Higher homological algebra

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PhD Dissertation

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Dedicated to my mother who always supports me
and to my father who has always encouraged me.

Abstract

In 2004 Iyama has introduced the concept of n-cluster tilting subcategories (under the name 'maximal $(n-1)$-orthogonal subcategories') and shown that these categories admit a "higher" version of Auslander–Reiten theory. Soon Geiss–Keller–Oppermann, Jasso and later Herschend–Liu–Nakaoka have introduced the axioms of "higher" structures, such as n-angulated, n-abelian, n-exact, and – generalising the former notions – n-exangulated categories. These give an abstract framework to the properties of n-cluster tilting subcategories and generalise their "classic" counterparts. This thesis is concerned with studying the categorical properties of such "higher" structures, with generalising "classic" result to the "higher" framework and with developing tools and techniques to do so.

One important result in "classic" homological algebra is that an extension-closed subcategory of e.g. an abelian category inherits an exact structure. One important "classic" application of this is in the construction of a Hom-finite Frobenius model for the singularity category of an Iwanaga–Gorenstein ring. We show a "higher" analogue of this result – twice, in different generality under the use of different techniques. While doing so we develop a technique to work with a "higher" version of the octahedral axiom and complete the characterisation of n-exangulated which are n-exact, which was started by Herschend–Liu–Nakaoka.

Another technical but important result in "classic" homological algebra is that every exact or triangulated category can be embedded into a minimal idempotent complete version of itself (with the same kind of structure). This is for example useful whenever categories are equivalent up to taking direct summands. We also show a "higher" analogue of this result.

We also prove "classic" homological results. Notably, we show that a weakly idempotent complete extriangulated category satisfies the technical but important condition (WIC) assumed in many theorems concerning extriangulated categories. As any extriangulated category can be (weakly) idempotent completed, this is a very convenient result.

Lastly, it is well-known that the Coxeter–transformation of a hereditary algebra extends the action of the Auslander–Reiten translation (on the non-projective indecomposable objects) linearly to the Grothendieck group. Originally motivated by the prospect of constructing cluster characters for non-hereditary algebras, we ask the question whether this phenomenon can be exhibited in other algebras. Unfortunately, we showed that for non-directed connected algebras this is only the case for Nakayama algebras.

Resumé

I 2004 introducerede Iyama konceptet n-klynge-vippe delkategorier (under navnet 'maksimal $(n-1)$-ortogonale delkategorier') og viste, at disse kategorier giver plads til en 'højere' version af Auslander-Reiten teori. Snart derefter introducerede Geiss-Keller-Oppermann, Jasso, og senere Herschend-Liu-Nakaoka, aksiomer for "højere" strukturer, så som n-anguleret, n-abelsk, n-eksakt og n-ekstrianguleret strukturer, hvoraf den sidste er en generalisering af de forgående. Disse giver en abstrakt ramme for egenskaberne af n-klynge-vippe delkategorier og generaliserer deres "klassiske" modparter. Denne afhandling handler om at studere de kategoriske egenskaber af sådanne "højere" strukturer ved at generalisere "klassiske" resultater til "højere" versioner, samt at udvikle egenskaber og teknikker til at gøre dette.

Et vigtigt resultat i "klassisk" homologisk algebra er, at en ekstensionsaflukket delkategori, for eksempel af en abelsk kategori, arver en eksakt struktur. En vigtig "klassisk" anvendelse af dette er ved konstruktionen af en Hom-endelig Frobenius model for singularitetskategorien af en Iwanaga–Gorenstein ring. Vi viser to "højere" analoger af dette resultat i forskellige grader af generalitet ved brug af forskellige teknikker. Mens vi gør dette, udvikler vi en teknik til at arbejde med en "højere" version af oktaederaksiomet og færdiggør en karakteristik af n-eksangulerede kategorier som er n-eksakte, som blev startet af Herchend-Liu-Nakaoka.

Et andet vigtigt teknisk resultat i "klassisk" homologisk algebra er, at enhver eksakt eller trianguleret kategori kan indlejres i en minimal idempotent-fuldstændig version af den selv (med den samme slags struktur). Dette er for eksempel brugbart, når ækvivalenser mellem kategorier kun holder op til at tage direkte summander. Vi viser også en "højere" analog af dette resultat.

Vi vil også vise et "klassisk" homologisk resultat. Vi viser, at en svag idempotent-fuldstændig ekstrianguleret kategori opfylder den tekniske, men vigtige, egenskab (WIC), som er antaget i mange sætninger, der omhandler ekstrianguleredе kategorier. Da enhver trianguleret kategori kan (svagt) idempotent-fuldstændiggøres, viser dette sig at være et meget praktisk resultat.

Det er velkendt, at Coxeter transformationen af en hereditær algebra udvider virkningen af Auslander-Reiten translationen (på ikke-projektive indekomposable objekter) lineært til Grothendieck-gruppen. Motiveret af muligheden for at konstruere klyngekarakterer for ikke-hereditære algebraer, spørger vi, om dette fænomen kan findes i andre algebraer. Vi viser desværre, at for ikke-rettet sammensatte algebraer, gælder dette kun i tilfældet af, at vi har en Nakayama-algebra.

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Preface

This dissertation consists of four papers, which are also publicly available on arXiv or published – with minor changes compared to the version included here:

Paper A: n-Exact categories arising from $(n+2)$-angulated categories
Paper B: Idempotent completions of n-exangulated categories
Paper C: n-Extension closed subcategories of n-exangulated categories
Paper D: When does the Auslander–Reiten translation operate linearly on the Grothendieck group? – Part I

Paper A is available on the arXiv and will probably undergo another revision before it will be submitted for review and publication. Paper B is joint-work with Amit Shah and Dixy Msapato and has been published in the journal 'Applied Categorical Structures'. Paper C is currently submitted for review. Paper D has been published in the 'Journal of Pure and Applied Algebra'.

All papers and this thesis have been written under the supervision of Peter Jørgensen whom I would like to greatly thank for the opportunity of doing this PhD and his guidance, advice, patience and mental and professional support throughout my PhD. I am also very grateful for his contributions towards my mathematical development, especially for his constant help with the presentation of my results.

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I would like to thank Martin Kalck for introducing me to algebraic geometry and for inviting me to his joint-project 'Obstructions to semiorthognal decompositions II', which is unfortunately not part of this thesis. Furthermore, I am very thankful to Steffen König for his support and encouragement throughout my master's and for his help in finding this PhD position. I would also like to thank my former teacher Dieter Schwarz for his encouragement and support throughout the final years of high school which ultimately made me study mathematics.

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Carlo Klapproth
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Introduction

1 Overview

One central aspect of homological algebra is the investigation of additive categories and structures put upon these. A classic example of such structures are exact categories, which are due to Quillen [Qui73]. The prototypical exact categories are extension closed subcategories of abelian categories equipped with the inherited exact structure [Büh10]. Indeed, a famous theorem by Gabriel and Quillen shows that every (small) exact category arises this way. The notion of exact categories is still extremely useful and versatile. Recently, a similar development has occurred with the formulation of axioms of extension-closed subcategories of triangulated categories, so called extriangulated categories [NP19]. Even though higher homological algebra is in an early stage of its development, its story seems to be parallel to the story of exact and extriangulated categories in the sense that it puts a set of axioms to the inherited structure on interesting subcategories of classic homological structures (such as abelian, exact and triangulated categories) which cannot be described by the existing language of (classic) homological algebra.

Let us consider an example. Suppose A is the algebra $\Bbbk\big(\overset{1}{\bullet} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \overset{2}{\bullet}\big)/(\alpha\beta, \beta\alpha)$. The category $\mathcal{T} \subseteq \mathsf{mod}\,A$ which is (as an additive category) generated by the blue vertices in the Auslander–Reiten quiver

$$\begin{tikzcd}
\cdots & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[dr] & & \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[dr] & & \cdots \\
\cdots \arrow[r, dotted, no head] & 2 \arrow[ul] \arrow[ur] & & 1 \arrow[ur] \arrow[ll, dotted, "\tau"] & & 2 \arrow[ur] \arrow[ll, dotted, "\tau"] & \cdots \arrow[l, dotted]
\end{tikzcd}$$

of $\mathsf{mod}\,A$, is rigid that is $\operatorname{Ext}^1_A(\mathcal{T}, \mathcal{T}) = 0$. Clearly, $\mathcal{T}$ is a maximally rigid subcategory of $\mathsf{mod}\,A$ as $\operatorname{Ext}^1_A(1, 2) \neq 0$. But in fact, $\mathcal{T}$ is a maximal rigid subcategory of $\mathsf{mod}\,A$ in a very strong sense, making it into a $2\mathbb{Z}$-*cluster-tilting subcategory* of $\mathsf{mod}\,A$. For $n \in \mathbb{N}_{\geq 1}$ we have the following definition.

Definition (cf. e.g. [Iya07], [GKO13] and [Jas16])**.** Let $\mathcal{T}$ be a functorially finite subcategory of a module category $\mathsf{mod}\,A$, where A is finite dimensional, (resp. of a triangulated category $\mathcal{C}$ with shift [1]). Then $\mathcal{T}$ is called *n-cluster tilting subcategory* if

$$\begin{aligned}
\mathcal{T} &= \left\{ X \;\middle|\; \operatorname{Ext}^i(X, \mathcal{T}) = 0 \text{ for all } i = 1, \dots, n-1 \right\} && (1.1)\\
&= \left\{ X \;\middle|\; \operatorname{Ext}^i(\mathcal{T}, X) = 0 \text{ for all } i = 1, \dots, n-1 \right\}, && (1.2)
\end{aligned}$$

where $\operatorname{Ext}^i(-,-) = \operatorname{Ext}^i_A(-,-)$ (resp. $\operatorname{Ext}^i(-,-) = \operatorname{Hom}_{\mathcal{C}}(-,-[i])$). If additionally $\operatorname{Ext}^i(\mathcal{T},\mathcal{T}) = 0$ for $i \notin n\mathbb{Z}$ then we call $\mathcal{T}$ an *$n\mathbb{Z}$-cluster tilting subcategory*. If $\mathcal{T}$ has an additive generator T, then we call T an *n-cluster tilting object*.

In particular, $\mathcal{T}$ (and in fact any n-cluster tilting subcategory of $\operatorname{mod} A$, where A is finite dimensional) does only inherit the split exact structure from $\operatorname{mod} A$ and one easily checks that $\mathcal{T}$ does not even have any non-split kernel-cokernel pairs, using that $\mathcal{T}$ is rigid with $A \in \mathcal{T}$. One now may ask whether there is some different structure on $\mathcal{T}$ inherited from $\operatorname{mod} A$. Following the arguments in [Jas16] we aim to describe this structure.

As $\mathcal{T}$ is Hom-finite and has an additive generator (or more generally, because it is functorially finite in $\operatorname{mod} A$, which has all kernels and cokernels), it has weak kernels (a.k.a. pseudokernels) and weak cokernels (a.k.a. pseudocokernels). But many categories (e.g. all abelian categories) do so and we have not used (1.1) and (1.2) yet.

Consider the non-trivial morphism $\begin{smallmatrix}2\\1\end{smallmatrix} \to 2$ in the diagram

$$\begin{tikzcd} 2 \arrow[r, tail] & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[rr] \arrow[dr, dotted, two heads] & & \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[r, two heads] & 2. \\ & & 1 \arrow[ur, dotted, tail] & & \end{tikzcd} \tag{$*$}$$

Its (real) kernel $1 \to \begin{smallmatrix}1\\2\end{smallmatrix}$ is not in $\mathcal{T}$. However, it has a weak kernel $\begin{smallmatrix}1\\2\end{smallmatrix} \to \begin{smallmatrix}2\\1\end{smallmatrix}$. A small diagram chase shows that every weak kernel factors through the (real) kernel $1 \to \begin{smallmatrix}2\\1\end{smallmatrix}$ and the factor ending at 1 is right $\mathcal{T}$-approximation (which is necessarily epic as $A \in \mathcal{T}$). However, this, the long exact Ext-sequence and (1.2) forces the kernel of $\begin{smallmatrix}2\\1\end{smallmatrix} \to \begin{smallmatrix}1\\2\end{smallmatrix}$ to be in $\mathcal{T}$ and indeed $2 \in \mathcal{T}$. That is the kernel of $\begin{smallmatrix}2\\1\end{smallmatrix} \to 2$ is not in $\mathcal{T}$ but we can "resolve" it in 2 steps within $\mathcal{T}$. We say that $(*)$ is a 2-*kernel diagram* of $\begin{smallmatrix}2\\1\end{smallmatrix} \to 2$. Generally, for $n \in \mathbb{N}_{\geq 1}$, we have the following definition, generalising the notions of kernels (=1-kernels) and cokernels (=1-cokernels).

Definition (cf. e.g. [Jas16])**.** An *n-kernel (diagram)* of f_n is a sequence

$$X_0 \overset{f_0}{\rightarrowtail} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_n (\xrightarrow{f_n} X_{n+1})$$

such that f_{i-1} is a weak kernel of f_i for $i = 2, \ldots, n$ and such that f_0 is a kernel of f_1. Dually, an *n-cokernel (diagram)* of f_0 is defined. A sequence which is both an n-kernel diagram of f_n and an n-cokernel diagram of f_0 is called *n-exact sequence*.

The argument above and its dual argument shows that every morphism in $\mathcal{T}$ has a 2-kernel and a 2-cokernels (and in fact, a similar argument can be applied to show that all n-cluster tilting subcategories admit all n-kernels and n-cokernels). However, the existence of 2-kernels and 2-cokernels does not fully capture the "nature" of $\mathcal{T}$ yet. Indeed, there are many categories with interesting exact structures, which admit all 2-kernels and 2-cokernels (for example $\mathsf{GP}(\Lambda)$ for a 3-Iwanga–Gorenstein algebra Λ has all 2-kernels and 2-cokernels building on work of Auslander–Buchweiz, cf. e.g. [Eno17, Proposition 4.6]).

Looking at the 2-kernel $(*)$ again, we see that it is also a 2-cokernel diagram of $2 \to \begin{smallmatrix}1\\2\end{smallmatrix}$ and hence a 2-exact sequence. This is not a coincidence but is owed to the condition (1.1) making the sequence $(*)$ *almost* into a resolution of 2 by *almost* $\operatorname{Hom}(-,\mathcal{T})$-acyclic objects.

The conditions (1.1) and (1.2) are just strong enough to show that every 2-kernel of an epimorphism and 2-cokernel of a monomorphism in $\mathcal{T}$ yields a 2-exact sequence without forcing them to split (and the same applies to general n-kernels and n-cokernels in general n-cluster tilting subcategories). Similarly to the axiom for abelian categories that every epimorphism is a cokernel and every monomorphism is a kernel, this symmetry is essential to understanding the (higher) structure on $\mathcal{T}$. Apart from the technical condition of idempotent completeness (which $\mathcal{T}$ clearly satisfies) we have checked the following axioms for $n = 2$, which generalise ordinary abelian (=1-abelian) categories.

Definition ([Jas16])**.** For $n \in \mathbb{N}_{\geq 1}$ an idempotent complete category is called *n-abelian* if

- every morphism has an n-kernel and n-cokernel, and
- every n-kernel diagram of an epimorphism and dually every n-cokernel diagram of a monomorphism is an n-exact sequence.

Hence, $\mathcal{T}$ has the structure of a 2-abelian category. It can easily be shown that every kernel-cokernel pair of an n-abelian category has to be split exact. Thus we have discovered a new structure on additive categories, which cannot coexist with any non-trivial exact or triangulated structure on the same category.

But the category $\mathcal{T}$ demonstrates more. Because $\mathcal{T}$ is an $2\mathbb{Z}$-cluster titling subcategory of $\mathsf{mod}\,A$ one can show that the image $\mathsf{D}^{\mathrm{sg}}(\mathcal{T})$ of $\mathcal{T}$ under the canonical functor $\mathsf{mod}\,A \to \mathsf{D}^{\mathrm{sg}}(A)(\cong \underline{\mathsf{mod}}(A))$ is a $2\mathbb{Z}$-cluster tilting subcategory in the triangulated sense [Kva21] (or [Jas16] using that A is self-injective). This yields a 2-*angulated structure* on $\mathsf{D}^{\mathrm{sg}}(\mathcal{T})$, i.e. an autoequivalence $[n]\colon \mathsf{D}^{\mathrm{sg}}(\mathcal{T}) \to \mathsf{D}^{\mathrm{sg}}(\mathcal{T})$ and an a collection $\pentagon$ of complexes

$$X_0 \to X_1 \to \cdots \to X_n \to X_{n+1} \rightsquigarrow X_0[n]$$

called *n-angles*, satisfying axioms similar to the axioms of a triangulated category.

Theorem (Cf. [GKO13])**.** *Let $\mathcal{T}$ be a $n\mathbb{Z}$-cluster tilting subcategory of a triangulated category $\mathcal{C}$ with suspension $[1]$. Then $\mathcal{T}$ inherits the structure of an $(n+2)$-angulated category. In this case the suspension of $\mathcal{T}$ is $[n]$ and the $(n+2)$-angles $\pentagon$ are direct summands of complexes*

$$X_\bullet\colon \qquad X_0 \to X_1 \to \cdots \to X_n \to X_{n+1} \rightsquigarrow X_0[n]$$

such that there are triangles $C_i \to X_{i+1} \to C_{i+1} \rightsquigarrow C_i[1]$ for $0 \leq i \leq n-1$ with $C_0 = X_0$ and $C_n = X_{n+1}$ making the apparent solid morphisms of the diagram

$$\begin{tikzcd}
& X_1 \arrow[rr] \arrow[dr] & & X_2 \arrow[r] \arrow[dr] & \cdots \arrow[r] & X_{n-1} \arrow[rr] \arrow[dr] & & X_n \arrow[dr] & \\
C_0 = X_0 \arrow[ur] & & C_1 \arrow[ll, dotted] \arrow[ur] & & \cdots \arrow[ll, dotted] \arrow[ur] & & C_{n-1} \arrow[ll, dotted] \arrow[ur] & & X_{n+1} = C_n \arrow[ll, dotted]
\end{tikzcd}$$

commutative and such that the last morphism $X_{n+1} \rightsquigarrow X_0[n]$ of $X_\bullet$ is given by the composite $X_{n+1} = C_n \rightsquigarrow C_{n-1}[1] \rightsquigarrow \cdots \rightsquigarrow C_1[n-1] \rightsquigarrow C_0[n] = X_0[n]$ of the apparent morphisms.

In our example we have $\mathsf{D}^{\mathrm{sg}}(\mathcal{T}) \cong \operatorname{\mathsf{mod}}\Bbbk \subseteq \operatorname{\mathsf{mod}}\Bbbk \times \Bbbk \cong \mathsf{D}^{\mathrm{sg}}(\operatorname{\mathsf{mod}} A)$ as additive categories and $\mathsf{D}^{\mathrm{sg}}(\mathcal{T})$ carries a trivial 4-angulated structure with $[2] \cong \mathrm{Id}_{\mathsf{D}^{\mathrm{sg}}(\mathcal{T})}$ and 4-angles given by the "contractible" complexes.

Apart from n-abelian and $(n+2)$-angulated categories, higher homological algebra puts an axiomatic framework to so-called *n-exact* categories [Jas16] which generalise exact categories. These n-exact categories have an additional datum consisting of a subclass of all n-exact sequences, called *conflations*, which are considered "good" and satisfy certain axioms. A monomorphism (epimorphism) appearing as the first (last) morphism of a conflation is then called an *inflation* (a *deflation*). As in the classic case an n-abelian category together with the class of all n-exact sequences forms an n-exact category.

Similarly to extriangulated categories there are *n-exangulated categories* [HLN21] unifying all of the above higher notions. These categories are triples $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$:

- $\mathcal{C}$ is an additive category.
- $\mathbb{E}$ "parametrises" extensions functorially. We have $\mathbb{E}(-,-) = \operatorname{Ext}^n(-,-)$ for an n-exact category and $\mathbb{E}(-,-) = \mathcal{C}(-,-[n])$ for a $(n+2)$-angulated category.
- $\mathfrak{s}$ "realises" an extension (up to homotopy). Morally, for an n-exact category $\mathfrak{s}$ just picks a representative (if we think of $\operatorname{Ext}^n(-,-)$ as Yoneda extensions) and for an $(n+2)$-angulated category $\mathfrak{s}$ completes to an $(n+2)$-angle (to the left).

All these higher structures have in common that one obtains the classic structures by letting $n = 1$. Informally speaking, classic homological algebra studies 3-term sequences (e.g. short exact sequences, triangles) whilst higher homological algebra is concerned with $(n+2)$-term sequences which are exact in a suitable sense (e.g. n-exact sequences [Jas16], $(n+2)$-*angles* [GKO13]). In particular, higher homological algebra is not "higher" in the sense of higher category theory.

Two major obstacles in the development of higher homological algebra are that many constructions (e.g. n-kernels) are only unique up to homotopy (making the theory quite technical) and the (current) lack of ability to produce easily accessible of examples. Indeed, admitting an n-cluster tilting subcategory (for $n \in \mathbb{N}_{\geq 2}$) appears to be a very strong property for a module category $\operatorname{\mathsf{mod}} A$ or a derived category $\mathsf{D}^{\mathrm{b}}(A)$, where $A = \Bbbk Q/I$ is a finite dimensional $\Bbbk$-algebra, say over an algebraically closed field $\Bbbk$.

Notably, let $n \geq 2$. Then if $\operatorname{rad}^2(A) = 0$ and $\operatorname{\mathsf{mod}} A$ has a n-cluster tilting subcategory then A is a representation-finite string algebra with strong restrictions on its quiver Q [Vas23]. If A is a gentle algebra with an n-cluster tilting subcategory in $\operatorname{\mathsf{mod}} A$ (resp. with an $n\mathbb{Z}$-cluster tilting subcategor in $\mathsf{D}^{\mathrm{b}}(A)$) then A is a Nakayama algebra with $\operatorname{rad}^2 A = 0$ (resp. derived equivalent to an algebra of Dynkin type A) [HJS22]. If A is self-injective and $\operatorname{\mathsf{mod}} A$ has a n-cluster subcategory admitting an additive generator then the complexity of A is at most 1 [EH08].

However, this thesis is mostly concerned with studying the categorical properties of higher homological structures, n-exangulated categories in particular, the development of tools and techniques to work with these and not so much with constructing explicit examples or giving immediate applications.

2 Summary of results

The following papers have been either uploaded as preprints on the arXiv or published. Apart from minor changes the papers included in this thesis agree with their publicly available version.

Paper A: n-Exact categories arising from $(n+2)$-angulated categories.
(Preprint, arXiv:2108.04596)

In this paper we show a higher analogue of an unpublished result by Dyer [Dye05]. He has shown that given a full additive extension-closed subcategory $\mathcal{E}$ of a triangulated category $(\mathcal{C}, [1], \Delta)$ with $\operatorname{Hom}_{\mathcal{C}}(\mathcal{E}[1], \mathcal{E})$ one can define an exact structure $(\mathcal{E}, \mathcal{X})$ on $\mathcal{E}$ by letting $\mathcal{X}$ be the class of all sequences $E' \to E \to E''$ in $\mathcal{E}$ which admit a completion to a distinguished triangle $E' \to E \to E'' \rightsquigarrow \Sigma E'$. It was natural to ask, whether this result has a higher analogue. Indeed, using the apparent higher reformulation of the class $\mathcal{X}$ we can show the following.

Theorem A. *Suppose that $\mathcal{E}$ is a full, additive and n-extension closed subcategory of an $(n+2)$-angulated Krull-Schmidt category $(\mathcal{C}, [n], \pentagon)$ with $\operatorname{Hom}_{\mathcal{C}}(\mathcal{E}[n], \mathcal{E}) = 0$. Then*

1. *$(\mathcal{E}, \mathcal{X})$ is an n-exact category and*
2. *there is a natural and bilinear isomorphism $\operatorname{Hom}_{\mathcal{C}}(-, -[n]) \to \operatorname{Ext}^n_{(\mathcal{E},\mathcal{X})}(-, -)$ of functors $\mathcal{E}^{\mathrm{op}} \times \mathcal{E} \to \mathsf{Ab}$.*

We also developed a method to mitigate the lack of control the higher version of the octahedral axiom offers. To do so, we (re)introduce the notion of minimal $(n+2)$-angles, which were also considered in [OT12] and [Jør16], in the context of Krull–Schmidt $(n+2)$-angulated categories.

Paper B: Idempotent completions of n-exangulated categories.
(With Dixy Msapato and Amit Shah: Applied Categorical Structures **32**, article 7, 2024.)

Msapato has shown that the (weak) idempotent completion of an extriangulated category has again a natural extriangulated structure [Msa22]. This generalises a well-known result for triangulated categories [BS01] and exact categories, cf. e.g. [Büh10]. In the higher case, Liu has shown a that the idempotent completion of an $(n+2)$-angulated category is again $(n+2)$-angulated [Liu21] . In this joint-paper with Amit Shah and Dixy Msapato we consider the case of n-exangulated categories and (essentially) unify all previous results.

Theorem B. *For any n-exangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ there is an n-exangulated functor $(\mathscr{I}_{\mathcal{C}}, \Gamma) \colon (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\mathcal{D}, \mathbb{F}, \mathfrak{t})$ to a (weakly) idempotent complete n-exangulated category $(\mathcal{D}, \mathbb{F}, \mathfrak{t})$ which is 2-universal among n-exangulated functors from $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ to (weakly) idempotent complete n-exangulated categories.*

Paper C: n-Extension closed subcategories of n-exangulated categories.
(Preprint, arXiv:2209.01128)

In this paper we consider a similar problem to Paper A but in a more general setting and under the use of different methods. This paper arose from the question, whether the Krull–Schmidt requirement could be removed from Theorem A and the question posed by Haugland, whether higher torsion classes are again n-exact (cf. Section 3).

Answering the first question, we show that the class of n-exangulated categories is closed under forming n-extension closed subcategories.

Theorem C. *Suppose that $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an n-exangulated category and $\mathcal{A} \subseteq \mathcal{C}$ is an n-extension closed additive subcategory. Then $\mathcal{A}$ inherits an n-exangulated structure from $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ in a natural way.*

Concerning classic homological algebra we also show that the condition (WIC) for extriangulated categories, which was introduced by [NP19] and which is similar to the obscure axiom for exact categories, is equivalent to the underlying category being weakly idempotent complete.

Proposition. *An extriangulated category is weakly idempotent complete if and only if f is an inflation whenever there is a morphism g with gf being an inflation. In an n-exangulated category with $n \geq 2$ it is always true that f is an inflation whenever there is a morphism g with gf being an inflation.*

This result also allows us to complete the characterisation of n-exact categories among n-exangulated categories which was started in [HLN21],

Theorem. *For an additive category $\mathcal{C}$ there is a one-to-one correspondence*

$$\left\{\begin{matrix} n\text{-exact structures } (\mathcal{C}, \mathcal{X}) \text{ with} \\ \text{small extension groups} \end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \frac{\left\{\begin{matrix} n\text{-exangulated structures } (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \text{ with} \\ \text{monic } \mathfrak{s}\text{-inflations and epic } \mathfrak{s}\text{-deflations} \end{matrix}\right\}}{\left\{\begin{matrix} \text{equivalences of } n\text{-exangulated categories} \\ \text{of the form } (\mathrm{Id}_{\mathcal{C}}, \Gamma) \end{matrix}\right\}}.$$

This was used to answer Haugland's question [AHJKPT23] (cf. also Section 3).

Paper D: When does the Auslander–Reiten translation operate linearly on the Grothendieck group? – Part I
(Journal of Pure and Applied Algebra **228**(6), 2024.)

For an hereditary finite-dimensional algebras A there is a linear map $\Phi\colon \mathsf{K}_0(A) \to \mathsf{K}_0(A)$, the Coxeter transformation, with $\Phi[X] = [\tau X]$ for all indecomposable non-projective modules $X \in \operatorname{\mathsf{mod}} A$, where τ is the Auslander–Reiten translation. Originally motivated by defining cluster characters for non-hereditary finite-dimensional algebras A, we call such a linear map Φ a *τ-map*, and investigate for which algebras such a map exists. For Nakayama algebras this is indeed the case.

Proposition. *Let A be a Nakayama algebra and fix $x_S \in \mathsf{K}_0(\operatorname{mod} A)$ for each simple projective module $S \in \operatorname{mod} A$. Then the unique morphism $\Phi \in \operatorname{End}_{\mathbb{Z}}(\mathsf{K}_0(\operatorname{mod} A))$ with*

$$\Phi[S] = \begin{cases} [\tau_A S] & \text{for } S \in \operatorname{mod} A \text{ simple and non-projective,} \\ x_S & \text{for } S \in \operatorname{mod} A \text{ simple and projective,} \end{cases} \tag{2.1}$$

is a τ-map for A. If A has no projective simple modules then A has a unique τ-map.

In general, the answer seems to be unfortunately disappointing. We show that for example for any admissibly presented finite dimensional connected algebra $A := \Bbbk Q/I$ with a cycle in Q such a map exists only if A is already a Nakayama algebra.

Theorem D. *Suppose the Ext^1-quiver of A is connected and non-acyclic. Then A has a τ-map if and only if A is a cyclic Nakayama algebra.*

3 Small Examples

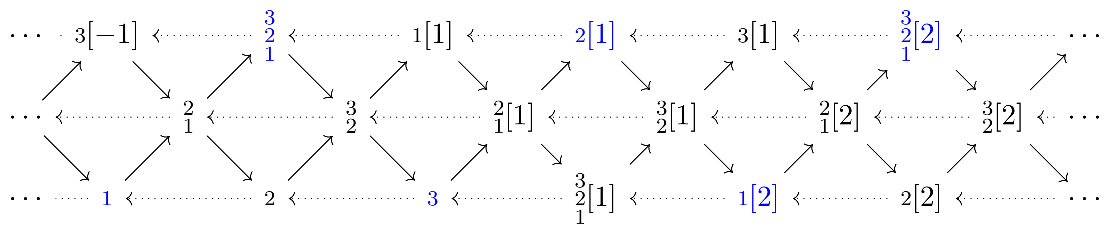
For a (1-)hereditary (1-)representation finite algebra A one can show that every object of $\mathsf{D}^{\mathrm{b}}(A)$ is the sum of suspensions of modules. Iyama has constructed in [Iya11] a "higher derived category" for certain n-abelian categories mimicking this observation.

Let A be a finite dimensional *n-hereditary n-representation finite* algebra in the sense of [IO11], that is $\operatorname{gldim} A \leq n$ and $\operatorname{mod} A$ has a n-cluster tilting object T. Then the subcategory

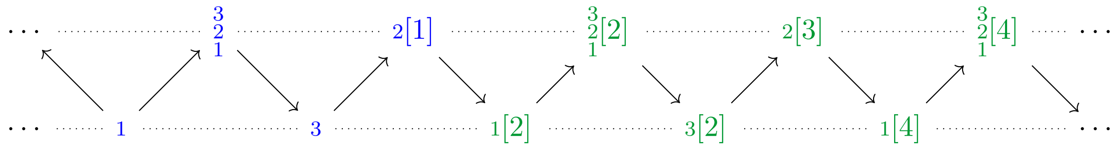
$$\mathcal{C} := \operatorname{add} \left\{ X \in \mathsf{D}^{\mathrm{b}}(A) \,\middle|\, X = T[ni] \text{ for some } i \in \mathbb{Z} \right\} \subseteq \mathsf{D}^{\mathrm{b}}(A)$$

is an $n\mathbb{Z}$-cluster tilting subcategory containing T (identified with its image in $\mathsf{D}^{\mathrm{b}}(A)$ under the canonical inclusion $T \in \operatorname{mod} A \hookrightarrow \mathsf{D}^{\mathrm{b}}(A)$) [Iya11, Theorem 1.21]. As in the classic case one can show that the subcategory $\mathcal{C}^I \subseteq \mathcal{C}$ of objects with homology concentrated in degrees $I \subseteq \mathbb{Z}$ is n-extension closed (showing this boils essentially down to homology mapping $(n+2)$-angles to exact sequences, as $\mathrm{H}^i(-) = \operatorname{Hom}_{\mathcal{C}}(A[i], -)\colon \mathcal{C} \to \mathcal{T} \subseteq \operatorname{mod} A$, and that every object in $\mathcal{C}$ isomorphic to its homology) and hence n-exangulated by Theorem C.

Example. Let us consider the example $A := \Bbbk(\overset{3}{\bullet} \xrightarrow{\alpha} \overset{2}{\bullet} \xrightarrow{\beta} \overset{1}{\bullet})$ and $B := A/(\alpha\beta)$. Then B is a so-called higher Auslander algebra [Iya11] and derived equivalent to A via the tilting module $1 \oplus {\scriptsize\begin{smallmatrix}3\\2\\1\end{smallmatrix}} \oplus 3 \in \operatorname{mod} A$. We may identify $\mathsf{D}^{\mathrm{b}}(A)$ and $\mathsf{D}^{\mathrm{b}}(B)$ via this equivalence. It follows that the additive subcategory $\mathcal{C} \subseteq \mathsf{D}^{\mathrm{b}}(A)$ spanned by all even suspensions of $1 \oplus {\scriptsize\begin{smallmatrix}3\\2\\1\end{smallmatrix}} \oplus 3 \oplus 2[1]$ is a cluster tilting subcategory. We can draw the Auslander–Reiten quiver



of $\mathsf{D}^{\mathrm{b}}(A)$, where the blue vertices indicate the indecomposable objects of the 2-cluster tilting subcategory $\mathcal{C} \subseteq \mathsf{D}^{\mathrm{b}}(A)$ and the dashed arrows indicate the Auslander–Reiten translation of $\mathsf{D}^{\mathrm{b}}(A)$. The (Gabriel-)quiver of $\mathcal{C}$ is given by



where we indicate zero relations by dotted arrows. By the above discussion the subcategory $\mathcal{A}$ spanned by the green vertices, i.e. by $1[2]$ and everything to the right of it, is closed under 2-extensions in $\mathcal{C}$ and hence 2-exangulated. The conflation $\begin{smallmatrix}3\\2\\1\end{smallmatrix}[2] \to 3[2] \to 2[3] \to 1[4] \dashrightarrow$ shows that $\mathcal{A}$ is not 2-exact as $\begin{smallmatrix}3\\2\\1\end{smallmatrix}[2] \to 3[2]$ is not a monomorphism in $\mathcal{A}$. However, $\mathcal{A}$ cannot be $(n+2)$-angulated either, because $1[2] \to \begin{smallmatrix}3\\2\\1\end{smallmatrix}[2]$ is a monomorphism that is not split.

Let $\mathcal{T} \subseteq \mathsf{mod}\, A$ be a n-cluster tilting subcategory, where A is a finite dimensional algebra. A subcategory $\mathcal{U} \subseteq \mathcal{T}$ is called *n-torsion class* if for every $T \in \mathcal{T}$ there exists an n-exact sequence

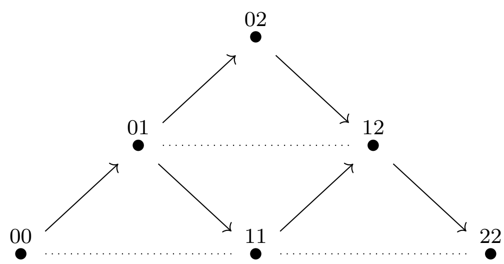
$$0 \to U_T \to T \to V_1 \to \cdots \to V_n \to 0$$

in (the n-abelian category) $\mathcal{T}$ such that $U_T \in \mathcal{U}$ and the induced sequence

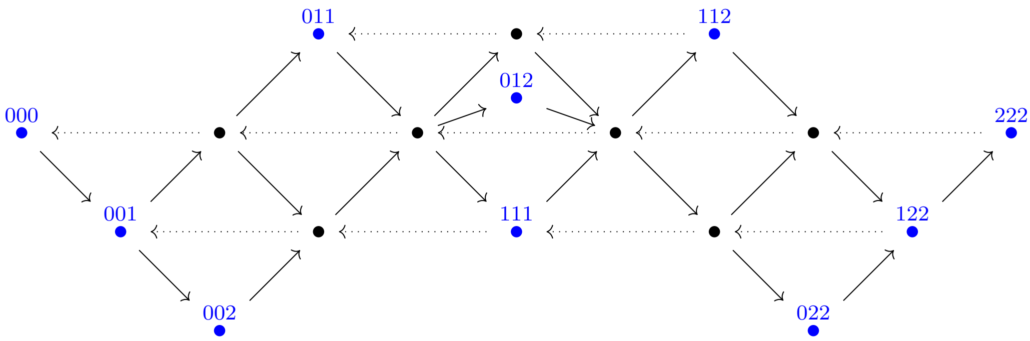
$$0 \to \mathrm{Hom}_{\mathcal{T}}(U, V_1) \to \cdots \to \mathrm{Hom}_{\mathcal{T}}(U, V_n)$$

is exact in Ab for every $U \in \mathcal{U}$ [Jør16]. It has been shown in [AHJKPT23] that such a category $\mathcal{U}$ is n-extension closed in $\mathcal{T}$ and therefore the class of all n-exact sequences in $\mathcal{T}$ with terms in $\mathcal{U}$ yields an n-exact structure on $\mathcal{U}$ by Theorem C and the characterisation of n-exact categories from Paper C.

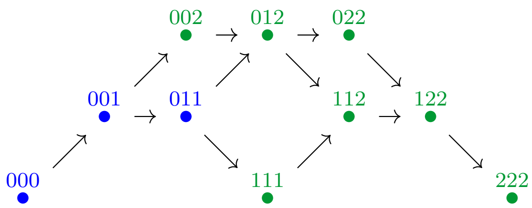
Example (Higher torsion classes, [AHJKPT23])**.** Let us consider a concrete example. Let A be the algebra of the quiver



modulo mesh relations, i.e. the dashed line in the square indicates anticommutativity and the other dashed lines indicate zero relations. This is also a higher Auslander algebra of type A. The Auslander–Reiten quiver[1] of A is



with the blue vertices corresponding to the indecomposable objects of a 2-cluster tilting subcategory of $\mathcal{T} := \mathsf{add}(A \oplus \mathsf{D}A \oplus S_{11}) \subseteq \mathsf{mod}\, A$, where S_{11} is the simple module corresponding to the vertex 11 of A (the labelling of the blue vertices is explained in [JKPK19]). The (Gabriel-)quiver of the 2-cluster-tilting subcategory is given by



with an example of a 2-torsion class[2] spanned by the green vertices.

As A is a higher Auslander algebra of type A we have gldim $A \leq 2$ and hence

$$\mathcal{C} := \mathsf{add}\left\{ X \in \mathsf{D}^{\mathrm{b}}(A) \,\middle|\, X = T[2i] \text{ for some } T \in \mathcal{T} \text{ and } i \in \mathbb{Z} \right\} \subseteq \mathsf{D}^{\mathrm{b}}(A)$$

is as above an $(n+2)$-angulated category with $\operatorname{Ext}^n(T, T') \cong \operatorname{Hom}_{\mathcal{C}}(T, T'[n])$ for $T, T' \in \mathcal{T}$. Hence, in this special case we could have also used Theorem A to show that all 2-torsion classes of $\mathcal{T}$ are 2-exact.

We have the following example in relationship to condition (WIC) and its higher analogue considered in Paper B.

Example. Let $\Bbbk$ be a field and consider the subcategory $\mathcal{V} \subseteq \mathsf{mod}\,\Bbbk$ of vector spaces with dimension unequal to 1. Clearly $\mathcal{V}$ is an extension closed subcategory of $\mathsf{mod}\,\Bbbk$ and hence an exact category. However, $\mathcal{V}$ is also an n-cluster tilting subcategory of itself for any $n \geq 1$ and hence an n-exact category (this is shown similarly to the n-abelian case). We may choose maps $r\colon \Bbbk^3 \to \Bbbk^2$ and $s\colon \Bbbk^2 \to \Bbbk^3$ with $rs = \mathrm{id}_{\Bbbk^2}$ and put $e := sr$. In $\mathcal{V}$ as a 1-exact category s is not an inflation as $\Bbbk \notin \mathcal{V}$, showing that condition (WIC) fails for $\mathcal{V}$. However if we consider $\mathcal{V}$ as an n-exact category, with $n \geq 2$, there is enough "space" resolve this problem. In fact, one easily checks that $\Bbbk^2 \xrightarrow{s} \Bbbk^3 \xrightarrow{1-e} \Bbbk^3 \xrightarrow{r} \Bbbk^2 \to 0 \to \cdots \to 0$ is a conflation in $\mathcal{V}$.

[1]Thanks to Jan Geuenich for providing the String Applet: `https://www.math.uni-bielefeld.de/~jgeuenich/string-applet/`

[2]Thanks to Jenny August, Johanne Haugland, Karin Jacobsen, Sondre Kvamme, Yann Palu and Hipolito Treffinger for providing code to calculate examples like these: `https://colab.research.google.com/drive/172Q-UZHvdPOhngGkl1T_xdYLzntg31dY`.

4 The author's contribution to joint papers

Paper B: Idempotent completions of n-exangulated categories.
(With Dixy Msapato and Amit Shah: Applied Categorical Structures **32**, article 7, 2024.)

This is a joint project and all the research and writing was done together without any clear work division. However, I played a major role during the research phase and proportional role during the writing-up phase of the paper.

References

[AHJKPT23] Jenny August, Johanne Haugland, Karin M. Jacobsen, Sondre Kvamme, Yann Palu and Hipolito Treffinger *A characterisation of higher torsion classes*, Preprint, arXiv:2301.10463, 2023.

[BS01] Paul Balmer and Marting Schlichting, *Idempotent completion of triangulated cate- gories*. J. Algebra, **236**(2):819–834, 2001.

[Büh10] Theo Bühler, *Exact categories*, Expositiones Mathematicae, **28**(1):1-69, 2010.

[Dye05] Matthew J Dyer, *Exact subcategories of triangulated categories*, https://www3.nd.edu/~dyer/papers/extri.pdf, preprint, 2005.

[EH08] Karin Erdmann and Thorsten Holm, *Maximal n-orthogonal modules for selfinjective algebras*, Proceedings of the American Mathematical Society **136**(9):3069–3078, 2008.

[Eno17] Haruhisa Enomoto, *Classifying exact categories via Wakamatsu tilting*, Journal of Algebra, **485**:1–44, 2017.

[GKO13] Christof Geiss, Bernhard Keller and Steffen Oppermann, *n-angulated categories*, Journal für die reine und angewandte Mathematik (Crelles Journal), **675**:101-120, 2013.

[HJS22] Johanne Haugland, Karin M. Jacobsen and Sibylle Schroll, *The role of gentle algebras in higher homological algebra*, Forum Mathematicum, **34**(5):1255-1275, 2022.

[HLN21] Martin Herschend, Yu Liu and Hiroyuki Nakaoka, *n-exangulated categories (I): Definitions and fundamental properties*, Journal of Algebra, **570**:531-586, 2021.

[IO11] Osamu Iyama and Steffen Oppermann, *n-representation-finite algebras and n-APR tilting*, Transactions of the American Mathematical Society, **363**6575–6614, 2011.

[Iya07] Osamu Iyama, *Higher-dimensional Auslander-Reiten theory on maximal orthogonal subcategories*, Advances in Mathematics **210**(1):22–50, 2007.

[Iya11] Osamu Iyama, *Cluster tilting for higher Auslander algebras*, Advances in Mathematics, **226**(1):1-61, 2011.

[Jas16] Gustavo Jasso, *n-Abelian and n-exact categories*, Mathematische Zeitschrift, **283**:703–759(3-4), 2016.

[JKPK19] Gustavo Jasso, Julian Külshammer, Chrysostomos Psaroudakis and Sondre Kvamme, *Higher Nakayama algebras I: Construction*, Advances in Mathematics **351**:1139-1200. 2019.

[Jør16] Peter Jørgensen, *Torsion classes and t-structures in higher homological algebra*, International Mathematics Research Notices, **13**:3880–3905, 2016.

[Kva21] Sondre Kvamme, *$d\mathbb{Z}$-Cluster tilting subcategories of singularity categories*, Mathematische Zeitschrift, **297**(1):803–825, 2021.

[Liu21] Zengqiang Lin, *Idempotent Completion of n-angulated Categories*, Applied Categorical Structures, **29**(6):1063–1071, 2021.

[Msa22] Dixy Msapato, *The Karoubi envelope and weak idempotent completion of an extriangulated category*, Applied Categegorical Structures, **30**(3):499—535, 2022.

[NP19] Hiroyuki Nakaoka and Yann Palu, *Extriangulated categories, Hovey twin cotorsion pairs and model structures*, Cahiers de Topologie et Géométrie Différentielle Catégoriques, **60**(3):117–193, 2019.

[OT12] Steffen Oppermann and Hugh Thomas. Higher-dimensional cluster combinatorics and representation theory. *Journal of the European Mathematical Society*, 14(6):1679–1737, 2012.

[Qui73] Daniel Quillen, *Higher algebraic K-theory I*, in Algebraic K-theory, I: Higher K-theories, Lecture Notes in Mathematics, **341**:85-147, Springer, Berlin-New York 1973.

[Vas23] Laertis Vaso, *n-cluster tilting subcategories for radical square zero algebras*, Journal of pure and applied algebra, **227**(1):Paper No. 107157, 2023

Paper A

n-Exact categories arising from $(n+2)$-angulated categories

Carlo Klapproth

Abstract

Let $(\mathscr{F}, \Sigma_n, \pentagon)$ be an $(n+2)$-angulated Krull-Schmidt category and $\mathscr{A} \subseteq \mathscr{F}$ an n-extension closed, additive and full subcategory with $\operatorname{Hom}_{\mathscr{F}}(\Sigma_n \mathscr{A}, \mathscr{A}) = 0$. Then $\mathscr{A}$ naturally carries the structure $(\mathscr{A}, \mathscr{E}_{\mathscr{A}})$ of an n-exact category in the sense of [Jas16, definition 4.2], arising from short $(n+2)$-angles in $(\mathscr{F}, \Sigma_n, \pentagon)$ with objects in $\mathscr{A}$ and there is a binatural and bilinear isomorphism $\operatorname{YExt}^n_{(\mathscr{A}, \mathscr{E}_{\mathscr{A}})}(A_{n+1}, A_0) \cong \operatorname{Hom}_{\mathscr{F}}(A_{n+1}, \Sigma_n A_0)$ for $A_0, A_{n+1} \in \mathscr{A}$. For $n = 1$ this has been shown in [Dye05] and we generalize this result to the case $n > 1$. On the journey to this result, we also develop a technique for harvesting information from the higher octahedral axiom (N4*) as defined in [BT13, section 4]. Additionally, we show that the axiom (F3) for pre-$(n+2)$-angulated categories, stating that a commutative square can be extended to a morphism of $(n+2)$-angles and defined in [GKO13, definition 2.1], implies a stronger version of itself.

Introduction

Let $\mathscr{A}$ be an additive extension-closed subcategory of a triangulated category $(\mathscr{F}, \Sigma, \triangle)$, with $\operatorname{Hom}_{\mathscr{F}}(\Sigma \mathscr{A}, \mathscr{A}) = 0$. Then $\mathscr{A}$ carries the structure $(\mathscr{A}, \mathscr{E}_{\mathscr{A}})$ of an exact category, where the conflations in $\mathscr{A}$ are precisely the sequences $A_0 \to A_1 \to A_2$ which can be completed to a triangle $A_0 \to A_1 \to A_2 \rightsquigarrow$ in $\triangle$. Moreover, there is an isomorphism $\operatorname{YExt}^1_{(\mathscr{A}, \mathscr{E}_{\mathscr{A}})}(A_2, A_0) \cong \operatorname{Hom}_{\mathscr{F}}(A_2, \Sigma A_0)$ which is natural in $A_0, A_2 \in \mathscr{A}$. This is the main theorem of [Dye05] and it was also rediscovered in [Jør20, proposition 2.5]. We show that this result has a higher counterpart:

Theorem A. *Let $\mathscr{A}$ be an full, additive and n-extension closed subcategory of an $(n+2)$-angulated Krull-Schmidt category $(\mathscr{F}, \Sigma_n, \pentagon)$ with $\operatorname{Hom}_{\mathscr{F}}(\Sigma_n \mathscr{A}, \mathscr{A}) = 0$. Then*

1. *$(\mathscr{A}, \mathscr{E}_{\mathscr{A}})$ is an n-exact category and*
2. *there is a natural and bilinear isomorphism $\operatorname{Hom}_{\mathscr{F}}(-, \Sigma_n(-)) \to \operatorname{YExt}^n_{(\mathscr{A}, \mathscr{E}_{\mathscr{A}})}(-,-)$ of functors $\mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathsf{Ab}$.*

Here n-exact categories are as defined in [Jas16, definition 4.2] and $\operatorname{YExt}^n(A_{n+1}, A_0)$ denotes the n-Yoneda-extensions of A_{n+1} by A_0 in $(\mathscr{A}, \mathscr{E}_{\mathscr{A}})$, that is the class of all sequences $A_0 \to A_1 \to \cdots \to A_n \to A_{n+1}$ in $\mathscr{E}_{\mathscr{A}}$ modulo the equivalence relation generated by the weak equivalences of n-exact sequences as defined in [Jas16, definition 2.9]. The n-exact structure $\mathscr{E}_{\mathscr{A}}$ is defined analogously to the theorem for triangulated categories. The proof of Theorem A has some new features. Notably, showing that the composition of two inflations is an inflation (see [Jas16, definition 4.2(E1)]) causes a problem as the (very technical) higher octahedral axiom [GKO13, definition 2.1(F4)] is required for this.

Hence, we reintroduce minimal $(n+2)$-angles (see [OT12, lemma 5.18] and [Fed19, lemma 3.14]), essentially assigning an, up to isomorphism and rotation, unique $(n+2)$-angle to a given morphism. We then use this and the version of the higher octahedral axiom (N4*) defined in [BT13, section 4] to show that the objects of the minimal $(n+2)$-angle of a composite $h_0 = f_0 g_0$ can be calculated by means of the objects of any $(n+2)$-angle

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}}$$

containing f_0, the objects of any $(n+2)$-angle

$$X'_0 \xrightarrow{g_0} X'_1 \xrightarrow{g_1} X'_2 \xrightarrow{g_2} \cdots \xrightarrow{g_n} X'_{n+1} \xrightarrow{g_{n+1}}$$

containing g_0 and the minimal $(n+2)$-angle of $g_{n+1}\Sigma_n f_1$ (see Lemma 2.4). This will equip us with the necessary leverage to show that [Jas16, definition 4.2(E1)] holds in the situation of Theorem A and it is likely very useful in other situations as well.

On the journey of showing part (2) of Theorem A we also show that the axiom (F3) from [GKO13, definition 2.1] implies a stronger version of itself: Suppose we are given a commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[d, "\phi_0"] & X_1 \arrow[r, "f_1"] \arrow[d, "\phi_1"] & \cdots \arrow[r, "f_{i-1}"] & X_i \arrow[r, "f_i"] \arrow[d, "\phi_i"] & X_{i+1} \arrow[r, "f_{i+1}"] & X_{i+2} \arrow[r, "f_{i+2}"] & \cdots \arrow[r, "f_n"] & X_{n+1} \arrow[r, squiggly, "f_{n+1}"] & \Sigma_n X_0 \arrow[d, "\Sigma_n \phi_0"] \\
Y_0 \arrow[r, "g_0"] & Y_1 \arrow[r, "g_1"] & \cdots \arrow[r, "g_{i-1}"] & Y_i \arrow[r, "g_i"] & Y_{i+1} \arrow[r, "g_{i+1}"] & Y_{i+2} \arrow[r, "g_{i+2}"] & \cdots \arrow[r, "g_n"] & Y_{n+1} \arrow[r, squiggly, "g_{n+1}"] & \Sigma_n Y_0
\end{tikzcd}$$

where the upper row is an $(n+2)$-angle X and the lower row is an $(n+2)$-angle Y and $i \geq 2$ is arbitrary. We show that we can complete the given morphisms to a morphism $\phi = (\phi_0, \ldots, \phi_{n+1})\colon X \to Y$ of $(n+2)$-angles. This will be used to show that the connecting morphism f_{n+1} of X is uniquely determined by $f_0, \ldots, f_n$ if $\operatorname{Hom}_{\mathscr{F}}(\Sigma_n X_0, X_{n+1}) = 0$ generalizing [BBD82, corollaire 1.1.10(ii)] to $(n+2)$-angulated categories (see Lemma 2.2).

1 Preliminaries and notations

Given morphisms $f\colon X \to X'$ and $g\colon X' \to X''$ we denote the composite $X \xrightarrow{f} X' \xrightarrow{g} X''$ by fg. Throughout this paper suppose that $(\mathscr{F}, \Sigma_n, \pentagon)$ is pre-$(n+2)$-angulated category in the sense of [GKO13, definition 2.1]. As in [GKO13] we assume that Σ_n is an automorphism of $\mathscr{F}$ and not just an autoequivalence. When no ambient category is mentioned we implicitly assume it to be $\mathscr{F}$. Further, all subcategories considered will be full subcategories, unless mentioned otherwise. Likewise to [GKO13, definition 2.1] we call a sequence of morphisms $X_0 \to X_1 \to \cdots \to X_{n+1} \to \Sigma_n X_0$ an $(n+2)$-Σ_n-sequence. Such an $(n+2)$-Σ_n-sequence is called $(n+2)$-angle if it belongs to $\pentagon$. In that case we may write the rightmost morphism squiggly, to emphasize that this $(n+2)$-Σ_n-sequence is a $(n+2)$-angle. We then may also drop writing the target $\Sigma_n X_0$ of the rightmost morphism. Suppose X is an $(n+2)$-angle of the shape

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_{n+1} \overset{f_{n+1}}{\rightsquigarrow} \quad .$$

Then we call f_i the morphism in *position* i of X. As we will often deal with trivial $(n+2)$-angles, we define a map $\operatorname{triv}\colon \mathscr{F} \times \{0,\dots,n+1\} \to \pentagon$ assigning to $(Z,i) \in \mathscr{F} \times \{0,\dots,n+1\}$ the trivial $(n+2)$-angle $0 \to 0 \to \cdots \to 0 \to Z \to Z \to 0 \to \cdots \to 0 \to 0 \rightsquigarrow$ having id_Z as the morphism in position i. We will always use subscript notation, i.e. we will write $\operatorname{triv}_i(Z) := \operatorname{triv}(Z,i)$ for $Z \in \mathscr{F}$ and $i \in \{0,\dots,n+1\}$.

For convenience we restate the following definitions:

Definition 1.1. A subcategory $\mathscr{A} \subseteq \mathscr{F}$ is called

1. *additive* if it is closed under direct summands and direct sums and

2. *n-extension closed* if for all morphisms $f_{n+1}\colon A_{n+1} \to \Sigma_n A_0$ with $A_0, A_{n+1} \in \mathscr{A}$ there is an $(n+2)$-angle

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_{n+1} \overset{f_{n+1}}{\rightsquigarrow}$$

   with terms $A_1, \dots, A_n \in \mathscr{A}$.

Recall that in pre-$(n+2)$-angulated categories as defined in [GKO13, definition 2.1(a)] the class $\pentagon$ of $(n+2)$-angles is closed under direct summands. However, since two isomorphic $(n+2)$-Σ_n-sequences are (trivial) direct summands of each other, this implies that $\pentagon$ is closed under isomorphisms inside the class of $(n+2)$-Σ_n-sequences. Similarly, any additive subcategory $\mathscr{A} \subseteq \mathscr{F}$ is closed under isomorphisms in $\mathscr{F}$.

We will often use this and the following trivial lemma to rename objects:

Lemma 1.2 (Replacement lemma)**.** *Suppose given an $(n+2)$-angle X of the shape*

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_{n+1} \overset{f_{n+1}}{\rightsquigarrow}$$

and a commutative diagram

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d, "\phi_i"] & X_{i+1} \arrow[r, "f_{i+1}"] \arrow[d, "\phi_{i+1}"] & \cdots \arrow[r, "f_{j-1}"] & X_j \arrow[d, "\phi_j"] \\ Y_i \arrow[r, "g_i"] & Y_{i+1} \arrow[r, "g_{i+1}"] & \cdots \arrow[r, "g_{j-1}"] & Y_j \end{tikzcd}$$

with $0 \le i \le j \le n+1$ and $\phi_i, \dots, \phi_j$ isomorphisms. Then the $(n+2)$-angle Y given by

$$X_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{i-2}} X_{i-1} \xrightarrow{f_{i-1}\phi_i} Y_i \xrightarrow{g_i} Y_{i+1} \xrightarrow{g_{i+1}} \cdots$$
$$\cdots \xrightarrow{g_{j-1}} Y_j \xrightarrow{\phi_j^{-1} f_j} X_{j+1} \xrightarrow{f_{j+1}} \cdots \xrightarrow{f_n} X_{n+1} \overset{f_{n+1}}{\rightsquigarrow}$$

is isomorphic to X via $\phi = (\mathrm{id}_{X_0}, \dots, \mathrm{id}_{X_{i-1}}, \phi_i, \dots, \phi_j, \mathrm{id}_{X_{j+1}}, \dots, \mathrm{id}_{X_{n+1}})\colon X \to Y$.

Proof. The proof is trivial, since the class of $(n+2)$-angles is closed under isomorphisms and the given ϕ is obviously an isomorphism of $(n+2)$-Σ_n-sequences. $\square$

Apart from pre-$(n+2)$-angulated categories, we will also have to deal with n-exact categories. Recall the following from [Jas16, definition 2.4]: An *n-exact sequence* in an additive cateogry $\mathscr{A}$ is a sequence of morphisms

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_{n+1}$$

such that f_0 is a monomorphism, f_n is a epimorphism, f_i is a weak cokernel of f_{i-1} for $i = 1, \ldots, n$ and f_i is a weak kernel of f_{i+1} for $i = 0, \ldots, n-1$. Notice, here f_0 being monic is equivalent to f_0 being a kernel of f_1 and f_n being epic is equivalent to f_n being a cokernel of f_{n-1}. Also recall from [Jas16, definition 4.1] that a morphism

$$\begin{tikzcd}
A_0 \arrow[r, "f_0"] \arrow[d, "\phi_0"] & A_1 \arrow[r, "f_1"] \arrow[d, "\phi_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "\phi_2"] & \cdots \arrow[r, "f_n"] & A_{n+1} \arrow[d, "\phi_{n+1}"] \\
B_0 \arrow[r, "g_0"] & B_1 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & \cdots \arrow[r, "g_n"] & B_{n+1}
\end{tikzcd}$$

from an n-exact sequences E to an n-exact sequence F is called *weak isomorphism* if ϕ_i and ϕ_{i+1} for an $i = 0, \ldots, n$ or ϕ_0 and ϕ_{n+1} are isomorphisms. The sequences E and F are then called *weakly isomorphic*. If $A_0 = B_0$ and $A_{n+1} = B_{n+1}$, as well as $\phi_0 = \mathrm{id}_{A_0}$ and $\phi_{n+1} = \mathrm{id}_{A_{n+1}}$ then E and F are called *equivalent* and the morphism between E and F is then called *equivalence of n-exact sequences* according to [Jas16, definition 2.9].

Recall, by [GKO13, proposition 2.5(a)], all $(n+2)$-angles are exact, that is applying $\mathrm{Hom}_{\mathscr{F}}(X,-)$ or $\mathrm{Hom}_{\mathscr{F}}(-,X)$ to an $(n+2)$-angle yields an (infinitely extended) long exact sequence. In particular, we will use the following and its dual regularly: Given an $(n+2)$-angle

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_{n+1} \overset{f_{n+1}}{\rightsquigarrow}$$

and a morphism $f\colon X' \to X_i$ for some $i \in \{0, \ldots, n+1\}$. Then f factors through f_{i-1}, where we use the convention $f_{-1} = \Sigma_n^{-1} f_{n+1}$, iff $f f_i = 0$. This also shows that f_{i-1} is a weak kernel of f_i for $i \in \{0, \ldots, n+1\}$. Hence, the following trivial lemma and its dual are not only useful in the context of n-exact sequences, but $(n+2)$-angulated categories as well:

Lemma 1.3. *Suppose we are given a commutative diagram as in Diagram 1.1 in an*

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[d, equal] & X_1 \arrow[r, "f_1"] \arrow[d, equal] & X_2 \arrow[r, "f_2"] \arrow[d, "\phi_2"] & \cdots \arrow[r, "f_{n-1}"] & X_n \arrow[d, "\phi_n"] \\
X'_0 \arrow[r, "g_0"] & X'_1 \arrow[r, "g_1"] & X'_2 \arrow[r, "g_2"] & \cdots \arrow[r, "g_{n-1}"] & X'_n
\end{tikzcd}$$

Diagram 1.1: Morphisms between weak cokernels.

arbitrary additive category $\mathscr{A}$. Suppose

1. the morphism f_i is a weak cokernel of f_{i-1} for all $i = 1, \dots, n-1$,

2. the morphism g_i is a weak cokernel of g_{i-1} for all $i = 1, \dots, n-1$ and

3. the morphisms $f_2, \dots, f_{n-1}$ and $g_2, \dots, g_{n-1}$ are in the radical.

Then $\phi_2, \dots, \phi_{n-1}$ are isomorphisms.

Notice, the construction of the inverses of $\phi_2, \dots, \phi_{n-1}$ is similar to the construction of the homotopy inverses in [Jas16, proposition 2.7] and to [Jas16, lemma 2.1]. However, we cannot directly deduce Lemma 1.3 from the statements in [Jas16], so we give a proof for the convenience of the reader:

Proof. We show the lemma under the additional assumption that $X_i = X_i'$ for $i = 0, \dots, n$ and $f_i = g_i$ for $i = 0, \dots, n-1$ first: We obtain a commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[d, "0"] & X_1 \arrow[r, "f_1"] \arrow[d, "0"] & X_2 \arrow[r, "f_2"] \arrow[d, "\mathrm{id}_{X_2} - \phi_2"] & \cdots \arrow[r, "f_{n-1}"] & X_n \arrow[d, "\mathrm{id}_{X_n} - \phi_n"] \\
X_0 \arrow[r, "f_0"] & X_1 \arrow[r, "f_1"] & X_2 \arrow[r, "f_2"] & \cdots \arrow[r, "f_{n-1}"] & X_n
\end{tikzcd}$$

by subtracting the identity from each vertical morphism. Then $\mathrm{id}_{X_2} - \phi_2$ factors through f_2, using $f_1(\mathrm{id}_{X_2} - \phi_2) = 0$ and f_2 being a weak cokernel of f_1. Say $\mathrm{id}_{X_2} - \phi_2 = f_2 h_3$ for some $h_3 \colon X_3 \to X_2$. In the same manner we can construct morphisms $h_i \colon X_i \to X_{i-1}$ for $i = 4, \dots, n$ with $\mathrm{id}_{X_{i-1}} - \phi_{i-1} = f_{i-1} h_i + h_{i-1} f_{i-2}$ inductively: Suppose all h_k have been constructed for $k = 3, \dots, i-1$ and $4 \le i \le n$. We have

$$f_{i-2}(\mathrm{id}_{X_{i-1}} - \phi_{i-1} - h_{i-1} f_{i-2}) = (\mathrm{id}_{X_{i-2}} - \phi_{i-2}) f_{i-2} - f_{i-2} h_{i-1} f_{i-2} \tag{1.1}$$

by commutativity of the above diagram. By assumption and because $f_{i-3} f_{i-2} = 0$ holds, or for $i = 4$ because $\mathrm{id}_{X_{i-2}} - \phi_{i-2} = f_2 h_3$ holds, the right hand side of Equation (1.1) vanishes. Since f_{i-1} is a weak cokernel of f_{i-2} the left side of Equation (1.1) vanishing tells us that h_i exists as desired, which completes the induction. Now, since $\mathrm{id}_{X_2} - \phi_2 = f_2 h_3$ and $\mathrm{id}_{X_i} - \phi_i = f_i h_{i+1} + h_i f_{i-1}$ for $i = 3, \dots, n-1$ are radical morphisms, this shows that ϕ_i is an isomorphism for $i = 2, \dots, n-1$. Hence, we have shown the lemma under our additional assumption $X_i = X_i'$ and $f_i = g_i$ for $i = 2, \dots, n-1$.

Now to the general case: Using that g_i is a weak cokernel of g_{i-1} for all $i = 1, \dots, n-1$ it is easy to construct the dashed morphisms $\{\phi_k' \colon X_k' \to X_k\}_{k=2}^n$ making the diagram in Diagram 1.2 commutative. By applying the already proven version of the lemma to the

$$\begin{tikzcd}
X_0' \arrow[r, "g_0"] \arrow[d, equal] & X_1' \arrow[r, "g_1"] \arrow[d, equal] & X_2' \arrow[r, "g_2"] \arrow[d, dashed, "\phi_2'"] & \cdots \arrow[r, "g_{n-1}"] & X_n' \arrow[d, dashed, "\phi_n'"] \\
X_0 \arrow[r, "f_0"] & X_1 \arrow[r, "f_1"] & X_2 \arrow[r, "f_2"] & \cdots \arrow[r, "f_{n-1}"] & X_n
\end{tikzcd}$$

Diagram 1.2: Construction of morphisms in the opposite direction as in Diagram 1.1.

composite of the vertical morphisms of Diagram 1.1 and Diagram 1.2 we obtain that $\phi_i \phi_i'$ and $\phi_i' \phi_i$ are isomorphisms for $i = 2, \dots, n-1$ and hence the general version of the lemma follows. □

2 Core lemmas

2.1 The completion lemma

We start with the observation that the axiom (F3) from [GKO13, definition 2.1] implies a stronger version of itself. Notice, the case $i = 1$ in the following lemma is precisely the axiom (F3):

Lemma 2.1 (Completion lemma)**.** *Suppose given $(n+2)$-angles X and Y as well as $i+1$ morphisms $\{\phi_k \colon X_k \to Y_k\}_{k=0}^{i}$ which make the diagram*

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[d, "\phi_0"] & X_1 \arrow[r, "f_1"] \arrow[d, "\phi_1"] & \cdots \arrow[r, "f_{i-1}"] & X_i \arrow[r, "f_i"] \arrow[d, "\phi_i"] & X_{i+1} \arrow[r, "f_{i+1}"] & X_{i+2} \arrow[r, "f_{i+2}"] & \cdots \arrow[r, "f_n"] & X_{n+1} \arrow[r, squiggly, "f_{n+1}"] & \Sigma_n X_0 \arrow[d, "\Sigma_n \phi_0"] \\
Y_0 \arrow[r, "g_0"] & Y_1 \arrow[r, "g_1"] & \cdots \arrow[r, "g_{i-1}"] & Y_i \arrow[r, "g_i"] & Y_{i+1} \arrow[r, "g_{i+1}"] & Y_{i+2} \arrow[r, "g_{i+2}"] & \cdots \arrow[r, "g_n"] & Y_{n+1} \arrow[r, squiggly, "g_{n+1}"] & \Sigma_n Y_0
\end{tikzcd}$$

commutative. If only one map $\phi_0 \colon X_0 \to Y_0$ is given, assume further $(\Sigma_n^{-1} f_{n+1})\phi_0 g_0 = 0$. Then there are morphisms $\{\phi_k \colon X_k \to Y_k\}_{k=i+1}^{n+1}$ such that $\phi = (\phi_0, \ldots, \phi_{n+1}) \colon X \to Y$ is a morphism of $(n+2)$-angles.

Proof. Let $i+1$ be the number of maps given. Notice, for $i = n+1$ there is nothing to show, as $\phi = (\phi_0, \ldots, \phi_{n+1})$ is then already a morphism of $(n+2)$-angles. Further, for $i = 1$ the lemma is precisely the axiom (F3) and therefore holds. For $i = 0$, by the additional assumption and because $(n+2)$-angles are exact, there is a map $\phi_1 \colon X_1 \to Y_1$ with $\phi_0 g_0 = f_0 \phi_1$. But then ϕ_0 and ϕ_1 satisfy the requirement of the lemma for $i = 1$ and in this case the completion has already been established. We show the remaining cases by induction on i. As we already established the start of the induction, suppose the lemma is true for some fixed $1 \leq i \leq n-1$. We show that we can complete any $i+2$ morphisms $\{\phi_k \colon X_k \to Y_k\}_{k=0}^{i+1}$ to a morphism of $(n+2)$-angles, as well:

By the induction hypothesis, we can apply the lemma to the morphisms $\{\phi_k\}_{k=0}^{i}$ and obtain a morphism $\phi' \colon X \to Y$ of $(n+2)$-angles satisfying $\phi_k = \phi'_k$ for $0 \leq k \leq i$. We obtain a commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[d, "0"] & X_1 \arrow[r, "f_1"] \arrow[d, "0"] & \cdots \arrow[r, "f_{i-1}"] & X_i \arrow[r, "f_i"] \arrow[d, "0"] & X_{i+1} \arrow[r, "f_{i+1}"] \arrow[d, "\phi_{i+1} - \phi'_{i+1}"] & X_{i+2} \arrow[r, "f_{i+2}"] & \cdots \arrow[r, "f_n"] & X_{n+1} \arrow[r, squiggly, "f_{n+1}"] & \Sigma_n X_0 \arrow[d, "0"] \\
Y_0 \arrow[r, "g_0"] & Y_1 \arrow[r, "g_1"] & \cdots \arrow[r, "g_{i-1}"] & Y_i \arrow[r, "g_i"] & Y_{i+1} \arrow[r, "g_{i+1}"] & Y_{i+2} \arrow[r, "g_{i+2}"] & \cdots \arrow[r, "g_n"] & Y_{n+1} \arrow[r, squiggly, "g_{n+1}"] & \Sigma_n Y_0
\end{tikzcd}$$

by subtraction of ϕ'_k from ϕ_k for $0 \leq k \leq i+1$. Therefore, $f_i(\phi_{i+1} - \phi'_{i+1}) = 0$ and because $(n+2)$-angles are exact, this shows that there is a morphism $h_{i+2} \colon X_{i+2} \to Y_{i+1}$ with $\phi_{i+1} - \phi'_{i+1} = f_{i+1} h_{i+2}$. Hence, $f_{i+1} h_{i+2} g_{i+1} = (\phi_{i-1} - \phi'_{i+1}) g_{i+1}$ and $h_{i+2} g_{i+1} g_{i+2} = 0$, so the diagram

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[d, "0"] & \cdots \arrow[r, "f_{i-1}"] & X_i \arrow[r, "f_i"] \arrow[d, "0"] & X_{i+1} \arrow[r, "f_{i+1}"] \arrow[d, "\phi_{i+1} - \phi'_{i+1}"] & X_{i+2} \arrow[r, "f_{i+2}"] \arrow[d, "h_{i+2} g_{i+1}"] & X_{i+3} \arrow[r, "f_{i+3}"] \arrow[d, "0"] & \cdots \arrow[r, "f_n"] & X_{n+1} \arrow[r, squiggly, "f_{n+1}"] \arrow[d, "0"] & \Sigma_n X_0 \arrow[d, "0"] \\
Y_0 \arrow[r, "g_0"] & \cdots \arrow[r, "g_{i-1}"] & Y_i \arrow[r, "g_i"] & Y_{i+1} \arrow[r, "g_{i+1}"] & Y_{i+2} \arrow[r, "g_{i+2}"] & Y_{i+3} \arrow[r, "g_{i+3}"] & \cdots \arrow[r, "g_n"] & Y_{n+1} \arrow[r, squiggly, "g_{n+1}"] & \Sigma_n Y_0
\end{tikzcd}$$

is commutative. This yields a morphism $\psi\colon X \to Y$ of $(n+2)$-angles defined by $\psi_k = 0$ for $k \notin \{i+1, i+2\}$ and $\psi_{i+1} = \phi_{i+1} - \phi'_{i+1}$ as well as $\psi_{i+2} = h_{i+2}g_{i+1}$. Notice, since $\phi'_k = \phi_k$ for $0 \le k \le i$ this means that the diagram

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[d, "\phi_0"] & \cdots \arrow[r, "f_{i-1}"] & X_i \arrow[r, "f_i"] \arrow[d, "\phi_i"] & X_{i+1} \arrow[r, "f_{i+1}"] \arrow[d, "\phi_{i+1}"] & X_{i+2} \arrow[r, "f_{i+2}"] \arrow[d, "\psi_{i+2}+\phi'_{i+2}"] & X_{i+3} \arrow[r, "f_{i+3}"] \arrow[d, "\phi'_{i+3}"] & \cdots \arrow[r, "f_n"] & X_{n+1} \arrow[r, squiggly, "f_{n+1}"] \arrow[d, "\phi'_{n+1}"] & \Sigma_n X_0 \arrow[d, "\Sigma_n \phi_0"] \\
Y_0 \arrow[r, "g_0"] & \cdots \arrow[r, "g_{i-1}"] & Y_i \arrow[r, "g_i"] & Y_{i+1} \arrow[r, "g_{i+1}"] & Y_{i+2} \arrow[r, "g_{i+2}"] & Y_{i+3} \arrow[r, "g_{i+3}"] & \cdots \arrow[r, "g_n"] & Y_{n+1} \arrow[r, squiggly, "g_{n+1}"] & \Sigma_n Y_0
\end{tikzcd}$$

which is obtained by the summation of ψ and ϕ', is commutative. Therefore $\psi + \phi'$ is the desired completion of the given $\{\phi_k\}_{k=0}^{i+1}$ to a morphism of $(n+2)$-angles and the lemma hence follows by induction. □

One immediate consequence of the completion Lemma 2.1 is the following, which is a generalization of [BBD82, corollaire 1.1.10(ii)] to the context of $(n+2)$-angulated categories. It will be used later to establish the isomorphism $\mathrm{YExt}^n_{(\mathscr{A},\mathscr{E}_{\mathscr{A}})}(A_{n+1}, A_0) \to \mathrm{Hom}_{\mathscr{F}}(A_{n+1}, \Sigma_n A_0)$ for $A_0, A_{n+1} \in \mathscr{A}$, similarly to [Jør20, proposition 2.5(ii)].

Lemma 2.2. *Suppose given a commutative diagram*

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[d, equal] & X_1 \arrow[r, "f_1"] \arrow[d, "\phi_1"] & \cdots \arrow[r, "f_{n-1}"] & X_n \arrow[r, "f_n"] \arrow[d, "\phi_n"] & X_{n+1} \arrow[r, squiggly, "f_{n+1}"] \arrow[d, equal] & \Sigma_n X_0 \\
Y_0 \arrow[r, "g_0"] & Y_1 \arrow[r, "g_1"] & \cdots \arrow[r, "g_{n-1}"] & Y_n \arrow[r, "g_n"] & Y_{n+1} \arrow[r, squiggly, "g_{n+1}"] & \Sigma_n Y_0
\end{tikzcd}$$

with $(n+2)$-angles as rows and $\mathrm{Hom}(\Sigma_n X_0, Y_{n+1}) = 0$*. Then $f_{n+1} = g_{n+1}$.*

Proof. The undashed maps of

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[d, equal] & X_1 \arrow[r, "f_1"] \arrow[d, "\phi_1"] & \cdots \arrow[r, "f_{n-1}"] & X_n \arrow[r, "f_n"] \arrow[d, "\phi_n"] & X_{n+1} \arrow[r, squiggly, "f_{n+1}"] \arrow[d, dashed, "\phi_{n+1}"] & \Sigma_n X_0 \arrow[d, equal] \\
Y_0 \arrow[r, "g_0"] & Y_1 \arrow[r, "g_1"] & \cdots \arrow[r, "g_{n-1}"] & Y_n \arrow[r, "g_n"] & Y_{n+1} \arrow[r, squiggly, "g_{n+1}"] & \Sigma_n Y_0
\end{tikzcd}$$

clearly form a commutative diagram. We claim there is at most one $\phi_{n+1}\colon X_{n+1} \to Y_{n+1}$ which makes the penultimate square commute, i.e. satisfies $f_n\phi_{n+1} = \phi_n g_n$: Suppose we are given two maps $\phi_{n+1}\colon X_{n+1} \to Y_{n+1}$ and $\phi'_{n+1}\colon X_{n+1} \to Y_{n+1}$, both having this property. Then we obtain $f_n(\phi_{n+1} - \phi'_{n+1}) = 0$ and hence, using that the upper row is an exact $(n+2)$-Σ_n-sequence, there is a morphism $h\colon \Sigma_n X_0 \to Y_{n+1}$ with $f_{n+1}h = \phi_{n+1} - \phi'_{n+1}$. However, $\mathrm{Hom}(\Sigma_n X_0, Y_{n+1}) = 0$ by assumption, hence $h = 0$ and $\phi_{n+1} = \phi'_{n+1}$. This shows that there is at most one ϕ_{n+1} making the penultimate square commute.

By the completion lemma, there is a choice of ϕ_{n+1} which makes the whole diagram commutative. But by the previous claim, this choice is unique and, by requirement of the lemma, its the identity on $X_{n+1} = Y_{n+1}$. This means that

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[d, equal] & X_1 \arrow[r, "f_1"] \arrow[d, "\phi_1"] & \cdots \arrow[r, "f_{n-1}"] & X_n \arrow[r, "f_n"] \arrow[d, "\phi_n"] & X_{n+1} \arrow[r, squiggly, "f_{n+1}"] \arrow[d, equal] & \Sigma_n X_0 \arrow[d, equal] \\
Y_0 \arrow[r, "g_0"] & Y_1 \arrow[r, "g_1"] & \cdots \arrow[r, "g_{n-1}"] & Y_n \arrow[r, "g_n"] & Y_{n+1} \arrow[r, squiggly, "g_{n+1}"] & \Sigma_n Y_0
\end{tikzcd}$$

is a commutative diagram, which shows $f_{n+1} = g_{n+1}$. □

2.2 Obtaining information from (N4*)

Throughout Section 2.2 we assume that $(\mathscr{F}, \Sigma_n, \pentagon)$ is an $(n+2)$-angulated Krull-Schmidt category. Recall from [BT13, theorem 4.4] that given any commutative diagram as in Diagram 2.1 with all three rows being $(n+2)$-angles, we can find the dashed morphisms

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[d, equal] & X_1 \arrow[r, "f_1"] \arrow[d, "g_0"] & X_2 \arrow[r, "f_2"] & \cdots \arrow[r, "f_n"] & X_{n+1} \arrow[r, squiggly, "f_{n+1}"] & {} \\
X_0'' \arrow[r, "h_0"] \arrow[d, "f_0"] & X_1'' \arrow[r, "h_1"] \arrow[d, equal] & X_2'' \arrow[r, "h_2"] & \cdots \arrow[r, "h_n"] & X_{n+1}'' \arrow[r, squiggly, "h_{n+1}"] & {} \\
X_0' \arrow[r, "g_0"] & X_1' \arrow[r, "g_1"] & X_2' \arrow[r, "g_2"] & \cdots \arrow[r, "g_n"] & X_{n+1}' \arrow[r, squiggly, "g_{n+1}"] & {}
\end{tikzcd}$$

Diagram 2.1: A factorization $h_0 = f_0 g_0$ and the three $(n+2)$-angles arising from the morphisms involved. Notice that $X_0' = X_1$.

of Diagram 2.2 such that each upright square commutes and so that the (Mayer-Vietoris-

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[d, equal] & X_1 \arrow[r, "f_1"] \arrow[d, "g_0"] & X_2 \arrow[r, "f_2"] \arrow[d, dashed] & X_3 \arrow[r, "f_3"] \arrow[d, dashed] \arrow[ddl, dashed] & \cdots \arrow[r, "f_{n-1}"] & X_n \arrow[r, "f_n"] \arrow[d, dashed] & X_{n+1} \arrow[r, squiggly, "f_{n+1}"] \arrow[d, dashed] \arrow[ddl, dashed] & \Sigma_n X_0 \arrow[d, equal] \\
X_0'' \arrow[r, "h_0"] \arrow[d, "f_0"] & X_1'' \arrow[r, "h_1"] \arrow[d, equal] & X_2'' \arrow[r, "h_2"] \arrow[d, dashed] & X_3'' \arrow[r, "h_3"] \arrow[d, dashed] & \cdots \arrow[r, "h_{n-1}"] & X_n'' \arrow[r, "h_n"] \arrow[d, dashed] & X_{n+1}'' \arrow[r, squiggly, "h_{n+1}"] \arrow[d, dashed] & \Sigma_n X_0'' \arrow[d, "\Sigma_n f_0"] \\
X_0' \arrow[r, "g_0"] & X_1' \arrow[r, "g_1"] & X_2' \arrow[r, "g_2"] & X_3' \arrow[r, "g_3"] & \cdots \arrow[r, "g_{n-1}"] & X_n' \arrow[r, "g_n"] & X_{n+1}' \arrow[r, squiggly, "g_{n+1}"] & \Sigma_n X_0'
\end{tikzcd}$$

Diagram 2.2: Morphisms arising from Diagram 2.1 via (N4*).

like) totalisation of the complex enclosed by the dashed rectangle in Diagram 2.2, shown in Diagram 2.3 is an $(n+2)$-angle. In this section we want to tackle the question, what

$$X_2 \to X_3 \oplus X_2'' \longrightarrow X_4 \oplus X_3'' \oplus X_2' \longrightarrow X_5 \oplus X_4'' \oplus X_3' \longrightarrow \cdots$$
$$\cdots \longrightarrow X_{n+1} \oplus X_n'' \oplus X_{n-1}' \longrightarrow X_{n+1}'' \oplus X_n' \longrightarrow X_{n+1}' \xrightarrow{g_{n+1}\Sigma_n f_1} \Sigma_n X_2$$

Diagram 2.3: Totalisation obtained from Diagram 2.2.

information about $X_2'', \ldots, X_{n+1}''$ can be obtained from this $(n+2)$-angle? Notice, for $n > 1$ the completion of a morphism to an $(n+2)$-angle is not unique, as we can always add a trivial $(n+2)$-angle to an existing completion to obtain another one. To circumvent this problem we reintroduce the following from [OT12, lemma 5.18] and [Fed19, lemma 3.14], which adds uniqueness up to isomorphism to $(n+2)$-angles:

Definition 2.3. Suppose given an $(n+2)$-angle X of the shape

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_{n+1} \overset{f_{n+1}}{\rightsquigarrow} \, .$$

Then X is called

1. a *minimal completion of* f_0 if $f_i \in \mathrm{rad}(X_i, X_{i+1})$ for $i = 2, \dots, n$ and
2. a *minimal* $(n+2)$*-angle for* f_i if its rotation

$$X_i \xrightarrow{f_i} \cdots \xrightarrow{f_{n+1}} \Sigma_n X_0 \xrightarrow{(-1)^n \Sigma_n f_0} \cdots \xrightarrow{(-1)^n \Sigma_n f_{i-2}} \Sigma_n X_{i-1} \overset{(-1)^n \Sigma_n f_{i-1}}{\rightsquigarrow}$$

is a minimal completion of f_i.

Notice, since the rotation of an $(n+2)$-angle is an $(n+2)$-angle again it does not really matter in what position i the morphism f_i is in (2). For sake of convenience and less heavy notation, we will state and prove the lemmas regarding minimal $(n+2)$-angles for minimal completions only and leave the rotated versions for minimal $(n+2)$-angles as an easy exercise for the reader.

Most of the properties of minimal $(n+2)$-angles can be shown without the Krull-Schmidt property of $\mathscr{F}$, but we need it for the existence of minimal $(n+2)$-angles and hence to use the main result of this section: A lemma, which enables us to compare the objects of the $(n+2)$-angle arising as in Diagram 2.3 and the objects of the minimal $(n+2)$-angle arising from the morphism $g_{n+1}\Sigma_n f_1$ in position $n+1$ of this $(n+2)$-angle.

Lemma 2.4. *Suppose $n \geq 3$ and we are given a commutative diagram as in Diagram 2.1, where the second row is additionally a minimal $(n+2)$-angle for h_0. Further let*

$$X_2 \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow \cdots \longrightarrow Y_{n-1} \longrightarrow Y_n \longrightarrow X'_{n+1} \overset{g_{n+1}\Sigma_n f_1}{\rightsquigarrow}$$

be a minimal $(n+2)$-angle for $g_{n+1}\Sigma_n f_1$. Then there are objects $\{Z_k\}_{k=1}^{n-1}$ satisfying

1. $Z_1 \in \mathsf{add}(X_3 \oplus X_4 \oplus X'_2)$,
2. $Z_k \in \mathsf{add}(X_{k+2} \oplus X'_k \oplus X_{k+3} \oplus X'_{k+1})$ *for* $2 \leq k \leq n-2$ *and*
3. $Z_{n-1} \in \mathsf{add}(X_{n+1} \oplus X'_{n-1} \oplus X'_n)$,

expressing the difference between the objects of Diagram 2.3 and the minimal $(n+2)$-angle of $g_{n+1}\Sigma_n f_1$ depicted above, in the sense that

4. $X_3 \oplus X''_2 = Y_1 \oplus Z_1$,
5. $X_{k+2} \oplus X''_{k+1} \oplus X'_k = Z_{k-1} \oplus Y_k \oplus Z_k$ *for* $2 \leq k \leq n-1$ *and*
6. $X''_{n+1} \oplus X'_n = Z_{n-1} \oplus Y_n$

hold.

This lemma seems very technical, though it is very useful, too. For example it can be used to show that $X''_2, \dots, X''_{n+1}$ lie in an additive n-extension closed subcategory of $\mathscr{F}$ if the upper and lower row of Diagram 2.1 do so.

Notice the case $n = 1$ of Lemma 2.4 is uninteresting, since triangles in a triangulated category are unique up to isomorphism. A similar form of Lemma 2.4 holds for $n = 2$. Here (1)-(3) need to be replaced by $Z_1 \in \mathsf{add}(X_3 \oplus X_2')$.

Our plan of proving this is as follows: We show that the $(n+2)$-angles of Diagram 2.3 and Lemma 2.4 differ by a direct sum of trivial complexes. The objects of these trivial complexes are our candidates for $\{Z_k\}_{k=1}^{n-1}$. We then use the matrix lemma (see Lemma 2.7) and $h_2, \dots, h_n$ being in the radical to obtain (1)–(3). We begin with the first part of this plan:

Lemma 2.5. *Let $f_0\colon X_0 \to X_1$ be a morphism. Suppose X is a minimal completion*

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_{n+1} \overset{f_{n+1}}{\rightsquigarrow}$$

of f_0. Then X is a direct summand of any $(n+2)$-angle containing f_0 in position 0. Moreover, the minimal completion of f_0 is unique up to isomorphism.

Proof. In a triangulated category the lemma is clearly true, as triangles are, up to isomorphism, uniquely determined by one morphism. Hence, we can assume that $n \geq 2$.

Suppose Y is an arbitrary $(n+2)$-angle

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} \cdots \xrightarrow{g_n} Y_{n+1} \overset{g_{n+1}}{\rightsquigarrow}$$

containing f_0 in the same position as X. Using the axiom [GKO13, defintion 2.1(F3)] we find the dashed morphisms $\phi_2, \dots, \phi_{n+1}$ and $\phi_2', \dots, \phi_{n+1}'$ making

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[d, equal] & X_1 \arrow[r, "f_1"] \arrow[d, equal] & X_2 \arrow[r, "f_2"] \arrow[d, dashed, "\phi_2"] & \cdots \arrow[r, "f_n"] & X_{n+1} \arrow[r, squiggly, "f_{n+1}"] \arrow[d, dashed, "\phi_{n+1}"] & \Sigma_n X_0 \arrow[d, equal] \\
X_0 \arrow[r, "f_0"] \arrow[d, equal] & X_1 \arrow[r, "g_1"] \arrow[d, equal] & Y_2 \arrow[r, "g_2"] \arrow[d, dashed, "\phi_2'"] & \cdots \arrow[r, "g_n"] & Y_{n+1} \arrow[r, squiggly, "g_{n+1}"] \arrow[d, dashed, "\phi_{n+1}'"] & \Sigma_n X_0 \arrow[d, equal] \\
X_0 \arrow[r, "f_0"] & X_1 \arrow[r, "f_1"] & X_2 \arrow[r, "f_2"] & \cdots \arrow[r, "f_n"] & X_{n+1} \arrow[r, squiggly, "f_{n+1}"] & \Sigma_n X_0
\end{tikzcd}$$

a commutative diagram. For short, that is to say, we have morphisms of $(n+2)$-angles $\phi = (\mathrm{id}_{X_0}, \mathrm{id}_{X_1}, \phi_2, \dots, \phi_{n+1})\colon X \to Y$ and $\phi' = (\mathrm{id}_{X_0}, \mathrm{id}_{X_1}, \phi_2', \dots, \phi_{n+1}')\colon Y \to X$. Now by Lemma 1.3 we know that $\phi_2\phi_2', \dots, \phi_n\phi_n'$ are isomorphisms. By the dual of Lemma 1.3, we obtain that $\phi_3\phi_3', \dots, \phi_{n+1}\phi_{n+1}'$ are isomorphisms, hence $\phi\phi'$ is an isomorphism and X is a direct summand of Y.

Now if Y is additionally a minimal $(n+2)$-angle for f_0 then the same argument using Lemma 1.3 and its dual shows that ϕ has to be an isomorphism already. This shows that any two minimal $(n+2)$-angles of f_0 are isomorphic. □

Lemma 2.6. *Each morphism $f_0\colon X_0 \to X_1$ has a minimal completion. Moreover, any completion of f_0 to an $(n+2)$-angle is the direct sum of the minimal completion of f_0 and trivial $(n+2)$-angles $\mathrm{triv}_2(Z_2), \dots, \mathrm{triv}_n(Z_n)$ for some $Z_2, \dots, Z_n \in \mathscr{F}$.*

Proof. As each morphism in $\mathscr{F}$ has a completion to an $(n+2)$-angle the first part of the lemma follows from the second part.

We show the second part: Suppose we are given an $(n+2)$-angle X of the form

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_{n+1} \overset{f_{n+1}}{\rightsquigarrow}$$

containing f_0. Suppose f_i is not in the radical for some $i = 2, \dots, n$. By definition there is a $0 \neq Z$ and $g\colon Z \to X_i$ and $h\colon X_{i+1} \to Z$ such that gf_ih is an isomorphism. Then commutativity of

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & Z \arrow[r, equal] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "gf_i"] & 0 \arrow[d] \\
X_{i-1} \arrow[r, "f_{i-1}"] \arrow[d] & X_i \arrow[r, "f_i"] \arrow[d, "f_ih(gf_ih)^{-1}"] & X_{i+1} \arrow[r, "f_{i+1}"] \arrow[d, "h(gf_ih)^{-1}"] & X_{i+2} \arrow[d] \\
0 \arrow[r] & Z \arrow[r, equal] & Z \arrow[r] & 0
\end{tikzcd}$$

shows that the trivial $(n+2)$-angle $\operatorname{triv}_i(Z)$ is a direct summand of X. Choose a complement X' for $\operatorname{triv}_i(Z)$ in X, that is $X = X' \oplus \operatorname{triv}_i(Z)$ as $(n+2)$-angles. Since $i \neq 0, 1, n+1$ we know that X' has f_0 in position 0. If X' is not a minimal completion of f_0 we can repeat this procedure. As any object $X_0, \dots, X_{n+1}$ has the descending chain condition on direct summands, the $(n+2)$-angle X has the descending chain condition on direct summands. This means, after finitely many steps this procedure must stop and hence X is the direct sum of a minimal completion of f_0 and trivial $(n+2)$-angles. Finally, noticing $\operatorname{triv}_i(Z) \oplus \operatorname{triv}_i(Z') = \operatorname{triv}_i(Z \oplus Z')$ for $i \in \{2, \dots, n\}$ and $Z, Z' \in \mathscr{F}$ and thus collecting all the trivial $(n+2)$-angles of the same shape in one summand yields the lemma. $\square$

Notice, by Lemma 2.6 minimal completions exist and by Lemma 2.5 they are unique up to isomorphism. We can therefore speak of *the* minimal completion of a morphism f_0. The same holds for minimal $(n+2)$-angles.

For the second part of our plan we show the following easy lemma, which holds in any additive Krull-Schmidt category:

Lemma 2.7 (Matrix lemma)**.** *Let $f\colon X_1 \oplus X_2 \to Y_1 \oplus Y_2$ be so that $\iota_1 f \pi_1' \in \operatorname{rad}(X_1, Y_1)$, where $\iota_1\colon X_1 \to X_1 \oplus X_2$ is the canonical inclusion and $\pi_1'\colon Y_1 \oplus Y_2 \to Y_1$ is the canonical projection. Then all objects Z which have $g\colon Z \to X_1 \oplus X_2$ and $h\colon Y_1 \oplus Y_2 \to Z$ such that $gfh \in \operatorname{Aut} Z$ satisfy $Z \in \mathsf{add}(X_2 \oplus Y_2)$.*

Proof. First assume that Z is indecomposable, with g and h as in the lemma. For $i = 1, 2$ let the morphisms $\iota_i\colon X_i \to X_1 \oplus X_2$ and $\iota_i'\colon Y_i \to Y_1 \oplus Y_2$ be the canonical inclusions and $\pi_i\colon X_1 \oplus X_2 \to X_i$ and $\pi_i'\colon Y_1 \oplus Y_2 \to Y_i$ be the canonical projections with respect to the given direct sum decomposition. We know that $\operatorname{rad}(Z, Z) = \operatorname{rad}(\operatorname{End} Z)$ is local, since Z is indecomposable. Now we have

$$\sum_{(i,j) \in \{1,2\}^2} g\pi_i\iota_i f \pi_j' \iota_j' h = gfh \in \operatorname{Aut}(Z),$$

so at least one of the four summands on the left is an isomorphism. However, since the radical is an ideal, we have $g\pi_1\iota_1 f\pi_1'\iota_1' h \in \operatorname{rad}(\operatorname{End} Z)$, so one of the other three summands needs to be an isomorphism. But then it must be a summand with $(i, j) \neq (1, 1)$, so $g\pi_2$ is split mono or $\iota_2' h$ is split epi. This shows the lemma for Z indecomposable.

Now let Z be arbitrary, with g and h as in the lemma. Let Z' be an indecomposable direct summand of Z, that is there are $\iota\colon Z' \to Z$ and $\pi\colon Z \to Z'$ with $\iota\pi = \mathrm{id}_{Z'}$.

Then $\mathrm{id}_{Z'} = g'fh'$ with $g' = \iota g$ and $h' = h(gfh)^{-1}\pi$. Hence $Z' \in \mathsf{add}(X_2 \oplus Y_2)$ by the first part. But Z decomposes, up to isomorphism and reordering, uniquely into finitely many indecomposable objects. As all indecomposable direct summands of Z belong to $\mathsf{add}(X_2 \oplus Y_2)$ we finally obtain $Z \in \mathsf{add}(X_2 \oplus Y_2)$. $\square$

We are now able to prove the main result of this section.

Proof of Lemma 2.4. We use the same notation as in Lemma 2.4. We apply the axiom (N4*) to Diagram 2.1. We obtain the $(n+2)$-angle in Diagram 2.3 and Diagram 2.4 where

$$X_2 \xrightarrow{k_0} X_3 \oplus X_2'' \xrightarrow{k_1} X_4 \oplus X_3'' \oplus X_2' \xrightarrow{k_2} X_5 \oplus X_4'' \oplus X_3' \xrightarrow{k_3} \cdots$$
$$\cdots \xrightarrow{k_{n-2}} X_{n+1} \oplus X_n'' \oplus X_{n-1}' \xrightarrow{k_{n-1}} X_{n+1}'' \oplus X_n' \xrightarrow{k_n} X_{n+1}' \overset{k_{n+1}}{\rightsquigarrow} \Sigma_n X_2$$

Diagram 2.4: Labeled version of the totalisation in Diagram 2.3.

the morphisms $k_0, \ldots, k_{n+1}$ can be written in the form

$$\begin{aligned}
k_0 &= \begin{bmatrix} f_2 \\ \phi_2 \end{bmatrix} \\
k_1 &= \begin{bmatrix} -f_3 & 0 \\ \phi_3 & -h_2 \\ \psi_3 & \phi_2' \end{bmatrix} \\
k_i &= \begin{bmatrix} -f_{i+2} & 0 & 0 \\ \phi_{i+2} & -h_{i+1} & 0 \\ \psi_{i+2} & \phi_{i+1}' & g_i \end{bmatrix} && (2 \le i \le n-2) \\
k_{n-1} &= \begin{bmatrix} \phi_{n+1} & -h_n & 0 \\ \psi_{n+1} & \phi_n' & g_{n-1} \end{bmatrix} \\
k_n &= \begin{bmatrix} \phi_{n+1}' & g_n \end{bmatrix} \\
k_{n+1} &= g_{n+1}\Sigma_n f_1
\end{aligned}$$

for some morphisms $\phi_k \colon X_k \to X_k''$ and $\phi_k' \colon X_k'' \to X_k'$ for $2 \le k \le n+1$ as well as morphisms $\psi_k \colon X_k \to X_{k-1}'$ for $3 \le k \le n+1$. By Lemma 2.6 we know that the $(n+2)$-angle shown in Diagram 2.4 is the direct sum of the minimal $(n+2)$-angle of k_{n+1} and trivial $(n+2)$-angles $\mathrm{triv}_1(Z_1), \ldots, \mathrm{triv}_{n-1}(Z_{n-1})$. This choice of $Z_1, \ldots, Z_{n-1}$ establishes (4)-(6) of Lemma 2.4.

It remains to show that (1)-(3) of Lemma 2.4 hold for this choice of $Z_1, \ldots, Z_{n-1}$. We only show (1), the other assertions are analogous. Since the trivial $(n+2)$-angle $\mathrm{triv}_1(Z_1)$ is a direct summand of the $(n+2)$-angle in Diagram 2.4 we know that id_{Z_1} factors through k_1. Since $-h_2$ is in the radical by assumption we must have $Z_1 \in \mathsf{add}(X_3 \oplus (X_4 \oplus X_2'))$ by the matrix Lemma 2.7. $\square$

3 Main results

Throughout Section 3 we assume that $(\mathscr{F}, \Sigma_n, \pentagon)$ is an $(n+2)$-angulated Krull-Schmidt category. Further, we will assume that $\mathscr{A}$ is an additive, n-extension closed subcategory of $\mathscr{F}$ with $\mathrm{Hom}_{\mathscr{F}}(\Sigma_n\mathscr{A}, \mathscr{A}) = 0$. We define an n-exact structure on $\mathscr{A}$:

Definition 3.1. For a subcategory $\mathscr{A} \subseteq \mathscr{F}$ an *$\mathscr{A}$-conflation* is a complex

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_{n+1}$$

with $A_0, \ldots, A_{n+1} \in \mathscr{A}$ for which there is a morphism $f_{n+1}\colon A_{n+1} \to \Sigma_n A_0$ such that

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_{n+1} \overset{f_{n+1}}{\rightsquigarrow}$$

is an $(n+2)$-angle in $(\mathscr{F}, \Sigma_n, \pentagon)$. The morphisms in position 0 of $\mathscr{A}$-conflations are called *$\mathscr{A}$-inflations*. Dually, the morphisms which appear in position n of $\mathscr{A}$-conflations are called *$\mathscr{A}$-deflations*. We denote by $\mathscr{E}_{\mathscr{A}}$ the class of all $\mathscr{A}$-conflations.

Notice, by the replacement Lemma 1.2, it follows immediately that any complex in $\mathscr{A}$ isomorphic to an $\mathscr{A}$-conflation is an $\mathscr{A}$-conflation itself.

We now want to present our main theorem:

Theorem 3.2. *Let $\mathscr{A} \subseteq \mathscr{F}$ be an additive, n-extension closed subcategory of $\mathscr{F}$ with* $\mathrm{Hom}_{\mathscr{F}}(\Sigma_n \mathscr{A}, \mathscr{A}) = 0$. *Then*

1. *$(\mathscr{A}, \mathscr{E}_{\mathscr{A}})$ is an n-exact category and*
2. *there is a natural bilinear isomorphism* $\mathrm{Hom}_{\mathscr{F}}(-, \Sigma_n(-)) \to \mathrm{YExt}^n_{(\mathscr{A}, \mathscr{E}_{\mathscr{A}})}(-,-)$ *of functors* $\mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathsf{Ab}$.

To check part (1) of Theorem 3.2 we need to verify the axioms of [Jas16, definition 4.2]. For the convenience of the reader this is split up into Lemma 3.3, Lemma 3.6, Lemma 3.7, Lemma 3.8 and Lemma 3.9. We will check part (2) separately in Lemma 3.10.

First we need to verify that $\mathscr{E}_{\mathscr{A}}$ really consist of n-exact sequences:

Lemma 3.3. *All $\mathscr{A}$-conflations are n-exact sequences in $\mathscr{A}$.*

Proof. Suppose we are given an arbitrary $\mathscr{A}$-conflation

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_{n+1}$$

in $\mathscr{E}_{\mathscr{A}}$. By definition there is a morphism $f_{n+1}\colon A_{n+1} \to \Sigma_n A_0$ such that

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_{n+1} \overset{f_{n+1}}{\rightsquigarrow}$$

is an $(n+2)$-angle in $(\mathscr{F}, \Sigma_n, \pentagon)$. Applying the functor $\mathrm{Hom}_{\mathscr{F}}(A, -)$ for $A \in \mathscr{A}$ to this $(n+2)$-angle and using that $(n+2)$-angles are exact, we obtain an exact sequence

$$\mathrm{Hom}_{\mathscr{F}}(A, \Sigma_n^{-1} A_{n+1}) \longrightarrow \mathrm{Hom}_{\mathscr{F}}(A, A_0) \xrightarrow{-\cdot f_0} \mathrm{Hom}_{\mathscr{F}}(A, A_1) \xrightarrow{-\cdot f_1} \cdots \xrightarrow{-\cdot f_n} \mathrm{Hom}_{\mathscr{F}}(A, A_{n+1})$$

where the leftmost term vanishes since $0 = \mathrm{Hom}_{\mathscr{F}}(\Sigma_n \mathscr{A}, \mathscr{A}) \cong \mathrm{Hom}_{\mathscr{F}}(\mathscr{A}, \Sigma_n^{-1} \mathscr{A})$. This statement and its dual statement obtained from applying $\mathrm{Hom}_{\mathscr{F}}(-, A)$ for $A \in \mathscr{A}$ to the same $(n+2)$-angle show that any sequence of $\mathscr{E}_{\mathscr{A}}$ is indeed n-exact. $\square$

Next, we need to check that an n-exact sequence in $\mathscr{A}$ weakly isomorphic to an n-exact sequence in $\mathscr{E}_{\mathscr{A}}$ is itself in $\mathscr{E}_{\mathscr{A}}$. We could prove this directly. However, for the sake of readability, we restate a version of Lemma 2.6 for n-exact sequences in $\mathscr{A}$:

Lemma 3.4. *Suppose we are given an n-exact sequence E in $\mathscr{A}$ of the shape*

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_{n+1}$$

and a fixed $i \in \{0, \dots, n\}$. Then E is the direct sum of an n-exact sequence E' of the shape

$$A'_0 \xrightarrow{g_0} \cdots \xrightarrow{g_{i-2}} A'_{i-1} \xrightarrow{g_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{g_{i+1}} A'_{i+2} \xrightarrow{g_{i+2}} \cdots \xrightarrow{g_n} A'_{n+1} ,$$

where $g_0, \dots, g_{i-2}, g_{i+2}, g_n$ are in the radical, and $\mathscr{A}$-conflations $F_0, \dots, F_{i-2}, F_{i+2}, \dots, F_n$ arising from trivial $(n+2)$-angles $\operatorname{triv}(B_0), \dots, \operatorname{triv}(B_{i-2}), \operatorname{triv}(B_{i+2}), \dots, \operatorname{triv}(B_n)$ with $B_0, \dots, B_{i-2}, B_{i+2}, \dots, B_n \in \mathscr{A}$. Further, if E' is an $\mathscr{A}$-conflation then so is E.

Proof. The $\mathscr{A}$-conflations $F_0, \dots, F_{i-2}, F_{i+2}, \dots, F_n$ can be constructed in the same way as in Lemma 2.6, using that $\mathscr{A} \subseteq \mathscr{F}$ is additive. Clearly a direct summand of an n-exact sequence in $\mathscr{A}$ is an n-exact sequence in $\mathscr{A}$, hence E' constructed similarly to Lemma 2.6 is indeed an n-exact sequence in $\mathscr{A}$. Since the direct sum of $\mathscr{A}$-conflations is again an $\mathscr{A}$-conflation, as the direct sum of $(n+2)$-angles is again an $(n+2)$-angle, we have that E is an $\mathscr{A}$-conflation if E' is an $\mathscr{A}$-conflation. □

Obviously, Lemma 3.4 also has the following version, where the end terms of the n-exact sequences are fixed. The proof is omitted as it is almost the same as of Lemma 3.4. Notice though, that here the n-Yoneda-extension class rather than a morphism is fixed.

Lemma 3.5. *Suppose we are given an n-exact sequence E in $\mathscr{A}$ of the shape*

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_{n+1} .$$

Then E is the direct sum of an n-exact sequence E' of the shape

$$A_0 \xrightarrow{g_0} A'_1 \xrightarrow{g_1} A'_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-2}} A'_{n-1} \xrightarrow{g_{n-1}} A'_n \xrightarrow{g_n} A_{n+1} ,$$

where $g_1, \dots, g_{n-1}$ are in the radical, and $\mathscr{A}$-conflations $F_1, \dots, F_{n-1}$ arising from trivial $(n+2)$-angles $\operatorname{triv}(B_1), \dots, \operatorname{triv}(B_{n-1})$ with $B_1, \dots, B_{n-1} \in \mathscr{A}$. Furthermore, if E' is an $\mathscr{A}$-conflation then so is E.

Lemma 3.6. *The class of $\mathscr{A}$-conflations $\mathscr{E}_{\mathscr{A}}$ is closed under weak isomorphisms in the class of n-exact sequences in $\mathscr{A}$.*

Proof. We show that for any weak isomorphism $\phi = (\phi_0, \dots, \phi_{n+1})$ of n-exact sequences

$$\begin{tikzcd}
A_0 \arrow[r, "f_0"] \arrow[d, "\phi_0"] & A_1 \arrow[r, "f_1"] \arrow[d, "\phi_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "\phi_2"] & \cdots \arrow[r, "f_n"] & A_{n+1} \arrow[d, "\phi_{n+1}"] \\
B_0 \arrow[r, "g_0"] & B_1 \arrow[r, "g_1"] & B_2 \arrow[r, "g_2"] & \cdots \arrow[r, "g_n"] & B_{n+1}
\end{tikzcd}$$

in $\mathscr{A}$ the upper n-exact sequence E belongs to $\mathscr{E}_{\mathscr{A}}$ if and only if the lower sequence F belongs to $\mathscr{E}_{\mathscr{A}}$. Therefore, we distinguish the following two cases:

Suppose ϕ_i and ϕ_{i+1} are isomorphisms for some $0 \leq i \leq n$. Then we may assume that E belongs to $\mathscr{E}_{\mathscr{A}}$ and show that F then belongs to $\mathscr{E}_{\mathscr{A}}$ as the proof with reverse roles of E and F is analogous. Further, by the replacement Lemma 1.2, we may assume $A_i = B_i$ and $A_{i+1} = B_{i+1}$, as well as $\phi_i = \mathrm{id}_{A_i}$ and $\phi_{i+1} = \mathrm{id}_{A_{i+1}}$. Now let $\iota \colon E' \to E$ be the inclusion of the minimal $(n+2)$-angle with f_i in position i, which is a direct summand of E by Lemma 2.6. Further let $\pi \colon F \to F'$ be the projection to the direct summand constructed in Lemma 3.4. By construction $\iota\phi\pi$ is still a weak isomorphism with identities in position i and $i+1$. Further, since the class $\pentagon$ of $(n+2)$-angles is closed under direct summands, E' is an $\mathscr{A}$-conflation and by Lemma 3.4 our sequence F is an $\mathscr{A}$-conflation if F' is so. By replacing (E, F, ϕ) by $(E', F', \iota\phi\pi)$, without changing any notation, we can assume that $f_0, f_1, \dots, f_{i-2}, f_{i+2}, \dots, f_n$ and $g_0, \dots, g_{i-2}, g_{i+2}, \dots, g_n$ are in the radical of $\mathscr{F}$, hence also in the radical of $\mathscr{A}$. If we can now show that ϕ is an isomorphism, then F is an $\mathscr{A}$-conflation by the replacement Lemma 1.2. Now notice, because E is n-exact, f_i is a weak cokernel of f_{i-1} for $i = 1, \dots, n$ and $0 \colon A_{n+1} \to 0$ is a weak cokernel of the epimorphism f_n in $\mathscr{A}$. Similarly g_i is a weak cokernel of g_{i-1} for $i = 1, \dots, n$ and $0 \colon B_{n+1} \to 0$ is a weak cokernel of g_n in $\mathscr{A}$. Hence, Lemma 1.3 shows that $\phi_{i+2}, \dots, \phi_{n+1}$ are isomorphisms. By the dual of Lemma 1.3 also $\phi_0, \dots, \phi_{i-1}$ are isomorphisms. Therefore ϕ is an isomorphism and the lemma follows if ϕ_i and ϕ_{i+1} are isomorphisms for some $0 \leq i \leq n$.

Suppose ϕ_0 and ϕ_{n+1} are isomorphisms. Without loss of generality, we can again assume that E belongs to $\mathscr{E}_{\mathscr{A}}$ and show that F then belongs to $\mathscr{E}_{\mathscr{A}}$. Therefore, we have a morphism $f_{n+1} \colon A_{n+1} \to \Sigma_n A_0$ completing E to an $(n+2)$-angle X. Further, by using the replacement Lemma 1.2 twice, we may assume that $A_0 = B_0$ and $A_{n+1} = B_{n+1}$ as well as $\phi_0 = \mathrm{id}_{A_0}$ and $\phi_{n+1} = \mathrm{id}_{A_{n+1}}$. Similarly to the first part of the proof, using Lemma 2.6 and Lemma 3.5, we may assume that the morphisms $f_1, \dots, f_{n-1}$ and $g_1, \dots, g_{n-1}$ are in the radical. We want to show that ϕ_n is then already an isomorphism, thus reducing the case of ϕ_0 and ϕ_{n+1} being isomorphisms to the already solved case of ϕ_n and ϕ_{n+1} being isomorphisms.

To obtain an inverse of ϕ_n we first show $g_n f_{n+1} = 0$. Following the idea of the last part of the proof in [Jas16, proposition 4.8] we look at Diagram 3.1. Since $\mathscr{A}$ is n-extension closed, we can find the dashed maps $h_0, \dots, h_n$ of Diagram 3.1 such that the upper row is

$$\begin{tikzcd}
A_0 \arrow[r, dashed, "h_0"] \arrow[d, equal] & B_1' \arrow[r, dashed, "h_1"] \arrow[d, dashed, "\psi_1"] & \cdots \arrow[r, dashed, "h_{n-1}"] & B_n' \arrow[r, dashed, "h_n"] \arrow[d, dashed, "\psi_n"] & B_n \arrow[r, squiggly, "g_n f_{n+1}"] \arrow[d, "g_n"] & \Sigma_n A_0 \arrow[d, equal] \\
A_0 \arrow[r, "f_0"] \arrow[d, equal] & A_1 \arrow[r, "f_1"] \arrow[d, "\phi_1"] & \cdots \arrow[r, "f_{n-1}"] & A_n \arrow[r, "f_n"] \arrow[d, "\phi_n"] & A_{n+1} \arrow[r, squiggly, "f_{n+1}"] \arrow[d, equal] & \Sigma_n A_0 \\
B_0 \arrow[r, "g_0"] & B_1 \arrow[r, "g_1"] & \cdots \arrow[r, "g_{n-1}"] & B_n \arrow[r, "g_n"] & B_{n+1} &
\end{tikzcd}$$

Diagram 3.1: An $(n+2)$-angle arising from $g_n f_{n+1}$.

an $(n+2)$-angle and $B_1', \dots, B_n' \in \mathscr{A}$. By the axiom (F3) from [GKO13, definition 2.1] we find $\psi_1, \dots, \psi_n$ such that Diagram 3.1 is a commutative diagram. However, composition of the vertical morphisms in Diagram 3.1 yields that the diagram

$$\begin{tikzcd}
A_0 \arrow[r, "h_0"] \arrow[d, equal] & B'_1 \arrow[r, "h_1"] \arrow[d, "\psi_1\phi_1"] & \cdots \arrow[r, "h_{n-2}"] & B'_{n-1} \arrow[r, "h_{n-1}"] \arrow[d, "\psi_{n-1}\phi_{n-1}"] & B'_n \arrow[r, "h_n"] \arrow[d, "\psi_n\phi_n - h_n"] & B_n \arrow[d, "0"] \\
B_0 \arrow[r, "g_0"] & B_1 \arrow[r, "g_1"] & \cdots \arrow[r, "g_{n-2}"] & B_{n-1} \arrow[r, "g_{n-1}"] & B_n \arrow[r, "g_n"] & B_{n+1}
\end{tikzcd}$$

is commutative, using $h_{n-1}h_n = 0$ for commutativity of the penultimate square. This chain map is homotopic to the zero chain map by the dual of [Jas16, lemma 2.1]. In particular, this shows that h_0 is a split-mono and hence h_n is split-epi by [Jas16, proposition 2.6]. Now, this yields $g_n f_{n+1} = 0$.

Since $(n+2)$-angles are exact, $g_n f_{n+1} = 0$ proves existence of a morphism $\phi'_n\colon B_n \to A_n$ as in Diagram 3.2 satisfying $\phi'_n f_n = g_n$. From this we obtain $(\phi_n\phi'_n - \mathrm{id}_{A_{n+1}})f_n = 0$. Yet

$$\begin{tikzcd}
A_0 \arrow[r, "f_0"] \arrow[d, equal] & A_1 \arrow[r, "f_1"] \arrow[d, "\phi_1"] & \cdots \arrow[r, "f_{n-2}"] & A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "\phi_{n-1}"] & A_n \arrow[r, "f_n"] \arrow[d, bend left, "\phi_n"] & A_{n+1} \arrow[r, squiggly, "f_{n+1}"] \arrow[d, equal] & {} \\
B_0 \arrow[r, "g_0"] & B_1 \arrow[r, "g_1"] & \cdots \arrow[r, "g_{n-2}"] & B_{n-1} \arrow[r, "g_{n-1}"] & B_n \arrow[r, "g_n"] \arrow[u, dashed, bend left, "\phi'_n"] & B_{n+1} &
\end{tikzcd}$$

Diagram 3.2: Construction of ϕ'_n.

again, by exactness of $(n+2)$-angles, the morphism $\phi_n\phi'_n - \mathrm{id}_{A_{n+1}}$ factors through the radical morphism f_{n-1}. Therefore, $\phi_n\phi'_n$ is an isomorphism. Similarly, $\phi'_n\phi_n - \mathrm{id}_{B_{n+1}}$ factors through g_{n-1} using that E' is an n-exact sequence in $\mathscr{A}$. As g_{n-1} is in the radical as well, $\phi'_n\phi_n$ is an isomorphism. This shows that ϕ_n is an isomorphism. Since ϕ_{n+1} is an isomorphism as well, E' is an $\mathscr{A}$-conflation by the first part of the proof. $\square$

The axiom (E0) of [Jas16, definition 4.2] is trivially satisfied:

Lemma 3.7. *The n-exact sequence $0 \to 0 \to \cdots \to 0$ is an $\mathscr{A}$-conflation.*

Proof. This follows from $0 \in \mathscr{A}$ as $\mathscr{A} \subseteq \mathscr{F}$ is additive. $\square$

That the axiom (E1) and (E1$^{\mathrm{op}}$) of [Jas16, defintion 4.2] are satisfied for $(\mathscr{A}, \mathscr{E}_{\mathscr{A}})$ follows almost immediately from our work in Section 2.2. We only present a proof of (E1) as the proof of (E1$^{\mathrm{op}}$) is completely analogous.

Lemma 3.8. *The composite $h_0 = g_0 f_0$ of two $\mathscr{A}$-inflations f_0 and g_0 is an $\mathscr{A}$-inflation.*

Proof. Let $f_0\colon X_0 \to X'_0$ and $g_0\colon X'_0 \to X'_1$ be $\mathscr{A}$-inflations and $h_0 = f_0 g_0\colon X_0 \to X'_1$ their composite. Because f_0 and g_0 are $\mathscr{A}$-inflations we can choose $X_2, \ldots, X_{n+1} \in \mathscr{A}$ and $X'_2, \ldots, X'_{n+1} \in \mathscr{A}$, as well as $X''_2, \ldots, X''_{n+1} \in \mathscr{F}$ so that all three rows in Diagram 2.1 are $(n+2)$-angles and the middle row in Diagram 2.1 is a minimal $(n+2)$-angle. In the notation of Diagram 2.1 and Lemma 2.4, using that $\mathscr{A}$ is n-extension closed and that $g_{n+1}\Sigma_n f_1$ is a morphism from an object in $\mathscr{A}$ to an object in $\Sigma_n\mathscr{A}$, we see that $Y_1, \ldots, Y_n$ belong to $\mathscr{A}$ as the minimal $(n+2)$-angle of $g_{n+1}\Sigma_n f_1$ depicted in Lemma 2.4 is a direct summand of any $(n+2)$-angle with $g_{n+1}\Sigma_n f_1$ in position $n+2$. However, this means that the objects $X''_2, \ldots, X''_{n+1}$ of the minimal $(n+2)$-angle of h_0 belong to $\mathscr{A}$ using (1)-(6) of Lemma 2.4. Hence h_0 is an $\mathscr{A}$-inflation. $\square$

Notice, the assumption $\operatorname{Hom}_{\mathscr{F}}(\Sigma_n\mathscr{A},\mathscr{A})=0$ can be dropped in Lemma 3.8. However, we still need $\mathscr{A}$ to be an additive, n-extension closed subcategory of $\mathscr{F}$ and $(\mathscr{F},\Sigma_n,\pentagon)$ to be an $(n+2)$-angulated Krull-Schmidt category to make our proof of Lemma 3.8 work.

Finally, that (E2) and (E2$^{\mathrm{op}}$) of [Jas16, definition 4.2] are satisfied for $(\mathscr{A},\mathscr{E}_{\mathscr{A}})$ follows easily from the definitions. Again, we present a proof for (E2) only, as the proof of (E2$^{\mathrm{op}}$) is completely analogous.

Lemma 3.9. *Suppose we are given a diagram*

$$\begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, "\phi_0"] & A_1 \arrow[r, "f_1"] & A_2 \arrow[r, "f_2"] & \cdots \arrow[r, "f_{n-1}"] & A_n \arrow[r, "f_n"] & A_{n+1} \\ B_0 \end{tikzcd}$$

where the upper row is a $\mathscr{A}$-conflation A. Then there is an n-pushout diagram Diagram 3.3 in the sense of [Jas16, definition 2.11] such that g_0 is an $\mathscr{A}$-inflation.

$$\begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, "\phi_0"] & A_1 \arrow[r, "f_1"] \arrow[d, dashed, "\phi_1"] & A_2 \arrow[r, "f_2"] \arrow[d, dashed, "\phi_2"] & \cdots \arrow[r, "f_{n-1}"] & A_n \arrow[d, dashed, "\phi_n"] \\ B_0 \arrow[r, dashed, "g_0"] & B_1 \arrow[r, dashed, "g_1"] & B_2 \arrow[r, dashed, "g_2"] & \cdots \arrow[r, dashed, "g_{n-1}"] & B_n \end{tikzcd}$$

Diagram 3.3: n-Pushout diagram of A along ϕ_0.

Proof. Let A be a $\mathscr{A}$-conflation as in the lemma. This means that we find a morphism $f_{-1}\colon \Sigma_n^{-1}A_{n+1}\to A_0$ completing A to an $(n+2)$-angle. This is to say, we are given the undashed morphisms of Diagram 3.4 so that the upper row is an $(n+2)$-angle. Since

$$\begin{tikzcd} \Sigma_n^{-1}A_{n+1} \arrow[r, "f_{-1}"] \arrow[d, equal] & A_0 \arrow[r, "f_0"] \arrow[d, "\phi_0"] & A_1 \arrow[r, "f_1"] \arrow[d, dashed, "\phi_1"] & A_2 \arrow[r, "f_2"] \arrow[d, dashed, "\phi_2"] & \cdots \arrow[r, "f_{n-1}"] & A_n \arrow[r, "f_n"] \arrow[d, dashed, "\phi_n"] & A_{n+1} \arrow[d, equal] \\ \Sigma_n^{-1}A_{n+1} \arrow[r, "f_{-1}\phi_0"] & B_0 \arrow[r, dashed, "g_0"] & B_1 \arrow[r, dashed, "g_1"] & B_2 \arrow[r, dashed, "g_2"] & \cdots \arrow[r, dashed, "g_{n-1}"] & B_n \arrow[r, dashed, "g_n"] & A_{n+1} \end{tikzcd}$$

Diagram 3.4: Construction of an n-pushout of A along ϕ_0.

$\mathscr{A}\subseteq\mathscr{F}$ is an n-extension closed subcategory, we find objects $B_1,\dots,B_n\in\mathscr{A}$ and the dashed morphisms $g_0,\dots,g_n$ as in Diagram 3.4 such that the lower row is an $(n+2)$-angle. Notice, g_0 is an $\mathscr{A}$-inflation, as the second row of Diagram 3.4 is an $(n+2)$-angle. By the axiom (F4) of [GKO13, defintion 2.1] there are dashed morphisms $\phi_1,\dots,\phi_n$ in Diagram 3.4 so that the depicted morphism $\phi=(\phi_0,\dots,\phi_n,\mathrm{id}_{A_{n+1}})$ of $(n+2)$-angles is a good morphism of $(n+2)$-angles, i.e. so that the mapping cone of ϕ is an $(n+2)$-angle as well. Then similarly to [BT13, lemma 4.1] we conclude that the $(n+2)$-Σ_n-sequence P shown in Diagram 3.5 is an $(n+2)$-angle, as $\pentagon$ is closed under direct summands and because P is a direct summand of the cone of ϕ, as shown in Diagram 3.6. Moreover, since all $\mathscr{A}$-conflations are n-exact by Lemma 3.3, we conclude from the $(n+2)$-angle P shown in Diagram 3.5 that the constructed objects $B_1,\dots,B_n$ and morphisms $g_0,\dots,g_{n-1}$ and $\phi_1,\dots,\phi_n$ make Diagram 3.3 an n-pushout diagram. $\square$

$$A_0 \xrightarrow{\begin{bmatrix} -f_0 \\ \phi_0 \end{bmatrix}} A_1 \oplus B_0 \xrightarrow{\begin{bmatrix} -f_1 & 0 \\ \phi_1 & g_0 \end{bmatrix}} A_2 \oplus B_1 \xrightarrow{\begin{bmatrix} -f_2 & 0 \\ \phi_2 & g_1 \end{bmatrix}} \cdots \xrightarrow{[\,\phi_{n-1}\ g_{n-1}\,]} B_n \overset{g_n \Sigma_n f_{-1}}{\rightsquigarrow} \Sigma_n A_0$$

Diagram 3.5: Direct summand P of the mapping cone of ϕ.

$$\begin{tikzcd}
A_0 \arrow[r, "{\begin{bmatrix} \mathrm{id}_{A_0} \\ 0 \end{bmatrix}}"] \arrow[d, "{\begin{bmatrix} -f_0 \\ \phi_0 \end{bmatrix}}"] & A_0 \oplus \Sigma_n^{-1} A_{n+1} \arrow[r, "{[\,\mathrm{id}_{A_0}\ f_{-1}\,]}"] \arrow[d, "{\begin{bmatrix} -f_0 & 0 \\ \phi_0 & f_{-1}\phi_0 \end{bmatrix}}"] & A_0 \arrow[d, "{\begin{bmatrix} -f_0 \\ \phi_0 \end{bmatrix}}"] \\
A_1 \oplus B_0 \arrow[r, equal] \arrow[d, "{\begin{bmatrix} -f_1 & 0 \\ \phi_1 & g_0 \end{bmatrix}}"] & A_1 \oplus B_0 \arrow[r, equal] \arrow[d, "{\begin{bmatrix} -f_1 & 0 \\ \phi_1 & g_0 \end{bmatrix}}"] & A_1 \oplus B_0 \arrow[d, "{\begin{bmatrix} -f_1 & 0 \\ \phi_1 & g_0 \end{bmatrix}}"] \\
\vdots \arrow[d, "{\begin{bmatrix} -f_{n-1} & 0 \\ \phi_{n-1} & g_{n-2} \end{bmatrix}}"] & \vdots \arrow[d, "{\begin{bmatrix} -f_{n-1} & 0 \\ \phi_{n-1} & g_{n-2} \end{bmatrix}}"] & \vdots \arrow[d, "{\begin{bmatrix} -f_{n-1} & 0 \\ \phi_{n-1} & g_{n-2} \end{bmatrix}}"] \\
A_n \oplus B_{n-1} \arrow[r, equal] \arrow[d, "{[\,\phi_n\ g_{n-1}\,]}"] & A_n \oplus B_{n-1} \arrow[r, equal] \arrow[d, "{\begin{bmatrix} -f_n & 0 \\ \phi_n & g_{n-1} \end{bmatrix}}"] & A_n \oplus B_{n-1} \arrow[d, "{[\,\phi_n\ g_{n-1}\,]}"] \\
B_n \arrow[r, "{\begin{bmatrix} -g_n \\ \mathrm{id}_{B_n} \end{bmatrix}}"] \arrow[d, rightsquigarrow, "g_n \Sigma_n f_{-1}"] & A_{n+1} \oplus B_n \arrow[r, "{[\,0\ \mathrm{id}_{B_n}\,]}"] \arrow[d, rightsquigarrow, "{\begin{bmatrix} -\Sigma_n f_{-1} & 0 \\ \mathrm{id}_{A_{n+1}} & g_n \end{bmatrix}}"] & B_n \arrow[d, rightsquigarrow, "g_n \Sigma_n f_{-1}"] \\
\Sigma_n A_0 \arrow[r, "{\begin{bmatrix} \Sigma_n \mathrm{id}_{A_0} \\ 0 \end{bmatrix}}"] & \Sigma_n A_0 \oplus A_{n+1} \arrow[r, "{[\,\Sigma_n \mathrm{id}_{A_0}\ \Sigma_n f_{-1}\,]}"] & \Sigma_n A_0
\end{tikzcd}$$

Diagram 3.6: Diagram showing that the $(n+2)$-Σ_n-sequence P given in Diagram 3.5 is a direct summand of the mapping cone of $\phi = (\phi_0, \ldots, \phi_n, \mathrm{id}_{A_{n+1}})$ as shown in Diagram 3.4.

Notice, Lemma 3.3, Lemma 3.6, Lemma 3.7, Lemma 3.8 and Lemma 3.9 show that part (1) of Theorem 3.2 holds. It remains to show part (2) of Theorem 3.2. In the following we will write $[E]$ for the equivalence class of E in $\mathrm{YExt}^n_{(\mathscr{A},\mathscr{E}_\mathscr{A})}(A_{n+1}, A_0)$, where a conflation $E \in \mathscr{E}_\mathscr{A}$ is viewed as an n-extension of an object $A_{n+1} \in \mathscr{A}$ by an object $A_0 \in \mathscr{A}$.

Lemma 3.10. *There is a natural bilinear isomorphism*

$$\mathrm{Hom}_\mathscr{F}(-, \Sigma_n(-)) \to \mathrm{YExt}^n_{(\mathscr{A},\mathscr{E}_\mathscr{A})}(-,-)$$

of functors $\mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathsf{Ab}$.

Proof. We construct $\Psi_{A_0,A_{n+1}} \colon \mathrm{YExt}^n_{(\mathscr{A},\mathscr{E}_\mathscr{A})}(A_{n+1}, A_0) \to \mathrm{Hom}_\mathscr{F}(A_{n+1}, \Sigma_n A_0)$ for any fixed pair $A_0, A_{n+1} \in \mathscr{A}$: Suppose $E \in \mathscr{E}_\mathscr{A}$ is an $\mathscr{A}$-conflation with end terms A_0 and A_{n+1}. Then there is a $(n+2)$-angle X_E of the shape

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \overset{\psi(E)}{\rightsquigarrow}$$

completing E to an $(n+2)$-angle. Notice that $\psi(E)$ is uniquely determined by E, using Lemma 2.2. Hence, this way a map $\psi \colon \mathscr{E}_\mathscr{A} \to \mathrm{Hom}_\mathscr{F}(\mathscr{A}, \Sigma_n \mathscr{A})$ is defined. Moreover, Lemma 2.2 shows that we have $\psi(E) = \psi(E')$ if E and E' are linked by a sequence of equivalences in the sense of [Jas16, defintion 2.9], i.e. if $[E] = [E']$. Hence, ψ induces a map $\Psi_{A_0,A_{n+1}}\colon \operatorname{YExt}^n_{(\mathscr{A},\mathscr{E}_\mathscr{A})}(A_{n+1}, A_0) \to \operatorname{Hom}_\mathscr{F}(A_{n+1}, \Sigma_n A_0)$, $[E] \mapsto \psi(E)$ for each pair $A_0, A_{n+1} \in \mathscr{A}$.

Conversely we construct $\Phi_{A_0,A_{n+1}}\colon \operatorname{Hom}_\mathscr{F}(A_{n+1}, \Sigma_n A_0) \to \operatorname{YExt}^n_{(\mathscr{A},\mathscr{E}_\mathscr{A})}(A_{n+1}, A_0)$ for each fixed pair $A_0, A_{n+1} \in \mathscr{A}$: Assume we are given an $f \in \operatorname{Hom}_\mathscr{F}(A_{n+1}, \Sigma_n A_0)$. Since $\mathscr{A}$ is n-extension closed, we can construct an $(n+2)$-angle X_f of the shape in Diagram 3.7 with $A_1, \ldots, A_n \in \mathscr{A}$. Denote the $\mathscr{A}$-conflation arising from such an $(n+2)$-angle X_f by

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \overset{f}{\rightsquigarrow}$$

Diagram 3.7: Completion of f to an $(n+2)$-angle.

E_{X_f}. We claim, the equivalence class $[E_{X_f}]$ does not depend on the choice of X_f: Suppose we are given another completion of f to an $(n+2)$-angle X'_f of the shape

$$A'_0 \xrightarrow{g_0} A'_1 \xrightarrow{g_1} \cdots \xrightarrow{g_{n-1}} A'_n \xrightarrow{g_n} A'_{n+1} \overset{f}{\rightsquigarrow}$$

with $A'_1, \ldots, A'_n \in \mathscr{A}$. Using the axiom (F3) from [GKO13, definition 2.1] we find the dashed morphisms of a commutative diagram

$$\begin{tikzcd}
A_0 \arrow[r, "f_0"] \arrow[d, equal] & A_1 \arrow[r, "f_1"] \arrow[d, dashed] & \cdots \arrow[r, "f_{n-1}"] & A_n \arrow[r, "f_n"] \arrow[d, dashed] & A_{n+1} \arrow[r, squiggly, "f"] \arrow[d, equal] & \Sigma_n A'_0 \arrow[d, equal] \\
A'_0 \arrow[r, "g_0"] & A'_1 \arrow[r, "g_1"] & \cdots \arrow[r, "g_{n-1}"] & A'_n \arrow[r, "g_n"] & A'_{n+1} \arrow[r, squiggly, "f"] & \Sigma_n A'_0
\end{tikzcd}$$

and hence obtain $[E_{X_f}] = [E_{X'_f}]$. Therefore, for $A_0, A_{n+1} \in \mathscr{A}$ the assignment

$$\Phi_{A_0,A_{n+1}}\colon \operatorname{Hom}_\mathscr{F}(A_{n+1}, \Sigma_n A_0) \to \operatorname{YExt}^n_{(\mathscr{A},\mathscr{E}_\mathscr{A})}(A_{n+1}, A_0),\ f \mapsto [E_{X_f}]$$

where X_f is an arbitrary $(n+2)$-angle as in Diagram 3.7 is well-defined. By construction it is clear that $\Phi_{A_0,A_{n+1}}$ and $\Psi_{A_0,A_{n+1}}$ for any fixed pair $A_0, A_{n+1} \in \mathscr{A}$ are mutually inverse to each other. It is straight forward to check, that this way a natural and bilinear isomorphism Φ is defined. $\square$

References

[BBD82] A. A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Astérisque **100** (1982) (proceedings of the conference "Analysis and topology on singular spaces", Vol. 1, Luminy, 1981).

[BT13] Petter Andreas Bergh and Marius Thaule. The axioms for n–angulated categories. *Algebraic & Geometric Topology*, 13(4):2405 – 2428, 2013.

[Dye05] Matthew J Dyer. Exact subcategories of triangulated categories. https://www3.nd.edu/~dyer/papers/extri.pdf, preprint (2005).

[Fed19] Francesca Fedele. Auslander–reiten $(d+2)$-angles in subcategories and a $(d+2)$-angulated generalisation of a theorem by brüning. *Journal of Pure and Applied Algebra*, 223(8):3554–3580, 2019.

[GKO13] Christof Geiss, Bernhard Keller, and Steffen Oppermann. n-angulated categories. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2013(675):101–120, 2013.

[Jas16] Gustavo Jasso. n-Abelian and n-exact categories. *Mathematische Zeitschrift*, 283(3-4):703–759, 2016.

[Jør20] Peter Jørgensen. Abelian subcategories of triangulated categories induced by simple minded systems. *Mathematische Zeitschrift*, 1-28, 2022.

[OT12] Steffen Oppermann and Hugh Thomas. Higher-dimensional cluster combinatorics and representation theory. *Journal of the European Mathematical Society*, 14(6):1679–1737, 2012.

Paper B

Idempotent completions of n-exangulated categories

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1 Introduction

Abstract

Suppose $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an n-exangulated category. We show that the idempotent completion and the weak idempotent completion of $\mathcal{C}$ are again n-exangulated categories. Furthermore, we also show that the canonical inclusion functor of $\mathcal{C}$ into its (resp. weak) idempotent completion is n-exangulated and 2-universal among n-exangulated functors from $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ to (resp. weakly) idempotent complete n-exangulated categories. We note that our methods of proof differ substantially from the extriangulated and $(n+2)$-angulated cases. However, our constructions recover the known structures in the established cases up to n-exangulated isomorphism of n-exangulated categories.

Idempotent completion began with Karoubi's work [Kar68] on additive categories. It was shown that an additive category embeds into an associated one which is *idempotent complete*, that is, in which all idempotent morphisms admit a kernel. Particularly nice examples of idempotent complete categories include Krull-Schmidt categories, which can be characterised as idempotent complete additive categories in which each object has a semi-perfect endomorphism ring (see Chen–Ye–Zhang [CYZ08, Thm. A.1], Krause [Kra15, Cor. 4.4]). Other examples include the vast class of pre-abelian categories (see e.g. [Sha20, Rem. 2.2]); e.g. a module category, or the category of Banach spaces (over the reals, say).

Suppose $\mathcal{C}$ is an additive category. The objects of the *idempotent completion* $\widetilde{\mathcal{C}}$ *of* $\mathcal{C}$ are pairs (X, e), where X is an object of $\mathcal{C}$ and $e\colon X \to X$ is an *idempotent* morphism in $\mathcal{C}$, i.e. $e^2 = e$. What is particularly nice is that if $\mathcal{C}$ has a certain kind of structure, then in several cases this induces the same structure on $\widetilde{\mathcal{C}}$. For example, Karoubi had already shown that the idempotent completion of an additive category is again additive (see [Kar68, (1.2.2)]). Furthermore, it has been shown for the following, amongst others, extrinsic structures that if $\mathcal{C}$ has such a structure, then so too does $\widetilde{\mathcal{C}}$:

(i) triangulated (see Balmer–Schlichting [BS01, Thm. 1.5]);

(ii) exact (see Bühler [Büh10, Prop. 6.13]);

(iii) extriangulated (see [Msa22, Thm. 3.1]); and

(iv) $(n+2)$-angulated, where $n \geq 1$ is an integer (see Lin [Lin21, Thm. 3.1]).

See also Liu–Sun [LS14] and Zhou [Zho22].

Idempotent complete exact and triangulated categories are verifiably important in algebra and algebraic geometry. As a classical example, in Neeman [Nee90] an idempotent complete exact category $\mathcal{E}$ is needed to give a clean description of the kernel of the localisation functor from the homotopy category of $\mathcal{E}$ to its derived category. And, more generally, many equivalences only hold up to direct summands, i.e. up to idempotents (see, for example, Orlov [Orl11, Thm. 2.11], or Kalck–Iyama–Wemyss–Yang [KIWY15, Thm. 1.1]). Therefore, it is usually helpful to view an algebraic structure as sitting inside its idempotent completion.

The idempotent completion $\widetilde{\mathcal{C}}$ comes equipped with an inclusion functor $\mathscr{I}_{\mathcal{C}} \colon \mathcal{C} \to \widetilde{\mathcal{C}}$ given by $\mathscr{I}_{\mathcal{C}}(X) = (X, \mathrm{id}_X)$ on objects. Moreover, in several of the cases above it has been shown that this functor is 2-*universal* in an appropriate sense; see e.g. Proposition 2.8 for a precise formulation. For example, without any assumptions other than additivity, the functor $\mathscr{I}_{\mathcal{C}}$ is additive and 2-universal amongst additive functors from $\mathcal{C}$ to idempotent complete additive categories. On the other hand, if e.g. $\mathcal{C}$ has an exact structure, then $\mathscr{I}_{\mathcal{C}}$ is exact and 2-universal amongst exact functors from $\mathcal{C}$ to idempotent complete exact categories.

In homological algebra two parallel generalisations have been made from the classical settings of exact and triangulated categories. One of these has been the introduction of *extriangulated* categories as defined by Nakaoka–Palu [NP19]. An extriangulated category is a triplet $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, where $\mathcal{C}$ is an additive category, $\mathbb{E} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Ab}$ is a biadditive functor to the category of abelian groups, and $\mathfrak{s}$ is a so-called additive realisation of $\mathbb{E}$. The realisation $\mathfrak{s}$ associates to each $\delta \in \mathbb{E}(Z, X)$ a certain equivalence class $\mathfrak{s}(\delta) = [\ X \xrightarrow{x} Y \xrightarrow{y} Z\ ]$ of a 3-term complex. As an example, each triangulated category $(\mathcal{C}, \Sigma, \triangle)$, where Σ is a suspension functor and $\triangle$ is a triangulation, is an extriangulated category. Indeed, one defines the corresponding bifunctor by $\mathbb{E}_{\Sigma}(Z, X) := \mathcal{C}(Z, \Sigma X)$. See [NP19, Prop. 3.22] for more details. In addition, each suitable exact category is extriangulated; see [NP19, Exam. 2.13]. A particular advantage of this theory is that the collection of extriangulated categories is closed under taking extension-closed subcategories. Although an extension-closed subcategory of an exact category is again exact, the same does not hold in general for triangulated categories.

We note here that, importantly, it was shown in [Msa22, Sec. 3.1] that the extriangulated structure on $\widetilde{\mathcal{C}}$ produced from case (iii) above is compatible with the more classical constructions of (i) and (ii). For instance, given a triangulated category $\mathcal{C}$, one can equip its idempotent completion $\widetilde{\mathcal{C}}$ with a triangulation by (i) or with an extriangulation by (iii), but these structures are the same in the sense of [NP19, Prop. 3.22]. Analogously, (iii) also recovers (ii) if one starts with an extriangulated category that is exact.

Let $n \geq 1$ be an integer. The other aforementioned generalisation in homological algebra has been the development of higher homological algebra. This includes the introduction of n-*exact* and n-*abelian* categories by Jasso [Jas16], and $(n+2)$-*angulated* categories by Geiss–Keller–Oppermann [GKO13]. Respectively, these generalise exact, abelian and triangulated categories, in that one recovers the classical notions by setting $n = 1$. For instance, an $(n+2)$-angulated category is a triplet $(\mathcal{C}, \Sigma, \pentagon)$ satisfying some axioms, where Σ is still an automorphism of $\mathcal{C}$, but now $\pentagon$ consists of a collection of $(n+2)$-*angles* each of which has $n+3$ terms.

The focal point of this paper is on the idempotent completion of an n-*exangulated* category. These categories were axiomatised by Herschend–Liu–Nakaoka [HLN21], and simultaneously generalise extriangulated, $(n+2)$-angulated, and suitable n-exact categories (see [HLN21, Sec. 4]). Like an extriangulated category, an n-exangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ consists of an additive category $\mathcal{C}$, a biadditive functor $\mathbb{E}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Ab}$, and a so-called exact realisation $\mathfrak{s}$ of $\mathbb{E}$, which satisfy some axioms (see Subsection 3.1). The realisation $\mathfrak{s}$ now associates to each $\delta \in \mathbb{E}(Z, X)$ a certain equivalence class (see Subsection 3.1)

$$\mathfrak{s}(\delta) = [\ X_0 \xrightarrow{d_0^X} X_1 \xrightarrow{d_1^X} \cdots \xrightarrow{d_n^X} X_{n+1}\ ]$$

of an $(n+2)$-term complex. In this case, the pair $\langle X_\bullet, \delta\rangle$ is called an $\mathfrak{s}$-*distinguished* n-*exangle*. We recall that structure-preserving functors between n-exangulated categories were defined in [BTS21, Def. 2.32]. They are known as n-*exangulated* functors and they send distinguished n-exangles to distinguished n-exangles.

Suppose that $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an n-exangulated category. Let $\widetilde{\mathcal{C}}$ denote the idempotent completion of $\mathcal{C}$ as an additive category. We define a biadditive functor $\mathbb{F}\colon \widetilde{\mathcal{C}}^{\mathrm{op}} \times \widetilde{\mathcal{C}} \to \mathsf{Ab}$ as follows. For any pair of objects $(X, e), (Z, e') \in \widetilde{\mathcal{C}}$, we let $\mathbb{F}((Z, e'), (X, e))$ consist of triplets (e, δ, e') where $\delta \in \mathbb{E}(Z, X)$ such that $\mathbb{E}(Z, e)(\delta) = \delta = \mathbb{E}(e', X)(\delta)$. On morphisms $\mathbb{F}$ is essentially a restriction of $\mathbb{E}$; see Definition 4.4 for details. Now we define a realisation $\mathfrak{t}$ of $\mathbb{F}$. For $(e, \delta, e') \in \mathbb{F}((Z, e'), (X, e))$, we have that $\mathfrak{s}(\delta) = [X_\bullet]$ for some $(n+2)$-term complex $X_\bullet$ with $X_0 = X$ and $X_{n+1} = Z$ since $\mathfrak{s}$ is a realisation of $\mathbb{E}$. We choose an idempotent morphism $e_\bullet\colon X_\bullet \to X_\bullet$ of complexes, such that $e_0 = e$ and $e_{n+1} = e'$; see Corollary 4.13. Lastly, we set $\mathfrak{t}((e, \delta, e'))$ to be the equivalence class of the complex

$$(X, e) \xrightarrow{e_1 d_0^X e_0} (X_1, e_1) \xrightarrow{e_2 d_1^X e_1} \cdots \xrightarrow{e_n d_{n-1}^X e_{n-1}} (X_n, e_n) \xrightarrow{e_{n+1} d_n^X e_n} (Z, e')$$

in $\widetilde{\mathcal{C}}$. We say that an n-exangulated category is *idempotent complete* if its underlying additive category is (see Definition 4.31).

Theorem 1.1 (Theorem 4.32, Theorem 4.39)**.** *The triplet $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$ is an idempotent complete n-exangulated category. Furthermore, the inclusion functor $\mathscr{I}_{\mathcal{C}}\colon \mathcal{C} \to \widetilde{\mathcal{C}}$ extends to an n-exangulated functor $(\mathscr{I}_{\mathcal{C}}, \Gamma)\colon (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$, which is 2-universal among n-exangulated functors from $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ to idempotent complete n-exangulated categories.*

An n-*exact category* $(\mathcal{C}, \mathcal{X})$ (see [Jas16, Def. 4.2]) induces an n-exangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ if, for each pair of objects $A, C \in \mathcal{C}$, the collection $\mathbb{E}(C, A) = \mathrm{Ext}^n_{\mathcal{C}}(C, A)$ of n-extensions of C by A forms a set; see [HLN21, Prop. 4.34]. As in [Kla23, Def. 4.6], we say that an n-exangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is n-*exact* if its n-exangulated structure arises in this way. Combining Theorem 1.1 with [Kla23, Cor. 4.12], we deduce the following.

Corollary 1.2 (Corollary 4.34)**.** *If $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an n-exangulated category that is n-exact, then the idempotent completion $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$ is n-exact.*

We explain in Remark 4.40 how Theorem 1.1 unifies the constructions in cases (i)–(iv) above. Furthermore, we comment on some obstacles faced in proving the n-exangulated case in Remark 4.41.

From Theorem 1.1 we deduce the following corollary, giving a way to produce Krull-Schmidt n-exangulated categories.

Corollary 1.3 (Corollary 4.33)**.** *If each object in* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ *has a semi-perfect endomorphism ring, then the idempotent completion* $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$ *is a Krull-Schmidt* n*-exangulated category.*

Finally, we note that analogues of Theorem 1.1 and Corollary 1.2 are shown for the weak idempotent completion in Section 5. The importance of being weakly idempotent complete for extriangulated categories was very recently demonstrated in [Kla23, Prop. 2.7]. It turns out that for an extriangulated category, the underlying category being weakly idempotent complete is equivalent to the condition (WIC) defined in [NP19, Cond. 5.8]. Moreover, (WIC) is a key assumption in many results on extriangulated categories, e.g. [NP19, §§5–7], [HLN22, §3], Zhao–Zhu–Zhuang [ZZZ21]. We remark that the analogue of (WIC) for n-exangulated categories is automatic if $n \geq 2$, but it is not equivalent to the weak idempotent completeness of the underlying category; see [Kla23, Thm. B] for more details.

2 On the splitting of idempotents

In this section we recall some key definitions regarding idempotents and idempotent completions of categories. We focus on the idempotent completion of an additive category in Subsection 2.1 and on the weak idempotent completion in Subsection 2.2. Throughout this section, we let $\mathcal{A}$ denote an additive category. For a more in-depth treatment, we refer the reader to [Büh10, Secs. 6–7].

2.1 Idempotent completion

Recall that by an *idempotent (in $\mathcal{A}$)* we mean a morphism $e\colon X \to X$ satisfying $e^2 = e$ for some object $X \in \mathcal{A}$.

The following definition is from Borceux [Bor94].

Definition 2.1. [Bor94, Defs. 6.5.1, 6.5.3] An idempotent $e\colon X \to X$ in $\mathcal{A}$ is said to *split* if there exist morphisms $r\colon X \to Y$ and $s\colon Y \to X$, such that $sr = e$ and $rs = \mathrm{id}_Y$. The category $\mathcal{A}$ is *idempotent complete*, or has *split idempotents*, if every idempotent in $\mathcal{A}$ splits.

If $\mathcal{A}$ has split idempotents and $e\colon X \to X$ is an idempotent in $\mathcal{A}$, then the object X admits a direct sum decomposition $X \cong \operatorname{Ker}(e) \oplus \operatorname{Ker}(\mathrm{id}_X - e)$ (see e.g. Auslander [Aus74, p. 188]). In particular, the idempotent e and its counterpart $\mathrm{id}_X - e$ each admit a kernel. Idempotent complete additive categories can be characterised by such a criterion and its dual.

Proposition 2.2. *[Bor94, Prop. 6.5.4] An additive category is idempotent complete if and only if every idempotent admits a kernel, if and only if every idempotent admits a cokernel.*

From this point of view, idempotent complete categories sit between additive categories and *pre-abelian* categories, the latter being additive categories in which every morphism admits a kernel and a cokernel; see for example Bucur–Deleanu [BD68, §5.4].

Every additive category can be viewed as a full subcategory of an idempotent complete one. This goes back to Karoubi [Kar68, Sec. 1.2], so the idempotent completion of $\mathcal{A}$ is also often referred to as the *Karoubi envelope of $\mathcal{A}$*.

Definition 2.3. The *idempotent completion $\widetilde{\mathcal{A}}$ of $\mathcal{A}$* is the category defined as follows. Objects of $\widetilde{\mathcal{A}}$ are pairs (X, e), where X is an object of $\mathcal{A}$ and $e \in \operatorname{End}_{\mathcal{A}}(X)$ is idempotent. For objects $(X, e), (Y, e') \in \operatorname{obj} \widetilde{\mathcal{A}}$, a morphism from (X, e) to (Y, e') is a triplet (e', r, e), where $r \in \mathcal{A}(X, Y)$ satisfies

$$re = r = e'r$$

in $\mathcal{A}$. Composition of morphisms is defined by

$$(e'', s, e') \circ (e', r, e) := (e'', sr, e),$$

whenever $(e', r, e) \in \widetilde{\mathcal{A}}((X, e), (Y, e'))$ and $(e'', s, e') \in \widetilde{\mathcal{A}}((Y, e'), (Z, e''))$. The identity of an object $(X, e) \in \operatorname{obj} \widetilde{\mathcal{A}}$ will be denoted $\widetilde{\mathrm{id}}_{(X,e)}$ and is the morphism (e, e, e).

A morphism $(e', r, e)\colon (X, e) \to (Y, e')$ in the idempotent completion $\widetilde{\mathcal{A}}$ of $\mathcal{A}$ is usually denoted more simply as r; see e.g. [BS01, Def. 1.2] and [Büh10, Rem. 6.3]. However, for precision in Sections 4–5, we use triplets for morphisms in $\widetilde{\mathcal{A}}$ so that we can easily distinguish morphisms in $\mathcal{A}$ from morphisms in its idempotent completion. Our choice of notation also has the added benefit of keeping track of the (co)domain of a morphism in $\widetilde{\mathcal{A}}$. This becomes important later when different morphisms in $\widetilde{\mathcal{A}}$ have the same underlying morphism; see Notation 4.37.

By a functor we always mean a covariant functor. The *inclusion functor* $\mathscr{I}_{\mathcal{A}}\colon \mathcal{A} \to \widetilde{\mathcal{A}}$ is defined as follows. An object $X \in \operatorname{obj} \mathcal{A}$ is sent to $\mathscr{I}_{\mathcal{A}}(X) := (X, \mathrm{id}_X) \in \operatorname{obj} \widetilde{\mathcal{A}}$ and a morphism $r \in \mathcal{A}(X, Y)$ is mapped to $\mathscr{I}_{\mathcal{A}}(r) := (\mathrm{id}_Y, r, \mathrm{id}_X) \in \widetilde{\mathcal{A}}(\mathscr{I}_{\mathcal{A}}(X), \mathscr{I}_{\mathcal{A}}(Y))$.

Lemma 2.4. *If $e \in \operatorname{End}_{\mathcal{A}}(X)$ is a split idempotent, with a splitting $e = sr$ where $r\colon X \to Y$ and $s\colon Y \to X$, then $(X, e) \cong \mathscr{I}_{\mathcal{A}}(Y)$.*

Proof. We have $re = rsr = \mathrm{id}_Y r = r$ and $es = srs = s\,\mathrm{id}_Y = s$. Hence, there are morphisms $\tilde{r} := (\mathrm{id}_Y, r, e)\colon (X, e) \to \mathscr{I}_{\mathcal{A}}(Y)$ and $\tilde{s} := (e, s, \mathrm{id}_Y)\colon \mathscr{I}_{\mathcal{A}}(Y) \to (X, e)$ in $\widetilde{\mathcal{A}}$ with $\tilde{s}\tilde{r} = \widetilde{\mathrm{id}}_{(X,e)}$ and $\tilde{r}\tilde{s} = \widetilde{\mathrm{id}}_{\mathscr{I}_{\mathcal{A}}(Y)}$. Hence, $\tilde{r}$ and $\tilde{s}$ are mutually inverse isomorphisms in $\widetilde{\mathcal{A}}$. ☐

If $\mathcal{A}$ is an idempotent complete category, then the functor $\mathscr{I}_{\mathcal{A}}$ is an equivalence of categories; see e.g. [Büh10, Rem. 6.5]. But more generally we have the following.

Proposition 2.5. *[Büh10, Rem. 6.3] The idempotent completion $\widetilde{\mathcal{A}}$ is an idempotent complete additive category with biproduct given by $(X, e) \oplus (Y, e') = (X \oplus Y, e \oplus e')$. The inclusion functor $\mathscr{I}_{\mathcal{A}}\colon \mathcal{A} \to \widetilde{\mathcal{A}}$ is fully faithful and additive.*

Remark 2.6. Let (X, e) be an arbitrary object of $\widetilde{\mathcal{A}}$. Then (X, e) is a direct summand of $\mathscr{I}_{\mathcal{A}}(X) = (X, \mathrm{id}_X)$. Indeed, there is an isomorphism $(X, \mathrm{id}_X) \cong (X, e) \oplus (X, \mathrm{id}_X - e)$. The canonical inclusion of (X, e) into (X, id_X) is given by the morphism (id_X, e, e), and the projection of (X, id_X) onto (X, e) by (e, e, id_X). Similarly for $(X, \mathrm{id}_X - e)$.

The functor $\mathscr{I}_{\mathcal{A}}\colon \mathcal{A} \to \widetilde{\mathcal{A}}$ is 2-universal in some sense; see Proposition 2.8. For this we recall the notion of *whiskering* a natural transformation by a functor. We will use Hebrew letters (e.g. ב (beth), צ (tsadi), ד (daleth), מ (mem)) for natural transformations. Suppose $\mathcal{B}, \mathcal{C}, \mathcal{D}$ are categories and that we have a diagram

$$\begin{tikzcd} \mathcal{B} \arrow[r, "\mathscr{F}"] & \mathcal{C} \arrow[r, bend left, "\mathscr{G}"] \arrow[r, bend right, "\mathscr{H}"'] \arrow[r, phantom, "\Downarrow \beth"] & \mathcal{D}, \end{tikzcd}$$

where $\mathscr{F}, \mathscr{G}, \mathscr{H}$ are functors and $\beth\colon \mathscr{G} \Rightarrow \mathscr{H}$ is a natural transformation.

Definition 2.7. The *whiskering of $\mathscr{F}$ and $\beth$* is the natural transformation $\beth_{\mathscr{F}}\colon \mathscr{G}\mathscr{F} \Rightarrow \mathscr{H}\mathscr{F}$ defined by $(\beth_{\mathscr{F}})_X := \beth_{\mathscr{F}(X)}\colon \mathscr{G}\mathscr{F}(X) \to \mathscr{H}\mathscr{F}(X)$ for each $X \in \mathcal{B}$.

The next proposition explains the 2-universal property satisfied by $\mathscr{I}_{\mathcal{A}}\colon \mathcal{A} \to \widetilde{\mathcal{A}}$.

Proposition 2.8. *[Büh10, Prop. 6.10] For any additive functor $\mathscr{F}\colon \mathcal{A} \to \mathcal{B}$ with $\mathcal{B}$ idempotent complete:*

(i) *there is an additive functor* $\mathscr{E}\colon \widetilde{\mathcal{A}} \to \mathcal{B}$ *and a natural isomorphism* $\text{צ}\colon \mathscr{F} \overset{\cong}{\Longrightarrow} \mathscr{E}\mathscr{I}_{\mathcal{A}}$; *and, in addition,*

(ii) *for any functor* $\mathscr{G}\colon \widetilde{\mathcal{A}} \to \mathcal{B}$ *and any natural transformation* $\text{ד}\colon \mathscr{F} \Rightarrow \mathscr{G}\mathscr{I}_{\mathcal{A}}$, *there exists a unique natural transformation* $\text{מ}\colon \mathscr{E} \Rightarrow \mathscr{G}$ *with* $\text{ד} = \text{מ}_{\mathscr{I}_{\mathcal{A}}}\text{צ}$.

2.2 Weak idempotent completion

A weaker notion than being idempotent complete is that of being weakly idempotent complete. This was introduced in the context of exact categories by Thomason–Trobaugh [TT90, Axiom A.5.1]. It is, however, a property of the underlying additive category and gives rise to the following definition.

Definition 2.9. [Büh10, Def. 7.2] An additive category is *weakly idempotent complete* if every retraction has a kernel.

Definition 2.9 is actually self-dual. Indeed, in an additive category, every retraction has a kernel if and only if every section has a cokernel; see e.g. [Büh10, Lem. 7.1].

If $r\colon X \to Y$ is a retraction in $\mathcal{A}$, with corresponding section $s\colon Y \to X$, and r admits a kernel k, then the split idempotent $e := sr \in \operatorname{End}_{\mathcal{A}}(X)$ also has kernel k. Conversely, if $e\colon X \to X$ is a split idempotent, with splitting given by $e = sr$ where $r\colon X \to Y$, then a kernel of e is also a kernel of r. Therefore, weakly idempotent complete categories are those additive categories in which split idempotents admit kernels, in contrast to idempotent complete categories in which all idempotents admit kernels (see Proposition 2.2).

Definition 2.10. The *weak idempotent completion* $\widehat{\mathcal{A}}$ *of* $\mathcal{A}$ is the full subcategory of $\widetilde{\mathcal{A}}$ consisting of all objects $(X, e) \in \widetilde{\mathcal{A}}$ such that $\mathrm{id}_X - e$ is a split idempotent in $\mathcal{A}$.

Remark 2.11. We note that Definition 2.10 above differs slightly from the definition of the weak idempotent completion of $\mathcal{A}$ suggested in [Büh10, Rem. 7.8]. If, as in [Büh10], we ask that objects of $\widehat{\mathcal{A}}$ are pairs (X, e) where $e\colon X \to X$ splits, then $\widehat{\mathcal{A}}$ is equivalent to $\mathcal{A}$. Indeed, if $sr = e$ and $rs = \mathrm{id}_Y$, where $r\colon X \to Y$ and $s\colon Y \to X$, then $(X, e) \cong (Y, \mathrm{id}_Y)$ in $\widetilde{\mathcal{A}}$ by Lemma 2.4. That is, we have not added any objects that are not already isomorphic to some object of $\mathscr{I}_{\mathcal{A}}(\mathcal{A})$. On the other hand, if we take objects in $\widehat{\mathcal{A}}$ to be pairs (X, e) where $\mathrm{id}_X - e$ splits (as in Definition 2.10), then we have $(X, \mathrm{id}_X) \cong (X, e) \oplus (Y', \mathrm{id}_{Y'})$ in $\widehat{\mathcal{A}}$, where $s'r' = \mathrm{id}_X - e$ and $r's' = \mathrm{id}_{Y'}$, where $r'\colon X \to Y'$ and $s'\colon Y' \to X$. In this case, since $(X, \mathrm{id}_X - e) \cong (Y', \mathrm{id}_{Y'})$ in $\widetilde{\mathcal{A}}$, we see that a "complementary" summand of $(X, \mathrm{id}_X - e)$ in (X, id_X) has been added. This discrepancy has been noticed previously; see e.g. Henrard–van Roosmalen [HR19, Prop. A.11].

It follows that $\widehat{\mathcal{A}}$ is an additive subcategory of $\widetilde{\mathcal{A}}$ and that it is weakly idempotent complete; see e.g. [Büh10, Rem. 7.8] or [HR19, Sec. A.2]. From this observation, we immediately have the next lemma.

Lemma 2.12. *Suppose* $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \widetilde{\mathcal{A}}$ *with* $\widetilde{X} \oplus \widetilde{Y} \cong \widetilde{Z}$. *Then any two of* $\widetilde{X}, \widetilde{Y}, \widetilde{Z}$ *being isomorphic to objects in* $\widehat{\mathcal{A}}$ *implies that the third object is also isomorphic to an object in* $\widehat{\mathcal{A}}$.

Analogously to the construction in Subsection 2.1, there exists an inclusion functor $\mathscr{K}_{\mathcal{A}} \colon \mathcal{A} \to \widehat{\mathcal{A}}$, given by $\mathscr{K}_{\mathcal{A}}(X) := (X, \mathrm{id}_X)$ on objects, which is 2-universal among additive functors from $\mathcal{A}$ to weakly idempotent complete categories; see e.g. [Nee90, Rem. 1.12] or [Büh10, Rem. 7.8].

Proposition 2.13. *For any additive functor $\mathscr{F} \colon \mathcal{A} \to \mathcal{B}$ with $\mathcal{B}$ weakly idempotent complete:*

(i) *there is an additive functor $\mathscr{E} \colon \widehat{\mathcal{A}} \to \mathcal{B}$ and a natural isomorphism $\text{צ} \colon \mathscr{F} \overset{\cong}{\Longrightarrow} \mathscr{E}\mathscr{K}_{\mathcal{A}}$; and, in addition,*

(ii) *for any additive functor $\mathscr{G} \colon \widehat{\mathcal{A}} \to \mathcal{B}$ and any natural transformation $\text{ד} \colon \mathscr{F} \Rightarrow \mathscr{G}\mathscr{K}_{\mathcal{A}}$, there exists a unique natural transformation $\text{מ} \colon \mathscr{E} \Rightarrow \mathscr{G}$ with $\text{ד} = \text{מ}_{\mathscr{K}_{\mathcal{A}}} \text{צ}$.*

Let $\mathscr{L}_{\widehat{\mathcal{A}}} \colon \widehat{\mathcal{A}} \to \widetilde{\mathcal{A}}$ denote the inclusion functor of the subcategory $\widehat{\mathcal{A}}$ into $\widetilde{\mathcal{A}}$. The functor $\mathscr{I}_{\mathcal{A}} \colon \mathcal{A} \to \widetilde{\mathcal{A}}$ factors through $\mathscr{K}_{\mathcal{A}}$ as $\mathscr{I}_{\mathcal{A}} = \mathscr{L}_{\widehat{\mathcal{A}}}\mathscr{K}_{\mathcal{A}}$. An additive functor $\mathscr{F} \colon \widehat{\mathcal{A}} \to \mathcal{B}$ to a weakly idempotent complete category $\mathcal{B}$ is determined up to unique natural isomorphism by its behaviour on the image $\mathscr{K}_{\mathcal{A}}(\mathcal{A})$ of $\mathcal{A}$ in $\widehat{\mathcal{A}}$; similarly, a natural transformation $\text{ב} \colon \mathscr{F} \Rightarrow \mathscr{G}$ of additive functors $\widehat{\mathcal{A}} \to \mathcal{B}$ is also completely determined by its action on $\mathscr{K}_{\mathcal{A}}(\mathcal{A})$; see [Büh10, Rems. 6.7, 6.9].

Remark 2.14. In [Büh10, Rem. 7.9], it is remarked that there is a subtle set-theoretic issue regarding the existence of the weak idempotent completion of an additive category. Let NBG denote von Neumann-Bernays-Gödel class theory (see Fraenkel–Bar-Hillel–Levy [FBHL73, p. 128]), and let (AGC) denote the Axiom of Global Choice [FBHL73, p. 133]. The combination NBG + (AGC) is a conservative extension of ZFC [FBHL73, p. 131–132, 134]. If one chooses an appropriate class theory to work with, such as NBG + (AGC), then the weak idempotent completion always exists as a category. This would follow from the Axiom of Predicative Comprehension for Classes (see [FBHL73, p. 123]); this is also known as the Axiom of Separation (e.g. Smullyan–Fitting [SF96, p. 15]). Furthermore, a priori it is not clear to the authors if Propositions 2.8 and 2.13 follow in an arbitrary setting without (AGC). This is because in showing that, for example, an additive functor $\mathscr{F} \colon \widetilde{\mathcal{A}} \to \mathcal{B}$, where $\mathcal{B}$ is idempotent complete, is determined by its values on $\mathscr{I}_{\mathcal{A}}(\mathcal{A})$, one must *choose* a kernel and an image of the idempotent $\mathscr{F}(e)$ for each idempotent e in $\mathcal{A}$.

3 n-Exangulated categories, functors and natural transformations

Let $n \geq 1$ be an integer. In this section we recall the theory of n-exangulated categories established in [HLN21], n-exangulated functors as defined in [BTS21], and n-exangulated natural transformations as recently introduced in [BTHSS23]. We also use this opportunity to set up some notation.

3.1 n-Exangulated categories

The definitions in this subsection and more details can be found in [HLN21, Sec. 2]. For this subsection, suppose that $\mathcal{C}$ is an additive category and that $\mathbb{E}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Ab}$ is a biadditive functor.

Let A, C be objects in $\mathcal{C}$. We denote by ${}_A0_C$ the identity element of the abelian group $\mathbb{E}(C, A)$. Suppose $\delta \in \mathbb{E}(C, A)$ and that $a\colon A \to B$ and $d\colon D \to C$ are morphisms in $\mathcal{C}$. We put $a_{\mathbb{E}}\delta := \mathbb{E}(C, a)(\delta) \in \mathbb{E}(C, B)$ and $d^{\mathbb{E}}\delta := \mathbb{E}(d, A)(\delta) \in \mathbb{E}(D, A)$. Since $\mathbb{E}$ is a bifunctor, we have that $d^{\mathbb{E}}a_{\mathbb{E}}\delta = \mathbb{E}(d, a)(\delta) = a_{\mathbb{E}}d^{\mathbb{E}}\delta$.

An $\mathbb{E}$-*extension* is an element $\delta \in \mathbb{E}(C, A)$ for some $A, C \in \mathcal{C}$. A *morphism of* $\mathbb{E}$-*extensions* from $\delta \in \mathbb{E}(C, A)$ to $\rho \in \mathbb{E}(D, B)$ is given by a pair (a, c) of morphisms $a\colon A \to B$ and $c\colon C \to D$ in $\mathcal{C}$ such that $a_{\mathbb{E}}\delta = c^{\mathbb{E}}\rho$.

Let $A \xleftarrow{p_A} A \oplus B \xrightarrow{p_B} B$ be a product and $C \xrightarrow{i_C} C \oplus D \xleftarrow{i_D} D$ be a coproduct in $\mathcal{C}$, and let $\delta \in \mathbb{E}(C, A)$ and $\rho \in \mathbb{E}(D, B)$ be $\mathbb{E}$-extensions. The *direct sum of* δ *and* ρ is the unique $\mathbb{E}$-extension $\delta \oplus \rho \in \mathbb{E}(C \oplus D, A \oplus B)$ such that the following equations hold.

$$\begin{aligned}
\mathbb{E}(i_C, p_A)(\delta \oplus \rho) &= \delta \\
\mathbb{E}(i_C, p_B)(\delta \oplus \rho) &= {}_B0_C \\
\mathbb{E}(i_D, p_A)(\delta \oplus \rho) &= {}_A0_D \\
\mathbb{E}(i_D, p_B)(\delta \oplus \rho) &= \rho
\end{aligned}$$

From the Yoneda Lemma, each $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$ induces two natural transformations. The first is ${}_{\mathbb{E}}\delta\colon \mathcal{C}(A, -) \Rightarrow \mathbb{E}(C, -)$ given by ${}_{\mathbb{E}}\delta_B(a) := a_{\mathbb{E}}\delta$ for all objects $B \in \mathcal{C}$ and all morphisms $a\colon A \to B$. The second is ${}^{\mathbb{E}}\delta\colon \mathcal{C}(-, C) \Rightarrow \mathbb{E}(-, A)$ and defined by ${}^{\mathbb{E}}\delta_D(d) := d^{\mathbb{E}}\delta$ for all objects $D \in \mathcal{C}$ and all morphisms $d\colon D \to C$.

Let $\mathsf{Ch}(\mathcal{C})$ be the category of complexes in $\mathcal{C}$. Its full subcategory consisting of complexes concentrated in degrees $0, 1, \ldots, n, n+1$ is denoted $\mathsf{Ch}(\mathcal{C})^n$. If $X_\bullet \in \mathsf{Ch}(\mathcal{C})^n$, we depict $X_\bullet$ as

$$X_0 \xrightarrow{d_0^X} X_1 \xrightarrow{d_1^X} \cdots \xrightarrow{d_{n-1}^X} X_n \xrightarrow{d_n^X} X_{n+1},$$

omitting the trails of zeroes at each end.

Definition 3.1. Let $X_\bullet, Y_\bullet \in \mathsf{Ch}(\mathcal{C})^n$ be complexes, and suppose that $\delta \in \mathbb{E}(X_{n+1}, X_0)$ and $\rho \in \mathbb{E}(Y_{n+1}, Y_0)$ are $\mathbb{E}$-extensions.

(i) The pair $\langle X_\bullet, \delta \rangle$ is known as an $\mathbb{E}$-*attached complex* if $(d_0^X)_{\mathbb{E}}\delta = 0$ and $(d_n^X)^{\mathbb{E}}\delta = 0$. An $\mathbb{E}$-attached complex $\langle X_\bullet, \delta \rangle$ is called an n-*exangle* (for $(\mathcal{C}, \mathbb{E})$) if, further, the sequences

$$\mathcal{C}(-,X_0) \xRightarrow{\mathcal{C}(-,d_0^X)} \mathcal{C}(-,X_1) \xRightarrow{\mathcal{C}(-,d_1^X)} \cdots \xRightarrow{\mathcal{C}(-,d_n^X)} \mathcal{C}(-,X_{n+1}) \xRightarrow{\mathbb{E}\delta} \mathbb{E}(-,X_0)$$

and

$$\mathcal{C}(X_{n+1},-) \xRightarrow{\mathcal{C}(d_n^X,-)} \mathcal{C}(X_n,-) \xRightarrow{\mathcal{C}(d_{n-1}^X,-)} \cdots \xRightarrow{\mathcal{C}(d_0^X,-)} \mathcal{C}(X_0,-) \xRightarrow{\mathbb{E}^\delta} \mathbb{E}(X_{n+1},-)$$

of functors are exact.

(ii) A *morphism* $f_\bullet\colon \langle X_\bullet,\delta\rangle \to \langle Y_\bullet,\rho\rangle$ *of* $\mathbb{E}$*-attached complexes* is given by a morphism $f_\bullet \in \mathsf{Ch}(\mathcal{C})^n(X_\bullet,Y_\bullet)$ such that $(f_0)_{\mathbb{E}}\delta = (f_{n+1})^{\mathbb{E}}\rho$. Such an $f_\bullet$ is called a *morphism of* n*-exangles* if $\langle X_\bullet,\delta\rangle$ and $\langle Y_\bullet,\rho\rangle$ are both n-exangles.

(iii) The *direct sum of the* $\mathbb{E}$*-attached complexes* (or *the* n*-exangles*) $\langle X_\bullet,\delta\rangle$ and $\langle Y_\bullet,\rho\rangle$ is the pair $\langle X_\bullet \oplus Y_\bullet, \delta\oplus\rho\rangle$.

From the definition above, one can form the additive category of $\mathbb{E}$-attached complexes, and its additive full subcategory of n-exangles.

Given a pair of objects $A,C\in\mathcal{C}$, we define a subcategory $\mathsf{Ch}(\mathcal{C})^n_{(A,C)}$ of $\mathsf{Ch}(\mathcal{C})^n$ in the following way. An object $X_\bullet \in \mathsf{Ch}(\mathcal{C})^n_{(A,C)}$ is an object of $\mathsf{Ch}(\mathcal{C})^n$ that satisfies $X_0 = A$ and $X_{n+1} = C$. For $X_\bullet, Y_\bullet \in \mathsf{Ch}(\mathcal{C})^n_{(A,C)}$, a morphism $f_\bullet \in \mathsf{Ch}(\mathcal{C})^n_{(A,C)}(X_\bullet,Y_\bullet)$ is a morphism $f_\bullet = (f_0,\dots,f_{n+1}) \in \mathsf{Ch}(\mathcal{C})^n(X_\bullet,Y_\bullet)$ with $f_0 = \mathrm{id}_A$ and $f_{n+1} = \mathrm{id}_C$. Note that this implies $\mathsf{Ch}(\mathcal{C})^n_{(A,C)}$ is not necessarily a full subcategory of $\mathsf{Ch}(\mathcal{C})^n$, nor necessarily additive.

Let $X_\bullet, Y_\bullet \in \mathsf{Ch}(\mathcal{C})^n_{(A,C)}$ be complexes. Two morphisms in $\mathsf{Ch}(\mathcal{C})^n_{(A,C)}(X_\bullet,Y_\bullet)$ are said to be *homotopic* if they are homotopic in the standard sense viewed as morphisms in $\mathsf{Ch}(\mathcal{C})^n$. This induces an equivalence relation $\sim$ on $\mathsf{Ch}(\mathcal{C})^n_{(A,C)}(X_\bullet,Y_\bullet)$. We define $\mathsf{K}(\mathcal{C})^n_{(A,C)}$ as the category with the same objects as $\mathsf{Ch}(\mathcal{C})^n_{(A,C)}$ and with

$$\mathsf{K}(\mathcal{C})^n_{(A,C)}(X_\bullet,Y_\bullet) := \mathsf{Ch}(\mathcal{C})^n_{(A,C)}(X_\bullet,Y_\bullet)/\!\sim.$$

A morphism $f_\bullet \in \mathsf{Ch}(\mathcal{C})^n_{(A,C)}(X_\bullet,Y_\bullet)$ is called a *homotopy equivalence* if its image in the category $\mathsf{K}(\mathcal{C})^n_{(A,C)}(X_\bullet,Y_\bullet)$ is an isomorphism. In this case, $X_\bullet$ and $Y_\bullet$ are said to be *homotopy equivalent*. The isomorphism class of $X_\bullet$ in $\mathsf{K}(\mathcal{C})^n_{(A,C)}$ (equivalently, its homotopy class in $\mathsf{Ch}(\mathcal{C})^n_{(A,C)}$) is denoted $[X_\bullet]$. Since the (usual) homotopy class of $X_\bullet$ in $\mathsf{Ch}(\mathcal{C})$ may differ from its homotopy class in $\mathsf{Ch}(\mathcal{C})^n_{(A,C)}$, we reserve the notation $[X_\bullet]$ specifically for its isomorphism class in $\mathsf{K}(\mathcal{C})^n_{(A,C)}$.

Notation 3.2. For $X\in\mathcal{C}$ and $i\in\{0,\dots,n\}$, we denote by $\mathrm{triv}_i(X)_\bullet$ the object in $\mathsf{Ch}(\mathcal{C})^n$ given by $\mathrm{triv}_i(X)_j = X$ for $j = i, i+1$ and $\mathrm{triv}_i(X)_j = 0$ for $0\le j\le i-1$ and $i+2\le j\le n+1$, as well as $d_i^{\mathrm{triv}_i(X)} = \mathrm{id}_X$.

Definition 3.3. Let $\mathfrak{s}$ be an assignment that, for each pair of objects $A,C\in\mathcal{C}$ and each $\mathbb{E}$-extension $\delta\in\mathbb{E}(C,A)$, associates to δ an isomorphism class $\mathfrak{s}(\delta) = [X_\bullet]$ in $\mathsf{K}(\mathcal{C})^n_{(A,C)}$. The correspondence $\mathfrak{s}$ is called an *exact realisation of* $\mathbb{E}$ if it satisfies the following conditions.

(R0) For any morphism $(a,c)\colon \delta\to\rho$ of $\mathbb{E}$-extensions with $\delta\in\mathbb{E}(C,A)$, $\rho\in\mathbb{E}(D,B)$, $\mathfrak{s}(\delta) = [X_\bullet]$ and $\mathfrak{s}(\rho) = [Y_\bullet]$, there exists $f_\bullet \in \mathsf{Ch}(\mathcal{C})^n(X_\bullet,Y_\bullet)$ such that $f_0 = a$ and $f_{n+1} = c$. In this setting, we say that $X_\bullet$ *realises* δ and $f_\bullet$ is a *lift* of (a,c).

(R1) If $\mathfrak{s}(\delta) = [X_\bullet]$, then $\langle X_\bullet, \delta\rangle$ is an n-exangle.

(R2) For each object $A \in \mathcal{C}$, we have $\mathfrak{s}({}_A 0_0) = [\mathrm{triv}_0(A)_\bullet]$ and $\mathfrak{s}({}_0 0_A) = [\mathrm{triv}_n(A)_\bullet]$.

In case $\mathfrak{s}$ is an exact realisation of $\mathbb{E}$ and $\mathfrak{s}(\delta) = [X_\bullet]$, the following terminology is used. The morphism d_0^X is said to be an $\mathfrak{s}$-*inflation* and the morphism d_n^X an $\mathfrak{s}$-*deflation*. The pair $\langle X_\bullet, \delta\rangle$ is known as an $\mathfrak{s}$-*distinguished* n-exangle.

Suppose $\mathfrak{s}$ is an exact realisation of $\mathbb{E}$ and $\mathfrak{s}(\delta) = [X_\bullet]$. We will often use the diagram

$$X_0 \xrightarrow{d_0^X} X_1 \xrightarrow{d_1^X} \cdots \xrightarrow{d_{n-1}^X} X_n \xrightarrow{d_n^X} X_{n+1} \overset{\delta}{\dashrightarrow}$$

to express that $\langle X_\bullet, \delta\rangle$ is an $\mathfrak{s}$-distinguished n-exangle. If we also have that $\mathfrak{s}(\rho) = [Y_\bullet]$ and $f_\bullet\colon \langle X_\bullet, \delta\rangle \to \langle Y_\bullet, \rho\rangle$ is a morphism of n-exangles, then we call $f_\bullet$ a *morphism of* $\mathfrak{s}$-*distinguished* n-*exangles* and we depict this by the following commutative diagram.

$$\begin{tikzcd}
X_0 \arrow[r, "d_0^X"] \arrow[d, "f_0"] & X_1 \arrow[r, "d_1^X"] \arrow[d, "f_1"] & \cdots \arrow[r, "d_{n-1}^X"] & X_n \arrow[r, "d_n^X"] \arrow[d, "f_n"] & X_{n+1} \arrow[r, dashed, "\delta"] \arrow[d, "f_{n+1}"] & {} \\
Y_0 \arrow[r, "d_0^Y"] & Y_1 \arrow[r, "d_1^Y"] & \cdots \arrow[r, "d_{n-1}^Y"] & Y_n \arrow[r, "d_n^Y"] & Y_{n+1} \arrow[r, dashed, "\rho"] & {}
\end{tikzcd}$$

We need one last definition before being able to define an n-exangulated category.

Definition 3.4. Suppose $f_\bullet\colon X_\bullet \to Y_\bullet$ is a morphism in $\mathsf{Ch}(\mathcal{C})^n$, such that $f_0 = \mathrm{id}_A$ for some $A = X_0 = Y_0$. The *mapping cone* $M_{f_\bullet}^{\mathcal{C}} \in \mathsf{Ch}(\mathcal{C})^n$ of $f_\bullet$ is the complex

$$X_1 \xrightarrow{d_0^{M_f^{\mathcal{C}}}} X_2 \oplus Y_1 \xrightarrow{d_1^{M_f^{\mathcal{C}}}} X_3 \oplus Y_2 \xrightarrow{d_2^{M_f^{\mathcal{C}}}} \cdots \xrightarrow{d_{n-1}^{M_f^{\mathcal{C}}}} X_{n+1} \oplus Y_n \xrightarrow{d_n^{M_f^{\mathcal{C}}}} Y_{n+1},$$

with $d_0^{M_f^{\mathcal{C}}} := [\, -d_1^X \;\; f_1 \,]^\top$, $d_n^{M_f^{\mathcal{C}}} := [\, f_{n+1} \;\; d_n^Y \,]$, and $d_i^{M_f^{\mathcal{C}}} := \begin{bmatrix} -d_{i+1}^X & 0 \\ f_{i+1} & d_i^Y \end{bmatrix}$ for $i \in \{1, \ldots, n-1\}$.

We are in position to state the main definition of this subsection.

Definition 3.5. An n-*exangulated category* is a triplet $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, consisting of an additive category $\mathcal{C}$, a biadditive functor $\mathbb{E}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Ab}$ and an exact realisation $\mathfrak{s}$ of $\mathbb{E}$, such that the following conditions are met.

(EA1) The collection of $\mathfrak{s}$-inflations is closed under composition. Dually, the collection of $\mathfrak{s}$-deflations is closed under composition.

(EA2) Suppose $\delta \in \mathbb{E}(D, A)$ and $c \in \mathcal{C}(C, D)$. If $\mathfrak{s}(c^{\mathbb{E}}\delta) = [Y_\bullet]$ and $\mathfrak{s}(\delta) = [X_\bullet]$, then there exists a morphism $f_\bullet\colon Y_\bullet \to X_\bullet$ lifting (id_A, c), such that $\mathfrak{s}((d_0^Y)_{\mathbb{E}}\delta) = [M_{f_\bullet}^{\mathcal{C}}]$. In this case, the morphism $f_\bullet$ is called a *good lift of* (id_A, c).

(EA2)$^{\mathrm{op}}$ The dual of (EA2).

Notice that the definition of an n-exangulated category is self-dual. In particular, the dual statements of several results in Sections 4–5 are used without proof.

3.2 n-Exangulated functors and natural transformations

In order to show that the canonical functor from an n-exangulated category $(\mathcal{C},\mathbb{E},\mathfrak{s})$ to its idempotent completion is 2-universal among structure-preserving functors from $(\mathcal{C},\mathbb{E},\mathfrak{s})$ to idempotent complete n-exangulated categories, we will need the notion of a morphism of n-exangulated categories and that of a morphism between such morphisms.

For this subsection, suppose $(\mathcal{C},\mathbb{E},\mathfrak{s})$, $(\mathcal{C}',\mathbb{E}',\mathfrak{s}')$ and $(\mathcal{C}'',\mathbb{E}'',\mathfrak{s}'')$ are n-exangulated categories. If $\mathscr{F}\colon\mathcal{C}\to\mathcal{C}'$ is an additive functor, then it induces several other additive functors, e.g. $\mathscr{F}^{\mathrm{op}}\colon\mathcal{C}^{\mathrm{op}}\to(\mathcal{C}')^{\mathrm{op}}$, or $\mathscr{F}_{\mathsf{Ch}}\colon\mathsf{Ch}(\mathcal{C})\to\mathsf{Ch}(\mathcal{C}')$ and obvious restrictions thereof. These are all defined in the usual way. However, by abuse of notation, we simply write $\mathscr{F}$ for each of these.

Definition 3.6. [BTS21, Def. 2.32] Suppose that $\mathscr{F}\colon\mathcal{C}\to\mathcal{C}'$ is an additive functor and that $\Gamma\colon\mathbb{E}(-,-)\Rightarrow\mathbb{E}'(\mathscr{F}-,\mathscr{F}-)$ is a natural transformation of functors $\mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathsf{Ab}$. The pair $(\mathscr{F},\Gamma)\colon(\mathcal{C},\mathbb{E},\mathfrak{s})\to(\mathcal{C}',\mathbb{E}',\mathfrak{s}')$ is called an *n-exangulated functor* if, for all $A,C\in\mathcal{C}$ and each $\delta\in\mathbb{E}(A,C)$, we have that $\mathfrak{s}'(\Gamma_{(C,A)}(\delta))=[\mathscr{F}(X_\bullet)]$ whenever $\mathfrak{s}(\delta)=[X_\bullet]$.

If we have a sequence $(\mathcal{C},\mathbb{E},\mathfrak{s})\xrightarrow{(\mathscr{F},\Gamma)}(\mathcal{C}',\mathbb{E}',\mathfrak{s}')\xrightarrow{(\mathscr{L},\Phi)}(\mathcal{C}'',\mathbb{E}'',\mathfrak{s}'')$ of n-exangulated functors, then the *composite* of $(\mathscr{F},\Gamma)$ and $(\mathscr{L},\Phi)$ is defined to be

$$(\mathscr{L},\Phi)\circ(\mathscr{F},\Gamma):=(\mathscr{L}\circ\mathscr{F},\Phi_{\mathscr{F}\times\mathscr{F}}\circ\Gamma).$$

This is an n-exangulated functor $(\mathcal{C},\mathbb{E},\mathfrak{s})\to(\mathcal{C}'',\mathbb{E}'',\mathfrak{s}'')$; see [BTHSS23, Lem. 3.19(ii)].

The next result implies that n-exangulated functors preserve finite direct sum decompositions of distinguished n-exangles. It will be used in the main result of Subsection 4.5.

Proposition 3.7. *Let $\mathscr{F}\colon\mathcal{C}\to\mathcal{C}'$ be an additive functor and $\Gamma\colon\mathbb{E}(-,-)\Rightarrow\mathbb{E}'(\mathscr{F}-,\mathscr{F}-)$ a natural transformation. Suppose $\delta\in\mathbb{E}(C,A)$ and $\rho\in\mathbb{E}(D,B)$ are $\mathbb{E}$-extensions, and $\langle X_\bullet,\delta\rangle$ and $\langle Y_\bullet,\rho\rangle$ are $\mathfrak{s}$-distinguished.*

(i) *If $f_\bullet\colon\langle X_\bullet,\delta\rangle\to\langle Y_\bullet,\rho\rangle$ is a morphism of $\mathbb{E}$-attached complexes, then the induced morphism $\mathscr{F}(f_\bullet)\colon\langle\mathscr{F}(X_\bullet),\Gamma_{(C,A)}(\delta)\rangle\to\langle\mathscr{F}(Y_\bullet),\Gamma_{(D,B)}(\rho)\rangle$ is a morphism of $\mathbb{E}'$-attached complexes.*

(ii) *We have $\langle\mathscr{F}(X_\bullet\oplus Y_\bullet),\Gamma_{(C\oplus D,A\oplus B)}(\delta\oplus\rho)\rangle\cong\langle\mathscr{F}(X_\bullet),\Gamma_{(C,A)}(\delta)\rangle\oplus\langle\mathscr{F}(Y_\bullet),\Gamma_{(D,B)}(\rho)\rangle$ as $\mathbb{E}'$-attached complexes.*

Proof. (i) Note that $(\mathscr{F}(d_0^X))_{\mathbb{E}'}(\Gamma_{(X_{n+1},X_0)}(\delta))=\Gamma_{(X_{n+1},X_1)}((d_0^X)_{\mathbb{E}}\delta)={}_{\mathscr{F}(X_1)}0_{\mathscr{F}(X_{n+1})}$ since Γ is natural and $\langle X_\bullet,\delta\rangle$ is an $\mathbb{E}$-attached complex. Similar computations show that both $\langle\mathscr{F}(X_\bullet),\Gamma_{(C,A)}(\delta)\rangle$ and $\langle\mathscr{F}(Y_\bullet),\Gamma_{(D,B)}(\rho)\rangle$ are $\mathbb{E}'$-attached complexes. As $\mathscr{F}(f_\bullet)$ is a morphism $\mathscr{F}(X_\bullet)\to\mathscr{F}(Y_\bullet)$ of complexes, it suffices to prove

$$\mathscr{F}(f_0)_{\mathbb{E}'}\Gamma_{(X_{n+1},X_0)}(\delta)=\mathscr{F}(f_{n+1})^{\mathbb{E}'}\Gamma_{(Y_{n+1},Y_0)}(\rho).$$

This follows immediately from $(f_0)_{\mathbb{E}}\delta=(f_{n+1})^{\mathbb{E}}\rho$ and the naturality of Γ.

(ii) This follows from applying (i) to the morphisms in the appropriate biproduct diagram of $\mathbb{E}$-attached complexes. □

Lastly, we recall the notion of a morphism of n-exangulated functors. The extriangulated version was defined in Nakaoka–Ogawa–Sakai [NOS22, Def. 2.11(3)].

Definition 3.8. [BTHSS23, Def. 4.1] Suppose $(\mathscr{F},\Gamma),(\mathscr{G},\Lambda)\colon(\mathcal{C},\mathbb{E},\mathfrak{s})\to(\mathcal{C}',\mathbb{E}',\mathfrak{s}')$ are n-exangulated functors. A natural transformation $\beth\colon\mathscr{F}\Rightarrow\mathscr{G}$ of functors is said to be *n-exangulated* if, for all $A,C\in\mathcal{C}$ and each $\delta\in\mathbb{E}(C,A)$, we have

$$(\beth_A)_{\mathbb{E}'}\Gamma_{(C,A)}(\delta)=(\beth_C)^{\mathbb{E}'}\Lambda_{(C,A)}(\delta). \tag{3.1}$$

We denote this by $\beth\colon(\mathscr{F},\Gamma)\Rightarrow(\mathscr{G},\Lambda)$. In addition, if $\beth$ has an n-exangulated inverse, then it is called an *n-exangulated natural isomorphism*. It is straightforward to check that $\beth$ has an n-exangulated inverse if and only if $\beth_X$ is an isomorphism for each $X\in\mathcal{C}$.

4 The idempotent completion of an n-exangulated category

Throughout this section we work with the following setup.

Setup 4.1. Let $n \geq 1$ be an integer. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an n-exangulated category. We denote by $\mathscr{I}_{\mathcal{C}}$ the inclusion of the category $\mathcal{C}$ into its idempotent completion $\widetilde{\mathcal{C}}$; see Section 2.

In this section, we will construct a biadditive functor $\mathbb{F}\colon \widetilde{\mathcal{C}}^{\mathrm{op}} \times \widetilde{\mathcal{C}} \to \mathsf{Ab}$ (see Subsection 4.1) and an exact realisation $\mathfrak{t}$ of $\mathbb{F}$ (see Subsection 4.2), and then show that $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$ is an n-exangulated category (see Subsections 4.3–4.5). For $n = 1$, we recover the main results of [Msa22]. First, we establish some notation to help our exposition.

Notation 4.2. We reserve notation with a tilde for objects and morphisms in $\widetilde{\mathcal{C}}$.

(i) If $\widetilde{X} \in \widetilde{\mathcal{A}}$ is some object, then we will denote the identity morphism of $\widetilde{X}$ by $\widetilde{\mathrm{id}}_{\widetilde{X}}$. Recall from Definition 2.3 that the identity of an object $(X, e) \in \widetilde{\mathcal{A}}$ is $\widetilde{\mathrm{id}}_{(X,e)} = (e, e, e)$.

(ii) Given a morphism $(e', r, e) \in \widetilde{\mathcal{C}}((X, e), (Y, e'))$, we call $r\colon X \to Y$ the *underlying morphism of* (e', r, e).

(iii) Suppose $(X, e), (Y, e') \in \widetilde{\mathcal{C}}$ and $r \in \mathcal{C}(X, Y)$ with $e'r = r = re$. Then there is a unique morphism $\tilde{r} \in \widetilde{\mathcal{C}}((X, e), (Y, e'))$ with underlying morphism r. This morphism $\tilde{r}$ is the triplet (e', r, e). Moreover, we will use this notation specifically for this correspondence. That is, we write $\tilde{s}\colon (X, e) \to (Y, e')$ is a morphism in $\widetilde{\mathcal{C}}$ if and only if we implicitly mean that the underlying morphism of $\tilde{s}$ is denoted s, i.e. we have $\tilde{s} = (e', s, e)$.

Remark 4.3. By Notation 4.2(iii), two morphisms $\tilde{r}, \tilde{s} \in \widetilde{\mathcal{C}}((X, e), (Y, e'))$ are equal if and only if their underlying morphisms r and s, respectively, are equal in $\mathcal{C}$. Thus, for all objects $\widetilde{X}, \widetilde{Y} \in \widetilde{\mathcal{C}}$, removing the tilde from morphisms in $\widetilde{\mathcal{C}}(\widetilde{X}, \widetilde{Y})$ defines an injective abelian group homomorphism $\widetilde{\mathcal{C}}(\widetilde{X}, \widetilde{Y}) \to \mathcal{C}(X, Y)$. In particular, a diagram in $\widetilde{\mathcal{C}}$ commutes if and only if its diagram of underlying morphisms commutes.

4.1 Defining the biadditive functor $\mathbb{F}$

The following construction is the higher version of the one given in [Msa22, Sec. 3.1] for extriangulated categories.

Definition 4.4. We define a functor $\mathbb{F}\colon \widetilde{\mathcal{C}}^{\mathrm{op}} \times \widetilde{\mathcal{C}} \to \mathsf{Ab}$ as follows. For objects (X_0, e_0) and (X_{n+1}, e_{n+1}) in $\widetilde{\mathcal{C}}$, we put

$$\mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0)) := \left\{ (e_0, \delta, e_{n+1}) \;\middle|\; \delta \in \mathbb{E}(X_{n+1}, X_0) \text{ and } (e_0)_{\mathbb{E}}\delta = \delta = (e_{n+1})^{\mathbb{E}}\delta \right\}.$$

For morphisms $\tilde{a}\colon (X_0, e_0) \to (Y_0, e'_0)$ and $\tilde{c}\colon (Z_{n+1}, e''_{n+1}) \to (X_{n+1}, e_{n+1})$ in $\widetilde{\mathcal{C}}$, we define

$$\begin{aligned} \mathbb{F}(\tilde{c}, \tilde{a})\colon \mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0)) &\longrightarrow \mathbb{F}((Z_{n+1}, e''_{n+1}), (Y_0, e'_0)) \\ (e_0, \delta, e_{n+1}) &\longmapsto (e'_0, \mathbb{E}(c, a)(\delta), e''_{n+1}). \end{aligned}$$

Remark 4.5. We make some comments on Definition 4.4.

(i) The assignment $\mathbb{F}$ on morphisms takes values where claimed due to the following. For morphisms $\tilde{a}\colon (X_0, e_0) \to (Y_0, e_0')$ and $\tilde{c}\colon (Z_{n+1}, e_{n+1}'') \to (X_{n+1}, e_{n+1})$, and an $\mathbb{F}$-extension $(e_0, \delta, e_{n+1}) \in \mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0))$, we have

$$\begin{aligned}\mathbb{E}(e_{n+1}'', e_0')\mathbb{E}(c, a)(\delta) &= \mathbb{E}(ce_{n+1}'', e_0'a)(\delta)\\ &= \mathbb{E}(c, a)(\delta).\end{aligned}$$

Therefore, $\mathbb{F}(\tilde{c}, \tilde{a})(e_0, \delta, e_{n+1}) = (e_{n+1}'', \mathbb{E}(c, a)(\delta), e_0')$ lies in $\mathbb{F}((Z_{n+1}, e_{n+1}''), (Y_0, e_0'))$. It is then straightforward to verify that $\mathbb{F}$ is indeed a functor.

(ii) The set $\mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0))$ is an abelian group by defining

$$(e_0, \delta, e_{n+1}) + (e_0, \rho, e_{n+1}) := (e_0, \delta + \rho, e_{n+1})$$

for $(e_0, \delta, e_{n+1}), (e_0, \rho, e_{n+1}) \in \mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0))$. The additive identity element of $\mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0))$ is ${}_{(X_0,e_0)}\widetilde{0}_{(X_{n+1},e_{n+1})} := (e_0, {}_{X_0}0_{X_{n+1}}, e_{n+1})$. The inverse of (e_0, δ, e_{n+1}) is $(e_0, -\delta, e_{n+1})$. Notice that we get an abelian group monomorphism:

$$\begin{aligned}\mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0)) &\longrightarrow \mathbb{E}(X_{n+1}, X_0)\\ (e_0, \delta, e_{n+1}) &\longmapsto \delta.\end{aligned}$$

This homomorphism plays a role later in the proof of Theorem 4.39.

(iii) It follows from the definition of $\mathbb{F}$ that it is biadditive since $\mathbb{E}$ is.

(iv) Given $(e_0, \delta, e_{n+1}) \in \mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0))$, the pair (e_0, e_{n+1}) is a morphism of $\mathbb{E}$-extensions $\delta \to \delta$. Indeed, we have that $(e_0)_{\mathbb{E}}\delta = \delta = (e_{n+1})^{\mathbb{E}}\delta$ from Definition 4.4.

Notation 4.6. As for objects and morphisms in $\widetilde{\mathcal{C}}$, we use tilde notation for $\mathbb{F}$-extensions, which gives us a way to pass back to $\mathbb{E}$-extensions.

(i) We will denote an $\mathbb{F}$-extension of the form $(e_0, \delta, e_{n+1}) \in \mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0))$ by $\tilde{\delta}$. We call $\delta \in \mathbb{E}(X_{n+1}, X_0)$ the *underlying* $\mathbb{E}$*-extension of* $\tilde{\delta}$.

(ii) For $(X_{n+1}, e_{n+1}), (X_0, e_0) \in \widetilde{\mathcal{C}}$ and $\delta \in \mathbb{E}(X_{n+1}, X_0)$ with $(e_0)_{\mathbb{E}}\delta = \delta = (e_{n+1})^{\mathbb{E}}\delta$, there exists a unique $\mathbb{F}$-extension $\tilde{\delta} \in \mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0))$ with underlying $\mathbb{E}$-extension δ. This $\mathbb{F}$-extension is $\tilde{\delta} = (e_0, \delta, e_{n+1})$. Again, we use this instance of the tilde notation for this correspondence: we write $\tilde{\rho} \in \mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0))$ if and only if the underlying $\mathbb{E}$-extension of $\tilde{\rho}$ is ρ, i.e. $\tilde{\rho} = (e_0, \rho, e_{n+1})$.

Remark 4.7. Analogously to our observations in Remark 4.3, we note that by Notation 4.6(ii) any two $\mathbb{F}$-extensions $\tilde{\delta}, \tilde{\rho} \in \mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0))$ are equal if and only if their underlying $\mathbb{E}$-extensions are equal. Hence, removing the tilde from $\mathbb{F}$-extensions defines an injective group homomorphism $\mathbb{F}((Y, e'), (X, e)) \to \mathbb{E}(Y, X)$ for $(X, e), (Y, e') \in \widetilde{\mathcal{C}}$.

4.2 Defining the realisation $\mathfrak{t}$

To define an exact realisation $\mathfrak{t}$ of the functor $\mathbb{F}$ defined in Section 4.1, given a morphism of extensions consisting of two idempotents, we will need to lift this morphism by an $(n+2)$-tuple of idempotents. That is, we require a higher version of the idempotent lifting trick (see [Msa22, Lem. 3.5] and [BS01, Lem. 1.13]). This turns out to be quite non-trivial and requires an abstraction of the case when $n = 1$ in order to understand the mechanics of why this trick is successful.

We start with two lemmas related to the polynomial ring $\mathbb{Z}[x]$. Recall that $\mathbb{Z}[x]$ has the universal property that for any (unital, associative) ring R and any element $r \in R$ there is a unique (identity preserving) ring homomorphism $\phi_r \colon \mathbb{Z}[x] \to R$ with $\phi_r(x) = r$. For $p = p(x) \in \mathbb{Z}[x]$, we denote $\phi_r(p)$ by $p(r)$ as is usual.

Lemma 4.8. *For each $m \in \mathbb{N}$, the ideals $(x^m) = (x)^m$ and $((x-1)^m) = (x-1)^m$ of $\mathbb{Z}[x]$ are coprime.*

Proof. The ideals $\sqrt{(x)^m} = (x)$ and $\sqrt{(x-1)^m} = (x-1)$ are coprime in $\mathbb{Z}[x]$. Hence, (x^m) and $((x-1)^m)$ are also coprime, by Atiyah–MacDonald [AM69, Prop. 1.16]. □

Lemma 4.9. *For each $m \in \mathbb{N}_{\geq 1}$, there is a polynomial $p_m \in (x^m) \trianglelefteq \mathbb{Z}[x]$, such that for every (unital, associative) ring R we have:*

(i) *$p_m(e) = e$ for each idempotent $e \in R$; and*

(ii) *the element $p_m(r) \in R$ is an idempotent for each $r \in R$ satisfying $(r^2 - r)^m = 0$.*

Proof. Fix an integer $m \geq 1$. By Lemma 4.8, we can write $1 = x^m p'_m + (x-1)^m q'_m$ for some polynomials p'_m and q'_m in $\mathbb{Z}[x]$. We set $p_m := x^m p'_m$.

Let R be a ring. For any idempotent $e \in R$, evaluating $x = x^{m+1}p'_m + x(x-1)^m q'_m$ at e and using $e(e-1) = 0$ yields $e = e^{m+1}p'_m(e) = e^m p'_m(e) = p_m(e)$, proving (i).

Now suppose $r \in R$ is an element with $(r^2 - r)^m = 0$. Evaluation of

$$p_m = (x^m p'_m) \cdot 1 = (x^m p'_m) \cdot (x^m p'_m + (x-1)^m q'_m) = p_m^2 + (x^2 - x)^m p'_m q'_m$$

at r shows $p_m(r)^2 = p_m(r)$ since $(r^2 - r)^m = 0$, which finishes the proof. □

The following is an abstract formulation of [Msa22, Lem. 3.5] and [BS01, Lem. 1.13].

Lemma 4.10. *Let $X_\bullet \colon\ X_0 \xrightarrow{d_0^X} X_1 \xrightarrow{d_1^X} X_2$ be a complex in an additive category $\mathcal{A}$ and suppose d_1^X is a weak cokernel of d_0^X. Suppose $(e_0, f_1, e_2) \colon X_\bullet \to X_\bullet$ is a morphism of complexes with $e_0 \in \operatorname{End}_{\mathcal{A}}(X_0)$ and $e_2 \in \operatorname{End}_{\mathcal{A}}(X_2)$ both idempotent. Then there exists a morphism $f'_1 \colon X_1 \to X_1$, such that the following hold.*

(i) *The triplet $(e_0, f'_1, e_2) \colon X_\bullet \to X_\bullet$ is a morphism of complexes.*

(ii) *The element $e_1 := f_1 f'_1 \in \operatorname{End}_{\mathcal{A}}(X_1)$ is idempotent and satisfies $e_1 = f'_1 f_1$.*

(iii) *The triplet $(e_0, e_1, e_2) \colon X_\bullet \to X_\bullet$ is an idempotent morphism of complexes.*

(iv) *If $(h_1, h_2) \colon (e_0, f_1, e_2) \sim 0_\bullet$ is a homotopy of morphisms $X_\bullet \to X_\bullet$, then the pair $(e_0 h_1, f'_1 h_2)$ yields a homotopy $(e_0, e_1, e_2) \sim 0_\bullet$.*

Proof. Choose a polynomial $p_2 = x^2 p_2' \in (x^2) \trianglelefteq \mathbb{Z}[x]$ as obtained in Lemma 4.9. Define $q := xp_2'$ and set $f_1' := q(f_1)\colon X_1 \to X_1$. We show this morphism satisfies the claims in the statement. For this, we will make use of the following. Let $p = p(x) \in \mathbb{Z}[x]$ be any polynomial. Since $(e_0, f_1, e_2)\colon X_\bullet \to X_\bullet$ is a morphism of complexes, we have that $(p(e_0), p(f_1), p(e_2))\colon X_\bullet \to X_\bullet$ is also a morphism of complexes, i.e. the diagram

$$\begin{tikzcd} X_0 \arrow[r, "d_0^X"] \arrow[d, "p(e_0)"'] & X_1 \arrow[r, "d_1^X"] \arrow[d, "p(f_1)"] & X_2 \arrow[d, "p(e_2)"] \\ X_0 \arrow[r, "d_0^X"'] & X_1 \arrow[r, "d_1^X"'] & X_2 \end{tikzcd} \tag{4.1}$$

commutes.

(i) Note that $q(e_0) = e_0 p_2'(e_0) = e_0^2 p_2'(e_0) = p_2(e_0) = e_0$, where the last equality follows from Lemma 4.9(i). Similarly, $q(e_2) = e_2$. Thus, using $p = q$ in the commutative diagram (4.1) shows that $(e_0, f_1', e_2) = (q(e_0), q(f_1), q(e_2))\colon X_\bullet \to X_\bullet$ is a morphism of complexes.

(ii) Since $f_1' = q(f_1)$ is a polynomial in f_1, we immediately have that $e_1 := f_1 f_1' = f_1' f_1$. Furthermore, we see that $e_1 = f_1 q(f_1) = p_2(f_1)$. Thus, to show that e_1 is idempotent, it is enough to show that $(f_1^2 - f_1)^2 = 0$ by Lemma 4.9(ii). Let $r(x) = x^2 - x$. We see that $r(e_0)$ and $r(e_2)$ vanish as e_0 and e_2 are idempotents. Therefore, by choosing $p = r$ in (4.1) we have $r(f_1)d_0^X = 0$ and so there is $h\colon X_2 \to X_1$ with $hd_1^X = r(f_1)$, because d_1^X is a weak cokernel of d_0^X. This implies $(f_1^2 - f_1)^2 = r(f_1)^2 = hd_1^X r(f_1) = hr(e_2)d_1^X = 0$ as $r(e_2) = 0$, and hence e_1 is idempotent.

(iii) Note that $(e_0, e_1, e_2)^2 = (e_0, e_1, e_2) = (p_2(e_0), p_2(f_1), p_2(e_2))\colon X_\bullet \to X_\bullet$ is a morphism of complexes using $p = p_2$ in (4.1).

(iv) Suppose $(h_1, h_2)\colon (e_0, f_1, e_2) \sim 0_\bullet$ is a homotopy. Then we see that

$$\begin{aligned}
(e_0, e_1, e_2) &= (e_0, f_1', e_2)(e_0, f_1, e_2) && \text{by (ii)} \\
&= (e_0, f_1', e_2)(h_1 d_0^X, h_2 d_1^X + d_0^X h_1, d_1^X h_2) && \text{as } (h_1, h_2)\colon (e_0, f_1, e_2) \sim 0_\bullet \\
&= (e_0 h_1 d_0^X, f_1' h_2 d_1^X + f_1' d_0^X h_1, e_2 d_1^X h_2) \\
&= ((e_0 h_1) d_0^X, (f_1' h_2) d_1^X + d_0^X (e_0 h_1), d_1^X (f_1' h_2)) && \text{by (i)}.
\end{aligned}$$

Hence, $(e_0 h_1, f_1' h_2)\colon (e_0, e_1, e_2) \sim 0_\bullet$ is a null homotopy as desired. □

Remark 4.11. Let $p_2' = -2x + 3$ and $q_2' = 2x + 1$. Then indeed $1 = x^2 p_2' + (x-1)^2 q_2'$. Hence, $p_2 = x^2 p_2' = 3x^2 - 2x^3$ is a possible choice for $m = 2$ in Lemma 4.9. Letting $h = x^2 - x$ and $i = x$, we see that $p_2 = i + h - 2ih$. Then the idempotent e_1 obtained in Lemma 4.10 is the idempotent obtained through the idempotent lifting trick in [Msa22, Lem. 3.5].

Lemma 4.12. *Suppose $\langle X_\bullet, \delta \rangle$ is an $\mathfrak{s}$-distinguished n-exangle and $e_0 \in \operatorname{End}_{\mathcal{C}}(X_0)$ is an idempotent with $(e_0)_{\mathbb{E}}\delta = 0$. Then e_0 can be extended to a null homotopic, idempotent morphism $e_\bullet\colon \langle X_\bullet, \delta \rangle \to \langle X_\bullet, \delta \rangle$ with $e_i = 0$ for $2 \leq i \leq n+1$. Further, the null homotopy of $e_\bullet$ can be chosen to be of the shape $h_\bullet = (h_1, 0, \ldots, 0)\colon e_\bullet \sim 0_\bullet$.*

Proof. We have $(e_0)_{\mathbb{E}}\delta = 0 = 0^{\mathbb{E}}\delta$ so $(e_0, 0)\colon \delta \to \delta$ is a morphism of $\mathbb{E}$-extensions. The solid morphisms of the diagram

$$\begin{tikzcd}
X_0 \arrow[r, "d_0^X"] \arrow[d, "e_0"] & X_1 \arrow[r, "d_1^X"] \arrow[d, dotted, "f_1"] & X_2 \arrow[r, "d_2^X"] \arrow[d, "0"] & \cdots \arrow[r, "d_{n-2}^X"] & X_{n-1} \arrow[r, "d_{n-1}^X"] \arrow[d, "0"] & X_n \arrow[r, "d_n^X"] \arrow[d, "0"] & X_{n+1} \arrow[r, dashed, "\delta"] \arrow[d, "0"] & {} \\
X_0 \arrow[r, "d_0^X"] & X_1 \arrow[r, "d_1^X"] & X_2 \arrow[r, "d_2^X"] & \cdots \arrow[r, "d_{n-2}^X"] & X_{n-1} \arrow[r, "d_{n-1}^X"] & X_n \arrow[r, "d_n^X"] & X_{n+1} \arrow[r, dashed, "\delta"] & {}
\end{tikzcd}$$

clearly commute, so we need to find a morphism $f_1\colon X_1 \to X_1$ making the two leftmost squares commute. Since $\langle X_\bullet, \delta\rangle$ is an $\mathfrak{s}$-distinguished n-exangle, there is an exact sequence

$$\mathcal{C}(X_1, X_0) \xrightarrow{\mathcal{C}(d_0^X, X_0)} \mathcal{C}(X_0, X_0) \xrightarrow{\ {}_{\mathbb{E}}\delta\ } \mathbb{E}(X_{n+1}, X_0).$$

The morphism e_0 is in the kernel of ${}_{\mathbb{E}}\delta$ as ${}_{\mathbb{E}}\delta(e_0) = (e_0)_{\mathbb{E}}\delta = 0$. Therefore, there exists $k_1\colon X_1 \to X_0$ with $e_0 = k_1 d_0^X$. If we put $f_1 := d_0^X k_1$, then $(e_0, f_1, 0, \dots, 0)\colon \langle X_\bullet, \delta\rangle \to \langle X_\bullet, \delta\rangle$ is morphism of $\mathfrak{s}$-distinguished n-exangles and $(k_1, 0, \dots, 0)\colon e_\bullet \sim 0_\bullet$ is a homotopy. By Lemma 4.10, using that e_0 and 0 are idempotents, there is an idempotent $e_1 \in \mathrm{End}_{\mathcal{C}}(X_1)$, such that $(e_0, e_1, 0, \dots, 0)\colon X_\bullet \to X_\bullet$ is an idempotent morphism of complexes and that $h_\bullet := (e_0 k_1, 0, \dots, 0)\colon e_\bullet \sim 0_\bullet$ is a homotopy. Finally, $e_\bullet$ is a morphism of $\mathfrak{s}$-distinguished n-exangles since $(e_0)_{\mathbb{E}}\delta = 0 = 0^{\mathbb{E}}\delta$. □

Corollary 4.13. *Suppose $\tilde{\delta} \in \mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0))$ and suppose that $\langle X_\bullet, \delta\rangle$ is an $\mathfrak{s}$-distinguished n-exangle. The morphism $(e_0, e_{n+1})\colon \delta \to \delta$ of $\mathbb{E}$-extensions has a lift $e_\bullet\colon \langle X_\bullet, \delta\rangle \to \langle X_\bullet, \delta\rangle$ that is idempotent and satisfies $e_i = \mathrm{id}_{X_i}$ for all $2 \le i \le n-1$, such that there is a homotopy $h_\bullet = (h_1, 0, \dots, 0, h_{n+1})\colon \mathrm{id}_{X_\bullet} - e_\bullet \sim 0_\bullet$.*

Proof. Put $e_0' := \mathrm{id}_{X_0} - e_0$ and $e_{n+1}'' := \mathrm{id}_{X_{n+1}} - e_{n+1}$. Since $\tilde{\delta} \in \mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0))$, we have $(e_0)_{\mathbb{E}}\delta = \delta = (e_{n+1})^{\mathbb{E}}\delta$ and so $(e_0')_{\mathbb{E}}\delta = 0 = (e_{n+1}'')^{\mathbb{E}}\delta$. Therefore, by Lemma 4.12 we can extend e_0' to an idempotent morphism $e_\bullet'\colon \langle X_\bullet, \delta\rangle \to \langle X_\bullet, \delta\rangle$ of $\mathfrak{s}$-distinguished n-exangles with $e_i' = 0$ for $i \in \{2, \dots, n+1\}$, having a homotopy $(k_1, 0, \dots, 0)\colon e_\bullet' \sim 0_\bullet$. Similarly, by the dual of Lemma 4.12, we can extend e_{n+1}'' to an idempotent morphism $e_\bullet''\colon \langle X_\bullet, \delta\rangle \to \langle X_\bullet, \delta\rangle$ with $e_i'' = 0$ for $i \in \{0, \dots, n-1\}$, such that there is a homotopy $(0, \dots, 0, k_{n+1})\colon e_\bullet'' \sim 0_\bullet$. Consider the morphism $f_\bullet := \mathrm{id}_{X_\bullet} - e_\bullet' - e_\bullet''\colon \langle X_\bullet, \delta\rangle \to \langle X_\bullet, \delta\rangle$. We have $\mathrm{id}_{X_\bullet} - f_\bullet = e_\bullet' + e_\bullet''$ and hence $(k_1, 0, \dots, 0, k_{n+1})\colon \mathrm{id}_{X_\bullet} - f_\bullet \sim 0_\bullet$ is a homotopy.

If $n = 1$, then $e_\bullet' + e_\bullet'' = (\mathrm{id}_{X_0} - e_0, e_1' + e_1'', \mathrm{id}_{X_2} - e_2)$ and $(k_1, k_2)\colon e_\bullet' + e_\bullet'' \sim 0_\bullet$ is a homotopy. Lemma 4.10 yields an idempotent $e_\bullet''' = (\mathrm{id}_{X_0} - e_0, e_1''', \mathrm{id}_{X_2} - e_2)\colon \langle X_\bullet, \delta\rangle \to \langle X_\bullet, \delta\rangle$ and a homotopy $(h_1, h_2)\colon e_\bullet''' \sim 0_\bullet$. Then $e_\bullet := \mathrm{id}_{X_\bullet} - e_\bullet'''$ and $h_\bullet := (h_1, h_2)$ are the desired idempotent morphism and homotopy, respectively.

If $n \ge 2$, then the compositions $e_\bullet' e_\bullet''$ and $e_\bullet'' e_\bullet'$ are zero. This implies that $f_\bullet = \mathrm{id}_{X_\bullet} - e_\bullet' - e_\bullet''$ is idempotent. Hence, $e_\bullet := f_\bullet$ and $(h_1, 0, \dots, 0, h_{n+1}) := (k_1, 0, \dots, 0, k_{n+1})$ are the desired idempotent morphism and homotopy, respectively. □

The following simple lemma will be used several times.

Lemma 4.14. *Suppose that $(X, e), (Y, e')$ are objects in $\widetilde{\mathcal{C}}$ and $r\colon X \to Y$ is a morphism in $\mathcal{C}$. Setting $s := e're$ yields a morphism $\tilde{s} = (e', s, e)\colon (X, e) \to (Y, e')$ in $\widetilde{\mathcal{C}}$.*

The previous result allows us to view a complex in $\mathcal{C}$ that is equipped with an idempotent endomorphism as a complex in the idempotent completion $\widetilde{\mathcal{C}}$, as follows.

Definition 4.15. Suppose $X_\bullet$ is a complex in $\mathcal{C}$ and $e_\bullet\colon X_\bullet \to X_\bullet$ is an idempotent morphism of complexes. We denote by $(X_\bullet, e_\bullet)$ the complex in $\widetilde{\mathcal{C}}$ with object (X_i, e_i) in degree i and differential $\tilde{d}_i^{(X,e)} := (e_{i+1}, e_{i+1}d_i^X e_i, e_i)\colon (X_i, e_i) \to (X_{i+1}, e_{i+1})$.

In the notation of Definition 4.15, the underlying morphism of the differential $\tilde{d}_i^{(X,e)}$ satisfies

$$d_i^{(X,e)} = e_{i+1}d_i^X e_i = d_i^X e_i = e_{i+1}d_i^X, \tag{4.2}$$

since $e_\bullet$ is a morphism of complexes and consists of idempotents. Furthermore, whenever we write $(X_\bullet, e_\bullet)$ to denote a complex in $\widetilde{\mathcal{C}}$, we always mean that $e_\bullet\colon X_\bullet \to X_\bullet$ is an idempotent morphism in $\mathsf{Ch}(\mathcal{C})$ and that $(X_\bullet, e_\bullet)$ is the induced object in $\mathsf{Ch}(\widetilde{\mathcal{C}})$ as described in Definition 4.15.

We make a further remark on the notation $(X_\bullet, e_\bullet)$. Because of the need to tweak the differentials in $X_\bullet$ according to (4.2), one cannot recover the original complex $X_\bullet \in \mathsf{Ch}(\mathcal{C})$ with differentials d_i^X from the pair $(X_\bullet, e_\bullet) \in \mathsf{Ch}(\widetilde{\mathcal{C}})$ defined in Definition 4.15. This is in contrast to the description of an object in $\widetilde{\mathcal{C}}$ as a pair (X, e) where one can recover $X \in \mathcal{C}$ uniquely. Thus, $(X_\bullet, e_\bullet)$ is an abuse of notation but should hopefully cause no confusion.

Lemma 4.14 allows us to induce morphisms of complexes in $\widetilde{\mathcal{C}}$ given a morphism between complexes in $\mathcal{C}$ if the complexes involved come with idempotent endomorphisms. The proof is also straightforward.

Lemma 4.16. *Suppose that $(X_\bullet, e_\bullet), (Y_\bullet, e'_\bullet)$ are objects in $\mathsf{Ch}(\widetilde{\mathcal{C}})$ and that $r_\bullet\colon X_\bullet \to Y_\bullet$ is a morphism in $\mathsf{Ch}(\mathcal{C})$. Then defining $s_i := e'_i r_i e_i$ for each $i \in \mathbb{Z}$ gives rise to a morphism $\tilde{s}_\bullet\colon (X_\bullet, e_\bullet) \to (Y_\bullet, e'_\bullet)$ in $\mathsf{Ch}(\widetilde{\mathcal{C}})$ with $\tilde{s}_i = (e'_i, s_i, e_i)$.*

Notation 4.17. In the setup of Lemma 4.16, the composite $e'_\bullet r_\bullet e_\bullet$ is a morphism of complexes $X_\bullet \to Y_\bullet$. In this case, we call $s_\bullet := e'_\bullet r_\bullet e_\bullet$ the *underlying morphism* of $\tilde{s}_\bullet$.

We need two more lemmas before we can define a realisation of the functor $\mathbb{F}$.

Lemma 4.18. *Assume $\tilde{\delta} \in \mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0))$. Further, suppose that $\langle X_\bullet, \delta\rangle$ is an $\mathfrak{s}$-distinguished n-exangle and $e_\bullet\colon \langle X_\bullet, \delta\rangle \to \langle X_\bullet, \delta\rangle$ is an idempotent lift of $(e_0, e_{n+1})\colon \delta \to \delta$. Then $\langle (X_\bullet, e_\bullet), \tilde{\delta}\rangle$ is an n-exangle for $(\widetilde{\mathcal{C}}, \mathbb{F})$.*

Proof. Let $(Y, e') \in \widetilde{\mathcal{C}}$ be arbitrary. We will show that the induced sequence

$$\begin{aligned}\widetilde{\mathcal{C}}((Y,e'),(X_0,e_0)) &\xrightarrow{(\tilde{d}_0^{(X,e)})_*} \widetilde{\mathcal{C}}((Y,e'),(X_1,e_1)) \xrightarrow{(\tilde{d}_1^{(X,e)})_*} \cdots \\ \cdots &\xrightarrow{(\tilde{d}_n^{(X,e)})_*} \widetilde{\mathcal{C}}((Y,e'),(X_{n+1},e_{n+1})) \xrightarrow{\mathbb{F}\tilde{\delta}} \mathbb{F}((Y,e'),(X_0,e_0)),\end{aligned}$$

where $(\tilde{d}_i^{(X,e)})_* = \widetilde{\mathcal{C}}((Y,e'), \tilde{d}_i^{(X,e)})$, is exact. The exactness of the dual sequence can be verified similarly. Checking the above sequence is a complex is straightforward using that $(d_n^X)^{\mathbb{E}}\delta = 0$ and that $d_i^{(X,e)} = d_i^X e_i = e_{i+1}d_i^X$.

To check exactness at $\widetilde{\mathcal{C}}((Y,e'),(X_i,e_i))$ for some $1 \leq i \leq n$, suppose we have a morphism $\tilde{r}\colon (Y,e') \to (X_i,e_i)$ with $\tilde{d}_i^{(X,e)}\tilde{r} = 0$, that is, $d_i^X e_i r = 0$. As $e_i r = r$, we see that $d_i^X r = 0$, whence there exists $s\colon Y \to X_{i-1}$ such that $d_{i-1}^X s = r$ because $\langle X_\bullet, \delta\rangle$ is an $\mathfrak{s}$-distinguished n-exangle. By Lemma 4.14, there is a morphism $\tilde{t}\colon (Y, e') \to (X_{i-1}, e_{i-1})$ with $t = e_{i-1} s e'$. Then we observe that $d_{i-1}^{(X,e)} t = d_{i-1}^X e_{i-1} e_{i-1} s e' = e_i d_{i-1}^X s e' = e_i r e' = r$, whence $\tilde{d}_{i-1}^{(X,e)} \tilde{t} = \tilde{r}$.

Lastly, suppose $\tilde{u} \in \widetilde{\mathcal{C}}((Y, e'), (X_{n+1}, e_{n+1}))$ is a morphism with ${}^{\mathbb{F}}\tilde{\delta}(\tilde{u}) = (\tilde{u})^{\mathbb{F}}\delta = 0$. Then we have ${}^{\mathbb{E}}\delta(u) = u^{\mathbb{E}}\delta = 0$. Hence, there is a morphism $v\colon Y \to X_n$ such that $d_n^X v = u$ as $\langle X_\bullet, \delta\rangle$ is an $\mathfrak{s}$-distinguished n-exangle. Then the morphism $\tilde{w}\colon (Y, e') \to (X_n, e_n)$ with $w = e_n v e'$ satisfies $\tilde{d}_n^{(X,e)} \tilde{w} = \tilde{u}$, as required. □

Lemma 4.19. *Suppose $\tilde{\delta} \in \mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0))$ and that $[Y_\bullet] = \mathfrak{s}(\delta) = [X_\bullet]$ in $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. Suppose further that $e_\bullet\colon \langle X_\bullet, \delta\rangle \to \langle X_\bullet, \delta\rangle$ and $e'_\bullet\colon \langle Y_\bullet, \delta\rangle \to \langle Y_\bullet, \delta\rangle$ are idempotent lifts of $(e_0, e_{n+1})\colon \delta \to \delta$. Then $(X_\bullet, e_\bullet)$ and $(Y_\bullet, e'_\bullet)$ are isomorphic in the category $\mathsf{K}(\widetilde{\mathcal{C}})^n_{((X_0,e_0),(X_{n+1},e_{n+1}))}$, i.e. $[(X_\bullet, e_\bullet)] = [(Y_\bullet, e'_\bullet)]$.*

Proof. We will use [HLN21, Prop. 2.21]. To this end, note that $\langle (X_\bullet, e_\bullet), \tilde{\delta}\rangle$ and $\langle (Y_\bullet, e'_\bullet), \tilde{\delta}\rangle$ are both n-exangles in $(\widetilde{\mathcal{C}}, \mathbb{F})$ by Lemma 4.18. Hence, we only have to show that

$$\mathsf{Ch}(\widetilde{\mathcal{C}})^n_{((X_0,e_0),(X_{n+1},e_{n+1}))}((X_\bullet, e_\bullet), (Y_\bullet, e'_\bullet)) \quad \text{and} \quad \mathsf{Ch}(\widetilde{\mathcal{C}})^n_{((X_0,e_0),(X_{n+1},e_{n+1}))}((Y_\bullet, e'_\bullet), (X_\bullet, e_\bullet))$$

are both non-empty. Since we have $[Y_\bullet] = \mathfrak{s}(\delta) = [X_\bullet]$, there are morphisms $f_\bullet\colon X_\bullet \to Y_\bullet$ and $g_\bullet\colon Y_\bullet \to X_\bullet$ in $\mathsf{Ch}(\mathcal{C})^n_{(X_0,X_{n+1})}$ (with $g_\bullet f_\bullet \sim \mathrm{id}_{X_\bullet}$ and $f_\bullet g_\bullet \sim \mathrm{id}_{Y_\bullet}$). We then obtain morphisms $\tilde{h}_\bullet\colon (X_\bullet, e_\bullet) \to (Y_\bullet, e'_\bullet)$ and $\tilde{k}_\bullet\colon (Y_\bullet, e'_\bullet) \to (X_\bullet, e_\bullet)$ in $\mathsf{Ch}(\widetilde{\mathcal{C}})^n$ with $h_\bullet = e'_\bullet f_\bullet e_\bullet$ and $k_\bullet = e_\bullet g_\bullet e'_\bullet$ by Lemma 4.16. Note that $e_0 = e'_0, e_{n+1} = e'_{n+1}, f_0 = g_0 = \mathrm{id}_{X_0}$ and $f_{n+1} = g_{n+1} = \mathrm{id}_{X_{n+1}}$. So, since $\mathrm{id}_{(X_i,e_i)} = e_i$, we have that $\tilde{h}_\bullet$ and $\tilde{k}_\bullet$ are morphisms in $\mathsf{Ch}(\widetilde{\mathcal{C}})^n_{((X_0,e_0),(X_{n+1},e_{n+1}))}$ and we are done. □

Hence, the following is well-defined.

Definition 4.20. For $\tilde{\delta} \in \mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0))$, pick $X_\bullet$ so that $\mathfrak{s}(\delta) = [X_\bullet]$ and, by Corollary 4.13, an idempotent morphism $e_\bullet\colon \langle X_\bullet, \delta\rangle \to \langle X_\bullet, \delta\rangle$ lifting $(e_0, e_{n+1})\colon \delta \to \delta$. We put $\mathfrak{t}(\tilde{\delta}) := [(X_\bullet, e_\bullet)]$.

Remark 4.21. For $\tilde{\delta} \in \mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0))$, the definition of $\mathfrak{t}(\tilde{\delta}) = [(X_\bullet, e_\bullet)]$ depends on neither the choice of $X_\bullet$ with $[X_\bullet] = \mathfrak{s}(\delta)$, nor on the choice of $e_\bullet$ lifting $(e_0, e_{n+1})\colon \delta \to \delta$ by Lemma 4.19. By Corollary 4.13, for each $\tilde{\delta} \in \mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0))$, we can find an $\mathfrak{s}$-distinguished n-exangle $\langle X_\bullet, \delta\rangle$ and an idempotent morphism $e_\bullet\colon \langle X_\bullet, \delta\rangle \to \langle X_\bullet, \delta\rangle$, such that $\mathfrak{t}(\tilde{\delta}) = [(X_\bullet, e_\bullet)]$ and $\mathrm{id}_{X_\bullet} - e_\bullet$ is null homotopic in $\mathsf{Ch}(\mathcal{C})^n$.

Proposition 4.22. *The assignment $\mathfrak{t}$ is an exact realisation of $\mathbb{F}$.*

Proof. (R0) Suppose $\tilde{\delta} \in \mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0))$ and $\tilde{\rho} \in \mathbb{F}((Y_{n+1}, e'_{n+1}), (Y_0, e'_0))$, and let $(\tilde{a}, \tilde{c})\colon \tilde{\delta} \to \tilde{\rho}$ be a morphism of $\mathbb{F}$-extensions. Suppose $\mathfrak{t}(\tilde{\delta}) = [(X_\bullet, e_\bullet)]$ and $\mathfrak{t}(\tilde{\rho}) = [(Y_\bullet, e'_\bullet)]$. Since (a, c) is a morphism of $\mathbb{E}$-extensions, there is a lift $f_\bullet\colon X_\bullet \to Y_\bullet$ of it using that $\mathfrak{s}$ is an exact realisation of $\mathbb{E}$. As $\tilde{a}\colon (X_0, e_0) \to (Y_0, e'_0)$ and $\tilde{c}\colon (X_{n+1}, e_{n+1}) \to (Y_{n+1}, e'_{n+1})$ are morphisms in $\widetilde{\mathcal{C}}$, we have that $e'_0 a = a = a e_0$ and $e'_{n+1} c = c = c e_{n+1}$. Hence, by Lemma 4.16, it follows that $\tilde{g}_\bullet\colon (X_\bullet, e_\bullet) \to (Y_\bullet, e'_\bullet)$ with $g_\bullet = e'_\bullet f_\bullet e_\bullet$ lifts $(\tilde{a}, \tilde{c})$.

(R1) This is Lemma 4.18.

(R2) Let $(X,e)\in\widetilde{\mathcal{C}}$ be arbitrary. By Remark 4.5(ii), we have that the zero element of $\mathbb{F}((0,0),(X,e))$ has the zero element ${}_X0_0$ of $\mathbb{E}(0,X)$ as its underlying $\mathbb{E}$-extension. Since $\mathfrak{s}$ is an exact realisation of $\mathbb{E}$, we know

$$\mathfrak{s}({}_X0_0)=[X_\bullet]=[\ X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0\ ].$$

The tuple $(e,e,0,\dots,0)\colon X_\bullet\to X_\bullet$ is an idempotent morphism lifting $(e,0)\colon {}_X0_0\to {}_X0_0$. Thus, by Definition 4.20 and using $\mathrm{id}_{(X,e)}=e$, we see that

$$\mathfrak{t}({}_{(X,e)}\widetilde{0}_{(0,0)})=[\ (X,e) \xrightarrow{\widetilde{\mathrm{id}}_{(X,e)}} (X,e) \longrightarrow (0,0) \longrightarrow \cdots \longrightarrow (0,0)\ ].$$

Dually, $\mathfrak{t}({}_{(0,0)}\widetilde{0}_{(X,e)})=[\ (0,0) \longrightarrow \cdots \longrightarrow (0,0) \longrightarrow (X,e) \xrightarrow{\widetilde{\mathrm{id}}_{(X,e)}} (X,e)\ ].$ □

4.3 The axiom (EA1) for $(\widetilde{\mathcal{C}},\mathbb{F},\mathfrak{t})$

Now that we have a biadditive functor $\mathbb{F}\colon \widetilde{\mathcal{C}}^{\mathrm{op}}\times\widetilde{\mathcal{C}}\to\mathsf{Ab}$ and an exact realisation $\mathfrak{t}$ of $\mathbb{F}$, we can begin to verify axioms (EA1), (EA2) and (EA2)$^{\mathrm{op}}$. In this subsection, we will check that the collection of $\mathfrak{t}$-inflations is closed under composition. One can dualise the results here to see that $\mathfrak{t}$-deflations compose to $\mathfrak{t}$-deflations.

The following result only needs that $\mathfrak{s}$ is an exact realisation of $\mathbb{E}\colon\mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathsf{Ab}$. It is an analogue of [Kla21, Lem. 2.1] for n-exangulated categories, allowing us to complete a "partial" lift of a morphism of extensions.

Lemma 4.23 (Completion Lemma)**.** *Let $\langle X_\bullet,\delta\rangle$ and $\langle Y_\bullet,\rho\rangle$ be $\mathfrak{s}$-distinguished n-exangles. Let l,r be integers with $0\le l\le r-2\le n-1$. Suppose there are morphisms $f_0,\dots,f_l$ and $f_r,\dots,f_{n+1}$, where $f_i\colon X_i\to Y_i$, such that $(f_0,f_{n+1})\colon\delta\to\rho$ is a morphism of $\mathbb{E}$-extensions and the solid part of the diagram*

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "f_0"] & \cdots \arrow[r] & X_l \arrow[r] \arrow[d, "f_l"] & X_{l+1} \arrow[r] \arrow[d, dotted, "f_{l+1}"] & \cdots \arrow[r] & X_{r-1} \arrow[r] \arrow[d, dotted, "f_{r-1}"'] & X_r \arrow[r] \arrow[d, "f_r"'] & \cdots \arrow[r] & X_{n+1} \arrow[r, dotted, "\delta"] \arrow[d, "f_{n+1}"'] & {} \\
Y_0 \arrow[r] & \cdots \arrow[r] & Y_l \arrow[r] & Y_{l+1} \arrow[r] & \cdots \arrow[r] & Y_{r-1} \arrow[r] & Y_r \arrow[r] & \cdots \arrow[r] & Y_{n+1} \arrow[r, dotted, "\rho"] & {}
\end{tikzcd} \tag{4.3}$$

commutes. Then there exist morphisms $f_i\in\mathcal{C}(X_i,Y_i)$ for $i\in\{l+1,\dots,r-1\}$ such that (4.3) *commutes.*

Proof. We proceed by induction on $l\ge 0$. Suppose $l=0$. We induct downwards on $r\le n+1$. If $r=n+1$, then the result follows from axiom (R0) for $\mathfrak{s}$ since $(f_0,f_{n+1})\colon\delta\to\rho$ is a morphism of $\mathbb{E}$-extensions. Now assume that the results holds for $l=0$ and some $3\le r\le n+1$. Suppose we are given morphisms f_0 and $f_{r-1},f_r,\dots,f_{n+1}$ such that $f_{i+1}d_i^X=d_i^Yf_i$ for $i\in\{r-1,\dots,n\}$. By the induction hypothesis, we obtain a morphism

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "f_0"] & X_1 \arrow[r] \arrow[d, dotted, "g_1"] & \cdots \arrow[r] & X_{r-1} \arrow[r] \arrow[d, dotted, "g_{r-1}"] & X_r \arrow[r] \arrow[d, "f_r"] & \cdots \arrow[r] & X_{n+1} \arrow[r, dashed, "\delta"] \arrow[d, "f_{n+1}"] & {} \\
Y_0 \arrow[r] & Y_1 \arrow[r] & \cdots \arrow[r] & Y_{r-1} \arrow[r] & Y_r \arrow[r] & \cdots \arrow[r] & Y_{n+1} \arrow[r, dashed, "\rho"] & {}
\end{tikzcd} \tag{4.4}$$

of $\mathfrak{s}$-distinguished n-exangles. We will denote this morphism by $g_\bullet$. Next, note that we have $d^Y_{r-1}(f_{r-1} - g_{r-1}) = (f_r - f_r)d^X_{r-1} = 0$. Since d^Y_{r-2} is a weak kernel of d^Y_{r-1}, there exists $h\colon X_{r-1} \to Y_{r-2}$ so that $f_{r-1} - g_{r-1} = d^Y_{r-2}h$. Set $f_i := g_i$ for $1 \leq i \leq r-3$ and $f_{r-2} := g_{r-2} + hd^X_{r-2}$. Notice that we have $f_i = g_i$ for $i \notin \{r-1, r-2\}$. We claim that (4.3) commutes. By construction, we only need to check commutativity of the two squares involving f_{r-2}. These indeed commute since

$$f_{r-2}d^X_{r-3} = (g_{r-2} + hd^X_{r-2})d^X_{r-3} = g_{r-2}d^X_{r-3} = d^Y_{r-3}g_{r-3} = d^Y_{r-3}f_{r-3}$$

and

$$d^Y_{r-2}f_{r-2} = d^Y_{r-2}(g_{r-2} + hd^X_{r-2}) = d^Y_{r-2}g_{r-2} + (f_{r-1} - g_{r-1})d^X_{r-2} = f_{r-1}d^X_{r-2},$$

using the commutativity of (4.4). This concludes the base case $l = 0$.

The inductive step for $l \geq 0$ is carried out in a similar way to the inductive step above on r, using that d^X_{l+1} is a weak cokernel of d^X_l. □

From the Completion Lemma 4.23 and some earlier results from this section we derive the following, which is used in the main result of this subsection.

Lemma 4.24. *Suppose $\langle X_\bullet, \delta\rangle$ is an $\mathfrak{s}$-distinguished n-exangle. Assume $e_0\colon X_0 \to X_0$ and $e_1\colon X_1 \to X_1$ are idempotents, such that $(e_0)_{\mathbb{E}}\delta = \delta$ and*

$$\begin{tikzcd} X_0 \arrow[r, "d^X_0"] \arrow[d, "e_0"] & X_1 \arrow[d, "e_1"] \\ X_0 \arrow[r, "d^X_0"] & X_1 \end{tikzcd}$$

commutes. Then e_0 and e_1 can be extended to an idempotent morphism $e_\bullet\colon \langle X_\bullet, \delta\rangle \to \langle X_\bullet, \delta\rangle$ with $e_i = \mathrm{id}_{X_i}$ for $3 \leq i \leq n+1$ and $\tilde{\delta} = (e_0, \delta, e_{n+1}) \in \mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0))$.

Proof. First, suppose $n = 1$. Then the solid morphisms of the diagram

$$\begin{tikzcd} X_0 \arrow[r, "d^X_0"] \arrow[d, "e_0"] & X_1 \arrow[r, "d^X_1"] \arrow[d, "e_1"] & X_2 \arrow[r, dashed, "\delta"] \arrow[d, dotted, "f_2"] & {} \\ X_0 \arrow[r, "d^X_0"] & X_1 \arrow[r, "d^X_1"] & X_2 \arrow[r, dashed, "\delta"] & {} \end{tikzcd}$$

form a commutative diagram, and by [HLN21, Prop. 3.6(1)] there is a morphism f_2 such that $(e_0, e_1, f_2)\colon \langle X_\bullet, \delta\rangle \to \langle X_\bullet, \delta\rangle$ is a morphism of $\mathfrak{s}$-distinguished 1-exangles. Recall the polynomial p_2 from Lemma 4.9. We will show that $e_\bullet = (e_0, e_1, e_2)$, where $e_2 := p_2(f_2)$, is the desired idempotent morphism of $\mathfrak{s}$-distinguished n-exangles.

Since $\langle X_\bullet, \delta\rangle$ is an $\mathfrak{s}$-distinguished 1-exangle, there is an exact sequence

$$\mathcal{C}(X_2, X_1) \xrightarrow{\mathcal{C}(X_2, d^X_1)} \mathcal{C}(X_2, X_2) \xrightarrow{\ ^{\mathbb{E}}\delta\ } \mathbb{E}(X_2, X_0).$$

As $^{\mathbb{E}}\delta(f_2^2 - f_2) = (f_2^2 - f_2)^{\mathbb{E}}\delta = (e_0^2 - e_0)_{\mathbb{E}}\delta = {}_{X_0}0_{X_2}$, there exists a morphism $h_2\colon X_2 \to X_1$ with $d^X_1 h_2 = f_2^2 - f_2$. This shows that $(f_2^2 - f_2)^2 = (f_2^2 - f_2)d^X_1 h_2 = d^X_1(e_1^2 - e_1)h_2 = 0$

because e_1 is idempotent. Hence, $e_\bullet = (e_0, e_1, e_2) = (e_0, e_1, p_2(f_2)) \colon X_\bullet \to X_\bullet$ is an idempotent morphism of complexes by Lemma 4.9(ii). Furthermore, $(e_2)^{\mathbb{E}}\delta = (p_2(f_2))^{\mathbb{E}}\delta = (p_2(e_0))_{\mathbb{E}}\delta = (e_0)_{\mathbb{E}}\delta = \delta$ using Lemma 4.9(i), so that $(e_0, e_2) \colon \delta \to \delta$ is a morphism of $\mathbb{E}$-extensions. This computation also shows the existence of $\tilde{\delta} \in \mathbb{F}((X_2, e_2), (X_0, e_0))$ with underlying $\mathbb{E}$-extension δ.

Now suppose $n \geq 2$. We have $(e_0)_{\mathbb{E}}\delta = \delta = (\mathrm{id}_{X_{n+1}})^{\mathbb{E}}\delta$. Therefore, $\tilde{\delta} = (e_0, \delta, \mathrm{id}_{X_{n+1}})$ is an element of $\mathbb{F}((X_{n+1}, \mathrm{id}_{X_{n+1}}), (X_0, e_0))$ with underlying $\mathbb{E}$-extension δ. The solid morphisms of the diagram

$$\begin{tikzcd}
X_0 \arrow[r, "d_0^X"] \arrow[d, "e_0"] & X_1 \arrow[r] \arrow[d, "e_1"] & X_2 \arrow[r] \arrow[d, dotted, "e_2"] & X_3 \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & X_n \arrow[r] \arrow[d, equal] & X_{n+1} \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\
X_0 \arrow[r, "d_0^X"] & X_1 \arrow[r] & X_2 \arrow[r] & X_3 \arrow[r] & \cdots \arrow[r] & X_n \arrow[r] & X_{n+1} \arrow[r, dashed, "\delta"] & {}
\end{tikzcd}$$

form a commutative diagram, and $(e_0, \mathrm{id}_{X_{n+1}}) \colon \delta \to \delta$ is a morphism of $\mathbb{E}$-extensions as $(e_0)_{\mathbb{E}}\delta = \delta$. Since the rows are the $\mathfrak{s}$-distinguished n-exangle $\langle X_\bullet, \delta\rangle$, by Lemma 4.23 we can find a morphism $e_2 \in \mathrm{End}_{\mathcal{C}}(X_2)$, so that the diagram above is a morphism $\langle X_\bullet, \delta\rangle \to \langle X_\bullet, \delta\rangle$. Furthermore, as $e_1 \in \mathrm{End}_{\mathcal{C}}(X_1)$ and $\mathrm{id}_{X_3} \in \mathrm{End}_{\mathcal{C}}(X_3)$ are idempotent, we may assume that e_2 is an idempotent by Lemma 4.10. □

Given a $\mathfrak{t}$-inflation $\tilde{f}$ that fits into a $\mathfrak{t}$-distinguished n-exangle $\langle \widetilde{Y}_\bullet, \tilde{\delta}\rangle$, we cannot a priori say too much about how $\widetilde{Y}_\bullet$ might look. This is one of the main issues in trying to prove (EA1) for $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$. The next lemma gives us a way to deal with this and is the last preparatory result we need before the main result of this subsection.

Lemma 4.25. *Let $\tilde{f} \colon (X_0, e_0) \to (X_1, e_1)$ be a $\mathfrak{t}$-inflation. Then there is an $\mathfrak{s}$-distinguished n-exangle $\langle X'_\bullet, \delta\rangle$ with $X'_0 = X_0$ and $X'_1 = X_1 \oplus C$ for some $C \in \mathcal{C}$, such that $(e_0)_{\mathbb{E}}\delta = \delta$ and $d_0^{X'} = \begin{bmatrix} f & f'(\mathrm{id}_{X_0} - e_0) \end{bmatrix}^\top \colon X_0 \to X_1 \oplus C$ for some $f' \colon X_0 \to C$.*

Proof. Since $f \colon (X_0, e_0) \to (X_1, e_1)$ is an $\mathfrak{t}$-inflation, there is a $\mathfrak{t}$-distinguished n-exangle $\langle \widetilde{Y}_\bullet, \tilde{\delta}'\rangle$ with $\widetilde{Y}_0 = (X_0, e_0)$, $\widetilde{Y}_1 = (X_1, e_1)$ and $d_0^{\widetilde{Y}} = \tilde{f}$. By definition of $\mathfrak{t}$, this means there is an $\mathfrak{s}$-distinguished n-exangle $\langle Y'_\bullet, \delta'\rangle$ with an idempotent morphism $e'_\bullet \colon \langle Y'_\bullet, \delta'\rangle \to \langle Y'_\bullet, \delta'\rangle$, such that $(Y'_0, e'_0) = \widetilde{Y}_0$ and $(Y'_{n+1}, e'_{n+1}) = \widetilde{Y}_{n+1}$, and there are mutually inverse homotopy equivalences $\tilde{r}_\bullet \colon \widetilde{Y}_\bullet \to (Y'_\bullet, e'_\bullet)$ and $\tilde{s}_\bullet \colon (Y'_\bullet, e'_\bullet) \to \widetilde{Y}_\bullet$ which satisfy $\tilde{r}_0 = \mathrm{id}_{\widetilde{Y}_0} = \tilde{s}_0$ and $\tilde{r}_{n+1} = \mathrm{id}_{\widetilde{Y}_{n+1}} = \tilde{s}_{n+1}$. Note that we, thus, have $Y'_0 = X_0$ and $e'_0 = e_0$. In particular, we have a commutative diagram

$$\begin{tikzcd}
(X_0, e_0) \arrow[r, "\tilde{f}"] \arrow[d, equal] & (X_1, e_1) \arrow[r] \arrow[d, "\tilde{r}_1"] & \widetilde{Y}_2 \arrow[r] \arrow[d, "\tilde{r}_2"] & \cdots \arrow[r] & \widetilde{Y}_n \arrow[r] \arrow[d, "\tilde{r}_n"] & \widetilde{Y}_{n+1} \arrow[r, dashed, "\tilde{\delta}'"] \arrow[d, equal] & {} \\
(X_0, e_0) \arrow[r, "\tilde{t}"] \arrow[d, equal] & (Y'_1, e'_1) \arrow[r] \arrow[d, "\tilde{s}_1"] & (Y'_2, e'_2) \arrow[r] \arrow[d, "\tilde{s}_2"] & \cdots \arrow[r] & (Y'_n, e'_n) \arrow[r] \arrow[d, "\tilde{s}_n"] & (Y'_{n+1}, e'_{n+1}) \arrow[r, dashed, "\tilde{\delta}'"] \arrow[d, equal] & {} \\
(X_0, e_0) \arrow[r, "\tilde{f}"] & (X_1, e_1) \arrow[r] & \widetilde{Y}_2 \arrow[r] & \cdots \arrow[r] & \widetilde{Y}_n \arrow[r] & \widetilde{Y}_{n+1} \arrow[r, dashed, "\tilde{\delta}'"] & {}
\end{tikzcd}$$

in $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$, where $t = e'_1 d_0^{Y'} e_0 = d_0^{Y'} e_0 = e'_1 d_0^{Y'}$.

Consider the complex $Y''_\bullet := \mathrm{triv}_1(X_1)_\bullet$ and the $\mathbb{E}$-extension $\delta'' := {}_0 0_{Y''_{n+1}}$. Note that if $n = 1$, then $Y''_{n+1} = X_1$; otherwise we have $Y''_{n+1} = 0$. In either case, we have an $\mathfrak{s}$-distinguished n-exangle $\langle Y''_\bullet, \delta''\rangle$ using the axiom (R2) for $\mathfrak{s}$, and hence also an $\mathfrak{s}$-distinguished n-exangle $\langle Y''_\bullet \oplus Y'_\bullet, \delta'' \oplus \delta'\rangle$ by [HLN21, Prop. 3.3]. Using the canonical isomorphism $u\colon 0 \oplus X_0 \to X_0$ we see that the complex

$$Z_\bullet\colon \qquad X_0 \xrightarrow{\left[\begin{smallmatrix} 0 \\ d_0^{Y'} \end{smallmatrix}\right]} X_1 \oplus Y'_1 \xrightarrow{\left[\begin{smallmatrix} \mathrm{id}_{X_1} & 0 \\ 0 & d_1^{Y'} \end{smallmatrix}\right]} X_1 \oplus Y'_2 \xrightarrow{\left[\begin{smallmatrix} 0 & 0 \\ 0 & d_2^{Y'} \end{smallmatrix}\right]} 0 \oplus Y'_3 \xrightarrow{\left[\begin{smallmatrix} 0 & 0 \\ 0 & d_3^{Y'} \end{smallmatrix}\right]} \cdots \xrightarrow{\left[\begin{smallmatrix} 0 & 0 \\ 0 & d_n^{Y'} \end{smallmatrix}\right]} 0 \oplus Y'_{n+1}$$

realises $\delta := u_{\mathbb{E}}(\delta'' \oplus \delta')$ in $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ by [HLN21, Cor. 2.26(2)]. Consider the diagram

$$\begin{tikzcd}
X_0 \arrow[r, "\left[\begin{smallmatrix} 0 \\ d_0^{Y'} \end{smallmatrix}\right]"] \arrow[d, equal] & X_1 \oplus Y'_1 \arrow[d, "a"] \\
X_0 \arrow[r, "\left[\begin{smallmatrix} f \\ d_0^{Y'} \end{smallmatrix}\right]"] \arrow[d, equal] & X_1 \oplus Y'_1 \arrow[d, "b"] \\
X_0 \arrow[r, "\left[\begin{smallmatrix} f \\ d_0^{Y'}(\mathrm{id}_{X_0} - e_0) \end{smallmatrix}\right]"] & X_1 \oplus Y'_1
\end{tikzcd}$$

in $\mathcal{C}$, where $a := \left[\begin{smallmatrix} \mathrm{id}_{X_1} & s_1 \\ 0 & \mathrm{id}_{Y'_1} \end{smallmatrix}\right]$ and $b := \left[\begin{smallmatrix} \mathrm{id}_{X_1} & 0 \\ -r_1 & \mathrm{id}_{Y'_1} \end{smallmatrix}\right]$. This diagram commutes since

$$\begin{bmatrix} \mathrm{id}_{X_1} & s_1 \\ 0 & \mathrm{id}_{Y'_1} \end{bmatrix} \begin{bmatrix} 0 \\ d_0^{Y'} \end{bmatrix} = \begin{bmatrix} s_1 d_0^{Y'} \\ d_0^{Y'} \end{bmatrix} = \begin{bmatrix} s_1 e'_1 d_0^{Y'} \\ d_0^{Y'} \end{bmatrix} = \begin{bmatrix} s_1 t \\ d_0^{Y'} \end{bmatrix} = \begin{bmatrix} f \\ d_0^{Y'} \end{bmatrix}$$

and

$$\begin{bmatrix} \mathrm{id}_{X_1} & 0 \\ -r_1 & \mathrm{id}_{Y'_1} \end{bmatrix} \begin{bmatrix} f \\ d_0^{Y'} \end{bmatrix} = \begin{bmatrix} f \\ d_0^{Y'} - r_1 f \end{bmatrix} = \begin{bmatrix} f \\ d_0^{Y'} - t \end{bmatrix} = \begin{bmatrix} f \\ d_0^{Y'}(\mathrm{id}_{X_0} - e_0) \end{bmatrix}.$$

Notice that the composition ba is an automorphism of $X_1 \oplus Y'_1$, and so the complex

$$X'_\bullet\colon \qquad X_0 \xrightarrow{\left[\begin{smallmatrix} f & d_0^{Y'}(\mathrm{id}_{X_0} - e_0) \end{smallmatrix}\right]^\top} X_1 \oplus Y'_1 \xrightarrow{d_1^Z (ba)^{-1}} Z_2 \xrightarrow{d_2^Z} Z_3 \xrightarrow{d_3^Z} \cdots \xrightarrow{d_n^Z} Z_{n+1}$$

forms part of an $\mathfrak{s}$-distinguished n-exangle $\langle X'_\bullet, \delta\rangle$ by [HLN21, Cor. 2.26(2)]. We have

$$(e_0)_{\mathbb{E}}\delta = (e_0)_{\mathbb{E}} u_{\mathbb{E}}(\delta'' \oplus \delta') = u_{\mathbb{E}}(0_{\mathbb{E}}\delta'' \oplus (e_0)_{\mathbb{E}}\delta') = u_{\mathbb{E}}(\delta'' \oplus \delta') = \delta$$

as $e_0 u = u(0 \oplus e_0)$, $\delta'' = {}_0 0_{Y''_{n+1}}$ and $\tilde{\delta}' \in \mathbb{F}(\widetilde{Y}_{n+1}, (X_0, e_0))$. Setting $f' := d_0^{Y'}$ and $C := Y'_1$ finishes the proof. $\square$

We close this subsection with the following result, which together with its dual demonstrates that axiom (EA1) holds for $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$.

Proposition 4.26. *Suppose $\tilde{f}\colon (X_0, e_0) \to (X_1, e_1)$ and $\tilde{g}\colon (Y_0, e'_0) \to (Y_1, e'_1)$ are $\mathfrak{t}$-inflations with $(X_1, e_1) = (Y_0, e'_0)$. Then $\tilde{g}\tilde{f}\colon (X_0, e_0) \to (Y_1, e'_1)$ is a $\mathfrak{t}$-inflation.*

Proof. By Lemma 4.25, there exists an $\mathfrak{s}$-distinguished n-exangle $\langle X'_\bullet, \delta\rangle$ with $X'_0 = X_0$ and $X'_1 = X_1 \oplus C$ for some $C \in \mathcal{C}$, so that $d_0^{X'} = \left[\begin{smallmatrix} f & f'(\mathrm{id}_{X_0} - e_0) \end{smallmatrix}\right]^\top$ for some $f'\colon X_0 \to C$ and $(e_0)_{\mathbb{E}}\delta = \delta$. Similarly, there is also an $\mathfrak{s}$-distinguished n-exangle $\langle Y'_\bullet, \delta'\rangle$ with $Y'_0 = Y_0 = X_1$

and $Y'_1 = Y_1 \oplus C'$ for some $C' \in \mathcal{C}$, so that $d_0^{Y'} = \left[\begin{smallmatrix} g & g'(\mathrm{id}_{X_1} - e'_0) \end{smallmatrix}\right]^\top$ for some $g' \colon X_1 \to C'$ and $(e'_0)_{\mathbb{E}}\delta' = \delta'$. Setting $Y''_\bullet := \mathrm{triv}_0(C)_\bullet$, we also have the n-exangle $\langle Y''_\bullet, {}_C0_0\rangle$ by axiom (R2) for $\mathfrak{s}$. Then $\langle Y'_\bullet \oplus Y''_\bullet, \delta' \oplus {}_C0_0\rangle$ is $\mathfrak{s}$-distinguished by [HLN21, Prop. 3.3]. We have $Y'_0 \oplus Y''_0 = X_1 \oplus C$ and $Y'_1 \oplus Y''_1 = Y_1 \oplus C' \oplus C$, and

$$d_0^{Y' \oplus Y''} = \begin{bmatrix} g & 0 \\ g'(\mathrm{id}_{X_1} - e'_0) & 0 \\ 0 & \mathrm{id}_C \end{bmatrix}$$

is the $\mathfrak{s}$-inflation of $\langle Y'_\bullet \oplus Y''_\bullet, \delta' \oplus {}_C0_0\rangle$ with respect to the given decompositions. Since $d_0^{X'}$ and $d_0^{Y' \oplus Y''}$ are $\mathfrak{s}$-inflations, by (EA1) for $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, we have that the morphism

$$d_0^{Y' \oplus Y''} d_0^{X'} = \begin{bmatrix} g & 0 \\ g'(\mathrm{id}_{X_1} - e'_0) & 0 \\ 0 & \mathrm{id}_C \end{bmatrix} \begin{bmatrix} f \\ f'(\mathrm{id}_{X_0} - e_0) \end{bmatrix} = \begin{bmatrix} gf \\ g'(\mathrm{id}_{X_1} - e'_0)f \\ f'(\mathrm{id}_{X_0} - e_0) \end{bmatrix} = \begin{bmatrix} gf \\ 0 \\ f'(\mathrm{id}_{X_0} - e_0) \end{bmatrix}$$

is an $\mathfrak{s}$-inflation, where we used that $e'_0 f = e_1 f = f$. Therefore, there is an $\mathfrak{s}$-distinguished n-exangle $\langle Z''_\bullet, \delta''\rangle$ with $Z''_0 = X_0$, $Z''_1 = Y_1 \oplus C' \oplus C$ and $d_0^{Z''} = \left[\begin{smallmatrix} gf & 0 & f'(\mathrm{id}_{X_0} - e_0) \end{smallmatrix}\right]^\top$.

Our next aim is to apply Lemma 4.24 to $\langle Z''_\bullet, \delta''\rangle$. Thus, we claim that $(e_0)_{\mathbb{E}}\delta'' = \delta''$. Since $d_0^{Y' \oplus Y''} d_0^{X'} = d_0^{Z''}$, we can apply [HLN21, Prop. 3.6(1)] to obtain a morphism

$$\begin{tikzcd}
X_0 \arrow[r, "d_0^{Z''}"] \arrow[d, "d_0^{X'}"'] & Y_1 \oplus C' \oplus C \arrow[r] \arrow[d, equal] & Z''_2 \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & Z''_n \arrow[r] \arrow[d, dotted] & Z''_{n+1} \arrow[r, dashed, "\delta''"] \arrow[d, dotted, "l_{n+1}"] & {} \\
X_1 \oplus C \arrow[r, "d_0^{Y' \oplus Y''}"] & Y_1 \oplus C' \oplus C \arrow[r] & Y'_2 \oplus 0 \arrow[r] & \cdots \arrow[r] & Y'_n \oplus 0 \arrow[r] & Y'_{n+1} \oplus 0 \arrow[r, dashed, "\delta' \oplus {}_C0_0"] & {}
\end{tikzcd}$$

of $\mathfrak{s}$-distinguished n-exangles. In particular, we have that

$$(d_0^{X'})_{\mathbb{E}}\delta'' = (l_{n+1})^{\mathbb{E}}(\delta' \oplus {}_C0_0). \tag{4.5}$$

As $e'_0 = e_1$, $e_1 f = f = f e_0$ and e_0 is idempotent, we see that

$$d_0^{X'} e_0 = \begin{bmatrix} fe_0 \\ 0 \end{bmatrix} = \begin{bmatrix} e_1 f \\ 0 \end{bmatrix} = \begin{bmatrix} e_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f \\ f'(\mathrm{id}_{X_0} - e_0) \end{bmatrix} = \begin{bmatrix} e'_0 & 0 \\ 0 & 0 \end{bmatrix} d_0^{X'}. \tag{4.6}$$

This implies that

$$\begin{aligned}
(d_0^{X'})_{\mathbb{E}}(\mathrm{id}_{X_0} - e_0)_{\mathbb{E}}\delta'' &= (d_0^{X'} - d_0^{X'} e_0)_{\mathbb{E}}\delta'' \\
&= \left(d_0^{X'} - \begin{bmatrix} e'_0 & 0 \\ 0 & 0 \end{bmatrix} d_0^{X'}\right)_{\mathbb{E}} \delta'' && \text{by (4.6)} \\
&= \left(\mathrm{id}_{X_1 \oplus C} - \begin{bmatrix} e'_0 & 0 \\ 0 & 0 \end{bmatrix}\right)_{\mathbb{E}} (d_0^{X'})_{\mathbb{E}}\delta'' \\
&= \begin{bmatrix} \mathrm{id}_{X_1} - e'_0 & 0 \\ 0 & \mathrm{id}_C \end{bmatrix}_{\mathbb{E}} (l_{n+1})^{\mathbb{E}}(\delta' \oplus {}_C0_0) && \text{by (4.5)} \\
&= (l_{n+1})^{\mathbb{E}} \left((\mathrm{id}_{X_1} - e'_0)_{\mathbb{E}}\delta' \oplus (\mathrm{id}_C)_{\mathbb{E}}\, {}_C0_0\right) \\
&= {}_{X_1 \oplus C}0_{Z''_{n+1}} && \text{as } (e'_0)_{\mathbb{E}}\delta' = \delta'.
\end{aligned}$$

Since $\langle X'_\bullet, \delta\rangle$ is an $\mathfrak{s}$-distinguished n-exangle, by [HLN21, Lem. 3.5] there is an exact sequence

$$\mathcal{C}(Z''_{n+1}, X'_{n+1}) \xrightarrow{\mathbb{E}_\delta} \mathbb{E}(Z''_{n+1}, X_0) \xrightarrow{(d_0^{X'})_{\mathbb{E}}} \mathbb{E}(Z''_{n+1}, X_1 \oplus C).$$

As seen above, $(\mathrm{id}_{X_0} - e_0)_{\mathbb{E}}\delta''$ vanishes under $(d_0^{X'})_{\mathbb{E}}$, so there is a morphism $r_{n+1}\colon Z''_{n+1} \to X'_{n+1}$ with $^{\mathbb{E}}\delta(r_{n+1}) = (r_{n+1})^{\mathbb{E}}\delta = (\mathrm{id}_{X_0} - e_0)_{\mathbb{E}}\delta''$. Since $(\mathrm{id}_{X_0} - e_0)_{\mathbb{E}}\delta = {}_{X_0}0_{X'_{n+1}}$, this implies

$$ {}_{X_0}0_{Z''_{n+1}} = (r_{n+1})^{\mathbb{E}}(\mathrm{id}_{X_0} - e_0)_{\mathbb{E}}\delta = (\mathrm{id}_{X_0} - e_0)_{\mathbb{E}}(r_{n+1})^{\mathbb{E}}\delta = \left((\mathrm{id}_{X_0} - e_0)^2\right)_{\mathbb{E}}\delta'' = (\mathrm{id}_{X_0} - e_0)_{\mathbb{E}}\delta'', $$

showing that $(e_0)_{\mathbb{E}}\delta'' = \delta''$.

Now consider the idempotent $e'_1 \oplus 0 \oplus 0 \in \mathrm{End}_{\mathcal{C}}(Y_1 \oplus C' \oplus C)$. A quick computation yields the equality $(e'_1 \oplus 0 \oplus 0)d_0^{Z''} = d_0^{Z''}e_0$. Therefore, by Lemma 4.24, there is an idempotent morphism $e''_\bullet\colon \langle Z''_\bullet, \delta''\rangle \to \langle Z''_\bullet, \delta''\rangle$ with $e''_0 = e_0$, $e''_1 = e'_1 \oplus 0 \oplus 0$ as well as an $\mathbb{F}$-extension $\tilde{\rho} \in \mathbb{F}((X_0, e_0), (Z''_n, e''_{n+1}))$ with underlying $\mathbb{E}$-extension $\rho = \delta''$. We obtain a $\mathfrak{t}$-distinguished n-exangle $\langle (Z''_\bullet, e''_\bullet), \tilde{\rho}\rangle$. Then the $\mathfrak{t}$-inflation of this n-exangle is given by the morphism $\tilde{d}_0^{(Z'',e'')}\colon (X_0, e_0) \to (Y_1 \oplus C' \oplus C, e'_1 \oplus 0 \oplus 0)$ satisfying

$$ d_0^{(Z'',e'')} = e''_1 d_0^{Z''} e''_0 = (e'_1 \oplus 0 \oplus 0)d_0^{Z''}e_0 = \begin{bmatrix} e'_1 gfe_0 & 0 & 0 \end{bmatrix}^\top = \begin{bmatrix} gf & 0 & 0 \end{bmatrix}^\top. $$

As $\tilde{s}\colon (Y_1 \oplus C' \oplus C, e'_1 \oplus 0 \oplus 0) \to (Y_1, e'_1)$ with $s = \begin{bmatrix} e'_1 & 0 & 0 \end{bmatrix}$ is an isomorphism in $\widetilde{\mathcal{C}}$, the complex

$$ \widetilde{X}''_\bullet\colon \qquad (X_0, e_0) \xrightarrow{\tilde{d}_0^{\widetilde{X}''}} (Y_1, e'_1) \xrightarrow{\tilde{d}_1^{\widetilde{X}''}} (Z''_2, e''_2) \xrightarrow{\tilde{d}_2^{(Z'',e'')}} \cdots \xrightarrow{\tilde{d}_n^{(Z'',e'')}} (Z''_{n+1}, e''_{n+1}) $$

with $\mathfrak{t}$-inflation $\tilde{d}_0^{\widetilde{X}''} := \tilde{s}\tilde{d}_0^{(Z'',e'')} = \tilde{g}\tilde{f}$ and $\tilde{d}_1^{\widetilde{X}''} = \tilde{d}_1^{(Z'',e'')}\tilde{s}^{-1}$ forms part of the $\mathfrak{t}$-distinguished n-exangle $\langle \widetilde{X}''_\bullet, \tilde{\rho}\rangle$ by [HLN21, Cor. 2.26(2)]. □

4.4 The axiom (EA2) for $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$

The goal of this subsection is to show that axiom (EA2) holds for the triplet $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$. Again, by dualising one can deduce that axiom (EA2)$^{\mathrm{op}}$ also holds. We need two key technical lemmas first.

Lemma 4.27. *Suppose that:*

(i) $\tilde{\delta} \in \mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0))$ *is an* $\mathbb{F}$*-extension;*

(ii) $\tilde{c}\colon (Y_{n+1}, e'_{n+1}) \to (X_{n+1}, e_{n+1})$ *is a morphism in* $\widetilde{\mathcal{C}}$ *for some* $(Y_{n+1}, e'_{n+1}) \in \widetilde{\mathcal{C}}$*;*

(iii) $\langle X_\bullet, \delta\rangle$ *and* $\langle Y_\bullet, c^{\mathbb{E}}\delta\rangle$ *are* $\mathfrak{s}$*-distinguished* n*-exangles with* $Y_0 = X_0$*;*

(iv) $e_\bullet\colon \langle X_\bullet, \delta\rangle \to \langle X_\bullet, \delta\rangle$ *is an idempotent morphism lifting* $(e_0, e_{n+1})\colon \delta \to \delta$*, such that* $\mathrm{id}_{X_\bullet} - e_\bullet$ *is null homotopic; and*

(v) $e'_\bullet\colon \langle Y_\bullet, c^{\mathbb{E}}\delta\rangle \to \langle Y_\bullet, c^{\mathbb{E}}\delta\rangle$ *is an idempotent morphism lifting* $(e_0, e'_{n+1})\colon c^{\mathbb{E}}\delta \to c^{\mathbb{E}}\delta$*, such that* $\mathrm{id}_{Y_\bullet} - e'_\bullet$ *is null homotopic.*

Then a good lift $g_\bullet \colon \langle Y_\bullet, c^{\mathbb{E}}\delta\rangle \to \langle X_\bullet, \delta\rangle$ of the morphism $(\mathrm{id}_{X_0}, c) \colon c^{\mathbb{E}}\delta \to \delta$ of $\mathbb{E}$-extensions exists, so that

$$\begin{tikzcd}
& Y_0 \arrow[rr, "d_0^Y"] \arrow[dl, "e_0"'] \arrow[dd, equal] & & Y_1 \arrow[rr, "d_1^Y"] \arrow[dl, "e_1'"'] \arrow[dd, dotted, "g_1" near start] & & \cdots \arrow[rr, "d_{n-1}^Y"] & & Y_n \arrow[rr, "d_n^Y"] \arrow[dl, "e_n'"'] \arrow[dd, dotted, "g_n" near start] & & Y_{n+1} \arrow[dl, "e_{n+1}'"'] \arrow[dd, "c"] \\
Y_0 \arrow[rr, "d_0^Y" near end] \arrow[dd, equal] & & Y_1 \arrow[rr, "d_1^Y" near end] \arrow[dd, dotted, "g_1" near start] & & \cdots \arrow[rr, "d_{n-1}^Y" near end] & & Y_n \arrow[rr, "d_n^Y" near end] \arrow[dd, dotted, "g_n" near start] & & Y_{n+1} \arrow[dd, "c" near start] & \\
& X_0 \arrow[rr, "d_0^X" near start] \arrow[dl, "e_0"] & & X_1 \arrow[rr, "d_1^X"] \arrow[dl, "e_1"] & & \cdots \arrow[rr, "d_{n-1}^X"] & & X_n \arrow[rr, "d_n^X" near start] \arrow[dl, "e_n"] & & X_{n+1} \arrow[dl, "e_{n+1}"] \\
X_0 \arrow[rr, "d_0^X"'] & & X_1 \arrow[rr, "d_1^X"'] & & \cdots \arrow[rr, "d_{n-1}^X"'] & & X_n \arrow[rr, "d_n^X"'] & & X_{n+1} &
\end{tikzcd} \tag{4.7}$$

is commutative in $\mathcal{C}$. In particular, we have $g_\bullet e_\bullet' = e_\bullet g_\bullet$ as morphisms $\langle Y_\bullet, c^{\mathbb{E}}\delta\rangle \to \langle X_\bullet, \delta\rangle$.

Remark 4.28. Notice that $(e_0)_{\mathbb{E}}\delta = \delta$ and $ce_{n+1}' = c$ imply

$$(e_0)_{\mathbb{E}}(c^{\mathbb{E}}\delta) = c^{\mathbb{E}}\delta = (e_{n+1}')^{\mathbb{E}}(c^{\mathbb{E}}\delta). \tag{4.8}$$

Therefore, $(e_0, e_{n+1}') \colon c^{\mathbb{E}}\delta \to c^{\mathbb{E}}\delta$ is indeed a morphism of $\mathbb{E}$-extensions and condition (v) makes sense. Condition (iv) makes sense due to Remark 4.5(iv).

Proof of Lemma 4.27. Since $(\mathrm{id}_{X_0}, c) \colon c^{\mathbb{E}}\delta \to \delta$ is a morphism of $\mathbb{E}$-extensions, it admits a good lift $g_\bullet' = (g_0', \dots, g_{n+1}') = (\mathrm{id}_{X_0}, g_1', \dots, g_n', c) \colon \langle Y_\bullet, c^{\mathbb{E}}\delta\rangle \to \langle X_\bullet, \delta\rangle$ using axiom (EA2) for the n-exangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. Define $g_i \coloneqq e_i g_i' e_i' + (\mathrm{id}_{X_i} - e_i) g_i' (\mathrm{id}_{Y_i} - e_i')$ for $0 \leq i \leq n+1$. Note that $e_0 = e_0'$ and $e_{n+1}c = c = ce_{n+1}'$ by assumption. For $i = 0$, we have $g_0 = e_0 \,\mathrm{id}_{X_0} e_0' + (\mathrm{id}_{X_0} - e_0)\,\mathrm{id}_{X_0}(\mathrm{id}_{X_0} - e_0') = \mathrm{id}_{X_0}$. On the other hand, for $i = n+1$ we have $g_{n+1} = e_{n+1}ce_{n+1}' + (\mathrm{id}_{X_{n+1}} - e_{n+1})c(\mathrm{id}_{X_{n+1}} - e_{n+1}') = c$. Therefore, the morphism $g_\bullet = e_\bullet g_\bullet' e_\bullet' + (\mathrm{id}_{X_\bullet} - e_\bullet) g_\bullet' (\mathrm{id}_{Y_\bullet} - e_\bullet') \colon Y_\bullet \to X_\bullet$ is of the form $(\mathrm{id}_{X_0}, g_1, \dots, g_n, c)$.

The squares on the top and bottom faces in (4.7) commute as $e_\bullet' \colon Y_\bullet \to Y_\bullet$ and $e_\bullet \colon X_\bullet \to X_\bullet$, respectively, are morphisms of complexes. The squares on the front and back faces in (4.7) commute because $g_\bullet$ is the sum of morphisms of complexes from $Y_\bullet$ to $X_\bullet$. This also implies $g_\bullet = (\mathrm{id}_{X_0}, g_1, \dots, g_n, c)$ is a lift of (id_{X_0}, c). Of the remaining squares, the leftmost clearly commutes and the rightmost commutes as $c \colon (Y_{n+1}, e_{n+1}') \to (X_{n+1}, e_{n+1})$ is a morphism in $\widetilde{\mathcal{C}}$. For $1 \leq i \leq n$, we have

$$\begin{aligned}
g_i e_i' &= e_i g_i' e_i' e_i' + (\mathrm{id}_{X_i} - e_i) g_i' (\mathrm{id}_{Y_i} - e_i') e_i' \\
&= e_i g_i' e_i' && \text{as } (e_i')^2 = e_i' \\
&= e_i e_i g_i' e_i' + e_i (\mathrm{id}_{X_i} - e_i) g_i' (\mathrm{id}_{Y_i} - e_i') && \text{as } (e_i)^2 = e_i \\
&= e_i g_i.
\end{aligned}$$

Therefore, diagram (4.7) commutes and, further, the last assertion follows.

It remains to show that $g_\bullet$ is a good lift of $(\mathrm{id}_{X_0}, c) \colon c^{\mathbb{E}}\delta \to \delta$. Recall that $\mathrm{id}_{X_\bullet} - e_\bullet$ and $\mathrm{id}_{Y_\bullet} - e_\bullet'$ are both null homotopic by assumption, and so $g_\bullet' - g_\bullet = (\mathrm{id}_{X_\bullet} - e_\bullet) g_\bullet' e_\bullet' + e_\bullet g_\bullet' (\mathrm{id}_{Y_\bullet} - e_\bullet')$ is also null homotopic. Then it follows from [HLN21, Rem. 2.33(1)] that $g_\bullet$ is a good lift of (id_{X_0}, c) since $g_\bullet'$ is. $\square$

The next result allows us to define a good lift in $\widetilde{\mathcal{C}}$ from the one we created in Lemma 4.27.

Lemma 4.29. *In the setup of* Lemma 4.27, *the morphism* $\tilde{h}_\bullet\colon (Y_\bullet, e'_\bullet) \to (X_\bullet, e_\bullet)$ *with underlying morphism* $h_\bullet = e_\bullet g_\bullet e'_\bullet$ *is a good lift of the morphism* $(\widetilde{\mathrm{id}}_{(X_0,e_0)}, \tilde{c})\colon \tilde{c}^{\mathbb{F}}\tilde{\delta} \to \tilde{\delta}$ *of* $\mathbb{F}$*-extensions.*

Proof. From (4.8), we see that $\tilde{c}^{\mathbb{F}}\tilde{\delta} \in \mathbb{F}((Y_{n+1}, e'_{n+1}), (X_0, e_0))$ is indeed an $\mathbb{F}$-extension and $(\widetilde{\mathrm{id}}_{(X_0,e_0)}, \tilde{c})\colon \tilde{c}^{\mathbb{F}}\tilde{\delta} \to \tilde{\delta}$ a morphism of $\mathbb{F}$-extensions. Using $h_0 = e_0 g_0 e'_0 = e_0 = \mathrm{id}_{(X_0,e_0)}$ and $h_{n+1} = e_{n+1} c e'_{n+1} = c$, as well as the commutativity of (4.7), we see that $\tilde{h}_\bullet$ is a morphism $\langle (Y_\bullet, e'_\bullet), \tilde{c}^{\mathbb{F}}\tilde{\delta} \rangle \to \langle (X_\bullet, e_\bullet), \tilde{\delta} \rangle$ of $\mathfrak{t}$-distinguished n-exangles, lifting $(\widetilde{\mathrm{id}}_{(X_0,e_0)}, \tilde{c})$.

Recall from Definition 3.4 that $M^{\mathcal{C}}_{g_\bullet}$ denotes the mapping cone of $g_\bullet\colon Y_\bullet \to X_\bullet$ in $\mathcal{C}$, and that $\langle M^{\mathcal{C}}_{g_\bullet}, (d_0^Y)_{\mathbb{E}}\delta \rangle$ is $\mathfrak{s}$-distinguished as $g_\bullet\colon \langle Y_\bullet, c^{\mathbb{E}}\delta \rangle \to \langle X_\bullet, \delta \rangle$ is a good lift of $(\mathrm{id}_{X_0}, c)\colon c^{\mathbb{E}}\delta \to \delta$. Using the commutativity of (4.7), that $e_\bullet\colon X_\bullet \to X_\bullet$ and $e'_\bullet\colon Y_\bullet \to Y_\bullet$ are morphisms of complexes, and that $e_{n+1}c = c = ce'_{n+1}$, one can verify that the diagram

$$\begin{tikzcd}
Y_1 \arrow[r, "d_0^{M_g^{\mathcal{C}}}"] \arrow[d, "e'_1"'] & Y_2 \oplus X_1 \arrow[r, "d_1^{M_g^{\mathcal{C}}}"] \arrow[d, "\left[\begin{smallmatrix} e'_2 & 0 \\ 0 & e_1 \end{smallmatrix}\right]"] & Y_3 \oplus X_2 \arrow[r, "d_2^{M_g^{\mathcal{C}}}"] \arrow[d, "\left[\begin{smallmatrix} e'_3 & 0 \\ 0 & e_2 \end{smallmatrix}\right]"] & \cdots \arrow[r, "d_{n-1}^{M_g^{\mathcal{C}}}"] & Y_{n+1} \oplus X_n \arrow[r, "d_n^{M_g^{\mathcal{C}}}"] \arrow[d, "\left[\begin{smallmatrix} e'_{n+1} & 0 \\ 0 & e_n \end{smallmatrix}\right]"'] & X_{n+1} \arrow[r, dashed, "(d_0^Y)_{\mathbb{E}}\delta"] \arrow[d, "e_{n+1}"'] & {} \\
Y_1 \arrow[r, "d_0^{M_g^{\mathcal{C}}}"'] & Y_2 \oplus X_1 \arrow[r, "d_1^{M_g^{\mathcal{C}}}"'] & Y_3 \oplus X_2 \arrow[r, "d_2^{M_g^{\mathcal{C}}}"'] & \cdots \arrow[r, "d_{n-1}^{M_g^{\mathcal{C}}}"'] & Y_{n+1} \oplus X_n \arrow[r, "d_n^{M_g^{\mathcal{C}}}"'] & X_{n+1} \arrow[r, dashed, "(d_0^Y)_{\mathbb{E}}\delta"'] & {}
\end{tikzcd} \tag{4.9}$$

commutes. Thus, the vertical morphisms form an idempotent morphism $e''_\bullet\colon M^{\mathcal{C}}_{g_\bullet} \to M^{\mathcal{C}}_{g_\bullet}$ of complexes. Furthermore, (4.9) is a morphism of $\mathfrak{s}$-distinguished n-exangles as

$$\begin{aligned}
(e'_1)_{\mathbb{E}}(d_0^Y)_{\mathbb{E}}\delta &= (d_0^Y)_{\mathbb{E}}(e_0)_{\mathbb{E}}\delta && \text{as (4.7) is commutative} \\
&= (d_0^Y)_{\mathbb{E}}\delta && \text{as } \tilde{\delta} \in \mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0)) \\
&= (d_0^Y)_{\mathbb{E}}(e_{n+1})^{\mathbb{E}}\delta && \text{as } \tilde{\delta} \in \mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0)) \\
&= (e_{n+1})^{\mathbb{E}}(d_0^Y)_{\mathbb{E}}\delta.
\end{aligned}$$

This calculation also shows that $\tilde{\rho} := (e'_1, (d_0^Y)_{\mathbb{E}}\delta, e_{n+1}) \in \mathbb{F}((X_{n+1}, e_{n+1}), (Y_1, e'_1))$. Thus, by definition of $\mathfrak{t}$, we have that $\mathfrak{t}(\tilde{\rho}) = [(M^{\mathcal{C}}_{g_\bullet}, e''_\bullet)]$, i.e. $\langle (M^{\mathcal{C}}_{g_\bullet}, e''_\bullet), \tilde{\rho} \rangle$ is $\mathfrak{t}$-distinguished.

It is straightforward to verify that the object $(M^{\mathcal{C}}_{g_\bullet}, e''_\bullet)$ is equal to the mapping cone $M^{\widetilde{\mathcal{C}}}_{\tilde{h}_\bullet}$ of $\tilde{h}_\bullet$ in $\mathsf{Ch}(\widetilde{\mathcal{C}})^n$, so $\langle M^{\widetilde{\mathcal{C}}}_{\tilde{h}_\bullet}, \tilde{\rho} \rangle$ is $\mathfrak{t}$-distinguished. Lastly, we note that $(\tilde{d}_0^{(Y,e')})_{\mathbb{F}}(\tilde{\delta}) = \tilde{\rho}$ because

$$(d_0^{(Y,e')})_{\mathbb{E}}\delta = (d_0^Y e_0)_{\mathbb{E}}\delta = (d_0^Y)_{\mathbb{E}}(e_0)_{\mathbb{E}}\delta = (d_0^Y)_{\mathbb{E}}\delta \ = \rho.$$

Hence, $\langle M^{\widetilde{\mathcal{C}}}_{\tilde{h}_\bullet}, (\tilde{d}_0^{(Y,e')})_{\mathbb{F}}\tilde{\delta} \rangle$ is a $\mathfrak{t}$-distinguished n-exangle. □

We are in position to prove axiom (EA2) for $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$. Axiom (EA2)$^{\mathrm{op}}$ can be shown dually.

Proposition 4.30 (Axiom (EA2) for $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$)**.** *Let* $\tilde{\delta} \in \mathbb{F}(\widetilde{X}_{n+1}, \widetilde{X}_0)$ *be an* $\mathbb{F}$*-extension and suppose* $\tilde{c}\colon \widetilde{Y}_{n+1} \to \widetilde{X}_{n+1}$ *is a morphism in* $\widetilde{\mathcal{C}}$*. Suppose* $\langle \widetilde{X}_\bullet, \tilde{\delta} \rangle$ *and* $\langle \widetilde{Y}_\bullet, \tilde{c}^{\mathbb{F}}\tilde{\delta} \rangle$ *are* $\mathfrak{t}$*-distinguished* n*-exangles. Then* $(\widetilde{\mathrm{id}}_{\widetilde{X}_0}, \tilde{c})$ *has a good lift* $\tilde{h}_\bullet\colon \widetilde{Y}_\bullet \to \widetilde{X}_\bullet$*.*

Proof. Notice that the underlying $\mathbb{E}$-extension of $\tilde{c}^{\mathbb{F}}\tilde{\delta}$ is $c^{\mathbb{E}}\delta$. By definition of $\mathfrak{t}$ and Remark 4.21, there are $\mathfrak{s}$-distinguished n-exangles $\langle X'_\bullet, \delta\rangle$ and $\langle Y'_\bullet, c^{\mathbb{E}}\delta\rangle$ and idempotent morphisms $e_\bullet\colon \langle X'_\bullet, \delta\rangle \to \langle X'_\bullet, \delta\rangle$ and $e'_\bullet\colon \langle Y'_\bullet, c^{\mathbb{E}}\delta\rangle \to \langle Y'_\bullet, c^{\mathbb{E}}\delta\rangle$, such that $\mathfrak{t}(\tilde{\delta}) = [(X'_\bullet, e_\bullet)]$ and $\mathfrak{t}(\tilde{c}^{\mathbb{F}}\tilde{\delta}) = [(Y'_\bullet, e'_\bullet)]$, and so that $\mathrm{id}_{X'_\bullet} - e_\bullet$ and $\mathrm{id}_{Y'_\bullet} - e'_\bullet$ are null homotopic in $\mathsf{Ch}(\mathcal{C})^n$. We note that since $[\widetilde{Y}_\bullet] = \mathfrak{t}(\tilde{c}^{\mathbb{F}}\tilde{\delta}) = [(Y'_\bullet, e'_\bullet)]$ and $[\widetilde{X}_\bullet] = \mathfrak{t}(\tilde{\delta}) = [(X'_\bullet, e_\bullet)]$, we have that $(Y'_0, e'_0) = \widetilde{Y}_0 = \widetilde{X}_0 = (X'_0, e_0)$ and, in particular, that $e_0 = e'_0$. Moreover, it follows that all the hypotheses of Lemma 4.27 are satisfied.

Therefore, by Lemma 4.29, the morphism $(\widetilde{\mathrm{id}}_{\widetilde{X}_0}, \tilde{c})\colon \tilde{c}^{\mathbb{F}}\tilde{\delta} \to \tilde{\delta}$ of $\mathbb{F}$-extensions has a good lift $\tilde{h}'_\bullet\colon (Y'_\bullet, e'_\bullet) \to (X'_\bullet, e_\bullet)$. Since $[(X'_\bullet, e_\bullet)] = [\widetilde{X}_\bullet]$ and $[(Y'_\bullet, e'_\bullet)] = [\widetilde{Y}_\bullet]$, there is homotopy equivalence $\tilde{a}_\bullet\colon (X'_\bullet, e_\bullet) \to \widetilde{X}_\bullet$ in $\mathsf{Ch}(\widetilde{\mathcal{C}})^n_{(\widetilde{X}_0, \widetilde{X}_{n+1})}$ and a homotopy equivalence $\tilde{b}_\bullet\colon \widetilde{Y}_\bullet \to (Y'_\bullet, e'_\bullet)$ in $\mathsf{Ch}(\widetilde{\mathcal{C}})^n_{(\widetilde{Y}_0, \widetilde{Y}_{n+1})}$. By [HLN21, Cor. 2.31], the composite $\tilde{a}_\bullet \tilde{h}'_\bullet \tilde{b}_\bullet\colon \widetilde{Y}_\bullet \to \widetilde{X}_\bullet$ is then also a good lift of $(\widetilde{\mathrm{id}}_{\widetilde{X}_0}, \tilde{c})$. □

4.5 Main results

In this subsection we present our main results regarding the idempotent completion and an n-exangulated structure we can impose on it.

Definition 4.31. We call an n-exangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ (resp. *weakly*) *idempotent complete* if the underlying additive category $\mathcal{C}$ is (resp. weakly) idempotent complete.

In [BHST22, Prop. 2.2] a characterisation of weakly idempotent complete extriangulated categories is given. Next we note that the first part of Theorem 1.1 from Section 1 summarises our work from Subsections 4.1–4.4.

Theorem 4.32. *Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an n-exangulated category. Then the triplet $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$ is an idempotent complete n-exangulated category.*

Proof. This follows from Propositions 2.5, 4.22, 4.26 and 4.30, and the duals of the latter two. □

And Corollary 1.3 from Section 1 is a nice consequence of this.

Corollary 4.33. *Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an n-exangulated category, such that each object in $\mathcal{C}$ has a semi-perfect endomorphism ring. Then the idempotent completion $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$ is a Krull-Schmidt n-exangulated category.*

Proof. By Theorem 4.32, the idempotent completion $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$ is an idempotent complete n-exangulated category. By [CYZ08, Thm. A.1] (or [Kra15, Cor. 4.4]), it is enough to show that endomorphism rings of objects in $\widetilde{\mathcal{C}}$ are semi-perfect rings. Let (X, e) be an object in $\widetilde{\mathcal{C}}$. We have that $\mathrm{End}_{\widetilde{\mathcal{C}}}((X, \mathrm{id}_X)) \cong \mathrm{End}_{\mathcal{C}}(X)$ is semi-perfect since $\mathscr{I}_{\mathcal{C}}$ is fully faithful (see Proposition 2.5). By Remark 2.6, we have that $(X, \mathrm{id}_X) \cong (X, e) \oplus (X, \mathrm{id}_X - e)$. In particular, we see that $\mathrm{End}_{\widetilde{\mathcal{C}}}((X, e))$ is an idempotent subring of the semi-perfect ring $\mathrm{End}_{\widetilde{\mathcal{C}}}((X, \mathrm{id}_X))$. Hence, by Anderson–Fuller [AF92, Cor. 27.7], we have that the endomorphism ring of each object in $\widetilde{\mathcal{C}}$ is semi-perfect. □

We recall that, by [Kla23, Cor. 4.12], an n-exangulated category is n-exact if and only if its inflations are monomorphisms and its deflations are epimorphisms.

Corollary 4.34. *Suppose* $(\mathcal{C},\mathbb{E},\mathfrak{s})$ *is* n*-exact. Then* $(\widetilde{\mathcal{C}},\mathbb{F},\mathfrak{t})$ *is* n*-exact.*

Proof. We use [Kla23, Cor. 4.12] and only show that $\mathfrak{t}$-inflations are monomorphisms; showing $\mathfrak{t}$-deflations are epimorphisms is dual. Let $\tilde{f}\colon (X_0,e_0)\to(X_1,e_1)$ be a $\mathfrak{t}$-inflation and suppose there is a morphism $\tilde{g}\colon (Y_0,e_0')\to(X_0,e_0)$ in $\widetilde{\mathcal{C}}$ with $\tilde{f}\tilde{g}=\widetilde{0}$. By Lemma 4.25, there is an $\mathfrak{s}$-inflation $d_0^{X'}=\left[\begin{smallmatrix} f & f'(\mathrm{id}_{X_0}-e_0)\end{smallmatrix}\right]^\top\colon X_0\to X_1\oplus C$, which is monic as $(\mathcal{C},\mathbb{E},\mathfrak{s})$ is n-exact. We have $fg=0$ as $\tilde{f}\tilde{g}=\widetilde{0}$, and we also have $f'(\mathrm{id}_{X_0}-e_0)g=0$ because the underlying morphism g of $\tilde{g}$ satisfies $g=e_0g$. Thus, we see that $d_0^{X'}g=0$ and this implies $g=0$ as $d_0^{X'}$ is monic. Hence, $\tilde{g}=0$ and we are done. □

The main aim of this subsection is to establish the relevant 2-universal property of the inclusion functor $\mathscr{I}_{\mathcal{C}}\colon\mathcal{C}\to\widetilde{\mathcal{C}}$. We will show that $\mathscr{I}_{\mathcal{C}}$ forms part of an n-exangulated functor $(\mathscr{I}_{\mathcal{C}},\Gamma)\colon(\mathcal{C},\mathbb{E},\mathfrak{s})\to(\widetilde{\mathcal{C}},\mathbb{F},\mathfrak{t})$, and that this is 2-universal in an appropriate sense. The next lemma is straightforward to check.

Lemma 4.35. *The family of abelian group homomorphisms*

$$\begin{aligned}\Gamma_{(X_{n+1},X_0)}\colon\mathbb{E}(X_{n+1},X_0)&\longrightarrow\mathbb{F}(\mathscr{I}_{\mathcal{C}}(X_{n+1}),\mathscr{I}_{\mathcal{C}}(X_0))\\ \delta&\longmapsto(\mathrm{id}_{X_0},\delta,\mathrm{id}_{X_{n+1}}),\end{aligned}$$

for $X_0,X_{n+1}\in\mathcal{C}$*, defines a natural isomorphism* $\Gamma\colon\mathbb{E}(-,-)\overset{\cong}{\Longrightarrow}\mathbb{F}(\mathscr{I}_{\mathcal{C}}-,\mathscr{I}_{\mathcal{C}}-)$.

Proposition 4.36. *The pair* $(\mathscr{I}_{\mathcal{C}},\Gamma)$ *is an* n*-exangulated functor from* $(\mathcal{C},\mathbb{E},\mathfrak{s})$ *to* $(\widetilde{\mathcal{C}},\mathbb{F},\mathfrak{t})$.

Proof. We show if $\langle X_\bullet,\delta\rangle$ is an $\mathfrak{s}$-distinguished n-exangle, then $\langle\mathscr{I}_{\mathcal{C}}(X_\bullet),\Gamma_{(X_{n+1},X_0)}(\delta)\rangle$ is $\mathfrak{t}$-distinguished, where $\Gamma_{(X_{n+1},X_0)}(\delta)=(\mathrm{id}_{X_0},\delta,\mathrm{id}_{X_{n+1}})\in\mathbb{F}(\mathscr{I}_{\mathcal{C}}(X_{n+1}),\mathscr{I}_{\mathcal{C}}(X_0))$. We have the idempotent morphism $\mathrm{id}_{X_\bullet}\colon\langle X_\bullet,\delta\rangle\to\langle X_\bullet,\delta\rangle$ that is a lift of $(\mathrm{id}_{X_0},\mathrm{id}_{X_{n+1}})\colon\delta\to\delta$, so from Definition 4.20 we see that $\mathfrak{t}(\Gamma_{(X_{n+1},X_0)}(\delta))=[(X_\bullet,\mathrm{id}_{X_\bullet})]=[\mathscr{I}_{\mathcal{C}}(X_\bullet)]$. □

We lay out some notation that will be used in the remainder of this section and also in Section 5.

Notation 4.37. Let (X,e) be an object in the idempotent completion $\widetilde{\mathcal{C}}$ of $\mathcal{C}$. Then (X,e) is a direct summand of $\mathscr{I}_{\mathcal{A}}(X)=(X,\mathrm{id}_X)$ by Remark 2.6. By $\tilde{i}_e:=(\mathrm{id}_X,e,e)\colon(X,e)\to(X,\mathrm{id}_X)$ and $\tilde{p}_e:=(e,e,\mathrm{id}_X)\colon(X,\mathrm{id}_X)\to(X,e)$, we denote the canonical inclusion and projection morphisms, respectively.

Recall that, for an additive category $\mathcal{C}'$ and a biadditive functor $\mathbb{E}'\colon(\mathcal{C}')^{\mathrm{op}}\times\mathcal{C}'\to\mathsf{Ab}$, the $\mathbb{E}'$-attached complexes and morphisms between them were defined in Definition 3.1, and together they form an additive category.

Lemma 4.38. *Let* $\tilde{\delta}\in\mathbb{F}((X_{n+1},e_{n+1}),(X_0,e_0))$ *be an* $\mathbb{F}$*-extension. Suppose* $\mathfrak{s}(\delta)=[X_\bullet]$ *and* $e_\bullet\colon\langle X_\bullet,\delta\rangle\to\langle X_\bullet,\delta\rangle$ *is an idempotent morphism. With* $e_\bullet':=\mathrm{id}_{X_\bullet}-e_\bullet$*, we have that*

$$\langle\mathscr{I}_{\mathcal{C}}(X_\bullet),\Gamma_{(X_{n+1},X_0)}(\delta)\rangle\cong\langle(X_\bullet,e_\bullet),\tilde{\delta}\rangle\oplus\langle(X_\bullet,e_\bullet'),{}_{(X_0,e_0')}\widetilde{0}_{(X_{n+1},e_{n+1}')}\rangle \tag{4.10}$$

as $\mathfrak{t}$*-distinguished* n*-exangles.*

Proof. Let $\tilde{\rho} := \Gamma_{(X_{n+1},X_0)}(\delta)$. Note that $\langle (X_\bullet, e_\bullet), \tilde{\delta}\rangle$ and $\langle (X_\bullet, e'_\bullet), {}_{(X_0,e'_0)}\widetilde{0}_{(X_{n+1},e'_{n+1})}\rangle$ are $\mathbb{F}$-attached complexes, and $\langle \mathscr{I}_{\mathcal{C}}(X_\bullet), \tilde{\rho}\rangle$ is an $\mathfrak{t}$-distinguished n-exangle since $(\mathscr{F}, \Gamma)$ is an n-exangulated functor.

Consider the morphisms $\tilde{i}_{e_\bullet} : (X_\bullet, e_\bullet) \to \mathscr{I}_{\mathcal{C}}(X_\bullet)$ and $\tilde{p}_{e_\bullet} : \mathscr{I}_{\mathcal{C}}(X_\bullet) \to (X_\bullet, e_\bullet)$ of complexes induced by $e_\bullet$, as well as the corresponding ones $\tilde{i}_{e'_\bullet}$ and $\tilde{p}_{e'_\bullet}$ for $e'_\bullet$. We claim that there is a biproduct diagram

$$\langle (X_\bullet, e_\bullet), \tilde{\delta}\rangle \underset{\tilde{p}_{e_\bullet}}{\overset{\tilde{i}_{e_\bullet}}{\rightleftarrows}} \langle \mathscr{I}_{\mathcal{C}}(X_\bullet), \tilde{\rho}\rangle \underset{\tilde{p}_{e'_\bullet}}{\overset{\tilde{i}_{e'_\bullet}}{\leftrightarrows}} \langle (X_\bullet, e'_\bullet), {}_{(X_0,e'_0)}\widetilde{0}_{(X_{n+1},e'_{n+1})}\rangle, \tag{4.11}$$

in the category of $\mathbb{F}$-attached complexes. To see that $\tilde{i}_{e_\bullet}$ is a morphism of $\mathbb{F}$-attached complexes, we just need to check that $(\tilde{i}_{e_0})_{\mathbb{F}}\tilde{\delta} = (\tilde{i}_{e_{n+1}})^{\mathbb{F}}\tilde{\rho}$. By Remark 4.7, it is enough to see that $(e_0)_{\mathbb{E}}\delta = (e_{n+1})^{\mathbb{E}}\delta$ holds, and this is indeed true because $\delta = (e_0)_{\mathbb{E}}\delta = (e_{n+1})^{\mathbb{E}}\delta$. Similarly, we see that $\tilde{p}_{e_\bullet}$ is a morphism of $\mathbb{F}$-attached complexes. To see that $\tilde{i}_{e'_\bullet}$ and $\tilde{p}_{e'_\bullet}$ are morphisms of $\mathbb{F}$-attached complexes, one uses that $(\mathrm{id}_{X_0} - e_0)_{\mathbb{E}}\delta = {}_{X_0}0_{X_{n+1}} = (\mathrm{id}_{X_{n+1}} - e_{n+1})^{\mathbb{E}}\delta$. Furthermore, we have the identities $\widetilde{\mathrm{id}}_{(X_\bullet,e_\bullet)} = \tilde{p}_{e_\bullet}\tilde{i}_{e_\bullet}$, $\widetilde{\mathrm{id}}_{(X_\bullet,e'_\bullet)} = \tilde{p}_{e'_\bullet}\tilde{i}_{e'_\bullet}$ and $\widetilde{\mathrm{id}}_{\mathscr{I}_{\mathcal{C}}(X_\bullet)} = \tilde{i}_{e_\bullet}\tilde{p}_{e_\bullet} + \tilde{i}_{e'_\bullet}\tilde{p}_{e'_\bullet}$, so (4.11) is a biproduct diagram in the additive category of $\mathbb{F}$-attached complexes. Therefore, we have that (4.10) is an isomorphism as $\mathbb{F}$-attached complexes.

Lastly, since $\langle \mathscr{I}_{\mathcal{C}}(X_\bullet), \tilde{\rho}\rangle$ is $\mathfrak{t}$-distinguished, it follows from [HLN21, Prop. 3.3] that (4.10) is an isomorphism of $\mathfrak{t}$-distinguished n-exangles. $\square$

Thus, we can now present and prove the main result of this section, which shows that the n-exangulated inclusion functor $(\mathscr{I}_{\mathcal{C}}, \Gamma)\colon (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$ is 2-universal amongst n-exangulated functors from $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ to idempotent complete n-exangulated categories.

Theorem 4.39. *Suppose $(\mathscr{F}, \Lambda)\colon (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ is an n-exangulated functor to an idempotent complete n-exangulated category $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$. Then the following statements hold.*

(i) *There is an n-exangulated functor $(\mathscr{E}, \Psi)\colon (\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t}) \to (\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ and an n-exangulated natural isomorphism $\text{צ}\colon (\mathscr{F}, \Lambda) \overset{\cong}{\Longrightarrow} (\mathscr{E}, \Psi) \circ (\mathscr{I}_{\mathcal{C}}, \Gamma)$.*

(ii) *In addition, for any n-exangulated functor $(\mathscr{G}, \Theta)\colon (\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t}) \to (\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ and any n-exangulated natural transformation $\daleth\colon (\mathscr{F}, \Lambda) \Rightarrow (\mathscr{G}, \Theta) \circ (\mathscr{I}_{\mathcal{C}}, \Gamma)$, there is a unique n-exangulated natural transformation $\text{מ}\colon (\mathscr{E}, \Psi) \Rightarrow (\mathscr{G}, \Theta)$ with $\daleth = \text{מ}_{\mathscr{I}_{\mathcal{C}}}\text{צ}$.*

Proof. (i) By Proposition 2.8(i), there exists an additive functor $\mathscr{E}\colon \widetilde{\mathcal{C}} \to \mathcal{C}'$ and a natural isomorphism $\text{צ}\colon \mathscr{F} \Rightarrow \mathscr{E}\mathscr{I}_{\mathcal{C}}$. It remains to show that $\mathscr{E}$ forms part of an n-exangulated functor $(\mathscr{E}, \Psi)$ and that צ is n-exangulated.

First, we define a natural transformation $\Psi\colon \mathbb{F}(-,-) \Rightarrow \mathbb{E}'(\mathscr{E}-, \mathscr{E}-)$ as the composition of several abelian group homomorphisms. For $X_0, X_{n+1} \in \mathcal{C}$, we set

$$T_{(X_{n+1},X_0)} := \mathbb{E}'(\text{צ}^{-1}_{X_{n+1}}, \text{צ}_{X_0})\colon \mathbb{E}'(\mathscr{F}(X_{n+1}), \mathscr{F}(X_0)) \to \mathbb{E}'(\mathscr{E}\mathscr{I}_{\mathcal{C}}(X_{n+1}), \mathscr{E}\mathscr{I}_{\mathcal{C}}(X_0)).$$

For $\widetilde{X}_0 = (X_0, e_0)$ and $\widetilde{X}_{n+1} = (X_{n+1}, e_{n+1})$ in $\widetilde{\mathcal{C}}$, we define an abelian group homomorphism

$$\begin{aligned} I_{(\widetilde{X}_{n+1},\widetilde{X}_0)} \colon \mathbb{F}(\widetilde{X}_{n+1}, \widetilde{X}_0) &\longrightarrow \mathbb{E}(X_{n+1}, X_0) \\ \tilde{\delta} = (e_0, \delta, e_{n+1}) &\longmapsto \delta, \end{aligned}$$

and put

$$P_{(\widetilde{X}_{n+1},\widetilde{X}_0)} := \mathbb{E}'(\mathscr{E}(\tilde{i}_{e_{n+1}}), \mathscr{E}(\tilde{p}_{e_0}))\colon \mathbb{E}'(\mathscr{E}\mathscr{I}_{\mathcal{C}}(X_{n+1}), \mathscr{E}\mathscr{I}_{\mathcal{C}}(X_0)) \to \mathbb{E}'(\mathscr{E}(\widetilde{X}_{n+1}), \mathscr{E}(\widetilde{X}_0)).$$

For morphisms $a\colon X_0 \to Y_0$ and $c\colon Y_{n+1} \to X_{n+1}$ in $\mathcal{C}$, we have

$$T_{(Y_{n+1},Y_0)}\mathbb{E}'(\mathscr{F}(c), \mathscr{F}(a)) = \mathbb{E}'(\mathscr{E}\mathscr{I}_{\mathcal{C}}(c), \mathscr{E}\mathscr{I}_{\mathcal{C}}(a))T_{(X_{n+1},X_0)} \tag{4.12}$$

as $\mathfrak{T}$ is natural. For morphisms $\tilde{a}\colon \widetilde{X}_0 \to \widetilde{Y}_0$ and $\tilde{c}\colon \widetilde{Y}_{n+1} \to \widetilde{X}_{n+1}$ in $\widetilde{\mathcal{C}}$, we have

$$I_{(\widetilde{Y}_{n+1},\widetilde{Y}_0)}\mathbb{F}(\tilde{c}, \tilde{a}) = \mathbb{E}(c, a)I_{(\widetilde{X}_{n+1},\widetilde{X}_0)}, \tag{4.13}$$

using how $\mathbb{F}$ is defined on morphisms (see Definition 4.4). We claim that the family of abelian group homomorphisms

$$\Psi_{(\widetilde{X}_{n+1},\widetilde{X}_0)} := P_{(\widetilde{X}_{n+1},\widetilde{X}_0)}T_{(X_{n+1},X_0)}\Lambda_{(X_{n+1},X_0)}I_{(\widetilde{X}_{n+1},\widetilde{X}_0)}$$

for $\widetilde{X}_0, \widetilde{X}_{n+1} \in \widetilde{\mathcal{C}}$ defines a natural transformation $\Psi\colon \mathbb{F}(-,-) \Rightarrow \mathbb{E}'(\mathscr{E}-, \mathscr{E}-)$. To this end, fix objects $\widetilde{X}_0 = (X_0, e_0), \widetilde{Y}_0 = (Y_0, e'_0), \widetilde{X}_{n+1} = (X_{n+1}, e_{n+1})$ and $\widetilde{Y}_{n+1} = (Y_{n+1}, e'_{n+1})$, and morphisms $\tilde{a}\colon \widetilde{X}_0 \to \widetilde{Y}_0$ and $\tilde{c}\colon \widetilde{Y}_{n+1} \to \widetilde{X}_{n+1}$ in $\widetilde{\mathcal{C}}$. First, note that we have

$$\tilde{p}_{e'_0}\mathscr{I}_{\mathcal{C}}(a) = \tilde{p}_{e'_0}\mathscr{I}_{\mathcal{C}}(a)\mathscr{I}_{\mathcal{C}}(e_0) = \tilde{p}_{e'_0}\mathscr{I}_{\mathcal{C}}(a)\tilde{i}_{e_0}\tilde{p}_{e_0} = \tilde{a}\tilde{p}_{e_0} \tag{4.14}$$

and, similarly,

$$\mathscr{I}_{\mathcal{C}}(c)\tilde{i}_{e'_{n+1}} = \tilde{i}_{e_{n+1}}\tilde{c}. \tag{4.15}$$

Therefore, we see that

$$\begin{aligned} &\Psi_{(\widetilde{Y}_{n+1},\widetilde{Y}_0)}\mathbb{F}(\tilde{c}, \tilde{a}) \\ &\quad = P_{(\widetilde{Y}_{n+1},\widetilde{Y}_0)}T_{(Y_{n+1},Y_0)}\Lambda_{(Y_{n+1},Y_0)}I_{(\widetilde{Y}_{n+1},\widetilde{Y}_0)}\mathbb{F}(\tilde{c}, \tilde{a}) \\ &\quad = P_{(\widetilde{Y}_{n+1},\widetilde{Y}_0)}T_{(Y_{n+1},Y_0)}\Lambda_{(Y_{n+1},Y_0)}\mathbb{E}(c, a)I_{(\widetilde{X}_{n+1},\widetilde{X}_0)} && \text{by (4.13)} \\ &\quad = P_{(\widetilde{Y}_{n+1},\widetilde{Y}_0)}T_{(Y_{n+1},Y_0)}\mathbb{E}'(\mathscr{F}(c), \mathscr{F}(a))\Lambda_{(X_{n+1},X_0)}I_{(\widetilde{X}_{n+1},\widetilde{X}_0)} && \text{as } \Lambda \text{ is natural} \\ &\quad = P_{(\widetilde{Y}_{n+1},\widetilde{Y}_0)}\mathbb{E}'(\mathscr{E}\mathscr{I}_{\mathcal{C}}(c), \mathscr{E}\mathscr{I}_{\mathcal{C}}(a))T_{(X_{n+1},X_0)}\Lambda_{(X_{n+1},X_0)}I_{(\widetilde{X}_{n+1},\widetilde{X}_0)} && \text{by (4.12)} \\ &\quad = \mathbb{E}'(\mathscr{E}(\mathscr{I}_{\mathcal{C}}(c)\tilde{i}_{e'_{n+1}}), \mathscr{E}(\tilde{p}_{e'_0}\mathscr{I}_{\mathcal{C}}(a)))T_{(X_{n+1},X_0)}\Lambda_{(X_{n+1},X_0)}I_{(\widetilde{X}_{n+1},\widetilde{X}_0)} \\ &\quad = \mathbb{E}'(\mathscr{E}(\tilde{i}_{e_{n+1}}\tilde{c}), \mathscr{E}(\tilde{a}\tilde{p}_{e_0}))T_{(X_{n+1},X_0)}\Lambda_{(X_{n+1},X_0)}I_{(\widetilde{X}_{n+1},\widetilde{X}_0)} && \text{by (4.14), (4.15)} \\ &\quad = \mathbb{E}'(\mathscr{E}(\tilde{c}), \mathscr{E}(\tilde{a}))P_{(\widetilde{X}_{n+1},\widetilde{X}_0)}T_{(X_{n+1},X_0)}\Lambda_{(X_{n+1},X_0)}I_{(\widetilde{X}_{n+1},\widetilde{X}_0)} \end{aligned}$$

$$= \mathbb{E}'(\mathscr{E}(\tilde{c}), \mathscr{E}(\tilde{a}))\Psi_{(\widetilde{X}_{n+1}, \widetilde{X}_0)}.$$

Next, we must show that $(\mathscr{E}, \Psi)$ sends $\mathfrak{t}$-distinguished n-exangles to $\mathfrak{s}'$-distinguished n-exangles. Thus, let $\widetilde{X}_0 = (X_0, e_0), \widetilde{X}_{n+1} = (X_{n+1}, e_{n+1}) \in \widetilde{\mathcal{C}}$ and $\tilde{\delta} \in \mathbb{F}(\widetilde{X}_{n+1}, \widetilde{X}_0)$, and suppose $\mathfrak{t}(\tilde{\delta}) = [\widetilde{X}_\bullet]$. We need that $\mathfrak{s}'(\Psi_{(\widetilde{X}_{n+1}, \widetilde{X}_0)}(\tilde{\delta})) = [\mathscr{E}(\widetilde{X}_\bullet)]$, which will follow from seeing that $\langle \mathscr{E}(\widetilde{X}_\bullet), \Psi_{(\widetilde{X}_{n+1}, \widetilde{X}_0)}(\tilde{\delta})\rangle$ is a direct summand of an $\mathfrak{s}'$-distinguished n-exangle.

By Remark 4.21, we may take a complex $X_\bullet$ in $\mathsf{Ch}(\mathcal{C})^n$ with $\mathfrak{s}(\delta) = [X_\bullet]$ and an idempotent morphism $e_\bullet\colon \langle X_\bullet, \delta\rangle \to \langle X_\bullet, \delta\rangle$ lifting $(e_0, e_{n+1})\colon \delta \to \delta$, such that $\mathfrak{t}(\tilde{\delta}) = [(X_\bullet, e_\bullet)]$. Note for later that we thus have $[(X_\bullet, e_\bullet)] = [\widetilde{X}_\bullet]$, and hence $[\mathscr{E}((X_\bullet, e_\bullet))] = [\mathscr{E}(\widetilde{X}_\bullet)]$. Let $\tilde{\rho} := \Gamma_{(X_{n+1}, X_0)}(\delta)$. Since $(\mathscr{F}, \Lambda)\colon (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ is an n-exangulated functor, the n-exangle $\langle \mathscr{F}(X_\bullet), \Lambda_{(X_{n+1}, X_0)}(\delta)\rangle$ is $\mathfrak{s}'$-distinguished. As we have an isomorphism of complexes $\text{צ}_{X_\bullet}\colon \mathscr{F}(X_\bullet) \to \mathscr{E}\mathscr{I}_{\mathcal{C}}(X_\bullet)$, the $\mathbb{E}'$-attached complex

$$\langle \mathscr{E}\mathscr{I}_{\mathcal{C}}(X_\bullet), T_{(X_{n+1}, X_0)}\Lambda_{(X_{n+1}, X_0)}(\delta)\rangle$$

is $\mathfrak{s}'$-distinguished by [HLN21, Cor. 2.26(2)]. Since

$$P_{(\mathscr{I}_{\mathcal{C}}(X_{n+1}), \mathscr{I}_{\mathcal{C}}(X_0))} = \mathbb{E}'(\mathscr{E}(\tilde{i}_{\mathrm{id}_{X_{n+1}}}), \mathscr{E}(\tilde{p}_{\mathrm{id}_{X_0}}))$$

is just the identity homomorphism, a quick computation yields

$$\Psi_{(\mathscr{I}_{\mathcal{C}}(X_{n+1}), \mathscr{I}_{\mathcal{C}}(X_0))}(\tilde{\rho}) = T_{(X_{n+1}, X_0)}\Lambda_{(X_{n+1}, X_0)}(\delta) = \mathbb{E}'(\text{צ}^{-1}_{X_{n+1}}, \text{צ}_{X_0})\Lambda_{(X_{n+1}, X_0)}(\delta). \tag{4.16}$$

In particular, this implies that $\langle \mathscr{E}\mathscr{I}_{\mathcal{C}}(X_\bullet), \Psi_{(\mathscr{I}_{\mathcal{C}}(X_{n+1}), \mathscr{I}_{\mathcal{C}}(X_0))}(\tilde{\rho})\rangle$ is $\mathfrak{s}'$-distinguished.

Note that $\langle \mathscr{I}_{\mathcal{C}}(X_\bullet), \tilde{\rho}\rangle \cong \langle (X_\bullet, e_\bullet), \tilde{\delta}\rangle \oplus \langle (X_\bullet, e'_\bullet), {}_{(X_0, e'_0)}\tilde{0}_{(X_{n+1}, e'_{n+1})}\rangle$ as $\mathbb{F}$-attached complexes by Lemma 4.38, where $e'_\bullet := \mathrm{id}_{X_\bullet} - e_\bullet$. We see that $\langle \mathscr{E}((X_\bullet, e_\bullet)), \Psi_{(\widetilde{X}_{n+1}, \widetilde{X}_0)}(\tilde{\delta})\rangle$ is a direct summand of the $\mathfrak{s}'$-distinguished n-exangle $\langle \mathscr{E}\mathscr{I}_{\mathcal{C}}(X_\bullet), \Psi_{(\mathscr{I}_{\mathcal{C}}(X_{n+1}), \mathscr{I}_{\mathcal{C}}(X_0))}(\tilde{\rho})\rangle$ by Proposition 3.7(ii). Hence, $\mathfrak{s}'(\Psi_{(\widetilde{X}_{n+1}, \widetilde{X}_0)}(\tilde{\delta})) = [\mathscr{E}((X_\bullet, e_\bullet))] = [\mathscr{E}(\widetilde{X}_\bullet)]$ by [HLN21, Prop. 3.3], and so $(\mathscr{E}, \Psi)$ is an n-exangulated functor.

Lastly, it follows immediately from (4.16) that צ is an n-exangulated natural transformation $(\mathscr{F}, \Lambda) \Rightarrow (\mathscr{E}, \Psi) \circ (\mathscr{I}_{\mathcal{C}}, \Gamma) = (\mathscr{E}\mathscr{I}_{\mathcal{C}}, \Psi_{\mathscr{I}_{\mathcal{C}} \times \mathscr{I}_{\mathcal{C}}}\Gamma)$.

(ii) By Proposition 2.8(ii), there exists a unique natural transformation $\text{מ}\colon \mathscr{E} \Rightarrow \mathscr{G}$ with $\daleth = \text{מ}_{\mathscr{I}_{\mathcal{C}}}\text{צ}$, so it remains to show that מ induces an n-exangulated natural transformation $(\mathscr{E}, \Psi) \Rightarrow (\mathscr{G}, \Theta)$. For this, let $\widetilde{X}_0 = (X_0, e_0), \widetilde{X}_{n+1} = (X_{n+1}, e_{n+1}) \in \widetilde{\mathcal{C}}$ and $\tilde{\delta} \in \mathbb{F}(\widetilde{X}_{n+1}, \widetilde{X}_0)$ be arbitrary. Note that we have

$$\tilde{\delta} = \mathbb{F}(\tilde{i}_{e_{n+1}}, \tilde{p}_{e_0})\Gamma_{(X_{n+1}, X_0)}(\delta). \tag{4.17}$$

Hence, we obtain

$$\begin{aligned}
&(\text{מ}_{\widetilde{X}_0})_{\mathbb{E}'}\Psi_{(\widetilde{X}_{n+1}, \widetilde{X}_0)}(\tilde{\delta}) \\
&\quad = (\text{מ}_{\widetilde{X}_0})_{\mathbb{E}'}\Psi_{(\widetilde{X}_{n+1}, \widetilde{X}_0)}\mathbb{F}(\tilde{i}_{e_{n+1}}, \tilde{p}_{e_0})\Gamma_{(X_{n+1}, X_0)}(\delta) && \text{by (4.17)} \\
&\quad = (\text{מ}_{\widetilde{X}_0})_{\mathbb{E}'}\mathbb{E}'(\mathscr{E}(\tilde{i}_{e_{n+1}}), \mathscr{E}(\tilde{p}_{e_0}))(\Psi_{\mathscr{I}_{\mathcal{C}} \times \mathscr{I}_{\mathcal{C}}}\Gamma)_{(X_{n+1}, X_0)}(\delta) && \text{as } \Psi \text{ is natural}
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}'(\mathscr{E}(\tilde{i}_{e_{n+1}}), \text{מ}_{\widetilde{X}_0}\mathscr{E}(\tilde{p}_{e_0}))(\Psi_{\mathscr{I}_\mathcal{C}\times\mathscr{I}_\mathcal{C}}\Gamma)_{(X_{n+1},X_0)}(\delta) \\
&= \mathbb{E}'(\mathscr{E}(\tilde{i}_{e_{n+1}}), \mathscr{G}(\tilde{p}_{e_0})\text{מ}_{\mathscr{I}_\mathcal{C}(X_0)})(\Psi_{\mathscr{I}_\mathcal{C}\times\mathscr{I}_\mathcal{C}}\Gamma)_{(X_{n+1},X_0)}(\delta) && \text{as } \text{מ} \text{ is natural} \\
&= \mathbb{E}'(\mathscr{E}(\tilde{i}_{e_{n+1}}), \mathscr{G}(\tilde{p}_{e_0})\,\text{ד}_{X_0}\,\text{צ}^{-1}_{X_0})(\Psi_{\mathscr{I}_\mathcal{C}\times\mathscr{I}_\mathcal{C}}\Gamma)_{(X_{n+1},X_0)}(\delta) && \text{as } \text{ד} = \text{מ}_{\mathscr{I}_\mathcal{C}}\text{צ} \\
&= \mathbb{E}'(\mathscr{E}(\tilde{i}_{e_{n+1}}), \mathscr{G}(\tilde{p}_{e_0}))(\text{ד}_{X_0})_{\mathbb{E}'}(\text{צ}^{-1}_{X_0})_{\mathbb{E}'}(\Psi_{\mathscr{I}_\mathcal{C}\times\mathscr{I}_\mathcal{C}}\Gamma)_{(X_{n+1},X_0)}(\delta) \\
&= \mathbb{E}'(\mathscr{E}(\tilde{i}_{e_{n+1}}), \mathscr{G}(\tilde{p}_{e_0}))(\text{ד}_{X_0})_{\mathbb{E}'}(\text{צ}^{-1}_{X_{n+1}})^{\mathbb{E}'}\Lambda_{(X_{n+1},X_0)}(\delta) && \text{as } \text{צ} \text{ is } n\text{-exan.} \\
&= \mathbb{E}'(\mathscr{E}(\tilde{i}_{e_{n+1}}), \mathscr{G}(\tilde{p}_{e_0}))(\text{צ}^{-1}_{X_{n+1}})^{\mathbb{E}'}(\text{ד}_{X_0})_{\mathbb{E}'}\Lambda_{(X_{n+1},X_0)}(\delta) \\
&= \mathbb{E}'(\mathscr{E}(\tilde{i}_{e_{n+1}}), \mathscr{G}(\tilde{p}_{e_0}))(\text{צ}^{-1}_{X_{n+1}})^{\mathbb{E}'}(\text{ד}_{X_{n+1}})^{\mathbb{E}'}(\Theta_{\mathscr{I}_\mathcal{C}\times\mathscr{I}_\mathcal{C}}\Gamma)_{(X_{n+1},X_0)}(\delta) && \text{as } \text{ד} \text{ is } n\text{-exan.} \\
&= \mathbb{E}'(\text{ד}_{X_{n+1}}\,\text{צ}^{-1}_{X_{n+1}}\,\mathscr{E}(\tilde{i}_{e_{n+1}}), \mathscr{G}(\tilde{p}_{e_0}))(\Theta_{\mathscr{I}_\mathcal{C}\times\mathscr{I}_\mathcal{C}}\Gamma)_{(X_{n+1},X_0)}(\delta) \\
&= \mathbb{E}'(\text{מ}_{\mathscr{I}_\mathcal{C}(X_{n+1})}\mathscr{E}(\tilde{i}_{e_{n+1}}), \mathscr{G}(\tilde{p}_{e_0}))(\Theta_{\mathscr{I}_\mathcal{C}\times\mathscr{I}_\mathcal{C}}\Gamma)_{(X_{n+1},X_0)}(\delta) && \text{as } \text{ד} = \text{מ}_{\mathscr{I}_\mathcal{C}}\text{צ} \\
&= \mathbb{E}'(\mathscr{G}(\tilde{i}_{e_{n+1}})\text{מ}_{\widetilde{X}_{n+1}}, \mathscr{G}(\tilde{p}_{e_0}))(\Theta_{\mathscr{I}_\mathcal{C}\times\mathscr{I}_\mathcal{C}}\Gamma)_{(X_{n+1},X_0)}(\delta) && \text{as } \text{מ} \text{ is natural} \\
&= (\text{מ}_{\widetilde{X}_{n+1}})^{\mathbb{E}'}\mathbb{E}'(\mathscr{G}(\tilde{i}_{e_{n+1}}), \mathscr{G}(\tilde{p}_{e_0}))(\Theta_{\mathscr{I}_\mathcal{C}\times\mathscr{I}_\mathcal{C}}\Gamma)_{(X_{n+1},X_0)}(\delta) \\
&= (\text{מ}_{\widetilde{X}_{n+1}})^{\mathbb{E}'}\Theta_{(\widetilde{X}_{n+1},\widetilde{X}_0)}\mathbb{F}(\tilde{i}_{e_{n+1}}, \tilde{p}_{e_0})\Gamma_{(X_{n+1},X_0)}(\delta) && \text{as } \Theta \text{ is natural} \\
&= (\text{מ}_{\widetilde{X}_{n+1}})^{\mathbb{E}'}\Theta_{(\widetilde{X}_{n+1},\widetilde{X}_0)}(\tilde{\delta}) && \text{by (4.17),}
\end{aligned}
$$

and the proof is complete. □

We close this section with some remarks on our main results and constructions.

Remark 4.40. Before commenting on how our results unify the constructions in cases in the literature and on how our proof methods compare, we set up and recall a little terminology. Suppose $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ and $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ are n-exangulated categories. We call an n-exangulated functor $(\mathscr{F}, \Gamma)\colon (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ an *n-exangulated isomorphism* if $\mathscr{F}$ is an isomorphism of categories and Γ is a natural isomorphism. This terminology is justified by [BTHSS23, Prop. 4.11]. Lastly, we recall that n-exangulated functors between $(n+2)$-angulated categories are *$(n+2)$-angulated* in the sense of [BTS21, Def. 2.7] (or *exact* as in Bergh–Thaule [BT14, Sec. 4]), and that n-exangulated functors between n-exact categories are *n-exact* in the sense of [BTS21, Def. 2.18]; see [BTS21, Thms. 2.33, 2.34].

It has been shown that a triplet $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is a 1-exangulated category if and only if it is extriangulated (see [HLN21, Prop. 4.3]). Suppose that $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category and consider the idempotent completion $\widetilde{\mathcal{C}}$ of $\mathcal{C}$. By [Msa22, Thm. 3.1], there is an extriangulated structure $(\mathbb{F}', \mathfrak{t}')$ on $\widetilde{\mathcal{C}}$. By our Theorem 4.32, there is a 1-exangulated (or extriangulated) category $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$. By direct comparison of the constructions, one can check that $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$ and $(\widetilde{\mathcal{C}}, \mathbb{F}', \mathfrak{t}')$ are n-exangulated isomorphic. Indeed, the bifunctors $\mathbb{F}$ and $\mathbb{F}'$ differ only by a labelling of the elements due to our convention in Definition 4.4; and, ignoring this re-labelling, the realisations $\mathfrak{s}$ and $\mathfrak{s}'$ are the same by Lemma 4.19. Furthermore, since $(\widetilde{\mathcal{C}}, \mathbb{F}', \mathfrak{t}')$ recovers the triangulated and exact category cases, we see that our construction agrees with the classical (i.e. $n = 1$) cases up to n-exangulated isomorphism.

For larger n, we just need to compare $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$ with the construction in [Lin21]. Thus, suppose $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is the n-exangulated category coming from an $(n+2)$-angulated category $(\mathcal{C}, \Sigma, \pentagon)$. Recall that in this case $\mathbb{E}(Z, X) = \mathcal{C}(Z, \Sigma X)$ for $X, Z \in \mathcal{C}$. Using [Lin21, Thm. 3.1], one obtains an $(n+2)$-angulated category $(\widetilde{\mathcal{C}}, \widetilde{\Sigma}, \widetilde{\pentagon})$, where $\widetilde{\Sigma}$ is induced by Σ. From this $(n+2)$-angulated category, just like above, we obtain an induced n-exangulated category $(\widetilde{\mathcal{C}}, \mathbb{F}', \mathfrak{t}')$. Notice that $\mathbb{F}'(-,-) = \widetilde{\mathcal{C}}(-, \widetilde{\Sigma}-)$. Comparing $(\widetilde{\mathcal{C}}, \mathbb{F}', \mathfrak{t}')$ to the n-exangulated category $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$ found from Theorem 4.32, again we see that $\mathbb{F}$ and $\mathbb{F}'$ differ by the labelling convention we chose in Definition 4.4. By [HLN21, Prop. 4.8] we have that $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$ induces an $(n+2)$-angulated category $(\widetilde{\mathcal{C}}, \widetilde{\Sigma}, \pentagon')$, and therefore the n-exangulated inclusion functor $\mathscr{I}_{\mathcal{C}} \colon \mathcal{C} \to \widetilde{\mathcal{C}}$ is, moreover, $(n+2)$-angulated. It follows from [Lin21, Thm. 3.1(2)] that $\widetilde{\pentagon}$ and $\pentagon'$ must be equal, and hence $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$ and $(\widetilde{\mathcal{C}}, \mathbb{F}', \mathfrak{t}')$ are n-exangulated isomorphic.

Remark 4.41. Our proofs in this article differ from the proofs in both the extriangulated and the $(n+2)$-angulated cases. First, the axioms for an n-exangulated category look very different from the axioms for an extriangulated category. Therefore, the proofs from [Msa22] cannot be directly generalised to the $n > 1$ case. Even of the results that seem like they might generalise nicely, one comes across immediate obstacles. Indeed, Lin [Lin21, p. 1064] already points out that lifting idempotent morphisms of extensions to idempotent morphisms of n-exangles is non-trivial. Despite this, we are able to overcome this here. This, amongst other problems, forces Lin to use another approach, and hence demonstrates why our methods are distinct.

Remark 4.42. He–He–Zhou [HHZ22] have considered idempotent completions of n-exangulated categories in a specific setup. In their setup, there is an ambient Krull-Schmidt $(n+2)$-angulated category $\mathcal{C}$ and an additive subcategory $\mathcal{A}$ that is *n-extension-closed* (see Definition 5.2) and closed under direct summands in $\mathcal{C}$. The main aim of [HHZ22] is to show that the idempotent completion $\widetilde{\mathcal{A}}$ of $\mathcal{A}$ is an n-exangulated subcategory of $\widetilde{\mathcal{C}}$.

Since $\mathcal{A}$ is an additive subcategory of and closed under direct summands in a Krull-Schmidt category, it is Krull-Schmidt itself. In particular, $\mathcal{A} \simeq \widetilde{\mathcal{A}}$ is already idempotent complete by [Kra15, Cor. 4.4]. Moreover, in the setup of [HHZ22], it already follows that $\mathcal{A}$ inherits an n-exangulated structure from $\mathcal{C} \simeq \widetilde{\mathcal{C}}$. Indeed, (EA1) is proven in [Kla21, Lem. 3.8], and (EA2) and (EA2)$^{\mathrm{op}}$ are straightforward to check directly. It is then clear that $\mathcal{A}$ inherits an n-exangulated structure from $\mathcal{C}$.

5 The weak idempotent completion of an n-exangulated category

Just as in Section 4, we assume $n \geq 1$ is an integer and that $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an n-exangulated category. By Theorems 4.32 and 4.39, the idempotent completion of $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an n-exangulated category $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$ and the inclusion functor $\mathscr{I}_{\mathcal{C}}$ of $\mathcal{C}$ into $\widetilde{\mathcal{C}}$ is part of an n-exangulated functor $(\mathscr{I}_{\mathcal{C}}, \Gamma)\colon (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$, which satisfies the 2-universal property from Theorem 4.39. In this section, we turn our attention to the weak idempotent completion $\widehat{\mathcal{C}}$ of $\mathcal{C}$ and we show that it forms part of a triplet $(\widehat{\mathcal{C}}, \mathbb{G}, \mathfrak{r})$ that is n-extension-closed (see Definition 5.2) in $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$. It will then follow that $(\widehat{\mathcal{C}}, \mathbb{G}, \mathfrak{r})$ is itself n-exangulated, and, moreover, there is an analogue of Theorem 4.39 for $(\widehat{\mathcal{C}}, \mathbb{G}, \mathfrak{r})$; see Theorem 5.5.

We begin with the following proposition, which is an analogue of Lemma 4.38 for the weak idempotent completion.

Proposition 5.1. *Suppose* $(X_0, e_0), (X_{n+1}, e_{n+1}) \in \widehat{\mathcal{C}}$, $\tilde{\delta} \in \mathbb{F}((X_{n+1}, e_{n+1}), (X_0, e_0))$ *and* $\mathfrak{s}(\delta) = [X_\bullet]$. *Then there is a* $\mathfrak{t}$*-distinguished* n*-exangle* $\langle \widetilde{Y}_\bullet, \tilde{\delta} \rangle$ *with* $\widetilde{Y}_\bullet \in \mathsf{Ch}(\widehat{\mathcal{C}})^n$ *and an* $\mathfrak{s}$*-distinguished* n*-exangle* $\langle Y'_\bullet, {}_{Y'_0}0_{Y'_{n+1}} \rangle$*, such that*

$$\langle \mathscr{I}_{\mathcal{C}}(X_\bullet), \Gamma_{(X_{n+1}, X_0)}(\delta) \rangle \cong \langle \widetilde{Y}_\bullet, \tilde{\delta} \rangle \oplus \langle \mathscr{I}_{\mathcal{C}}(Y'_\bullet), \Gamma_{(Y'_{n+1}, Y'_0)}({}_{Y'_0}0_{Y'_{n+1}}) \rangle$$

as $\mathfrak{t}$*-distinguished* n*-exangles.*

Proof. By Corollary 4.13, there exists an idempotent morphism $e_\bullet\colon \langle X_\bullet, \delta \rangle \to \langle X_\bullet, \delta \rangle$ with $e_i = \mathrm{id}_{X_i}$ for $2 \leq i \leq n-1$, as well as a homotopy $h_\bullet = (h_1, 0, \ldots, 0, h_{n+1})\colon e'_\bullet \sim 0_\bullet$, where $e'_\bullet := \mathrm{id}_{X_\bullet} - e_\bullet$. Notice $(X_i, e_i) \in \widehat{\mathcal{C}}$ for $i = 0, n+1$ by assumption. Furthermore, $(X_i, e_i) = (X_i, \mathrm{id}_{X_i}) \in \mathscr{I}_{\mathcal{C}}(\mathcal{C}) \subseteq \widehat{\mathcal{C}}$ for $2 \leq i \leq n-1$. Set $\tilde{\rho} := \Gamma_{(X_{n+1}, X_0)}(\delta)$. By Lemma 4.38 we have $\langle \mathscr{I}_{\mathcal{C}}(X_\bullet), \tilde{\rho} \rangle \cong \langle (X_\bullet, e_\bullet), \tilde{\delta} \rangle \oplus \langle (X_\bullet, e'_\bullet), {}_{(X_0, e'_0)}\widetilde{0}_{(X_{n+1}, e'_{n+1})} \rangle$ as $\mathfrak{t}$-distinguished n-exangles. We will show that there is an isomorphism $\tilde{s}_\bullet\colon (X_\bullet, e_\bullet) \to \widetilde{Y}_\bullet$ in $\mathsf{Ch}(\widetilde{\mathcal{C}})^n_{((X_0, e_0), (X_{n+1}, e_{n+1}))}$ for some $\widetilde{Y}_\bullet \in \mathsf{Ch}(\widehat{\mathcal{C}})^n$, as well as an isomorphism $\tilde{s}'_\bullet\colon (X_\bullet, e'_\bullet) \to \mathscr{I}_{\mathcal{C}}(Y'_\bullet)$ in $\mathsf{Ch}(\widetilde{\mathcal{C}})^n$ for some object $Y'_\bullet \in \mathsf{Ch}(\mathcal{C})^n$.

If $i = 0, n+1$, then $e'_i = \mathrm{id}_{X_i} - e_i$ is split by assumption, so by Lemma 2.4 there are objects $Y'_i \in \mathcal{C}$ and isomorphisms $\tilde{s}'_i\colon (X_i, e'_i) \to \mathscr{I}_{\mathcal{C}}(Y'_i)$. For $2 \leq i \leq n-1$, we see that $e'_i = \mathrm{id}_{X_i} - e_i = 0$, so by Lemma 2.4 again we have isomorphisms $\tilde{s}'_i\colon (X_i, e'_i) \to \mathscr{I}_{\mathcal{C}}(Y'_i)$, but now where $Y'_i = 0 \in \mathcal{C}$. Since $(X_0, e_0), (X_{n+1}, e_{n+1}) \in \widehat{\mathcal{C}}$ by assumption and because $(X_i, e_i) = (X_i, \mathrm{id}_{X_i}) \in \mathscr{I}_{\mathcal{C}}(\mathcal{C}) \subseteq \widehat{\mathcal{C}}$ for $2 \leq i \leq n-1$, we put $\widetilde{Y}_i := (X_i, e_i)$ and $\tilde{s}_i := \widetilde{\mathrm{id}}_{(X_i, e_i)}$ for $i \in \{0, n+1\} \cup \{2, \ldots, n-1\}$. It remains to find appropriate isomorphisms $\tilde{s}_i$ and $\tilde{s}'_i$ for $i = 1, n$.

We have a morphism $\tilde{k}_1\colon (X_1, e'_1) \to (X_0, e'_0)$ with underlying morphism $k_1 := e'_0 h_1 e'_1$ and $\tilde{k}_{n+1}\colon (X_{n+1}, e'_{n+1}) \to (X_n, e'_n)$ with underlying morphism $k_{n+1} := e'_n h_{n+1} e'_{n+1}$ by Lemma 4.14. Since $(h_1, 0, \ldots, 0, h_{n+1})\colon e'_\bullet \sim 0_\bullet$ is a homotopy, we see that $h_1 d_0^X = e'_0$. This implies

$$k_1 d_0^{(X,e')} = e'_0 h_1 e'_1 d_0^X e'_0 = e'_0 h_1 d_0^X e'_0 = e'_0 = \mathrm{id}_{(X_0, e'_0)}, \tag{5.1}$$

and so $\tilde{k}_1 \tilde{d}_0^{(X,e')} = \widetilde{\mathrm{id}}_{(X_0, e'_0)}$. Similarly, we also have $\tilde{d}_n^{(X,e')} \tilde{k}_{n+1} = \widetilde{\mathrm{id}}_{(X_{n+1}, e'_{n+1})}$.

1. If $n=1$, then (5.1) shows that $\tilde{d}_0^{(X,e')}$ is a section in the complex $(X_\bullet, e'_\bullet)$, and hence this complex is a split short exact sequence by [HLN21, Claim 2.15]. In particular, we have that $(X_1, e'_1) \cong (X_0, e'_0) \oplus (X_2, e'_2) \cong \mathscr{I}_\mathcal{C}(Y'_0) \oplus \mathscr{I}_\mathcal{C}(Y'_2) \cong \mathscr{I}_\mathcal{C}(Y'_0 \oplus Y'_2)$. So we put $Y'_1 := Y'_0 \oplus Y'_2$ and define $\tilde{s}'_1 \colon (X_1, e'_1) \to \mathscr{I}_\mathcal{C}(Y'_1)$ to be this composition of isomorphisms. As $\mathscr{I}_\mathcal{C}(X_1) \cong (X_1, e_1) \oplus (X_1, e'_1)$, and $\mathscr{I}_\mathcal{C}(X_1)$ and (X_1, e'_1) are isomorphic to objects in $\widehat{\mathcal{C}}$, by Lemma 2.12 there is an isomorphism $\tilde{s}_1 \colon (X_1, e_1) \to \widetilde{Y}_1$ for some $\widetilde{Y}_1 \in \widehat{\mathcal{C}}$.

2. If $n \geq 2$, then the form of the homotopy $h_\bullet$ implies that the identities $d_0^X h_1 = e'_1$ and $h_{n+1} d_{n+1}^X = e'_n$ hold. Therefore, we see that

$$d_0^{(X,e')} k_1 = d_0^X e'_0 h_1 e'_1 = e'_1 d_0^X h_1 e'_1 = e'_1 = \mathrm{id}_{(X_1, e'_1)},$$

which shows that $\tilde{k}_1$ and $\tilde{d}_0^{(X,e')}$ are mutually inverse isomorphisms. We now define $Y'_1 := Y'_0$ and $\tilde{s}'_1 := \tilde{s}'_0 \tilde{k}_1 \colon (X_1, e'_1) \to \mathscr{I}_\mathcal{C}(Y'_1)$. Because there are isomorphisms $\mathscr{I}_\mathcal{C}(X_1) \cong (X_1, e_1) \oplus (X_1, e'_1) \cong (X_1, e_1) \oplus (X_0, e'_0)$, and $\mathscr{I}_\mathcal{C}(X_1)$ and (X_0, e'_0) are isomorphic to objects in $\widehat{\mathcal{C}}$, by Lemma 2.12 there is an isomorphism $\tilde{s}_1 \colon (X_1, e_1) \to \widetilde{Y}_1$ for some $\widetilde{Y}_1 \in \widehat{\mathcal{C}}$.

   In a similar way, one can show that $\tilde{k}_{n+1}$ and $\tilde{d}_n^{(X,e')}$ are mutually inverse isomorphisms. We set $Y'_n := Y'_{n+1}$ and $\tilde{s}'_n := \tilde{s}'_{n+1} \tilde{d}_n^{(X,e')} \colon (X_n, e'_n) \to \mathscr{I}_\mathcal{C}(Y'_n)$. In addition, there is an isomorphism $\tilde{s}_n \colon (X_n, e_n) \to \widetilde{Y}_n$ for some $\widetilde{Y}_n \in \widehat{\mathcal{C}}$.

The complex $\widetilde{Y}_\bullet$ with object $\widetilde{Y}_i$ in degree $0 \leq i \leq n+1$ and differential $d_i^{\widetilde{Y}} := \tilde{s}_{i+1} \tilde{d}_i^{(X,e)} \tilde{s}_i^{-1}$ in degree $0 \leq i \leq n$ is isomorphic to $(X_\bullet, e_\bullet)$ via $\tilde{s}_\bullet$. Furthermore, as $\tilde{s}_0$ and $\tilde{s}_{n+1}$ are identity morphisms, we have that $\langle \widetilde{Y}_\bullet, \tilde{\delta} \rangle$ is $\mathfrak{t}$-distinguished by [HLN21, Cor. 2.26(2)]. The complex $\widetilde{Y}'_\bullet$ with object $\widetilde{Y}'_i := \mathscr{I}_\mathcal{C}(Y'_i)$ in degree $0 \leq i \leq n+1$ and differential $d_i^{\widetilde{Y}'} := \tilde{s}'_{i+1} \tilde{d}^{(X,e')}{}_i \tilde{s}_i'^{-1}$ in degree $0 \leq i \leq n$ is isomorphic to $(X_\bullet, e'_\bullet)$ via $\tilde{s}'_\bullet$. Moreover, this induces an isomorphism $\tilde{s}'_\bullet \colon \langle (X_\bullet, e'_\bullet), {}_{(X_0, e'_0)}\widetilde{0}_{(X_{n+1}, e'_{n+1})} \rangle \to \langle \widetilde{Y}'_\bullet, {}_{\widetilde{Y}'_0}\widetilde{0}_{\widetilde{Y}'_{n+1}} \rangle$ of $\mathbb{F}$-attached complexes, and hence of $\mathfrak{t}$-distinguished n-exangles. It is clear that

$$\langle \widetilde{Y}'_\bullet, {}_{\widetilde{Y}'_0}\widetilde{0}_{\widetilde{Y}'_{n+1}} \rangle \cong \langle \mathscr{I}_\mathcal{C}\left(\mathrm{triv}_0(Y'_0)_\bullet \oplus \mathrm{triv}_n(Y'_{n+1})_\bullet \right), \Gamma_{(Y'_{n+1}, Y'_0)}({}_{Y'_0}0_{Y'_{n+1}}) \rangle$$

by the construction of $\widetilde{Y}'_\bullet$. Lastly, $\langle \mathrm{triv}_0(Y'_0)_\bullet \oplus \mathrm{triv}_n(Y'_{n+1})_\bullet, {}_{Y'_0}0_{Y'_{n+1}} \rangle$ is $\mathfrak{s}$-distinguished using [HLN21, Prop. 3.3] and that $\mathfrak{s}$ is an exact realisation of $\mathbb{E}$. □

From Proposition 5.1 we see that $\widehat{\mathcal{C}}$ is n-extension-closed in $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$ in the following sense.

Definition 5.2. [HLN22, Def. 4.1] Let $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ be an n-exangulated category. A full subcategory $\mathcal{D} \subseteq \mathcal{C}'$ is said to be *n-extension-closed* if, for all $A, C \in \mathcal{D}$ and each $\mathbb{E}'$-extension $\delta \in \mathbb{E}'(C, A)$, there is an object $X_\bullet \in \mathsf{Ch}(\mathcal{C}')^n$ such that $X_i \in \mathcal{D}$ for all $1 \leq i \leq n$ and $\mathfrak{s}'(\delta) = [X_\bullet]$.

Let us now define the biadditive functor and realisation with which we wish to equip $\widehat{\mathcal{C}}$.

Definition 5.3. (i) Let $\mathbb{G} := \mathbb{F}|_{\widehat{\mathcal{C}}^{\mathrm{op}}\times\widehat{\mathcal{C}}}\colon \widehat{\mathcal{C}}^{\mathrm{op}} \times \widehat{\mathcal{C}} \to \mathsf{Ab}$ be the restriction of $\mathbb{F}\colon \widetilde{\mathcal{C}}^{\mathrm{op}} \times \widetilde{\mathcal{C}} \to \mathsf{Ab}$.

(ii) For a $\mathbb{G}$-extension $\tilde{\delta} \in \mathbb{G}(\widetilde{X}_{n+1}, \widetilde{X}_0)$, there is a $\mathfrak{t}$-distinguished n-exangle $\langle \widetilde{X}_\bullet, \tilde{\delta}\rangle$ with $\widetilde{X}_\bullet \in \mathsf{Ch}(\widehat{\mathcal{C}})^n$ by Proposition 5.1. We put $\mathfrak{r}(\tilde{\delta}) = [\widetilde{X}_\bullet]$, the isomorphism class of $\widetilde{X}_\bullet$ in $\mathsf{K}(\widehat{\mathcal{C}})^n_{(\widetilde{X}_0,\widetilde{X}_{n+1})}$.

(iii) Recall from Subsection 2.2 that $\mathscr{K}_{\mathcal{C}}\colon \mathcal{C} \to \widehat{\mathcal{C}}$ is the inclusion functor defined by $\mathscr{K}_{\mathcal{C}}(X) = (X, \mathrm{id}_X)$ on objects $X \in \mathcal{C}$. Let $\Delta\colon \mathbb{E}(-,-) \Rightarrow \mathbb{G}(\mathscr{K}_{\mathcal{C}}-, \mathscr{K}_{\mathcal{C}}-)$ be the restriction of the natural transformation $\Gamma\colon \mathbb{E}(-,-) \Rightarrow \mathbb{F}(\mathscr{I}_{\mathcal{C}}-, \mathscr{I}_{\mathcal{C}}-)$ defined in Lemma 4.35. This means $\Delta(\delta) := (\mathrm{id}_{X_0}, \delta, \mathrm{id}_{X_{n+1}}) \in \mathbb{G}(\mathscr{K}_{\mathcal{C}}(X_{n+1}), \mathscr{K}_{\mathcal{C}}(X_0))$ for $\delta \in \mathbb{E}(X_{n+1}, X_0)$.

Since $\widehat{\mathcal{C}}$ is an n-extension closed subcategory of $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$ by Proposition 5.1, one can use [HLN22, Prop. 4.2(1)] to deduce axioms (EA2) and (EA2)$^{\mathrm{op}}$ hold for the triplet $(\widehat{\mathcal{C}}, \mathbb{G}, \mathfrak{r})$. The difficult part is then to show that (EA1) is satisfied; this follows from Lemma 5.4 below. We note here, however, that it has been shown more generally in [Kla23, Thm. A] that any n-extension-closed subcategory of an n-exangulated category inherits an n-exangulated structure in the expected way.

Lemma 5.4. *Let $\tilde{f}\colon \widetilde{X}_0 \to \widetilde{X}_1$ be a $\mathfrak{t}$-inflation with $\widetilde{X}_0 = (X_0, e_0), \widetilde{X}_1 = (X_1, e_1) \in \widehat{\mathcal{C}}$. Then there is a $\mathfrak{t}$-distinguished n-exangle $\langle \widetilde{X}_\bullet, \tilde{\delta}\rangle$ with $\widetilde{X}_\bullet \in \mathsf{Ch}(\widehat{\mathcal{C}})^n$ and $\tilde{d}_0^{\widetilde{X}} = \tilde{f}$.*

Proof. By Lemma 4.25 there is an object $C \in \mathcal{C}$, a morphism $f'\colon X_0 \to C$ and an $\mathfrak{s}$-distinguished n-exangle $\langle Z_\bullet, \rho\rangle$ with $Z_0 = X_0$, $Z_1 = X_1 \oplus C$, $d_0^Z = \left[\begin{smallmatrix} f & f'(\mathrm{id}_{X_0} - e_0)\end{smallmatrix}\right]^\top$ and $(e_0)_{\mathbb{E}}\rho = \rho$. The solid morphisms of the diagram

$$\begin{tikzcd}
X_0 \arrow[r, "d_0^Z"] \arrow[d, "e_0"] & X_1 \oplus C \arrow[r] \arrow[d, "\left[\begin{smallmatrix} e_1 & 0 \\ 0 & 0\end{smallmatrix}\right]"] & Z_2 \arrow[r] \arrow[d, dotted, "e_2'"] & Z_3 \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & Z_n \arrow[r] \arrow[d, equal] & Z_{n+1} \arrow[r, dashed, "\rho"] \arrow[d, equal] & {} \\
X_0 \arrow[r, "d_0^Z"] & X_1 \oplus C \arrow[r] & Z_2 \arrow[r] & Z_3 \arrow[r] & \cdots \arrow[r] & Z_n \arrow[r] & Z_{n+1} \arrow[r, dashed, "\rho"] & {}
\end{tikzcd}$$

form a commutative diagram. By Lemma 4.24 there exists an idempotent morphism of n-exangles $e'_\bullet\colon \langle Z_\bullet, \rho\rangle \to \langle Z_\bullet, \rho\rangle$ with $e'_0 = e_0$, $e'_1 = \left[\begin{smallmatrix} e_1 & 0 \\ 0 & 0\end{smallmatrix}\right]$ and $e'_i = \mathrm{id}_{Z_i}$ for $3 \leq i \leq n+1$, which makes the diagram above commute. Let $\tilde{\rho} := \Gamma_{(Z_{n+1}, X_0)}(\rho) \in \mathbb{F}(\mathscr{I}_{\mathcal{C}}(Z_{n+1}), \mathscr{I}_{\mathcal{C}}(X_0))$ and $\tilde{\rho}' := \mathbb{F}(\tilde{i}_{e'_{n+1}}, \tilde{p}_{e'_0})(\tilde{\rho}) \in \mathbb{F}((Z_{n+1}, e'_{n+1}), (X_0, e_0))$. Notice that the underlying $\mathbb{E}$-extension of $\tilde{\rho}'$ is ρ. Set $e''_\bullet := \mathrm{id}_{Z_\bullet} - e'_\bullet$. Then

$$\langle \mathscr{I}_{\mathcal{C}}(Z_\bullet), \tilde{\rho}\rangle \cong \langle (Z_\bullet, e'_\bullet), \tilde{\rho}'\rangle \oplus \langle (Z_\bullet, e''_\bullet), {}_{(Z_0, e''_0)}\widetilde{0}_{(Z_{n+1}, e''_{n+1})}\rangle$$

as $\mathfrak{t}$-distinguished n-exangles by Lemma 4.38.

We claim that there is a split short exact sequence

$$0 \longrightarrow (Z_0, e''_0) \xrightarrow{\tilde{d}_0^{(Z,e'')}} (Z_1, e''_1) \xrightarrow{\tilde{d}_1^{(Z,e'')}} (Z_2, e''_2) \longrightarrow 0. \tag{5.2}$$

Since $(Z_\bullet, e''_\bullet)$ realises the trivial $\mathbb{F}$-extension ${}_{(Z_0, e''_0)}\widetilde{0}_{(Z_{n+1}, e''_{n+1})}$, we have that $\tilde{d}_0^{(Z,e'')}$ is a section by [HLN21, Claim 2.15]. If $n = 1$, then this is enough to see that (5.2) is split short exact. For $n \geq 2$ we notice that there is an isomorphism $(Z_3, e_3'') \cong 0$ in $\widetilde{\mathcal{C}}$, so $\tilde{d}_2^{(Z,e'')} = 0$. Thus, since $\tilde{d}_1^{(Z,e'')}$ is a weak kernel of $\tilde{d}_2^{(Z,e'')}$, we see that $\widetilde{\mathrm{id}}_{(Z_2,e_2'')}$ factors through $\tilde{d}_1^{(Z,e'')}$. In particular, this implies $\tilde{d}_1^{(Z,e'')}$ is a cokernel of $\tilde{d}_0^{(Z,e'')}$. Again, (5.2) is split short exact.

In particular, we have an isomorphism $(Z_1, e_1'') \cong (Z_0, e_0'') \oplus (Z_2, e_2'')$. We know that the objects $(Z_0, e_0'') = (X_0, \mathrm{id}_{X_0} - e_0)$ and $(Z_1, e_1'') = (X_1 \oplus C, (\mathrm{id}_{X_1} - e_1) \oplus \mathrm{id}_C)$ are isomorphic to objects in $\mathscr{I}_{\mathcal{C}}(\mathcal{C}) \subseteq \widehat{\mathcal{C}} \subseteq \widetilde{\mathcal{C}}$ by Lemma 2.4, as $\mathrm{id}_{X_0} - e_0$ and $(\mathrm{id}_{X_1} - e_1) \oplus \mathrm{id}_C$ are split idempotents. This implies that (Z_2, e_2'') is isomorphic to an object in $\widehat{\mathcal{C}}$ by Lemma 2.12. Again Lemma 2.12 and the isomorphism $\mathscr{I}_{\mathcal{C}}(Z_2) \cong (Z_2, e_2') \oplus (Z_2, e_2'')$ imply that there is an isomorphism $\tilde{s}_2 \colon (Z_2, e_2') \to \widetilde{X}_2$ for some $\widetilde{X}_2 \in \widehat{\mathcal{C}}$.

The morphism $\tilde{s}_1 \colon \left(X_1 \oplus C, \left[\begin{smallmatrix} e_1 & 0 \\ 0 & 0 \end{smallmatrix}\right]\right) \to \widetilde{X}_1$ with underlying morphism $s_1 = \left[\begin{smallmatrix} e_1 & 0 \end{smallmatrix}\right]$ is an isomorphism. Finally, put $\tilde{s}_i = \widetilde{\mathrm{id}}_{(Z_i,e_i')}$ for $i = 0$ and $3 \leq i \leq n+1$. Then the complex

$$\widetilde{X}_\bullet\colon \qquad \widetilde{X}_0 \xrightarrow{\tilde{f}} \widetilde{X}_1 \xrightarrow{\tilde{s}_2 \tilde{d}_1^{(Z,e')} \tilde{s}_1^{-1}} \widetilde{X}_2 \xrightarrow{\tilde{d}_2^{(Z,e')} \tilde{s}_2^{-1}} (Z_3, e_3') \xrightarrow{\tilde{d}_3^{(Z,e')}} \cdots \xrightarrow{\tilde{d}_n^{(Z,e')}} (Z_{n+1}, e_{n+1}')$$

is isomorphic to $(Z_\bullet, e_\bullet')$ via $\tilde{s}_\bullet \colon (Z_\bullet, e_\bullet') \to \widetilde{X}_\bullet$ in $\mathsf{Ch}(\widetilde{\mathcal{C}})^n$. With $\tilde{\delta} := (\tilde{s}_{n+1}^{-1})^{\mathbb{F}} \tilde{\rho}'$, we see that $\langle \widetilde{X}_\bullet, \tilde{\delta} \rangle$ is $\mathfrak{t}$-distinguished by [HLN21, Cor. 2.26(2)], as desired. □

We may state and prove our main result of this section.

Theorem 5.5. *Suppose that $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an n-exangulated category. Then $(\widehat{\mathcal{C}}, \mathbb{G}, \mathfrak{r})$ is a weakly idempotent complete n-exangulated category, and $(\mathscr{K}_{\mathcal{C}}, \Delta) \colon (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\widehat{\mathcal{C}}, \mathbb{G}, \mathfrak{r})$ is an n-exangulated functor, such that the following 2-universal property is satisfied. Suppose $(\mathscr{F}, \Lambda) \colon (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ is an n-exangulated functor to a weakly idempotent complete n-exangulated category $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$. Then the following statements hold.*

(i) *There is an n-exangulated functor $(\mathscr{E}, \Psi) \colon (\widehat{\mathcal{C}}, \mathbb{G}, \mathfrak{r}) \to (\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ as well as an n-exangulated natural isomorphism $\text{ץ} \colon (\mathscr{F}, \Lambda) \overset{\cong}{\Longrightarrow} (\mathscr{E}, \Psi) \circ (\mathscr{K}_{\mathcal{C}}, \Delta)$.*

(ii) *In addition, for any n-exangulated functor $(\mathscr{G}, \Theta) \colon (\widehat{\mathcal{C}}, \mathbb{G}, \mathfrak{r}) \to (\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ and any n-exangulated natural transformation $\text{ד} \colon (\mathscr{F}, \Lambda) \Rightarrow (\mathscr{G}, \Theta) \circ (\mathscr{K}_{\mathcal{C}}, \Delta)$, there is a unique n-exangulated natural transformation $\text{ב} \colon (\mathscr{E}, \Psi) \Rightarrow (\mathscr{G}, \Theta)$ with $\text{ד} = \text{ב}_{\mathscr{K}_{\mathcal{C}}} \text{ץ}$.*

Proof. Since $\widehat{\mathcal{C}}$ is a full subcategory of $\widetilde{\mathcal{C}}$ and because $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$ is n-exangulated, we can apply [HLN22, Prop. 4.2] and Definition 5.2. We showed above that $\widehat{\mathcal{C}}$ is n-extension-closed in $\widetilde{\mathcal{C}}$; see Proposition 5.1. Moreover, it follows immediately from Lemma 5.4 and its dual that $(\widehat{\mathcal{C}}, \mathbb{G}, \mathfrak{r})$ satisfies (EA1). Therefore, we deduce that $(\widehat{\mathcal{C}}, \mathbb{G}, \mathfrak{r})$ is an n-exangulated category.

One may argue that $(\mathscr{K}_{\mathcal{C}}, \Delta)$ is an n-exangulated functor as in Proposition 4.36, by using the definition of $\mathfrak{r}$ and noting that $(X, \mathrm{id}_X) = \mathscr{K}_{\mathcal{C}}(X)$ lies in $\widehat{\mathcal{C}}$ for all $X \in \mathcal{C}$.

(i) One argues like in the proof of Theorem 4.39(i), but using Proposition 2.13 instead of Proposition 2.8, and Proposition 5.1 instead of Lemma 4.38. In particular, we note that the isomorphism $\langle \mathscr{I}_{\mathcal{C}}(X_\bullet), \Gamma_{(X_{n+1},X_0)}(\delta) \rangle \cong \langle \widetilde{Y}_\bullet, \tilde{\delta} \rangle \oplus \langle \mathscr{I}_{\mathcal{C}}(Y_\bullet'), \Gamma_{(Y_{n+1}',Y_0')}({}_{Y_0'}0_{Y_{n+1}'}) \rangle$ of $\mathfrak{t}$-distinguished n-exangles from the statement of Proposition 5.1 induces an isomorphism

$$\langle \mathscr{K}_{\mathcal{C}}(X_\bullet), \Delta_{(X_{n+1},X_0)}(\delta) \rangle \cong \langle \widetilde{Y}_\bullet, \tilde{\delta} \rangle \oplus \langle \mathscr{K}_{\mathcal{C}}(Y_\bullet'), \Delta_{(Y_{n+1}',Y_0')}({}_{Y_0'}0_{Y_{n+1}'}) \rangle$$

of $\mathfrak{r}$-distinguished n-exangles.

(ii) Similarly, one adapts the proof of Theorem 4.39(ii), using Proposition 2.13 instead of Proposition 2.8. □

Finally, we have an analogue of Corollary 4.34 as a consequence.

Corollary 5.6. *Suppose* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ *is* n*-exact. Then* $(\widehat{\mathcal{C}}, \mathbb{G}, \mathfrak{r})$ *is* n*-exact.*

Proof. The n-exangulated category $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$ is n-exact by Corollary 4.34. As $(\widehat{\mathcal{C}}, \mathbb{G}, \mathfrak{r})$ inherits its structure as an n-extension closed subcategory of $(\widetilde{\mathcal{C}}, \mathbb{F}, \mathfrak{t})$, the result follows from the proof of [Kla23, Cor. 4.15]. □

References

[AF92] F. W. Anderson and K. R. Fuller. *Rings and categories of modules*, volume 13 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1992.

[AM69] M. F. Atiyah and I. G. MacDonald. *Introduction to commutative algebra*. Addison-Wesley-Longman, 1969.

[Aus74] M. Auslander. *Representation theory of Artin algebras. I*. Comm. Algebra, **1**:177–268, 1974.

[BD68] I. Bucur and A. Deleanu. *Introduction to the theory of categories and functors*. With the collaboration of Peter J. Hilton and Nicolae Popescu. Pure and Applied Mathematics, Vol. XIX. Interscience Publication John Wiley & Sons, Ltd., London-New York-Sydney, 1968.

[BHST22] T. Brüstle, S. Hassoun, A. Shah, and A. Tattar. *Stratifying systems and Jordan-Hölder extriangulated categories*. Preprint, 2022. https://arxiv.org/abs/2208.07808v2.

[Bor94] F. Borceux. *Handbook of categorical algebra. 1. Basic category theory*, volume 50 of Encyclopedia of Mathematics and its Applications. Cambridge Univ. Press, Cambridge, 1994.

[BS01] P. Balmer and M. Schlichting. *Idempotent completion of triangulated categories*. J. Algebra, **236**(2):819–834, 2001.

[BT14] P. A. Bergh and M. Thaule. *The Grothendieck group of an n-angulated category*. J. Pure Appl. Algebra, **218**(2):354–366, 2014.

[BTHSS23] R. Bennett-Tennenhaus, J. Haugland, M. H. Sandøy, and A. Shah. *The category of extensions and a characterisation of n-exangulated functors*. Math. Z., **305**(3):44, 2023.

[BTS21] R. Bennett-Tennenhaus and A. Shah. *Transport of structure in higher homological algebra*. J. Algebra, **574**:514–549, 2021.

[Büh10] T. Bühler. *Exact categories*. Expo. Math., **28**(1):1–69, 2010.

[CYZ08] X.-W. Chen, Y. Ye, and P. Zhang. *Algebras of derived dimension zero*. Comm. Algebra, **36**(1):1–10, 2008.

[FBHL73] A. A. Fraenkel, Y. Bar-Hillel, and A. Levy. *Foundations of set theory.* Studies in Logic and the Foundations of Mathematics, Vol. 67. North-Holland Publishing Co., Amsterdam-London, revised edition, 1973. With the collaboration of Dirk van Dalen.

[GKO13] C. Geiss, B. Keller, and S. Oppermann. *n-angulated categories.* J. Reine Angew. Math., **675**:101–120, 2013.

[HHZ22] J. He, J. He, and P. Zhou. *Idempotent completion of certain n-exangulated categories.* Preprint, 2022. https://arxiv.org/abs/2207.01232v1.

[HLN21] M. Herschend, Y. Liu, and H. Nakaoka. *n-exangulated categories (I): Definitions and fundamental properties.* J. Algebra, **570**:531–586, 2021.

[HLN22] M. Herschend, Y. Liu, and H. Nakaoka. *n-exangulated categories (II): Constructions from n-cluster tilting subcategories.* J. Algebra, **594**:636–684, 2022.

[HR19] R. Henrard and A.-C. van Roosmalen. *Derived categories of (one-sided) exact categories and their localizations.* Preprint, 2019. https://arxiv.org/abs/1903.12647v3.

[Jas16] G. Jasso. *n-abelian and n-exact categories.* Math. Z., **283**(3-4):703–759, 2016.

[Kar68] M. Karoubi. *Algèbres de Clifford et K-théorie.* Ann. Sci. École Norm. Sup. (4), **1**:161–270, 1968.

[KIWY15] M. Kalck, O. Iyama, M. Wemyss, and D. Yang. *Frobenius categories, Gorenstein algebras and rational surface singularities.* Compos. Math., **151**(3):502–534, 2015.

[Kla21] C. Klapproth. *n-exact categories arising from $(n+2)$-angulated categories.* Preprint, 2021. https://arxiv.org/abs/2108.04596v2.

[Kla23] C. Klapproth. *n-extension closed subcategories of n-exangulated categories.* Preprint, 2023. https://arxiv.org/abs/2209.01128v3.

[Kra15] H. Krause. *Krull–Schmidt categories and projective covers.* Expo. Math., **33**(4):535–549, 2015.

[Lin21] Z. Lin. *Idempotent Completion of n-Angulated Categories.* Appl. Categ. Structures, **29**(6):1063–1071, 2021.

[LS14] J. Liu and L. Sun. *Idempotent completion of pretriangulated categories.* Czechoslovak Math. J., **64(139)**(2):477–494, 2014.

[Msa22] D. Msapato. *The Karoubi envelope and weak idempotent completion of an extriangulated category.* Appl. Categ. Structures, **30**(3):499–535, 2022.

[Nee90] A. Neeman. *The derived category of an exact category.* J. Algebra, **135**(2):388–394, 1990.

[NOS22] H. Nakaoka, Y. Ogawa, and A. Sakai. *Localization of extriangulated categories.* J. Algebra, **611**:341–398, 2022.

[NP19] H. Nakaoka and Y. Palu. *Extriangulated categories, Hovey twin cotorsion pairs and model structures.* Cah. Topol. Géom. Différ. Catég., **60**(2):117–193, 2019.

[Orl11] D. Orlov. *Formal completions and idempotent completions of triangulated categories of singularities.* Adv. Math., **226**(1):206–217, 2011.

[SF96] R. M. Smullyan and M. Fitting. *Set theory and the continuum problem*, volume 34 of Oxford Logic Guides. The Clarendon Press, Oxford University Press, New York, 1996. Oxford Science Publications.

[Sha20] A. Shah. *Auslander-Reiten theory in quasi-abelian and Krull-Schmidt categories.* J. Pure Appl. Algebra, **224**(1):98–124, 2020.

[TT90] R. W. Thomason and T. Trobaugh. *Higher algebraic K-theory of schemes and of derived categories.* In The Grothendieck Festschrift, Vol. III, volume 88 of Progr. Math., pages 247–435. Birkhäuser Boston, Boston, MA, 1990.

[Zho22] Y. Zhou. *Idempotent completion of right suspended categories.* AIMS Math., **7**(7):13442–13453, 2022.

[ZZZ21] T. Zhao, B. Zhu, and X. Zhuang. *Tilting pairs in extriangulated categories.* Proc. Edinb. Math. Soc. (2), **64**(4):947–981, 2021.

Paper C

n-Extension closed subcategories of n-exangulated categories

Carlo Klapproth

Abstract

Let n be a positive integer. We show that an n-extension closed subcategory of an n-exangulated category naturally inherits an n-exangulated structure through restriction of the ambient n-exangulated structure. Furthermore, we show that a strong version of the Obscure Axiom holds for n-exangulated categories, where $n \geq 2$. This allows us to characterize n-exact categories as n-exangulated categories with monic inflations and epic deflations. We also show that for an extriangulated category condition (WIC), which was introduced by Nakaoka and Palu, is equivalent to the underlying additive category being weakly idempotent complete. We then apply our results to show that n-extension closed subcategories of an n-exact category are again n-exact. Furthermore, we recover and improve results of Klapproth and Zhou.

Introduction

Generalisation and abstraction are very useful tools as they allow us to understand an area of mathematics as a whole rather than locally. Furthermore, they are very efficient as proofs need to be carried out only once. To this end, Herschend, Liu and Nakaoka recently introduced the notion of n-exangulated categories, see [HLN21] and [HLN22]. These categories generalize n-exact and $(n + 2)$-angulated categories simultaneously, in a similar way as extriangulated categories generalize exact and triangulated categories. These structures also allow us to compare the recently emerged field of higher homological algebra to classic homological algebra.

Extension closed subcategories are an important part of homological algebra and representation theory and they appear naturally. For example the torsion and torsion free class of a torsion pair or the aisle and coaisle of a t-structure are extension closed subcategories. Therefore, we are interested in studying properties of them. It is well known that any extension closed subcategory of an exact category inherits an exact structure from the exact structure of the ambient category in a natural way, see for example Bühler [Büh10, Lemma 10.20]. The same does not hold for triangulated categories, but the larger class of extriangulated categories is again closed under taking extension closed subcategories by [NP19, Remark 2.18]. The proof of this result is straightforward.

For n-exangulated categories, where $n \in \mathbb{N}_{\geq 2}$, the situation is more difficult. It was shown in He–Zhou [HZ21, Theorem 1.1] that an n-extension closed subcategory of a Krull–Schmidt n-exangulated category inherits an n-exangulated structure from the n-exangulated structure of the ambient category in a natural way. However, this is not completely satisfying, as a large class of categories, for example the category of finitely generated abelian groups $\mathsf{mod}\,\mathbb{Z}$ or its bounded derived category $\mathsf{D}^b(\mathsf{mod}\,\mathbb{Z})$, are not Krull-Schmidt. Indeed the Krull–Schmidt property restricts endomorphism rings to be semiperfect, see Krause [Kra15, Corollary 4.4]. Using a completely different method than employed in He–Zhou [HZ21] we are able to show the following theorem in full generality.

Theorem A (See Theorem 3.3)**.** *Suppose that* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ *is an* n*-exangulated category and* $\mathcal{A} \subseteq \mathcal{C}$ *is an* n*-extension closed additive subcategory. Then* $\mathcal{A}$ *inherits an* n*-exangulated structure from* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ *in a natural way.*

On the journey to this result technical obstacles need to be overcome. We prove the following useful theorem.

Theorem B (Strong Obscure Axiom, see Corollary 2.5)**.** *Let* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ *be an* n*-exangulated category and* $\mathcal{C}$ *weakly idempotent complete or* $n \in \mathbb{N}_{\geq 2}$*. If* gf *is an* $\mathfrak{s}$*-inflation then so is* f*.*

It is now well known that for exact categories the strong version of the Obscure Axiom presented here is equivalent to the underlying additive category being weakly idempotent complete, see for example [Büh10, Propsition 7.6]. However, when Quillen first defined exact categories, a weaker version of Theorem B with the assumption that f admits a cokernel, was an important part of the original definition of exact categories, see Quillen [Qui73, Section 2]. It was discovered by Yoneda and later rediscovered by Keller that this axiom is a consequence of the other axioms for exact categories.

It is remarkable that the strong Obscure Axiom holds for $n \in \mathbb{N}_{\geq 2}$ without further assumption on the underlying category. The result can be interpreted as n-exangulated categories being more detached from the exact structure induced by the underlying additive category for $n \in \mathbb{N}_{\geq 2}$ than they are for $n = 1$.

For extriangulated categories the strong Obscure Axiom corresponds to condition (WIC) introduced in [NP19, Condition 5.8]. This condition seems to be very important for several results in the theory of extriangulated categories, see for example [ES22], [NP19] and [WW22]. For example in [NP19, Section 5] it is used as a technical ingredient to show a bijection between Hovey twin cotorsion pairs and admissible model structures for extriangulated categories. It turns out that not only for exact but also for extriangulated categories the strong Obscure Axiom is equivalent to the underlying additive category being weakly idempotent complete.

Proposition C (See Proposition 2.7)**.** *An extriangulated category is weakly idempotent complete if and only if it satisfies condition* (WIC)*.*

This is particularly interesting, since any extriangulated category can be weakly idempotent completed, see for example [Msa22, Theorem 3.31] for small extriangulated categories, or Remark 1.17 and [KMS22, Theorem 5.5]. Hence, any extriangulated category is a full subcategory of an extriangulated category where condition (WIC) is satisfied.

Another application of the strong Obscure Axiom is to characterise n-exangulated categories which arise from n-exact categories completely by properties of their inflations and deflations. We prove the following theorem.

Theorem D (See Theorem 4.11)**.** *For an additive category $\mathcal{C}$ there is a one-to-one correspondence*

$$\left\{\begin{array}{c} n\text{-exact structures } (\mathcal{C},\mathcal{X}) \text{ with} \\ \text{small extension groups} \end{array}\right\} \overset{1:1}{\longleftrightarrow} \frac{\left\{\begin{array}{c} n\text{-exangulated structures } (\mathcal{C},\mathbb{E},\mathfrak{s}) \text{ with} \\ \text{monic } \mathfrak{s}\text{-inflations and epic } \mathfrak{s}\text{-deflations} \end{array}\right\}}{\left\{\begin{array}{c} \text{equivalences of } n\text{-exangulated categories} \\ \text{of the form } (\mathrm{Id}_{\mathcal{C}},\Gamma) \end{array}\right\}}.$$

For the relationship of extriangulated and exact categories a similar result is well-known, see [NP19, Example 2.13] and [NP19, Corollary 3.18]. However, for n-exangulated categories, where $n \in \mathbb{N}_{\geq 2}$, our strong Obscure Axiom is the missing ingredient to make the bijection constructed in [HLN21, Section 4.3] complete.

The combination of Theorem A and Theorem D allows us to show the following theorem.

Theorem E (See Corollary 4.15)**.** *Suppose $(\mathcal{C},\mathcal{X})$ is an n-exact category with small extension groups and $\mathcal{A} \subseteq \mathcal{C}$ is an n-extension closed additive subcategory. Then $\mathcal{A}$ inherits an n-exact structure in a natural way.*

Furthermore, we improve results [Zho22, Theorem 1.2] and [Kla21, Theorem I] about n-extension closed subcategories of $(n+2)$-angulated categories, see Corollary 5.4 and Corollary 5.5.

1 Conventions and notation

Throughout this paper we will use the notion of n-exangulated categories. We refer to [HLN21, Section 2] for the definition and [HLN21] and [HLN22] for an introduction. In Section 4 and Section 5 we consider n-exact categories. We refer to Jasso [Jas16, Section 4] for an introduction. In Section 5 we will study $(n+2)$-angulated categories. We refer to Geiss–Keller–Oppermann [GKO13, Section 2] for the definition.

For the rest of this section suppose we are given an arbitrary additive category $\mathcal{D}$ and $n \in \mathbb{N}_{\geq 1}$. We recall the following definitions.

Definition 1.1. A subcategory $\mathcal{B} \subseteq \mathcal{D}$ is called an *additive subcategory* if $\mathcal{B} \subseteq \mathcal{D}$ is full and closed under finite direct sums. That means $\mathcal{B}$ is closed under isomorphisms, $0 \in \mathcal{B}$ and for $B, B' \in \mathcal{B}$ the biproduct $B \oplus B'$ is in $\mathcal{B}$.

Notice that for the scope of this paper, we do *not* require an additive subcategory to be closed under direct summands. Note, that we assume additive subcategories to be full.

Remark 1.2. We will use cohomological degrees for complexes, i.e. the differentials go from lower to higher degree. On the other hand, we use homological notation, i.e. we use subscripts instead of superscripts, to avoid confusion with powers of morphisms.

Definition 1.3 ([HLN21, Definition 2.7])**.** Let $\mathbf{C}^{n+2}_{\mathcal{D}}$ be the full subcategory of the category of complexes over $\mathcal{D}$ consisting of all complexes concentrated in degrees $0, \ldots, n+1$. We denote objects in $\mathbf{C}^{n+2}_{\mathcal{D}}$ by

$$X_\bullet\colon \qquad X_0 \xrightarrow{d_0^X} X_1 \xrightarrow{d_1^X} \cdots \xrightarrow{d_{n-1}^X} X_n \xrightarrow{d_n^X} X_{n+1}$$

and morphisms in $\mathbf{C}^{n+2}_{\mathcal{D}}(X_\bullet, Y_\bullet)$ by $f_\bullet = (f_0, f_1, \ldots, f_n, f_{n+1})\colon X_\bullet \to Y_\bullet$.

We will use the following special complexes.

Definition 1.4. For $X \in \mathcal{D}$ and $i = 0, \ldots, n$ let $\mathrm{triv}_i(X)_\bullet \in \mathbf{C}^{n+2}_{\mathcal{D}}$ denote the complex

$$\mathrm{triv}_i(X)_\bullet\colon \qquad 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow X =\!= X \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0$$

with $\mathrm{triv}_i(X)_j = 0$ for $j \neq i, i+1$ and $\mathrm{triv}_i(X)_i = X = \mathrm{triv}_i(X)_{i+1}$ and $d_i^{\mathrm{triv}_i(X)} = \mathrm{id}_X$.

Definition 1.5 ([HLN21, Definition 2.27])**.** Let $f_\bullet \in \mathbf{C}^{n+2}_{\mathcal{D}}(X_\bullet, Y_\bullet)$ be a morphism of complexes. If $A := X_0 = Y_0$ and $f_0 = \mathrm{id}_A$ we denote by

$$M^f_\bullet\colon \qquad X_1 \xrightarrow{\begin{bmatrix} -d_1^X \\ f_1 \end{bmatrix}} X_2 \oplus Y_1 \xrightarrow{\begin{bmatrix} -d_2^X & 0 \\ f_2 & d_1^Y \end{bmatrix}} \cdots \xrightarrow{\begin{bmatrix} -d_n^X & 0 \\ f_n & d_{n-1}^Y \end{bmatrix}} X_{n+1} \oplus Y_n \xrightarrow{\begin{bmatrix} f_{n+1} & d_n^Y \end{bmatrix}} Y_{n+1}$$

the *mapping cone of* $f_\bullet$. Dually, if $C := X_{n+1} = Y_{n+1}$ and $f_{n+1} = \mathrm{id}_C$ we denote by

$$N^f_\bullet\colon \qquad X_0 \xrightarrow{\begin{bmatrix} d_0^X \\ f_0 \end{bmatrix}} X_1 \oplus Y_0 \xrightarrow{\begin{bmatrix} d_1^X & 0 \\ f_1 & -d_0^Y \end{bmatrix}} \cdots \xrightarrow{\begin{bmatrix} d_{n-1}^X & 0 \\ f_{n-1} & -d_{n-2}^Y \end{bmatrix}} X_n \oplus Y_{n-1} \xrightarrow{\begin{bmatrix} f_n & -d_{n-1}^Y \end{bmatrix}} Y_n$$

the *mapping cocone of* $f_\bullet$.

Definition 1.6 ([HLN21, Defintion 2.17])**.** For $A, C \in \mathcal{D}$ let $\mathbf{C}^{n+2}_{(\mathcal{D};A,C)}$ be the subcategory of $\mathbf{C}^{n+2}_{\mathcal{D}}$ where objects are complexes $X_\bullet$ with $X_0 = A$ and $X_{n+1} = C$ and where morphisms $f_\bullet \in \mathbf{C}^{n+2}_{(\mathcal{D};A,C)}(X_\bullet, Y_\bullet)$ are morphisms $f_\bullet \in \mathbf{C}^{n+2}_{\mathcal{D}}(X_\bullet, Y_\bullet)$ with $f_0 = \mathrm{id}_A$ and $f_{n+1} = \mathrm{id}_C$.

Definition 1.7 ([HLN21, Definition 2.17])**.** For $A, C \in \mathcal{D}$ and $X_\bullet, Y_\bullet \in \mathbf{C}^{n+2}_{(\mathcal{D};A,C)}$ we say $f_\bullet, g_\bullet \in \mathbf{C}^{n+2}_{(\mathcal{D};A,C)}(X_\bullet, Y_\bullet)$ are *homotopy equivalent* if $f_\bullet - g_\bullet$ is zero homotopic as a morphism of complexes $X_\bullet \to Y_\bullet$.

Homotopy equivalence induces an equivalence relation on morphisms and objects of $\mathbf{C}^{n+2}_{(\mathcal{D};A,C)}$ for $A, C \in \mathcal{D}$, see [HLN21, Definition 2.17]. We use a different notation than [HLN21] to denote the equivalence class of an object in $\mathbf{C}^{n+2}_{(\mathcal{D};A,C)}$.

Notation 1.8. For any pair of objects $A, C \in \mathcal{D}$ and any $X_\bullet \in \mathbf{C}^{n+2}_{(\mathcal{D};A,C)}$ we denote by $[X_\bullet]_{\mathcal{D}}$ the homotopy equivalence class of $X_\bullet$ in $\mathbf{C}^{n+2}_{(\mathcal{D},A,C)}$.

Now, suppose we are additionally given an arbitrary biadditive functor $\mathbb{G}\colon \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathsf{Ab}$.

Notation 1.9. For any pair $C, A \in \mathcal{D}$ the elements of $\mathbb{G}(C,A)$ are called $\mathbb{G}$-*extensions*. We will denote by ${}_A0_C$ the neutral element of $\mathbb{G}(C,A)$. We will often simply write 0 instead of ${}_A0_C$. Furthermore, if there is no risk of confusion, we will denote

$$a_* := \mathbb{G}(C,a)\colon \mathbb{G}(C,A) \to \mathbb{G}(C,B) \text{ and } c^* := \mathbb{G}(c,A)\colon \mathbb{G}(C,A) \to \mathbb{G}(D,A)$$

for $A,B,C,D \in \mathcal{D}$, $a \in \mathcal{D}(A,B)$ and $c \in \mathcal{D}(D,C)$

Definition 1.10 ([HLN21, Definition 2.3])**.** For $\mathbb{G}$-extensions $\delta \in \mathbb{G}(C,A)$ and $\rho \in \mathbb{G}(D,B)$ a *morphism of* $\mathbb{G}$-*extensions* $(a,c)\colon \delta \to \rho$ is a tuple consisting of a morphism $a \in \mathcal{D}(A,B)$ and $c \in \mathcal{D}(C,D)$ with $a_*\delta = c^*\rho$.

Definition 1.11 ([HLN21, Definition 2.9])**.** We define a category $Æ^{n+2}_{(\mathcal{D},\mathbb{G})}$ as follows.

1. Objects are tuples $\langle X_\bullet, \delta\rangle$ consisting of $X_\bullet \in \mathbf{C}^{n+2}_{\mathcal{D}}$ and $\delta \in \mathbb{G}(X_{n+1}, X_0)$ satisfying $(d_0^X)_*\delta = 0$ and $(d_n^X)^*\delta = 0$. We call $\langle X_\bullet, \delta\rangle$ a $\mathbb{G}$-*attached complex* and denote
$$\langle X_\bullet, \delta\rangle\colon \qquad X_0 \xrightarrow{d_0^X} X_1 \xrightarrow{d_1^X} \cdots \xrightarrow{d_{n-1}^X} X_n \xrightarrow{d_n^X} X_{n+1} \overset{\delta}{\dashrightarrow} .$$
2. Morphisms $f_\bullet \in Æ^{n+2}_{(\mathcal{D},\mathbb{G})}(\langle X_\bullet, \delta\rangle, \langle Y_\bullet, \rho\rangle)$ are morphisms $f_\bullet \in \mathbf{C}^{n+2}_{\mathcal{D}}(X_\bullet, Y_\bullet)$ such that $(f_0, f_{n+1})\colon \delta \to \rho$ is a morphism of $\mathbb{G}$-extensions.
3. Composition is the same as in $\mathbf{C}^{n+2}_{\mathcal{D}}$.

We want to remark that $Æ^{n+2}_{(\mathcal{D},\mathbb{G})}$ is an additive category.

Notation 1.12 ([HLN21, Definition 2.11])**.** If there is no risk of confusion we denote

$$\delta_\sharp\colon \mathcal{D}(X,C) \to \mathbb{G}(X,A),\ f \mapsto f^*\delta \text{ and } \delta^\sharp\colon \mathcal{D}(A,X) \to \mathbb{G}(C,X),\ g \mapsto g_*\delta$$

for any $X \in \mathcal{D}$ and $\delta \in \mathbb{G}(C,A)$.

Definition 1.13 ([HLN21, Definition 2.13])**.** An object $\langle X_\bullet, \delta\rangle \in Æ^{n+2}_{(\mathcal{D},\mathbb{G})}$ is called an n-*exangle* if for all $X \in \mathcal{D}$ the sequences

$$\mathcal{D}(X,X_0) \xrightarrow{\mathcal{D}(X,d_0^X)} \mathcal{D}(X,X_1) \xrightarrow{\mathcal{D}(X,d_1^X)} \cdots \xrightarrow{\mathcal{D}(X,d_n^X)} \mathcal{D}(X,X_{n+1}) \xrightarrow{\delta_\sharp} \mathbb{G}(X,X_0)$$

and

$$\mathcal{D}(X_{n+1},X) \xrightarrow{\mathcal{D}(d_n^X,X)} \mathcal{D}(X_n,X) \xrightarrow{\mathcal{D}(d_{n-1}^X,X)} \cdots \xrightarrow{\mathcal{D}(d_0^X,X)} \mathcal{D}(X_0,X) \xrightarrow{\delta^\sharp} \mathbb{G}(X_{n+1},X)$$

are exact in Ab. A morphism of n-exangles is just a morphism in $Æ^{n+2}_{(\mathcal{D},\mathbb{G})}$.

In particular, if $\langle X_\bullet, \delta\rangle$ is an n-exangle then d_{i+1}^X is a weak cokernel of d_i^X for any $i = 0, \dots, n-1$. Dually, d_{i-1}^X is a weak kernel of d_i^X for any $i = 1, \dots, n$.

Definition 1.14 ([HLN21, Definition 2.22])**.** An *exact realisation of* $\mathbb{G}$ is a map $\mathfrak{r}$ that assigns to each pair $A, C \in \mathcal{D}$ and each $\delta \in \mathbb{G}(A,C)$ a homotopy equivalence class $\mathfrak{r}(\delta)$ of an object in $\mathbf{C}^{n+2}_{(\mathcal{D},A,C)}$ such that the following axioms hold.

(R0) For any objects $A, B, C, D \in \mathcal{D}$, any $\mathbb{G}$-extensions $\delta \in \mathbb{G}(C, A)$ and $\rho \in \mathbb{G}(D, B)$, any complexes $X_\bullet \in \mathcal{C}^{n+2}_{(\mathcal{D};A,C)}$ with $[X_\bullet]_\mathcal{D} = \mathfrak{r}(\delta)$ and $Y_\bullet \in \mathcal{C}^{n+2}_{(\mathcal{D};B,D)}$ with $[Y_\bullet]_\mathcal{D} = \mathfrak{r}(\rho)$ and any morphism $(a, c) \colon \delta \to \rho$ of $\mathbb{G}$-extensions there exists an $f_\bullet \colon X_\bullet \to Y_\bullet$ with $f_0 = a$ and $f_{n+1} = c$. We call $f_\bullet$ a *lift* of (a, c).

(R1) If $\mathfrak{r}(\delta) = [X_\bullet]_\mathcal{D}$ for some $A, C \in \mathcal{D}$, $\delta \in \mathbb{G}(C, A)$ and $X_\bullet \in \mathbf{C}^{n+2}_{(\mathcal{D};A,C)}$ then $\langle X_\bullet, \delta\rangle$ is an n-exangle.

(R2) For $A, C \in \mathcal{D}$ we have $\mathfrak{r}({}_A0_0) = [\mathrm{triv}_0(A)_\bullet]_\mathcal{D}$ and $\mathfrak{r}({}_00_C) = [\mathrm{triv}_n(C)_\bullet]_\mathcal{D}$.

For any pair $A, C \in \mathcal{D}$ and $\mathbb{G}$-extension $\delta \in \mathbb{G}(A, C)$ we call any $X_\bullet \in \mathbf{C}^{n+2}_{(\mathcal{D},A,C)}$ with $[X_\bullet] = \mathfrak{r}(\delta)$ an $\mathfrak{r}$-*realisation* of δ.

Definition 1.15. Suppose $\mathfrak{r}$ is an exact realisation of $\mathbb{G}$. For any objects $A, C \in \mathcal{D}$, any $\mathbb{G}$-extension $\delta \in \mathbb{G}(C, A)$ and any $\mathfrak{r}$-realisation $X_\bullet$ of δ we call

1. the n-exangle $\langle X_\bullet, \delta\rangle$ an $\mathfrak{r}$-*distinguished* n-*exangle* and
2. $X_\bullet$ an $\mathfrak{r}$-*conflation*, d_0^X an $\mathfrak{r}$-*inflation* as well as d_n^X an $\mathfrak{r}$-*deflation*.

A *morphism of* $\mathfrak{r}$-*distinguished* n-*exangles* is just a morphism of n-exangles.

In the following definition we divide [HLN21, Definition 2.32(EA1)] into two separate statements (EA1) and (EA1$^{\mathrm{op}}$).

Definition 1.16 ([HLN21, Definition 2.32])**.** An n-*exangulated category* is a 3-tuple $(\mathcal{D}, \mathbb{G}, \mathfrak{r})$ where $\mathcal{D}$ is an additive category, $\mathbb{G} \colon \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathsf{Ab}$ is a biadditive functor and $\mathfrak{r}$ is an exact realisation of $\mathbb{G}$ such that the following axioms hold.

(EA1) If $f \in \mathcal{D}(X, Y)$ and $g \in \mathcal{D}(Y, Z)$ are $\mathfrak{r}$-inflations then so is $gf \in \mathcal{D}(X, Z)$.

(EA1$^{\mathrm{op}}$) If $f \in \mathcal{D}(X, Y)$ and $g \in \mathcal{D}(Y, Z)$ are $\mathfrak{r}$-deflations then so is $gf \in \mathcal{D}(X, Z)$.

(EA2) For any morphism $c \in \mathcal{D}(X_{n+1}, Y_{n+1})$ and any pair of $\mathfrak{r}$-distinguished n-exangles $\langle X_\bullet, c^*\rho\rangle$ and $\langle Y_\bullet, \rho\rangle$ with $A := Y_0 = X_0$ there is a lift $f_\bullet \colon X_\bullet \to Y_\bullet$ of (id_A, c) such that $\langle M^f_\bullet, (d_0^X)_*\rho\rangle$ is $\mathfrak{r}$-distinguished. We call $f_\bullet$ a *good lift* of (id_A, c).

(EA2$^{\mathrm{op}}$) For any morphism $a \in \mathcal{D}(X_0, Y_0)$ and any pair of $\mathfrak{r}$-distinguished n-exangles $\langle X_\bullet, \delta\rangle$ and $\langle Y_\bullet, a_*\delta\rangle$ with $C := Y_{n+1} = X_{n+1}$ there is a lift $f_\bullet \colon X_\bullet \to Y_\bullet$ of (a, id_C) such that $\langle N^f_\bullet, (d_n^Y)^*\rho\rangle$ is $\mathfrak{r}$-distinguished. We call $f_\bullet$ a *good lift* of (a, id_C).

Remark 1.17. By [HLN21, Proposition 4.3] a triplet $(\mathcal{D}, \mathbb{G}, \mathfrak{r})$ is a 1-exangulated category if and only if it is an extriangulated category in the sense of [NP19]. We therefore may use the term extriangulated category synonymously with the term 1-exangulated category.

Definition 1.18 ([HLN22, Definition 4.1])**.** An additive subcategory $\mathcal{B} \subseteq \mathcal{D}$ of an n-exangulated category $(\mathcal{D}, \mathbb{G}, \mathfrak{r})$ is called n-*extension closed* if for all $A, C \in \mathcal{B}$ and $\delta \in \mathbb{G}(C, A)$ there is an $\mathfrak{r}$-distinguished n-exangle $\langle X_\bullet, \delta\rangle$ with $X_i \in \mathcal{B}$ for $i = 0, \ldots, n+1$.

Remark 1.19. The notion of 1-extension closed additive subcategories coincides with the notion of extension closed subcategories of [NP19, Definition 2.17] as any two extriangles realizing the same extension have isomorphic terms by [HLN21, Lemma 4.1] and additive subcategories are closed under isomorphisms.

We recall the notion of an n-exangulated functor from Bennett-Tennenhaus–Shah.

Definition 1.20 ([BTS21, Definition 2.32])**.** Let $(\mathcal{D}, \mathbb{G}, \mathfrak{r})$ and $(\mathcal{D}', \mathbb{G}', \mathfrak{r}')$ be n-exangulated categories. An *n-exangulated functor* $(\mathscr{F}, \Gamma)\colon (\mathcal{D}, \mathbb{G}, \mathfrak{r}) \to (\mathcal{D}', \mathbb{G}', \mathfrak{r}')$ is a tuple consisting of an additive functor $\mathscr{F}\colon \mathcal{D} \to \mathcal{D}'$ and a natural transformation $\Gamma\colon \mathbb{G}(-,-) \Rightarrow \mathbb{G}'(\mathscr{F}-, \mathscr{F}-)$, such that $[X_\bullet]_{\mathcal{D}} = \mathfrak{r}(\delta)$ implies $[\mathscr{F}(X_\bullet)]_{\mathcal{D}'} = \mathfrak{r}'(\Gamma_{C,A}(\delta))$ for all $A, C \in \mathcal{D}$, $\delta \in \mathbb{G}(A, C)$ and $X_\bullet \in \mathbf{C}^{n+2}_{(\mathcal{D};A,C)}$.

Bennett-Tennenhaus–Haugland–Sandøy–Shah [BTHSS22, Definition 4.9] introduced the notion of n-exangulated equivalences. Using [BTHSS22, Proposition 4.11] we obtain the following equivalent definition.

Definition 1.21. A functor $(\mathscr{F}, \Gamma)\colon (\mathcal{D}, \mathbb{G}, \mathfrak{r}) \to (\mathcal{D}', \mathbb{G}, \mathfrak{r}')$ of n-exangulated categories $(\mathcal{D}, \mathbb{G}, \mathfrak{r})$ and $(\mathcal{D}', \mathbb{G}', \mathfrak{r}')$ is called an *n-exangulated equivalence* if $\mathscr{F}\colon \mathcal{D} \to \mathcal{D}'$ is an equivalence and $\Gamma\colon \mathbb{G}(-,-) \Rightarrow \mathbb{G}'(\mathscr{F}-, \mathscr{F}-)$ is a natural isomorphism.

From now we assume the following global Setup 1.22, unless *explicitly* stated otherwise.

Setup 1.22. Suppose $n \in \mathbb{N}_{\geq 1}$. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an n-exangulated category and $\mathcal{A} \subseteq \mathcal{C}$ be an n-extension closed additive subcategory. Let $\mathscr{I}_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{C}$ denote the canonical inclusion.

In the situation of Setup 1.22 one can define a functor $\mathbb{F}$ on $\mathcal{A}$ and an exact realisation $\mathfrak{t}$. We will use the following notation for the rest of this paper.

Definition 1.23 ([HLN22, Proposition 4.2])**.** We define $\mathbb{F}(-,-) := \mathbb{E}(\mathscr{I}_{\mathcal{A}}-, \mathscr{I}_{\mathcal{A}}-)$ to be the restriction of $\mathbb{E}$. For $A, C \in \mathcal{A}$ we define $(\Theta_{\mathcal{A}})_{C,A}\colon \mathbb{F}(C, A) \to \mathbb{E}(C, A)$, $\delta \mapsto \delta$ as the canonical inclusion. This yields a natural isomorphism $\Theta_{\mathcal{A}}\colon \mathbb{F}(-,-) \to \mathbb{E}(\mathscr{I}_{\mathcal{A}}-, \mathscr{I}_{\mathcal{A}}-)$. For $A, C \in \mathcal{A}$ and $\delta \in \mathbb{F}(C, A)$ we define $\mathfrak{t}(\delta) := [X_\bullet]_{\mathcal{A}}$ where $\langle X_\bullet, \delta \rangle$ is an $\mathfrak{s}$-distinguished n-exangle with $X_i \in \mathcal{A}$ for $i = 0, \ldots, n+1$.

Notice that $\mathfrak{t}$ is well-defined since for any pair $A, C \in \mathcal{A}$ and $X_\bullet, Y_\bullet \in \mathbf{C}^{n+2}_{(\mathcal{A};A,C)}$ we have $[X_\bullet]_{\mathcal{A}} = [Y_\bullet]_{\mathcal{A}}$ if and only if $[X_\bullet]_{\mathcal{C}} = [Y_\bullet]_{\mathcal{C}}$, as homotopy equivalence are preserved and reflected under $\mathscr{I}_{\mathcal{A}}$, since $\mathcal{A} \subseteq \mathcal{C}$ is additive.

Recall also that $\mathfrak{t}$ is an exact realisation of $\mathbb{F}$ and that $(\mathcal{A}, \mathbb{F}, \mathfrak{t})$ satisfies axioms (EA2), (EA2$^{\mathrm{op}}$) by [HLN22, Propsition 4.2(1)]. We have the following important remark which we will make extensive use of.

Remark 1.24. An $\mathbb{F}$-attached complex $\langle X_\bullet, \delta \rangle$ with $\delta \in \mathbb{F}(X_{n+1}, X_0)$ is a $\mathfrak{t}$-distinguished n-exangle if and only if $\langle \mathscr{I}_{\mathcal{A}}(X_\bullet), (\Theta_{\mathcal{A}})_{X_{n+1},X_0}(\delta) \rangle = \langle X_\bullet, \delta \rangle$ is an $\mathfrak{s}$-distinguished n-exangle with $X_i \in \mathcal{A}$ for $i = 0, \ldots, n+1$. Indeed, if $\langle X_\bullet, \delta \rangle$ is $\mathfrak{t}$-distinguished then $X_0, \ldots, X_{n+1} \in \mathcal{A}$ and $[X_\bullet]_{\mathcal{A}} = [Y_\bullet]_{\mathcal{A}}$ for an $\mathfrak{s}$-distinguished n-exangle $\langle Y_\bullet, \delta \rangle$ with $Y_0, \ldots, Y_{n+1} \in \mathcal{A}$, by definition. However, then $[X_\bullet]_{\mathcal{C}} = [Y_\bullet]_{\mathcal{C}} = \mathfrak{s}(\delta)$, since $\mathscr{I}_{\mathcal{A}}$ preserves homotopy equivalences and hence $\langle X_\bullet, \delta \rangle$ is $\mathfrak{s}$-distinguished. On the other hand, if $\langle X_\bullet, \delta \rangle$ is $\mathfrak{s}$-distinguished and $X_0, \ldots, X_{n+1} \in \mathcal{A}$, then $\langle X_\bullet, \delta \rangle$ is $\mathfrak{t}$-distinguished, since $\mathfrak{t}$ is well-defined.

2 The Obscure Axiom

Recall Setup 1.22 and Definition 1.23. Before we can start, we need an easy but crucial lemma, which is similar to [HLN21, Corollary 3.4].

Lemma 2.1. *Suppose $n \in \mathbb{N}_{\geq 2}$. If $\left[\begin{smallmatrix} 0 & f \end{smallmatrix}\right]^\top : X_0 \to A \oplus X'_1$ is a $\mathfrak{t}$-inflation with $A, X'_1 \in \mathcal{A}$ then $f\colon X_0 \to X'_1$ is a $\mathfrak{t}$-inflation.*

Proof. Suppose $\langle X_\bullet, \delta \rangle$ is a $\mathfrak{t}$-distinguished n-exangle with $X_1 = A \oplus X'_1$ and $d_0^X = \left[\begin{smallmatrix} 0 & f \end{smallmatrix}\right]^\top$. We construct a commutative diagram

$$
\begin{tikzcd}
X_0 \arrow[r, "d_0^X = \left[\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right]"] & A \oplus X'_1 \arrow[r, "{[\,g\ h\,] := d_1^X}"] \arrow[d, "{p := [\,\mathrm{id}_A\ 0\,]}"] & X_2 . \arrow[dl, dotted, bend left, "p'"] \\
 & A &
\end{tikzcd}
$$

Let $p := \left[\begin{smallmatrix} \mathrm{id}_A & 0 \end{smallmatrix}\right] : A \oplus X'_1 \to A$. Then $pd_0^X = 0$ and because d_1^X is a weak cokernel of d_0^X there is a $p'\colon X_2 \to A$ with $p'd_1^X = p$. Denote $d_1^X : A \oplus X'_1 \to X_2$ by $\left[\begin{smallmatrix} g & h \end{smallmatrix}\right]$. Then $p'g = \mathrm{id}_A$ and $p'h = 0$. Hence, p' is a retraction with section g and $e := gp'$ and $e' := \mathrm{id}_{X_2} - gp'$ are orthogonal idempotents. The n-exangle $\langle \mathrm{triv}_2(A)_\bullet, 0 \rangle$ is $\mathfrak{s}$-distinguished using $n \in \mathbb{N}_{\geq 2}$ and [Hau21, Proposition 2.14]. Notice that this crucially depends on $n \in \mathbb{N}_{\geq 2}$ as for $n = 1$ there is not enough space to define $\mathrm{triv}_2(A)$, compare Definition 1.4. Hence, the n-exangle $\langle X''_\bullet, \delta' \rangle := \langle X_\bullet \oplus \mathrm{triv}_2(A)_\bullet, \delta \oplus 0 \rangle$

$$
\begin{tikzcd}
X_0 \arrow[r, "\left[\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right]"] & A \oplus X'_1 \arrow[r, "\left[\begin{smallmatrix} g & h \\ 0 & 0 \end{smallmatrix}\right]"] & X_2 \oplus A \arrow[r, "\left[\begin{smallmatrix} d_2^X & 0 \\ 0 & \mathrm{id}_A \end{smallmatrix}\right]"] & X_3 \oplus A \arrow[r, "{[\,d_3^X\ 0\,]}"] & X_4 \arrow[r, "d_4^X"] & \cdots \arrow[r, "d_n^X"] & X_{n+1} \arrow[r, dashed, "\delta'"] & {}
\end{tikzcd}
$$

is $\mathfrak{s}$-distinguished, by Remark 1.24 and [HLN21, Corollary 2.26(2)]. It is easy to check that Diagram 2.1

$$
\begin{tikzcd}
X_0 \arrow[r, "f"] \arrow[d, equal] & X'_1 \arrow[r, "h"] \arrow[d, shift left, "\left[\begin{smallmatrix} 0 \\ \mathrm{id}_{X'_1} \end{smallmatrix}\right]"] & X_2 \arrow[r, "\left[\begin{smallmatrix} d_2^X e' \\ p' \end{smallmatrix}\right]"] \arrow[d, shift left, "\left[\begin{smallmatrix} e' \\ p' \end{smallmatrix}\right]"] & X_3 \oplus A \arrow[r, "d_3^{X''}"] \arrow[d, equal] & X_4 \arrow[r, "d_4^X"] \arrow[d, equal] & \cdots \arrow[r, "d_n^X"] & X_{n+1} \arrow[r, dashed, "\delta'"] \arrow[d, equal] & {} \\
X_0 \arrow[r, "d_0^{X''}"] \arrow[d, shift right] & A \oplus X'_1 \arrow[r, "d_1^{X''}"] \arrow[u, shift left, "{[\,0\ \mathrm{id}_{X'_1}\,]}"] \arrow[d, shift right, "{[\,\mathrm{id}_A\ 0\,]}"'] & X_2 \oplus A \arrow[r, "d_2^{X''}"] \arrow[u, shift left, "{[\,e'\ g\,]}"] \arrow[d, shift right, "{[\,p'\ 0\,]}"'] & X_3 \oplus A \arrow[r, "d_3^{X''}"] \arrow[d, shift right] & X_4 \arrow[r, "d_4^{X''}"] \arrow[d, shift right] & \cdots \arrow[r, "d_n^{X''}"] & X_{n+1} \arrow[r, dashed, "\delta'"] \arrow[d, shift right] & {} \\
0 \arrow[r] \arrow[u, shift right] & A \arrow[r, equal] \arrow[u, shift right, "\left[\begin{smallmatrix} \mathrm{id}_A \\ 0 \end{smallmatrix}\right]"'] & A \arrow[r] \arrow[u, shift right, "\left[\begin{smallmatrix} g \\ 0 \end{smallmatrix}\right]"'] & 0 \arrow[r] \arrow[u, shift right] & 0 \arrow[r] \arrow[u, shift right] & \cdots \arrow[r] & 0 \arrow[r, dashed, "0"] \arrow[u, shift right] & {} ,
\end{tikzcd}
$$

Diagram 2.1: Biproduct diagram in $\text{Æ}^{n+2}_{(\mathcal{C},\mathbb{E})}$.

where the middle row is $\langle X''_\bullet, \delta' \rangle$, is a biproduct diagram in the additive category $\text{Æ}^{n+2}_{(\mathcal{C},\mathbb{E})}$, see Calculation A.1. By [HLN21, Proposition 3.3] for $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ this means that the upper row of Diagram 2.1 is an $\mathfrak{s}$-distinguished n-exangle $\langle X'_\bullet, \delta' \rangle$. All terms of $\langle X'_\bullet, \delta' \rangle$ are in $\mathcal{A}$. This shows that $\langle X'_\bullet, \delta' \rangle$ is a $\mathfrak{t}$-distinguished n-exangle, by Remark 1.24. Hence f is a $\mathfrak{t}$-inflation. $\square$

The proof of Lemma 2.1 depends on $n \in \mathbb{N}_{\geq 2}$. However, Lemma 2.1 still holds for $n = 1$ if $\mathcal{A}$ is weakly idempotent complete. Indeed, we can then just remove a trivial summand $\langle \mathrm{triv}_1(A)_\bullet, 0\rangle$ from $\langle X_\bullet, \delta\rangle$. For the case where $\mathcal{A} = \mathcal{C}$ this has been shown by Tattar, see [Tat22, Lemma II.1.43]. We provide a proof for convenience of the reader.

Lemma 2.2. *Suppose $\mathcal{A}$ is weakly idempotent complete and $n = 1$. If $\left[\begin{smallmatrix} 0 & f \end{smallmatrix}\right]^\top : X_0 \to A \oplus X_1'$ is a $\mathfrak{t}$-inflation with $A, X_1' \in \mathcal{A}$ then $f \colon X_0 \to X_1'$ is a $\mathfrak{t}$-inflation.*

Proof. As $\left[\begin{smallmatrix} 0 & f \end{smallmatrix}\right]^\top : X_0 \to A \oplus X_1'$ is a $\mathfrak{t}$-inflation there is a $\mathfrak{t}$-distinguished 1-exangle $\langle X_\bullet, \delta\rangle$ with $X_1 = A \oplus X_1'$ and $d_0^X = \left[\begin{smallmatrix} 0 & f \end{smallmatrix}\right]^\top$. We construct the following diagram

$$\begin{tikzcd}
X_0 \arrow[r, "d_0^X = \left[\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right]"] & A \oplus X_1' \arrow[r, "d_1^X"] \arrow[d, shift left, "p := \left[\begin{smallmatrix} \mathrm{id}_A & 0 \end{smallmatrix}\right]"] & X_2 \arrow[dl, dotted, "p'"] \\
 & A \arrow[u, shift left, "i := \left[\begin{smallmatrix} \mathrm{id}_A \\ 0 \end{smallmatrix}\right]"] &
\end{tikzcd}$$

in $\mathcal{A}$. Let $p := \left[\begin{smallmatrix} \mathrm{id}_A & 0 \end{smallmatrix}\right] : A \oplus X_1' \to A$. Then $pd_0^X = 0$ and because d_1^X is a weak cokernel of d_0^X there is $p' \colon X_2 \to A$ with $p'd_1^X = p$. Now, $i := \left[\begin{smallmatrix} \mathrm{id}_A & 0 \end{smallmatrix}\right]^\top : A \to A \oplus X_1'$ is a section for p. Hence, $d_1^X i$ is a section for the retraction p' and $e := d_1^X ip' \in \mathrm{End}_{\mathcal{C}}(X_2)$ is a split idempotent in $\mathcal{A}$. As $\mathcal{A}$ is weakly idempotent complete there is a splitting of $e' := \mathrm{id}_{X_2} - e$, say with retraction $q' \colon X_2 \to X_2'$ and section $j' \colon X_2' \to X_2$ such that $e' = j'q'$ and $X_2' \in \mathcal{A}$. Put $\delta' := (j')^*\delta$, $q := \left[\begin{smallmatrix} 0 & \mathrm{id}_{X_1'} \end{smallmatrix}\right]$ and $j := \left[\begin{smallmatrix} 0 & \mathrm{id}_{X_1'} \end{smallmatrix}\right]^\top$. It is easy to check that Diagram 2.2

$$\begin{tikzcd}
X_0 \arrow[r, "f"] \arrow[d, equal] & X_1' \arrow[r, "q'd_1^X j"] \arrow[d, shift right, "j"'] & X_2' \arrow[r, dashed, "\delta'"] \arrow[d, shift right, "j'"'] & {} \\
X_0 \arrow[r, "\left[\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right]"] \arrow[d, shift right] & A \oplus X_1' \arrow[r, "d_1^X"] \arrow[u, shift right, "q"'] \arrow[d, shift right, "p"'] & X_2 \arrow[r, dashed, "\delta"] \arrow[u, shift right, "q'"'] \arrow[d, shift right, "p'"'] & {} \\
0 \arrow[r] \arrow[u, shift right] & A \arrow[r, equal] \arrow[u, shift right, "i"'] & A \arrow[r, dashed, "{}_0 0_A"] \arrow[u, shift right, "d_1^X i"'] & {}
\end{tikzcd},$$

Diagram 2.2: Biproduct diagram in $\text{Æ}^3_{(\mathcal{C},\mathbb{E})}$.

where the middle row is the 1-exangle $\langle X_\bullet, \delta\rangle$, is a biproduct diagram in the additive category $\text{Æ}^3_{(\mathcal{C},\mathbb{E})}$, see Calculation A.2. By [HLN21, Proposition 3.3] for $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ this means that the upper row of Diagram 2.2 is an $\mathfrak{s}$-distinguished n-exangle $\langle X'_\bullet, \delta'\rangle$. All terms of $X'_\bullet$ are in $\mathcal{A}$, so $\langle X', \delta'\rangle$ is $\mathfrak{t}$-distinguished, by Remark 1.24. Hence, f is a $\mathfrak{t}$-inflation. $\square$

Lemma 2.3. *If $g \colon X_0 \to X_1$ is a $\mathfrak{t}$-inflation and $f \colon X_0 \to A$ is a morphism with $A \in \mathcal{A}$ then $\left[\begin{smallmatrix} f & g \end{smallmatrix}\right]^\top : X_0 \to A \oplus X_1$ is a $\mathfrak{t}$-inflation.*

Proof. We can complete the $\mathfrak{t}$-inflation g to a $\mathfrak{t}$-distinguished n-exangle $\langle X_\bullet, \delta\rangle$ with $d_0^X = g$. As $A, X_{n+1} \in \mathcal{A}$ there is a $\mathfrak{t}$-distinguished n-exangle $\langle Y_\bullet, f_*\delta\rangle$ with $Y_0 = A$ and $Y_{n+1} = X_{n+1}$. The solid morphisms f and $\mathrm{id}_{X_{n+1}}$ in the diagram

$$\begin{tikzcd}
X_0 \arrow[r, "g"] \arrow[d, "f"] & X_1 \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & X_n \arrow[r] \arrow[d, dotted] & X_{n+1} \arrow[r, dashed, "\delta"] \arrow[d, equal] & {} \\
A \arrow[r] & Y_1 \arrow[r] & \cdots \arrow[r] & Y_n \arrow[r] & X_{n+1} \arrow[r, dashed, "f_*\delta"] & {}
\end{tikzcd}$$

form a morphism of $\mathbb{F}$-extensions $(f, \mathrm{id}_{X_{n+1}})\colon \delta \to f_*\delta$. Since $(\mathcal{A}, \mathbb{F}, \mathfrak{t})$ satisfies axiom (EA2$^{\mathrm{op}}$) we can find a good lift $f_\bullet\colon \langle X_\bullet, \delta\rangle \to \langle Y_\bullet, f_*\delta\rangle$ of $(f, \mathrm{id}_{X_{n+1}})\colon \delta \to f_*\delta$ such that the mapping cocone

$$\langle N^f_\bullet, (d^Y_n)^*\delta\rangle\colon \qquad X_0 \xrightarrow{\left[\begin{smallmatrix} g \\ f \end{smallmatrix}\right]} X_1 \oplus A \xrightarrow{d_1^{N^f}} X_2 \oplus Y_1 \to \cdots \to X_n \oplus Y_{n-1} \to Y_n \overset{(d^Y_n)^*\delta}{\dashrightarrow}$$

of $f_\bullet$ is $\mathfrak{t}$-distinguished. Now there is an isomorphism

$$s := \begin{bmatrix} 0 & \mathrm{id}_A \\ \mathrm{id}_{X_1} & 0 \end{bmatrix} : X_1 \oplus A \to A \oplus X_1$$

so $\langle N^f_\bullet, (d^Y_n)^*\delta\rangle$ is isomorphic to

$$\langle N_\bullet, \delta\rangle\colon \qquad X_0 \xrightarrow{\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]} A \oplus X_1 \xrightarrow{d_1^{N^f} s^{-1}} X_2 \oplus Y_1 \to \cdots \to X_n \oplus Y_{n-1} \to Y_n \overset{(d^Y_n)^*\delta}{\dashrightarrow} .$$

By [HLN21, Corollary 2.26(2)] this is a $\mathfrak{t}$-distinguished n-exangle. The result follows. □

Proposition 2.4 (Relative Obscure Axiom)**.** *Suppose $\mathcal{A}$ is weakly idempotent complete or $n \in \mathbb{N}_{\geq 2}$. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two morphisms with $Y \in \mathcal{A}$. If $gf\colon X \to Z$ is a $\mathfrak{t}$-inflation, then so is f.*

Proof. We have $Y \in \mathcal{A}$. Therefore, $[\, f \;\, gf \,]^\top : X \to Y \oplus Z$ is a $\mathfrak{t}$-inflation by applying Lemma 2.3 to the $\mathfrak{t}$-inflation $gf\colon X \to Z$ and the morphism $f\colon X \to Y$. Hence, there is a $\mathfrak{t}$-distinguished n-exangle $\langle X_\bullet, \delta\rangle$ with $X_0 = X$, $X_1 = Y \oplus Z$ and $d_0^X = [\, f \;\, gf \,]^\top$. Consider the isomorphism

$$s := \begin{bmatrix} 0 & \mathrm{id}_Z \\ \mathrm{id}_Y & 0 \end{bmatrix} \begin{bmatrix} \mathrm{id}_Y & 0 \\ -g & \mathrm{id}_Z \end{bmatrix} : Y \oplus Z \to Z \oplus Y.$$

This isomorphism satisfies

$$s \begin{bmatrix} f \\ gf \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{id}_Z \\ \mathrm{id}_Y & 0 \end{bmatrix} \begin{bmatrix} \mathrm{id}_Y & 0 \\ -g & \mathrm{id}_Z \end{bmatrix} \begin{bmatrix} f \\ gf \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{id}_Z \\ \mathrm{id}_Y & 0 \end{bmatrix} \begin{bmatrix} f \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

Using s and [HLN21, Corollary 2.26(2)], the $\mathfrak{t}$-distinguished n-exangle $\langle X_\bullet, \delta\rangle$ gives rise to an $\mathfrak{t}$-distinguished n-exangle

$$X \xrightarrow{\left[\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right]} Z \oplus Y \xrightarrow{d_1^X s^{-1}} X_2 \longrightarrow \cdots \longrightarrow X_n \longrightarrow X_{n+1} \overset{\delta}{\dashrightarrow} .$$

Hence, $[\, 0 \;\, f \,]^\top$ is a $\mathfrak{t}$-inflation. Notice, $Y, Z \in \mathcal{A}$. Hence, f is a $\mathfrak{t}$-inflation by Lemma 2.2 if $\mathcal{A}$ is weakly idempotent complete and $n = 1$ and by Lemma 2.1 if $n \in \mathbb{N}_{\geq 2}$. □

It is remarkable that, for $n \in \mathbb{N}_{\geq 2}$, we do not need to assume that $\mathcal{C}$ is weakly idempotent complete for the following to hold.

Corollary 2.5 (Strong Obscure Axiom)**.** *Suppose $\mathcal{C}$ is weakly idempotent complete or $n \in \mathbb{N}_{\geq 2}$. Let $f\colon X \to Y$ and $g\colon Y \to Z$ are two morphisms. If $gf\colon X \to Z$ is an $\mathfrak{s}$-inflation, then so is f.*

Proof. This follows immediately from Proposition 2.4. □

Indeed, the converse of Corollary 2.5 is true as well. We recall the following definition.

Definition 2.6 ([NP19, Condition 5.8])**.** An extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfies *condition* (WIC) if for any two morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ the following hold.

1. If gf is a $\mathfrak{s}$-inflation, then f is a $\mathfrak{s}$-inflation.
2. If gf is a $\mathfrak{s}$-deflation, then g is a $\mathfrak{s}$-deflation.

Proposition 2.7. *An extriangulated category satisfies condition* (WIC) *if and only if it is weakly idempotent complete.*

Proof. That a weakly idempotent complete extriangulated category satisfies condition (WIC) follows from Corollary 2.5 and its dual using Remark 1.17. That an extriangulated category which satisfies condition (WIC) is weakly idempotent complete follows from [Msa22, Proposition 3.33] or [Tat22, Corollary II.1.41]. □

3 n-Extension closed subcategories of n-exangulated categories

Recall Setup 1.22 and Definition 1.23. By [HLN22, Proposition 4.2(1)], we know that $\mathfrak{t}$ is an exact realisation for $\mathbb{F}$ and that $(\mathcal{A}, \mathbb{F}, \mathfrak{t})$ satisfies axioms (EA2) and (EA2$^{\mathrm{op}}$). To show that $(\mathcal{A}, \mathbb{F}, \mathfrak{t})$ is n-exangulated we only need to show that $(\mathcal{A}, \mathbb{F}, \mathfrak{t})$ satisfies axioms (EA1) and (EA1$^{\mathrm{op}}$), by [HLN22, Proposition 4.2(2)]. We will show that $\mathfrak{t}$-inflations are closed under composition, the remaining axiom (EA1$^{\mathrm{op}}$) follows dually.

If $f\colon X_0 \to X_1$ and $g\colon X_1 \to Y_1$ are $\mathfrak{t}$-inflations then gf is an $\mathfrak{s}$-inflation by Remark 1.24 and axiom (EA1) for $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. By completing the inflations f, gf and g to distinguished n-exangles, we may obtain the solid morphism of Diagram 3.1

$$\begin{tikzcd}
X_0 \arrow[r, "f"] \arrow[d, equal] & X_1 \arrow[r, "d_1^X"] \arrow[d, "g"] & X_2 \arrow[r] \arrow[d, dotted, "\phi_2"] & \cdots \arrow[r] & X_n \arrow[r] \arrow[d, dotted, "\phi_n"] & X_{n+1} \arrow[r, dashed, "\delta"] \arrow[d, dotted, "\phi_{n+1}"] & {} \\
Y_0 \arrow[r, "gf"] \arrow[d, "f"] & Y_1 \arrow[r] \arrow[d, equal] & Y_2 \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & Y_n \arrow[r] \arrow[d, dotted] & Y_{n+1} \arrow[r, dashed, "\rho"] \arrow[d, dotted] & {} \\
Z_0 \arrow[r, "g"] & Z_1 \arrow[r] & Z_2 \arrow[r] & \cdots \arrow[r] & Z_n \arrow[r] & Z_{n+1} \arrow[r, dashed, "\gamma"] & {}
\end{tikzcd}$$

Diagram 3.1: The n-exangles arising from $\mathfrak{t}$-inflations f and g.

such that the upper and lower row are $\mathfrak{t}$-distinguished n-exangles and the middle row is an $\mathfrak{s}$-distinguished n-exangle. Our plan is to replace the object Y_{n+1} by an object $Y' \in \mathcal{A}$ and $\rho \in \mathbb{E}(Y_{n+1}, Y_0)$ by an $\mathbb{F}$-extension $\varepsilon \in \mathbb{F}(Y', Y_0)$, see Lemma 3.1. Then we want to realise ε by a $\mathfrak{t}$-distinguished n-exangle and replace the $\mathfrak{t}$-inflation of this n-exangle by gf using the relative Obscure Axiom.

Lemma 3.1. *Suppose we are given the solid morphisms of Diagram 3.1 such that the upper and lower row, respectively, form $\mathfrak{t}$-distinguished n-exangles $\langle X_\bullet, \delta\rangle$ and $\langle Z_\bullet, \gamma\rangle$, and* *such that the middle row forms an $\mathfrak{s}$-distinguished n-exangle $\langle Y_\bullet, \rho\rangle$. Then there is an object $Y' \in \mathcal{A}$ and morphisms $s\colon Y' \to Y_{n+1}$ and $t\colon Y_{n+1} \to Y'$ such that $(st)^*\rho = \rho$.*

Proof. It follows from [HLN21, Proposition 3.6(2)] applied in $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ that there is a morphism $\phi_\bullet\colon \langle X_\bullet, \delta\rangle \to \langle Y_\bullet, \rho\rangle$ with $\phi_0 = \mathrm{id}_{X_0}$ and $\phi_1 = g$ such that the mapping cone $\langle M^\phi_\bullet, f_*\rho\rangle$ of $\phi_\bullet$ is $\mathfrak{s}$-distinguished. Notice that $Z_0 = X_1, Y_1 = Z_1, X_2 \in \mathcal{A}$. By Lemma 2.3 for the $\mathfrak{t}$-inflation $g\colon X_1 \to Y_1$ and the morphism $-d^X_1\colon X_1 \to X_2$ we have that $\left[\begin{smallmatrix} -d^X_1 & g\end{smallmatrix}\right]^\top : X_1 \to X_2 \oplus Y_1$ is a $\mathfrak{t}$-inflation. Hence, there is a $\mathfrak{t}$-distinguished n-exangle $\langle Z'_\bullet, \gamma'\rangle$ with $Z'_0 = X_1$, $Z'_1 = X_2 \oplus Y_1$ and $d^{Z'}_0 = \left[\begin{smallmatrix} -d^X_1 & g\end{smallmatrix}\right]^\top$. We obtain the solid morphisms of a diagram

$$\begin{tikzcd}
X_1 \arrow[r, "\left[\begin{smallmatrix} -d^X_1 \\ g\end{smallmatrix}\right]"] \arrow[d, equal] & X_2 \oplus Y_1 \arrow[r] \arrow[d, equal] & Z'_2 \arrow[r] \arrow[d, dotted, shift left, "s'_2"] & \cdots \arrow[r] & Z'_n \arrow[r] \arrow[d, dotted, shift left, "s'_n"] & Z'_{n+1} \arrow[r, dashed, "\gamma'"] \arrow[d, dotted, shift left, "s'_{n+1}"] & {} \\
X_1 \arrow[r, "\left[\begin{smallmatrix} -d^X_1 \\ g\end{smallmatrix}\right]"'] & X_2 \oplus Y_1 \arrow[r] & X_3 \oplus Y_2 \arrow[r] \arrow[u, dotted, shift left, "t'_2"] & \cdots \arrow[r] & X_{n+1} \oplus Y_n \arrow[r, "\left[\begin{smallmatrix} \phi_{n+1} & d^Y_n\end{smallmatrix}\right]"] \arrow[u, dotted, shift left, "t'_n"] & Y_{n+1} \arrow[r, dashed, "f_*\rho"] \arrow[u, dotted, shift left, "t'_{n+1}"] & {}
\end{tikzcd}$$

where the upper row is the $\mathfrak{t}$-distinguished n-exangle $\langle Z'_\bullet, \gamma'\rangle$ and the lower row is the $\mathfrak{s}$-distinguished n-exangle $\langle M^\phi_\bullet, f_*\rho\rangle$. By [HLN21, Propostion 3.6(1)], this gives rise to morphisms $s'_\bullet\colon \langle Z'_\bullet, \gamma'\rangle \to \langle M^\phi_\bullet, f_*\rho\rangle$ and $t'_\bullet\colon \langle M^\phi_\bullet, f_*\rho\rangle \to \langle Z'_\bullet, \gamma'\rangle$ with $s'_0 = \mathrm{id}_{X_1} = t'_0$ and $s'_1 = \mathrm{id}_{X_2 \oplus Y_1} = t'_1$. This implies that $(\mathrm{id}_{X_1}, s'_{n+1})\colon \gamma' \to f_*\rho$ and $(\mathrm{id}_{X_1}, t'_{n+1})\colon f_*\rho \to \gamma'$ are morphisms of $\mathbb{E}$-extensions. Hence, $(\mathrm{id}_{X_1}, s'_{n+1}t'_{n+1})\colon f_*\rho \to f_*\rho$ is also a morphism of $\mathbb{E}$-extensions. Therefore,

$$(\mathrm{id}_{Y_{n+1}} - s'_{n+1}t'_{n+1})^* f_*\rho = f_*\rho - (s'_{n+1}t'_{n+1})^* f_*\rho = f_*\rho - (\mathrm{id}_{X_1})_* f_*\rho = 0$$

holds. Because the sequence

$$\mathcal{C}(Y_{n+1}, X_{n+1} \oplus Y_n) \xrightarrow{\mathcal{C}(Y_{n+1}, \left[\begin{smallmatrix} \phi_{n+1} & d^Y_n\end{smallmatrix}\right])} \mathcal{C}(Y_{n+1}, Y_{n+1}) \xrightarrow{(f_*\rho)_\sharp} \mathbb{E}(Y_{n+1}, X_1)$$

is exact, there is a morphism $\left[\begin{smallmatrix} h & h'\end{smallmatrix}\right]^\top : Y_{n+1} \to X_{n+1} \oplus Y_n$ with

$$\mathrm{id}_{Y_{n+1}} - s'_{n+1}t'_{n+1} = \left[\begin{smallmatrix} \phi_{n+1} & d^Y_n\end{smallmatrix}\right] \left[\begin{smallmatrix} h \\ h'\end{smallmatrix}\right] = \phi_{n+1}h + d^Y_n h'.$$

Now we define $Y' := X_{n+1} \oplus Z'_{n+1}$ and $s := \left[\begin{smallmatrix} \phi_{n+1} & s'_{n+1}\end{smallmatrix}\right] : X_{n+1} \oplus Z'_{n+1} \to Y_{n+1}$ as well as $t := \left[\begin{smallmatrix} h & t'_{n+1}\end{smallmatrix}\right]^\top : Y_{n+1} \to X_{n+1} \oplus Z'_{n+1}$. We claim that these are the desired morphisms. Indeed, $\mathrm{id}_{Y_{n+1}} - st = \mathrm{id}_{Y_{n+1}} - s'_{n+1}t'_{n+1} - \phi_{n+1}h = d^Y_n h'$. Therefore, we obtain $(\mathrm{id}_{Y_{n+1}} - st)^*\rho = (d^Y_n h')^*\rho = (h')^*(d^Y_n)^*\rho = 0$ since already $(d^Y_n)^*\rho = 0$, as all n-exangles are $\mathbb{E}$-attached complexes. Since $\langle X_\bullet, \delta\rangle$ and $\langle Z'_\bullet, \gamma'\rangle$ were $\mathfrak{t}$-distinguished, we have $Y' = X_{n+1} \oplus Z'_{n+1} \in \mathcal{A}$ and the result follows. □

We are ready to prove that $\mathfrak{t}$-inflations are closed under composition.

Lemma 3.2. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are $\mathfrak{t}$-inflations, then so is $gf\colon X \to Z$.*

Proof. If $n = 1$ then [NP19, Remark 2.18] and Remark 1.17 imply that the triplet $(\mathcal{A}, \mathbb{F}, \mathfrak{t})$ is 1-exangulated. The Lemma follows from (EA1) in this case.

Let $n \in \mathbb{N}_{\geq 2}$. Define $X_0 := X =: Y_0$, $X_1 := Y =: Z_0$ and $Y_1 := Z =: Z_1$. Since f and g are $\mathfrak{t}$-inflations, hence $\mathfrak{s}$-inflations, we know that gf is a $\mathfrak{s}$-inflation by (EA1) for $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$.

This shows that we can construct the solid morphisms of Diagram 3.1 such that the upper row and lower row, respectively, are $\mathfrak{t}$-distinguished n-exangles $\langle X_\bullet, \delta\rangle$ and $\langle Z_\bullet, \gamma\rangle$ and such that the middle row is an $\mathfrak{s}$-distinguished n-exangle $\langle Y_\bullet, \rho\rangle$. By Lemma 3.1 there is an object $Y' \in \mathcal{A}$ and morphisms $s\colon Y' \to Y_{n+1}$ and $t\colon Y_{n+1} \to Y'$ with $(st)^*\rho = \rho$. We have $s^*\rho \in \mathbb{E}(Y', Y_0)$. Since $\mathcal{A}$ is n-extension closed there is a $\mathfrak{t}$-distinguished n-exangle $\langle Y'_\bullet, s^*\rho\rangle$. We obtain the solid morphisms of a commutative diagram

$$\begin{tikzcd}
Y_0 \arrow[r, "gf"] \arrow[d, equal] & Y_1 \arrow[r] \arrow[d, dotted, "t_1"] & Y_2 \arrow[r] \arrow[d, dotted, "t_2"] & \cdots \arrow[r] & Y_n \arrow[r] \arrow[d, dotted, "t_n"] & Y_{n+1} \arrow[r, dashed, "\rho"] \arrow[d, "t"] & {} \\
Y'_0 \arrow[r, "d_0^{Y'}"] & Y'_1 \arrow[r] & Y'_2 \arrow[r] & \cdots \arrow[r] & Y'_n \arrow[r] & Y' \arrow[r, dashed, "s^*\rho"] & {}
\end{tikzcd}$$

where the upper row is the $\mathfrak{s}$-distinguished n-exangle $\langle Y_\bullet, \rho\rangle$ and the lower row is the $\mathfrak{t}$-distinguished n-exangle $\langle Y'_\bullet, s^*\rho\rangle$. Since $(st)^*\rho = \rho$, the morphism $(\mathrm{id}_{Y_0}, t)\colon \rho \to s^*\rho$ is a morphism of $\mathbb{E}$-extension and hence can be lifted to a morphism $t_\bullet\colon \langle Y_\bullet, \rho\rangle \to \langle Y'_\bullet, s^*\rho\rangle$ of n-exangles. This gives us $d_0^{Y'} = t_1 gf$. Since $d_0^{Y'} = t_1(gf)$ is a $\mathfrak{t}$-inflation and $Y_1 \in \mathcal{A}$, Proposition 2.4 shows that gf is a $\mathfrak{t}$-inflation. □

The following theorem proves [HZ21, Theorem 1.1] in a more general setting.

Theorem 3.3. *Suppose that $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an n-exangulated category with an n-extension closed additive subcategory $\mathcal{A} \subseteq \mathcal{C}$. Then $(\mathcal{A}, \mathbb{F}, \mathfrak{t})$ is an n-exangulated category and $(\mathscr{I}_\mathcal{A}, \Theta_\mathcal{A})\colon (\mathcal{A}, \mathbb{F}, \mathfrak{t}) \to (\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is a fully faithful n-exangulated functor.*

Proof. The first part follows from [HLN22, Proposition 4.2(2)], Lemma 3.2 and its dual. The second part is clear by definition of $(\mathscr{I}_\mathcal{A}, \Theta_\mathcal{A})$ and Remark 1.24. □

Remark 3.4. Notice that $(\mathcal{A}, \mathbb{F}, \mathfrak{t})$ is an n-exangulated subcategory of $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ in the sense of Haugland [Hau21, Definition 3.7].

4 n-Extension closed subcategories of n-exact categories

Throughout this section we also assume that $\mathcal{D}$ is an additive category. We recall the following definition.

Definition 4.1 ([Jas16, Defintion 2.4])**.** An object $X_\bullet \in \mathbf{C}^{n+2}_\mathcal{D}$ is called an *n-exact sequence* if for all $X \in \mathcal{D}$ the sequences

$$0 \longrightarrow \mathcal{D}(X, X_0) \xrightarrow{\mathcal{D}(X, d_0^X)} \mathcal{D}(X, X_1) \xrightarrow{\mathcal{D}(X, d_1^X)} \cdots \xrightarrow{\mathcal{D}(X, d_n^X)} \mathcal{D}(X, X_{n+1})$$

and

$$0 \longrightarrow \mathcal{D}(X_{n+1}, X) \xrightarrow{\mathcal{D}(d_n^X, X)} \mathcal{D}(X_n, X) \xrightarrow{\mathcal{D}(d_{n-1}^X, X)} \cdots \xrightarrow{\mathcal{D}(d_0^X, X)} \mathcal{D}(X_0, X)$$

are exact in Ab.

Notation 4.2 ([HLN21, Definition 4.12])**.** We denote by $\Lambda^{n+2}_{(\mathcal{D};A,C)}$ the class of all homotopy equivalence classes of n-exact sequences in $\mathbf{C}^{n+2}_{(\mathcal{D};A,C)}$.

We recall the construction of n-exangulated categories from n-exact categories defined in [HLN21, Section 4.3]. Suppose that $(\mathcal{D}, \mathcal{X})$ is an n-exact category in the sense of [Jas16, Definition 4.2]. For any pair $A, C \in \mathcal{D}$ we define a class

$$\mathbb{G}_{\mathcal{X}}(C, A) := \left\{ [X_\bullet]_{\mathcal{D}} \in \Lambda^{n+2}_{(\mathcal{D};A,C)} \,\middle|\, X_\bullet \in \mathcal{X} \right\} \tag{4.1}$$

as in [HLN21, Defintion 4.24]. This does not have to be a set in general.

Definition 4.3. We say that an n-exact category $(\mathcal{D}, \mathcal{X})$ has *small extension groups* if $\mathbb{G}_{\mathcal{X}}(C, A)$ as defined in (4.1) is a set for all $A, C \in \mathcal{D}$.

We recall the following construction from [HLN21, Definition 4.16]. For $A, B, C \in \mathcal{D}$, $[X_\bullet]_{\mathcal{D}} \in \mathbb{G}_{\mathcal{X}}(C, A)$ and $a \colon A \to B$ we define

$$\mathbb{G}_{\mathcal{X}}(C, a)([X_\bullet]_{\mathcal{D}}) := [Y_\bullet]_{\mathcal{D}} \tag{4.2}$$

by picking an $Y_\bullet \in \mathcal{X} \cap \mathbf{C}^{n+2}_{(\mathcal{D};B,C)}$ such that there is a morphism $f_\bullet \in \mathbf{C}^{n+2}_{\mathcal{D}}(X_\bullet, Y_\bullet)$ with $f_0 = a$ and $f_{n+1} = \mathrm{id}_C$ making the solid part of the diagram

$$\begin{tikzcd}
A \arrow[r, "d_0^X"] \arrow[d, "a"] & X_1 \arrow[r, "d_1^X"] \arrow[d, "f_1"] & \cdots \arrow[r, "d_{n-2}^X"] & X_{n-1} \arrow[r, "d_{n-1}^X"] \arrow[d, "f_{n-1}"] & X_n \arrow[r, dotted, "d_n^X"] \arrow[d, "f_n"] & C \arrow[d, dotted, equal] \\
B \arrow[r, "d_0^Y"] & Y_2 \arrow[r, "d_1^Y"] & \cdots \arrow[r, "d_{n-2}^Y"] & Y_{n-1} \arrow[r, "d_{n-1}^Y"] & X_n \arrow[r, dotted, "d_n^Y"] & C
\end{tikzcd} \tag{4.3}$$

an n-pushout diagram as defined in [Jas16, Definition 2.11]. Such a $Y_\bullet$ exists by using [Jas16, Definition 4.2] and [Jas16, Proposition 4.8] and the assignment (4.2) is well-defined by [HLN21, Proposition 4.18]. Dually, we can define $\mathbb{G}_{\mathcal{X}}(c, A)([X_\bullet]_{\mathcal{D}})$ for $A, C, D \in \mathcal{D}$, $[X_\bullet]_{\mathcal{D}} \in \mathbb{G}_{\mathcal{X}}(C, A)$ and $c \colon D \to C$.

If $(\mathcal{D}, \mathcal{X})$ has additionally small extension groups, a bifunctor $\mathbb{G}_{\mathcal{X}} \colon \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathsf{Ab}$ can be defined this way, see [HLN21, Definition 4.24, Lemma 4.26 and Proposition 4.32].

Recall that the additive structure on $\mathbb{G}_{\mathcal{X}}(C, A)$ for $C, A \in \mathcal{D}$ is defined through Baer sums as follows. For $[X_\bullet]_{\mathcal{D}}, [Y_\bullet]_{\mathcal{D}} \in \mathbb{G}_{\mathcal{X}}(C, A)$ we have $[X_\bullet \oplus Y_\bullet]_{\mathcal{D}} \in \mathbb{G}_{\mathcal{X}}(C \oplus C, A \oplus A)$ by [Jas16, Proposition 4.6] and we can define $[X_\bullet]_{\mathcal{D}} + [Y_\bullet]_{\mathcal{D}} := \mathbb{G}_{\mathcal{X}}(\Delta_C, \nabla_A)([X_\bullet \oplus Y_\bullet]_{\mathcal{D}})$, where $\Delta_C = \left[\, \mathrm{id}_C \ \mathrm{id}_C \,\right]^\top : C \to C \oplus C$ is the diagonal and $\nabla_A = \left[\, \mathrm{id}_A \ \mathrm{id}_A \,\right] : A \oplus A \to A$ is the codiagonal, see [HLN21, Definition 4.28].

Notation 4.4. For an n-exact category $(\mathcal{D}, \mathcal{X})$ with small extension groups we denote by $\mathbb{G}_{\mathcal{X}}$ the functor constructed above and by $\mathfrak{r}_{\mathcal{X}}$ the assigment $\mathfrak{r}_{\mathcal{X}}(\delta) = [X_\bullet]_{\mathcal{D}}$ for $A, C \in \mathcal{D}$ and $\delta = [X_\bullet]_{\mathcal{D}} \in \mathbb{G}_{\mathcal{X}}(C, A)$.

Proposition 4.5. *If $(\mathcal{D}, \mathcal{X})$ is an n-exact category with small extension groups then $(\mathcal{D}, \mathbb{G}_{\mathcal{X}}, \mathfrak{r}_{\mathcal{X}})$ is an n-exangulated category with monic $\mathfrak{r}_{\mathcal{X}}$-inflations and epic $\mathfrak{r}_{\mathcal{X}}$-deflations.*

Proof. The proof is given in [HLN21, Propsition 4.34] and [HLN21, Remark 4.35]. □

Definition 4.6. We say an n-exangulated category $(\mathcal{D}, \mathbb{G}, \mathfrak{r})$ is *n-exact* if there exists an n-exact structure $\mathcal{X} \subseteq \mathbf{C}^{n+2}_{\mathcal{D}}$ on $\mathcal{D}$ and an equivalence of n-exangulated categories $(\mathrm{Id}_{\mathcal{D}}, \Gamma) \colon (\mathcal{D}, \mathbb{G}, \mathfrak{r}) \to (\mathcal{D}, \mathbb{G}_{\mathcal{X}}, \mathfrak{r}_{\mathcal{X}})$.

Conversely, we can construct n-exact categories from n-exangulated categories using [HLN21, Propsition 4.37] and the strong Obscure Axiom.

Notation 4.7 ([HLN21, Lemma 4.36]). For an n-exangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, denote by $\mathcal{X}_{(\mathbb{E},\mathfrak{s})}$ the class of all $\mathfrak{s}$-conflations.

Proposition 4.8. *Suppose $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an n-exangulated category such that all $\mathfrak{s}$-inflations are monic and all $\mathfrak{s}$-deflations are epic, then $(\mathcal{C}, \mathcal{X}_{(\mathbb{E},\mathfrak{s})})$ is an n-exact category.*

Proof. For $n = 1$ this follows from Remark 1.17 and [NP19, Corollary 3.18]. For $n \in \mathbb{N}_{\geq 2}$ this follows from [HLN21, Proposition 4.37(2)] because the two conditions (a) and (b) of [HLN21, Proposition 4.37(2)] are satisfied by Corollary 2.5 and its dual. □

Showing that the construction of Proposition 4.5 and Proposition 4.8 are inverse to each other relies on the following Lemmas 4.9 and 4.10. We need some setup.

Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an n-exangulated category in which all $\mathfrak{s}$-inflations are monic and all $\mathfrak{s}$-deflations are epic. Then $(\mathcal{C}, \mathcal{X}_{(\mathbb{E},\mathfrak{s})})$ is an n-exact structure by Proposition 4.8. We obtain a class $\mathbb{G}_{\mathcal{X}_{(\mathbb{E},\mathfrak{s})}}(C, A)$ for $C, A \in \mathcal{C}$ through the assignment (4.1). We define a map

$$\Gamma_{(C,A)} \colon \mathbb{E}(C, A) \to \mathbb{G}_{\mathcal{X}_{(\mathbb{E},\mathfrak{s})}}(C, A),\ \delta \mapsto \mathfrak{s}(\delta)$$

for $C, A \in \mathcal{C}$, which is bijective by [HLN21, Lemma 4.36(3)]. In particular, $(\mathcal{C}, \mathcal{X}_{(\mathbb{E},\mathfrak{s})})$ has small extension groups by the Axiom of Replacement using that $\mathbb{E}(C, A)$ is a set. Hence, $(\mathcal{C}, \mathbb{G}_{\mathcal{X}_{(\mathbb{E},\mathfrak{s})}}, \mathfrak{r}_{\mathcal{X}_{(\mathbb{E},\mathfrak{s})}})$ is an n-exangulated category by Proposition 4.5.

Lemma 4.9. *Under the above assumptions the following hold.*

1. *For $A, B, C \in \mathcal{C}$, $a \colon A \to B$ and $\delta \in \mathbb{E}(C, A)$ we have*

$$\Gamma_{(C,B)}(\mathbb{E}(C, a)(\delta)) = \mathbb{G}_{\mathcal{X}_{(\mathbb{E},\mathfrak{s})}}(C, a)(\Gamma_{(C,A)}(\delta)).$$

2. *For $A, C, D \in \mathcal{C}$, $c \colon D \to C$ and $\delta \in \mathbb{E}(C, A)$ we have*

$$\Gamma_{(D,A)}(\mathbb{E}(c, A)(\delta)) = \mathbb{G}_{\mathcal{X}_{(\mathbb{E},\mathfrak{s})}}(c, A)(\Gamma_{(C,A)}(\delta)).$$

3. *For $A, C \in \mathcal{C}$ and $\delta, \rho \in \mathbb{E}(C, A)$ we have*

$$\Gamma_{(C,A)}(\delta + \rho) = \Gamma_{(C,A)}(\delta) + \Gamma_{(C,A)}(\rho).$$

Hence, $\Gamma \colon \mathbb{E} \Rightarrow \mathbb{G}_{\mathcal{X}_{(\mathbb{E},\mathfrak{s})}}$ is a natural isomorphism and

$$(\mathrm{Id}_{\mathcal{C}}, \Gamma) \colon (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\mathcal{C}, \mathbb{G}_{\mathcal{X}_{(\mathbb{E},\mathfrak{s})}}, \mathfrak{r}_{\mathcal{X}_{(\mathbb{E},\mathfrak{s})}})$$

is an n-exangulated equivalence.

Proof. (1): Let $\rho := \mathbb{E}(C, a)(\delta)$ and $\langle X_\bullet, \delta\rangle, \langle Y_\bullet, \rho\rangle$ be $\mathfrak{s}$-distinguished n-exangles. Notice that $(a, \mathrm{id}_C) \colon \delta \to \rho$ is a morphism of $\mathbb{E}$-extensions. There exists a lift $f_\bullet \colon X_\bullet \to Y_\bullet$ with $f_0 = a$ and $f_{n+1} = \mathrm{id}_C$. By the dual of [HLN21, Lemma 4.36(1)], this means that the solid part of (4.3) is an n-pushout diagram. We have $X_\bullet \in \mathcal{X}_{(\mathbb{E},\mathfrak{s})} \cap \mathbf{C}^{n+2}_{(\mathcal{C};A,C)}$ and $Y_\bullet \in \mathcal{X}_{(\mathbb{E},\mathfrak{s})} \cap \mathbf{C}^{n+2}_{(\mathcal{C};B,C)}$. Hence,

$$\Gamma_{(C,B)}(\mathbb{E}(C,a)(\delta)) = \mathfrak{s}(\rho) = [Y_\bullet]_{\mathcal{C}} = \mathbb{G}_{\mathcal{X}_{(\mathbb{E},\mathfrak{s})}}(C,a)([X_\bullet]_{\mathcal{C}}) = \mathbb{G}_{\mathcal{X}_{(\mathbb{E},\mathfrak{s})}}(C,a)(\Gamma_{C,A}(\delta))$$

by definition of $\mathbb{G}_{\mathcal{X}_{(\mathbb{E},\mathfrak{s})}}(C,a)$. In the same way (2) can be shown.

(3): Let $\delta, \rho \in \mathbb{E}(C,A)$ for $A, C \in \mathcal{C}$. Then we have $\delta + \rho = \mathbb{E}(\Delta_C, \nabla_A)(\delta \oplus \rho)$, where $\Delta_C := \left[\,\mathrm{id}_C \ \mathrm{id}_C\,\right]^\top : C \to C \oplus C$ and $\nabla_A = \left[\,\mathrm{id}_A \ \mathrm{id}_A\,\right] : A \oplus A \to A$ as mentioned in [HLN21, Definition 2.6]. We have

$$\Gamma_{(A,C)}(\delta + \rho) = \Gamma_{(C,A)}(\mathbb{E}(\Delta_C, \nabla_A)(\delta \oplus \rho)) = \mathbb{G}_{\mathcal{X}_{(\mathbb{E},\mathfrak{s})}}(\Delta_C, \nabla_A)(\Gamma_{(C\oplus C, A\oplus A)}(\delta \oplus \rho))$$

by using (1) and (2). Let $X_\bullet$ be an $\mathfrak{s}$-realisation of δ and $Y_\bullet$ be an $\mathfrak{s}$-realisation of ρ. Then $[X_\bullet \oplus Y_\bullet]_{\mathcal{C}} = \mathfrak{s}(\delta \oplus \rho)$, by [HLN21, Proposition 3.3]. Hence,

$$\begin{aligned} \mathbb{G}_{\mathcal{X}_{(\mathbb{E},\mathfrak{s})}}(\Delta_C, \nabla_A)(\Gamma_{(C\oplus C, A\oplus A)}(\delta \oplus \rho)) &= \mathbb{G}_{\mathcal{X}_{\mathbb{E},\mathfrak{s}}}(\Delta_C, \nabla_A)([X_\bullet \oplus Y_\bullet]_{\mathcal{C}}) \\ &= [X_\bullet]_{\mathcal{C}} + [Y_\bullet]_{\mathcal{C}} \\ &= \Gamma_{(C,A)}(\delta) + \Gamma_{(C,A)}(\rho) \end{aligned}$$

as addition in $\mathbb{G}_{\mathcal{X}_{(\mathbb{E},\mathfrak{s})}}(C,A)$ is defined through Baer sums. Therefore, (3) holds.

Finally, $\Gamma\colon \mathbb{E} \Rightarrow \mathbb{G}_{\mathcal{X}_{(\mathbb{E},\mathfrak{s})}}$ is a natural isomorphism of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Ab}$ by (1), (2) and (3). It is clear that $\mathfrak{s}(\delta) = \mathfrak{r}_{\mathcal{X}_{(\mathbb{E},\mathfrak{s})}}(\Gamma_{(C,A)}(\delta))$ for $A, C \in \mathcal{C}$ and $\delta \in \mathbb{E}(C,A)$, by definition. Hence, $(\mathrm{Id}_{\mathcal{C}}, \Gamma)\colon (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\mathcal{C}, \mathbb{G}_{\mathcal{X}_{(\mathbb{E},\mathfrak{s})}}, \mathfrak{r}_{\mathcal{X}_{(\mathbb{E},\mathfrak{s})}})$ is an n-exangulated equivalence. □

Suppose that $(\mathcal{D}, \mathcal{X})$ is an n-exact category with small extension groups. The n-exangulated category $(\mathcal{D}, \mathbb{G}_{\mathcal{X}}, \mathfrak{r}_{\mathcal{X}})$, as defined in Proposition 4.5, has monic $\mathfrak{r}_{\mathcal{X}}$-inflations and epic $\mathfrak{r}_{\mathcal{X}}$-deflations. Therefore, an n-exact category $(\mathcal{D}, \mathcal{X}_{(\mathbb{G}_{\mathcal{X}}, \mathfrak{r}_{\mathcal{X}})})$ can be defined using Proposition 4.8.

Lemma 4.10. *Under the above assumptions we have $\mathcal{X} = \mathcal{X}_{(\mathbb{G}_{\mathcal{X}}, \mathfrak{r}_{\mathcal{X}})}$.*

Proof. Let $X_\bullet \in \mathcal{X}$ and $\delta := [X_\bullet]_{\mathcal{D}} \in \mathbb{G}_{\mathcal{X}}(X_{n+1}, X_0)$. Then $\mathfrak{r}_{\mathcal{X}}(\delta) = [X_\bullet]_{\mathcal{D}}$, by definition. Hence, $X_\bullet$ is an $\mathfrak{r}_{\mathcal{X}}$-conflation. This means $X_\bullet \in \mathcal{X}_{(\mathbb{G}_{\mathcal{X}}, \mathfrak{r}_{\mathcal{X}})}$, again by definition.

Conversely, let $X_\bullet \in \mathcal{X}_{(\mathbb{G}_{\mathcal{X}}, \mathfrak{r}_{\mathcal{X}})}$. Then $X_\bullet$ is an $\mathfrak{r}_{\mathcal{X}}$-conflation. By definition of $\mathbb{G}_{\mathcal{X}}$ and $\mathfrak{r}_{\mathcal{X}}$ this means there is a $Y_\bullet \in \mathcal{X}$ such that $[Y_\bullet]_{\mathcal{D}} = [X_\bullet]_{\mathcal{D}}$. This implies the existence of an equivalence $X_\bullet \to Y_\bullet$ of n-exact sequences in the sense of [Jas16, Definition 2.9]. By [Jas16, Definition 4.2] the class $\mathcal{X}$ is closed under weak isomorphisms and hence $X_\bullet \in \mathcal{X}$. □

We can now summarize [HLN21, Section 4.3] and show that the two constructions given are inverse to each other.

Theorem 4.11. *Proposition 4.5 and 4.8 induce a one-to-one correspondence*

$$\left\{\begin{matrix} n\text{-exact structures } (\mathcal{D}, \mathcal{X}) \text{ with} \\ \text{small extension groups} \end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \frac{\left\{\begin{matrix} n\text{-exangulated structures } (\mathcal{D}, \mathbb{G}, \mathfrak{r}) \text{ with} \\ \text{monic } \mathfrak{r}\text{-inflations and epic } \mathfrak{r}\text{-deflations} \end{matrix}\right\}}{\left\{\begin{matrix} \text{equivalences of } n\text{-exangulated categories} \\ \text{of the form } (\mathrm{Id}_{\mathcal{D}}, \Gamma) \end{matrix}\right\}}$$

$$(\mathcal{D}, \mathcal{X}) \longmapsto (\mathcal{D}, \mathbb{G}_{\mathcal{X}}, \mathfrak{r}_{\mathcal{X}})$$

$$(\mathcal{D}, \mathcal{X}_{(\mathbb{G},\mathfrak{r})}) \longleftarrow (\mathcal{D}, \mathbb{G}, \mathfrak{r}).$$

Proof. The map from left to right is well-defined by Proposition 4.5.

For any n-exangulated equivalence $(\mathrm{Id}_{\mathcal{D}}, \Gamma)\colon (\mathcal{D}, \mathbb{G}, \mathfrak{r}) \to (\mathcal{D}, \mathbb{G}', \mathfrak{r}')$ we have a natural isomorphism $\Gamma\colon \mathbb{G} \Rightarrow \mathbb{G}'$. Moreover, any $\mathfrak{r}$-realisation of any $\mathbb{G}$-extension is an $\mathfrak{r}'$-realisation of its image under Γ as $(\mathrm{Id}_{\mathcal{D}}, \Gamma)$ is an n-exangulated functor. Hence, the classes of conflations $\mathcal{X}_{(\mathbb{G},\mathfrak{r})}$ and $\mathcal{X}_{(\mathbb{G}',\mathfrak{r}')}$ coincides and the map from right to left is well-defined.

The Theorem follows now from Lemma 4.9 and Lemma 4.10. □

Corollary 4.12. *For any n-exangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ the following are equivalent.*

1. *$(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is n-exact.*
2. *Every $\mathfrak{s}$-inflation is monic and every $\mathfrak{s}$-deflation is epic.*

Definition 4.13. An additive subcategory $\mathcal{B} \subseteq \mathcal{D}$ of an n-exact category $(\mathcal{D}, \mathcal{X})$ is called *n-extension closed* if for all $X_\bullet \in \mathcal{X}$ with $X_0, X_{n+1} \in \mathcal{B}$ there exists a $Y_\bullet \in \mathbf{C}^{n+2}_{\mathcal{B}} \cap \mathcal{X}$ with $[X_\bullet]_{\mathcal{D}} = [Y_\bullet]_{\mathcal{D}}$.

The two notions of n-extension closed given in Definitions 1.18 and 4.13 coincide.

Lemma 4.14. *Suppose $(\mathcal{D}, \mathcal{X})$ is an n-exact category with an additive subcategory $\mathcal{B} \subseteq \mathcal{D}$. Then $\mathcal{B}$ is n-extension closed in $(\mathcal{D}, \mathcal{X})$ if and only if it is n-extension closed in $(\mathcal{D}, \mathbb{G}_{\mathcal{X}}, \mathfrak{r}_{\mathcal{X}})$.*

Proof. We show only that if $\mathcal{B}$ is n-extension closed in $(\mathcal{D}, \mathcal{X})$ in the sense of Definition 4.13 then $\mathcal{B}$ is n-extension closed in $(\mathcal{D}, \mathbb{G}_{\mathcal{X}}, \mathfrak{r}_{\mathcal{X}})$ in the sense of Definition 1.18, the reverse statement follows similarly. Let $\delta \in \mathbb{G}_{\mathcal{X}}(C, A)$ for $C, A \in \mathcal{B}$. We have $\mathcal{X} = \mathcal{X}_{(\mathbb{G}_{\mathcal{X}},\mathfrak{r}_{\mathcal{X}})}$ by Theorem 4.11 and hence $X_\bullet \in \mathcal{X}$ for any $X_\bullet \in \mathbf{C}^{n+2}_{\mathcal{D}}$ with $[X_\bullet] = \mathfrak{r}_{\mathcal{X}}(\delta)$. By Definition 4.13 we can pick $Y_\bullet \in \mathbf{C}^{n+2}_{\mathcal{B}}$ with $[Y_\bullet] = [X_\bullet] = \mathfrak{s}(\delta)$. □

We have the following corollary which is a higher analogue of [Büh10, Lemma 10.20].

Corollary 4.15. *Suppose that $(\mathcal{D}, \mathcal{X})$ is an n-exact category with small extension groups and $\mathcal{B} \subseteq \mathcal{D}$ is an n-extension closed additive subcategory. Then $(\mathcal{B}, \mathcal{X}_{\mathcal{B}})$ is an n-exact category with small extension groups, where $\mathcal{X}_{\mathcal{B}} := \mathcal{X} \cap \mathbf{C}^{n+2}_{\mathcal{B}}$.*

Proof. By Lemma 4.14 we know that $\mathcal{B}$ is n-extension closed in $(\mathcal{D}, \mathbb{G}_{\mathcal{X}}, \mathfrak{r}_{\mathcal{X}})$. Theorem 3.3 $(\mathcal{D}, \mathbb{G}_{\mathcal{X}}, \mathfrak{r}_{\mathcal{X}})$ induces an n-exangulated structure $(\mathcal{B}, \mathbb{F}_{\mathcal{B}}, \mathfrak{t}_{\mathcal{B}})$ on $\mathcal{B}$. By Theorem 4.11, any $\mathfrak{r}_{\mathcal{X}}$-inflation is monic in $\mathcal{D}$. The $\mathfrak{t}_{\mathcal{B}}$-conflations are precisely the $\mathfrak{r}_{\mathcal{X}}$-conflations with terms in $\mathcal{B}$, see Remark 1.24. Therefore, any $\mathfrak{t}_{\mathcal{B}}$-inflation is monic in $\mathcal{D}$ and hence in $\mathcal{B} \subseteq \mathcal{D}$. Dually, any $\mathfrak{t}_{\mathcal{B}}$-deflation is epic in $\mathcal{B}$. Hence, $(\mathcal{B}, \mathbb{F}_{\mathcal{B}}, \mathfrak{t}_{\mathcal{B}})$ is n-exact, by Corollary 4.12.

It follows from Theorem 4.11 that $(\mathcal{B}, \mathbb{F}_{\mathcal{B}}, \mathfrak{t}_{\mathcal{B}})$ induces an n-exact structure $\mathcal{X}_{(\mathbb{F}_{\mathcal{B}},\mathfrak{t}_{\mathcal{B}})}$ with small extension groups on $\mathcal{B}$. Remark 1.24 and $\mathcal{X} = \mathcal{X}_{(\mathbb{G}_{\mathcal{X}},\mathfrak{r}_{\mathcal{X}})}$ imply $\mathcal{X}_{\mathcal{B}} = \mathcal{X}_{(\mathbb{F}_{\mathcal{B}},\mathfrak{t}_{\mathcal{B}})}$. □

5 n-Extension closed subcategories of $(n+2)$-angulated categories

Throughout this section let $\mathcal{D}$ be an additive category, $\Sigma\colon \mathcal{D} \to \mathcal{D}$ be an additive automorphism of $\mathcal{D}$ and $\mathbb{G}_\Sigma(-,-) := \mathcal{D}(-, \Sigma-)$ be the induced biadditive bifunctor, see [HLN21, Section 4.2]. We recall the following constructions from [HLN21, Section 4.2].

Suppose $(\mathcal{D}, \Sigma, \pentagon)$ is an $(n+2)$-angulated category in the sense of [GKO13]. Define a realisation $\mathfrak{r}_{\pentagon}$ of $\mathbb{G}_\Sigma$ as follows. For $C, A \in \mathcal{D}$ and $\delta \in \mathbb{G}_\Sigma(C, A)$ pick an $(n+2)$-angle

$$\widehat{X}_\bullet\colon \qquad A \xrightarrow{d_0^X} X_1 \xrightarrow{d_1^X} \cdots \xrightarrow{d_{n-1}^X} X_n \xrightarrow{d_n^X} C \xrightarrow{\delta} \Sigma A \tag{5.1}$$

in $\pentagon$. Let $X_\bullet \in \mathbf{C}_{\mathcal{D}}^{n+2}$ be the truncated complex

$$X_\bullet\colon \qquad A \xrightarrow{d_0^X} X_1 \xrightarrow{d_1^X} \cdots \xrightarrow{d_{n-1}^X} X_n \xrightarrow{d_n^X} C \tag{5.2}$$

and define $\mathfrak{r}_{\pentagon}(\delta) := [X_\bullet]_{\mathcal{D}}$. This is independent of the $(n+2)$-angle chosen in (5.1), by [HLN21, Lemma 4.4]. Then $(\mathcal{D}, \mathbb{G}_\Sigma, \mathfrak{r}_{\pentagon})$ is n-exangulated, see [HLN21, Proposition 4.5]

Conversely, let $(\mathcal{D}, \mathbb{G}_\Sigma, \mathfrak{r})$ be n-exangulated. Let $\pentagon_{\mathfrak{r}}$ be the class of all complexes $\widehat{X}_\bullet$ as in (5.1) such that $\langle X_\bullet, \delta\rangle$ is $\mathfrak{r}$-distinguished, where $X_\bullet$ is the corresponding complex in (5.2), then $(\mathcal{D}, \Sigma, \pentagon_{\mathfrak{r}})$ is $(n+2)$-angulated, see [HLN21, Proposition 4.8].

Indeed this gives us a bijective correspondence.

Theorem 5.1 ([HLN21, Section 4.2])**.** *There is a one-to-one correspondence*

$$\{(n+2)\text{-}\textit{angulated structures } (\mathcal{D}, \Sigma, \pentagon)\} \overset{1:1}{\longleftrightarrow} \{n\text{-}\textit{exangulated structures } (\mathcal{D}, \mathbb{G}_\Sigma, \mathfrak{r})\}$$

$$(\mathcal{D}, \Sigma, \pentagon) \longmapsto (\mathcal{D}, \mathbb{G}_\Sigma, \mathfrak{r}_{\pentagon})$$

$$(\mathcal{D}, \Sigma, \pentagon_{\mathfrak{r}}) \longleftarrow\!\shortmid (\mathcal{D}, \mathbb{G}_\Sigma, \mathfrak{r}).$$

Proof. By [HLN21, Proposition 4.5], every $(n+2)$-angulated structure $(\mathcal{D}, \Sigma, \pentagon)$ yields an n-exangulated structure $(\mathcal{D}, \mathbb{G}_\Sigma, \mathfrak{r}_{\pentagon})$. Conversely, by [HLN21, Propsition 4.8], every n-exangulated structure $(\mathcal{D}, \mathbb{G}_\Sigma, \mathfrak{r})$ yields an $(n+2)$-angulated structure $(\mathcal{D}, \Sigma, \pentagon_{\mathfrak{r}})$. We only need to show $\pentagon = \pentagon_{\mathfrak{r}_{\pentagon}}$ for any $(n+2)$-angulated structure $(\mathcal{D}, \Sigma, \pentagon_{\mathfrak{r}})$ and $\mathfrak{r} = \mathfrak{r}_{\pentagon_{\mathfrak{r}}}$ for any n-exangulated structure $(\mathcal{D}, \mathbb{G}_\Sigma, \mathfrak{r})$.

Let $(\mathcal{D}, \mathbb{G}_\Sigma, \mathfrak{r})$ be n-exangulated, $A, C \in \mathcal{D}$, $\delta \in \mathbb{G}_\Sigma(C, A)$ and $\langle X_\bullet, \delta\rangle$ be an $\mathfrak{r}$-distinguished n-exangle. Then $X_\bullet$ is of the shape of (5.2) and hence $\widehat{X}_\bullet$ as in (5.1) is in $\pentagon_{\mathfrak{r}}$. Using the independence [HLN21, Lemma 4.4] provides, we have that $\langle X_\bullet, \delta\rangle$ is $\mathfrak{r}_{\pentagon_{\mathfrak{r}}}$-distinguished. Therefore, $\mathfrak{r} = \mathfrak{r}_{\pentagon_{\mathfrak{r}}}$.

For the rest of this proof denote for any complex $\widehat{X}_\bullet$ as in (5.1) the corresponding complex as in (5.2) by $X_\bullet$.

Let $(\mathcal{D}, \Sigma, \pentagon_{\mathfrak{r}})$ be $(n+2)$-angulated. We show the two inclusions of $\pentagon = \pentagon_{\mathfrak{r}_{\pentagon}}$ separately. Let $\widehat{X}_\bullet \in \pentagon$ be as in (5.1). Then $[X_\bullet]_{\mathcal{D}} = \mathfrak{r}_{\pentagon}(\delta)$, using [HLN21, Lemma 4.4]. Hence, $\widehat{X}_\bullet \in \pentagon_{\mathfrak{r}_{\pentagon}}$. Conversely, let $\widehat{X}_\bullet \in \pentagon_{\mathfrak{r}_{\pentagon}}$ be as in (5.1). Then $\langle X_\bullet, \delta\rangle$ is $\mathfrak{r}_{\pentagon}$-distinguished, by definition. This means that there is a $\widehat{Y}_\bullet \in \pentagon$ of the shape

$$\widehat{Y}_\bullet\colon \qquad A \xrightarrow{d_0^Y} Y_1 \xrightarrow{d_1^Y} \cdots \xrightarrow{d_{n-1}^Y} Y_n \xrightarrow{d_n^Y} C \xrightarrow{\delta} \Sigma A$$

such that $[Y_\bullet]_{\mathcal{D}} = [X_\bullet]_{\mathcal{D}}$. Hence, there is a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "d_0^Y"] \arrow[d, equal] & Y_1 \arrow[r, "d_1^Y"] \arrow[d, dotted] & \cdots \arrow[r, "d_{n-1}^Y"] & Y_n \arrow[r, "d_n^Y"] \arrow[d, dotted] & C \arrow[r, "\delta"] \arrow[d, equal] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "d_0^X"] & X_1 \arrow[r, "d_1^X"] & \cdots \arrow[r, "d_{n-1}^X"] & X_n \arrow[r, "d_n^X"] & C \arrow[r, "\delta"] & \Sigma A
\end{tikzcd}$$

where the dotted morphisms are obtained through the homotopy equivalence $[X_\bullet]_{\mathcal{D}} = [Y_\bullet]_{\mathcal{D}}$. By [GKO13, Lemma 2.4] we have $\widehat{X}_\bullet \in \pentagon$. This shows $\pentagon = \pentagon_{\mathfrak{r}_\pentagon}$ □

We recall the following definition.

Definition 5.2. An additive subcategory $\mathcal{B} \subseteq \mathcal{D}$ of an $(n+2)$-angulated category $(\mathcal{D}, \Sigma, \pentagon)$ is called *n-extension closed* if for all $A, C \in \mathcal{B}$ and all $\delta \in \mathcal{D}(C, \Sigma A)$ there is an $(n+2)$-angle $\widehat{X}_\bullet$ as in (5.1) with $X_1, \dots, X_n \in \mathcal{B}$.

Remark 5.3. Suppose $(\mathcal{D}, \Sigma, \pentagon)$ is an $(n+2)$-angulated category with an additive subcategory $\mathcal{B} \subseteq \mathcal{D}$. Then $\mathcal{B}$ is n-extension closed in $(\mathcal{D}, \Sigma, \pentagon)$ in the sense of Definition 5.2 if and only if it is n-extension closed in $(\mathcal{D}, \mathbb{G}_\Sigma, \mathfrak{r}_\pentagon)$ in the sense of Definition 1.18.

Suppose $(\mathcal{D}, \Sigma, \pentagon)$ is an $(n+2)$-angulated category and $\mathcal{B} \subseteq \mathcal{D}$ is n-extension closed. For each $A, C \in \mathcal{B}$ and $\delta \in \mathcal{D}(C, \Sigma A)$ pick an $(n+2)$-angle $\widehat{X}_\bullet$ as in (5.1) with $X_1, \dots, X_n \in \mathcal{B}$ and define $\mathfrak{r}_\mathcal{B}(\delta) = [X_\bullet]_\mathcal{B}$, where $X_\bullet$ is the corresponding complex from (5.2). This is well-defined using that for $A, C \in \mathcal{B}$ and $X_\bullet, Y_\bullet \in \mathcal{C}^{n+2}_{(\mathcal{B};A,C)}$ the equality $[X_\bullet]_\mathcal{D} = [Y_\bullet]_\mathcal{D}$ implies $[X_\bullet]_\mathcal{B} = [Y_\bullet]_\mathcal{B}$ and using that $[X_\bullet]_\mathcal{D}$ is independent of the choice of $\widehat{X}_\bullet \in \pentagon$ completing $\delta \colon C \to \Sigma A$ by [HLN21, Lemma 4.4]. The following corollary proves [Zho22, Theorem 1.2] in a more general setting.

Corollary 5.4. *Suppose that $(\mathcal{D}, \Sigma, \pentagon)$ is an $(n+2)$-angulated category and $\mathcal{B} \subseteq \mathcal{D}$ is an n-extension closed additive subcategory. Then $(\mathcal{B}, \mathbb{G}_\mathcal{B}, \mathfrak{r}_\mathcal{B})$ is an n-exangulated category, where $\mathbb{G}_\mathcal{B} = \mathbb{G}_\Sigma|_{\mathcal{B}^{\mathrm{op}} \times \mathcal{B}}$ and $\mathfrak{r}_\mathcal{B}$ is as defined above.*

Proof. By Theorem 5.1 there is an n-exangulated structure $(\mathcal{D}, \mathbb{G}_\Sigma, \mathfrak{r})$ on $\mathcal{D}$ with $\pentagon = \pentagon_\mathfrak{r}$. By Remark 5.3 we know that $\mathcal{B} \subseteq \mathcal{D}$ is n-extension closed in $(\mathcal{D}, \mathbb{G}_\Sigma, \mathfrak{r})$. By Theorem 3.3 there is an n-exangulated structure $(\mathcal{B}, \mathbb{F}_\mathcal{B}, \mathfrak{t}_\mathcal{B})$ on $\mathcal{B}$, where $\mathbb{F}_\mathcal{B} = \mathbb{G}_\Sigma|_{\mathcal{B}^{\mathrm{op}} \times \mathcal{B}}$. It is clear that $\mathfrak{t}_\mathcal{B}$ and $\mathfrak{r}_\mathcal{B}$ coincide. □

Suppose $(\mathcal{D}, \Sigma, \pentagon)$ is an $(n+2)$-angulated category and $\mathcal{B} \subseteq \mathcal{D}$ is an n-extension closed additive subcategory with $\mathcal{D}(\Sigma\mathcal{B}, \mathcal{B}) = 0$. Let $\mathcal{X}_\mathcal{B}$ be the class of all sequences $X_\bullet$ as in (5.2) with $A, X_1, \dots, X_n, C \in \mathcal{B}$ such that there exists a corresponding $(n+2)$-angle $\widehat{X}_\bullet$ as in (5.1). The following corollary proves [Kla21, Theorem I] in a more general setting.

Corollary 5.5. *Suppose that $(\mathcal{D}, \Sigma, \pentagon)$ is an $(n+2)$-angulated category and $\mathcal{B} \subseteq \mathcal{D}$ is an n-extension closed additive subcategory with $\mathcal{D}(\Sigma\mathcal{B}, \mathcal{B}) = 0$. Then $(\mathcal{B}, \mathcal{X}_\mathcal{B})$ is an n-exact category with small extension groups, where $\mathcal{X}_\mathcal{B}$ is as defined above.*

Proof. Let $(\mathcal{D}, \mathbb{G}_\Sigma, \mathfrak{r}_\pentagon)$ be the n-exangulated structure induced on $\mathcal{D}$ via Theorem 5.1 and $(\mathcal{B}, \mathbb{G}_\mathcal{B}, \mathfrak{r}_\mathcal{B})$ be the n-exangulated structure induced on $\mathcal{B}$ via Corollary 5.4 or equivalently Theorem 3.3.

The class of $\mathfrak{r}_\mathcal{B}$-conflations is the class of $\mathfrak{r}_\pentagon$-conflations with terms in $\mathcal{B}$, by Remark 1.24. By Theorem 5.1, the class of $\mathfrak{r}_\pentagon$-conflations is the class of all sequences $X_\bullet$ as in (5.2) such that there exists a corresponding $(n+2)$-angle $\widehat{X}_\bullet$ as in (5.1), which is in $\pentagon$. We conclude that the class of $\mathfrak{r}_\mathcal{B}$-conflations is $\mathcal{X}_\mathcal{B}$.

We show that all $\mathfrak{r}_\mathcal{B}$-inflations are monic in $\mathcal{B}$. Indeed let $A, X_1 \in \mathcal{B}$ and $d_0^X \colon A \to X_1$ be an $\mathfrak{r}_\mathcal{B}$-inflation. Then there is an $(n+2)$-angle as in (5.1) with $X_2, \dots, X_n, C \in \mathcal{B}$.

Applying the functor $\mathcal{D}(-,B)$ for $B \in \mathcal{B}$ yields an exact sequence

$$\mathcal{D}(B,\Sigma^{-1}C) \xrightarrow{\mathcal{D}(B,\Sigma^{-1}\delta)} \mathcal{D}(B,A) \xrightarrow{\mathcal{D}(B,d_0^X)} \mathcal{D}(B,X_1)$$

by [GKO13, Propositon 2.5]. Using that $\mathcal{B} \subseteq \mathcal{D}$ is full and $\mathcal{D}(B,\Sigma^{-1}C) \cong \mathcal{D}(\Sigma B,C) = 0$ because $\mathcal{D}(\Sigma\mathcal{B},\mathcal{B}) = 0$, we conclude that d_0^X is monic in $\mathcal{B}$. Similarly, one can show that $\mathfrak{r}_{\mathcal{B}}$-deflations are epic.

By Theorem 4.11 we conclude that $(\mathcal{B},\mathcal{X}_{\mathcal{B}})$ is n-exact with small extension groups. $\square$

A Calculations

Calculation A.1. Diagram 2.1 is a biproduct diagram in $Æ^{n+2}_{(\mathcal{C},\mathbb{E})}$.

Proof. We first prove identities, which will be used later in the proof. We have

$$\mathrm{id}_A = p'g, \qquad e = gp', \qquad e' = \mathrm{id}_{X_2} - gp' = \mathrm{id}_{X_2} - e, \qquad p'h = 0$$

by definition. Notice that e and e' are idempotents. The above identities imply $eg = g$ and $p'e = p'$ as well as $eh = 0$ which imply

$$e'g = 0, \qquad p'e' = 0, \qquad e'h = h.$$

Finally, $d_2^{X''}d_1^{X''} = 0$ since X'' is a complex and hence

$$d_2^X g = 0, \qquad d_2^X h = 0, \qquad d_2^X e' = d_2^X(\mathrm{id}_{X_2} - gp') = d_2^X.$$

Now, it is clear that all columns of Diagram 2.1 except the third one are biproduct diagrams in $\mathcal{A}$. Concerning the third column, we have

$$\begin{bmatrix} p' & 0 \end{bmatrix}\begin{bmatrix} g \\ 0 \end{bmatrix} = p'g = \mathrm{id}_A \text{ and } \begin{bmatrix} e' & g \end{bmatrix}\begin{bmatrix} e' \\ p' \end{bmatrix} = e' + gp' = e' + e = \mathrm{id}_{X_2}$$

as well as

$$\begin{bmatrix} e' \\ p' \end{bmatrix}\begin{bmatrix} e' & g \end{bmatrix} + \begin{bmatrix} g \\ 0 \end{bmatrix}\begin{bmatrix} p' & 0 \end{bmatrix} = \begin{bmatrix} (e')^2 & e'g \\ p'e' & p'g \end{bmatrix} + \begin{bmatrix} gp' & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e'+e & 0 \\ 0 & \mathrm{id}_A \end{bmatrix} = \mathrm{id}_{X_2 \oplus A}.$$

Hence, all columns of Diagram 2.1 are biproduct diagrams in $\mathcal{A}$.

To conclude that Diagram 2.1 is a biproduct in $Æ^{n+2}_{(\mathcal{C},\mathbb{E})}$ we only need to show that that all squares commute. The two upper left squares commute since

$$\begin{bmatrix} 0 \\ \mathrm{id}_{X_1'} \end{bmatrix} f = \begin{bmatrix} 0 \\ f \end{bmatrix} = d_0^{X''} \text{ and } \begin{bmatrix} 0 & \mathrm{id}_{X_1'} \end{bmatrix} d_0^{X''} = \begin{bmatrix} 0 & \mathrm{id}_{X_1'} \end{bmatrix}\begin{bmatrix} 0 \\ f \end{bmatrix} = f.$$

The two lower left squares commute since

$$\begin{bmatrix} \mathrm{id}_A & 0 \end{bmatrix} d_0^{X''} = \begin{bmatrix} \mathrm{id}_A & 0 \end{bmatrix}\begin{bmatrix} 0 \\ f \end{bmatrix} = 0$$

and any morphism starting in the zero object is 0. The two, second to left, upper squares commute since

$$\begin{bmatrix} e' \\ p' \end{bmatrix} h = \begin{bmatrix} e'h \\ p'h \end{bmatrix} = \begin{bmatrix} h \\ 0 \end{bmatrix} = \begin{bmatrix} g & h \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ \mathrm{id}_{X_1'} \end{bmatrix} = d_1^{X''}\begin{bmatrix} 0 \\ \mathrm{id}_{X_1'} \end{bmatrix}$$

and

$$\begin{bmatrix} e' & g \end{bmatrix} d_1^{X''} = \begin{bmatrix} e' & g \end{bmatrix} \begin{bmatrix} g & h \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e'g & e'h \end{bmatrix} = \begin{bmatrix} 0 & h \end{bmatrix} = h \begin{bmatrix} 0 & \mathrm{id}_{X_1'} \end{bmatrix}.$$

The two, second to left, lower squares commute since

$$\begin{bmatrix} p' & 0 \end{bmatrix} d_1^{X''} = \begin{bmatrix} p' & 0 \end{bmatrix} \begin{bmatrix} g & h \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} p'g & p'h \end{bmatrix} = \begin{bmatrix} \mathrm{id}_A & 0 \end{bmatrix} \text{ and } \begin{bmatrix} g \\ 0 \end{bmatrix} = \begin{bmatrix} g & h \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{id}_A \\ 0 \end{bmatrix} = d_1^{X''} \begin{bmatrix} \mathrm{id}_A \\ 0 \end{bmatrix}.$$

The two, third to left, upper squares commute, since

$$\begin{bmatrix} d_2^X e' \\ p' \end{bmatrix} = \begin{bmatrix} d_2^X & 0 \\ 0 & \mathrm{id}_A \end{bmatrix} \begin{bmatrix} e' \\ p' \end{bmatrix} = d_2^{X''} \begin{bmatrix} e' \\ p' \end{bmatrix} \text{ and } d_2^{X''} = \begin{bmatrix} d_2^X & 0 \\ 0 & \mathrm{id}_A \end{bmatrix} = \begin{bmatrix} d_2^X (e')^2 & d_2^X e' g \\ p'e' & p'g \end{bmatrix} = \begin{bmatrix} d_2^X e' \\ p' \end{bmatrix} \begin{bmatrix} e' & g \end{bmatrix}.$$

Finally, the two, third to left, lower squares commute because any morphism ending in the zero object is 0 and

$$0 = \begin{bmatrix} d_2^X g & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} d_2^X & 0 \\ 0 & \mathrm{id}_A \end{bmatrix} \begin{bmatrix} g \\ 0 \end{bmatrix} = d_2^{X''} \begin{bmatrix} g \\ 0 \end{bmatrix}.$$

It is clear that the remaining squares commute. It follows immediately that Diagram 2.1 is a biproduct diagram in $\text{Æ}^{n+2}_{(\mathcal{C},\mathbb{E})}$. □

Calculation A.2. Diagram 2.2 is a biproduct diagram in $\text{Æ}^{3}_{(\mathcal{C},\mathbb{E})}$.

Proof. By construction, all columns of Diagram 2.2 are biproduct diagrams in $\mathcal{A}$, all squares except for two upper right squares commute and $(j')^*\delta = \delta'$. We only need to show that the two upper right squares commute, that the upper row of Diagram 2.2 is an $\mathbb{E}$-attached complex, and that $(d_1^X i)^*\delta = 0$ and $(q')^*\delta' = \delta$ hold. As the columns of Diagram 2.2 are biproduct diagrams the identities

$$qj = \mathrm{id}_{X_1'}, \quad ip + jq = \mathrm{id}_{A\oplus X_1'}, \quad e = d_1^X ip', \quad q'j' = \mathrm{id}_{X_2'}, \quad e' = \mathrm{id}_{X_2} - e = j'q'$$

hold. We have $ed_1^X = (d_1^X ip')d_1^X = d_1^X(ip)$ as the two lower left squares commute. Hence,

$$j'(q'd_1^X j) = e'd_1^X j = (\mathrm{id}_{X_2} - e)d_1^X j = d_1^X(\mathrm{id}_{A\oplus X_1'} - ip)j = d_1^X jqj = d_1^X j$$

as well as

$$q'd_1^X = q'j'q'd_1^X = q'e'd_1^X = q'(\mathrm{id}_{X_2} - e)d_1^X = q'd_1^X(\mathrm{id}_{A\oplus X_1'} - ip) = (q'd_1^X j)q$$

show that the two upper right squares commute. We have $(d_1^X)^*\delta = 0$ as $\langle X_\bullet, \delta\rangle$ is a 1-exangle. Hence, $(q'd_1^X j)^*\delta' = (j'q'd_1^X j)^*\delta = (d_1^X j)^*\delta = 0$ and $(d_1^X i)^*\delta = 0$. In particular, the upper row of Diagram 2.2 is an $\mathbb{E}$-attached complex. Finally, $(d_1^X)^*\delta = 0$ implies

$$(q')^*\delta' = (j'q')^*\delta = (\mathrm{id}_{X_2} - e)^*\delta = \delta - (d_1^X ip')^*\delta = \delta$$

which completes the proof. □

References

[BTHSS22] Raphael Bennett-Tennenhaus, Johanne Haugland, Mads Hustad Sandøy and Amit Shah, *The category of extensions and a characterisation of n-exangulated functors*, Preprint, 2022. https://arxiv.org/abs/2205.03097.

[BTS21] Raphael Bennett-Tennenhaus and Amit Shah, *Transport of structure in higher homological algebra*, Journal of Algebra, **574**:514-549, 2021.

[Büh10] Theo Bühler, *Exact categories*, Expositiones Mathematicae, **28**(1):1-69, 2010.

[ES22] Haruhisa Enomoto and Shunya Saito, *Grothendieck monoids of extriangulated categories* Preprint, 2022. https://arxiv.org/abs/2208.02928.

[GKO13] Christof Geiss, Bernhard Keller and Steffen Oppermann, *n-angulated categories*, Journal für die reine und angewandte Mathematik (Crelles Journal), **675**:101-120, 2013.

[Hau21] Johanne Haugland, *The Grothendieck Group of an n-exangulated Category*, Applied Categorical Structures **29**(3):431–446, 2021.

[HLN21] Martin Herschend, Yu Liu and Hiroyuki Nakaoka, *n-exangulated categories (I): Definitions and fundamental properties*, Journal of Algebra, **570**:531-586, 2021.

[HLN22] Martin Herschend, Yu Liu and Hiroyuki Nakaoka, *n-Exangulated categories (II): Constructions from n-cluster tilting subcategories*, Journal of Algebra, **594**:636-684, 2022.

[HZ21] Jian He, Panyue Zhou, *n-exact categories arising from n-exangulated categories*, Preprint, 2021. https://arxiv.org/abs/2109.12954.

[Jas16] Gustavo Jasso, *n-Abelian and n-exact categories*, Mathematische Zeitschrift, **283**:703–759(3-4), 2016.

[Kla21] Carlo Klapproth, *n-Exact categories arising from $(n+2)$-angulated categories*, Preprint, 2021. https://arxiv.org/abs/2108.04596.

[Kra15] Henning Krause, *Krull–Schmidt categories and projective covers*, Expositiones Mathematicae, **33**(4):535-549, 2015.

[KMS22] Carlo Klapproth, Dixy Msapato and Amit Shah, *Idempotent completions of n-exangulated categories*, Preprint, 2022. https://arxiv.org/abs/2207.04023.

[Msa22] Dixy Msapato, *The Karoubi envelope and weak idempotent completion of an extriangulated category*, Applied Categorical Structures, **30**(3):499–535, 2022.

[NP19] Hiroyuki Nakaoka and Yann Palu, *Extriangulated categories, Hovey twin cotorsion pairs and model structures*, Cahiers de Topologie et Géométrie Différentielle Catégoriques, **60**(3):117–193, 2019.

[Qui73] Daniel Quillen, *Higher algebraic K-theory I*, in Algebraic K-theory, I: Higher K-theories, Lecture Notes in Mathematics, **341**:85-147, Springer, Berlin-New York 1973.

[Tat22] Aran Tattar, *Torsion structures, subobjects and unique filtrations in non-abelian categories*, PhD thesis, Universität zu Köln, 2022. http://kups.ub.uni-koeln.de/id/eprint/61013.

[WW22] Yucheng Wang and Jiaqun Wei, *Ideal approximation in n-exangulated categories*, Preprint, 2022. https://arxiv.org/abs/2207.13465.

[Zho22] Panyue Zhou, *n-extension closed subcategories of $(n+2)$-angulated categories*, Archiv der Mathematik, **118**:375–382, 2022.

Paper D

When does the Auslander–Reiten translation operate linearly on the Grothendieck group? – Part I

Carlo Klapproth

Abstract

For a hereditary, finite-dimensional algebra A the Coxeter transformation extends the action of the Auslander–Reiten translation on the non-projective indecomposable modules to a linear endomorphism of the Grothendieck group of the category of finitely generated A-modules. It is natural to ask whether other algebras admit a similar linear extension. We show that this is indeed the case for all Nakayama algebras. Conversely, we show that finite-dimensional algebras with non-acyclic and connected quiver admitting such a linear extension are already cyclic Nakayama algebras.

1 Introduction

Let A be a basic, finite-dimensional $\mathbb{k}$-algebra over a field $\mathbb{k}$. If $\mathbb{k}$ is algebraically closed and A has finite global dimension, then the bounded derived category $\mathsf{D}^b(\operatorname{mod} A)$ of A has Auslander–Reiten triangles by [Hap88, Theorem 4.6], and hence a Serre functor $\mathbb{S}$ by [RVdB02, Theorem I.2.4]. It is well-known that $\tau := \mathbb{S}[-1]$ is then a triangulated functor which induces the *Coxeter transformation* Φ, that is a linear endomorphism of the Grothendieck group $\mathsf{K}_0(\mathsf{D}^b(\operatorname{mod} A)) \cong \mathsf{K}_0(\operatorname{mod} A)$, see e.g. [Hap88, Section III.1.2], with $\Phi[X] = [\tau X]$ for $X \in \mathsf{D}^b(\operatorname{mod} A)$.

The situation is more difficult for the category $\operatorname{mod} A$ of finitely generated A-modules. The Auslander–Reiten translation τ_A on $\operatorname{mod} A$ does not need to be a functor. However, inspired by the situation for hereditary algebras one can define the following.

Definition 1.1. An endomorphism $\Phi \in \operatorname{End}_{\mathbb{Z}}(\mathsf{K}_0(\operatorname{mod} A))$ with $\Phi[M] = [\tau_A M]$ for all non-projective indecomposable A-modules M is called *τ-map for A*. Similarly, an endomorphism $\Phi' \in \operatorname{End}_{\mathbb{Z}}(\mathsf{K}_0(\operatorname{mod} A))$ with $\Phi'[M] = [\tau_A^{-1} M]$ for all non-injective indecomposable A-modules M is called *τ^{-1}-map for A*.

It is natural to ask which algebras do admit a τ-map. Recall the following.

Definition 1.2. A finite-dimensional $\mathbb{k}$-algebra A is called *Nakayama algebra* if the indecomposable projective and indecomposable injective A-modules are uniserial. A Nakayama algebra is called *cyclic*, if it has no projective and no injective simple modules.

We have the following examples of algebras admitting a τ-map.

1. If Q is an acyclic quiver and $\Bbbk Q$ is its (hereditary) quiver algebra then $\Bbbk Q$ has a τ-map given by the Coxeter transformation, see [ARS95, Proposition VIII.2.2(b)].

2. For a Nakayama algebra A there is a unique $\Phi \in \mathrm{End}_{\mathbb{Z}}(\mathsf{K}_0(\mathsf{mod}\, A))$ with

$$\Phi[S] = \begin{cases} [\tau_A S] & \text{for } S \in \mathsf{mod}\, A \text{ simple and non-projective,} \\ x_S & \text{for } S \in \mathsf{mod}\, A \text{ simple and projective,} \end{cases}$$

   where x_S is an arbitrary, fixed element in $\mathsf{K}_0(\mathsf{mod}\, A)$ for each simple projective A-module S. This map Φ is a τ-map by Proposition 4.7.

Notice, a τ-map can look very different from the Coxeter transformation. For example the algebra $A := \Bbbk Q/I$, where Q is the quiver

$$\overset{1}{\bullet} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \overset{2}{\bullet}$$

and $I := \langle \alpha\beta \rangle$, is a Nakayama algebra without simple projective modules and hence has a unique τ-map Φ, by Proposition 4.7. This map transposes the two simple A-modules. On the other hand, the Cartan matrix C of A is symmetric and hence the Coxeter matrix $C^t C^{-1}$ of A is the identity matrix. So, the Coxeter transformation of A is the identity map on $\mathsf{K}_0(\mathsf{mod}\, A)$.

We suspect that the class of algebras which have a τ-map admits a classification. Because of very different behaviour with regards to the existence of a τ-map, we want to treat algebras with an acyclic Ext^1-quiver (see Definition 2.2) separately from those with a non-acyclic Ext^1-quiver. This paper is mainly concerned with algebras which admit a τ-map and have a non-acyclic Ext^1-quiver. We show the following.

Theorem A (Theorem 4.1). *Suppose the* Ext^1*-quiver of A is connected and non-acyclic. Then A has a τ-map if and only if A is a cyclic Nakayama algebra.*

In the language of quotients of path algebras Theorem A can be reformulated as follows.

Corollary B (Corollary 4.2). *Let $\Bbbk$ be an algebraically closed field. Suppose $A := \Bbbk Q/I$, where Q is a connected and non-acyclic quiver and $I \lhd \Bbbk Q$ is an admissible ideal. Then A has a τ-map if and only if A is a cyclic Nakayama algebra.*

This shows that for an algebra with a non-acyclic Ext^1-quiver the existence of a τ-map restricts the shape of the underlying quiver significantly. This is not the case for algebras with an acyclic Ext^1-quiver as indeed all hereditary algebras admit a τ-map.

2 Preliminaries and Notation

For morphisms $f\colon X \to X'$ and $g\colon X' \to X''$ we write gf for the composite $g \circ f\colon X \to X''$. Similarly, for functors $F\colon \mathcal{C} \to \mathcal{C}'$ and $G\colon \mathcal{C}' \to \mathcal{C}''$ we write GF for the composite functor $G \circ F\colon \mathcal{C} \to \mathcal{C}''$. For a category $\mathcal{C}$ we denote by $\mathsf{ind}\mathcal{C}$ the full subcategory of indecomposable objects in $\mathcal{C}$. We fix a field $\Bbbk$, which is not necessarily algebraically closed.

For arbitrary finite-dimensional $\Bbbk$-algebras A and B we refer to right A-modules as A-modules and to right $A^{\mathrm{op}} \otimes_{\Bbbk} B$-modules as A-B-bimodules. We denote by $\operatorname{\mathsf{mod}} A$ the category of finitely generated A-modules. We write $\operatorname{\mathsf{proj}} A$ and $\operatorname{\mathsf{inj}} A$ for the full subcategories of $\operatorname{\mathsf{mod}} A$ consisting of projective and injective A-modules, respectively. For $P \in \operatorname{\mathsf{proj}} A$ we denote by $\operatorname{\mathsf{simp}} P$ the isomorphism classes of simple quotients of P. Abusing notation, we view the elements of $\operatorname{\mathsf{simp}} P$ as simple A-modules by picking a representative. We denote the $\Bbbk$-duality $\operatorname{Hom}_{\Bbbk}(-, \Bbbk)$ by D and write $\nu_A(-) := \mathsf{D} \operatorname{Hom}_A(-, A) \colon \operatorname{\mathsf{mod}} A \to \operatorname{\mathsf{mod}} A$ for the Nakayama functor, $\nu_A^{-1}(-) := \operatorname{Hom}_{A^{\mathrm{op}}}(\mathsf{D}(-), A) \colon \operatorname{\mathsf{mod}} A \to \operatorname{\mathsf{mod}} A$ for the inverse Nakayama functor as well as τ_A and τ_A^{-1} for the Auslander–Reiten translations on $\operatorname{\mathsf{mod}} A$. We want to point out that the restriction $\nu_A \colon \operatorname{\mathsf{proj}} A \to \operatorname{\mathsf{inj}} A$ is inverse to the restriction $\nu_A^{-1} \colon \operatorname{\mathsf{inj}} A \to \operatorname{\mathsf{proj}} A$. However, the unrestricted functors ν_A and ν_A^{-1} are generally not mutually inverse equivalences on $\operatorname{\mathsf{mod}} A$. We refer the reader to [ARS95, Chapter IV] for a more detailed exposition.

For a simple module $S \in \operatorname{\mathsf{mod}} A$ and an arbitrary module $M \in \operatorname{\mathsf{mod}} A$ we denote by $[M : S]$ how often S appears as a composition factor in any composition series of M. By the Jordan–Hölder theorem this is independent of the chosen composition series. Recall that $[M : S] \neq 0$ if and only if $\operatorname{Hom}_A(P, M) \neq 0$ where P is a projective cover of S, see for example [ARS95, Exercise II.1(a)]. Compare also [ASS06, Corollary III.3.6] for the case where $\Bbbk$ is algebraically closed. The following result is folklore, see e.g. [Iwa79, Lemma 1].

Lemma 2.1. *Let* $\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M$ *be an augmented minimal projective resolution of some A-module M. Then we have* $\operatorname{Ext}_A^i(M, S) \cong \operatorname{Hom}_A(P_i, S)$ *for all* $i \in \mathbb{N}$ *and all simple A-modules S.*

Proof. By the definition of (augmented) minimal projective resolutions the morphisms $\varepsilon \colon P_0 \to M$ and $P_i \to \operatorname{Im} d_i$ for $i \geq 1$ are projective covers and hence the submodules $\operatorname{Ker} \varepsilon \subseteq P_0$ and $\operatorname{Ker} d_i \subseteq P_i$ are superfluous. By exactness, the submodules $\operatorname{Im} d_i \subseteq P_{i-1}$ for $i \geq 1$ are superfluous and hence $\operatorname{Im} d_i \subseteq \operatorname{rad} P_{i-1}$ for $i \geq 1$. On the other hand, for any $N \in \operatorname{\mathsf{mod}} A$ the submodule $\operatorname{rad} N \subseteq N$ can be characterized as the intersection of all kernels of morphisms from N to simple A-modules. Therefore, $\operatorname{Hom}_A(d_i, S)$ vanishes for any given simple module S and all $i \geq 1$. This means that applying $\operatorname{Hom}_A(-, S)$ to the unaugmented complex $\cdots \to P_1 \to P_0 =: P_\bullet$ yields a complex $\operatorname{Hom}_A(P_\bullet, S)$ with zero differentials and as a result of this we obtain $\operatorname{Ext}_A^i(M, S) \cong \mathrm{H}^i \operatorname{Hom}_A(P_\bullet, S) \cong \operatorname{Hom}_A(P_i, S)$ for all $i \in \mathbb{N}$. $\square$

Throughout this paper we assume that A is a basic, finite-dimensional $\Bbbk$-algebra and that $e \in A$ is an idempotent. We write $\Gamma_e := (1-e)A(1-e)$ for the idempotent subalgebra obtained from A by deleting e. By projectivisation, cf. [ARS95, Proposition II.2.1], the functor $\operatorname{Hom}_A((1-e)A, -) \colon \operatorname{\mathsf{add}}(1-e)A \to \operatorname{\mathsf{proj}} \Gamma_e$ is an equivalence and maps the basic object $(1-e)A$ to Γ_e, showing that Γ_e is also a basic finite-dimensional $\Bbbk$-algebra. Recall that the simple Γ_e-modules are in correspondence with the modules in $\operatorname{\mathsf{simp}}(1-e)A$, see Lemma 2.4(7). We define the two functors

$$\begin{aligned} F_e(-) &:= - \otimes_A A(1-e) \colon \operatorname{\mathsf{mod}} A \to \operatorname{\mathsf{mod}} \Gamma_e \text{ and} \\ G_e(-) &:= - \otimes_{\Gamma_e} (1-e)A \colon \operatorname{\mathsf{mod}} \Gamma_e \to \operatorname{\mathsf{mod}} A. \end{aligned}$$

Notice that $A(1-e)$ is a projective A^{op}-module. Hence, $- \otimes_A A(1-e) \cong \operatorname{Hom}_A((1-e)A, -)$, see Lemma 2.4(4). This means that F_e is right adjoint to G_e by Tensor-Hom adjunction.

If $\Bbbk$ is algebraically closed then it is well-known that we have an isomorphism $A \cong \Bbbk Q/I$ where Q is a quiver and $I \subseteq \Bbbk Q$ is an admissible ideal. However, as we do normally not assume $\Bbbk$ to be algebraically closed, we define the following as a replacement.

Definition 2.2. The Ext^1-*quiver* $Q = (Q_0, Q_1)$ of A is the quiver with

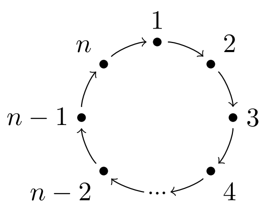
1. vertices $Q_0 = \mathsf{simp}\, A$ and
2. arrows $Q_1 = \{ S \to S' \mid S, S' \in Q_0 \text{ and } \operatorname{Ext}^1_A(S, S') \neq 0 \}$.

Notice, the Ext^1-quiver of A can have 1-cycles, that is loops. Furthermore, it has *precisely one* arrow from S to S' if S' has non-trivial extensions by S and *no arrow* from S to S' otherwise. For an admissible quotient of a path algebra the Ext^1-quiver relates to the underlying quiver of the path algebra as follows.

Remark 2.3. Suppose $\Bbbk$ is algebraically closed. Let Q be a quiver and $I \lhd \Bbbk Q$ be an admissible ideal. Then by [ASS06, Proposition III.2.12(a)] the Ext^1-quiver of $\Bbbk Q/I$ is obtained from Q by replacing all vertices with their corresponding simple $\Bbbk Q/I$-modules and by identifying parallel arrows. In particular, acyclicity and connectedness hold for Q if and only if they hold for the Ext^1-quiver of $\Bbbk Q/I$.

Recall that a vertex of a quiver is called *source vertex* if there are no arrows ending in it. Dually a vertex is called *sink vertex* if there is no arrow starting in it. Notice, this does not imply that there is any arrow starting in a source vertex or ending in a sink vertex. Indeed, a vertex is *isolated* if and only if it is both a source and a sink vertex.

For $n = 1$ we define C_n to be a single vertex with a loop attached to it. For $n \geq 2$ we denote the oriented cycle on n vertices



by C_n. For quivers Q' and Q'' we denote the disjoint union of Q' and Q'' by $Q' \sqcup Q''$. If $Q = Q' \sqcup Q''$ and Q' is connected, then we call Q' a *component* of Q.

For a basic algebra A and a decomposition $1 = e_1 + \cdots + e_n$ into primitive idempotents, the A-modules $e_i A / \operatorname{rad} e_i A$ for $1 \leq i \leq n$ form a full system of representatives of $\mathsf{simp}\, A$.

For convenience, we recall some well-known facts about F_e and G_e. These functors have been studied for example in [Aus74, Section 5 and 6], [Gre07, Section 6.2] and [Psa14].

Lemma 2.4. *The following statements hold.*

1. *There is an equivalence* $\mathsf{add}(1-e)A \leftrightarrow \mathsf{proj}\,\Gamma_e$ *induced by* F_e *and* G_e.
2. *The functor* F_e *is exact. If* $(1-e)Ae \in \mathsf{proj}\,\Gamma_e^{\mathrm{op}}$ *then* G_e *is exact.*
3. *The functor* F_e *is left inverse to* G_e *that is* $F_e G_e \cong \mathrm{Id}_{\mathsf{mod}\,\Gamma_e}$.
4. *We have* $F_e(-) = \operatorname{Hom}_A((1-e)A, -)$.

5. *The functor* F_e *is right adjoint to* G_e.

6. *The functor* G_e *preserves augmented (minimal) projective presentations. Hence,* G_e *preserves the properties of being non-projective and being indecomposable.*

Let $1-e = e_1 + \cdots + e_n$ *be a decomposition into primitive idempotents in* $\Gamma_e \subseteq A$.

7. *We have* $F_e(e_iA/\operatorname{rad} e_iA) \cong e_i\Gamma_e/\operatorname{rad} e_i\Gamma_e$ *for* $1 \leq i \leq n$. *Hence,* F_e *induces a bijection between* $\mathsf{simp}(1-e)A$ *and* $\mathsf{simp}\Gamma_e$.

8. *If* $(1-e)Ae = 0$ *then* $G_e(e_i\Gamma_e/\operatorname{rad} e_i\Gamma_e) \cong e_iA/\operatorname{rad} e_iA$ *for* $1 \leq i \leq n$. *Hence,* G_e *induces a bijection between* $\mathsf{simp}\Gamma_e$ *and* $\mathsf{simp}(1-e)A$.

9. *If* $(1-e)Ae = 0$ *then* $\operatorname{Ext}^1_{\Gamma_e}(S,S') \cong \operatorname{Ext}^1_A(G_e(S), G_e(S'))$ *for* $S, S' \in \mathsf{simp}\Gamma_e$.

Proof. Item (1) follows from projectivisation, cf. [ARS95, Proposition II.2.1]. Item (2) is a consequence of $A(1-e) \in \mathsf{proj}\, A^{\mathrm{op}}$ and $(1-e)A \cong \Gamma_e \oplus (1-e)Ae$ as Γ_e-modules, implying that $(1-e)A \in \mathsf{proj}\,\Gamma_e$ if and only if $(1-e)Ae \in \mathsf{proj}\,\Gamma_e$. Item (3) follows, because $(1-e)A \otimes_A A(1-e) \cong \Gamma_e$ holds as Γ_e-Γ_e-bimodules. Item (4) follows from

$$F_e(-) = - \otimes_A A(1-e) \cong - \otimes_A \operatorname{Hom}_A((1-e)A, A) \cong \operatorname{Hom}_A((1-e)A, -)$$

because $A(1-e) \in \mathsf{proj}\, A^{\mathrm{op}}$, see [ARS95, Proposition II.4.4] for the last equation. Then (5) follows from (4) using Tensor-Hom adjunction.

To show (6) notice first that $G_e(-) = - \otimes_{\Gamma_e} (1-e)A$ is right exact. Hence, G_e maps augmented projective presentations to augmented projective presentations using (1).

We show that G_e preserves minimality. A projective presentation $f\colon P_1 \to P_0$ is minimal if and only if the two term complex $f\colon P_1 \to P_0$ has no non-trivial direct summand of the form $f'\colon P_1' \to P_0'$ with f' being a split epimorphism, cf. [Aus74, Corollary 4.10]. Since equivalences preserve direct summands and split epimorphisms, it follows from (1) that G_e preserves minimal projective resolutions.

A module with a minimal projective presentation $f\colon P_1 \to P_0$ is non-projective if and only if P_1 is non-zero, and indecomposable if and only if $f\colon P_1 \to P_0$ is indecomposable as a 2-term complex. Both properties are preserved by G_e by (1) and the first part of (6).

We show (7). Let $P \to e_iA \to e_iA/\operatorname{rad} e_iA \to 0$ be an augmented minimal projective presentation. Since F_e is exact there is an exact sequence

$$F_e(P) \to F_e(e_iA) \to F_e(e_iA/\operatorname{rad} e_iA) \to 0 \tag{2.1}$$

in $\mathsf{mod}\,\Gamma_e$ and $F_e(e_iA) \cong e_i\Gamma_e$ is an indecomposable projective Γ_e-module. Furthermore, $F_e(e_iA/\operatorname{rad} e_iA)$ is simple by [Aus74, Corollary 6.3d)] and therefore the sequence (2.1) implies $F_e(e_iA/\operatorname{rad} e_iA) \cong \operatorname{top} e_i\Gamma_e = e_i\Gamma_e/\operatorname{rad} e_i\Gamma_e$.

We show (8). Assume $(1-e)Ae = 0$ and consider again the exact sequence arising as in (2.1). We have $\operatorname{Hom}_A(eA, e_iA) \subseteq \operatorname{Hom}_A(eA, (1-e)A) \cong (1-e)Ae = 0$, that is every map from an object in $\mathsf{add}\, eA$ to e_iA is trivial. Then $P \in \mathsf{add}(1-e)A$, by minimality of the chosen projective resolution of $e_iA/\operatorname{rad} e_iA$. Hence, $F_e(P)$ is projective and (2.1) is an augmented projective presentation of $F_e(e_iA/\operatorname{rad} e_iA) \cong e_i\Gamma_e/\operatorname{rad} e_i\Gamma_e$. Applying the functor G_e to (2.1) shows that $G_eF_e(P) \to G_eF_e(e_iA)$ is a projective presentation of $G_e(e_i\Gamma_e/\operatorname{rad} e_i\Gamma_e)$, by (6). However, as e_iA and P are in $\mathsf{add}(1-e)A$ this projective presentation is equivalent to $P \to e_iA$ by (1) and hence $G_e(e_i\Gamma_e/\operatorname{rad} e_i\Gamma_e) \cong e_iA/\operatorname{rad} e_iA$.

We show (9). Suppose $P_1 \to P_0 \to S \to 0$ is an augmented minimal projective presentation of the simple Γ_e-module S. Then $G_e(P_1) \to G_e(P_0) \to G_e(S) \to 0$ is an augmented minimal projective presentation of $G_e(S)$ by (6). We know that $G_e(S')$ is a simple A-module as S' is a simple Γ_e-module, using (8). Applying Lemma 2.1 we obtain $\operatorname{Ext}^1_{\Gamma_e}(S,S') \cong \operatorname{Hom}_{\Gamma_e}(P_1,S')$ and $\operatorname{Ext}^1_A(G_e(S),G_e(S')) \cong \operatorname{Hom}_A(G_e(P_1),G_e(S'))$. By (3) and (5) we have $\operatorname{Hom}_A(G_e(P_1),G_e(S')) \cong \operatorname{Hom}_{\Gamma_e}(P_1,S')$. Finally, combining all of the above equations we obtain $\operatorname{Ext}^1_{\Gamma_e}(S,S') \cong \operatorname{Ext}^1_A(G_e(S),G_e(S'))$. □

Next, we want to recall how the Ext^1-quiver of A and Γ_e are related if $(1-e)Ae = 0$.

Lemma 2.5. *Let Q_A be the Ext^1-quiver of A and Q_{Γ_e} be the Ext^1-quiver of Γ_e. Suppose that $(1-e)Ae = 0$. Then the morphism of quivers $Q_{\Gamma_e} \to Q_A$ mapping*

1. *a vertex S to the vertex $G_e(S)$ and*
2. *an arrow $S \to S'$ to the arrow $G_e(S) \to G_e(S')$*

is the inclusion of the full subquiver of Q_A having the elements of $\mathsf{simp}(1-e)A$ as vertices.

Proof. By Lemma 2.4(8) the functor G_e induces a bijection between the vertices of Q_{Γ_e} and the vertices of Q_A which are in $\mathsf{simp}(1-e)A$. By Definition 2.2 and Lemma 2.4(9) there is an arrow $S \to S'$ in Q_{Γ_e} if and only if there is an arrow $G_e(S) \to G_e(S')$ in Q_A. □

Lemma 2.6. *We have $\operatorname{Ext}^1_A(\operatorname{top} M, S) \neq 0$ for any A-module M and any non-trivial direct summand S of $\operatorname{top}(\operatorname{rad} M)$.*

Proof. Consider the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{rad} M \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[r] \arrow[d, dashed, "g'"'] & \operatorname{top} M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & S \arrow[r, dashed, "f'"] \arrow[ur, phantom, "\lrcorner", very near end] & M' \arrow[r, dashed] & \operatorname{top} M \arrow[r] & 0 \end{tikzcd}$$

where g is a projection onto the direct summand S and the lower row is obtained by a pushout from the upper row. We have $\operatorname{Im} f' = \operatorname{Im} f'g = \operatorname{Im} g'f = g'(\operatorname{rad} M) \subseteq \operatorname{rad} M'$ using that g is epic. If the lower row was a split sequence then M' would be semisimple and hence $\operatorname{rad} M' = 0$ implying $\operatorname{Im} f' = 0$. However, $\operatorname{Im} f' \cong S$ is non-zero, so the lower sequence defines a non-trivial element in $\operatorname{Ext}^1_A(\operatorname{top} M, S)$. □

The Ext^1-quiver contains information about the composition series of projective modules.

Lemma 2.7. *Let $S, S' \in \mathsf{simp}\,A$ and $P' \to S'$ be a projective cover. If there is $k \in \mathbb{N}$ so that $S \in \mathsf{add}(\operatorname{top}(\operatorname{rad}^k P'))$ then there is an oriented path of length k from S' to S in the Ext^1-quiver Q of A. So, all composition factors S of P' admit a path from S' to S in Q.*

Proof. If S is a composition factor of P' then S is a direct summand of $\operatorname{top}(\operatorname{rad}^k P')$ for some $k \geq 0$, so the second part of the lemma follows from the first part.

We use induction on $k \in \mathbb{N}$ to prove the first part of the lemma. If $k = 0$ then we have $S \in \mathsf{add}(\operatorname{top} P')$ and so $S \cong S'$. Hence, the constant path at S' gives the desired path of length 0. For the induction step suppose S is a simple summand of $\operatorname{top}(\operatorname{rad}^k P')$ for some $k \geq 1$. Then $\operatorname{Ext}^1_A(\operatorname{top}(\operatorname{rad}^{k-1} P'), S) \neq 0$, by Lemma 2.6. Therefore, there is a simple summand S'' of $\operatorname{top}(\operatorname{rad}^{k-1} P_i)$ with $\operatorname{Ext}^1_A(S'', S) \neq 0$. Hence, there is an arrow from S'' to S in Q and by induction hypothesis there is an oriented path of length $k-1$ from S' to S'' in Q. Concatenation gives a path of length k from S' to S in Q. $\square$

In particular, connectedness of Q corresponds to indecomposability of A as an algebra, and projective and injective simple modules correspond to sink and source vertices of Q.

Remark 2.8. A component C of the Ext^1-quiver Q of A induces a central idempotent e and an idempotent subalgebra $\Gamma_C := \Gamma_{1-e}$. Indeed, let eA be the projective cover of the direct sum of all vertices in C. Any vertex $S \in C$ does not appear in any composition series of $(1-e)A$, by Lemma 2.7. Therefore, $\operatorname{Hom}_A(eA, (1-e)A) = 0$ and similarly one can show, $\operatorname{Hom}_A((1-e)A, eA) = 0$. Hence, $A \cong \operatorname{End}_A((1-e)A \oplus eA) \cong \Gamma_e \times \Gamma_{1-e}$ as $\Bbbk$-algebras, showing that $e \in A$ is a central.

Remark 2.9. Suppose S is a sink vertex of Q. Let $P \to S$ be a projective cover. Lemma 2.7 implies $\operatorname{rad} P = 0$ as it has no composition factors. Hence, $S = P$ is projective. On the other hand, a non-sink vertex in Q has non-trivial extensions and is hence non-projective. We see that the projective simple A-modules are precisely the sink vertices of Q. Using $\Bbbk$-duality we see that source vertices in Q are precisely the injective simple A-modules.

Using Remark 2.8 we can easily extend the following result to algebras A where only some of the simple A-modules have τ_A-orbits consisting of simple modules only.

Theorem 2.10 ([ARS95, Theorem IV.2.10]). *Suppose all modules in the τ_A-orbits of simple A-modules are simple themselves. Then A is a Nakayama algebra.*

Corollary 2.11. *Suppose $\mathsf{simp}(1-e)A$ is a union of τ_A-orbits. Then we have a decomposition $A = \Gamma_e \times \Gamma_{1-e}$ where Γ_e is a Nakayama algebra.*

Proof. Let $S \in \mathsf{simp}(1-e)A$ and $\operatorname{Ext}^1_A(S, S') \neq 0$ for some simple module $S' \in \mathsf{mod}\, A$. There is a non-split short exact sequence $0 \to S' \to M' \to S \to 0$. In particular, $S \notin \mathsf{proj}\, A$ and we have an Auslander–Reiten sequence $0 \to \tau_A S \to M \to S \to 0$. As $M \to S$ is a right almost split morphism we obtain the dashed morphism of the diagram

$$\begin{tikzcd} 0 \arrow[r] & S' \arrow[r] & M' \arrow[r] \arrow[d, dashed] & S \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \tau_A S \arrow[r] & M \arrow[r] & S \arrow[r] & 0 \end{tikzcd}$$

using that $M' \to S$ is not a split epimorphism and that M' is indecomposable. But because M' and M must be uniserial of length 2 the dashed morphism must be an isomorphism and therefore $S' \cong \tau_A S$ holds. Hence, $\operatorname{Ext}^1_A(S, S') \neq 0$ can hold only if $S' \in \mathsf{simp}(1-e)A$. In the same way we can show that $\operatorname{Ext}^1_A(S'', S) \neq 0$ for $S \in \mathsf{simp}(1-e)A$ and $S'' \in \mathsf{simp}\, A$ implies $S'' \in \mathsf{simp}(1-e)A$. But the Ext^1-quiver of A is then disconnected and $A = \Gamma_e \times \Gamma_{1-e}$ by Remark 2.8 for a central idempotent e. Using for example Proposition 3.2 one can show that $\tau_{\Gamma_e} = F_e \tau_A G_e$. This shows that all Γ_e-modules in the τ_{Γ_e}-orbits of simple Γ_e-modules are simple using Lemma 2.4(7) and (8). Now, Theorem 2.10 applied to Γ_e shows that Γ_e is a Nakayama algebra. $\square$

Next, recall the following well-known extension of [ARS95, Proposition II.4.4(a)].

Lemma 2.12. *The family of morphisms*

$$\begin{aligned}\psi_{M,N}\colon N\otimes_A \operatorname{Hom}_A(M,A) &\to \operatorname{Hom}_A(M,N)\\ n\otimes f &\mapsto (m\mapsto nf(m))\end{aligned}$$

for $M\in \mathsf{mod}\,A$ *and* $N\in\mathsf{mod}\,A$, *defines a natural transformation*

$$\psi\colon -_2\otimes_A \operatorname{Hom}_A(-_1,A)\to \operatorname{Hom}_A(-_1,-_2)$$

of functors $\mathsf{mod}\,A\times\mathsf{mod}\,A\to\mathsf{mod}\,\Bbbk$. *Furthermore,*

1. *if* ${}_BN_A$ *is an* B-A-*bimodule then* $\psi_{M,N}$ *is also a* B^{op}-*module morphism from* ${}_BN\otimes_A \operatorname{Hom}_A(M,A)$ *to* $\operatorname{Hom}_A(M,{}_BN)$ *for any* $M\in\mathsf{mod}\,A$ *and*

2. *if* $N_A\in\mathsf{proj}\,A$ *then* $\psi_{M,N}$ *is an isomorphism for all* $M\in\mathsf{mod}\,A$.

In particular, if the conditions of (1) and (2) are satisfied then there is natural isomorphism $\phi\colon {}_BN\otimes_A\operatorname{Hom}_A(-,A)\to\operatorname{Hom}_A(-,{}_BN)$ *of functors* $\mathsf{mod}\,A\to\mathsf{mod}\,B^{\mathrm{op}}$.

Proof. By [ARS95, Proposition II.4.4(a)] the family of morphisms ψ is a natural transformation. Item (1) follows since $(\psi_{M,N}(bn,f))(m)=bnf(m)=(b\psi_{M,N}(n,f))(m)$ for $m\in M$, $n\in N$, $b\in B$ and $f\in\operatorname{Hom}_A(M,A)$.

The morphism $\psi_{M,A}$ is the canonical isomorphism between $A\otimes_A\operatorname{Hom}_A(M,A)$ and $\operatorname{Hom}_A(M,A)$, so (2) holds for $N_A=A_A$. Since ψ is natural and both $-_2\otimes_A\operatorname{Hom}_A(-_1,A)$ and $\operatorname{Hom}_A(-_1,-_2)$ are additive in $-_2$, this implies (2) for $N_A\in\mathsf{add}\,A=\mathsf{proj}\,A$.

For any B-A-bimodule ${}_BN_A$ which is projective as an A-module the natural transformation $\phi\colon {}_BN\otimes_A\operatorname{Hom}_A(-,A)\to\operatorname{Hom}_A(-,{}_BN)$ defined by $\phi_M:=\psi_{M,N}$ for $M\in\mathsf{mod}\,A$ is a natural isomorphism by (1) and (2) since a morphism of B^{op}-modules is an isomorphism if and only if it is an isomorphism of $\Bbbk$-modules. This shows the last part of the statement. □

Finally, recall the following from linear algebra.

Lemma 2.13. *Let* $X,X'\in\mathsf{M}_n(\mathbb{N})$ *be matrices satisfying* $XX'=1_{n\times n}$. *Then* X *and* X' *are permutation matrices (cf. e.g. [Ser02, Section 1.1.2]).*

Proof. In the ring $\mathsf{M}_n(\mathbb{Q})\supset\mathsf{M}_n(\mathbb{N})$ we have $X'=X^{-1}$ by [Ser02, Proposition 2.2.1]. Hence, $X'X=1_{n\times n}$ holds in $\mathsf{M}_n(\mathbb{N})$, so X and X' define bijective maps from $\mathbb{N}^n$ to $\mathbb{N}^n$. They map $\mathbb{N}^n\setminus\{0\}$ into $\mathbb{N}^n\setminus\{0\}$ and hence form semigroup automorphisms of $(\mathbb{N}^n\setminus\{0\},+)$. The only elements of $(\mathbb{N}^n\setminus\{0\},+)$ that are not the sum of any two other elements are the standard basis vectors. But then X and X' have to permute those elements. □

3 Core lemmas

In this Section we want to build the core lemmas for our classification result. The main results of this section are the formula of Proposition 3.2 showing how to calculate τ_{Γ_e} using τ_A as well as Lemma 3.10.

We want to show that τ-maps are inherited by idempotent subalgebras arising from the removal of injective simple modules.

Lemma 3.1. *There is a natural isomorphism $F_e\nu_A G_e \cong \nu_{\Gamma_e}$.*

Proof. We have $F_e(-) \cong \mathrm{Hom}_A((1-e)A, -)$ by Lemma 2.4(4) and therefore

$$F_e\mathsf{D}(-) \cong \mathrm{Hom}_A((1-e)A, \mathrm{Hom}_{\Bbbk}(-,\Bbbk)) \cong \mathrm{Hom}_{\Bbbk}((1-e)A\otimes_A -, \Bbbk) = \mathsf{D}((1-e)A\otimes_A -)$$

by Tensor-Hom adjunction. This, Lemma 2.12 and Tensor-Hom adjunction give a sequence

$$\begin{aligned} F_e(\mathsf{D}\,\mathrm{Hom}_A(-\otimes_{\Gamma_e}(1-e)A, A)) &\cong \mathsf{D}((1-e)A\otimes_A \mathrm{Hom}_A(-\otimes_{\Gamma_e}(1-e)A, A)) \\ &\cong \mathsf{D}\,\mathrm{Hom}_A(-\otimes_{\Gamma_e}(1-e)A, (1-e)A) \\ &\cong \mathsf{D}\,\mathrm{Hom}_{\Gamma_e}(-, \mathrm{Hom}_A((1-e)A, (1-e)A)) \end{aligned}$$

of natural isomorphisms, where the first functor is $F_e\nu_A G_e$ and the last functor is naturally isomorphic to ν_{Γ_e} as $\Gamma_e \cong \mathrm{Hom}_A((1-e)A, (1-e)A)$. □

Proposition 3.2. *There is an isomorphism $\tau_{\Gamma_e} M \cong F_e\tau_A G_e(M)$ for $M \in \mathsf{mod}\,\Gamma_e$.*

Proof. Let $P_1 \to P_0 \to M \to 0$ be an augmented minimal projective presentation. Lemma 2.4(6) implies that $G_e(P_1) \to G_e(P_0) \to G_e(M) \to 0$ is an augmented minimal projective presentation of $G_e(M) \in \mathsf{mod}\,A$. Applying ν_A yields an exact sequence

$$0 \to \tau_A G_e(M) \to \nu_A G_e(P_1) \to \nu_A G_e(P_0) \to \nu_A G_e(M) \to 0$$

which yields an exact sequence

$$0 \to F_e\tau_A G_e(M) \to \nu_{\Gamma_e} P_1 \to \nu_{\Gamma_e} P_0 \to \nu_{\Gamma_e} M \to 0$$

upon applying the exact functor F_e and using Lemma 3.1 for the three terms on the right. But, because $P_1 \to P_0 \to M \to 0$ was an augmented minimal projective presentation we have $F_e\tau_A G_e(M) \cong \tau_{\Gamma_e} M$. □

Corollary 3.3. *If A has a τ-map and $(1-e)Ae \in \mathsf{proj}\,\Gamma_e^{\mathrm{op}}$ then Γ_e has a τ-map.*

Proof. Let $\Phi_A \colon \mathsf{K}_0(\mathsf{mod}\,A) \to \mathsf{K}_0(\mathsf{mod}\,A)$ be a τ-map of A. Since F_e and G_e are exact by Lemma 2.4(2), they induce linear maps $F_e^* := \mathsf{K}_0(F_e) \colon \mathsf{K}_0(\mathsf{mod}\,A) \to \mathsf{K}_0(\mathsf{mod}\,\Gamma_e)$ and $G_e^* := \mathsf{K}_0(G_e) \colon \mathsf{K}_0(\mathsf{mod}\,\Gamma_e) \to \mathsf{K}_0(\mathsf{mod}\,A)$, respectively. Proposition 3.2 now implies that $\Phi_{\Gamma_e} := F_e^*\Phi_A G_e^*$ is a τ-map for Γ_e. Indeed, for $M \in \mathsf{ind}\,\mathsf{mod}\,\Gamma_e$ non-projective

$$\Phi_{\Gamma_e}[M] = F_e^*\Phi_A G_e^*[M] = F_e^*\Phi_A[G_e(M)] = F_e^*[\tau_A G_e(M)] = [F_e\tau_A G_e(M)] = [\tau_{\Gamma_e} M]$$

holds, because $G_e(M) \in \mathsf{ind}\,\mathsf{mod}\,A$ is non-projective by Lemma 2.4(6). □

Corollary 3.4. *If A has a τ-map and $e \in A$ is a central idempotent then Γ_e has a τ-map.*

Proof. We have $(1-e)Ae = 0$ since e is a central idempotent, so $(1-e)Ae \in \mathsf{mod}\,\Gamma_e^{\mathrm{op}}$ is projective. The result follows from Corollary 3.3. □

Remark 3.5. If $\mathsf{D}(Ae)$ is simple then the requirements of Corollary 3.3 are satisfied. Indeed, the A^{op}-module Ae is not a direct summand of $A(1-e)$, as A^{op} is basic. But, Ae is a simple and projective A^{op}-module. Hence, $0 = \mathrm{Hom}_{A^{\mathrm{op}}}(A(1-e), Ae) \cong (1-e)Ae$.

If A has simple injective modules, then Remark 3.5 and Corollary 3.3 allow us to study a τ-map on A by studying a τ-map on an idempotent subalgebra. The following Lemma and Remark are useful as they allow us the same reduction, if A has no injective simple modules but at least one projective simple module.

Lemma 3.6. *Suppose A has a τ-map Φ. If A has no simple injective modules then Φ has an inverse Φ'. Furthermore, Φ' is a τ^{-1}-map for A.*

Proof. Let Φ be a τ-map for A. Since $\operatorname{\mathsf{mod}} A$ has no injective simple modules and since $\mathsf{K}_0(\operatorname{\mathsf{mod}} A)$ is free on the simple A-modules, we can define a $\mathbb{Z}$-linear morphism $\Phi' \colon \mathsf{K}_0(\operatorname{\mathsf{mod}} A) \to \mathsf{K}_0(\operatorname{\mathsf{mod}} A)$ through $\Phi'[S] = [\tau_A^{-1} S]$ for $S \in \operatorname{\mathsf{mod}} A$ simple. Further, since $\tau_A^{-1} S$ is not projective and indecomposable for $S \in \operatorname{\mathsf{mod}} A$ simple and as Φ is a τ-map, we have $\Phi\Phi'([S]) = [S]$ for $S \in \operatorname{\mathsf{mod}} A$ simple. This means that Φ' is a right inverse for Φ, since $\mathsf{K}_0(\operatorname{\mathsf{mod}} A)$ is the free abelian group on the simple A-modules. Notice that $\operatorname{End}_{\mathbb{Z}}(\mathsf{K}_0(\operatorname{\mathsf{mod}} A))$ is a subring of a matrix ring over $\mathbb{Q}$ and as a result Φ and Φ' are indeed inverse to each other by [Ser02, Proposition 2.2.1].

We claim that $\Phi'[M] = [\tau_A^{-1} M]$ holds for any $M \in \operatorname{\mathsf{ind}}\operatorname{\mathsf{mod}} A$ which is non-injective. Indeed if $M \in \operatorname{\mathsf{ind}}\operatorname{\mathsf{mod}} A$ is non-injective, then $\tau_A^{-1} M$ is not projective and indecomposable. Therefore, we have $\Phi[\tau_A^{-1} M] = [\tau_A \tau_A^{-1} M] = [M]$, using that Φ is a τ-map. Finally, applying Φ' to both sides, we obtain that $[\tau_A^{-1} M] = \Phi'[M]$ holds. $\square$

Remark 3.7. Notice, we have $\mathsf{D}\tau_{A^{\mathrm{op}}}\mathsf{D}(M) \cong \tau_A^{-1} M$ for any $M \in \operatorname{\mathsf{mod}} A$. Since the $\Bbbk$-dualities are exact, this implies that A^{op} has a τ-map Φ^{op} if and only if A has τ^{-1}-map $\Phi' := \mathsf{K}_0(\mathsf{D})\Phi^{\mathrm{op}}\mathsf{K}_0(\mathsf{D})$. Hence, Lemma 3.6 shows that if A has a τ-map and no injective simple modules, then A^{op} has a τ-map as well.

Finally, we want to investigate how the Auslander–Reiten translation behaves on modules with a simple socle and a first cosyzygy with decomposable socle.

Lemma 3.8. *Suppose A has a τ-map Φ. If $M \in \operatorname{\mathsf{ind}}\operatorname{\mathsf{mod}} A$ satisfies $\Phi[M] \neq [N]$ for every $N \in \operatorname{\mathsf{mod}} A$ then any submodule of M is projective.*

Proof. It suffices to show the lemma for all indecomposable submodules $M_0 \subseteq M$. First, if $M_0 = M$ then $\Phi[M_0] \neq [\tau_A M_0]$ and hence M_0 is projective.

If $M_0 \subsetneq M$ let $M/M_0 = M_1 \oplus \cdots \oplus M_k$ be a decomposition with $M_1, \ldots, M_k \in \operatorname{\mathsf{ind}}\operatorname{\mathsf{mod}} A$. For $1 \leq i \leq k$ the modules M_i cannot be projective as they are proper quotients of the indecomposable module M. We have $[M] = [M_0] + \cdots + [M_k]$ in $\mathsf{K}_0(\operatorname{\mathsf{mod}} A)$. If M_0 is also not projective then we have

$$\Phi[M] = \Phi[M_0] + \cdots + \Phi[M_k] = [\tau_A M_0] + \cdots + [\tau_A M_k] = [\tau_A M_0 \oplus \cdots \oplus \tau_A M_k]$$

in $\mathsf{K}_0(\operatorname{\mathsf{mod}} A)$ as Φ is a τ-map, showing the lemma by contraposition. $\square$

Lemma 3.9. *Let $0 \to M \to I_0 \xrightarrow{f} I_1$ be an augmented minimal injective copresentation of an A-module M. Suppose I_0 is indecomposable and $I_1 = I_1' \oplus I_1''$ is a non-trivial decomposition yielding a decomposition $f = [f', f'']^{\mathsf{T}}$. Then there is an exact sequence*

$$0 \to M \to M' \to I_1'' \to \operatorname{Cok} f \to \operatorname{Cok} f' \to 0 \tag{3.1}$$

and an exact sequence

$$0 \to \nu_A^{-1} M \to \nu_A^{-1} M' \to \nu_A^{-1} I_1'' \to \tau_A^{-1} M \to \tau_A^{-1} M' \to 0 \tag{3.2}$$

with $M' = \operatorname{Ker}(f')$. *Further,* M *and* M' *are indecomposable non-injective with* $M \ncong M'$.

Proof. We can draw the undashed arrows of the commutative diagram in Diagram 3.1 where the row $0 \to I_1'' \to I_1 \to I_1' \to 0$ is the split short exact sequence defined through

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r, dashed] \arrow[d] & M \arrow[r, dashed] \arrow[d] & M' \arrow[d] \arrow[dddll, dashed] & \\
 & 0 \arrow[r] \arrow[d] & I_0 \arrow[r, equal] \arrow[d, "f"] & I_0 \arrow[r] \arrow[d, "f'"] & 0 \\
0 \arrow[r] & I_1'' \arrow[r] \arrow[d, equal] & I_1 \arrow[r] \arrow[d] & I_1' \arrow[r] \arrow[d] & 0 \\
 & I_1'' \arrow[r, dashed] \arrow[d] & \operatorname{Cok} f \arrow[r, dashed] \arrow[d] & \operatorname{Cok} f' \arrow[r, dashed] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Diagram 3.1: Exact sequences induced by the decomposition $I_1 = I_1' \oplus I_1''$.

the direct sum decomposition $I_1 = I_1' \oplus I_1''$ and M' is the kernel of f'. The snake lemma shows that the dashed sequence in Diagram 3.1 is exact, which yields the desired sequence (3.1).

That $f\colon I_0 \to I_1$ is a minimal injective presentation means that the inclusions $M \to I_0$ and $\operatorname{Im} f \to I_1$ are essential extensions. Now, the inclusion $M' \to I_0$ is an essential extension as the composite $M \to M' \to I_0$ is an essential extension and the inclusion $\operatorname{Im} f' \to I_1'$ is an essential extension as the composite $\operatorname{Im} f \to \operatorname{Im} f' \oplus \operatorname{Im} f'' \to I_1' \oplus I_1''$ is an essential extension. Hence, $f'\colon I_0 \to I_1'$ is minimal injective copresentation of M'. In particular, M and M' are indecomposable as I_0 is so and both M and M' are non-injective as they have non-trivial minimal injective copresentations. Additionally, $M \ncong M'$ as M and M' have different minimal injective copresentations.

We can apply the inverse Nakayama functor to the upper three rows of Diagram 3.1 and obtain a commutative diagram in Diagram 3.2 where the two rightmost columns are exact by the definition of τ_A^{-1} and the two middle rows are exact because functors preserve split exact sequences. The dashed sequence in Diagram 3.2 is then exact by the snake lemma. This is the sequence (3.2). $\square$

Lemma 3.10. *Suppose that* A *has a* τ*-map* Φ. *Let* $0 \to M \to I_0 \xrightarrow{f} I_1$ *be an augmented minimal injective copresentation of an* A*-module* M. *If* I_0 *is indecomposable and there is a non-trivial decomposition* $I_1 = I_1' \oplus I_1''$, *where* I_1'' *is indecomposable, then any submodule of* $\nu_A^{-1} I_1''$ *is projective.*

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r, dashed] \arrow[d] & \nu_A^{-1}M \arrow[r, dashed] \arrow[d] & \nu_A^{-1}M' \arrow[d] \arrow[dddll, dashed, rounded corners, to path={-- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart |- \tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] & \\
 & 0 \arrow[r] \arrow[d] & \nu_A^{-1}I_0 \arrow[r, equal] \arrow[d] & \nu_A^{-1}I_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \nu_A^{-1}I_1'' \arrow[r] \arrow[d, equal] & \nu_A^{-1}I_1 \arrow[r] \arrow[d] & \nu_A^{-1}I_1' \arrow[r] \arrow[d] & 0 \\
 & \nu_A^{-1}I_1'' \arrow[r, dashed] \arrow[d] & \tau_A^{-1}M \arrow[r, dashed] \arrow[d] & \tau_A^{-1}M' \arrow[r, dashed] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Diagram 3.2: Exact sequences induced by applying ν_A^{-1} to Diagram 3.1.

Proof. Lemma 3.9 applied to the decomposition yields a monomorphism $g\colon M \to M'$ coming from (3.1) and an exact sequence

$$\nu_A^{-1}I_1'' \to \tau_A^{-1}M \xrightarrow{g'} \tau_A^{-1}M' \to 0 \tag{3.3}$$

coming from (3.2). Notice, M and M' are indecomposable, non-injective and not isomorphic by Lemma 3.9 and hence $\tau_A^{-1}M$ and $\tau_A^{-1}M'$ are indecomposable, non-projective and not isomorphic. In particular, $\operatorname{Cok} g \neq 0$ and $\operatorname{Ker} g' \neq 0$.

We have $[M] = [M'] - [\operatorname{Cok} g]$ and $[\tau_A^{-1}M] = [\operatorname{Ker} g'] + [\tau_A^{-1}M']$. Applying the τ-map Φ to the latter equation yields $[M] = \Phi[\operatorname{Ker} g'] + [M']$. Hence, $\Phi[\operatorname{Ker} g'] = -[\operatorname{Cok} g]$. If there was $N \in \mathsf{mod}\, A$ such that $\Phi[\operatorname{Ker} g'] = [N]$ then $0 = [\operatorname{Cok} g \oplus N]$, which is absurd.

As $\nu_A^{-1}I_1''$ is an indecomposable projective module and $\operatorname{Ker} g' \neq 0$, the sequence (3.3) gives rise to a projective cover $\nu_A^{-1}I_1'' \to \operatorname{Ker} g'$. But this implies that $\operatorname{Ker} g'$ is indecomposable, as $\nu_A^{-1}I_1''$ is so. Hence, any submodule of $\operatorname{Ker} g'$ is projective, by Lemma 3.8. But then $\operatorname{Ker} g'$ is also projective and therefore isomorphic to its projective cover $\nu_A^{-1}I_1''$, showing the lemma as any submodule of $\operatorname{Ker} g'$ is projective. ☐

4 Main results

4.1 Algebras with connected and non-acyclic Ext^1-quiver admitting a τ-map

In this Section we want to prove our main result, a classification of finite-dimensional $\Bbbk$-algebras with connected and non-acyclic Ext^1-quiver admitting a τ-map.

Theorem 4.1. *Suppose the* Ext^1*-quiver of* A *is connected and non-acyclic. Then* A *has a* τ*-map if and only if* A *is a cyclic Nakayama algebra.*

The proof of Theorem 4.1 is split up. In Theorem 4.6 we show that all algebras with a connected and non-acyclic Ext^1-quiver admitting a τ-map are cyclic Nakayama algebras and in Proposition 4.7 we show that all Nakayama algebras admit a τ-map. Using Remark 2.3 we can reformulate this as the following result for quotients of path algebras.

Corollary 4.2. *Let $\Bbbk$ be an algebraically closed field. Suppose $A := \Bbbk Q/I$, where Q is a connected and non-acyclic quiver and $I \lhd \Bbbk Q$ is an admissible ideal. Then A has a τ-map if and only if A is a cyclic Nakayama algebra.*

The key to Theorem 4.1 lies in Lemma 4.3 and Lemma 4.4.

Lemma 4.3. *Suppose A admits a τ-map Φ. If the Ext^1-quiver Q of A has no source and no sink vertices then A is a cyclic Nakayama algebra.*

Proof. Since there are no source and sink vertices in Q there are no simple injective and no simple projective modules in $\operatorname{mod} A$, by Remark 2.9. Then A has a τ^{-1}-map Φ' which is the inverse of Φ by Lemma 3.6. Suppose $S_1, \dots, S_n$ are the simple A-modules. Fix the basis $\mathscr{B}$ of $\mathsf{K}_0(\operatorname{mod} A)$ given by $[S_1], \dots, [S_n]$. Then the matrix representation of Φ with respect to $\mathscr{B}$ has the dimension vectors of $\tau_A S_1, \dots, \tau_A S_n$ as columns and the matrix representation of Φ' with respect $\mathscr{B}$ has the dimension vectors of $\tau_A^{-1} S_1, \dots, \tau_A^{-1} S_n$ as columns. But then, by Lemma 2.13, the maps Φ and Φ' permute the elements of $\mathscr{B}$. Hence, τ_A permutes the simple A-modules and A is a cyclic Nakayama algebra by Theorem 2.10. $\square$

Lemma 4.4. *Suppose that*

1. *the Ext^1-quiver Q_A of A is connected,*
2. *there is a source vertex $S' := \mathsf{D}(Ae)$ in Q_A for some idempotent $e \in A$ and*
3. *the Ext^1-quiver Q_{Γ_e} of Γ_e has a component isomorphic to C_m for some $m \in \mathbb{N}$.*

Then A does not admit a τ-map.

Proof. Suppose to the contrary that A admits a τ-map Φ. We may identify C_m with a component of Q_{Γ_e} and, by Lemma 2.5, we can identify Q_{Γ_e} with the full subquiver of Q_A containing all vertices except S'. As Q_A is connected and as C_m is a component of Q_{Γ_e} there must be an arrow $S' \to S$ in Q_A ending at some $S \in C_m$. There is a vertex $S'' \in C_m$ and an arrow $S'' \to S$ in $C_m \subseteq Q_A$.

We have $\mathrm{Ext}^1_A(S', S) \neq 0$ and $\mathrm{Ext}^1_A(S'', S) \neq 0$. Let $0 \to S \to I_0 \to I_1$ be an augmented minimal injective copresentation of S. We have $\mathrm{Hom}_A(S', I_1) \neq 0$ and $\mathrm{Hom}_A(S'', I_1) \neq 0$, by [Iwa79, Lemma 1] or the dual of Lemma 2.1. Since S' and S'' are simple, this implies that S' and S'' are submodules of I_1. Since I_1 is injective, there is a non-trivial decomposition $I_1 = I_1' \oplus I_1''$, where I_1' has the injective envelope of S' as a direct summand and I_1'' is the injective envelope of S''. Notice, $\nu_A^{-1} I_1''$ is indecomposable and hence $\mathrm{soc}(\nu_A^{-1} I_1'')$ is projective by Lemma 3.10.

But, $\nu_A^{-1} I_1''$ is the projective cover of S''. As there is no path in Q_A from S'' to any simple A-module not in C_m, we obtain that all composition factors of $\nu_A^{-1} I_1''$ lie in C_m, by Lemma 2.7. Hence, all composition factors of $\nu_A^{-1} I_1''$ are non-projective, by Remark 2.9. This includes all the direct summands of $\mathrm{soc}(\nu_A^{-1} I_1'')$ which is a contradiction. $\square$

Remark 4.5. Suppose the Ext^1-quiver Q of a Nakayama algebra A has an oriented cycle and is connected. For a fixed simple A-module S there exists at most one other simple A-module S' with $\operatorname{Ext}^1_A(S, S') \neq 0$, otherwise the projective cover of S would not be uniserial. Dually, there exists at most one other simple A-module S'' with $\operatorname{Ext}^1_A(S'', S) \neq 0$. Hence, there starts and ends at most one arrow in each vertex of Q. Because Q is connected it must already be isomorphic to its oriented cycle.

Theorem 4.6. *Suppose A admits a τ-map. If the Ext^1-quiver Q_A of A is connected and has a subquiver isomorphic to C_m for some $m \geq 1$ then A is a cyclic Nakayama algebra.*

Proof. We show the theorem by induction on the number n of vertices of Q_A. For $n = 1$ the Ext^1-quiver Q_A of A can only contain a cycle if $Q_A \cong C_1$. Hence, A is a cyclic Nakayama algebra by Lemma 4.3. Now, assume that Q_A has n vertices and the theorem holds for all algebras with less than n vertices in its Ext^1-quiver.

We show that A cannot have any injective simple module. Suppose A has an injective simple module $S' = \mathsf{D}(Ae)$. Then we have $(1 - e)Ae = 0$ by Remark 3.5 and Γ_e has a τ-map by Corollary 3.3. Let Q_{Γ_e} be the Ext^1-quiver of Γ_e. Then Q_{Γ_e} is obtained from Q_A by removing the source vertex S' and the arrows incident to it, by Lemma 2.5. Notice, Q_{Γ_e} contains an oriented cycle as Q_A did so. Hence, there is a component C of Q_{Γ_e} which contains this oriented cycle. This component induces an idempotent subalgebra Γ_C corresponding to a central idempotent, by Remark 2.8. Then Γ_C admits a τ-map, by Corollary 3.4. Furthermore, C has fewer than n vertices. Hence, Γ_C must be a Nakayama algebra by induction hypothesis. Therefore, C is a component of Q_{Γ_e} which is isomorphic to an oriented cycle C_m for some $m \geq 1$, by Remark 4.5. Applying Lemma 4.4 yields that A does not admit a τ-map. Therefore, this case cannot occur and A cannot have any injective simple modules.

If A has no injective simple modules but projective simple modules, then A^{op} has a τ-map by Remark 3.7 and, by $\Bbbk$-duality, the Ext^1-quiver of A^{op} is Q_A^{op}, which has n vertices and an oriented cycle. But A^{op} then has injective simple modules, which cannot happen by the above.

Therefore, A cannot have any injective or projective simple modules. But then A is a cyclic Nakayama algebra by Lemma 4.3. □

Finally, we show that every Nakayama algebra admits a τ-map.

Proposition 4.7. *Let A be a Nakayama algebra and fix $x_S \in \mathsf{K}_0(\operatorname{\mathsf{mod}} A)$ for each simple projective module $S \in \operatorname{\mathsf{mod}} A$. Then the unique morphism $\Phi \in \operatorname{End}_{\mathbb{Z}}(\mathsf{K}_0(\operatorname{\mathsf{mod}} A))$ with*

$$\Phi[S] = \begin{cases} [\tau_A S] & \text{for } S \in \operatorname{\mathsf{mod}} A \text{ simple and non-projective,} \\ x_S & \text{for } S \in \operatorname{\mathsf{mod}} A \text{ simple and projective,} \end{cases} \tag{4.1}$$

is a τ-map for A. If A has no projective simple modules then A has a unique τ-map.

Proof. every module $M \in \operatorname{\mathsf{ind}} \operatorname{\mathsf{mod}} A$ is uniserial by [ARS95, Theorem VI.2.1] and hence of the form $P/\operatorname{rad}^l P$ for $P \in \operatorname{\mathsf{ind}} \operatorname{\mathsf{proj}} A$ and $l \in \mathbb{N}$ the Loewy length of M. Let Φ be the unique $\mathbb{Z}$-linear endomorphism of $\mathsf{K}_0(\operatorname{\mathsf{mod}} A)$ defined through (4.1). We claim by induction on $l \in \mathbb{N}$ that $[\tau_A M] = \Phi[M]$ for all non-projective $M \in \operatorname{\mathsf{ind}} \operatorname{\mathsf{mod}} A$ of Loewy length l.

For $l = 1$ this follows from the definition of Φ, as an indecomposable module of Loewy length 1 is simple. For the induction step suppose $l \geq 2$. We have a short exact sequence

$$0 \to \operatorname{rad} M \to M \to M/\operatorname{rad} M \to 0. \tag{4.2}$$

Write $M \cong P/\operatorname{rad}^l P$ for some $P \in \operatorname{ind}\operatorname{proj} A$. Notice, $\operatorname{rad}^l P \neq 0$ because $M \cong P/\operatorname{rad}^l P$ is non-projective by assumption. Therefore, $\operatorname{rad} M \cong \operatorname{rad} P/\operatorname{rad}^l P$ is non-projective and uniserial as a non-trivial quotient of the uniserial module $\operatorname{rad} P$. Similarly, $M/\operatorname{rad} M$ is non-projective and simple.

We can now calculate the Auslander–Reiten translates of the modules in (4.2). Because $\operatorname{rad} P/\operatorname{rad}^{l+1} P$ is uniserial of Loewy length l we have $M' := \operatorname{rad} P/\operatorname{rad}^{l+1} P \cong P'/\operatorname{rad}^l P'$ for some $P' \in \operatorname{ind}\operatorname{proj} A$. Applying [ARS95, Proposition IV.2.6(c)] and the third isomorphism theorem for modules yields

$$\begin{aligned}
\tau_A(M/\operatorname{rad} M) &\cong \tau_A(P/\operatorname{rad} P) \cong \operatorname{rad} P/\operatorname{rad}^2 P \cong M'/\operatorname{rad} M', \\
\tau_A M &\cong \tau_A(P/\operatorname{rad}^l P) \cong \operatorname{rad} P/\operatorname{rad}^{l+1} P \cong M' \text{ and} \\
\tau_A(\operatorname{rad} M) &\cong \tau_A(M'/\operatorname{rad}^{l-1} M') \cong \tau_A(P'/\operatorname{rad}^{l-1} P') \cong \operatorname{rad} P'/\operatorname{rad}^l P' \cong \operatorname{rad} M'.
\end{aligned}$$

But those Auslander–Reiten translates also fit into a short exact sequence

$$0 \to \operatorname{rad} M' \to M' \to M'/\operatorname{rad} M' \to 0$$

which shows

$$[\tau_A M] = [M'] = [\operatorname{rad} M'] + [M'/\operatorname{rad} M'] = [\tau_A(\operatorname{rad} M)] + [\tau_A(M/\operatorname{rad} M)]$$

which is by the induction hypothesis equal to

$$\Phi[\operatorname{rad} M] + \Phi[M/\operatorname{rad} M] = \Phi([\operatorname{rad} M] + [M/\operatorname{rad} M]) = \Phi[M].$$

This completes the induction step.

Finally, if A has no projective simple module, then each τ-map Φ satisfies $\Phi[S] = [\tau_A S]$ for each simple module $S \in \operatorname{mod} A$. Hence, the τ-map is uniquely determined. $\square$

5 Algebras not covered by this paper

There are examples of algebras which admit a τ-map but are neither hereditary nor Nakayama algebras. Such algebras can arise from certain kinds of gluings of Nakayama algebras to hereditary algebras. For instance the two algebras of the form $\Bbbk Q/\operatorname{rad}^2 \Bbbk Q$ where Q is

$$\begin{tikzcd}
 & & 3 \\
1 \arrow[r] & 2 \arrow[ur] \arrow[dr] & \\
 & & 3'
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
1 \arrow[r] & 2 \arrow[dr] & \\
 & & 3 \\
 & 2' \arrow[ur] &
\end{tikzcd}$$

both admit a τ-map, even though they are neither Nakayama algebras nor hereditary themselves. That these algebras indeed admit τ-maps can be read of their Auslander–Reiten quivers which are given by

$$\begin{tikzcd}[column sep=small]
3 \arrow[dr] & & \begin{smallmatrix}2\\3'\end{smallmatrix} \arrow[ll, dotted, "\tau"'] \arrow[dr] & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[dr] & \\
& \begin{smallmatrix}&2&\\3&&3'\end{smallmatrix} \arrow[ur] \arrow[dr] & & 2 \arrow[ll, dotted, "\tau"'] \arrow[ur] & & 1 \arrow[ll, dotted, "\tau"'] \\
3' \arrow[ur] & & \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[ll, dotted, "\tau"'] \arrow[ur] & & &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}[column sep=small]
& \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[dr] & & 2' \arrow[ll, dotted, "\tau"'] & & \\
3 \arrow[ur] \arrow[dr] & & \begin{smallmatrix}2&&2'\\&3&\end{smallmatrix} \arrow[ll, dotted, "\tau"'] \arrow[ur] \arrow[dr] & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[dr] & \\
& \begin{smallmatrix}2'\\3\end{smallmatrix} \arrow[ur] & & 2 \arrow[ll, dotted, "\tau"'] \arrow[ur] & & 1. \arrow[ll, dotted, "\tau"']
\end{tikzcd}$$

Notice, the Auslander–Reiten quiver arises in both cases as the gluing of the Auslander–Reiten quiver of a hereditary algebra and an acyclic Nakayama algebra.

One can ask the question whether all algebras which admit a τ-map and have an acyclic Ext^1-quiver can be constructed by a similar gluing construction. We want to put this question for further research.

References

[ARS95] Maurice Auslander, Idun Reiten, and Sverre O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, vol. 36, Cambridge University Press, Cambridge, 1995.

[ASS06] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński, *Elements of the representation theory of associative algebras. Vol. 1*, London Mathematical Society Student Texts, vol. 65, Cambridge University Press, Cambridge, 2006.

[Aus74] Maurice Auslander, *Representation theory of Artin algebras. I,* Comm. Algebra **1** (1974), 177–268.

[Gre07] J. A. Green, *Polynomial representations of* GL_n, Lecture Notes in Mathematics, vol. 830, Springer, Berlin, 2007.

[Hap88] Dieter Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, vol. 119, Cambridge University Press, Cambridge, 1988.

[Iwa79] Yasuo Iwanaga, *On rings with finite self-injective dimension,* Comm. Algebra **7** (1979), no. 4, 393–414.

[Psa14] Chrysostomos Psaroudakis, *Homological theory of recollements of abelian categories*, J. Algebra **398** (2014), 63–110.

[RVdB02] I. Reiten and M. Van den Bergh, *Noetherian hereditary abelian categories satisfying Serre duality*, J. Amer. Math. Soc. **15** (2002), no. 2, 295–366.

[Ser02] Denis Serre, *Matrices*, Graduate Texts in Mathematics, vol. 216, Springer-Verlag, New York, 2002.