PhD Dissertation

Equivariant intersection theory on the curvilinear Hilbert scheme

Jonas Møller Svendsen

Department of Mathematics
Aarhus University
July, 2023

Equivariant intersection theory on the curvilinear Hilbert scheme

PhD Dissertation by
Jonas Møller Svendsen

Department of Mathematics, Aarhus University
Ny Munkegade 118, 8000 Aarhus C, Denmark

Supervised by
Associate Professor Gergely Bérczi

Submitted to the Graduate School of Natural Sciences, Aarhus University, on July 31, 2023

Table of Contents

Preface

This dissertation is the result of my Ph.D. studies at the Department of Mathematics at Aarhus University. The dissertation contains material from the preprint

paper *Fixed point distribution on Hilbert scheme of points.* Paper draft

This paper is a result of the detailed study of the curvilinear Hilbert scheme initiated by my supervisor Ass. Professor Gergely Bérczi in which I joined along, and is cowritten by the two of us. The main result is that all monomial ideals are contained in the curvilinear component. The curvilinear Hilbert scheme admits many different interpretations and open questions are thus possible to address from many directions. The paper uses a birational model coming from global singularity theory for which much of my understanding goes through another model, *The non-associative Hilbert scheme* [42], which was introduced to me by Prof. András Szenes at my three month visit at Université de Genève. For that and many discussions I am very thankful.

The initial goal of this Ph.D. project was another than that adressed in the paper. Namely, to calculate tautological integrals on the curvilinear Hilbert scheme. A description of how to perform such integration was already given in [3], but contains an unknown ingredient: the polynomial Q_d. This polynomial can be related to the non-associative Hilbert scheme, and I give an account of the relevant story of the non-associative Hilbert scheme in the relevant case in this thesis, containing also new results, in particular, Proposition 9.10.

The other original results of main interest are the aforementioned statement of the monomial ideals containment in the curvilinear Hilbert scheme, here Theorem 10.9, and formulas for integration on the curvilinear Hilbert scheme in Section 8 , in particular, Theorem 8.5 and its improvement 8.7. The methods used for obtaining integration formulas build on an original resolution model described in Section 7 using setup described in Section 6. The core of the proof of Theorem 10.9 is twofold: Proving the statement for a subclass of monomial ideals and the reduction from general monomials ideals to this subclass. This is adressed in Chapters 10 and 11, which are rewritten parts of tha paper mentioned above.

The paper is co-written with my supervisor Ass. Professor Gergely Bérczi with whom I have shared many great moments. I am truly grateful that you gave me the opportunity to explore the academic world of mathematics with you as my supervisor! But the Ph.D. project has not been all sunshine and roses. It has been a roller coaster ride of a magnitude, which was for me unforeseen when I initiated this project. The returning state of being “almost there” and then loosing it all on the ground revealed new and unknown psychological states. On the other hand, the state achieved when one succeeds with ideas can be rather addictive. To any new Ph.D. student out there: These are occassions not to take for granted.

Many people have dealt with and supported me through these roller coaster rides. I thank in particular my friends in oldekolle, and I thank my friends from u-klassen. To my flatmate and best mate Simon, thank you for bearing with me during the time of this project and for a relaxing environment – just perfect for brain recharging ! At last, a very big and warm thank you to my two brothers, Lasse and Simon, and to my mom and dad for their never ending support.

Abstract

The aim of this project has been to develop a method to integrate and calculate intersection numbers on the curvilinear Hilbert scheme. Such integration on the curvilinear Hilbert scheme is interesting in its own right, but even more relevant due to a new integration technique described in [4, 5] reducing integration on larger subsets of the Hilbert scheme of points to integration on the curvilinear Hilbert scheme. These subsets include for instance the geometric subsets defined in [54], and the method provides machinery for counting hypersurfaces with prescribed singularities.

The curvilinear Hilbert scheme is highly singular, and determining the exact elements of it is a question in deformation theory of algebras. A full understanding of such deformation theory is far out of reach at the moment. Even defining the meaning of intersections on singular spaces is a diffuclt task, let alone compute such. The modern approach is often via virtual intersection theory, proving existence of so-called virtual fundamental classes allowing one to define integration. However, such virtual intersections theory lags the geometric nature of intersections as it is often not possible to give the virtual fundamental class any geometric meaning. In any case such virtual techniques are hopeless for Hilbert schemes of points on a space of large dimension.

We construct instead an explicit resolution of a birational model for the curvilinear Hilbert scheme, and use the newly developed non-reductive geometric invariant theory to calculate intersection numbers. The primary method of calculation is via equivariant localization formulas using various torus-actions, and in particular using also a localization formula in non-reductive geometric invariant theory. With these methods we are able to calculate specific intersection numbers, and in particular we state Conjecture 12.1 possibly linking the Hilbert scheme of points on surfaces to the numbers of Cayley's formula for counting trees.

The curvilinear Hilbert scheme is an irreducible component of the punctual Hilbert scheme of points, whose points are ideals in a fixed polynomial ring. An important ingredient in the localization techniques are the torus-fixed ideals in the polynomial ring; these are exactly the monomial ideals. We use the explicit resolution of the birational model to show that all monomial ideals are contained in the same irreducible component, the curvilinear Hilbert scheme. This opens up for a new direction of further studies applying similar techniques as those applied in this work: To any monomial singularity in the punctual Hilbert scheme of points is associated a birational model as the one studied here, and via almost the same methods, one will in principle be able to obtain a complete picture of the hierarchy of monomial singularities; that is, a complete picture of the deformation theory of the algebras formed by taking the quotient with these monomial ideals.

Resumé

Målet for dette projekt har været at konstruere en metode til at integrere og udregne snittal på det kurvelineære Hilbert skema. En sådan integreringsprocedure på det kurvelineære Hilbert skema er interesant i sig selv, men endnu mere relevant på grund af en ny integreringsteknik beskrevet i [4, 5], hvor integrering på større delmængder af et Hilbert skema af punkter reduceres til integrering på det kurvelineære Hilbert skema. Disse delmængder inkluderer for eksempel de geometriske delmængder defineret i [54], og teknikken giver dermed en mulighed for at tælle hyperflader med givne singulariteter.

Det kurvelineære Hilbert skema er meget singulært, og dét at bestemme dets elementer er et problem i deformationsteori for algebraer. En fuldstændig forståelse af sådanne deformationer er langt uden for rækkevidde på nuværende tidspunkt. Bare det at definere skæringer i singulære rum er svært, for slet ikke at tale om at udregne sådanne. Den moderne tilgang er ofte via virtuel snitteori, hvor en såkaldt virtual fundamentalklasse vises at eksistere, hvormed snitteorien kan defineres. En sådan virtuel snitteori mangler dog geometrisk mening, idet den virtuelle fundamentalklasse ofte ikke kan tillægges nogen geometrisk fortolkning. Under alle omstændigheder er det håbløst at forsøge at anvende sådanne virtuelle teknikker for Hilbert skemaer af punkter på højdimensionelle rum.

Vi konstruerer i stedet en eksplicit opløsning af en birationel model for det kurvelineære Hilbert skema og anvender den nyudviklede ikke-reduktive geometriske invariantteori til at udregne snittal. Den primært anvendte metode er ækvivariant lokalisering med forskellige torusvirkninger, og specielt også en lokaliseringsformel i ikke-reduktiv geometrisk invariantteori. Ved hjælp af disse metoder kan specifikke snittal udregnes, og vi formulerer formodningen 12.1 som indikerer en sammenhæng mellem Hilbert skemaer af punkter på flader og tallene i Cayleys formel for graf-træer.

Det kurvelineære Hilbert skema er en irreducibel komponent i det punktlige Hilbert skema af punkter, der har idealer i en fast polynomiumsring som sine punkter. En vigtig ingrediens i lokaliseringsteknikkerne er de torus-fastholdte idealer; disse er netop monomiumsidealerne. Vi udnytter den eksplicitte opløsning af den birationelle model til at vise at alle monomiumsidealer er indeholdt i det kurvelineære Hilbert skema. Dette åbner op for en ny retning af videre studier, hvor der anvendes lignende teknikker: Til enhver monomiumssingularitet ie det punktlige Hilbert skema af punkter kan der associeres en birationel model som den, der er benyttet her, og ved at benytte stort set samme metoder, kan man i princippet opnå et komplet billede af hierarkiet blandt monomiumssingulariteter; det vil sige et komplet billede af deformationsteorien for algebraer dannet ved at kvotientere med disse monomiumsidealer.

Introduction

The main purpose of this thesis is to study the topology of Hilbert schemes of k points $\mathrm{Hilb}^k(X)$ on a smooth projective variety X of any dimension over an algebraically closed base field $\Bbbk$. The points of $\mathrm{Hilb}^k(X)$ consist of 0-dimensional closed subschemes of X of length k. In particular, we will study the punctual curvilinear Hilbert scheme $\mathrm{CHilb}^k_p(X)$ at $p \in X$, whose generic points consist of k distinct points coming together at p along a smooth curve in X. It is an irreducible component of the punctual Hilbert scheme $\mathrm{Hilb}^k_p(X) \subset \mathrm{Hilb}^k(X)$, the subset of subschemes with support in $p \in X$. The main goals are twofold: To show that all torus-fixed points of $\mathrm{Hilb}^k(\mathbb{C}^n)$ are in the curvilinear component $\mathrm{CHilb}^k_0(\mathbb{C}^n)$, and to develop an integration procedure on the curvilinear Hilbert scheme $\mathrm{CHilb}^k_0(\mathbb{C}^n)$.

The first goal on distribution of torus-fixed points is of independent importance and addresses the

> **Problem** ([53, Problem 1.7], [37]). What is the distribution of torus fixed points among the components of $\mathrm{Hilb}^k_0(\mathbb{C}^n)$?

This part appears already in the preprint [11].

The second goal is part of a larger integration theory on the Hilbert scheme of points initiated by the authors supervisor in [4], reducing tautological integration on the main component $\mathrm{GHilb}^k(X)$ – the closure in $\mathrm{Hilb}^k(X)$ of the set of k-tuples of k distinct points in X – to integration on the curvilinear component. The second goal is to describe integration on the curvilinear component $\mathrm{CHilb}^k_p(X)$ in $\mathrm{Hilb}^k_p(X)$.

In general the Hilbert scheme of k points $\mathrm{Hilb}^k(X)$ on a smooth projective variety X can be quite badly behaved being neither smooth nor irreducible, and contain irreducible components of large dimension (larger than the expected $k\dim(X)$) as first observed by Iarrobino [36]. Moreover, Jelisiejew [38] showed that a certain form of Vakil's Murphy's law [60] holds for the Hilbert scheme of points on $\mathbb{A}^n_{\mathbb{Z}}$ for $n \geq 16$ implying, in particular, non-reducedness of the Hilbert scheme of points on $\mathbb{A}^n_{\mathbb{Z}}$ for $n \geq 16$. The main property knowing to hold for all dimensions of X is the classical result of Hartshorne [33] on connectedness of the Hilbert scheme with fixed Hilbert polynomial (the Hilbert scheme of points being a special case).

In the 2-dimensional case with $X = S$ a surface, the behaviour of $\mathrm{Hilb}^k(S)$ is nice in the sense that $\mathrm{Hilb}^k(S)$ is both smooth and irreducible of dimension $2k$; it is in fact a resolution of singularities of the k'th symmetric product $\mathrm{Sym}^k S$ [24]. It is by far the most studied case in the literature, see e.g. [20, 21, 30, 51, 32] and the book [52].

The Hilbert scheme of points $\mathrm{Hilb}^k(X)$ contains the *punctual* Hilbert scheme $\mathrm{Hilb}^k_p(X)$ of k points on X (at p) as the subset formed by subschemes with support in the point $p \in X$.

The punctual Hilbert scheme of points $\mathrm{Hilb}^k_p(X)$ is, in general, neither smooth nor irreducible (again by Iarrobino [36]). For $X = \mathbb{C}^2$ Briancon [16] showed irreducibility of the punctual Hilbert scheme $\mathrm{Hilb}^k_p(\mathbb{C}^2)$.

We define the curvilinear locus (a quasi-projective subscheme) in $\mathrm{Hilb}^k_p(X)$

$$\mathrm{Curv}^k_p(X) = \{\xi \in \mathrm{Hilb}^k_p(X) \mid \mathcal{O}_\xi \simeq \Bbbk[t]/(t^k)\}.$$

The closure $\mathrm{CHilb}^k_p(X)$ of $\mathrm{Curv}^k_p(X)$ in $\mathrm{Hilb}^k_p(X)$ is an irreducible component of dimension $(k-1)(\dim(X)-1)$ of the punctual Hilbert scheme $\mathrm{Hilb}^k_p(X)$. It was only recently shown in [57] that there can exist components of $\mathrm{Hilb}^k_p(X)$ of smaller dimension than $(k-1)(\dim(X)-1)$.

Taking $X = \mathbb{A}^n_\Bbbk$ with an action of a maximal torus $T \subset \mathrm{GL}(\Bbbk^n)$, the T-fixed locus $\mathrm{Hilb}^k(\mathbb{A}^n_\Bbbk)^T \subset \mathrm{Hilb}^k_0(\mathbb{A}^n_\Bbbk)$ is contained in the punctual part at 0. Writing $x_1, \dots, x_n$ for the coordinates of $\mathbb{A}^n_\Bbbk$, there is a 1-to-1 correspondence between points of $\mathrm{Hilb}^k(\mathbb{A}^n_\Bbbk)$ and ideals I in $\mathcal{O}_{\mathbb{A}^n_\Bbbk} \simeq \Bbbk[x_1, \dots, x_n]$ with $\dim(\mathcal{O}_{X,0}/I) = k$, and under this correspondence the set of T-fixed points $\mathrm{Hilb}^k(\mathbb{A}^n_\Bbbk)^T$ correspond to the monomial ideals in $\Bbbk[x_1, \dots, x_n]$. Our main result in the direction of distribution of the T-fixed points is

Theorem 10.9. Let $m \in \mathrm{Hilb}^k_0(\mathbb{C}^n)$ be a monomial ideal. Then $m \in \mathrm{CHilb}^k_0(\mathbb{C}^n)$.

Since every ideal deforms to its initial ideal (with respect to some chosen term ordering), which is monomial, we obtain immediately the

Corollary. The punctual Hilbert scheme $\mathrm{Hilb}^k_0(\mathbb{C}^n)$ is connected for all positive integers n and k.

This is a trivial result in the case that $n \geq k$, since any algebra deforms to an algebra isomorphic to $\mathbb{C}[x_1, \dots, x_k]/(x_1 \cdots x_k)^2$.

By [29] $\mathrm{Hilb}^k(X)$ admits a universal family $\mathcal{Z}_k = \{(\xi, x) \mid \; x \in \xi\} \subset \mathrm{Hilb}^k(X) \times X$, and for a vector bundle $F \to X$, we define the tautological bundle $F^{[k]} = p_* q^*(F)$, where p and q are the projections in the diagram:

$$\begin{array}{ccc} \mathrm{Hilb}^k(X) \times X \supset \mathcal{Z} & \xrightarrow{\quad q \quad} & X \\ {\scriptstyle p}\big\downarrow & & \\ \mathrm{Hilb}^k(X) & & \end{array}$$

In [54] Rennemo defines so-called geometric subsets of $\mathrm{Hilb}^k(X)$ of which $\mathrm{CHilb}^k_p(X)$ is a very special example. Integrals of forms in the Chern classes of $F^{[k]}$ over such geometric subsets are referred to as *tautological integrals*. Rennemo shows that for fixed polynomial P in the Chern classes of $F^{[k]}$ and geometric subset Z, there exists a universal polynomial R depending only on P, the rank of F and Z such that $\int_{\overline{Z}} M = R(\{x_M\})$, where $\{x_M\}$ denotes the set of mixed Chern numbers of F and the tangent bundle TX on X. This result is a generalization of the surface case $\dim(X) = 2$ proven first in [20]. In [5] it is shown that these tautological integrals (on geometric subsets) can be reduced to integration on the curvilinear component $\mathrm{CHilb}^k_p(X)$.

The second goal of this work is to describe intersection theory on the curvilinear Hilbert scheme of points $\mathrm{CHilb}^k_p(X)$, more specifically we seek to obtain a procedure for calculating integrals on the curvilinear component $\mathrm{CHilb}_0^k(\mathbb{C}^n)$. We describe in Section 7 an algorithm to resolve the Berczi-Szenes model [13] constructed in Section 2 near torus-fixed points. The algorithm is a sequence of blow ups and to such sequence we associate a tree $\mathcal{T}$ with nodes corresponding to affine charts on the exceptional divisor, and edges decorated by the non-zero projective coordinate. We denote the end nodes (leaves) of such tree by $\mathcal{L}$. Using equivariant localization together with a non-reductive GIT description we obtain the general result in this direction

Theorem 8.5. Let z_0 denote a generic coordinate on the Lie algebra of $\lambda(\mathbb{C}^*) \subset \mathrm{Diff}_k$. Fix positive integers n and k, write $m = \min(n, k)$, and denote by $\mathcal{L}_{n,k}$ the leaves of the blow up tree $\mathcal{T}_{n,k}$. Let F be a vector bundle on $\mathbb{C}^n$ of rank r, and $c_1, \dots, c_{kr}$ the Chern classes of $F^{[k]}$. Give c_j the weight j, and let $\alpha \in H^*(\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n))$ a polynomial in the Chern classes of F of weighted degree $k(n-1)$ we have

$$\int_{\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)} \alpha = \operatorname*{Res}_{z_0=\infty} \operatorname*{Res}_{\boldsymbol{z}=\infty} \frac{\prod_{1\le i<j\le m}(z_i - z_j)(k-1)! z_0^{k-1}}{\prod_{j=1}^{m}\prod_{i=1}^{n}(\lambda_i - z_j)} \cdot \sum_{L\in\mathcal{L}_{n,k}} \frac{\alpha(\theta_1^L(\boldsymbol{z}), \dots, \theta_k^L(\boldsymbol{z}))}{c_{\mathrm{top}}(T_{0_L}L)(\boldsymbol{z})} d\boldsymbol{z} dz_0,$$

where $\alpha(\theta_1^L, \dots, \theta_k^L)$ means substituting in the polynomial expression of α the i'th elementary symmetric polynomial for c_i, and then evaluating in the Chern roots at 0_L, θ_i^L, of the pullback bundle $\phi_k[k]^*\mathcal{E}$ of the tautological bundle $\mathcal{E} \to \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i(\mathbb{C}^n))$.

Integration on $\mathrm{Hilb}^k(X)$ plays an important role in physics and many areas of mathematics, such as curve counting [31, 54] and other enumerative problems [44, 46], and the Segre-Verlinde correspondence [28]. Other applications include complexity theory (see e.g. [17, 39]) and global singularity theory [12, 13, 42, 23]. For X of dimension 2 this intersection theory has been long studied [21, 22, 30, 44, 51]. As reflected by the references the classical work has studied extensively the cohomology of the Hilbert scheme of points on surfaces leading to the famous Nakajima calculus and link to representation theory and physics. In general there are no explicit formulas for tautological integrals, however on surfaces there is a recursive method which in principle computes the universal polynomial R described above [20].

In the case of threefolds $\dim(X) = 3$ a symmetric obstruction theory on $X^{[n]}$ exists [2], implying existence of a virtual fundamental class. Also for Calabi-Yau 4-folds a virtual fundamental class exists as explained in [18]. In recent years these virtual techniques have gained much interest [28, 45, 46], however the requirement of existence of perfect obstruction theories makes it impossible to extend these techniques to Hilbert schemes of points on X of large dimension.

In this work we compute tautological integrals on the curvilinear component $\mathrm{CHilb}_p^k(X)$ of the punctual Hilbert scheme of points $\mathrm{Hilb}_p^k(X)$. Formulas for such tautological integrals appeared already in [3], but these formulas depend on a polynomial Q_{k-1}, which is unknown for $k > 7$. The approach in this project is to resolve instead a birational model for $\mathrm{CHilb}_p^k(X)$, and in this way obtain formulas not involving the polynomial Q_{k-1}. The procedure relies on the following three key ingredients

(1) The Berczi-Szenes model for curvilinear subschemes via test jets (Theorem 2.5).

(2) Realizing $\mathrm{CHilb}_0^k(\mathbb{C}^n)$ as a compactification of a non-reductive quotient by $\mathrm{Diff}_k(1)$, the polynomial reparameterization group of holomorphic maps $(\mathbb{C},0) \to (\mathbb{C}^n,0)$

(3) Equivariant localization

The Berczi-Szenes model described in Theorem 2.5 is a GL_n-equivariant rational map $\phi : J_k(1,n) \dashrightarrow \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$ constant on orbits of the action of the reparameterization group $\mathrm{Diff}_k(1) \subset J_k(1,1)$ and satisfying $\overline{\mathrm{Im}\,\phi} \simeq \mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$. Moreover, the induced map $J_k(1,n)/\mathrm{Diff}_k(1) \dashrightarrow \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$ is injective. We resolve the indeterminacies of ϕ by blowing up $J_k(1,n)$ equivariantly with respect to $\mathrm{Diff}_k(1)$ and the maximal torus $T^n \subset \mathrm{GL}_n$ resulting in a space $\mathbb{P}(\widehat{J_k(1,n)} \oplus \mathbb{C})$, and obtain a diagram

$$\begin{tikzcd}
\mathbb{P}(\widehat{J_k(1,n)} \oplus \mathbb{C})^s \arrow[d] \arrow[dr, "\tilde{\phi}"] & \\
\mathbb{P}(J_k(1,n) \oplus \mathbb{C}) \arrow[r, dashed, "\phi"] & \mathrm{CHilb}_0^k(\mathbb{C}^n) \subset \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)
\end{tikzcd}$$

where $\tilde{\phi}$ is $\mathrm{Diff}_k(1)$-invariant, T^n-equivariant and defined on the Diff_k-stable (here stability=semistability in the sense of (5.1)) part of $\mathbb{P}(\widehat{J_k(1,n)} \oplus \mathbb{C})$. Since the non-reductive GIT-quotient is categorical, we obtain an induced surjective map $\mathbb{P}(\widehat{J_k(1,n)} \oplus \mathbb{C}) /\!\!/ \mathrm{Diff}_k(1) \twoheadrightarrow \mathrm{CHilb}_0^k(\mathbb{C}^n)$.

The tautological integrals on $\mathrm{CHilb}_0^k(\mathbb{C}^n)$ are then calculated using localization techniques on the non-reductive GIT-quotient $\mathbb{P}(\widehat{J_k(1,n)} \oplus \mathbb{C}) /\!\!/ \mathrm{Diff}_k(1)$. In order to perform the equivariant localization, we need information on the tangent bundle of $\mathbb{P}(\widehat{J_k(1,n)} \oplus \mathbb{C})^s$, and as such it is necessary to have a description of the resolution of the indeterminacy locus of ϕ; we calculate the blow up $\mathbb{P}(\widehat{J_k(1,n)} \oplus \mathbb{C}) \to J_k(1,n)$ explicitly, and *all blow up centers will locally be ideals generated only by coordinates.*

The main historical motivations for studying tautological integrals is their application to enumerative geometry. In [31] Göttsche showed that certain tautological integrals correspond to counting nodal curves on smooth surfaces. This result was generalized by Rennemo in [54] to express counts of hypersurfaces with any given singularity type on a smooth projective variety of any dimension.

Also the famous conjecture of Lehn [44] on integration of top Segre classes in the case of $X = S$ a surface has attracted much attention. It was proven for K3 surfaces in [46] and extended to all surfaces in [48] using methods of [61]. Lehn's conjecture (for line bundles) was generalized to higher rank bundles in [47] in which some results are obtained. Related to this work is the Segre-Verlinde correspondence conjectured also in [47] based on [40], and further generalized and analyzed in [28].

Another application is global singularity theory for holomorphic maps. In short, a singularity of a holomorphic map $f : M \to N$ between complex manifolds correspond to a nilpotent algebra (the local algebra of f at the singularity). The locus Σ_A of singularities of a given singularity type (nilpotent algebra) A is for N compact and f sufficiently generic an analytic submanifold. Thom's work in [59] showed that the the equivariant poincare dual $[\Sigma_A] \in H^*(M,\mathbb{Z})$ is a polynomial in the "Chern classes of the bundle" $TM - f^*TN$, nowadays denoted $Tp_A^{m,n}$ and called the Thom polynomial of the singularity A. A fundamental problem in enumerative geometry is to calculate these polynomials, which is indeed a difficult task (see e.g. [42, 55]). The Thom polynomials of Morin singularities (the case $A = \mathbb{C}[t]/t^k$) can be expressed as tautological integrals on $\mathrm{CHilb}_p^k(M)$ as described and analyzed in [13].

A generalization of these ideas is the global singularity theory of multipoint loci [41, 56] for which not many explicit general formulas are known, but certain ones are computed in [12] by using a sieve method to reduce the computation to integration on $\mathrm{CHilb}_p^k(M)$.

At last, the idea of using such sieve-type argument together with residue vanishing has been used in [4] to reduce tautological integration on the main component of $\mathrm{Hilb}^k(X)$

$$\mathrm{GHilb}^k(X) = \overline{\{\xi \in \mathrm{Hilb}^k(X) : |\operatorname{Supp}\xi| = k\}},$$

where a generic point is a scheme $\xi \in \mathrm{Hilb}^k(X)$ with support at k distinct points of X, completely to tautological integration on the punctual curvilinear component $\mathrm{CHilb}_p^k(X) \simeq \mathrm{CHilb}_0^k(\mathbb{C}^n)$. More generally, for any geometric subset $Z \subset \mathrm{Hilb}^k(X)$ as defined in [54], tautological integration on the closure $\overline{Z} \subset \mathrm{Hilb}^k(X)$ can be reduced also to tautological integration only on the punctual curvilinear component $\mathrm{CHilb}_0^k(\mathbb{C}^n)$ [5].

Chapter 1

The Hilbert scheme of points and the integration process – an overview

In this section we provide an overview of how to integrate on the curvilinear Hilbert scheme $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^k)$. We emphasize that integration formulas on $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^k)$ will imply integration formulas on $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$ for all values of n. The details are left to other sections describing the different aspects of the birational model of $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^k)$, and the process of (partially) resolving the indeterminacy locus of this model. In particular all results written here are stated in Section 8. Before describing the process of integration, we set the scene for the Hilbert scheme of points.

1.1 The Hilbert scheme of points

We describe here briefly the punctual curvilinear Hilbert scheme $\mathrm{CHilb}_0^k(X)$ a subscheme in the Hilbert scheme of points $\mathrm{Hilb}^k(X)$ on a nonsingular projective scheme X over some base field $\Bbbk$.

We define the Hilbert scheme of k points on X

$$\mathrm{Hilb}^k(X) = \{\xi \subset X \text{ closed subscheme} \mid \dim \xi = 0, \text{length}\xi = k\}$$

as the closed 0-dimensional subschemes in X of length k. In this scheme we define the punctual Hilbert scheme of k points supported at $p \in X$

$$\mathrm{Hilb}_p^k(X) = \{\xi \in \mathrm{Hilb}^k(X) \mid \mathrm{Supp}\, \xi = \{p\}\}.$$

We define the the set of curvilinear subschemes

$$\mathrm{Curv}_p^k(X) = \{\xi \in \mathrm{Hilb}_p^k(X) \mid \mathcal{O}_\xi \simeq \Bbbk[\epsilon]/(\epsilon^k)\}$$

and its closure $\mathrm{CHilb}_p^k(X)$ in $\mathrm{Hilb}_p^k(X)$. The elements of $\mathrm{CHilb}_p^k(X)$ are also refered to with the adjective *allignable* in the litterature.

Lemma 1.1. The closure $\mathrm{CHilb}_p^k(X)$ of $\mathrm{Curv}_p^k(X)$ is an irreducible component of $\mathrm{Hilb}_p^k(X)$ of dimension

$$\dim \mathrm{CHilb}_p^k(X) = (\dim X - 1)(k - 1)$$

Proof. The first part follows since

$$\xi \in \mathrm{Curv}_p^k(X) \iff \mathcal{O}_\xi \text{ contains an element of degree } k-1,$$

and the second part will follow from the Berczi-Szenes model in Theorem 2.4. □

Observe that choosing coordinates near the point $p \in X$, we have identification $\mathrm{Hilb}_p^k(X) \simeq \mathrm{Hilb}_0^k(\Bbbk^n)$ for $n = \dim X$, and in particular

$$\mathrm{CHilb}_p^k(X) \simeq \mathrm{CHilb}_0^k(\Bbbk^n).$$

Moreover, write $\Bbbk^n = \operatorname{Spec} \Bbbk[x_1, \dots, x_n]$ then an ideal I in $\Bbbk[x_1, \dots, x_n]$ has codimension $\operatorname{codim} I = k$ if $\dim \Bbbk[x_1, \dots, x_n]/I = k$ as a vector space over $\Bbbk$.

$$\begin{aligned} \mathrm{Hilb}_0^k(\Bbbk^n) &\simeq \{\Bbbk[x_1, \dots, x_n]/I \mid I \text{ is an ideal in } \Bbbk[x_1, \dots, x_n] \text{ with } \operatorname{codim} I = k\} \\ &\simeq \{I \mid I \text{ is an ideal in } \Bbbk[x_1, \dots, x_n] \text{ with } \operatorname{codim} I = k\} \end{aligned}$$

We will use these two interpretations of $\mathrm{Hilb}_0^k(\Bbbk^n)$ interchangibly, speaking of an elements as either an ideal I in $\Bbbk[x_1, \dots, x_n]$ or as an (quotient) algebra $\Bbbk[x_1, \dots, x_n]/I$.

1.2 An overview of the integration process on $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^k)$

The model of Berczi and Szenes provides a birational model for the curvilinear Hilbert scheme $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$ and is described in section 2. This model can be expressed as the following rational map

$$\begin{aligned} \phi : J_k(1,n) &\dashrightarrow \mathrm{Grass}_k(J_k(n,1)^*) = \mathrm{Grass}_k\Big(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n\Big) \\ \gamma = (0 \neq \nu_1, \nu_2, \dots \nu_k) &\longmapsto \mathrm{Span}_{\mathbb{C}}(\nu_1, \nu_2 + \nu_1^2, \dots, \sum_{\sigma \in \mathcal{P}_k} |\mathrm{perm}(\sigma)|\, \nu_\sigma), \end{aligned}$$

where the sum is over all partitions $\sigma = \sigma_1 + \cdots + \sigma_r$ of k, $|\mathrm{perm}(\sigma)|$ is the number of compositions representing the partition σ, and $\nu_\sigma = \nu_{\sigma_1} \cdots \nu_{\sigma_r}$ is the product of the normed derivatives $\nu_i = \gamma^{(i)}(0)/i!$. The map ϕ has several important properties

(1) ϕ is well-defined on $J_k^{\mathrm{reg}}(1,n) := \{[f] \in J_k(1,n) \mid f'(0) \neq 0\}$.

(2) ϕ is invariant with respect to the action of the reparameterization group $\mathrm{Diff}_k = J_k^{\mathrm{reg}}(1,1)$.

(3) ϕ is injective on the Diff_k-orbits in $J_k^{\mathrm{reg}}(1,n)$.

(4) ϕ is GL_n-equivariant with respect to the naturally induced actions on $J_k^{\mathrm{reg}}(1,n)$ and $\mathrm{Grass}_k(J_k(n,1)^*)$

(5) $\mathrm{Im}\,\phi \simeq \mathrm{Curv}_0^{k+1}(\mathbb{C}^n)$, and moreover $\overline{\mathrm{Im}\,\phi} \simeq \mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$ embeds GL_n-equivariantly in $\mathrm{Grass}_k(J_k(n,1)^*)$

The reparameterization group Diff_k is a non-reductive group, and we shall thus apply the theory of non-reductive geometric invariant theory (GIT) to obtain a quotient on the source space. Throughout this work we consider mainly the case $k = n$, since integration on $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$ for a general n can be reduced to integration on $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^k)$.

The general procedure for computing integrals on $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^k)$ will be to construct a (more or less) explicit resolution of the indeterminacy locus of the model ϕ composed with the Plücker embedding $\mu : \mathrm{Grass}_k(J_k(n,1)^*) \to \mathbb{P}(\bigwedge^k J_k(n,1)^*)$ obtaining a map well-defined near all T-fixed points of the source space for the maximal torus $T \subset \mathrm{GL}_n$, and then apply localization techniques. Having obtained such partial resolution of ϕ, we take a GL_n-equivariant and Diff_k-invariant map full resolution $\widehat{J_k(1,n)} \to \mathbb{P}(\bigwedge^k J_k(n,1)^*)$ with some blow up of $J_k(1,n)$ as its source space. Since this blow up is GL_n-equivariant the localization contributions of T-fixed points in $\widehat{J_k(1,n)}$ can be described via the explicitly given partial resolution of ϕ.

We will make sure that this source space admits a certain stability condition such that the non-reductive GIT quotient $\widehat{J_k(1,n)}/\!\!/\mathrm{Diff}_k$ is defined – as in classical GIT it is a categorical quotient, and a compactification of the geometric quotient on the semistable (which will coincide with the stable locus, cf. (5.1)) locus $\widehat{J_k(1,n)}^s/\mathrm{Diff}_k$. It follows that it is enough to resolve the map ϕ on the semistable locus and thus obtain a Diff_k-invariant map

$$\varphi : \widehat{J_k(1,n)}^s \to \mathbb{P}\Big(\bigwedge^k J_k(n,1)^*\Big),$$

and which further induces the map

$$\tilde{\varphi} : \widehat{J_k(1,n)}/\!\!/\mathrm{Diff}_k \to \mathbb{P}\Big(\bigwedge^k J_k(n,1)^*\Big)$$

on the categorical non-reductive GIT quotient, which maps surjectively onto its image to $\mathrm{Im}\,\tilde{\varphi} = \mathrm{CHilb}_0^{k+1}(\mathbb{C}^k)$.

We proceed to describe the resolution of ϕ. It consist of an initial partial resolution, making the pullback of ϕ well-defined near T-fixed points of the source space $\tilde{\mathbb{A}}[k]$ (constructed in Section 7), and a second theoretical non-explicit resolution which has no effect on the equivariant localization. We describe now in general terms the partial resolution process of the indeterminacy locus of $\mu \circ \phi$.

We will first construct a fibered version of the model ϕ. We will drop the Plücker embedding μ from the notation.

$$\phi : J_k(1,n) \dashrightarrow \mathbb{P}\Big(\bigwedge^k J_k(n,1)^*\Big).$$

We assume now $n \geq k$ (although this is not necessary, see Section 2.3) containing the case described here $n = k$. We write

$$E = [\mathrm{Span}(e_1) \subset \mathrm{Span}(e_1, e_2) \subset \cdots \subset \mathrm{Span}(e_1, \ldots, e_k) = \mathbb{C}^k] \in \mathrm{Flag}_k(\mathbb{C}^n)$$

and $P_{k,n} \subset \mathrm{GL}_n$ for the parabolic subgroup preserving E. We define the space

$$\begin{aligned} \widetilde{J_k(1,n)} := \mathrm{GL}_n \times_{P_{k,n}} \overline{P_{k,n}.E} \to \overline{\mathrm{GL}_n .E} \to J_k(1,n) \\ (g, [M_1 \subset \cdots \subset M_k]) \mapsto [g.M_1 \subset \cdots \subset g.M_k] \mapsto g.M_k \end{aligned}$$

which has the fibration

$$\pi_{k,n} : \widetilde{J_k(1,n)} \to \mathrm{GL}_n / P_{k,n} = \mathrm{Flag}_k(\mathbb{C}^n).$$

We write $\widetilde{J_k(1,n)}_E := \pi_{k,n}^{-1}(E)$ for the fiber over $E \in \mathrm{Flag}_k(\mathbb{C}^n)$ on which we have for all $1 \leq i \leq k$

$$\nu_i \in \mathrm{Span}(e_1, \ldots, e_i).$$

For localization applications it is enough to consider only the contribution on the fiber $\widetilde{J_k(1,n)}_E$ (as we will see below the diagram). By abuse of notation, we write also ϕ for the induced map $\phi : \widetilde{J_k(1,n)} \to \mathbb{P}(\bigwedge^k J_k(n,1)^*)$. The picture to obtain is then

$$\begin{tikzcd}
\mathrm{Bl}(\widetilde{J_k(1,n)}_E) \arrow[d] & \mathrm{Bl}(\widetilde{J_k(1,n)}_E)^s \arrow[l, hook'] \arrow[dr, "\varphi_E"] & \\
\widetilde{J_k(1,n)}_E \arrow[rr, dashed, "\phi_E"] \arrow[d, hook] & & \mathbb{P}(\bigwedge^k J_k(n,1)^*) \\
\widetilde{J_k(1,n)} \arrow[urr, dashed, "\phi"] & &
\end{tikzcd}$$

Write $\lambda_1, \ldots, \lambda_n$ for the weights of the diagonal action of $T \subset \mathrm{GL}_n$ on $\mathbb{C}^n$. Applying the localization formula in equivariant cohomology of Atiyah-Bott [1] and Berline-Vergne [14], we obtain already, the integration formula

$$\int_{\widetilde{J_k(1,n)}} \alpha = \sum_{\sigma \in \mathcal{S}_n/\mathcal{S}_k} \frac{\alpha_{\sigma.E}}{\prod_{m=1}^k \prod_{i=m+1}^n (\lambda_{\sigma.i} - \lambda_{\sigma.m})}, \tag{1.1}$$

where $\mathcal{S}_n/\mathcal{S}_k$ denotes the set of injective mappings $\{1, \ldots, k\} \hookrightarrow \{1, \ldots, n\}$, and

$$\alpha_{\sigma.E} = \Big(\int_{\widetilde{J_k(1,n)}_{\sigma.E}} \alpha \Big)^{[0]} (\sigma.E) \in \mathrm{Sym}^\bullet \mathfrak{t}^*$$

and here $\sigma.E$ is the corresponding flag, and moreover we have

$$\alpha_{\sigma.E} = \sigma.\alpha_E = \alpha_E(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}).$$

Thus, for integration purposes it is enough to consider and resolve the map $\phi_E : \widetilde{J_k(1,n)}_E \dashrightarrow \mathbb{P}(\bigwedge^k J_k(n,1)^*)$.

For fixed $n \geq k$ we define the vanishing ideal $I_{n,k}$ of the map ϕ together with the monomial ideal $M_{n,k}$ generated by the monomials of the generators of $I_{n,k}$. We perform a sequence of blow ups of $\widetilde{J_k(1,n)}_E$ in GL_n- and Diff_k-invariant centers – these will be ideals generated by coordinates – until the pullback ideal sheaf of $M_{n,k}$ associated to the final blow up space is a prime ideal on each chart. The sheaf of ideals of these principal ideals correspond to the space $\tilde{\mathbb{A}}[k]$ appearing in Theorem 7.1, and we obtain a partial resolution $\phi[k] : \tilde{\mathbb{A}}[k] \dashrightarrow \mathbb{P}(\bigwedge^k J_k(n,1)^*)$ well-defined near all T-fixed points (here we assume that T-fixed points are isolated; this is a technicality. See Section 6.6). One takes at last a GL_n- and Diff_k-equivariant resolution $\varphi_E : \widehat{J_k(1,n)}_E \to \mathbb{P}(\bigwedge^k J_k(n,1)^*)$ which is everywhere defined (this exists by [34]). This construction glues over $\mathrm{Flag}_k(\mathbb{C}^n)$ to obtain $\varphi : \widehat{J_k(1,n)} \to \mathbb{P}(\bigwedge^k J_k(n,1)^*)$.

The fact that we always blow up in (GL_n- and) Diff_k-invariant centers implies that φ is also (GL_n-equivariant and) Diff_k-invariant, and so φ induces a map on the categorical quotient $\tilde{\varphi} : \widehat{J_k(1,n)} /\!\!/ \mathrm{Diff}_k \dashrightarrow \mathbb{P}(\bigwedge^k J_k(n,1)^*)$ — To be precise, we must replace the jet space $\widetilde{J_k(1,n)}_E$ with its projective completion $\mathbb{P}(\widetilde{J_k(1,n)}_E \oplus \mathbb{C})$, but we leave this out for now for simplicity of the exposition in this overview (see Section 6.4).

Since the non-reductive quotient is a quotient of the stable part only, we need only consider the stable locus $\widehat{J_k(1,n)}^s$, and this implies that we need only consider charts of minimal Diff_k-weight throughout the sequence of blow ups.

We apply the theory of non-reductive GIT in a particularly nice case, where the non-reductive group is of the form $\mathrm{Diff}_k = U \rtimes \mathbb{C}^*$(see (2.5) and Section 6.4). We write z_0 for the standard coordinate of the Lie algebra of $\mathbb{C}^*$, $\mathfrak{u}$ for the Lie algebra of U, i for the inclusion of the minimal weight space $\widehat{J_k(1,n)}_{\min}$ of the $\mathbb{C}^*$-action in $\widehat{J_k(1,n)}$, and $N_{\min}$ for its corresponding normal bundle. We apply first a localization theorem of Berczi and Kirwan for non-reductive quotients (cf. Theorem 5.5) to obtain a formula

$$\int_{\widehat{J_k(1,n)}/\!/\mathrm{Diff}_k} \kappa(\alpha) = \operatorname*{Res}_{z_0=\infty} \int_{(\widehat{J_k(1,n)})_{\min}} \frac{i^*\big(\alpha(z_0) \cup c_{\mathrm{top}}(V_{\mathfrak{u}})(z_0)\big)}{c_{\mathrm{top}}(N_{\min})(z_0)} dz_0, \tag{1.2}$$

where $\kappa : H^*_{\mathrm{Diff}_k}(\widehat{J_k(1,n)}) \to H^*(\widehat{J_k(1,n)} /\!/ \mathrm{Diff}_k)$ is surjective (cf. Section 5.2), and $V_{\mathfrak{u}}$ is an associated bundle isomorphic to the conormal bundle of the inclusion $\widehat{J_k(1,n)} /\!/ \mathrm{Diff}_k \hookrightarrow \widehat{J_k(1,n)} /\!/ \mathbb{C}^*$. Moreover, the notation c_{top} of a bundle, means taking the equivariant top Chern class, equal to the product of the weights. For fixed k we have

$$c_{\mathrm{top}}(V_{\mathfrak{u}})(z_0) = z_0(2z_0) \cdots (k-1)z_0 = (k-1)! z_0^{k-1}.$$

The denominator $c_{\mathrm{top}}(N_{\min})(z_0)$ is the most difficult ingredient to calculate since it requires explicit knowledge of the resolution of ϕ.

We proceed by applying the Atiyah-Bott, Berline-Vergne localization formula in equivariant cohomology for the action of the standard maximal torus $T \subset \mathrm{GL}_n$ to obtain a localization formula on stable part of the blow up $\widehat{J_k(1,n)}^s$. Recall that $\lambda_1, \ldots, \lambda_n$ are the weights of the diagonal action of T on $\mathbb{C}^n$. Combining the localizations (1.1) and (1.2), the T-fixed points of $\widehat{J_k(1,n)}_{\min}$ can be assumed isolated (cf. Section 6.6), and are then exactly the 0's of the affine charts of this minimal weight space.

We will encode the sequence of blow ups of the fiber$\widetilde{J_k(1,n)}_E$ in a tree $\mathcal{T}_E$ with each node corresponding to a chart on the exceptional divisor, and thus the charts on the source space $\tilde{\mathbb{A}}[k]$ of the partial resolution $\phi[k] : \tilde{\mathbb{A}}[k] \dashrightarrow \mathbb{P}(\bigwedge^k J_k(n,1)^*)$ correspond exactly to the leaves (final nodes) of the $\mathcal{T}_E$; denote by $\mathcal{L}_E$ the set of leaves, and for $L \in \mathcal{L}_E$ the corresponding unique T-fixed point by 0_L with inclusion $i_L : \{0_L\} \hookrightarrow \tilde{\mathbb{A}}[k]$. The localization formula takes the form

$$\begin{aligned} \int_{\widehat{J_k(1,n)}/\!/\mathrm{Diff}_k} \kappa(\alpha) = & \sum_{\sigma \in \mathcal{S}_n/\mathcal{S}_k} \frac{1}{\prod_{m=1}^k \prod_{i=m+1}^n (\lambda_{\sigma.i} - \lambda_{\sigma.m})} \\ & \cdot \operatorname*{Res}_{z_0=\infty} (k-1)! z_0^{k-1} \sum_{L \in \mathcal{L}_E} \frac{(i_L)_* i_L^* p^* \alpha(z_0, \lambda_1, \ldots, \lambda_n)}{c_{\mathrm{top}}(T_{0_L} L)} dz_0, \end{aligned}$$

where $T_{0_L} L$ denotes the tangent space at the origin 0_L in the affine chart L of $\tilde{\mathbb{A}}[k]$, and $\pi : \tilde{\mathbb{A}}[k] \to \widetilde{J_k(1,n)}_E$ is the sequence of blow ups used to partially resolve the model ϕ. In this case $c_{\mathrm{top}}(T_{0_L} L)$ denotes simply the product of the weights of the variables of the chart L in $\tilde{\mathbb{A}}[k]$.

At last, consider a T-equivariant form α on $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$. To perform the integration of α, we simply pullback along the induced proper map $\tilde{\varphi} : \widehat{J_k(1,n)} /\!/ \mathrm{Diff}_k \to \mathbb{P}(\bigwedge^k J_k(n,1)^*)$ to obtain a form $\tilde{\varphi}^*(\alpha) \in H^*(\widehat{J_k(1,n)} /\!/ \mathrm{Diff}_k)$. Since as already stated

$\kappa : H^*_{\mathrm{Diff}_k}(\widehat{J_k(1,n)}) \to H^*(\widehat{J_k(1,n)} /\!\!/ \mathrm{Diff}_k)$ is surjective (cf. Section 5.2), the form $\tilde{\varphi}^*(\alpha)$ extends to a Diff_k-equivariant form.

Write $\theta_1, \dots, \theta_k$ for the Chern roots of the tautological bundle on $\mathrm{Grass}_k(J_k(n,1)^*)$. By $\theta_1^L, \dots, \theta_k^L$ we denote the specialization of these weights at the image point $\varphi(0_L) \in \mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$ of $0_L \in L \subset \tilde{\mathbb{A}}[k]$. Each θ_i^L is thus a linear form in $\lambda_1, \dots, \lambda_n$. We obtain the integration formula (this is the result of Theorem 8.1 for $n \geq k$)

$$\begin{aligned}\int_{\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)} \alpha = &\sum_{\sigma \in \mathcal{S}_n/\mathcal{S}_k} \frac{1}{\prod_{i=1}^n \prod_{j=i+1}^k (\lambda_\sigma(i) - \lambda_\sigma(j))} \\ &\cdot \sum_{L \in \mathcal{L}_E} \operatorname*{Res}_{z_0=\infty} (k-1)! z_0^{k-1} \frac{\alpha(\theta_1^L(\sigma.\boldsymbol{\lambda}), \dots, \theta_k^L(\sigma.\boldsymbol{\lambda}))}{c_{\mathrm{top}}(T_{0_L} L)(z_0, \sigma.\boldsymbol{\lambda})} dz_0.\end{aligned} \tag{1.3}$$

This integration formula in principle ends the overview of the process of integration on $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$. However, rewriting the sum over the flag $\mathrm{Flag}_k(\mathbb{C}^n)$ as an iterated residue (see Proposition 8.3, originally [13, Proposition 5.4]), we obtain the formula

$$\begin{aligned}\int_{\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)} \alpha = &\sum_{L \in \mathcal{L}_E} \operatorname*{Res}_{\boldsymbol{z}=\infty} \operatorname*{Res}_{z_0=\infty} (k-1)! z_0^{k-1} \\ &\cdot \frac{\prod_{1 \leq i < j \leq k} (z_i - z_j)}{\prod_{i=1}^n \prod_{j=1}^k (\lambda_i - z_j)} \frac{(i_L)_* \alpha(\theta_1^L(\boldsymbol{z}), \dots, \theta_k^L(\boldsymbol{z}))}{c_{\mathrm{top}}(T_{0_L} L)(\boldsymbol{z})} dz_0 d\boldsymbol{z}.\end{aligned}$$

From this expression, we argue that only one T-fixed point in the image $\varphi(\widetilde{J_k(1,n)_E})$ contributes to the integration formula (see Theorem 8.6 and 8.7) : Only the isomorphism class of the algebra $\mathbb{C}[x_1, \dots, x_n]/(x_1, \dots, x_n)^2$ contributes to localization. This argument depends on the fact that $n \geq k$. The most simplified result in this direction is

Theorem 8.7. If $n \geq k$, then the integration formula of Theorem 8.5 reduces to

$$\int_{\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)} \alpha = \operatorname*{Res}_{z_0=\infty} \operatorname*{Res}_{\boldsymbol{z}=\infty} \frac{\prod_{1 \leq i < j \leq k} (z_i - z_j) \cdot \alpha(z_1, \dots, z_k)}{\prod_{j=1}^k \prod_{i=1}^n (\lambda_i - z_j) \cdot \omega_{1,1}^{\mathrm{Port}} \cdot \prod_{2 \leq i \leq j \leq k} \omega_{i,j}^{\mathrm{Port}}} d\boldsymbol{z} dz_0.$$

In the case $n < k$, we are one the one hand able to obtain an integration formula (see Theorem 8.9) from equation (1.3). On the other hand, the above construction using a fibration over $\mathrm{Flag}(1, \dots, k; \mathbb{C}^n)$ when $n \geq k$ can be adjusted to the case $n < k$ obtaining a fibration over $\mathrm{Flag}(1, \dots, n; \mathbb{C}^n)$, and one arrives in the same way at an integration formula like that of (1.3) (see Theorem 8.1). In this case, some cancellation symmetry is lost, and it is *not* the case that only a single fixed point contributes.

Chapter 2

The Berczi-Szenes model

In this section we describe the testcurve model of Berczi and Szenes [13], an explicit parameterization of the natural embedding

$$\begin{aligned} \mathrm{Hilb}_0^{k+1}(\mathbb{C}^n) &\hookrightarrow \mathrm{Grass}_k\Big(\bigoplus_{i=1}^{k}\mathrm{Sym}^i(\mathbb{C}^n)\Big) \\ I &\longmapsto m_{\mathcal{O}_{\mathbb{C}^n}}/I \subset m_{\mathcal{O}_{\mathbb{C}^n}}/m_{\mathcal{O}_{\mathbb{C}^n}}^{k+1} \simeq \bigoplus_{i=1}^{k}\mathrm{Sym}^i(\mathbb{C}^n) \end{aligned} \tag{2.1}$$

restricted to the curvilinear Hilbert scheme $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n) \subset \mathrm{Hilb}_0^{k+1}(\mathbb{C}^n)$. Here if we write $\mathcal{O}_{\mathbb{C}^n} = \mathbb{C}[x_1,\dots,x_n]$ for the structure sheaf of $\mathbb{C}^n$, we denote by $m_{\mathcal{O}_{\mathbb{C}^n}} = (x_1,\dots,x_n)$ the maximal ideal. Recall that $\mathrm{CHilb}_0^k(\mathbb{C}^n)$ is the compactification of the open locus $\mathrm{Curv}_0^k = \{\xi \in \mathrm{Hilb}_0^k(\mathbb{C}^n) \mid \mathcal{O}_\xi \simeq \mathbb{C}[\varepsilon]/(\varepsilon^k)\}$ in the punctual Hilbert scheme $\mathrm{Hilb}_0^k(\mathbb{C}^n)$.

2.1 Jet spaces of holomorphic maps

We start by introducing jet spaces, which are vector spaces, that in special cases are also groups. These are crucial preliminaries for the construction. Let $u, v > 0$ be positive integers and denote by $J_k(u,v)$ the vector space of equivalence classes of maps $f : (\mathbb{C}^u, 0) \to (\mathbb{C}^v, 0)$, where $f \sim g$ if and only if all derivatives of order $\leq k$ of f and g at 0 agree. These are called *k-jets of holomorphic maps at the origin.* We note that $J_k(u,v) = J_k(u,1) \otimes \mathbb{C}^v$ and thus $\dim J_k(u,v) = v\big(\binom{u+k}{k} - 1\big)$.

We can define a structure on these sets in the sense that, we may compose k-jets and take the result modulo terms of degree $\geq k+1$ (the square brackets indicate that we take the result modulo these terms)

$$J_k(u,v) \times J_k(v,w) \to J_k(u,w), \qquad (\Psi_1, \Psi_2) \mapsto [\Psi_2 \circ \Psi_1] \in J_k(u,w). \tag{2.2}$$

Eliminating terms of degree k yields a morphism of $\mathbb{C}$-algebras $J_k(u,1) \twoheadrightarrow J_{k-1}(u,1)$, yielding further a chain of such $J_k(u,1) \twoheadrightarrow \cdots \twoheadrightarrow J_1(u,1)$, which in turn induces a filtration of the dual $J_k(u,1)^*$

$$J_1(u,1)^* \subset \cdots \subset J_k(u,1)^*.$$

The elements of $J_k(u,1)^*$ are interpreted as differential operators on $\mathbb{C}^u$ of degree $\leq k$

$$J_k(u,1)^* \simeq \bigoplus_{l=1}^{k}\mathrm{Sym}^l\,\mathbb{C}^u, \tag{2.3}$$

where Sym^l is the l'th symmetric tensor power and the isomorphism is as filtered $\mathrm{GL}(u)$-modules.

Choosing coordinates on $\mathbb{C}^u$ and $\mathbb{C}^v$, we can identify a k-jet $[f] \in J_k(u,v)$ with its derivatives of order $\leq k$ at the origin, i.e. with the vector $[f] = (f'(0)/1!, \dots, f^{(k)}(0)/k!)$. We have the correspondence

$$J_k(u,v) \simeq J_k(u,1) \otimes \mathbb{C}^v.$$

We define the set of *regular* k-jets to be

$$J_k^{\mathrm{reg}}(u,v) = \{[f] \in J_k(u,v) \mid f'(0) \text{ has maximal rank}\}.$$

When $u=1$, we speak of k-jets of curves and the set of *regular* k-jets of curves is then

$$J_k^{\mathrm{reg}}(1,n) = \{[\gamma] \in J_k(1,n) \mid \gamma'(0) \neq 0\}.$$

We note that the composition (2.2) induces a natural group structure on the set

$$\mathrm{Diff}_k(u) := J_k^{\mathrm{reg}}(u,u)$$

of regular k-jets $(\mathbb{C}^u,0) \to (\mathbb{C}^u,0)$. We call this group the *diffeomorphism group of k-jets*. We introduce the even shorter notation $\mathrm{Diff}_k := \mathrm{Diff}_k(1)$.

We end by discussing the composition described in (2.2) for Diff_k acting on regular k-jets of curves, since this case is of special interest to us. Let

$$f_\xi(z) = f'(0)z + \cdots + \frac{f^{(k)}(z)}{k!} z^k \in J_k^{\mathrm{reg}}(1,n)$$

be the k-jet of a germ parameterizing a smooth germ $\mathcal{C}_0 \supset \xi$, and let

$$\phi(z) = \alpha_1 z + \cdots + \alpha_k z^k \in \mathrm{Diff}_k.$$

The composition described in (2.2) yields then

$$\begin{aligned} f_\xi \circ \phi(z) = \\ & \big(f'(0)\alpha_1\big)z + \big(f'(0)\alpha_2 + f''(0)\alpha_1^2/2\big)z^2 + \cdots + \Big(\sum_{i_1+\cdots+i_l=k} \frac{f^{(l)}(0)}{l!}\alpha_{i_1}\cdots\alpha_{i_l}\Big)z^k \\ & =(f'(0),\dots,f^{(l)}(0)/l!) \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \cdots & \sum\limits_{i_1+i_2=k} \alpha_{i_1}\alpha_{i_2} \\ 0 & 0 & \alpha_1^3 & \cdots & \sum\limits_{i_1+i_2+i_3=k} \alpha_{i_1}\alpha_{i_2}\alpha_{i_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_1^k \end{pmatrix} \begin{pmatrix} z \\ z^2 \\ \vdots \\ z^k \end{pmatrix}. \end{aligned} \tag{2.4}$$

Thus we obtain an explicit linearization of the Diff_k-action on $J_k^{\mathrm{reg}}(1,n)$

$$\begin{aligned} &\mathrm{Diff}_k \hookrightarrow \mathrm{GL}_k \\ &(\alpha_1,\dots,\alpha_k) \longmapsto \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \cdots & \sum\limits_{i_1+i_2=k} \alpha_{i_1}\alpha_{i_2} \\ 0 & 0 & \alpha_1^3 & \cdots & \sum\limits_{i_1+i_2+i_3=k} \alpha_{i_1}\alpha_{i_2}\alpha_{i_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_1^k \end{pmatrix}, \end{aligned} \tag{2.5}$$

where $\alpha_1 \neq 0$ and the (i,j)'th entrance is the sum of degree i monomials in $\alpha_1, \dots, \alpha_k$ with exponent vectors given by the partitions of k of length j. This illustrates directly the semi-direct product $\mathrm{Diff}_k = U \rtimes \mathbb{C}^*$ with U the unipotent group obtained by setting $\alpha_1 = 1$, and thus Diff_k *is not reductive.*

Observe also that naturally we have embeddings $\mathrm{Diff}_{k-1} \hookrightarrow \mathrm{Diff}_k$, which in the representation (2.5) manifests itself in the fact that Diff_{k-1} corresponds to the upper left $(k-1)$-by-$(k-1)$ matrix.

2.2 Test curves for curvilinear subschemes

We give the necessary definitions and results in order to describe the model ϕ for $\mathrm{CHilb}_0^k(\mathbb{C}^n)$ of Berczi and Szenes. For more details one might consult [13].

Let $\xi \in \mathrm{Curv}_0^{k+1}(\mathbb{C}^n)$ be a curvilinear subscheme supported at the origin. By definition

$$\xi \subset \mathcal{C}_0 \subset \mathbb{C}^n$$

for a germ $\mathcal{C}_0$ of a smooth curve in $\mathbb{C}^n$ parameterized by f. Such curve is defined only up to polynomial reparameterization, and we have the

Lemma 2.1. The curvilinear locus $\mathrm{Curv}_0^{k+1}(\mathbb{C}^n)$ inside $\mathrm{Hilb}_0^{k+1}(\mathbb{C}^n)$ is in bijection with the set of regular k-jets of curves in $\mathbb{C}^n$ modulo polynomial reparameterizations

$$\mathrm{Curv}_0^{k+1}(\mathbb{C}^n) = J_k^{\mathrm{reg}}(1,n)/\mathrm{Diff}_k.$$

Fix now $N \geq 1$ and define the set of k-jets vanishing on a regular curve

$$\Theta_k = \{\Psi \in J_k(n,N) \mid \exists \gamma \in J_k^{\mathrm{reg}}(1,n) : \Psi \circ \gamma = 0\}.$$

Such a curve γ satisfying $\Psi \circ \gamma = 0$ is called a test curve for Ψ.

Lemma 2.2 (Gaffney [26], Bérczi-Szenes [13]). Let $\gamma \in J_k^{\mathrm{reg}}(1,n)$ and $\Psi \in J_k(n,N)$ be k-jets, and write $\nu_i = \gamma^{(i)}(0)/i!$. The equating $\Psi \circ \gamma = 0$ is equivalent to the system of k linear equations

$$\sum_{\ell \in \mathcal{P}(m)} \Psi^{\ell}(\boldsymbol{\nu}_\ell) = 0, \qquad m = 1, \dots, k \tag{2.6}$$

where $\mathcal{P}(m)$ denotes the set of partitions $\ell = 1^{\ell_1} \dots m^{\ell_m}$ of m, $\boldsymbol{\nu}_\ell = \nu_1^{\ell_1} \dots \nu_m^{\ell_m}$, $|\ell|$ denotes the length of ℓ and Ψ^ℓ is the $|\ell|$'th derivative of Ψ.

On the other hand, we define for a given $\gamma \in J_k^{\mathrm{reg}}(1,n)$ the set of Ψ such that γ is a test curve for Ψ – at least up to order i, that is, for $m = 1, \dots, i$

$$\mathcal{S}_\gamma^{i,N} = \{\Psi \in J_k(n,N) \mid (\gamma, \Psi) \text{ solve (2.6) up to order } i\}.$$

Since the equations (2.6) are linear in Ψ, we see that $\mathcal{S}_\gamma^{i,N} \subset J_k(n,N)$ is a linear subspace of codimension iN, that is $\mathcal{S}_\gamma^{i,N} \in \mathrm{Grass}_{\mathrm{codim}=iN}(J_k(n,N))$ with orthogonal complement $(\mathcal{S}_\gamma^{i,N})^\perp$ an iN-dimensional subspace of the dual $J_k(n,N)^*$. In fact the interpretation (2.3) yields $J_k(n,N)^* \simeq \bigoplus_{i=1}^k \mathrm{Sym}^i\, \mathbb{C}^n \otimes \mathbb{C}^N$, and we see that $(\mathcal{S}_\gamma^{i,N})^\perp = (\mathcal{S}_\gamma^{i,1})^\perp \otimes \mathbb{C}^N$.

Remark 2.3. From Lemma 2.2 it follows directly that for $\gamma \in J_k^{\mathrm{reg}}(1,n)$, the orthogonal complement of $\mathcal{S}_\gamma^{i,N}$ is given by

$$(\mathcal{S}_\gamma^{i,N})^\perp = \mathrm{Span}_{\mathbb{C}}(\nu_1, \nu_2 + \nu_1^2, \ldots, \sum_{j_1+\cdots+j_l=i} \nu_{j_1}\cdots\nu_{j_l}),$$

where $\nu_i = \gamma^{(i)}(0)/i!$ denotes the normed i'th derivative. ♦

At last we note that when $N \geq n$

$$\tilde{\mathcal{S}}_\gamma^{i,N} = \{\Psi \in \mathcal{S}_\gamma^{i,N} \mid \dim \operatorname{Ker} D_0\Psi = 1\}$$

is an open dense subset of the space $\mathcal{S}_\gamma^{i,N}$. In fact, $\mathcal{S}_\gamma^{i,N} \backslash \tilde{\mathcal{S}}_\gamma^{i,N} = \{\Psi \in \mathcal{S}_\gamma^{i,N} | \dim \operatorname{Ker} D_0\Psi > 1\}$ since the determinantal variety

$$\{A \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^N) \mid \mathrm{rk}(A) \leq r\} = \{A \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^N) \mid \dim \operatorname{Ker} A \geq n - r\}$$

is closed of dimension $r(n+N-r)$, we obtain that the complement $\mathcal{S}_\gamma^{i,N} \setminus \tilde{\mathcal{S}}_\gamma^{i,N} = \{\Psi \in \mathcal{S}_\gamma^{i,N} \mid \dim \operatorname{Ker} D_0\Psi \geq 2\}$ is closed and of codimension $N - n + 3$ in $\mathcal{S}_\gamma^{i,N}$.

Now, if γ is a test curve for $\Psi \in \Theta_k$, observe trivially that any $\mathrm{Diff}_k(1) = J_k^{\mathrm{reg}}(1,1)$-reparameterization $\delta = \gamma \circ \phi$ of γ is again a test curve for Ψ. The converse is not true in general, but for Ψ with linear part $D_0\Psi$ satisfying $\dim \operatorname{Ker} D_0\Psi = 1$, the converse does indeed hold: Any test curve δ of Ψ is of the form $\delta = \gamma \circ \phi$ for some $\phi \in \mathrm{Diff}_k$. In other words

$$\tilde{\mathcal{S}}_\gamma^{i,N} = \tilde{\mathcal{S}}_\delta^{i,N} \iff \delta = \gamma \circ \phi \text{ for some } \phi \in \mathrm{Diff}_k.$$

This can be proven by an inductive argument (see [13, Proof of Theorem 4.3]). Since $\tilde{\mathcal{S}}_\gamma^{i,N}$ is dense in $\mathcal{S}_\gamma^{i,N}$, we obtain the following theorem using the interpretation of (2.3)

$$\bigoplus_{l=1}^{k} \mathrm{Sym}^l\, \mathbb{C}^n \simeq J_k(n,1)^*$$

Theorem 2.4. The map

$$\phi_{n,k} : J_k^{\mathrm{reg}}(1,n) \to \mathrm{Grass}_k(\bigoplus_{i=1}^{k} \mathrm{Sym}^i\, \mathbb{C}^n), \quad \gamma \mapsto (\mathcal{S}_\gamma^{k,1})^\perp \tag{2.7}$$

is $\mathrm{Diff}_k(1)$-invariant and induces an injective map on orbits

$$\bar{\phi}_{n,k} : J_k^{\mathrm{reg}}(1,n)/\mathrm{Diff}_k(1) \hookrightarrow \mathrm{Grass}_k(\bigoplus_{i=1}^{k} \mathrm{Sym}^i\, \mathbb{C}^n).$$

Moreover, $\phi_{n,k}$ and $\bar{\phi}_{n,k}$ are $\mathrm{GL}(n)$-equivariant with respect to the standard action on $J_k^{\mathrm{reg}}(1,n) \subset \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ and its induced action on $\mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i\, \mathbb{C}^n)$.

Recall the natural embedding $\mathrm{Hilb}_0^{k+1}(\mathbb{C}^n) \hookrightarrow \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i\, \mathbb{C}^n)$ defined in (2.1). The image of this embedding restricted to $\mathrm{Curv}_0^{k+1}(\mathbb{C}^n)$ coincides with the image

$$\mathrm{Im}(\phi_{n,k}) \simeq \mathrm{Curv}_0^{k+1}(\mathbb{C}^n) \subset \mathrm{Grass}_k(\bigoplus_{i=1}^{k} \mathrm{Sym}^i\, \mathbb{C}^n),$$

of the map $\phi_{n,k} : J_k^{\mathrm{reg}}(1,n) \to \mathrm{Grass}(\bigoplus_{i=1}^k \mathrm{Sym}^i\, \mathbb{C}^n)$. We obtain thus

Theorem 2.5 (Bérczi-Szenes model for $\mathrm{CHilb}_0^k(\mathbb{C}^n)$). For any k and n we have

$$\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n) \simeq \overline{\mathrm{Im}(\phi_{n,k})} \subset \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$$

for the map

$$\phi_{n,k} : J_k^{\mathrm{reg}}(1,n) \to \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n), \quad \gamma \mapsto (\mathcal{S}_\gamma^{k,1})^\perp$$

of Theorem 2.4.

2.3 A fibered version

In this section we construct a fibration $\widetilde{J_k(1,n)}$ of the jet space $J_k(1,n)$ over a flag manifold, and obtain thus a version of the Berczi-Szenes model $\phi_{n,k} : \widetilde{J_k(1,n)} \dashrightarrow \mathrm{CHilb}_0^{k+1}(\mathbb{C}^n) \subset \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$. The constructions of the space $\widetilde{J_k(1,n)}$ in the two ranges $n \leq k$ and $k \leq n$ are distinct; in the first case the fibration will be over the flag manifold $\mathrm{Flag}(1,\dots,n;\mathbb{C}^n)$, whereas the latter case will be over $\mathrm{Flag}(1,\dots,k;\mathbb{C}^n)$.

We write $m = \min(n,k)$, denote by $e_1,\dots,e_n$ the canonical basis of $\mathbb{C}^n$ and consider the flag

$$E_{n,k} = [\mathrm{Span}(e_1) \subset \dots \subset \mathrm{Span}(e_1,\dots,e_m)] \in \mathrm{Flag}(1,\dots,m;\mathbb{C}^n).$$

By $P_{n,k} \subset \mathrm{GL}_n$ we denote the parabolic subgroup fixing the flag $E_{n,k}$.

Under the identification $J_k(1,n) \simeq \mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^n) \ni (e_1,\dots,e_m,\nu_{m+1},\dots,\nu_k)$, we construct the associated bundle

$$\widetilde{J_k(1,n)} = \mathrm{GL}_n \times_{P_{n,k}} P_{n,k}.(e_1,\dots,e_m,\nu_{m+1},\dots,\nu_k)$$

with the fibration

$$\widetilde{J_k(1,n)} \to \mathrm{GL}_n / P_{n,k} = \mathrm{Flag}(1,\dots,m;\mathbb{C}^n).$$

Observe that the set $J_k^{\mathrm{nondeg}}(1,n) \subset J_k(1,n)$ of jets of curves γ with $\gamma'(0),\dots\gamma^{(m)}(0)$ linearly independent has the fibration

$$\begin{aligned} J_k^{\mathrm{nondeg}}(1,n) &\to \mathrm{Flag}(1,\dots,m;\mathbb{C}^n) \\ [\gamma'(0),\dots,\gamma^{(k)}(0)] &\mapsto [\mathrm{Span}(\gamma'(0)),\dots,\mathrm{Span}(\gamma'(0),\dots,\gamma^{(m)}(0))]. \end{aligned}$$

The space $\widetilde{J_k(1,n)}$ is a fiberwise compactification of $J_k^{\mathrm{nondeg}}(1,n)$.

We obtain an induced map from the Berczi-Szenes model, and we will abuse notation and write simply

$$\phi_{n,k} : \widetilde{J_k(1,n)} \dashrightarrow \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n),$$

which on each fiber over $\mathrm{Flag}(1,\dots,m;\mathbb{C}^n)$ is well-defined on the regular part. For localization purposes later in this work it will be enough to consider only the fiber over $E_{n,k}$.

We will sometimes drop the indices on this flag $E_{n,k}$ in the notation and write $\widetilde{J_k(1,n)}_E$ for the fiber over $E_{n,k}$. Moreover, for the corresponding map obtained by restricting $\phi_{n,k}$ to the fiber $\widetilde{J_k(1,n)}_E$ we will usually just write

$$\phi_E : \widetilde{J_k(1,n)}_E \dashrightarrow \mathrm{Grass}_k(\bigoplus_{i=1}^{k} \mathrm{Sym}^i \mathbb{C}^n)$$

dropping the dependence on n and k.

2.4 The toric submodel

We describe here a toric subvariety of the image closure $\overline{\mathrm{Im}\,\phi_{n,k}} \subset \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$.

The Berczi-Szenes model takes the form (cf. Theorem 2.4)

$$\begin{aligned}\phi_{n,k} : J_k(1,n) \dashrightarrow \mathrm{Grass}_k(\bigoplus_{i=1}^{k} \mathrm{Sym}^i \mathbb{C}^n)& \\ \gamma \mapsto \mathrm{Span}_{\mathbb{C}}(\nu_1, \nu_2 + \nu_1^2, \dots, \sum_{\rho \in \mathcal{P}_k} |\mathrm{perm}(\rho)|\, \nu_\rho),&\end{aligned}$$

where $\mathcal{P}_k$ is the set of partitions of k, and $|\mathrm{perm}(\rho)|$ is the number of compositions representing the partition ρ.

Recall that we have constructed a fibration $\widetilde{J_k(1,n)}$ over the space $\mathrm{Flag}(1,\dots,m;\mathbb{C}^n)$ for $m = \min(k,n)$, such that on the fiber $\widetilde{J_k(1,n)}_E$ over the flag

$$E = E_{n,k} = [\mathrm{Span}(e_1) \subset \mathrm{Span}(e_1,e_2) \subset \dots \subset \mathrm{Span}(e_1,\dots,e_m)] \in \mathrm{Flag}(1,\dots,m;\mathbb{C}^n)$$

we have $\nu_i \in \mathrm{Span}(e_1,\dots,e_i)$ for $i = 1,\dots,m$. Here $e_1,\dots,e_n$ is the standard basis of $\mathbb{C}^n$.

Denote by $\widetilde{J_k(1,n)}_E^0$ the subvariety in $\widetilde{J_k(1,n)}_E$ determined by the conditions $\nu_i \in \mathrm{Span}(e_i)$ for $i = 1,\dots,m$ and $\nu_i = 0$ for $i = m+1,\dots,k$. These fibers together constitute a closed subvariety in $\widetilde{J_k(1,n)}$

$$\widetilde{J_k(1,n)} \supset \widetilde{J_k(1,n)}^0 := \mathrm{GL}_n \times_{P_{n,k}} \widetilde{J_k(1,n)}_E^0 \to \mathrm{Flag}_m(\mathbb{C}^n)$$

admitting also a fibration over the flag manifold. We the restricted map

$$\phi_{n,k}^0 : \widetilde{J_k(1,n)}^0 \dashrightarrow \mathrm{Grass}_k(\bigoplus_{i=1}^{k} \mathrm{Sym}^i \mathbb{C}^n).$$

the *toric* model. The image closure $\overline{\mathrm{Im}\,\phi_{n,k}^0} \subset \overline{\mathrm{Im}\,\phi_{n,k}} = \mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$ is a toric variety.

Chapter 3

Tautological Bundles and Integrals

In this section we associate to a bundle $F \to X$ a so-called tautological bundle $F^{[k]} \to \mathrm{Hilb}^k(X)$ on the Hilbert scheme of k points on X. A tautological integral will then be defined as the integration of a form expressed in the Chern classes of $F^{[k]}$ over certain geometric subsets of $\mathrm{Hilb}^k(X)$.

3.1 Tautological Bundle on Hilbert Scheme of Points

Let X be a smooth projective variety of dimension n with a rank r vector bundle $F \to X$. We define the rank rk *tautological bundle on* $\mathrm{Hilb}^k(X)$ to be $F^{[k]} = p_* q^*(F)$, where p and q are the projections of the universal family $\mathcal{Z}$, the diagram is the following

$$\begin{tikzcd} \mathrm{Hilb}^k(X) \times X \supset \mathcal{Z} \arrow[r, "q"] \arrow[d, "p"'] & X \\ \mathrm{Hilb}^k(X). & \end{tikzcd}$$

The fiber over a subscheme $\xi \in \mathrm{Hilb}^k(X)$ is

$$F^{[k]}|_\xi = H^0(\xi, F|_\xi) = H^0(X, F \otimes \mathcal{O}_\xi),$$

which has dimension rk.

In particular with $X = \mathbb{C}^n$ and $F = \mathcal{O}_{\mathbb{C}^n}$ the structure sheaf (line bundle), one has that $\xi \in \mathrm{Hilb}^k(\mathbb{C}^n)$ corresponds to a colength k ideal I in $\mathcal{O}_{\mathbb{C}^n}$ and the fiber over ξ is

$$\mathcal{O}_{\mathbb{C}^n}^{[k]}|_\xi = \mathcal{O}_\xi = \mathbb{C}[x_1, \ldots, x_n]/I,$$

which has rank k.

On the other hand, the restriction of this tautological bundle to the punctual Hilbert scheme $\mathrm{Hilb}_p^{k+1}(X) \simeq \mathrm{Hilb}_0^{k+1}(\mathbb{C}^n)$ can also be described via the pullback of the tautological bundle $\mathcal{E}$ on the Grassmannian $\mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$ by the natural embedding φ (cf. (2.1)), and we have

$$\begin{tikzcd} O_{\mathbb{C}^n}^{[k+1]}/\mathcal{O}_{\mathbb{C}^n} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathrm{Hilb}_0^{k+1}(\mathbb{C}^n) \arrow[r, hook, "\varphi"] & \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n). \end{tikzcd}$$

From this description we obtain

$$F^{[k+1]}|_{\mathrm{Hilb}_0(\mathbb{C}^n)} = \mathcal{O}_{\mathbb{C}^n}^{[k+1]} \otimes F = (\mathcal{O}_{\mathbb{C}^n} \oplus \varphi^* E) \otimes F.$$

It follows that writing $c(F) = \prod_{j=1}^r (1+\eta_j)$ and $c(\mathcal{E}) = \prod_{j=1}^k (1+\theta_j)$ for the total Chern classes, we get

$$c(F^{[k+1]}) = \prod_{l=1}^{r}(1+\eta_l)\prod_{j=1}^{k}\prod_{i=1}^{r}(1+\theta_j+\eta_i). \tag{3.1}$$

In particular we see that each Chern class $c_i(F^{[k+1]})$ is a polynomial function in the Chern roots of F and $\mathcal{E}$.

3.2 Tautological integrals

Following Rennemo in [54] we define what is meant by a tautological integral on any geometric subset. We start by defining geometric subsets of the Hilbert scheme of points $\mathrm{Hilb}^k(X)$ of a smooth projective variety X of dimension n.

By a *punctual geometric set* we mean a constructible subset $Q \subset \mathrm{Hilb}_0^k(\mathbb{C}^n)$ satisfying that if $\xi \in Q$ and $\xi \simeq \xi' \in \mathrm{Hilb}_0^k(\mathbb{C}^n)$ as schemes then also $\xi' \in Q$.

For fixed punctual geometric sets $Q_1, \dots, Q_r$ with $Q_i \in \mathrm{Hilb}_0^{k_i}(\mathbb{C}^n)$ and $\sum k_i = k$. We define the set

$$P(Q_1, \dots, Q_r) = \{\xi \in \mathrm{Hilb}^k(X) \mid \xi = \xi_1 \sqcup \dots \sqcup \xi_r,\ \xi_i \in Q_i\}.$$

We define then the *geometric subsets* of $\mathrm{Hilb}^k(X)$ as the sets, which can be constructed by applying a finite number of unions, intersections and complements to sets of the form $P(Q_1, \dots, Q_r)$.

Suppose now X is equipped with a vector bundle F of rank r, and associated tautological bundle $F^{[k]} \to \mathrm{Hilb}^k(X)$ of rank rk. Let $P \subset \mathrm{Hilb}^k(X)$ be a geometric subset and $\overline{P}$ its Zariski closure in $\mathrm{Hilb}^k(X)$. Let $M = M(c_1, \dots c_{rk})$ be a monomial in the Chern classes $c_i := c_i(F^{[k]})$ of the tautological bundle $F^{[k]}$, such that M has weighted degree $\dim \overline{P}$, where the weight of c_i is $2i$. Now if $\alpha_M \in \Omega^*(P)$ is a closed compactly supported differential form representing the cohomology class of M then the Chern numbers

$$M \cap [\overline{P}] = \int_{\overline{P}} \alpha_M$$

are called tautological integrals of $F^{[k]}$. The main theorem of [54, Theorem 1.1] is that the tautological integrals can be expressed in terms of the Chern numbers of X and F, and that this is true in a universal way.

Observe that the punctual curvilinear locus $\mathrm{Curv}_p^k(X)$ at p is nothing but $P(Q_1)$ for Q_1 the punctual geometric subset consisting of the isomorphism class $\mathcal{O}_\xi \simeq \mathbb{C}[z]/z^k$. It follows that integrals of the form

$$\int_{\mathrm{CHilb}_0^k(\mathbb{C}^n)} \alpha_M$$

are a very special type of tautological integrals.

Chapter 4

Equivariant Cohomology

We setup in this part notation for equivariant cohomology. We choose to introduce it via principal bundles, associated bundles and a universal bundle as was done by Borel [15]. We will follow mainly [25]. The reference for theory of fiber bundles is [35] – especially chapter 4 in this case. Our main goal is to define equivariant pushforwards and to state the localization theorem of Atiyah-Bott [1] and Berline-Vergne [14], Theorem 4.6).

Let G be a linear algebraic group and X a left G-variety. Let $EG \to BG$ be a universal principal G-bundle (see [35, The Milnor Construction]) with EG right G-space. Such a space EG must be contractible (in fact the principal G-bundle is universal if and only if EG is contractible, [35, chapter 4, Exercise 13]), and $BG = EG/G$ is unique up to homotopy. We form the associated bundle with fiber X

$$X_G = EG \times^G X := EG \times X/\sim,$$

the equivalence relation being $(e{\cdot}g, x) \sim (e, g{\cdot}x)$. It turns out (see [15, Chapter IV, Application 3.4]) that the singular cohomology of X_G is independent of the choice of EG, and we define the *equivariant cohomology of* X as the singular cohomology of X_G in a coefficient ring R, and write

$$H^*_G(X) := H^*(X_G) = H^*(EG \times^G X).$$

The equivariant cohomology theory is functorial for equivariant maps, and so we have in particular pullbacks.

We observe that taking some fixed point $p \in X^G$, we have homotopically $EG \times^G \{p\} \simeq BG$, and so the inclusion map $i_p : \{p\} \hookrightarrow X$ yields the pullback map of cohomology

$$i_p^* : H^*_G(X) \to H^*_G(p) \simeq H^*(BG)$$

making $H^*_G(X)$ into a $H^*(BG)$-module. We write $\Lambda_G := H^*(BG)$ since it will show up often.

The spaces EG and BG will usually be of infinite dimension. However, there are always finite dimensional approximations of these spaces $E_m \to B_m = E_m/G$, and it suffices to work with these. In fact this has nothing to do with the bundle being universal. The result is

Theorem 4.1. Let $E_m \to B_m$ be a principal G-bundle such that $H^i(E_m) = 0$ for $i < N$. Then there is a canonical isomorphism

$$H^i(EG \times^G X) \simeq H^i(E_m \times^G X),$$

for $i < N$, respecting cup products.

Proof. The proof can be found in [25, Proposition 2.2]. □

We give an explicit construction of approximation spaces $E_m \to B_m$ for $G = T$ a torus in Example 4.3. This is a special case of the construction in [25, Chapter 2, Sections 4,5] showing that for a linear algebraic group G acting algebraically on X such approximation space $E_m \to B_m$ exists, and E_m can be taken to be a nonsingular variety (see also [25, Proposition 2.6]).

Remark 4.2. The equivariant cohomology can also be constructed using the Cartan-De Rham complex of equivariant forms; we follow the overview of Berline, Getzler and Vergne in [14, Chapter 7]. Let M be a smooth manifold with an action of a Lie group G with Lie algebra $\mathfrak{g}$. The equivariant forms are exactly the equivariant polynomial maps on the Lie algebra $\mathfrak{g}$ with values in the G-equivariant ordinary differential forms on M

$$\begin{aligned}\Omega^*_G(M) &= \{\text{polynomial } \alpha : \mathfrak{g} \to \Omega^*(M) \mid \alpha(g.X) = g.\alpha(X)\} \\ &= (\mathrm{Sym}^* \mathfrak{g} \otimes \Omega^*(M))^G,\end{aligned}$$

where the action on forms is $g.\alpha(X) = g.(\alpha(g^{-1}.X))$. In this sense an equivariant form $\alpha \in \Omega^d_G(M)$ of degree d can be written

$$\alpha = \alpha_d + p_1\alpha_{d-1} + \cdots + p_{d-1}\alpha_1 + p_d, \qquad \alpha_i \in \Omega^i(M), p_j \in \mathrm{Sym}^j(\mathfrak{g}^*).$$

The equivariant exterior differential $d_{\mathfrak{g}}$ is defined by

$$(d_{\mathfrak{g}}\alpha)(X) = (d - \iota(X_M)) = \alpha(X),$$

where $\iota(X_M)$ is the contraction of the infinitesimal vector field X_M on M defined by

$$(X_M.\phi)(x) = \left.\frac{d}{dt}\right|_{t=0} \phi(e^{-tX}.x)$$

for $\phi \in \Omega^0(M)$. One may calculate $d_{\mathfrak{g}}^2 = 0$, and so one obtains indeed a complex. The cohomology of this complex is (isomorphic to) the equivariant cohomology $H^*_G(M)$. ♦

One of the strengths of equivariant cohomology theory is the existence of equivariant Chern classes and equivariant fundamental classes, which may be defined using the approximation spaces $E_m \to B_m$.

For an equivariant vector bundle $V \to X$, we get induced bundles $E_m \times^G V \to E_m \times^G X$ and define *the equivariant Chern classes of* V

$$c_i^G(V) = c_i(E_m \times^G V) \in H^{2i}_G(X), \quad m \gg 0.$$

Similarly, when X is a nonsingular variety so is $E_m \times^G X$, and further any G-invariant subvariety $Y \subset X$ of codimension d yields a codimension d subvariety $E_m \times^G Y \subset E_m \times X$, and we define *the equivariant fundamental form of* Y

$$[Y]^G = [E_m \times^G Y] \in H^{2d}_G(X), \quad m \gg 0,$$

where $[\,\cdot\,]$ denotes the usual fundamental class of a subvariety of a nonsingular variety. Defining the classes in this way, it is of course necessary to check compatibility with varying parameter m and independence of approximation spaces E_m.

Example 4.3. Let a maximal torus $T = (\mathbb{C}^*)^n$ act diagonally on $X = \mathbb{C}^n$ with weights $(t_1, \ldots, t_n)$. We may take

$$E_m = (\mathbb{C}^m \setminus \{0\})^n \to (\mathbb{P}^{m-1})^n = B_m.$$

It follows that $\Lambda_T := H^*((\mathbb{P}^\infty)^n) = R[t_1, \ldots, t_n]$, $t_i = c_1(\mathcal{O}_i(-1))$ admitting an interpretation as the usual first Chern class of the pullback of $\mathcal{O}(-1)$ of the i'th projection.

Observing that

$$V := E_m \times^T X \simeq \bigoplus_{i=1}^n \mathcal{O}(-1)$$

as vector bundles on $B_m = (\mathbb{P}^{m-1})^n$, we obtain that the equivariant Chern classes on X (considered as a bundle $X \to \{pt\}$ over a fixed point) are elementary symmetric polynomials e_i in the weights t_j

$$c_i^T(\mathbb{C}^n) = e_i(t_1, \ldots, t_n) \in \Lambda_T = R[t_1, \ldots, t_n].$$

In particular, we observe that the chern roots are exactly the weights.

The induced action of T on $\mathbb{P}^{n-1}$ makes $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ into a T-equivariant line bundle, and we write $\zeta = c_1^T(\mathcal{O}_{\mathbb{P}^{n-1}}(1))$. The induced vector bundles are

$$E_m \times^T \mathbb{P}^{n-1} = \mathbb{P}(V) \quad \text{and} \quad E_m \times^T \mathcal{O}_{\mathbb{P}^{n-1}}(1) = \mathcal{O}_{\mathbb{P}(V)}(1)$$

the projectivization and its Serre twist. It follows that

$$c_i^T(\mathbb{C}^n) = c_i(V) = e_i \quad \text{and} \quad \zeta = c_1^T(\mathcal{O}_{\mathbb{P}^{n-1}}(1)) = c_1(\mathcal{O}_{\mathbb{P}(V)}(1)),$$

and since the cohomology of $\mathbb{P}(V)$ is known in terms of the cohomology of X, we obtain

$$\begin{aligned} H_T^*(\mathbb{P}^{n-1}) = H^*(\mathbb{P}(V)) &= R[t_1, \ldots, t_n][\zeta] \,/\, (\zeta^n + e_1(\boldsymbol{t})\zeta^{n-1} + \cdots + e_{n-1}(\boldsymbol{t})\zeta + e_n(\boldsymbol{t})) \\ &= \Lambda_T[\zeta] \Big/ \Big(\prod_{i=1}^n (\zeta + t_i)\Big). \end{aligned}$$

using the description of equivariant Chern classes of $\mathbb{C}^n$ from above. ♦

Let $V \to X$ be an equivariant bundle of a T-space X of rank r. Further let $p \in X^T$ be a fixed point and denote the weights of the action on the fiber V_p by $t_1, \ldots, t_r$. It follows then by the calculation of Chern classes in Example 4.3 above and by functoriallity that

$$i_p^*(c_i^T(V)) = c_i^T(V_p) = \sigma_i(t_1, \ldots, t_r), \tag{4.1}$$

where σ_i is the i'th elementary symmetric polynomial. It follows in particular that the restrictions of the Chern roots of V to a point p are exactly the weights $t_1, \ldots, t_r$ on the fiber V_p.

4.1 Localization in equivariant cohomology

We shall now briefly discuss the concept of localization in equivariant cohomology. By a localization, we mean a restriction map in cohomology, that is, a pullback of an embedding (usually the injection of the fixed point locus). We shall only discuss the case of torus actions and shall in particular describe the localization theorem by Atiyah and Bott [1], and Berline and Vergne [14], which under assumption of finite fixed point locus yields an integration formula.

For this discussion we shall need the Gysin pushforward, which exists for proper maps $f : X \to Y$,

$$f_* : H_T^* X \to H_T^{*+2d} Y$$

with $d = \dim Y - \dim X$. Two special cases of such equivariant pushforwards of main interest to us are

(1) *Closed embeddings.* For a T-invariant closed embedding $i : Y \hookrightarrow X$, there is a Gysin pushforward $i_* : H_T^* Y \to H_T^{*+2d} X$ satisfying

$$i_*(1) = i_*[Y]^T = [Y]^T \quad \text{and} \quad i^* i_*(\alpha) = c_d^T(N_{X/Y}) \cdot \alpha,$$

where $N_{Y/X}$ is the normal bundle of Y in X.

(2) *Integration along fibers.* For a complete nonsingular variety X of dimension n and p a fixed point, the map $\rho : X \to \{p\}$ has the Gysin pushforward called integration along p

$$\int_X := \rho_* : H_T^* X \to H_T^{*-2n}(p).$$

For more general properties and definitions we refer again to [25].

Remark 4.4. Following the description of equivariant cohomology via equivariant forms as described in Remark 4.2, we can describe integration in the following way: Let $\alpha \in \Omega_G^*(M)$ with $d \geq \dim M$, and write

$$\alpha = p_{d-\dim M} \alpha_{\dim M} + \cdots + p_{d-1}\alpha_1 + p_d.$$

One may define the equivariant pushforward ρ_*^G via the usual pushforward ρ_*

$$\begin{aligned} \int_M \alpha = \rho_*^G \alpha &= p_{d-\dim M}\, \rho_*(\alpha_{\dim M}) \\ &= p_{d-\dim M} \Big(\int_M \alpha_{\dim M} \Big) \in \mathrm{Sym}^{d-\dim M} \mathfrak{t}^*, \end{aligned}$$

yielding a polynomial in $H_G^0(p) = \mathrm{Sym}^* \mathfrak{t}^*$ in general. ♦

Remark 4.5. For an equivariant vector bundle $V \to M$, we observe that by the very definition of the equivariant Chern classes $c_i^G(V)$, it is an *equivariant extension* of the ordinary Chern class $c_i(V)$ in the following sense: Fixing a base point $b \in BG$, the fiber of an associated bundle $E_m \times^G V$ over b is isomorphic to V, and restricting $c_i^G(V)$ to this fiber yields $c_i(V)$.

Equivalently in the setting where G is a Lie group acting on a manifold M, in terms of the description of the equivariant cohomology as equivariant forms we have

$$c_i^G(V) = c_i(V) + p_1\alpha_{2i-1} + \cdots + p_{2i-1}\alpha_1 + p_{2i} \in H_G^{2i}(M)$$

for suitable forms $\alpha_j \in H^j(X)$ and polynomials $p_j \in \mathrm{Sym}^j(\mathfrak{g}^*)$.

We have thus from Remark 4.4 the equality

$$\rho_*^G c_i^G(V) = \rho_* c_i(V)$$

of equivariant integration on M on the one hand, and ordinary integration on M on the other, and more generally this holds equality is true when the Chern classes are replaced by a polynomial of Chern classes. ♦

We end this section with the integration formula proven independently by Atiyah and Bott [1], and by Berline and Vergne [14], which applies to compact nonsingular varieties with finite fixed point locus, and is key to the concept of localization.

Theorem 4.6. Let X be a compact nonsingular T-variety of dimension n with finite fixed point locus X^T, and denote by $i : p \to X$ the inclusion of p in X. Then

$$\int_X \alpha = \sum_{p \in X^T} \frac{i_* i^* \alpha}{c_n^T(T_p X)}$$

for all $\alpha \in H_T^* X$.

Proof. Let $\alpha = \frac{\alpha}{1} \in H_T^* X$, and observe that since $S^{-1} i_*$ is surjective, we obtain that the (partially defined) map $S^{-1} H_T^* X^T \to H_T^* X$ is surjective, and it follows that we can assume $\alpha = (i_p)_*(\beta)$ for some $\beta \in H_T^*(p) = \Lambda_T$. We get on the left hand side

$$\int_X \alpha = \int_X (i_q)_* \beta = \beta,$$

since $\Lambda_T = H_T^*(p) \xrightarrow{(i_p)_*} H_T^* X \xrightarrow{\int_X} \Lambda_T$ is an isomorphism. On the other hand, the right side yields

$$\sum_{q \in X^T} \frac{\alpha(q)}{c_n^T(T_q X)} = \sum_{q \in X^T} \frac{i_q^*(i_p)_*\beta}{c_n^T(T_q X)} = \frac{i_p^*(i_p)_*\beta}{c_n^T(T_p X)} = \beta,$$

yielding the equality. □

Remark 4.7. There is a generalization of this localization formula, when the fixed point locus X^T is not necessarily finite. In this case the integral splits into a sum over each connected component F of the fixed point locus X^T. The tangent space at a point is replaced by the normal bundle of the component in X. The formula is

$$\int_X \alpha = \sum_{F \subset X^T} \int_F \frac{\alpha|_F}{c_n^T(N_{X/F})},$$

and we refer the reader to [25, Theorem 2.1] for further details. ♦

Chapter 5

Non-reductive geometric invariant theory

This section aim at giving a short introduction to non-reductive geometric invariant theory as introduced by Berczi, Doran, Hawes and Kirwan in [6, 7]. The theory includes also the general geometric invariant theory (GIT) of Mumford for which the reference is [50].

As usual we fix our ground field to be $\mathbb{C}$. Recall that in this case any linear algebraic group G has a Levi decomposition $G = U \rtimes R$ where U is the unipotent radical and R is a reductive (Levi) subgroup. The original GIT of Mumford deals with the problem of constructing quotients of algebraic varieties when the group $G = R$ is reductive. The paper [7] deals with extending the GIT to the case of linear algebraic groups $G = U \rtimes R$ with internally graded unipotent radical reducing to the classical GIT of Mumford, when $G = R$ is reductive. Here a group $G = U \rtimes R$ is said to have internally graded unipotent radical if there is a central 1-parameter subgroup $\lambda : \mathbb{C}^* \to Z(R)$ whose weights for the adjoint action on the Lie algebra of U are all strictly positive.

The idea of non-reductive GIT is to do a two-step quotient construction, the second of which is by a reductive group so is already understood in the sense of Mumford. The first quotient is by a linear algebraic group of type $\hat{U} = U \rtimes \lambda(\mathbb{C}_m) \subset G$ where $\lambda : \mathbb{C}^* \to Z(R)$ is the central 1-parameter subgroup $\lambda : \mathbb{C}_m^* \to Z(R)$ mentioned before. Our focus will be here to describe quotients by groups of this form $\hat{U} = U \rtimes \lambda(\mathbb{C}_m)$, and in particular we remark that the diffeomorphism group of specific interest to us is $\mathrm{Diff}_k = U \rtimes \mathbb{C}^*$ is already of this form.

5.1 Non-reductive Geometric Invariant Theory for $\hat{U} = U \rtimes \lambda(\mathbb{C}^*)$-groups

Let $G = U \rtimes R$ be a linear algebraic group with unipotent radical U and $G/U = R$ the reductive quotient.

Definition 5.1. We say that G has *internally graded unipotent radical* U if there is a central 1-parameter subgroup $\lambda : \mathbb{C}^* \to Z(R)$, such that the adjoint action of $\mathbb{C}^*$ on the Lie algebra of U has strictly positive weights. We write $\hat{U} = U \rtimes \lambda(\mathbb{C}^*)$.

If G has internally graded unipotent radical then $\hat{U}$ is normal in G and $G/\hat{U} \simeq R/\lambda(\mathbb{C}^*)$ is a reductive group.

Let now G act linearly on an irreducible projective variety X with respect to an ample line bundle L, i.e. that the action lifts to an action via automorphisms of L, and write $V := H^0(X, L)^*$. Suppose the induced action of $\lambda : \mathbb{C}^* \leq \hat{U} \leq G$ on the fibers of the tautological line bundle $\mathcal{O}_{\mathbb{P}(V)}(-1)$ over (fixed point) components of $\mathbb{P}(V)^{\mathbb{C}^*}$ has weights $\omega_{\min} = \omega_0 < \omega_1 < \cdots < \omega_{\max}$. We can assume that at least two weights are different, since if all weights ω_i are equal the action of U on X is trivial , and the action of H on X is really just an action of the reductive group $R = G/U$ so classical GIT applies.

Consider a character $\chi : G \to \mathbb{C}^*$ of G then one may pick a positive integer c such that

$$\omega_0 < \frac{\chi}{c} < \omega_1.$$

Such a rational character χ/c is called *adapted* with respect the linear action of G. This condition actually ensures the existence of a geometric quotient (if it replaces “well-adapted” in Definition 5.2 and this definition is employed in Theorem 5.3 instead) on the stable $\hat{U}$-locus.

To achieve finite generation of algebras of invariants, one must impose a stronger condition. We say that the rational character χ/c is *well-adapted* for the linear action of G if there exists $\epsilon > 0$ small enough such that

$$\omega_0 < \frac{\chi}{c} < \omega_0 + \epsilon.$$

Exactly how small ϵ must be depends on the property that one wants to hold. We refer to [7] and to [9] for the fact that in our situation it is enough to have $0 < \epsilon < 1$.

We can then twist the linearization by χ in such a way that the weights ω_i are replaced by $\omega_i c - \chi$; we denote by $L_\chi^{\otimes c}$ this twisted bundle. We observe that $U \subset \operatorname{Ker}\chi$, so the restriction of the linearized action to U is unaffected by the twist. We write $X^{s,\mathbb{C}^*}$ for the set of stable points of X for the linear action of $\mathbb{C}^*$ with respect to the twisted line bundle $L_\chi^{\otimes c}$. Further, we write $V_{\min}$ for the minimal weight space of V, and define

$$Z_{\min} = X \cap \mathbb{P}(V_{\min}) = \{x \in X^{\mathbb{C}^*} \mid \mathbb{C}^* \text{ acts on } L^*|_x \text{ with weight } \omega_{\min}\}$$

and

$$X^0_{\min} = \{x \in X \mid \lim_{t\to 0} t.x \in Z_{\min}\}.$$

The crucial condition for non-reductive GIT is the following, which is referred to as 'semistability coincides with stability' for the $\hat{U}$-action (for short, we shall write ss=s for $\hat{U}$)

$$\operatorname{Stab}_U(z) = \{e\} \qquad \text{for all } z \in Z_{\min}. \tag{5.1}$$

One observes that this condition is equivalent to the same being true for all $z \in X^0_{\min}$.

At last, before giving the result on existence of a non-reductive GIT quotient, we collect some of the definitions in

Definition 5.2. The data $(X, L, G, \hat{U}, \chi)$ of a linear algebraic group G acting on an irreducible projective variety X is said to be *well-adapted* if

(1) G has internally graded unipotent radical U,

(2) G acts linearly on X with respect to a very ample line bundle L

(3) $\chi : G \to \lambda(\mathbb{C}^*)$ is a character and there is a positive integer c such that χ/c is well-adapted for the linear action of $\hat{U} = U \rtimes \mathbb{C}^*$ on X.

We refer to such data simply as a well-adapted action of G on X.

When ss=s for $\hat{U}$ and χ/c is well-adapted for $\hat{U}$ then the min-stable locus of the $\hat{U}$-action satisfies the equalities

$$X^{s,\hat{U}} = X^{ss,\hat{U}} = \bigcap_{u\in U} uX^{s,\mathbb{C}^*} = X^0_{\min} \setminus UZ_{\min}.$$

Theorem 5.3. Let $(X, L, G, \hat{U}, \chi)$ be a well-adapted action satisfying ss=s for $\hat{U}$. Then

(1) The algebras of invariants

$$\bigoplus_{m\geq 0} H^0(X, L^{\otimes cm}_{m\chi})^{\hat{U}}$$

and

$$\bigoplus_{m\geq 0} H^0(X, L^{\otimes cm}_{m\chi})^G = \Big(\bigoplus_{m\geq 0} H^0(X, L^{\otimes cm}_{m\chi})^{\hat{U}} \Big)^R$$

are finitely generated.

(2) The projective variety $X/\!\!/\hat{U}$ associated to the algebra of invariants $\bigoplus_{m\geq 0} H^0(X, L^{\otimes cm}_{m\chi})^{\hat{U}}$ is a geometric quotient of the open subset $X^{s,\hat{U}}$ by $\hat{U}$.

(3) The projective variety $X/\!\!/G$ associated to the algebra of invariants $\bigoplus_{m\geq 0} H^0(X, L^{\otimes cm}_{m\chi})^G$ is the classical GIT quotient of $X/\!\!/\hat{U}$ by the induced action of the reductive group $R/\lambda(\mathbb{C}^*)$ with respect to the linearization induced by $L^{\otimes c}$.

The varieties $X/\!\!/G$ and $X/\!\!/\hat{U}$ are referred to as the non-reductive GIT-quotient of X by G and $\hat{U}$, respectively.

Remark 5.4. In [8] it is shown that if equation (5.1) is not satisfied, there exists a sequence of blow ups along $\hat{U}$-invariant projective subvarieties resulting in a projective variety $\tilde{X}$ with a well-adapted linear action of $\hat{U}$, which satisfies the equation (5.1) of ss=s for $\hat{U}$ so that

$$\tilde{X}/\!\!/\hat{U} = \tilde{X}^{s,\hat{U}}/\hat{U}.$$

In this sense the non-reductive GIT quotient can always be constructed, when $\hat{U}$ has internally graded unipotent radical. ♦

5.2 Integration on non-reductive GIT quotients

In this section we give a very brief account of the theory of moment maps and cohomology of the non-reductive GIT quotients described in the above section, as discussed by Bérczi and Kirwan in [10] – especially we refer to section 7. Our goal of this section is to state a formula for integration on non-reductive GIT-quotients. This is the content of Theorem 5.5 at the end of this section. The non-reductive group appearing in this work is $\mathrm{Diff}_k = \hat{U} \rtimes \mathbb{C}^*$, and for this reason we restrict ourselves to the case of linear algebraic groups of the form $\hat{U} = U \rtimes \lambda(\mathbb{C}^*)$ where $\lambda : \mathbb{C}^* \to Z(R)$ is a 1-parameter subgroup from Definition 5.1.

Let $G = U \rtimes R$ be a complex linear algebraic group with internally graded unipotent radical U (and 1-parameter subgroup $\lambda : \mathbb{C}^* \to Z(R)$ of Definition 5.1) acting on a nonsingular projective variety X with respect to an ample line bundle L, such that semistability coincides with stability for the action of $\hat{U}$ (that is, satisfying (5.1)) Then $\lambda(S^1) \subset \hat{U}$ is a maximal compact subgroup.

Using the embedding $X \hookrightarrow \mathbb{P}^n$ defined by a very ample power of L, a G-moment map $\mu_G : X \to \mathfrak{g}^*$ is defined by composing the $\mathrm{GL} = \mathrm{GL}_{n+1}$-moment map $\mu_{GL} : X \to \mathfrak{gl}^*$ with the dual of the representation $G \to \mathrm{GL}$. Here $\mathfrak{g}$ and $\mathfrak{gl}$ are the Lie algebras of G and GL, respectively. When the action of $\hat{U}$ on X is well-adapted one has the map

$$X /\!\!/ \hat{U} \simeq \mu_{\hat{U}}^{-1}(0)/S^1 \overset{i}{\hookrightarrow} \mu_{S^1}^{-1}(0)/S^1 \simeq X /\!\!/ \lambda(\mathbb{C}^*).$$

Denote by $N(i)$ the normal bundle of $X /\!\!/ \hat{U}$ in $X /\!\!/ \lambda(\mathbb{C}^*)$, and write $V_{\mathfrak{u}} = \mu_{S^1}^{-1}(0) \times_{S^1} \mathfrak{u}$ for the associated vector bundle, where $\mathfrak{u}$ is the Lie algebra of U. Then one has equality of bundles $N(i) \simeq V_{\mathfrak{u}}^*$, and a natural ring-isomorphism in cohomology (see [10, Theorem 7.13]).

$$H^*(X /\!\!/ \hat{U}, \mathbb{Q}) \simeq \frac{H^*(X /\!\!/ \lambda(\mathbb{C}^*), \mathbb{Q})}{\mathrm{ann}(c_{\mathrm{top}}(V_{\mathfrak{u}}))},$$

where ann denotes the annihilator, and c_{top} denotes the top equivariant Chern class.

Finally, this isomorphism induces two surjective ring homomorphisms

$$\kappa_{\mathbb{C}^*} : H^*_{S^1}(X, \mathbb{Q}) \longrightarrow H^*(X /\!\!/ \lambda(\mathbb{C}^*), \mathbb{Q})$$

and

$$\kappa_{\hat{U}} : H^*_{\hat{U}}(X, \mathbb{Q}) = H^*_{S^1}(X, \mathbb{Q}) \longrightarrow H^*(X /\!\!/ \hat{U}, \mathbb{Q})$$

relating equivariant cohomology of X to cohomology of the non-reductive GIT quotients.

Theorem 5.5 (Corollary 7.16, [10]). Let X be a nonsingular projective variety with a well-adapted action of $\hat{U} = U \rtimes \lambda(\mathbb{C}^*)$ such that semistability coincides with stability for $\hat{U}$, and let z denote a generic coordinate on the Lie algebra of $\mathbb{C}^*$. Given any $\eta \in H^*_{\hat{U}}(X, \mathbb{Q})$ represented by an equivariant differential form $\eta(z) \in H^*(X /\!\!/ \hat{U}, \mathbb{Q})$ of degree $\dim X /\!\!/ \hat{U}$, one has

$$\int_{X/\!\!/\hat{U}} \kappa_{\hat{U}}(\eta) = n_{\mathbb{C}^*} \operatorname*{Res}_{z=\infty} \int_{F_{\min}} \frac{i^*_{F_{\min}}\big(\eta(z) \cup c_{\mathrm{top}}(V_{\mathfrak{u}})(z)\big)}{c_{\mathrm{top}}(N_{F_{\min}/X})(z)} dz,$$

where $F_{\min}$ is the part of the fixed point locus $X^{\mathbb{C}^*}$, μ_{S^1} takes minimal value $\omega_{\min}$, $n_{\mathbb{C}^*}$ is the size of the stabilizer in $\lambda(\mathbb{C}^*)$ of a generic $x \in X$, and c_{top} denotes the top equivariant Chern class.

Chapter 6

Setup

6.1 Bases and partitions

We start by setting up some notation for partitions and particular sequences of partitions that we will be interested in. The goal is to describe the polynomial generators of the vanishing ideal of the Berczi-Szenes model

$$\phi_E : \widetilde{J_k(1,n)}_E \dashrightarrow \mathbb{P}\Big(\bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n\Big)$$

described in Section 2.3.

6.1.1 Partitions and sequences of partitions

We write $\mathcal{P}_s$ for the partitions of $s \in \mathbb{Z}_{>0}$, and depict such partition $p \in \mathcal{P}_s$ by symbols

$$p = i_1^{l_1} \cdots i_r^{l_r} \in \mathcal{P}_s, \quad \text{where } i_1 < \cdots < i_r \ . \tag{6.1}$$

and we define for such partition p the sum

$$|p| := l_1 i_1 + \cdots + l_r i_r = s.$$

By $l(p) = l_1 + \cdots + l_r$ we denote the *length of* p, and by $\mathrm{Parts}(p)$ we denote the multiset

$$\mathrm{Parts}(p) = \{\underbrace{i_1, \ldots, i_1}_{l_1}, \ldots, \underbrace{i_r, \ldots, i_r}_{l_r}\},$$

while $\underline{\mathrm{Parts}}(p) = (i_1, \ldots, i_r)$ denotes the underlying *ordered* (we chose from smallest to largest) set of $\mathrm{Parts}(p)$. If $|\mathrm{Parts}(p)| = 1$ we shall say that p is *linear.*

For another partition q with $l(q) = l(p)$, writing $\mathrm{Parts}(p) = (i_1, \ldots, i_{l(p)})$ and $\mathrm{Parts}(q) = (j_1, \ldots, j_{l(q)})$ we define the relation

$$q \leq p \iff j_m \leq i_m \quad \text{for all } m \ .$$

For a sequence of k partitions $\pi = (\pi_1, \ldots, \pi_k)$ we extend the sum-notation $|\pi| = |\pi_1| + \cdots + |\pi_k|$, and define the length vector together with its sum

$$l(\pi) = (l(\pi_1), \ldots, l(\pi_k)) \quad \text{and} \quad L(\pi) = l(\pi_1) + \cdots + l(\pi_k).$$

For another sequence of partitions π' with $l(\pi') = l(\pi)$ we define the relation

$$\pi' \leq \pi \iff \pi'_i \leq \pi_i \quad \text{for all } i.$$

We define the multiset of parts of π

$$\mathrm{Parts}(\pi) = \mathrm{Parts}(\pi_1) \cup \cdots \cup \mathrm{Parts}(\pi_k)$$

together with it's underlying *ordered* set $\underline{\mathrm{Parts}}(\pi)$. At last the action of permutations on π:

Definition 6.1. For a fixed dimension $n \in \mathbb{Z}_{>0}$, a sequence $\pi = (\pi_1, \ldots, \pi_k)$ of partitions is *n-admissible* if the following bullets hold

- The underlying set of $\underline{\mathrm{Parts}}(\pi_i)$ is a subset of $\{1, \ldots, n\}$ for all i,
- $\pi_i \in \mathcal{P}_j$ is a partition of some $j \leq i$, and
- $\pi_i \neq \pi_j$ for $i \neq j$.

We will often leave out the dependence on n and write just *admissible*.

Definition 6.2. Consider a sequence of partitions $\pi = (\pi_1, \ldots, \pi_k)$. We say that π_i is *l-defect* if $\pi_i \in \mathcal{P}_{i-l}$ and write $\mathrm{def}(\pi_i) = l$.

We define the *defect of* π as

$$\mathrm{def}(\pi) := \mathrm{def}(\pi_1) + \cdots + \mathrm{def}(\pi_k) = k(k+1) - |\pi|.$$

If π is 0-defect, we say that π is *toric*.

We adopt here the same definition for a sequence of *ordered* partitions (a.k.a. compositions), which we will need below, but do not give much attention.

By $\mathrm{perm}(\pi)$ we denote the set of reorderings σ of any subpartitions such that the reordered $\sigma.\pi$ – after reordering each new partition $(\sigma.\pi)_i$ in the form of (6.1) – is again an admissible sequence of partitions. In particular, the magnitude of lengths is preserved $L(\pi) = L(\sigma.\pi)$.

For a sequence of ordered partitions, we define the set of permutations and their action in the same way – only we do not reorder parts by size in the end.

Example 6.3. We give some examples of reorderings.

(1) Take $\pi = (1, 2, 3, 13, 2^2)$. Then $\pi' = (1, 2, 3, 12, 23)$ is a reordering obtained by swapping the part 3 in π_4 with one of the parts in π_5.

(2) Take $\pi = (1, 2, 12, 13, 2^2)$. Then $\pi' = (1, 2, 3, 12, 12^2)$ is a reordering obtained by cyclically permuting subpartitions $12, 3, 2$ in π_3, π_4, π_5, respectively.

(3) Take $\pi = (1, 2, 3, 12, 13)$. One checks that $\pi' = (1, 1^2, 2, 3, 23)$ is a reordering of π.

♦

Definition 6.4. We say that an admissible sequence π is *complete* if for every subpartition $\rho \subset \pi_i$ there exists j such that $\pi_j = \rho$.

We denote by $\mathcal{A}_{n,k}$ the set of n-admissible sequences of partitions of length k. Further, we define the set $\mathcal{T}_k$ of *ordered* toric partitions of length k.

We shall be particularly interested in a certain class of pairs $(\pi, \tau) \in \mathcal{A}_{n,k} \times \mathcal{T}_k$.

Definition 6.5. A pair $(\pi, \tau) \in \mathcal{A}_{n,k} \times \mathcal{T}_k$ is *n-admissible* if

- $l(\pi) = l(\tau)$, and
- $\pi \leq \tau$.

We say that an admissible pair (π, τ) is l-defect if π is l-defect, and also that (π, τ) is toric if π is toric.

As for admissible sequences of partitions, we will often leave out the dependence on n, and simply say that a pair (π, τ) is *admissible*.

We introduce also an equivalence relation on the set of admissible pairs in $\mathcal{A}_{n,k} \times \mathcal{T}_k$. First we extend the notation of permutations to such pairs (π, τ). For a reordering of the pair to make sense, we must require that the reordering $\sigma.\pi$ is admissible and that $\sigma.\tau$ is toric (it will then follow that $(\sigma.\pi, \sigma.\tau)$ is an admissible pair).

A reordering $\sigma \in \operatorname{perm}(\pi)$ acts on the pair

$$\sigma.(\pi, \tau) = (\sigma.\pi, \sigma.\tau),$$

and we define then $\operatorname{perm}(\pi, \tau) \subset \operatorname{perm}(\pi)$ to be the subset of reorderings σ such that the reordered sequence $\sigma.\tau$ is toric.

Example 6.6. We extend on the examples of Example 6.3.

(1) Take $(\pi, \tau) = ((1, 2, 3, 13, 2^2), (1, 2, 3, 13, 23))$. Then

$$(\pi', \tau') = ((1, 2, 3, 12, 23), (1, 2, 3, 13, 23))$$

is a reordering obtained by swapping the the pair of subpartitions $(3, 3)$ in (π_4, τ_4) with the pair $(2, 3)$ in (π_5, τ_5).

(2) Take $(\pi, \tau) = ((1, 2, 12, 13, 2^2), (1, 2, 12, 13, 23))$. Then

$$(\pi', \tau') = ((1, 2, 3, 12, 12^2), (1, 2, 3, 13, 12^2))$$

is a reordering obtained by cyclically permuting the pairs of subpartitions $(12, 12)$,$(3, 3)$, $(2, 3)$ in (π_3, τ_3),(π_4, τ_4),(π_5, τ_5), respectively.

(3) Take $(\pi, \tau) = ((1, 2, 3, 12, 13), (1, 2, 3, 13, 14))$. One checks that

$$(\pi', \tau') = ((1, 1^2, 2, 3, 23), (1, 1^2, 3, 4, 23))$$

is a reordering of (π, τ).

♦

The induced equivalence relation is

$$(\pi, \tau) \sim (\pi', \tau') \iff \begin{array}{l}\text{There exists a permutation } \sigma \in \operatorname{perm}(\pi, \tau) \\ \text{such that } \sigma.(\pi, \tau) = (\pi', \tau'),\end{array}$$

and we drop the soft brackets in favor of the square ones to denote equivalence classes: $[\pi, \tau] = [\pi', \tau']$. We define then

$$\mathcal{Q}_{n,k} = \{(\pi, \tau) \in \mathcal{A}_{n,k} \times \mathcal{T}_k \mid (\pi, \tau) \text{ is admissible}\}/\sim .$$

We shall abuse the language slightly and talk simply about admissible pairs $[\pi, \tau]$. The notion of l-defect carries over to equivalence classes. In particular, we say that $[\pi, \tau]$ is toric if π is toric.

6.1.2 Basis elements and torus actions

We proceed to define basis elements for some vector spaces and introduce notation for actions on these spaces. We denote by $e_1, \dots, e_n$ the standard basis of $\mathbb{C}^n$ and consider always the maximal torus $T \leq \mathrm{GL}_n$ acting diagonally on this basis with distinct weights $\lambda_1, \dots, \lambda_n$.

Throughout we work with the Berczi-Szenes model of which we consider many variants (that is, changing the source and target space a little bit). To set up the notation here, we consider the model

$$\begin{aligned} \phi_E : \widetilde{J_k(1,n)}_E &\dashrightarrow \mathbb{P}(\bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n) \\ (v_1, \dots, v_k) &\longmapsto \big[v_1 \wedge (v_2 + v_1^2) \wedge (v_3 + 2v_1v_2 + v_1^3) \wedge \cdots \wedge \sum_{\sigma \in \mathcal{P}_k} |\mathrm{perm}(\sigma)|\, v_\sigma\big], \end{aligned}$$

where $|\mathrm{perm}(\sigma)|$ is the number of compositions representing the partition σ and $v_i \in \mathrm{Span}(e_1, \dots, e_i)$ for $i = 1, \dots, m = \min(n, k)$, so the elements of $\widetilde{J_k(1,n)}_E \simeq \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ are upper triangular on the first m columns, and the induced action of T acts with weight λ_i on the a basis element $E_{i,j}$.

The image of the map ϕ_E is a subset of the subspace

$$\begin{aligned} \big\{v_1 \wedge (v_2 + v_1^2) \wedge \cdots \wedge \sum_{\sigma \in \mathcal{P}_k} |\mathrm{perm}(\sigma)|\, v_\sigma : v_i \in \mathrm{Span}(e_1, \dots, e_i) \text{ for } i = 1, \dots, m\big\} \\ \subset \bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n, \end{aligned}$$

which is spanned by the basis elements

$$e_\pi = e_{\pi_1} \wedge \cdots \wedge e_{\pi_k}, \qquad \pi \in \mathcal{A}_{n,k} \text{ admissible},$$

where for a partition $p = i_1^{l_1} \cdots i_r^{l_r} \in \mathcal{P}_j$ we have written

$$e_p = e_{i_1}^{l_1} \cdots e_{i_r}^{l_r} \in \mathrm{Sym}^j \mathbb{C}^n.$$

We write $[e_\pi] \in \mathbb{P}(\bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$ for the point corresponding to the line spanned by $e_\pi \in \bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n$.

The induced action of T gives e_p the T-weight $\lambda_p = l_1\lambda_{i_1} + \cdots + l_r\lambda_{i_r}$, and further the weight of e_π is then

$$(\lambda_{\pi_1}, \dots, \lambda_{\pi_k}).$$

6.2 Monomial notation

Here we will introduce notation associated to the equations of the vanishing of the Berczi-Szenes model

$$\phi_{n,k} : \widetilde{J_k(1,n)_E} \dashrightarrow \mathbb{P}(\bigwedge^k \bigoplus_{i=1}^k \operatorname{Sym}^i \mathbb{C}^n).$$

First, we write

$$\widetilde{J_k(1,n)_E} = \operatorname{Spec} R_{n,k}, \qquad R_{n,k} = \mathbb{C}[b_{i,j} : 1 \leq i \leq n, i \leq j \leq k].$$

The ideal $I_{n,k} \subset R_{n,k}$ denotes the vanishing ideal of $\phi_{n,k}$ generated by the polynomial coefficients (in the variables $b_{i,j}$) of the basis elements

$$e_\pi = e_{\pi_1} \wedge \cdots \wedge e_{\pi_k} \in \bigwedge^k \bigoplus_{i=1}^k \operatorname{Sym}^i \mathbb{C}^n, \qquad \pi \in \mathcal{A}_{n,k} \text{ admissible},$$

and $M_{n,k}$ denotes the ideal generated by the monomials of these polynomials. When there is no confusion, we will leave out the indices n, k.

For partitions π_i and τ_i with (multiset) parts $\operatorname{Parts}(\pi_i) = \{\pi_{i,1}, \ldots, \pi_{i,r}\}$ and $\operatorname{Parts}(\tau_i) = \{\tau_{i,1}, \ldots, \tau_{i,r}\}$ coming from an admissible pair (π, τ), we write

$$b_{\pi_i,\tau_i} = b_{\pi_{i,1},\tau_{i,1}} \cdots b_{\pi_{i,r},\tau_{i,r}}$$

as a product of variables, and extend to the pair (π, τ) by writing

$$m_\pi^\tau = b_{\pi_1,\tau_1} \cdots b_{\pi_k,\tau_k}.$$

Note that (π, τ) is toric if and only if $m_\pi^\tau \in \mathbb{C}[b_{i,i} : 1 \leq i \leq k]$, and we may refer to the variables $b_{i,i}$ as being *toric*. Observe moreover, that for π toric there is only a single associated monomial m_π^τ, namely with $\tau = \pi$.

The reason for introducing the equivalence relation on admissible pairs (π, τ) is

Lemma 6.7. For admissible pairs (π, τ) and (π', τ') we have

$$m_\pi^\tau = m_{\pi'}^{\tau'} \iff [\pi, \tau] = [\pi', \tau'] \in \mathcal{Q}_{n,k}$$

Proof. Expanding m_π^τ and $m_{\pi'}^{\tau'}$ to their product of variables it is evident that equality is equivalent to a reordering of pairs of subpartitions. □

The monomial generators $m \in M_{n,k}$ are thus in 1-1 correspondence with equivalence classes of admissible pairs $[\pi, \tau] \in \mathcal{Q}_{n,k}$, and we will abuse notation and write just m_π^τ for these generators.

6.3 Interplay between models for $\mathrm{CHilb}_0^k(\mathbb{C}^n)$ for different values of n and k

For $j \le k \le n$, we choose an embedding

$$\mathbb{C}^j = \mathrm{Span}(e_1, \dots, e_j) \hookrightarrow \mathbb{C}^n,$$

and define the embeddings given by extending with $e_{j+1}, \dots, e_k$, e.g.

$$\begin{aligned} \bigwedge^j \bigoplus_{i=1}^j \mathrm{Sym}^i \mathbb{C}^n &\hookrightarrow \bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n \\ v &\mapsto v \wedge e_{j+1} \wedge \dots \wedge e_k. \end{aligned}$$

We have the commutative diagram

$$\begin{tikzcd} \widetilde{J_j(1,j)}_{E|_j} \arrow[r, hook] \arrow[d, "\phi_{j,j}"] & \widetilde{J_j(1,n)}_E \arrow[r, hook] \arrow[d, "\phi_{n,j}"] & \widetilde{J_k(1,n)}_E \arrow[d, "\phi_{n,k}"] \\ \mathbb{P}(\bigwedge^j \bigoplus_{i=1}^j \mathrm{Sym}^i \mathbb{C}^j) \arrow[r, hook] & \mathbb{P}(\bigwedge^j \bigoplus_{i=1}^j \mathrm{Sym}^i \mathbb{C}^n) \arrow[r, hook] & \mathbb{P}(\bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n). \end{tikzcd}$$

where the symbol $(\,\cdot\,)|_j$ indicates a *truncation to* $j \le k$. We adapt the notation to $\pi|_j = (\pi_1, \dots, \pi_j)$, $\tau|_j = (\tau_1, \dots, \tau_j)$ and

$$m_\pi^\tau|_j = m_{\pi|_j}^{\tau|_j} = b_{\pi_1,\tau_1} \cdots b_{\pi_j,\tau_j}.$$

The diagram is compatible with inclusions $\mathrm{Diff}_j \hookrightarrow \mathrm{Diff}_k$, and we see that $I_{n,j} = I_{j,j}$ is generated by the polynomials obtained by restricting the polynomial generators of $I_{n,k}$ to j. Furthermore, the monomial ideal $M_{n,j} = M_{j,j}$ is generated the by the monomials $m_\pi^\tau|_j$. In other words

$$I_{n,k}|_j = I_{n,j} = I_{j,j} \quad \text{and} \quad m_{n,k}|_j = m_{n,j} = m_{j,j}.$$

These monomial and ideals live in the polynomial ring $R_{j,j}$, and we have naturally also the inclusions

$$R_{j,j} \subset R_{n,j} \subset R_{n,k}.$$

Observe that considering "the diagonal" $n = k$ we obtain infinite increasing systems of and can in principle define in the limit the spaces

$$R_{\infty,\infty}, \ \ \widetilde{J_\infty(1,\infty)}_\infty, \ \ \bigwedge^\infty \bigoplus_{i=1}^\infty \mathrm{Sym}^i \mathbb{C}^\infty$$

and the model

$$\phi_{\infty,\infty} : \widetilde{J_\infty(1,\infty)}_\infty \to \mathbb{P}(\bigwedge^\infty \bigoplus_{i=1}^\infty \mathrm{Sym}^i \mathbb{C}^\infty).$$

We shall not make use of these limit objects as such, but we as the resolution algorithm for $\phi_{k,k}$ is defined iteratively by working with the ideals $M_{k,k}, I_{k,k} \subset R_{k,k}$, it can be beneficial to consider all these ideals as ideals in the same space, namely $R_{\infty,\infty}$. This is purely formal.

6.4 Non-reductive GIT setup

To ensure the existence of the non-reductive GIT quotient of the source space $\widetilde{J_k(1,n)_E}$ of the Berczi-Szenes model

$$\phi_E : \widetilde{J_k(1,n)}_E \dashrightarrow \mathbb{P}\Big(\bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i\,\mathbb{C}^n\Big),$$

we must have that ss=s for the Diff_k-action, where we recall the description of the embedding $\mathrm{Diff}_k \hookrightarrow \mathrm{GL}_k$ in (2.5)

$$U = \Big\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_k \\ 0 & \alpha_1^2 & \cdots & \sum\limits_{i_1+i_2=k} \alpha_{i_1}\alpha_{i_2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_1^k \end{pmatrix} \in \mathrm{Diff}_k \,\Big|\, \alpha_1 = 1, \alpha_2, \ldots, \alpha_k \in \mathbb{C} \Big\}.$$

First, we work with the projective completion of the source space $J_k(1,n)$ in accordance with Theorem 5.3. As in Section 2.3 we write $m = \min(n,k)$ and $M = \max(n,k)$, and taking fiberwise projective completions over $\mathrm{Flag}(1,\ldots,m;\mathbb{C}^n)$, we write

$$\mathbb{P} := \mathbb{P}(\widetilde{J_k(1,n)} \oplus \mathbb{C}) = \mathrm{GL}_n \times_{P_{n,k}} \mathbb{P}(\widetilde{J_k(1,n)}_E \oplus \mathbb{C}) \to \mathrm{GL}_M / P_{m,M} = \mathrm{Flag}(1,\ldots,m;\mathbb{C}^n),$$

where the first $\mathbb{C}$ is the trivial bundle and $E = [\mathrm{Span}(e_1) \subset \cdots \subset \mathrm{Span}(e_1,\ldots,e_m)] \in \mathrm{Flag}(1,\ldots,m;\mathbb{C}^n)$ is the standard flag. The Berczi-Szenes model is then homogenized by the coordinate of $\mathbb{C}$.

For purposes of localization it is enough to study the the fiber over $E \in \mathrm{Flag}(1,\ldots,m;\mathbb{C}^n)$, which we denote by $\mathbb{P}_E = \mathbb{P}(\widetilde{J_k(1,n)}_E \oplus \mathbb{C})$ where $\widetilde{J_k(1,n)}_E = \pi_{n,k}^{-1}(E)$ is the fiber over E of the fibration $\pi_{n,k} : \widetilde{J_k(1,n)} \to \mathrm{Flag}(1,\ldots,m;\mathbb{C}^M)$ defined in Section 2.3.

We pick the 1-parameter subgroup

$$\lambda : \mathbb{C}^* \to \mathrm{Diff}_k, \quad t \mapsto \mathrm{diag}(t^{-1},\ldots,t^{-k}), \tag{6.2}$$

and observe that $\mathfrak{u}$ in accordance with the expression of U above is obtained by taking the derivative $\frac{d}{dt}|_{t=0}$ of expressions $\alpha_{i_1}\cdots\alpha_{i_r}(t)$. We calculate the adjoint action of λ on $\mathfrak{u}$ where we write $a_j = \alpha_j'(0)$

$$\begin{aligned} &\lambda(t) \begin{pmatrix} 1 & a_2 & \cdots & a_k \\ 0 & 1 & \cdots & \sum\limits_{i_1+i_2=k} (a_{i_1}\alpha_{i_2}(0) + \alpha_{i_1}(0)a_{i_2}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \lambda^{-1}(t) \\ &= \begin{pmatrix} 1 & ta_2 & \cdots & t^{k-1}a_k \\ 0 & 1 & \cdots & t^{k-2} \sum\limits_{i_1+i_2=k} (a_{i_1}\alpha_{i_2}(0) + \alpha_{i_1}(0)a_{i_2}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \end{aligned} \tag{6.3}$$

from which we see that the weights are $1,2,\ldots,(k-1)$ on the generators $a_2,\ldots,a_k$, so are all strictly positive as is needed for $\hat{U} = \mathrm{Diff}_k$ to have internally graded unipotent radical (cf. Definition 5.1).

Moreover, λ acts on the $\mathbb{C}$-coordinate x with weight 0, and on the coordinate $b_{i,j}$ with weight j. The action of the subgroup λ on $\mathbb{P}_E$ thus has the 1-point minimal weight space $Z_{\min} = \{[x : \mathbf{0}]\} \subset \mathbb{P}_E$ with full stabilizer U, and so ss$\neq$s for Diff_k (cf. condition (5.1)). To remedy this situation, we blow up $\mathbb{P}_E$ in this point

$$\begin{aligned}\tilde{\mathbb{P}}_E &= \mathrm{Bl}_{[x:\mathbf{0}]}(\mathbb{P}_E)\\ &= \{([x : (\nu_1, \dots, \nu_k)], [\omega_1, \dots \omega_k]) \mid \\ &\qquad \nu_i \otimes \omega_j = \nu_j \otimes \omega_i,\ \nu_i, \omega_i \in \mathrm{Span}(e_1, \dots, e_i) \text{ for } i = 1, \dots, m\},\end{aligned}$$

and write $D = \{([x : \mathbf{0}], \omega) \in \tilde{\mathbb{P}}\}$ for the exceptional divisor. Fix the linearization to be the pullback of

$$L = \mathcal{O}_{\mathbb{P}_E}(1) \otimes \mathcal{O}_D(1)$$

under the embedding $\tilde{\mathbb{P}}_E \hookrightarrow \mathbb{P}_E \times D$, and observe that the minimal weight space is then

$$\tilde{Z}_{\min} = \{([x : \mathbf{0}], [\omega_1 : \mathbf{0} \cdots : \mathbf{0}]) \mid x, \omega_{1,1} \neq 0\} \subset \tilde{\mathbb{P}}_E,$$

where $\omega_1 = (\omega_{1,1}, 0, \dots, 0) \in \mathrm{Span}(e_1)$, with contracting set

$$(\tilde{\mathbb{P}}_E)^0_{\min} = \{([x : (\nu_1, \dots, \nu_k)], [\omega_1, \dots \omega_k]) \in \tilde{\mathbb{P}}_E \mid x, \omega_{1,1} \neq 0\}.$$

Moreover, $\tilde{\mathbb{P}}_E$ satisfies ss=s for Diff_k as defined in (5.1), that is,

$$\mathrm{Stab}_U(z) = \{\mathbf{1}_k\} \subset \mathrm{Diff}_k \subset \mathrm{Mat}_k,$$

and it follows that the non-reductive GIT quotient exists, see Theorem 5.3. Since the non-reductive GIT quotient is a quotient of the semistable part, which is contained in the contracting set $(\tilde{\mathbb{P}}_E)^0_{\min}$, we are only interested in the chart where $\omega_{1,1} \neq 0$ and denote this chart by $\tilde{\mathbb{A}}_{n,k}$. Pulling back the Berczi-Szenes model along the blow up, we obtain a map

$$\tilde{\mathbb{A}}_{n,k} \dashrightarrow \mathbb{P}\Big(\bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n\Big),$$

which serves as the starting point for the blow up algorithm, which will be described in Section 7.

We have already noted that a basis element $E_{i,j}$ of $\widetilde{J_k(1,n)}_E$ (with $i \leq j$) has T-weight λ_i where $T \subset \mathrm{GL}_n$ is the maximal torus in the Subsection 6.1.2, which translates to the fact that the T-weight of the coordinate $b_{i,j}$ in $\widetilde{J_k(1,n)}_E$ is λ_i. Thus the total $(T \times \mathrm{Diff}_k)$ weight of the coordinate $b_{i,j}$ in $\widetilde{J_k(1,n)}_E$ is

$$\omega_{i,j} = \lambda_i + j z_0.$$

After the initial blow up just described, we consider the chart $\tilde{\mathbb{A}}$, and we abuse notation and write still $b_{i,j}$ for the affine coordinates in $\tilde{\mathbb{A}}$ and $\omega_{i,j}$ for its total weight. We have then

$$\omega_{i,j} = \lambda_i - \lambda_1 + (j-1) z_0$$

for $(i, j \neq (1,1)$ and $\omega_{1,1} = \lambda_1 + z_0$ is unchanged.

6.5 A slice of the Diff_k-action and a branched covering

For computational reasons it will be beneficial to choose a slice of the Diff_k-action on $\widetilde{J_k(1,n)_E}$ on which each coordinate $b_{i,j}$ is Diff_k-invariant. In addition, we will use a branched covering to obtain that all $b_{i,j}$ share the same Diff_k-weight.

The slice is found by considering the algebraic expression of the action of Diff_k on $J_k(1,n)$, (see (2.4)). Write $P_{i,j}$ for the (i,j)'th entrance of the generic point of Diff_k – a polynomial expression in the coordinates $\alpha_1,\dots,\alpha_k$. The action of Diff_k on a coordinate $b_{i,j}$ is then generically

$$\mathrm{Diff}_k.b_{i,j} = P_{i,j}b_{i,i} + \cdots + P_{j,j}b_{i,j},$$

where specifically $P_{j,j} = \alpha_1^j$. Performing the substitution $\alpha_j \rightsquigarrow b_{1,j}$, and putting $\alpha_1 \rightsquigarrow b_{1,1} = 1$ since we consider the induced action on $\mathbb{A}$, we obtain the change of coordinates

$$b_{i,j} \mapsto b_{i,j} - P_{i,j}(b_{1,2},\dots,b_{1,k})b_{i,i} - \cdots - P_{j,j}(b_{1,2},\dots,b_{1,k})b_{i,j}.$$

The change of coordinates has two implications

(1) Each coordinate $b_{i,j}$ is Diff_k-invariant.

(2) The vanishing ideal I is transformed accordingly by evaluating the generating polynomials at $b_{1,2},\dots,b_{1,k} = 0$.

   (1) The monomial ideal M transforms by removing all generators divisible by some of $b_{1,2},\dots,b_{1,k}$.

We finish these transformations by taking a branched k-covering

$$b_{i,j} \mapsto b_{i,j}b_{1,1}^{k-j},\ b_{1,1} \mapsto b_{1,1}^k,$$

and abusing notation we write still $\mathbb{A}$ for the corresponding preimage. Pulling back further along the blow up $\tilde{\mathbb{P}}_F = \mathrm{Bl}_{[x:\mathbf{0}]}(\mathbb{P}_F)$ described above in Section 6.4, the map ϕ of the Berczi-Szenes model takes the form (abusing notation now both for $\mathbb{A}$ and ϕ)

$$\phi_{n,k} : \tilde{\mathbb{A}}_{n,k} \dashrightarrow \mathbb{P}\Big(\bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n\Big).$$

The implication of introducing this covering is

3. All coordinates have the same Diff_k-weight 1.

   (1) All polynomial generators of the vanishing ideal I are homogeneous of the same degree.

In Section 6.4 we gave a description of the total weight of each coordinate $b_{i,j}$ on the space $\tilde{\mathbb{A}}_{n,k}$ to be $\omega_{i,j} = \lambda_i - \lambda_1 + (j-1)z_0$ for $(i,j) \neq (1,1)$ and $\omega_{1,1} = \lambda_1 + z_0$. After choosing this slice and covering, we have

$$\omega_{i,j} = \lambda_i - \lambda_1 + z_0$$

for $(i,j) \neq (1,1)$ and still $\omega_{1,1} = \lambda_1 + z_0$.

6.5.1 The monomial generators of M

In this subsection we describe the monomial generators of the monomial ideal M of the vanishing ideal I of the Berczi-Szenes model ϕ on the slice of the Diff_k-action described above. We recall that all polynomial generators of I are now homogeneous of the same degree, and so all monomial generators of M are of the same degree; it can be found e.g. by maximizing the number of 2's appearing in $\pi = (\pi_1, \dots, \pi_k)$. We denote this degree by

$$d_k = \begin{cases} l^2 & \text{for } k = 2l, \\ l(l+1) & \text{for } k = 2l+1, \end{cases} \tag{6.4}$$

and observe that for a monomial m_π^τ each factor $b_{i,j} \neq b_{1,1}$ is in 1-1 correspondence with pairs of parts $(\pi_{l,m}, \tau_{l,m}) = (i,j)$, whereas $b_{1,1}$ serves as a homogenizing coordinate.

Indeed, following the notation and structure of Section 6.3 of restriction to $j \leq k$, we see that $\deg m_\pi^\tau|_j = d_j$, or even more

$$\deg b_{\pi_j, \tau_j} = d_j - d_{j-1} = \lfloor j/2 \rfloor$$

with $\lfloor \cdot \rfloor$ denoting the integer part.

6.6 Isolated T-fixed points on the blow up source space

In this section we discuss the issue of non-isolated T-fixed points on the source space of the map $\phi_{n,k} : \mathrm{Bl}(\tilde{\mathbb{A}}_{n,k})^s \dashrightarrow \mathbb{P}(\bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$, where T is the maximal torus in GL_k. In order to work around this issue we will introduce yet another torus-action. For the actual construction of the source space $\mathrm{Bl}(\tilde{\mathbb{A}}_{n,k})^s$ we refer to Section 7, but we emphasize here that it is performed via a sequence of blow ups in *ideals generated by coordinates*.

The issue to understand and overcome in the scenario of non-isolated fixed points is that the weights $\omega_{i,j}$, as we have described them in Section 6.5, are not linearly independent. This implies that after (a sequence of) blow ups the weights of two coordinates $b_{r,s}$ and $b_{t,u}$ on a chart of the blow up might satisfy $\omega_{r,s} = \omega_{t,u}$. In this case all points on the line defined by $b_{i,j} = 0$ for all $(i,j) \neq (r,s), (t,u)$ are T-fixed.

Lemma 6.8. Let T be a torus acting diagonally on $\mathbb{C}^n = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]$ with weights $t_1, \dots, t_n$. Let $m < n$ and consider the blow up

$$\mathrm{Bl}_I(\mathbb{C}^n) = \{((x_1, \dots, x_n), [y_1, \dots, y_m]) \mid x_i y_j = x_j y_i\} \subset \mathbb{C}^n \times \mathbb{P}^{m-1}$$

in the ideal $I = (x_1, \dots, x_m)$. Let T act on $\mathrm{Bl}_I(\mathbb{C}^n)$ via the naturally induced action on $\mathbb{C}^n \times \mathbb{P}^{m-1}$. Write X for the affine chart of the blow up $\mathrm{Bl}_I(\mathbb{C}^n)$ where $y_1 \neq 0$.

(1) The induced diagonal weights on X are

$$u_1 = t_1, u_2 = t_2 - t_1, \dots, u_m = t_m - t_1, u_{m+1} = t_{m+1}, \dots, u_n = t_n.$$

(2) If $t_1, \dots, t_n$ are linearly independent, then $u_1, \dots, u_n$ are linearly independent.

Proof. On the affine chart $X = \operatorname{Spec}\mathbb{C}[z_1, \dots, z_n]$ the coordinates are given by $(z_1, \dots, z_n) = (x_1, y_2, \dots, y_m, x_{m+1}, \dots, x_n)$. For $i \in \{2, \dots, m\}$ we have on X the equation $x_i = y_i x_1$ and hence the weight u_i of y_i is

$$u_i = t_i - t_1,$$

and since for $j \notin \{2, \dots, m\}$ we have $z_j = x_j$ the weights are not changed $u_j = t_j$.

Suppose

$$0 = \sum_{i=1}^{n} a_i u_i = (a_1 - a_2 - \cdots - a_m)t_1 + a_2 t_2 + \cdots + a_n t_n.$$

If $t_1, \dots, t_n$ are linearly independent, then

$$a_2 = \cdots = a_n = 0 = a_1 - a_2 - \cdots - a_m,$$

and so also $a_1 = 0$, proving the linear independence of $u_1, \dots, u_n$. □

We remedy the issue of non-isolated weights by introducing yet another torus action on $\tilde{\mathbb{A}}_{n,k}$ to make sure that the weights $\omega_{i,j}$ are linearly independent. We simply let also the maximal torus $T' \subset \mathrm{GL}_{\dim \tilde{\mathbb{A}}}$ act diagonally on $\widetilde{J_k(1,n)}_E$ with distinct (linearly independent) weights $t_{i,j}$. This induces actions of T' on the projective completion $\mathbb{P}_E$ and further on the chart $\tilde{\mathbb{A}}$ such that the total weights (of the action of $T \times T' \times \mathrm{Diff}_k$) $\omega_{i,j}$ are transformed to

$$\omega_{i,j} = \lambda_i - \lambda_1 + z_0 + t_{i,j}.$$

Since the $t_{i,j}$ are linearly independent, the weights $\omega_{i,j}$ are also linearly independent, and it follows that upon picking charts through a sequence of blow ups, the weights remain linearly independent by Lemma 6.8, and so the $T \times T'$-fixed points on $\mathrm{Bl}(\tilde{\mathbb{A}}_{n,k})^s$ are linearly independent. Thus with $\lambda(\mathbb{C}^*)$ the 1-parameter subgroup defined in (6.2), the $T \times T' \times \lambda(\mathbb{C}^*)$-fixed points on the blow up (still to be constructed in the Section 7) $\mathrm{Bl}(\tilde{\mathbb{A}}_{n,k})^s$ are isolated.

Remark 6.9. We observe that although the Berczi-Szenes model $\phi_{n,k} : \mathrm{Bl}(\tilde{\mathbb{A}}_{n,k})^s \dashrightarrow \mathbb{P}(\bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$ is T-equivariant, it is certainly not T'-equivariant.

This is also not necessary for us. The goal is to integrate on $\mathrm{CHilb}_0^k(\mathbb{C}^n)$, and this is done by pulling back the integral along a resolution of $\phi_{n,k}$. In order to calculate this integral on the source space, we apply equivariant localization (cf. Remark 4.5). ♦

Chapter 7

The blow up algorithm for $\mathrm{CHilb}^{k+1}(\mathbb{C}^k)$

In this chapter we describe a blow up algorithm partly resolving the Berczi-Szenes model in the case $n = k$. In general throughout this section we will simply attach the index k instead of the pair $(n, k) = (k, k)$. The variant of the Berczi-Szenes model to be resolved is

$$\begin{aligned} \phi_k : \tilde{\mathbb{A}}_k &\dashrightarrow \mathbb{P}(\bigwedge^k J_k(k,1)^*) = \mathbb{P}(\bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^k) \\ \gamma &\longmapsto \nu_1 \wedge (\nu_2 + \nu_1^2) \wedge \cdots \wedge \sum_{\sigma \in \mathcal{P}(k)} \mathrm{sym}(\sigma)\, \nu_\sigma, \end{aligned}$$

which was defined in Section 6.5.

The algorithm will be divided into k many steps $A_1, \ldots, A_k$, such that performing the steps $A_1, \ldots, A_j$ partly resolves the restricted model

$$\phi_j : \tilde{\mathbb{A}}_j \dashrightarrow \mathbb{P}\Big(\bigwedge^j \bigoplus_{i=1}^j \mathrm{Sym}^i \mathbb{C}^j\Big).$$

To be more precise, considering a T-action coming from the natural GL_k-action on the source space $J_k(1, k)$, the goal will be to blow up $J_k(1, k)$ both GL_k- and Diff_k-equivariantly to obtain a partial resolution map $\phi_k[k] : \tilde{\mathbb{A}}_k[k] \dashrightarrow \mathbb{P}(\bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^k)$, where $\tilde{\mathbb{A}}_k[k]$ is the blow up, such that $\phi_k[k]$ is *well-defined in a neighborhood of the T-fixed points.*

7.1 Choice of blow up centers

Seeking to obtain a (partial) resolution φ defined on a blow up of $J_k(1, n)$ to use for localization purposes, we emphazise that φ must be Diff_k-invariant to obtain a map on the categorical non-reductive GIT of this blow up. Secondly, φ must be T-equivariant to apply equivariant localization.

In order to make sure that φ stays Diff_k-invariant and T-equivariant it is sufficient to take all blow up centers invariant with respect to these actions, since it follows that the blow up will then be *equivariant* with respect to the and Diff_k- and T-action, and φ thus Diff_k-invariant and T-equivariant. In fact, since all coordinates $b_{i,j}$ are Diff_k-invariant by the choices in Section 6.5, the blow up centers will be chosen as *ideals generated by coordinates.* Moreover, the centers chosen in the steps $A_1, \ldots, A_i$ depend only on the coordinates $b_{i,j}$ with $j \leq i$, that is, are centers in $J_i(1, i) \hookrightarrow J_k(1, k)$.

Since all coordinates $b_{i,j}$ share the same Diff_k-weight, it follows that each blow up has two possible outcomes for the change of the Diff_k-weights. Recall that we are only interested in the semistable locus, and hence only charts corresponding to coordinates of minimal weight are of interest, and thus

(1) If all coordinates of the blow up center have non-zero Diff_k-weight, then on any chart of the exceptional divisor, say corresponding to $b_{i,j}$, the Diff_k-weight of all other coordinates transform to 0, while the Diff_k-weight of $b_{i,j}$ remains invariant.

(2) If some of the coordinates of the blow up center have Diff_k-weight 0, then only charts corresponding to these coordinates are considered, and all Diff_k-weights are invariant under this blow up.

7.1.1 Notation for charts of exceptional divisors

Throughout the blow up procedure we shall often refer to centers and charts of the exceptional divisor of a blow up by some shorthand notation. In particular, in general terms consider an affine space $\mathbb{A} = \operatorname{Spec}\mathbb{C}[x_1, \dots, x_m, x_{m+1}, \dots, x_n]$ and the ideal $C = (x_1, \dots, x_m)$, which will serve as out center for the blow up. The description of the blow up of $\mathbb{A}$ in C is

$$\mathrm{Bl}_C(\mathbb{A}) = \{((x_1, \dots, x_m, x_{m+1}, \dots, x_n), [y_1 : \dots : y_m]) \mid x_i y_j = y_i x_j\},$$

and an affine chart of the exceptional divisor $D = \{((x_1, \dots, x_m, 0, \dots 0), [\mathbf{y}]) \in \mathrm{Bl}_C(\mathbb{A})\}$, say we take $y_1 \neq 0$, has description

$$\begin{aligned}\mathbb{A}_1 &= \{((x_1, \dots, x_m, x_{m+1}, \dots, x_n), (\tilde{y}_2 : \dots, \tilde{y}_m)) \mid x_i = x_1 \tilde{y}_i\} \\ &\simeq \operatorname{Spec}\mathbb{C}[x_1, \tilde{y}_2 : \dots, \tilde{y}_m, x_{m+1}, \dots, x_n],\end{aligned}$$

where $\tilde{y}_i = y_i / y_1$. We will abuse notation and altogether neglect the projective notation, since we will always consider the affine charts. As such, on $\tilde{\mathbb{A}}_1$, we will keep the notation $x_1, \dots, x_n$ for all variables, and we will merely refer to a chart $\mathbb{A}_1$ by writing $x_1 \neq 0$, or simply refer to it as "the x_1-chart".

Moreover, we are interested in induced torus-actions on such a blow up. So suppose that a torus T acts diagonally on $\mathbb{A}$ with weights $t_1, \dots, t_n$, we will then describe the corresponding induced weights on $\mathbb{A}_1$. We will always fix the linearization to be the pullback of

$$\mathcal{O}_{\mathbb{A}}(1) \otimes \mathcal{O}_D(1),$$

so that the we have equalities of weights $w_T(y_i) = w_T(x_i) = t_i$ for $1 \leq i \leq m$. It follows that on the chart $\mathbb{A}_1$, we will obtain weights

$$w_T(\tilde{y}_i) = w_T(y_i) - w_T(y_1) = w_T(x_i) - w_T(x_1) = t_i - t_1 \quad \text{for} \quad 2 \leq i \leq m.$$

Altogether, considering directly a chart $x \neq 0$ of a blow up of $\tilde{\mathbb{A}}_k = \operatorname{Spec}\mathbb{C}[\{b_{i,j} : 1 \leq i \leq n, i \leq j \leq k\}]$ in a (monomial) center C with generators forming the set $\mathcal{C} \ni x$, the transformation from $\tilde{\mathbb{A}}$ to the chart $x \neq 0$ is described by a change of coordinates and change of the weight $w_{i,j}$ on $\tilde{\mathbb{A}}_k$ (write w_x for the weight of x)

$$b_{i,j} \longmapsto \begin{cases} b_{i,j} & \text{for } b_{i,j} \in \mathcal{C}^c \cup \{x\} \\ x b_{i,j} & \text{for } b_{i,j} \in \mathcal{C} \setminus x, \end{cases}$$

$$w_{i,j} \longmapsto \begin{cases} w_{i,j} & \text{for } b_{i,j} \in \mathcal{C}^c \cup \{x\} \\ w_{i,j} - w_x & \text{for } b_{i,j} \in \mathcal{C} \setminus x, \end{cases}$$

Blow up tree of a sequence of blow ups

A sequence of blow ups on an affine space may be visualized by a *rooted tree* with each node corresponding to an affine chart: Given a node corresponding to an affine space $\mathbb{A}$, performing a blow up in a center with ideal generated by coordinates to obtain $\mathrm{Bl}_C(\mathbb{A})$ we associate new nodes for each coordinate in C of minimal Diff_k-weight. We connect the original node $\mathbb{A}$ with each of the new nodes (corresponding to a chart of the blow up) by an edge labeled with the corresponding coordinate.

We refer in this language to a successive choice of charts on each exceptional divisor as choosing a *branch* of the blow up. Moreover, the charts of the final space obtained after the sequence of blow ups correspond exactly to the final nodes of the blow up tree, and we refer to these as *leaves*. We will always denote the set of leaves by $\mathcal{L}$.

7.1.2 The algorithmic steps A_i

We define the algorithm $A = (A_j)_{j \in \mathbb{Z}_{\geq 0}}$ inductively in j such that each step A_j can be described from the Berczi-Szenes model (from sections 6.4 and 6.5)

$$\phi_{j,j} : \tilde{\mathbb{A}}_j \dashrightarrow \mathbb{P}\Big(\bigwedge^j \bigoplus_{i=1}^j \mathrm{Sym}^i \mathbb{C}^j \Big).$$

Each step A_k will consist of a sequence of blow ups. In this setup we are assuming $n = k$, which is in fact not a restriction. Recall that in this case we write only a single index. We write thus I_k for the vanishing ideal of ϕ_k and M_k for the monomial ideal generated by the monomials of the generators of I_k. Here

$$\tilde{\mathbb{A}}_k = \mathrm{Spec}\, R_k, \qquad R_k = \mathbb{C}[b_{i,j} : 1 \leq i \leq j \leq k],$$

and we have chosen a slice of the Diff_k-action such that all monomial generators m_π^τ of M_k all share the same degree d_k and are not divisible by $b_{1,j}$ for $j \geq 2$, and such that all the remaining $b_{i,j}$ are Diff_k-invariant. We define for this reason the subalgebra

$$S_k = \mathbb{C}[b_{1,1}, b_{i,j} : 1 < i \leq j \leq k] \subset R_k,$$

and have $I_k \subset S_k$.

Since each A_k is a sequence of blow ups, we introduce inductively the notation $(\,\cdot\,)[k]$ for the pullback of $(\,\cdot\,)[k-1]$, through all the blow ups of A_k, where we define also $(\,\cdot\,)[0] = (\cdot)$. This notation applies to monomials m_π^τ, ideals I and M, the rings R and S, the source space $\tilde{\mathbb{A}}$ as well as the map ϕ. In principle we want to consider $M[k]$ on any branch after performing $A_1, \ldots, A_{k-1}$, and should thus include such branch in the notation. It turns out, as will be apparent from the arguments and in particular Proposition 7.2, that in order to describe the next step A_k (up to little ambiguity) we need only the information of the last chart of the branch, and we will use this to simplify the notation:

Suppose $A_1, \ldots, A_{k-1}$ have been performed and consider a pullback $(\,\cdot\,)[k-1]_B$ on a branch B_{k-1} (recall, this means a choice of successive charts on every exceptional divisor of the blow ups constituting $A_1, \ldots, A_{k-1}$). Performing A_k and considering a branch B_k extending B_{k-1} and ending with the chart $x \neq 0$ we ought to write $(\,\cdot\,)[k]_{B_k}$, but since the description of A_{k+1} depends (up to little ambiguity) only on x, we will merely write $(\,\cdot\,)[k]_x := (\,\cdot\,)[k]_{B_k}$. Again, the notation applies e.g. to monomials, ideal and rings of the setup. The result is the following

Theorem 7.1. After performing the steps $A_1, \ldots, A_k$ the monomial ideal $M_{k,k}[k]$ is a principal ideal on any branch.

In particular, the map $\phi_{k,k}[k]$ is well-defined in a sufficiently small neighborhood of every T-fixed point of $\tilde{\mathbb{A}}_{k,k}[k]$.

We proceed to define the steps A_k. The justification for this definition is Proposition 7.2, and the procedure of performing the steps is illustrated in Example 7.3.

Since $\phi_{1,1} \equiv e_1$ is always defined, we take A_1 to be an empty process (we do nothing). However, due to the initially performed blow up (to obtain semistability=stability for U as in Section 6.4), we shall in fact write artificially $x = b_{1,1} \neq 0$ for the chart (this is only to allow an easy statement in Proposition 7.2, not distinguishing the case $k = 1$).

Suppose the $A_1, \ldots, A_{k-1}$ have been performed and consider a branch B with the last chart $x \neq 0$, so that $M_{k-1}[k-1]_x$ is a principal ideal. It will follow that the ideal $C_{k,x} = (x, b_{2,k}, \ldots, b_{k,k})$ – after removing common factors of generators – is a minimal prime of $M_k[k-1]_x$ (essentially since any monomial $m_\pi^\tau[k-1]_x$ has nonzero Diff_k-weight, and these variables are the only ones with nonzero Diff_k-weight, see Proposition 7.2).

We define now the *preparation of* A_k to be a sequence of blow ups such that the pullback $\widetilde{M_k[k-1]_x}$ of $M_k[k-1]_x$ through these blow ups satisfies $\widetilde{M_k[k-1]_x} = C_{k,x}$ – we will be specific after this paragraph; each blow up center in this preparation contains at least one coordinate not in $C_{k,x}$, which thus has Diff_k-weight 0, and hence no Diff_k-weights are changed in the preparation. The preparation depends on the chosen branch B in the sequence of blow ups $A_1, \ldots, A_{k-1}$ (not just on the last chart $x \neq 0$).

At last, we blow up in $\widetilde{M_k[k-1]_x} = C_{k,x} = (x, b_{2,k}, \ldots, b_{k,k})$ finishing step A_k, and obtain that on any chart $y \neq 0$ of this last blow up, the pullback ideal $M_k[k]_y = (y)$ of $\widetilde{M_k[k-1]_x}$ is principal.

Proposition 7.2. Consider a branch of the steps $A_1, \ldots, A_k$ with final chart $x \neq 0$ then

(1) The coordinates with positive Diff_{k+1}-weights in the ring $S_{k+1}[k]_x$ are exactly variables $x, b_{2,k+1}, \ldots b_{k+1,k+1}$,

(2) The ideal $C_{k+1,x} = (x, b_{2,k+1}, \ldots, b_{k+1,k+1})$ is a minimal prime of the ideal obtained by dividing all generators of $M_{k+1}[k]_x$ through by their common factor,

(3) The ideal $M_k[k]_x$ in $S_k[k]_x$ is a principal ideal.

In the following proof, we will without mentioning think of the monomial ideals $M_1, M_2, M_3, \ldots$ as being ideals in the same ring. One may take the ring $R_{\infty,\infty}$ discussed briefly in Section 6.3. In a finite setting with fixed k, we may take the ring $R_{k,k}$.

Proof. We argue by induction on k. For $k = 1$, recall that A_1 is an empty process, but we write artificially $x = b_{1,1} \neq 0$ for the chart. We have $S_2[1]_{b_{1,1}} = \mathbb{C}[b_{1,1}, b_{2,2}]$ and both coordinates indeed have Diff_2-weight 1. Also there are only two monomial generators of type m_π^τ of the monomial ideal $M_2[1]_{b_{1,1}}$. These correspond to

$$[\pi, \tau] \in \{[(1,2),(1,2)], [(1,1^2).(1,1^2)]\}$$

yielding (on the chosen slice of the Diff_2-action) the monomials

$$m_{(1,2)}^{(1,2)} = b_{1,1} \cdot b_{2,2} \quad \text{and} \quad m_{(1,1^2)}^{(1,1^2)} = b_{1,1} \cdot b_{1,1},$$

and we see that indeed – after dividing away the common factor $b_{1,1}$ – the ideal $C_{2,b_{1,1}} = (b_{1,1}, b_{2,2})$ is a minimal prime of $M_2[1]_{b_{1,1}}$. Here we see also that A_2 consists of the single blow up of $M_2[1]_{b_{1,1}}$ in $C_{2,b_{1,1}}$. We give the next few steps $k = 3$ and part of $k = 4$ in the example following this proof.

Consider the subbranch of the branch in question of steps $A_1, \ldots, A_{k-1}$ with final chart $y \neq 0$. By induction the coordinates of positive Diff_k-weight in $\operatorname{Spec} S_k[k-1]_y$ are exactly $y, b_{2,k}, \ldots b_{k,k}$, and the ideal $C_{k,y} = (y, b_{2,k}, \ldots b_{k,k})$ is a minimal prime of $M_k[k]_y$. We describe now the step A_k utilizing this hypothesis.

We start by making the crucial observation that the Diff_k-weight of a monomial generator m_π^τ is unchanged under blow ups. We consider generators $m_\pi^\tau \in M_{k+1}$, monomials of (degree and) Diff_{k+1}-weight $D = d_1 + \cdots + d_{k+1}$. It follows that a pullback $m_\pi^\tau[k-1]_y$ has Diff_{k+1}-weight D as well, whereas the restriction $m_\pi^\tau|_k[k-1]_y$ has (Diff_{k+1}- and) Diff_k-weight $D - d_{k+1}$.

Since any such m_π^τ, and also $m_\pi^\tau|_k[k-1]_y$, is divisible at most once by only one of $b_{i,k}$, it follows that any $m_\pi^\tau|_k[k-1]_y$ is divisible by y at least $D - d_{k+1} - 1$ times. Since the map ϕ is projective we may simply divide through by the power $y^{D-d_{k+1}-1}$ without changing the map; we have reduced to the case that all $m_\pi^\tau|_k[k-1]_y$ have Diff_k-weight 1.

The preparation process of A_k consists first of grouping these generators $m_\pi^\tau|_k[k-1]_y$ by their factor $y, b_{2,k}, \ldots b_{k,k}$ of Diff_k-weight 1. We divide the preparation process into a preparation process for each of these coordinates.

Let $z \in \{y, b_{2,k-1}, \ldots b_{k-1,k-1}\}$, and consider the set Z of monomial generators $m_\pi^\tau|_k[k-1]_y$ divisible by z, where we divide through by the common factor z; that is, the elements of Z take the form $m_\pi^\tau|_k[k-1]_y/z$, and have thus Diff_k-weight 0. The preparation $\mathscr{P}_z$ of z is then a sequence of blow ups, such that the ideal generated by Z pulls back on all branches to a principal ideal. Such preparation process is far from unique.

By abuse of language we call also the sequence of blow ups $\mathscr{P}_z$ applied (in the obvious way) to the ideal generated by $Z \cup \{y, b_{2,k}, \ldots b_{k,k}\} \setminus \{z\}$ the preparation of z. It follows that the pullback of the ideal $M_k[k-1]_y$ through $\mathscr{P}_z$ has (after dividing through by a common factor, as we did with a power of y above) z as one of its generators on any branch of $\mathscr{P}_z$.

Now, perform $\mathscr{P}_z$ for all $z \in \{y, b_{2,k}, \ldots b_{k,k}\}$ and denote the pullback ideal of $M_k[k-1]$ by $M_k[k-1]$ – the order of chosen z is irrelevant. It follows that on any branch of these preparation blow ups this pullback ideal takes the form $C_{k,y} = (y, b_{2,k}, \ldots, b_{k,k})$. So in particular on the branch in question, the pullback of $M_k[k-1]_y$ through the full preparation process of A_k leaves us with the ideal $C_{k,y} = (y, b_{2,k}, \dots, b_{k,k})$. The step A_k on this branch is now finished by blowing up in the ideal $C_{k,y}$ and taking the chart $x \neq 0$, where by construction $x \in \{y, b_{2,k}, \dots, b_{k,k}\}$, to obtain $M_{k+1}[k]_x$. We observe that the ideal $M_k[k]_x = (x)$ is principal. This proves (3).

It is clear that in $S_k[k]$ only x has positive Diff_k-weight, and hence $x, b_{2,k+1}, \dots b_{k,k+1}$ are the coordinates of $S_{k+1}[k]$ of positive Diff_{k+1}-weight, proving (1).

We consider still $m_\pi^\tau \in M_{k+1}$ and recall

$$m_\pi^\tau = m_\pi^\tau|_k \cdot b_{\pi_{k+1},\tau_{k+1}}.$$

Still $b_{\pi_{k+1},\tau_{k+1}}[k]_x$ has Diff_{k+1}-weight d_{k+1}, and since any $b_{\pi_{k+1},\tau_{k+1}}$ can at most be divisible once by one of $b_{i,k+1}$, it follows that $b_{\pi_{k+1},\tau_{k+1}}[k]_x$ is divisible by $x^{d_{k+1}-1}$. Moreover, since $M_k[k]_x = (x)$ (after performing the required divisions, described earlier for y in this proof), we obtain that any $m_\pi^\tau[k]_x$ is divisible by x^{d_k}. Dividing through by this common factor in the monomials $m_\pi^\tau[k]_x$ yields monomial generators of Diff_{k+1}-weight 1, which are thus divisible by one of $b_{2,k+1}, \dots b_{k+1,k+1}, x$. It is clear that any of these factors actually appear; corresponding to $\pi_{k+1} = 2, \dots, k+1$ and π_k not linear, respectively. Whence $C_{k+1,x} = (x, b_{2,k+1}, \dots, b_{k+1,k+1})$ is a minimal prime of the ideal generated by the generators of $M_{k+1}[k]_x$ after removing their common factors, which proves (2). □

Example 7.3. In the proof of Proposition 7.2 above we have already described the case $k = 2$ of the proposition. In this case the step A_2 consisted of the blow up in $C_{2,b_{1,1}} = (b_{1,1}, b_{2,2})$. Recall also that A_1 is empty. We proceed to describe the $k = 3$ and part of $k = 4$.

Take now $n = k = 3$. On the chosen slice of the Diff_3-action (recall that no monomials are divisible by $b_{1,j}$ for $j \geq 2$) we have the following table of fixed points e_π with their associated monomials underneath

$$\begin{array}{ccccc} e_1 \wedge e_2 \wedge e_3 & e_1 \wedge e_2 \wedge e_1e_2 & e_1 \wedge e_1^2 \wedge e_2 & e_1 \wedge e_1^2 \wedge e_3 & e_1 \wedge e_1^2 \wedge e_1^3 \\ b_{2,2}b_{3,3} & b_{2,2}^2 & b_{1,1}b_{2,3} & b_{1,1}b_{3,3} & b_{1,1}^2 \end{array}$$

Performing the empty step A_1 and then A_2, there are two charts to consider: $b_{1,1} \neq 0$ and $b_{2,2} \neq 0$.

Consider first the chart $b_{1,1} \neq 0$. The generators (by Proposition 7.9 the fixed points must contain e_1 and e_1^2, and by Corollary 7.8 they must be complete) of $M_3[2]_{b_{1,1}}$ – after dividing through by common factors – are the monomials

$$\begin{array}{ccc} e_1 \wedge e_1^2 \wedge e_2 & e_1 \wedge e_1^2 \wedge e_3 & e_1 \wedge e_1^2 \wedge e_1^3 \\ b_{2,3} & b_{3,3} & b_{1,1} \end{array}$$

and there is no preparation process for A_3 on this branch, since we have $M_3[2]_{b_{1,1}} = C_{2,b_{1,1}} = (b_{1,1}, b_{2,3}, b_{3,3})$ already. Step A_3 on this branch is concluded by blowing up $M_3[2]_{b_{1,1}}$ in $C_{2,b_{1,1}}$, such that the pullback ideal $M_3[3]_{b_{1,1}}$ is a principal ideal on each chart.

On the chart $b_{2,2} \neq 0$, the generators (by Proposition 7.9 the fixed points must contain e_1 and e_2, and by Corollary 7.8 they must be complete) of $M_3[2]_{b_{2,2}}$ – after dividing through by common factors – are the monomials

$$\begin{array}{ccc} e_1 \wedge e_2 \wedge e_3 & e_1 \wedge e_2 \wedge e_1e_2 & e_1 \wedge e_1^2 \wedge e_2 \\ b_{3,3} & b_{2,2} & b_{1,1}b_{2,3} \end{array}$$

We observe that $C_{2,b_{2,2}} = (b_{2,2}, b_{2,3}, b_{3,3})$ is indeed now a minimal prime of $M_3[2]_{b_{2,2}}$. We perform the preparation process on this branch: This requires a blow up of $M_3[2]_{b_{2,2}}$ in the center $(b_{1,1}, b_{2,2}, b_{3,3})$, where only $b_{1,1}$ has Diff_3-weight 0, so we consider only the chart $b_{1,1} \neq 0$. We denote the pullback ideal by $\widetilde{M_3[2]}$, and it has generators

$$\begin{array}{ccc} e_1 \wedge e_2 \wedge e_3, & e_1 \wedge e_2 \wedge e_1e_2, & e_1 \wedge e_1^2 \wedge e_2 \\ b_{3,3}, & b_{2,2}, & b_{2,3}, \end{array}$$

that is, $\widetilde{M_3[2]}_{b_{2,2}} = C_{2,b_{2,2}} = (b_{2,2}, b_{2,3}, b_{3,3})$. At last, we conclude the step A_3 on this branch by blowing up $\widetilde{M_3[2]}_{b_{2,2}}$ in $C_{2,b_{2,2}}$, such that the pullback ideal $M_3[3]_{b_{2,2}}$ is a principal ideal on each chart.

For $n = k = 4$, we finish this example by describing the extensions of the branch B in A_1, A_2, A_3 determined by taking the chart $b_{2,2} \neq 0$ in A_2 and the chart $b_{3,3} \neq 0$ in A_3. By Proposition 7.9 the generators of $M_4[3]_{b_{3,3}} = M_4[3]_B$ must contain e_1, e_2 and e_3, and by Corollary 7.8 they must be complete. Hence the T-fixed points e_π must have π of the form

$$\begin{cases} (1, \pi_2, 2, 3) & \text{for} \quad \pi_2 \in \{1^2\} \\ (1, \pi_2, 3, 2) & \text{for} \quad \pi_2 \in \{1^2\} \\ (1, 2, \pi_3, 2) & \text{for} \quad \pi_3 \in \{12\} \\ (1, 2, 3, \pi_4) & \text{for} \quad \pi_4 \in \{4, 12, 13, 2^2\} \end{cases}$$

We give again the fixed points and their associated monomials – this time on the right. We start by giving them before any blow ups were performed, that is, in M_4.

$e_1 \wedge e_1^2 \wedge e_2 \wedge e_3$	$b_{1,1}^2 b_{2,3} b_{3,4}$, $b_{1,1}^2 b_{3,3} b_{2,4}$
$e_1 \wedge e_2 \wedge e_1e_2 \wedge e_3$	$b_{1,1} b_{2,2}^2 b_{3,4}$
$e_1 \wedge e_2 \wedge e_3 \wedge e_4$	$b_{1,1} b_{2,2} b_{3,3} b_{4,4}$
$e_1 \wedge e_2 \wedge e_3 \wedge e_1e_2$	$b_{1,1} b_{2,2} b_{2,3} b_{3,3}$
$e_1 \wedge e_2 \wedge e_3 \wedge e_1e_3$	$b_{1,1} b_{2,2} b_{3,3}^2$
$e_1 \wedge e_2 \wedge e_3 \wedge e_2^2$	$b_{2,2}^3 b_{3,3}$

Pulling the monomials back to $M_4[3]_{b_{3,3}}$ and removing the common factor $b_{1,1}^4 b_{2,2}^2 b_{3,3}^3$, we get

$e_1 \wedge e_1^2 \wedge e_2 \wedge e_3$	$b_{2,3} b_{3,4}$, $b_{1,1} b_{2,4}$
$e_1 \wedge e_2 \wedge e_1e_2 \wedge e_3$	$b_{2,2} b_{3,4}$
$e_1 \wedge e_2 \wedge e_3 \wedge e_4$	$b_{4,4}$
$e_1 \wedge e_2 \wedge e_3 \wedge e_1e_2$	$b_{2,3} b_{3,3}$
$e_1 \wedge e_2 \wedge e_3 \wedge e_1e_3$	$b_{1,1} b_{3,3}$
$e_1 \wedge e_2 \wedge e_3 \wedge e_2^2$	$b_{2,2} b_{3,3}$

It is clear that $C_{3,b_{3,3}} = (b_{3,3}, b_{2,4}, b_{3,4}, b_{4,4})$ is a minimal prime of $M_4[3]_{b_{3,3}}$. We perform the preparation process for each generator-coordinate of $C_{3,b_{3,3}}$, which we denote by $\mathscr{P}$ with an index of the corresponding coordinate.

In this case all the preparations are given by a single blow up, and we give the corresponding (description of) the center together with the coordinates of relevant charts, we need to consider (those of minimal Diff_4-weight). Here $\emptyset$ means that there is no preparation to be performed.

Preparation	Center	Charts
$\mathscr{P}_{3,3}$:	$(b_{1,1}, b_{2,2}, b_{2,3}, b_{2,4}, b_{3,4}, b_{4,4})$	$b_{1,1}, b_{2,2}, b_{2,3}$
$\mathscr{P}_{2,4}$:	$(b_{1,1}, b_{3,3}, b_{3,4}, b_{4,4})$	$b_{1,1}$
$\mathscr{P}_{3,4}$:	$(b_{2,2}, b_{2,3}, b_{3,3}, b_{2,4}, b_{4,4})$	$b_{2,2}, b_{2,3}$
$\mathscr{P}_{4,4}$:	$\emptyset$	

Choosing a branch in the full preparation of A_4 corresponds to a choice of chart in each of these four (that is three, since $\mathscr{P}_{4,4}$ is empty; and really only choices in two, since $\mathscr{P}_{2,4}$ leaves no choice) preparations of coordinates. Take any such branch B_4 and observe that the pullback through B_4 is $\widetilde{M_4[3]}_{b_{3,3}} = C_{3,b_{3,3}}$.

Writing B' for the branch obtained when extending B by B_4, we conclude A_4 on B' by blowing up in $C_{3,b_{3,3}}$ to obtain that the pullback ideal is principal on each chart. ♦

Proof of Theorem 7.1. The first part of Theorem 7.1 is already part of Proposition 7.2. For the second part, consider a T-fixed point of the pullback $\tilde{\mathbb{A}}_{k,k}[k]$. Since these fixed points are isolated (cf. Section 6.6), they are exactly the 0's of the final charts of the blow ups $A_1, \ldots, A_k$, and so we consider $\phi_{k,k}$ restricted to a branch, say ending on the chart $x \neq 0$, such that the monomial ideal $M_{k,k}$ pulls back to the principal ideal $M_{k,k}[k] = (x)$ (as in the proof of Proposition 7.2). Since $\phi_{k,k}$ is projective, we may divide through by x in the expression of the map, and observe upon this that one of the coordinate projections takes the form

$$f_j = 1 + P(\{b_{i,j}\}), \quad P(\{b_{i,j}\}) \in S_{k,k}[k],$$

where $P(\{b_{i,j}\})$ has Diff_k-weight 0 (i.e. no monomial is divisible by x) and contains no constant term. It follows that near 0 of the chart $x \neq 0$ the polynomial f_j is non-zero, and hence $\phi_{k,k}$ is defined near this fixed point. □

Remark 7.4. In the proof of Proposition 7.2 we described the preparation process for each step A_k, which accomplishes that the pullback $\widetilde{M_k[k-1]}$ of $M_k[k-1]$ is generated by coordinates. We accomplished this by dividing the preparation process into several preparations $\mathscr{P}_z$, which we find to be a quite transparent method. One should however note that this is in general not the most efficient (more efficient meaning in this case fewer blow ups) way of achieving that the pullback ideal is generated by coordinates. For example, if the preparation processes $\mathscr{P}_z$ and $\mathscr{P}_{z'}$ on a branch ending with the chart $x \neq 0$ have some blow ups (that is, blow up centers) $\mathscr{P}_{z,z'}$ in common, it could be more efficient to perform these blow ups on the union of these ideals.

Moreover, we noted already in the proof that also each single preparation $\mathscr{P}_z$, also is not naturally given. ♦

7.1.3 Properties of the blow up algorithm

Observe that performing the steps $A_1, \ldots, A_k$ and considering a branch B the result of Proposition 7.2 is (among other things) that $M_k[k]$ is a principal ideal, so is generated by a single monomial $m_{\pi^B}^{\tau^B}[k]$, which may also be represented as an equivalence class $[\pi^B, \tau^B] \in \mathcal{Q}_k$. The following proposition establishes in essence that the admissible pair (π^B, τ^B) is *uniquely determined* (to be precise, the monomial $m_{\pi^B}^{\tau^B} \in M_k$ occurs only as a monomial – with certain coefficient – of a single polynomial generator of the vanishing ideal I_k of the model ϕ_k). We introduce some notation first.

Definition 7.5. For a sequence of partitions π and a partition p, we define $p \in \pi$ if there exists i such that $p = \pi_i$. If there is no such i, we write $p \notin \pi$.

For a sequence of partitions $(p_1, \ldots, p_r)$, we write $(p_1, \ldots, p_r) \subset \pi$ if for each i, we have $p_i \in \pi$

Proposition 7.6. Let $(\pi, \tau), (\pi', \tau') \in \mathcal{A}_{n,k} \times \mathcal{T}_k$ be admissible pairs with $[\pi, \tau] = [\pi', \tau'] \in \mathcal{Q}_k$ (so that $m_\pi^\tau = m_{\pi'}^{\tau'}$).

If there exists a branch B of $A_1, \ldots, A_k$ such that the monomial pullback $m_\pi^\tau[k]_B$ restricted to the branch B is the generator of the principal ideal $M_k[k]_B$, then $e_\pi = e_{\pi'}$.

Proof. If m_π^τ pulls back to a generator $m_\pi^\tau[k]_B$, then m_π^τ must be a vertex of the Newton polytope of the monomial ideal M_k.

Since $[\pi, \tau] = [\pi', \tau']$ there is a reordering $\sigma \in \operatorname{perm}(\pi, \tau)$ (of subpartitions) such that $\sigma.(\pi, \tau) = (\pi', \tau')$. Assuming that $e_\pi \neq e_{\pi'}$, there are non-empty (maximal) index sets I and J of equal size $r := |I| = |J|$, such that $\pi_i \notin \pi'$ for $i \in I$ and $\pi'_j \notin \pi$ for $j \in J$. We will swap partitions π_l and π'_l in order to obtain two other admissible sequences $\hat{\pi}$ and $\hat{\pi}'$. The procedure is illustrated in the example following this proof.

Take $i \in I$, and start by swapping π_i and π'_i in the two sequences π and π' to obtain $\bar{\pi}$ and $\bar{\pi}'$, respectively. If both $\bar{\pi}$ and $\bar{\pi}'$ are admissible, we take $\hat{\pi}$ and $\hat{\pi}'$ to be these. If not, then $\bar{\pi}'$ is admissible, while $\bar{\pi}$ is not admissible since already $\pi'_i = \pi_j$ and hence $\bar{\pi}_i = \pi'_i = \bar{\pi}_j$. Define $\bar{I}$ and $\bar{J}$ in the same way for $\bar{\pi}$ and $\bar{\pi}'$, and observe that $|\bar{I}| = I + 1$ increases and $|\bar{J}| = J + 1$. Proceed now to make the swap on $\bar{\pi}_j$ ("the partition of inadmissibility for $\bar{\pi}$") and $\bar{\pi}'_j$. Again, if the resulting sequences $\bar{\bar{\pi}}$ and $\bar{\bar{\pi}}'$ are admissible, we take $\hat{\pi}$ and $\hat{\pi}'$ to be these. Otherwise, observe that the resulting index sets $\bar{\bar{I}}$ and $\bar{\bar{J}}$ satisfy $|\bar{\bar{I}}| = |\bar{I}|$ and $|\bar{\bar{J}}| = |\bar{J}|$, and again inadmissibility occurs because $\pi'_j \in \bar{\pi}$ already. It is clear that this procedure terminates for finite π.

Now, observe that all swappings have been performed, such that we may swap pairs (π_l, τ_l) and (π'_l, τ'_l) obtaining admissible pairs $(\hat{\pi}, \hat{\tau})$ and $(\hat{\pi}', \hat{\tau}')$. It is now an easy check that m_π^τ is the midpoint on the line between $m_{\hat{\pi}}^{\hat{\tau}}$ and $m_{\hat{\pi}'}^{\hat{\tau}'}$, that is

$$(m_\pi^\tau)^2 = m_{\hat{\pi}}^{\hat{\tau}} \cdot m_{\hat{\pi}'}^{\hat{\tau}'},$$

and so m_π^τ is not a vertex of the Newton polytope associated to M_k, hence can never pull back to a generator. $\square$

Example 7.7. We consider one of the examples of Example 6.6. Consider namely the admissible pairs

$$(\pi,\tau)=((1,2,3,12,13),(1,2,3,13,14))$$

and

$$(\pi',\tau')=((1,1^2,2,3,23),(1,1^2,3,4,23))$$

satisfying $m_\pi^\tau = m_{\pi'}^{\tau'}$, or equivalently $[\pi,\tau]=[\pi',\tau']$, together with $e_\pi \neq e_{\pi'}$. In the notation of the proof of Lemma 7.6 above, we have $I=\{4,5\}$ and $J=\{2,5\}$. Select e.g. $i=4\in I$ and perform the swapping of (π_4,τ_4) and (π'_4,τ'_4) as described in the proof to obtain

$$(\bar\pi,\bar\tau)=((1,2,3,3,13),(1,2,3,4,14))$$

and

$$(\bar\pi',\bar\tau')=((1,1^2,2,12,23),(1,1^2,3,13,23)),$$

in which $\bar\pi$ is not admissible, and we have $j=3$. We observe that $\bar I=\{3,4,5\}$ and $\bar J=\{2,4,5\}$.

We continue by swapping $(\bar\pi_3,\bar\tau_3)$ and $(\bar\pi'_3,\bar\tau'_3)$ to obtain the pairs

$$(\bar{\bar\pi},\bar{\bar\tau})=((1,2,2,3,13),(1,2,3,4,14))$$

and

$$(\bar{\bar\pi}',\bar{\bar\tau}')=((1,1^2,3,12,23),(1,1^2,3,13,23)).$$

Again, $\bar{\bar\pi}$ is not admissible, and we observe that $\bar{\bar I}=\{2,3,5\}$ and $\bar{\bar J}=\{2,4,5\}$. This time "inadmissibility occurs at" $\bar{\bar\pi}_2=2$, so we swap $(\bar{\bar\pi}_2,\bar{\bar\tau}_2)$ and $(\bar{\bar\pi}'_2,\bar{\bar\tau}'_2)$, and obtain pairs

$$(\hat\pi,\hat\tau)=((1,1^2,2,3,13),(1,1^2,3,4,14))$$

and

$$(\hat\pi',\hat\tau')=((1,2,3,12,23),(1,2,3,13,23))$$

with $\hat\pi$ and $\hat\pi'$ both admissible. One verifies easily that

$$(m_\pi^\tau)^2 = m_{\hat\pi}^{\hat\tau}\cdot m_{\hat\pi'}^{\hat\tau'}.$$

♦

We mention another criterion for a monomial m_π^τ to pull back to a generator of the pullback of M_k in the algorithm.

Proposition 7.8. Consider a branch B of the sequence of blow ups $A_1,\dots,A_k$ and write $M_k[k]_B=(m_\pi^\tau[k]_B)$ (by Proposition 7.2) for some $\pi\in\mathcal{A}_k$. Then π is a complete sequence of partitions.

Proof. Since $M_k[k]_B=(m_\pi^\tau[k]_B)$, it follows that $[e_\pi]\in\overline{\operatorname{Im}\phi}$ is in the image closure of the Berczi-Szenes model. By Proposition 9.15 it follows that π is complete. □

We continue the discussion from before Proposition 7.6. The lemma shows that on any branch B of $A_1,\dots,A_k$ there is an admissible and unique pair $[\pi^B,\tau^B]$ such that $m_{\pi^B}^{\tau^B}$ satisfies $M_k[k]_B=(m_{\pi^B}^{\tau^B}[k]_B)$.

Proposition 7.9. Let $m_\pi^\tau \in M_k$ and consider a branch B of $A_1, \ldots, A_{k-1}$. A necessary condition for $m_\pi^\tau[k-1]_B$ to be a generator of $M_k[k]_B$ is that $\pi^B \subset \pi$.

Proof. Denote by I the (maximal) index set such that for $i \in I$ we have $\pi_i^B \notin \pi$. Let $i \in I$ so $\pi_i^B \notin \pi$. We claim that we can find a pair $[\bar{\pi}, \bar{\tau}]$, such that defining $\bar{I}$ for $\bar{\pi}$ in the same way as I for π, we have $|\bar{I}| = |I| - 1$ and $m_{\bar{\pi}}^{\bar{\tau}}[k-1]_B$ divides $m_\pi^\tau[k-1]_B$. Consider the replacement

$$[(\pi_1, \ldots, \pi_{i-1}, \pi_i^B, \pi_{i+1}, \ldots, \pi_k), (\tau_1, \ldots, \tau_{i-1}, \tau_i^B, \tau_{i+1}, \ldots, \tau_k)].$$

If $\pi_i \notin \pi^B$ we define $[\bar{\pi}, \bar{\tau}]$ to be this pair for which $|\bar{I}| = |I| - 1$ holds. Otherwise, $\pi_i = \pi_{i_2}^B$ for some $i_2 \neq i$, and we write also $i_1 = i$. To ease notation we assume $i_1 < i_2$ (the other case is similar), and replace again to obtain

$$\big[(\pi_1, \ldots, \pi_{i_1-1}, \pi_{i_1}^B, \pi_{i_1+1}, \ldots, \pi_{i_2-1}, \pi_{i_2}^B, \pi_{i_2+1}, \ldots, \pi_k),$$
$$(\tau_1, \ldots, \tau_{i_1-1}, \tau_{i_1}^B, \tau_{i_1+1}, \ldots, \tau_{i_2-1}, \tau_{i_2}^B, \tau_{i_2+1}, \ldots, \tau_k)\big].$$

Again, if $\pi_{i_2} \notin \pi^B$, we define $[\bar{\pi}, \bar{\tau}]$ to be this pair and $|\bar{I}| = |I| - 2$ holds. Otherwise continue this procedure, which surely terminates since π^B is finite. This yields a sequence of indices $i_1, \ldots, i_r$ such that $\pi_{i_l} \in \pi^B$ for $l = 1, \ldots, r-1$ and $\pi_{i_r} \notin \pi^B$ together with a pair $[\bar{\pi}, \bar{\tau}]$ with $|\bar{I}| = |I| - r$. We denote this process by $\mathscr{P}_i$ – The insertion of π_i^B in π.

Now the claim that $m_{\bar{\pi}}^{\bar{\tau}}[k-1]_B$ divides $m_\pi^\tau[k-1]_B$ is by construction equivalent to (by removing common factors)

$$(b_{\pi_{i_1}^B, \tau_{i_1}^B} \cdots b_{\pi_{i_r}^B, \tau_{i_r}^B})[k-1]_B \text{ divides } (b_{\pi_{i_1}, \tau_{i_1}} \cdots b_{\pi_{i_r}, \tau_{i_r}})[k-1]_B. \tag{7.1}$$

To this end, apply the reverse process of $\mathscr{P}_i$ on the pair $[\pi^B, \tau^B]$ (observe that $\mathscr{P}_i$ never alters π_k, and that the reverse process yields an *admissible* π') to obtain a pair $[\pi', \tau']$. Since $m^B[k-1]_B$ generates $M_{k-1}[k-1]_B$, we surely have $m_{\pi^B}^{\tau^B}[k-1]_B$ divides $m_{\pi'}^{\tau'}[k-1]$, and by construction this is equivalent to (7.1). □

Example 7.10. Consider the branch B in $A_1, \ldots, A_4$ with

$$[\pi^B, \tau^B] = [(1, 2, 3, 12), (1, 2, 3, 13)],$$

and the pair

$$[\pi, \tau] = [(1, 1^2, 3, 2, 23), (1, 1^2, 3, 4, 23)].$$

The index set is $I = \{4\}$ since $\pi_4^B \notin \pi$. We pick then $i_1 = 4$ and following the proof of Proposition 7.9, we consider the substitution

$$[(1, 1^2, 3, 12, 23), (1, 1^2, 3, 13, 23)].$$

Since $\pi_4 = 2 = \pi_2^B \in \pi^B$, we continue by putting $i_2 = 2$ and consider

$$[\bar{\pi}, \bar{\tau}] = [(1, 2, 3, 12, 23), (1, 2, 3, 13, 23)].$$

In this case $\pi_2 = 1^2 \notin \pi^B$, and the process $\mathcal{P}_4$ has terminated.

Applying the reverse process of $\mathcal{P}_4$ to $[\pi^B, \tau^B]$ yields the pair

$$[\pi', \tau'] = [(1, 1^2, 3, 2), (1, 1^2, 3, 4)],$$

an so by removing common factors in $[\pi^B, \tau^B]$ and $[\pi', \tau']$, we obtain that

$$(b_{\pi_2^B,\tau_2^B} \cdot b_{\pi_4^B,\tau_4^B})[4]_B \text{ divides } (b_{\pi_2',\tau_2'} \cdot b_{\pi_4',\tau_4'})[4]_B$$

which is equivalent to

$$m_{\bar{\pi}}^{\bar{\tau}}[4]_B \text{ divides } m_\pi^\tau[4]_B.$$

♦

7.1.4 Toricity and image points of the Berczi-Szenes model $\phi_{n,k}$

In this subsection we determine some points in the image closure $\overline{\operatorname{Im}\phi_{n,k}} = \mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$ (Proposition 7.11) and characterize points in the same image closure for varying k under an assumption of 0-defectness (toricity) of a sequence of partitions π (Proposition 7.14).

Recall that for a toric sequence $\pi \in \mathcal{A}_k$ there is only a single associated monomial $m_\pi^\tau = m_\pi^\pi$, namely that with $\tau = \pi$.

Proposition 7.11. Let $\pi \in \mathcal{A}_k$ be toric. Then there exists a branch B of $A_1, \dots, A_k$ on which $m_\pi^\tau[k]_B$ is the generator of $M_k[k]_B$ if and only if π is complete.

In particular, if $\pi \in \mathcal{A}_{n,k}$ is toric, then $[e_\pi] \in \mathrm{CHilb}^{k+1}(\mathbb{C}^n)$.

Proof. That π must be complete follows from Proposition 7.8.

Assume that π is complete then also the restricted π_{k-1} is complete and toric, and by induction there is a branch B of the sequence of blow ups $A_1, \dots, A_{k-1}$ such that

$$m_\pi^\tau|_{k-1}[k-1]_B \text{ is the generator of } M_{k-1}[k-1]_B.$$

By Proposition 7.9 any generator $m_{\pi'}^{\tau'}[k-1]_B$ of $M_k[k-1]_B$ must satisfy that $\pi|_{k-1} \subset \pi'$.

At this point, if there is no extension B' of B on which $m_\pi^\tau|_k[k]_{B'}$ is the generator of the principal ideal $M_k[k]_{B'}$, there must in fact exist $\pi' = (\pi_1, \dots, \pi_{k-1}, \pi_k')$ satisfying

$$m_{\pi',\tau'}[k-1]_B \text{ divides } m_{\pi,\tau}[k-1]_B,$$

but then in particular

$$b_{\pi_k',\tau_k'}[k-1]_B \text{ divides } b_{\pi_k,\tau_k}[k-1]_B.$$

At this point we must have $\pi_k \neq k$. Furthermore, there exist pairs of subpartitions (γ, δ) and (γ', δ') of (π_k, τ_k) and (π_k', τ_k'), respectively, with $|\gamma| = |\gamma'| < k$ satisfying

$$b_{\gamma',\delta'}[k-1]_B \text{ divides } b_{\gamma,\delta}[k-1]_B.$$

Since $m_\pi^\tau|_{k-1}[k-1]_B$ generates $M_{k-1}[k-1]_B$ we see that $\gamma \neq \pi_1, \dots, \pi_{k-1}$ as we wanted.

For the last part we argue as follows: Since $m_\pi^\pi[k]_B$ generates $M_k[k]_B$, we have $[e_\pi] \in \operatorname{Im}\phi_k[k] \subset \overline{\operatorname{Im}\phi_{k,k}}$. But $\pi \in \mathcal{A}_{n,k}$ is assumed n-admissible, so in fact $[e_\pi] \in \overline{\operatorname{Im}\phi_{n,k}}$, and at last by Theorem 2.5, we have

$$\overline{\operatorname{Im}\phi_{n,k}} = \mathrm{CHilb}^{k+1}(\mathbb{C}^n).$$

□

Remark 7.12. In fact, basically the same proof as given for Proposition 7.11 yields the following stronger result:

Let $\pi = (\pi_1, \dots, \pi_k) \in \mathcal{A}_{n,k}$ be a complete sequence with $\pi_k \in \mathcal{P}_k$ a partition of k. Then the following implication holds:

$$[e_{(\pi_1,\dots,\pi_{k-1})}] \in \mathrm{CHilb}_0^k(\mathbb{C}^n) \implies [e_\pi] \in \mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$$

♦

The converse of the implication in the above remark is also true.

Lemma 7.13. Let $\pi = (\pi_1, \dots, \pi_k) \in \mathcal{A}_{n,k}$ be a complete sequence with $\pi_k \in \mathcal{P}_k$ a partition of k. Then

$$[e_\pi] \in \mathrm{CHilb}_0^{k+1}(\mathbb{C}^n) \implies [e_{(\pi_1,\dots,\pi_{k-1})}] \in \mathrm{CHilb}_0^k(\mathbb{C}^n)$$

Proof. We have $[e_\pi] \in \mathrm{CHilb}_0^{k+1}(\mathbb{C}^n) = \overline{\mathrm{Im}\,\phi_{n,k}}$ for the original model $\phi_{n,k} : J_k(1,n) \dashrightarrow \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$. Since $\pi \in \mathcal{A}_{n,k}$ is n-admissible it follows that in particular $[\epsilon_\pi] \in \overline{\mathrm{Im}\,\phi_E}$ is in the image closure of the fibered version $\phi_E : \widetilde{J_k(1,n)}_E \dashrightarrow \mathrm{Grass}_k(\mathrm{Sym}^{\leq k} \mathbb{C}^n)$ as defined in Section 2.3:

To see this, use the language of the mentioned section and construct the associated bundle

$$\begin{aligned}\widetilde{\mathrm{CHilb}}_0^{k+1}(\mathbb{C}^n) &= \mathrm{GL}_m \times_{P_{m,M}} \overline{P_{m,M} . \phi_{n,k}(e_1, \dots, e_m, \nu_{m+1}, \dots, \nu_k)} \\ &= \mathrm{GL}_m \times_{P_{m,M}} \overline{\mathrm{Im}\,\phi_E},\end{aligned}$$

which is a partial resolution

$$\widetilde{\mathrm{CHilb}}_0^{k+1}(\mathbb{C}^n) \longrightarrow \overline{\mathrm{GL}_M . \phi_{n,k}(e_1, \dots, e_m, \nu_{m+1}, \dots, \nu_k)} = \mathrm{CHilb}_0^{k+1}(\mathbb{C}^n).$$

Similarly, construct the fibered version of the Grassmannian by setting

$$\widetilde{\mathrm{Grass}}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)_E := \{v_1 \wedge (v_2 + v_1^2) \wedge \dots \wedge \sum_{\sigma \in \mathcal{P}_k} |\mathrm{perm}(\sigma)|\, v_\sigma \mid$$
$$v_i \in \mathrm{Span}(e_1, \dots, e_i) \text{ for } i = 1, \dots, m\},$$

and defining

$$\widetilde{\mathrm{Grass}}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n) = \mathrm{GL}_n \times_{P_{n,k}} \widetilde{\mathrm{Grass}}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)_E \xrightarrow{\rho} \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n).$$

Then the partial resolution $\widetilde{\mathrm{CHilb}}_0^{k+1}(\mathbb{C}^n)$ is the (fiberwise) compactification of the curvilinear locus in $\widetilde{\mathrm{Grass}}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$. One checks that

$$\rho^{-1}(\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n) = \widetilde{\mathrm{CHilb}}_0^{k+1}(\mathbb{C}^n). \tag{7.2}$$

Now, $\pi \in \mathcal{A}_{n,k}$ implies $[e_\pi] \in \widetilde{\mathrm{Grass}}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)_E$, and since $[e_\pi] \in \mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$ we get from (7.2) that $[e_\pi] \in \overline{\mathrm{Im}\,\phi_E}$, the fiber of $\widetilde{\mathrm{CHilb}}_0^{k+1}(\mathbb{C}^n)$ over E.

Moreover, since $[e_\pi]$ is T-fixed the partial resolution $\phi_{n,k}[k]$ has $[e_\pi] \in \mathrm{Im}\,\phi_{n,k}[k]$. Since π_k is toric, it follows automatically upon forgetting the k'th piece that

$$[e_{(\pi_1,\dots,\pi_{k-1})}] \in \mathrm{Im}\,\phi_{n,k-1}[k] = \overline{\mathrm{Im}\,\phi_{n,k-1}} = \mathrm{CHilb}_0^k(\mathbb{C}^n).$$

□

For completeness we state the full result combining Remark 7.12 and Lemma 7.13

Proposition 7.14. Let $\pi = (\pi_1, \ldots, \pi_k) \in \mathcal{A}_{n,k}$ be a complete sequence with $\pi_k \in \mathcal{P}_k$ a partition of k. Then the following implication holds

$$[e_\pi] \in \mathrm{CHilb}_0^{k+1}(\mathbb{C}^n) \iff [e_{(\pi_1,\ldots,\pi_{k-1})}] \in \mathrm{CHilb}_0^k(\mathbb{C}^n).$$

7.2 The blow up trees of the algorithms

Recall that $\tilde{\mathbb{A}}_k[k]^s$ has been constructed via a sequence of blow ups partially resolving the map $\phi_k : \tilde{\mathbb{A}}_k \dashrightarrow \mathbb{P}(\bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^k)$, and that this a sequences of blow ups can be visualized as a rooted tree with each node corresponding to a chart of the blow up (see Subsection 7.1.1). The final nodes are called leaves of the tree; these correspond exactly to the charts of the space $\tilde{\mathbb{A}}_k[k]^s$. We denote by $\mathcal{T}_{k,k}$ the blow up tree and by $\mathcal{L}$ the set of leaves.

The T-fixed points of $\tilde{\mathbb{A}}_k[k]$ are isolated by the arguments of Section 6.6, and we observe that the T-fixed points of $\tilde{\mathbb{A}}_k[k]^s$ are the 0's of the affine charts corresponding to leaves in the blow up tree. As such we label the fixed points of $\tilde{\mathbb{A}}_k[k]$ by 0_L for $L \in \mathcal{L}$:

$$(\tilde{\mathbb{A}}_k[k]^s)^T = \{0_L \mid L \in \mathcal{L}\}.$$

From the resolution of the map $\phi_{k,k}$, we will now describe how to obtain resolutions for the models $\phi_{n,k}$ of $\mathrm{CHilb}^{k+1}(\mathbb{C}^n)$ for all values of n and k. These resolutions will also be described by a blow up tree.

Definition 7.15. We denote by $\mathcal{T}_{n,k}$ the blow up tree associated to the partial resolution of $\phi_{n,k} : \tilde{\mathbb{A}}_{n,k} \to \mathbb{P}(\bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$.

Observe first that from Theorem 7.1, we have the following inclusion of (abstract) rooted trees we have

$$\mathcal{T}_{1,1} \subset \mathcal{T}_{2,2} \subset \mathcal{T}_{3,3} \subset \ldots.$$

Now fix the parameter k. For $n \geq k$ we have $\widetilde{J_k(1,n)}_E \simeq \widetilde{J_k(1,k)}_E$, and under this isomorphism $\phi_{n,k} = \phi_{k,k}$. Hence we can take $\mathcal{T}_{n,k} = \mathcal{T}_{k,k}$ when $n \geq k$.

For the values $n < k$, we have that $\mathrm{Spec}\,\mathbb{C}[\{b_{i,j} \mid 1 \leq i \leq n, i \leq j \leq k\}] = \widetilde{J_k(1,n)}_{E_{n,k}} \subset \widetilde{J_k(1,k)}_{E_{k,k}} = \mathrm{Spec}\,\mathbb{C}[\{b_{i,j} \mid 1 \leq i \leq j \leq k\}]$ is the subspace cut out by equations $b_{i,j} = 0$ for all $i > n$. Thus, we observe simply that $\phi_{n,k}$ is resolved by the corresponding subtree of $\mathcal{T}_{k,k}$ obtained by removing all branches containing edges labeled by $b_{i,j}$ with $i > n$. We denote this subtree of $\mathcal{T}_{k,k}$ by $\mathcal{T}_{n,k}$; then $\mathcal{T}_{n,k}$ partially resolves $\phi_{n,k}$, such that $\phi_{n,k}[k]$ is well-defined near all T-fixed points of $\tilde{\mathbb{A}}_{n,k}[k]$.

Proposition 7.16. The blow up trees $\mathcal{T}_{n,k}$ associated to the partial resolution of $\phi_{n,k}$: $\tilde{\mathbb{A}}_{n,k} \dashrightarrow \mathbb{P}(\bigwedge^k \bigoplus_{i=1}^k \operatorname{Sym}^i \mathbb{C}^n)$ sit in the following grid as abstract rooted trees

$$\begin{array}{ccccccc}
\mathcal{T}_{1,1} & \subset & \mathcal{T}_{1,2} & \subset & \mathcal{T}_{1,3} & \subset & \cdots \\
\parallel & \rotatebox{-45}{$\subset$} & \cap & & \cap & & \\
\mathcal{T}_{2,1} & \subset & \mathcal{T}_{2,2} & \subset & \mathcal{T}_{2,3} & \subset & \cdots \\
\parallel & & \parallel & \rotatebox{-45}{$\subset$} & \cap & & \\
\mathcal{T}_{3,1} & \subset & \mathcal{T}_{3,2} & \subset & \mathcal{T}_{3,3} & \subset & \cdots \\
\parallel & & \parallel & & \parallel & \rotatebox{-45}{$\subset$} & \\
\vdots & & \vdots & & \vdots & & \ddots
\end{array}$$

Remark 7.17. We write $\mathcal{T}_{n,k} \subset \mathcal{T}_{k,k}$ as abstract rooted tress (that is, with no embedding of the trees). The inclusion $\mathcal{T}_{n,k} \subset \mathcal{T}_{k,k}$ associated to a leaf (affine chart) $\operatorname{Spec} \mathbb{C}[\{b_{i,j} \mid 1 \leq i \leq n, i \leq j \leq k\}] = L_{n,k} \in \mathcal{L}_{n,k}$ a corresponding leaf $\mathbb{C}[\{b_{i,j} \mid 1 \leq i \leq j \leq k\}] = L_{k,k} \in \mathcal{L}_{k,k}$. As such these two leaves correspond to "the same" node in $\mathcal{T}_{n,k}$ and $\mathcal{T}_{k,k}$, but correspond to two different spaces. For instance

$$\begin{aligned}
n < k \implies \dim L_{n,k} &= \frac{k(k+1)}{2} - \frac{(k-n)(k-n+1)}{2} \\
&< \frac{k(k+1)}{2} = \dim L_{k,k}.
\end{aligned}$$

♦

Chapter 8

Integration formulas on $\mathrm{CHilb}_0^k(\mathbb{C}^n)$

In this section we seek to obtain an integration formula on $\mathrm{CHilb}_0^k(\mathbb{C}^n)$ for any n and k. We then rewrite this formula using iterated residues. From the iterated residues we are able to see that in the range $n \geq k$ only a single T-fixed point of $\mathrm{CHilb}_0^k(\mathbb{C}^n)$ contributes to the integration (that is, in the localization formula).

8.1 Localization on $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$

In this section we give a localization formula for integration on $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$ for any parameters n and k. The strategy is to pull back the integration via a (yet to be constructed) well-defined map $\tilde{\varphi}_{n,k} : \widehat{J_k(1,n)}/\!\!/\mathrm{Diff}_k \to \mathbb{P}(\bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$ where $\widehat{J_k(1,n)}$ is a Diff_k- and T-equivariant blow up of $J_k(1,n)$ equipped also with a fibration over $\mathrm{Flag}(1,\dots,m;\mathbb{C}^n)$. We then apply integration (cf. Theorem 5.5) of the non-reductive GIT quotient $\widehat{J_k(1,n)}/\!\!/\mathrm{Diff}_k$ to reduce to integration on the resolution space $\widehat{J_k(1,n)}$, and follow up with equivariant localization (cf. Theorem 4.6) on the fibration $\widehat{J_k(1,n)} \to \mathrm{Flag}(1,\dots,m;\mathbb{C}^n)$. We can then, at last, reduce to integration on $\tilde{\mathbb{A}}_{n,k}[k]$, which we can again describe via equivariant localization (again Theorem 4.6) using the description of the blow up tree associated to $\tilde{\mathbb{A}}_{n,k}[k]$. We setup some notation to remedy these ideas.

We start by defining the well-defined map $\tilde{\varphi}_{n,k} : \widehat{J_k(1,n)}/\!\!/\mathrm{Diff}_k \to \mathbb{P}(\bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$. Take a resolution $\varphi_{n,k,E} : \widehat{J_k(1,n)}_E \to \mathbb{P}(\bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$ of the map $\phi_{n,k}[k] : \tilde{\mathbb{A}}_{n,k}[k] \dashrightarrow \mathbb{P}(\bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$ which is already well-defined at every T-fixed point. Here we choose $\widehat{J_k(1,n)}$ as a Diff_k- and T-equivariant blow up of $\tilde{\mathbb{A}}_{n,k}[k]$ (this is possible by [34]).

As we have done earlier, we write $m = \min(n,k)$. Define then

$$\widehat{J_k(1,n)} = \mathrm{GL}_n \times_{P_{m,n}} \widehat{J_k(1,n)}_E,$$

where $P_{m,n} \subset \mathrm{GL}_n$ is the parabolic subgroup fixing the flag

$$E_{n,k} = [\mathrm{Span}(e_1) \subset \cdots \subset \mathrm{Span}(e_1,\dots,e_m)] \in \mathrm{Flag}(1,\dots,m;\mathbb{C}^n).$$

Observe that this resolution space is equipped with the T-equivariant fibration

$$\widehat{J_k(1,n)} \to \mathrm{GL}_n / P_{m,n} = \mathrm{Flag}(1,\dots,m;\mathbb{C}^n)$$

as well. We have an induced map $\varphi_{n,k} : \widehat{J_k(1,n)} \to \mathbb{P}(\bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$ restricting to $\varphi_{n,k,E}$ on the fiber $\widehat{J_k(1,n)}_E$. Furthermore, $\varphi_{n,k}$ induces a map on the categorical non-reductive

GIT quotient $\tilde{\varphi}_{n,k} : \widehat{J_k(1,n)} /\!\!/ \mathrm{Diff}_k \to \mathbb{P}(\bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$. Recall that the space $\tilde{\mathbb{A}}_{n,k}[k]$ is constructed as a sequence of blow ups which we encode as a rooted tree with every node corresponding to an affine chart (see the description in Section 7.2). The final nodes – corresponding to the affine charts of $\tilde{\mathbb{A}}_{n,k}[k]$ – are called leaves, and we denote by $\mathcal{L}_{n,k}$ the set of leaves. Since the T-fixed points of $\tilde{\mathbb{A}}_{n,k}[k]$ are isolated, they are exactly the 0's of the affine charts $L \in \mathcal{L}_{n,k}$ of $\tilde{\mathbb{A}}_{n,k}[k]$; we denote such fixed point by 0_L:

$$\tilde{\mathbb{A}}_{n,k}[k]^T = \{0_L \mid L \in \mathcal{L}_{n,k}\}.$$

At last, recall that $\mathrm{Diff}_k = U \rtimes \lambda(\mathbb{C}^*)$ with $\lambda : \mathbb{C}^* \to \mathrm{Diff}_k$ defined in (6.2). There is an induced action of Diff_k on $\widehat{J_k(1,n)}$ satisfying that semistability=stability for $\hat{U} = \mathrm{Diff}_k$ as defined in (5.1), and hence the non-reductive GIT quotient $\widehat{J_k(1,n)} /\!\!/ \mathrm{Diff}_k$ exists.

Theorem 8.1. Let z_0 denote a generic coordinate on the Lie algebra of $\lambda(\mathbb{C}^*) \subset \mathrm{Diff}_k$. Fix positive integers n and k, write $m = \min(n,k)$, and denote by $\mathcal{L}_{n,k}$ the leaves of the blow up tree $\mathcal{T}_{n,k}$. Let F be a vector bundle on $\mathbb{C}^n$ of rank r, and $c_1, \ldots, c_{kr}$ the Chern classes of $F^{[k]}$. Give c_j the weight j, and let $\alpha \in H^*(\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n))$ a polynomial in the Chern classes of F of weighted degree $k(n-1)$ we have

$$\int_{\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)} \alpha = \operatorname*{Res}_{z_0=\infty} \sum_{\sigma \in \mathcal{S}_n/\mathcal{S}_{n-m}} \frac{(k-1)! z_0^{k-1}}{\prod_{j=1}^m \prod_{i=j+1}^n (\lambda_{\sigma.i} - \lambda_{\sigma.j})} \sum_{L \in \mathcal{L}} \frac{\sigma.\alpha(\theta_1^L, \ldots, \theta_k^L)}{\sigma.c_{\mathrm{top}}(T_{0_L} L)} dz_0,$$

where $\alpha(\theta_1^L, \ldots, \theta_k^L)$ means substituting in the polynomial expression of α the i'th elementary symmetric polynomial for c_i, and then evaluating in the Chern roots at 0_L, θ_i^L, of the pullback bundle $\phi_k[k]^*\mathcal{E}$ of the tautological bundle $\mathcal{E} \to \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i(\mathbb{C}^n))$, and $\mathcal{S}_n/\mathcal{S}_{n-m}$ denotes the set of injections $\{1, \ldots, m\} \hookrightarrow \{1, \ldots, n\}$.

Proof. Since we have fixed the values of n and k, we drop all indices referring to these.

Start by pulling back via the resolved model $\tilde{\varphi} : \widehat{J_k(1,n)} /\!\!/ \mathrm{Diff}_k \to \mathbb{P}(\bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$ to obtain

$$\int_{\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)} \alpha = \int_{\widehat{J_k(1,n)})/\!\!/\mathrm{Diff}_k} \tilde{\varphi}^*(\alpha).$$

We apply the integration formula on a non-reductive GIT quotient Theorem 5.5 and obtain

$$\int_{\widehat{J_k(1,n)}/\!\!/\mathrm{Diff}_k} \tilde{\varphi}^*(\alpha) = \operatorname*{Res}_{z_0=\infty} \int_{(\widehat{J_k(1,n)})_{\min}} \frac{i^*_{(\widehat{J_k(1,n)})_{\min}} \left(\tilde{\varphi}^*\alpha(z_0) \cup c_{\mathrm{top}}(V_{\mathfrak{u}})(z_0)\right)}{c_{\mathrm{top}}(N_{\min})(z_0)} dz_0,$$

where $(\widehat{J_k(1,n)})_{\min}$ denotes the minimal weight space of $\widehat{J_k(1,n)}$ with respect to the $\lambda(\mathbb{C}^*)$-action, and $N_{\min}$ denotes the normal bundle of $(\widehat{J_k(1,n)})_{\min}$ in $\widehat{J_k(1,n)}$.

In Section 6.4 we identified the weights of the adjoint action of $\lambda(\mathbb{C}^*)$ on the Lie algebra $\mathfrak{u}$ of $U \subset \mathrm{Diff}_k$ to be $1, \ldots, k-1$, and it follows that (the bundle $V_{\mathfrak{u}}$ is defined in Section 5.2)

$$c_{\mathrm{top}}(V_{\mathfrak{u}})(z_0) = (k-1)! z_0^{k-1}.$$

We take an equivariant extension of the form being integrated on $(\widehat{J_k(1,n)})_{\min}$, and apply equivariant localization techniques. Using the Atiyah-Bott, Berline-Vergne localization formula (Theorem 4.6) on the fibration $\widehat{J_k(1,n)} \to \mathrm{Flag}(1, \ldots, m; \mathbb{C}^n)$, we reduce to integration on the fiber $\widehat{J_k(1,n)}_E$

$$\int_{\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)} \alpha = \sum_{\sigma \in \mathcal{S}_n/\mathcal{S}_{n-m}} \frac{\sigma.\left(\operatorname*{Res}_{z_0=\infty} \int_{(\widehat{J_k(1,n)}_E)_{\min}} (k-1)! z_0^{k-1} \frac{i^*_{(\widehat{J_k(1,n)}_E)_{\min}} \tilde{\varphi}^*\alpha(z_0)}{c_{\mathrm{top}}(N_{\min,E})(z_0)} dz_0\right)}{\prod_{j=1}^m \prod_{i=j+1}^n (\lambda_{\sigma.i} - \lambda_{\sigma.j})},$$

where $(\widehat{J_k(1,n)_E})_{\min}$ denotes the minimal weight with respect to the $\lambda(\mathbb{C}^*)$-action on the fiber $\widehat{J_k(1,n)_E}$, and $N_{\min,E}$ is the normal bundle of $(\widehat{J_k(1,n)_E})_{\min}$ in $\widehat{J_k(1,n)_E}$.

Since $\widehat{J_k(1,n)_E}$ is chosen as a T-equivariant blow up of $\tilde{\mathbb{A}}[k]$, we have that any T-fixed point of $\widehat{J_k(1,n)_E}$ is in the preimage of a T-fixed point of $\tilde{\mathbb{A}}[k]$. Applying yet again the Atiyah-Bott,Berline-Vergne localization formula and identifying the T-fixed points of $\tilde{\mathbb{A}}[k]$ with the leaves $L \in \mathcal{L}$ of the associated blow up tree $\mathcal{T}$, we obtain thus

$$\int_{\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)} \alpha = \sum_{\sigma \in \mathcal{S}_n/\mathcal{S}_{n-m}} \frac{\sigma.\left(\operatorname*{Res}_{z_0=\infty}(k-1)! z_0^{k-1} \sum_{L\in\mathcal{L}} \frac{(i_L)_* i_L^* \phi[k]^* \alpha(\boldsymbol{\lambda}, z_0)}{c_{\mathrm{top}}(T_{0_L} L)(\boldsymbol{\lambda}, z_0)} dz_0\right)}{\prod_{j=1}^m \prod_{i=j+1}^n (\lambda_{\sigma.i} - \lambda_{\sigma.j})},$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ denotes the weights of the action of T on $\mathbb{C}^m$.

Observe then

$$(i_L)_* i_L^* \phi[k]^* \alpha = \alpha(\theta_1^L, \ldots, \theta_k^L)$$

with θ_i^L the restriction to $\varphi(0_L) = \phi[k](0_L) \in \mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$ of the i'th Chern root of the tautological bundle $\mathcal{E} \to \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i(\mathbb{C}^n))$ and obtain the desired formula.

Identify at last, that the action of $\sigma \in \mathcal{S}_n/S_{n-m}$ on the forms is to permute $\lambda_i \mapsto \lambda_{\sigma(i)}$, and use the fact that the residue commutes with the the summation to obtain the desired formula. $\square$

8.2 Residue vanishing theorem and the Porteous point

The obtained integration formula in Theorem 8.1 can be rewritten into an iterated residue using [13, Proposition 5.4] (and here Proposition 8.3). Having done this, we will analyze the degrees in the residue formula, and from this conclude that in the range $n \geq k$ only a single T-fixed image point $[e_{(1,\ldots,k)}]$ of the map $\phi_{n,k}[k] : \tilde{\mathbb{A}}_{n,k}[k] \dashrightarrow \mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$ contributes to the localization; in other words, only T-fixed points of $\tilde{\mathbb{A}}_{n,k}[k]$ in the preimage of $\phi_{n,k}^{-1}([e_{(1,\ldots,k)}])$ contribute to the integration formula in Theorem 8.1 when $n \geq k$.

8.2.1 Iterated residues

We start by introducing iterated residues (cf. e.g. [58] or [13]) at infinity. Consider an affine space $\mathbb{C}^m$ with coordinates $z_1, \ldots, z_m$ and affine linear forms $\omega_1, \ldots, \omega_N$ with

$$\omega_i = a_i^0 + a_i^1 z_1 + \cdots + a_i^m z_m.$$

We define the symbols $h(\boldsymbol{z}) = h(z_1, \ldots, z_m)$ for an entire function h and $d\boldsymbol{z} = dz_1 \wedge \cdots \wedge dz_m$ for the holomorphic d-form. We will also write $\mathrm{Res}_{\boldsymbol{z}=\infty} = \mathrm{Res}_{z_1=\infty} \cdots \mathrm{Res}_{z_m=\infty}$.

Definition 8.2. Let h be an entire function on $\mathbb{C}^m$ and $\omega_1, \ldots, \omega_N$ linear forms as above

$$\operatorname*{Res}_{\boldsymbol{z}=\infty} \frac{h(\boldsymbol{z})}{\prod_{i=1}^N \omega_i} = \left(\frac{1}{2\pi i}\right)^m \int_{|z_1|=R_1} \cdots \int_{|z_m|=R_m} \frac{h(\boldsymbol{z}) d\boldsymbol{z}}{\prod_{i=1}^N \omega_i},$$

where $1 \ll R_1 \ll \cdots \ll R_m$. Here the torus $\{\boldsymbol{z} \mid |z_j| = R_j \text{ for } j = 1, \ldots, m\}$ is oriented such that $\mathrm{Res}_{\boldsymbol{z}=\infty} d\boldsymbol{z}/(z_1 \cdots z_m)^m = (-1)^d$.

In general the iterated residue depends on the order. There are a few ways to calculate such iterated residues. One method involves Laurent expanding $1/\prod i = 1^N \omega_i$, then multiplying by $(-1)^m h(\boldsymbol{z})$ and taking the coefficient of $(z_1 \dots z_m)^{-1}$. The Laurent expansion can easily be performed by geometrically expanding each linear form $1/\omega_i$ (in the range $z_1 \ll \cdots \ll z_m$). Another way to calculate the iterated residue is to simply, iteratively, sum over the poles as in the usual case of residues of a meromorphic function on $\mathbb{C}$.

The iterated residues are related to our integration formulas via

Proposition 8.3 ([13, Proposition 5.4]). Let $n \geq m$ and $Q(\boldsymbol{z})$ be a polynomial on $\mathbb{C}^m$, then

$$\sum_{\sigma \in S_n/S_{n-m}} \frac{Q(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(m)})}{\prod_{i=1}^m \prod_{j=i+1}^n (\lambda_{\sigma(j)} - \lambda_{\sigma(i)})} = \operatorname*{Res}_{\boldsymbol{z}=\infty} \frac{\prod_{1 \leq i \leq j \leq m} (z_i - z_j) Q(\boldsymbol{z})}{\prod_{i=1}^m \prod_{j=1}^n (\lambda_i - z_j)}.$$

Remark 8.4 ([13, Remark 5.5]). As stated already, the iterated residue in general depends on the order of the residues. However, the poles in the formula of Proposition 8.3 are normal crossings, and it follows that the iterated residue in this case does not depend on the order of the poles. ♦

We note that the restricted Chern roots θ_i^L of the pullback bundle under $\phi_{n,k}[k]$ of the tautological bundle $\mathcal{E} \to \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$ are linear forms in the weights $\lambda_1, \dots, \lambda_m$ U(and in the weights $t_{i,j}$): If $\phi_{n,k}[k](0_L) = [e_{(\pi_1,\dots,\pi_k)}]$ and we write $\pi_i = 1^{a_1^i} \cdots m^{a_i^m}$ (cf. Definition 6.1), then

$$\theta_i^L(\lambda_1, \dots, \lambda_m) = a_i^1 \lambda_1 + \cdots + a_i^m \lambda_m + \ell_i^L(\{t_{i,j} \mid 1 \leq i \leq n, i \leq j \leq k\}) \tag{8.1}$$

for some linear form ℓ_i^L. In order to apply Proposition 8.3 it is important to note that integration on the fiber $\tilde{\mathbb{A}}[k] /\!\!/ \mathrm{Diff}_k$ over $E \in \mathrm{Flag}(1, \dots, m; \mathbb{C}^M)$ takes the form (see the proof of Theorem 8.1)

$$I_E(\lambda_1, \dots, \lambda_m) = \operatorname*{Res}_{z_0=\infty} (k-1) z_0^{k-1} \sum_{L \in \mathcal{L}_{n,k}} \frac{\alpha(\theta_1^L, \dots, \theta_k^L)}{c_{\mathrm{top}}(T_{0_L} L)} dz_0 \in H_T^{n(n-1)/2}(E) = \mathbb{Z}[\lambda_1, \dots, \lambda_m],$$

which is a polynomial of degree $n(n-1)/2$ in the weights $\lambda_1, \dots, \lambda_m$. Moreover, we observe that for $\sigma \in \mathcal{S}_M/\mathcal{S}_{M-m}$ we have

$$I_{\sigma.E}(\lambda_1, \dots, \lambda_m) = \sigma.I_E(\lambda_1, \dots, \lambda_m) = I_E(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(m)}).$$

We obtain then directly

Theorem 8.5. Let z_0 denote a generic coordinate on the Lie algebra of $\lambda(\mathbb{C}^*) \subset \mathrm{Diff}_k$. Fix positive integers n and k, write $m = \min(n, k)$, and denote by $\mathcal{L}_{n,k}$ the leaves of the blow up tree $\mathcal{T}_{n,k}$. Let F be a vector bundle on $\mathbb{C}^n$ of rank r, and $c_1, \dots, c_{kr}$ the Chern classes of $F^{[k]}$. Give c_j the weight j, and let $\alpha \in H^*(\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n))$ a polynomial in the Chern classes of F of weighted degree $k(n-1)$ we have

$$\int_{\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)} \alpha = \operatorname*{Res}_{z_0=\infty} \operatorname*{Res}_{\boldsymbol{z}=\infty} \frac{\prod_{1\le i<j\le m}(z_i - z_j)(k-1)! z_0^{k-1}}{\prod_{j=1}^{m}\prod_{i=1}^{n}(\lambda_i - z_j)} \cdot \sum_{L \in \mathcal{L}_{n,k}} \frac{\alpha(\theta_1^L(\boldsymbol{z}), \dots, \theta_k^L(\boldsymbol{z}))}{c_{\mathrm{top}}(T_{0_L}L)(\boldsymbol{z})} d\boldsymbol{z} dz_0,$$

where $\alpha(\theta_1^L, \dots, \theta_k^L)$ means substituting in the polynomial expression of α the i'th elementary symmetric polynomial for c_i, and then evaluating in the Chern roots at 0_L, θ_i^L, of the pullback bundle $\phi_k[k]^*\mathcal{E}$ of the tautological bundle $\mathcal{E} \to \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i(\mathbb{C}^n))$.

8.2.2 Residue vanishing for $n \ge k$

We will at last show that most terms vanish in the iterated residue formula in Theorem 8.5 describing integration on the curvilinear Hilbert scheme $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$ under the additional assumption $n \ge k$.

Recall that the T-fixed points of $\mathbb{P}(\bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$ are written as $[e_\pi]$ for π a sequence of partitions (see Section 6.1.2. In this setting the image closure of $\phi_{n,k}[k] : \tilde{\mathbb{A}}_{n,k}[k] \dashrightarrow \mathbb{P}(\bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$ can contain only T-fixed points $[e_\pi]$ where $\pi = (\pi_1, \dots, \pi_k)$ is n-admissible (cf. Definition 6.1). In the range $n \ge k$ there is a special fixed point, namely the *Porteous point*

$$[e_{(1,\dots,k)}] \in \mathrm{CHilb}_0^{k+1}(\mathbb{C}^n),$$

which is easily seen to be in fact in the closure of any unital algebra of length $k+1$, so in particular in the curvilinear Hilbert scheme. Recall that $\mathcal{L}_{n,k}$ denotes the set of leaves in the blow up tree $\mathcal{T}_{n,k}$ associated to $\tilde{\mathbb{A}}_{n,k}[k]$. We define then the subset of leaves

$$\mathcal{L}_{n,k}^{(1,\dots,k)} = \{L \in \mathcal{L}_{n,k} \mid \phi_{n,k}[k](0_L) = [e_{(1,\dots,k)}]\}$$

whose T-fixed point map to the Porteous point $[e_{(1,\dots,k)}] \in \mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$

Theorem 8.6. If $n \ge k$, then the integration formula of Theorem 8.5 reduces to

$$\int_{\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)} \alpha = \operatorname*{Res}_{z_0=\infty} \operatorname*{Res}_{\boldsymbol{z}=\infty} \frac{\prod_{1\le i<j\le k}(z_i - z_j)(k-1)! z_0^{k-1}}{\prod_{j=1}^{k}\prod_{i=1}^{n}(\lambda_i - z_j)} \sum_{L \in \mathcal{L}_{n,k}^{(1,\dots,k)}} \frac{\alpha(z_1, \dots, z_k)}{c_{\mathrm{top}}(T_{0_L}L)(\boldsymbol{z})} d\boldsymbol{z} dz_0,$$

Proof. Fix a leaf $L \in \mathcal{L}_{n,k}$ and write $\phi_{n,k}[k](0_L) = [e_{(\pi_1,\dots,\pi_k)}]$. We start by showing that if $\pi_k \ne k$, then the contribution of $L \in \mathcal{L}_{n,k}$ in the iterated residue of Theorem 8.5 vanishes. The residue to evaluate in this case is

$$\operatorname*{Res}_{z_k=\infty} \frac{\prod_{1\le i<j\le k}(z_i - z_j)(k-1)! z_0^{k-1}}{\prod_{j=1}^{k}\prod_{i=1}^{n}(\lambda_i - z_j)} \frac{\alpha(\theta_1^L(\boldsymbol{z}), \dots, \theta_k^L(\boldsymbol{z}))}{c_{\mathrm{top}}(T_{0_L}L)(\boldsymbol{z})} dz_k.$$

For a linear form $\omega(\boldsymbol{z}) = a_{-1} + a_0 z_0 + a_1 z_1 + \cdots + a_k z_k$ we write $\operatorname{coeff}(\omega; z_i) = a_i$, and we make the following observations (see the description (8.1))

$$\operatorname{coeff}(\theta_i^L(\boldsymbol{z}); z_j) = 0 \qquad \text{for } j = i+1, \dots, k, \tag{8.2}$$

and

$$\operatorname{coeff}(\theta_k^L(\boldsymbol{z}); z_k) \neq 0 \iff \pi_k = k.$$

In particular, writing $\deg(\,\cdot\,; z_i)$ for the degree in the variable z_i, we have that if $\pi_k \neq k$ then $\deg(\alpha(\theta_1^L(\boldsymbol{z}), \dots, \theta_k^L(\boldsymbol{z})); z_k) = 0$.

From the description of the blow up algorithm to obtain $\tilde{\mathbb{A}}_{k,k}[k]$ in Section 7.1.2, and since the blow up algorithm is the same for $\tilde{\mathbb{A}}_{n,k}[k]$ by Proposition 7.16, we see that the pullback weight $\omega_{k,k}[k]$ of $\omega_{k,k} = \lambda_k - \lambda_1 + z_0$ as described in Section 6.5 remains to have $\operatorname{coeff}(\omega_{k,k}[k], \lambda_k) \neq 0$. This implies that $\deg(c_{\mathrm{top}}(T_{0_L}L)(\boldsymbol{z}); z_k) \geq 1$ (in fact when $\pi_k \neq k$ it is possible to argue that this is an equality). Collecting z_k-degrees we have

$$\deg\Big(\prod_{1 \leq i < j \leq k} (z_i - z_j) \cdot \alpha(\theta_1^L(\boldsymbol{z}), \dots, \theta_k^L(\boldsymbol{z})); z_k \Big) = k - 1$$

and

$$\deg\Big(\prod_{j=1}^{k} \prod_{i=1}^{n} (\lambda_i - z_j) \cdot c_{\mathrm{top}}(T_{0_L}L)(\boldsymbol{z}); z_k \Big) \geq n + 1 > k,$$

and so it follows for degree reasons that the residue vanishes. Thus we can assume $\pi_k = k$.

Assume iteratively that we have shown that we must have $\pi_{l+1} = l+1, \dots, \pi_k = k$ for L to contribute in the iterated residue of Theorem 8.5, then $\theta_j^L(\boldsymbol{z}) = j$ for $j = l+1, \dots, k$. Observe that by Remark 8.4 we can reorder the poles in this iterated residue, such that the first residue to evaluate is that of z_l. We show now that we must have $\pi_l = l$ for L to contribute in the iterated residue of Theorem 8.5. The residue to evaluate is

$$\operatorname*{Res}_{z_l = \infty} \frac{\prod_{1 \leq i < j \leq k} (z_i - z_j)(k-1)! z_0^{k-1}}{\prod_{j=1}^{k} \prod_{i=1}^{n} (\lambda_i - z_j)} \frac{\alpha(\theta_1^L(\boldsymbol{z}), \dots, \theta_l^L(\boldsymbol{z}), z_{l+1}, \dots, z_k)}{c_{\mathrm{top}}(T_{0_L}L)(\boldsymbol{z})} dz_l.$$

From the observation (8.2) we have now that if $\pi_l \neq l$, then $\operatorname{coeff}(\theta_l^L(\boldsymbol{z}); z_l) = 0$ and thus $\deg(\alpha(\theta_1^L(\boldsymbol{z}), \dots, \theta_l^L(\boldsymbol{z}), z_{l+1}, \dots, z_k); z_l) = 0$. In total we get

$$\deg\Big(\prod_{1 \leq i < j \leq k} (z_i - z_j) \cdot \alpha(\theta_1^L(\boldsymbol{z}), \dots, \theta_l^L(\boldsymbol{z}), z_{l+1}, \dots, z_k); z_l \Big) = l - 1$$

and

$$\deg\Big(\prod_{j=1}^{k} \prod_{i=1}^{n} (\lambda_i - z_j) \cdot c_{\mathrm{top}}(T_{0_L}L)(\boldsymbol{z}); z_l \Big) \geq n > l,$$

and again the residue vanishes for degree reasons.

We obtain that for L to contribute in the iterated residue of Theorem 8.5, we must have $\pi = (1, \dots, k)$ or in other words $L \in \mathcal{L}_{n,k}^{(1,\dots,k)}$. $\square$

8.2.3 The blow up model revisited

We assume still $n \geq k$. From Theorem 8.6 we obtain that only the fixed point $[e_{(1,\dots,k)}] = [e_1 \wedge \dots \wedge e_k]$ contributes to the integration on $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$. The blow up tree in which we worked was the full tree $\mathcal{T}_{n,k}$ associated to the (partial) resolution algorithm described in Chapter 7, and with leaves $\mathcal{L}_{n,k}$ and the subset of leaves $\mathcal{L}^{(1,\dots,k)}$ whose fixed point maps to $[e_{(1,\dots,k)}]$ under the Berczi-Szenes model $\phi_{n,k}$. Since we are only interested in this subset of leaves, we can *drastically improve* the blow up algorithm in Chapter 7:

It is enough to prepare the coordinate $b_{i,i}$ in step A_i, and in the concluding blow up with center $(b_{i-1,i-1}, b_{2,i}, \dots, b_{i,i})$ one considers only the chart of $b_{i,i}$.

Assume for now that $n = k$. The blow up process for the Porteous point $[e_{(1,\dots,k)}]$ has (as in described Section 7.2) an associated blow up tree, which we denote by $\mathcal{T}_{k,k}^{\mathrm{Port}}$, and its set of leaves $\mathcal{L}_{k,k}^{\mathrm{Port}}$. We are able to give an explicit description of the these blow ups. Recall the description of the monomial generators m_π^τ of the monomomial ideal M_k in subsection 6.5.1. We write $m_{\mathrm{Port}} = m_{(1,\dots,k)}^{(1,\dots,k)}$ for the (only) monomial associated to the Porteous point $[e_{(1,\dots,k)}]$, and factor in expressions b_{π_j,τ_j} such that

$$m_{\mathrm{Port}} = b_{2,2} \cdot b_{3,3} \cdot b_{1,1} b_{4,4} \cdot b_{1,1} b_{5,5} \cdot b_{1,1}^2 b_{6,6} \cdots b_{1,1}^{\lfloor k/2 \rfloor - 1} b_{k,k}$$

with $\lfloor \cdot \rfloor$ denoting the integer part.

Assume that $m_{\mathrm{Port}}|_{k-1}$ generates $M_{k-1}[k-1]_B$ on a branch B, then by Proposition 7.9 we have that any generator $m_\pi^\tau[k-1]_B$ of $M_k[k]_B$ satisfies $(1, \dots, k-1) \subset \pi$. It follows that if $m_\pi^\tau[k-1]_B$ is divisible by $b_{k,k}$ then $\pi_k = k$ and $[e_\pi] = [e_(1, \dots, k)]$ is the Porteous point. Since m_{Port} is the only generator of $M_k[k]_B$ divisible by $b_{k,k}$, the preparation process $\mathcal{P}_{k,k}$ of $b_{k,k}$ can be described as blow ups with centers only of the form

$$(b_{k-1,k-1}, b_{2,k}, \dots, b_{k-1,k}, y), \qquad y \text{ of } \mathrm{Diff}_k\text{-weight } 0,$$

that is, $y \in \{b_{i,j} \mid 1 \leq i \leq k-2, i \leq j \leq k-1\}$. Here only the y-chart needs to be considered (since the other charts are not in the Diff_k-stable locus). At last, the step A_k is concluded by the blow up in $(b_{k-1,k-1}, b_{2,k}, \dots, b_{k,k})$, where we consider only the $b_{k,k}$-chart, since the others charts produce leaves L, whose fixed points 0_L *do not* map to the Porteous point under the blow up map composed with the Berczi-Szenes model ϕ_k.

In conclusion in every blow up forming the blow up sequence A_k only as single chart is considered, and so it follows that

$$|\mathcal{L}_{k-1,k-1}^{\mathrm{Port}}| = |\mathcal{L}_{k,k}^{\mathrm{Port}}|.$$

It is clear that $|\mathcal{L}_{2,2}^{\mathrm{Port}}| = 1$, and so $\mathcal{T}_{k,k}^{\mathrm{Port}}$ has only a single leaf for any k; we denote it L_k^{Port}

$$\mathcal{L}_{k,k}^{\mathrm{Port}} = \{L_k^{\mathrm{Port}}\}.$$

We give an explicit description of $\mathcal{T}_{k,k}^{\mathrm{Port}}$, that is, of the blow up centers describing it. By the description of m_{Port}, we see (inductively) that denoting the pullback of $b_{1,1}$ through $A_1, \dots, A_{j+1}$ by a_{j+1}, we have

$$a_{j+1} = \left(\frac{a_j}{b_{j,j}}\right)^{\lfloor j/2 \rfloor} a_j b_{j+1,j+1} \tag{8.3}$$

with $a_1 = b_{1,1}$. The pullback of $b_{\pi_k,\tau_k} = b_{1,1}^{\lfloor k/2 \rfloor -1} b_{k,k}$ through $A_1, \dots, A_{k-1}$ thus has the form

$$b_{\pi_k,\tau_k}[j-1] = (a_{j-1})^{\lfloor k/2 \rfloor -1} \cdot b_{k,k}$$

Complementary to this description is the description of blow up centers: We have that a_{k-1} is a monomial in the variables $b_{i,i}$ with $1 \le i \le k-1$, and we write thus

$$b_{\pi_k,\tau_k}[j-1] = b_{1,1}^{r_1} \cdots b_{k-1,k-1}^{r_{k-1}} b_{k,k},$$

where we observe that $r_{k-1} = \lfloor k/2 \rfloor - 1$. Recall, that by Proposition 7.2 every other monomial is also divisible by $b_{k-1,k-1}^{r_{k-1}}$, and so in fact this factor can be disregarded. We can describe the step A_k (this is not a unique choice, and in fact not minimal with respect to the number of blow ups; instead it is very explicit) as the following sequence of blow up centers (the order is irrelevant except for the fact that the last blow up in the list, must in fact be the concluding blow up of A_k)

$$\begin{gathered}
\underbrace{(b_{k-1,k-1}, b_{2,k}, \dots, b_{k-1,k}, b_{1,1}), \dots, (b_{k-1,k-1}, b_{2,k}, \dots, b_{k-1,k}, b_{1,1})}_{\text{Preparation of } b_{k,k}\text{: } r_1 \text{ many}}, \\
\underbrace{(b_{k-1,k-1}, b_{2,k}, \dots, b_{k-1,k}, b_{1,1}), \dots, (b_{k-1,k-1}, b_{2,k}, \dots, b_{k-1,k}, b_{2,2})}_{\text{Preparation of } b_{k,k}\text{: } r_2 \text{ many}}, \\
\vdots \\
\underbrace{(b_{k-1,k-1}, b_{2,k}, \dots, b_{k-1,k}, b_{1,1}), \dots, (b_{k-1,k-1}, b_{2,k}, \dots, b_{k-1,k}, b_{k-2,k-2})}_{\text{Preparation of } b_{k,k}\text{: } r_{k-2} \text{ many}}, \\
\underbrace{(b_{k-1,k-1}, b_{2,k}, \dots, b_{k,k})}_{\text{Concluding blow up}}
\end{gathered} \tag{8.4}$$

These blow ups are given iteratively in k by the description in equation (8.3).

We write $\omega_{i,j}^{\mathrm{Port}}$ for the weight $\omega_{i,j}$ (defined in Section 6.6) pulled back through the sequence of blow ups associated to $\mathcal{T}_{k,k}^{\mathrm{Port}}$ to the chart corresponding to the leaf L^{Port}. Since for $n \ge k$ the models Berczi-Szenes models $\phi_{n,k}$ and $\phi_{k,k}$ the above description is the same for any $n \ge k$, and following the same arguments as in Section 7.2, the Porteous tree $\mathcal{T}_{n,k}^{\mathrm{Port}}$ is the same as that of $\mathcal{T}_{k,k}^{\mathrm{Port}}$. We get a directly the much simplified formula via localization on $\mathcal{T}_{n,k}^{\mathrm{Port}}$ since there is only the single fixed point L^{Port}.

Theorem 8.7. If $n \ge k$, then the integration formula of Theorem 8.5 reduces to

$$\int_{\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)} \alpha = \operatorname*{Res}_{z_0=\infty} \operatorname*{Res}_{\boldsymbol{z}=\infty} \frac{\prod_{1 \le i < j \le k}(z_i - z_j) \cdot \alpha(z_1, \dots, z_k)}{\prod_{j=1}^k \prod_{i=1}^n (\lambda_i - z_j) \cdot \omega_{1,1}^{\mathrm{Port}} \cdot \prod_{2 \le i \le j \le k} \omega_{i,j}^{\mathrm{Port}}} d\boldsymbol{z} dz_0.$$

In this case we chose to give the factor $c_{\mathrm{top}}(T_{0_L} L)(\boldsymbol{z}) = \prod_{1 \le i \le j \le k} \omega_{i,j}^L$ explicitly, since these can in fact be found iteratively by the description (8.4).

8.2.4 Residue vanishing in general

We state here a result regarding, which torus fixed points of $\mathrm{CHilb}_0^k(\mathbb{C}^n)$ contribute in localization formulas. The following proposition should be seen as an extension of Theorem 8.6. Denote by

$$\mathcal{L}_{n,k}^{\mathrm{eDim}=n} = \{ L \in \mathcal{L}_{n,k} \mid \mathrm{eDim}\big(\phi_{n,k}[k](0_L)\big) = n \}$$

the set of leaves $L \in \mathcal{L}_{n,k}$ whose torus-fixed point 0_L maps to an algebra $\phi_{n,k}[k](0_L)$, which as a singularity has embedding dimension n. Equivalently, the minimal number of generators of $\phi_{n,k}[k](0_L)$ as an algebra is n.

Proposition 8.8. The integration formula of Theorem 8.5 reduces to

$$\int_{\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)} \alpha = \operatorname*{Res}_{z_0=\infty} \operatorname*{Res}_{\boldsymbol{z}=\infty} \frac{\prod_{1\le i<j\le k}(z_i - z_j)(k-1)!z_0^{k-1}}{\prod_{j=1}^k \prod_{i=1}^n (\lambda_i - z_j)} \sum_{L\in\mathcal{L}_{n,k}^{\mathrm{eDim}=n}} \frac{\alpha(z_1,\dots,z_k)}{c_{\mathrm{top}}(T_{0_L}L)(\boldsymbol{z})} d\boldsymbol{z} dz_0,$$

Proof. Taking any $L \in \mathcal{L}_{n,k} \setminus \mathcal{L}_{n,k}^{\mathrm{eDim}=n}$ with $\phi_{n,k}[k](0_L) = [e_\pi]$, there exists $j \in \{1,\dots,n\}$ with $j \notin \underline{\mathrm{Parts}}(\pi)$. Using Remark 8.4, we can reorder the iterated residue, such that the innermost residue (the first one to be taken) is

$$\operatorname*{Res}_{z_j=\infty},$$

and the same argument as that applied in the proof of Theorem 8.6 (which is just a degree count) yields no contribution from L. □

8.3 Integration on $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$ in terms of integration on $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^k)$ for $n < k$

In the previous sections we have given an integration formula for integration on $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$ for all n and k in Theorem 8.1, and rewritten this formula in Theorem 8.5 from which we could prove vanishing of most terms (cf. Theorem 8.6) in the range $n \ge k$. In this section we complete the picture by relating via equivariant Poincare duals the integration on $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$ with integration on $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^k)$.

Fix $n < k$. As we have noted many times already (see e.g. Section 7.2) the space $\tilde{\mathbb{A}}_{n,k} \hookrightarrow \tilde{\mathbb{A}}_{k,k} = \operatorname{Spec} \mathbb{C}[\{b_{i,j} \mid 1 \le i \le j \le k\}]$ is cut out by the equations $b_{i,j} = 0$ for $i > n$. This manifests itself in the blow up tree $\mathcal{T}_{k,k}$ associated to $\tilde{\mathbb{A}}_{k,k}[k]$ in the sense that $\mathcal{T}_{n,k} \subset \mathcal{T}_{k,k}$ is a subtree, where each node $N_{n,k} = \operatorname{Spec}\mathbb{C}[\{b_{i,j}^N \mid 1 \le i \le n, i \le j \le k\}]$ of $\mathcal{T}_{n,k}$ corresponds to "the same node" $N_{k,k} = \operatorname{Spec}\mathbb{C}[\{b_{i,j}^N \mid 1 \le i \le j \le k\}]$ in $\mathcal{T}_{k,k}$ (see also Remark 7.17).

Here the notation $b_{i,j}^N$ means the pullback of the coordinate $b_{i,j}$ to the node $N_{k,k}$ through the relevant sequence of blow ups. As usual we denote the weight of $b_{i,j}$ by $\omega_{i,j}$, and we write $\omega_{i,j}^N$ for the pullback to N. We have then

$$\mathrm{ePD}_{T\times\lambda(\mathbb{C}^*)}(N_{n,k} \subset N_{k,k}) = \prod_{n+1\le i\le j\le k} \omega_{i,j}^N \tag{8.5}$$

Using these equivariant Poincare duals we obtain

Theorem 8.9. Fix $n < k$. In the situtation of Theorem 8.5

$$\int_{\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)} \alpha = \operatorname*{Res}_{z_0=\infty} \operatorname*{Res}_{\boldsymbol{z}=\infty} \frac{\prod_{1\le i<j\le n}(z_i - z_j)(k-1)!z_0^{k-1}}{\prod_{j=1}^n \prod_{i=1}^k (\lambda_i - z_j)} \cdot \sum_{L\in\mathcal{L}_{k,k}} \frac{\alpha(\theta_1^L(\boldsymbol{z}),\dots,\theta_k^L(\boldsymbol{z})) \cdot \prod_{n+1\le i\le j\le k} \omega_{i,j}^L(\boldsymbol{z})}{c_{\mathrm{top}}(T_{0_L}L)(\boldsymbol{z})} d\boldsymbol{z} dz_0.$$

Proof. At first, note that for $L \in \mathcal{L}_{n,k}$ we have

$$c_{\mathrm{top}}(T_{0_L}L) = \prod_{1\le i\le n}\prod_{i\le j\le k}\omega_{i,j}^L.$$

Using the expression of the equivariant Poincare dual in (8.5) in the case of a leaf $N = L$ we calculate in the L-dependent part of the residue form

$$\sum_{L\in\mathcal{L}_{n,k}} \frac{\alpha(\theta_1^L(\boldsymbol{z}),\dots,\theta_k^L(\boldsymbol{z}))}{c_{\mathrm{top}}(T_{0_L}L)(\boldsymbol{z})} \frac{\prod_{n+1\le i\le j\le k}\omega_{i,j}^L}{\prod_{n+1\le i\le j\le k}\omega_{i,j}^L} d\boldsymbol{z}dz_0$$
$$= \sum_{L\in\mathcal{L}_{k,k}} \frac{\alpha(\theta_1^L(\boldsymbol{z}),\dots,\theta_k^L(\boldsymbol{z}))\cdot\prod_{n+1\le i\le j\le k}\omega_{i,j}^L(\boldsymbol{z})}{c_{\mathrm{top}}(T_{0_L}L)(\boldsymbol{z})} d\boldsymbol{z}dz_0,$$

as we wanted. □

Remark 8.10. There are other ways to relate the integration on $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$ with $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^k)$ via equivariant multiplicities. One different approach is to use the smooth non-associative Hilbert scheme $M_{\mathbf{1}_k,n}$ (see Section 9.3) as embedding space. Using the equivariant Poincare dual in (9.6) and the description of the Poincare dual of $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$ in $M_{\mathbf{1}_k,n}$ in Proposition 9.8, one obtains immediately

$$\int_{\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)} \alpha = \int_{\mathrm{CHilb}_0^{k+1}(\mathbb{C}^k)} \alpha \prod_{i=1}^{k}\prod_{j=n+1}^{k}(z_i - \lambda_j),$$

where $z_1,\dots,z_k$ denotes the Chern roots of the tautological bundle $V_{\mathbf{1}_k,n} \to M_{\mathbf{1}_k,n}$ as in Section 9.3.1. Equivalently, with $\psi_{\mathbf{1}_k,n} : M_{\mathbf{1}_k,n} \to \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i\,\mathbb{C}^n)$ the morphism defined in (9.7) and θ_i the i'th Chern root of the tautological bundle $\mathcal{E} \to \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i\,\mathbb{C}^n)$, we have $\psi^*_{\mathbf{1}_k,n}(\theta_i) = z_i$. ♦

Chapter 9

The non-associative Hilbert scheme

The purpose of this section is to introduce a smooth and proper embedding space for a filtered version of the punctual Hilbert scheme $\mathrm{Hilb}_0^k(\mathbb{C}^n)$ called the non-associative Hilbert scheme constructed by Kazarian in [42]. This construction is as such not needed for the other parts of this thesis, but some of the reasons for including this construction are

- To offer a comparison with the model of Berczi and Szenes set up in Section 2; this is Subsection 9.4.
- To explain in a very clean way that if a point e_π is a point in $\mathrm{CHilb}_0^k(\mathbb{C}^n) = \overline{\phi} \subset \mathbb{P}(\bigwedge^k \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$ then π is complete; this is Proposition 9.15
- To give a new result (Proposition 9.10) relating the polynomial Q_d introduced in [13] with Q_{d-1}. This is the Poincare dual of (a filtered version of) $\mathrm{CHilb}_0^k(\mathbb{C}^n)$ inside the non-associative Hilbert scheme (see Section 9.3).

For the construction we will follow closely the arguments of Kazarian [42].

9.1 Filtered commutative algebra structures

Fix a sequence of non-negative integers $\boldsymbol{d} = (d_1, \ldots, d_l)$ and write $k = d_1 + \cdots + d_l$. We will consider algebra structures on a given vector space V of dimension $\dim V = k$ with some extra property on a fixed flag $V_\bullet$

$$V = V_1 \supset V_2 \supset \cdots \supset V_l \supset V_{l+1} = 0$$

with dim $V_i/V_{i+1} = d_i$. A *filtered commutative algebra structure* on $V_\bullet$ is then a linear mapping $\xi \in \mathrm{Hom}(\mathrm{Sym}^2 V, V)$ satisfying $\xi(V_r \cdot V_s) \subset V_{r+s}$. Denote by $\mathrm{Alg}_{\boldsymbol{d}}$ the vector space of these filtered commutative algebra structures.

Note that $\dim V = k$, and pick a basis $e_1, \ldots, e_k$ of V such that V_r/V_{r+1} is spanned by $e_{d_1+\cdots+d_{r-1}+i}$ for $1 \le i \le d_r$. To any basis vector e_s of V_r/V_{r+1} we assign the weight $w(s) = r$. Denote by c_{rs}^t the structure constants forming the coordinates of $\mathrm{Alg}_{\boldsymbol{d}}$, then by the filtration property and by commutativity, respectively, we have

$$\begin{cases} c_{rs}^t = 0 & \text{for } w(t) < w(r) + w(s), \\ c_{rs}^t = c_{sr}^t & \text{otherwise.} \end{cases} \tag{9.1}$$

The structure $\xi \in \mathrm{Alg}_{\boldsymbol{d}}$ is thus given by

$$\xi(e_r, e_s) = \sum_{t:w(t)\geq w(r)+w(s)} c_{rs}^t e_t.$$

Denote by $\mathrm{Hilb}_{\boldsymbol{d}}^{\mathrm{loc}}$ the subvariety in $\mathrm{Alg}_{\boldsymbol{d}}$ consisting of *associative* filtered commutative algebra structures, that is, filtered commutative structures satisfying for all quadruples (r, s, t, u) with $w(r) + w(s) + w(t) \leq w(u)$ the associativity equations

$$\sum_v c_{rs}^v c_{vt}^u - \sum_v c_{rv}^u c_{st}^v = 0. \tag{9.2}$$

Moreover, define for a given isomorphism class μ of a local algebra of dimension $k+1$ the subvariety $\mathrm{Hilb}_\mu^{\mathrm{loc}} \subset \mathrm{Hilb}_{\boldsymbol{d}}^{\mathrm{loc}}$ as the closure of the locus of filtered structures, which – upon forgetting the filtered structure – is in the isomorphism class μ.

Remark 9.1. This is one place where we differ from Kazarian's original notation in [42]. Kazarian works with singularity types, and defines the subvariety $\mathrm{Hilb}_\mu^{\mathrm{loc}}$ for a singularity type μ of a map germ $f : (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$. The definition of a singularity type as an isomorphism class is in some sense too naive (see [42][Definition 2.7]). However, for our purposes (we will work with the isomorphism classes corresponding to monomial ideals) the notion of isomorphism classes suffices.

In this case $\mathrm{Hilb}_\mu^{\mathrm{loc}}$ is a special case of a filtered version of the (punctual) geometric subsets defined by Rennemo [54]. Again, one could broaden the definition of $\mathrm{Hilb}_\mu^{\mathrm{loc}}$ to $\mathrm{Hilb}_Q^{\mathrm{loc}}$ for a punctual geometric subset Q, such that the open dense locus consists of all filtered structures which upon forgetting the filtration are algebras in Q (see [54][Definition 2.3] for the definition of such Q). ♦

Denote by $B_{\boldsymbol{d}} \subset \mathrm{GL}_k$ the group of of automorphisms of V fixing the flag $V_\bullet$. Observe that both $\mathrm{Hilb}_\mu^{\mathrm{loc}}$ and $\mathrm{Hilb}_{\boldsymbol{d}}^{\mathrm{loc}}$ are invariant with respect to the action of $B_{\boldsymbol{d}}$. The maximal torus $(\mathbb{C}^*)^k \subset B_{\boldsymbol{d}}$ acts by re-scaling the basis vectors e_r in V with weight λ_r, and the weight λ_{rs}^t of the coordinate c_{rs}^t for the induced action on $\mathrm{Alg}_{\boldsymbol{d}}$ is then given by

$$\lambda_{rs}^t = \lambda_t - \lambda_r - \lambda_s. \tag{9.3}$$

We define for the singularity type μ the polynomial $P_\mu(t_1, \ldots, t_k)$ to be the equivariant Poincare dual of $\mathrm{Hilb}_\mu^{\mathrm{loc}}$ in $\mathrm{Alg}_{\boldsymbol{d}}$ with respect to the natural action of $B_{\boldsymbol{d}}$ expressed in the weight (equivalently, Chern roots) of this action.

Example 9.2. Take $\boldsymbol{d} = (1, \ldots, 1)$ with $k = 1 + \cdots + 1$ and μ to be the singularity type $A_k = \varepsilon\mathbb{C}[\varepsilon]/(\varepsilon^{k+1})$. For $k = 1, 2, 3$ the set of associativity relations (9.2) is empty, and $P_{A_1} = P_{A_2} = P_{A_3} = 1$.

For $k = 4$ there is a single associativity relation (9.2)

$$c_{11}^2 c_{22}^4 = c_{13}^4 c_{12}^3$$

cutting out $\mathrm{Hilb}_{\boldsymbol{d}}^{\mathrm{loc}} \subset \mathrm{Alg}_{\boldsymbol{d}}$. One checks that a generic algebra in $\mathrm{Hilb}_{\boldsymbol{d}}^{\mathrm{loc}}$ is isomorphic to A_4 and hence $\mathrm{Hilb}_\mu^{\mathrm{loc}} = \mathrm{Hilb}_{\boldsymbol{d}}^{\mathrm{loc}}$. It follows that

$$P_{A_4}(t_1, t_2, t_3, t_4) = \lambda_{11}^2 \lambda_{22}^4 = \lambda_{13}^4 \lambda_{12}^3 = t_4 - t_2 - 2t_1.$$

In general the polynomial P_{A_d} is much more complicated and is essentially the polynomial Q_d defined in [13]. These polynomials have been computed only up to $k = 6$ with the help of computers. ♦

For the construction of the non-associative Hilbert scheme it is convenient to consider the dual point of view of such filtered algebra structures: The coalgebra structure on the dual space $D = V^\vee$ is defined by the adjoint filtered morphism $\xi^\vee : D \to \operatorname{Sym}^2 D$, where D has the induced filtration

$$0 = D_0 \subset D_1 \subset \cdots \subset D_l = D = V^\vee, \qquad D_i = (V/V_{i+1})^\vee = \operatorname{Ann}(V_{i+1}).$$

Here, writing $S_m = \sum_{i+j\leq m} D_i \cdot D_j \subset \operatorname{Sym}^2 D$, the comultiplication satisfies the dual filtration property $\xi^\vee(D_m) \subset S_m$. The subvariety $\operatorname{Hilb}^{\mathrm{loc}}_{\boldsymbol{d}} \subset \operatorname{Alg}_{\boldsymbol{d}}$ is then cut out by the associativity relations of the coalgebra structures $\xi^\vee$, and $\operatorname{Hilb}^{\mathrm{loc}}_{\mu} \subset \operatorname{Hilb}^{\mathrm{loc}}_{\boldsymbol{d}}$ is the closure of the locus of coalgebras isomorphic to the coalgebra μ.

9.1.1 The d-nested punctual Hilbert scheme of points

We define a nested version of the punctual Hilbert scheme $\operatorname{Hilb}^k_0(\mathbb{C}^n)$ depending on a given vector of non-negative integers $\boldsymbol{d} = (d_1, \ldots, d_l)$. As usual, we write $k = d_1 + \cdots + d_l$. Denote namely by $\operatorname{Hilb}^{\boldsymbol{d}}_0(\mathbb{C}^n)$ the variety parameterizing flags of ideals in $\mathcal{O}_{\mathbb{C}^n}$

$$\boldsymbol{I_\bullet} := [\boldsymbol{m} = I_1 \supset I_2, \supset \cdots \supset I_{l+1}]$$

satisfying $I_r I_s \subset I_{r+s}$ for $r + s \leq l + 1$ and $\dim I_r/I_{r+1} = d_r$ as a $\mathbb{C}$-module. Here $\boldsymbol{m}$ denotes the maximal ideal of $\mathcal{O}_{\mathbb{C}^n}$.

Observe that $\boldsymbol{m}^{l+1} \subset I_{l+1}$ and there is a natural embedding

$$\begin{aligned} \varphi : \operatorname{Hilb}^{\boldsymbol{d}}_0(\mathbb{C}^n) &\longrightarrow \operatorname{Flag}(d_1, d_1 + d_2, \ldots, k; \boldsymbol{m}/\boldsymbol{m}^{l+1}) \\ [I_1 \supset I_2, \supset \cdots \supset I_{l+1}] &\longmapsto [\boldsymbol{m}/I_{l+1} \supset \cdots \supset \boldsymbol{m}/I_2 \supset \boldsymbol{m}/I_1 = 0]. \end{aligned}$$

This is the $\boldsymbol{d}$-nested version of the natural embedding described in (2.1). There is a canonical subbundle $I = I_{l+1}/\boldsymbol{m}^{l+1}$ on $\operatorname{Hilb}^{\boldsymbol{d}}_0(\mathbb{C}^n)$ of the trivial vector bundle $\boldsymbol{m}/\boldsymbol{m}^{l+1} \times \operatorname{Hilb}^{\boldsymbol{d}}_0(\mathbb{C}^n) \to \operatorname{Hilb}^{\boldsymbol{d}}_0(\mathbb{C}^n)$, and we denote by $V = \boldsymbol{m}/I_{l+1}$ the quotient bundle of rank $\operatorname{rk} V = k$ and further by $D = V^\vee$ its dual. The non-associative Hilbert scheme will be a smooth and proper variety on which V extends. For an isomorphism class μ of a local algebra of dimension $k+1$ denote by $\operatorname{Hilb}^{\boldsymbol{d}}_\mu(\mathbb{C}^n)$ the closure of the locus of flags $\boldsymbol{I_\bullet}$ such that adjoining a unit element to $\boldsymbol{m}/I_{l+1}$ yields an algebra in the isomorphism class μ.

With this notation one observes that $\operatorname{Hilb}^{(k)}_0(\mathbb{C}^n)$ is the set of codimension k ideals I in $\boldsymbol{m}$ corresponding to k-dimensional algebras $\boldsymbol{m}/I$, or after adjoining a unit element to an algebra $\mathcal{O}_{\mathbb{C}^n}/I$ of dimension $k+1$, that is, to an element of $\operatorname{Hilb}^{k+1}_0(\mathbb{C}^n)$. In total $\operatorname{Hilb}^{(k)}_0(\mathbb{C}^n) \simeq \operatorname{Hilb}^{k+1}_0(\mathbb{C}^n)$ under this identification. At last write

$$\pi_{\boldsymbol{d},n} : \operatorname{Hilb}^{\boldsymbol{d}}_0(\mathbb{C}^n) \to \operatorname{Hilb}^{(k)}_0(\mathbb{C}^n) \tag{9.4}$$

for the forgetful morphism ignoring the filtration. Since there is only one way to complete an ideal $I \in \operatorname{Curv}^{k+1}(\mathbb{C}^n)$ to a flag $\boldsymbol{I_\bullet}$, we see that $\pi_{(1,\ldots,1),n} : \operatorname{Hilb}^{(1,\ldots,1)}_0(\mathbb{C}^n) \to \operatorname{Hilb}^{(k)}_0(\mathbb{C}^n)$ is a bijection over the curvilinear locus $\{\xi \simeq \epsilon\mathbb{C}[\epsilon]/(\epsilon^{k+1})\}$.

9.2 Construction of the non-associative Hilbert scheme

In this section we construct the non-associative Hilbert scheme – a smooth and proper embedding space for $\mathrm{Hilb}_0^{\boldsymbol{d}}(\mathbb{C}^n)$.

At first we give another characterization of points $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$. We state it in terms of the associated nilpotent algebra obtained by quotienting by the unit element. We continue to write $\boldsymbol{m}$ for the maximal ideal in $\mathcal{O}_{\mathbb{C}^n}$.

Lemma 9.3. The set of k-codimensional ideals in $\boldsymbol{m}$ is in one-to-one correspondence with the set of isomorphism classes of pairs (ψ_1, ψ_2) where $\psi_2 : \mathrm{Sym}^2 V \to V$ is an associative commutative nilpotent algebra structure on a k-dimensional vector space V, and $\psi_1 : (\mathbb{C}^n)^\vee \to V$ is a linear map such that $\psi_1 \oplus \psi_2$ is surjective.

Proof. If I is an ideal with $V = \boldsymbol{m}/I$ of dimension k, then ψ_1 is the restriction of the canonical projection $\boldsymbol{m} \to \boldsymbol{m}/I$ to the subspace $(\mathbb{C}^n)^\vee \subset \boldsymbol{m}$, and ψ_2 is the multiplication in the nilpotent algebra V.

If ψ_1 and ψ_2 are given, one observes, since $(\mathbb{C}^n)^\vee$ generates $\boldsymbol{m}$, that $\psi_1 : (\mathbb{C}^n)^\vee \to V$ extends to a morphism of algebras $\overline{\psi_1} : \boldsymbol{m} \to V$ which is surjective. One takes $I = \mathrm{Ker}\,\overline{\psi_1}$. □

This identification is extended to flags of ideals and correspondingly the nilpotent structure $(\psi_1, \psi_2) \in \mathrm{Hom}((\mathbb{C}^n)^\vee \oplus \mathrm{Sym}^2 V, V)$ must be compatible with a given filtration on V

$$V_\bullet = [V = V_1 \supset V_2 \supset \cdots \supset V_l \supset V_{l+1} = 0], \qquad \dim V_i/V_{i+1} = d_i.$$

Consider the locally closed subspace $\widetilde{M_{\boldsymbol{d},n}} \subset \mathrm{Hom}((\mathbb{C}^n)^\vee \oplus \mathrm{Sym}^2 V, V)$ consisting of pairs (ψ_1, ψ_2) satisfying

(1) $\psi_2(V_i \cdot V_j) \subset V_{i+j}$ for all i and j, and

(2) $\psi_1 \oplus \psi_2$ is surjective.

The second condition determines an open and dense subset in $\mathrm{Hom}((\mathbb{C}^n)^\vee, V) \oplus \mathrm{Alg}_{\boldsymbol{d}}$, while the first condition determines a closed subspace in this open subset.

The group $B_{\boldsymbol{d}} \subset \mathrm{GL}_k$ of automorphisms of V fixing the flag $V_\bullet$ acts naturally on $\widetilde{M_{\boldsymbol{d},n}}$.

Definition 9.4. The non-associative Hilbert scheme is the quotient space $M_{\boldsymbol{d},n} = \widetilde{M_{\boldsymbol{d},n}}/B_{\boldsymbol{d}}$.

For the following result an independent construction of $M_{\boldsymbol{d},n}$ as a flag space is given.

Proposition 9.5. The action of $B_{\boldsymbol{d}}$ on $\widetilde{M_{\boldsymbol{d},n}}$ is free, and so the non-associative Hilbert scheme $M_{\boldsymbol{d},n}$ is smooth and compact.

Proof. We will use the dual picture of coalgebras and construct a moduli space of flags

$$0 = D_0 \subset D_1 \subset \cdots \subset D_l = D = V^\vee, \qquad \dim D_m/D_{m-1} = d_m$$

equipped with an *injective* map $D \to E \oplus \mathrm{Sym}^2 D$ such that $D_m \subset E \oplus S_m$, where $S_m = \sum_{i+j\leq m} D_i \cdot D_j$.

The construction goes by induction on l. For $l = 1$, we simply take $M_{(d_1),n} = \mathrm{Grass}_{d_1}(\mathbb{C}^n)$ since $S_1 = 0$. The Grassmannian $M_{(d_1),n}$ is equipped with its tautological bundle D_1 with $\mathrm{rk}\, D_1 = d_1$.

Assume now that $M_{(d_1,\dots,d_{l-1}),n}$ has been constructed with the flag of tautological bundles $D_1 \subset \cdots \subset D_{l-1}$, where $D_{l-1} \subset E \oplus S_{l-1}$. The subbundle S_l is determined by $D_1, \dots, D_{l-1}$ and can thus in fact be regarded as a bundle over $M_{(d_1,\dots,d_{l-1}),n}$. Thus $(E \oplus S_l)/D_{l-1}$ can be regarded as a bundle on $M_{(d_1,\dots,d_{l-1}),n}$, and we define $M_{\boldsymbol{d},n} = \mathrm{Grass}_{d_l}((E \oplus S_l)/D_{l-1})$.

By construction $M_{\boldsymbol{d},n}$ is thus the total space of a tower of fibrations with smooth and compact fibers, which implies the result. □

From the construction of $M_{\boldsymbol{d},n}$ as a tower of Grassmannian bundles exhibited in the proof above, we see that $M_{\boldsymbol{d},n}$ is equipped with bundles $D_m \subset E \oplus S_m$ satisfying also $D_m \subset D_{m+1}$, and taking the dual picture, we obtain the diagram of bundles

$$\begin{tikzcd}
V_l \arrow[r, two heads] & V_{l-1} \arrow[r, two heads] & \cdots \arrow[r, two heads] & V_1 \\
(\mathbb{C}^n)^\vee \oplus S_l^\vee \arrow[u, two heads] \arrow[r, two heads] & (\mathbb{C}^n)^\vee \oplus S_{l-1}^\vee \arrow[u, two heads] \arrow[r, two heads] & \cdots \arrow[r, two heads] & (\mathbb{C}^n)^\vee \oplus S_1^\vee = (\mathbb{C}^n)^\vee, \arrow[u, two heads]
\end{tikzcd}$$

where $S_m = \sum_{i+j\le m} D_i \cdot D_j$ as in the proof above. We will write $V_{\boldsymbol{d},n} = V_l$, since we should really indicate that this bundle depends on the vector $\boldsymbol{d}$ and the dimension n, not only its length l. The restrictions of the vertical maps to each summand correspond to a linear map $(\mathbb{C}^n)^\vee \to V_l$ and a canonical commutative filtered algebra structure on the fibers over $M_{\boldsymbol{d},n}$.

From Lemma 9.3 we obtain immediately

Proposition 9.6. (1) The Hilbert scheme $\mathrm{Hilb}_0^{\boldsymbol{d}}(\mathbb{C}^n)$ is isomorphic to the locus in $M_{\boldsymbol{d},n}$ consisting of points with *associative* canonical filtered algebra structure on the fiber of $V_{\boldsymbol{d},n}$.

(2) The subvariety $\mathrm{Hilb}_\mu^{\boldsymbol{d}}(\mathbb{C}^n)$ consists of points for which the canonical filtered algebra structure on the fiber is isomorphic to μ.

Writing $M_{|\boldsymbol{d}|,n}$ for the space obtained by forgetting the filtration structure on the algebras of $M_{\boldsymbol{d},n}$ (alternatively, view $M_{\boldsymbol{d},n}$ as a flag space and projection to the largest space, then $M_{|\boldsymbol{d}|,n}$ can be viewed as a Grassmannian), the forgetful map $\pi_{\boldsymbol{d},n}$ in (9.4) extends to

$$\pi_{\boldsymbol{d},n} : M_{\boldsymbol{d},n} \to M_{|\boldsymbol{d}|,n},$$

for which we use the same symbol.

9.3 Poincare duals in $M_{\boldsymbol{d},n}$

In this section we relate the Poincaré dual of a variety $\mathrm{Hilb}_\mu^{\boldsymbol{d}}(\mathbb{C}^n)$ in $M_{\boldsymbol{d},n}$ to a certain equivariant Poincaré dual. This relation is implicit in Kazarian's construction [42], but used. We will then give a few results – new at least to the author – on these equivariant Poincaré duals, in particular on the polynomial Q_d appearing in [13].

Recall that $M_{\boldsymbol{d},n} = \widetilde{M_{\boldsymbol{d},n}}/B_{\boldsymbol{d}}$ by definition, and $\widetilde{M_{\boldsymbol{d},n}} \subset \mathrm{Hom}((\mathbb{C}^n)^\vee, V) \oplus \mathrm{Alg}_{\boldsymbol{d}}$ is an open dense subspace. Moreover, we have defined subvarieties $\mathrm{Hilb}_\mu^{\boldsymbol{d}}(\mathbb{C}^n) \subset \mathrm{Hilb}_0^{\boldsymbol{d}}(\mathbb{C}^n) \subset M_{\boldsymbol{d},n}$ as well as subvarieties $\mathrm{Hilb}_\mu^{\mathrm{loc}} \subset \mathrm{Hilb}_{\boldsymbol{d}}^{\mathrm{loc}} \subset \mathrm{Alg}_{\boldsymbol{d}}$. The non-associative Hilbert scheme $M_{\boldsymbol{d},n}$ is equipped with the associated vector bundle

$$V_{\boldsymbol{d},n} \times_{B_{\boldsymbol{d}}} \left(\mathrm{Hom}((\mathbb{C}^n)^\vee, V_{\boldsymbol{d},n}) \oplus \mathrm{Alg}_{\boldsymbol{d}}\right) \to M_{\boldsymbol{d},n},$$

which has the canonical section $\sigma : B_{\boldsymbol{d}}.(\psi_1,\psi_2) \mapsto (0, \psi_1 \oplus \psi_2)$. Write $\mathbf{0} \subset V_{\boldsymbol{d},n}$ for the image of the 0-section in $V_{\boldsymbol{d},n}$. Observe that the meaning of Proposition 9.6 is

$$\begin{aligned}
&\sigma^{-1}(V_{\boldsymbol{d},n} \times_{B_{\boldsymbol{d}}} \mathrm{Hilb}^{\mathrm{loc}}_{\boldsymbol{d}}) = \mathrm{Hilb}^{\boldsymbol{d}}_0(\mathbb{C}^n) \quad \text{and} \quad \sigma(\mathrm{Hilb}^{\boldsymbol{d}}_0(\mathbb{C}^n)) = \mathbf{0} \times_{B_{\boldsymbol{d}}} \mathrm{Hilb}^{\mathrm{loc}}_{\boldsymbol{d}},\\
&\sigma^{-1}(V_{\boldsymbol{d},n} \times_{B_{\boldsymbol{d}}} \mathrm{Hilb}^{\mathrm{loc}}_{\mu}) = \mathrm{Hilb}^{\boldsymbol{d}}_\mu(\mathbb{C}^n) \quad \text{and} \quad \sigma(\mathrm{Hilb}^{\boldsymbol{d}}_\mu(\mathbb{C}^n)) = \mathbf{0} \times_{B_{\boldsymbol{d}}} \mathrm{Hilb}^{\mathrm{loc}}_{\mu}.
\end{aligned} \tag{9.5}$$

We have the following transversality property of the canonical section σ

Proposition 9.7. The following equalities hold

$$\mathrm{codim}(\mathrm{Hilb}^{\mathrm{loc}}_{\boldsymbol{d}} \subset \mathrm{Alg}_{\boldsymbol{d}}) = \mathrm{codim}(\mathrm{Hilb}^{\boldsymbol{d}}_0(\mathbb{C}^n) \subset M_{\boldsymbol{d},n})$$

$$\mathrm{codim}(\mathrm{Hilb}^{\mathrm{loc}}_{\mu} \subset \mathrm{Alg}_{\boldsymbol{d}}) = \mathrm{codim}(\mathrm{Hilb}^{\boldsymbol{d}}_\mu(\mathbb{C}^n) \subset M_{\boldsymbol{d},n})$$

Proof. We prove only the first statement. The proof of the second statement is exactly the same.

One observes trivially that

$$\begin{aligned}
&\mathrm{codim}(\mathrm{Hilb}^{\mathrm{loc}}_{\boldsymbol{d}} \subset \mathrm{Alg}_{\boldsymbol{d}})\\
&\quad = \mathrm{codim}\big(\mathrm{Hom}((\mathbb{C}^n)^\vee, V_{\boldsymbol{d},n}) \oplus \mathrm{Hilb}^{\mathrm{loc}}_{\boldsymbol{d}} \subset \mathrm{Hom}((\mathbb{C}^n)^\vee, V_{\boldsymbol{d},n}) \oplus \mathrm{Alg}_{\boldsymbol{d}}\big).
\end{aligned}$$

Define $\widetilde{M_{0,n}}$ as the pullback in the diagram

$$\begin{tikzcd}
\widetilde{M_{0,n}} \arrow[r, hook, "\text{dense}"] \arrow[d, hook] & \mathrm{Hom}((\mathbb{C}^n)^\vee, V_{\boldsymbol{d},n}) \oplus \mathrm{Hilb}^{\mathrm{loc}}_{\boldsymbol{d}} \arrow[d, hook] \\
\widetilde{M_{\boldsymbol{d},n}} \arrow[r, hook, "\text{dense}"] & \mathrm{Hom}((\mathbb{C}^n)^\vee, V_{\boldsymbol{d},n}) \oplus \mathrm{Alg}_{\boldsymbol{d}},
\end{tikzcd}$$

that is, requiring associativity and surjectivity of the pair (ψ_1, ψ_2). But $\widetilde{M_{0,n}}$ fits also into

$$\begin{tikzcd}
\widetilde{M_{0,n}} \arrow[r, hook] \arrow[d, hook] & \widetilde{M_{\boldsymbol{d},n}} \arrow[d, hook] \\
\mathrm{Hilb}^{\boldsymbol{d}}_0(\mathbb{C}^n) \arrow[r, hook] & M_{\boldsymbol{d},n},
\end{tikzcd}$$

and we find thus

$$\mathrm{codim}(\widetilde{M_{0,n}} \subset \widetilde{M_{\boldsymbol{d},n}}) = \mathrm{codim}(\mathrm{Hilb}^{\boldsymbol{d}}_0(\mathbb{C}^n) \subset M_{\boldsymbol{d},n})$$

concluding the proof. □

By the universal property of equivariant Poincare duals [13][Proposition 2.8] we obtain

Proposition 9.8.

(1) The Poincare dual of the subvariety $\mathrm{Hilb}^{\boldsymbol{d}}_0(\mathbb{C}^n)$ in $M_{\boldsymbol{d},n}$ equals the equivariant Poincare dual $\mathrm{ePD}[\mathrm{Hilb}^{\mathrm{loc}}_{\boldsymbol{d}}, \mathrm{Alg}_{\boldsymbol{d}}]_{B_{\boldsymbol{d}}}(V_{\boldsymbol{d},n})$ of $\mathrm{Hilb}^{\mathrm{loc}}_{\boldsymbol{d}}$ in $\mathrm{Alg}_{\boldsymbol{d}}$ evaluated in the Chern roots of the bundle $V_{\boldsymbol{d},n}$.

(2) The Poincare dual of the subvariety $\mathrm{Hilb}^{\boldsymbol{d}}_\mu(\mathbb{C}^n)$ in $M_{\boldsymbol{d},n}$ equals the equivariant Poincare dual $\mathrm{ePD}[\mathrm{Hilb}^{\mathrm{loc}}_{\mu}, \mathrm{Alg}_{\boldsymbol{d}}]_{B_{\boldsymbol{d}}}(V_{\boldsymbol{d},n})$ of $\mathrm{Hilb}^{\mathrm{loc}}_{\mu}$ in $\mathrm{Alg}_{\boldsymbol{d}}$ evaluated in the Chern roots of the bundle $V_{\boldsymbol{d},n}$.

We remark in the above Proposition that the construction of $\mathrm{Hilb}^{\mathrm{loc}}_{\boldsymbol{d}}$ and $\mathrm{Alg}_{\boldsymbol{d}}$ does not depend on the dimension n of the linear space $(\mathbb{C}^n)^\vee$. We give an example of an application of this observation to the equivariant Poincare dual $\mathrm{ePD}[\mathrm{Hilb}^{\mathrm{loc}}_{A_k}, \mathrm{Alg}_{\mathbf{1}_k}]_{B_{\mathbf{1}_k}}(V_{\mathbf{1}_k})$ where $A_k = \epsilon\mathbb{C}[\epsilon]/(\epsilon^{k+1})$. This equivariant Poincare dual is essentially the same as the polynomial Q_d appearing in [13] – there is only a distinguishing in some sign choices of the weights.

9.3.1 An example study of $A_k = \epsilon\mathbb{C}[\epsilon]/(\epsilon^{k+1})$

In this subsection we study briefly the isomorphism class of $A_k = \epsilon\mathbb{C}[\epsilon]/(\epsilon^{k+1})$ in $M_{\boldsymbol{d},n}$. In this case we must have $\boldsymbol{d} = (1, \dots, 1) = \mathbf{1}_k$. The general idea is to apply Proposition 9.8 and study the equivariant Poincare dual $\mathrm{ePD}[\mathrm{Hilb}^{\mathrm{loc}}_{A_k}, \mathrm{Alg}_{\mathbf{1}_k}]_{B_{\mathbf{1}_k}}(V_{\mathbf{1}_k})$ as the ordinary Poincare dual of $\mathrm{Hilb}^{\mathbf{1}_k}_{A_k}(\mathbb{C}^n)$ in $M_{\mathbf{1}_k,n}$, and relate these ordinary Poincare duals for varying n.

We write $q_{\mathbf{1}_k,n} : M_{\mathbf{1}_k,n} \to M_{\mathbf{1}_{k-1},n}$ for the projections.

Lemma 9.9. For a positive integer j denote by $p_2(j)$ the number of ways to partition j into two positive integers.

(1) $\dim \mathrm{Alg}_{\mathbf{1}_k} = \displaystyle\sum_{2 \le i \le j \le k} p_2(j)$

(2) $\dim M_{\mathbf{1}_k,n} = kn - \dfrac{k(k+1)}{2} + \displaystyle\sum_{2 \le i \le j \le k} p_2(j)$

(3) The dimension of the projective fiber of $q_{\mathbf{1}_k,n} : M_{\mathbf{1}_k,n} \to M_{\mathbf{1}_{k-1},n}$ is

$$\begin{cases} n - k + i(i+1), & k = 2i+1 \\ n - k + i^2, & k = 2i \end{cases}$$

Proof. By definition $M_{\mathbf{1},k,n}$ is the $B_{\mathbf{1}_k}$-quotient of $\widetilde{M_{\mathbf{1},k,n}} \subset \mathrm{Hom}((\mathbb{C}^n)^\vee, V) \oplus \mathrm{Alg}_{\mathbf{1}_k}$. The vector space $\mathrm{Alg}_{\mathbf{1}_k}$ is spanned by c^t_{rs} with $t \le r + s$ as explained in (9.1), and one counts thus

$$\dim \mathrm{Alg}_{\mathbf{1}_k} = \sum_{t=2}^{k} \sum_{r+s \le t} 1 = \sum_{2 \le s \le t \le k} p_2(s).$$

Moreover $B_{\mathbf{1}_k} \subset \mathrm{GL}_k$ is just the usual set of upper triangular matrices of dimension $k(k+1)/2$. Hence

$$\dim M_{\mathbf{1}_k,n} = kn - \frac{k(k+1)}{2} + \sum_{2 \le s \le t \le k} p_2(s).$$

At last, the dimension of a fiber of $q_{\mathbf{1}_k,n}$ is then

$$\dim M_{\mathbf{1}_k,n} - \dim M_{\mathbf{1}_{k-1},n} = n - k + \sum_{2 \le s \le t = k} p_2(s),$$

and one checks that

$$\sum_{2 \le s \le k} p_2(s) = \begin{cases} i(i+1), & k = 2i+1 \\ i^2, & k = 2i. \end{cases}$$

□

We write z_i for the Chern root of the line bundle $V_{\mathbf{1}_i,n}/V_{\mathbf{1}_{i-1},n}$. Observe that $M_{\mathbf{1}_k,n-1} \subset M_{\mathbf{1}_k,n}$ naturally, and we have

$$\mathrm{ePD}[M_{\mathbf{1}_k,n-1}, M_{\mathbf{1}_k,n}](V_{\mathbf{1}_k,n}) = \prod_{i=1}^{d}(z_i - \lambda_n). \tag{9.6}$$

Write also $R_k = \mathrm{ePD}[\mathrm{Hilb}^{\mathrm{loc}}_{A_k}, \mathrm{Alg}_{\mathbf{1}_k}]_{B_{\mathbf{1}_k}}(V_{\mathbf{1}_k})$ for the equivariant Poincare dual, and denote by $q_{\mathbf{1}_k,n,*} : H^*_{T_k}(M_{\mathbf{1}_k,n}) \to H^*_{T_k}(M_{\mathbf{1}_{k-1},n})$ the "truncation" map. The polynomial R_k is essentially the same polynomial as Q_k defined in [13] – up to signs due to the choice of signs on the weights λ^t_{rs} described in (9.3).

Proposition 9.10. The $T_k \subset B_{\mathbf{1}_k}$ equivariant pushforward $q_{\mathbf{1}_k,n,*}$ satisfies

$$q_{\mathbf{1}_k,n,*}R_k z_k^{n-1} = R_{k-1}, \quad k \geq 2.$$

For $k = 1$ we have $q_{(1),n,*}R_1 z_1^{n-1} = 1$.

Proof. For any equivariant form $\alpha \in H^*_{T_k}(M_{\mathbf{1}_k,n})$ we have

$$\prod_{i=1}^{k-1}(z_i - \lambda_n) q_{\mathbf{1}_k,n,*}\alpha = q_{\mathbf{1}_k,n-1,*}\prod_{i=1}^{k}(z_i - \lambda_n)\alpha.$$

Clearly the first $k-1$ factors are constant under these pushforwards (on either side) since they do not depend on z_k, and we obtain thus

$$q_{\mathbf{1}_k,n,*}\alpha = q_{\mathbf{1}_k,n-1,*}(z_k - \lambda_n)\alpha.$$

From Lemma 9.9, we have that the codimension of the fiber bundle $q_{\mathbf{1}_k,1}$ in $M_{\mathbf{1}_k,n}$ is

$$\mathrm{codim}(q^{-1}_{\mathbf{1}_k,1}(M_{\mathbf{1}_{k-1},1}) \subset q^{-1}_{\mathbf{1}_k,n}(M_{\mathbf{1}_{k-1},1})) = n-1.$$

Moreover, in $M_{\mathbf{1}_k,1}$ every associative algebra is isomorphic to $\epsilon\mathbb{C}[\epsilon]/(\epsilon^{k+1})$, which has only a single subalgebra of length i for $1 \leq i \leq k$. It follows that

$$\begin{aligned}\deg(R_k; z_k) &= \dim M_{\mathbf{1}_k,n} - \dim M_{\mathbf{1}_{k-1},n} - (n+1) \\ &= \begin{cases} 1-k+i(i+1), & k = 2i+1 \\ 1-k+i^2, & k = 2i, \end{cases}\end{aligned}$$

and that

$$q_{\mathbf{1}_k,1,*}R_k = R_{k-1}.$$

We have then iteratively

$$\begin{aligned} q_{\mathbf{1}_k,n,*}z_k^{n-1}R_k &= q_{\mathbf{1}_k,n,*}(z_k - \lambda_n)^{n-1}R_k \\ &= q_{\mathbf{1}_k,n-1,*}(z_k - \lambda_n)^{n-2}R_k \\ &\vdots \\ &= q_{\mathbf{1}_k,1,*}R_k \\ &= R_{k-1}, \end{aligned}$$

where the first equality follows since upon a binomial expansion of the factor $(z_k - \lambda_n)^{n-1}$ every other term vanishes in the pushforward due to having too small z_k-degree.

The equality for $k=1$ follows by identifying $M_{(1),n} \simeq \mathrm{Grass}_1(\mathbb{C}^n) \simeq \mathbb{P}^{n-1}$ as in the construction in Proposition 9.5, and calculating (here $R_1 = 1$)

$$q_{(1),n,*} R_1 z_1^{n-1} = \int_{\mathbb{P}^{n-1}} c_1^{n-1} = 1,$$

where c_1 denotes the Chern class of the tautological line bundle on $\mathbb{P}^{n-1}$. □

In terms of direct information of the polynomials R_k, Proposition 9.10 tells us that the coefficient of the top-degree z_k-term of R_k is exactly the polynomial R_{k-1}.

Remark 9.11. The arguments in the proof of Proposition 9.10 above give also directly that

$$\deg R_k = \mathrm{codim}(\mathrm{Hilb}^{\mathbf{1}_k}_{A_k}(\mathbb{C}^1) \subset M_{\mathbf{1}_k,1}) = \dim M_{\mathbf{1}_k,1},$$

and so directly from Lemma 9.9 we get

$$\deg R_k = -\frac{k(k-1)}{2} + \sum_{2\le i\le j\le k} p_2(j).$$

♦

From Proposition 9.10 we get immediately some intersection numbers.

Corollary 9.12. Suppose F is vector bundle on $\mathbb{C}^n$ of rank $\mathrm{rk}\, F = r$, and write $c_j = c_j(F^{[k+1]})$ for the j'th Chern class of the tautological bundle $F^{[k+1]}$ of rank $\mathrm{rk}\, F^{[k+1]} = r(k+1)$, and give c_j the weighted degree j. Let $M(c_1, \ldots, c_{r(k+1)})$ be a monomial in $c_1, \ldots, c_{r(k+1)}$ of weighted degree $\dim \mathrm{CHilb}^{k+1}_0(\mathbb{C}^n) = k(n-1)$. Then

(1) If M is divisible by some c_j with $j > rk$, then

$$\int_{\mathrm{CHilb}^{k+1}_0(\mathbb{C}^n)} M(c_1, \ldots, c_{r(k+1)}) = 0$$

(2) Suppose r divides $n-1$, then for $M = c_{rk}^{(n-1)/r}$ we have

$$\int_{\mathrm{CHilb}^{k+1}_0(\mathbb{C}^n)} c_{rk}^{(n-1)/r} = 1.$$

In particular, this formula holds for any line bundle F where $r=1$.

Proof. Recall from equation (3.1) that the total Chern class of the bundle $F^{[k+1]}$ is

$$c(F^{[k+1]}) = \prod_{j=1}^{r}(1+\theta_j) \prod_{i=1}^{k}\prod_{j=1}^{r}(1+z_i+\theta_j),$$

where $\theta_1, \ldots, \theta_r$ are the Chern roots of F and z_i is still the Chern root of the line bundle $V_{\mathbf{1}_i,n}/V_{\mathbf{1}_{i-1},n}$ as defined earlier.

Identify birationally the curvilinear Hilbert scheme $\mathrm{CHilb}^{k+1}_0(\mathbb{C}^n)$ with the filtered $\mathrm{Hilb}^{\mathbf{1}_k}_{A_k}(\mathbb{C}^n) \subset M_{\mathbf{1}_k,n}$ (the forgetful map $\pi_{\mathbf{1}_k,n}$ defined in (9.4) is a bijection over the dense open part $\mathrm{Curv}^{k+1}_0(\mathbb{C}^n) \subset \mathrm{CHilb}^{k+1}_0(\mathbb{C}^n)$) and observe under this identification

$$\int_{\mathrm{CHilb}^{k+1}_0(\mathbb{C}^n)} M = \int_{M_{\mathbf{1}_k,n}} R_k M.$$

For degree reasons we have

$$q_{\mathbf{1}_k,n,*}R_k z_k^j = 0, \qquad j = 1, \dots, n-1,$$

and it follows that for a term of M to contribute to the integral it must be divisible by z_k^{n-1}. From the description of $c(F^{[k+1]})$ we see that the largest power of z_k, by which a given term of any c_i is divisible, is z_k^r.

For the first part, write $M = c_j M'$ for a monomial M' of weighted degree $k(n-1) - j < k(n-r-1)$. Then M' is divisible by a power z_k^s for some $0 \le s < n - r - 1$. In total M is divisible by z_k^t for some $t = s + r < n - 1$ proving that these intersection numbers vanish.

For $M = c_{rk}^{(n-1)/r}$, we see that for a term of c_{rk} to contribute to the integral it must be divisible by z_k^r, yielding the first equality below

$$\begin{aligned}
\int_{M_{\mathbf{1}_k,n}} R_k c_{rk}(F^{[k+1]})^{(n-1)/r} &= \int_{M_{\mathbf{1}_k,n}} R_k (c_{r(k-1)}(F^{[k]}) \cdot z_k^r)^{(n-1)/r} \\
&= \int_{M_{\mathbf{1}_k,n}} R_k z_k^{n-1} c_{r(k-1)}(F^{[k]})^{(n-1)/r} \\
&= \int_{M_{\mathbf{1}_{k-1},n}} R_{k-1} c_{r(k-1)}(F^{[k]})^{(n-1)/r} \\
&\vdots \\
&= \int_{M_{(1),n}} R_1 c_r(F^{[2]})^{(n-1)/r} \\
&= \int_{M_{(1),n}} z_1^{n-1} \\
&= 1
\end{aligned}$$

and the remaining equalities follow from Proposition 9.10. $\square$

9.4 Comparison with the Berczi-Szenes model

In this section we compare Kazarian's non-associative Hilbert scheme with the model 2.5 of Berczi and Szenes. From this comparison we obtain also an easy criterion for torus-fixed points in $\mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$ not to be in the image closure of the Berczi-Szenes model which is isomorphic the curvilinear Hilbert scheme $\mathrm{CHilb}_0^{k+1}(\mathbb{C}^n)$. This criterion was already shown [13][Proposition 6.14], but proven by different methods.

First, we construct a map $M_{\boldsymbol{d},n} \to \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$. The construction is the same as that in the proof of Lemma 9.3: Since $(\psi_1, \psi_2) \in \widetilde{M_{\boldsymbol{d},n}}$ the map $\psi_1 : (\mathbb{C}^n)^\vee \to V$ extends to the maximal ideal $\boldsymbol{m}_{\psi_2}$ generated under the product of ψ_2 by the basis of $(\mathbb{C}^n)^\vee$, and thus yields a map $\overline{\psi_1} : m_{\psi_2} \to V$ by which we define the algebraic structure on V. The map $\overline{\psi_1}$ is surjective since $\psi_1 \oplus \psi_2$ is surjective, and writing $I = \mathrm{Ker}\, \overline{\psi_1}$ we have $V \simeq \boldsymbol{m}_{\psi_2}/I$ as algebras. In particular $\dim_{\mathbb{C}} \boldsymbol{m}_{\psi_2}/I = k$, that is, I has codimension k in $\boldsymbol{m}_{\psi_2}$, so $I \subset \widetilde{\boldsymbol{m}_{\psi_2}^{k+1}}$ and thus $\boldsymbol{m}_{\psi_2}/I \subset \boldsymbol{m}_{\psi_2}/\boldsymbol{m}_{\psi_2}^{k+1} \simeq \bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n$. In total, we associate to $(\psi_1, \psi_2) \in \widetilde{M_{\boldsymbol{d},n}}$ the subspace

$$\boldsymbol{m}_{\psi_2} / \mathrm{Ker}\, \overline{\psi_1} \in \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n).$$

At last, observe that acting by the filtration preserving $B_{\boldsymbol{d}} \subset \mathrm{GL}_k$ preserves this vector space, and so we have indeed defined a map

$$\begin{aligned} \psi_{\boldsymbol{d},n} : M_{\boldsymbol{d},n} &\to \mathrm{Grass}_k(\bigoplus_{i=1}^{k} \mathrm{Sym}^i \mathbb{C}^n) \\ (\psi_1, \psi_2) &\mapsto \boldsymbol{m}_{\psi_2} / \operatorname{Ker} \overline{\psi_1} \end{aligned} \tag{9.7}$$

factoring through the forgetful map $\pi_{\boldsymbol{d}} : M_{\boldsymbol{d},n} \to M_{|\boldsymbol{d}|,n}$. Also, $\psi_{\boldsymbol{d},n}$ is equivariant with respect to the natural action of GL_n on $M_{\boldsymbol{d},n}$ and $\mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$.

Take $\boldsymbol{d} = (1, \ldots, 1) = \mathbf{1}_k$ of size $1 + \cdots + 1 = k$. From the construction of the non-associative Hilbert scheme $M_{\mathbf{1}_k,n}$ we know that the subvariety $\mathrm{Hilb}^{\mathbf{1}_k}_{A_k}(\mathbb{C}^n) \subset M_{\mathbf{1}_k,n}$ with $A_k = \epsilon\mathbb{C}[\epsilon]/(\epsilon^{k+1})$ satisfies $\pi_{\mathbf{1}_k}(\mathrm{Hilb}^{\mathbf{1}_k}_{A_k})$ is birationally equivalent to $\mathrm{CHilb}^{k+1}_0(\mathbb{C}^n)$, since $\pi_{\mathbf{1}_k}$ is bijective over the locus $\mathrm{Curv}^{k+1}_0(\mathbb{C}^n)$. We obtain immediately

Proposition 9.13. For the model $\phi_{n,k} : J_k^{\mathrm{reg}}(1, n) \to \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$ of Theorem 2.4 and the model $\psi_{\mathbf{1}_k,n} : M_{\mathbf{1}_k,n} \to \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$, we have

$$\psi_{\mathbf{1}_k,n}(\mathrm{Hilb}^{\mathbf{1}_k}_{A_k}) = \overline{\mathrm{Im}(\phi_{n,k})} \simeq \mathrm{CHilb}^{k+1}_0(\mathbb{C}^n).$$

9.4.1 Complete sequences and associative algebras

In this section we relate the notion of complete sequences of partitions to the the notion of associative torus-fixed algebras.

Let $T \subset \mathrm{GL}_n$ be a torus acting with distinct weights on $\mathbb{C}^n$, and consider the natural action on $\mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$. Recall that for a partition $p = i_1^{l_1} \cdots i_r^{l_r} \in \mathcal{P}_j$ of j, we write $e_p = e_{i_1}^{l_1} \cdots e_{i_r}^{l_r} \in \mathrm{Sym}^j \mathbb{C}^n$. Then the T-fixed elements of $\mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$ can be indexed (up to permutations of basis elements)

$$W_\pi = \mathrm{Span}(e_{\pi_1}, \ldots, e_{\pi_k})$$

with $\pi = (\pi_1, \ldots, \pi_k)$ a sequence of partitions of integers $\leq k$. Recall that such a sequence is called complete (see Definition 6.4) if for every $i \in \{1, \ldots, k\}$ and every subpartition $\rho \subset \pi_i$ there exists j such that $\pi_j = \rho$

Lemma 9.14. Let $A \in M_{|\boldsymbol{d}|,n}$ be a T-fixed algebra with $\psi_{\boldsymbol{d}}(A) = W_\pi$. Then A is associative if and only if π is a complete sequence of partitions.

Proof. Denote by $\boldsymbol{m}$ the maximal ideal of the polynomial algebra $\mathcal{O}_{\mathbb{C}^n}$. For a T-fixed algebra $A \in M_{|\boldsymbol{d}|}$ we observe that the sequence of partitions π of the basis elements $e_{\pi_1}, \ldots, e_{\pi_k}$ of $\psi_{\mathbf{1}_k}$ is complete if and only if $A \simeq \boldsymbol{m}/I$ for a monomial ideal I. But an algebra $A \in M_{|\boldsymbol{d}|}$ is associative if and only if $A \simeq \boldsymbol{m}/I$ for some ideal I in $\boldsymbol{m}$, and moreover A is T-fixed if and only if I is a monomial ideal. $\square$

We obtain immediately

Proposition 9.15. The following statements hold.

(1) A T-fixed point $W_\pi \in \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$ is in the image of the restricted map $\psi_{\mathbf{1}_k,n} : \mathrm{Hilb}_0^{\mathbf{1}_k} \to \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$ only if π is complete.

(2) For the model $\phi : J_k^{\mathrm{reg}}(1,n) \to \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$ of Theorem 2.4, if $W_\pi \in \overline{\mathrm{Im}(\phi)}$ then π is complete.

Proof. Since $\pi_{\boldsymbol{d}}$ factors through $\pi_{\mathbf{1}_k} : M_{\mathbf{1}_k,n} \to M_{(k)}$ the first statement follows directly from Lemma 9.14. The last statement follows since

$$\overline{\mathrm{Im}(\phi)} \simeq \mathrm{CHilb}_0^{k+1}(\mathbb{C}^n) \simeq \psi_{\mathbf{1}_k,n}(\mathrm{Hilb}_{A_k}^{\mathbf{1}_k})$$

by Proposition 9.13. $\square$

Chapter 10

Distribution of monomial fixed points in $\mathrm{Hilb}_0^k(\mathbb{C}^n)$

In this chapter we will discuss the monomial fixed points of the punctual Hilbert scheme of points $\mathrm{Hilb}_0^k(\mathbb{C}^n)$. We shall apply so-called trivial extensions of algebras (to be defined in the next section) to reduce the question of whether all monomial ideals lie in the irreducible curvilinear component $\mathrm{CHilb}_0^k(\mathbb{C}^n)$ to the same question only for a special class of monomial ideals. This special class of monomial ideals is handled by Proposition 7.11, and so we end this section with proving

Theorem 10.9. Let $m \in \mathrm{Hilb}_0^k(\mathbb{C}^n)$ be a monomial ideal. Then $m \in \mathrm{CHilb}_0^k(\mathbb{C}^n)$.

10.1 Trivial extensions of algebras

In this section we will define what we mean by a *trivial extension* of a given algebra structure $\xi \in \mathrm{Hilb}_0^k(\mathbb{C}^n)$. This is merely a matter of adjoining basis elements to ξ without any multiplicative structure whatsoever (that is, every element multiplies to 0 with such adjoined basis element). The goal of this section is to obtain a descent property for trivial extensions:

If $\xi' \in \mathrm{CHilb}_0^{k+m}(\mathbb{C}^{n+m})$ is a trivial extension of $\xi \in \mathrm{Hilb}_0^k(\mathbb{C}^n)$, then $\xi \in \mathrm{CHilb}_0^k(\mathbb{C}^n)$.

This is one half of Proposition 10.2.

Choosing a basis $E_1, \dots, E_n$ for the underlying vector space $\mathbb{C}^n \simeq \xi$, the algebra structure of $\xi \in \mathrm{Hilb}_0^k(\mathbb{C}^n)$ is determined by its *structure constants* c_{ij}^k in the equations

$$E_i E_j = \sum_k c_{ij}^k E_k.$$

In this context we may write $\xi = \langle E_1, \dots, E_n \rangle$ (the parameter space of structure constants corresponds essentially to algebra structures with basis, see [27]). Recall that the socle of such algebra is defined as

$$\mathrm{soc}(\xi) = \{x \in \xi \mid \xi x = 0\}.$$

Definition 10.1. Let $m \in \mathbb{Z}_{\geq 0}$. We say that $\xi' \in \mathrm{Hilb}_0^{k+m}(\mathbb{C}^{n+m})$ is a *trivial extension of* $\xi \in \mathrm{Hilb}_0^k(\mathbb{C}^n)$ if $\xi' = \langle E_1, \dots, E_n, F_1, \dots, F_m \rangle$ is an algebra structure on $\mathbb{C}^{n+m}$ satisfying that ξ' restricted to $\mathrm{Span}(E_1, \dots, E_n)$ is that of ξ together with $F_1, \dots, F_m \in \mathrm{soc}(\xi')$

Concretely, the structure constants for ξ' are given by

$$E_i \cdot_{\xi'} E_j = E_i \cdot_{\xi} E_j \in \xi = \langle E_1, \dots, E_n \rangle, \quad E_i \cdot_{\xi'} F_j = 0, \quad F_i \cdot_{\xi'} F_j = 0.$$

Proposition 10.2. Let $\xi' \in \mathrm{Hilb}_0^{k+m}(\mathbb{C}^{n+m})$ be a trivial extension of a monomial algebra $\xi \in \mathrm{Hilb}_0^k(\mathbb{C}^n)$. Then

$$\xi' \in \mathrm{CHilb}_0^{k+m}(\mathbb{C}^{n+m}) \iff \xi \in \mathrm{CHilb}_0^k(\mathbb{C}^n).$$

Proof. Considering the algebra ξ as a point in $\mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$, we write $\xi = [e_\pi]$ with $\pi \in \mathcal{A}_{n,k}$. Consider then the sequence of partitions

$$\pi' = (\pi_1, \dots, \pi_k, k+1, \dots, k+m) \in \mathcal{A}_{k+m,k+m}$$

with $[e'_\pi] \simeq \xi'$. Observing that $l = \pi'_l \in \mathcal{P}_l$ is toric for $l = m+1, \dots, k+m$ we obtain by repeated application of Proposition 7.14 that

$$[e_{\pi'}] \in \mathrm{CHilb}_0^{k+m}(\mathbb{C}^{k+m}) \iff [e_\pi] \in \mathrm{CHilb}_0^k(\mathbb{C}^{k+m}).$$

At last, since $[e_\pi] \in \mathrm{Hilb}_0^k(\mathbb{C}^n)$ by assumption, we have also

$$[e_\pi] \in \mathrm{CHilb}_0^k(\mathbb{C}^{n+m}) \iff [e_\pi] \in \mathrm{CHilb}_0^k(\mathbb{C}^n)$$

finishing the proof. □

In the above proof we use both that ξ is a monomial algebra and that ξ' is a trivial extension of ξ. The latter to make sure that the extra basis elements adjoined to ξ can be put in a toric position; i.e. such that π'_l is toric for $l = k+1, \dots, k+m$ in the notation of the proof. The author definitely believes these assumptions can be relaxed, but certainly some assumptions are required, as we illustrate in the following

Example 10.3. By [19] there exists an algebra $A \notin \mathrm{CHilb}_0^8(\mathbb{A}^4)$ with Hilbert function $H_A = (1,4,3)$ (in fact, A can be chosen not even smoothable).

Pick d maximal so that there exists an algebra A with Hilbert function $H_A = (1,4,d)$ such that $A \notin \mathrm{CHilb}_0^{d+5}(\mathbb{A}^4)$. If $d = 10$ (which is the maximal possible choice) we have $A \simeq \mathbb{C}[x_1,x_2,x_3,x_4]/m^3 \in \mathrm{CHilb}_0^{15}(\mathbb{A}^4)$ for $m = (x_1,x_2,x_3,x_4)$ the maximal ideal in $\mathbb{C}[x_1,x_2,x_3,x_4]$, whence it follows that $3 \leq d < 10$.

Let now $A \notin \mathrm{CHilb}_0^{d+5}(\mathbb{A}^4)$ be such algebra with $H_A = (1,4,d)$ for this maximal d. Then A may be written as $A = B/(s)$ for some some B with $H_B = (1,4,d+1)$ and some socle element $s \in \mathrm{soc}(B)$, so that by the choice of d we have $B \in \mathrm{CHilb}_0^{d+6}(\mathbb{A}^4)$. Observe however that in this case A is not a subalgebra of B since $s \in m_A^2$ where m_A is the maximal ideal in A. ♦

10.2 Associated 0-defect algebra of monomial ideal

In this section we describe how to associate to a monomial ideal $m \in \mathrm{Hilb}_0^k(\mathbb{C}^n)$ another monomial ideal $m' \in \mathrm{Hilb}_0^K(\mathbb{C}^N)$ with $K \geq k, N \geq n$ such that the algebra $\mathcal{O}_{\mathbb{C}^N}/m'$ is a trivial extension in the sense of the previous Section 10.1 of a subalgebra isomorphic to $\mathcal{O}_{\mathbb{C}^n}/m$. In this case it will be clear that by Proposition 7.11 we have $m' \in \mathrm{CHilb}_0^K(\mathbb{C}^N)$, and it will follow further from Proposition 10.2 that $m \in \mathrm{CHilb}_0^k(\mathbb{C}^n)$ as well.

10.2.1 The sequence π^m associated to a monomial ideal m

Recall that a monomial ideal $m \in \mathrm{Hilb}_0^{k+1}(\mathbb{C}^n)$ corresponds to an algebra $\mathcal{O}_{\mathbb{C}^n}/m$ with underlying vector space of dimension $k+1$ and with basis, say $\langle 1, E_1, \ldots, E_k \rangle$. We write A^m for the algebra obtained by quotienting by the ideal generated by 1, so A^m is a nilpotent, associative and commutative algebra with basis $\langle E_1, \ldots, E_k \rangle$. Write $E_{i_1}, \ldots, E_{i_r}$ for the generators of the maximal ideal $\mathfrak{m}_{A^m}$ of A^m where $r \leq n$. Since m is a monomial ideal and one observes that for all $1 \leq j \leq k$

$$E_j = E_{i_1}^{a_j^1} \cdots E_{i_r}^{a_j^r}$$

for some $a_j^l \in \mathbb{Z}_{\geq 0}$. To such expression we associate the partition (recall our notation from the Subsection 6.1.1)

$$\pi_j = 1^{a_j^1} \cdots r^{a_j^r}$$

where if $a_j^l = 0$, there are no parts l in the partition π_j.

Introduce the following ordering on $\mathbb{Z}_{\geq 0}^k$

$$(a_1, \ldots, a_k) \leq (b_1, \ldots, b_k) \iff a_d < b_d, \qquad d := \max\{i \mid a_i \neq b_i\}. \tag{10.1}$$

This induces an ordering on the partitions

$$\pi_i \leq \pi_j \iff (a_i^1, \ldots, a_i^r) \leq (a_j^1, \ldots, a_j^r),$$

and further on the basis elements

$$E_i \leq E_j \iff \pi_i \leq \pi_j.$$

We denote the ordered sequence of partitions by $\pi^m = (\pi_1^m, \ldots, \pi_k^m)$. Since A is associative π^m is also complete by Lemma 9.14.

We note that we are *not* claiming that π^m is admissible. The author believes that for any isomorphism class of a monomial algebra there exists a representative, such that the associated sequence is admissible, but has not been able to prove this.

Example 10.4. Consider $m = (x^3, xy, y^3) \in \mathrm{Hilb}_0^4(\mathbb{C}^2)$ with associated algebra $\mathbb{C}[x,y]/m$ with basis elements $\langle 1, x, y, x^2, y^2 \rangle$ as $\mathbb{C}$-module. Quotienting by 1 we get the nilpotent algebra A generated as $\mathbb{C}$-module by $\langle x, y, x^2, y^2 \rangle$. As written we see that $(\pi_1, \ldots, \pi_4) = (1, 2, 1^2, 2^2)$, which is not ordered according to (10.1). Rearranging the basis elements and the partitions, we obtain an isomorphic algebra structure with the associated sequence of partitions

$$\pi^m = (1, 1^2, 2, 2^2),$$

which is complete. ♦

Observe that under the natural embedding described in (2.1) composed with the Plücker embedding

$$\mathrm{Hilb}_0^{k+1}(\mathbb{C}^n) \hookrightarrow \mathrm{Grass}_k\Big(\bigoplus_{i=1}^{k} \mathrm{Sym}^i(\mathbb{C}^n)\Big) \hookrightarrow \mathbb{P}\Big(\bigwedge^k \bigoplus_{i=1}^{k} \mathrm{Sym}^i\,\mathbb{C}^n\Big)$$

we have $m \mapsto [e_{\pi^m}]$. We proceed to explain in the next subsection how to extend such admissible sequence π^m to a toric sequence τ^m, which in a certain sense contains π^m.

10.2.2 The toric sequence τ^m associated to a monomial ideal m

We assume still that $m \in \mathrm{Hilb}_0^{k+1}(\mathbb{C}^n)$ is a monomial ideal with $\mathcal{O}_{\mathbb{C}^n}/m$ an algebra of dimension $k+1$ and A^m the associated nilpotent algebra obtained by quotienting by 1. In the previous subsection we associated to A^m a complete sequence of partitions π^m. We describe now how to associate to π^m a toric sequence $\tau^m = (\tau_1^m, \ldots, \tau_K^m)$ for some $K \geq k$ containing π^m as a subsequence in the following sense: There is an injection $\sigma : \{1, \ldots, k\} \hookrightarrow \{1, \ldots, K\}$ acting on a partition by acting on each part, and such that the partitions $\sigma.\pi_1^m, \ldots, \sigma.\pi_k^m$ are in the sequence τ^m.

Recall that $A^m = \langle E_1, \ldots, E_k\rangle$ and π^m are ordered by (10.1), and that $E_{i_1}, \ldots, E_{i_r}$ generate the maximal ideal $\mathfrak{m}_{A^m}$ of A^m. The appropriate extension of A^m to a larger algebra B^m is made by embedding the underlying vector space $\mathbb{C}^k = \mathrm{Span}(E_1, \ldots, E_k)$ into a (sufficiently) large space $\mathbb{C}^K$, as we will now explain. We will iterate on the number of generators r.

To each number $j \in \{1, \ldots, k\}$ we will associate a new number $\sigma^m(j) \geq j$ such that $\sigma^m : \{1, \ldots, k\} \to \{1, \ldots, K\}$ is an injection, and where $K \geq k$ is still to be defined.

For $j < i_2$ we put $\sigma^m(j) = j$. Assume now that for any $j < i_l$ the number $\sigma^m(j)$ is given; in particular, $\sigma^m(i_1), \ldots, \sigma^m(i_{l-1})$ are given. Define then $\sigma^m(i_l) = \sigma^m(i_{l-1}) + 1$. For $j < i_{l+1}$ we have the expression $E_j = E_{i_1}^{a_j^1} \cdots E_{i_l}^{a_j^l}$ in A^m, and we define

$$\sigma^m(j) := \sum_{p=1}^{l} \sigma^m(i_p) a_j^p.$$

Define $K := \sigma^m(k)$. Moreover, we define the action of σ^m on partitions: Let $p = 1^{l_1} \cdots r^{l_r}$ be a partition with parts $\underline{\mathrm{Parts}}(\pi) \subset \{1, \ldots, r\}$, then σ^m acts on p by

$$\sigma^m.p = \sigma(i_1)^{l_1} \cdots \sigma^m(i_r)^{l_r},$$

and we define the action on a sequence of partitions $\sigma^m.(\pi_1, \ldots, \pi_k) = (\sigma^m.\pi_1, \ldots, \sigma^m.\pi_k)$.

Lemma 10.5. The map $\sigma^m : \{1, \ldots, k\} \to \{1, \ldots, K\}$ defined above satisfies

(1) $\pi_i > \pi_j \implies \sigma^m.\pi_i > \sigma^m.\pi_j$,

(2) $i > j \implies \sigma^m(i) > \sigma^m(j)$, and in particular σ^m is injective,

(3) $\sigma^m.\pi_j^m \in \mathcal{P}_{\sigma^m(j)}$ is a partition of $\sigma^m(j)$.

Proof. We write just $\sigma := \sigma^m$. Since $\sigma(i_l) = \sigma(i_{l-1}) + 1$ by definition, in order to show (2), it is enough to show that $\sigma(j) < \sigma(j+1)$ for $i_l \le j < j+1 < i_{l+1}$. By the ordering of (10.1), we have $(a_j^1, \dots, a_j^l, 0 \dots, 0) \le (a_{j+1}^1, \dots, a_{j+1}^l, 0, \dots, 0)$ with $a_j^l, a_{j+1}^l \neq 0$. In particular, (2) follows from (1).

Thus, consider $(a_j^1, \dots, a_j^k) \le (a_{j+1}^1, \dots, a_{j+1}^k)$ for some $1 \le j \le k-1$. If $a_j^k < a_{j+1}^k$, we compare simply $(a_j^1, \dots, a_j^{k-1}, 0) \le (a_{j+1}^1, \dots, a_{j+1}^{k-1}, a_{j+1}^k - a_j^k)$. Since π^m is complete the partitions corresponding to these vectors are also in π^m, and we see directly (again since $\sigma(i_k) = \sigma(i_{k-1}) + 1$) that $\sigma(j) < s(j+1)$ in this case. If $a_j^k = a_{j+1}^k$, one iterates to largest l such that $a_j^l \neq a_{j+1}^l$ in which case we must have $a_j^l < a_{j+1}^l$, and the above argument applies again. This proves (1) and (2).

For (3), we have defined $\sigma(j) := \sum_{p=1}^{l} \sigma(i_p) a_j^p$ and $\sigma . \pi_j^m = \sigma(i_1)^{a_j^1} \cdots \sigma(i_k)^{a_j^k}$, and so by construction

$$|\sigma . \pi_j^m| = \sigma(j).$$

In other words $\sigma . \pi_j^m \in \mathcal{P}_{\sigma(j)}$ is a partition of $\sigma(j)$. $\square$

Definition 10.6. The toric sequence $\tau^m = (\tau_1^m, \dots, \tau_K^m)$ associated to $m \in \mathrm{Hilb}_0^{k+1}(\mathbb{C}^n)$ is defined by

$$\tau_i^m = \begin{cases} \sigma^m . \pi_j & i = \sigma^m(j) \\ i & i \in \{1, \dots, K\} \setminus \ \mathrm{Im}\, \sigma^m \end{cases}$$

The sequence τ^m is well-defined and toric by Lemma 10.5. Such sequence corresponds again to a nilpotent algebraic structure $B^m = \langle E_1', \dots, E_K' \rangle$ determined by the description

$$E_i' = \begin{cases} E_{\sigma^m(i_1)}^{a_j^1} \cdots E_{\sigma^m(i_r)}^{a_j^r} & i = \sigma^m(j) \\ E_i' & i \in \{1, \dots, K\} \setminus \mathrm{Im}\, \sigma^m. \end{cases}$$

Clearly, restricting B^m to $\mathrm{Span}(E_{\sigma^m(i_1)}, \dots, E_{\sigma^m(i_r)})$ we clearly obtain an algebraic structure isomorphic to the original algebra A^m. Moreover, for $i \in \{1, \dots, K\} \setminus \mathrm{Im}\, \sigma^m$ we have $E_i' \in \mathrm{soc}(B^m)$.

At last, we remark that taking $N = \max(\{1, \dots, K\} \setminus \mathrm{Im}\, \sigma^m \cup \{j_1, \dots, j_r\})$ to be the maximal index among the generators of the maximal ideal of B^m, we have that after adjoining again the unit element $(\mathbb{C}\langle 1 \rangle \oplus B^m) \in \mathrm{Hilb}_0^{K+1}(\mathbb{C}^N)$.

These observations together with the definition of B^m via τ^m yield

Proposition 10.7. The algebra $B^m \in \mathrm{Hilb}_0^{K+1}(\mathbb{C}^N)$ is a trivial extension of $A^m \in \mathrm{Hilb}_0^{k+1}(\mathbb{C}^n)$ with $B^m \mapsto [e_{\tau^m}]$ under the natural embedding defined in (2.1)

$$\mathrm{Hilb}_0^{K+1}(\mathbb{C}^N) \longrightarrow \mathrm{Grass}_K\Big(\bigoplus_{i=1}^{K} \mathrm{Sym}^i(\mathbb{C}^N)\Big) \longrightarrow \mathbb{P}\Big(\bigwedge^{K} \bigoplus_{i=1}^{K} \mathrm{Sym}^i\, \mathbb{C}^N\Big)$$

composed with the Plücker embedding.

Example 10.8. We extend upon the Example 10.4, where $m = (x^3, xy, y^3) \in \mathrm{Hilb}_0^4(\mathbb{C}^2)$ and we found $\pi^m = (1, 1^2, 2, 2^2)$ so that $i_1 = 1$ and $i_2 = 3$ are the indices of the generators of the algebra A^m.

To obtain the toric sequence τ^m, we consider the map σ^m used to define τ^m. In this case $\sigma^m . \pi_1^m = 1 = \sigma^m(1)$ and $\sigma^m . \pi_2^m = 1^2$ with $\sigma^m(2) = 2$. We get then $\sigma^m(3) = 3$

and $\sigma^m.\pi_3^m = 3$, and it follows that $\sigma^m.\pi_4^m = \sigma(3)^2 = 3^2$ and $\sigma^m(4) = 3 \cdot 2 = 6 =: K$. Thus, $\sigma^m : \{1, \dots, 4\} \to \{1, \dots, 6\}$, and we have defined

$$\tau^m = (1, 1^2, 3, 4, 5, 3^2),$$

which is indeed toric. We finish by noting that the algebraic substructure of the associated algebra $B^m = \langle E_1', \dots, E_6' \rangle$ on $\mathrm{Span}(E_1', E_2', E_3', E_6')$ is isomorphic to A^m, and that B^m is indeed a trivial extension of A^m in the sense of Section 10.1. ♦

10.2.3 All monomial ideals are in the curvilinear component

In this subsection we simply put the puzzle together from constructions and results from previous sections. The main result is the following

Theorem 10.9. Let $m \in \mathrm{Hilb}_0^k(\mathbb{C}^n)$ be a monomial ideal. Then $m \in \mathrm{CHilb}_0^k(\mathbb{C}^n)$.

Proof. Write A^m for the $(k-1)$-dimensional nilpotent algebra obtained from the algebra $\mathcal{O}_{\mathbb{C}^n}/m$ by quotienting by the unit 1. We choose a filtration as in Subsection 10.2.1; that is a choice of an ordered basis $E_1, \dots, E_{k-1}$ of A^m ordered according to (10.1), which associates to m also an admissible and complete sequence π^m.

Following Subsection 10.2.2 we extend A^m to a larger K-dimensional algebra B^m for some $K \geq k-1$, which is a trivial extension of A^m in the sense of Section 10.1 by Proposition 10.7. The larger algebra is constructed by a choice of ordered basis, which immediately associates to B^m a toric sequence τ^m such that $B^m \mapsto [e_{\tau^m}]$ under the natural embedding defined in (2.1)

$$\mathrm{Hilb}_0^{K+1}(\mathbb{C}^N) \hookrightarrow \mathrm{Grass}_K\Big(\bigoplus_{i=1}^{K} \mathrm{Sym}^i(\mathbb{C}^N)\Big) \hookrightarrow \mathbb{P}\Big(\bigwedge^{K} \bigoplus_{i=1}^{K} \mathrm{Sym}^i\,\mathbb{C}^N\Big)$$

composed with the Plücker embedding (cf. Proposition 10.7).

It follows then from Proposition 7.11 that the algebra $[e_{\tau^m}] \in \overline{\mathrm{Im}\,\phi_{n,k}}$, or equivalently by Theorem 2.5 after augmenting B^m with a unit element $\mathbb{C}\langle 1 \rangle \oplus B^m \in \mathrm{CHilb}_0^{K+1}(\mathbb{C}^N)$.

Since as noted B^m is a trivial extension of A^m, we have that $\mathbb{C}\langle 1 \rangle \oplus B^m \in \mathrm{CHilb}_0^{K+1}(\mathbb{C}^N)$ is a trivial extension of $1 \oplus A = \mathcal{O}_{\mathbb{C}^n}/m$, and we obtain immediately from Proposition 10.2 that $\mathcal{O}_{\mathbb{C}^n}/m \in \mathrm{CHilb}_0^k(\mathbb{C}^n)$, which is what we wanted to conclude. □

Chapter 11

The toric submodel

In this section our goal is to give another proof of the fact that $[e_\pi] \in \overline{\operatorname{Im}\phi^0_{n,k}} \subset \mathrm{CHilb}^{k+1}_0(\mathbb{C}^n)$ for all toric sequences of partitions π than given in the proof of Proposition 7.11. We offer here a different viewpoint using general methods of toric geometry for which we refer to [43]. The cloncluding result in this chapter is Proposition 11.3.

In this section we consider the toric submodel described in Section 2.4. Writing as usual

$$\widetilde{J_k(1,n)}_E = \operatorname{Spec} R_{n,k}, \quad R_{n,k} = \mathbb{C}[b_{i,j} : 1 \le i \le n, i \le j \le k]$$

we have on the subvariety (defined in Section 2.4), after easing the notation $b_i := b_{i,i}$,

$$\widetilde{J_k(1,n)}^0_E = \operatorname{Spec} R^T_{n,k}, \quad R^T_{n,k} = \mathbb{C}[b_i : 1 \le i \le m] \subset R_{n,k},$$

where $m = \min(n,k)$. Recall the toric submodel defined in section 2.4

$$\phi^0_{n,k} : \widetilde{J_k(1,n)}^0_E \dashrightarrow \mathbb{P}\Big(\bigwedge^k \bigoplus_{i=1}^k \operatorname{Sym}^i \mathbb{C}^n\Big),$$

and write $T_{n,k} = \overline{\operatorname{Im}\phi^0_{n,k}}$ for the toric variety in $\mathbb{P}(\bigwedge^k \bigoplus_{i=1}^k \operatorname{Sym}^i \mathbb{C}^n)$. In fact, recall that a sequence of partitions $\pi = (\pi_1, \dots, \pi_k)$ is *toric* (or 0-defect) if each π_i is a partition of i, i.e. $\pi_i \in \mathcal{P}_i$, then it is clear that the image of $\phi^0_{n,k}$ satisfies

$$\operatorname{Im}\phi^0_{n,k} \subset \mathbb{P}(\operatorname{Span}\{e_\pi \mid \pi \text{ is toric}\}).$$

We write $[e_\pi]$ for the point in $\mathbb{P}(\operatorname{Span}\{e_\pi \mid \pi \text{ is toric}\})$ corresponding to the line in $\operatorname{Span}\{e_\pi \mid \pi \text{ is toric}\}$ spanned by e_π.

Our goal is to show that in fact $[e_\pi] \in \overline{\operatorname{Im}\phi^0_{n,k}} \subset \mathrm{CHilb}^{k+1}_0(\mathbb{C}^n)$ for all toric sequences of partitions π.

Writing the map $\phi^0_{n,k}$ explicitly, we see that

$$(b_1, \dots, b_k) \xmapsto{\phi^0_{n,k}} [\sum_\pi b_\pi e_\pi], \quad \pi \text{ is toric}$$

where $b_\pi = b_{\pi_1} \cdots b_{\pi_k}$ is a monomial (in the notation of Section 6.2, we have $b_\pi = m^\pi_\pi$), and we have defined $b_\rho = b^{r_1}_{\rho_1} \cdots b^{r_l}_{\rho_l}$ for a partition $\rho = \rho_1^{r_1} \cdots \rho_l^{r_l}$.

11.1 Divisibility relations of monomials and hyperplanes

In this section we discuss and setup language in order to construct a resolution of the map $\phi^0_{n,k}$ whose sheaf of (vanishing) ideal is principal on every chart of the source space generated by the pullback of a monomial b_π with π 0-defect. More importantly we argue that for every monomial b_π there exists a chart of the source space on which the pullback of b_π generates the vanishing ideal restricted to this chart. This is the content of Proposition 11.3. We shall use language from toric geometry such as fans and toric varieties associated to such fans. We refer for a general treatment of this subject to [43]. The arguments and results of this section appeared already in [11].

We fix n and k, and write still $m = \min(n, k)$. Obviously $b_\pi \in R^T_{n,k} = \mathbb{C}[b_i : 1 \leq i \leq m]$ and we consider its vector of exponents v_π in the integer lattice $M \simeq \mathbb{Z}^m \subset \mathbb{R}^m$ with dual space $N \simeq \mathbb{Z}^m \subset (\mathbb{R}^m)^*$. Via the canonical pairing

$$M \times N \to \mathbb{Z}, \quad (v, \mu) \mapsto \langle v, \mu \rangle$$

we introduce hyperplanes $H_{\pi,\pi'}$ in the dual space $(\mathbb{R}^m)^*$

$$H_{\pi,\pi'} = \{\mu \in (\mathbb{R}^m)^* \mid \langle v_\pi - v_{\pi'}, \mu \rangle = 0\}$$

and write $H^-_{\pi,\pi'}$ for the half space of μ with $\langle v_\pi - v_{\pi'}, \mu \rangle \leq 0$. The hyperplanes $H_{\pi,\pi'}$ go through the origin and yield a division of $(\mathbb{R}^m_{\geq 0})^*$ into strictly convex rational polyhedral cones (meaning that each cone contains no positive dimensional linear subspace, and is bounded by *finitely many* hyperplanes)

Definition 11.1. We denote by $F_{n,k}$ the fan with support $(\mathbb{R}^m_{\geq 0})^*$ defined by the hyperplanes $H^-_{\pi,\pi'}$, and $\mathcal{T}(F)$ denotes the corresponding toric variety.

Every such fan F has a regular (meaning that all rays are in the integral lattice $(\mathbb{Z}^m)^*$) refinement $\tilde{F}$ [43][Chapter 1, Theorem 11]). Equivalently, the associated toric variety $\mathcal{T}(\tilde{F})$ is nonsingular. We obtain the following resolution picture

$$\begin{tikzcd}
\mathcal{T}(\tilde{F}) \arrow[d] \arrow[ddr, "\tilde{\phi}^0_{n,k}"] & \\
\mathcal{T}(F) \arrow[d] \arrow[dr, dashed, "\bar{\phi}^0_{n,k}"] & \\
\widetilde{J_k(1,n)}^0_E \arrow[r, dashed, "\phi^0_{n,k}"] & T_{n,k},
\end{tikzcd}$$

where all maps are torus equivariant with respect to the torus T^m acting on $R^T_{n,k}$ diagonally.

We denote by $\mathcal{A}^T_{n,k}$ the set of toric n-admissible sequences, and write just $\mathcal{A}^T_k := \mathcal{A}^T_{k,k}$. Recall that a sequence of partitions $\pi = (\pi_1, \ldots, \pi_k)$ is complete (cf. Definition 6.4) if for every i and every subpartition $\delta \subset \pi_i$ there exists j such that $\delta = \pi_j$ (when π is 0-defect, we must have $j \leq i$).

Proposition 11.2. Let $\pi = (\pi_1, \ldots, \pi_k)$ be a toric sequence of partitions, and suppose that

(1) $\bigcap_{(\pi'_1,\ldots,\pi'_{k-1})\in\mathcal{A}^T_{k-1}} H^-_{(\pi_1,\ldots,\pi_{k-1}),(\pi'_1,\ldots,\pi'_{k-1})} \cap (\mathbb{R}^k_{\geq 0})^* \neq \emptyset$

(2) $\bigcap_{\pi'\in\mathcal{A}^T_k} H^-_{\pi,\pi'} \cap (\mathbb{R}^k_{\geq 0})^* = \emptyset$.

Then π is not complete.

Proof. Assume for contradiction that the two conditions of the proposition hold, and π is complete. There must exist a toric sequence $\pi' = (\pi_1, \ldots, \pi_{k-1}, \pi'_k)$ with $\pi'_k \neq \pi_k$ such that

$$\mu \in \bigcap_{(\pi'_1,\ldots,\pi'_{k-1})} H^-_{(\pi_1,\ldots,\pi_{k-1}),(\pi'_1,\ldots,\pi'_{k-1})} \cap (\mathbb{R}^k_{\geq 0})^* \implies \langle \mu, v_{\pi_k} - v_{\pi'_k}\rangle > 0.$$

It follows that in π_k and π'_k we have subpartitions δ in π_k and δ' in π'_k with $s(\delta) = s(\delta') < k$ and satisfying

$$\mu \in \bigcap_{(\pi'_1,\ldots,\pi'_{k-1})} H^-_{(\pi_1,\ldots,\pi_{k-1}),(\pi'_1,\ldots,\pi'_{k-1})} \cap (\mathbb{R}^k_{\geq 0})^* \implies \langle \mu, v_{\delta'} - v_{\delta}\rangle < 0.$$

and we conclude that $\delta \neq \pi_1, \ldots, \pi_{k-1}$. □

We conclude with the following result, which was already proven in Proposition 7.11.

Proposition 11.3. If $\pi \in \mathcal{A}^T_{n,k}$ is a complete, toric and n-admissible sequence of partitions, then $[e_\pi] \in \operatorname{Im} \bar{\phi}^0_{n,k} \subset \mathrm{CHilb}^{k+1}_0(\mathbb{C}^n)$.

Proof. For a complete, toric and n-admissible sequence π, we show that there is a nonempty cone σ_π in the fan $F_{k,k}$ satisfying

$$\sigma_\pi \subset \bigcap_{\pi'\in\mathcal{A}^T_k} H^-_{\pi,\pi'},$$

and this is enough since $F_{k,k}$ is a refinement of $F_{n,k}$ considered in the space $(\mathbb{R}^k_{\geq 0})^*$. The argument goes by induction on k. The statement is trivial for $k = 1$ since $\pi = (1)$ is the only toric sequence of one partition. The induction hypothesis is now that

$$\bigcap_{(\pi'_1,\ldots,\pi'_{k-1})\in\mathcal{A}^T_{k-1}} H^-_{(\pi_1,\ldots,\pi_{k-1}),(\pi'_1,\ldots,\pi'_{k-1})} \cap (\mathbb{R}^k_{\geq 0})^* \neq \emptyset.$$

Since π is complete, it follows from Proposition 11.2 that also

$$\bigcap_{\pi'\in\mathcal{A}^T_k} H^-_{\pi,\pi'} \cap (\mathbb{R}^k_{\geq 0})^* \neq \emptyset,$$

yielding in particular the existence of such cone σ_π in F as we wanted. □

Chapter 12

Calculations

We provide here some examples of calculations of integration on curvilinear Hilbert schemes $\mathrm{CHilb}_0^k(\mathbb{C}^n)$ via the methods described. We will always consider forms on the tautological bundle of the trivial line $\mathcal{O}_{\mathbb{C}^n}^{[k]}$, which essentially means that the Chern roots $\eta_i = 0$ vanish in the description given in equation (3.1) in Section 3.1.

12.1 The cases $k \leq n \leq 4$

We consider in this section examples of calculations for $k \leq n \leq 4$. We will gradually increase the parameter n to illustrate the increasing difficulty in calculations as well. Since $k \leq n$, we calculate always via Theorem 8.7.

Via the description given in Subsection 8.2.3 we find the following description of blow ups

$$\begin{array}{rc}
A_2: & (b_{1,1}, \underline{b_{2,2}}) \\
A_3: & (b_{2,2}, b_{2,3}, \underline{b_{3,3}}) \\
A_4: & (b_{3,3}, b_{2,4}, b_{3,4}, \underline{b_{1,1}}), (b_{3,3}, b_{2,4}, b_{3,4}, \underline{b_{2,2}}), (b_{3,3}, b_{2,4}, b_{3,4}, \underline{b_{4,4}})
\end{array}$$

with $\underline{(\,\cdot\,)}$ describing the chart. Denoting by $\omega_{i,j}^l$ the pullback of the weight $\omega_{i,j}$ (described in Section 6.5) through $A_1, \ldots, A_l$ in the Porteous tree $\mathcal{T}_{n,k}^{\mathrm{Port}}$, we obtain the following diagram – Here we give only the weights which actually change:

A_2 :	$\omega_{1,1}^2 = \omega_{1,1} - \omega_{2,2} = 2\lambda_1 - \lambda_2$
A_3 :	$\omega_{2,2}^3 = \omega_{2,2} - \omega_{3,3} = \lambda_2 - \lambda_3$ $\omega_{2,3}^3 = \omega_{2,3} - \omega_{3,3} = \lambda_2 - \lambda_3$
A_4 :	$\omega_{3,3}^4 = 2\omega_{3,3} - \omega_{1,1} - \omega_{4,4} = 2\lambda_3 - 2\lambda_1 - \lambda_4$ $\omega_{2,4}^4 = \omega_{2,4} + \omega_{3,3} - \omega_{1,1} - \omega_{4,4} = \lambda_2 + \lambda_3 - 2\lambda_1 - \lambda_4$ $\omega_{3,4}^4 = \omega_{3,4} + \omega_{3,3} - \omega_{1,1} - \omega_{4,4} = 2\lambda_3 - 2\lambda_1 - \lambda_4$

The weights are here described in terms of the weights $\omega_{i,j} = z_0 + \lambda_i - \lambda_1$ on $\tilde{\mathbb{A}}_{n,k}$.

12.1.1 The case $k = n = 2$

In this case there are only two relevant monomials in the Chern classes c_1, c_2 of the tautological bundle $\mathcal{O}_{\mathbb{C}^n}^{[k]}$ to calculate integrate, since they must have weighted degree $k(n-1) = 2$. These are c_2 and c_1^2. We integrate both.

We start by integrating the form $\alpha = c_2$ via Theorem 8.7. We use the description of weights given above, and will be very detailed in this first calculation.

$$\begin{aligned}
\int_{\mathrm{CHilb}_0^3(\mathbb{C}^2)} c_2 &= \operatorname*{Res}_{z_0=\infty} \operatorname*{Res}_{\boldsymbol{z}=\infty} \frac{(z_1 - z_2)\cdot c_2(z_1, z_2)}{\prod_{j=1}^k \prod_{i=1}^n (\lambda_i - z_j)\cdot \omega_{1,1}^2 \omega_{2,2}^2} \\
&= \operatorname*{Res}_{\boldsymbol{z}=\infty} \operatorname*{Res}_{z_0=\infty} \frac{(z_1 - z_2)\cdot z_1 z_2}{\prod_{j=1}^k \prod_{i=1}^n (\lambda_i - z_j)\cdot (2z_1 - z_2)(z_2 - z_1 + z_0)} \\
&= \operatorname*{Res}_{z_1=\infty} \operatorname*{Res}_{z_2=\infty} \frac{(z_1 - z_2)\cdot z_1 z_2}{\prod_{j=1}^{k-1} \prod_{i=1}^n (\lambda_i - z_j)} \frac{-1}{z_2^3}(1 + \frac{2z_1}{z_2} + \Big(\frac{2z_1}{z_2}\Big)^2 + \cdots) \\
&\qquad\qquad \cdot (1 + \frac{\lambda_1}{z_2} + \Big(\frac{\lambda_1}{z_2}\Big)^2 + \cdots)(1 + \frac{\lambda_2}{z_2} + \Big(\frac{\lambda_2}{z_2}\Big)^2 + \cdots) \\
&= \operatorname*{Res}_{z_1=\infty} \frac{z_1}{(\lambda_1 - z_1)(\lambda_2 - z_1)} \cdot (1) \\
&= \operatorname*{Res}_{z_1=\infty} -z_1 \cdot \frac{1}{z_1^2}(1 + \frac{\lambda_1}{z_1} + \Big(\frac{\lambda_1}{z_1}\Big)^2 + \cdots)(1 + \frac{\lambda_2}{z_1} + \Big(\frac{\lambda_2}{z_1}\Big)^2 + \cdots) \\
&= 1.
\end{aligned}$$

This result validates our Corollary 9.12, where the intersection number 1 was obtained via complete different methods.

We proceed by integration the form $\alpha = c_1^2$ via the same methods.

$$\begin{aligned}
\int_{\mathrm{CHilb}_0^2(\mathbb{C}^2)} c_1^2 &= \operatorname*{Res}_{z_0=\infty} \operatorname*{Res}_{\boldsymbol{z}=\infty} \frac{(z_1 - z_2)\cdot (z_1 + z_2)^2}{\prod_{j=1}^k \prod_{i=1}^n (\lambda_i - z_j)\cdot \omega_{1,1}^2 \omega_{2,2}^2} \\
&= \operatorname*{Res}_{z_1=\infty} \operatorname*{Res}_{z_2=\infty} \frac{z_1^3 + z_1^2 z_2 - z_1 z_2^2 - z_2^3}{\prod_{j=1}^{k-1} \prod_{i=1}^n (\lambda_i - z_j)} \frac{-1}{z_2^3}(1 + \frac{2z_1}{z_2} + \Big(\frac{2z_1}{z_2}\Big)^2 + \cdots) \\
&\qquad\qquad \cdot (1 + \frac{\lambda_1}{z_2} + \Big(\frac{\lambda_1}{z_2}\Big)^2 + \cdots)(1 + \frac{\lambda_2}{z_2} + \Big(\frac{\lambda_2}{z_2}\Big)^2 + \cdots) \\
&= \operatorname*{Res}_{z_1=\infty} \frac{z_1 \cdot 1 + (2z_1 + \lambda_1 + \lambda_2)}{(\lambda_1 - z_1)(\lambda_2 - z_1)} \\
&= \operatorname*{Res}_{z_1=\infty} \frac{3z_1}{(\lambda_1 - z_1)(\lambda_2 - z_1)} \\
&= 3.
\end{aligned}$$

12.1.2 Some cases $k = n = 3, 4$

We start by fixing $k = n = 3$ and integrate the form $\alpha = \chi_3^2$.

$$\begin{aligned}
\int_{\mathrm{CHilb}_0^4(\mathbb{C}^3)} c_3^2 &= \operatorname*{Res}_{z_0=\infty} \operatorname*{Res}_{\boldsymbol{z}=\infty} \frac{(z_1-z_2)(z_1-z_3)(z_2-z_3)\cdot(z_1z_2z_3)^2}{\prod_{j=1}^k\prod_{i=1}^n(\lambda_i-z_j)\cdot\omega_{1,1}^3\omega_{2,2}^3\omega_{2,3}^3\omega_{3,3}^3} \\
&= \operatorname*{Res}_{\boldsymbol{z}=\infty} \frac{(z_1-z_2)(z_1-z_3)(z_2-z_3)\cdot z_1^2z_2^2z_3^2}{\prod_{j=1}^k\prod_{i=1}^n(\lambda_i-z_j)\cdot(2z_1-z_2)(z_2-z_3)^2} \\
&= \operatorname*{Res}_{z_1=\infty} \operatorname*{Res}_{z_2=\infty} \frac{(-1)(z_1-z_2)z_1^2z_2^2}{\prod_{j=1}^{k-1}\prod_{i=1}^n(\lambda_i-z_j)\cdot(2z_1-z_2)} \\
&= \operatorname*{Res}_{z_1=\infty} \frac{(-1)z_1^2}{\prod_{i=1}^n(\lambda_i-z_1)} \\
&= 1
\end{aligned}$$

validating Corollary 9.12. The calculation of the case of $n = k = 4$ and $\alpha = c_4^3$ follows the same lines, and one obtains also

$$\int_{\mathrm{CHilb}_0^5(\mathbb{C}^4)} c_4^3 = 1,$$

validating again Corollary 9.12.

12.2 Some cases $n < k \leq 4$

In this section we focus on examples where $n < k$. We offer both a calculation via a full blow up trees $\mathcal{T}_{n,k}$ and a calculation using equivariant Poincare duals.

12.2.1 Blow up trees for $n \leq k = 3$

We have already described the blow up procedure in the case $n = k = 3$ in Example 7.3. We know from Section 7.2 that the tree $\mathcal{T}_{2,3}$ is obtained from $\mathcal{T}_{3,3}$ by removing branches containing some $b_{3,j}$. The blow up tree $\mathcal{T}_{3,3}$ then takes the following form

$$\begin{tikzcd}
& & & \tilde{\mathbb{A}}_{3,3} \arrow[d] & & & \\
& & & (b_{1,1},b_{2,2}) \arrow[dll, "b_{1,1}"'] \arrow[drr, "b_{2,2}"] & & & \\
& (b_{1,1},b_{2,3},b_{3,3}) \arrow[dl] \arrow[d] \arrow[dr] & & & & (b_{1,1},b_{2,2},b_{3,3}) \arrow[d, "b_{1,1}"] & \\
{} & {} & {} & & & (b_{2,2},b_{2,3},b_{3,3}) \arrow[dl] \arrow[d] \arrow[dr] & \\
& & & & {} & {} & {}
\end{tikzcd}$$

Here empty leaves in the end illustrate that the pullback of the Berczi-Szenes model $\phi_{3,3}$ to the leaves is well-defined. We choose not to label the edges since there is an edge for every generator of the ideal.

From this blow up tree $\mathcal{T}_{3,3}$ we calculate via Theorem 8.1. In the above illustration of the blow up tree, we the leaves by by a double index (l, m) with $l \in \{1,2\}$ and $m \in \{1,2,3\}$ according the branch. For m the numbers $1,2,3$ are ordered by the order given in the ideal of the last blow up. In this sense, we denote the pullback of the weights $\omega_{i,j}$ by $\omega_{i,j}^{l,m}$. These weights are illustrated in the follwing table. Since certain weights will pull back to 0, we insert weights $t_{i,j}$ the relevant variables $b_{2,2}$ and $b_{2,3}$ (see Section 6.6):

	$b_{1,1}$	$b_{2,2}$	$b_{2,3}$	$b_{3,3}$
$(1,1)$	$z_0+\lambda_1$	$\lambda_2-2\lambda_1+t_{2,2}$	$\lambda_2-2\lambda_1+t_{2,3}$	$\lambda_3-2\lambda_1$
$(1,2)$	$2\lambda_1-\lambda_2-b_{2,3}$	$\lambda_2-2\lambda_1+t_{2,2}-t_{2,3}$	$z_0+\lambda_2-\lambda_1+t_{2,3}$	$\lambda_3-\lambda_2-b_{2,3}$
$(1,3)$	$2\lambda_1-\lambda_3$	$\lambda_2-2\lambda_1+t_{2,2}$	$\lambda_2-\lambda_3+t_{2,3}$	$z_0+\lambda_3-\lambda_1$
$(2,1)$	$2\lambda_1-\lambda_2-t_{2,2}$	$z_0+\lambda_2-\lambda_1+t_{2,2}$	$t_{2,3}-t_{2,2}$	$\lambda_3-2\lambda_1-t_{2,2}$
$(2,2)$	$2\lambda_1-\lambda_2-t_{2,3}$	$t_{2,2}-t_{2,3}$	$z_0+\lambda_2-\lambda_1+t_{2,3}$	$\lambda_3-\lambda_2-t_{2,3}$
$(2,3)$	$2\lambda_1-\lambda_2$	$\lambda_2-2\lambda_1+t_{2,2}$	$\lambda_2-\lambda_3+t_{2,3}$	$z_0+\lambda_3-\lambda_1$

We recall that $\omega_{1,j} = (j-1)z_0$ for $j = 2, \ldots, k$ and by construction (in particular, by Section 6.5) we have also $\omega_{1,j}^{l,m} = (j-1)z_0$, so that

$$\prod_{j=2}^{k} \omega_{1,j}^{l,m} = (k-1)! z_0^{k-1}.$$

Furthermore for a leaf $L = L^{l,m}$ in $\mathcal{T}_{3,3}$ associated to an index (l, m) we have

$$c_{\mathrm{top}}(T_{0_{L^{l,m}}} L^{l,m}) = \prod_{1=i\leq j\leq k} \omega_{i,j}^{l,m}.$$

Now, fix the form $\alpha = c_2^3$. We calculate then

$$\begin{aligned} I_E(z_1,\ldots,z_k) :&= \operatorname*{Res}_{z_0=\infty}(k-1)! z_0^{k-1} \sum_{L\in\mathcal{L}} \frac{c_2^3(\theta_1^L,\ldots,\theta_k^L)}{c_{\mathrm{top}}(T_{0_L}L)} \\ &= -66\lambda_1^3 - 40\lambda_1^2\lambda_2 - 1\lambda_1\lambda_2^2 - \lambda_2^3 - 13\lambda_1^2\lambda_3 - 5\lambda_1\lambda_2\lambda_3 - \lambda_2^2\lambda_3 \\ &\quad + \text{terms depending on } t_{2,2} \text{ or } t_{2,3} \end{aligned}$$

and then by Theorem 8.1 we get

$$\begin{aligned} \int_{\mathrm{CHilb}_0^4(\mathbb{C}^3)} c_2^3 &= \sum_{\sigma\in\mathcal{S}_3} \frac{1}{\prod_{j=1}^3 \prod_{i=j+1}^3 (\lambda_{\sigma.i} - \lambda_{\sigma.j})} I_E(z_{\sigma.1}, z_{\sigma.2}, z_{\sigma.3}) \\ &= 17 \end{aligned}$$

For $n = 2$ and $k = 3$ we obtain the blow up tree $\mathcal{T}_{2,3}$ from $\mathcal{T}_{3,3}$ by removing all branches containing an edge labeled with some $b_{3,j}$, and removing these variables from the description of the ideals. Hence $\mathcal{T}_{2,3}$ has the shape

$$\begin{tikzcd}
 & & \tilde{\mathbb{A}}_{2,3} \arrow[d] & & \\
 & & (b_{1,1}, b_{2,2}) \arrow[dl, "b_{1,1}"'] \arrow[dr, "b_{2,2}"] & & \\
 & (b_{1,1}, b_{2,3}) \arrow[dl] \arrow[d] & & (b_{1,1}, b_{2,2}) \arrow[d, "b_{1,1}"] & \\
 {} & {} & & (b_{2,2}, b_{2,3}) \arrow[dl] \arrow[d] & \\
 & & {} & {} &
\end{tikzcd}$$

It follows that the table of weights $\omega_{i,j}^{l,m}$ is also just the relevant subtable from that of the $n = k = 3$ case

	$b_{1,1}$	$b_{2,2}$	$b_{2,3}$
$(1,1)$	$z_0 + \lambda_1$	$\lambda_2 - 2\lambda_1 + t_{2,2}$	$\lambda_2 - 2\lambda_1 + t_{2,3}$
$(1,2)$	$2\lambda_1 - \lambda_2 - t_{2,3}$	$\lambda_2 - 2\lambda_1 + t_{2,2} - t_{2,3}$	$z_0 + \lambda_2 - \lambda_1 + t_{2,3}$
$(2,1)$	$2\lambda_1 - \lambda_2 - t_{2,2}$	$z_0 + \lambda_2 - \lambda_1 + t_{2,2}$	$t_{2,3} - t_{2,2}$
$(2,2)$	$2\lambda_1 - \lambda_2 - t_{2,3}$	$t_{2,2} - t_{2,3}$	$z_0 + \lambda_2 - \lambda_1 + t_{2,3}$

Fix this time the form $\alpha = c_1^3$ and calculate

$$\begin{aligned}
I_E(z_1, \dots, z_k) : &= \operatorname*{Res}_{z_0=\infty} (k-1)! z_0^{k-1} \sum_{L \in \mathcal{L}} \frac{c_1^3(\theta_1^L, \dots, \theta_k^L)}{c_{\mathrm{top}}(T_{0_L} L)} \\
&= 22\lambda_1 + 6\lambda_2 + \text{terms depending on } t_{2,2} \text{ or } t_{2,3}
\end{aligned}$$

We obtain then from Theorem 8.1

$$\int_{\mathrm{CHilb}_0^4(\mathbb{C}^2)} c_1^3 = -16.$$

12.3 The cases $n < k = 4$

In Example 7.3 we have given part of the blow up table $\mathcal{T}_{4,4}$. We present this branching here. Due to the size of the full blow up tree $\mathcal{T}_{4,4}$, we will not present it fully. The First part of the tree $\mathcal{T}_{4,4}$ is exactly $\mathcal{T}_{3,3}$ given above. One then has to consider the six branch extensions of $\mathcal{T}_{3,3}$. The following is the extension of the edge labeled $b_{3,3}$. We have omitted the very last arrows; one interprets here that each generator coordinate of the verylast ideals correspond to an affine chart (a leaf).

$$
\begin{tikzcd}[column sep=small]
 & & (b_{2,2}, b_{2,3}, b_{3,3}) \arrow[d, "b_{3,3}"] & & & \\
 & & (b_{1,1}, b_{3,3}, b_{3,4}, b_{4,4}) \arrow[d, "b_{1,1}"] & & & \\
 & & (b_{2,2}, b_{2,3}, b_{3,3}, b_{2,4}, b_{4,4}) \arrow[dl, "b_{2,2}"'] \arrow[drr, "b_{2,3}"] & & & \\
 & (b_{1,1}, b_{2,2}, b_{2,3}, b_{2,4}, b_{3,4}, b_{4,4}) \arrow[dl, "b_{1,1}"'] \arrow[d, "b_{2,2}"] \arrow[dr, "b_{2,3}"] & & & (b_{1,1}, b_{2,2}, b_{2,3}, b_{2,4}, b_{3,4}, b_{4,4}) \arrow[dl, "b_{1,1}"'] \arrow[d, "b_{2,2}"] \arrow[dr, "b_{2,3}"] & \\
(b_{3,3}, b_{2,4}, b_{3,4}, b_{4,4}) & (b_{3,3}, b_{2,4}, b_{3,4}, b_{4,4}) & (b_{3,3}, b_{2,4}, b_{3,4}, b_{4,4}) & (b_{3,3}, b_{2,4}, b_{3,4}, b_{4,4}) & (b_{3,3}, b_{2,4}, b_{3,4}, b_{4,4}) & (b_{3,3}, b_{2,4}, b_{3,4}, b_{4,4})
\end{tikzcd}
$$

The other five extensions are about the same size (that is, the same number of blow ups). Via these blow up trees, we calculate for instance

$$\int_{\mathrm{CHilb}_0^5(\mathbb{C}^2)} c_1^4 = 125.$$

Recall that $\mathrm{Hilb}_0^k(\mathbb{C}^2) = \mathrm{CHilb}_0^k(\mathbb{C}^2)$ by [16]. We have thus verified the following conjecture, which was proposed to the author by Marcel Bökstedt, in the cases $k = 3, 4, 5$ in this chapter.

Conjecture 12.1. For all positive integers k, we have

$$\int_{\mathrm{Hilb}_0^k(\mathbb{C}^2)} c_1\big(\mathcal{O}_{\mathbb{C}^2}^{[k]}\big)^{k-1} = (-1)^{k-1} k^{k-2}.$$

Chapter 13

Final comments and further studies

In this concluding chapter we discuss some further directions of study extending upon results in this work. Some of these ideas are also noted in the preprint [11].

13.1 Positive characteristic $\operatorname{char} \Bbbk > 0$

We have assumed throughout this work that the base field k is algebraically closed and of characteristic $\operatorname{char} \Bbbk = 0$, but essentially the same methods used in this work can be applied to the positive characteristic case $\operatorname{char} \Bbbk > 0$ as well.

The first thing to understand in this case is the change of the Berczi-Szenes $\phi : J_k(1,n) \to \mathrm{Grass}_k(\bigoplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^n)$ model described in the Chapter 2. Explicitly, under the composition with the Plücker embedding into projective space, we have written ϕ with polynomial coefficients on basis elements e_π in Sections 6.1.2 and 6.2, and one must study which monomials vanish in case of characteristic $\operatorname{char} \Bbbk > 0$. This amounts to studying the number of compositions $|\mathrm{perm}(\rho)|$ representing the partition ρ and products of such (which is the coefficient of the monomials). In particular, since these numbers are bounded from above (far from being optimal by e.g. $1! \cdots k!$ for fixed n and k, we obtain that for very large prime $\operatorname{char} \Bbbk = p \gg 0$ no terms vanish in the expression of the model.

With these considerations in mind one may immediately study again which monomial ideals $m \in \mathrm{Hilb}_0^k(\Bbbk^n)$ satisfy that $\mu \in \mathrm{CHilb}_0^k(\Bbbk^n)$ via the Berczi-Szenes model ϕ. Since no terms vanish in the model for $\operatorname{char} \Bbbk \gg 0$, we obtain directly the result

Theorem 13.1. Let $\Bbbk$ be an algebraically closed field with $\operatorname{char} \Bbbk = 0$ or $\operatorname{char} \Bbbk \gg 0$.
If $m \in \mathrm{Hilb}_0^k(\Bbbk^n)$ is a monomial ideal, then $m \in \mathrm{CHilb}_0^k(\Bbbk^n)$.
A far from optimal bound for which the statement holds is $\operatorname{char} \Bbbk > 1! \cdots k!$.

As stated in the introduction, we obtain immediately the

Corollary 13.2. Let $\Bbbk$ be an algebraically closed field with $\operatorname{char} \Bbbk = 0$ or $\operatorname{char} \Bbbk \gg 0$.
The punctual Hilbert scheme $\mathrm{Hilb}_0^k(\mathbb{C}^n)$ is connected.

Regarding the results on integration on the curvilinear Hilbert scheme $\mathrm{CHilb}_0^k(\mathbb{C}^n)$ the main issue is that the non-reductive GIT theory developed in [6, 7], which our integration methods rely heavily upon, requires $\operatorname{char} \Bbbk = 0$. Should this non-reductive GIT theory be extended to positive characteristic in the future work, then these methods will once again apply more or less directly. To be precise, in case of char $\Bbbk > 0$ it may happen that some terms vanish in the model, and in this case the exact blow up algorithm described in Chapter 7 still applies, and if some monomials vanish in the model, then the blow up algorithm may even be reduced to a smaller blow up tree (see Section 7.2).

13.2 Hierarchy of singularities

To any monomial ideal with corresponding algebra $A^m \in \mathrm{Hilb}_0^{k+1}(\mathbb{C}^n)$ the ideas of the Berczi-Szenes model constructed in Chapter 2 extends to construct a similar model $\phi^m : J_{\boldsymbol{\lambda}}(u,v) \dashrightarrow \mathrm{Grass}_k(\mathrm{Sym}^{\leq \omega(\boldsymbol{\lambda})} \mathbb{C}^n)$ for suitable $u \geq m$ and v, where $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \dots, \lambda_k)$ represents the n-dimensional Young diagram formed by $k+1$ boxes (or equivalently coordinates in $\mathbb{Z}^n_{\geq 0}$) each corresponding to a basis element of A^m (the usual visualization of boxes under the staircase formed by m), and $\omega(\boldsymbol{\lambda}) = 1 + \max(i : \boldsymbol{m}^i/\boldsymbol{m}^{i+1})$ with $\boldsymbol{m}$ the maximal ideal of A^m. At last, $J_{\boldsymbol{\lambda}}(u,v)$ is the set of equivalence classes of holomorphic maps $f : (\mathbb{C}^u, 0) \to (\mathbb{C}^v, 0)$ where $f \sim g$ if and only if $f^(\boldsymbol{j})(0) = g^(\boldsymbol{j})(0)$ for all $\boldsymbol{j} \in \boldsymbol{\lambda}$ This construction is explained in [5][Section 4.1]. Writing the map ϕ^m explicitly with coefficients on basis elements e_π, it is the case that some of the monomials appearing in the model ϕ of Chapter 2 are simply left out of the model ϕ^m (as was the case in positive characteristic described above, but for different reasons).

One may then apply the same techniques as have been used here. In particular, one finds again that for all complete "toric" (a special class of sequences of partitions, but now quite with the same meaning as otherwise used in this work) π, the basis elements $[e_\pi]$ are in the closure of the set $\{\xi \in \mathrm{Hilb}_0^{k+1}(\mathbb{C}^n) \mid \xi \simeq A^m\} \subset \mathrm{Hilb}_0^{k+1}(\mathbb{C}^n)$. We offer an

Example 13.3. Consider the monomial ideal $m = (x^4, x^2y, y^2) \in \mathbb{C}[x,y]$ with algebra $A^m = \mathbb{C}[x,y]/m \in \mathrm{Hilb}_0^6(\mathbb{C}^2)$ represented by the basis elements $A^m = \langle 1, x, x^2, x^3, y, xy \rangle$. Give x and y the coordinates $(1,0)$ and $(0,1)$, respectively, and associate accordingly coordinates

$$(0,0), (1,0), (2,0), (3,0), (0,1), (1,1).$$

Let $\lambda_0, \dots, \lambda_5$ be unit boxes with lower left corner in each of these coordinates (say, in the given order) forming the Young diagram $\boldsymbol{\lambda}$ of A^m with $\omega(\boldsymbol{\lambda}) = 4$. The model $\phi^m : J_{\boldsymbol{\lambda}}(u,v) \dashrightarrow \mathrm{Grass}_5(\mathrm{Sym}^{\leq \omega(\boldsymbol{\lambda})} \mathbb{C}^n)$ takes the form

$$\begin{aligned} &(v_{10}, v_{20}, v_{30}, v_{01}, v_{11}) \\ &\qquad \longmapsto \mathrm{Span}(v_{10}, v_{20} + v_{10}^2, v_{30} + 2v_{10}v_{20} + v_{10}^3, v_{01}, v_{11} + 2v_{10}v_{01}) \end{aligned}$$

(observe how these expressions are determined via the "partitions" of each coordinate inside λ, as explained in [5][Section 4.1]). Being a model for A^m it satisfies (among many other things) that $\overline{\mathrm{Im}\,\phi^m} \simeq \mathrm{Hilb}^6_{A^m}(\mathbb{C}^5) := \overline{\{\xi \in \mathrm{Hilb}_0^6(\mathbb{C}^5) \mid \xi \simeq A^m\}} \subset \mathrm{Hilb}_0^6(\mathbb{C}^5)$.

Denote by $e_{10}, e_{20}, e_{30}, e_{01}, e_{11}$ the corresponding basis vectors of $\mathbb{C}^5$ and consider a diagonal torus T acting on these. Define a T-invariant $W \in \mathrm{Grass}_5(\mathrm{Sym}^{\leq \omega(\boldsymbol{\lambda})} \mathbb{C}^5)$ to be *toric* if

$$W \in \{\phi^m(v_{10}, v_{20}, v_{30}, v_{01}, v_{11}) \mid v_{\boldsymbol{j}} \in \mathrm{Span}(e_{\boldsymbol{j}}) \text{ for all } \boldsymbol{j} \in \boldsymbol{\lambda}\},$$

where ϕ^m is given above.

To a T-invariant W we associate also a diagram of boxes – each box with coordinate corresponding to a basis element of W. We say that W is complete if the diagram if in fact a Young diagram (that is, the diagram has "no holes". As in Lemma refassComplete this property corresponds to associativity of the corresponding algebra).

An argument similar to e.g. that of Chapter 11 yields that every T-invariant *toric and complete* $W \in \mathrm{Grass}_5(\mathrm{Sym}^{\leq\omega(\boldsymbol{\lambda})}\mathbb{C}^5)$ satisfies $W \in \mathrm{Hilb}^6_{A^m}(\mathbb{C}^5)$.

We consider the polynomial ring $R = \mathbb{C}[x_1, \ldots, x_5]$ and let I be a monomial ideal in R with corresponding algebra local algebra $(A, \boldsymbol{m})$.

If $\boldsymbol{m}$ has 5 generators then we may take $W = \mathrm{Span}(e_{10}, e_{20}, e_{30}, e_{01}, e_{11})$ which is toric and complete, and hence $W \in \mathrm{Hilb}^6_{A^m}(\mathbb{C}^5)$.

If $\boldsymbol{m}$ has 4 generators there are two isomorphism classes of monomial algebras in $\mathrm{Hilb}^6_0(\mathbb{C}^5)$, and we may take the toric and complete spaces

$$\mathrm{Span}(e_{10}, e_{10}^2, e_{30}, e_{01}, e_{11}), \mathrm{Span}(e_{10}, e_{20}, e_{10}e_{20}, e_{01}, e_{11}) \in \mathrm{Hilb}^6_{A^m}(\mathbb{C}^5).$$

When $\boldsymbol{m}$ has 3 generators there are four isomorphism classes of which only three are apparent as toric and complete spaces

$$\mathrm{Span}(e_{10}, e_{10}^2, e_{10}^3, e_{01}, e_{11}), \mathrm{Span}(e_{10}, e_{10}^2, e_{30}, e_{01}, e_{10}e_{01}),$$
$$\mathrm{Span}(e_{10}, e_{20}, e_{10}e_{20}, e_{01}, e_{10}e_{01}) \in \mathrm{Hilb}^6_{A^m}(\mathbb{C}^5).$$

At last, we note that obviously also $A^m \in \mathrm{Hilb}^6_{A^m}(\mathbb{C}^5)$.

We show that the remaining possible monomial algebras (up to isomorphism) are not in $A^m \in \mathrm{Hilb}^6_{A^m}(\mathbb{C}^5)$. Suppose that the algebra $A = R/I$ with basis $\langle 1, E_1, \ldots, E_5\rangle$ contains two pure squares; by which we mean that e.g. $E_3 = E_1^2$ and $E_4 = E_2^2$. We claim that $A \notin \mathrm{Hilb}^6_{A^m}(\mathbb{C}^5)$. if $W \in \overline{\mathrm{Im}\,\phi^m}$ is T-invariant then $v_{10} \in \mathrm{Span}(e_{\boldsymbol{i}})$ for some fixed $\boldsymbol{i} \in \lambda$. Observe that any monomial expression of the basis elements (in the image of ϕ^m)

$$v_{10}, v_{20} + v_{10}^2, v_{30} + 2v_{10}v_{20} + v_{10}^3, v_{01}, v_{11} + 2v_{10}v_{01}$$

in $\mathrm{Sym}^2(\mathbb{C}^5)$ is divisible by v_{10}. Thus, if W has a basis element $w \in \mathrm{Sym}^2(\mathbb{C}^5)$, we must have $w = e_{\boldsymbol{i}}e_{\boldsymbol{j}}$ for some $\boldsymbol{j} \in \boldsymbol{\lambda}$. We see that W cannot contain two pure squares as claimed.

Consider now the ideal $I = (x_1^3, x_2^2, x_3, x_4, x_5)$ for which the algebra A contains a non-pure elements of $\mathrm{Sym}^3\,\mathbb{C}^5$ corresponding to $x_1^2x_2 \in A$. By analyzing again the expression of ϕ^m, we see again that this is not possible so $R/I \notin \mathrm{Hilb}^6_{A^m}(\mathbb{C}^5)$. These two arguments together show that if $\boldsymbol{m}$ has 2 generators then $A \in \mathrm{Hilb}^6_{A^m}(\mathbb{C}^5) \implies A = A^m$.

A similar analysis shows that if $\boldsymbol{m}$ has 1 generator then $A \notin \mathrm{Hilb}^6_0(\mathbb{C}^5)$.

These considerations completely determines the hierarchy of monomial singularities with respect to A^m, i.e. determines which monomial ideal are – and which are not – in $\mathrm{Hilb}^6_{A^m}(\mathbb{C}^5)$. ♦

In the above example no extension of algebras were needed. However, this was a crucial trick for constructing T-fixed points in Morin case $A^m = \mathbb{C}[t]/(t^n)$, and so such extension are probably needed in the hope of characterizing T-fixed points in $\overline{\{\xi \in \mathrm{Hilb}^{k+1}_0(\mathbb{C}^n) \mid \xi \simeq A^m\}}$ via the midel ϕ^m. On the other hand, the issue is only that of obtaining a *toric* algebra, and in this respect trivial extensions (as described in Section 10.1) should always be enough. The problem would then be reduced to characterize the (isomorphism classes of) monomial ideals in

$$\{\phi^m(v_{\lambda_1}, \ldots, v_{\lambda_k}) \mid v_{\boldsymbol{j}} \in \mathbb{C}^n \text{ for all } \boldsymbol{j} \in \boldsymbol{\lambda}\},$$

which is simply combinatorics.

13.3 Cayley's formula – counting graphs

In Chapter 12, we showcased some calculations for relatively small parameters n and k. We ended by stating the Conjecture 12.1

$$\int_{\mathrm{Hilb}_0^k(\mathbb{C}^2)} c_1\big(\mathcal{O}_{\mathbb{C}^2}^{[k]}\big)^{k-1} = (-1)^{k-1}k^{k-2}$$

relating a certain integral on the punctual Hilbert scheme on surfaces with the numbers of Cayley's formula for counting trees. It would be very interesting to verify the conjecture for larger numbers k. In the positive, a geometric understanding of this relation in which $\mathrm{Hilb}_0^k(\mathbb{C}^2)$ is related to graphs would – to the author's best knowledge – be a new direction.

Bibliography

[1] M. Atiyah and R. Bott. The moment map and equivariant cohomology. *Topology*, 23(1):1–28, 1984.

[2] K. Behrend and B. Fantechi. Symmetric obstruction theories and Hilbert schemes of points on threefolds. *Algebra Number Theory 2*, pages 313–345, 2008.

[3] G. Bérczi. Tautological integrals on curvilinear Hilbert schemes. *Geometry & Topology*, 21(5):2897–2944, aug 2017.

[4] G. Bérczi. Tautological integrals on Hilbert scheme of points I. 2023.

[5] G. Bérczi. Tautological integrals on Hilbert scheme of points II: Geometric subsets, 2023.

[6] G. Bérczi, B. Doran, T. Hawes, and F. Kirwan. Constructing quotients of algebraic varieties by linear algebraic group actions, 2015.

[7] G. Bérczi, B. Doran, T. Hawes, and F. Kirwan. Geometric invariant theory for graded unipotent groups and applications. *Journal of Topology*, 11(3):826–855, Jul 2018.

[8] G. Bérczi, B. Doran, T. Hawes, and F. Kirwan. Projective completions of graded unipotent quotients, 2020.

[9] G. Bérczi and F. Kirwan. Graded unipotent groups and Grosshans theory, 2015.

[10] G. Bérczi and F. Kirwan. Moment maps and cohomology of non-reductive quotients, 2022.

[11] G. Bérczi and J. M. Svendsen. Fixed point distribution on Hilbert scheme of points, 2023.

[12] G. Bérczi and A. Szenes. Multiple-point residue formulas for holomorphic maps. arXiv:2112.15502.

[13] G. Bérczi and A. Szenes. Thom polynomials of Morin singularities. *Annals of Mathematics*, 175(2):567–629, 2012.

[14] N. Berline, E. Getzler, and M. Vergne. *Heat Kernels and Dirac Operators*. Grundlehren Text Editions. Springer Berlin Heidelberg, 2004.

[15] A. Borel, G. Bredon, E. E. Floyd, D. Montgomery, and R. Palais. *Seminar on Transformation Groups. (AM-46)*. Princeton University Press, 1960.

[16] J. Briancon. Description de $Hilb^n C\{x, y\}$. *Inventiones mathematicae*, 41:45–90, 1977.

[17] W. Buczyńska and J. Buczyński. Apolarity, border rank and multigraded Hilbert scheme, 2020.

[18] Y. Cao and M. Kool. Zero-dimensional Donaldson–Thomas invariants of Calabi–Yau 4-folds. *Advances in Mathematics*, 338:601–648, 2018.

[19] D. Cartwright, D. Erman, M. Velasco, and B. Viray. Hilbert schemes of 8 points. *Algebra & Number Theory*, 3(7):763 – 795, 2009.

[20] G. Ellingsrud, L. Göttsche, and M. Lehn. On the Cobordism Class of the Hilbert Scheme of a Surface. *J. Algebraic Geom*, 10, 05 1999.

[21] G. Ellingsrud and S.A. Stromme. On the homology of the Hilbert scheme of points in the plane. *Inventiones mathematicae*, 87:343–352, 1987.

[22] G. Ellingsrud and S.A. Stromme. An Intersection Number for the Punctual Hilbert Scheme of a Surface. *Transactions of the American Mathematical Society*, 350(6):2547–2552, 1998.

[23] L. M. Fehér and R. Rimányi. Thom series of contact singularities. *Annals of Mathematics*, 176(3):1381–1426, 2012.

[24] J. Fogarty. Algebraic Families on an Algebraic Surface. *American Journal of Mathematics*, 90(2):511–521, 1968.

[25] W. Fulton and D. Anderson. Equivariant Cohomology in Algebraic Geometry, 2021.

[26] T. Gaffney. The Thom polynomial of $\overline{\Sigma^{1111}}$. *Proc. Symp. Pure Math*, 40:399–408, 1983.

[27] M. Gerstenhaber. On the Deformation of Rings and Algebras. *Annals of Mathematics*, 79(1):59–103, 1964.

[28] L. Göttsche and M. Kool. Virtual Segre and Verlinde numbers of projective surfaces. *Journal of the London Mathematical Society*, 106(3):2562–2608, 2022.

[29] A. Grothendieck. Techniques de construction et théorèmes d'existence en géométrie algébrique IV : les schémas de Hilbert. Number 6 in Séminaire Bourbaki, pages 205–222. Société mathématique de France, 1961. talk:221.

[30] L. Göttsche. The Betti numbers of the Hilbert scheme of ponts on a smooth projective surface. *Mathematische Annalen*, 286(1-3):193–208, 1990.

[31] L. Göttsche. A Conjectural Generating Function for Numbers of Curves on Surfaces. *Communications in Mathematical Physics*, 196(3):523–533, Sep 1997.

[32] L. Göttsche. Hilbert schemes of points on surfaces. 2003.

[33] R. Hartshorne. Connectedness of the Hilbert scheme. *Publications Mathématiques de l'IHÉS*, 29:5–48, 1966.

[34] H. Hironaka. Resolution of Singularities of an Algebraic Variety Over a Field of Characteristic Zero: I. *Annals of Mathematics*, 79(1):109–203, 1964.

[35] D. Husemöller. *Fibre Bundles*. Graduate Texts in Mathematics. Springer, 1994.

[36] A. Iarrobino. Reducibility of the families of 0-dimensional schemes on a variety. *Inventiones mathematicae*, 15:72–77, 1972.

[37] J. Jelisiejew. Open problems in deformations of Artinian algebras, Hilbert schemes and around. to appear.

[38] J. Jelisiejew. Pathologies on the Hilbert scheme of points. *Inventiones mathematicae*, 220:581–610, 2020.

[39] J. Jelisiejew, J. M. Landsberg, and A. Pal. Concise tensors of minimal border rank, 2023.

[40] D. Johnson. Universal Series for Hilbert Schemes and Strange Duality. *International Mathematics Research Notices*, 2020(10):3130–3152, 05 2018.

[41] M. Kazarian. Thom Polynomials for Lagrange, Legendre, and Critical Point Function Singularities. *Proceedings of the London Mathematical Society. Third Series*, 86, 05 2003.

[42] M. Kazarian. Non-associative Hilbert scheme and Thom polynomials, 2017.

[43] G. R. Kempf, F. F. Knudsen, D. Mumford, and B. Saint-Donat. *Toroidal Embeddings I*. Springer, 1973.

[44] M. Lehn. Chern classes of tautological sheaves on Hilbert schemes of points on surfaces. *Inventiones Mathematicae*, 136(1):157–207, Mar 1999.

[45] J. Li. Zero dimensional Donaldson–Thomas invariants of threefolds. *Geometry & Topology*, 10(4):2117 – 2171, 2006.

[46] A. Marian, D. Oprea, and R. Pandharipande. Segre classes and Hilbert schemes of points. *Annales scientifiques de l'École normale supérieure*, 50, 07 2015.

[47] A. Marian, D. Oprea, and R. Pandharipande. Higher rank Segre integrals over the Hilbert scheme of points. *Journal of the European Mathematical Society*, 2017.

[48] A. Marian, D. Oprea, and R. Pandharipande. The combinatorics of Lehn's conjecture. *Journal of the Mathematical Society of Japan*, 71(1):299 – 308, 2019.

[49] E. Miller and B. Sturmfels. *Combinatorial Commutative Algebra*. Graduate Texts in Mathematics. Springer New York, 2005.

[50] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric Invariant Theory*. Ergebnisse der Mathematik und Ihrer Grenzgebiete, 3 Folge/A Series of Modern Surveys in Mathematics Series. Springer Berlin Heidelberg, 1994.

[51] H. Nakajima. Heisenberg Algebra and Hilbert Schemes of Points on Projective Surfaces. *Annals of Mathematics*, 145(2):379–388, 1997.

[52] H. Nakajima. *Lectures on Hilbert Schemes of Points on Surfaces*. University lecture series. American Mathematical Society, 1999.

[53] American Institute of Mathematics Problem List. Components of Hilbert Schemes, 2010.

[54] J.V. Rennemo. Universal polynomials for tautological integrals on Hilbert schemes. *Geometry & Topology*, 21(1):253–314, Feb 2017.

[55] R. Rimányi. Thom polynomials, symmetries and incidences of singularities. *Inventiones mathematicae*, 143:499–521, 03 2001.

[56] R. Rimányi. Multiple-point formulas - A new point of view. *PACIFIC JOURNAL OF MATHEMATICS*, 202, 02 2002.

[57] M. Satriano and A. P. Staal. Small elementary components of Hilbert schemes of points. *Forum of Mathematics, Sigma*, 11:e45, 2023.

[58] A. Szenes. Iterated residues and multiple Bernoulli polynomials. *International Mathematics Research Notices*, 1998(18):937–956, 01 1998.

[59] R. Thom. Les singularités des applications différentiables. In *Séminaire Bourbaki : années 1954/55 - 1955/56, exposés 101-136*, number 3 in Séminaire Bourbaki. Société mathématique de France, 1956. talk:134.

[60] R. Vakil. Murphy's law in algebraic geometry: Badly-behaved deformation spaces. *Inventiones mathematicae*, 164:569–590, 06 2006.

[61] C. Voisin. Segre classes of tautological bundles on Hilbert schemes of surfaces. *Algebraic Geometry*, March 2019.