ where ψ is a non-trivial character of N_H. Using induction in stages, we get that

$$\mathrm{Ind}_S^H(\omega)\simeq\mathrm{Ind}_S^H(\mathrm{Ind}_{N_H}^S(\psi))\simeq\mathrm{Ind}_{N_H}^H(\psi_\alpha)\simeq L^2(H/N_H,\psi),$$

where the right hand side can be recognized as the Whittaker model. The Whittaker Plancherel formula for reductive groups tell us how this space decomposes as [W92, p.423]

$$L^2(H/N_H,\psi)\simeq\bigoplus_{[P]\in\mathcal{P}}\bigoplus_{\tau\in\widehat{M}_{P,ds}}\int_{i\mathfrak{a}_P^*}^{\oplus}\pi_{\tau,\lambda}\otimes\pi_{\tau,\lambda}^{*,(N,\psi)}d\lambda.$$

Here $\mathcal{P}$ are the parabolic subgroups of H up to conjugation, and in this case we have $\mathcal{P}=\{[H],[P_H]\}$. Every element P of $\mathcal{P}$ has a Langlands decomposition $P=MAN$. In the case $[H]$, we have the decomposition $M = \mathrm{SL}^{\pm}(2,\mathbb{R})$, $A = \mathbb{R}_+ I$, $N = I$ and in the case $[P_H]$, we have $M = M_H$, $A = A_H$ and $N = N_H$. The space $\widehat{M_{P,ds}}$ is the discrete series representations in the unitary dual of M. In the case where $P = H$, we get $\widehat{M_P}_{,ds} = \{\rho^{ds}_{\varepsilon,\mu} : \mu \in 1+\varepsilon_+ - 2\mathbb{N}\}$. In the case where $P = P_H$, we get $\widehat{M_P}_{,ds} = \{0,1\}^2$ given as characters of $\mathbb{Z}/2\mathbb{Z}$. The space is $\mathfrak{a}_P \simeq A_P$ is the Lie algebra of A_P and $\mathfrak{a}_H = \mathbb{R}$ and $\mathfrak{a}_{P_H} = \mathbb{R}^2$. We can then find characters of A_P as $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ by e^{λ}. Lastly, we have

$$\pi_{\tau,\lambda} = \mathrm{Ind}_P^H(\tau \otimes e^{\lambda} \otimes 1).$$

In the case where $P = H$, we get that $\pi_{\rho^{ds}_{\varepsilon,\mu},\lambda} \simeq \rho^{ds}_{\varepsilon,\mu} \otimes e^{\lambda} \simeq \tau^{ds}_{\varepsilon,\frac{1}{2}(\lambda+\mu,\lambda-\mu)}$. In the case where $P = P_H$ and $\tau = \eta \in \{0,1\}^2$ and $\lambda = \nu \in \mathbb{C}^2$, we get that $\pi_{\eta,\nu} = \tau_{\eta,\nu}$. Here we think of $\pi^{*,(N,\psi_\alpha)}_{\tau,\lambda}$ as a multiplicity space and by [GW80] it is of dimension 1. We can now rewrite the Whittaker Plancherel formula

$$L^2(H/N,\psi_\alpha) = \bigoplus_{\eta \in (\mathbb{Z}/2\mathbb{Z})^2} \int_{i\mathbb{R}^2}^{\oplus} \tau_{\tau,\nu}\, d\nu \oplus \bigoplus_{\eta \in \mathbb{Z}/2\mathbb{Z}} \bigoplus_{\nu_- \in 1+\eta-2\mathbb{N}} \int_{i\mathbb{R}}^{\oplus} \tau^{ds}_{\eta,\nu} d\nu_+,$$

giving us the desired result. We note that the unitary principal series, the complementary series and the unitarily induced generalized principal series are all induced from infinite dimensional unitary representations of $\mathrm{SL}^{\pm}(2,\mathbb{R})$, and thus by Lemma 10.1 the result follows. □

A Restriction representations of $\mathrm{SL}^{\pm}(2,\mathbb{R})$ to the $ax+b$-group

Consider the $ax+b$-group which is the semi-direct product of $\mathbb{R}^{\times}$ and $\mathbb{R}$ with group law

$$(a,b)(a',b') = (aa', ab'+b), \quad (a,a' \in \mathbb{R}^{\times}, b,b' \in \mathbb{R}).$$

This group can be realized as a subgroup of $\mathrm{GL}(2,\mathbb{R})$ by the isomorphism s given by

$$(a,b) \mapsto \begin{pmatrix} a & b \\ & 1 \end{pmatrix},$$

or alternatively as a subgroup of $\mathrm{SL}^{\pm}(2,\mathbb{R})$ by the isomorphism ϕ given by

$$\begin{pmatrix} a & b \\ & 1 \end{pmatrix} \mapsto |a|^{-\frac{1}{2}} \begin{pmatrix} a & b \\ & 1 \end{pmatrix} = \begin{pmatrix} \mathrm{sgn}(a)|a|^{\frac{1}{2}} & |a|^{-\frac{1}{2}} b \\ & |a|^{-\frac{1}{2}} \end{pmatrix}.$$

The following Proposition is a common exercise in the Mackey machine and can be found in [F15, Example 6.7]

Proposition A.1 (The unitary dual of the $ax+b$-group)**.** *The following list of non-isomorphic representations exhausts all the irreducible unitary representations of the* $ax+b$*-group:*

1. *The one-dimensional representations, parameterized by* $\nu \in i\mathbb{R}$ *and* $\eta \in \mathbb{Z}/2\mathbb{Z} \simeq \{0,1\}$ *given by the characters* $\chi_{\eta,\nu}((a,b)) = |a|^{\nu}_{\eta}$.

2. *There is one infinite-dimensional representation ω, which for any $\alpha \in \mathbb{C}$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ can be realized on $L^2(\mathbb{R}, |x|^{2\operatorname{Re}(\alpha)-1}dx)$ by the action*

$$\omega((a,b))f(x) = |a|_\varepsilon^\alpha e^{ibx} f(ax),$$

where $a \in \mathbb{R}^\times, b \in \mathbb{R}$.

The goal is now to describe how all the unitary representations of $\mathrm{SL}^\pm(2,\mathbb{R})$ restrict as representations of the $ax+b$-group. To do so we first describe an L^2-model for the complementary series. Consider the restriction of $\sigma_{\varepsilon,\mu}$ to the non-compact picture $L^2(\mathbb{R})$ by $f \mapsto f|_{\overline{N}}$, where $\overline{N} \simeq \mathbb{R}$. On $L^2(\mathbb{R})$ we use the inner product given by $(T^w_{\varepsilon,\mu} f | f')$ where

$$T^w_{\varepsilon,\mu} f(\overline{n}_y) = \int_{\mathbb{R}} f(\overline{n}_y w \overline{n}_x) dx, \quad \text{where} \quad \overline{n}_t = \begin{pmatrix} 1 & \\ t & 1 \end{pmatrix}, \quad w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

and $(f|f')$ being the usual L^2 inner product on $\overline{N} \simeq \mathbb{R}$. Now decomposing

$$w\overline{n}_x = \begin{pmatrix} 1 & \\ -\frac{1}{x} & 1 \end{pmatrix} \begin{pmatrix} x & \\ & \frac{1}{x} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{x} \\ & 1 \end{pmatrix},$$

and changing variables $x \to \frac{1}{x}$, we see that

$$T^w_{\varepsilon,\mu} f(\overline{n}_y) = \int_{\mathbb{R}} |x|^{\mu-1}_{\varepsilon_+} f(\overline{n}_{y-x}) dx.$$

We notice that this is a convolution and thus applying the Fourier transform, we see the convolution becomes multiplication

$$\mathcal{F}[T^w_{\varepsilon,\mu} f](\xi) = \mathcal{F}\big[|x|^{\mu-1}_{\varepsilon_+}\big](\xi) \mathcal{F}[f](\xi).$$

By [GS64, p.170] we get

$$\mathcal{F}\big[|x|^{\mu-1}_{\varepsilon_+}\big](\xi) = \sqrt{\pi} \frac{\Gamma(\frac{\mu+\varepsilon_+}{2})}{\Gamma(\frac{1-\mu+\varepsilon_+}{2})} i^{\varepsilon_+} |\xi|^{-\mu}_{\varepsilon_+}.$$

As we are trying to get an L^2-model for the complementary series, we restrict ourselves to the case where $\varepsilon_+ = 0$ or equivalently $\varepsilon_1 = \varepsilon_2$. In this case we denote the constant in front of $|\xi|^{-\mu}$ by $c(\mu)$ and notice that $c(\mu) > 0$ when $\mu \in (0,1)$. Thus, we have used the Fourier transform to change $L^2(\mathbb{R})$ with the convoluted convolution inner product to $L^2(\mathbb{R}, |\xi|^{-\mu} d\xi)$ with the regular L^2 inner product multiplied by $c(\mu)$. We define the action $\widetilde{\pi}_\mu$ of $\mathrm{SL}^\pm(2,\mathbb{R})$ on $L^2(\mathbb{R}, |\xi|^{-\mu})$ by $\widetilde{\pi}_\mu \mathcal{F}[f] = \mathcal{F}[\pi_{0,\mu} f]$. We specify the action for certain elements of $\mathrm{SL}^\pm(2,\mathbb{R})$:

$$\pi_{0,\mu}\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} f(\overline{n}_y) = f(\overline{n}_{y-x}), \quad \pi_{0,\mu}\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} f(\overline{n}_y) = |a|^{\mu+1} f(\overline{n}_{a^2 y}), \quad (x \in \mathbb{R}, a \in \mathbb{R}_+),$$

$$\pi_{0,\mu}\begin{pmatrix} \delta_1 & \\ & \delta_2 \end{pmatrix} f(\overline{n}_y) = \delta_1^{\varepsilon_1} \delta_2^{\varepsilon_2} f(\overline{n}_{\delta_1\delta_2 y}), \quad (\delta_i \in \{\pm 1\}).$$

Using basic properties of the Fourier transform we get similar actions in the Fourier model

$$\tilde{\pi}_\mu \begin{pmatrix} 1 & \\ b & 1 \end{pmatrix} \mathcal{F}[f](\xi) = e^{ib\xi}\mathcal{F}[f](\xi), \quad \tilde{\pi}_\mu \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \mathcal{F}[f](\xi) = |a|^{\mu-1}\mathcal{F}[f](a^{-2}\xi)$$

$$\tilde{\pi}_\mu \begin{pmatrix} \delta_1 & \\ & \delta_2 \end{pmatrix} \mathcal{F}[f](\xi) = \delta_1^{\varepsilon_1}\delta_2^{\varepsilon_2}\mathcal{F}[f](\delta_1\delta_2\xi).$$

Using the embedding of the $ax+b$–group in the lower corner of $\mathrm{SL}^{\pm}(2,\mathbb{R})$

$$(a,b) \mapsto |a|^{-\frac{1}{2}} \begin{pmatrix} 1 & \\ b & a \end{pmatrix},$$

we have the action

$$\tilde{\pi}_\mu\Big(|a|^{-\frac{1}{2}} \begin{pmatrix} 1 & \\ b & a \end{pmatrix}\Big)\mathcal{F}[f](\xi) = e^{ibx}|a|_{\varepsilon_2}^{-\frac{\mu-1}{2}}\mathcal{F}[f](a\xi),$$

which is exactly the L^2-model of ω from Proposition A.1 with $\alpha = \frac{1-\mu}{2}$ and $\varepsilon = \varepsilon_2$. As the copy of the $ax+b$-group in the lower corner of $\mathrm{SL}^{\pm}(2,\mathbb{R})$ is conjugate to the one in the upper corner by

$$w' = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix},$$

it follows that $\tilde{\pi}_\mu(w')$ is an intertwiner between $\tilde{\pi}_\mu$ restricted to the $ax+b$-group in the upper corner and $\tilde{\pi}_\mu$ restricted to the $ax+b$-group in the lower corner.

Proposition A.2. *Restricting any of the infinite-dimensional representations of Proposition 3.1 to the $ax+b$-group is isomorphic to the representation ω from Proposition A.1.*

Proof. In the case of the unitary principal series and discrete series representations this has been done in [FWZ23, Lemma 3.5] and for the complementary series we have just argued above. □

B Complex analysis

Proposition B.1. *Suppose g is an analytic function on a domain U with $g'(b) \neq 0$ for some $b \in U$. If f is an analytic function on $g(U)$ that has a simple pole at $g(b)$ then*

$$\mathrm{Res}(f \circ g, b) = \frac{\mathrm{Res}(f, g(b))}{g'(b)}.$$

Proposition B.2.

$$\mathrm{Res}(\Gamma, -n) = \frac{(-1)^n}{n!}.$$

Example B.3. Consider the multi-variable meromorphic function

$$p(\lambda,\nu,\delta) = \Gamma\Big(\frac{\lambda-\nu+\frac{1}{2}+\delta}{2}\Big)\Gamma\Big(\frac{\nu-\lambda+\frac{1}{2}+\delta}{2}\Big), \quad (\delta \in \{0,1\}),$$

which has poles at

$$\nu = \lambda + \frac{1}{2} + \delta + 2n, \quad \text{or} \quad \lambda = \nu - \frac{1}{2} - \delta - 2n,$$
$$\nu = \lambda - \frac{1}{2} - \delta - 2n, \quad \text{or} \quad \lambda = \nu + \frac{1}{2} + \delta + 2n,$$

where $n \in \mathbb{N}_0$. For $\mathrm{Re}(\lambda) \in (-\frac{1}{2}, \frac{1}{2})$ let

$$I(\lambda, \delta, f) = \int_{i\mathbb{R}} f(\lambda, \nu, \delta) p(\lambda, \nu, \delta)\, d\nu,$$

where f is a holomorphic function with sufficiently strong decay at imaginary infinity. In the case where $\delta = 1$ we can actually extend the definition of $I(\lambda, \delta, f)$ to $\mathrm{Re}(\lambda) \in (-\frac{3}{2}, \frac{3}{2})$ but we will refrain from doing so as this example is presented to showcase a simpler version of the analytic continuation in Section 7. Restricting the domain of $I(\lambda, \delta, f)$ to $\mathrm{Re}(\lambda) \in (0, \frac{1}{2})$ we have that the poles are at

$$\mathrm{Re}(\nu) \in (\tfrac{1}{2}, 1) + \delta + 2\mathbb{N}_0, \quad \& \quad \mathrm{Re}(\nu) \in (-\tfrac{1}{2}, 0) - \delta - 2\mathbb{N}_0,$$

and so shifting the contour of integration we get

$$I(\lambda, \delta, f) = \int_{i\mathbb{R}+\frac{1}{2}} f(\lambda, \nu, \delta) p(\lambda, \nu, \delta)\, d\nu. \tag{B.1}$$

When $\mathrm{Re}(\nu) = \frac{1}{2}$ the poles in terms of λ are at

$$\mathrm{Re}(\lambda) \in -\delta - 2\mathbb{N}_0, \quad \& \quad \mathrm{Re}(\lambda) \in 1 + \delta + 2\mathbb{N}_0,$$

and so the right hand side of (B.1) is defined for $\mathrm{Re}(\lambda) \in (0, 1)$ meaning we have an analytic continuation of $I(\lambda, \delta, f)$. On this new domain we restrict $I(\lambda, \delta, f)$ to $\mathrm{Re}(\lambda) \in (\frac{1}{2}, 1)$ then the poles are at

$$\mathrm{Re}(\nu) \in (1, \tfrac{3}{2}) + \delta + 2\mathbb{N}_0, \quad \& \quad \mathrm{Re}(\nu) \in (0, \tfrac{1}{2}) - \delta - 2\mathbb{N}_0,$$

so in the case of $\delta = 0$ we pass the through a pole when we shift the contour of integration back

$$\int_{i\mathbb{R}+\frac{1}{2}} f(\lambda, \nu, \delta) p(\lambda, \nu, \delta)\, d\nu$$
$$= \int_{i\mathbb{R}} f(\lambda, \nu, \delta) p(\lambda, \nu, \delta)\, d\nu - 2\pi(1 - \delta) \,\mathrm{Res}\Big(f(\lambda, \nu, \delta) p(\lambda, \nu, \delta), \nu = \lambda - \tfrac{1}{2}\Big). \tag{B.2}$$

Here the residue is only non-zero if $\delta = 0$ (as otherwise the function does not have a pole) so the factor $1 - \delta$ is just here to remind us of this. Calculating the residue we get

$$\mathrm{Res}(f(\lambda, \nu, 0) p(\lambda, \nu, 0), \nu = \lambda - \tfrac{1}{2}) = 2\sqrt{\pi} f(\lambda, \lambda - \tfrac{1}{2}, 0),$$

which is holomorphic in λ. Looking at the poles of $p(\lambda, \delta, \nu)$ when $\mathrm{Re}(\nu) = 0$ we see that the interval that overlaps with $\mathrm{Re}(\lambda) \in (\frac{1}{2}, 1)$ is $\mathrm{Re}(\lambda) \in (\frac{1}{2}, \frac{5}{2})$ if $\delta = 0$ and $\mathrm{Re}(\lambda) \in (-\frac{3}{2}, \frac{3}{2})$

if $\delta = 1$. In either case the interval $\mathrm{Re}(\lambda) \in (\frac{1}{2}, \frac{3}{2})$ works. Thus we have an analytic continuation of $I(\lambda, \delta, f)$ to $\mathrm{Re}(\lambda) \in (\frac{1}{2}, \frac{3}{2})$ given by

$$\int_{i\mathbb{R}} f(\lambda, \nu, \delta) p(\lambda, \nu, \delta)\, d\nu - 4\pi^{\frac{3}{2}}(1-\delta) f(\lambda, \lambda - \tfrac{1}{2}, 0).$$

Here we note that in the case $\delta = 1$ this is not a continuation as the original domain is $\mathrm{Re}(\lambda) \in (-\frac{3}{2}, \frac{3}{2})$, but this "continuation" coincides with the original expression in this case.

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Appendix

Jonathan Ditlevsen

1 A non-result

In this section we show different approaches to solve the problem of evaluating the symmetry breaking operator of $(\mathrm{GL}(n+1,\mathbb{R}), \mathrm{GL}(n,\mathbb{R}))$ on the K-invariant vector. None of these approaches worked out in the end but we still include them here as we think it is a fun and interesting problem. Let the notation be as in Paper B.

One of the result we have worked hard on is trying to evaluate the symmetry breaking operator $\mathbf{A}^{\eta,\nu}_{\xi,\lambda}$ on the K-invariant vector $\mathbb{1}_\lambda$. For $n = 2$ the integral to be computed is

$$\int_{\mathbb{R}^3} |z|^{s_1}|x|^{t_1}|z-xy|^{s_2}(1+x^2+z^2)^{-\frac{s_1+t_1+2}{2}}(1+y^2+(z-xy)^2)^{-\frac{s_2+t_2+2}{2}}\, d(x,y,z), \quad (1.1)$$

this was done by Frahm and Su in [FS21]. The computation is quite involved and contains lots of tricks along the way. We found a more direct way to compute this integral and we tried to generalize it such that we would either compute it via induction or directly.

Lemma 1.1. *Let* $\varphi(x,y,z) = (1+x^2+z^2)^a(1+y^2+(z-xy)^2)^b$ *and* $\tilde{\varphi}(x,y,z) = (1+x^2)^a (1+y^2)^b(1+z^2)^{a+b+\frac{1}{2}}$ *then*

$$\int_{\mathbb{R}^3} f(x,y,z)\varphi(x,y,z)d(x,y,z) = \int_{\mathbb{R}^3} f(x, \tfrac{y\sqrt{1+z^2}+xz}{\sqrt{1+x^2}}, z\sqrt{1+x^2})\tilde{\varphi}(x,y,z)d(x,y,z). \quad (1.2)$$

Proof. By completing the square write

$$1+y^2+(z-xy)^2 = (1+x^2)\Big[(y-\frac{xz}{1+x^2})^2 + \frac{1+x^2+z^2}{(1+x^2)^2}\Big].$$

Doing the substitutions $y \to y + \frac{xz}{1+x^2}$ then $y \to y\frac{\sqrt{1+x^2+z^2}}{1+x^2}$ and lastly $z \to z\sqrt{1+x^2}$ we arrive at the result. $\square$

We can then use this substitution to compute (1.1).

Proposition 1.2. *The integral* (1.1) *evaluates to*

$$\frac{\Gamma\Big(\frac{\lambda_1-\nu_2+\frac{1}{2}}{2}\Big)\Gamma\Big(\frac{\lambda_1-\nu_1+\frac{1}{2}}{2}\Big)\Gamma\Big(\frac{\lambda_2-\nu_1+\frac{1}{2}}{2}\Big)\Gamma\Big(\frac{\nu_2-\lambda_2+\frac{1}{2}}{2}\Big)\Gamma\Big(\frac{\nu_2-\lambda_3+\frac{1}{2}}{2}\Big)\Gamma\Big(\frac{\nu_1-\lambda_3+\frac{1}{2}}{2}\Big)}{\pi^{-\frac{1}{2}}\Gamma\Big(\frac{\lambda_1-\lambda_2+1}{2}\Big)\Gamma\Big(\frac{\lambda_1-\lambda_3+1}{2}\Big)\Gamma\Big(\frac{\lambda_2-\lambda_3+1}{2}\Big)\Gamma\Big(\frac{\nu_2-\nu_1+1}{2}\Big)}$$

Proof. Use (1.2) to change (1.1) into

$$\int_{\mathbb{R}^3} |x|^{\nu_2-\nu_1-1}(1+x^2)^{\frac{\nu_1-\nu_2-1}{2}}\Big|\frac{z}{x\sqrt{1+z^2}} - y\Big|^{\lambda_2-\nu_1-\frac{1}{2}}(1+y^2)^{\frac{\lambda_3-\lambda_2-1}{2}}$$
$$\times |z|^{\lambda_1-\nu_2-\frac{1}{2}}(1+z^2)^{\frac{\lambda_2+\lambda_3-\nu_1-\lambda_1-\frac{3}{2}}{2}}\, d(x,y,z).$$

Now only considering the y-integral we can apply Corollary 1.6 and obtain

$$\int_{\mathbb{R}} \Big|\frac{z}{x\sqrt{1+z^2}} - y\Big|^{\lambda_2-\nu_1-\frac{1}{2}}(1+y^2)^{\frac{\lambda_3-\lambda_2-1}{2}}dy = \frac{\Gamma(\frac{\lambda_2-\nu_1+\frac{1}{2}}{2})\Gamma(\frac{\nu_1-\lambda_3+\frac{1}{2}}{2})}{\Gamma(\frac{\lambda_2-\lambda_3+1}{2})}F(-\tfrac{t}{x^2}),$$

where $F(w) = {}_2F_1(\frac{\nu_1-\lambda_3+\frac{1}{2}}{2}, \frac{\nu_1-\lambda_2+\frac{1}{2}}{2}; \frac{1}{2}; w)$ and $t = \frac{z^2}{1+z^2}$. Note the remaining integrands are invariant under $(x,z)\mapsto(-x,-z)$, so it suffices to integrate x and z over $(0,\infty)$. Applying Proposition 1.7 to the x-integral after the substitution $s = t/x^2$ we get

$$t^{\frac{\nu_2-\nu_1}{2}}\int_0^\infty s^{-1/2}(t+s)^{\frac{\nu_1-\nu_2-1}{2}}F(-s)ds = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{\nu_2-\lambda_3+\frac{1}{2}}{2})\Gamma(\frac{\nu_2-\lambda_2+\frac{1}{2}}{2})}{\Gamma(\frac{\nu_1-\nu_2+1}{2})\Gamma(\frac{\nu_1+\nu_2-\lambda_2-\lambda_3+1}{2})}t^{\frac{\nu_2-\nu_1}{2}}G(t),$$

where $G(w) = {}_2F_1(\frac{\nu_2-\lambda_3+\frac{1}{2}}{2}, \frac{\nu_2-\lambda_2+\frac{1}{2}}{2}; \frac{\nu_1+\nu_2-\lambda_2-\lambda_3+1}{2}; 1-w)$. Now apply (1.4) to the remaining z-integral after the substitution $t = \frac{z^2}{1+z^2}$ to arrive at

$$\int_0^1 t^{\frac{\lambda_1-\nu_1-\frac{3}{2}}{2}}(1-t)^{\frac{\nu_1+\nu_2-\lambda_2-\lambda_3-1}{2}}G(1-t)dt = \frac{\Gamma(\frac{\lambda_1-\nu_1+\frac{1}{2}}{2})\Gamma(\frac{\nu_1+\nu_2-\lambda_2-\lambda_3+1}{2})}{\Gamma(\frac{\lambda_1+\nu_2-\lambda_2-\lambda_3+\frac{3}{2}}{2})}H(1),$$

where

$$\begin{aligned} H(1) &= {}_2F_1\Big(\frac{\nu_2-\lambda_3+\frac{1}{2}}{2}, \frac{\nu_2-\lambda_2+\frac{1}{2}}{2}; \frac{\lambda_1+\nu_2-\lambda_2-\lambda_3+\frac{3}{2}}{2}; 1\Big) \\ &= \frac{\Gamma(\frac{\lambda_1+\nu_2-\lambda_2-\lambda_3+\frac{3}{2}}{2})\Gamma(\frac{\lambda_1-\nu_2+\frac{1}{2}}{2})}{\Gamma(\frac{\lambda_1-\lambda_2+1}{2})\Gamma(\frac{\lambda_1-\lambda_3+1}{2})}. \end{aligned}$$

This follows from the ${}_3F_2$ we got from (1.4) collapses into an ${}_2F_1$ and then using (1.5). Now collecting all the Gamma-factors we picked up along the way we arrive at the result.
□

This might seem very complicated but we just did a smart substitution and then computed the integrals one at a time. This smart choice of substitution (1.2) comes from the following Lemma found in [K16].

Lemma 1.3. *Let MAN, MAN' and MAN'' be parabolic subgroups with the same MA, and suppose $\mathfrak{n}''\cap\mathfrak{n} \subseteq \mathfrak{n}'\cap\mathfrak{n}$. If $\overline{N}$, $\overline{N'}$, and $\overline{N''}$ denote the Cartan involution of N, N' and N'' and if Haar measures are suitably normalized, then*

$$\int_{\overline{N}\cap N''} f(\overline{n})\, d\overline{n} = \int_{\overline{N'}\cap N''}\Big[\int_{\overline{N}\cap N'} f(\overline{n'}\overline{n})\, d\overline{n}\Big]\, d\overline{n'},$$

for every nonnegative measurable function f.

The general integral we are trying to compute is

$$\int_{\overline{N}_G} K^\nu_\lambda(\overline{n})\mathbb{1}_\lambda(\overline{n})\, d\overline{n},$$

by using [K16, (5.25)] we can change it to

$$\int_{K_G} K^\nu_\lambda(k)\, dk = \int_{K_G} K^\nu_\lambda(k'k)\, dk,$$

for any $k' \in K_G$ and then change it back

$$\int_{\overline{N}_G} K_\lambda^\nu(k'\overline{n})\mathbb{1}_\lambda(\overline{n})\,d\overline{n}.$$

If $k' = w_0^H$ the longest Weyl-group element of H then if we write $\overline{n}$ in block form

$$\overline{n} = \begin{pmatrix} \overline{n}_H & \\ x^\intercal & 1 \end{pmatrix} = \overline{n}_H\overline{n}_x$$

where $x \in \mathbb{R}^n$ and $\overline{n}_H \in \overline{N}_H$.The kernel then has the particularly simple form

$$K_\lambda^\nu(w_0^H\overline{n}) = |x_1|^{s_1}\dots|x_n|^{s_n}.$$

If $k' = w_0$ the longest Weyl-group element of G then the kernel has the form

$$K_\lambda^\nu(w_0\overline{n}) = |\Psi_1(w_0\overline{n}_H)|^{t_1}\dots|\Psi_{n-1}(w_0\overline{n}_H)|^{t_{n-1}}|\Psi_n(w_0\overline{n})|^{t_n}.$$

In either case one would think this simplifies the integral significantly but it turns out it does not, as computing integrals containing $\mathbb{1}(\overline{n})$ is complicated as it is.

Another trick to apply is using Lemma 1.3 we can write the integral as

$$\int_{\mathbb{R}^n}\int_{\overline{N}_H} K_\lambda^\nu(\overline{n}_x\overline{n}_H)\mathbb{1}_\lambda(\overline{n}_x\overline{n}_H)d\overline{n}_H dx.$$

We can recognize the inner integral as the Knapp–Stein intertwining operator

$$T_{-\rho_G}^{w_0^H}(K_\lambda^\nu\mathbb{1}_\lambda)(\overline{n}_x w_0^H).$$

If we decompose $\overline{n}_x w_0^H$ into $k(\overline{n}_x w_0^H)e^{H(\overline{n}_x w_0^H)}n_0$ from KAN we can use the equivariance properties of the Knapp–Stein intertwining operator

$$T_{-\rho_G}^{w_0^H}(K_\lambda^\nu\mathbb{1}_\lambda)(\overline{n}_x w_0^H) = e^{-(-w_0^H\rho_G+\rho_G)H(\overline{n}_x w_0^H)}T_{-\rho_G}^{w_0^H}(K_\lambda^\nu\mathbb{1}_\lambda)(k(\overline{n}_x w_0^H)),$$

reducing the inner integral to

$$\int_{\overline{N}_H} K_\lambda^\nu(\overline{n}_x w_0^H n_0^{-1}e^{-H(\overline{n}_x w_0^H)}\overline{n}_H)\mathbb{1}_\lambda(\overline{n}_H)\,d\overline{n}_H.$$

Which at first glance might not seem like a reduction but notice how we have split $\mathbb{1}_\lambda$ into two pieces, one that only depends on $\overline{n}_H$ and the other on $\overline{n}_x$. Another thing we can note is that if $\det(g) = 1$ then

$$\mathbb{1}_\lambda(g) = \Phi_1(w_0 g^\intercal g)^{-\frac{s_1+t_1+2}{2}}\dots\Phi_n(w_0 g^\intercal g)^{-\frac{s_n+t_n+2}{2}},$$

where $s_i + t_i + 2 = \lambda_i - \lambda_{i+1} + 1$. This follows from the age old trick where if we want to decompose with the Iwasawa decomposition $g = kan$ then writing $g^\intercal g = n^\intercal a^2 n$ we just have to find the $\overline{N}MAN$ decomposition of $g^\intercal g$. In classical linear algebra this corresponds to finding the LDU-decomposition of $g^\intercal g$ and the diagonal part has entries given by quotients of consecutive principal minor determinants which is exactly what Φ_i describes.

This comes in handy if we are trying to make a rank one reduction. Let $\overline{n}_1(y) = e^{yE_{2,1}}$ then using Lemma 1.3 we get

$$\int_{\overline{N}'}\int_{\mathbb{R}} K^{\nu}_{\lambda}(\overline{n}\,\overline{n}_1(y))\mathbb{1}_{\lambda}(\overline{n}\,\overline{n}_1(y))dyd\overline{n},$$

and considering only the inner integral we get

$$\int_{\mathbb{R}} |\Phi_1(\overline{n}) + y\Phi_1(\overline{n}w_1)|^{s_1}|\Psi_1(\overline{n}) + y\Psi_1(\overline{n}w_1)|^{t_1} \times (\Phi_1(w_0\overline{n}^{\intercal}\overline{n}) + 2y\Phi_1(w_0\overline{n}^{\intercal}\overline{n}w_1) + \Phi_1(w_1w_0\overline{n}^{\intercal}\overline{n}w_1))^{-\frac{s_1+t_1+2}{2}}dy,$$

which we miraculously can evaluate but the answer contains a ${}_2F_1$ with an argument that contains Φ_1 and Ψ_1 and the next integral then becomes hard to evaluate. This is where our story ends because no matter what fancy tricks we use to rewrite the integral after evaluating the first integral the second one becomes impossible for us to evaluate.

1.1 Integral formulas

Proposition 1.4. *For* $\mathrm{Re}(\frac{u}{\beta}) > 0$, $\mathrm{Re}(\lambda) > 0$ *and* $\mathrm{Re}(\mu) > 0$ *we have*

$$\int_0^u x^{\lambda-1}(u-x)^{\mu-1}(\beta^2+x^2)^{\nu}\,dx = \beta^{2\nu}u^{\lambda+\mu-1}B(\lambda,\mu){}_3F_2(-\nu, \tfrac{\lambda}{2}, \tfrac{\lambda+1}{2}; \tfrac{\lambda+\mu}{2}, \tfrac{\lambda+\mu+1}{2}; \tfrac{-u^2}{\beta^2}).$$

For $\varepsilon \in \{0,1\}$, $\mathrm{Re}(\frac{\beta}{u}) > 0$ *and* $\mathrm{Re}(\mu) > 0$ *we have*

$$\int_u^{\infty} x^{\varepsilon}(x-u)^{\mu-1}(\beta^2+x^2)^{\nu}\,dx = u^{2\nu+\mu+\varepsilon}B(-\mu-\varepsilon-2\nu,\mu){}_2F_1(\tfrac{-\mu-\varepsilon-2\nu}{2}, \tfrac{1-\mu-\varepsilon-2\nu}{2}; \tfrac{1}{2}-\nu-\varepsilon; -\tfrac{\beta^2}{u^2}).$$

Proof. The first formula comes from [GR94, 3.254]. The second integral follows from the first one by considering

$$\int_0^u x^{\lambda-1}(u-x)^{\mu-1}(\beta^2+x^2)^{-\frac{\lambda+\mu+\varepsilon}{2}}\,dx,$$

and then doing the change of variables $x \to x^{-1}$. Here we used that

$${}_3F_2(\tfrac{\lambda+\mu+\varepsilon}{2}, \tfrac{\lambda}{2}, \tfrac{\lambda+1}{2}; \tfrac{\lambda+\mu}{2}, \tfrac{\lambda+\mu+1}{2}; -\tfrac{u^2}{\beta^2}) = {}_2F_1(\tfrac{\lambda}{2}, \tfrac{\lambda+1}{2}; \tfrac{\lambda+\mu+1-\varepsilon}{2}; -\tfrac{u^2}{\beta^2}).$$

□

Following [FS21, Appendix A] we can analytically extend the second formula by

$${}_2F_1(a,b;c;x) = \tfrac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}(-x)^{-a}{}_2F_1(a, a-c+1; a-b+1; x^{-1}) + \tfrac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}(-x)^{-b}{}_2F_1(b, b-c+1; b-a+1; x^{-1}).$$

By the duplication formula for the Gamma-function we then obtain

Proposition 1.5. *For $u \in \mathbb{R}$, $\varepsilon \in \{0,1\}$, $\mathrm{Re}(\mu) > 0$ and $\beta > 0$*

$$\int_u^\infty x^\varepsilon (x-u)^{\mu-1}(\beta^2+x^2)^\nu dx$$
$$= \frac{\beta^{\mu+2\nu+\varepsilon}}{2\Gamma(-\nu)}\Big[\Gamma(\tfrac{\mu+\varepsilon}{2})\Gamma(\tfrac{-\mu-2\nu-\varepsilon}{2}){}_2F_1\Big(\tfrac{-\mu-2\nu-\varepsilon}{2},\tfrac{1+\varepsilon-\mu}{2};\tfrac{1}{2};-\tfrac{u^2}{\beta^2}\Big)$$
$$-\tfrac{u}{2\beta}(\tfrac{\mu-1}{2})\Gamma(\tfrac{\mu-1+\varepsilon}{2})\Gamma(\tfrac{-\varepsilon-\mu-2\nu}{2}){}_2F_1\Big(\tfrac{1-\varepsilon-\mu-2\nu}{2},\tfrac{2+\varepsilon-\mu}{2};\tfrac{3}{2};-\tfrac{u^2}{\beta^2}\Big)\Big]. \quad (1.3)$$

Proposition 1.6. *Let $u \in \mathbb{R}$, $\varepsilon \in \{0,1\}$, $\mathrm{Re}(\mu) > 0$ and $\beta > 0$. If $\varepsilon + \sigma \equiv_2 0$ then*

$$\int_{\mathbb{R}} x^\varepsilon |u-x|_\sigma^{\mu-1}(\beta^2+x^2)^\nu dx = (-1)^\sigma \beta^{\mu+2\nu+\varepsilon} B(\tfrac{\mu+\varepsilon}{2},\tfrac{-\mu-2\nu-\varepsilon}{2}){}_2F_1(\tfrac{-\mu-2\nu-\varepsilon}{2},\tfrac{1+\varepsilon-\mu}{2};\tfrac{1}{2};-\tfrac{u^2}{\beta^2}).$$

If $\varepsilon + \sigma \equiv_2 1$ then

$$\int_{\mathbb{R}} x^\varepsilon |u-x|_\sigma^{\mu-1}(1+x^2)^\nu dx = (-1)^{\sigma+1} u\beta^{\mu+2\nu+\varepsilon-1}(\tfrac{\mu-1}{2})$$
$$\times \frac{\Gamma(\frac{\mu-1+\varepsilon}{2})\Gamma(\frac{-\varepsilon-\mu}{2}-\nu)}{2\Gamma(-\nu)}{}_2F_1(\tfrac{-\mu-2\nu+1-\varepsilon}{2},\tfrac{2+\varepsilon-\mu}{2};\tfrac{3}{2};-u^2).$$

Proof. If we consider the right hand side of (1.3) as a function $F(u)$ then we see that the first term is even in u and the second term is odd. Writing the integral as

$$\int_{-\infty}^u x^\varepsilon (u-x)^{\mu-1}(\beta^2+x^2)\nu\, dx + (-1)^\sigma \int_u^\infty x^\varepsilon (x-u)^{\mu-1}(\beta^2+x^2)^\nu dx,$$

and changing variables $x \to -x$ in the first integral we see that it has the form $(-1)^\varepsilon F(-u) + (-1)^\sigma F(u)$ and thus depending on ε and σ either the odd or the even part of F cancels out. □

Proposition 1.7 ([GR94, 7.512])**.** *We have for $\mathrm{Re}(\gamma) > 0$, $\mathrm{Re}(\alpha-\gamma+\nu) > 0$, $\mathrm{Re}(\beta-\gamma+\nu) > 0$ and $|\arg z| > 0$:*

$$\int_0^\infty x^{\gamma-1}(z+x)^{-\nu}{}_2F_1(\alpha,\beta;\gamma;-x)dx$$
$$= \frac{\Gamma(\gamma)\Gamma(\alpha+\nu-\gamma)\Gamma(\beta+\nu-\gamma)}{\Gamma(\nu)\Gamma(\alpha+\beta+\nu-\gamma)}{}_2F_1(\alpha+\nu-\gamma,\beta+\nu-\gamma;\alpha+\beta+\nu-\gamma;1-z).$$

Proposition 1.8 ([GR94, 7.512])**.** *We have for $\mathrm{Re}(\mu) > 0$, $\mathrm{Re}(\nu) > 0$ and $p \le q+1$:*

$$\int_0^1 t^{\mu-1}(1-t)^{\nu-1}{}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;tx)dt$$
$$= B(\mu,\nu){}_{p+1}F_{q+1}(a_1,\dots,a_p,\mu;b_1,\dots,b_q,\mu+\nu;x).$$

Setting $x = 1$ and using the substitution $s = 1-t$ we get

$$\int_0^1 (1-s)^{\mu-1}s^{\nu-1}{}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;1-s)ds$$
$$= B(\mu,\nu){}_{p+1}F_{q+1}(a_1,\dots,a_p,\mu;b_1,\dots,b_q,\mu+\nu;1). \quad (1.4)$$

Lastly, for completeness sake we include the Gauss formula for the hypergeometric function

$${}_2F_1(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}. \quad (1.5)$$

2 Extending the Plancherel formula

The attentive reader will have noticed that in Paper A we prove a Plancherel formula for $\mathrm{SL}(2,\mathbb{R})/MA$ but in Paper C we apply a Plancherel formula for $\mathrm{GL}(2,\mathbb{R})/MA$. In this section we will describe how to get from the one to the other. We first recall the central parts of Paper A.

2.1 Principal series for $\mathrm{SL}(2,\mathbb{R})$

In this section we recall some results about the representation theory of $\mathrm{SL}(2,\mathbb{R})$ following [C20]. Let $S = \mathrm{SL}(2,\mathbb{R})$ and consider the following subgroups

$$M_S = \{\pm I\}, \quad A_S = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{R}_{>0} \right\}, \quad N_S = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\},$$

then $P_S = M_S A_S N_S$ is a minimal parabolic subgroup of S. Identify $\widehat{M_S} \cong \mathbb{Z}/2\mathbb{Z}$ by mapping $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ to the character

$$M_S \to \{\pm 1\}, \quad \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \mapsto (\pm 1)^{\varepsilon}.$$

Further, we identify $\mathfrak{a}_{\mathbb{C}}^* \cong \mathbb{C}$ by mapping $\lambda \mapsto \lambda(\mathrm{diag}(1,-1))$. We then observe that any character of $D_S := M_S A_S$ is of the form $\chi_{\varepsilon,\lambda} = \varepsilon \otimes e^{\lambda}$ where

$$\chi_{\varepsilon,\lambda} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = |t|_{\varepsilon}^{\lambda} := \mathrm{sgn}(t)^{\varepsilon} |t|^{\lambda}, \qquad (t \in \mathbb{R}^{\times}).$$

As the commutator subgroup of P_S is N_S the characters of P_S is of the form $\varepsilon \otimes e^{\mu} \otimes 1$ and these characters are unitary exactly when $\lambda \in i\mathbb{R}$.

Let $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ and $\mu \in \mathbb{C}$. For any character $\varepsilon \otimes e^{\mu} \otimes 1$ of P_S define the principal series representation $\pi_{\varepsilon,\mu}$ to be the left regular representation of S on

$$\mathrm{Ind}_{P_S}^{S}(\varepsilon \otimes e^{\mu} \otimes 1) = \left\{ f \in C^{\infty}(S) \mid f(gman) = |t|_{\varepsilon}^{-\mu-1} f(g),\ m \in M_S,\ a \in A_S,\ n \in N_S \right\},$$

where $ma = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \in M_S A_S$. We introduce the notation

$$k_{\theta} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

and $\zeta_m(k_\theta) = e^{im\theta}$. According to the theory of Fourier series we have the K-type decomposition

$$\mathrm{Ind}_{P_S}^{S}(\varepsilon \otimes e^{\mu} \otimes 1) \cong \widehat{\bigoplus_{m \in 2\mathbb{Z}+\varepsilon}} \mathbb{C}\zeta_m.$$

We let $\mathrm{Ind}_{P_S}^{S}(\varepsilon \otimes e^{\mu} \otimes 1)_m$ denote the set of functions contained in the K-type given by $m \in \mathbb{Z}$, that is $\mathrm{Ind}_{P_S}^{S}(\varepsilon \otimes e^{\mu} \otimes 1)_m = \mathbb{C}\zeta_m$.

Proposition 2.1. *The representation $\mathrm{Ind}_{P_S}^{S}(\varepsilon \otimes e^{\mu} \otimes 1)$ is irreducible except when $\mu \in 1-\varepsilon-2\mathbb{Z}$. If $\mu \in 1-\varepsilon-2\mathbb{N}$ then $\mathrm{Ind}_{P_S}^{S}(\varepsilon \otimes e^{\mu} \otimes 1)$ decomposes as $V_0 \oplus V_1 \oplus V_2$ where V_0 is an irreducible representation containing exactly the K-types with $|m| \leq -\mu$. The quotient $\pi_{\varepsilon,\mu}^{ds}$ is a direct sum of two infinite dimensional representations $\pi_{\varepsilon,\mu}^{\mathrm{hol}}$ and $\pi_{\varepsilon,\mu}^{\mathrm{ahol}}$.*

Let $w_0 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, a representative of the longest Weyl group element of S. Recall the definition of the Knapp–Stein intertwining operator

$$T_{\varepsilon,\mu} : \mathrm{Ind}_{P_S}^S(\varepsilon \otimes e^{\mu} \otimes 1) \to \mathrm{Ind}_{P_S}^S(\varepsilon \otimes e^{-\mu} \otimes 1), \quad T_{\varepsilon,\mu} f(g) = \frac{1}{\Gamma(\frac{\mu+\varepsilon}{2})} \int_{\overline{N}} f(g w_0 \overline{n}) d\overline{n},$$

for $\mathrm{Re}\,(\mu) > 0$. The normalization is chosen such that $T_{\varepsilon,\mu}$ extends holomorphically to $\mu \in \mathbb{C}$.

Proposition 2.2. *For $f \in \mathrm{Ind}_{P_S}^S(\varepsilon \otimes e^{\mu} \otimes 1)_m$ we have*

$$T_{\varepsilon,\mu} f = b_m^{\varepsilon}(\mu) f,$$

where

$$b_m^{\varepsilon}(\mu) = \sqrt{\pi} i^{[\varepsilon]_2} (-1)^{\frac{m+|m|}{2} - [\varepsilon]_2} \frac{\left(\frac{1+\varepsilon-\mu}{2}\right)_{\frac{|m|-\varepsilon}{2}}}{\Gamma\left(\frac{\mu+1+|m|}{2}\right)}.$$

For $\varepsilon = 0$ and $\mu \in 1 - 2\mathbb{N}$ we have $b_m^0(\mu) \geq 0$ for all $m \in 2\mathbb{Z}$. Whereas for $\varepsilon = 1$, m odd and $\mu \in -2\mathbb{N}$ we have $-ib_m^1(\mu) \geq 0$ for $m > 0$ and $ib_m^1(\mu) \geq 0$ for $m < 0$.

For $\mu \in i\mathbb{R}$ we equip the space $\mathrm{Ind}_{P_S}^S(\varepsilon \otimes e^{\mu} \otimes 1)$ with the usual L^2-norm. Using Proposition 2.2 we can for $\varepsilon = 0$ and $\mu \in 1 - 2\mathbb{N}$ equip $\mathrm{Ind}_{P_S}^S(0 \otimes e^{\mu} \otimes 1)$ with the norm

$$\|f\|^2 = \int_{K_S/M_S} f(k) \overline{T_{0,\mu} f(k)}\, dk.$$

Similarly for $\varepsilon = 1$ and $\mu \in -2\mathbb{N}$ we can equip $\mathrm{Ind}_{P_S}^S(1 \otimes e^{\mu} \otimes 1)$ with the norm

$$\|f\|^2 = \int_{K_S/M_S} f(k) \overline{\widehat{T}_{1,\mu} f(k)}\, dk$$

where

$$\widehat{T}_{1,\mu} f = \begin{cases} iT_{1,\mu} f, & \text{for } m > 0, \\ -iT_{1,\mu} f, & \text{for } m < 0 \end{cases}$$

for $f \in \mathrm{Ind}_{P_S}^S(1 \otimes e^{\mu} \otimes 1)_m$. The operator $\widehat{T}_{1,\mu}$ is still an intertwining operator as it vanishes on V_0 per Proposition 2.2 and thus we just altered it by a scalar on each of the summands in Proposition 2.1.

2.2 The homogeneous space S/D_S and H/D

Let $D_S = M_S A_S$ and $D = M_H A_H$. For a unitary character $\chi_{\varepsilon,\lambda} = \varepsilon \otimes e^{\lambda}$ with $\lambda \in i\mathbb{R}$ the left-regular action $\tau_{\varepsilon,\lambda}$ of S on the space of L^2-sections associated to the line bundle $S \times_{D_S} \mathbb{C}_{\varepsilon,\lambda} \to S/D_S$, given by

$$\mathrm{Ind}_{D_S}^S(\varepsilon \otimes e^{\lambda}) = \left\{ f : S \to \mathbb{C}, \mid f(gh) = \chi_{\varepsilon,\lambda}(h)^{-1} f(g), \int_{S/D_S} |f(g)|^2 d(gD_S) < \infty \right\},$$

defines a unitary representation of S. The goal of this paper is to decompose this space. We introduce the notation

$$b_u = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix}, \quad \overline{n}_x = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}.$$

Using the decomposition $S = K_S B A_S$, where $B = \{b_u \mid u \in \mathbb{R}\}$, we consider S/D_S in the global coordinates $(\theta, u) \in [0, \pi) \times \mathbb{R}$ where $xD_S = k_\theta b_u D_S$ and the invariant measure is $d(xD_S) = \cosh(2u) du d\theta$, see e.g. [M84]. Now in terms of these coordinates we have the K-type decomposition

$$C_c^\infty\text{-}\operatorname{Ind}_{D_S}^S(\varepsilon \otimes e^\lambda) = \widehat{\bigoplus_{m \in 2\mathbb{Z}+\varepsilon}} \mathbb{C}\zeta_m \otimes C_c^\infty(\mathbb{R}) \tag{2.1}$$

with $\zeta_m(k_\theta) = e^{im\theta}$. We let $\operatorname{Ind}_{D_S}^S(\varepsilon \otimes e^\lambda)_m$ denote the set of functions contained in the K-type given by $m \in 2\mathbb{Z} + \varepsilon$, that is $\operatorname{Ind}_{D_S}^S(\varepsilon \otimes e^\lambda)_m = \mathbb{C}\zeta_m \otimes C_c^\infty(\mathbb{R})$.

Another set of coordinates can be obtained by using the Iwasawa decomposition $S = K_S A_S \overline{N}_S$ with $(\theta, y) \in [0, \pi) \times \mathbb{R}$ where $xD_S = k_\theta \overline{n}_y D_S$. The invariant measure is given by $d(xD_S) = \frac{1}{2} dy d\theta$ see [K16, Chap. 5, §6].

Similarly we can define the space

$$\operatorname{Ind}_D^H(\varepsilon \otimes e^\mu) = \left\{ f : H \to \mathbb{C}, \mid f(gh) = \chi_{\varepsilon,\mu}(h)^{-1} f(g), \int_{H/D} |f(g)|^2 d(gD) < \infty \right\},$$

where $\chi_{\varepsilon,\mu} = \varepsilon \otimes e^\mu$ is a unitary character on D with $\mu \in i\mathbb{R}^2$ and $\varepsilon \in (\mathbb{Z}/2\mathbb{Z})^2$. We note that $S/D_S \simeq H/D$ and for $f \in \operatorname{Ind}_D^H(\varepsilon \otimes e^\mu)$ we have

$$\int_{H/D} f(xD)\, d(xD) = \int_{S/D_S} f(xD_S)\, d(xD_S).$$

This can be seen by noticing $N_S = N_H$ and considering $S = \overline{N}_S N_S M_S A_S$ and $H = \overline{N}_H N_H M_H A_H$ and the integral formula by writing both integrals in $\overline{N}N$-coordinates.

2.3 Principal series representations for $\mathrm{GL}(2, \mathbb{R})$ and the Extension map

2.4 Principal series for $\mathrm{GL}(2, \mathbb{R})$

In this section we recall some results about the representation theory of $\mathrm{GL}(2, \mathbb{R})$. Let $H = \mathrm{GL}(2, \mathbb{R})$ and consider the following subgroups of H

$$M_H = \left\{ \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_1 \end{pmatrix} : \varepsilon_1, \varepsilon_2 \in \{\pm 1\} \right\}, \quad A_H = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} : a_1, a_2 \in \mathbb{R}_{>0} \right\}, \quad N_H = N_S.$$

Then $P_H = M_H A_H N_H$ is a minimal parabolic subgroup for H. In a similar fashion as for S we can identify $\widehat{M_H} \simeq (\mathbb{Z}/2\mathbb{Z})^2$ and $\mathfrak{a}_\mathbb{C}^* \simeq \mathbb{C}^2$. The characters of P_H are then of the form $\eta \otimes e^\nu \otimes 1$ where $\eta \in (\mathbb{Z}/2\mathbb{Z})^2$, $\nu \in \mathbb{C}^2$ and for any such character we define the principal series representations given as the left regular representation of H on

$$\operatorname{Ind}_{P_H}^H(\eta \otimes e^\nu \otimes 1) = \{ f \in C^\infty(H) \mid f(gman) = |b|_{\eta_1}^{-\nu_1 - \frac{1}{2}} |c|_{\eta_2}^{-\nu_2 + \frac{1}{2}} f(g),\, man \in M_H A_H N_H \},$$

where $ma = \operatorname{diag}(b, c)$. Consider the map

$$L_{\delta,s} : \operatorname{Ind}_{P_S}^S(\varepsilon \otimes e^t \otimes 1) \to \operatorname{Ind}_{P_H}^H((\varepsilon + \delta, \delta) \otimes e^{(t+s,s)} \otimes 1), \quad f \mapsto F,$$

where $F(h) = |\det(h)|_\delta^{\frac{1}{2}-s} f(h\,\mathrm{diag}(1,\det(h))^{-1})$. This map creates a family of natural extensions as $L_s f|_S = f$ and it intertwines the S action. As $K_H/M_H \simeq K_S/M_S$ we can use the pairing

$$\langle f, f'\rangle = \int_{K_S/M_S} f(k)\overline{f'(k)}\, d(kM_S),$$

on both $\mathrm{Ind}_{P_H}^H(\eta\otimes e^\nu\otimes 1)\times \mathrm{Ind}_{P_H}^H(\eta\otimes e^{-\overline{\nu}}\otimes 1)$ and $\mathrm{Ind}_{P_S}^S(\varepsilon\otimes e^t\otimes 1)\times \mathrm{Ind}_{P_S}^S(\varepsilon\otimes e^{-\overline{t}}\otimes 1)$. Furthermore we can pick a representative,

$$w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

for the longest Weyl group element in both H and S. As $N_H = N_S$ we can use the same expression for the Knapp–Stein intertwiners $T_{\eta,\nu}$ for $\mathrm{Ind}_{P_H}^H(\eta\otimes e^\nu\otimes 1)$ and $T_{\varepsilon,t}$ for $\mathrm{Ind}_{P_S}^S(\varepsilon\otimes e^t\otimes 1)$, namely

$$f\mapsto Tf, \quad \text{where} \quad Tf(g) = \int_{\overline{N}_S} f(gw_0\overline{n})\, d\overline{n}.$$

Lemma 2.3. *For $f\in \mathrm{Ind}_{P_S}^S(\varepsilon\otimes e^t\otimes 1)$ we have the following*

1. *If $F\in \mathrm{Ind}_{P_H}^H(\eta\otimes e^\nu\otimes 1)$ satisfies $F|_S = f$ then $F = L_{\eta_2,\nu_2} f$ and $(\varepsilon, t) = (\eta_1+\eta_2, \nu_1-\nu_2)$.*
2. *$\langle L_{\delta,s} f, L_{\delta,-\overline{s}} f'\rangle = \langle f, f'\rangle$, where $f'\in \mathrm{Ind}_{P_S}^S(\varepsilon\otimes e^{-\overline{t}}\otimes 1)$*
3. *$T_{(\varepsilon+\delta,\delta),(t+s,s)} L_{\delta,s} = L_{\delta+\varepsilon,s+t} T_{\varepsilon,t}$.*
4. *$\langle f, T_{\varepsilon,t} f\rangle = \langle L_{\delta,s} f, T_{(\delta,\delta+\varepsilon),(-\overline{s},-\overline{s}-t)} L_{\delta+\varepsilon,-\overline{s}-t} f\rangle$ for $t\in\mathbb{R}$.*

Proof. **(1):** The first results follows from

$$\begin{aligned} F(g) &= F\big(g\,\mathrm{diag}(1,\det(g))^{-1}\,\mathrm{diag}(1,\det(g))\big) \\ &= |\det(g)|_{\eta_2}^{\frac{1}{2}-\nu_2} F\big(g\,\mathrm{diag}(1,\det(g))^{-1}\big) = L_{\delta,s} f(g). \end{aligned}$$

For $a\in\mathbb{R}$ we have

$$|a|_{\eta_1+\eta_2}^{\nu_2-\nu_1-1} F(1) = F\big(\mathrm{diag}(a,a^{-1})\big) = f\big(\mathrm{diag}(a,a^{-1})\big) = |a|_\varepsilon^{-t-1} f(1),$$

and as $F(1) = f(1)$ this completes the proof of (1).
(2): Follows from the pairing being done as a integral over a subgroup of S.
(3): By definition we have

$$T_{(\varepsilon+\delta,\delta),(t+s,s)} L_{\delta,s} f(g) = \int_{\overline{N}_H} |\det(gw_0\overline{n})|_\delta^{\frac{1}{2}-s} f(gw_0\overline{n}\,\mathrm{diag}(1,\det(gw_0\overline{n}))^{-1})\, d\overline{n}.$$

Now we move $d := \mathrm{diag}(1,\det(g))^{-1}$ past $\overline{n}$, by doing the change of variables $\overline{n}\to d^{-1}\overline{n}d$ and by conjugation of d by w_0 we get

$$= |\det(g)|_\delta^{-\frac{1}{2}-s} \int_{\overline{N}_H} f(g\,\mathrm{diag}(\det(g),1)^{-1} w_0\overline{n})\, d\overline{n}.$$

We recognize the integral as being the Knapp–Stein intertwiner $T_{\varepsilon,t}f$ evaluated at

$$g\,\mathrm{diag}(\det(g),1)^{-1} = g\,\mathrm{diag}(1,\det(g))^{-1}\,\mathrm{diag}(\det(g)^{-1},\det(g)).$$

Using the equivariance properties of $T_{\varepsilon,t}f$ we get that

$$= |\det(g)|_{\delta+\varepsilon}^{\frac{1}{2}-s-t} T_{\varepsilon,t}f(g\,\mathrm{diag}(1,\det(g))^{-1}) = L_{\delta+\varepsilon,s+t}T_{\varepsilon,t}.$$

(4): Follows directly from (2) and (3). □

Consider the restriction map

$$R_{\eta,\nu} : \mathrm{Ind}_{P_H}^H(\eta\otimes e^{\nu}\otimes 1) \to \mathrm{Ind}_{P_S}^S(\eta_1+\eta_2\otimes e^{\nu_1-\nu_2}\otimes 1), \quad f\to f|_S,$$

which satisfies

$$R_{(\varepsilon+\delta,\delta),(s+t,s)}\circ L_{\delta,s} = \mathrm{id}, \quad \text{and} \quad L_{\eta_2,\nu_2}\circ R_{\eta,\nu} = \mathrm{id},$$

as S-intertwining maps. We can now consider the K_H-types for H, for $\zeta_m \in \mathrm{Ind}_{P_S}^S(\eta_1+\eta_2\otimes e^{\nu_1-\nu_2}\otimes 1)_m$ let $\tilde{\zeta}_m := L_{\eta_2,\nu_2}\zeta_m \in \mathrm{Ind}_{P_H}^H(\eta\otimes e^{\nu}\otimes 1)$. As

$$K_H = K_S\cup K_S\kappa, \quad \text{where} \quad \kappa = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

and $\mathbb{C}\zeta_m$ is fixed on the action of K_S so we need to consider what happens when acting with κ. Considering the two cases of the argument being in each connected components of K_H at once, we let $\varepsilon = 0,1$ and get

$$\tilde{\zeta}_m(\kappa k_\theta\kappa^\varepsilon) = (-1)^{\eta_2(1+\varepsilon)}\zeta_m(\kappa k_\theta\kappa) = (-1)^{\eta_2(1+\varepsilon)}\tilde{\zeta}_{-m}(k_\theta) = (-1)^{\eta_2}\tilde{\zeta}_{-m}(k_\theta\kappa^\varepsilon),$$

as $\kappa k_\theta\kappa = k_{-\theta}$. Therefore we get that the K_H-types are $\mathbb{C}\tilde{\zeta}_m\oplus\mathbb{C}\tilde{\zeta}_{-m}$ as κ maps $\mathbb{C}\zeta_m$ to $\mathbb{C}\zeta_{-m}$ giving us the K_H-type decomposition

$$\mathrm{Ind}_{P_H}^H(\eta\otimes e^{\nu}\otimes 1) \simeq \bigoplus_{m\in 2\mathbb{N}_0+[\eta_1+\eta_2]} \mathbb{C}\tilde{\zeta}_m\oplus\mathbb{C}\tilde{\zeta}_{-m}.$$

For $\nu\in(i\mathbb{R})^2$ we equip $\mathrm{Ind}_{P_H}^H(\eta\otimes e^{\nu}\otimes 1)$ with the norm coming from $\langle\,\cdot\,,\,\cdot\,\rangle$. For $\eta_1+\eta_2=0$ and $\nu_1-\nu_2\in 1-2\mathbb{N}$ we equip $\mathrm{Ind}_{P_H}^H(\eta\otimes e^{\nu}\otimes 1)/\ker(T_{\eta,\nu})$ with the norm

$$\|f\|^2 = \langle f, T_{\eta,\nu}f\rangle,$$

which is positive as by (3) in Lemma 2.3 and Proposition 2.2 we have

$$T_{\eta,\nu}\tilde{\zeta}_m = T_{\eta,\nu}L_{\eta_2,\nu_2}\zeta_m = L_{\eta_1,\nu_1}T_{0,\nu_1-\nu_2}\zeta_m = b_m^0(\nu_1-\nu_2)L_{\eta_1,\nu_1}\zeta_m = b_m^0(\nu_1-\nu_2)\tilde{\zeta}_m,$$

where the last equality follows from $\eta_1=\eta_2$ and that the second parameter in the extension map does not matter in the K_H-picture. For $\eta_1+\eta_2=1$ and $\nu_1-\nu_2\in -2\mathbb{N}$ consider the S-intertwining operator

$$\widehat{T}_{\eta\nu} = L_{\eta_2,\nu_1}\widehat{T}_{1,\nu_1-\nu_2}R_{\eta,\nu} : \mathrm{Ind}_{P_H}^H(\eta\otimes e^{\nu}\otimes 1)\to \mathrm{Ind}_{P_H}^H(\eta\otimes e^{(\nu_2,\nu_1)}\otimes 1),$$

to see that this is a H-intertwining operator we need to check that it intertwinines the center of H and the action of κ. The centre action follows directly by looking at the equivariance of the domain and the codomain. To see that it intertwines the action of κ we get by Proposition 2.2 that

$$\widehat{T}_{\eta,\nu}(c_0\tilde{\zeta}_m + c_1\tilde{\zeta}_{-m}) = L_{\eta_1,\nu_2} a_m(\nu_1-\nu_2)(c_0\zeta_m + c_1\zeta_{-m}) = a_m(\nu_1-\nu_2)(c_0\tilde{\zeta}_m + c_1\tilde{\zeta}_{-m}),$$

where $c_0, c_1 \in \mathbb{C}$, $m \in 2\mathbb{N}_0 + 1$ and

$$a_m(\nu_1-\nu_2) = \sqrt{\pi}\frac{(\frac{2-\nu_1+\nu_2}{2})_{\frac{m-1}{2}}}{\Gamma(\frac{\nu_1-\nu_2+1+m}{2})}.$$

This also shows that we can equip $\mathrm{Ind}_{P_H}^H(\eta\otimes e^\nu\otimes 1)/\ker(\widehat{T}_{\eta,\nu})$ with the norm

$$\|f\|^2 = \langle f, \widehat{T}_{\eta,\nu} f\rangle.$$

2.5 Extending the Plancherel formula from $\mathrm{SL}(2,\mathbb{R})$ to $\mathrm{GL}(2,\mathbb{R})$.

In Paper A we introduced the S-intertwining operators

$$A^{\sigma,t}_{\varepsilon,\mu} : \mathrm{Ind}_{D_S}^S(\varepsilon\otimes e^\mu) \to \mathrm{Ind}_{P_S}^S(\varepsilon\otimes e^t\otimes 1), \quad A^{\sigma,t}_{\varepsilon,\mu} f(g) = \int_{S/D_S} K^{\sigma,t}_{\varepsilon,\mu}(x^{-1}g) f(x)\, d(xD_S),$$

for $\sigma = 0,1$ where

$$K^{\sigma,t}_{\varepsilon,\mu}(g) = |g_{11}|_{\varepsilon+\sigma}^{\frac{-\mu-t-1}{2}} |g_{21}|_{\sigma}^{\frac{\mu-t-1}{2}}.$$

We then proved the following Plancherel formula for the hyperboloid S/D_S.

Proposition 2.4. *For $f \in C^\infty - \mathrm{Ind}_{D_S}^S(\varepsilon\otimes e^\mu)$ we have*

$$\|f\|^2 = \int_{i\mathbb{R}} \sum_{\sigma=0}^{1} \|\mathbf{A}^{\sigma,t}_{\varepsilon,\mu} f\|^2 \frac{dt}{|a(\varepsilon,t)|^2} + \sum_{t\in 1-\varepsilon-2\mathbb{N}} d(\varepsilon,t)\|\mathbb{A}^t_{\varepsilon,\mu} f\|^2,$$

where

$$a(\varepsilon,t) = 2^{3/2}\pi\frac{\Gamma(\frac{t}{2})}{\Gamma(\frac{1+t+\varepsilon}{2})\Gamma(\frac{1+t-\varepsilon}{2})}, \quad \Lambda(t) = \frac{1}{\Gamma(\frac{1+t}{4})^2\Gamma(\frac{1-t}{4})^2} + \frac{1}{\Gamma(\frac{3+t}{4})^2\Gamma(\frac{3-t}{4})^2},$$

and

$$d(0,t) = \frac{\Gamma(1-t)}{8\pi^3\Gamma(\frac{-t}{2})\Lambda(t)}, \quad d(1,t) = \frac{t^2\Gamma(-t)}{2\pi\Gamma(\frac{1-t}{2})}.$$

The operators are given by

$$\mathbb{A}^t_{0,\mu} = \frac{2^{\frac{1+t}{2}}\sqrt{\pi}\Gamma(\frac{1+t}{4}+\frac{\mu}{4})}{\Gamma(\frac{1-t}{4})\Gamma(\frac{1+t}{4})\Gamma(\frac{1-t}{4}-\frac{\mu}{4})} A^{0,t}_{0,\mu} + \frac{2^{\frac{1+t}{2}}\sqrt{\pi}\Gamma(\frac{3+t}{4}+\frac{\mu}{4})}{\Gamma(\frac{3-t}{4})\Gamma(\frac{3+t}{4})\Gamma(\frac{3-t}{4}-\frac{\mu}{4})} A^{1,t}_{0,\mu},$$

$$\mathbf{A}^{\sigma,t}_{\varepsilon,\mu} = \frac{A^{\sigma,t}_{\varepsilon,\mu}}{\Gamma(\frac{1-t}{2})}, \quad \textit{and} \quad \mathbb{A}^t_{1,\mu} = \frac{2^{\frac{1+t}{2}}\sqrt{\pi}\Gamma(\frac{1+t}{4}+\frac{\mu}{4})}{\Gamma(\frac{1-t}{4})\Gamma(\frac{3+t}{4})\Gamma(\frac{1-t}{4}-\frac{\mu}{4})} A^{0,t}_{1,\mu}.$$

Here we can notice for $t \in 1-\varepsilon-2\mathbb{N}$ that $K^{\sigma,t}_{\varepsilon,\mu}$ is locally integrable as g_{11} and g_{21} cannot vanish simultaneously and $\operatorname{Re}\left(\frac{-\lambda-\mu-1}{2}\right), \operatorname{Re}\left(\frac{-\lambda-\mu-1}{2}\right) > -1$. So in the case that $\varepsilon = 0$ we have

$$\Gamma\Big(\frac{1+t}{4}\Big)^{-1}\Big|_{t=1-2n} = \Gamma\Big(\frac{1-n}{2}\Big)^{-1}, \quad \& \quad \Gamma\Big(\frac{3+t}{4}\Big)^{-1}\Big|_{t=1-2n} = \Gamma\Big(1-\frac{n}{2}\Big)^{-1},$$

meaning that one of the two terms in $\mathbb{A}^t_{0,\mu}$ vanishes in the sum over $t \in 1-2\mathbb{N}$ in the Plancherel formula. Allowing us to rewrite it in the following way

$$\|f\|^2 = \int_{i\mathbb{R}} \sum_{\sigma=0}^{1} \|\mathbf{A}^{\sigma,t}_{0,\mu} f\|^2 \frac{dt}{|a(0,t)|^2} + \sum_{\sigma=0}^{1} \sum_{t\in 1+2\sigma-4\mathbb{N}} d'(0,t)\|\mathbb{A}^{\sigma,t}_{0,\mu} f\|^2,$$

where

$$\mathbb{A}^{\sigma,t}_{\varepsilon,\mu} = \frac{\Gamma(\frac{1+2\sigma+t+\mu}{4})}{\Gamma(\frac{1+2\sigma-t-\mu}{4})} A^{\sigma,t}_{\varepsilon,\mu}, \quad d'(0,t) = \frac{2^t\Gamma(1-t)}{4\pi^2\Gamma(-\frac{t}{2})}.$$

To reduce the amount of different A's flying around we also express the Plancherel formula for $\varepsilon = 1$ for the newly introduced A:

$$\|f\|^2 = \int_{i\mathbb{R}} \sum_{\sigma=0}^{1} \|\mathbf{A}^{\sigma,t}_{1,\mu} f\|^2 \frac{dt}{|a(1,t)|^2} + \sum_{t\in 1-\varepsilon-2\mathbb{N}} d'(1,t)\|\mathbb{A}^{0,t}_{1,\mu} f\|^2,$$

where

$$d'(1,t) = \frac{2^{1+t}t^2\Gamma(-t)}{\pi^2\Gamma(\frac{1-t}{2})},$$

where we used here that for $t \in -2\mathbb{N}$ we get $\Gamma(\frac{1-t}{4})^2\Gamma(\frac{3+t}{4})^2 = \frac{1}{2}$.

2.6 Extending the intertwining operators

Now we wish to extend the operators $A^{\sigma,t}_{\varepsilon,\mu}$ in the following way

$$\begin{array}{ccc}
\operatorname{Ind}_D^H(\varepsilon\otimes e^{\mu}) & & \operatorname{Ind}_{P_H}^H((\varepsilon_1+\varepsilon_2+\delta,\delta)\otimes e^{(t+s,s)}\otimes 1) \\
\big\downarrow \operatorname{Res}_S & & \big\uparrow L_{\delta,s} \\
\operatorname{Ind}_{D'}^S(\varepsilon_1+\varepsilon_2\otimes e^{\mu_1-\mu_2}) & \xrightarrow{A^{\sigma,t}_{\varepsilon_1+\varepsilon_2,\mu_1-\mu_2}} & \operatorname{Ind}_{P_S}^S(\varepsilon_1+\varepsilon_2\otimes e^{t}\otimes 1)
\end{array}$$

where the maps Res_S is the map $f \mapsto f|_S$. Following the diagram we get

$$B^{\eta,\nu}_{\varepsilon,\mu} f(g) := (L_{\delta,s}\circ A^{\sigma,t}_{\varepsilon_1+\varepsilon_2,\mu_1-\mu_2}\circ \operatorname{Res}_S) f(g) = \int_{S/D_S} K^{\eta,\nu}_{\varepsilon,\mu}(x^{-1}g)f(x)\,d(xD_S),$$

where

$$K^{\eta,\nu}_{\varepsilon,\mu}(g) = |g_{11}|^{\frac{\mu_2-\mu_1-t-1}{2}}_{\sigma+\varepsilon_1+\varepsilon_2}\,|g_{21}|^{\frac{\mu_1-\mu_2-t-1}{2}}_{\sigma}\,|\det(g)|^{\frac{1}{2}-s}_{\delta},$$

and $(\eta,\nu) = \big((\varepsilon_1+\varepsilon_2+\delta,\delta),(t+s,s)\big)$. The operator $B^{\eta,\nu}_{\varepsilon,\mu}$ maps between the right spaces but is not necessarily intertwining, to ensure that we need to change the integral from S/D_S to H/D meaning we need to find (δ,s) such that $x \mapsto K^{\eta,\nu}_{\varepsilon,\mu}(x^{-1}g)f(x)$ is a map on H/D. For $d = \operatorname{diag}(d_1,d_2)$ we have

$$K^{\eta,\nu}_{\varepsilon,\mu}(d^{-1}g) = |d_1|^{\frac{\mu_1-\mu_2+t}{2}+s}_{\sigma+\varepsilon_1+\varepsilon_2+\delta}|d_2|^{\frac{\mu_2-\mu_1+t}{2}+s}_{\sigma+\delta}K^{\eta,\nu}_{\varepsilon,\mu}(g), \quad f(gd) = |d_1|^{-\mu_1}_{\varepsilon_1}|d_2|_{\varepsilon_2}^{-\mu_2}f(g),$$

implying that $s = \frac{\mu_1+\mu_2-t}{2}$ and $\delta = \varepsilon_2 + \sigma$. Thus changing the coordinates we arrive at the operator

$$B^{\eta,\nu}_{\varepsilon,\mu} : \operatorname{Ind}^H_D(\varepsilon \otimes e^\mu) \to \operatorname{Ind}^H_{P_H}(\eta \otimes e^\nu \otimes 1), \quad B^{\eta,\nu}_{\varepsilon,\mu} f(g) = \int_{H/D} \hat{K}^{\eta,\nu}_{\varepsilon,\mu}(x^{-1}g) f(x)\, d(xD),$$

where

$$\hat{K}^{\eta,\nu}_{\varepsilon,\mu}(g) = |g_{11}|^{\frac{\nu_2-\nu_1-(\mu_1-\mu_2)-1}{2}}_{\eta_1+\varepsilon_2} |g_{21}|^{\frac{\nu_2-\nu_1+(\mu_1-\mu_2)-1}{2}}_{\eta_2+\varepsilon_2} |\det(g)|^{\frac{1}{2}-\nu_2}_{\eta_2}.$$

By acting by the centre of H we see that

$$\varepsilon_1 + \varepsilon_2 = \eta_1 + \eta_2, \quad \mu_1 + \mu_2 = \nu_1 + \nu_2.$$

Note here that $B^{\eta,\nu}_{\varepsilon,\mu}$ maps into different spaces for different σ's whereas this was not the case for $A^{\sigma,t}_{\varepsilon,\mu}$.

We are now ready to extend the Plancherel formulas to $\mathrm{GL}(2,\mathbb{R})$. For $\mu \in i\mathbb{R}^2$, $t \in i\mathbb{R}$ and $f \in \operatorname{Ind}^H_{P_H}(\varepsilon \otimes e^\mu)$ we have

$$\|A^{\sigma,t}_{\varepsilon_1+\varepsilon_2,\mu_1-\mu_2} \operatorname{Res} f\|^2 = \|L_{\eta_2,\nu_2} A^{\sigma,t}_{\varepsilon_1+\varepsilon_2,\mu_1-\mu_2} \operatorname{Res} f\|^2 = \|B^{\eta,\nu}_{\varepsilon,\mu} f\|^2,$$

by Lemma 2.3, if instead $t \in \mathbb{R}$ we have

$$\begin{aligned} &\langle A^{\sigma,t}_{\varepsilon_1+\varepsilon_2,\mu_1-\mu_2} \operatorname{Res} f, T_{\varepsilon_1+\varepsilon_2,t} A^{\sigma,t}_{\varepsilon_1+\varepsilon_2,\mu_1-\mu_2} \operatorname{Res} f\rangle \\ &\qquad = \langle L_{\eta_2,\nu_2} A^{\sigma,t}_{\varepsilon_1+\varepsilon_2,\mu_1-\mu_2} \operatorname{Res} f, L_{\eta_2,\nu_1} T_{\varepsilon_1+\varepsilon_2,t} A^{\sigma,t}_{\varepsilon_1+\varepsilon_2,\mu_1-\mu_2} \operatorname{Res} f\rangle \\ &\qquad = \langle B^{\eta,\nu}_{\varepsilon,\mu} f,\ L_{\eta_2,\nu_1} T_{\varepsilon_1+\varepsilon_2,t} R_{\eta,\nu} L_{\eta_2,\nu_2} A^{\sigma,t}_{\varepsilon_1+\varepsilon_2,\mu_1-\mu_2} \operatorname{Res} f\rangle \\ &\qquad = \langle B^{\eta,\nu}_{\varepsilon,\mu} f,\ L_{\eta_2,\nu_1} T_{\varepsilon_1+\varepsilon_2,t} R_{\eta,\nu} B^{\eta,\nu}_{\varepsilon,\mu} f\rangle \end{aligned} \tag{2.2}$$

In the case where $\varepsilon_1 + \varepsilon_2 = 0$ we have that

$$L_{\eta_2,\nu_1} T_{\varepsilon_1+\varepsilon_2,t} R_{\eta,\nu} B^{\eta,\nu}_{\varepsilon,\mu} = T_{(\eta_2,\eta_2+\varepsilon_1+\varepsilon_2),\nu} L_{\eta_2+\varepsilon_1+\varepsilon_2,\nu_2} R_{\eta,\nu} = T_{\eta,\nu}$$

by Lemma 2.3 and the fact that $\eta_1 = \eta_2$. For $\varepsilon_1 + \varepsilon_2 = 1$ and $T_{\varepsilon_1+\varepsilon_2,t}$ replaced by $\widehat{T}_{1,t}$ in (2.2) we have

$$L_{\eta_2,\nu_1} \widehat{T}_{1,t} R_{\eta,\nu} = \widehat{T}_{\eta,\nu}.$$

Theorem 2.5. *Let $\mu \in (i\mathbb{R})^2$ and put $\mu_\pm = \mu_1 \pm \mu_2$. For $f \in \operatorname{Ind}^H_D(\varepsilon \otimes e^\mu)$ when $\varepsilon_1 + \varepsilon_2 = 0$*

$$\|f\|^2 = \int_{i\mathbb{R}} \sum_{\sigma=0}^{1} \|\boldsymbol{B}^{\eta(\sigma),\nu(t)}_{\varepsilon,\mu} f\|^2 \frac{dt}{|a(0,t)|^2} + \sum_{\sigma=0}^{1} \sum_{t\in 1+2\sigma-4\mathbb{N}} b(0,t) \|\mathbb{B}^{\eta(\sigma),\nu(t)}_{\varepsilon,\mu} f\|^2,$$

whereas if $\varepsilon_1 + \varepsilon_2 = 1$

$$\|f\|^2 = \int_{i\mathbb{R}} \sum_{\sigma=0}^{1} \|\mathbf{B}^{\eta(\sigma),\nu(t)}_{\varepsilon,\mu} f\|^2 \frac{dt}{|a(1,t)|^2} + \sum_{t\in -2\mathbb{N}} b(1,t) \|\mathbb{B}^{\eta(0),\nu(t)}_{\varepsilon,\mu} f\|^2,$$

where

$$\boldsymbol{B}^{\eta,\nu}_{\varepsilon,\mu} = \frac{B^{\eta,\nu}_{\varepsilon,\mu}}{\Gamma(\frac{1-t}{2})}, \quad \mathbb{B}^{\eta,\nu}_{\varepsilon,\mu} = \frac{\Gamma(\frac{1+2\sigma+t+\mu_-}{4})}{\Gamma(\frac{1+2\sigma-t-\mu_-}{4})} B^{\eta,\nu}_{\varepsilon,\mu}, \quad (\eta(\sigma),\nu(t)) = \Big(\varepsilon + \sigma(1,1), (\tfrac{\mu_++t}{2}, \tfrac{\mu_+-t}{2})\Big),$$

and

$$b(\delta,t) = \frac{2^{t-2+3\delta}(-t)^\delta \Gamma(1-t)}{\pi^2 \Gamma(\frac{\delta-t}{2})}, \quad a(\delta,t) = \frac{2^{\frac{3}{2}}\pi\Gamma(\frac{t}{2})}{\Gamma(\frac{1+t+\delta}{2})\Gamma(\frac{1+t-\delta}{2})}, \quad (\delta \in \mathbb{Z}/2\mathbb{Z}).$$

We can slightly reformulate this result which is more useful for computations. As the normalization used for $\mathbf{B}^{\eta,\nu}_{\varepsilon,\mu}$ does not contain any zeroes for $t \in i\mathbb{R}$ the normalization can be moved to $a(\varepsilon, t)$. The normalization used for $\mathbb{B}^{\eta,\nu}_{\varepsilon,\mu}$ does not have zeroes or poles for t being summed over in the discrete part

Theorem 2.6. *Let $\mu \in i\mathbb{R}^2$ and put $\mu_\pm = \mu_1 \pm \mu_2$ and $\varepsilon_* = \varepsilon_1 + \varepsilon_2$. For $f \in \operatorname{Ind}_D^H(\varepsilon \otimes e^\mu)$*

$$\|f\|^2 = \int_{i\mathbb{R}} \sum_{\sigma=0}^{1} \|B^{\eta(\sigma),\nu(t)}_{\varepsilon,\mu} f\|^2 \frac{dt}{|a(\varepsilon_*, t)|^2} + \sum_{\sigma=0}^{1} \sum_{t \in 1-\varepsilon_*+2\sigma-4\mathbb{N}} b(\varepsilon_*, \sigma(1-\varepsilon_*), t) \|B^{\eta(\sigma(1-\varepsilon_*)),\nu(t)}_{\varepsilon,\mu} f\|^2,$$

where

$$(\eta(\sigma), \nu(t)) = \Big(\varepsilon + \sigma(1,1), (\frac{\mu_+ + t}{2}, \frac{\mu_+ - t}{2})\Big),$$

and

$$b(\delta, \sigma, t, \mu_-) = \frac{2^{t-2+3\delta}(-t)^\delta \Gamma(1-t) \big|\Gamma(\frac{1+2\sigma+t+\mu_-}{4})\big|^2}{\pi^2 \Gamma(\frac{\delta-t}{2}) \big|\Gamma(\frac{1+2\sigma-t-\mu_-}{4})\big|^2}, \quad a(\delta, t) = \frac{2^{\frac{3}{2}} \pi \Gamma(\frac{t}{2}) \Gamma(\frac{1-t}{2})}{\Gamma(\frac{1+t+\delta}{2}) \Gamma(\frac{1+t-\delta}{2})},$$

where $\delta \in (\mathbb{Z}/2\mathbb{Z})^2$.

This is the result presented in Paper C.

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