Homological Algebra of Proper Abelian Subcategories

Anders Sten Kortegård

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Supervised by Peter Jørgensen

Department of Mathematics
Aarhus University

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Abstract

This thesis consists of four main parts. In Paper A we work with a Frobenius category $\mathscr{E}$, whose stable category $\mathscr{C}$ is 2 Calabi–Yau, Hom-finite, and idempotent complete. Motivating examples of these stable categories are cluster categories. We show that, given two maximal rigid objects $x, y \in \mathscr{C}$ with self-injective endomorphism algebras, the endomorphism algebras are derived equivalent. Furthermore, we provide a method to construct a 2-sided tilting complex which induces such an equivalence.

In the rest of the projects, we focus on proper abelian categories, which are abelian categories, nicely embedded into an ambient triangulated category. In Paper B we define intermediate categories with respect to a proper abelian category. This generalizes a similar definition by Enomoto and Saito. We show that, under mild assumption, these intermediate categories are in bijection with torsion-free classes in the corresponding proper abelian categories. This mirrors the relationship between intermediate t-structures and torsion-free classes in a certain heart.

In Paper C we discuss torsion triples, a way to filter objects in a proper abelian subcategory, which generalized that of torsion pairs. We describe a method to construct torsion triples from two appropriately 'close' proper abelian categories. This generalizes work by Jensen, Madsen and Su, who has done similar work for the heart of a standard t-structure.

In Paper B we came across a snake lemma for proper abelian subcategories, closely resembling homology of two-term complexes. In Project D, we generalize this snake lemma into a theory of homology for proper abelian subcategoires. We show that for a proper abelian subcategory in a triangulated category $\mathscr{T}$, we can define functors, on a subcategory of $\mathscr{T}$, that looks like homology functors in the sense that we get long exact sequences "of homology".

We end the thesis on a computational note, describing some code that implements parts of a combinatorial model for the negative cluster categories of type A_n.

Resumé

Denne afhandling er opdelt i fire hoveddele. I Paper A arbejder vi med en Frobenius-kategori $\mathscr{E}$, hvis stabile kategori $\mathscr{C}$ er 2 Calabi–Yau, Hom-endelige og idempotent fuldstændig. Motiverende eksempler af sådanne stabile kategorier er klyngekategorier. Vi viser, at givet to maksimalt rigide objekter $x, y \in \mathscr{C}$, hvis endomorfialgebraer er selvinjektive, så er de deriveret ækvivalente. Vi fremlægger en metode til at konstruere et 2-sidet tilting kompleks, som inducerer en sådanne ækvivalens.

I resten af projekterne fokuserer vi på proper abelske delkategorier, som er abelske kategorier, der er pænt indlejret i en omgivende trianguleret kategori. I Paper B definerer vi, hvad det vil sige at være en intermediær kategori i forhold til en proper abelsk delkategori. Dette generaliserer en lignende definition af Enomoto og Saito. Vi viser, at under milde antagelser er disse intermediære kategorier i bijektion med torsionsfrie klasser i den tilsvarende proper abelske delkategori. Dette spejler forholdet mellem intermediære t-strukturer og torsionsfrie klasser i et bestemt hjerte.

I Paper C diskuterer vi torsionstripler, som er en måde at filtrere objekter i en proper abelsk delkategori, der generaliserer torsionspar. Vi giver en metode til at konstruere torsionstriple fra to passende 'tætte' proper abelske delkategorier. Dette generaliserer arbejde af Jensen, Madsen og Su, som har lavet lignende arbejde for hjertet af en standard t-struktur.

I Paper B stødte vi på et slangelemma for proper abelske delkategorier, som ligner homologi for to-leds komplekser. I Project D generaliserer vi dette slangelemma til en teori om homologi. Vi viser at for en proper abelsk delkategori i en trianguleret kategori $\mathscr{T}$, kan vi definere funktorer, på en delkategori af $\mathscr{T}$, der ligner homologifunktorene, i den forstand at vi får lange eksakte sekvenser "af homologi".

Vi afslutter afhandlingen på en beregningsmæssig note, hvor vi beskriver noget kode, der implementerer dele af en kombinatorisk model for de negative klyngekategorier af type A_n.

Contents

Preface

This thesis concludes my studies at Aarhus university. I have spent eight years Aarhus University pursuing my bachelor's, master's, and now my PhD.

The main part of this thesis consists of self-contained 3 papers, an ongoing project, and an ongoing coding project.

- Paper A: Derived Equivalences of Self-injective 2-Calabi–Yau Tilted Algebras.
- Paper B: Intermediate Categories for Proper Abelian Subcategories.
- Paper C: Filtrations of Torsion Classes in Proper Abelian Subcategories.
- Project D: Homology for Proper Abelian Subcategories.
- Project E: Code – Negative Cluster Categories.

A version of Paper A along with parts of Paper B was included in my progress report for the qualifying examination. Paper A has been published in the Bulletin of the London Mathematical Society. Paper B and Paper C are preprints available on arXiv. Some of the papers are slightly modified, including extra details, and minor changes to tailor them to the format of this thesis. Paper A contains two extra proofs than the published versions, however, these proofs that are part of the arXiv version. Paper A also includes an extra section demonstrating an application of the main result.

In the first part of my PhD I was mainly focussed on working with tilting in Frobenius categories and their stable categories, which is reflected by Paper A. For the remaining part of my PhD, my research shifted toward the topic of proper abelian categories, a relatively new and unexplored area. Paper B, Paper C, Project D and Project E, all focus on this topic.

I would like to thank my supervisor Peter Jørgensen for all his guidance and support throughout my PhD, as well as for plenty of interesting discussions and meetings. Also a big thanks to the whole of Aarhus homological algebra group and related colleagues, including Amit, Carlo, Charley, Cyril, David, Davide, Esther, Jenny, Karin, and Raphael. I feel lucky to have been a part of this group, and it has made my time as a PhD student very enjoyable. A special thanks to Carlo, with whom I have shared an office and spent a lot of time with over the last couple of years, and who have always been helpful and supportive. Also a special thanks to Charley and David whose support I have been grateful for.

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Introduction

We will start by briefly introducing the field and present the essential definitions and results. This means there will not be provided proof of the results stated. For an in-depth introduction to the different topics or proof of the results, we refer you to the relevant material, which is referenced throughout the chapter.

For the rest of this chapter, let k be a field and $\mathscr{A}$ an abelian category.

1 The derived category

The derived category of an abelian category is a category that encodes some of the homological properties of $\mathscr{A}$, such as Ext and Tor. To describe the derived category, we first need to talk about complexes.

Definition 1.1 ([Wei94, def. 1.1.1])**.** A *chain complex* in $\mathscr{A}$ is a pair

$$X = ((X_i)_{i\in\mathbb{Z}}, (d_i)_{i\in\mathbb{Z}}),$$

where $(X_i)_{i\in\mathbb{Z}}$ is a collection of objects $X_i \in \mathscr{A}$, and $(d_i)_{i\in\mathbb{Z}}$ a collection of morphisms $d_i : X_i \to X_{i-1}$, such that $d_{i-1}d_i = 0$ for all $i \in \mathbb{Z}$.

$$\cdots \longrightarrow X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} X_{-1} \xrightarrow{d_{-1}} X_{-2} \longrightarrow \cdots .$$

The maps d_i are referred to as *differential maps*.

Given two chain complexes $X = ((X_i)_{i\in\mathbb{Z}}, (d_i^X)_{i\in\mathbb{Z}})$ and $Y = ((Y_i)_{i\in\mathbb{Z}}, (d_i^Y)_{i\in\mathbb{Z}})$, a *morphism of chain complexes* $f : X \to Y$ is a collection $f = (f_i)_{i\in\mathbb{Z}}$ of morphisms $f_i : X_i \to Y_i$, such that $d_i^Y f_i = f_{i-1}d_i^X$ for all $i \in \mathbb{Z}$, i.e. making the following diagram commute.

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & X_2 & \xrightarrow{d_2^X} & X_1 & \xrightarrow{d_1^X} & X_0 & \xrightarrow{d_0^X} & X_{-1} & \xrightarrow{d_{-1}^X} & X_{-2} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_{-1}} & & \downarrow{\scriptstyle f_{-2}} & & \\ \cdots & \longrightarrow & Y_2 & \xrightarrow{d_2^Y} & Y_1 & \xrightarrow{d_1^Y} & Y_0 & \xrightarrow{d_0^Y} & Y_{-1} & \xrightarrow{d_{-1}^Y} & Y_{-2} & \longrightarrow & \cdots . \end{array}$$

Denote the category of chain complexes $\mathsf{C}(\mathscr{A})$.

Remark 1.2. A complex can also be thought of as a pair (X, d), where X is a $\mathbb{Z}$-graded object, and $d : X \to X$ is a morphims of degree 1, such that $d^2 = 0$.

Dual to the notion of chain complexes is that of cochain complexes. A *cochain complex* is a pair $X = ((X^i)_{i\in\mathbb{Z}}, (d^i)_{i\in\mathbb{Z}})$ where $(X^i)_{i\in\mathbb{Z}}$ is a collection of objects $X^i \in \mathscr{A}$, and $(d^i)_{i\in\mathbb{Z}}$ a collection of morphisms $d^i : X^i \to X^{i+1}$, such that $d^{i+1}d^i = 0$ for all $i \in \mathbb{Z}$.

$$\cdots \longrightarrow X^{-2} \xrightarrow{d^{-2}} X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \longrightarrow \cdots .$$

Given an object $X \in \mathsf{C}(\mathscr{A})$, we can represent X both as a chain complex and as a cochain complex. The only difference is that the indexing is done in opposite directions, i.e., a difference of notation. Thus, everything that can be done for chain complexes can also be done for cochain complexes. By default, we will work with chain complexes.

Notation 1.3. Let $X \in \mathsf{C}(\mathscr{A})$. Unless otherwise stated, we will assume that the corresponding collection of objects is denoted $(X_i)_{i\in\mathbb{Z}}$, and the differential maps are denoted $(d_i^X)_{i\in\mathbb{Z}}$. We may denote $d_i = d_i^X$ if the corresponding complex easily can be identified by context.

Given a chain complex $X \in \mathsf{C}(\mathscr{A})$, a natural operation we can do on X is to shift all the objects a number of times. That is, given an integer $n \in \mathbb{Z}$ define the complex $X[n]$ by $(X[n])_i = X_{i-n}$. Visually, we can see this as shifting the complex n degrees to the left, as follows:

$$X: \quad \cdots \longrightarrow X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} X_{-1} \longrightarrow \cdots ,$$

$$X[n]: \quad \cdots \longrightarrow X_{2-n} \xrightarrow{d_{2-n}} X_{1-n} \xrightarrow{d_{1-n}} X_{-n} \xrightarrow{d_{-n}} X_{-1-n} \longrightarrow \cdots ,$$

where the degrees are aligned vertically.

Definition 1.4. Given two complexes $X, Y \in \mathsf{C}(\mathscr{A})$, and a morphism $f : X \to Y$. Define the complex $\operatorname{cone}(f) = ((C_i)_{i\in\mathbb{Z}}, (d_i^{\operatorname{cone}(f)})_{i\in\mathbb{Z}})$, where $C_i = X_{i-1} \oplus Y_i$ and

$$d_i^{\operatorname{cone}(f)} = \begin{pmatrix} -d_{i-1}^X & 0 \\ -f_{i-1} & d_i^Y \end{pmatrix},$$

for $i \in \mathbb{Z}$.

Example 1.5. Let $\mathscr{A}$ be an abelian category, and consider $X, Y \in \mathscr{A}$ and $f \in \operatorname{Hom}(X, Y)$. If we consider X, Y as complexes concentrated in degree 0 then f becomes a morphism of chain complexes.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & Y & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Denote the cone by $C := \operatorname{cone}(f)$. Then $C_0 = 0 \oplus Y \cong Y$ and $C_1 = X \oplus 0 \cong X$. For $i \neq 0, 1$ we have that $C_i = 0$. Calculating the differential d_1^C, we get that

$$d_1^C = \begin{pmatrix} 0 & 0 \\ -f & 0 \end{pmatrix}.$$

Hence we get that $\operatorname{cone}(f)$ is the complex

$$\cdots \longrightarrow 0 \longrightarrow X \xrightarrow{-f} Y \longrightarrow 0 \longrightarrow \cdots .$$

Definition 1.6.

- A complex $X \in \mathsf{C}(\mathscr{A})$ is called *left bounded* (resp. *right bounded*) if there exists $N \in \mathbb{Z}$ such that $X_i = 0$ for $i > N$ (resp. $i < N$). Denote the category of left bounded (resp. right bounded) chain complexes $\mathsf{C}^{\mathrm{lb}}(\mathscr{A})$ (resp. $\mathsf{C}^{\mathrm{rb}}(\mathscr{A})$)

- A complex $X \in \mathsf{C}(\mathscr{A})$ is called *bounded* if it is both left- and right bounded. Denote the category of bounded chain complexes $\mathsf{C}^{b}(\mathscr{A})$.

Lemma 1.7 (cf. [Wei94, thm. 1.2.3])**.** *The categories $\mathsf{C}(\mathscr{A}), \mathsf{C}^{\mathrm{lb}}(\mathscr{A}), \mathsf{C}^{\mathrm{rb}}(\mathscr{A})$ and $\mathsf{C}^{b}(\mathscr{A})$ are abelian categories, in which kernels, cokernels, and images are computed degreewise.*

1.1 Double complexes

Definition 1.8. A *double complex* X over $\mathscr{A}$ consists of a collection $\{X_{i,j}\}_{i,j\in\mathbb{Z}}$ of objects $X_{i,j} \in \mathscr{A}$, together with horizontal differentials $d^h_{i,j} : X_{i,j} \to X_{i-1,j}$ for each $i, j \in \mathbb{Z}$ and vertical differentials $d^v_{i,j} : X_{i,j} \to X_{i,j-1}$ for each $i, j \in \mathbb{Z}$, such that $d^h d^h = 0$ and $d^v d^v = 0$, and $d^v d^h + d^h d^v = 0$.

Essentially, this means that we have an anti-commutative diagram.

$$
\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & X_{2,1} & \xrightarrow{d^h_{2,1}} & X_{1,1} & \xrightarrow{d^h_{1,1}} & X_{0,1} & \xrightarrow{d^h_{0,1}} & X_{-1,1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle d^v_{2,1}} & & \downarrow{\scriptstyle d^v_{1,1}} & & \downarrow{\scriptstyle d^v_{0,1}} & & \downarrow{\scriptstyle d^v_{-1,1}} & & \\
\cdots & \longrightarrow & X_{2,0} & \xrightarrow{d^h_{2,0}} & X_{1,0} & \xrightarrow{d^h_{1,0}} & X_{0,0} & \xrightarrow{d^h_{0,0}} & X_{-1,0} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle d^v_{2,0}} & & \downarrow{\scriptstyle d^v_{1,0}} & & \downarrow{\scriptstyle d^v_{0,0}} & & \downarrow{\scriptstyle d^v_{-1,0}} & & \\
\cdots & \longrightarrow & X_{2,-1} & \xrightarrow{d^h_{2,-1}} & X_{1,-1} & \xrightarrow{d^h_{1,-1}} & X_{0,-1} & \xrightarrow{d^h_{0,-1}} & X_{-1,-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle d^v_{2,-1}} & & \downarrow{\scriptstyle d^v_{1,-1}} & & \downarrow{\scriptstyle d^v_{0,-1}} & & \downarrow{\scriptstyle d^v_{-1,-1}} & & \\
\cdots & \longrightarrow & X_{2,-2} & \xrightarrow{d^h_{2,-2}} & X_{1,-2} & \xrightarrow{d^h_{1,-2}} & X_{0,-2} & \xrightarrow{d^h_{0,-2}} & X_{-1,-2} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

in which each row and column is a complex.

There are multiple ways to produce complexes from double complexes; we will use two of them.

Definition 1.9. Let $X = \{X_{i,j}\}_{i,j\in\mathbb{Z}}$ be a double complex with differential maps $\{d^h_{i,j}\}_{i,j\in\mathbb{Z}}$ and $\{d^v_{i,j}\}_{i,j\in\mathbb{Z}}$. Then define the *total complexes* $\mathrm{Tot}^{\oplus}$ and Tot^{Π} with objects

$$(\mathrm{Tot}^{\oplus})_n := \bigoplus_{i+j=n} X_{i,j} \qquad \text{and} \qquad (\mathrm{Tot}^{\Pi})_n := \prod_{i+j=n} X_{i,j},$$

for $n \in \mathbb{Z}$ and differential maps $d = d^h + d^v$.

Example 1.10 (Hom complex)**.** Given a chain complex $X \in \mathsf{C}(\mathscr{A})$ and a cochain complex $Y \in \mathsf{C}(\mathscr{A})$, we can define a double complex $\operatorname{Hom}(X,Y)$, with objects $\operatorname{Hom}(X,Y)_{i,j} = \operatorname{Hom}(X_i, Y^j)$, horizontal differentials

$$d^h_{i,j} : \operatorname{Hom}(X_i, Y^j) \to \operatorname{Hom}(X_{i+1}, Y_j), \quad \text{defined by} \quad d^h_{i,j} f = f d^X_{i+1},$$

and vertical differentials

$$d^v_{i,j} : \operatorname{Hom}(X_i, Y^j) \to \operatorname{Hom}(X_i, Y^{j+1}), \quad \text{defined by} \quad d^v_{i,j} f = (-1)^{i+j-1} d^j_Y f.$$

From this we get the total complex $\operatorname{Tot}^{\Pi} \operatorname{Hom}(X,Y) \in \mathsf{C}(\mathsf{Ab})$. From now on, we will denote by $\operatorname{Hom}(X,Y)$ the total complex $\operatorname{Tot}^{\Pi} \operatorname{Hom}(X,Y)$.

Example 1.11 (Tensor complex)**.** Let R be a ring. Then given two complexes $X \in \mathsf{C}(\operatorname{Mod} R)$ and $Y \in \mathsf{C}(\operatorname{Mod} R^{\mathrm{op}})$, we can define a double complex $X \otimes Y$, with objects $(X \otimes Y)_{i,j} = X_i \otimes Y_j$, horizontal differentials

$$d^h_{i,j} : X_i \otimes Y_i \to X_{i-1} \otimes Y_i, \quad \text{generated by} \quad d^h_{i,j}(x \otimes y) = d^X_i x \otimes y$$

and vertical differentials

$$d^v_{i,j} : X_i \otimes Y_i \to X_i \otimes Y_{i-1}, \quad \text{generated by} \quad d^v_{i,j}(x \otimes y) = (-1)^i x \otimes d^Y_i y.$$

From this we get the complex $\operatorname{Tot}^{\oplus}(X \otimes Y) \in \mathsf{C}(\mathsf{Ab})$. From now on we will denote by $X \otimes Y$ the total complex $\operatorname{Tot}^{\oplus}(X \otimes Y)$.

1.2 The homotopy category

Definition 1.12. Let $X, Y \in \mathsf{C}(\mathscr{A})$ and $f, g \in \operatorname{Hom}_{\mathsf{C}(\mathscr{A})}(X,Y)$. Then a *chain homotopy from f to g* is a collection of morphisms $\{s_i : X_i \to Y_{i+1}\}_{i \in \mathbb{Z}}$ such that $f_i - g_i = s_{i-1} d^X_i + d^Y_{i+1} s_i$ for all $i \in \mathbb{Z}$.

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & X_2 & \xrightarrow{d^X_2} & X_1 & \xrightarrow{d^X_1} & X_0 & \xrightarrow{d^X_0} & X_{-1} & \xrightarrow{d^X_{-1}} & X_{-2} & \longrightarrow \cdots \\
\swarrow s_2 & f_2 \downarrow\downarrow g_2 & \swarrow s_1 & f_1 \downarrow\downarrow g_1 & \swarrow s_0 & f_0 \downarrow\downarrow g_0 & \swarrow s_{-1} & f_{-1} \downarrow\downarrow g_{-1} & \swarrow s_{-2} & f_{-2} \downarrow\downarrow g_{-2} & \\
\cdots \longrightarrow & Y_2 & \xrightarrow[d^Y_2]{} & Y_1 & \xrightarrow[d^Y_1]{} & Y_0 & \xrightarrow[d^Y_0]{} & Y_{-1} & \xrightarrow[d^Y_{-1}]{} & Y_{-2} & \longrightarrow \cdots .
\end{array}$$

In this case we say that f and g are *homotopic*, and write $f \sim g$. If $f \sim 0$, then f is said to be *null-homotopic*.

Lemma 1.13. *For each $X, Y \in \mathsf{C}^*(\mathscr{A})$, with $* \in \{\emptyset, \mathrm{lb}, \mathrm{rb}, b\}$, the map $\sim$ defines an equivalence relation on $\operatorname{Hom}_{\mathsf{C}^*(\mathscr{A})}(X,Y)$.*

Definition 1.14. The *homotopy category* $\mathsf{K}^*(\mathscr{A})$ for $* \in \{\emptyset, \mathrm{lb}, \mathrm{rb}, b\}$ if defined by

- $\operatorname{ob}(\mathsf{K}^*(\mathscr{A})) = \operatorname{ob}(\mathsf{C}^*(\mathscr{A}))$,
- $\operatorname{Hom}_{\mathsf{K}^*(\mathscr{A})}(X,Y) = \operatorname{Hom}_{\mathsf{C}^*(\mathscr{A})}(X,Y)/\sim$.

1.3 The derived category

Definition 1.15. Let $X \in \mathsf{C}(\mathscr{A})$ and $i \in \mathbb{Z}$ then define the *i'th homology of X* as $H_i(X) = \operatorname{Ker} d_i / \operatorname{Im} d_{i+1}$.

Lemma 1.16. *Let $X, Y \in \mathsf{C}(\mathscr{A})$ and $f \in \operatorname{Hom}(X, Y)$, then for each $i \in \mathbb{Z}$ there is an induced morphism on the homology $H_i(f) : H_i(X) \to H_i(Y)$.*

Definition 1.17. Let $X, Y \in \mathsf{C}(\mathscr{A})$, then a morphism $f : X \to Y$ is called a *quasi-isomorphism* if $H_i(f)$ is an isomorphism for each $i \in \mathbb{Z}$. Denote the collection of quasi-isomorphisms by Qu.

Definition 1.18. Define the *derived category* $\mathsf{D}(\mathscr{A})$ *of* $\mathscr{A}$ as the localization $\mathsf{D}(\mathscr{A}) := \mathsf{K}(\mathscr{A})[\mathrm{Qu}^{-1}]$. That is, a category $\mathsf{D}(\mathscr{A})$ equipped with a *localization functor* $q : \mathsf{K}(\mathscr{A}) \to \mathsf{D}(\mathscr{A})$, such that

1. q sends quasi-isomorphisms $f \in \mathrm{Qu}$ to isomorphisms $q(f)$.

2. For each functor $F : \mathsf{K}(\mathscr{A}) \to C$ sending quasi-isomorphisms to isomorphisms, there exists a unique functor $\widehat{F} : \mathsf{D}(\mathscr{A}) \to C$ such that $F = \widehat{F}q$.

$$\begin{array}{ccc} \mathsf{K}(\mathscr{A}) & \xrightarrow{F} & C. \\ {\scriptstyle q}\downarrow & \nearrow_{\exists! \widehat{F}} & \\ \mathsf{D}(\mathscr{A}) & & \end{array}$$

Similarly we can define $\mathsf{D}^*(\mathscr{A})$ as $\mathsf{K}^*(\mathscr{A})[\mathrm{Qu}^{-1}]$ for $* \in \{\mathrm{lb}, \mathrm{rb}, b\}$.

Remark 1.19. It is important to note that there might be some set-theoretical issues arising in the definition given above. If the abelian category in question is small, we are guaranteed that the resulting derived category is a locally small. However, if the abelian category is not small, this becomes a bit more tricky. For the purposes of this thesis, we will not delve into these set-theoretical issues.

It is important to note that even though $\mathscr{A}$ and $\mathsf{C}(\mathscr{A})$ are abelian categories, it is not true that the homotopy category $\mathsf{K}(\mathscr{A})$ nor the derived category $\mathsf{D}(\mathscr{A})$ are abelian. The problem is that kernels and cokernels no longer exist. Instead, we get weak kernels and weak cokernels. Hence, we get a different structure that, to some extent, resembles an abelian structure. More precisely, they are *triangulated categories.*

2 Triangulated categories

Definition 2.1. Let $\mathscr{T}$ be a category, and $\Sigma : \mathscr{T} \to \mathscr{T}$ an automorphism, then a triple of morphisms (u, v, w)

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X, \tag{2.1}$$

is called a *triangle* on the triple (X, Y, Z) of objects. A *morphism between triangles*, is a triple of morphisms (f, g, h) such that the following diagram commute

$$\begin{CD} X @>u>> Y @>v>> Z @>w>> \Sigma X \\ @VfVV @VgVV @VhVV @V\Sigma fVV \\ X' @>u'>> Y' @>v'>> Z' @>w'>> \Sigma X', \end{CD}$$

where each row is a triangle. The triple is called an *isomorphism* if f, g, h are isomorphisms.

Typically, we refer to a triangle by its corresponding diagram, such as (2.1).

Definition 2.2 ([Wei94, def. 10.2.1], [Nee14, prop 1.4.6])**.** Let $\mathscr{T}$ be an additive category, $\Sigma : \mathscr{T} \to \mathscr{T}$ an automorphism, and Δ a collection of triangles, called *exact triangles*. Then the triple $\mathscr{T} = (\mathscr{T}, \Sigma, \Delta)$ is a *triangulated category* if it satisfies the following axioms

TR1
- For each morphism $u : X \to Y$, there exists a triangle $(u, v, w) \in \Delta$.
- For each $X \in \mathscr{A}$, we have that $(\mathrm{id}_X, 0, 0) \in \Delta$.
- Given two isomorphic triangles η and η', then $\eta \in \Delta$ if and only if $\eta' \in \Delta$.

TR2 Given a triangle $(u, v, w) \in \Delta$ then $(v, w, -\Sigma u) \in \Delta$ and $(-\Sigma^{-1} w, u, v) \in \Delta$. These two triangles are called *rotations* of the original triangle.

TR3 Given exact triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$, together with morphism $f : X \to X'$ and $g : Y \to Y'$ such that $u'f = gu$, then there exists a morphism $h : Z \to Z'$ making (f, g, h) a morphism of triangles.

$$\begin{CD} X @>u>> Y @>v>> Z @>w>> \Sigma X \\ @VfVV @VgVV @V\exists hVV @V\Sigma fVV \\ X' @>u'>> Y' @>v'>> Z' @>w'>> \Sigma X' \end{CD}$$

TR4 Given exact triangles

$$X \xrightarrow{u} Y \longrightarrow Z \longrightarrow \Sigma X$$

$$Y \xrightarrow{\alpha} Y' \longrightarrow Y'' \longrightarrow \Sigma Y \ ,$$

$$X \xrightarrow{\alpha u} Y' \longrightarrow Z' \longrightarrow \Sigma X \ ,$$

then we can complete these into a commutative diagram

$$\begin{CD} X @>u>> Y @>>> Z @>>> \Sigma X \\ @| @V\alpha VV @VVV @VVV \\ X @>\alpha u>> Y' @>>> Z' @>>> \Sigma X \\ @VVV @VVV @VVV @VVV \\ 0 @>>> Y'' @= Z'' @>>> 0 \\ @VVV @VVV @VVV @VVV \\ \Sigma X @>\Sigma u>> \Sigma Y @>>> \Sigma Z @>>> \Sigma^w X, \end{CD}$$

where each row and column is an exact triangle.

Σ is called the *suspension functor*.

Notation 2.3. From now on, when we say triangle, we mean exact triangles.

Proposition 2.4 ([Wei94, prop. 10.2.4, cor. 10.2.5, cor. 10.4.3])**.** *For* $* \in \{\emptyset, \mathrm{lb}, \mathrm{rb}, b\}$*. the categories* $\mathsf{K}^*(\mathscr{A})$ *and* $\mathsf{D}^*(\mathscr{A})$ *are triangulated categories with suspension functor* [1]*.*

Proposition 2.5 ([Wei94, ex. 10.4.9, 1.5.2])**.** *Given two objects* $X, Y \in \mathsf{D}(\mathscr{A})$*, and a morphism* $f : X \to Y$*, then* f *fits into a triangle*

$$X \xrightarrow{f} Y \longrightarrow \mathrm{cone}(f) \longrightarrow \Sigma X.$$

Remark 2.6. Given a triangulated category $\mathscr{T}$, objects $X, Y \in \mathscr{T}$ and a morphism $f : X \to Y$, there exists a triangle $X \xrightarrow{f} Y \to Z \to \Sigma X$. Motivated by Proposition 2.5 Z is often denoted by $\mathrm{cone}(f)$.

Definition 2.7. Let $\mathscr{T}, \mathscr{T}'$ be triangulated categories. A pair $F = (F, \eta)$ of a functor $F : \mathscr{T} \to \mathscr{T}'$, and a natural isomorphism $\eta : F\Sigma \to \Sigma F$, is called a *triangulated functor* if given a triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

in $\mathscr{T}$ then

$$FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \xrightarrow{\eta_X Fh} \Sigma FX.$$

is a triangle in $\mathscr{T}'$. If F is an equivalence, we say that $\mathscr{T}$ and $\mathscr{T}'$ are *equivalent as triangulated categories.*

Definition 2.8. Let $\mathscr{B}$ be an abelian category, and $F : \mathsf{D}(\mathscr{A}) \to \mathsf{D}(\mathscr{B})$ a triangulated equivalence. Then we call F a *derived equivalence* and say that $\mathscr{A}$ and $\mathscr{B}$ are *derived equivalent.*

Given two noetherian ring R, S, then we say they are *derived equivalent* if their module categories $\mathrm{mod}(R), \mathrm{mod}(S)$ are *derived equivalent.*

2.1 Derived functors

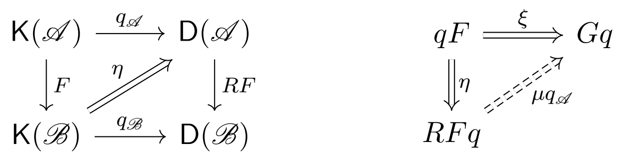
Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories. If we are given a functor $F : \mathscr{A} \to \mathscr{B}$, it would be very convenient if we could construct a functor $F' : \mathsf{D}(\mathscr{A}) \to \mathsf{D}(\mathscr{B})$ on their derived categories. Since $\mathscr{B}$ embeds nicely into its own derived category, we get a functor $\mathscr{A} \to \mathsf{D}(\mathscr{B})$. The naive way to get the functor F' defined on $\mathsf{D}(\mathscr{A})$ would be to apply F component-wise. However, this will not necessarily provide us with a well-defined triangulated functor. To fix this, we will introduce *derived functors*. For a more in-depth introduction see [Kra21, sec. 4.3] and [Wei94, sec. 10.5].

Proposition 2.9 ([Wei94, sec. 10.5])**.** *Let* $\mathscr{A}, \mathscr{B}$ *be abelian categories, and let* $F : \mathscr{A} \to \mathscr{B}$ *be a functor. Then there are induced functors* $F' : \mathsf{C}(\mathscr{A}) \to \mathsf{C}(\mathscr{B})$ *and* $F'' : \mathsf{K}(\mathscr{A}) \to \mathsf{K}(\mathscr{B})$*.*

Definition 2.10 ([Wei94, def. 10.5.1])**.** Let $\mathscr{A}, \mathscr{B}$ be abelian categories, with $q_{\mathscr{A}} : \mathsf{K}(\mathscr{A}) \to \mathsf{D}(\mathscr{A})$ and $q_{\mathscr{B}} : \mathsf{K}(\mathscr{B}) \to \mathsf{D}(\mathscr{B})$ the corresponding localization functors. Given a functor $F : \mathsf{K}(\mathscr{A}) \to \mathsf{K}(\mathscr{B})$, a *right derived functor of* F is a functor $RF : \mathsf{D}(\mathscr{A}) \to \mathsf{D}(\mathscr{B})$

together with a natural transformation $\eta : q_{\mathscr{B}}F \Rightarrow RFq_{\mathscr{A}}$ satisfying the following universal property.

For each pair (G, ξ) where $G : \mathsf{D}(\mathscr{A}) \to \mathsf{D}(\mathscr{B})$ is a functor, and $\xi : q_{\mathscr{B}}F \Rightarrow Gq_{\mathscr{A}}$ a natural transformation, there exists a unique natural transformation $\mu : RF \Rightarrow G$ such that $(\mu q_{\mathscr{A}})\eta = \xi$.



Left derived functors are defined dually.

Since the right- and left-derived functors are defined by a universal property, we are guaranteed that if they exist, then they are unique. However, we are not guaranteed that they always exist.

Theorem 2.11 ([CFH, constr. 7.2.7])**.** *Given rings S and R, and a functor*

$$F : \mathsf{K}(\mathrm{Mod}(R)) \to \mathsf{K}(\mathrm{Mod}(S)),$$

then both the right derived functor RF and left derived functor LF exists.

Example 2.12 (Hom functor)**.** Let R be a ring. Recall that given complexes $X, Y \in \mathsf{C}(\mathrm{Mod}(R))$, we can construct a complex $\mathrm{Hom}(X, Y)$, see Example 1.10. This construction induces a functor

$$\mathrm{Hom}(X, -) : \mathsf{K}(\mathrm{Mod}(R)) \to \mathsf{K}(\mathsf{Ab}).$$

Theorem 2.11 now says that there exists a *right derived hom functor*

$$R\,\mathrm{Hom}(X, -) : \mathsf{D}(\mathrm{Mod}(R)) \to \mathsf{D}(\mathsf{Ab}).$$

This functor can be extended to a bifunctor

$$R\,\mathrm{Hom} : \mathsf{D}(\mathrm{Mod}(R))^{\mathrm{op}} \times \mathsf{D}(\mathrm{Mod}(R)) \to \mathsf{D}(\mathsf{Ab}),$$

cf. [Wei94, thm. 10.7.4].

Now let A, B, C be finite-dimension k-algebras. If $\mathrm{gl.dim}(A) < \infty$ then the functor above induces a functor

$$R\,\mathrm{Hom} : \mathsf{D}^b(\mathrm{mod}(B^{\mathrm{op}} \otimes A))^{\mathrm{op}} \times \mathsf{D}^b(\mathrm{mod}(C^{\mathrm{op}} \otimes A)) \to \mathsf{D}^b(\mathrm{mod}(B \otimes C^{\mathrm{op}})).$$

Example 2.13 (Tensor product)**.** Let R be a ring, and let $X \in \mathsf{C}(\mathrm{Mod}(R))$ and $Y \in \mathsf{C}(\mathrm{Mod}(R^{\mathrm{op}}))$. In Example 1.11 we defined a complex $X \otimes Y$. This construction induces a functor

$$X \otimes - : \mathsf{K}(\mathrm{Mod}(R^{\mathrm{op}})) \to \mathsf{K}(\mathsf{Ab}),$$

Theorem 2.11 now says that a left derived functor

$$X \overset{\mathrm{L}}{\otimes} - : \mathsf{D}(\mathrm{Mod}(R^{\mathrm{op}})) \to \mathsf{D}(\mathsf{Ab})$$

exists. Furthermore, this functor can then be extended to a bifunctor

$$- \overset{\mathrm{L}}{\otimes} - : \mathsf{D}(\mathrm{Mod}(R)) \times \mathsf{D}(\mathrm{Mod}(R^{\mathrm{op}})) \to \mathsf{D}(\mathsf{Ab}),$$

cf. [Wei94, thm. 10.6.3].

Now let A, B, C be finite-dimensional k-algebras. If $\mathrm{gl.dim}(B) < \infty$ then the functor above induces a functor on the following form.

$$- \overset{\mathrm{L}}{\otimes} - : \mathsf{D}^b(\mathrm{mod}(A^{\mathrm{op}} \otimes B)) \times \mathsf{D}^b(\mathrm{mod}(B^{\mathrm{op}} \otimes C)) \to \mathsf{D}^b(\mathrm{mod}(A^{\mathrm{op}} \otimes C)).$$

Definition 2.14 ([Ric91, def. 3.4])**.** Let R, S be rings. A triangulated functor $F : \mathsf{D}(\mathrm{Mod}(R)) \to \mathsf{D}(\mathrm{Mod}(S))$ is called a *standard derived equivalence* if it is isomorphic to $R\,\mathrm{Hom}(T, -)$ for some $T \in \mathsf{D}(\mathrm{Mod}(R))$.

2.2 Serre functors

Some triangulated categories have an especially interesting functor, called a *Serre functor*.

Definition 2.15. Let $\mathscr{T}$ be a k-linear triangulated category. A triangulated functor $\mathbb{S} : \mathscr{T} \to \mathscr{T}$ is called a *Serre functor* if it is an autoequivalence, and for all $X, Y \in \mathscr{T}$

$$\mathrm{Hom}(X, Y) \cong D\,\mathrm{Hom}(Y, \mathbb{S}X),$$

where D refers to the k-dual $D = \mathrm{Hom}(-, k)$.

Definition 2.16. Let $\mathscr{T}$ be a k-linear triangulated category, and $d \in \mathbb{Z}$. Then $\mathscr{T}$ is said to be d-Calabi–Yau (or d-CY), if $(\Sigma, -\mathrm{id})^d$ is a Serre functor.

3 Frobenius and stable categories

Homotopy categories and derived categories are good sources of triangulated categories. However, this section will introduce another way to find triangulated categories through Frobenius's exact categories.

Definition 3.1. An exact category $\mathscr{E}$ is called a *Frobenius category* if

1. $\mathscr{E}$ has enough projective objects.
2. $\mathscr{E}$ has enough injective objects.
3. The projective and injective objects coincide, i.e. $\mathrm{proj}(\mathscr{E}) = \mathrm{inj}(\mathscr{E})$.

Example 3.2. Let A be a finite-dimensional self-injective k-algebra, then the module category $\mathrm{mod}(A)$ is a Frobenius category. For a specific example, let $n \in \mathbb{N}$ and choose the self-injective algebra $A = k[x]/x^n$. Then, $\mathrm{mod}(A)$ is a Frobenius category.

Definition 3.3. Let $\mathscr{E}$ be a Frobenius category, and define its corresponding *stable category* $\underline{\mathscr{E}}$ as follows:

- $\mathrm{ob}(\underline{\mathscr{E}}) := \mathrm{ob}(\mathscr{E})$,

- $\mathrm{Hom}_{\underline{\mathscr{E}}}(x,y) \coloneqq \mathrm{Hom}_{\mathscr{E}}(x,y)/\mathscr{I}(x,y)$, where $\mathscr{I}(x,y)$ is the ideal of morphisms that factor through a projective object.

Proposition 3.4 ([Hap98, thm. I.2.6]). *Given a Frobenius category $\mathscr{E}$, then the corresponding stable category $\underline{\mathscr{E}}$ is a triangulated category.*

4 The cluster category

The cluster category is a triangulated category defined as the orbit category of the derived category. Thus, to define the cluster category, we must start by defining orbit categories.

Throughout this section let $\mathscr{T}$ be a triangulated category.

Definition 4.1 ([Kel05, sec. 1]). Let $F : \mathscr{T} \to \mathscr{T}$ be an automorphism, define the *orbit category* $\mathscr{T}/F$ by

- $\mathrm{ob}(\mathscr{T}/F) \coloneqq \mathrm{ob}(\mathscr{T})$
- $\mathrm{Hom}_{\mathscr{T}/F}(X,Y) \coloneqq \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{T}}(X,F^nY)$.

Notice that the orbit category comes together with a canonical projection functor $\pi : \mathscr{T} \to \mathscr{T}/F$. As stated by Keller, the orbit category does not need to be triangulated in such a way that π becomes a triangulated functor. However, there are still some cases in which this happens.

Theorem 4.2 ([Kel05, thm. 1]). *Let $\mathscr{T} = \mathsf{D}^b(\mathrm{mod}\,A)$ for some finite-dimensional algebra A, and let F a standard equivalence, and assume the following*

1. *There exists a hereditary abelian category $\mathcal{H}$, such that $\mathsf{D}^b(\mathrm{mod}\,A) \cong \mathsf{D}^b(\mathcal{H})$ as triangulated categories. From now on, we will identify $\mathscr{T}$ as $\mathsf{D}^b(\mathcal{H})$.*
2. *For each indecomposable $X \in \mathcal{H}$, we have that $F^i(X) \in \mathcal{H}$ for only finitely many $i \in \mathbb{Z}$.*
3. *There exists $N \in \mathbb{N}$ such that for each $X \in \mathscr{T}$ there exists an $i \in \mathbb{Z}$, and $n \in \mathbb{N}_0$ with $0 \le n \le N$ such that $F^iX \in \Sigma^n\mathcal{H}$. In other words, each F-orbit has an object contained in $\Sigma^n\mathcal{H}$ for some integer n with $0 \le n \le N$.*

Then $\mathscr{T}/F$ has a triangulated structure, which makes the corresponding projection functor $\pi : \mathscr{T} \to \mathscr{T}/F$ a triangulated functor.

This result says that the orbit category sometimes has a canonical triangulated structure. If this is the case, then Keller also showed that it arises as a stable category.

Theorem 4.3 ([Kel05, sec. 9.6]). *Let $\mathscr{T}$ be a triangulated category, and $F : \mathscr{T} \to \mathscr{T}$ a standard equivalence satisfying the conditions from Theorem 4.2, the orbit category $\mathscr{T}/F$ is the stable category of some Frobenius category.*

With the framework of orbit categories set up, it is now possible to define cluster categories. Buan, Marsh, Reineke, Reiten, and Todorov defined the cluster category as a categorization of cluster algebras.

Definition 4.4 ([BMRRT06])**.** Let H be a finite dimensional hereditary k-algebra, and consider the derived category $\mathsf{D}^b(\text{mod}\, H)$, with Auslander-Reiten translation τ, then define the *cluster category* is $\mathcal{C}(H) := \mathsf{D}^b(\text{mod}\, H)/\tau^{-1}\Sigma$.

Proposition 4.5. *Let H be a finite-dimensional hereditary k-algebra, then the cluster category $\mathcal{C}(H)$ is 2-Calabi–Yau.*

There also exist generalizations of this definition, which produces n-Calabi–Yau categories, for $n \in \mathbb{Z}\backslash\{0, 1\}$.

Definition 4.6. Let H be a finite-dimensional hereditary k-algebra.

- m-cluster categories [Tho07]: Let $m \in \mathbb{N}$ with $m \geq 2$, then define the *m-cluster category* $\mathcal{C}_m(H) := \mathsf{D}^b(\text{mod}\, H)/\tau^{-1}\Sigma^{m-1}$. This is an m-Calabi–Yau category.

- Negative cluster categories [CPP22]: Let $w \in \mathbb{N}$ with $w \geq 1$, then define the *negative cluster category* $\mathcal{C}_{-w}(H) := \mathsf{D}^b(\text{mod}\, H)/\Sigma^{w+1}\tau$. This is a $-w$-Calabi–Yau category.

5 Abelian categories inside triangulated categories

An abelian category $\mathscr{A}$ can be embedding into its derived category $\mathsf{D}^b(\mathscr{A})$ as stalk complexes, that is complexes that are concentrated in degree 0. Thus, from now on, we will associate $\mathscr{A}$ with its embedding in $\mathsf{D}^b(\mathscr{A})$. This embedding has a very nice property. Each short exact sequence $a \longrightarrow b \longrightarrow c$, induces a triangle $a \longrightarrow b \longrightarrow c \longrightarrow \Sigma a$. Conversely, given a triangle $a \longrightarrow b \longrightarrow c \longrightarrow \Sigma a$, in $\mathsf{D}^b(\mathscr{A})$ with objects $a, b, c \in \mathscr{A}$, then $a \longrightarrow b \longrightarrow c$ is a short exact sequence in $\mathscr{A}$. This is a property that will be important to us.

This embedding of an abelian category into its derived category is an example of the heart of a t-structure.

Proposition/Definition 5.1 ([BBD83, def. 1.3])**.** Let $(\mathcal{T}, \mathcal{F})$ be a pair subcategories in in $\mathscr{T}$, then it is called a *torsion pair* if it satisfies the following.

1. $\text{Hom}(\mathcal{T}, \mathcal{F}) = 0$.

2. $\mathscr{T} = \mathcal{T} * \mathcal{F}$.

Furthermore, a torsion pair is called a *t-structure*, if it also satisfies

3. $\Sigma\mathcal{T} \subseteq \mathcal{T}$ and $\Sigma^{-1}\mathcal{F} \subseteq \mathcal{F}$.

Associated to a t-structure is an abelian category $\mathcal{H} = \mathcal{T} \cap \Sigma\mathcal{F}$ called its *heart*. Furthermore, $\mathcal{T}$ is called the *aisle*, and $\mathcal{F}$ the *coaisle* of the t-structure.

Remark 5.2. Notice that this definition of a t-structure differs slightly from the original definition. In the original definition, $\mathcal{T}$ and $\mathcal{F}$ overlap, with the intersection being the heart. Thus, if we have a t-structure $(\mathcal{T}, \mathcal{F})$ with respect to our definition, then $(\mathcal{T}, \Sigma\mathcal{F})$ would be a t-structure with respect to the original definition. Both these definitions are widely used in the literature; thus, one should be aware of this difference.

Example 5.3. Consider the triangulated category $\mathsf{D}^b(\mathscr{A})$. Given an integer $n \in \mathbb{Z}$ consider the following collections

$$\mathsf{D}_{\geq n} = \{X \in \mathsf{D}^b(\mathscr{A}) \mid H_i(X) = 0 \text{ for } i < n\},$$
$$\mathsf{D}_{<n} = \{X \in \mathsf{D}^b(\mathscr{A}) \mid H_i(X) = 0 \text{ for } i \geq n\}.$$

Then $(\mathsf{D}_{\geq n}, \mathsf{D}_{<n})$ is a t-structure. We call the t-structure $(\mathsf{D}_{\geq 0}, \mathsf{D}_{<0})$ the *standard t-structure.*

Let $(\mathcal{T}, \mathcal{F})$ be a t-structure in $\mathscr{T}$, then given an object $x \in \mathscr{T}$ there exists a unique triangle

$$\tau_{\mathcal{T}} x \longrightarrow x \longrightarrow \tau_{\mathcal{F}} x \longrightarrow \Sigma \tau_{\mathcal{T}} x, \tag{5.1}$$

with $\tau_{\mathcal{T}} x \in \mathcal{T}$ and $\tau_{\mathcal{F}} x \in \mathcal{F}$. This defines two functors.

$$\begin{aligned} \tau_{\mathcal{T}} : \mathscr{T} &\longrightarrow \mathcal{T} \\ x &\longmapsto \tau_{\mathcal{T}} x \end{aligned} \qquad\qquad \begin{aligned} \tau_{\mathcal{F}} : \mathscr{T} &\longrightarrow \mathcal{F} \\ x &\longmapsto \tau_{\mathcal{F}} x \end{aligned}$$

We call these functors *truncation functors*, and the corresponding triange (5.1) a *truncation triangle.* The reason for this name will be made clear in the following example.

Example 5.4. Given an integer $i \in \mathbb{Z}$ consider the t-structure $(D_{\geq i}, D_{<i})$, introduced in Example 5.3, in the derived category $\mathsf{D}^b(\mathscr{A})$. Denote $\tau_{\geq i} \coloneqq \tau_{D_{\geq i}}$ and $\tau_{<i} \coloneqq \tau_{D_{<i}}$. Let $X \in \mathsf{D}^b(\mathscr{A})$ be given by the following complex.

$$\cdots \longrightarrow X_{i+2} \xrightarrow{d_{i+2}} X_{i+1} \xrightarrow{d_{i+1}} X_i \xrightarrow{d_i} X_{i-1} \xrightarrow{d_{i-1}} X_{i-2} \longrightarrow \cdots.$$

Then we have the truncation triangle

$$\tau_{\geq i} X \longrightarrow X \longrightarrow \tau_{<i} X \longrightarrow \Sigma \tau_{\geq i} X.$$

We can also describe the objects of this triangle using complexes.

$$\begin{array}{cccccccccccc}
\tau_{\geq i} & : \cdots & \longrightarrow & X_{i+1} & \xrightarrow{d_{i+1}} & \operatorname{Coker} d_{i+1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
x & : \cdots & \longrightarrow & X_{i+1} & \xrightarrow{d_{i+1}} & X_i & \xrightarrow{d_i} & X_{i-1} & \xrightarrow{d_{i-1}} & X_{i-2} & \longrightarrow & \cdots \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\tau_{<i} & : \cdots & \longrightarrow & 0 & \longrightarrow & \operatorname{Ker} d_i & \xrightarrow{d_i} & X_{i-1} & \xrightarrow{d_{i-1}} & X_{i-2} & \longrightarrow & \cdots.
\end{array}$$

It is straightforward to check that

$$H_j \tau_{\geq i} X = \begin{cases} H_j X & j \geq i \\ 0 & j < 0 \end{cases} \qquad \text{and} \qquad H_j \tau_{<i} X = \begin{cases} 0 & j \geq i \\ H_j X & j < 0. \end{cases}$$

Notice that $H_i X = \tau_{<i+1} \tau_{\geq i} X$, which motivates the following definition

Definition 5.5. Let $\tau = (\mathcal{T}, \mathcal{F})$ be a t-structure in $\mathscr{T}$, then define the *0'th homology functor* $H_0^\tau \coloneqq \tau_{\Sigma \mathcal{F}} \tau_{\mathcal{T}} : \mathscr{T} \to \mathcal{H}$. From this define the *i'th homology functor* $H_i^\tau = H_0^\tau \Sigma^{-i}$.

Given a t-structure $\tau = (\mathcal{T}, \mathcal{F})$, with corresponding heart $\mathcal{H}$, there is a method to generate other t-structures. This method was described by Happel, Reiten, and Smalø, and therefore is known as *HRS-tilting*. Let $(\mathcal{X}, \mathcal{Y})$ be a torsion pair in $\mathcal{H}$, and define

$$\mathcal{T}' = \{X \in \mathcal{T} \mid H_0^t(X) \in \mathcal{X}\},$$
$$\mathcal{F}' = \{X \in \Sigma\mathcal{F} \mid H_0^t(X) \in \mathcal{Y}\}.$$

Then $(\mathcal{T}', \mathcal{F}')$ is a t-structure, and is called the *HRS-Tilt* of τ with respect to $(\mathcal{X}, \mathcal{Y})$.

Example 5.6 (HRS tilt)**.** Let $Q = 1 \to 2 \to 3 \to 4$ and consider $\mathscr{A} := \text{mod}(kQ)$, with the corresponding derived category $D := \mathsf{D}^b(\text{mod}(kQ))$. In D we have the standard t-structure $(D_{\geq 0}, D_{<0})$, see Figure 5.1.

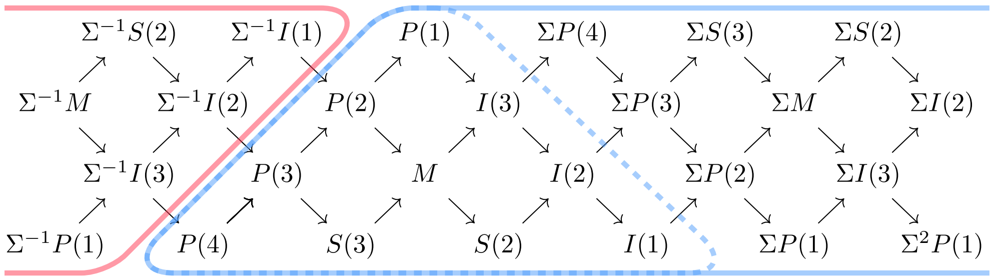


Figure 5.1: AR quiver of D, the aisle $D_{\geq 0}$ is surrounded by a blue line and the coaisle $D_{<0}$ surrounded by a red line. The heart of the t-structure $(D_{\geq 0}, D_{<0})$ is surrounded by a blue dashed line.

In $\mathscr{A}$ consider the torsion pair $(\mathcal{T}, \mathcal{F})$ given by

$$\mathcal{T} = \text{add}(P(4) \oplus P(3) \oplus S(3) \oplus I(1)) \quad \text{and} \quad \mathcal{F} = \text{add}(S(2) \oplus I(2)).$$

Tilting the standard t-structure $(D_{\geq 0}, D_{<0})$ with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$ we obtain a new t-structure $t = (\mathcal{X}, \mathcal{Y})$, which can be seen in Figure 5.2. From Figure 5.2 we also observe that the heart of $(\mathcal{X}, \mathcal{Y})$ is equivalent to $\mathcal{H}_t \cong \text{mod}(kA_2) \oplus \text{mod}(kA_2)$. In particular notice that $\mathcal{H}_t \cong \Sigma\mathcal{F} * \mathcal{T}$.

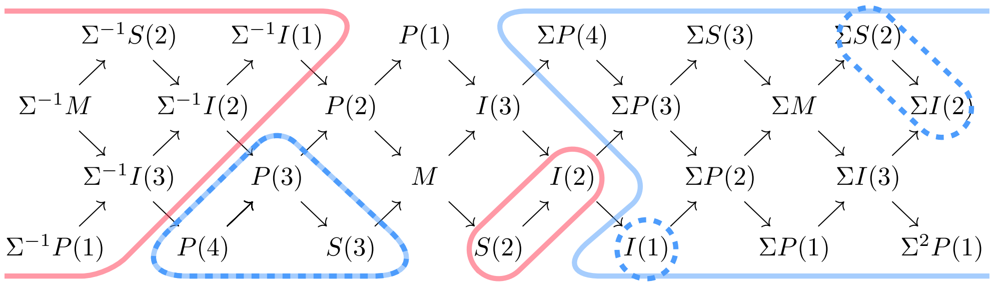


Figure 5.2: AR quiver of D, the aisle $\mathcal{X}$ is surrounded by a blue line and the coaisle $\mathcal{Y}$ surrounded by a red line. The heart of the t-structure $(\mathcal{X}, \mathcal{Y})$ is surrounded by a dashed blue line.

One of the important properties of hearts of t-structures is the correspondence between its short exact sequences and the triangles in the ambient triangulated category.

Proposition 5.7 ([BBD83, thm. 1.3.6]). *Let $\mathcal{H}$ be the heart of a t-structure in a triangulated category $\mathscr{T}$. Then $0 \longrightarrow x \longrightarrow y \longrightarrow z \longrightarrow 0$ is a short exact sequence in $\mathcal{H}$ if and only if $x, y, z \in \mathcal{H}$ and there is a triangle $x \longrightarrow y \longrightarrow z \longrightarrow \Sigma x$ in $\mathscr{T}$.*

Proposition 5.8. *Let $\mathcal{H}$ be the heart of a t-structure in the triangulated category $\mathscr{T}$. Then $\mathcal{H}$ has no negative self-extensions in $\mathscr{T}$, i.e. given $h, h' \in \mathcal{H}$ then $\mathrm{Hom}_{\mathscr{T}}(h, \Sigma^{-i}h') = 0$ for $i > 0$.*

5.1 Proper abelian subcategories

Definition 5.9. Let $\mathscr{A} \subseteq \mathscr{T}$ be an additive subcategory, then $\mathscr{A}$ is a *proper abelian subcategory of $\mathscr{T}$* if it is abelian in such a way that $0 \longrightarrow x \longrightarrow y \longrightarrow z \longrightarrow 0$ is a short exact sequence if and only if $x, y, z \in \mathscr{A}$ and there is a triangle $x \longrightarrow y \longrightarrow z \longrightarrow \Sigma x$ in $\mathscr{T}$.

Definition 5.10. Let $\mathscr{A} \subseteq \mathscr{T}$ be a proper abelian subcategory and $n \in \mathbb{N}$, then $\mathscr{A}$ is said to *satisfy E_n* if $\mathrm{Hom}_{\mathscr{A}}(a, \Sigma^{-i}a') = 0$ for $0 < i \leq n$. If $\mathscr{A}$ satisfies E_n for all $n \in \mathbb{N}$, then we say it *satisfies E_∞*.

Example 5.11. By Propositions 5.7 and 5.8 hearts of t-structures are all proper abelian subcategories which satisfy E_∞, however, there are also proper abelian subcategories that are not hearts. Consider the algebra kA_4, where A_4 is the quiver $A_4 : 1 \to 2 \to 3 \to 4$. The AR-quiver of the derived category $\mathsf{D}^b(\mathrm{mod}\, kA_4)$ can be seen in Figure 5.1.

1. Consider an object $X \in \mathsf{D}^b(kA_4)$ then the category $\mathscr{B} := \mathrm{add}(X)$ is a proper abelian subcategory. It is straightforward to see that $\mathscr{B}$ satisfies E_∞. Generally, given a proper abelian subcategory $\mathscr{B}$ of the triangulated category $\mathscr{T}$, then every wide subcategory $\mathscr{B}' \subseteq \mathscr{B}$ is also a proper abelian subcategory of $\mathscr{T}$.

2. Consider the objects $S(2), S(3) \in \mathsf{D}^b(kA_4)$, then $\mathscr{B} := \mathrm{add}(S(2) \oplus S(3))$ is a proper abelian subcategory. However, it is not closed under extensions. It is straightforward to see that $\mathscr{B}$ satisfies E_∞.

3. Consider the object $P(4) \in \mathsf{D}^b(kA_4)$. Then $\mathscr{B} = \mathrm{add}(P(4) \oplus \Sigma^2 P(2))$ is a proper abelian subcategory, which satisfies E_1 but not E_2.

4. For a non-example, consider $\mathcal{X} = \mathrm{add}(P(4) \oplus P(1) \oplus I(1))$, then $\mathcal{X}$ is abelian and is equivalent to $\mathrm{mod}(kA_2)$. However, $\mathcal{X}$ is **NOT** a proper abelian subcategory. This can be seen by realizing that $0 \to P(4) \to P(1) \to I(1) \to 0$ is a short exact sequence in $\mathcal{X}$, but there exists no triangle $P(4) \to P(1) \to I(1) \to \Sigma P(4)$ in $D^b(kA_4)$.

6 Results

In this section, we will present an overview of the results in this thesis, comparing them to the results they build on top of.

The papers in this thesis are slightly modified versions of the published and preprint versions of the same papers.

Paper A: Derived equivalences of self-injective 2-Calabi–Yau tilted algebras

(Bulletin of the London Mathematical Society **56** (2024), no. 3, 1071–1094, [Kor24b])

In this paper, we generalize a result by August [Aug20]. Let $\operatorname{Spec}(R)$ be a complete local isolated cDV singularity. By $\operatorname{CM}(R)$, we denote the full subcategory of $\operatorname{mod}(R)$ consisting of all the maximal Cohen-Macaulay modules, see [Aug20, def. 2.2] for a definition. Then $\operatorname{CM}(R)$ is a Frobenius category, and the stable category $\underline{\operatorname{CM}}(R)$ is a 2-CY, Hom-finite, Krull-Schmidt triangulated category, with suspension functor Σ_R, in which $\Sigma_R^2 \cong \operatorname{id}$, see [Aug20, prop. 2.3].

Consider a basic rigid object $M = \bigoplus_{i=0}^n M_i \in \operatorname{CM}(M)$, with $M_0 \cong R$. In $\operatorname{CM}(R)$, there is a notation of mutation, meaning that we can mutate this kind of rigid object to obtain a different rigid object, see [Aug20, sec. 2.2]. A mutation of M in the i'th summand is done by replacing it with another summand. We will denote this mutation $\nu_i M = M/M_i \oplus V_i$, for some correctly chosen $V_i \in \operatorname{CM}(R)$. Denote algebras

$$\Lambda := \operatorname{Hom}_{\operatorname{CM}(R)}(M, M) \qquad \Gamma := \operatorname{Hom}_{\operatorname{CM}(R)}(\nu_i M, \nu_i M)$$
$$\Lambda_{con} := \operatorname{Hom}_{\underline{\operatorname{CM}}(R)}(M, M) \qquad \Gamma_{con} := \operatorname{Hom}_{\underline{\operatorname{CM}}(R)}(\nu_i M, \nu_i M).$$

Define $\mathscr{T}_i := \tau_{\leq 1}(\Gamma_{con} \otimes_\Gamma^{\mathrm{L}} \operatorname{Hom}(M, \nu_i M) \otimes_\Lambda^{\mathrm{L}} \Lambda_{con})$ where $\tau_{\leq 1}$ refers to a soft truncation in cohomological degrees ≤ 1. With this setup, August proved the following statement.

Theorem 6.1 ([Aug20, cor. 3.3])**.** *$\mathscr{T}_i$ is a 2-sided tilting complex inducing an equivalence $- \otimes_{\Gamma_{con}}^{\mathrm{L}} \mathscr{T}_i : \mathsf{D}^b(\Gamma_{con}) \to \mathsf{D}^b(\Lambda_{con})$.*

In particular, this means that the endomorphism algebras Γ_{con} and Λ_{con} are derived equivalent. August then showed that if instead of Γ_{con} and Λ_{con} being separated by one mutation, they are separated by several mutations, then they still are derived equivalent, see [Aug20, cor. 5.11].

Let k be an algebraically closed field. Paper A generalizes the above result, replacing $\operatorname{CM}(R)$ with a general k-linear Frobenius category $\mathscr{E}$ whose stable category $\mathscr{C} := \underline{\mathscr{E}}$ is a 2-CY, Hom-finite and Krull-Schmidt triangulated category, with suspension functor $\Sigma_{\mathscr{C}}$. This allows for an extensive range of new categories, which we will discuss later. Furthermore, we relax the assumption that the objects must be connected by a sequence of mutations. We were able to show the following.

Theorem 6.2 (= Corollary A.3.11)**.** *Let $\mathscr{E}$ be k-linear Frobenius category such that the associated stable category $\mathscr{C} := \underline{\mathscr{E}}$ is 2-CY, hom-finite and Krull-Schmidt.*

Let $l, m \in \mathscr{C}$ be maximal rigid objects. Denote $\underline{A} = \operatorname{Hom}_{\mathscr{C}}(l, l)$ and $\underline{B} = \operatorname{Hom}_{\mathscr{C}}(m, m)$. Assume that $\underline{A}$ and $\underline{B}$ are self-injective. Then, they are derived equivalent.

Furthermore, there exist objects $l', m' \in \mathscr{C}$ with $l \cong l'$ and $m \cong m'$, such that by denoting $A = \operatorname{Hom}_{\mathscr{E}}(l', l')$, $B = \operatorname{Hom}_{\mathscr{E}}(m', m')$ and $T = \operatorname{Hom}_{\mathscr{E}}(l', m')$ there is a two-sided tilting complex of $\underline{B} \otimes \underline{A}^{\mathrm{op}}$-modules

$$_{\underline{B}}\mathscr{T}_{\underline{A}} = \tau_{\leq 1}\left(\underline{B} \overset{\mathrm{L}}{\underset{B}{\otimes}} T \overset{\mathrm{L}}{\underset{A}{\otimes}} \underline{A}\right),$$

inducing this derived equivalence. The subscript $\tau_{\leq 1}$ denotes the soft truncation to homological degrees ≤ 1.

Notice that in this result, we have the extra assumption that $\underline{\mathscr{A}}$ and $\underline{\mathscr{B}}$ need to be self-injective. To get an idea of what this means, consider the following lemma.

Lemma 6.3 (= Lemma A.2.5)**.** *Let $x \in \mathscr{C}$ be a maximal rigid object, then $\Sigma^2_{\mathscr{C}} x \cong x$ if and only if $\mathscr{C}(x,x)$ is a self-injective algebra.*

If we look at the setup from [Aug20], as discussed above, we see that due to the fact that $\Sigma^2_R \cong \mathrm{id}$, it automatically follows that Λ_{con} and Γ_{con} are self-injective. They are even symmetric. Essentially, this means that we have removed the global assumption of $\Sigma^2_R \cong \mathrm{id}$, and replaced it with a local assumption on the endomorphism algebras.

Paper B: Intermediate categories for proper abelian subcategories

(Preprint, arXiv:2310.12045, [Kor24a])

Paper B generalizes one of the results in [ES22] by Enomoto and Saito. Given a skeletally small abelian category $\mathscr{A}$, they define *intermediate categories* in $\mathsf{D}^b(\mathscr{A})$.

Definition 6.4 ([ES22, def. 5.2])**.** A subcategory $\mathcal{C} \subseteq \mathsf{D}^b(\mathscr{A})$ is called an *intermediate category* if it satisfies the following conditions.

1. $\mathscr{A} \subseteq \mathcal{C} \subseteq \mathscr{A}[1] * \mathscr{A}$.

2. $\mathcal{C}$ is closed under extensions in $\mathsf{D}^b(\mathscr{A})$.

3. $\mathcal{C}$ is closed under direct summands in $\mathsf{D}^b(\mathscr{A})$.

This definition can be considered a local version of the coaisle of an intermediate t-structure. This intuition also aligns with the result they have shown, which is the following.

Theorem 6.5 ([ES22, thm. 5.3])**.** *The following statements hold.*

1. *Given a torsion-free class $\mathcal{F} \subseteq \mathscr{A}$, then $\mathcal{F}[1] * \mathscr{A}$ is an intermediate category in $\mathsf{D}^b(\mathscr{A})$*

2. *Let $\mathcal{C} \subseteq \mathsf{D}^b(\mathscr{A})$ be an intermediate category, then $H^{-1}(\mathcal{C})$ is an torsion-free class in $\mathscr{A}$, and $\mathcal{C} = H^{-1}(\mathcal{C})[1] * \mathscr{A}$*

The assignments given by (1) and (2) induce a bijection between the set of torsion-free classes in $\mathscr{A}$ and the set of intermediate categories in $\mathsf{D}^b(\mathscr{A})$.

Now assume $\mathscr{A}$ is an extension-closed proper abelian subcategory of some triangulated category $\mathscr{T}$ satisfying E_2, then it is not guaranteed that $\mathscr{A}$ is the heart of a t-structure. There might not even exist any non-trivial t-structures in $\mathscr{T}$. Thus we define intermediate categories with respect to a proper abelian category, instead of in relation to a heart.

Definition 6.6. A subcategory $\mathcal{C} \subseteq \mathscr{T}$ is called an *$\mathscr{A}$-intermediate category* if

1. $\mathscr{A} \subseteq \mathcal{C} \subseteq \Sigma\mathscr{A} * \mathscr{A}$,
2. $\mathcal{C}$ is extension-closed,
3. $\mathcal{C}$ is closed under direct summands.

Given an object $x \in \Sigma\mathscr{A} * \mathscr{A}$ there exists a unique triangle

$$\Sigma a_1^x \longrightarrow x \longrightarrow a_0^x \longrightarrow \Sigma^2 a_1^x$$

with $a_i^x \in \mathscr{A}$, thus the assignment $x \mapsto a_i^x$ induces a functor $F_i : \Sigma\mathscr{A} * \mathscr{A} \to \mathscr{A}$. The functor F_1 will take the place of H^{-1} from Theorem 6.5. With this, we can show the following.

Theorem 6.7 (= Theorem B.4.2 & Corollary B.4.3)**.** *The following statements hold.*

1. *If $\mathscr{F} \subseteq \mathscr{A}$ is a torsion-free class then $\Sigma\mathscr{F} * \mathscr{A}$ is an $\mathscr{A}$-intermediate category. Furthermore, $F_1(\Sigma\mathscr{F} * \mathscr{A}) = \mathscr{F}$.*
2. *Let $\mathcal{C}$ be an $\mathscr{A}$-intermediate category such that $\mathcal{C} \subseteq \mathscr{A} * \Sigma\mathscr{A}$. Then $F_1(\mathcal{C})$ is torsion-free. Furthermore, we have that $\mathcal{C} = \Sigma F_1(\mathcal{C}) * \mathscr{A}$.*

Furthermore, if $\Sigma\mathscr{A} * \mathscr{A} = \mathscr{A} * \Sigma\mathscr{A}$ then there is a bijection

$$\begin{aligned} \{\mathcal{C} \subseteq \mathscr{T} \mid \mathcal{C} \text{ is } \mathscr{A}\text{-intermediate}\} &\overset{1:1}{\longleftrightarrow} \{\mathscr{F} \subseteq \mathscr{A} \mid \mathscr{F} \text{ torsion-free}\} \\ \mathcal{C} &\longmapsto F_1(\mathcal{C}) \\ \Sigma\mathscr{F} * \mathscr{A} &\longleftarrow\!\shortmid \mathscr{F} \end{aligned}$$

Thus, we have generalized the result to not only hold for any t-structures, but to hold for any proper abelian subcategory satisfying E_2.

A thing that is worth noting is that if $\mathscr{A}$ is the heart of a t-structure, then the functors F_i are exactly the homology functors restricted to the subcategory $\Sigma\mathscr{A} * \mathscr{A}$. In Paper B, we even show that we can get long exact sequences "of homology", similar to those in the derived category.

Lemma 6.8 (= Lemma B.3.4)**.** *Let $c \xrightarrow{f} c' \xrightarrow{g} c'' \longrightarrow \Sigma c$ be a triangle in $\mathscr{T}$ with $c, c', c'' \in \Sigma\mathscr{A} * \mathscr{A}$ then there exists a morphism $\delta : F_1(c'') \to F_0(c)$, such that*

$$0 \longrightarrow F_1(c) \xrightarrow{F_1(f)} F_1(c') \xrightarrow{F_1(g)} F_1(c'') \xrightarrow{\delta} F_0(c) \xrightarrow{F_0(f)} F_0(c') \xrightarrow{F_0(g)} F_0(c'') \longrightarrow 0$$

is an exact sequence in $\mathscr{A}$.

Paper C: Filtrations of Torsion Classes in Proper Abelian Subcategories

(Preprint, arXiv:2406.13418, [Kor24])

In Paper C, we generalize a result from [JMS13] where Jensen, Madsen, and Su show their result using what essentially is, the standard t-structure on the derived category. We will give a proof of a similar result using proper abelian subcategories, in some fitting triangulated category, leading to a more general result.

Let k be a field, and $\mathscr{A}$ an abelian k-category similar to one of those from [JMS13, sec. 0], as an example one could choose $\operatorname{mod}(A)$ for a finite-dimensional k-algebra A. Let $T \in \mathscr{A}$ be a tilting object of projective dimension ≤ 2, i.e., it induces a derived equivalence

$$F = R\operatorname{Hom}(T, -) : \mathsf{D}^b(\mathscr{A}) \longrightarrow \mathsf{D}^b(\operatorname{mod}\operatorname{Hom}_{\mathscr{A}}(T,T)^{op}).$$

Denote $\widehat{\mathscr{B}} \coloneqq \operatorname{mod}\operatorname{Hom}_{\mathscr{A}}(T,T)^{op}$, and define subcategories

$$\mathcal{F}^i \coloneqq \{x \in \mathscr{A} \mid H^j F(x) = 0 \text{ for } j \neq i\}.$$

It is a well-known result, that if $\operatorname{pd} T = 1$ (think of tilting modules), then T induces a torsion pair $(\operatorname{Gen}(T), {}^{\perp}\operatorname{Gen}(T))$, however, this need not be the case if $\operatorname{pd} T > 1$. For $\operatorname{pd} T = 2$ Jensen, Madsen and Su define three subcategories of $\mathscr{A}$

$$\mathcal{E}^0 \coloneqq \langle \operatorname{Gen}_{\mathscr{A}}(\mathcal{F}^0) \rangle_{\mathscr{A}}, \qquad \mathcal{E}^1 \coloneqq \mathcal{F}^1, \qquad \text{and} \quad \mathcal{E}^2 \coloneqq \langle \operatorname{Sub}_{\mathscr{A}}(\mathcal{F}^2) \rangle_{\mathscr{A}},$$

which function as a kind of *torsion triple.*

Theorem 6.9 ([JMS13, thm. 2])**.** *Using the notation from above, assume* $\operatorname{pd} T \leq 2$*. Then for each object* $x \in \mathscr{A}$ *there exists a unique filtration* $0 = x_0 \subseteq x_1 \subseteq x_2 \subseteq x_3 = x$*, such that* $x_{i+1}/x_i \in \mathcal{E}^i$*, for* $i = 0, 1, 2$*.*

Since F is an equivalence of triangulated categories, we can pull back $\widehat{\mathscr{B}}$ through the equivalence to obtain a heart $\mathscr{B} \coloneqq F^{-1}(\widehat{\mathscr{B}})$ in the derived category $\mathsf{D}^b(\mathscr{A})$. This is the kind of setup we work with in Paper C.

Let $\mathscr{T}$ be a triangulated category, and let $\mathscr{A}, \mathscr{B} \subseteq \mathscr{T}$ be proper abelian categories satisfying E_5. Assume that $\mathscr{A}$ is noetherian, and that $\mathscr{A} \subseteq \mathscr{B} * \Sigma^{-1}\mathscr{B} * \Sigma^{-2}\mathscr{B}$ and $\mathscr{B} \subseteq \Sigma^2\mathscr{A} * \Sigma\mathscr{A} * \mathscr{A}$. Define

- $\mathcal{E}_0 = \langle \operatorname{Gen}_{\mathscr{A}}(\mathscr{A} \cap \mathscr{B}) \rangle_{\mathscr{A}}$,
- $\mathcal{E}_1 = \mathscr{A} \cap \Sigma^{-1}\mathscr{B}$,
- $\mathcal{E}_2 = \langle \operatorname{Sub}_{\mathscr{A}}(\mathscr{A} \cap \Sigma^{-2}\mathscr{B}) \rangle_{\mathscr{A}}$.

This allows us to show the following result.

Theorem 6.10 (= Corollary C.3.6)**.** *Let* $x \in \mathscr{A}$*. Then, up to isomorphism, there exists a unique filtration of subobjects* $0 = x_0 \subseteq x_1 \subseteq x_2 \subseteq x_3 = x$ *such that each quotient* $x_{i+1}/x_i \in \mathcal{E}_i$*.*

Project D: Homology for Proper Abelian Subcategories

In [BBD83] Beilinson, Bernstein and Deligne define t-structures of triangulated categories, see Proposition/Definition 5.1. Given a t-structures $\sigma = (\mathcal{T}, \mathcal{F})$ with heart $\mathcal{H}$ in a triangulated category $\mathscr{T}$ they define homological functors $H_i^\sigma : \mathscr{T} \to \mathcal{H}$ for $i \in \mathbb{Z}$, see Definition 5.5. They show the following result

Theorem 6.11 ([BBD83, thm. 1.3.6])**.** *Using the notation from above, given a triangle $x \to y \to z \to \Sigma x$ in $\mathscr{T}$, then there exists a long exact sequence*

$$\cdots \longrightarrow H_{i+1}^\sigma(z) \longrightarrow H_i^\sigma(x) \longrightarrow H_i^\sigma(y) \longrightarrow H_i^\sigma(z) \longrightarrow H_{i-1}^\sigma(x) \longrightarrow \cdots .$$

This result is very powerful, and it is one of the reasons why t-structures are important. However, there are many situations in which a similar result could be useful, but where there is not enough space for a t-structure. As an example, consider the negative cluster category $\mathcal{C}_{-3}(A_4) = \mathcal{C}_{-3}(\operatorname{mod} kA_4)$ for some field k, see Figure 6.1.

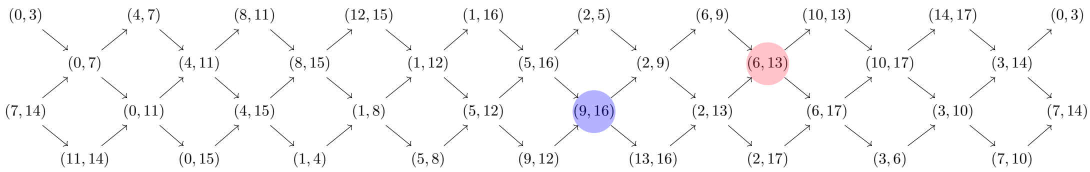


Figure 6.1: AR quiver for $C_{-3}(A_4)$.

Consider the object $(9, 16) \in \mathcal{C}_{-3}(A_4)$, see Figure 6.1. If we were to look for a t-structure $(\mathcal{T}, \mathcal{F})$ with heart $\mathcal{H}$ such that $(9, 16) \in \mathcal{H}$, then we would have that $(9, 16) \in \mathcal{T}$ and $\Sigma^{-1}(9, 16) = (8, 15) \in \mathcal{F}$. Similarly $\Sigma^{-i}(9, 16) \in \mathcal{F}$ for $i \geq 1$. However, this means that $\Sigma^{-3}(9, 16) = (6, 13) \in \mathcal{F}$, which is a problem since $\operatorname{Hom}((9, 16), (6, 13)) \neq 0$. Thus, there is no t-structure with a heart containing $(9, 16)$ in $\mathcal{C}_{-3}(A_4)$.

In fact it has been shown that no negative cluster category contains a non-trivial t-structure, see [HJY13, lem. 3.1]. The problem here is somehow that the aisle of a t-structure needs to be closed under suspension, and the coaisle under cosuspension, this is problematic when the category is "cylindrical" (see Figure 6.1). To solve this problem, instead of covering the whole triangulated category with a t-structure, we work locally around a given proper abelian subcategory.

Let $\mathscr{T}$ be a triangulated category, $n, m \in \mathbb{Z}$ with $n > m$, and consider a proper abelian subcategory $\mathscr{A} \subseteq \mathscr{T}$ such that $\operatorname{Hom}(\mathscr{A}, \Sigma^{-i}\mathscr{A}) = 0$ for $1 \leq i \leq n - m + 3$. Then for $k \in \mathbb{Z}$ such that $n \geq k \geq m$ there exists functors

$$\tau_{\geq k} : \Sigma^n \mathscr{A} * \cdots * \Sigma^m \mathscr{A} \to \Sigma^n \mathscr{A} * \cdots * \Sigma^k \mathscr{A},$$

and

$$\tau_{<k} : \Sigma^n \mathscr{A} * \cdots * \Sigma^m \mathscr{A} \to \Sigma^{k-1} \mathscr{A} * \cdots * \Sigma^m \mathscr{A}.$$

These functors resembles the truncation functor for a t-structure, but they are only defined locally around $\mathscr{A}$. With this we can define functors

$$H_i^{\mathscr{A}} : \Sigma^n \mathscr{A} * \cdots * \Sigma^m \mathscr{A} \to \mathscr{A}$$

for $i = m, \dots, n$ by $H_i^{\mathscr{A}} x = \Sigma^{-k} \tau_{<k+1} \tau_{\geq k} x$. This resembles the definition of homology defined using t-structures, see Definition 5.5. We can then show the following result.

Theorem 6.12 (= Theorem D.4.5)**.** *Assume the notation from above. Given a triangle* $c \to c' \to c'' \to \Sigma c$ *with object in* $c, c', c'' \in \Sigma^n \mathscr{A} * \cdots * \Sigma^m \mathscr{A}$*. Then there exists a long exact sequence*

$$0 \to H_n^{\mathscr{A}} c \to H_n^{\mathscr{A}} c' \to H_n^{\mathscr{A}} c'' \to H_{n-1}^{\mathscr{A}} c \to \cdots \to H_{m+1}^{\mathscr{A}} c'' \to H_m^{\mathscr{A}} c \to H_m^{\mathscr{A}} c' \to H_m^{\mathscr{A}} c'' \to 0.$$

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Paper A

Derived Equivalences of Self-injective 2-Calabi–Yau Tilted Algebras

Abstract

Consider a k-linear Frobenius category $\mathscr{E}$ such that the corresponding stable category $\mathscr{C}$ is 2-Calabi–Yau, Hom-finite with split idempotents. Let $l, m \in \mathscr{C}$ be maximal rigid objects with self-injective endomorphism algebras. We will show that their endomorphism algebras $\mathscr{C}(l, l)$ and $\mathscr{C}(m, m)$ are derived equivalent. Furthermore we will give a description of the two-sided tilting complex which induces this derived equivalence.

1 Introduction

In [Aug20b] August showed that given two objects $M, N \in \underline{\mathrm{CM}}(R)$ with $\operatorname{Spec} R$ being a complete local isolated cDV singularity, such that M and N are maximal rigid objects connected through a number of mutations, the contraction algebras $\underline{\mathrm{End}}(M)$ and $\underline{\mathrm{End}}(N)$ are derived equivalent. Note that in this setting $\underline{\mathrm{End}}(M)$ and $\underline{\mathrm{End}}(N)$ are symmetric algebras.

In this paper we generalize the result mentioned above to the setting of a more general Frobenius category than $\mathrm{CM}(R)$. Our general course of action and a number of the proofs will be based on those in [Aug20b]. We shall use a result from [ZZ11] that will give a conflation which is able to replace the exchange sequences you would get from mutations. This will allow us to prove that $\underline{\mathrm{End}}(M)$ and $\underline{\mathrm{End}}(N)$ are derived equivalent if they are self-injective and M and N are maximal rigid objects. This will therefore lead to significantly more general results, allowing for categories without the condition $\Sigma^2 \cong \mathrm{id}$ and categories that may have infinitely many maximal-rigid objects.

Two-sided tilting complex. The definition of a tilting complex was introduced by Rickard in [Ric89]. He showed that given two derived equivalent algebras, there exists a tilting complex inducing such an equivalence.

Let k be an algebraically closed field, A a k-algebra. Let $\operatorname{proj} A$ denote the category of f.g. projective left A-modules, $\mathcal{K}^b(\operatorname{proj} A)$ its bounded homotopy category.

Definition 1.1. A complex $T \in \mathcal{K}^b(\operatorname{proj} A)$ is called a *tilting complex* if the following are satisfied.

- $\mathrm{Hom}(T, T[i]) = 0$ for all $i \neq 0$.
- $\mathrm{thick}(T) = \mathcal{K}^b(\operatorname{proj} A)$, where $\mathrm{thick}(T)$ indicates the thick closure of $\mathrm{add}(T)$.

The notion of a two-sided tilting complex is also due to Rickard [Ric91]. We use the following form of the definition due to Keller [Kel98, 8.1.4]. Let unadorned tensor products be over k and let $\mathcal{D}$ denote the derived category.

Definition 1.2. Let ${}_BT_A \in \mathcal{D}(B \otimes A^{\mathrm{op}})$ be a complex of (B, A)-bimodules. Let ${}_BT$ (resp. T_A) be ${}_BT_A$ seen as a complex in $\mathcal{D}(B)$ (resp. $\mathcal{D}(A^{\mathrm{op}})$). ${}_BT_A$ is a *two-sided tilting complex* if the following are satisfied:

1. The canonical map $B \to \mathrm{Hom}_{\mathcal{D}(A^{\mathrm{op}})}(T_A, T_A)$ is bijective and $\mathrm{Hom}_{D(A^{\mathrm{op}})}(T_A, T_A[i]) = 0$ for $i \neq 0$.

2. T_A is quasi-isomorphic to a complex in $\mathcal{K}^b(\mathrm{proj}\, A^{\mathrm{op}})$.

3. $\mathrm{thick}(T_A) = \mathcal{K}^b(\mathrm{proj}\, A^{\mathrm{op}})$.

Let $\mathscr{E}$ be a k-linear Frobenius category. Then given two maximal rigid objects $l, m \in \mathscr{E}$ with suitable projective summands, the complex $T = \mathscr{E}(l, m)$ is a two-sided tilting complex in $\mathcal{D}(\mathscr{E}(m, m) \otimes \mathscr{E}(l, l)^{op})$ (see [JY19, prop 5.1]), making $A = \mathscr{E}(l, l)$ and $B = \mathscr{E}(m, m)$ derived equivalent. Looking at the stable category $\mathscr{C}$ of $\mathscr{E}$, a similar choice of the module $\mathscr{C}(l, m)$ does not necessarily give a tilting module, and it is not necessarily true that $\underline{A} = \mathscr{C}(l, l)$ and $\underline{B} = \mathscr{C}(m, m)$ are derived equivalent. However, we are able to prove the following main result.

Theorem A (Corollary 3.11)**.** *Let $\mathscr{E}$ be k-linear Frobenius category such that the associated stable category $\mathscr{C} := \underline{\mathscr{E}}$ is 2-CY and Hom-finite with split idempotents.*

Let $l, m \in \mathscr{C}$ be maximal rigid objects. Denote $\underline{A} = \mathscr{C}(l, l)$ and $\underline{B} = \mathscr{C}(m, m)$. Assume that $\underline{A}$ and $\underline{B}$ are self-injective. Then they are derived equivalent. Furthermore, there exist objects $l', m' \in \mathscr{C}$ with $l \cong l'$ and $m \cong m'$, such that by denoting $A = \mathscr{E}(l', l')$, $B = \mathscr{E}(m', m')$ and $T = \mathscr{E}(l', m')$ there is a two-sided tilting complex of $\underline{B} \otimes \underline{A}^{\mathrm{op}}$-modules

$$ {}_{\underline{B}}\mathscr{T}_{\underline{A}} = \left(\underline{B} \overset{\mathrm{L}}{\underset{B}{\otimes}} T \overset{\mathrm{L}}{\underset{A}{\otimes}} \underline{A} \right)_{\subseteq 1}, $$

inducing this derived equivalence. The subscript $\subseteq 1$ denotes the soft truncation to homological degrees ≤ 1.

In the fourth section we will focus on two examples. The first example is on cluster-tilting objects in the cluster category $\mathcal{C}(D_{2n})$. In [Rin08] Ringel has listed the cluster-tilting objects of $\mathcal{C}(D_{2n})$ with self-injective endomorphism algebras. It has already been shown in [BHL14, lem. 4.5] that these endomorphism algebras are derived equivalent. This result was achieved by supplying a tilting complex ad hoc. We will use our result to recover this tilting complex by manually calculating the tilting complex $\mathscr{T}_{\underline{A}}$.

The second example will be on a class of examples based on Postnikov diagrams. We will use the work of Pasquali [Pas20], which describes how reduced and symmetric (k, n)-Postnikov diagrams give rise to cluster tilting objects with self-injective endomorphism algebras. This will lead to the following result:

Corollary B (Corollary 4.4)**.** *Let $k, n \in \mathbb{N}$, with $k < n$. Let $\hat{B}$ be the completion of the so-called boundary algebra (see section 4). Let D, D' be two symmetric and reduced (k, n)-Postnikov diagrams, with associated cluster tilting objects T, T' (resp.) in $\underline{\mathrm{CM}}(\hat{B})$, the stable category of Cohen–Macaulay modules. Then the self-injective algebras $\underline{\mathrm{End}}(T)$ and $\underline{\mathrm{End}}(T')$ are derived equivalent.*

In the setting of Frobenius categories, derived equivalences between endomorphism algebras have previously been studied by several authors. Assume that $\mathscr{E}$ is a sufficiently nice Frobenius category with stable category $\mathscr{C}$ and choose two cluster tilting objects in $\mathscr{C}$. In a 2007 article, Iyama showed that if you look at these objects as objects in $\mathscr{E}$ then they have derived equivalent endomorphism algebras (see [Iya07, cor. 5.3.3]). Furthermore, Iyama constructs a two-sided tilting complex that induces the derived equivalence. This result was extended by Palu to a more general Frobenius category (see [Pal09, prop. 4]). Both of these results concerns derived equivalences of endomorphism algebras in the Frobenius category. A natural question to ask is whether this extends to derived equivalences of endomorphism algebras in the stable category. Although this is not always true, under certain conditions it will be. Dugas showed that if we look at the Frobenius category $\mathrm{CM}(R)$ for some odd-dimensional Gorenstein hypersurface R that is an isolated singularity, the endomorphism algebras of cluster-tilting objects in the stable category are derived equivalent (see [Dug15, cor. 5.5]). This was done by providing a one-sided tilting complex. If we instead have a complete local isolated cDV singularity R and two cluster-tilting objects linked by a path of mutations, a result by August provides us with a two-term tilting complex between the two stable endomorphism algebras (see [Aug20b, thm. 3.2, cor 3.3]). This result was shown to hold for maximal rigid objects, a generalization of cluster-tilting objects. The two latter articles consider Frobenius categories whose stable categories satisfy $\Sigma^2 \cong \mathrm{id}$. This article aims to generalize these results to a more general Frobenius category and to replace the global assumption of $\Sigma^2 \cong \mathrm{id}$ with local assumptions on the maximal rigid objects considered. Further, we do not assume that there exists a path of mutations between the objects.

2 Preliminaries

Setup 2.1. Let k be an algebraically closed field. Let $\mathscr{E}$ be a k-linear Frobenius category. Let $\mathscr{C} := \underline{\mathscr{E}}$ be the associated stable category. We will assume that $\mathscr{C}$ is a 2-Calabi–Yau, Hom-finite category with split idempotents. Observe that $\mathscr{C}$ has the same objects as $\mathscr{E}$ but different morphisms.

It is well-known that $\mathscr{C}$ is a triangulated category, whose suspension functor will be denoted Σ.

Definition 2.2.

- $x \in \mathscr{C}$ is called *rigid* if $\mathscr{C}(x, \Sigma x) = 0$.
- $x \in \mathscr{C}$ is called *maximal rigid* if it is rigid, and $\mathscr{C}(x \oplus y, \Sigma(x \oplus y)) = 0$ implies $y \in \mathrm{add}_{\mathscr{C}}(x)$.

The following result is due to Zhou and Zhu [ZZ11]. It generalizes a similar result from [GLS06], which then can be applied to our setup. We will use it to construct conflations which will then connect maximal rigid objects in a way that can "replace" exchange sequences of mutations.

Theorem 2.3 ([ZZ11, cor. 2.5])**.** *Let $x \in \mathscr{C}$ be maximal rigid, and let $y \in \mathscr{C}$ be rigid, then $y \in \operatorname{add}_{\mathscr{C}}(x) * \operatorname{add}_{\mathscr{C}}(\Sigma x)$, i.e. there exists a triangle*

$$x_1 \longrightarrow x_0 \longrightarrow y \longrightarrow \Sigma x_1,$$

with $x_i \in \operatorname{add}_{\mathscr{C}}(x)$.

The following lemma is a collection of useful results when working in the context of Setup 2.1. See also [JY19, lem. A.1].

Lemma 2.4. *Let $x, y, z \in \mathscr{E}$. Define $A \coloneqq \mathscr{E}(x,x)$ and $\underline{A} \coloneqq \mathscr{C}(x,x)$.*

(a) If $x \cong 0$ in $\mathscr{C}$, then x is a projective object in $\mathscr{E}$.

(b) For each triangle $x \to y \to z \to \Sigma x$ in $\mathscr{C}$ there is a conflation $0 \to x \to y' \to z' \to 0$ in $\mathscr{E}$ such that $y \cong y'$ and $z \cong z'$ in $\mathscr{C}$.

(c) $x \cong y$ in $\mathscr{C}$ if and only if there exist projective objects $p, p' \in \mathscr{E}$ such that $x \oplus p \cong y \oplus p'$ in $\mathscr{E}$.

(d) If $x \in \operatorname{add}_{\mathscr{E}}(y)$ or $z \in \operatorname{add}_{\mathscr{E}}(y)$ then composition of morphisms induces a k-linear bijection

$$\mathscr{E}(y,z) \otimes_B \mathscr{E}(x,y) \to \mathscr{E}(x,z)$$

which is natural in x, z, where $B = \mathscr{E}(y,y)$.

(e) If $\tilde{x} \in \operatorname{add}_{\mathscr{E}}(x)$ then composition of morphisms induces a k-linear bijection

$$\mathscr{C}(x,y) \otimes_A \mathscr{E}(\tilde{x},x) \to \mathscr{C}(\tilde{x},y)$$

which is natural in $\tilde{x}, y$.

(f) If $y \in \operatorname{add}_{\mathscr{C}}(x)$ then the canonical map

$$\mathscr{C}(x,-) : \mathscr{C}(y,z) \to \operatorname{Hom}_{\underline{A}^{\mathrm{op}}}(\mathscr{C}(x,y), \mathscr{C}(x,z))$$

is a bijection.

(g) If x is maximal rigid and y, z are rigid then the map

$$\mathscr{C}(x,-) : \mathscr{C}(y,z) \to \operatorname{Hom}_{\underline{A}^{\mathrm{op}}}(\mathscr{C}(x,y), \mathscr{C}(x,z))$$

is surjective.

Proof. (a) If $x \cong 0$ in $\mathscr{C}$, then $\underline{\mathrm{id}_x} = 0$ and therefore id_x factors through a projective object $\mathrm{id}_x = p'p : x \to P \to x$ in $\mathscr{E}$. This makes x projective (see [Büh10, cor 11.4]).

(b) Since $\mathscr{E}$ is a Frobenius category, there are enough injectives. Hence there is an inflation $\alpha : x \to I$, with I being an injective object. It follows that

$$\begin{pmatrix} u \\ \alpha \end{pmatrix} : x \to y \oplus I,$$

is an inflation. Thus there is a conflation $0 \to x \to y \oplus I \to z' \to 0$. Since I is injective, $y \oplus I \cong y$ in $\mathscr{C}$, and it then follows that $z \cong z'$ in $\mathscr{C}$.

(c) The 'if' part is straightforward. For the 'only if' part assume that $x \cong y$ in $\mathscr{C}$. This implies the existence of a triangle $x \to y \to 0 \to \Sigma x$, which by (b) is induced by a conflation $0 \to x \to y \oplus I \to P \to 0$, with $P \cong 0$ in $\mathscr{C}$. Now P is a projective object by (a), giving that the conflation splits. Hence $x \oplus P \cong y \oplus I$.

(d) Consider the morphism $\tilde{\circ}(-) : \mathscr{E}(y,z) \otimes_B \mathscr{E}(x,y) \to \mathscr{E}(x,z)$ induced by composition. We need to check that this is bijective. We show the case $x \in \operatorname{add}(y)$, the case for $z \in \operatorname{add}(y)$ is similar. Assume that $x \in \operatorname{add}(y)$. This means that there exist diagrams

$$x \underset{\nu}{\overset{\eta}{\rightleftarrows}} y^n \underset{\iota_i}{\overset{\pi_i}{\rightleftarrows}} y,$$

such that $\nu\eta = \mathrm{id}_x$, and where π_i is the projection to the i'th component, and ι_i is the inclusion of the i'th component.

To show surjectivity, let $\psi \in \mathscr{E}(x,z)$. Since $\mathrm{id}_{y^n} = \sum_i \iota_i \pi_i$, the element mapped to ψ can be found as follows:

$$\psi = \psi\nu\Big(\sum_i \iota_i\pi_i\Big)\eta = \sum_i (\psi\nu\iota_i) \circ (\pi_i\eta) = \sum_i \tilde{\circ}(\psi\nu\iota_i \otimes_B \pi_i\eta) = \tilde{\circ}\Big(\sum_i \psi\nu\iota_i \otimes_B \pi_i\eta\Big).$$

For injectivity assume that $\tilde{\circ}(\sum_j f_j \otimes g_j) = 0$, for $f_j \in \mathscr{E}(y,z)$ and $g_j \in \mathscr{E}(x,y)$. Then

$$\sum_j f_j \underset{B}{\otimes} g_j = \sum_j f_j \underset{B}{\otimes} \Big(\sum_i g_j\nu\iota_i\pi_i\eta\Big) = \sum_i\sum_j (f_j g_j \nu\iota_i) \underset{B}{\otimes} (\pi_i\eta) = \sum_i 0 \underset{B}{\otimes} (\pi_i\eta) = 0.$$

(e) Consider the diagram

$$\begin{array}{ccc} \mathscr{E}(x,y) \otimes_A \mathscr{E}(\tilde{x},x) & \xrightarrow{\ \phi\ } & \mathscr{E}(\tilde{x},y) \\ \big\downarrow{\scriptstyle \mathrm{pr}\otimes\mathrm{id}} & & \big\downarrow{\scriptstyle \mathrm{pr}} \\ \mathscr{C}(x,y) \otimes_A \mathscr{E}(\tilde{x},x) & \xrightarrow{\ \psi\ } & \mathscr{C}(\tilde{x},y) \end{array}$$

where the horizontal morphisms are induced by composition, and where pr denotes the projection. Since ϕ is an isomorphism and since pr is surjective, we get that ψ is surjective. To check that ψ is injective, suppose that $\psi(\sum_i \underline{g_i} \otimes_A f_i) = \underline{\sum_i g_i f_i} = 0$, with $g_i \in \mathscr{E}(x,y)$, and $f_i \in \mathscr{E}(\tilde{x},x)$. As in the proof of (d) consider the diagram

$$\tilde{x} \underset{\nu}{\overset{\eta}{\rightleftarrows}} x^n \underset{\iota_i}{\overset{\pi_i}{\rightleftarrows}} x,$$

such that $\nu\eta = \mathrm{id}_{\tilde{x}}$, and where π_i is the projection to the i'th component, and ι_i is the inclusion of the i'th component. Now

$$\begin{aligned} \sum_i \underline{g_i} \underset{A}{\otimes} f_i &= \sum_i \underline{g_i} \underset{A}{\otimes} f_i \sum_j \nu\iota_j\pi_j\eta = \sum_j \Big(\sum_i \underline{g_i} \underset{A}{\otimes} (f_i\nu\iota_j\pi_j\eta)\Big) \\ &= \sum_j \Big(\sum_i (\underline{g_i f_i})\underline{\nu\iota_j} \underset{A}{\otimes} (\pi_j\eta)\Big) = 0. \end{aligned}$$

(f) consider the following diagram

$$y \underset{\nu}{\overset{\eta}{\rightleftarrows}} x^n \underset{\iota_i}{\overset{\pi_i}{\rightleftarrows}} x,$$

such that $\nu\eta = \mathrm{id}_y$, and where π_i is the projection to the i'th component, and ι_i is the inclusion of the i'th component. First we show injectivity. Let $\psi \in \mathscr{C}(y, z)$, such that $\mathscr{C}(x, \psi) = 0$. Assume for the sake of a contradiction that $\psi \neq 0$.

$$0 \neq \psi \ \mathrm{id}_y = \psi\nu \sum_i \iota_i\pi_i\eta = \sum_i \psi\nu\iota_i\pi_i\eta.$$

Therefore there exists $k \in \{1, \dots, n\}$ such that $\psi\nu\iota_k\pi_k\eta \neq 0$. Thus $\mathscr{C}(x, \psi)(\nu\iota_k) = \psi\nu\iota_k \neq 0$, which is a contradiction.

For surjectivity let $\Phi \in \mathrm{Hom}_{\underline{A}^{\mathrm{op}}}(\mathscr{C}(x, y), \mathscr{C}(x, z))$. Let $\phi = \sum_i \Phi(\nu\iota_i)\pi_i\eta$, then we claim that $\Phi = \mathscr{C}(x, \phi)$. For each $f \in \mathscr{C}(x, y)$ calculate

$$\Phi(f) = \Phi(\mathrm{id}_y f) = \Phi\Big(\sum_i \nu\iota_i\pi_i\eta f\Big) = \sum_i \Phi(\nu\iota_i)\pi_i\eta f = \phi f = \mathscr{C}(x, \phi)(f).$$

(g) Since x is maximal rigid, and y, z are rigid, Theorem 2.3 says that there exist triangles

$$x_1^y \to x_0^y \xrightarrow{\alpha} y \to \Sigma x_1^y \qquad \text{and} \qquad x_1^z \to x_0^z \xrightarrow{\beta} z \to \Sigma x_1^z,$$

with $x_i^z, x_i^y \in \mathrm{add}_{\mathscr{C}}(x)$. Since x is rigid this induces two exact sequences

$$\mathscr{C}(x, x_1^y) \to \mathscr{C}(x, x_0^y) \xrightarrow{\alpha_*} \mathscr{C}(x, y) \to 0 \qquad \text{and} \qquad \mathscr{C}(x, x_1^z) \to \mathscr{C}(x, x_0^z) \xrightarrow{\beta_*} \mathscr{C}(x, z) \to 0.$$

Now let $f \in \mathrm{Hom}_{\underline{A}^{\mathrm{op}}}(\mathscr{C}(x, y), \mathscr{C}(x, z))$. Since $x_i^y, x_i^z \in \mathrm{add}_{\mathscr{C}}(x)$, (f) says that there exist morphisms $g_i \in \mathscr{C}(x_i^y, x_i^z)$ making the following diagram commute:

$$\begin{array}{ccccccc}
\mathscr{C}(x, x_1^y) & \longrightarrow & \mathscr{C}(x, x_0^y) & \xrightarrow{\alpha_*} & \mathscr{C}(x, y) & \longrightarrow & 0 \\
\downarrow{\scriptstyle g_{1*}} & & \downarrow{\scriptstyle g_{0*}} & & \downarrow{\scriptstyle f} & & \\
\mathscr{C}(x, x_1^z) & \longrightarrow & \mathscr{C}(x, x_0^z) & \xrightarrow{\beta_*} & \mathscr{C}(x, z) & \longrightarrow & 0.
\end{array}$$

By the axioms of triangulated categories, there exists a morphism $g \in \mathscr{C}(y, z)$, making (g_1, g_0, g) a morphism of triangles

$$\begin{array}{ccccccc}
x_1^y & \longrightarrow & x_0^y & \xrightarrow{\alpha} & y & \longrightarrow & \Sigma x_1^y \\
\downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \Sigma g_1} \\
x_1^z & \longrightarrow & x_0^z & \xrightarrow{\beta} & z & \longrightarrow & \Sigma x_1^z.
\end{array}$$

Thus we get a diagram

$$\begin{array}{ccccccc}
\mathscr{C}(x, x_1^y) & \longrightarrow & \mathscr{C}(x, x_0^y) & \xrightarrow{\alpha_*} & \mathscr{C}(x, y) & \longrightarrow & 0 \\
\downarrow{\scriptstyle g_{1*}} & & \downarrow{\scriptstyle g_{0*}} & & {\scriptstyle f}\downarrow\downarrow{\scriptstyle g_*} & & \\
\mathscr{C}(x, x_1^z) & \longrightarrow & \mathscr{C}(x, x_0^z) & \xrightarrow{\beta_*} & \mathscr{C}(x, z) & \longrightarrow & 0,
\end{array}$$

with $f\alpha_* = \beta_* g_{0*} = g_*\alpha_*$. Since α_* is an epimorphism this implies that $f = g_*$. □

Let $\mathsf{D}(-) = \operatorname{Hom}_k(-,k)$ denote k-duality.

Lemma 2.5. *Let $x \in \mathscr{C}$ be a maximal rigid object, then $\Sigma^2 x \cong x$ if and only if $\mathscr{C}(x,x)$ is a self-injective algebra.*

Proof. Let $x' \in \mathscr{C}$, and assume that $\Sigma^2 x \cong x$. Since $\mathscr{C}$ is 2-Calabi–Yau there are isomorphisms

$$\mathsf{D}\,\mathscr{C}(x',x) \cong \mathsf{D}\,\mathscr{C}(x',\Sigma^2 x) \cong \mathscr{C}(x,x'),$$

which are functorial in x'. This gives an isomorphism $\mathsf{D}\,\mathscr{C}(x,x) \cong \mathscr{C}(x,x)$ of left $\mathscr{C}(x,x)$-modules making $\mathscr{C}(x,x)$ self-injective. For the opposite implication one can do an argument similar to that of [IO13, prop. 3.6]. □

Let Ω denote the syzygy, Ω^{-1} the cosyzygy in $\mathscr{E}$. With this the following is a standard result.

Lemma 2.6. *Let $x, y \in \mathscr{E}$, then $\operatorname{Ext}^1_{\mathscr{E}}(x,\Omega y) \cong \mathscr{C}(x,y) \cong \operatorname{Ext}^1_{\mathscr{E}}(\Omega^{-1}x, y)$.*

Proof. We will prove the first isomorphism. The second one follows by a dual argument. Recall that $\mathscr{E}$ being Frobenius requires that there are enough projective objects. Hence there exists a conflation

$$0 \longrightarrow \Omega y \xrightarrow{f} P \xrightarrow{g} y \longrightarrow 0$$

with P being projective and injective. Denote by pr the projection $\mathscr{E}(x,y) \to \mathscr{C}(x,y)$. Since $\mathscr{C}(x,y) = \mathscr{E}(x,y)/\operatorname{Ker}(\mathrm{pr})$, and since $\operatorname{Ext}^1(x,\Omega y) = \mathscr{E}(x,y)/\operatorname{Im}(g_*)$, it is enough to show that $\operatorname{Ker}(\mathrm{pr}) = \operatorname{Im}(g_*)$. Since $\mathrm{pr}\, g_* = 0$ we have the inclusion $\operatorname{Im}(g_*) \subseteq \operatorname{Ker}(\mathrm{pr})$. Thus it is enough to show that $\operatorname{Ker}(\mathrm{pr}) \subseteq \operatorname{Im}(g_*)$.

Let $h \in \operatorname{Ker}(\mathrm{pr})$, then h factors through a projective object $\tilde{P}$, say $h = p'p$ where $p : x \to \tilde{P}$, $p' : \tilde{P} \to y$. Using that g is an epimorphism, there exists a morphism $w : \tilde{P} \to P$ making the following diagram commute:

$$\begin{array}{ccc} \tilde{P} & \overset{w}{\dashrightarrow} & P \\ {\scriptstyle p}\uparrow & \searrow^{p'} & \downarrow{\scriptstyle g} \\ x & \xrightarrow{h} & y. \end{array}$$

In other words, $h = g_*(wp)$, thus $h \in \operatorname{Im} g_*$. □

Condition 2.7. *Let $x \in \mathscr{C}$ be rigid such that $\Sigma^2 x \cong x$. Then x is said to* satisfy Condition 2.7 with projective objects q_1, q_2 *if there are conflations in $\mathscr{E}$:*

$$\begin{aligned} 0 \longrightarrow \Omega x \xrightarrow{g} q_1 \xrightarrow{f} x \longrightarrow 0, \\ 0 \longrightarrow x \xrightarrow{g'} q_2 \xrightarrow{f'} \Omega x \longrightarrow 0, \end{aligned} \tag{2.1}$$

with $q_i \in \operatorname{add}_{\mathscr{E}}(x)$ being projective objects.

Lemma 2.8. *Let $x \in \mathscr{C}$ be rigid such that $\Sigma^2 x \cong x$, and denote $A = \mathscr{E}(x,x)$. Assume that x satisfies Condition 2.7 with projective objects $q_1, q_2 \in \operatorname{add}_{\mathscr{E}}(x)$, then the corresponding conflations induce augmented projective resolutions:*

$$0 \longrightarrow \mathscr{E}(x,x) \longrightarrow \mathscr{E}(x,q_2) \longrightarrow \mathscr{E}(x,q_1) \longrightarrow \mathscr{E}(x,x) \longrightarrow \mathscr{C}(x,x) \tag{2.2}$$

$$0 \longrightarrow \mathscr{E}(x,x) \longrightarrow \mathscr{E}(q_1,x) \longrightarrow \mathscr{E}(q_2,x) \longrightarrow \mathscr{E}(x,x) \longrightarrow \mathscr{C}(x,x). \tag{2.3}$$

Here the first sequence is a projective resolution of right A-modules, and the second sequence is a projective resolution of left A-modules.

Proof. Since x satisfies Condition 2.7 with projective objects q_1, q_2 there are conflations

$$0 \longrightarrow \Omega x \xrightarrow{g} q_1 \xrightarrow{f} x \longrightarrow 0,$$
$$0 \longrightarrow x \xrightarrow{g'} q_2 \xrightarrow{f'} \Omega x \longrightarrow 0.$$

Hence there is an exact sequence spliced from these two conflations

$$0 \longrightarrow x \xrightarrow{g'} q_2 \longrightarrow q_1 \xrightarrow{f} x \longrightarrow 0.$$
$$q_2 \overset{f'}{\twoheadrightarrow} \Omega x \overset{g}{\rightarrowtail} q_1$$

Applying the functor $\mathscr{E}(x,-)$ gives the exact sequence

$$0 \longrightarrow \mathscr{E}(x,x) \xrightarrow{g'_*} \mathscr{E}(x,q_2) \longrightarrow \mathscr{E}(x,q_1) \xrightarrow{f_*} \mathscr{E}(x,x).$$
$$\mathscr{E}(x,q_2) \overset{f'_*}{\twoheadrightarrow} \mathscr{E}(x,\Omega x) \overset{g_*}{\rightarrowtail} \mathscr{E}(x,q_1)$$

That f'_* is surjective follows directly from x being rigid and therefore $\operatorname{Ext}^1(x,x) = 0$, see Lemma 2.6. Since $\operatorname{Ext}^1(x,q_1) = 0$ it follows directly from Lemma 2.6 that this is a projective resolution of $\mathscr{C}(x,x)$ over right A-modules. This shows that we have the resolution from (2.2). The method for finding the resolution from (2.3) is similar. □

Lemma 2.9. *Let $x \in \mathscr{C}$ be rigid such that $\Sigma^2 x \cong x$ and assume it satisfies Condition 2.7 with projective objects $q_1, q_2 \in \operatorname{add}_{\mathscr{E}}(x)$. Let $p \in \mathscr{E}$ be a projective object. Then $x \oplus p$ satisfies Condition 2.7 with projective objects $q_1 \oplus p, q_2 \oplus p$.*

Proof. Consider the conflations

$$0 \longrightarrow 0 \longrightarrow p = p \longrightarrow 0,$$
$$0 \longrightarrow p = p \longrightarrow 0 \longrightarrow 0.$$

By adding these to those in (2.1) one obtains the needed criteria to satisfy Condition 2.7. □

Lemma 2.10. *Let $x \in \mathscr{C}$ be rigid, such that $\Sigma^2 x \cong x$. Then there exists a projective object $p \in \mathscr{E}$ such that $x' = x \oplus p$ satisfies Condition 2.7 with projective objects $q_1, q_2 \in \operatorname{add}_{\mathscr{E}}(p)$.*

Proof. Let $0 \to \Omega x \to p_1 \to x \to 0$ and $0 \to \Omega^2 x \to p_2' \to \Omega x \to 0$ be conflations in $\mathscr{E}$ where p_1, p_2' are projective objects. Since $\Omega^2 x \cong \Sigma^{-2} x \cong x$ in $\mathscr{C}$, Lemma 2.4(c) says that there are projective objects $p_3, q \in \mathscr{E}$ such that $\Omega^2 x \oplus q \cong x \oplus p_3$. Hence there are two exact sequences

$$0 \longrightarrow \Omega x \xrightarrow{g} p_1 \xrightarrow{f} x \longrightarrow 0,$$
$$0 \longrightarrow x \oplus p_3 \xrightarrow{g'} p_2 \xrightarrow{f'} \Omega x \longrightarrow 0,$$

where $p_2 = p_2' \oplus q$. By adding some 'trivial' conflations we obtain two different conflations.

$$0 \longrightarrow \Omega x \xrightarrow{\binom{g}{0}} p_1 \oplus (p_1 \oplus p_2 \oplus p_3) \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & \mathrm{id} \end{pmatrix}} x \oplus (p_1 \oplus p_2 \oplus p_3) \longrightarrow 0,$$

$$0 \longrightarrow x \oplus p_3 \oplus (p_1 \oplus p_2) \xrightarrow{\begin{pmatrix} g' & 0 \\ 0 & \mathrm{id} \end{pmatrix}} p_2 \oplus (p_1 \oplus p_2) \xrightarrow{(f' \ 0)} \Omega x \longrightarrow 0.$$

Let $x' = x \oplus p_1 \oplus p_2 \oplus p_3$, $q_1 = p_1 \oplus (p_1 \oplus p_2 \oplus p_3)$, $q_2 = p_2 \oplus (p_1 \oplus p_2)$. Using this the two conflations above can be written as follows:

$$\begin{aligned} 0 \longrightarrow \Omega x' \longrightarrow q_1 \longrightarrow x' \longrightarrow 0, \\ 0 \longrightarrow x' \longrightarrow q_2 \longrightarrow \Omega x' \longrightarrow 0. \end{aligned} \tag{2.4}$$

This implies that x' satisfies Condition 2.7 with projective objects $q_1, q_2 \in \mathrm{add}_{\mathscr{E}}(p)$ for $p = p_1 \oplus p_2 \oplus p_3$. □

Lemma 2.11. *Let $x \in \mathscr{C}$ be rigid such that $\Sigma^2 x \cong x$ and assume it satisfies Condition 2.7. Let $y \in \mathscr{C}$. Define $A := \mathscr{E}(x, x)$, and $\underline{A} := \mathscr{C}(x, x)$. Then* $\mathrm{Tor}_2^A(\mathscr{C}(x, y), {}_A\underline{A}) \cong 0$.

Proof. By Lemma 2.8 there is an augmented projective resolution

$$0 \longrightarrow \mathscr{E}(x, x) \longrightarrow \mathscr{E}(p_1, x) \longrightarrow \mathscr{E}(p_2, x) \longrightarrow \mathscr{E}(x, x) \longrightarrow \mathscr{C}(x, x) \longrightarrow 0,$$

of ${}_A\underline{A}$ over left A-modules, where $p_i \in \mathrm{add}_{\mathscr{E}}(x)$ are projective objects. Using Lemma 2.4(e) gives that

$$\begin{aligned} \mathscr{C}(x, y) \overset{\mathrm{L}}{\underset{A}{\otimes}} \underline{A} &\cong \mathscr{C}(x, y) \underset{A}{\otimes} \big(\, 0 \longrightarrow \mathscr{E}(x, x) \longrightarrow \mathscr{E}(p_1, x) \longrightarrow \mathscr{E}(p_2, x) \longrightarrow \mathscr{E}(x, x) \longrightarrow 0 \, \big) \\ &\cong \; 0 \longrightarrow \mathscr{C}(x, y) \longrightarrow \mathscr{C}(p_1, y) \longrightarrow \mathscr{C}(p_2, y) \longrightarrow \mathscr{C}(x, y) \longrightarrow 0 \\ &\cong \; 0 \longrightarrow \mathscr{C}(x, y) \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathscr{C}(x, y) \longrightarrow 0. \end{aligned}$$

Hence $\mathrm{Tor}_2^A(\mathscr{C}(x, y), {}_A\underline{A}) \cong 0$. □

Lemma 2.12. *Let $x \in \mathscr{C}$ be rigid such that $\Sigma^2 x \cong x$ and such that it satisfies Condition 2.7. Define $A := \mathscr{E}(x, x)$, and $\underline{A} := \mathscr{C}(x, x)$. Let $\mathfrak{a}$ be the ideal in A of morphisms factoring through a projective object. Let $h : M \to Q$ be a morphism of right A-modules with Q being a projective object, such that* $\mathrm{Coker}\, h \cong \mathscr{C}(x, y)$ *for some $y \in \mathscr{C}$. Then* $M\mathfrak{a} \cap \mathrm{Ker}\, h = (\mathrm{Ker}\, h)\mathfrak{a}$.

Proof. Using the Tor_2 vanishing of Lemma 2.11, this can be proved by the same argument as [Aug20b, lem. 3.7]. □

Definition 2.13. Let $x, y \in \mathscr{C}$ be maximal rigid such that $\Sigma^2 x \cong x$, $\Sigma^2 y \cong y$ and assume that x, y satisfy Condition 2.7 with projective objects $p_1^x, p_2^x \in \mathscr{E}$ and $p_1^y, p_2^y \in \mathscr{E}$ respectively, then we say that (x, y) is a *compatible pair* if

1. $p_i^x, p_i^y \in \operatorname{add}_{\mathscr{E}}(x) \cap \operatorname{add}_{\mathscr{E}}(y)$.
2. There are conflations

$$0 \longrightarrow x_1 \longrightarrow x_0 \longrightarrow y \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow x \longrightarrow y_0 \longrightarrow y_1 \longrightarrow 0,$$

   where $x_i \in \operatorname{add}_{\mathscr{E}}(x)$ and $y_i \in \operatorname{add}_{\mathscr{E}}(y)$.

Remark 2.14. Given $x, y \in \mathscr{C}$ such that (x, y) is a compatible pair, notice that this does not necessarily mean that (y, x) is a compatible pair.

Lemma 2.15. *Let $x', y' \in \mathscr{C}$ be maximal rigid objects such that $\Sigma^2 x' \cong x'$ and $\Sigma^2 y' \cong y'$. Then there exist objects $x, y \in \mathscr{C}$ with $x \cong x'$ and $y \cong y'$ such that (x, y) is a compatible pair.*

Proof. By Lemma 2.10 there exist projective objects $P_{x'}, P_{y'} \in \mathscr{E}$ such that $x' \oplus P_{x'}$ and $y' \oplus P_{y'}$ satisfy Condition 2.7 with projective objects in $\operatorname{add}_{\mathscr{E}}(P_{x'})$ and $\operatorname{add}_{\mathscr{E}}(P_{y'})$ respectively. Let $\tilde{x} = x' \oplus P_{x'} \oplus P_{y'}$ and $\tilde{y} = y' \oplus P_{x'} \oplus P_{y'}$. By Lemma 2.9 $\tilde{x}$ and $\tilde{y}$ satisfy Condition 2.7 with projective objects in $\operatorname{add}_{\mathscr{E}}(P_{x'} \oplus P_{y'})$, and thereby satisfy condition (1) of Definition 2.13.

By Theorem 2.3 there exist triangles

$$\tilde{x}_1 \longrightarrow \tilde{x}_0 \longrightarrow \tilde{y} \longrightarrow \Sigma\tilde{x}_1$$

$$\tilde{y}_0 \longrightarrow \tilde{y}_1 \longrightarrow \Sigma\tilde{x} \longrightarrow \Sigma\tilde{y}_0,$$

with $\tilde{x}_i \in \operatorname{add}_{\mathscr{C}}(\tilde{x})$ and $\tilde{y}_i \in \operatorname{add}_{\mathscr{C}}(\tilde{y})$. By rotating the latter triangle we get that there also is a triangle

$$\tilde{x} \longrightarrow \tilde{y}_0 \longrightarrow \tilde{y}_1 \longrightarrow \Sigma\tilde{x}.$$

By Lemma 2.4(b) there exist conflations

$$0 \longrightarrow \tilde{x}_1' \longrightarrow \tilde{x}_0' \longrightarrow \tilde{y} \longrightarrow 0$$

$$0 \longrightarrow \tilde{x} \longrightarrow \tilde{y}_0' \longrightarrow \tilde{y}_1' \longrightarrow 0,$$

with $\tilde{x}_i', \tilde{y}_i' \in \mathscr{E}$ such that $\tilde{x}_i \cong \tilde{x}_i'$ and $\tilde{y}_i \cong \tilde{y}_i'$ in $\mathscr{C}$. Thus there exist projective objects $p_i^{\tilde{x}}, p_i^{\tilde{x}'}, p_i^{\tilde{y}}, p_i^{\tilde{y}'} \in \mathscr{E}$ such that

$$\tilde{x}_i \oplus p_i^{\tilde{x}} \cong \tilde{x}_i' \oplus p_i^{\tilde{x}'} \quad \text{and} \quad \tilde{y}_i \oplus p_i^{\tilde{y}} \cong \tilde{y}_i' \oplus p_i^{\tilde{y}'} \tag{2.5}$$

in $\mathscr{E}$, see Lemma 2.4(c).

Since $\tilde{x}_i \in \operatorname{add}_{\mathscr{C}}(\tilde{x})$, there exist $n \in \mathbb{N}$ and $t \in \mathscr{C}$ such that $\tilde{x}_i \oplus t \cong \tilde{x}^n$. Therefore, by Lemma 2.4(c) there exist projective objects $q_i, r_i \in \mathscr{E}$ such that $\tilde{x}_i \oplus t \oplus q_i \cong \tilde{x}^n \oplus r_i$ in $\mathscr{E}$. This means that $\tilde{x}_i \in \operatorname{add}_{\mathscr{E}}(\tilde{x} \oplus r_i)$, and by (2.5) $\tilde{x}_i' \in \operatorname{add}_{\mathscr{E}}(\tilde{x} \oplus r_i \oplus p_i^{\tilde{x}})$. Let $Q_i^x = r_i \oplus p_i^{\tilde{x}}$, and similarly we can find projective objects $Q_i^y \in \mathscr{E}$ such that $\tilde{y}_i' \in \operatorname{add}_{\mathscr{E}}(\tilde{y} \oplus Q_i^y)$.

Let $P = Q_0^x \oplus Q_1^x \oplus Q_0^y \oplus Q_1^y$, $x = \tilde{x} \oplus P$, and $y = \tilde{y} \oplus P$. By Lemma 2.9 x, y satisfy Condition 2.7 with projective objects in $\operatorname{add}_{\mathscr{E}}(P_{x'} \oplus P_{y'} \oplus P)$, and therefore satisfy condition (1) of Definition 2.13. By adding 'trivial' conflations to the ones above we get following conflations:

$$0 \longrightarrow \tilde{x}'_1 \longrightarrow \tilde{x}'_0 \oplus P \longrightarrow y \longrightarrow 0,$$

$$0 \longrightarrow x \longrightarrow \tilde{y}'_0 \oplus P \longrightarrow \tilde{y}'_1 \longrightarrow 0.$$

Letting $x_1 = \tilde{x}'_1$, $x_0 = \tilde{x}'_0 \oplus P$, $y_1 = \tilde{y}'_1$, $y_0 = \tilde{y}'_0 \oplus P$, we get that (x, y) is a compatible pair. □

3 Derived equivalences

In this section, Setup 2.1 together with the following setup will be assumed.

Setup 3.1. Let $l, m \in \mathscr{C}$ be maximal rigid objects, such that $\Sigma^2 l \cong l$, and $\Sigma^2 m \cong m$ in $\mathscr{C}$. Without loss of generality, by Lemma 2.15, we may assume that (l, m) is a compatible pair. Let $A = \mathscr{E}(l, l)$, $\underline{A} = \mathscr{C}(l, l)$, $B = \mathscr{E}(m, m)$, $\underline{B} = \mathscr{C}(m, m)$ and ${}_B T_A = \mathscr{E}(l, m)$. The following construction of a two-sided tilting complex is inspired by [Miz19, p. 5123] and [Aug20b, thm. 1.1].

$${}_{\underline{B}}\mathscr{T}_{\underline{A}} = \left(\underline{B} \overset{\mathrm{L}}{\underset{B}{\otimes}} T \overset{\mathrm{L}}{\underset{A}{\otimes}} \underline{A} \right)_{\subseteq 1},$$

where $\subseteq 1$ refers to taking a soft truncation, keeping the homological degrees ≤ 1.

The main goal of this section is to show that ${}_{\underline{B}}\mathscr{T}_{\underline{A}}$ is a 2-sided tilting complex, making $\underline{A}$ and $\underline{B}$ derived equivalent (Corollary 3.11).

Lemma 3.2 ([JY19, prop 5.1])**.** *${}_B T_A$ is a two-sided tilting complex viewed as a $B \otimes A^{\mathrm{op}}$-complex.*

Proof. Given that (l, m) is a compatible pair, the proof is similar to that of [JY19, prop 5.1]. □

Since (l, m) is a compatible pair there are conflations in $\mathscr{E}$:

$$\begin{array}{c} 0 \longrightarrow \Omega m \xrightarrow{g} p_1 \xrightarrow{f} m \longrightarrow 0, \\ 0 \longrightarrow m \xrightarrow{g'} p_2 \xrightarrow{f'} \Omega m \longrightarrow 0 \end{array} \tag{3.1}$$

where $p_i \in \operatorname{add}_{\mathscr{E}}(m) \cap \operatorname{add}_{\mathscr{E}}(l)$ are projective objects, giving a projective resolution

$$Q_B : \quad 0 \longrightarrow \mathscr{E}(m, m) \longrightarrow \mathscr{E}(m, p_2) \longrightarrow \mathscr{E}(m, p_1) \longrightarrow \mathscr{E}(m, m) \tag{3.2}$$

of $\underline{B}_B$ (see Lemma 2.8).

Lemma 3.3. *In $\mathcal{D}(A^{\mathrm{op}})$ the object $\underline{B} \otimes_B^{\mathrm{L}} T_A$ is quasi-isomorphic to the complex*

$$0 \longrightarrow \mathscr{E}(l, m) \xrightarrow{g'_*} \mathscr{E}(l, p_2) \xrightarrow{(gf')_*} \mathscr{E}(l, p_1) \xrightarrow{f_*} \mathscr{E}(l, m), \tag{3.3}$$

with homology

$$H_i(\underline{B} \overset{\mathrm{L}}{\underset{B}{\otimes}} T_A) = \begin{cases} \mathscr{C}(l,m) & i=0, \\ \mathscr{C}(l,\Omega m) & i=1, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Using the projective resolution Q_B of $\underline{B}_B$ from (3.2), calculate

$$\begin{aligned} \underline{B} \overset{\mathrm{L}}{\underset{B}{\otimes}} T_A &\cong Q \underset{B}{\otimes} T_A \\ &\cong\ 0 \to \mathscr{E}(m,m) \underset{B}{\otimes} T_A \to \mathscr{E}(m,p_2) \underset{B}{\otimes} T_A \to \mathscr{E}(m,p_1) \underset{B}{\otimes} T_A \to \mathscr{E}(m,m) \underset{B}{\otimes} T_A \\ &\cong\ 0 \to \mathscr{E}(l,m) \to \mathscr{E}(l,p_2) \to \mathscr{E}(l,p_1) \to \mathscr{E}(l,m), \end{aligned}$$

where the last isomorphism follows from Lemma 2.4(d).

This complex can also be seen as the result of applying the functor $\mathscr{E}(l,-)$ to the concatenation of the conflations from (3.1):

$$0 \longrightarrow \mathscr{E}(l,m) \xrightarrow{g'_*} \mathscr{E}(l,p_2) \xrightarrow{(gf')_*} \mathscr{E}(l,p_1) \xrightarrow{f_*} \mathscr{E}(l,m).$$

$$\mathscr{E}(l,p_2) \xrightarrow{f'_*} \mathscr{E}(l,\Omega m) \overset{g_*}{\rightarrowtail} \mathscr{E}(l,p_1)$$

Thus $H_i(\underline{B} \otimes^{\mathrm{L}}_B T_A) = 0$ for $i \neq 0,1$. Since $\mathrm{Ext}^1_{\mathscr{E}}(l,p_1) \cong 0$, it follows from Lemma 2.6 together with the long exact Ext sequence of the conflation

$$0 \longrightarrow \Omega m \longrightarrow p_1 \longrightarrow m \longrightarrow 0,$$

that $H_0(\underline{B} \otimes^{\mathrm{L}}_B T_A) \cong \mathscr{C}(l,m)$.

To calculate H_1, notice that $\mathrm{Ker}(f_*) \cong \mathscr{E}(l,\Omega m)$, and $\mathrm{Im}((gf')_*) \cong \mathrm{Im}(f'_*)$, due to g_* being injective. Therefore $H_1(\underline{B} \otimes^{\mathrm{L}}_B T_A) \cong \mathscr{E}(l,\Omega m)/\,\mathrm{Im}(f'_*)$. Using that $\mathrm{Ext}^1_{\mathscr{E}}(l,p_2) \cong 0$, the long exact Ext sequence of the conflation

$$0 \longrightarrow m \longrightarrow p_2 \longrightarrow \Omega m \longrightarrow 0$$

gives that $H_1(\underline{B} \otimes^{\mathrm{L}}_B T_A) \cong \mathrm{Ext}^1(l,m) \cong \mathscr{C}(l,\Omega^{-1}m)$, with the last isomorphism coming from Lemma 2.6. However, $\Sigma^2 m \cong m$ means $\Omega^{-2}m \cong m$ so $\Omega^{-1}m \cong \Omega m$. □

Corollary 3.4. *There is a quasi-isomorphism in* $\mathcal{D}(A^{\mathrm{op}})$ *from* $\underline{B} \otimes^{\mathrm{L}}_B T_A$ *to the complex*

$$\mathscr{E}(l,p_1)/\,\mathrm{Im}((gf')_*) \xrightarrow{f_*} \mathscr{E}(l,m). \tag{3.4}$$

Proof. This follows directly from Lemma 3.3 by using soft truncation. □

The next goal is to obtain an alternative description of $\underline{B} \otimes^{\mathrm{L}}_B T_A$. Since l,m are maximal rigid and (l,m) is a compatible pair, there exists a conflation

$$0 \longrightarrow l_1 \xrightarrow{\phi_1} l_0 \xrightarrow{\phi_0} m \longrightarrow 0,$$

where $l_i \in \operatorname{add}_{\mathscr{E}}(l)$. For the rest of this section denote by $P_{\underline{A}}$ the complex of $\underline{A}^{\mathrm{op}}$-modules

$$P_{\underline{A}} : \ \mathscr{C}(l, l_1) \xrightarrow{\phi_{1*}} \mathscr{C}(l, l_0),$$

concentrated in degrees 0,1. The claim is that $\underline{B} \otimes^{\mathrm{L}}_{B} T_A \cong P_A$ in $\mathcal{D}(A^{\mathrm{op}})$. To show this is the case, we will find a complex of projective objects which is quasi-isomorphic to the complex from Corollary 3.4, and show that it is also quasi-isomorphic to P_A.

Firstly, notice that since p_1 is projective there is a push-out diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega m & \xrightarrow{g} & p_1 & \xrightarrow{f} & m & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma_1} & \lrcorner & \downarrow{\scriptstyle \gamma_0} & & \| & & \\
0 & \longrightarrow & l_1 & \xrightarrow{\phi_1} & l_0 & \xrightarrow{\phi_0} & m & \longrightarrow & 0,
\end{array}$$

which gives a conflation (see [Büh10, lem. 2.12])

$$0 \longrightarrow \Omega m \xrightarrow{\left(\begin{smallmatrix} g \\ \gamma_1 \end{smallmatrix}\right)} p_1 \oplus l_1 \xrightarrow{\left(\begin{smallmatrix} -\gamma_0 & \phi_1 \end{smallmatrix}\right)} l_0 \longrightarrow 0. \tag{3.5}$$

Lemma 3.5. *There is a complex $\mathscr{P}_A$ in $\mathcal{D}(A^{\mathrm{op}})$ of projective objects, given by*

$$\mathscr{E}(l, l_1) \xrightarrow{\phi_{1*}} \mathscr{E}(l, l_0) \xrightarrow{(g'\phi_0)_*} \mathscr{E}(l, p_2) \xrightarrow{\left(\begin{smallmatrix} gf' \\ \gamma_1 f' \end{smallmatrix}\right)_*} \mathscr{E}(l, p_1 \oplus l_1) \xrightarrow{(-\gamma_0 \ \phi_1)_*} \mathscr{E}(l, l_0), \tag{3.6}$$

that is quasi-isomorphic to the complex from (3.4).

Proof. Notice that the complex from (3.4) is isomorphic to $\operatorname{cone}(f_*)$. We will show that $\operatorname{cone}(f_*)$ is quasi-isomorphic to the complex in (3.6).

By Lemma 2.6 there is an exact sequence

$$0 \longrightarrow \mathscr{E}(l, l_1) \xrightarrow{\phi_{1*}} \mathscr{E}(l, l_0) \xrightarrow{\phi_{0*}} \mathscr{E}(l, m) \longrightarrow 0. \tag{3.7}$$

This is an augmented projective resolution of $\mathscr{E}(l, m)$, and we write

$$Q : \ \mathscr{E}(l, l_1) \xrightarrow{\phi_{1*}} \mathscr{E}(l, l_0).$$

By concatenating the sequences from (3.3) and (3.7), followed by a truncation and adding the cokernel, the following exact sequence is obtained:

$$0 \longrightarrow \mathscr{E}(l, l_1) \xrightarrow{\phi_{1*}} \mathscr{E}(l, l_0) \xrightarrow{(g'\phi_0)_*} \mathscr{E}(l, p_2) \xrightarrow{(gf')_*} \mathscr{E}(l, p_1) \xrightarrow{\mathrm{pr}} \mathscr{E}(l, p_1)/\operatorname{Im}(gf')_* \longrightarrow 0.$$

This is an augmented projective resolution of $\mathscr{E}(l, p_1)/\operatorname{Im}(gf')_*$, and we write

$$Q' : \ \mathscr{E}(l, l_1) \xrightarrow{\phi_{1*}} \mathscr{E}(l, l_0) \xrightarrow{(g'\phi_0)_*} \mathscr{E}(l, p_2) \xrightarrow{(gf')_*} \mathscr{E}(l, p_1).$$

Now lift f_* to a morphism of chain complexes $\bar{f}_* : Q' \to Q$. This lift is illustrated in the following diagram.

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \mathscr{E}(l,l_1) & \xrightarrow{\phi_{1*}} & \mathscr{E}(l,l_0) & \xrightarrow{(g'\phi_0)_*} & \mathscr{E}(l,p_2) & \xrightarrow{(gf')_*} & \mathscr{E}(l,p_1) & \xrightarrow{\mathrm{pr}} & \mathscr{E}(l,p_1)/\operatorname{Im}(gf')_* & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle (\gamma_1 f')_*} & & \downarrow{\scriptstyle \gamma_{0*}} & & \downarrow{\scriptstyle f_*} & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathscr{E}(l,l_1) & \xrightarrow{\phi_{1*}} & \mathscr{E}(l,l_0) & \xrightarrow{\phi_{0*}} & \mathscr{E}(l,m) & \longrightarrow & 0,
\end{array}$$

where each row is an augmented projective resolution of A^{op}-modules. Thus there is a quasi-isomorphism

$$\operatorname{cone}(\bar{f}_*) \xrightarrow{\sim} \operatorname{cone}(f_*).$$

Using [Wei94, ex. 1.2.8] $\operatorname{cone}(\bar{f}_*)$ is seen to be the total complex $\operatorname{Tot}^{\oplus}(C)$, of the double complex C induced by $\bar{f}_*$, which exactly is the complex from (3.6). □

For the rest of this section we will use the complex $\mathscr{P}_A$ from the previous lemma.

Lemma 3.6. *The degreewise projection* $\Phi : \mathscr{P}_A \to P_A$

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathscr{E}(l,l_1) & \xrightarrow{\phi_{1*}} & \mathscr{E}(l,l_0) & \xrightarrow{(g'\phi_0)_*} & \mathscr{E}(l,p_2) & \xrightarrow{\binom{gf'}{\gamma_1 f'}_*} & \mathscr{E}(l,p_1\oplus l_1) & \xrightarrow{(-\gamma_0\ \phi_1)_*} & \mathscr{E}(l,l_0) \\
 & & & & & & & & \downarrow{\scriptstyle \mathrm{pr}\circ(0\ \mathrm{id}_{l_1})_*} & & \downarrow{\scriptstyle \mathrm{pr}} \\
 & & & & & & & & \mathscr{C}(l,l_1) & \xrightarrow[\phi_{1*}]{} & \mathscr{C}(l,l_0)
\end{array}$$

is a quasi-isomorphism.

Proof. By Lemma 3.5 the top row $\mathscr{P}_A$ is quasi-isomorphic to $\underline{B} \otimes^{\mathrm{L}}_B T_A$ and hence is exact in all degrees other than 0,1 by Lemma 3.3. Thus to prove the lemma, it is enough to show that $H_i(\Phi)$ is an isomorphism for $i = 0, 1$. First we check that the homology of the two complexes agree, after which it is enough to check that $H_i(\Phi)$ is surjective because the homology spaces are finite dimensional over k.

Since there is a triangle $\Omega m \to l_1 \to l_0 \to m$, and since l is rigid with $\Omega l \cong \Sigma^{-1} l \cong \Sigma l$, there is an exact sequence

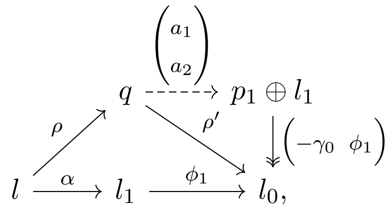
$$0 \longrightarrow \mathscr{C}(l,\Omega m) \longrightarrow \mathscr{C}(l,l_1) \longrightarrow \mathscr{C}(l,l_0) \longrightarrow \mathscr{C}(l,m) \longrightarrow 0.$$

In combination with Corollary 3.4 and Lemmas 3.3 and 3.5 this gives that $H_i(P_A) \cong H_i(\mathscr{P}_A)$, for every $i \in \mathbb{Z}$.

Next we show that $H_i(\Phi)$ is surjective. For $i = 0$ this is straightforward. For $i = 1$ consider the diagram

$$\begin{array}{ccc}
\mathscr{E}(l,p_1\oplus l_1) & \xrightarrow{(-\gamma_0\ \phi_1)_*} & \mathscr{E}(l,l_0) \\
\downarrow{\scriptstyle \mathrm{pr}(0\ \mathrm{id}_{l_1})_*} & & \downarrow{\scriptstyle \mathrm{pr}} \\
\mathscr{C}(l,l_1) & \xrightarrow{\phi_{1*}} & \mathscr{C}(l,l_0).
\end{array}$$

Let $\alpha \in \mathscr{E}(l, l_1)$ such that the projection $\underline{\alpha} \in \mathscr{C}(l, l_1)$ lies in $\operatorname{Ker} \phi_{1*}$. Thus $\underline{\phi_1 \alpha} = 0$, which means that $\phi_1\alpha$ factors through a projective object $q \in \mathscr{E}$, say by $\phi_1\alpha = \rho'\rho$, where $\rho : l \to q$ and $\rho' : q \to l_0$. By using (3.5) there is a commutative diagram



where a_1, a_2 exist by using the fact that q is projective. Now calculate

$$\phi_1\alpha = \begin{pmatrix} -\gamma_0 & \phi_1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \rho = \phi_1 a_2 \rho - \gamma_0 a_1 \rho.$$

Therefore

$$0 = \phi_1(a_2\rho - \alpha) - \gamma_0 a_1 \rho = \begin{pmatrix} -\gamma_0 & \phi_1 \end{pmatrix} \begin{pmatrix} a_1\rho \\ a_2\rho - \alpha \end{pmatrix} = \begin{pmatrix} -\gamma_0 & \phi_1 \end{pmatrix}_* \begin{pmatrix} a_1\rho \\ a_2\rho - \alpha \end{pmatrix}.$$

This means that $\begin{pmatrix} a_1\rho \\ a_2\rho - \alpha \end{pmatrix} \in \operatorname{Ker} \begin{pmatrix} -\gamma_0 & \phi_1 \end{pmatrix}_*$, making $\begin{pmatrix} a_1\rho \\ a_2\rho - \alpha \end{pmatrix} + \operatorname{Im} \begin{pmatrix} gf' \\ \gamma_1 f' \end{pmatrix}_*$ an element of $H_1(\mathscr{P}_A)$. Now

$$H_1(\Phi)\left(- \begin{pmatrix} a_1\rho \\ a_2\rho - \alpha \end{pmatrix} + \operatorname{Im} \begin{pmatrix} gf' \\ \gamma_1 f' \end{pmatrix}_* \right) = \underline{\alpha} - \underline{a_2\rho} = \underline{\alpha},$$

which means that $H_1(\Phi)$ is surjective, and therefore Φ is surjective on homology. □

Corollary 3.7. *There are two isomorphisms*

(a) $\mathscr{T}_{\underline{A}} = \left(\underline{B} \otimes^{\mathrm{L}}_{B} T \otimes^{\mathrm{L}}_{A} \underline{A}\right)_{\subseteq 1} \cong P_{\underline{A}}$ *in* $\mathcal{D}(\underline{A}^{\mathrm{op}})$.

(b) $\mathscr{T} \otimes^{\mathrm{L}}_{\underline{A}} \underline{A}_A \cong \underline{B} \otimes^{\mathrm{L}}_{B} T_A$ *in* $\mathcal{D}(A^{\mathrm{op}})$.

Proof. (a) Corollary 3.4 and Lemma 3.5 imply $\mathscr{P}_A \cong \underline{B} \otimes^{\mathrm{L}}_{B} T_A$. Lemma 2.4(e) implies that $\mathscr{P} \otimes_A \underline{A} \cong P_{\underline{A}} \oplus P_{\underline{A}}[3]$. Thus

$$\underline{B} \overset{\mathrm{L}}{\underset{B}{\otimes}} T \overset{\mathrm{L}}{\underset{A}{\otimes}} \underline{A} \cong \mathscr{P} \underset{A}{\otimes} \underline{A} \cong P_{\underline{A}} \oplus P_{\underline{A}}[3],$$

giving that

$$\mathscr{T}_{\underline{A}} \cong (P_{\underline{A}} \oplus P_{\underline{A}}[3])_{\subseteq 1} \cong P_{\underline{A}}.$$

(b) It follows from (a) combined with Corollary 3.4 and Lemmas 3.5 and 3.6 that

$$\mathscr{T} \overset{\mathrm{L}}{\underset{\underline{A}}{\otimes}} \underline{A}_A \cong P \overset{\mathrm{L}}{\underset{\underline{A}}{\otimes}} \underline{A}_A \cong P_A \cong \mathscr{P}_A \cong \underline{B} \overset{\mathrm{L}}{\underset{B}{\otimes}} T_A.$$ □

We now have a one-sided isomorphism $\mathscr{T} \otimes^{\mathrm{L}}_{\underline{A}} \underline{A}_A \cong \underline{B} \otimes^{\mathrm{L}}_B T_A$ in $\mathcal{D}(A^{\mathrm{op}})$. Next we see that this isomorphism can be lifted to a two-sided isomorphism, i.e. an isomorphism in $\mathcal{D}(\underline{B} \otimes A^{\mathrm{op}})$.

Theorem 3.8. *In the derived category $\mathcal{D}(\underline{B} \otimes A^{\mathrm{op}})$ (resp. $\mathcal{D}(B \otimes \underline{A}^{\mathrm{op}})$), there is an isomorphism*

$$ {}_{\underline{B}}\mathscr{T} \overset{\mathrm{L}}{\underset{\underline{A}}{\otimes}} \underline{A}_A \cong \underline{B} \overset{\mathrm{L}}{\underset{B}{\otimes}} T_A \qquad \left(\textit{resp. } {}_B\underline{B} \overset{\mathrm{L}}{\underset{\underline{B}}{\otimes}} \mathscr{T}_{\underline{A}} \cong {}_B T \overset{\mathrm{L}}{\underset{A}{\otimes}} \underline{A} \right). $$

Proof. We will prove the first isomorphism, the second one can be done in a symmetric fashion. Let ${}_{\underline{B}}Q_A \xrightarrow{\sim} \underline{B} \otimes^{\mathrm{L}}_B T_A$ be a projective resolution over $\underline{B} \otimes A^{\mathrm{op}}$. Denote by $\alpha : {}_{\underline{B}}Q_A \to {}_{\underline{B}}Q \otimes_A \underline{A}_A$ the morphism defined by $q \mapsto q \otimes 1$. Consider the composition of morphisms

$$ \mathrm{pr} \circ \alpha : {}_{\underline{B}}Q_A \xrightarrow{\alpha} {}_{\underline{B}}Q \underset{A}{\otimes} \underline{A}_A \xrightarrow{\mathrm{pr}} ({}_{\underline{B}}Q \underset{A}{\otimes} \underline{A}_A)_{\subseteq 1}. $$

By construction this is isomorphic to

$$ \underline{B} \overset{\mathrm{L}}{\underset{B}{\otimes}} T_A \longrightarrow \underline{B} \overset{\mathrm{L}}{\underset{B}{\otimes}} T \overset{\mathrm{L}}{\underset{A}{\otimes}} \underline{A}_A \longrightarrow \left(\underline{B} \overset{\mathrm{L}}{\underset{B}{\otimes}} T \overset{\mathrm{L}}{\underset{A}{\otimes}} \underline{A}_A \right)_{\subseteq 1} \cong {}_{\underline{B}}\mathscr{T} \overset{\mathrm{L}}{\underset{\underline{A}}{\otimes}} \underline{A}_A. $$

Writing out the composition gives the following, which we will show to be a quasi-isomorphism:

$$ \begin{array}{ccccccccc} \cdots \longrightarrow & Q_2 & \xrightarrow{d_2} & Q_1 & \xrightarrow{d_1} & Q_0 & \longrightarrow & 0 & \longrightarrow \cdots \\ & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow & \\ \cdots \longrightarrow & Q_2 \otimes_A \underline{A} & \xrightarrow{d_2 \otimes \mathrm{id}} & Q_1 \otimes_A \underline{A} & \xrightarrow{d_1 \otimes \mathrm{id}} & Q_0 \otimes_A \underline{A} & \longrightarrow & 0 & \longrightarrow \cdots \\ & \downarrow & & \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow & \\ \cdots \longrightarrow & 0 & \longrightarrow & Q_1 \otimes_A \underline{A} / \operatorname{Im}(d_2 \otimes \mathrm{id}) & \xrightarrow{d_1 \otimes \mathrm{id}} & Q_0 \otimes_A \underline{A} & \longrightarrow & 0 & \longrightarrow \cdots . \end{array} $$

The projection $\mathrm{pr} : {}_{\underline{B}}Q \otimes_A \underline{A}_A \to ({}_{\underline{B}}Q \otimes_A \underline{A}_A)_{\subseteq 1}$ of soft truncation is an isomorphism on homology in degrees ≤ 1. By Lemma 3.3 we have $H_i({}_{\underline{B}}Q_A) = 0$ for $i \neq 0, 1$. Therefore it is enough to check that α is an isomorphism on homology in degrees 0 and 1. Moreover, since we are dealing with finite dimensional vector spaces, Corollary 3.7(b) gives that for $i = 0, 1$ the dimensions of $H_i(\underline{B} \otimes^{\mathrm{L}}_B T_A) \cong H_i({}_{\underline{B}}Q_A)$ and $H_i({}_{\underline{B}}\mathscr{T} \otimes^{\mathrm{L}}_{\underline{A}} \underline{A}_A) \cong H_i({}_{\underline{B}}Q \otimes_A \underline{A}_A)$ are the same, making it enough to check that $H_i(\alpha)$ is injective, for $i = 0, 1$.

Let $i \in \{0, 1\}$, and let $h \in \operatorname{Ker}(d_i)$ be such that $h + \operatorname{Im}(d_{i+1}) \in \operatorname{Ker}(H_i(\alpha))$. Then there exist morphisms $h_j \in Q_{i+1}$ and $a_j \in A$ such that

$$ \begin{aligned} \operatorname{Im}(d_{i+1} \underset{A}{\otimes} \mathrm{id}) \ni \alpha_i(h) = h \underset{A}{\otimes} 1 & \\ &= (d_{i+1} \underset{A}{\otimes} \mathrm{id}) \Big(\sum_j h_j \underset{A}{\otimes} \underline{a_j} \Big) \\ &= \sum_j (d_{i+1}(h_j) \underset{A}{\otimes} \underline{a_j}) \\ &= \sum_j (d_{i+1}(h_j a_j) \underset{A}{\otimes} 1) \\ &= \Big(\sum_j d_{i+1}(h_j a_j) \Big) \underset{A}{\otimes} 1. \end{aligned} $$

Thus $(h - \sum_j d_{i+1}(h_j a_j)) \otimes_A 1 = 0$. Letting $\mathfrak{a}$ be the ideal of morphisms in A factoring through a projective object gives that $h - \sum_j d_{i+1}(h_j a_j) \in Q_i\mathfrak{a} \cap \operatorname{Ker} d_i$. However, this intersection equals $(\operatorname{Ker} d_i)\mathfrak{a}$, by Lemma 2.12. Lemma 2.12 applies for $i = 0$ because the cokernel of $Q_0 \to 0$ is 0 which has the form $\mathscr{C}(l, y)$ for $y = 0$. Lemma 2.12 applies for $i = 1$ because the cokernel of $Q_1 \to Q_0$ is $H_0(Q_A) \cong H_0(\underline{B} \otimes^{\mathrm{L}}_B T_A)$ which has the form $\mathscr{C}(l, m)$ by Lemma 3.3. Since $h - \sum_j d_{i+1}(h_j a_j) \in (\operatorname{Ker} d_i)\mathfrak{a}$ we have

$$h - \sum_j d_{i+1}(h_j a_j) = \sum_j \tilde{h}_j \tilde{a}_j,$$

for some $\tilde{h}_j \in \operatorname{Ker} d_i$ and $\tilde{a}_j \in \mathfrak{a}$. By Lemma 3.3 there are isomorphisms over A^{op}:

$$\xi_i : H_i(Q_A) \longrightarrow \begin{cases} \mathscr{C}(l, m) & i = 0, \\ \mathscr{C}(l, \Omega m) & i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\mathfrak{a}$ annihilates $\operatorname{Im}(\xi_i)$, meaning that

$$\xi_i\Big(\sum_j \tilde{h}_j \tilde{a}_j + \operatorname{Im}(d_{i+1})\Big) = \sum_j \xi_i(\tilde{h}_j + \operatorname{Im}(d_{i+1}))\tilde{a}_j = 0.$$

Since ξ_i is injective $h - \sum_j d_{i+1}(h_j a_j) = \sum_j \tilde{h}_j \tilde{a}_j \in \operatorname{Im}(d_{i+1})$. Thus $h \in \operatorname{Im} d_{i+1}$. □

Theorem 3.9. *The canonical morphisms*

$$\underline{B} \longrightarrow \operatorname{End}_{\mathcal{D}(\underline{A}^{\mathrm{op}})}({}_{\underline{B}}\mathscr{T}_{\underline{A}}) \qquad \textit{and} \qquad \underline{A} \longrightarrow \operatorname{End}_{\mathcal{D}(\underline{B})}({}_{\underline{B}}\mathscr{T}_{\underline{A}}),$$

induced by $- \otimes^{\mathrm{L}}_{\underline{B}} \mathscr{T}_{\underline{A}}$ *and* ${}_{\underline{B}}\mathscr{T} \otimes^{\mathrm{L}}_{\underline{A}} -$ *are isomorphisms.*

Proof. We will show the first isomorphism. The second one is done similarly in a symmetric fashion. From the isomorphism in Theorem 3.8 the following commutative diagram is produced:

$$\begin{array}{ccc} \operatorname{End}_{\mathcal{D}(\underline{B}^{\mathrm{op}})}(\underline{B}_{\underline{B}}) & \xrightarrow{-\otimes^{\mathrm{L}}_{\underline{B}} \underline{B}_B} & \operatorname{End}_{\mathcal{D}(B^{\mathrm{op}})}(\underline{B}_B) \\ {\scriptstyle -\otimes^{\mathrm{L}}_{\underline{B}} \mathscr{T}_{\underline{A}}}\downarrow & & \downarrow{\scriptstyle -\otimes^{\mathrm{L}}_B T_A} \\ \operatorname{End}_{\mathcal{D}(\underline{A}^{\mathrm{op}})}(\mathscr{T}_{\underline{A}}) & \xrightarrow{-\otimes^{\mathrm{L}}_{\underline{A}} \underline{A}_A} & \operatorname{End}_{\mathcal{D}(A^{\mathrm{op}})}(\mathscr{T}_A). \end{array}$$

The morphism induced by $- \otimes^{\mathrm{L}}_{\underline{B}} \underline{B}_B$ is an isomorphism, and since ${}_BT_A$ is a tilting complex (by Lemma 3.2), the morphism induced by $- \otimes^{\mathrm{L}}_B T_A$ is also an isomorphism. Hence the map induced by $- \otimes^{\mathrm{L}}_{\underline{A}} \underline{A}_A$ is surjective. Thus to show $- \otimes^{\mathrm{L}}_{\underline{B}} \mathscr{T}_{\underline{A}}$ is an isomorphism, it is enough to show that

$$\operatorname{End}_{\mathcal{D}\underline{A}^{\mathrm{op}}}(\mathscr{T}_{\underline{A}}) \cong \operatorname{End}_{\mathcal{D}A^{\mathrm{op}}}(\mathscr{T}_A),$$

making $-\otimes^{\mathrm{L}}_{\underline{A}} \underline{A}_A$ an isomorphism. This is seen using the following calculation:

$$\begin{aligned}
\mathrm{RHom}_{A^{\mathrm{op}}}(\mathscr{T}_A, \mathscr{T}_A) &\cong \mathrm{RHom}_{A^{\mathrm{op}}}\left(\mathscr{T} \overset{\mathrm{L}}{\underset{\underline{A}}{\otimes}} \underline{A}_A, \mathrm{RHom}_{\underline{A}^{\mathrm{op}}}({}_A\underline{A}, \mathscr{T}_{\underline{A}})\right) \\
&\cong \mathrm{RHom}_{\underline{A}^{\mathrm{op}}}\left((\mathscr{T} \overset{\mathrm{L}}{\underset{\underline{A}}{\otimes}} \underline{A}) \overset{\mathrm{L}}{\underset{A}{\otimes}} \underline{A}, \mathscr{T}_{\underline{A}}\right) \\
&\cong \mathrm{RHom}_{\underline{A}^{\mathrm{op}}}\left(\underline{B} \overset{\mathrm{L}}{\underset{B}{\otimes}} T \overset{\mathrm{L}}{\underset{A}{\otimes}} \underline{A}, \mathscr{T}_{\underline{A}}\right) && \text{(by Corollary 3.7(b))} \\
&\cong \mathrm{RHom}_{\underline{A}^{\mathrm{op}}}\left(\mathscr{P} \underset{A}{\otimes} \underline{A}, \mathscr{T}_{\underline{A}}\right) && \text{(by Corollary 3.4 and Lemma 3.5)} \\
&\cong \mathrm{RHom}_{\underline{A}^{\mathrm{op}}}\left(P_{\underline{A}} \oplus P_{\underline{A}}[3], \mathscr{T}_{\underline{A}}\right). && \text{(by Lemma 2.4(e))}
\end{aligned}$$

By taking the 0'th homology one obtains that

$$\begin{aligned}
\mathrm{End}_{\mathcal{D}(A^{\mathrm{op}})}(\mathscr{T}_A) &\cong \mathrm{Hom}_{\mathcal{D}(\underline{A}^{\mathrm{op}})}\left(P_{\underline{A}} \oplus P_{\underline{A}}[3], \mathscr{T}_{\underline{A}}\right) \\
&\cong \mathrm{Hom}_{\mathcal{D}(\underline{A}^{\mathrm{op}})}\left(P_{\underline{A}}, \mathscr{T}_{\underline{A}}\right) \\
&\cong \mathrm{End}_{\mathcal{D}(\underline{A}^{\mathrm{op}})}\left(\mathscr{T}_{\underline{A}}\right),
\end{aligned}$$

where the second isomorphism comes from $\mathscr{T}_{\underline{A}}$ being isomorphic in $\mathcal{D}(\underline{A}^{\mathrm{op}})$ to a two term complex of projective objects, see Corollary 3.7(a), and the last isomorphism is also by Corollary 3.7(a). $\square$

Lemma 3.10. *The complex* $P_{\underline{A}}$*,*

$$\mathscr{C}(l, l_1) \xrightarrow{\phi_{1*}} \mathscr{C}(l, l_0),$$

is a two-term tilting complex in $\mathcal{D}(\underline{A}^{\mathrm{op}})$*.*

Proof. $P_{\underline{A}}$ is a silting complex by [Aug20a, thm 2.18 and rmk 2.19(2)]. Thus to show that $P_{\underline{A}}$ is a tilting complex it is enough to show that $\mathrm{Hom}_{\mathcal{K}^b(\mathrm{proj}\,\underline{A}^{\mathrm{op}})}(P_{\underline{A}}, P_{\underline{A}}[-1]) = 0$.

Recall that $P_{\underline{A}}$ is defined using the triangle

$$l_1 \xrightarrow{\phi_1} l_0 \xrightarrow{\phi_0} m \xrightarrow{\Sigma\phi_2} \Sigma l_1.$$

By applying $\mathscr{C}(l, -)$ to this triangle we obtain an exact sequence

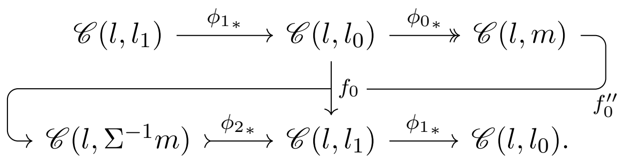
$$0 \longrightarrow \mathscr{C}(l, \Sigma^{-1}m) \xrightarrow{\phi_{2*}} \mathscr{C}(l, l_1) \xrightarrow{\phi_{1*}} \mathscr{C}(l, l_0) \xrightarrow{\phi_{0*}} \mathscr{C}(l, m) \longrightarrow 0.$$

This shows that $\mathrm{Ker}(\phi_{1*}) = \mathscr{C}(l, \Sigma^{-1}m)$, and $\mathrm{Coker}(\phi_{1*}) = \mathscr{C}(l, m)$. Let a chain map $f \in \mathrm{Hom}_{\mathcal{K}^b(\mathrm{proj}\,\underline{A}^{\mathrm{op}})}(P_{\underline{A}}, P_{\underline{A}}[-1])$ be given:

$$\begin{array}{ccccc}
\mathscr{C}(l, l_1) & \xrightarrow{\phi_{1*}} & \mathscr{C}(l, l_0) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle f_0} & & \downarrow \\
0 & \longrightarrow & \mathscr{C}(l, l_1) & \xrightarrow{\phi_{1*}} & \mathscr{C}(l, l_0).
\end{array}$$

f being a chain map implies that $f_0\phi_{1*} = 0$. Thus f_0 factors through $\mathrm{Coker}(\phi_{1*})$, say $f_0 = f_0'\phi_{0*}$. If we use that f is a chain map again we get that $\phi_{1*}f_0'{=}0$, giving that

f_0' factors through $\operatorname{Ker}(\phi_{1*})$, say $f_0' = \phi_{2*}f_0''$, with $f_0'' \in \operatorname{Hom}_{\underline{A}}(\mathscr{C}(l,m), \mathscr{C}(l,\Sigma^{-1}m))$. Therefore $f_0 = \phi_{2*}f_0''\phi_{0*}$:



Since m is rigid and $\Sigma^2 m \cong m$ we have that $\mathscr{C}(m, \Sigma^{-1}m) = 0$. Thus Lemma 2.4(g) gives that $f_0'' = 0$, and thereby $f_0 = 0$. Hence $f = 0$, making $P_{\underline{A}}$ a tilting complex. □

Corollary 3.11. ${}_{\underline{B}}\mathscr{T}_{\underline{A}}$ *is a two-sided tilting complex. In particular*

$$- \overset{\mathrm{L}}{\underset{\underline{B}}{\otimes}} \mathscr{T}_{\underline{A}} : \mathcal{D}(\underline{B}^{\mathrm{op}}) \longrightarrow \mathcal{D}(\underline{A}^{\mathrm{op}})$$

is a triangulated equivalence, making $\underline{A}$ and $\underline{B}$ derived equivalent.

Proof. Recall that $\mathscr{T}_{\underline{A}} \cong P_{\underline{A}}$ in $\mathcal{D}(\underline{A}^{\mathrm{op}})$ (by Corollary 3.7(a)), and that $P_{\underline{A}}$ is a tilting complex (by Lemma 3.10). This establishes the second half of part (1) as well as parts (2) and (3) of Definition 1.2. The first half of part (1) holds by Theorem 3.9. The triangulated equivalence follows from [Kel98, prop. 8.1.4]. □

4 Examples

4.1 Cluster-tilting objects from $\mathcal{C}(D_{2n})$

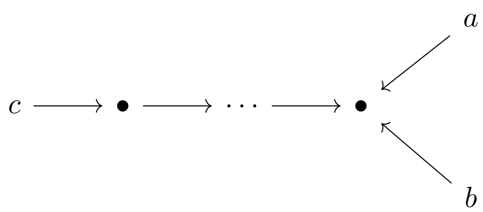
It was shown in [BHL14, lem. 4.5] that the self-injective cluster tilted algebras of the cluster category $\mathscr{C} = \mathcal{C}(D_{2n})$ are derived equivalent. This was done by finding a tilting complex ad hoc. In this example we will see how our results can be used to find such a tilting complex, and thereby recover the tilting complex from [BHL14].

To apply the results from the previous section on this example, we need to ensure that $\mathscr{C}$ has a Frobenius model $\mathscr{E}$ that satisfies Setup 2.1.

Theorem 4.1. *There exists a Frobenius category $\mathscr{E}$, such that $\mathscr{C} = \underline{\mathscr{E}}$. Furthermore the pair of $\mathscr{E}$ and $\mathscr{C}$ satisfy Setup 2.1.*

Proof. Let I be the direct sum of all injective indecomposable objects in $\operatorname{mod}(kD_{2n})$, and $M = I \oplus \tau I$. This M has the needed properties to apply [GLS07, thm. 2.1], giving a Frobenius category $\mathscr{E}$ such that $\mathscr{C} = \underline{\mathscr{E}}$, which also satisfies Setup 2.1. □

Let $n \in \mathbb{N}$, with $n \geq 4$. Consider the quiver D_{2n}.

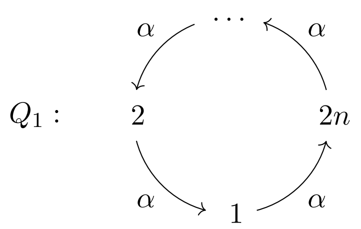


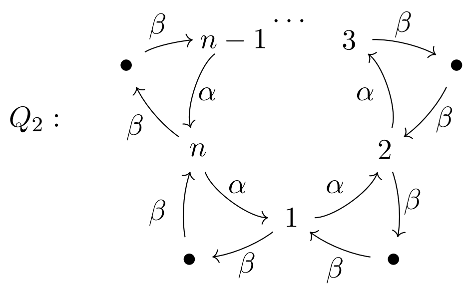
There are indecomposable projective representations $P(a), P(b), P(c)$ corresponding to the vertices a, b, c, and these can be viewed as objects of $\mathscr{C}$. In [Rin08], Ringel described two cluster-tilting objects in $\mathscr{C}$, see Figure 4.1:

$$T_1 = \left(\bigoplus_{i=0}^{n-1} \tau^{-2i} P(a)\right) \oplus \left(\bigoplus_{i=0}^{n-1} \tau^{-2i-1} P(b)\right)$$

$$T_2 = \left(\bigoplus_{i=0}^{n-1} \tau^{-2i} P(a)\right) \oplus \left(\bigoplus_{i=0}^{n-1} \tau^{-2i-1} P(c)\right).$$

Denote their endomorphism algebras by $A_i = \mathrm{End}_{\mathscr{C}}(T_i) = \mathscr{C}(T_i, T_i)$. To describe the endomorphism algebras we define the following quivers:





Then $A_i = kQ_i/I_i$, with $I_1 = \langle \alpha^{2n-1} \rangle$, and $I_2 = \langle \alpha\beta, \beta\alpha, \beta^2 - \alpha^{n-1} \rangle$. Both of these algebras are self-injective by Lemma 2.5, but neither of them is symmetric. Now Corollary 3.11 says that A_1 and A_2 are derived equivalent. Furthermore, there is a direct way to calculate the associated one-sided tilting complexes.

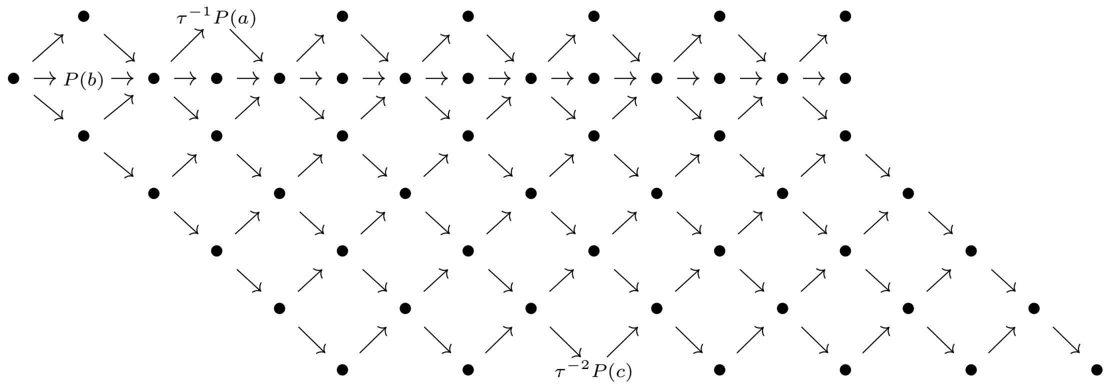


Figure 4.1: Auslander-Reiten quiver of $\mathrm{mod}(kD_8)$.

Using the Auslander-Reiten quiver of $\mathrm{mod}(kD_{2n})$ (see Figure 4.1 for an example in the case $n = 4$), one can use the dimension vectors to see that there is an exact sequence (see [Ass06, prop. IX.3.1, lem. IX.1.1(a)])

$$0 \longrightarrow P(b) \xrightarrow{\phi} \tau^{-1}P(a) \xrightarrow{\psi} \tau^{-2}P(c) \longrightarrow 0$$

inducing a triangle

$$P(b) \xrightarrow{\phi} \tau^{-1}P(a) \xrightarrow{\psi} \tau^{-2}P(c) \longrightarrow \Sigma P(b)$$

in the cluster category $\mathscr{C}$. This implies that for each i there is a triangle

$$\tau^{i}P(b) \xrightarrow{\phi_i} \tau^{i-1}P(a) \xrightarrow{\psi_{i-1}} \tau^{i-2}P(c) \longrightarrow \Sigma\tau^{i}P(b)$$

in $\mathscr{C}$. With this we construct the following triangle. Let $\Phi' = \oplus_{i=0}^{n-1}\phi_{-2i-1}$ and let $\Psi' = \oplus_{i=1}^{n}\psi_{-2i}$, then there exists a triangle

$$\begin{array}{ccccccc}
\left(\bigoplus_{i=0}^{n-1}\tau^{-2i-1}P(b)\right) & \xrightarrow{\begin{pmatrix}0\\ \Phi'\end{pmatrix}} & \begin{array}{c}\left(\bigoplus_{i=1}^{n}\tau^{-2i}P(a)\right)\\ \oplus \\ \left(\bigoplus_{i=1}^{n}\tau^{-2i}P(a)\right)\end{array} & \xrightarrow{\begin{pmatrix}\mathrm{id} & 0\\ 0 & \Psi'\end{pmatrix}} & \begin{array}{c}\left(\bigoplus_{i=1}^{n}\tau^{-2i}P(a)\right)\\ \oplus \\ \left(\bigoplus_{i=1}^{n}\tau^{-2i-1}P(c)\right)\end{array} & \longrightarrow & \\
\| & & \| & & \| & & \\
T_1^2 & \xrightarrow{\Phi} & T_1^1 & \xrightarrow{\Psi} & T_2 & \longrightarrow &
\end{array}$$

with $T_1^j \in \mathrm{add}(T_1)$. Notice that we have used that $\tau^{2n} \cong \mathrm{id}$ when describing T_2. It follows from T_1 being rigid that Ψ is an $\mathrm{add}(T_1)$ pre-cover, namely if there is a morphism $\tilde{\Psi} : S \to T_2$, with $S \in \mathrm{add}(T_1)$ then $\mathscr{C}(S, \Sigma T_1^2) = 0$, and $\tilde{\Psi}$ will therefore factor through Ψ. To see that Ψ is a cover, it is enough to check that $\Phi \in \mathrm{rad}_{\mathscr{C}}$ (see [Fed19, lem 3.12]). All the components ϕ_{-2i-1} of Φ are morphisms between two different indecomposable objects, and are therefore all in the radical. Thus Φ is in the radical.

Define the following complex concentrated in degrees 0,1:

$$P_1 : \ \mathscr{C}(T_1, T_1^2) \xrightarrow{\Phi_*} \mathscr{C}(T_1, T_1^1).$$

This complex is a tilting complex over A_1 by Lemma 3.10. By Theorem 3.9, there is an isomorphism $A_2 \cong \mathrm{End}_{\mathcal{D}(A_1^{\mathrm{op}})}(P_1)$. Similarly a tilting complex P_2 could be found such that $A_1 \cong \mathrm{End}_{\mathcal{D}(A_2^{\mathrm{op}})}(P_2)$.

To verify that this is indeed the case we check the isomorphism $A_2 \cong \mathrm{End}_{\mathcal{D}(A_1^{\mathrm{op}})}(P_1)$. For each $T' \in \mathrm{add}(T_1)$, the right A_1-module $\mathscr{C}(T_1, T')$ is projective. We can therefore identify the Hom-space from T_1 to indecomposable summands of T_1 with indecomposable projective modules over A_1. We will do that as follows:

$$P_{A_1}(2i) \cong \mathscr{C}(T_1, \tau^{-2i}P(a)) \qquad \text{and} \qquad P_{A_1}(2j+1) \cong \mathscr{C}(T_1, \tau^{-(2j+1)}P(b))$$

for $0 < i \leq n$, and $0 \leq j < n$. The morphisms between projective objects can be described as follows:

$$\dim \mathrm{Hom}(P_{A_1}(i), P_{A_1}(j)) = \begin{cases} 0 & \text{if } j = i-1 \\ 0 & \text{if } i = 1,\ j = 2n \\ 1 & \text{otherwise.} \end{cases}$$

Now $\operatorname{End}_{\mathcal{D}(A_1^{\mathrm{op}})}(P_1)$ can be calculated. There are $2n$ indecomposable components in P_1:

$$\begin{array}{lrcl} B_i: & P_{A_1}(2i+1) & \xrightarrow{\phi_{-2i-1*}} & P_{A_1}(2i+2), \\ C_j: & 0 & \longrightarrow & P_{A_1}(2j), \end{array}$$

with $0 \leq i < n$, and $0 < j \leq n$. To describe the morphisms we split it into cases.

(i) $\gamma \in \operatorname{Hom}(C_i, B_{i-1}) \neq 0$, for $0 < i \leq n$. Here γ is the inclusion.

(ii) $\beta \in \operatorname{Hom}(B_i, C_i) \neq 0$, for $0 < i \leq n-1$ and $\beta \in \operatorname{Hom}(B_0, C_n) \neq 0$. Here β is induced by the morphism $P(a) \to \tau^2 P(a)$.

(iii) $\alpha \in \operatorname{Hom}(C_i, C_{i+1}) \neq 0$, for all $0 < i \leq n-1$, and $\alpha \in \operatorname{Hom}(C_n, C_1) \neq 0$. Here α is induced by the morphism $P(a) \to \tau^{-2} P(a)$.

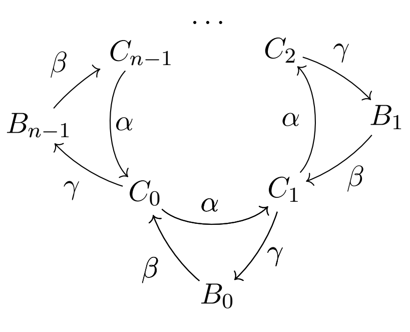
(iv) $\operatorname{Hom}(B_i, B_{i+1}) = 0$, for all $0 \leq i < n-1$, and $\operatorname{Hom}(B_{n-1}, B_0) = 0$. This is due to the 'potential' morphisms being null-homotopic.

(v) $\delta \in \operatorname{Hom}(C_i, C_{i-1}) \neq 0$, for all $1 < i \leq n$, and $\delta \in \operatorname{Hom}(C_1, C_n) \neq 0$. But notice that δ factors through B_{i-1}:

$$\begin{array}{c} C_i \\ \big\downarrow \gamma \\ B_{i-1} \\ \big\downarrow \beta \\ C_{i-1} \end{array} \quad : \quad \begin{array}{ccc} 0 & \longrightarrow & P_{A_1}(2i) \\ \big\downarrow & & \big\downarrow \\ P_{A_1}(2i-1) & \longrightarrow & P_{A_1}(2i) \\ \big\downarrow & & \big\downarrow \\ 0 & \longrightarrow & P_{A_1}(2i-2). \end{array}$$

From this we can determine the quiver of the endomorphism algebra $\operatorname{End}_{\mathcal{D}(A_1^{\mathrm{op}})}(P_1)$:

$Q:$



It is straightforward to check that $\beta\gamma \cong \alpha^{n-1}$ using (v). It follows from the form of I_1 that $\alpha\beta = 0$. Lastly there is the relation $\gamma\alpha = 0$, which is due to $\gamma\alpha$ being null-homotopic. It follows from $\alpha^{n-1} \neq 0$ and $\beta\gamma \neq 0$, that there are no more relations. This now means that

$$\operatorname{End}_{\mathcal{D}(A_1^{\mathrm{op}})}(P_1) \cong Q/\langle \alpha^{n-1} - \beta\gamma, \alpha\beta, \gamma\alpha \rangle \cong A_2.$$

4.2 Example from symmetric (k, n)-Postnikov diagrams

It turns out that a good source of examples for finding derived equivalent algebras using Corollary 3.11 are Postnikov diagrams as shown in Figures 4.2 and 4.3. A (k, n)-Postnikov diagram D is a Postnikov diagram with n vertices, and strands going from vertices i to vertices $i + k$. Such a diagram is called *symmetric* if it is invariant under rotation by k vertices, see Figures 4.2 and 4.3. Furthermore, D is called *reduced* if no "untwisting" moves can be applied. For a detailed description of these properties see [Pas20, sec. 4].

To each Postnikov diagram D one can associate an ice quiver with potential (Q, W, F), where

- Q is the quiver associated to D. The vertices of Q corresponds to the alternating regions of D. There is an arrow between two vertices if their corresponding regions meet at an intersection of strands, the arrow will point with the 'flow' of those intersecting stands. See Figures 4.2 and 4.3.
- W is the potential given by the sum of clockwise cycles in Q minus anti-clockwise cycles in Q, and
- F is the frozen vertices, which is the set of vertices on the boundary of D.

For further details see [Pas20, sec. 4]. Denote the associated frozen Jacobian algebra $\mathcal{P}(Q, W, F)$.

Next we construct the boundary algebra as described in [Pas20, sec. 6]. Given $k, n \in \mathbb{N}$, with $k < n$, consider a $\mathbb{Z}/n\mathbb{Z}$-grading on $\mathbb{C}[x, y]$ given by $\deg x = 1$, and $\deg y = -1$. Now let $R = \mathbb{C}[x, y]/(x^k - y^{n-k})$ and define the boundary algebra

$$B = \operatorname{End}_R^{\mathbb{Z}/n\mathbb{Z}}\left(\bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} R(i)\right),$$

where (i) indicates the shift in degrees by i in $\operatorname{mod}^{\mathbb{Z}/n\mathbb{Z}}(R)$. Let $\hat{B}$ be the completion of B with respect to the ideal (x, y).

The following two theorems are a collection of results due to Geiß-Jensen-King-Leclerc-Pasquali-Schröer-Su. The first theorem shows Setup 2.1 is satisfied. The category CM of Cohen-Macaulay modules and its stabilisation $\underline{\mathrm{CM}}$ were introduced in [JKS16, secs. 3, 4].

Theorem 4.2. *Let $k, n \in \mathbb{N}$ with $k < n$. We then have the following results*

1. $\mathrm{CM}(\hat{B})$ *is a Frobenius category ([JKS16, cor.3.7])*
2. $\underline{\mathrm{CM}}(\hat{B})$ *is 2-Calabi–Yau ([JKS16, cor. 4.6] and [GLS08, prop. 3.4])*
3. $\underline{\mathrm{CM}}(\hat{B})$ *is Hom-finite ([JKS16, cor. 4.6] and [GLS08, sec. 3.1, 3.2]).*
4. $\underline{\mathrm{CM}}(\hat{B})$ *has split idempotents.*

Proof. That $\underline{\mathrm{CM}}(\hat{B})$ has split idempotents comes from the fact that $\operatorname{Sub} Q_k$ from [JKS16, cor. 4.6] and [GLS08, sec. 3] has split idempotents. This is due to it being the full subcategory of submodules of sums of Q_k, in a module category which has split idempotents. Since $\operatorname{Sub} Q_k$ has split idempotents, $\underline{\operatorname{Sub}}\, Q_k$ has split idempotents, and by [JKS16, cor. 4.6] there is a triangle equivalence $\underline{\operatorname{Sub}}\, Q_k \cong \underline{\mathrm{CM}}(\hat{B})$. □

The next collection of result describes the objects we want to work with.

Theorem 4.3. *We have the following results*

1. *For every reduced (k, n)-Postnikov diagram D, there is an associated cluster-tilting object $T(D)$ of $\mathrm{CM}(\hat{B})$ ([Pas20, thm. 7.2]).*

2. *Given a reduced (k, n)-Postnikov diagram D, then D is symmetric if and only if the endomorphism ring $\mathrm{End}(T(D))$ is self-injective ([Pas20, thm. 8.2, lem 7.7]).*

3. *If D is a (k, n)-Postnikov diagram, T the associated cluster tilting object, (Q,W,F) the associated ice quiver with potential, then $\underline{\mathrm{End}}_{\hat{B}}(T) \cong \mathcal{P}(Q, W, F)/\langle F\rangle$, where $\langle F\rangle$ is the ideal generated by the frozen vertices ([Pas20, lem. 7.5, prop 7.6, sec. 3]).*

Corollary 4.4. *Let $k, n \in \mathbb{N}$, with $k < n$. Let D, D' be two symmetric and reduced (k, n)-Postnikov diagrams, with associated cluster tilting objects $T = T(D)$ and $T' = T(D')$ in $\underline{\mathrm{CM}}(\hat{B})$. Then $\underline{\mathrm{End}}(T)$ and $\underline{\mathrm{End}}(T')$ are derived equivalent.*

Proof. Since D and D' are symmetric Postnikov diagrams Theorem 4.3(2) gives that $\underline{\mathrm{End}}\,T$ and $\underline{\mathrm{End}}\,T'$ are self-injective. Therefore Corollary 3.11 gives that $\underline{\mathrm{End}}(T)$ and $\underline{\mathrm{End}}(T')$ are derived equivalent. □

As an example of the use of Corollary 4.4, see Figures 4.2 and 4.3 for two symmetric and reduced $(3, 9)$-Postnikov diagrams and their associated ice quivers. The frozen vertices are exactly the ones on the boundary. Denote these ice quivers with potential by (Q, W, F) and (Q', W', F') respectively. Using Theorems 4.3(2) and 4.3(3) together with Corollary 4.4 we get that the associated algebras $\mathcal{P}(Q, W, F)/\langle F\rangle$ and $\mathcal{P}(Q', W', F')/\langle F'\rangle$ are self-injective and derived equivalent. Note that in contrast to [Aug20b] the 2-CY-tilted algebras considered here are not symmetric.

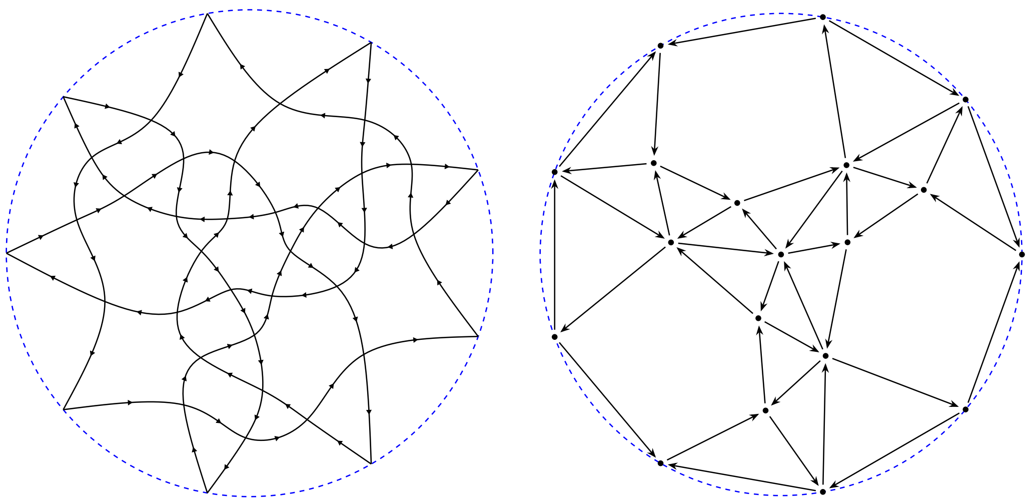


Figure 4.2: To the left is a (3, 9)-Postnikov diagram, and to the right is its associated quiver, with the vertices on the outer boundary being the frozen vertices.

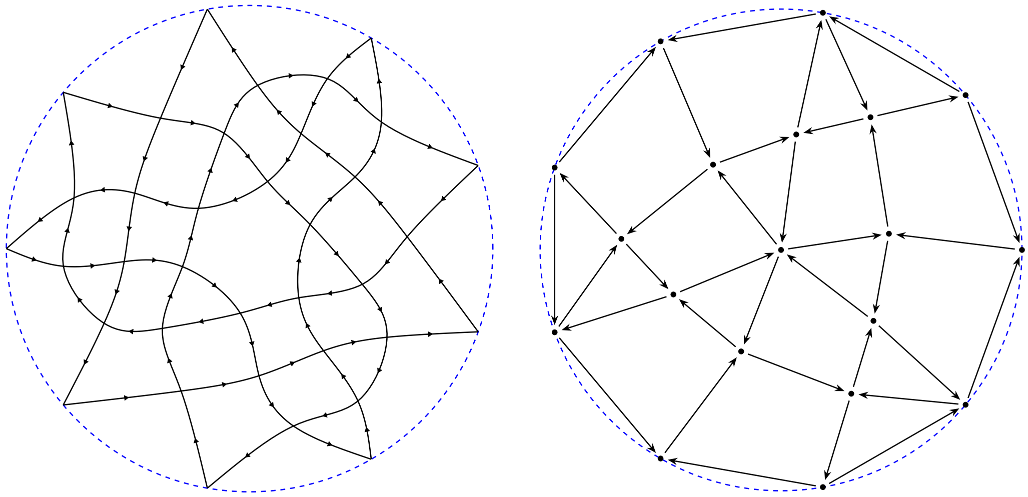


Figure 4.3: To the left is a (3, 9)-Postnikov diagram, and to the right is its associated quiver, with the vertices on the outer boundary being the frozen vertices.

5 An application

For the rest up this section, let k be an algebraically closed field.

We have already breifly discussed ice quivers with potential, and in this section we will work with quivers with potential. For an introduction into quiver with potential we refer the reader to [DWZ08]. However, we will include the essential definitions and results. First, we need to define what a quiver with potential is, and what the corresponding Jacobian algebra is.

Definition 5.1.

- Let Q be a quiver, then an element $W \in \prod_{i>1} kQ_{cyc,i}$, where $kQ_{cyc,i}$ is the collection of cycles of length i, is called a *potential* in kQ. A pair (Q, W) of a quiver Q and a potential W is called a *quiver with potential*, or *QP* for short.
- Let W be a potential for a quiver Q. Then W is said to be *reduced* if $W \in \prod_{i\geq 3} kQ_{cyc,i}$.
- [DWZ08, def 3.1] Let (Q, W) be a quiver with potential. Given an arrow $a \in Q_1$, define the *cyclic derivative* $\partial_a : \prod_{i>1} kQ_{cyc,i} \to \widehat{kQ}$ generated by

$$\partial_a(a_1 \cdots a_n) = \sum_{a_i=a} a_{i+1} \cdots a_n a_1 \cdots a_{i-1}.$$

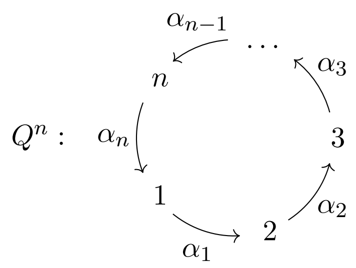
- [DWZ08, def 3.1] Let (Q, W) be QP. The *Jacobian ideal* is an ideal, of the completed path algebra $\widehat{kQ}$, defined by $\mathcal{J}(W) := \overline{\langle \partial_a(w) \mid a \in Q_1, w \in W \rangle}$.
- [DWZ08, def 3.1] Let (Q, W) be a QP. Then the *Jacobian algebra* is defined by $\mathcal{P}(Q, W) = \widehat{kQ}/\mathcal{J}(W)$.

Given a QP (Q, W), we will typically identify properties of the Jacobian algebra $\mathcal{P}(Q, W)$ with that of (Q, W). To that end we have the following definition.

Definition 5.2.

- [HI11, def. 3.6(b)] A QP (Q, W) is said to be *self-injective* if $\mathcal{P}(Q, W)$ is self-injective.
- [Ami09, sec. 3.3] A QP (Q, W) is called *Jacobi-finite* if $\mathcal{P}(Q, W)$ is finite-dimensional.

Example 5.3. Let $n \in \mathbb{N}$, and consider the following quiver.



In this quiver we have a cycle of length n. We will let this cycle be the potential $W = \alpha_n \cdots \alpha_2\alpha_1$. The Jacobian algebra corresponding to the QP (Q^n, W) is therefore $\mathcal{P}(Q^n, W) = kQ/\operatorname{rad}^{n-1}$.

Notice that this algebra is both self-injective and finite-dimensional. Thus, the QP (Q^n, W) is both self-injective and Jacobi-finite. Also notice that for $n > 2$ we do not have any loops or 2-cycles, making (Q, W) reduced.

Quivers with potential have a notion of mutation, which given a QP offers a way to generate another QP.

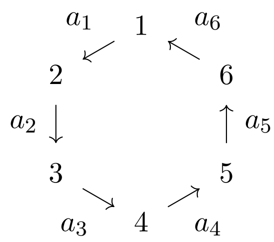
Definition 5.4 ([DWZ08, sec. 5] or [Miz15, def. 2.2])**.** Let (Q, W) be a QP, let $i \in Q_0$ not contained in a 2-cycle. For $x \in Q_0$ define a new QP $\tilde{\mu}_x(Q, W) = (\widetilde{Q}, \widetilde{W})$ where

- $\widetilde{Q}_0 \coloneqq Q_0$.
- $\widetilde{Q}_1$ is constructed in 3 steps:
  1. Set $\widetilde{Q}_1 \coloneqq Q_1$.
  2. For each pair of arrows $a, b \in Q_1$ with $t(a) = s(b) = x$ add an arrow $[ba] : s(a) \to t(b)$ to $\widetilde{Q}_1$.
  3. For each arrow $a \in Q_1$ such that $s(a) = x$ or $t(a) = x$, add an arrow $a^* : t(a) \to s(a)$ to $\widetilde{Q}_1$, and remove a from $\widetilde{Q}_1$.
- Let $[W]$ be the object coming from W by replacing compositions ba, where $t(a) = s(b) = x$, with the new direct arrow $[ba]$, i.e. with $s([ba]) = s(a)$ and $t([ba]) = t(b)$. Let $\Delta = \sum_{t(a)=s(b)=x} b^*[ba]a^*$. Define $\widetilde{W} = [W] + \Delta$.

Define the *mutation* $\mu_x(Q, W)$ of (Q, W) at x as the reduced part of $\tilde{\mu}_x(Q, W)$, i.e., we remove the all summands which are 2-cycles.

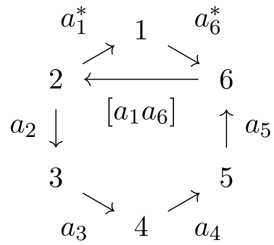
Example 5.5. Consider the QP from Example 5.3 for $n = 6$. That is the quiver

Q^6:



together with the potential $W = a_1a_6a_5a_4a_3a_2$. If we now mutate this QP at the vertex 1 we would get the QP $\mu_1(Q^6, W) = (\widehat{Q}^6, \widehat{W})$ where

$\widehat{Q}^6$:



and $\widehat{W} = [a_1a_6]a_5a_4a_3a_2 + a_1^*[a_1a_6]a_6^*$.

For our purposes, we are only interested in Jacobi-finite QPs. Therefore we would like to know whether or not the mutation of a QP will be Jacobi-finite. However, the class of Jacobi-finite QPs are closed under mutation.

Theorem 5.6 ([DWZ08, cor. 6.6])**.** *Let (Q, W) be a reduced QP, and let $x \in Q_0$ not contained in a 2-cycle, then (Q, W) is Jacobi-finite if and only if $\mu_x(Q, W)$ is Jacobi-finite.*

5.1 Nakayama permutation

Let Q be a finite quiver, then each simple module S over kQ is associated to a vertex $i \in Q_0$, and therefore we denote it by $S(i)$. Denote $P(i)$ (resp. I(i)) as the projective cover (resp. injective envolope) of $S(i)$.

Proposition/Definition 5.7. Let Q be a quiver and I an admissible ideal of kQ, such that kQ/I is a self-injective finite dimensional algebra. Then there exists a map $\sigma : Q_0 \to Q_0$ defined by the property that $P(i) = I(\sigma i)$. Since kQ/I is self-injective, this map is a permutation. This is called the *Nakayama permutation.*

Example 5.8. Consider the QP (Q, W) given in Example 5.3. There we get that $P(i) = I(i-2)$ modulo n. Thus, the corresponding Nakayama permutation $\sigma : Q_0 \to Q_0$ is given by $\sigma(i) = i - 2$ modulo n.

5.2 Derived equivalence via mutation

Our goal is to give an application of Corollary 3.11. We will show it generalizes a similar result in [Miz15] in the setting of quivers with potentials and Jacobian algebras.

Let Q be a quiver, and let $I \subseteq Q_0$ be a subset of vertices. Similar to [Miz15, sec. 3], we give names to the following conditions on I:

(a1) No vertex in I is contained in a 2-cycle.

(a2) There are no arrows between any two vertices in I.

Theorem 5.9 ([Miz15, thm. 3.1, cor. 3.2])**.** *Let (Q, W) be a jacobi-finite, self-injective QP and $\Lambda := \mathcal{P}(Q, W)$. Let $I \subseteq Q_0$ satisfying (a1) and (a2), and denote the corresponding Okuyama–Rickard complex $\mu_I(\Lambda) \in \mathcal{K}^b(\operatorname{proj}\Lambda)$ (see [Miz15, p. 1745]). Then there is an isomorphism of k-algebras*

$$\operatorname{End}_{\mathcal{K}^b(\operatorname{proj}\Lambda)}(\mu_I(\Lambda)) \cong \mathcal{P}(\mu_I(Q, W)).$$

Here $\mu_I(Q, W)$ refers to the QP obtained by mutating (Q, W) at each vertex in I. Let σ denote the Nakayama permutation for Λ. Furthermore if $\sigma I = I$ then $\mu_I(\Lambda)$ is tilting object making $\mathcal{P}(Q, W)$ and $\mathcal{P}(\mu_I(Q, W))$ derived equivalent.

We will show that the derived equivalence statement of Theorem 5.9 follows by Corollary 3.11. To use our result we need to know that the algebras in question arise as endomorphism algebras of maximal rigid objects in some algebraic category satisfying a few conditions. For this we need to turn to cluster categories. A result by Amiot shows that every Jacobi finite QP is isomorphic to the endomorphism algebra of a cluster tilting object in a certain cluster category.

Theorem 5.10 ([Ami11, cor. 3.11] or [Ami09, thm. 3.5])**.** *Let (Q, W) be a Jacobi-finite QP, then there exists a category $\mathcal{C}_{(Q,W)}$ which is* Hom*-finite, 2-CY, and Krull-Schmidt, and has a cluster-tilting object $\ell \in \mathcal{C}_{(Q,W)}$ such that* $\operatorname{End}(\ell) \cong \mathcal{P}(Q, W)$.

The construction of these cluster categories stems from the Ginzburg DG algebra in a way that ensures it has a Frobenius model.

Theorem 5.11. *Using notation from Theorem 5.10 there is a Frobenius category $\mathscr{E}$ such that the associated stable category is $\mathcal{C}_{(Q,W)}$.*

Notice that given two different QPs, this does not tell us whether the corresponding cluster categories coincide. However, Amiot has shown that they coincide when the QPs are related by mutation.

Theorem 5.12 ([Ami11, thm. 3.13(a)])**.** *Let (Q, W) be a QP without loops, and let $i \in Q_0$ not contained in a 2-cycle. Then there is a triangle equivalence $\mathcal{C}_{(Q,W)} \cong \mathcal{C}_{\mu_i(Q,W)}$.*

To use Theorem 5.12 we need to avoid loops, and therefore mutate in vertices not part of a 2-cycle. This motivated the next definition.

Definition 5.13. Given $n \in \mathbb{N}$ let $x = \{x_1, \dots, x_n\}$ be a collection where $x_i \subseteq Q_0$, be a sequence of vertices of Q. We say that (Q, W) is *mutable* over x if x_i is not in a 2-cycle of $\mu_{x_{i-1}} \cdots \mu_{x_1}(Q, W)$. Define $\mu_x(Q, W) = \mu_{x_n} \cdots \mu_{x_1}(Q, W)$.

Example 5.14. Consider the QP (Q^6, W) from Example 5.5, an example of mutable sequences would be $x = (1\,3\,5)$ or $x = (2\,4\,6)$.

Lemma 5.15. *Let (Q, W) be a QP, and let $I \subseteq Q_0$ satisfy (a1) and (a2). Then each sequence $\{x_j\} \subseteq I$, with $x_j \neq x_k$ for $j \neq k$, is mutable.*

Proof. Follows directly from the definition of mutation. □

Now we can combine Theorem 5.12 and Corollary 3.11 to get the following result.

Proposition 5.16. *Let (Q, W) be a self-injective QP without loops. Let $x = \{x_1, ..., x_n\}$ with $x_i \in Q_0$ be a mutable sequence in Q. If $\mu_x(Q, W)$ is self-injective, then $\mathcal{P}(Q, W)$ and $\mathcal{P}(\mu_x(Q, W))$ are derived equivalent.*

Proof. By Theorems 5.10 to 5.12 there is a Frobenius category $\mathscr{E}$ whose associated stable category is 2-CY, Hom-finite and Krull-Schmidt, such that there are cluster-tilting objects $\ell, m \in \mathscr{C}$ satisfying

$$\mathscr{C}(\ell, \ell) \cong \mathcal{P}(Q, W) \qquad \text{and} \qquad \mathscr{C}(m, m) \cong \mathcal{P}(\mu_x(Q, W)).$$

Now Corollary 3.11 gives that $\mathcal{P}(Q, W)$ and $\mathcal{P}(\mu_x(Q, W))$ are derived equivalent. □

This can be combined with the following result by Herschend and Iyama.

Theorem 5.17 ([HI11, thm. 4.2])**.** *Let (Q, W) be a self-injective QP without loops with Nakayama permutation σ. Let $x = \{x_1, ..., x_n\} \subseteq Q_0$ be a σ orbit satisfying (a1) and (a2), i.e., $x = \{x = \sigma^n x, \sigma x, \sigma^2 x, ..., \sigma_{n-1}\}$. Then $\mu_x(Q, W)$ is self-injective.*

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Paper B

Intermediate Categories for Proper Abelian Subcategories

Abstract

Let $\mathscr{A}$ be an extension closed proper abelian subcategory of a triangulated category $\mathscr{T}$, with no negative 1 and 2 extensions. From this, two functors from $\Sigma\mathscr{A} * \mathscr{A}$ to $\mathscr{A}$ can be constructed giving a snake lemma mirroring that of homology without needing a t-structure.

We generalize the concept of intermediate categories, which originates from a paper by Enomoto and Saito, to the setting of proper abelian subcategories and show that under certain assumptions this collection is in bijection with torsion-free classes in $\mathscr{A}$.

1 Introduction

In [ES22], Enomoto and Saito introduce and study Grothendieck monoids of extriangulated categories as a generalization of the Grothendieck group. They study what happens to the Grothendieck monoid when an extriangulated category is localized, and furthermore, they ask when this becomes the localization of the Grothendieck monoid of that original extriangulated category. That is, given an extriangulated category $\mathcal{C}$ and a class of morphisms $S \in \mathrm{Mor}(\mathcal{C})$, when does the Grothendieck monoid $M(\mathcal{C}_S)$ of the localization $\mathcal{C}_S$ become a localization of the Grothendieck monoid $M(\mathcal{C})$ of $\mathcal{C}$? As an example of when this happens, they study *intermediate categories* of a derived category $D^b(\mathcal{A})$. A category $\mathcal{C} \subseteq D^b(\mathcal{A})$ is called intermediate in $D^b(\mathcal{A})$ if it is closed under extensions and direct summands, and $\mathcal{A} \subseteq \mathcal{C} \subseteq \Sigma\mathcal{A} * \mathcal{A}$.

In this paper we generalize the notion of intermediate categories to the setting of an arbitrary triangulated category $\mathscr{T}$ containing a proper abelian subcategory $\mathscr{A}$. Proper abelian subcategories were introduced by Jørgensen in [Jør22] as full additive subcategories $\mathscr{A}$ of a triangulated category $\mathscr{T}$, with conflations exactly those coming from $\mathscr{T}$ as short triangles, giving $\mathscr{A}$ the structure of an abelian category. In this setting an *$\mathscr{A}$-intermediate category* $\mathcal{C} \subseteq \mathscr{T}$ is closed under extensions and direct summands, and satisfies $\mathscr{A} \subseteq \mathcal{C} \subseteq \Sigma\mathscr{A} * \mathscr{A}$. Notice that this is a generalization of the former notion since $\mathcal{A}$ sits inside $D^b(\mathcal{A})$ as a proper abelian subcategory. With this new definition we can consider, as an example, hearts of t-structures which also sit inside a triangulated category as proper abelian subcategories. When working with a t-structure in a triangulated category $\mathscr{T}$ one will always have a homology functor from $\mathscr{T}$ to the heart $\mathcal{H}$ of that t-structure. This homology functor can to some extent describe objects in $\mathscr{T}$ in terms of $\mathcal{H}$, using long exact sequences.

It is not always the case that proper abelian subcategories are hearts of t-structures; for an example of this look no further than Example 5.2. Therefore we loose the feature of having a homology functor. We can make a substitute of this, using a version of the snake lemma.

Lemma A (= Lemma 3.4)**.** *Let $\mathscr{A}$ be a proper abelian subcategory in a triangulated category $\mathscr{T}$. Assume $\mathscr{T}$ is Krull-Schmidt and that $\mathscr{T}(\mathscr{A}, \Sigma^{-i}\mathscr{A}) = 0$ for $i = 1, 2$. Let $c \in \Sigma\mathscr{A} * \mathscr{A}$, then there exists a unique minimal right $\Sigma\mathscr{A}$-approximation $\Sigma a_1 \to c$, and a unique minimal left $\mathscr{A}$-approximation $c \to a_0$. Defining $G, F : \Sigma\mathscr{A} * \mathscr{A} \to \mathscr{A}$ by $F(c) = a_1$ and $G(c) = a_0$ induces two functors with the following property: Let*

$$c \xrightarrow{f} c' \xrightarrow{g} c'' \longrightarrow \Sigma c \tag{1.1}$$

be a triangle in $\mathscr{T}$ with $c, c', c'' \in \Sigma\mathscr{A} * \mathscr{A}$, then there exists a morphism $\delta : F(c'') \to G(c)$, such that

$$0 \longrightarrow F(c) \xrightarrow{F(f)} F(c') \xrightarrow{F(g)} F(c'') \xrightarrow{\delta} G(c) \xrightarrow{G(f)} G(c') \xrightarrow{G(g)} G(c'') \longrightarrow 0$$

is an exact sequence in $\mathscr{A}$.

Using this snake lemma as a substitute for homology, we can similarly to Enomoto and Saito classify all the $\mathscr{A}$-intermediate subcategories of $\mathscr{T}$ as follows.

Theorem B (= Theorem 4.2 & Corollary 4.3)**.** *Let $\mathscr{A}$ be an extension closed proper abelian subcategory in a triangulated category $\mathscr{T}$. Assume $\mathscr{T}$ is Krull-Schmidt and that $\mathscr{T}(\mathscr{A}, \Sigma^{-i}\mathscr{A}) = 0$ for $i = 1, 2$.*

1. *If $\mathscr{F} \subseteq \mathscr{A}$ is a torsion-free class then $\Sigma\mathscr{F} * \mathscr{A}$ is an $\mathscr{A}$-intermediate category. Furthermore, $F(\Sigma\mathscr{F} * \mathscr{A}) = \mathscr{F}$.*

2. *Let $\mathcal{C}$ be an $\mathscr{A}$-intermediate category which also satisfies that $\mathcal{C} \subseteq \mathscr{A} * \Sigma\mathscr{A}$. Then $F(\mathcal{C})$ is a torsion-free class. Furthermore $\mathcal{C} = \Sigma F(\mathcal{C}) * \mathscr{A}$.*

If $\mathscr{A} * \Sigma\mathscr{A} = \Sigma\mathscr{A} * \mathscr{A}$, this gives a bijection between the collection of $\mathscr{A}$-intermediate categories and that of torsion-free classes of $\mathscr{A}$.

In the last section we give an example of applying this theory in a negative cluster category. This is a triangulated category in which proper abelian subcategories are not hearts of t-structures. We give examples of when the assumptions made in the article are satisfied, together with applying the results.

Conventions

Let $\mathscr{A}$ be a proper abelian subcategory of a triangulated category $\mathscr{T}$. We will denote monomorphisms (resp. epimorphisms) in $\mathscr{A}$ by $\rightarrowtail$ (resp. $\twoheadrightarrow$). Conversely, if one of these arrows is used, it will be with respect to a proper abelian subcategory.

All subcategories will by default be assumed to be full.

2 Preliminaries

Definition 2.1. Let $\mathscr{T}$ be a triangulated category and let $\mathscr{A}, \mathscr{B} \subseteq \mathscr{T}$ be subcategories. Then define the subcategory

$$\mathscr{A} * \mathscr{B} = \{x \mid \text{there exists a triangle } a \to x \to b \to \Sigma a \text{ in } \mathscr{T}, \text{ with } a \in \mathscr{A}, b \in \mathscr{B}\}.$$

Definition 2.2. Let $\mathscr{A}$ be an additive category. A subcategory $\mathscr{B} \subseteq \mathscr{A}$ of $\mathscr{A}$ is called an *additive subcategory of* $\mathscr{A}$ if it is closed under isomorphisms, direct sums and direct summands.

2.1 Proper abelian subcategories

Proper abelian subcategories of a triangulated category were introduced by Jørgensen in [Jør22], which also serves as a good introduction to the subject. Here we will state the properties that are needed for this article.

For the rest of this section let $\mathscr{T}$ be a triangulated category.

Definition 2.3. Let $\mathscr{A} \subseteq \mathscr{T}$ be an additive subcategory. $\mathscr{A}$ is a *proper abelian subcategory* if it is abelian in such a way that $a \overset{\alpha}{\rightarrowtail} a' \overset{\alpha'}{\twoheadrightarrow} a''$ is a short exact sequence in $\mathscr{A}$ if and only if there is a triangle $a \xrightarrow{\alpha} a' \xrightarrow{\alpha'} a'' \to \Sigma a$ in $\mathscr{T}$.

Definition 2.4 ([Jør21, def. 0.2])**.** Let $\mathscr{A} \subseteq \mathscr{T}$ be a proper abelian subcategory, $n \in \mathbb{N}$. We say that $\mathscr{A}$ *satisfies* E_n if $\mathscr{T}(\mathscr{A}, \Sigma^{-i}\mathscr{A}) = 0$ for $0 < i \leq n$.

Definition 2.5. Let $\mathscr{A}$ be an abelian category. An additive subcategory $\mathscr{F} \subseteq \mathscr{A}$ is called a *torsion-free* class if it is closed under extensions and subobjects.

Lemma 2.6. *Let $\mathscr{A} \subseteq \mathscr{T}$ be an extension closed proper abelian subcategory, $\mathscr{F} \subseteq \mathscr{A}$ a torsion-free class then the following statements hold.*

1. $\mathscr{A} * \Sigma\mathscr{F} \subseteq \Sigma\mathscr{F} * \mathscr{A}$*,*
2. $\Sigma\mathscr{F} * \mathscr{A}$ *is extension closed,*
3. *If $\mathscr{T}$ is Krull-Schmidt and $\mathscr{A}$ satisfies E_1, then $\Sigma\mathscr{F} * \mathscr{A}$ is an additive subcategory of $\mathscr{T}$.*

Proof. (1) Since $\mathscr{F}$ is closed under subobjects, this follows from a similar argument as [Jør22, lem. 5.2].

(2) Follows directly from the following calculation:

$$(\Sigma\mathscr{F} * \mathscr{A}) * (\Sigma\mathscr{F} * \mathscr{A}) \subseteq (\Sigma\mathscr{F} * \Sigma\mathscr{F}) * (\mathscr{A} * \mathscr{A}) \subseteq \Sigma(\mathscr{F} * \mathscr{F}) * (\mathscr{A} * \mathscr{A}) \subseteq \Sigma\mathscr{F} * \mathscr{A}.$$

(3) Since $\mathscr{F}$ and $\mathscr{A}$ are both additive subcategories of $\mathscr{T}$, it follows directly from [IY08, 2.1(i)] that $\Sigma\mathscr{F} * \mathscr{A}$ is an additive subcategory of $\mathscr{T}$. □

2.2 Extriangulated categories

The concept of extriangulated categories is a simultaneous generalization of exact categories and triangulated categories. These were first introduced by Nakaoka and Palu in [NP19]. For the concrete definition see [NP19, def. 2.12]. An *extriangulated category* is a triple $\mathscr{C} = (\mathscr{C}, \mathbb{E}, \mathfrak{s})$, where $\mathscr{C}$ is an additive category, $\mathbb{E} : \mathscr{C}^{\text{op}} \times \mathscr{C} \to \mathsf{Ab}$ is a bifunctor to the category of abelian groups, and $\mathfrak{s}$ is a "realization" of $\mathbb{E}$.

Example 2.7. 1. Let $\mathscr{E}$ be an exact category, then $\mathscr{E}$ can be described as an extriangulated category $(\mathscr{E}, \operatorname{Ext}^1_{\mathscr{E}}(-,-), \mathfrak{s})$ where the realization of $\delta \in \operatorname{Ext}^1_{\mathscr{E}}(Z, X)$ is its corresponding conflation, i.e. $\mathfrak{s}(\delta) = (X \rightarrowtail Y \twoheadrightarrow Z)$.

2. Let $\mathscr{T}$ be a triangulated category, then $\mathscr{T}$ can be described as an extriangulated category $(\mathscr{T}, \operatorname{Hom}_{\mathscr{T}}(-, \Sigma-), \mathfrak{s})$ where the realization of $\delta \in \operatorname{Hom}_{\mathscr{E}}(Z, \Sigma X)$ is its corresponding short triangle, i.e. $\mathfrak{s}(\delta) = (X \to Y \to Z)$.

3. Let $\mathscr{A}$ be an extension closed proper abelian subcategory of a triangulated category $\mathscr{T}$. The canonical extriangulated structure of $\mathscr{A}$ is the restriction of the extriangulated structure on $\mathscr{T}$ to $\mathscr{A}$.

There are also extriangulated categories that are neither exact nor triangulated. The following lemma can give an example of such a category.

Lemma 2.8. *Let $\mathscr{A}$ be an extension closed proper abelian subcategory of a triangulated category $\mathscr{T}$. Then $\Sigma\mathscr{A} * \mathscr{A} = (\Sigma\mathscr{A} * \mathscr{A}, \operatorname{Hom}_{\mathscr{T}}(-, \Sigma-), \mathfrak{s})$ is an extriangulated category obtained by restriction of the extriangulated structure of $\mathscr{T}$.*

Proof. By Lemma 2.6(2), $\Sigma\mathscr{A} * \mathscr{A}$ is extension-closed. Therefore the claim follows directly from [NP19, rmk. 2.18]. □

2.3 Monoids

For an introduction to monoids and localization of monoids we refer to [ES22, app. A]. Here we give a very short introduction including the results we need.

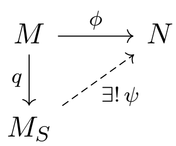
Definition 2.9. A *monoid* is a pair $(M, \cdot)$, where M is a set, and $\cdot : M \times M \to M$ is a binary operation satisfying the following:

1. The operation "$\cdot$" is associative,

2. There is an identity element in M w.r.t. "$\cdot$".

In other words, a monoid can be viewed as a group without inverses.

Definition 2.10. Given a monoid M and a subset $S \subseteq M$, a *localization* of M w.r.t. S is a pair (M_S, q) with a monoid M_S and a morphism $q : M \to M_S$ satisfying the following universal property: For each monoid N and morphism $\phi : M \to N$ where $\phi(s)$ is invertible for all $s \in S$, there exists a unique morphism $\psi : M_S \to N$ such that $\phi = \psi \circ q$. That is, there exists a commutative diagram



Lemma 2.11 ([ES22, under def. A.11])**.** *Given a monoid M and a subset $S \subseteq M$, the localization M_S exists.*

2.4 The Grothendieck monoid

Definition 2.12 ([ES22, def. 2.5])**.** Let $\mathcal{C}$ be a skeletally small extriangulated category. Then a *Grothendieck monoid* of $\mathcal{C}$ is a pair $M(\mathcal{C}) = (M(\mathcal{C}), \pi)$, with a monoid $M(\mathcal{C})$, and a map $\pi : \operatorname{Iso}(\mathcal{C}) \to M(\mathcal{C})$, such that

1. π *respects conflations*, i.e. $\pi([0]) = 0$ and for every conflation $X \to Y \to Z \dashrightarrow$ in $\mathcal{C}$ one has that $\pi([X]) + \pi([Z]) = \pi([Y])$.

2. Given a monoid N and map $\mu : \operatorname{Iso}(\mathcal{C}) \to N$ which respects conflations, there exists a unique morphism $\overline{\mu} : M(\mathcal{C}) \to N$ such that $\overline{\mu} \circ \pi = \mu$.

$$\begin{array}{ccc} \operatorname{Iso}(\mathcal{C}) & \xrightarrow{\mu} & N \\ {\scriptstyle \pi}\downarrow & \nearrow_{\exists!\, \overline{\mu}} & \\ M(\mathcal{C}). & & \end{array}$$

Proposition 2.13 ([ES22, prop. 2.7])**.** *Let $\mathcal{C}$ be an skeletally small extriangulated category, then the Grothendieck monoid exists.*

2.5 Intermediate categories

Definition 2.14. Let $\mathscr{A}$ be an extension closed proper abelian subcategory of a triangulated category $\mathscr{T}$. A subcategory $\mathcal{C} \subseteq \mathscr{T}$ is called an *$\mathscr{A}$-intermediate category* if

1. $\mathscr{A} \subseteq \mathcal{C} \subseteq \Sigma\mathscr{A} * \mathscr{A}$,

2. $\mathcal{C}$ is extension-closed,

3. $\mathcal{C}$ closed under direct summands.

Lemma 2.15. *Let $\mathscr{A}$ be an extension closed proper abelian subcategory of a triangulated category $\mathscr{T}$, then an $\mathscr{A}$-intermediate category $\mathcal{C}$ has an extriangulated structure inherited from $\mathscr{T}$ by restriction.*

Proof. By definition $\mathcal{C}$ will be extension-closed, hence it follows directly from [NP19, rmk. 2.18] that $\mathcal{C}$ has the claimed extriangulated structure. □

3 The snake lemma

The following setup will be assumed throughout the rest of this section.

Setup 3.1. Let $\mathscr{T}$ be a Krull-Schmidt triangulated category, and $\mathscr{A}$ an extension closed proper abelian subcategory satisfying E_2.

Lemma 3.2. *We have the following.*

1. *Up to isomorphism each $x \in \Sigma\mathscr{A} * \mathscr{A}$ permits a unique short triangle $\Sigma a_1^x \xrightarrow{\phi_x} x \xrightarrow{\psi_x} a_0^x$, with ϕ_x a minimal right $\Sigma\mathscr{A}$-approximation, and ψ_x a minimal left $\mathscr{A}$-approximation. Furthermore, ϕ_x, ψ_x are natural in x.*

2. *The assignments $x \mapsto a_1^x$ and $x \mapsto a_0^x$ have canonical augmentations to functors F and G.*

Proof. Given $x \in \Sigma\mathscr{A} * \mathscr{A}$ there is a short triangle $\Sigma a_1^x \xrightarrow{\phi_x} x \xrightarrow{\psi_x} a_0^x$, with $a_0^x, a_1^x \in \mathscr{A}$. Since $\mathscr{A}$ satisfies E_1 we get that ϕ_x is a right $\Sigma\mathscr{A}$-approximation, and ψ_x is a left $\mathscr{A}$-approximation. Since $\mathscr{T}$ is Krull-Schmidt there are, up to isomorphism, unique $a_0^x, a_1^x, \psi_x, \phi_x$ such that ϕ_x, ψ_x are minimal. Before showing that ϕ_x and ψ_x are natural in x, we will show that the assignments $F(x) = a_1^x$ and $G(x) = a_0^x$ induce functors. We will show that F is a functor. Showing that G is a functor follows by a similar argument.

It is enough to check that given $x, y \in \Sigma\mathscr{A} * \mathscr{A}$, and $f \in \mathrm{Hom}(x, y)$, there exists a unique morphism $\alpha : \Sigma F(x) \to \Sigma F(y)$ such that the following diagram commutes,

$$\begin{array}{ccc} \Sigma F(x) & \xrightarrow{\alpha} & \Sigma F(y) \\ \downarrow{\scriptstyle \phi_x} & & \downarrow{\scriptstyle \phi_y} \\ x & \xrightarrow{f} & y, \end{array} \tag{3.1}$$

where ϕ_x, ϕ_y are the minimal right $\Sigma\mathscr{A}$-approximations from above. Existence of α follows directly from ϕ_y being a right $\Sigma\mathscr{A}$-approximation. To prove uniqueness, assume there is another morphism $\beta : \Sigma F(x) \to \Sigma F(y)$ satisfying that $\phi_y \circ \beta = f \circ \phi_x$. Then $\phi_y \circ (\alpha - \beta) = 0$, thus $\alpha - \beta$ factors though $\Sigma^{-1} G(y)$:

$$\begin{array}{ccccc} & & \Sigma F(x) & \xrightarrow{\phi_x} & x \\ & \swarrow & \downarrow{\scriptstyle \alpha-\beta} & & \downarrow{\scriptstyle 0} \\ \Sigma^{-1} G(y) & \longrightarrow & \Sigma F(y) & \xrightarrow{\phi_y} & y. \end{array}$$

The assumption that $\mathscr{A}$ satisfies E_2 implies that $\alpha - \beta = 0$, thus $\alpha = \beta$. This shows that F is a functor, and that ϕ_x is natural in x follows directly from this (see (3.1)). $\square$

Notation 3.3. For the rest of the paper we let F, G denote the functors from Lemma 3.2, and given $x \in \Sigma\mathscr{A} * \mathscr{A}$ we let ϕ_x, ψ_x denote the morphisms from Lemma 3.2.

Lemma 3.4. *Let*

$$c \xrightarrow{f} c' \xrightarrow{g} c'' \longrightarrow \Sigma c \tag{3.2}$$

be a triangle with $c, c', c'' \in \Sigma\mathscr{A} * \mathscr{A}$ then there exists a morphism $\delta : F(c'') \to G(c)$, such that

$$0 \longrightarrow F(c) \xrightarrow{F(f)} F(c') \xrightarrow{F(g)} F(c'') \xrightarrow{\delta} G(c) \xrightarrow{G(f)} G(c') \xrightarrow{G(g)} G(c'') \longrightarrow 0$$

is an exact sequence in $\mathscr{A}$.

Proof. Given a triangle

$$c \xrightarrow{f} c' \xrightarrow{g} c'' \xrightarrow{\Sigma h} \Sigma c$$

with $c, c', c'' \in \Sigma\mathscr{A} * \mathscr{A}$, we can get the following diagram.

$$
\begin{array}{ccccccccccc}
F(c) & \xrightarrow{F(f)} & F(c') & \xrightarrow{F(g)} & F(c'') & & \Sigma F(c) & \xrightarrow{\Sigma F(f)} & \Sigma F(c') & \xrightarrow{\Sigma F(g)} & \Sigma F(c'') \\
\downarrow{\scriptstyle \Sigma^{-1}\phi_c} & & \downarrow{\scriptstyle \Sigma^{-1}\phi_{c'}} & & \downarrow{\scriptstyle \Sigma^{-1}\phi_{c''}} & & \downarrow{\scriptstyle \phi_c} & & \downarrow{\scriptstyle \phi_{c'}} & & \downarrow{\scriptstyle \phi_{c''}} \\
\Sigma^{-1}c & \xrightarrow{\Sigma^{-1}f} & \Sigma^{-1}c' & \xrightarrow{\Sigma^{-1}g} & \Sigma^{-1}c'' & \xrightarrow{h} & c & \xrightarrow{f} & c' & \xrightarrow{g} & c'' \\
\downarrow{\scriptstyle \Sigma^{-1}\psi_c} & & \downarrow{\scriptstyle \Sigma^{-1}\psi_{c'}} & & \downarrow{\scriptstyle \Sigma^{-1}\psi_{c''}} & & \downarrow{\scriptstyle \psi_c} & & \downarrow{\scriptstyle \psi_{c'}} & & \downarrow{\scriptstyle \psi_{c''}} \\
\Sigma^{-1}G(c) & \xrightarrow{\Sigma^{-1}G(f)} & \Sigma^{-1}G(c') & \xrightarrow{\Sigma^{-1}G(g)} & \Sigma^{-1}G(c'') & & G(c) & \xrightarrow{G(f)} & G(c') & \xrightarrow{G(g)} & G(c'').
\end{array}
$$

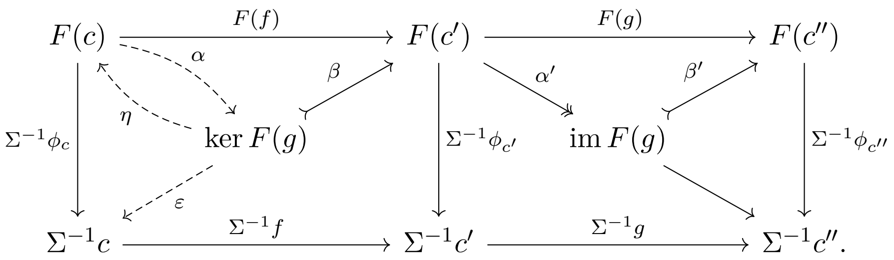
Composing the blue arrows in the diagram above defines a function

$$\delta \coloneqq \psi_c \circ h \circ \Sigma^{-1}\phi_{c''} \colon F(c'') \to G(c).$$

Exact at $F(c)$. Since $\mathscr{A}$ is an abelian category there exists a conflation

$$\ker F(g) \overset{\beta}{\rightarrowtail} F(c') \overset{\alpha'}{\twoheadrightarrow} \operatorname{im} F(g).$$

By definition this conflation comes from a triangle in $\mathscr{T}$. Consider the following commutative diagram of solid arrows.

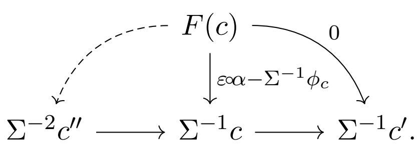


Using the axiom TR3 we get that ε exists. Since $\Sigma^{-1}\phi_c$ is a right $\mathscr{A}$-approximation, η exists such that $\Sigma^{-1}\phi_c \circ \eta = \varepsilon$. From the fact that $F(g) \circ F(f) = 0$ it follows that α exists in such a way that $\beta \circ \alpha = F(f)$.

We will check that $\varepsilon \circ \alpha = \Sigma^{-1}\phi_c$. Notice that

$$\Sigma^{-1}f \circ \varepsilon \circ \alpha = \Sigma^{-1}\phi_{c'} \circ \beta \circ \alpha = \Sigma^{-1}\phi_{c'} \circ F(f) = \Sigma^{-1}f \circ \Sigma^{-1}\phi_c.$$

Therefore $\Sigma^{-1}f(\varepsilon \circ \alpha - \Sigma^{-1}\phi_c) = 0$. This means that $\varepsilon \circ \alpha - \Sigma^{-1}\phi_c$ factors through $\Sigma^{-2}c''$.



Since $\mathscr{A}$ satisfies E_2, we have that $\operatorname{Hom}(\mathscr{A}, \Sigma^{-1}\mathscr{A} * \Sigma^{-2}\mathscr{A}) = 0$, thus $\varepsilon \circ \alpha - \Sigma^{-1}\phi_c = 0$, giving that $\varepsilon \circ \alpha = \Sigma^{-1}\phi_c$.

We now check that $\alpha \circ \eta = \mathrm{id}$. Firstly we have that

$$\Sigma^{-1}\phi_{c'} \circ \beta \circ \alpha \circ \eta = \Sigma^{-1}\phi_{c'} \circ F(f) \circ \eta = \Sigma^{-1}f \circ \Sigma^{-1}\phi_c \circ \eta = \Sigma^{-1}f \circ \varepsilon = \Sigma^{-1}\phi_{c'} \circ \beta.$$

Hence $\Sigma^{-1}\phi_{c'} \circ (\beta - \beta \circ \alpha \circ \eta) = 0$. A similar argument as before shows that $\beta - \beta \circ \alpha \circ \eta = 0$ since it would have to factor through $\Sigma^{-2}G(c')$. Therefore $\beta(\mathrm{id} - \alpha \circ \eta) = 0$, and since β is a monomorphism in $\mathscr{A}$, this implies that $\mathrm{id} - \alpha \circ \eta = 0$, and therefore $\mathrm{id} = \alpha \circ \eta$.

Lastly we see that $\eta \circ \alpha = \mathrm{id}$. Notice that

$$\Sigma^{-1}\phi_c \circ \eta \circ \alpha = \varepsilon \circ \alpha = \Sigma^{-1}\phi_c.$$

Thus $\Sigma^{-1}\phi_c \circ (\eta \circ \alpha - \mathrm{id}) = 0$, meaning that $\eta \circ \alpha - \mathrm{id}$ factors though $\Sigma^{-2}G(c)$ and therefore $\eta \circ \alpha - \mathrm{id} = 0$ by the assumption that $\mathscr{A}$ satisfies E_2. This concludes the proof that $\eta \circ \alpha = \mathrm{id}$. In particular, α is an isomorphism making $F(f)$ a monomorphism.

Exact at $\boldsymbol{F(c')}$. We just proved that $F(c) \cong \ker F(g)$. This gives the following diagram.

$$F(c) \overset{F(f)}{\rightarrowtail} F(c') \xrightarrow{F(g)} F(c'')$$

But then $\ker F(g) \cong F(c) \cong \operatorname{im} F(f)$, making the sequence exact at $F(c')$.

Exact at $\boldsymbol{F(c'')}$. Since $F(f)$ is a monomorphism with cokernel $a = \operatorname{im} F(g)$, we get the following triangle.

$$F(c) \xrightarrow{F(f)} F(c') \xrightarrow{\eta} a \xrightarrow{\mu} \Sigma F(c) \xrightarrow{\Sigma F(f)} \Sigma F(c').$$

Using the 3×3 lemma (see [BBD, prop. 1.1.11] or [May01, lem. 2.6]) on the commutative square

$$\begin{array}{ccc} F(c) & \xrightarrow{F(f)} & F(c') \\ \downarrow{\scriptstyle \Sigma^{-1}\phi_c} & & \downarrow{\scriptstyle \Sigma^{-1}\phi_{c'}} \\ \Sigma^{-1}c & \xrightarrow{\Sigma^{-1}f} & \Sigma^{-1}c' \end{array}$$

gives that Diagram 3.1 is a commutative diagram where each row and column is a triangle. In the Diagram 3.1, the monomorphism τ exists such that $F(g) = \tau\eta$ because η is a cokernel of $F(f)$ while $F(g) \circ F(f) = 0$. Recall that $\delta = \psi_c \circ h \circ \Sigma^{-1}\phi_{c''}$ (i.e. the composition of the blue arrows in the diagram above). We check that the solid triangle commutes, that is $\Sigma^{-1}\phi_{c''} \circ \tau = \gamma$. Since $\tau\eta = F(g)$, we get that

$$(\gamma - \Sigma^{-1}\phi_{c''} \circ \tau)\eta = \gamma \circ \eta - \Sigma^{-1}\phi_{c''} \circ F(g) = \Sigma^{-1}g \circ \Sigma^{-1}\phi_{c'} - \Sigma^{-1}g \circ \Sigma^{-1}\phi_{c'} = 0.$$

Therefore $\gamma - \Sigma^{-1}\phi_{c''} \circ \tau$ factors through μ:

$$\begin{array}{ccccc} F(c') & \xrightarrow{\eta} & a & \xrightarrow{\mu} & \Sigma F(c) \\ & {\scriptstyle 0}\searrow & \downarrow{\scriptstyle \gamma - \Sigma^{-1}\phi_{c''} \circ \tau} & \swarrow{\scriptstyle \exists\lambda} & \\ & & \Sigma^{-1}(c'') & & \end{array}$$

But notice that $\lambda \in \operatorname{Hom}(\Sigma\mathscr{A}, \mathscr{A} * \Sigma^{-1}\mathscr{A})$, thus $\lambda = 0$ since $\mathscr{A}$ satisfies E_2. This implies that $\gamma - \Sigma^{-1}\phi_{c''} \circ \tau = 0$, giving that $\gamma = \Sigma^{-1}\phi_{c''} \circ \tau$.

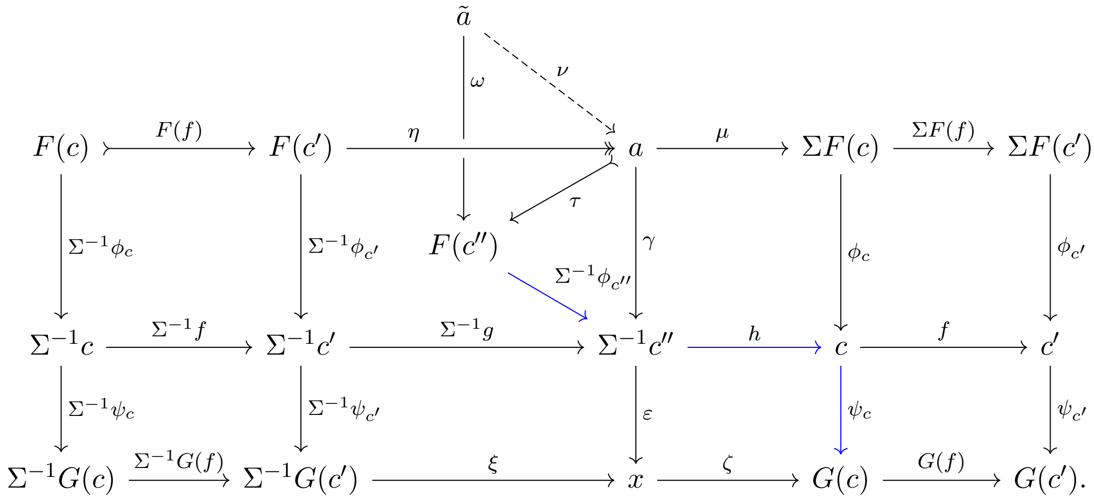


Diagram. 3.1: A commutative diagram used to show exactness at $F(c'')$.

We now show that τ satisfies the universal property of being a kernel of δ. Let $\tilde{a} \in \mathscr{A}$ be given together with a morphism $\omega : \tilde{a} \to F(c'')$ such that $\delta \circ \omega = 0$. We want to show that ω factors through τ.

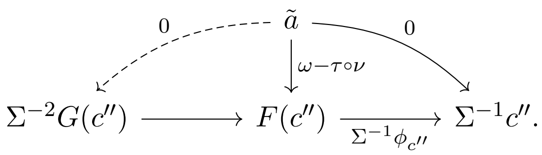
First we notice that

$$0 = \delta \circ \omega = \psi_c \circ h \circ \Sigma^{-1}\phi_{c''} \circ \omega = \zeta \circ (\varepsilon \circ \Sigma^{-1}\phi_{c''} \circ \omega).$$

This means that $\varepsilon \circ \Sigma^{-1}\phi_{c''} \circ \omega$ factors through ξ. But $\mathrm{Hom}(\tilde{a}, \Sigma^{-1}G(c')) = 0$ and therefore $\varepsilon \circ \Sigma^{-1}\phi_{c''} \circ \omega = 0$. This gives that $\Sigma^{-1}\phi_{c''} \circ \omega$ factors through γ, say $\Sigma^{-1}\phi_{c''} \circ \omega = \gamma \circ \nu$ (ν is the dashed morphism in the diagram). Notice that

$$\Sigma^{-1}\phi_{c''}(\omega - \tau \circ \nu) = \Sigma^{-1}\phi_{c''} \circ \omega - \gamma\nu = 0.$$

Thus $\omega - \tau \circ \nu$ factors through $\Sigma^{-2}G(c'')$.



This means that $\omega - \tau \circ \nu = 0$, thus $\omega = \tau \circ \nu$. The uniqueness part of the universal property follows from τ being a monomorphism in $\mathscr{A}$. This concludes the proof that $\ker \delta \cong a$.

Exact at $G(c), G(c'), G(c'')$. These can be done in a dual way to showing that the sequence is exact in $F(c), F(c'), F(c'')$. □

Lemma 3.5. *Let $c \in \mathscr{A} * \Sigma\mathscr{A}$, and thus $c \in \Sigma\mathscr{A} * \mathscr{A}$ by Lemma 2.6(1). Given two triangles $a_0 \xrightarrow{f} c \to \Sigma a_1 \to \Sigma a_0$ and $\Sigma b_0 \to c \xrightarrow{\beta} b_1 \to \Sigma^2 b_0$, with $a_i, b_i \in \mathscr{A}$, the composition $\beta \circ f : a_0 \to b_1$ is an epimorphism in $\mathscr{A}$.*

Proof. There exists a diagram

$$\begin{array}{ccccccc} & & a_0 & & & & \\ & & \downarrow{\scriptstyle f} & & & & \\ \Sigma b_0 & \xrightarrow{\alpha} & c & \xrightarrow{\beta} & b_1 & \xrightarrow{\gamma} & \Sigma^2 b_0 \\ & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle h} & \swarrow{\scriptstyle \varepsilon = 0} & \\ & & \Sigma a_1 & \xrightarrow{\xi = 0} & d & & \end{array}$$

where the row and column containing c are triangles. We claim that $\beta \circ f : a_0 \to b_1$ is an epimorphism. To show this let $d \in \mathscr{A}$ and a morphism $h : b_1 \to d$ be given such that $h \circ \beta \circ f = 0$. It is enough to show that $h = 0$. Since $h \circ \beta \circ f = 0$ we get that $h \circ \beta$ factors through m, i.e. there exists a morphism $\xi : \Sigma a_1 \to d$, such that $\xi \circ m = h \circ \beta$. But notice that $\xi = 0$ since $\mathscr{A}$ satisfies E_1. Thus $h \circ \beta = 0$. This means that h factors through γ, i.e. there exists a morphism $\varepsilon : \Sigma^2 b_0 \to d$ such that $h = \varepsilon \circ \gamma$. But since $\mathscr{A}$ satisfies E_2, we get that $\varepsilon = 0$, hence $h = 0$. □

Proposition 3.6. *The following are equivalent.*

1. *For all* $a, a' \in \mathscr{A}$ *and* $f \in \mathscr{T}(a, \Sigma^2 a')$ *there exist* $d \in \mathscr{A}$, $g_1 \in \mathscr{T}(a, \Sigma d)$, *and* $g_2 \in \mathscr{T}(\Sigma d, \Sigma^2 a')$ *such that* $f = g_2 \circ g_1$.
2. $\Sigma\mathscr{A} * \mathscr{A} = \mathscr{A} * \Sigma\mathscr{A}$.
3. *For* $c \in \Sigma\mathscr{A} * \mathscr{A}$, *there exist* $a \in \mathscr{A}$ *and* $f \in \mathscr{T}(a, c)$, *such that* $\psi_c \circ f : a \to G(c)$ *is an epimorphism.*

Proof. **(1) ⇒ (2)**: By Lemma 2.6(1), $\Sigma\mathscr{A} * \mathscr{A} \supseteq \mathscr{A} * \Sigma\mathscr{A}$. To check the other inclusion let $c \in \Sigma\mathscr{A} * \mathscr{A}$, which means that there is a triangle

$$\Sigma F(c) \longrightarrow c \longrightarrow G(c) \xrightarrow{\gamma} \Sigma^2 F(c).$$

By (1) there exists $d \in \mathscr{A}$ and morphisms $g_1 : G(c) \to \Sigma d$, $g_2 : \Sigma d \to \Sigma^2 F(c)$, such that $g_2 \circ g_1 = \gamma$. Using the octahedral axiom (see [Nee14, prop 1.4.6]) we get a diagram

$$\begin{array}{ccccccc} d & \longrightarrow & M & \longrightarrow & G(c) & \xrightarrow{g_1} & \Sigma d \\ \downarrow & & \downarrow & & \| & & \downarrow{\scriptstyle g_2} \\ \Sigma F(c) & \longrightarrow & c & \longrightarrow & G(c) & \xrightarrow{\gamma} & \Sigma^2 F(c) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ N & = & N & \longrightarrow & 0 & \longrightarrow & \Sigma N \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma d & \longrightarrow & \Sigma M & \longrightarrow & \Sigma G(c) & \longrightarrow & \Sigma^2 d, \end{array}$$

where each row and column is a triangle. Since $\mathscr{A}$ and $\Sigma\mathscr{A}$ are extension closed in $\mathscr{T}$ we get that $M \in \mathscr{A}$ and $N \in \Sigma\mathscr{A}$. The second column in the diagram now implies that $c \in \mathscr{A} * \Sigma\mathscr{A}$.

(2) ⇒ (1): Let $f \in \mathscr{T}(a, \Sigma^2 a')$, and consider the triangle

$$\Sigma a' \longrightarrow c \longrightarrow a \xrightarrow{f} \Sigma^2 a'.$$

Now $c \in \Sigma\mathscr{A} * \mathscr{A} = \mathscr{A} * \Sigma\mathscr{A}$, hence there exist $b_0, b_1 \in \mathscr{A}$ fitting into a triangle

$$b_1 \longrightarrow b_0 \longrightarrow c \longrightarrow \Sigma b_1.$$

The octahedral axiom gives the following commutative diagram with rows and columns being triangles.

$$\begin{array}{ccccccc}
d & \longrightarrow & b_0 & \xrightarrow{h} & a & \xrightarrow{g_1} & \Sigma d \\
\downarrow & & \downarrow & & \| & & \downarrow{\scriptstyle g_2} \\
\Sigma a' & \longrightarrow & c & \longrightarrow & a & \xrightarrow{f} & \Sigma^2 a' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma b_1 & = & \Sigma b_1 & \longrightarrow & 0 & \longrightarrow & \Sigma^2 b_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma d & \longrightarrow & \Sigma b_0 & \longrightarrow & \Sigma a & \longrightarrow & \Sigma^2 d.
\end{array}$$

It follows from Lemma 3.5 that h is an epimorphism, and therefore $d \in \mathscr{A}$. This concludes the argument since now $f = g_2 \circ g_1$.

(2) ⇒ (3): Follows directly from Lemma 3.5.

(3) ⇒ (2): By Lemma 2.6(1), $\mathscr{A} * \Sigma\mathscr{A} \subseteq \Sigma\mathscr{A} * \mathscr{A}$, hence there is only left to check that $\mathscr{A} * \Sigma\mathscr{A} \supseteq \Sigma\mathscr{A} * \mathscr{A}$. Let $c \in \Sigma\mathscr{A} * \mathscr{A}$, and assume we have an $a \in \mathscr{A}$ and a morphism $f : a \to c$ such that $\psi_c \circ f$ is an epimorphism. Using the octahedral axiom we can get the following commutative diagram, with rows and columns being triangles.

$$\begin{array}{ccccccc}
F(c) & \longrightarrow & 0 & \longrightarrow & \Sigma F(c) & = & \Sigma F(c) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1} M & \longrightarrow & a & \xrightarrow{f} & c & \longrightarrow & M \\
\downarrow & & \| & & \downarrow{\scriptstyle \psi_c} & & \downarrow \\
\ker(\psi_c \circ f) & \rightarrowtail & a & \xrightarrow{\psi_c \circ f} & G(c) & \longrightarrow & \Sigma \ker(\psi_c \circ f)
\end{array}$$

Since the right column is a triangle, we get that $M \in \Sigma\mathscr{A}$, and therefore since the middle row is a triangle $c \in \mathscr{A} * \Sigma\mathscr{A}$. □

4 intermediate subcategories

Setup 3.1 will be assumed throughout this section.

Lemma 4.1. *Let $\mathscr{F} \subseteq \mathscr{A}$ be a torsion-free class. Let $c \in \Sigma\mathscr{F} * \mathscr{A}$, then $F(c) \in \mathscr{F}$.*

Proof. Let $c \in \Sigma\mathscr{F} * \mathscr{A}$, then there is a triangle $\Sigma f \xrightarrow{\alpha} c \to a$, where $f \in \mathscr{F}$ and $a \in \mathscr{A}$. Since α is a right $\Sigma\mathscr{A}$-approximation we get that the object $F(c)$ from the minimal right $\Sigma\mathscr{A}$-approximation is a direct summand of f, and therefore $F(c) \in \mathscr{F}$. □

Theorem 4.2 (cf. [ES22, thm. 5.3])**.** *The following statements hold.*

1. *If $\mathscr{F} \subseteq \mathscr{A}$ is a torsion-free class then $\Sigma\mathscr{F} * \mathscr{A}$ is an $\mathscr{A}$-intermediate category. Furthermore, $F(\Sigma\mathscr{F} * \mathscr{A}) = \mathscr{F}$.*

2. *Let $\mathcal{C}$ be an $\mathscr{A}$-intermediate category such that $\mathcal{C} \subseteq \mathscr{A} * \Sigma\mathscr{A}$. Then $F(\mathcal{C})$ is torsion-free. Furthermore, we have that $\mathcal{C} = \Sigma F(\mathcal{C}) * \mathscr{A}$.*

Proof. (1) It follows directly from Lemma 2.6 that $\Sigma\mathscr{F} * \mathscr{A}$ is an $\mathscr{A}$-intermediate category. Furthermore, the claim that $F(\Sigma\mathscr{F} * \mathscr{A}) = \mathscr{F}$ follows from Lemma 4.1, together with the fact that $F(\Sigma f) = f$ for all $f \in \mathscr{F}$.

(2) That $\mathcal{C} \subseteq \Sigma F(\mathcal{C}) * \mathscr{A}$ follows directly from the fact that there for each $c \in \mathcal{C}$ is a triangle

$$\Sigma F(c) \longrightarrow c \longrightarrow G(c) \longrightarrow \Sigma^2 F(c).$$

To see that $\mathcal{C} \supseteq \Sigma F(\mathcal{C}) * \mathscr{A}$ it is enough to check that $\Sigma F(\mathcal{C}) \subseteq \mathcal{C}$. Let $c \in \mathcal{C}$. We will check that $\Sigma F(c) \in \mathcal{C}$. There is a triangle

$$\Sigma F(c) \xrightarrow{\phi_c} c \xrightarrow{\psi_c} G(c) \longrightarrow \Sigma^2 F(c).$$

Since $\mathcal{C} \subseteq \mathscr{A} * \Sigma\mathscr{A}$, Lemma 3.5 says that there exists an object $b \in \mathscr{A}$ and a morphism $\alpha : b \to c$ such that $\psi_c \circ \alpha : b \to G(c)$ is an epimorphism. Consider the following commutative square

$$\begin{array}{ccc}
\Sigma F(c) & = & \Sigma F(c) \\
\big\downarrow{\scriptstyle \binom{0}{1}} & & \big\downarrow{\scriptstyle \phi_c} \\
b \oplus \Sigma F(c) & \xrightarrow{(\alpha\ \phi_c)} & c.
\end{array}$$

Using the octahedral axiom we can complete this into a commutative diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & \Sigma F(c) & = & \Sigma F(c) \\
\big\downarrow & & \big\downarrow{\scriptstyle \binom{0}{1}} & & \big\downarrow{\scriptstyle \phi_c} \\
M & \longrightarrow & b \oplus \Sigma F(c) & \xrightarrow{(\alpha\ \phi_c)} & c \\
\| & & \big\downarrow{\scriptstyle (1\ 0)} & & \big\downarrow{\scriptstyle \psi_c} \\
M & \longrightarrow & b & \xrightarrow{\delta} & G(c),
\end{array}$$

where the rows and columns are short triangles. Since the lower right square commutes we get that $\delta = \psi_c \circ \alpha$ is an epimorphism. This implies that $M \in \mathscr{A}$. Using that the middle row in the diagram is a triangle we get that $b \oplus \Sigma F(c) \in \mathcal{C}$ due to $\mathcal{C}$ being extension closed. In particular, this means that $\Sigma F(c) \in \mathcal{C}$ given that $\mathcal{C}$ is closed under direct summands. This implies that $\mathcal{C} = \Sigma F(c) * \mathscr{A}$.

To show that $F(\mathcal{C})$ is a torsion-free class, we first need to check that it is extension closed. Let $c, c' \in \mathcal{C}$ and assume that there is a conflation

$$F(c) \rightarrowtail d \twoheadrightarrow F(c')$$

in $\mathscr{A}$. We just saw that $\Sigma F(c), \Sigma F(c') \in \mathcal{C}$, and since $\mathcal{C}$ is extension closed this implies that $\Sigma d \in \mathcal{C}$. Since $d \in \mathscr{A}$ we have $d = F(\Sigma d)$, thus $d \in F(\mathcal{C})$. Lastly to show that $F(\mathcal{C})$ is closed under subobjects, let $F(c) \in F(\mathcal{C})$, and let $d' \in \mathscr{A}$ be a subobject. This gives a triangle

$$a \longrightarrow \Sigma d' \longrightarrow \Sigma F(c) \longrightarrow \Sigma a$$

with $a \in \mathscr{A}$. Since $\mathcal{C}$ is closed under extensions, $\Sigma d' \in \mathcal{C}$. Given that $\Sigma d' \in \Sigma\mathscr{A}$, we get that $G(\Sigma d') = 0$ and thus $\Sigma F(\Sigma d') = \Sigma d'$. Therefore $d' = F(\Sigma d') \in F(\mathcal{C})$. □

Corollary 4.3. *If $\Sigma\mathscr{A} * \mathscr{A} = \mathscr{A} * \Sigma\mathscr{A}$ then there is a bijection*

$$\begin{aligned} \{\mathcal{C} \subseteq \mathscr{T} \mid \mathcal{C} \textit{ is } \mathscr{A}\textit{-intermediate}\} &\overset{1:1}{\longleftrightarrow} \{\mathscr{F} \subseteq \mathscr{A} \mid \mathscr{F} \textit{ torsion-free}\} \\ \mathcal{C} &\longmapsto F(\mathcal{C}) \\ \Sigma\mathscr{F} * \mathscr{A} &\longleftarrow\!\shortmid \mathscr{F} \end{aligned}$$

Proof. Follows directly from Theorem 4.2. □

Remark 4.4. Corollary 4.3 uses the assumption that $\Sigma\mathscr{A} * \mathscr{A} = \mathscr{A} * \Sigma\mathscr{A}$. Notice that Proposition 3.6 gives some statements that are equivalent to this.

Theorem 4.5. *Assume that $\mathscr{T}$ is skeletally small. Let $\mathscr{F} \subseteq \mathscr{A}$ be a torsion-free class, then the monoid morphism $i : M(\mathscr{A}) \to M(\Sigma\mathscr{F} * \mathscr{A})$ induced by the inclusion $\mathscr{A} \to \Sigma\mathscr{F} * \mathscr{A}$ induces an isomorphism*

$$M(\mathscr{A})_{M_\mathscr{F}} \to M(\Sigma\mathscr{F} * \mathscr{A}),$$

where $M_\mathscr{F} = \{[x] \in M(\mathscr{A}) \mid x \in \mathscr{F}\}$.

Proof. Using Lemma 3.4, this proof is exactly the same as in [ES22, thm. 5.4]. □

5 Examples

Lemma 5.1 ([HJY13, lem. 3.1])**.** *Let $\mathscr{C}$ be a $(-w)$-CY category for $w \in \mathbb{N}$. Then for each t-structure (X, Y) the associated heart $H = X \cap \Sigma Y = 0$.*

Given integers $w \geq 1$, $n \geq 1$, one can define the negative cluster category as the orbit category

$$\mathcal{C}_{-w}(A_n) := D^b(kA_n)/\Sigma^{w+1}\tau.$$

This is a triangulated category, and it is $(-w)$-Calabi–Yau, i.e. there is a Serre functor given by $\mathbb{S} = \Sigma^{-w}$. In [CSP16, sec. 10] they give a complete combinatorial model of this category. The indecomposable objects in $\mathcal{C}_{-w}(A_n)$ can be matched with certain diagonals in an N-gon, where $N = (w+1)(n+1)-2$. That is, if we label the vertices in the N-gon by $0, ..., N-1$, we can identify an indecomposable object $X \in \mathcal{C}_{-w}(A_n)$ as a pair of numbers $X = (a, b)$, where a and b correspond to the endpoints of the appropriate diagonal. A diagonal (a, b), with $a < b$, corresponds to an indecomposable if $w + 1 \mid b - a + 1$. We call these *admissible diagonals*. See Figure 5.1 for an example of such admissible diagonals.

There is also a combinatorial method to find proper abelian subcategories in $\mathcal{C}_{-w}(A_n)$, using simple-minded systems. A collection $\mathscr{S}$ consisting of n non-crossing admissible diagonals, with no two diagonals sharing an endpoint, corresponds to a w-simple-minded system in $\mathcal{C}_{-w}(A_n)$ (see [Jør22, def. 1.2] for a definition). given such a simple-minded system $\mathscr{S}$, one can obtain the corresponding abelian category $\langle \mathscr{S} \rangle$ by closing it under extensions. See Figure 5.1 for an example of such a collection.

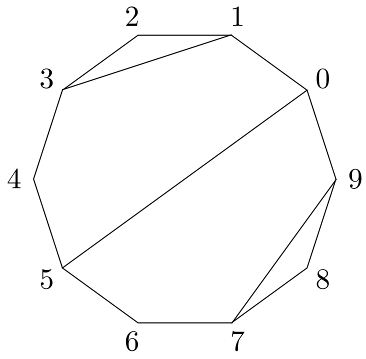


Figure 5.1: A simple-minded systems for $\mathcal{C}_{-2}(A_3)$.

Notice that due to Lemma 5.1 there is no t-structure with a non-trivial heart in any of the negative cluster categories. This means that the proper abelian subcategories obtained from simple-minded systems in $\mathcal{C}_{-w}(A_n)$ are not hearts of t-structures.

Example 5.2. Consider $\mathcal{C}_{-3}(A_4)$, the AR-quiver of which can be seen in Figure 5.2. The labels of the objects correspond to admissible diagonals of an 18-gon. This is a category where we can find proper abelian subcategories that satisfy the properties needed to use Corollary 4.3. Consider the collection of indecomposable objects

$$\mathscr{S} = \{(0,3), (4,11), (5,8), (12,15)\}.$$

This is a 3-simple-minded system. Consider the proper abelian subcategory $\mathscr{A} := \langle \mathscr{S} \rangle$ induced by $\mathscr{S}$. In Figure 5.2 we can see the indecomposable objects of $\mathscr{A}$ as those marked with red discs. Similarly, we can see $\Sigma\mathscr{A}$ in Figure 5.2 marked with blue discs. It is straightforward to check that $\mathscr{A}$ satisfies E_2 and that $\Sigma\mathscr{A} * \mathscr{A} = \mathscr{A} * \Sigma\mathscr{A}$.

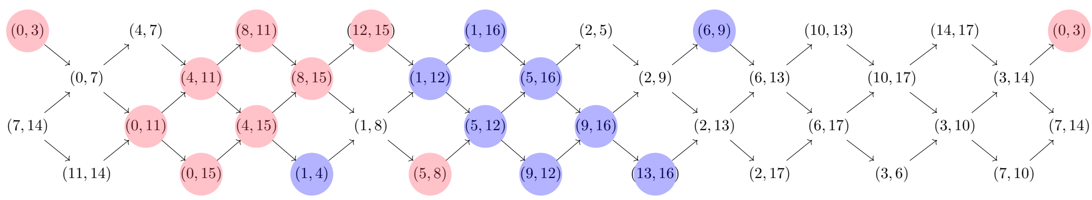
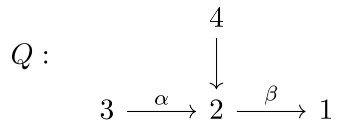


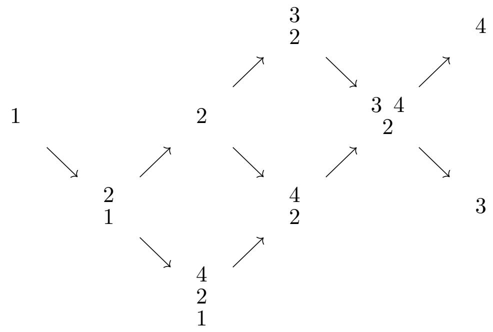
Figure 5.2: AR quiver for $C_{-3}(A_4)$. The red discs indicate $\mathscr{A}$, and the blue discs indicate $\Sigma\mathscr{A}$.

Using [Jør22, thm. 4.6] gives $\mathscr{A} \cong \operatorname{mod}(A)$, where $A = kQ/I$ with



$$I = \langle \alpha\beta \rangle.$$

Calculating the AR quiver of A results in the following:



where

$1 \sim (0,3)$ $\quad \begin{smallmatrix}2\\1\end{smallmatrix} \sim (0,11)$ $\quad \begin{smallmatrix}4\\2\\1\end{smallmatrix} \sim (0,15)$

$2 \sim (4,11)$ $\quad \begin{smallmatrix}4\\2\end{smallmatrix} \sim (4,15)$ $\quad \begin{smallmatrix}3\\2\end{smallmatrix} \sim (8,11)$

$\begin{smallmatrix}3\ 4\\2\end{smallmatrix} \sim (8,15)$ $\quad 4 \sim (12,15)$ $\quad 3 \sim (5,8).$

Using this we can calculate the torsion-free classes of $\mathscr{A}$ (see Figure 5.4). By Corollary 4.3 these torsion-free classes will correspond exactly to the $\mathscr{A}$-intermediate subcategories in $\mathcal{C}_{-3}(A_4)$. As an example, choose $\mathscr{F} = \operatorname{add}((0,3),(0,11),(4,11),(8,11))$, then $\mathcal{C} = \Sigma\mathscr{F} * \mathscr{A}$ is an $\mathscr{A}$-intermediate subcategory (see Figure 5.3).

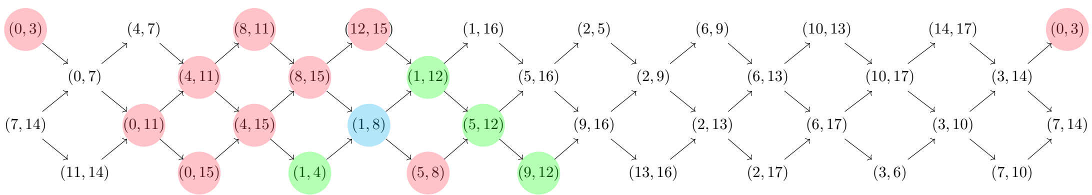
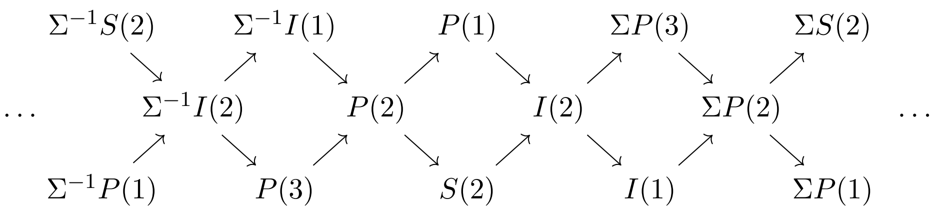


Figure 5.3: An $\mathscr{A}$-intermediate category of $C_{-3}(A_4)$. The red discs indicate $\mathscr{A}$, the green discs indicate $\Sigma\mathscr{F}$, and the cyan disc is not in either, but is contained in $\Sigma\mathscr{F} * \mathscr{A}$.

Example 5.3. Consider the algebra $A \coloneqq kQ$, where

$$Q: \quad 1 \longrightarrow 2 \longrightarrow 3.$$

The AR-quiver of $D^b(A)$ is as follows



Note that $\operatorname{mod} A$ sits inside $D^b(A)$ as a proper abelian subcategory since $\operatorname{mod} A$ is the heart of the standard t-structure in $D^b(A)$, but one can also find other proper abelian subcategories in $D^b(A)$ which are not hearts of t-structures.

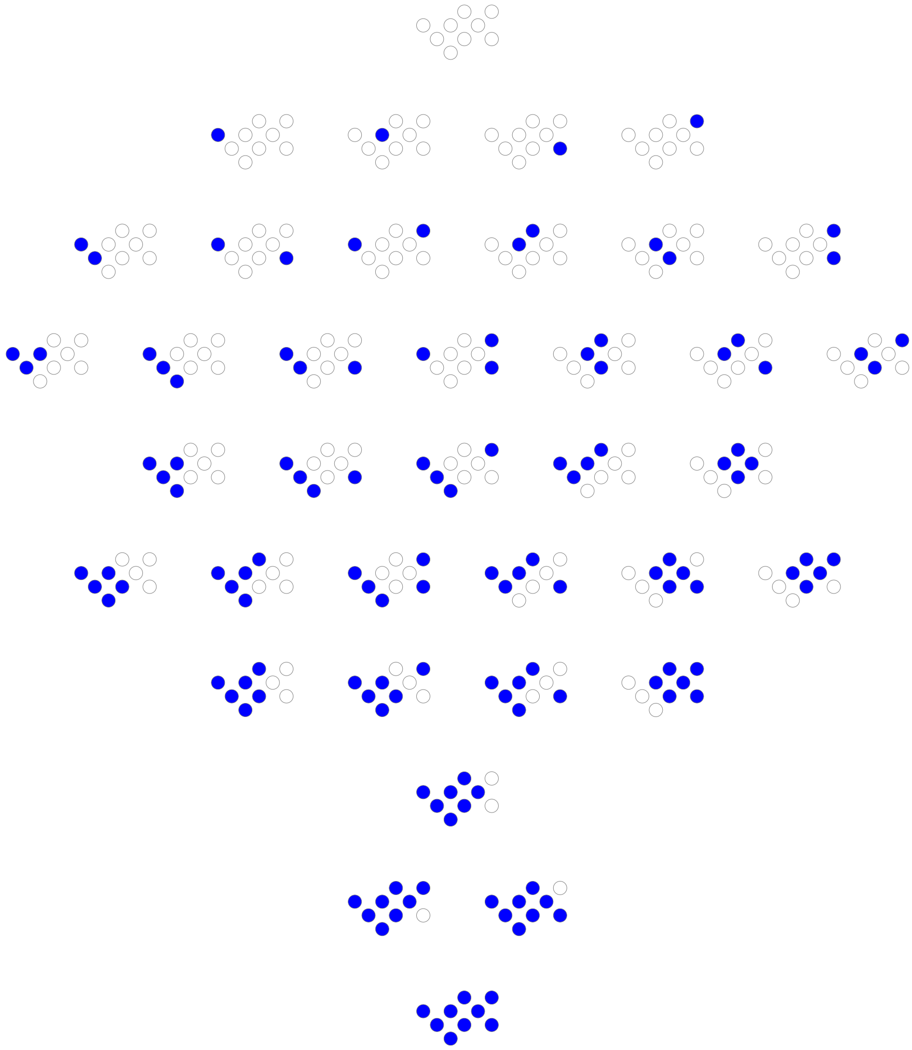


Figure 5.4: Torsion-free classes of $\mathscr{A}$. Each small figure is in the shape of the AR-quiver for $\mathscr{A}$ as described above, and the blue discs indicate the indecomposable objects of the torsion-free class.

For example, $\mathscr{S} = \{P(3), S(2)\}$ is a 2-orthogonal collection (see [Jør22, def. 1.2] for a definition). Thus the extension closure $\langle \mathscr{S} \rangle = \operatorname{add}(P(3), S(2), P(2))$ is a proper abelian subcategory by [Jør22, thm. A]. Notice that $\langle \mathscr{S} \rangle \cong \operatorname{mod} kA_2$. It is easy to check that $\langle \mathscr{S} \rangle$ satisfies E_2. This means that the proper abelian subcategory $\langle \mathscr{S} \rangle$ satisfies Setup 3.1. Furthermore,

$$\Sigma\langle \mathscr{S} \rangle * \langle \mathscr{S} \rangle = \Sigma\langle \mathscr{S} \rangle \oplus \langle \mathscr{S} \rangle = \langle \mathscr{S} \rangle * \Sigma\langle \mathscr{S} \rangle.$$

Since we know that $\langle \mathscr{S} \rangle \cong \operatorname{mod} kA_2$ we can find torsion-free classes. As an example we can choose $\mathscr{F} = \operatorname{add}(P(3), P(2))$, which would give the corresponding $\langle \mathscr{S} \rangle$-intermediate category $\mathcal{C} = \Sigma\mathscr{F} * \langle \mathscr{S} \rangle = \operatorname{add}(\Sigma P(3), \Sigma P(2), P(3), P(2), S(2))$.

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Paper C

Filtrations of Torsion Classes in Proper Abelian Subcategories

Abstract

In an abelian category $\mathscr{A}$, we can generate torsion pairs from tilting objects of projective dimension ≤ 1. However, when we look at tilting objects of projective dimension 2, there is no longer a natural choice of an associated torsion pair. Instead of trying to generate a torsion pair, Jensen, Madsen and Su generated a triple of extension closed classes that can filter any objects of $\mathscr{A}$. We generalize this result to proper abelian subcategories.

1 Introduction

Let k be a field. Given a finite dimensional k-algebra Λ let $\mathscr{A} = \operatorname{mod}\Lambda$. Given a tilting object $T \in \mathscr{A}$ of projective dimension $\operatorname{pd}(T) \leq 1$ (see [HRS96, chap. I.4] for a definition), we can construct a torsion pair $(\mathcal{T}, \mathcal{F})$ where $\mathcal{T} = \operatorname{Gen}(T)$ and $\mathcal{F} = \mathcal{T}^{\perp}$. For each $x \in \mathscr{A}$ there exists, by definition, a short exact sequence $t \rightarrowtail x \twoheadrightarrow f$ with $t \in \mathcal{T}$ and $f \in \mathcal{F}$. Another way to describe this is by saying that there exists a filtration $0 \subseteq t \subseteq x$ of x, where the quotient of the first inclusion $\operatorname{Cok}(0 \rightarrowtail t) \in \mathscr{T}$ and the quotient of the second inclusion $\operatorname{Cok}(t \rightarrowtail x) \in \mathscr{F}$. The reason to formulate it in this way will become clear a bit later.

If T is a tilting object of projective dimension $\operatorname{pd}(T) = 2$ there is no longer a natural choice for a torsion pair in $\mathscr{A}$ associated to T, but there will be an associated t-structure in the derived category $\mathsf{D}^b(\mathscr{A})$ whose heart is equivalent to $\mathscr{B} := \operatorname{mod}(\operatorname{End}(T)^{op})$ (under the right assumptions, see [BR07, sec. III.4]). Furthermore, $\mathscr{B}$ will be derived equivalent to $\mathscr{A}$. In [JMS13] Jensen, Madsen and Su use this derived equivalence to construct three extension closed classes $\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2 \subseteq \mathscr{A}$ with $\operatorname{Hom}(\mathcal{E}_i, \mathcal{E}_j) = 0$ for $i < j$, such that each object in $\mathscr{A}$ can be filtered with quotients in $\mathcal{E}_i$.

Theorem ([JMS13, thm. 2])**.** *Given $x \in \mathscr{A}$ there exists a unique filtration $0 = x_0 \subseteq x_1 \subseteq x_2 \subseteq x_3 = x$ such that $\operatorname{Cok}(x_i \rightarrowtail x_{i+1}) \in \mathcal{E}_i$, for $i = 0, 1, 2$.*

In this article we will give a different construction of the classes $\mathcal{E}_i$ and generalize this theorem to the setting of proper abelian subcategories. The concept of proper abelian subcategories is a generalization of hearts of t-structures, introduced by Jørgensen in [Jør22]. A proper abelian subcategory $\mathscr{A}$ is an abelian category that sits inside a triangulated category $\mathscr{T}$ in such a way that short exact sequences in $\mathscr{A}$ correspond exactly to short triangles in $\mathscr{T}$ whose objects are in $\mathscr{A}$, see Definition 2.18.

Instead of using the derived equivalence to construct the classes $\mathcal{E}_i$, we will construct them by using proper abelian subcategories $\mathscr{A}$ and $\mathscr{B}$ that satisfy the property that $\mathscr{B} \subseteq \Sigma^2\mathscr{A} * \Sigma\mathscr{A} * \mathscr{A}$ and $\mathscr{A} \subseteq \mathscr{B} * \Sigma^{-1}\mathscr{B} * \Sigma^{-2}\mathscr{B}$. With this we can show the following statement.

Theorem A (=Corollary 3.6)**.** *Let $\mathscr{T}$ be a triangulated category, and let $\mathscr{A}, \mathscr{B}$ be proper abelian subcategories, where $\mathscr{A}$ is a noetherian abelian category, satisfying the property $\mathscr{T}(\mathscr{A}, \Sigma^{-i}\mathscr{A}) = \mathscr{T}(\mathscr{B}, \Sigma^{-i}\mathscr{B}) = 0$ for $1 \leq i \leq 5$. Assume that $\mathscr{B} \subseteq \Sigma^2\mathscr{A} * \Sigma\mathscr{A} * \mathscr{A}$ and $\mathscr{A} \subseteq \mathscr{B} * \Sigma^{-1}\mathscr{B} * \Sigma^{-2}\mathscr{B}$. Then we can define extension closed classes $\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2 \subseteq \mathscr{A}$, with $\operatorname{Hom}(\mathcal{E}_i, \mathcal{E}_j) = 0$ for $i < j$, such that given $x \in \mathscr{A}$, there is a filtration of subobjects $0 = x_0 \subseteq x_1 \subseteq x_2 \subseteq x_3 = x$ such that each quotient $x_{i+1}/x_i = \operatorname{Cok}(x_i \rightarrowtail x_{i+1}) \in \mathcal{E}_i$.*

Notice that the condition that $\mathscr{T}(\mathscr{A}, \Sigma^{-i}\mathscr{A}) = 0$ for $1 \leq i \leq 5$ is satisfied if $\mathscr{A}$ is the heart of a t-structure (see [HJY13, lem. 3.1]). Furthermore, the condition that $\mathscr{B} \subseteq \Sigma^2\mathscr{A} * \Sigma\mathscr{A} * \mathscr{A}$ and $\mathscr{A} \subseteq \mathscr{B} * \Sigma^{-1}\mathscr{B} * \Sigma^{-2}\mathscr{B}$ essentially says that $\mathscr{A}$ and $\mathscr{B}$ are not too far apart, which as an example would be the case if $\mathscr{B}$ was induced from a tilting object of projective dimension ≤ 2 as in [JMS13].

In Section 4 we will apply Theorem A to an example of proper abelian subcategories that cannot be seen as hearts of t-structures.

2 Background

2.1 Abelian Categories

Definition 2.1. Let $\mathscr{T}$ be a triangulated category. Given full subcategories $\mathcal{X}, \mathcal{Z} \subseteq \mathscr{T}$ define the full subcategory

$$\mathcal{X} *_{\mathscr{T}} \mathcal{Z} = \{y \in \mathscr{T} \mid \text{there exists a triangle } x \to y \to z \to \Sigma x \text{ with } x \in \mathcal{X}, z \in \mathcal{Z}\}.$$

Similarly we can define $*$ for an abelian category.

Definition 2.2. Let $\mathscr{A}$ be an abelian category. Given full subcategories $\mathcal{X}, \mathcal{Z} \subseteq \mathscr{A}$ define the full subcategory

$$\mathcal{X} *_{\mathscr{A}} \mathcal{Z} = \{y \in \mathscr{A} \mid \text{there exists a short exact sequence } x \rightarrowtail y \twoheadrightarrow z \text{ with } x \in \mathcal{X}, z \in \mathcal{Z}\}.$$

Notation 2.3. We will omit the subscript of $*$ if it is clear in which category the operation is performed.

It is well-known that the operation $*$ is associative, both in the context of triangulated categories and abelian categories.

Notation 2.4. Let $\mathscr{A}$ be an abelian category. Given objects $x, y \in \mathscr{A}$ and a monomorphism $f : x \rightarrowtail y$, we write $y/x := \operatorname{Cok}(f)$.

Definition 2.5. Let $\mathscr{A}$ be an abelian category, and let $S \subseteq \mathscr{A}$ be a full subcategory.

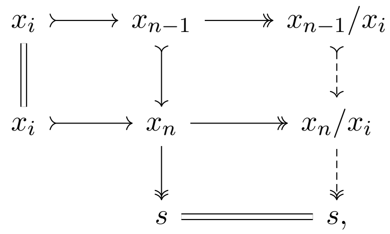
- $\operatorname{Gen}_{\mathscr{A}}(S) = \{x \in \mathscr{A} \mid \text{there exists an epimorphism } s \twoheadrightarrow x \text{ with } s \in S\}$.
- $\operatorname{Sub}_{\mathscr{A}}(S) = \{x \in \mathscr{A} \mid \text{there exists a monomorphism } x \rightarrowtail s \text{ with } s \in S\}$.
- $S^{\perp_{\mathscr{A}}} = \{x \in \mathscr{A} \mid \operatorname{Hom}(S, x) = 0\}$.
- ${}^{\perp_{\mathscr{A}}}S = \{x \in \mathscr{A} \mid \operatorname{Hom}(x, S) = 0\}$.
- S is said to be *extension closed* if $S * S \subseteq S$.

- Given $n \in \mathbb{N}$ let $(S)_n$ be the following full subcategory
$$(S)_n \coloneqq \left\{ a \in \mathscr{A} \;\middle|\; \begin{array}{l} a \text{ has a filtration } 0 = a_0 \subseteq a_1 \subseteq \cdots \subseteq a_n = a \\ \text{s.t. } a_{i+1}/a_i \in S \cup \{0\} \end{array} \right\}.$$
Then define the *extension closure of* S by $\langle S \rangle_{\mathscr{A}} \coloneqq \bigcup_{n \in \mathbb{N}} (S)_n$.

Notation 2.6. The subscripts of $\langle - \rangle, \operatorname{Gen}(-), \operatorname{Sub}(-), (-)^{\perp}$ and ${}^{\perp}(-)$ will be omitted if it is clear in which abelian category the operation is taking place.

Lemma 2.7. *Let $\mathscr{A}$ be an abelian category, and let $S \subseteq \mathscr{A}$ be a full subcategory. Given $x \in (S)_n$, with corresponding filtration $0 = x_0 \subseteq x_1 \subseteq \cdots \subseteq x_n = x$, then for all $0 \leq i < n$ the inclusion $x_i \subseteq x$ has cokernel $x/x_i \in (S)_{n-i}$.*

Proof. Consider the following diagram of solid arrows, where $\rightarrowtail$ represents the inclusions given by the filtration, which we then complete into short exact sequences.



with $s \in S \cup \{0\}$. Using [Buh10, lem. 3.5] we can fill out this diagram with the dashed arrows, such that the third column is a short exact sequence. In particular we get an inclusion $x_{n-1}/x_i \subseteq x_n/x_i$, with cokernel $(x_n/x_i)/(x_{n-1}/x_i) \cong s$. Using induction we can construct a filtration

$$0 = x_i/x_i \subseteq x_{i+1}/x_i \subseteq \cdots \subseteq x_{n-1}/x_n \subseteq x_n/x_i,$$

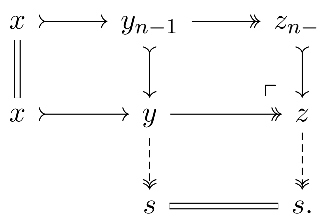
where $(x_j/x_i)/(x_{j-1}/x_i) \in S$ for $i < j \leq n$. In particular, we get that $x_n/x_i \in (S)_{n-i}$. $\square$

Lemma 2.8. *Let $\mathscr{A}$ be an abelian category, and let $S \subseteq \mathscr{A}$ be a full subcategory. Given $n, m \in \mathbb{N}$ then $(S)_{m+n} = (S)_m * (S)_n$.*

Proof. It follows from Lemma 2.7 that $(S)_{m+n} \subseteq (S)_m * (S)_n$, thus to show that they are equal, it is enough to show that $(S)_{m+n} \supseteq (S)_m * (S)_n$. Let $y \in (S)_m * (S)_n$, this means that there is a short exact sequence

$$x \rightarrowtail y \twoheadrightarrow z, \tag{2.1}$$

with $x \in (S)_m$ and $z \in (S)_n$. Since $z \in (S)_n$, there is a filtration $0 = z_0 \subseteq z_1 \subseteq \cdots \subseteq z_n = z$, where $z_i/z_{i-1} \in S \cup \{0\}$. Using the inclusion $z_{n-1} \subseteq z_n$ together with (2.1), we can construct the following pullback diagram of solid arrows



By [Buh10, prop. 2.12], the upper right square is bicartesian, meaning that the columns can be completed to short exact sequences, as illustrated by the dashed lines, such that $y/y_{n-1} \cong z/z_{n-1} \in S \cup \{0\}$. Notice that $x \subset y_{n-1}$. Using the same trick, an induction argument will construct a filtration $y_0 \subseteq y_1 \subseteq \cdots \subseteq y_n = y$, where $y_i/y_{i-1} \in S \cup \{0\}$. Furthermore, for each i we get a short exact sequence

$$x \rightarrowtail y_i \twoheadrightarrow z_i.$$

In particular we get such a short exact sequence for $i = 0$, and since $z_i = 0$ this gives that $x \cong y_0$. Notice that this short exact sequence also gives that $x \subseteq y_i$ for all i. Combining the filtration we have of y so far, together with that of x we get a filtration

$$0 = x_0 \subseteq x_1 \subseteq \cdots \subseteq x_m \subseteq y_1 \subseteq \cdots \subseteq y_n = y,$$

where the cokernel of each inclusion is contained in $S \cup \{0\}$, meaning that $y \in (S)_{m+n}$. With this we can conclude that $(S)_{m+n} = (S)_m * (S)_n$. □

Corollary 2.9. *Let $\mathscr{A}$ be an abelian category, and let $S \subseteq \mathscr{A}$ be a full subcategory. Then $\langle S \rangle$ is an extension-closed subcategory.*

Proof. Let $y \in \langle S \rangle * \langle S \rangle$. This means there is a short exact sequence

$$x \rightarrowtail y \twoheadrightarrow z$$

with $x, z \in \langle S \rangle = \cup_{n \in \mathbb{N}} (S)_n$. Thus there exists $n, m \in \mathbb{N}$ such that $x \in (S)_n$ and $z \in (S)_m$. Lemma 2.8 now gives that $y \in (S)_{n+m} \subseteq \langle S \rangle$. □

Lemma 2.10. *Let $\mathscr{A}$ be an abelian category, and let $\mathcal{X}, \mathcal{Y} \subseteq \mathscr{A}$ be full subcategories.*

1. *If $\mathcal{X}, \mathcal{Z}$ are closed under quotients, then so is $\mathcal{X} * \mathcal{Z}$.*
2. *If $\mathcal{X}, \mathcal{Z}$ are closed under subobjects, then so is $\mathcal{X} * \mathcal{Z}$.*

Proof. (1) Assume we have an epimorphism $v : y \twoheadrightarrow a$ with $a \in \mathscr{A}$ and $y \in \mathcal{X} * \mathcal{Z}$. That means there is a diagram with $x \in \mathcal{X}$ and $z \in \mathcal{Z}$ and the row short exact.

$$\begin{array}{ccccc} x & \overset{f}{\rightarrowtail} & y & \overset{g}{\twoheadrightarrow} & z \\ & & \downarrow{\scriptstyle v} & & \\ & & a & & \end{array}$$

Notice that vf factors over its own image, thus giving a commutative diagram of solid arrows

$$\begin{array}{ccccc} x & \overset{f}{\rightarrowtail} & y & \overset{g}{\twoheadrightarrow} & z \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \vdots{\scriptstyle w} \\ \mathrm{Im}(vf) & \overset{\alpha}{\rightarrowtail} & a & \overset{\beta}{\twoheadrightarrow} & \mathrm{Cok}(\alpha). \end{array}$$

Since $\beta v f = \beta \alpha u = 0$ there exists a morphism $w : z \to \mathrm{Cok}(\alpha)$, making the diagram above commute. Notice that w is an epimorphism since βv is an epimorphism. Hence $\mathrm{Im}(vf) \in \mathcal{X}$ and $\mathrm{Cok}(\alpha) \in \mathcal{Z}$ and thus $a \in \mathcal{X} * \mathcal{Z}$.

(2) Follows by a similar argument to (1). ☐

Lemma 2.11. *Let $\mathscr{A}$ be an abelian category, and let $S \subseteq \mathscr{A}$ be a full subcategory. Then $\langle \mathrm{Gen}(S) \rangle$ is closed under quotients, and $\langle \mathrm{Sub}(S) \rangle$ is closed under subobjects.*

Proof. This follows directly from Lemma 2.10 by the use of induction. ☐

Lemma 2.12. *Let $\mathscr{A}$ be an abelian category, and let $S \subseteq \mathscr{A}$, then the following hold.*

1. $S^{\perp} = \mathrm{Gen}(S)^{\perp}$,
2. $S^{\perp} = \langle S \rangle^{\perp}$,
3. ${}^{\perp}S = {}^{\perp}\mathrm{Sub}(S)$,
4. ${}^{\perp}S = {}^{\perp}\langle S \rangle$.

Proof. (1) The inclusion $S^{\perp} \supseteq \mathrm{Gen}(S)^{\perp}$ follows directly from the fact that $S \subseteq \mathrm{Gen}(S)$. For the other inclusion let $x \in S^{\perp}$, and let $z \in \mathrm{Gen}(S)$. This means that there exists an epimorphism $f : s \twoheadrightarrow z$, with $s \in S$. Now assume there is a morphism $h : z \to x$. Since $x \in S^{\perp}$ we get that $hf = 0$, and since f is an epimorphism, this implies that $h = 0$, and therefore $x \in \mathrm{Gen}(S)^{\perp}$.

$$\begin{array}{ccccc} s & \xrightarrow{f} & z & & \in \mathrm{Gen}(S) \\ & \overset{0}{\searrow} & \downarrow{\scriptstyle h} & & \\ & & x & & \in S^{\perp}. \end{array}$$

(2) The inclusion $S^{\perp} \supseteq \langle S \rangle^{\perp}$ follows directly from the fact that $S \subseteq \langle S \rangle$. To show the other inclusion, notice that $S^{\perp} = (S)_1^{\perp}$, and therefore it will be enough to show that $(S)_{n-1}^{\perp} \subseteq (S)_n^{\perp}$. Let $x \in (S)_{n-1}^{\perp}$, and let $z \in (S)_n$. By Lemma 2.8 there is a short exact sequence $s_1 \rightarrowtail z \twoheadrightarrow s_2$, with $s_1 \in S$ and $s_2 \in (S)_{n-1}$. Assume we have a morphism $f : z \to x$, that gives us the following diagram

$$\begin{array}{ccccc} s_1 & \overset{\alpha}{\rightarrowtail} & z & \overset{\beta}{\twoheadrightarrow} & s_2 \\ & & \downarrow{\scriptstyle f} & \overset{\gamma}{\swarrow} & \\ & & x. & & \end{array}$$

Since $x \in (S)_{n-1}^{\perp}$ we get that $f\alpha = 0$, and thus there exists a morphism $\gamma : s_2 \to x$ such that $f = \gamma\beta$. However, $s_2 \in (S)_{n-1}$ which implies that $\gamma = 0$, hence $f = 0$.

The proof of (3) and (4) is similar to that of (1) and (2). ☐

Definition 2.13. Let $\mathscr{A}$ be an abelian category, and let $\mathcal{F}, \mathcal{T} \subseteq \mathscr{A}$ be full subcategories. we say that $(\mathcal{T}, \mathcal{F})$ is a *torsion pair* if

1. $\mathrm{Hom}_{\mathscr{A}}(\mathcal{T}, \mathcal{F}) = 0$,
2. $\mathscr{A} = \mathcal{T} * \mathcal{F}$.

Here $\mathcal{T}$ is called the *torsion part*, and $\mathcal{F}$ is called the *torsion-free part*.

We can also define the torsion part and torsion-free part by themselves.

Definition 2.14. Let $\mathscr{A}$ be an abelian category, $\mathcal{X} \subseteq \mathscr{A}$ a full subcategory, then

1. $\mathcal{X}$ is called a *torsion class* if it is closed under quotients and extensions,
2. $\mathcal{X}$ is called a *torsion-free class* if it is closed under subobjects and extensions.

Given a torsion pair $(\mathcal{T}, \mathcal{F})$ in an abelian category $\mathscr{A}$, it is straightforward to see that the torsion part $\mathcal{T}$ will be a torsion class, and that the torsion-free part $\mathcal{F}$ will be a torsion-free class. However, it is important to note that given a torsion class $\mathcal{T}' \subseteq \mathscr{A}$, it does not need to be part of a torsion pair, and similar for torsion-free classes. If we want every torsion class to be part of a torsion pair, we need some assumptions on $\mathscr{A}$.

Definition 2.15. Let $\mathscr{A}$ be an abelian category, then $\mathscr{A}$ is said to be *noetherian* if for all objects $x \in \mathscr{A}$, ascending chains of subobjects $x_1 \subseteq x_2 \subseteq x_3 \subseteq \cdots$ of x stabilise. That is, there exists $n \in \mathbb{N}$ such that $x_n = x_{n+i}$ for all $i \in \mathbb{N}$.

Theorem 2.16 ([Pol07, lem. 1.1.3])**.** *Let $\mathscr{A}$ be a noetherian abelian category, and let $\mathcal{T} \subseteq \mathscr{A}$ be a torsion class, then $(\mathcal{T}, \mathcal{T}^{\perp})$ is a torsion pair. Similarly, given a torsion-free class $\mathcal{F} \subseteq \mathscr{A}$ then $({}^{\perp}\mathcal{F}, \mathcal{F})$ is a torsion pair.*

Corollary 2.17. *Let $\mathscr{A}$ be a noetherian abelian category, and $S \subseteq \mathscr{A}$. Then the pairs $(\langle \mathrm{Gen}(S) \rangle, S^{\perp})$ and $({}^{\perp}S, \langle \mathrm{Sub}(S) \rangle)$ are torsion pairs.*

Proof. This follows directly from Theorem 2.16 and lemmas 2.11 and 2.12. □

2.2 Proper Abelian Subcategories

Given an abelian category $\mathscr{A}$, it sits inside its derived category $\mathsf{D}^b(\mathscr{A})$ in such a way that short exact sequences in $\mathscr{A}$ induce triangles in $\mathsf{D}^b(\mathscr{A})$, and short triangles in $\mathsf{D}^b(\mathscr{A})$ with objects in $\mathscr{A}$ come from short exact sequences in $\mathscr{A}$. This situation can also be found many other places. Given a t-structure, the corresponding heart sits inside the associated triangulated category with this property. Simple-minded systems is another way to construct examples with this property that are neither hearts of t-structures nor contained in a derived category. To formalize this property we need the following definition.

Definition 2.18 ([Jør22, def. 1.2])**.** Let $\mathscr{T}$ be a triangulated category, and let $\mathscr{A} \subseteq \mathscr{T}$ be a full additive subcategory. $\mathscr{A}$ is called a *proper abelian subcategory of $\mathscr{T}$*, if it is an abelian category in such a way that $x \overset{\alpha}{\rightarrowtail} y \overset{\beta}{\twoheadrightarrow} z$ is a short exact sequence in $\mathscr{A}$ if and only if there is a triangle $x \xrightarrow{\alpha} y \xrightarrow{\beta} z \to \Sigma x$ in $\mathscr{T}$.

In a derived category $D = \mathsf{D}^b(\mathscr{A})$ the standard heart $\mathscr{A}$ has no negative self extensions, i.e. given $a, a' \in \mathscr{A}$ then $\mathrm{Hom}_D(a, \Sigma^{-n} a') = 0$ for $n \geq 1$. In general, this property is true for an arbitrary heart in a triangulated category (see [HJY13, lem. 3.1]). This is a very useful property of hearts, but it does not hold true for all proper abelian subcategories. Therefore we need the following definition.

Definition 2.19. Let $\mathscr{A} \subseteq \mathscr{T}$ be a proper abelian subcategory of a triangulated category $\mathscr{T}$. For $n \in \mathbb{N}$, we say that $\mathscr{A}$ satisfies E_n if $\operatorname{Hom}_{\mathscr{A}}(a, \Sigma^{-i}a') = 0$ for all $a, a' \in \mathscr{A}$ and $1 \leq i \leq n$.

Given two equivalent triangulated categories we can move proper abelian subcategories between them.

Proposition 2.20. *Let $\mathscr{T}, \mathscr{D}$ be triangulated categories, and $F : \mathscr{T} \to \mathscr{D}$ an equivalence of triangulated categories. Given a proper abelian subcategory $\mathscr{B} \subseteq \mathscr{D}$, then $F^{-1}(\mathscr{B}) \subseteq \mathscr{T}$ is a proper abelian subcategory.*

Proof. This is straightforward to check, thus we omit the proof. □

3 Filtrations of torsion classes

The following setup will be assumed throughout this section.

Setup 3.1. Let $\mathscr{T}$ be a triangulated category. Let $\mathscr{A}, \mathscr{B} \subseteq \mathscr{T}$ be proper abelian subcategories that satisfy E_5, such that $\mathscr{A} \subseteq \mathscr{B} * \Sigma^{-1}\mathscr{B} * \Sigma^{-2}\mathscr{B}$ and $\mathscr{B} \subseteq \Sigma^2\mathscr{A} * \Sigma\mathscr{A} * \mathscr{A}$. Furthermore, assume that $\mathscr{A}$ is noetherian. Define

- $\mathcal{E}_0 = \langle \operatorname{Gen}_{\mathscr{A}}(\mathscr{A} \cap \mathscr{B}) \rangle_{\mathscr{A}}$,
- $\mathcal{E}_1 = \mathscr{A} \cap \Sigma^{-1}\mathscr{B}$,
- $\mathcal{E}_2 = \langle \operatorname{Sub}_{\mathscr{A}}(\mathscr{A} \cap \Sigma^{-2}\mathscr{B}) \rangle_{\mathscr{A}}$.

Lemma 3.2. $\mathcal{E}_0^{\perp_{\mathscr{A}}} = (\mathscr{A} \cap \mathscr{B})^{\perp_{\mathscr{A}}} = \mathscr{A} \cap (\Sigma^{-1}\mathscr{B} * \Sigma^{-2}\mathscr{B})$.

Proof. **First equality:** This follows directly from Lemma 2.12.

Second equality: That $(\mathscr{A} \cap \mathscr{B})^{\perp_{\mathscr{A}}} \supseteq \mathscr{A} \cap (\Sigma^{-1}\mathscr{B} * \Sigma^{-2}\mathscr{B})$ follows directly from the fact that $\mathscr{B}$ satisfies E_2. For the other inclusion let $a \in (\mathscr{A} \cap \mathscr{B})^{\perp_{\mathscr{A}}}$. Since $\mathscr{A} \subseteq \mathscr{B} * \Sigma^{-1}\mathscr{B} * \Sigma^{-2}\mathscr{B}$, there exists a triangle

$$\Sigma^{-1}x \longrightarrow b \xrightarrow{f} a \longrightarrow x,$$

with $b \in \mathscr{B}$ and $x \in \Sigma^{-1}\mathscr{B} * \Sigma^{-2}\mathscr{B}$. We want to show that $f = 0$, which would give that a is a direct summand of x, and thus $a \in \Sigma^{-1}\mathscr{B} * \Sigma^{-2}\mathscr{B}$ by [IY08, prop. 2.1(1)], implying the inclusion we are seeking. Since $b \in \mathscr{B} \subseteq \Sigma^2\mathscr{A} * \Sigma\mathscr{A} * \mathscr{A}$ there exists a triangle

$$z \xrightarrow{g} b \longrightarrow \tilde{a} \longrightarrow \Sigma z,$$

where $\tilde{a} \in \mathscr{A}$ and $z \in \Sigma^2\mathscr{A} * \Sigma\mathscr{A}$. This gives the combined diagram

$$\begin{array}{ccccccc} & \swarrow\!\!\!\cdots & z & & & & \\ & & \downarrow g & & & & \\ \Sigma^{-1}x & \longrightarrow & b & \xrightarrow{f} & a & \longrightarrow & x. \\ & & \downarrow & & & & \\ & & \tilde{a} & & & & \end{array}$$

Since $\mathscr{A}$ satisfies E_2 we get that $fg = 0$, and therefore g factors through $\Sigma^{-1}x$, but

$$\Sigma^{-1}x \in \Sigma^{-2}\mathscr{B} * \Sigma^{-3}\mathscr{B} \subseteq (\mathscr{A} * \Sigma^{-1}\mathscr{A} * \Sigma^{-2}\mathscr{A}) * (\Sigma^{-1}\mathscr{A} * \Sigma^{-2}\mathscr{A} * \Sigma^{-3}\mathscr{A}).$$

Using that $\mathscr{A}$ satisfies E_5 we get that $\operatorname{Hom}(z, \Sigma^{-1}x) = 0$, and therefore $g = 0$. Thus b is a direct summand of $\tilde{a}$, in particular $b \in \mathscr{A}$. By assumption $a \in (\mathscr{A} \cap \mathscr{B})^{\perp_{\mathscr{A}}}$ meaning that $f = 0$, giving the result we want. □

Lemma 3.3. ${}^{\perp_{\mathscr{A}}}\mathcal{E}_2 = {}^{\perp_{\mathscr{A}}}(\mathscr{A} \cap \Sigma^{-2}\mathscr{B}) = \mathscr{A} \cap (\mathscr{B} * \Sigma^{-1}\mathscr{B})$.

Proof. The proof of this is very similar to that of Lemma 3.2 and is therefore omitted. □

Corollary 3.4. *There are torsion pairs*

- $(\mathcal{E}_0, \mathscr{A} \cap (\Sigma^{-1}\mathscr{B} * \Sigma^{-2}\mathscr{B}))$,
- $(\mathscr{A} \cap (\mathscr{B} * \Sigma^{-1}\mathscr{B}), \mathcal{E}_2)$.

Proof. This follows directly from Lemmas 3.2 and 3.3 and Theorem 2.16. □

Lemma 3.5. *There is a filtration of torsion classes* $0 \subseteq \mathcal{E}_0 \subseteq \mathscr{A} \cap (\mathscr{B} * \Sigma^{-1}\mathscr{B}) \subseteq \mathscr{A}$.

Proof. The only inclusion that it is necessary to check is the second inclusion. Notice that $\mathscr{A} \cap \mathscr{B} \subseteq \mathscr{A} \cap (\mathscr{B} * \Sigma^{-1}\mathscr{B})$. By Corollary 3.4 we get that $\mathscr{A} \cap (\mathscr{B} * \Sigma^{-1}\mathscr{B})$ is a torsion class, and is therefore closed under taking quotients and extensions. Thus it follows directly that $\mathcal{E}_0 = \langle \operatorname{Gen}_{\mathscr{A}}(\mathscr{A} \cap \mathscr{B}) \rangle_{\mathscr{A}} \subseteq \mathscr{A} \cap (\mathscr{B} * \Sigma^{-1}\mathscr{B})$. □

Corollary 3.6. *Let $x \in \mathscr{A}$, then up to isomorphism there exists a unique filtration of subobjects $0 = x_0 \subseteq x_1 \subseteq x_2 \subseteq x_3 = x$ such that each quotient $x_{i+1}/x_i = \operatorname{cok}(x_i \rightarrowtail x_{i+1}) \in \mathcal{E}_i$.*

Proof. By Corollary 3.4 we get that $(\mathscr{A} \cap (\mathscr{B} * \Sigma^{-1}\mathscr{B}), \mathcal{E}_2)$ is a torsion pair in $\mathscr{A}$. Thus there exists a short exact sequence

$$x_2 \rightarrowtail x \twoheadrightarrow e_2$$

with $x_2 \in \mathscr{A} \cap (\mathscr{B} * \Sigma^{-1}\mathscr{B})$ and $e_2 \in \mathcal{E}_2$. Corollary 3.4 also says that there is a torsion pair $(\mathcal{E}_0, A \cap (\Sigma^{-1}\mathscr{B} * \Sigma^{-2}\mathscr{B}))$. Therefore, there exists a short exact sequence

$$x_1 \rightarrowtail x_2 \twoheadrightarrow e_1$$

with $x_1 \in \mathcal{E}_0$ and $e_1 \in \mathscr{A} \cap (\Sigma^{-1}\mathscr{B} * \Sigma^{-2}\mathscr{B})$. Notice that since $\mathscr{A} \cap (\mathscr{B} * \Sigma^{-1}\mathscr{B})$ is a torsion class we get that $e_1 \in \mathscr{A} \cap (\mathscr{B} * \Sigma^{-1}\mathscr{B}) \cap (\Sigma^{-1}\mathscr{B} * \Sigma^{-2}\mathscr{B}) = \mathscr{A} \cap \Sigma^{-1}\mathscr{B} = \mathcal{E}_1$ by [Jør21, lem. 2.2(ii)].

The uniqueness follows directly from the uniqueness of the short exact sequences corresponding to torsion pairs. □

Definition 3.7. A tuple $(\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2)$ of full subcategories in an abelian category $\mathcal{A}$ is called a *torsion triple* if

1. $\operatorname{Hom}_{\mathcal{A}}(\mathcal{S}_i, \mathcal{S}_j) = 0$ for $i < j$,

2. $\mathcal{A} = \mathcal{S}_0 * \mathcal{S}_1 * \mathcal{S}_2$.

Corollary 3.8. *$(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2)$ is a torsion triple.*

Proof. That $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j) = 0$ for $i < j$ follows directly from the fact that $\mathscr{B}$ satisfies E_2. For the second condition, let $x \in \mathcal{A}$, then by Corollary 3.6 there is a filtration $0 = x_0 \subseteq x_1 \subseteq x_2 \subseteq x_3 = x$ such that each quotient $x_{i+1}/x_i \in \mathcal{E}_i$. Thus there exist short exact sequences

$$x_2 \rightarrowtail x \rightarrowtail e_2 \qquad \text{and} \qquad e_0 \rightarrowtail x_2 \rightarrowtail e_1,$$

where $e_i \in \mathcal{E}_i$. This means that $x_2 \in \mathcal{E}_0 * \mathcal{E}_1$ and thus $x \in \mathcal{E}_0 * \mathcal{E}_1 * \mathcal{E}_2$. □

Proposition 3.9. *There is a bijection*

$$\{\textit{torsion triples in } \mathscr{A}\} \xrightarrow{\Phi} \left\{ \begin{array}{l} \textit{Pairs of torsion pairs } [(\mathcal{T}, \mathcal{F}), (\tilde{\mathcal{T}}, \tilde{\mathcal{F}})] \\ \textit{in } \mathscr{A} \textit{ satisfying } \mathcal{T} \subseteq \tilde{\mathcal{T}} \end{array} \right\}$$

$$(\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2) \longmapsto (\mathcal{S}_0, \mathcal{S}_1 * \mathcal{S}_2), (\mathcal{S}_0 * \mathcal{S}_1, \mathcal{S}_2)$$

$$(\mathcal{T}, \mathcal{F} \cap \tilde{\mathcal{T}}, \tilde{\mathcal{F}}) \longleftarrow [(\mathcal{T}, \mathcal{F}), (\tilde{\mathcal{T}}, \tilde{\mathcal{F}})].$$

Proof. Denote the potential inverse for Φ by Φ'. It is straightforward to check that the maps take values in the relevant sets.

Let us check that $\Phi\Phi' = \mathrm{id}$. Let $[(\mathcal{T}, \mathcal{F}), (\mathcal{T}', \mathcal{F}')]$ be a pair of torsion pairs such that $\mathcal{T} \subseteq \mathcal{T}'$. Then

$$\Phi\Phi'((\mathcal{T}, \mathcal{F}), (\mathcal{T}', \mathcal{F}')) = \Phi(\mathcal{T}, \mathcal{F} \cap \mathcal{T}', \mathcal{F}') = [(\mathcal{T}, (\mathcal{F} \cap \mathcal{T}') * \mathcal{F}'), (\mathcal{T} * (\mathcal{F} \cap \mathcal{T}'), \mathcal{F}')].$$

Thus we need to check that $(\mathcal{F} \cap \mathcal{T}') * \mathcal{F}' = \mathcal{F}$ and $\mathcal{T} * (\mathcal{F} \cap \mathcal{T}') = \mathcal{T}'$. We check the first one of these, and the other one can be shown by a similar argument. Since $\mathcal{F}' \subseteq \mathcal{F}$ we get that $(\mathcal{F} \cap \mathcal{T}') * \mathcal{F}' \subseteq \mathcal{F}$. To see the other inclusion, let $x \in \mathcal{F}$, then since $(\mathcal{T}', \mathcal{F}')$ is a torsion pair, there is a short exact sequence

$$t' \rightarrowtail x \twoheadrightarrow f',$$

with $f' \in \mathcal{F}'$ and $t' \in \mathcal{T}'$. However, since torsion-free classes are closed under subobjects, we get that $t' \in \mathcal{F}$. Thus $t' \in \mathcal{F} \cap \mathcal{T}'$, and therefore $x' \in (\mathcal{F} \cap \mathcal{T}') * \mathcal{F}'$.

Let us check that $\Phi'\Phi = \mathrm{id}$. Let $(\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2)$ be a torsion triple. Then

$$\Phi'\Phi(\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2) = \Phi'((\mathcal{S}_0, \mathcal{S}_1 * \mathcal{S}_2), (\mathcal{S}_0 * \mathcal{S}_1, \mathcal{S}_2)) = (\mathcal{S}_0, (\mathcal{S}_1 * \mathcal{S}_2) \cap (\mathcal{S}_0 * \mathcal{S}_1), \mathcal{S}_2).$$

We therefore need to check if $(\mathcal{S}_1 * \mathcal{S}_2) \cap (\mathcal{S}_0 * \mathcal{S}_1) = \mathcal{S}_1$. It is straightforward to see that the inclusion $\supseteq$ is satisfied. For the other inclusion, let $x \in (\mathcal{S}_1 * \mathcal{S}_2) \cap (\mathcal{S}_0 * \mathcal{S}_1)$. Thus there are two short exact sequences $s_1 \rightarrowtail x \twoheadrightarrow s_2$ and $s_0 \rightarrowtail x \twoheadrightarrow s_1'$, with $s_i \in \mathcal{S}_i$ and $s_1' \in \mathcal{S}_1$. Consider the following diagram of solid arrows.

$$\begin{array}{ccccc} s_0 & \xrightarrow{f} & x & \xrightarrow{g} & s_1' \\ {\scriptstyle \eta} \downarrow & & \| & & \\ s_1 & \xrightarrow{\alpha} & x & \xrightarrow{\beta} & s_2. \end{array}$$

Notice that $\beta f = 0$ due to the definition of a torsion triple. Thus there exists a morphism $\eta : s_0 \to s_1$ such that $f = \alpha\eta$. However, $\eta = 0$ due to the same definition, implying that $f = 0$. Thus, g is an isomorphism, making $x \cong s_1' \in S_1$. □

4 Examples

4.1 Jensen-Madsen-Su

Let k be a field. Let $\mathscr{A}$ be a noetherian abelian category of the type studied in [JMS13], that is, $\mathscr{A}$ is either the module category of a finite-dimensional k-algebra, or it is a noetherian abelian k-category with finite homological dimension and Hom-finite derived category $\mathsf{D}^b(\mathscr{A})$, such that there is a locally noetherian abelian Grothendieck k-category $\mathscr{A}'$ with finite homological dimension such that $\mathscr{A} \subseteq \mathscr{A}'$ is the subcategory of noetherian objects (see [JMS13, sec. 0]).

Now consider a tilting object $T \in \mathscr{A}$ of homological dimension 2. By this, we mean an object $T \in \mathscr{A}$ that induces a derived equivalence

$$F = R\operatorname{Hom}(T, -) : \mathsf{D}^b(\mathscr{A}) \to \mathsf{D}^b(\widehat{\mathscr{B}}),$$

where $\widehat{\mathscr{B}} = \operatorname{mod}(\operatorname{End}(T)^{op})$, such that $H^iFX = \operatorname{Ext}^i(T, X) = 0$ for all $X \in \mathscr{A}$ and $i \geq 3$. Denote the quasi-inverse functor by $G : \mathsf{D}^b(\widehat{\mathscr{B}}) \to \mathsf{D}^b(\mathscr{A})$. Notice that $\widehat{\mathscr{B}}$ sits inside $\mathsf{D}^b(\widehat{\mathscr{B}})$ as a proper abelian subcategory, and since F is a derived equivalence we can pull $\widehat{\mathscr{B}}$ back to be considered as a proper abelian subcategory of $\mathsf{D}^b(\mathscr{A})$, see Proposition 2.20. Let $\mathscr{B} = F^{-1}(\widehat{\mathscr{B}})$.

To show that we are in a setup similar to that of Setup 3.1, we need the following lemma.

Lemma 4.1. *Using the notation from above, we have that* $\mathscr{A} \subseteq \mathscr{B} * \Sigma^{-1}\mathscr{B} * \Sigma^{-2}\mathscr{B}$ *and* $\mathscr{B} \subseteq \Sigma^2\mathscr{A} * \Sigma\mathscr{A} * \mathscr{A}$.

Proof. We start by proving the first inclusion. Let $x \in \mathscr{A}$. Using the assumption that T has projective dimension 2, it follows that $H^iR\operatorname{Hom}(T, x) = \operatorname{Ext}^i(T, x) = 0$ for $i \neq 0, 1, 2$. Thus by the use of soft truncations one can see that $R\operatorname{Hom}(T, x)$ is equivalent to a three term complex concentrated in cohomological degrees $0, 1, 2$, i.e. $x \in \mathscr{B} * \Sigma^{-1}\mathscr{B} * \Sigma^{-2}\mathscr{B}$.

To show the second inclusion, recall that $\mathscr{A}$ is the heart of the standard t-structure $(\mathcal{D}_{\geq 0}, \mathcal{D}_{<0})$ in $\mathsf{D}^b(\mathscr{A})$. For $i \in \mathbb{Z}$ denote $\mathcal{D}_{\geq i} = \Sigma^i\mathcal{D}_{\geq 0}$, and $\mathcal{D}_{<i} = \Sigma^i\mathcal{D}_{<0}$. Similarly, $\widehat{\mathscr{B}}$ is the heart of the standard t-structure $(\mathcal{P}_{\geq 0}, \mathcal{P}_{<0})$ in $\mathsf{D}^b(\widehat{\mathscr{B}})$. For $i \in \mathbb{Z}$ denote $\mathcal{P}_{\geq i} = \Sigma^i\mathcal{P}_{\geq 0}$, and $\mathcal{P}_{<i} = \Sigma^i\mathcal{P}_{<0}$. Since T has projective dimension 2 is follows that $F(\mathcal{D}_{\geq 0}) \subseteq \mathcal{P}_{\geq -2}$. Thus

$$\begin{aligned} F(\mathcal{D}_{\geq 0})^{\perp} \supseteq \mathcal{P}_{\geq -2}^{\perp} \quad &\Longrightarrow \quad F(\mathcal{D}_{\geq 0}^{\perp}) \supseteq \mathcal{P}_{\geq -2}^{\perp} \quad \Longrightarrow \quad F(\mathcal{D}_{<0}) \supseteq \mathcal{P}_{<-2} \\ &\Longrightarrow \quad \mathcal{D}_{<0} \supseteq G(\mathcal{P}_{<-2}) \quad \Longrightarrow \quad \mathcal{D}_{<3} \supseteq G(\mathcal{P}_{<1}), \end{aligned} \tag{4.1}$$

where the second implication follows from the fact that t-structures are torsion pairs, and the last implication follows by applying Σ^3. A similar calculation can be done for the torsion-free parts.

$$\begin{aligned} F(\mathcal{D}_{<0}) \subseteq \mathcal{P}_{<0} \quad &\Longrightarrow \quad {}^{\perp}F(\mathcal{D}_{<0}) \supseteq {}^{\perp}\mathcal{P}_{<0} \quad \Longrightarrow \quad F({}^{\perp}\mathcal{D}_{<0}) \supseteq {}^{\perp}\mathcal{P}_{<0} \\ &\Longrightarrow \quad F(\mathcal{D}_{\geq 0}) \supseteq \mathcal{P}_{\geq 0}. \quad \Longrightarrow \quad \mathcal{D}_{\geq 0} \supseteq G(\mathcal{P}_{\geq 0}). \end{aligned} \tag{4.2}$$

Combining (4.1) and (4.2) now gives that

$$G(\widehat{\mathscr{B}}) \subseteq G(\mathcal{P}_{\geq 0}) \cap G(\mathcal{P}_{<1}) \subseteq \mathcal{D}_{\geq 0} \cap \mathcal{D}_{<3}.$$

This means that objects in $\mathscr{B} = G(\widehat{\mathscr{B}})$ have homology concentrated in homological degrees $0, 1, 2$. Thus, $\mathscr{B} \subseteq \Sigma^2\mathscr{A} * \Sigma\mathscr{A} * \mathscr{A}$. □

In [JMS13] Jensen, Madsen and Su define collections of objects

$$\mathcal{F}^i = \{x \in \mathscr{A} \mid H^j F(x) = 0 \text{ for } j \neq i\},$$

for $i \geq 0$. Note that $\mathcal{F}^i = 0$ for $i \geq 3$. It is straightforward to check that $\mathcal{F}^i = \mathscr{A} \cap \Sigma^{-i}\mathscr{B}$. From this they build three other collections of objects $\mathscr{E}^i$, which by [JMS13, lem. 17 & 22] can be described as $\mathcal{E}^0 = \langle \mathrm{Gen}(\mathcal{F}^0)\rangle$, $\mathcal{E}^1 = \mathcal{F}^1$ and $\mathcal{E}^2 = \langle \mathrm{Sub}(\mathcal{F}^2)\rangle$. This places us in the situation of Setup 3.1 so [JMS13, thm. 2] follows from our Corollary 3.6.

4.2 Negative Cluster Categories

Negative cluster categories are examples of triangulated categories in which there are proper abelian subcategories, none of which are hearts of t-structures. Furthermore, these categories also have the advantage that there is a full combinatorial model describing some of them, see [CSP16, sec. 10]. Here we will give a short introduction.

Let $n, w \in \mathbb{N} = \{1, 2, 3, \ldots\}$, and define the *negative cluster category* as the orbit category

$$\mathcal{C}_{-w}(A_n) := \mathsf{D}^b(kA_n)/\Sigma^{w+1}\tau,$$

where τ refers to the Auslander-Reiten translation. This triangulated category is $-w$-Calabi–Yau (see [Kel05, sec. 4, sec. 8.4]), which means that Σ^{-w} is a Serre functor. Let $N = (w+1)(n+1)-2$ and consider the N-gon, say $\mathcal{P}_N$. Labelling the corners by $0, ..., N-1$ anticlockwise we can denote each diagonal in $\mathcal{P}_N$ by a pair of numbers (a, b) with $a < b$. We say that the diagonal (a, b) is *admissible* if $w + 1 \mid b - a + 1$. There is a one to one correspondence between indecomposable objects in $\mathcal{C}_{-w}(A_n)$ and admissible diagonals in the N-gon.

One way to find proper abelian subcategories in this setting is to create them from simple minded systems (see [Jør22, def. 1.2] for a definition). Given a simple minded system $\mathscr{S}$, a proper abelian subcategory is generated by taking the extension closure $\langle\mathscr{S}\rangle$, see [Jør22, thm. A]. In the setting of negative cluster categories, all simple minded systems can be classified combinatorially, see [CSP20, prop. 2.13] and [CS15, thm. 6.5]. In $\mathcal{C}_{-w}(A_n)$ a collection of n admissible diagonals corresponds to a w-simple minded system if no two diagonals in the collection cross and there are no two diagonals sharing an endpoint. See Figure 4.1 for two examples of such collections.

Furthermore, it is possible to describe triangles and morphisms between objects using the combinatorial model. All of this can be found in [CSP16, sec. 10].

The example. Let $w = 6$, $n = 5$, and consider the negative cluster category $\mathcal{C}_{-w}(A_n)$ which we can work with combinatorially as described above, using an N-gon, where $N = (w+1)(n+1) - 2 = 40$. In Figure 4.2 we can see a segment of the AR quiver of $\mathcal{C}_{-w}(A_n)$. Now consider the following two simple minded systems

$$\mathscr{S}_{\mathscr{A}} = \{(28, 34), (14, 20), (21, 27), (1, 7), (0, 13)\}$$
$$\mathscr{S}_{\mathscr{B}} = \{(23, 29), (7, 13), (22, 35), (1, 14), (15, 21)\}.$$

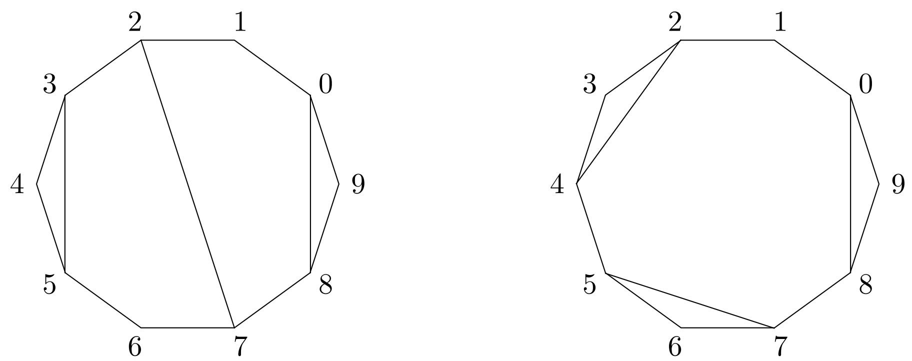


Figure 4.1: two simple-minded systems in $\mathcal{C}_{-2}(A_3)$.

Using these we construct proper abelian subcategories $\mathscr{A} = \langle \mathscr{S}_{\mathscr{A}} \rangle$ and $\mathscr{B} = \langle \mathscr{S}_{\mathscr{B}} \rangle$, see Figure 4.2. It is straightforward to check that both $\mathscr{A}$ and $\mathscr{B}$ satisfy E_5. Similarly, it is straightforward to check that $\mathscr{A} \subseteq \mathscr{B} * \Sigma^{-1}\mathscr{B} * \Sigma^{-2}\mathscr{B}$ and $\mathscr{B} \subseteq \Sigma^2\mathscr{A} * \Sigma^1\mathscr{A} * \mathscr{A}$. Lastly for Setup 3.1 to be satisfied, $\mathscr{A}$ needs to be noetherian. However, notice that $\mathscr{A}$ is isomorphic to the module category of a path algebra, that is $\mathscr{A} \cong \operatorname{mod} kQ$ where Q is the quiver

$$Q: \quad 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longleftarrow 5.$$

Thus we get that $\mathscr{A}$ is a noetherian abelian category, and thereby Setup 3.1 is satisfied. With this, the intersections needed can be described as

$$\begin{aligned}
\mathcal{E}_0 &= \langle \operatorname{Gen}_{\mathscr{A}}(\mathscr{A} \cap \mathscr{B}) \rangle_{\mathscr{A}} &&= \{(1,7), (7,13)\}, \\
\mathcal{E}_1 &= \mathscr{A} \cap \Sigma^{-1}\mathscr{B} &&= \{(0,34), (0,20), (14,20), (14,34), (21,34), (0,13), (28,34)\}, \\
\mathcal{E}_2 &= \langle \operatorname{Sub}_{\mathscr{A}}(\mathscr{A} \cap \Sigma^{-2}\mathscr{B}) \rangle_{\mathscr{A}} &&= \{(21,27)\},
\end{aligned}$$

see Figure 4.2. This means that we are in a setup where Corollary 3.6 can be used. To see a specific example of a filtration of an element consider $x = (7,27) \in \mathscr{A}$. This element has the filtration

$$0 \subsetneq (7,13) \subsetneq (7,20) \subsetneq (7,27) = x.$$

Using the combinatorial model for $\mathcal{C}_{-w}(A_n)$ we can see that there are short triangles,

$$(7,13) \longrightarrow (7,20) \longrightarrow (14,20) \quad \text{and} \quad (7,20) \longrightarrow (7,27) \longrightarrow (21,27).$$

Since all the elements of the two short triangles are in $\mathscr{A}$, this implies that we have corresponding short exact sequences

$$(7,13) \rightarrowtail (7,20) \twoheadrightarrow (14,20) \quad \text{and} \quad (7,20) \rightarrowtail (7,27) \twoheadrightarrow (21,27),$$

where $(7,13) \in \mathcal{E}_0$, $(14,20) \in \mathcal{E}_1$, and $(21,27) \in \mathcal{E}_2$. Thus the given filtration of x is indeed of the form stated by Corollary 3.6.

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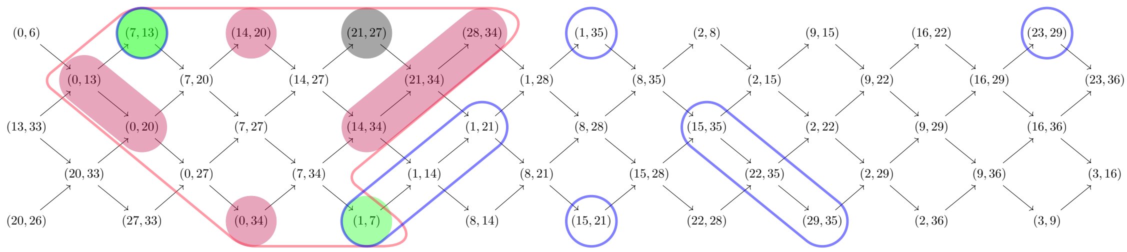

Figure 4.2: This is a segment of the AR quiver of $\mathscr{C}_{-6}(A_5)$. The objects in $\mathscr{A}$ are the ones surrounded by a red line, and the objects of $\mathscr{B}$ are the ones surrounded by a blue line. $\mathcal{E}_0$ is indicated by a green fill, $\mathcal{E}_1$ is indicated by a red fill, and $\mathcal{E}_2$ is indicated by a gray fill.

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Project D

Homology for Proper Abelian Subcategories

1 Introduction

Consider the derived category $\mathsf{D}(\mathscr{A})$ of an abelian category $\mathscr{A}$. Then each object $X \in \mathsf{D}(\mathscr{A})$ can be represented by a complex

$$\cdots \longrightarrow X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} X_{-1} \xrightarrow{d_{-1}} X_{-2} \longrightarrow \cdots .$$

From such a complex, we can define its homology groups. That is, given some $i \in \mathbb{Z}$ define the i'th homology $H_i(X) = \operatorname{Ker} d_i / \operatorname{Im} d_{i+1}$. One of the main features of homology is that each triangle $X \to Y \to Z \to \Sigma X$ induces a long exact sequence of homology

$$\cdots \longrightarrow H_{i+1}Z \longrightarrow H_iX \longrightarrow H_iY \longrightarrow H_iZ \longrightarrow H_{i-1}X \longrightarrow \cdots .$$

Using t-structures, the concept of homology has been generalized to a setting beyond what we have just seen. Given a triangulated category $\mathscr{T}$, let $\tau = (\mathcal{T}, \mathcal{F})$ be a t-structure with heart $\mathcal{H}_\tau$. A t-structure comes with two truncation functors $\tau_{\geq 0} : \mathscr{T} \to \mathcal{T}$ and $\tau_{<0} : \mathscr{T} \to \mathcal{F}$. Furthermore, define $\tau_{\geq i} = \Sigma^i \tau_{\geq 0} \Sigma^{-i}$, and $\tau_{<i} = \Sigma^i \tau_{<0} \Sigma^{-i}$. Another way to think about these, is that $\tau_{\geq i}$ and $\tau_{<i}$ are the truncation functors corresponding to the t-structure $(\Sigma^i \mathcal{T}, \Sigma^i \mathcal{F})$.

Using these functors we can define the i'th homology of an object $X \in \mathscr{T}$ as $H_i(X) = \Sigma^{-i} \tau_{\geq i} \tau_{<i+1} X = \tau_{\geq 0} \tau_{<1} \Sigma^{-i} X$. Notice that different t-structures can give different homology.

Essentially, the homology functor is a very nice functor $H_0 : \mathscr{T} \to \mathscr{A}$ from a triangulated category $\mathscr{T}$ to an embedded abelian category $\mathscr{A}$. However, there are many triangulated categories with nicely embedded abelian subcategories for which there is no corresponding t-structure. As an example, consider the abelian category $\mathscr{A} := \operatorname{mod}(kA_3)$ for some field k, together with the wide subcategory $\operatorname{mod}(kA_2) \subseteq \mathscr{A}$. We know that $\mathscr{A}$ is a heart of the standard t-structure inside $\mathsf{D}^b(\mathscr{A})$, however, there exists no t-structure whose heart is equivalent to $\operatorname{mod}(kA_2)$.

There are also triangulated categories in which there are no non-trivial t-structures. An example of such a category is the negative cluster category. However, we can usually find many nicely embedded abelian categories in a negative cluster category, such as those induced by simple-minded systems. In the examples above, generalizing any of the above methods to construct something that acts like homology is not straightforward.

But before we generalize homology to these cases, we must discuss what we mean by "nicely embedded". The categories from above are all examples of *proper abelian subcategories.*

Definition 1.1. Let $\mathscr{T}$ be a triangulated category and $\mathscr{A} \subseteq \mathscr{T}$ be a full additive subcategory. Then $\mathscr{A}$ is called a *proper abelian subcategory* if it is an abelian category in such a way that $x \rightarrowtail y \twoheadrightarrow z$ is a conflation in $\mathscr{A}$ if and only if there exists a triangle $x \to y \to z \to \Sigma x$ in $\mathscr{T}$ with objects $x, y, z \in \mathscr{A}$.

Notice that proper abelian subcategories are not extension-closed by definition; however, we will only be working with proper abelian subcategories that are extension-closed.

Consider an abelian category $\mathscr{A}$ and its bounded derived category $\mathsf{D}^b(\mathscr{A})$, then the standard t-structure is given by $\tau = (\mathcal{T}, \mathcal{F})$ where

$$\mathcal{T} = \cdots * \Sigma^2\mathscr{A} * \Sigma\mathscr{A} * \mathscr{A} \qquad \text{and} \qquad \mathcal{F} = \Sigma^{-1}\mathscr{A} * \Sigma^{-2}\mathscr{A} * \Sigma^{-3}\mathscr{A} * \cdots .$$

Now assume that the only objects in $\mathsf{D}^b(\mathscr{A})$ we care about, are complexes which are concentrated in degrees $-2, \dots, 2$, i.e. objects in $\mathcal{C} = \Sigma^2\mathscr{A} * \Sigma\mathscr{A} * \cdots * \Sigma^{-2}\mathscr{A}$. Then, to do the kinds of truncations associated with the t-structure τ on these objects, we do not need $\mathcal{T}$ and $\mathcal{F}$ to "extend" infinitely to either side. We just need it to cover $\mathcal{C}$. With this, we can prove the following result:

Theorem 1.2 (= Theorem 4.5)**.** *Let $\mathscr{A}$ be a proper abelian subcategory of a triangulated category $\mathscr{T}$. Let $n, m \in \mathbb{Z}$ with $n \geq m$, and assume that that $\operatorname{Hom}(\mathscr{A}, \Sigma^{-i}\mathscr{A}) = 0$ for $1 \leq i \leq n - m + 3$. Then there exist functors $H_i : \mathscr{T} \to \mathscr{A}$, such that given a triangle $c \to c' \to c'' \to \Sigma c$ with object $c, c', c'' \in \Sigma^n\mathscr{A} * \cdots * \Sigma^m\mathscr{A}$. Then the functors H_i induce a long exact sequence*

$$0 \to H_n c \to H_n c' \to H_n c'' \to H_{n-1} c \to \cdots \to H_{m+1} c'' \to H_m c \to H_m c' \to H_m c'' \to 0.$$

The notion of t-structures, and the homology of t-structures is due to Beilinson, Bernstein, and Deligne [BBD83]. Throughout this paper we will use a similar strategy as used in [BBD83], and some of the proofs will look similar.

Notation

Let $\mathcal{X}$ be a full subcategory category of the triangulated category $\mathscr{T}$ and let $n, m \in \mathbb{Z}$. Define $\Sigma^{[n,m]}\mathcal{X} = \Sigma^n\mathcal{X} * \cdots * \Sigma^m\mathcal{X}$, by convention if $n < m$ we let $\Sigma^{[n,m]}\mathcal{X} = 0$.

2 Background

Throughout this section, let $\mathscr{T}$ be a triangulated category.

2.1 Extriangulated categories

Extriangulated categories were defined by Nakaoka and Palu in [NP19]. It is a simultaneous generalization of exact and triangulated categories. For the full definition, see [NP19, Def. 2.12].

An *extriangulated category* is a triple $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$, where $\mathscr{C}$ is an additive category, $\mathbb{E} : \mathscr{C}^{op} \times \mathscr{C} \to \mathsf{Ab}$ is a biadditive functor, and $\mathfrak{s}$ is an additive realization of $\mathbb{E}$.

Remark 2.1. Let $\mathscr{C}$ be an extriangulated category, and let $\mathcal{X} \subseteq \mathscr{C}$ be a subcategory, that is closed under extensions. By [NP19, rem. 2.18], $\mathcal{X}$ will again be an extriangulated category if one defines the extriangulated structure as a restriction of the structure on $\mathscr{C}$. For the rest of this article, we will assume that it is given that an extension-closed subcategory has an extriangulated structure, which is given by restriction.

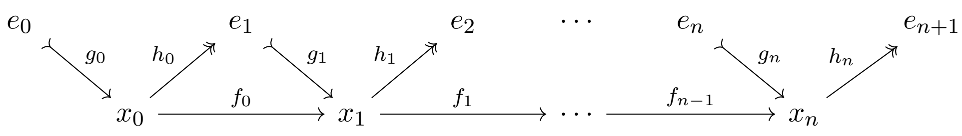
Definition 2.2 (Exact sequences)**.** Let $\mathscr{C}$ be an extriangulated category. A sequence

$$x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} x_n$$

is called *exact* if there exist objects $e_i \in \mathscr{C}$ for $0 \leq i \leq n+1$ and conflations

$$e_i \overset{g_i}{\rightarrowtail} x_i \overset{h_i}{\twoheadrightarrow} e_{i+1}$$

for $0 \leq i \leq n$, such that $f_i = g_{i+1} \circ h_i$. This can also be illustrated as the following commutative diagram.



where consecutive $\rightarrowtail \twoheadrightarrow$ are conflations in $\mathscr{C}$.

Remark 2.3. Given an abelian category $\mathscr{A}$, the exact sequences from the above definition align with the exact sequences we are used to.

3 Defining Homology functors

Throughout this section let $\mathscr{T}$ be a triangulated category, and let $n, m \in \mathbb{Z}$.

3.1 The E condition.

Definition 3.1. Let $\mathcal{X} \subseteq \mathscr{T}$ be a subcategory, $n \in \mathbb{N}$ or $n = \infty$ then $\mathcal{X}$ is said to *satisfy* E_n if $\operatorname{Hom}(\mathcal{X}, \Sigma^{-i}\mathcal{X}) = 0$ for $0 < i \leq n$.

An example of a category that satisfies all E_n properties is the heart of a t-structure.

Lemma 3.2. *Let $\mathcal{H}$ be the heart of a t-structure $(\mathcal{T}, \mathcal{F})$ in $\mathscr{T}$, then $\mathcal{H}$ satisfies E_∞.*

Proof. Recall that $\mathcal{H} \subseteq \Sigma\mathcal{F}$, and $\Sigma^i\mathcal{H} \subseteq \Sigma\mathcal{T}$ for all $i > 0$. Since $(\Sigma\mathcal{T}, \Sigma\mathcal{F})$ is a t-structure, we get that $\operatorname{Hom}(\Sigma\mathcal{T}, \Sigma\mathcal{F}) = 0$. Thus, it follows that $\mathcal{H}$ satisfies E_∞. □

Proper abelian subcategories do not, in general, satisfy E_∞. However, when working with proper abelian subcategories, we quickly realize that the vanishing of negative extensions is very useful, and many results require an E_n condition to be satisfied.

Lemma 3.3. *Let $\mathcal{X} \subseteq \mathscr{T}$ be a subcategory let $n > m$, and assume that $\mathcal{X}$ satisfies E_{n-m}, then $\Sigma^{[n,m]}\mathcal{X}$ is closed under direct summands.*

Proof. This follows directly from [IY08, prop. 2.1(1)] with the use of induction. □

Lemma 3.4. *Assume that $n > 1$, let $\mathcal{X}$ be an extension-closed subcategory of $\mathscr{T}$ satisfying E_{n-1} then $\mathcal{X} * \Sigma^n \mathcal{X} \subseteq \Sigma^n \mathcal{X} * \mathcal{X}$.*

Proof. Let $c \in \mathcal{X} * \Sigma^n \mathcal{X}$, meaning that there exists a triangle

$$x \longrightarrow c \longrightarrow \Sigma^n x' \xrightarrow{\alpha} \Sigma x,$$

with $x, x' \in \mathcal{X}$. Since $\mathcal{X}$ satisfies E_{n-1} we get that $\alpha = 0$, and thus $c \cong x \oplus \Sigma^n x' \in \mathcal{X} \oplus \Sigma^n \mathcal{X} \subseteq \Sigma^n \mathcal{X} * \mathcal{X}$. □

For proper abelian subcategories, the above lemma is also true for $n = 1$

Lemma 3.5 ([Jør21, lem. 4.1])**.** *Let $\mathscr{A} \subseteq \mathscr{T}$ be a proper abelian subcategory, then $\mathscr{A} * \Sigma\mathscr{A} \subseteq \Sigma\mathscr{A} * \mathscr{A}$.*

Corollary 3.6 (cf. [Jør21, thm. D])**.** *Let $\mathcal{X} \subseteq \mathscr{T}$ be an additive and extension-closed subcategory satisfying E_2. Then $\mathcal{X}$ is a proper abelian subcategory if and only if $\Sigma\mathcal{X} * \mathcal{X}$ is extension-closed.*

Proof. The only if part is given by Lemma 3.5. For the other direction, assume that $\Sigma\mathcal{X} * \mathcal{X}$ is extension-closed. Then

$$\mathcal{X} * \Sigma\mathcal{X} \subseteq (\Sigma\mathcal{X} * \mathcal{X}) * (\Sigma\mathcal{X} * \mathcal{X}) \subseteq \Sigma\mathcal{X} * \mathcal{X},$$

Thus [Jør21, thm. D] gives that $\mathcal{X}$ is a proper abelian subcategory. □

Lemma 3.7. *Let $m < n$ and let $\mathscr{A}$ be an extension-closed proper abelian category of $\mathscr{T}$, satisfying E_{n-m}, then $\Sigma^{[n,m]}\mathscr{A}$ is an extension-closed additive subcategory of $\mathscr{T}$. Especially, $\Sigma^{[n,m]}\mathscr{A}$ is an extriangulated category.*

Proof. Since $\Sigma^{[n,m]}\mathscr{A} = \Sigma^m \Sigma^{[n-m,0]}\mathscr{A}$, we can assume without loss of generality that $m = 0$. By Lemmas 3.4 and 3.5 we know that $\mathscr{A} * \Sigma^k\mathscr{A} \subseteq \Sigma^k\mathscr{A} * \mathscr{A}$ for all $n \geq k \geq 0$, thus by using this multiple times we get that

$$\Sigma^k\mathscr{A} * \cdots * \mathscr{A} * \Sigma^k\mathscr{A} \subseteq \Sigma^k\mathscr{A} * \Sigma^k\mathscr{A} * \Sigma^{k-1}\mathscr{A} * \cdots * \mathscr{A}.$$

Using this fact multiple times, we get that

$$\begin{aligned} \Sigma^{[n,0]}\mathscr{A} * \Sigma^{[n,0]}\mathscr{A} &= (\Sigma^n\mathscr{A} * \cdots * \mathscr{A}) * (\Sigma^n\mathscr{A} * \cdots * \mathscr{A}) \\ &\subseteq (\Sigma^n\mathscr{A} * \Sigma^n\mathscr{A}) * \cdots * (\mathscr{A} * \mathscr{A}) \\ &\subseteq \Sigma^n\mathscr{A} * \cdots * \mathscr{A} \\ &= \Sigma^{[n,0]}\mathscr{A}. \end{aligned}$$

Hence, $\Sigma^{[n,0]}\mathscr{A}$ is extension-closed and thereby also closed under direct sums. Therefore, $\Sigma^{[n,0]}\mathscr{A}$ is a additive subcategory of $\mathscr{T}$. It follows by [NP19, rem. 2.18] that $\Sigma^{[n,0]}\mathscr{A}$ is an extriangulated category. □

3.2 Truncation functors

For the rest of this section, we will assume the following setup.

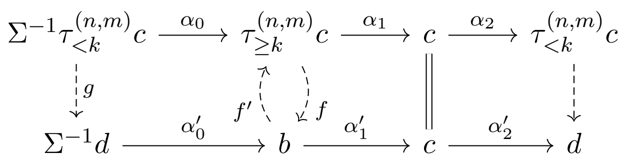
Setup 3.8. Let $\mathscr{T}$ be a triangulated category and $\mathscr{A} \subseteq \mathscr{T}$ be an extension-closed proper abelian subcategory. Let $m, n, k, l \in \mathbb{Z}$.

Lemma 3.9. *Let $m \leq k \leq n$, and assume that $\mathscr{A}$ satisfies E_{n-m+1}. Given $c \in \Sigma^{[n,m]}\mathscr{A}$, then up to isomorphism there exists a unique triangle*

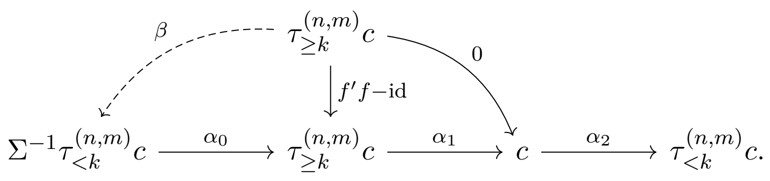
$$\tau^{(n,m)}_{\geq k} c \longrightarrow c \longrightarrow \tau^{(n,m)}_{<k} c \longrightarrow \Sigma\tau^{(n,m)}_{\geq k} c,$$

with $\tau^{(n,m)}_{\geq k} c \in \Sigma^{[n,k]}\mathscr{A}$ and $\tau^{(n,m)}_{<k} c \in \Sigma^{[k-1,m]}\mathscr{A}$. Furthermore, the assignments $c \mapsto \tau^{(n,m)}_{\geq k} c$ and $c \mapsto \tau^{(n,m)}_{<k} c$ induce functors.

Proof. The existence follows directly by the definition of $*$. Assume there is another such triangle $b \to c \to d \to \Sigma b$, and consider the following diagram of solid arrows.

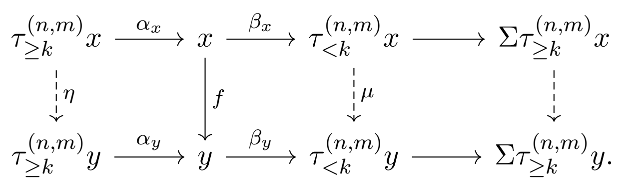


Since $\alpha_2'\alpha_1 = 0$, the dashed arrows in the diagram above exist, making the diagram commute. That the diagram is commutative means that $\alpha_1 f' f = \alpha_1' f = \alpha_1$, and thus $\alpha_1(f'f - \mathrm{id}) = 0$. Consider the following diagram of solid arrows.



Since $\alpha_1(f'f - \mathrm{id}) = 0$ there exists a morphism β making the above diagram commute. Due to $\mathscr{A}$ satisfying E_{n-m+1} we get that $\beta \in \mathrm{Hom}(\tau^{(n,m)}_{\geq k} c, \Sigma^{-1}\tau^{(n,m)}_{<k} c) = 0$, thus $f'f = \mathrm{id}$. A similar argument shows that $ff' = \mathrm{id}$, which means that f is an isomorphism. Using the 5-lemma gives that $d \cong \tau^{(n,m)}_{<k} c$.

To check that $\tau^{(n,m)}_{\geq k}$ and $\tau^{(n,m)}_{<k}$ induce functors, let $x, y \in \Sigma^{[n,m]}\mathscr{A}$ and a morphism $f : x \to y$ be given. Consider the following diagram of solid arrows.



Since $\mathscr{A}$ satisfies E_{n-m+1} it follows that $\beta_y f \alpha_x = 0$, hence the dashed morphisms η, μ in the diagram above exist. We want to check that η and μ are unique. The argument for the uniqueness of the two morphisms is very similar; therefore, we will only show that η is unique.

Assume that there exists another morphism $\eta' : \tau_{\geq k}^{(n,m)} x \to \tau_{\geq k}^{(n,m)} y$, such that $\alpha_y \eta' = f\alpha_x$, and consider the following diagram of solid arrows.

$$\Sigma^{-1}\tau_{<k}^{(n,m)} y \xrightarrow{\gamma_y} \tau_{\geq k}^{(n,m)} y \xrightarrow{\alpha_y} y \xrightarrow{\beta_y} \tau_{<k}^{(n,m)} y.$$

$$\eta - \eta' : \tau_{\geq k}^{(n,m)} x \to \tau_{\geq k}^{(n,m)} y, \quad 0 : \tau_{\geq k}^{(n,m)} x \to y, \quad \xi : \tau_{\geq k}^{(n,m)} x \dashrightarrow \Sigma^{-1}\tau_{<k}^{(n,m)} y$$

Since $\alpha_y(\eta - \eta') = \alpha_y \eta - \alpha_y \eta' = f\alpha_x - f\alpha_x = 0$ there exists a morphism ξ making the above diagram commute. However $\operatorname{Hom}(\tau_{\geq k}^{(n,m)} x, \Sigma^{-1}\tau_{<k}^{(n,m)} y) = 0$ since $\mathscr{A}$ satisfies E_{n-m+1}. This means that $\eta - \eta' = \gamma_y \xi = 0$, hence $\eta = \eta'$.

With this, we can now define $\tau_{\geq k}^{(n,m)} f = \eta$ and $\tau_{<k}^{(n,m)} f = \mu$. □

Notation 3.10. Using the notation from Lemma 3.9, for the rest of the article we will denote by $\tau_{\geq k}^{(n,m)} : \Sigma^{[n,m]}\mathscr{A} \to \Sigma^{[n,k]}\mathscr{A}$ the functor that assigns $c \mapsto \tau_{\geq k}^{(n,m)} c$. Similar, we will denote by $\tau_{<k}^{(n,m)} : \Sigma^{[n,m]}\mathscr{A} \to \Sigma^{[k-1,m]}\mathscr{A}$ the functor that assigns $c \mapsto \tau_{<k}^{(n,m)} c$.

For convenience, we will also define the functors $\tau_{>k}^{(n,m)} := \tau_{\geq k+1}^{(n,m)}$ and $\tau_{\leq k}^{(n,m)} := \tau_{<k+1}^{(n,m)}$.

Corollary 3.11. *Given integers $m' < n'$ and $m < n$, let $N = \max(n, n') - \min(m, m') + 1$, and assume that $\mathscr{A}$ satisfies E_N. If $x \in \Sigma^{[n,m]}\mathscr{A} \cap \Sigma^{[n',m']}\mathscr{A}$ then $\tau_{\geq k}^{(n,m)} x \cong \tau_{\geq k}^{(n',m')} x$ and $\tau_{<k}^{(n,m)} x \cong \tau_{<k}^{(n',m')} x$.*

Proof. Notice that the truncation triangles

$$\Sigma^{-1}\tau_{<k}^{(n,m)} x \longrightarrow \tau_{\geq k}^{(n,m)} x \xrightarrow{\alpha} x \longrightarrow \tau_{<k}^{(n,m)} x$$

and

$$\Sigma^{-1}\tau_{<k}^{(n',m')} x \longrightarrow \tau_{\geq k}^{(n',m')} x \xrightarrow{\alpha'} x \longrightarrow \tau_{<k}^{(n',m')} x.$$

both are truncation triangles for x in $\Sigma^{[\max(n,n'),\min(m,m')]}\mathscr{A}$ at degree k. Thus, the result follows by Lemma 3.9. □

Remark 3.12. So far, superscripts have been used for truncations to indicate where the truncations are taking place. Corollary 3.11 essentially says that under the correct E condition these truncations will coincide. Thus, from now on, we will omit the superscripts of truncations wherever they are not essential.

Lemma 3.13. *Let $n \geq k \geq m \geq l$ and assume that $\mathscr{A}$ satisfies E_{n-l+1}, then*

$$(\Sigma^{[n,m]}\mathscr{A}) \cap (\Sigma^{[k,l]}\mathscr{A}) = (\Sigma^{[k,m]}\mathscr{A}).$$

Proof. Let $x \in (\Sigma^{[n,m]}\mathscr{A}) \cap (\Sigma^{[k,l]}\mathscr{A})$. Since $x \in \Sigma^{[n,m]}\mathscr{A}$ there is a truncation triangle

$$\tau_{>k} x \xrightarrow{\alpha} x \xrightarrow{\beta} \tau_{\leq k} x \longrightarrow \Sigma\tau_{>k} x,$$

with $\tau_{>k}x \in \Sigma^{[n,k+1]}\mathscr{A}$ and $\tau_{\leq k}x \in \Sigma^{[k,m]}\mathscr{A}$. However, since $x \in \Sigma^{[k,l]}\mathscr{A}$ and $\mathscr{A}$ satisfies E_{n-l} we have that $\alpha = 0$. Thus x is a direct summand of $\tau_{\leq k}x$, and since $\Sigma^{[k,m]}\mathscr{A}$ is closed under direct summands by Lemma 3.3, we get that $x \in \Sigma^{[k,m]}\mathscr{A}$. ☐

Lemma 3.14. *Assume that $\mathscr{A}$ satisfies E_{n-m+1}, and $x \in \Sigma^{[n-1,m]}\mathscr{A}$, then $\tau_{\geq k+1}\Sigma x \cong \Sigma\tau_{\geq k}x$ and $\tau_{<k+1}\Sigma x \cong \Sigma\tau_{<k}x$.*

Proof. Let $x \in \Sigma^{[n-1,m]}\mathscr{A}$, then there is a truncation triangle

$$\tau^{(n-1,m)}_{\geq k}x \longrightarrow x \longrightarrow \tau^{(n-1,m)}_{<k}x \longrightarrow \Sigma\tau^{(n-1,m)}_{\geq k}x$$

in $\Sigma^{[n-1,m]}\mathscr{A}$. By applying Σ this induces a triangle in $\Sigma^{[n,m+1]}\mathscr{A}$

$$\Sigma\tau^{(n-1,m)}_{\geq k}x \longrightarrow \Sigma x \longrightarrow \Sigma\tau^{(n-1,m)}_{<k}x \longrightarrow \Sigma^2\tau^{(n-1,m)}_{\geq k}x.$$

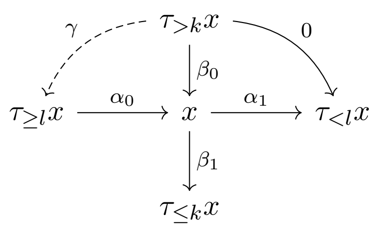
Since $\Sigma x \in \Sigma^{[n,m+1]}\mathscr{A}$, and $\Sigma\tau^{(n-1,m)}_{\geq k}x \in \Sigma^{[n,k+1]}\mathscr{A}$, and $\Sigma\tau^{(n-1,m)}_{<k}x \in \Sigma^{[k,m+1]}\mathscr{A}$, this triangle is isomorphic to the truncation triangle

$$\tau^{(n,m+1)}_{\geq k+1}\Sigma x \longrightarrow \Sigma x \longrightarrow \tau^{(n,m+1)}_{<k+1}\Sigma x \longrightarrow \Sigma\tau^{(n,m+1)}_{\geq k+1}\Sigma x,$$

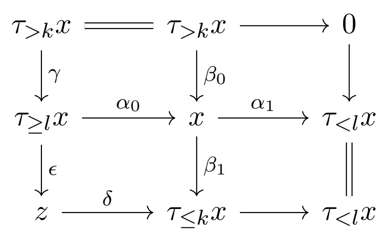
by Lemma 3.9, thereby proving the statement. ☐

Lemma 3.15. *Assume that $\mathscr{A}$ satisfies E_{n-m+2}, and that $m \leq l \leq k \leq n$. If $x \in \Sigma^{[n,m]}\mathscr{A}$ then $\tau_{\geq l}\tau_{\leq k}x = \tau_{\leq k}\tau_{\geq l}x$.*

Proof. Consider the following diagram.



where the middle row and the middle column are truncation triangles. Since $\mathscr{A}$ satisfies E_{n-m+1} and $l \leq k$ the composition $\alpha_1\beta_0 = 0$. Thus β_0 factors through α_0, meaning that there exists a morphism $\gamma : \tau_{>k}x \to \tau_{\geq l}x$ such that $\beta_0 = \alpha_0\gamma$. Next, use the octahedral axiom on the composition $\alpha_0\gamma$ to get the following diagram of triangles.



We now claim that $\tau_{\geq l}\tau_{\leq k}x = z = \tau_{\leq k}\tau_{\geq l}x$. Notice that

$$z \in (\Sigma^{[n,l]}\mathscr{A}) * (\Sigma^{[n+1,k+2]}\mathscr{A}) \subseteq \Sigma^{[n+1,l]}\mathscr{A},$$

and

$$z \in (\Sigma^{[l-2,m-1]}\mathscr{A}) * (\Sigma^{[k,m]}\mathscr{A}) \subseteq \Sigma^{[k,m-1]}\mathscr{A}.$$

By Lemma 3.13

$$z \in \Sigma^{[n+1,l]}\mathscr{A} \cap \Sigma^{[k,m-1]}\mathscr{A} = \Sigma^{[k,l]}\mathscr{A}.$$

However, this implies that the triangle

$$\tau_{>k}x \xrightarrow{\gamma} \tau_{\geq l}x \xrightarrow{\epsilon} z$$

is the unique truncation triangle (see Lemma 3.9) that splits $\tau_{\geq l}x$ at degree k. i.e., the following triangle

$$\tau_{>k}\tau_{\geq l}x \xrightarrow{\gamma'} \tau_{\geq l}x \xrightarrow{\epsilon} \tau_{\leq k}\tau_{\geq l}x.$$

Thus $z \cong \tau_{\leq k}\tau_{\geq l}x$. Using a similar argument, one obtains that $z \cong \tau_{\geq l}\tau_{\leq k}x$. □

Definition 3.16 (Homology)**.** Assume that $\mathscr{A}$ satisfies E_{n-m+1}, and let $m \leq k \leq n$. Given $x \in \Sigma^{[n,m]}\mathscr{A}$ define $H_k^{(n,m)}(x) = \Sigma^{-k}\tau_{\geq k}\tau_{\leq k}x \in \mathscr{A}$.

Remark 3.17. The superscript of $H_k^{(n,m)}$ is omitted if the interval is given or implied.

Lemma 3.18. *Assume that $\mathscr{A}$ satisfies E_{n-m+1} for $m < k < n$. Given $x \in \Sigma^{[n-1,m]}\mathscr{A}$ then $H_k^{(n,m)}\Sigma x \cong H_{k-1}^{(n,m)}x$.*

Proof. Let $x \in \Sigma^{[n-1,m]}\mathscr{A}$ and calculate

$$\begin{aligned}
H_k^{(n,m)}\Sigma x &= \Sigma^{-k}\tau_{\geq k}^{(n,m)}\tau_{\leq k}^{(n,m)}\Sigma x \\
&\cong \Sigma^{-k}\tau_{\geq k}^{(n,m+1)}\tau_{\leq k}^{(n,m+1)}\Sigma x && \text{by Corollary 3.11} \\
&\cong \Sigma^{-k}\tau_{\geq k}^{(n,m+1)}\Sigma\tau_{\leq k-1}^{(n-1,m)}x && \text{by Lemma 3.14} \\
&\cong \Sigma^{-(k-1)}\tau_{\geq k-1}^{(n-1,m)}\tau_{\leq k-1}^{(n-1,m)}x && \text{by Lemma 3.14} \\
&\cong \Sigma^{-(k-1)}\tau_{\geq k-1}^{(n,m)}\tau_{\leq k-1}^{(n,m)}x && \text{by Corollary 3.11} \\
&= H_{k-1}^{(n,m)}x.
\end{aligned}$$

□

4 Exactness of homology

The following setup will be assumed throughout this section.

Setup 4.1. Let $\mathscr{T}$ be a triangulated category and $\mathscr{A} \subseteq \mathscr{T}$ an extension-closed proper abelian subcategory. Let $n, m, k \in \mathbb{Z}$ with $n \geq m$ and assume that $\mathscr{A}$ satisfies E_{n-m+2}.

Lemma 4.2. *Assume that $m \leq k \leq n$, and let $c \to c' \to c'' \to \Sigma c$ be a triangle with elements in $\Sigma^{[n,m]}\mathscr{A}$ such that $\tau_{<k}c = 0$, then the induced sequence*

$$\tau_{\geq k}c \rightarrowtail \tau_{\geq k}c' \twoheadrightarrow \tau_{\geq k}c'$$

is a conflation in $\Sigma^{[n,k]}\mathscr{A}$.

Proof. The triangle induces the following diagram of solid arrows, where each column is a truncation triangle.

$$\begin{array}{ccccccccccc}
\tau_{\geq k}c & \xrightarrow{\tau_{\geq k}f} & \tau_{\geq k}c' & \longrightarrow & \tau_{\geq k}c'' & \dashrightarrow & \Sigma\tau_{\geq k}c & \longrightarrow & \Sigma\tau_{\geq k}c' & \longrightarrow & \Sigma\tau_{\geq k}c'' \\
\| \phi & & \downarrow \phi' & & \downarrow \phi'' & & \| & & \downarrow & & \downarrow \\
c & \xrightarrow{f} & c' & \xrightarrow{g} & c'' & \xrightarrow{h} & \Sigma c & \longrightarrow & \Sigma c' & \longrightarrow & \Sigma c'' \\
\downarrow & & \downarrow \psi' & & \downarrow \psi'' & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \tau_{<k}c' & \xrightarrow{\tau_{<k}g} & \tau_{<k}c'' & & 0 & \longrightarrow & \Sigma\tau_{<k}c' & \longrightarrow & \Sigma\tau_{<k}c''
\end{array}$$

Since $\tau_{<k}c = 0$ we have that ϕ is the identity, and thus it is clear that the dashed arrow exists (the dashed arrow is the composition $h\phi''$). Considering the octahedral axiom used on the composition $\phi'\tau_{\geq k}f$, the following diagram is obtained.

$$\begin{array}{ccccccc}
\tau_{\geq k}c & \xrightarrow{\tau_{\geq k}f} & \tau_{\geq k}c' & \xrightarrow{\gamma} & z & \xrightarrow{\delta} & \Sigma\tau_{\geq k}c \\
\| & & \downarrow \phi' & & \downarrow \alpha & & \| \\
c & \xrightarrow{f} & c' & \xrightarrow{g} & c'' & \longrightarrow & \Sigma c \\
\downarrow & & \downarrow \psi' & & \downarrow \beta & & \downarrow \\
0 & \longrightarrow & \tau_{<k}c' & = & \tau_{<k}c' & \longrightarrow & 0,
\end{array} \tag{4.1}$$

where each row and column is a triangle. From the top row, we can deduce that

$$z \in (\Sigma^{[n,k]}\mathscr{A}) * (\Sigma^{[n+1,k+1]}\mathscr{A}) \subseteq \Sigma^{[n+1,k]}\mathscr{A}.$$

Thus the triangle from the third column of (4.1) is a truncation triangle in $\Sigma^{[n+1,m]}\mathscr{A}$. Since $c'' \in \Sigma^{[n,m]}\mathscr{A}$ Corollary 3.11 gives that $\tau_{\geq k}c'' \cong z$ and $\tau_{<k}c' \cong \tau_{<k}c''$. That is, we have a commutative diagram

$$\begin{array}{ccccc}
\tau_{\geq k}c'' & \xrightarrow{\phi''} & c'' & \xrightarrow{\psi''} & \tau_{<k}c'' \\
\downarrow \xi_0 & & \| & & \downarrow \xi_1 \\
z & \xrightarrow{\alpha} & c'' & \xrightarrow{\beta} & \tau_{<k}c',
\end{array}$$

where ξ_0 is an isomorphism. Considering this diagram, we can construct Diagram 4.1, in which every square, and the middle triangle, commutes. For

$$\tau_{\geq k}c \xrightarrow{\tau_{\geq k}f} \tau_{\geq k}c' \xrightarrow{\tau_{\geq k}g} \tau_{\geq k}c'' \longrightarrow \Sigma\tau_{\geq k}c$$

to be a triangle, it is enough to show that the top triangle commutes, i.e., that $\gamma = \xi_0\tau_{\geq k}g$. Calculate

$$\alpha\xi_0\tau_{\geq k}g = \mathrm{id}\,\phi''\tau_{\geq k}g = g\phi' = \alpha\gamma,$$

and thus $\alpha(\xi_0\tau_{\geq k}g - \gamma) = 0$. Therefore, $\xi_0\ \tau_{\geq k}g - \gamma$ factors through $\Sigma^{-1}\tau_{<k}c'$.

$$\begin{array}{ccccccc}
 & \overset{\eta}{\swarrow\text{- - -}} & \tau_{\geq k}c' & \overset{0}{\searrow} & & & \\
 & & \downarrow \xi_0\tau_{\geq k}g-\gamma & & & & \\
\Sigma^{-1}\tau_{<k}c' & \longrightarrow & z & \xrightarrow{\alpha} & c'' & \xrightarrow{\beta} & \tau_{<k}c'
\end{array}$$

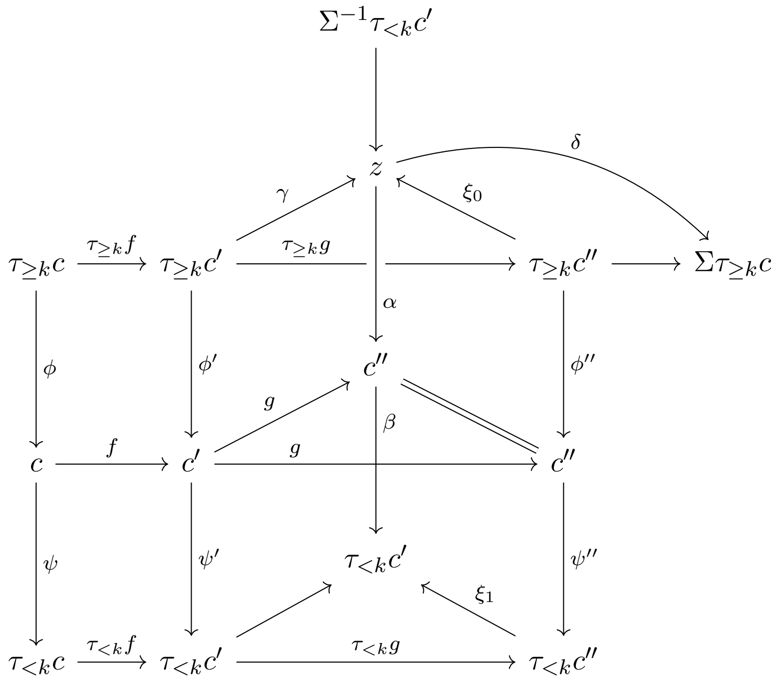
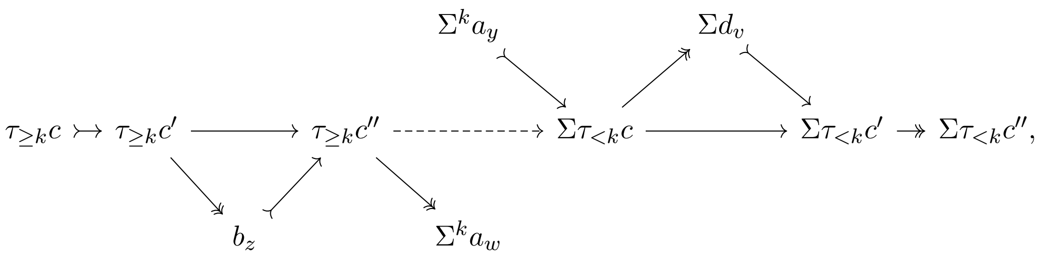


Diagram. 4.1: A diagram using in the proof of Lemma 4.2

Since $\mathscr{A}$ satisfies E_{n-m+1} we get that $\operatorname{Hom}(\tau_{\geq k}c', \Sigma^{-1}\tau_{<k}c') = 0$ and thus $\eta = 0$. With this we get that $\gamma = \xi_0 \tau_{\geq k} g$. $\square$

Lemma 4.3. *Let $c \to c' \to c'' \to \Sigma c$ be a triangle with $c, c', c'' \in \Sigma^{[n,m]}\mathscr{A}$. Then, there are exact sequences*



where $b_z \in \Sigma^{[n,k]}\mathscr{A}$, and $\Sigma^k a_w, \Sigma^k a_w \in \Sigma^k\mathscr{A}$, and $\Sigma d_v \in \Sigma^{[k,m+1]}\mathscr{A}$. Consecutive $\rightarrowtail \twoheadrightarrow$ are conflations in $\Sigma^{[n,m+1]}\mathscr{A}$.

Proof. Consider the following diagram in which each column is a truncation triangle.

$$
\begin{array}{lccccccccccc}
\Sigma^{[n,k]}\mathscr{A} \ni & \tau_{\geq k}c & \xrightarrow{\tau_{\geq k}f} & \tau_{\geq k}c' & \xrightarrow{\tau_{\geq k}g} & \tau_{\geq k}c'' & & \Sigma\tau_{\geq k}c & \longrightarrow & \Sigma\tau_{\geq k}c' & \longrightarrow & \Sigma\tau_{\geq k}c'' \\
& \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi''} & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{[n,m]}\mathscr{A} \ni & c & \xrightarrow{f} & c' & \xrightarrow{g} & c'' & \xrightarrow{h} & \Sigma c & \longrightarrow & \Sigma c' & \longrightarrow & \Sigma c'' \\
& \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \psi''} & & \downarrow{\scriptstyle \Sigma\psi} & & \downarrow & & \downarrow \\
\Sigma^{[k-1,m]}\mathscr{A} \ni & \tau_{<k}c & \xrightarrow{\tau_{<k}f} & \tau_{<k}c' & \xrightarrow{\tau_{<k}g} & \tau_{<k}c'' & & \Sigma\tau_{<k}c & \longrightarrow & \Sigma\tau_{<k}c' & \longrightarrow & \Sigma\tau_{<k}c''
\end{array}
$$

Using the octahedral axiom on the composition $f\phi$ gives the diagram

$$\begin{array}{ccccccc}
\tau_{\geq k}c & = & \tau_{\geq k}c & \longrightarrow & 0 & \longrightarrow & \Sigma\tau_{\geq k}c \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle f\phi} & & \downarrow & & \downarrow \\
c & \xrightarrow{f} & c' & \longrightarrow & c'' & \xrightarrow{h} & \Sigma c \\
\downarrow & & \downarrow & & \| & & \downarrow{\scriptstyle \Sigma\psi} \\
\tau_{<k}c & \longrightarrow & z & \xrightarrow{\nu} & c'' & \xrightarrow{\alpha} & \Sigma\tau_{<k}c.
\end{array} \tag{4.2}$$

Considering the second column will give the following diagram of truncation triangles.

$$\begin{array}{cccccccc}
\Sigma^{[n,k]}\mathscr{A} \ni & \tau_{\geq k}c & \longrightarrow & \tau_{\geq k}c' & \longrightarrow & \tau_{\geq k}z & & \Sigma\tau_{\geq k}c \\
& \| & & \downarrow & & \downarrow & & \| \\
\Sigma^{[n,m]}\mathscr{A} \ni & \tau_{\geq k}c & \longrightarrow & c' & \longrightarrow & z & \dashrightarrow & \Sigma\tau_{\geq k}c \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{[k-1,m]}\mathscr{A} \ni & 0 & \longrightarrow & \tau_{<k}c' & = & \tau_{<k}z & & 0
\end{array}$$

By Lemma 4.2 the top row is a conflation.

$$\tau_{\geq k}c \rightarrowtail \tau_{\geq k}c' \twoheadrightarrow \tau_{\geq k}z.$$

Consider the morphism $\nu : z \to c''$ from (4.2), this induces a morphism $\tau_{\geq k}\nu : \tau_{\geq k}z \to \tau_{\geq k}c''$, giving the commutative diagram

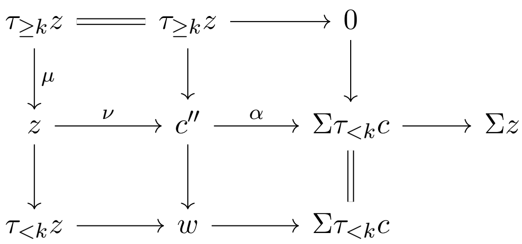
$$\begin{array}{ccccc}
\tau_{\geq k}c & \rightarrowtail & \tau_{\geq k}c' & \dashrightarrow & \tau_{\geq k}c''. \\
& & \searrow\!\!\!\!\searrow & & \nearrow{\scriptstyle \tau_{\geq k}\nu} \\
& & & \tau_{\geq k}z &
\end{array} \tag{4.3}$$

However, we do not currently know what $\mathrm{cone}(\tau_{\geq k}\nu)$ is. To figure this out, consider a rotation of the last row in (4.2), and find the corresponding truncation triangles. That is,

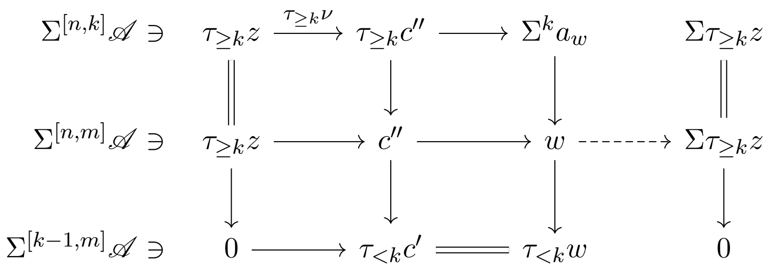
$$\begin{array}{cccccccc}
\Sigma^{[n,k]}\mathscr{A} \ni & \tau_{\geq k}z & \xrightarrow{\tau_{\geq k}\nu} & \tau_{\geq k}c'' & \longrightarrow & \Sigma^k a_d & & \Sigma\tau_{\geq k}z \\
& \downarrow{\scriptstyle \mu} & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{[n,m]}\mathscr{A} \ni & z & \xrightarrow{\nu} & c'' & \xrightarrow{\alpha} & \Sigma\tau_{<k}c & \dashrightarrow & \Sigma z \\
& \downarrow & & \downarrow{\scriptstyle \psi''} & & \downarrow & & \downarrow \\
\Sigma^{[k-1,m]}\mathscr{A} \ni & \tau_{<k}z & \longrightarrow & \tau_{<k}c'' & \longrightarrow & \tau_{<k}\Sigma\tau_{<k}c & & \Sigma\tau_{<k}z,
\end{array}$$

where $\Sigma^k a_d = \tau_{\geq k}\Sigma\tau_{<k}c \in \Sigma^k\mathscr{A}$. It follows directly from the commutativity of (4.2) that $\alpha = \Sigma\psi h$ Using the same strategy as before, apply the octahedral axiom on the

composition $\nu\mu$ using the same strategy as before.



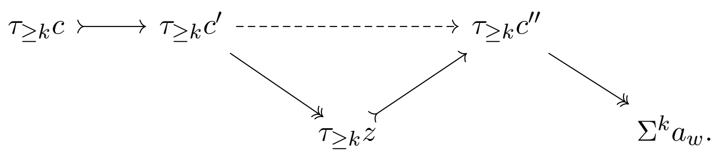
Recall that $\tau_{<k} c, \tau_{<k} z \in \Sigma^{[k-1,m]}\mathscr{A}$ implying that $\Sigma\tau_{<k} c \in \Sigma^{[k,m+1]}\mathscr{A}$, and therefore $w \in \Sigma^{[k-1,m]}\mathscr{A} * \Sigma^{[k,m+1]}\mathscr{A}$ which means that $w \in \Sigma^{[k,m]}\mathscr{A} = \Sigma^k\mathscr{A} * \Sigma^{[k-1,m]}\mathscr{A}$. Especially we can split w up as $\Sigma^k a_w \to w \to \tau_{<k} w$, with $\Sigma^k a_w = \tau_{\geq k} w \in \Sigma^k\mathscr{A}$. Thus, by considering the truncation diagram of the second column, we get



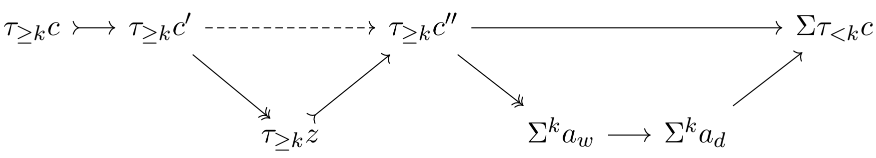
Lemma 4.2 gives that the top row is a triangle. Thus

$$\tau_{\geq k} z \overset{\tau_{\geq k}\nu}{\rightarrowtail} \tau_{\geq k} c'' \twoheadrightarrow \Sigma^k a_w.$$

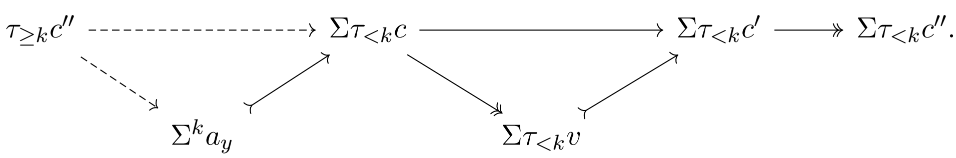
is a conflation in $\Sigma^{[n,k]}\mathscr{A}$. Using this, we can build upon the exact sequence (4.3) in $\Sigma^{[n,m+1]}\mathscr{A}$



Recall that we have a morphism $w \to \Sigma\tau_{<k} c$ which induces a morphism $\Sigma^k a_w \to \Sigma^k a_d$. This gives



By a similar argument, we get an exact sequence



Combining these two diagrams, we get the result. □

Corollary 4.4. *Let $c \to c' \to c'' \to \Sigma c$ be a triangle with $c, c', c'' \in \Sigma^{[n,m]}\mathscr{A}$. If $m \leq k \leq n$ then the induced sequence*

$$H_k c \longrightarrow H_k c' \longrightarrow H_k c''$$

is exact in $\mathscr{A}$.

Proof. Consider the following diagram, in which each column is a truncation triangle.

$$\begin{array}{ccccc}
\tau_{\geq k}c & \longrightarrow & \tau_{\geq k}c' & \longrightarrow & \tau_{\geq k}c'' \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \phi''} \\
c & \xrightarrow{f} & c' & \xrightarrow{g} & c'' \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \psi''} \\
\tau_{<k}c & \xrightarrow{\tau_{<k}f} & \tau_{<k}c' & \xrightarrow{\tau_{<k}g} & \tau_{<k}c''.
\end{array}$$

By Lemma 4.3, the top row is an exact sequence, with decomposition

$$\begin{array}{ccccccc}
 & & & b_z & & & \Sigma^k a_w \\
 & & \nearrow\!\!\!\!\twoheadrightarrow & & \searrow & \nearrow\!\!\!\!\twoheadrightarrow & \\
\tau_{\geq k}c & \rightarrowtail & \tau_{\geq k}c' & \longrightarrow & \tau_{\geq k}c'' & &
\end{array} \tag{4.4}$$

in $\Sigma^{[n,k]}\mathscr{A}$. Using Lemma 4.3, the first conflation induces the diagram

$$\begin{array}{ccccc}
\tau_{\geq k}c & \longrightarrow & \tau_{\geq k}c' & \longrightarrow & b_z \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^k H_k c & \longrightarrow & \Sigma^k H_k c' & \twoheadrightarrow & \tau_{\leq k} b_z \\
 & \searrow\!\!\twoheadrightarrow & & \nearrow\!\!\rightarrowtail & \\
 & & \Sigma^k a' & &
\end{array}$$

with consecutive $\rightarrowtail \twoheadrightarrow$ being a conflation in $\Sigma^k\mathscr{A}$. Using Lemma 4.2 with the second conflation from (4.4) we obtain the following commutative diagram

$$\begin{array}{ccccc}
b_z & \rightarrowtail & \tau_{\geq k}c'' & \twoheadrightarrow & \Sigma^k a_w \\
\downarrow & & \downarrow & & \| \\
\tau_{\leq k} b_z & \rightarrowtail & \Sigma^k H_k c'' & \twoheadrightarrow & \Sigma^k a_w.
\end{array}$$

Concatenating these two exact sequences, we get the following exact sequence in $\Sigma^k\mathscr{A}$.

$$\begin{array}{ccccccc}
\Sigma^k H_k c & & \longrightarrow & \Sigma^k H_k c', & \longrightarrow & & \Sigma^k H_k c''. \\
 & \searrow\!\!\twoheadrightarrow & & \nearrow\!\!\rightarrowtail \quad \searrow\!\!\twoheadrightarrow & & \nearrow\!\!\rightarrowtail & \\
 & & \Sigma^k a' & & \tau_{\leq k} b_z & &
\end{array}$$

By shifting this sequence by $-k$, we get the result as stated. □

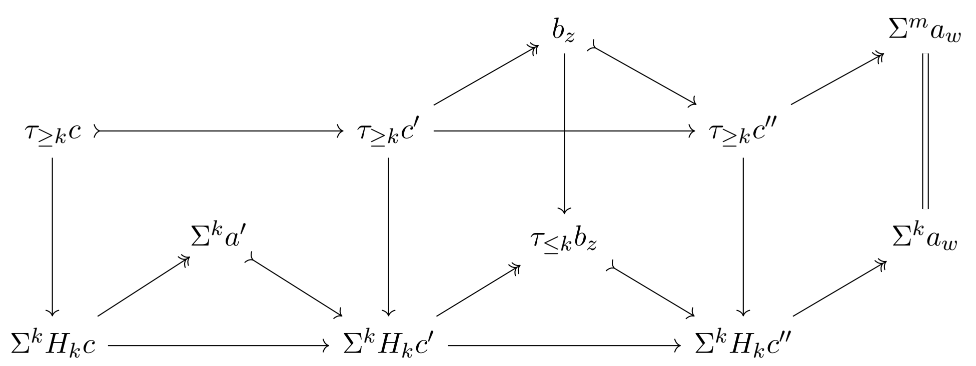
Theorem 4.5 (cf. [BBD83, thm. 1.3.6])**.** *Assume that* $\mathscr{A}$ *satisfies* E_{n-m+3}*. Given a triangle* $c \to c' \to c'' \to \Sigma c$ *with object in* $c, c', c'' \in \Sigma^{[n,m]}\mathscr{A}$*. Then there exists a long exact sequence of homology*

$$0 \to H_n c \to H_n c' \to H_n c'' \to H_{n-1} c \to \cdots \to H_{m+1} c'' \to H_m c \to H_m c' \to H_m c'' \to 0.$$

Proof. Let $m < k < n$, and consider the diagram

$$\begin{array}{ccccc} \tau_{\geq k} c & \longrightarrow & \tau_{\geq k} c' & \longrightarrow & \tau_{\geq k} c'' \\ \downarrow & & \downarrow & & \downarrow \\ c & \xrightarrow{f} & c' & \xrightarrow{g} & c'' \\ \downarrow & & \downarrow & & \downarrow \\ \tau_{<k} c & \longrightarrow & \tau_{<k} c' & \longrightarrow & \tau_{<k} c'', \end{array}$$

in which each column is a truncation triangle. By Lemma 4.3 the top row is exact in $\Sigma^{[n,k]}\mathscr{A}$, and the decomposition can be seen in the top row of the following diagram.



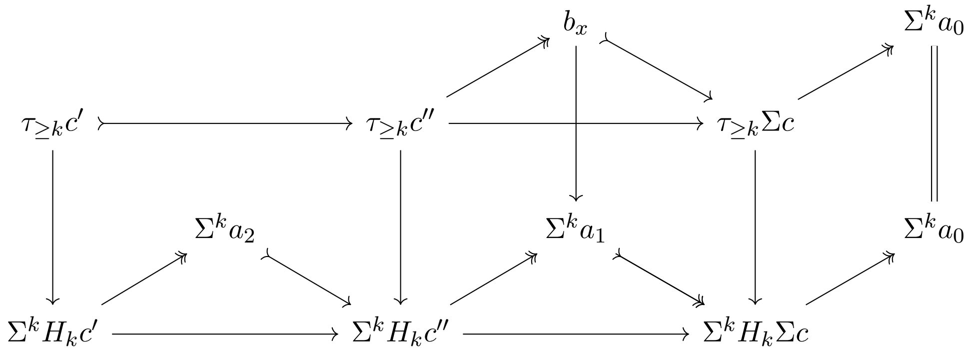
(4.5)

where consecutive $\rightarrowtail \twoheadrightarrow$ describes conflations in $\Sigma^{[n,k]}\mathscr{A}$, with $a_w \in \mathscr{A}$ and $b_z \in \Sigma^{[n,k]}\mathscr{A}$. Using Lemmas 4.2 and 4.3 again on these conflations we get the lower part of the digram (4.5), where consecutive $\rightarrowtail \twoheadrightarrow$ describes conflations in $\Sigma^k\mathscr{A}$.

Now consider the once-shifted triangle $c' \to c'' \to \Sigma c$. We can do a similar thing to this triangle as done above to the unshifted version, which would give the truncation diagram

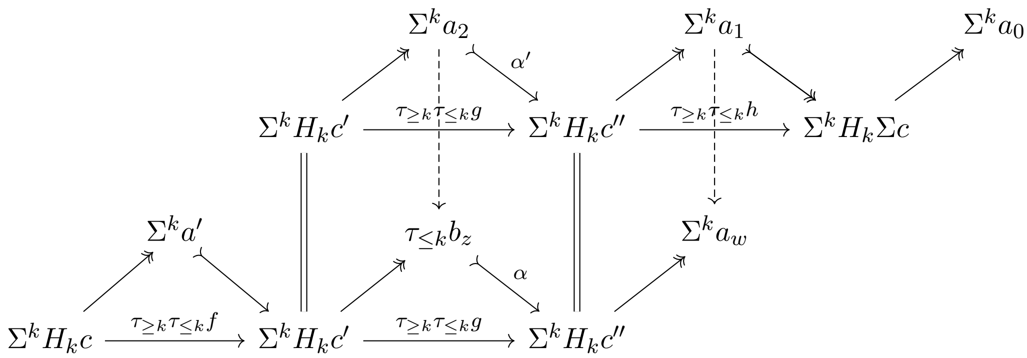
$$\begin{array}{ccccc} \tau_{\geq k} c' & \longrightarrow & \tau_{\geq k} c'' & \longrightarrow & \tau_{\geq k} \Sigma c \\ \downarrow & & \downarrow & & \downarrow \\ c' & \xrightarrow{g} & c'' & \xrightarrow{h} & \Sigma c \\ \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \psi''} & & \downarrow \\ \tau_{<k} c' & \xrightarrow{\tau_{<k} g} & \tau_{<k} c'' & \xrightarrow{\tau_{<k} h} & \tau_{<k} \Sigma c. \end{array}$$

However, notice that $c' \to c'' \to \Sigma c$ is no longer a triangle with objects in $\Sigma^{[n,m]}\mathscr{A}$ but with objects in $\Sigma^{[n+1,m]}\mathscr{A}$. Given that E_{n-m+3} is satisfied, we know that the top row is an exact sequence in $\Sigma^{[n+1,k]}\mathscr{A}$. Thus, similar to before, using Lemma 4.3, the top row gives us the top part a diagram



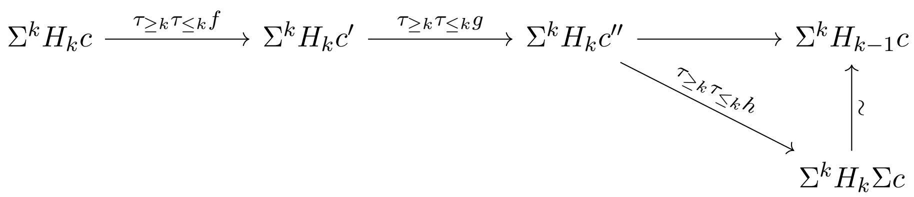
(4.6)

where $a_0 \in \mathscr{A}$ and $b_x \in \Sigma^{[n+1,k]}\mathscr{A}$. Using the same method twice, we get the bottom part of the above diagram. Comparing the bottom parts of the diagrams in (4.5) and (4.6), we obtain the following diagram:



Due to [Bü10, lem. 8.4] (see also [Hel58, prop. 3.4]) it follows that $\Sigma^k a_2 \cong \tau_{\leq k} b_z$. In particular, this means that $\alpha \cong \alpha'$ and thus $\operatorname{cok}(\alpha) \cong \operatorname{cok}(\alpha')$ meaning that $\Sigma^k a_1 \cong \Sigma^k a_w$.

With this, we have the exact sequence



The equivalence comes from Lemma 3.18. Shifting the sequence by $-k$ we get an exact sequence $H_k c \to H_k c' \to H_k c'' \to H_{k-1} c$. Using a similar argument, we can get an exact sequence

$$H_{k+1} c \to H_k c \to H_k c' \to H_k c''.$$

Notice that we can do the same thing for the endpoints.

$$0 \to H_n c \to H_n c' \to H_n c'' \to H_{n-1} c \qquad \text{and} \qquad H_{m+1} c \to H_m c' \to H_m c'' \to H_m c \to 0.$$

With this, the result follows. □

5 Examples

Example 5.1 (t-structures)**.** Let $\mathscr{A}$ be an abelian category, and consider the associated derived category $\mathsf{D}^b(\mathscr{A})$. This is a triangulated category that comes with a canonical t-structure, the standard t-structure, $\sigma = (D_{\geq 0}, D_{<0})$. Denote the corresponding truncation functors by $\sigma_{\geq 0}$ and $\sigma_{<0}$. From this we typically define truncation functors $\sigma_{\geq i}$ and $\sigma_{<i}$ which correspond to the t-structures $(D_{\geq i}, D_{<i}) \coloneqq (\Sigma^i D_{\geq 0}, \Sigma^i D_{<0})$

Since we are working in the bounded derived category, we can write these t-structures as

$$D_{\geq i} = \Sigma^{[\infty,i]}\mathscr{A} = \cdots * \Sigma^{i+2}\mathscr{A} * \Sigma^{i+1}\mathscr{A} * \Sigma^{i}\mathscr{A}$$
$$D_{<i} = \Sigma^{[i-1,-\infty]}\mathscr{A} = \Sigma^{i-1}\mathscr{A} * \Sigma^{i-2}\mathscr{A} * \Sigma^{i-3}\mathscr{A} * \cdots .$$

We want to verify that the truncation functors we have used for proper abelian subcategories match those from t-structures, at least when we are in a setup where both can be used. Consider a complex $X \in D^b(\mathscr{A})$

$$\cdots \longrightarrow X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{d_0} X_{-1} \xrightarrow{d_{-1}} X_{-2} \longrightarrow \cdots$$

Using the truncation functors $\sigma_{\geq i}$ and $\sigma_{<i}$ we obtain the triangle

$$\sigma_{\geq i}X \longrightarrow X \longrightarrow \sigma_{<i}X \longrightarrow \Sigma\sigma_{\geq i}X. \tag{5.1}$$

In this case, we know what the truncations look like:

$$\sigma_{\geq i} = \qquad \cdots \longrightarrow X_{i+2} \xrightarrow{d_{i+2}} X_{i+1} \xrightarrow{d_{i+1}} \operatorname{Ker} d_i \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

$$\sigma_{<i} = \qquad \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \operatorname{Cok} d_i \xrightarrow{d_{i-1}} X_{i-2} \longrightarrow \cdots .$$

Now assume that X is concentrated in degrees m to n for some integers $m < n$. Then $X \in \Sigma^{[n,m]}\mathscr{A}$, meaning that Lemma 3.9 tells us that up to isomorphism, there is a unique truncation triangle

$$\tau_{\geq i}X \longrightarrow X \longrightarrow \tau_{<i}X \longrightarrow \Sigma\tau_{\geq i}X,$$

with $\tau_{\geq i}X \in \Sigma^{[n,i]}\mathscr{A}$ and $\tau_{<i}X \in \Sigma^{[i-1,m]}\mathscr{A}$. However, (5.1) is another triangle that satisfies this. Hence $\sigma_{\geq i}X \cong \tau_{\geq i}X$ and $\sigma_{<i}X \cong \tau_{<i}X$. Using this, it follows directly from Definition 3.16, together with the definition of homology from t-structures, that $H_i(X) \cong H_i^\sigma(X)$, where $H_i(X)$ refers to the homology from Definition 3.16 and $H_i^\sigma(X)$ refers to homology with respect to the t-structure σ.

Example 5.2 (Negative cluster categories)**.** Let $w, n \in \mathbb{N} = \{1, 2, \dots\}$, in Section 5 we described a combinatorial model for the negative cluster category

$$\mathcal{C}_{-w}(A_n) = \mathsf{D}^b(kA_n)/\Sigma^{w+1}\tau.$$

For this example, we will use that same model. Let $w = 6$ and $n = 4$, then we can represent objects of $\mathcal{C}_{-6}(A_4)$ as admissible diagonals in an 33-gon.

Consider the 3-simple-minded system $\mathscr{S} = \{(0,6),(7,20),(8,14),(21,27)\}$ also considered in Example B.5.2, see Figure 5.1. For the object $(1,14) \in \Sigma^{[1,0]}\mathscr{A}$ there exists a triangle

$$(1,7) \longrightarrow (1,14) \longrightarrow (8,14) \longrightarrow (2,8),$$

where $(1,7) \in \Sigma\mathscr{A}$ and $(8,14) \in \Sigma\mathscr{A}$. Thus we may conclude that $H_1((1,14)) = \Sigma^{-1}(1,7) = (0,6)$ and $H_0((1,14)) = (8,14)$.

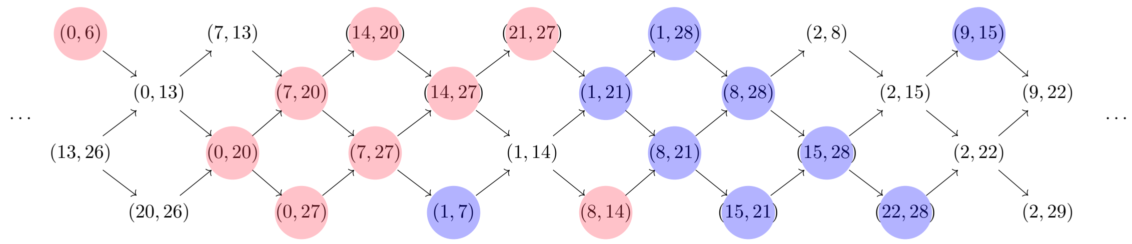


Figure 5.1: AR quiver for $C_{-6}(A_4)$. The red discs indicate $\mathscr{A}$, and the blue discs indicate $\Sigma\mathscr{A}$.

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Project E

Code – Negative Cluster Categories

1 Introduction

The negative cluster category is a triangulated category which, as seen in Lemma B.5.1, has no non-trivial t-structures. If we look at negative cluster categories over A_n, then there is a combinatorial model, that makes computations in the category much easier. To help make computations even easier, we have made some code that can do many of these calculations for us (see [Kor24]). By coding parts of the combinatorial model, we can quickly test an idea on a wide range of examples. It can also help us find examples for results that we already know. Later, we will give some examples of use cases.

We will only work with negative cluster categories of type A_n, therefore, for the rest of this chapter, we will use negative cluster categories refer to negative cluster categories of type A_n.

The code is written in the programming language typescript, which is built on top of javascript. There are a few reasons for this choice.

1. Typescript is a very accessible language, making it easier for other people to use the code even though they do not have experience with it.
2. Typescript can be compiled into javascript, which runs in the browser. Thus, there is the possibility of creating a web application that could be a useful tool.

The code is available online and can be found on GitHub [Kor24].

Setup 1.1. Let $e, w \in \mathbb{N}$. We will consider the negative cluster category $\mathcal{C}_{-w}A_e$.

Many methods will use numbers w and e as parameters throughout the code. These will always refer to the numbers from the setup.

We give a short overview of the methods available. The code and a more in-depth explanation of the methods can be found later.

`isNOrdered(a,b,c,N)`	Checks if $a < b < c$ in $\mathbb{Z}/N\mathbb{Z}$.
`Ndist`(a,b,N)	Calculates clockwise distance from a to b in polygon $\mathcal{P}_N$.
`sigma`(x,N,p)	Calculates $\Sigma^p x$.
`homDim`(x,y,w,e)	Calculates $\dim_k \operatorname{Hom}(x, y)$.
`isWDiagonal`(x, w)	Checks whether x is an admissible diagonal, i.e., if it corresponds to an object in $\mathcal{C}_{-w}A_e$.

`getDiagonalDifferenece`	Calculates the *diagonal difference* which is used to find triangles, and to calculate Ext.
`ext(z,x,w,N)`	Calculates a y such that there is a triangle $x \to y \to z \to \Sigma x$.
`extensionClose(A)`	Calculates the extension closure of a collection A of diagonals.
`isHomBetweenCollections(A,B)`	Checks if $\mathrm{Hom}(A, B)$ is non-empty.
`isEn(A, n)`	Checks if the collections A of objects satisfies E_n
`extension(A,B)`	Calculates $A * B$
`leftPerp(A,B)`	Calculates ${}^{\perp_B}A$
`rightPerp(A,B)`	Calculates $A^{\perp_B}$
`filtGen(A,S)`	Calculates $\langle \mathrm{Gen}_S(A) \rangle_S$
`filtSub(A,S)`	Calculates $\langle \mathrm{Sub}_S(A) \rangle_S$
`findRandomTorsionFreeClass(A)`	Generates a random torsion-free class in a proper abelian subcategory A.
`tilt(A,F)`	HRS-tilts a proper abelian category A in a torsion-free subcategory F.
`randomSimpleMindedSystem(w,e)`	Generates a random simple-minded system in $\mathcal{C}_{-w}(A_e)$.

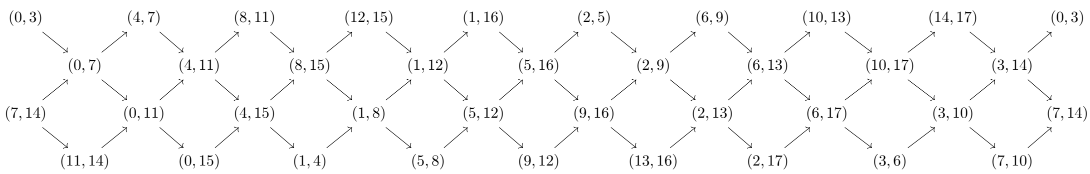


Figure 1.1: AR quiver for $C_{-3}(A_4)$.

2 Code for diagonals

2.1 Types and Classes

The combinatorial model for negative cluster categories associates to each object a diagonal of some N-gon $\mathcal{P}$, where $N \in \mathbb{N}$. Label the vertices of $\mathcal{P}_N$ anticlockwise by $0, \dots N-1$. Then, each diagonal can be represented by a pair of numbers. Thus, we create the following type:

code/src/DiagonalCollection.ts

```
export type Diagonal = [number, number]
```

Similar to the model for negative cluster categories, quite a few combinatorial models use diagonals. Therefore, we start by implementing a class representing a collection of diagonals.

code/src/DiagonalCollection.ts

```
export class DiagonalCollection {
    diagonals: Diagonal[] = [];

    constructor(objs: Diagonal[]) {
        for (var k of objs) { this.add(k); }
    }

    clone(filter: (d: Diagonal) => boolean = () => { return true; }) {
        return new (<any>this.constructor)([...this.diagonals].filter(filter)
            ↪ );
    }

    toString(): string {
        return this.diagonals.map(a => "(" + a.toString() + ")").toString()
    }

    add(obj: Diagonal) {
        this.diagonals.push(obj);
    }

    containsSet(objs: Diagonal[]) {
        for (let v of objs) {
            if (!this.contains(v)) { return false; }
        }
        return true;
    }

    contains(obj: Diagonal) {
        const i = this.find(obj);
        if (i < 0) { return null; }
        return this.diagonals[i];
    }

    find(obj: Diagonal) {
        for (var _i = 0; _i < this.diagonals.length; _i++) {
            if (this.diagonals[_i][0] == obj[0] && this.diagonals[_i][1] ==
                ↪ obj[1]) { return _i; }
        }
        return -1;
    }
```

```

    equal(A: DiagonalCollection) {
        return this.containsSet(A.diagonals) && A.containsSet(this.diagonals)
            ↪ ;
    }
}
```

2.2 Methods

Now that we have a framework to work with collections of diagonals, we can implement some standard operations for such collections.

Method – Union
`union<T extends DiagonalCollection>(...args : T[])`

Calculates the union of two collections of objects.

Implementation

code/src/DiagonalCollectionFcts.ts

```
export function union<T extends DiagonalCollection>(...args : T[]): T{
    if(args.length == 0){ return null; }
    if(args.length == 1){ return args[0]; }
    let unionColl = args[0].clone() as T
    for (let i = 1; i < args.length; i++) {
        for(let v of args[i].diagonals){
            if(!unionColl.contains(v)){
                unionColl.add(v);
            }
        }
    }
    return unionColl;
}
```

Method — Intersection
`intersect<T extends DiagonalCollection>(...args : T[])`

Calculates the intersection of two collections of objects.

Implementation

code/src/DiagonalCollectionFcts.ts

```
export function intersect<T extends DiagonalCollection>(...args : T[]): T{
    if(args.length == 0){ return null; }
    if(args.length == 1){ return args[0]; }
    return args[0].clone((diag) => {
        for (let i = 1; i < args.length; i++) {
            if(!args[i].contains(diag)){ return false }
        }
```

```
        return true
    }) as T
}
```

Method – Subtract
`subtract<T extends DiagonalCollection>(A: T, B: T)`

Calculates the difference between two collections of objects.

Implementation

code/src/DiagonalCollectionFcts.ts

```
// A - B
export function subtract<T extends DiagonalCollection>(A: T, B: T): T{
    return A.clone((diag)=>{ return !B.contains(diag) }) as T
}
```

Method – Are diagonals crossing
`isCrossing(a: Diagonal, b: Diagonal)`

This method checks if two diagonals cross. Two diagonals (a, b) and (c, d) cross if $a < c < b < d$ or $a < d < b < c$ with a cyclic order.

Example 2.1. See Figure 2.1, where the diagonal $(1, 3)$ and $(2, 8)$ cross, since $1 < 2 < 3 < 8$. However, the diagonals $(1, 3)$ and $(4, 8)$ do not cross. Similarly, the diagonals $(2, 8)$ and $(4, 8)$ are not seen as crossing, even though they share an endpoint.

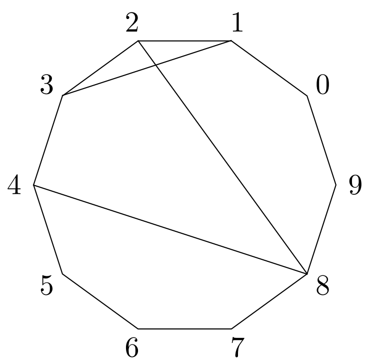


Figure 2.1: Diagonals in a 10-gon.

Implementation

code/src/DiagonalCollectionFcts.ts

```
export function isCrossing(a: Diagonal, b: Diagonal): boolean{
    return !(a[0]>=b[1] || a[1]<=b[0] || (a[1] <= b[1] && a[0] >= b[0]) || (b
        ↪ [1] <= a[1] && b[0] >= a[0]));
}
```

Method – Get a shared endpoint

`getSharedEndpoint(a: Diagonal, b: Diagonal)`

This method finds the shared endpoint between two diagonals, and if no such endpoint exists, it will return -1.

Example 2.2. Consider the diagonals $(2, 8)$ and $(4, 8)$ from Figure 2.1. These share the endpoint 8. However, if we look at the diagonals $(1, 3)$ and $(4, 8)$, we can see that they do not share an endpoint, thus, the method would return -1.

```
getSharedEndpoint((2,8), (4,8)) = 8,
getSharedEndpoint((1,3), (4,8)) = -1.
```

Implementation

code/src/DiagonalCollectionFcts.ts

```
export function getSharedEndpoint(a: Diagonal, b: Diagonal){
    if(a[0] == b[0] || a[0] == b[1]){ return a[0]; }
    if(a[1] == b[0] || a[1] == b[1]){ return a[1]; }
    return -1
}
```

Method – Calculate N-Distance

`Ndist(a:number, b:number, N:number)`

This function measures the distance from a vertex a to a vertex b by moving anticlockwise in the polygon.

Example 2.3. Consider the 10-gon, which can be seen in Figure 2.1. Then, we would want the following distances

```
Ndist(1,2,10) = 1,      Ndist(2,1,10) = 9,
Ndist(2,6,10) = 4,      Ndist(7,1,10) = 4.
```

Implementation

code/src/NCC.ts

```
export function Ndist(a:number,b:number, N:number){
    if(b > a){ return b-a }
    if(b < a){ return N - a + b }
    return 0;
}
```

Method – Is triple N-ordered

`isNOrdered(n1: number, n2: number, n3: number, N: number)`

Working with vertices in an anticlockwise order in a polygon is practically the same as working with a cyclic order in $\mathbb{Z}/N\mathbb{Z}$. Therefore, we need a method to check if a triple of numbers is ordered. The intuition behind the order is that $a < b < c$, if we follow the vertices of the polygon anticlockwise, starting at a, we will encounter b before we see c.

Example 2.4. For $N = 10$ we have that $1 < 4 < 7$ and $7 < 3 < 5$. For non-examples the following is **NOT** true: $1 < 4 < 3$ and $4 < 1 < 8$.

```
isNOrdered(1,4,7,10)= true        isNOrdered(7,3,5,10)= true
isNOrdered(1,4,3,10)= false       isNOrdered(4,1,8,10)= false
```

Implementation

code/src/NCC.ts

```
export function isNOrdered(n1: number, n2: number, n3: number, N: number):
    ↪ Boolean{
    let d = Ndist(n1, n3, N);
    return (Ndist(n1, n2, N) < d && Ndist(n2, n3, N) < d);
}
```

3 Code for Negative cluster categories

Now that we have a foundation to work with diagonals, we will create a class that inherits from the `DiagonalCollection`, in which we can put the methods that are special to the negative cluster category.

3.1 Classes

This class contains some basic methods to construct and clone the class. Besides that is contained a function to check whether a system of diagonals is a simple-minded system, and a function to check whether a system is extension closed.

code/src/NegativeCCDiagonalCollection.ts

```
export class NCCDiagonalCollection extends DiagonalCollection{

    w: number = 0;
    e: number = 0;
    N: number = 0;

    constructor(objs: Diagonal[], w:number, e:number){
        super(objs);
        this.w = w;
        this.e = e;
        this.N = (w+1)*(e+1)-2;
```

```
    }

    clone(filter: (d:Diagonal) => boolean = () => { return true; }): NCCDiagonalCollection{
        return new NCCDiagonalCollection([...this.diagonals].filter(filter), this.w, this.e);
    }

   isSimpleMindedSystem(){
        for(var _i = 0; _i < this.diagonals.length; _i++){
            if(this.diagonals[_i][1] <= this.diagonals[_i][0]){ return false }
            if(NCC.isWDiagonal(this.diagonals[_i], this.w))
            for(var _j = _i+1; _j < this.diagonals.length; _j++){
                if(this.diagonals[_i][0] == this.diagonals[_j][0] ||
                   this.diagonals[_i][1] == this.diagonals[_j][0] ||
                   this.diagonals[_i][1] == this.diagonals[_j][1] ||
                   this.diagonals[_i][0] == this.diagonals[_j][1] ){
                    return false
                }
                if(isCrossing(this.diagonals[_i],this.diagonals[_j])){
                    return false;
                }
            }
        }
        return true;
    }

    isExtensionClosed(){
        return this.equal(NCC.extensionClose(this))
    }
}
```

3.2 Methods

Method – Suspension

`Sigma(s:Diagonal, N:number, power:number)`

Given a diagonal (a, b), the suspension $\Sigma(a, b)$ is calculated by rotating the diagonal one step anticlockwise, that means $\Sigma^p(a, b) = (a + p, b + p)$ calculated modulo N. The code underneath takes a lot of different cases into account. It can apply the suspension to an arbitrary power, either of a diagonal or a collection of diagonals. Furthermore, it also ensures that the resulting diagonal (a', b') is sorted, that is, $a' < b'$.

Example 3.1. Consider $\mathcal{C}_{-2}(A_3)$, which corresponds to $w = 2$ and $e = 3$, and thereby $N = 10$. Then $(4, 9)$ is an admissible diagonal. Using Σ should result in $\Sigma(4, 9) = (0, 5)$, see Figure 3.1. Similarly $\Sigma^3(4, 9) = (2, 7)$. Thus, we want the function to output

`Sigma([4,9], 10) = [0,5],     Sigma([4,9],10,3) = [2,7],`

Implementation

code/src/NCC.ts

```
export function Sigma(s:NCCDiagonalCollection):NCCDiagonalCollection;
export function Sigma(s:NCCDiagonalCollection, power:number):
    ↪ NCCDiagonalCollection;
export function Sigma(s:Diagonal, N:number, power:number):Diagonal;
export function Sigma(s:Diagonal, N:number):Diagonal;
export function Sigma(s:NCCDiagonalCollection | Diagonal, N?:number, power?:
    ↪ number):unknown{
    if(N && !(s instanceof NCCDiagonalCollection) && typeof N == "number"){
        if(power === undefined){ power = 1 }
        if(!(s instanceof NCCDiagonalCollection)){
            const n: Diagonal = [(s[0]+power) % N, (s[1]+power) % N];
            if(n[0]<n[1]){ return n; }
            return [n[1], n[0]];
        }
    }
    if(s instanceof NCCDiagonalCollection){
        const objs = []
        for(let diag of s.diagonals){
            objs.push(Sigma(diag, s.N, N)) // Notice N here is the power
        }
        let a = new NCCDiagonalCollection(objs, s.w, s.e)
        return a
    }
}
```

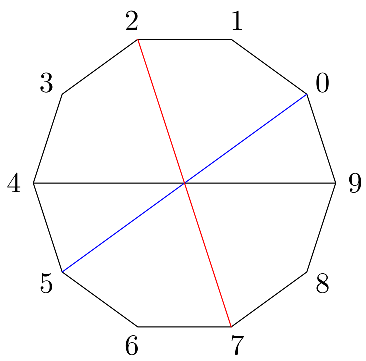


Figure 3.1: 10-gon, $\mathcal{C}_{-2}(A_3)$ corresponding to $e = 3$ and $w = 2$. The blue diagonal is the suspension of the black diagonal, and the red diagonal is Σ^3 applied to the black diagonal.

Method – Hom Space Dimension

`homDim(s1:Diagonal, s2:Diagonal, w:number, e:number)`

Let $x, y \in \mathcal{C}_{-w}A_e$, then by [CSP16, prop. 10.8] $\dim_k \operatorname{Hom}(x, y) \in \{0, 1\}$, and [CSP16, cor. 10.6] describes a combinatorial method to check when $\operatorname{Hom}(x, y) \neq 0$. This is implemented in the method below.

Example 3.2. Consider the negative cluster category $\mathcal{C}_{-3}A_4$. Using the AR-quiver (see Figure 1.1), we can visualize which objects have morphisms to which. From this, we can see that the following should be the case:

$$\texttt{homDim([1,4],[1,16],3,4)} = 1, \qquad \texttt{homDim([1,12],[9,16],3,4)} = 1,$$
$$\texttt{homDim([1,12],[1,12],3,4)} = 1, \qquad \texttt{homDim([1,12],[8,15],3,4)} = 0.$$

Implementation

code/src/NCC.ts

```
export function homDim(s1:Diagonal, s2:Diagonal, w:number, e:number){
    const N = (w+1)*(e+1)-2;
    const sig: Diagonal = Sigma(s1, N);
    if(s1[0] === s2[0] && s1[1] === s2[1]){ return 1; }
    if(s1[0] === s2[0] && (Ndist(s1[1], s2[1], N)) % (w+1) === 0 && !diag.
        ↪ isCrossing(s2, sig)){ return 1; }
    if(s1[1] === s2[0] && (Ndist(s1[0], s2[1], N)) % (w+1) === 0 && !diag.
        ↪ isCrossing(s2, sig)){ return 1; }
    if(s1[0] === s2[1] && (Ndist(s1[1], s2[0], N)) % (w+1) === 0 && !diag.
        ↪ isCrossing(s2, sig)){ return 1; }
    if(s1[1] === s2[1] && (Ndist(s1[0], s2[0], N)) % (w+1) === 0 && !diag.
        ↪ isCrossing(s2, sig)){ return 1; }

    if(diag.isCrossing(s2, sig) && diag.getSharedEndpoint(s2,s1)==-1){
        if(
            isNOrdered(sig[0],sig[1], s2[0], N) && (Ndist(s2[0], sig[0], N))
                ↪ % (w+1) === 0 &&
            (Ndist(s2[1], sig[1], N)) % (w+1) === 0
        ){ return 1; }
        if(
            isNOrdered(sig[0],sig[1], s2[1], N) && (Ndist(s2[1], sig[0], N))
                ↪ % (w+1) === 0 &&
            (Ndist(s2[0], sig[1], N)) % (w+1) === 0
        ){ return 1; }
    }
    return 0;
}
```

Method – Is the diagonal admissible

`isWDiagonal(a: Diagonal, w: number)`

To each object in $\mathcal{C}_{-w}A_e$, a diagonal is associated. However, not all diagonals correspond to an object. It is only admissible diagonal that does. This method checks whether a diagonal is admissible.

Example 3.3. Consider the negative cluster category $\mathcal{C}_{-2}A_3$, with $w = 2$ and $e = 3$. Then the diagonal $(4, 9)$ is admissible since $w + 1 = 3 \mid (9 - 4) + 1 = 6$. Similarly, the diagonals $(1, 3)$ and $(1, 9)$ are admissible, but $(5, 9)$ and $(6, 9)$ are not, see Figure 3.2. Thus, we should get the following.

`isWDiagonal([4,9], 2)` = true, `isWDiagonal([1,3], 2)` = true,

`isWDiagonal([1,9], 2)` = true, `isWDiagonal([5,9], 2)` = false,

`isWDiagonal([6,9], 2)` = false.

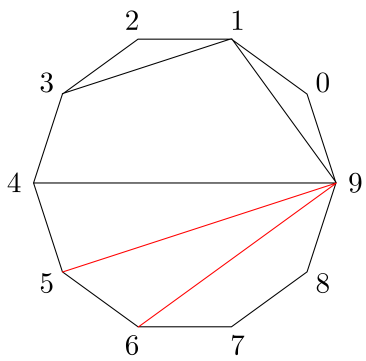


Figure 3.2: 10-gon, $\mathcal{C}_{-2}(A_3)$ corresponding for $e = 3$ and $w = 2$. The black diagonals are admissible, whereas the red diagonals are not admissible.

Implementation

code/src/NCC.ts

```
export function isWDiagonal(a: Diagonal, w: number): boolean{
    return (((a[1] - a[0]) + 1 )% (w + 1) == 0);
}
```

Method – Calculate the diagonal difference

`getDiagonalDifferenece(c: Diagonal, a: Diagonal, N:number)`

This is a helper function for when to calculate Ext. Consider two diagonals $c = (c_0, c_1)$ and $a = (a_0, a_1)$, see Figure 3.3. Then, if there is a triangle $a \to b \to c \to \Sigma b$, with $a \neq 0$, the diagonals representing direct summands of b can be described combinatorially, see Figure 3.3. b has two diagonal $b = b^0 \oplus b^1$, possibly one or both being 0. Loosely speaking, to find the diagonal b^i, we start a point b_i and follow the polygon anticlockwise until we get to a point x that is either part of the diagonal c or a if it is part of a we let $b^i = 0$. However, if it is part of c, we let $b^i = (a_i, x)$. We will call the result b the *diagonal difference* of (c, a). It is this the method `getDiagonalDifference` calculates. Notice that this method does not ensure that we get a triangle or that the resulting diagonals are admissible.

Example 3.4. The diagonal difference is somewhat easy to read directly from seeing the diagonals drawn in a polygon. Here are some examples for $N = 10$.

`getDiagonalDifferenece([2,7],[0,5],10) = [[0,2], [5,7]],`

`getDiagonalDifferenece([0,2],[0,5],10) = [],`

`getDiagonalDifferenece([0,5],[0,2],10) = [[2,5]].`

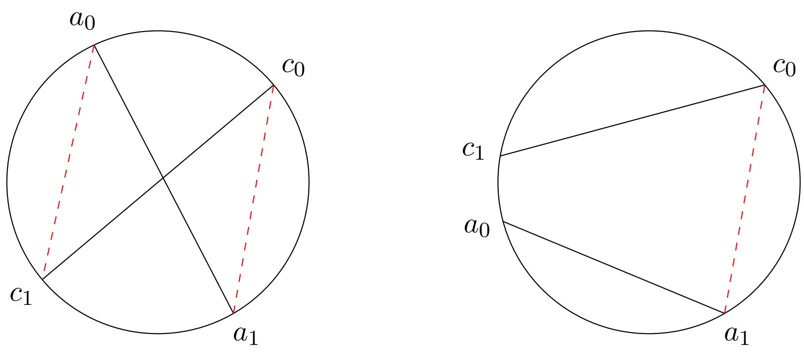


Figure 3.3: Given diagonals $c = (c_0, c_1)$ and $a = (a_0, a_1)$, the middle object of a triangle $a \to b \to c \to \Sigma a$ can be found combinatorially, here b is represented by a dashed red line.

See Figure 3.4 for an illustration of these examples.

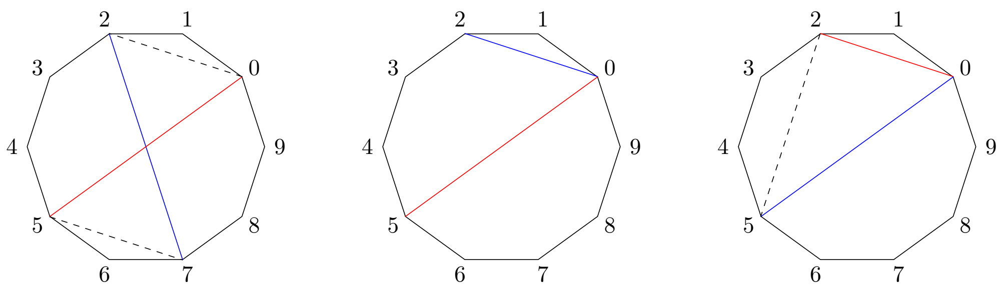


Figure 3.4: For the 10-gon, this figure illustrates three examples of the function `getDiagonalDifference(a,b,10)`, where a is blue, b is red, and the result is dashed

Implementation

code/src/NCC.ts

```
export function getDiagonalDifferenece(a: Diagonal, b: Diagonal, N:number){
    const objs:Diagonal[] = [];

    let currIndex = 0;
    if(a.includes(b[currIndex])){
        currIndex = 1;
    }

    for(var _j = currIndex; _j < 2; _j++){
        for(var _i = b[_j] + 1; _i < b[_j] + N; _i++){
            if( b.includes(_i % N) ){
                break;
            }
            if( a.includes(_i % N) ){
                objs.push([b[_j], _i % N].sort((n1,n2) => n1 - n2) as Diagonal
                    ↪ );
                break;
            }
```

```
        }
    }

    return objs;
}
```

Method – Ext between diagonals
`ext(c: Diagonal, a:Diagonal, w:number, N:number)`

Given two (admissible) diagonals c, a, if $\mathrm{Ext}(c, a) \neq 0$ then there exists a triangle $a \to b \to c \to \Sigma a$. The following method will use `getDiagonalDifference`, described above, and return a list of the direct summands of b. If no such triangle exists, the method will return an empty list.

Example 3.5. Using *Figure* 1.1 we can easily find triangles in the category $\mathcal{C}_{-3}(A_4)$. As an example, there are

$$(1,8) \to (5,8) \oplus (1,12) \to (5,12) \to \Sigma(1,8)$$
$$(8,15) \to (5,8) \oplus (12,15) \to (5,12) \to \Sigma(8,15)$$
$$(5,8) \to (2,5) \to (2,9) \to \Sigma(5,8)$$

However, there are no non-trivial triangles $(5,12) \to x \to (8,15) \to \Sigma(5,12)$. Thus, we will get the following.

```
ext((5,12),(1,8),3,18) = [(5,8), (1,12)] ,
ext((5,12),(8,15),3,18) = [(5,8), (12,15)],
ext((2,9),(5,8),3,18) = [(2,5)],
ext((8,15),(5,12),3,18) = [].
```

Implementation

code/src/NCC.ts

```
export function ext(c: Diagonal, a:Diagonal, w:number, N:number){
    if( diag.isCrossing(c,a) || ( !diag.isCrossing(c,a) && diag.
        ↪ getSharedEndpoint(Sigma(a, N), c) >= 0 ) ){
        //There is an extension
        let diff:Diagonal[] = getDiagonalDifferenece(a,c, N);

        // Check if it is w diagonals
        if(diff.every((curVal) => isWDiagonal(curVal, w))){
            return diff;
        }
    }
    return [];
}
```

Method – Extensions between collections

`extension(A: NCCDiagonalCollection, B: NCCDiagonalCollection)`

Given two collections of diagonals A and B, this method will return the collection $A * B$ of extension between them, i.e. all the diagonals c such that there exists a triangle $a \to c \to b \to \Sigma a$ with $a \in A$ and $b \in B$.

Implementation

code/src/NCC.ts

```
export function extension(A: NCCDiagonalCollection, B: NCCDiagonalCollection
    ↪ ): NCCDiagonalCollection{
    let a: NCCDiagonalCollection = diag.union(A, B);

    for(let x of A.diagonals){
        for(let y of B.diagonals){
            if(diag.diagonalEqual(x,y)){ continue; }
            // Find e's such that: x ---> e ---> y, or rather a ---> e ---> b
            let e = ext(y, x, A.w, A.N);
            for(let z of e){
                if(!a.contains(z)){ a.add(z); }
            }
        }
    }
    return a;
}
```

Method – Extension-close NCC Diagonal Collection

`extensionClose(A: NCCDiagonalCollection)`

Given a collection of object A the following method return a collection $\langle A \rangle$ which is the closure of A under extensions. Setting $(A)_0 = A$ and $(A)_i = (A)_{i-1} * (A)_{i-1}$ for $i > 0$, we can calculate the extension-closure as $\langle A \rangle = \cup_{i \in \mathbb{N}} (A)_i$. This is essentially what this method does.

Notice that since the number of objects in $\mathcal{C}_{-w}A_e$ is finite, there exists a natural number $M \in \mathbb{N}$ such that $\langle A \rangle = (A)_M$. This means that our method will always terminate.

Example 3.6. Consider the negative cluster category $\mathcal{C}_{-3}(A_4)$, and define the collection of objects $A = \{(1,4), (5,8), (9,12)\}$, i.e.

```
A = new NCCDiagonalCollection([[1,4], [5,8], [9,12]], 3,4).
```

Using Figure 1.1, we can calculate

$$(A)_1 = \{(1,4), (5,8), (9,12), (1,8), (5,12)\},$$
$$(A)_2 = \{(1,4), (5,8), (9,12), (1,8), (5,12), (1,12)\},$$
$$(A)_3 = \{(1,4), (5,8), (9,12), (1,8), (5,12), (1,12)\}.$$

Since $(A)_2 = (A)_3$ we may conclude that $\langle A \rangle = (A)_2$. In code, this can be computed using the following method:

```
extensionClose(A)= [[1,4], [5.8], [9,12], [1,8], [5,12], [1,12]].
```

Implementation

code/src/NCC.ts

```
export function extensionClose(A: NCCDiagonalCollection){
    let res: NCCDiagonalCollection = new NCCDiagonalCollection([...A.
        ↪ diagonals], A.w, A.e);
    let somethingAdded: Boolean = false;
    while(true){
        somethingAdded = false;
        for(let x of res.diagonals){
            for(let y of res.diagonals){
                if(x == y){ continue; }
                let e = ext(x, y, A.w, A.N);
                for(let z of e){
                    if(!res.contains(z)){
                        res.add(z);
                        somethingAdded = true;
                    }
                }
            }
        }
        if(!somethingAdded){ break; }
    }
    return res;
}
```

Method – Morphisms between collections

`isHomBetweenCollections(from: NCCDiagonalCollection, to:`
`↪ NCCDiagonalCollection)`

Given two collections A and B of diagonals, the following method checks if $\mathrm{Hom}(A, B) \neq 0$. This can be done by looping through the objects in A and the B until a pair of objects (a, b) is found such that $\mathrm{Hom}(a, b) \neq 0$. If no such pair exists, then it will return `false`

Example 3.7. Using Figure 1.1, it is possible to see which indecomposable objects have morphisms between each other. As an example, we can see that $\mathrm{Hom}((9, 12), (6, 9)) \neq 0$. Thus if we define two collections

```
let A = new NCCDiagonalCollection([[5,8],[9,12]], 3,4)
let B = new NCCDiagonalCollection([[6,9],[10,13]], 3,4)
```

This means that there should be morphisms from A to B but not the other way around. Thus we would get the following.

```
isHomBetweenCollections(A,B) = true
isHomBetweenCollections(B,A) = false
```

Implementation

code/src/NCC.ts

```
export function isHomBetweenCollections(from: NCCDiagonalCollection, to:
    NCCDiagonalCollection): boolean{
    for(let x of from.diagonals){
        for(let y of to.diagonals){
            if(homDim(x, y, from.w, to.e) > 0){
                return true;
            }
        }
    }
    return false;
}
```

Method – The E_n condition

Given a collection A of diagonals, this method checks if A satisfies E_n. That is, checking if $\mathrm{Hom}(x, \Sigma^{-i}y) = 0$ for $0 < i \leq n$.

Example 3.8. Consider the objects $(1,4)$, then by looking at Figure 1.1, we can obverse that

$$\mathrm{Hom}((1,4), \Sigma^{-1}(1,4)) = \mathrm{Hom}((1,4),(0,3)) = 0,$$
$$\mathrm{Hom}((1,4), \Sigma^{-2}(1,4)) = \mathrm{Hom}((1,4),(2,17)) = 0,$$
$$\mathrm{Hom}((1,4), \Sigma^{-3}(1,4)) = \mathrm{Hom}((1,4),(1,16)) \neq 0.$$

Thus $\{(1,4)\}$ satisfies E_2 but not E_3. Hence if we set

```
let A = new NCCDiagonalCollection([[1,4]], 3,4),
```

we will get that

`ncc.isEn(A,1)= true`, `ncc.isEn(A,2)= true`, `ncc.isEn(A,3)= false`.

Implementation

code/src/NCC.ts

```
export function isEn(coll: NCCDiagonalCollection, n: number){
    let a = new NCCDiagonalCollection(coll.diagonals, coll.w, coll.e);

    for(let i = 0; i<n; i++){
        a = Sigma(a);
        if(isHomBetweenCollections(a, coll)){ return false; }
    }
    return true
}
```

Method – Left orthogonal collection

`leftPerp(of:NCCDiagonalCollection, inColl:NCCDiagonalCollection)`

Given two collections $of, inColl$ of diagonals, this method finds all the diagonals in $inColl$, with no morphisms to of. That is ${}^{\perp_{inColl}}of = \{x \in inColl \mid \mathrm{Hom}(x, of) = 0\}$.

Example 3.9. Consider the negative cluster category $\mathcal{C}_{-3}(A_4)$, see Figure 1.1, and let $A = \mathrm{add}((1,4),(1,8),(1,12),(5,8),(5,12),(9,12))$. Then A is a proper abelian category equivalent to $\mathrm{mod}(kA_3)$, see Figure 3.5

```
let A = new NCCDiagonalCollection([[1,4], [1,8], [1,12], [5,8],
    [5,12], [9,12]], 3,4),
```

Letting $B = \mathrm{add}((9,12))$, i.e.

```
B = new NCCDiagonalCollection([[9,12]], 3,4),
```

we get that ${}^{\perp_A}B = \mathrm{add}((1,4),(1,8),(5,8))$, Which is also what the code shows

```
leftPerp(B,A)= [[1,4], [1,8], [5,8]].
```

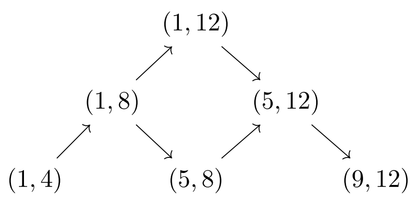


Figure 3.5: AR-quiver for the proper abelian subcategory A considered in Example 3.9

Implementation

code/src/NCC.ts

```
export function leftPerp(of:NCCDiagonalCollection, inColl:
    NCCDiagonalCollection){
    return inColl.clone((diag) => {
        for(let ofDiag of of.diagonals){
            if(homDim(diag, ofDiag, inColl.w, inColl.e) > 0){
                return false
            }
        }
        return true
    })
}
```

Method – Right orthogonal collection

`rightPerp(of:NCCDiagonalCollection, inColl:NCCDiagonalCollection)`

Given two collections $of, inColl$ of diagonals, this method find all the diagonals in $inColl$, with no morphisms from of. That is $of^{\perp_{inColl}} = \{x \in inColl \mid \mathrm{Hom}(of, x) = 0\}$.

Example 3.10. Using the collection A and B from Example 3.9, we can see that $B^{\perp_A} = A \backslash B$, and therefore we will get that

`rightPerp(B,A)= [[1,4], [1,8], [1,12], [5,8], [5,12]].`

Implementation

code/src/NCC.ts

```
export function rightPerp(of:NCCDiagonalCollection, inColl:
  ↪ NCCDiagonalCollection){
    return inColl.clone((diag) => {
        for(let ofDiag of of.diagonals){
            if(homDim(ofDiag, diag, inColl.w, inColl.e) > 0){ return false }
        }
        return true
    })
}
```

Method – Closure under extension and quotient objects

`filtGen(set: NCCDiagonalCollection, alg: NCCDiagonalCollection)`

Given a noetherian abelian category alg, together with a collection $A \subseteq alg$ of diagonals, the $\langle \mathrm{Gen}_{alg}(A) \rangle_{alg}$ is the smallest subcategory of alg containing A that is closed under extensions and quotients. However, this can also be calculated as $\langle \mathrm{Gen}_{alg}(A) \rangle_{alg} = {}^{\perp_{alg}}(A^{\perp_{alg}})$. This is what this method calculates.

Example 3.11. Consider the proper abelian algebra A from Example 3.9. And let $B = \mathrm{add}((1,8),(9,12))$, ie.

`B = new NCCDiagonalCollection([[1,8], [9,12]], 3,4).`

Since $(9,12)$ is simple, there is no epimorphism from it. However, we do have an epimorphism $(1,8) \twoheadrightarrow (5,8)$, thus $\mathrm{Gen}_A(B) = \mathrm{add}((1,8),(9,12),(5,8))$. However, this is not closed under extensions since there are short exact sequences

$$(1,8) \rightarrowtail (1,12) \twoheadrightarrow (9,12) \qquad \text{and} \qquad (5,8) \rightarrowtail (5,12) \twoheadrightarrow (9,12).$$

Including the object we get from these extensions, we get an extension-closed collection $\langle \mathrm{Gen}_A(B) \rangle_A = \mathrm{Gen}_A(B) = \mathrm{add}((1,8),(1,12),(5,8),(5,12),(9,12))$. Thus

`filtGen(B,A)= [[1,8], [1,12], [5,8], [5,12], [9,12]].`

Implementation

code/src/NCC.ts

```
export function filtGen(set: NCCDiagonalCollection, alg:
  ↪ NCCDiagonalCollection){
    return leftPerp(rightPerp(set, alg), alg)
}
```

Method – Closure under extension and sub objects
`filtSub(set: NCCDiagonalCollection, alg: NCCDiagonalCollection)`

Given a noetherian abelian category alg, together with a collection $A \subseteq alg$ of diagonals the $\langle \mathrm{Sub}_{alg}(A)\rangle_{alg}$ is the smallest subcategory of alg containing A that is closed under extensions and subobjects. However, this can also be calculated as $\langle \mathrm{Sub}_{alg}(A)\rangle_{alg} = ({}^{\perp_{alg}}A)^{\perp_{alg}}$. This is what this method calculates.

Example 3.12. Consider the proper abelian algebra A from Example 3.9. And let $B = \mathrm{add}((1,8),(9,12))$, ie.

```
B = new NCCDiagonalCollection([[1,8], [9,12]], 3,4).
```

Closing B under subobjects we get $\mathrm{add}(B \cup \{(1,4)\})$, and closing that under extensions, we get $\langle \mathrm{Sub}_A(b)\rangle_A = \mathrm{add}((1,4),(1,8),(1,12),(9,12))$. In code, this can be calculated as:

```
filtSub(B,A)= [[1,4], [1,8], [1,12], [9,12]].
```

Implementation

code/src/NCC.ts

```
export function filtSub(set: NCCDiagonalCollection, alg:
  ↪ NCCDiagonalCollection){
    return rightPerp(leftPerp(set, alg), alg)
}
```

Method – Finding random torsion class
`findRandomTorsionFreeClass(alg: NCCDiagonalCollection)`

Consider an abelian category $\mathscr{A}$, if we are given a collection of objects $\mathcal{X} \subseteq \mathscr{A}$, then, we can construct a torsion-free class by closing it under subobjects and extensions. Thus $\langle \mathrm{Sub}(\mathcal{X})\rangle$ would be the smallest torsion-free class containing $\mathcal{X}$. This method picks a random number of random objects in $\mathscr{A}$ and then closes it under subobjects and extensions.

Implementation

code/src/NCC.ts

```
export function findRandomTorsionFreeClass(alg: NCCDiagonalCollection){
    const num = Math.floor(Math.random() * alg.diagonals.length + 1)
    const shuffled = alg.diagonals.sort(() => 0.5 - Math.random());
    let selected = shuffled.slice(0, num);
    return filtSub(new NCCDiagonalCollection(selected, alg.w, alg.e), alg)
}
```

Method – HRS tilt
`tilt(alg:NCCDiagonalCollection, torsionFree:NCCDiagonalCollection)`

Given a torsion pair $(\mathcal{T},\mathcal{F})$ in a proper abelian subcategory $\mathscr{A}$, we can, under the right assumptions, tilt $\mathscr{A}$ with respect to this torsion pair. The resulting proper abelian subcategory is then $\Sigma\mathcal{F} * \mathcal{T}$.

Example 3.13. Working with $\mathcal{C}_{-3}(A_4)$, consider the proper abelian subcategory A from Example 3.9,

```
let A = new NCCDiagonalCollection([[1,4], [1,8], [1,12], [5,8],
↪ [5,12], [9,12]], 3,4).
```

Then $X = \text{add}((1,4),(1,8))$ is a torsion-free class. That is

```
let X = new NCCDiagonalCollection([[1,4], [1,8]], 3,4).
```

From this torsion-free class, we can determine the corresponding torsion class

$$Y = \text{add}((1,12),(5,8),(5,12),(9,12)).$$

Now we calculate the tilt by

$$B = \Sigma X * Y = \text{add}((2,5),(2,9),(1,12),(5,8),(5,12),(9,12)).$$

This calculation can be verified using Figure 1.1. We get the following if we try to do this calculation using the code.

```
tilt(A, X)= [[2,5], [2,9], [1,12], [5,8], [5,12], [9,12]].
```

Implementation

code/src/NCC.ts

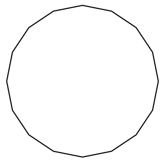
```
export function tilt(alg:NCCDiagonalCollection, torsionFree:
    ↪ NCCDiagonalCollection){
    let torsion = leftPerp(torsionFree, alg)
    return extension(Sigma(torsionFree), torsion);
}
```

Method – Finding random simple-minded system
`randomSimpleMindedSystem(w:number, e:number)`

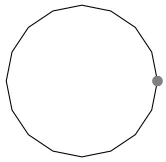
When testing ideas, it helps to be able to test something on many different proper abelian subcategories. A special class of proper abelian subcategories in negative cluster categories comes from simple-minded systems. In $\mathcal{C}_{-w}(A_e)$, a collection of diagonals S forms a simple-minded system if

1. $|S| = e$,
2. no two diagonals in S are crossing, and
3. no two diagonals in S share an endpoint.

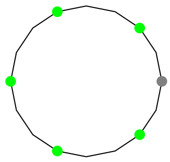
This is the same as saying S is a maximal set of diagonals satisfying 2 and 3. We will create some code to find such a collection of diagonals at random. We will go through the process we will implement step by step, with a running example in $\mathcal{C}_{-2}(A_5)$, which we can use to visualize the steps. We start with an empty polygon, in this case, a 16-gon.



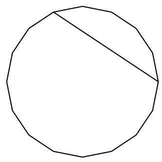
Step 1. Choose a random vertex. This is illustrated by a gray ball.



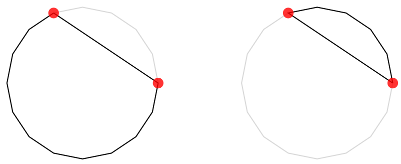
Step 2. Identify all the other vertices that, combined with v, make an admissible diagonal. These are illustrated with green balls.



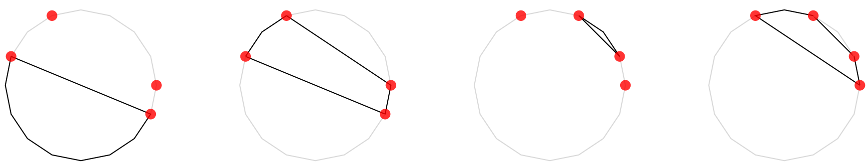
Step 3. Choose one of the vertices at random and complete to a diagonal



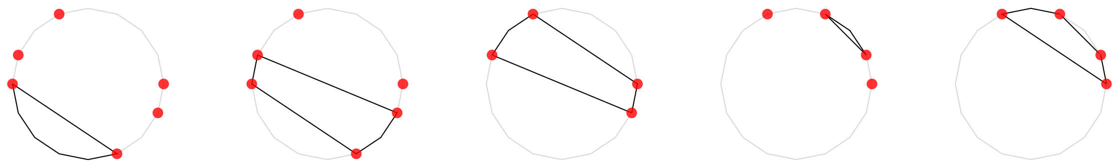
Step 4. Split the polygon into two polygons along this diagonal, and mark the points already taken as an endpoint. These endpoints are marked with red in the illustration.



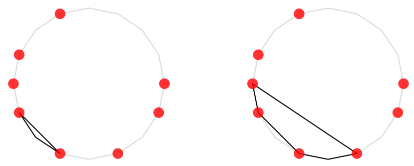
Step 5. Repeat steps 1 through 4 on the new restricted polygons until no more admissible diagonals can be found, however, do not allow the marked points to be chosen as endpoints again. In the example, we need to repeat the process three more time. First time:



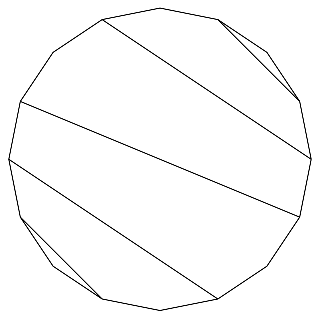
Second time:



The Third time, we split the leftmost into two:



If we collect all the diagonals into the original polygon, we end up with the following:



Implementation. First, we need some helper functions that will make the code a bit nicer to look at. These should be self-explanatory.

code/src/NCC.ts

```
function getRandomInteger(min: number, max: number) {
    return Math.floor(Math.random() * (max - min)) + min;
}

function random_array_value(arr: any[]){
    return arr[Math.floor(Math.random() * arr.length)]
}

function numberArray(from: number, to:number){
    return Array.from({length: to-from+1}, (_, index) => index + from)
}

function random_shuffle(array: any[]) {
    let currentIndex = array.length
    let randomIndex: number;

    while (currentIndex > 0) {
        randomIndex = getRandomInteger(0, currentIndex)
        currentIndex -= 1;

        [array[currentIndex], array[randomIndex]] = [array[randomIndex], array[currentIndex]];
```

```
    }

    return array;
}
```

With this, we are ready to see the implementation of the method. The code is commented and follows the steps described above.

code/src/NCC.ts

```
export function randomSimpleMindedSystem(w:number, e:number): NCCDiagonalCollection{
    const N: number = (e+1) * (w+1) - 2;

    function helper(polygon: number[], taken: number[]){
        // checks if there is enoguh space to have a diagonal
        if(polygon.length < (w+1)*2-2){ return [] }

        // Find all the vertices in ‘polygon‘ that are not already
        // enpoints of diagonals
        let available_nodes = random_shuffle(polygon.filter((d) => taken.indexOf(d) == -1))
        if(available_nodes.length < 2){ return [] }

        // Shuffles available vertices, to randomize pick of diagonal
        let randomized_available_nodes = random_shuffle(available_nodes)
        let random_partner:number = -1
        let i = 0
        let found_one = false

        // Goes throguh each of the available vertices, and try to match it
        // with another vertex to construct a diagonal
        for (i = 0; i < randomized_available_nodes.length; i++) {
            // Find possible partners to construct a diagonal with
            // randomized_available_nodes[i]
            let possiblePartners = available_nodes.filter((n) => {
                if(n == randomized_available_nodes[i]){ return false }
                return (Math.abs(n-randomized_available_nodes[i]) + 1) % (w + 1) == 0
            })
            if(possiblePartners.length == 0){ continue }

            // Picks a random partner
            found_one = true
            random_partner = random_array_value(possiblePartners)
            break
        }

        if(random_partner == -1){ return []}
        if(!found_one){
```

```
            console.log("error:␣Diagonal␣not␣found")
            return []
        }

        // The chosen random diagonal
        let diag = [randomized_available_nodes[i], random_partner].sort((a, b
            ↪ )=>{return a-b})

        // Splitting the polygon up into two part,
        // one on each side of the diagonal
        const pol1 = polygon.filter((n) => {
            return isNOrdered(diag[0], diag[1],n, N) || n == diag[0] || n ==
                ↪ diag[1]
        })
        const taken1 = taken.filter((n) => { return pol1.indexOf(n)>=0 })
        taken1.push(diag[0], diag[1])

        const pol2 = polygon.filter((n) => {
            return !isNOrdered(diag[0], diag[1],n, N) || n == diag[0] || n ==
                ↪ diag[1]
        })
        const taken2 = taken.filter((n) => { return pol2.indexOf(n)>=0 })
        taken2.push(diag[0], diag[1])

        // recursivly finding diagonal in the two parts the polygon is split
            ↪ into
        return [diag, ...helper(pol1, taken1), ...helper(pol2, taken2)]
    }

    let h = helper(numberArray(0,N-1), [])
    return new NCCDiagonalCollection(h,w,e)
}
```

4 Examples of usage

4.1 Testing the E_n properties of proper abelian subcategories

Let us assume that we have come up with the hypothesis that given a proper abelian subcategory $\mathscr{A}$ generated by a simple-minded system, in a negative cluster category $\mathcal{C}_{-w}(A_e)$, that $\mathscr{A}$ always will satisfy E_{w-1}. Let us come up with some code to test this. We will do this as follows.

1. Generate random numbers for w and e with $2 \leq w, e \leq 20$.
2. Generate a random simple-minded system in $\mathcal{C}_{-w}A_e$ and construct the corresponding algebra.
3. Check if it satisfies E_{w-1}.
4. Repeat (step 1 - step 3) 100,000 times.

We have implemented this:

code/examples/example1.ts

```
import * as ncc from "../src/NCC"

var isMyGuessCorrect = true
let times = 100000
for (var i = 0; i < times; i++){
    // Finding two random numbers
    var random_w = Math.floor(Math.random() * 18) + 2;
    var random_e = Math.floor(Math.random() * 18) + 2;

    // Generate a random proper abelian subcategory from an sms.
    var sms = ncc.randomSimpleMindedSystem3(random_w, random_e)
    var A = ncc.extensionClose(sms)

    // Finding
    isMyGuessCorrect &&= ncc.isEn(A, random_w-1)
}
console.log("Is␣my␣guess␣correct?", isMyGuessCorrect);
```

Running the code, we find that `isMyGuessCorrect = true`, meaning that it is true for all the cases in which the hypothesis has been checked.

Since we have found this experiment successful, we might think that all these proper abelian categories might satisfy E_w. However, changing the code above to check that, we will get a negative result, meaning that there are natural numbers e, w and proper abelian subcategories coming from simple-minded systems in $\mathcal{C}_{-w}(A_e)$, that do not satisfy E_w.

4.2 Finding torsion triples

Let $w, e \in \mathbb{N}$, and consider the negative cluster category $\mathcal{C}_{-w}(A_e)$. In Part Paper C, we defined torsion triples and found a way to construct such torsion triples. To do this, we needed two proper abelian subcategories $\mathscr{A}, \mathscr{B}$ satisfying E_5 such that $\mathscr{B} \subseteq \Sigma^2\mathscr{A} * \Sigma\mathscr{A} * \mathscr{A}$ and $\mathscr{A} \subseteq \Sigma^{-2}\mathscr{B} * \Sigma^{-1}\mathscr{B} * \mathscr{B}$.

The naive way to test this would be to generate two random simple-minded systems and testing if they satisfy the properties we need. However, given a proper abelian subcategory $\mathscr{A}$, one way to construct another proper abelian subcategory is by doing an HRS-tilt. This process was described in [Jør21] for proper abelian subcategories. The plan is, therefore, the following:

1. Generate a random simple-minded system and find the corresponding algebra `alg`.
2. Tilt `alg` two times with respect to random torsion-free classes to get `alg3`.
3. Check if the pair (`alg`, `alg3`) satisfies the wanted properties.
4. Repeat (step 1 - step 3) 10,000 times, or until step 3 says we have found an appropriate pair.

This we have implemented in the following code.

code/examples/example2.ts

```
import * as ncc from "../src/NCC"
import * as n from "../src/NegativeCCDiagonalCollection"

// Tilt an proper abelian subcategory at a random torsion class
function randomTilt(alg: n.NCCDiagonalCollection){
    var T = ncc.findRandomTorsionFreeClass(alg)
    return ncc.tilt(alg, T)
}

var A: null | n.NCCDiagonalCollection = null
var B: null | n.NCCDiagonalCollection = null

for(var i = 0; i< 10000; i++){

    // Find random simple-minded system
    var sms = ncc.randomSimpleMindedSystem3(6,6);
    var alg = ncc.extensionClose(sms);
    if(!ncc.isEn(alg, 5)){ continue; }

    // Tilt twice at random
    var alg2 = randomTilt(alg)
    var alg3 = randomTilt(alg2)

    // Check if the pair (alg, alg3) satisfies the wanted criteria
    if(!ncc.isEn(alg3, 5)){ continue }

    // Check if the needed properties are satisfied
    var SSA_SA_A = ncc.extension(ncc.Sigma(alg, 2), ncc.extension(ncc.Sigma(
        ↪ alg), alg))
    var B_SB_SSB = ncc.extension(alg3, ncc.extension(ncc.Sigma(alg3,-1), ncc.
        ↪ Sigma(alg3, -2)))

    if(!SSA_SA_A.containsSet(alg3.diagonals)){ continue }
    if(!B_SB_SSB.containsSet(alg.diagonals)) { continue }

    // Ensure not boring
    var SA_A = ncc.extension(ncc.Sigma(alg), alg)
    if(SA_A.containsSet(alg3.diagonals)){ continue }

    A = alg
    B = alg3
    break
}

console.log(A)
console.log(B)
```

References

[CSP16] R. Coelho Simões and D. Pauksztello, *Torsion pairs in a triangulated category generated by a spherical object*, J. Algebra, **448** (2016), 1–47. doi: 10.1016/j.jalgebra.2015.09.011

[Jør21] P. Jørgensen, *Proper abelian subcategories of triangulated categories and their tilting theory*, preprint (2021), doi: 10.48550/arXiv.2109.01395

[Kor24] A. S. Kortegaard, *Negative Cluster Categories*, Github (2024). https://github.com/Kortegaard/NegativeClusterCategories (Accessed: 2024-07-29).