Graduate School of Natural Sciences

Some constructions for canonical non-Kähler metrics

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Abstract

This thesis is about constructions of canonical metrics in complex non-Kähler geometry, focusing in particular on balanced metrics satisfying special hermitian curvature conditions.

More specifically, we adapt gluing strategies from Kähler geometry to obtain families of balanced metrics with special curvature properties, with particular relevance for the constructions of solutions of the Hull-Strominger system and the geometrization of balanced classes. More specifically, we show that: crepant resolutions of orbifolds with isolated singularities admitting singular Chern-Ricci flat balanced metrics can also be endowed with Chern-Ricci flat balanced metrics; small resolutions of smoothable Calabi-Yau singular threefolds with a finite family of Ordinary Double Points admit an approximately Chern-Ricci flat balanced metric; the blowup at a finite family of points of a compact Chern-Ricci flat balanced manifold always admits Chern-scalar constant balanced metrics. In all three cases we have a control on the Bott-Chern cohomology class of metrics constructed.

Furthermore, we use representation theory techniques to construct special balanced metrics on the class of real simple Lie groups of inner type, as well as on the corresponding compact homogeneous spaces, on which we obtain that the metrics constructed are Chern-scalar with non-vanishing Chern-Ricci tensor, providing a family of compact complex manifolds with vanishing first Chern class and non-vanishing first Bott-Chern class. Moreover, we show that for this class of homogeneous spaces the Fino-Vezzoni conjecture holds.

Abstract

Denne afhandling omhandler konstruktioner af kanoniske metrikker i kompleks ikke-Kähler geometri med særligt fokus på balancerede metrikker, der opfylder specielle hermitiske krumningsbetingelser.

Mere specifikt tilpasser vi limningsstrategier fra Kähler geometri til at opnå familier af balancerede metrikker med specielle krumningsegenskaber med særlig relevans for konstruktioner af løsninger til Hull-Strominger systemet og geometriseringen af balancerede klasser. Mere specifikt viser vi at: ikke-afvigende opløsninger af orbifolde med isolerede singulariteter, som tillader singulære Chern-Ricci-flad balancerede metrikker, også kan udstyres med Chern-Ricci-flad balancerede metrikker; små opløsninger af smoothable Calabi-Yau singulære 3-flader med en endelig familie af ordinære dobbeltpunkter tillader en approksimativ Chern-Ricci-flad balanceret metrik; blowuppet i en endelig familie af punkter på en kompakt Chern-Ricci-flad balanceret mangfoldighed tillader altid Chern-skalar-konstant balancerede metrikker. I alle tre tilfælde har vi kontrol over Bott-Chern cohomologiklassen for de konstruerede metrikker.

Ydermere anvender vi teknikker fra representationsteori til at konstruere specielle balancerede metrikker på klassen af reelle simple Lie-grupper af indre type og desuden på de tilhørende kompakte homogene rum, hvorpå vi opnår at de konstruerede metrikker er Chern-skalar med ikke-forsvindende Chern-Ricci-tensor, hvilket giver en familie af kompakte komplekse mangfoldigheder med forsvindende første Chern-klasse og ikke-forsvindende første Bott-Chern-klasse. Desuden viser vi at for denne klasse af homogene rum er Fino-Vezzoni-formodningen sand.

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Contents

Introduction

With the ultimate aim of geometrizing and classifying, one of the most studied problems in complex geometry is the existence of hermitian metrics that can be regarded as *canonical*. Through the years, the Kähler case is the one that has been studied and understood the most. However, in the last decades the interest towards the non-Kähler world has been increasing more and more, leading to the search for special metrics also in this particular context. While in the Kähler case special metrics arise naturally, the non-Kähler scenario is too wild to guide us directly towards some central notion of special metric. Nevertheless, one can have indications on the path to follow by watching the Kähler world. More specifically, given an n-dimensional complex manifold (M, J), if it is Kähler the obvious class of special (on a first level) metrics is given exactly by Kähler metrics - which we recall being hermitian metrics h whose fundamental form $\omega := h(J\cdot, \cdot)$ is d-closed. In addition, this condition can also be combined with the notion of *Einstein metric* (thanks to the properties of Kähler metrics) from the general riemannian case, giving rise to the notion of Kähler-Einstein metrics, which are universally regarded as the "most canonical" in the Kähler world. Likewise, other notions of special Kähler metrics have been introduced and studied (some of them are still central in the study of Kähler geometry), like *constant scalar curvature Kähler* (cscK) metrics, or the more general class of *extremal Kähler* metrics (introduced by Calabi in [C]), however they all share the fact that they are giving a curvature condition on the metric, thus this suggests that when searching for special metrics in the non-Kähler case we shall ask for these metrics to be special under two aspects: the cohomological one (satisfying a condition possibly generalizing the Kähler one) and the curvature one.

Regarding the cohomological aspect, several conditions have been introduced that generalize the Kähler one, and one of the most studied is given by $d\omega^{n-1} = 0$, identifying the class of *balanced metrics* (originally introduced by Michelsohn [M], and also considered by Gauduchon in [Ga1] as *semi-Kähler* metrics), which is the class of metrics we are interested in working with. Balanced metrics carry many interesting properties such as the coincidence between the Hodge laplacian and the Dolbeault laplacian on scalar functions (showed by Gauduchon in [Ga1]), and the class of balanced manifolds (i.e. manifolds admitting balanced metrics) was shown to be closed under holomorphic submersions proved in [M] (showing a sort of duality between the Kähler condition and the balanced condition). Also in [M], Michelsohn proved a characterization of balanced metrics in terms of currents, which leads to the celebrated result from Alessandrini and Bassanelli (see [AB1]) showing that the class of compact balanced manifolds is closed under proper modifications (condition not satisfied by the class of Kähler manifolds). Moreover, balanced metrics ended up being central in many interesting currently open problems, such as the conjecture from Fino and Vezzoni (see [FV]), regarding the coexistence of balanced and pluriclosed metrics - described by the condition $\partial\overline{\partial}\omega = 0$ - on compact non-Kähler manifolds, and the Gauduchon conjecture for balanced metrics (see [Tos] and [STW], in which was solved in its original version for Gauduchon metrics - identified by the condition $\partial\overline{\partial}\omega^{n-1} = 0$, which weakens the balanced condition - posed by Gauduchon). Moving instead on the curvature aspect, there are several known notions of special metrics in the non-Kähler world such as Chern-Ricci flat metrics, Bismut-Ricci flat metrics (which in the balanced case are equivalent to Chern-Ricci flat metrics, see [AI]), Chern-Einstein metrics and many more. As we will see, a class of metrics on which we will be focusing is the one of Chern-Ricci flat balanced metrics. Our interest towards said metrics comes actually from the realm of Calabi-Yau geometry. Indeed, for a not necessarily Kähler Calabi-Yau manifold (i.e. a complex manifold endowed with a holomorphic volume form) it was introduced by Hull and Strominger (respectively in [Hu] and [S]) a system coming from heterotic superstring theory known as the *Hull-Strominger system* whose solutions have proved to be extremely hard to construct (see [GF] for a full presentation of the system and some known solutions, together with several other references such as [AGF], [FuY], [LY3], [P], [TY] and the very recent [CPY2], [FeY] for the invariant case, [PPZ] for a flow approach, and the recent moment map picture from [GFGM]). The problem of solving this system, apart from its physical meaning, carries great geometric interest, since it generalizes the Calabi-Yau condition to the non-Kähler framework, and it holds a central role in the geometrization conjecture for compact Calabi-Yau threefolds known as *Reid's Fantasy* (see [R]). This last conjecture, in particular, states that all compact Kähler Calabi-Yau threefolds can be connected through a finite number of *conifold transitions* (introduced by Clemens and Friedman, see [F]). These framework motivates further our interest towards Chern-Ricci flat balanced metrics, since it is directly related to one of the equation of the Hull-Strominger system, namely the *conformally balanced equation*, which on a compact Calabi-Yau manifold (X, Ω) - where Ω is the holomorphic volume form - is an equation for hermitian metrics ω given by $d(||\Omega||_\omega \omega^{n-1}) = 0$ which is clearly satisfied by balanced Chern-Ricci flat hermitian metrics.

The main goal of this thesis is to construct examples of balanced metrics satisfying some curvature conditions, in the attempt of constructing good *candidate canonical met-*

rics for the non-Kähler setting. The approach focuses on the use of gluing techniques - which, as far as we know, were never used before in the non-Kähler setting to combine cohomological and curvature conditions (in [FLY] a gluing approach was used to construct balanced metrics on conifold transitions) - and through symmetries in the homogeneous case.

Regarding the gluing approach, our first result - which constitutes the main theorem of the paper [GS] - is a construction for Chern-Ricci flat balanced metrics on the crepant resolutions of certain non-Kähler Calabi-Yau orbifolds endowed with a singular Chern-Ricci flat balanced metric, hence an adaptation to the balanced setting of the Kümmer construction from Biquard and Minerbe in [BM]. The statement of this result is the following, and it is proved in Chapter 2.

Theorem (2.0.1)**.** *Let $(\tilde{M}, \tilde{\omega})$ be an n-dimensional non-Kähler Calabi-Yau orbifold with a finite family of isolated singularities, endowed with $\tilde{\omega}$ a singular Chern-Ricci flat balanced metric, and let M be a crepant resolution of $\tilde{M}$. Then M admits a Chern-Ricci flat balanced metric $\hat{\omega}$ such that*

$$[\hat{\omega}^{n-1}] = [\tilde{\omega}^{n-1}] + (-1)^{n-1}\varepsilon^{(2n-2)}(\sum_{i=1}^{k_j}\sum_{j=1}^{k} a_j^i PD[E_j^i])^{n-1},$$

where $PD[E_j^i]$ denotes the Poincaré dual of the class $[E_j^i]$.

This results hence produces many new examples of Chern-Ricci flat balanced metrics and shows that under crepant resolutions they behave as Kähler Ricci-flat metrics; also, our result takes a first step towards solving the problem proposed by Becker, Tseng and Yau in (Section 6 of) [BTY], about extending orbifold solutions of the Hull-Strominger system through crepant resolutions. A natural question that arises from this construction, in the setting of the Hull-Strominger system and Reid's Fantasy, is if this strategy can be adapted to the case of singular threefolds with a finite family of ordinary double points aiming (in some sense) towards "reversing the arrow" in the construction done by Fu, Li and Yau in [FLY] and Collins, Picard and Yau in [CPY1]. Our strategy in this scenario unfortunately carries a complication that is hidden in the asymptotic behaviour of the standard Calabi-Yau metric $\omega_{co,a}$ (introduced by Candelas and de la Ossa, see [CO]) on the small resolution of the standard conifold. We are however able to achieve some partial result, which is also part of [GS], and its proof is presented in Chapter 3, together with the difficulties arising in the gluing approach, along with a discussion on a gluing attempt to understand if the metric produced in the result below might make the holomorphic tangent bundle into a slope-stable bundle. The result is the following.

Proposition (3.0.1). *Let $(\tilde{M}, \tilde{\omega})$ be a smoothable projective Kähler Calabi-Yau nodal threefold (with $\tilde{\omega}$ a singular Calabi-Yau metric), and let M be a compact (not necessarily Kähler) small resolution of $\tilde{M}$. Then M admits a balanced approximately Chern-Ricci flat metric ω such that*

$$[\omega^2] = [\tilde{\omega}^2] + \varepsilon^4[\mathbb{P}^1].$$

Hence, despite not being able to produce a Chern-Ricci flat balanced metric, we construct a very explicit family of balanced metrics that are approximately Kähler and approximately Chern-Ricci flat, which can be extremely helpful to study the Hull-Strominger system on the considered small resolutions.

The third and last gluing result is instead focused on Chern-scalar constant balanced metrics, as an attempt to extend to the balanced case the celebrated result from Arezzo and Pacard (see [AP] and [Sz]); it is a joint work with Elia Fusi and its proof is given in Chapter 4. Our interest towards constructing metrics with this curvature property comes from their importance in the Kähler setting, where they have been central in the last decades, fueled by the famous Yau-Tian-Donaldson conjecture. The statement of the theorem is the following.

Theorem (4.0.1). *Let M be a compact complex manifold of dimension n, endowed with $\tilde{\omega}$ a Chern-Ricci flat balanced metric. Then, the blowup $\hat{M}$ at a point $x \in M$ admits Chern-scalar constant balanced metric $\hat{\omega}$ such that*

$$[\omega^{n-1}]_{BC} = [\tilde{\omega}^{n-1}]_{BC} + (-1)^{n-1}\varepsilon^{(2n-2)}[\mathbb{P}^{n-1}].$$

Moving instead to the homogeneous realm, our work focuses on searching for balanced metrics in the class of semisimple real non-compact Lie groups and on their compact (non-Kähler) quotients by a cocompact lattice, and it is joint work with Fabio Podestà, and it is all contained in the paper [GiPo]. The reason why we focus on the non-compact semisimple case is because, while it appears that, despite invariant complex structures on semisimple (reductive) Lie algebras being fully classified in [Sn] (after the special case of compact Lie algebras had been considered by Samelson ([Sam]) and later in [Pi]), they have never been deeply investigated from this point of view. In contrast, the case where the Lie group is compact is fully understood, as in this case it is very well known that every invariant complex structure can be deformed to an invariant one for which the opposite of the Cartan-Killing form is a pluriclosed Hermitian metric h, i.e. it satisfies $\partial\overline{\partial}\omega_h = 0$. Moreover it has been proved in [FGV] that a compact semisimple Lie group does not carry *any* balanced metric at all, in accordance with the conjecture from Fino and Vezzoni. More specifically, in this work we have focused on a large class of simple non-compact real Lie algebras $\mathfrak{g}_o$ of even dimension, namely those which are of inner type, i.e. when the maximal compactly embedded subalgebra $\mathfrak{k}$ in a Cartan decomposition of $\mathfrak{g}_o$ contains a Cartan subalgebra. In these algebras we construct standard invariant complex structures (named regular in [Sn]) and write down the balanced condition for invariant Hermitian metrics. A careful analysis of the resulting equation together with some general argument on root systems allows us to show the existence of a suitable invariant complex structure and a corresponding Hermitian metric satisfying the balanced equation, and having vanishing Chern-scalar curvature. Our result is the following.

Theorem (5.0.1)**.** *Every non-compact simple Lie group* $\mathrm{G_o}$ *of even dimension and of inner type admits an invariant complex structure* J *and* ω *an invariant balanced* J*-Hermitian metric. Moreover, if* Γ *is a cocompact lattice, the quotient* $M = \Gamma\backslash G_o$ *inherits the balanced structure.*

The existence of cocompact lattices is guaranteed by Borel's Theorem, stating that every semisimple Lie group $\mathrm{G_o}$ admits a cocompact lattice Γ. As a consequence of the theorem and with some further work we also obtain the following proposition.

Proposition (5.0.2)**.** *Let* $\mathrm{G_o}$ *be a non-compact simple group of even dimension and of inner type together with a co-compact lattice* $\Gamma \subset \mathrm{G_o}$*. If* $\mathcal{M} = \Gamma\backslash\mathrm{G_o}$ *is endowed with a standard complex structure and a Hermitian balanced metric* h*, then the Chern Ricci form* ρ *of* h *never vanishes and the Kodaira dimension* $\kappa(M) = -\infty$*.*

We note here that the resulting metrics come in families and moreover the same kind of arguments can be applied to show the existence of balanced structures on quotients $\mathrm{G_o}/\mathrm{S}$, where $\mathrm{G_o}$ is any simple non-compact Lie group of inner type of any dimension and S is a suitable abelian closed subgroup. It is also significant to highlight that, as a consequence of the theorem, we obtain that the compact quotients constructed have non-vanishing firs Bott-Chern class, hence giving a nice class of spaces such that $c_1(M) = 0$ but $c_1^{BC}(M) \neq 0$.

Our second result concerns the non-existence of pluriclosed metrics on the compact quotients of the complex manifolds we have constructed in Theorem 5.0.1. Namely, we prove the following

Theorem (5.0.3)**.** *Let* $\mathrm{G_o}$ *be a non-compact simple even-dimensional Lie group of inner type endowed with the invariant complex structure* J *as in Theorem 5.0.1. If* Γ *is a co-compact lattice of* $\mathrm{G_o}$*, then the complex manifold* $(\mathcal{M}, J)$ *with* $\mathcal{M} = \Gamma\backslash\mathrm{G_o}$ *does not carry any pluriclosed metric.*

This result is in accordance with the above mentioned conjecture by Fino and Vezzoni, that has been already verified in several cases, and in some sense reflects a kind of duality between the compact and non-compact case, switching the existence of balanced and pluriclosed Hermitian metrics.

Structure of the thesis

In Chapter 1, we give a general discussion on canonical metrics in complex geometry. In particular, after briefly describing the Kähler case, we see how the it suggests the approach in the non-Kähler setting, and recall some of the main families of special metrics in non-Kähler geometry and their interactions, focusing in particular on balanced metrics. We also shortly recall the Hitchin-Kobayashi correspondence in the non-Kähler case, and conclude with a discussion on the Hull-Strominger system, involving several concepts introduced through the chapter.

In Chapter 2, after giving examples, we present the first step of our work, consisting of the construction of a balanced metric on the crepant resolution. More specifically we wish to work on orbifolds $\tilde{M}$ whose singular set is made of a finite number of isolated singularities admitting crepant resolutions, and are endowed with a balanced Chern-Ricci flat singular metric $\tilde{\omega}$. Then, performing a cut-off on the singular metric to the flat one around the singularities, together with what it is known on orbifold singularities (i.e. Joyce's theory on ALE spaces) and its crepant resolutions to build, with a gluing construction (inspired by, for example, [AP], [BM] and [J]), Chern-Ricci flat balanced metrics on the crepant resolutions of the orbifold. The strategy of the proof consists of two main steps: (1) a metric "rough" gluing between the singular Chern-Ricci flat balanced metric $\tilde{\omega}$ with the (rescaled) Joyce's ALE metrics ω_{ALE} (that are Kähler Calabi-Yau metrics on the crepant resolution of the singularity model, see [J]), and (2) an Implicit Function Theorem deformation argument, where the deformation preserves the balanced class (introduced in [FWW], here chosen with a particular ansatz) and all the analysis is performed in suitable weighted Hölder spaces, in order to obtain the proof of Theorem 2.0.1. By the way, the choice of the deformation shows also that on this class of manifolds a Calabi-Yau-type Theorem for balanced metrics holds for some classes in the balanced cone.

In Chapter 3, we take a look at the case of Ordinary Double Points on threefolds, walk through the gluing process from Chapter 1 to produce again an approximately Chern-Ricci flat balanced metrics, obtain Proposition 3.0.1, and discuss the difficulties that arise if we try to repeat the deformation argument in this case. We also describe an attempt to adapt Collins-Picard-Yau's approach in [CPY1] to obtain Hermite-Einstein metrics with respect to the approximately Chern-Ricci flat balanced through a gluing process, but we again meet difficulties related to the ones found to construct Chern-Ricci flat balanced metrics.

In Chapter 4, we consider the case of blowups of Chern-Ricci flat balanced metrics, and show Theorem 4.0.1, repeating the strategy in Chapter 2 with the necessary adaptations - mostly with the substitution of Joyce's ALE metrics with the Burns-Simanca metric on the bubble - that they always admit Chern-scalar constant balanced metrics.

Finally, in Chapter 5, we first review basic facts on simple real non-compact Lie algebras with invariant complex structures and we consider a class of invariant Hermitian metrics for which we write down the balanced condition in terms of roots. Then we prove our main result, namely Theorem 5.0.1, in several stages: ee first rewrite the balanced equation in terms of simple roots and then the key Lemma 5.2.2 allows us to select an invariant complex structure so that the relative balanced equation admits solutions. We then move on to prove Theorem 5.0.3 by using the properties of the Weyl basis for root spaces, and conclude by proving Proposition 5.0.2.

Chapter 1

Canonical metrics in complex geometry

As highlighted in the introduction, the study of *special metrics* in geometry has always been at the center of the research in the field, as said metrics arise as a natural tool to *geometrize* classes of manifolds by helping describe their moduli spaces. In complex geometry, the class of manifolds that have been most investigated in this direction is the class of Kähler manifolds, for which many important and deep results have been obtained through the years. Hence, in this preliminary chapter, we shall first recall what is known in the Kähler case, and then move on to the non-Kähler case, describing how the many results from the Kähler world have somehow inspired and guided towards the definitions, the results and the conjectures that are currently the most studied in non-Kähler geometry, focusing in particular on the ones on which the work of this thesis builds on.

Throughout this chapter, (M, J, ω) will be an n-dimensional complex manifold with J an integrable almost-complex structure and ω (the fundamental form associated to g) a hermitian metric.

1.1 Kähler geometry

We shall start with a definition.

Definition 1.1.1. The metric ω is said to be *Kähler* if it holds $d\omega = 0$.

The reason why this metrics have been studied so much through time is clear by the many interesting characterizations that they have, which portray them as metrics with really special properties. For the various basic results and properties we will recall in this section, the references will always be [Bes] and [Sz], unless differently stated.

The first characterization can be given without recalling any other object, stating that in each point, a Kähler metric osculates the flat metric at second order.

Proposition 1.1.2. *The metric ω is Kähler if and only if for every $p \in M$ there exist coordinates z_i centered at p such that*

$$g_{j\bar{k}}(z) = \delta_{j\bar{k}} + O(|z|^2),$$

where $O(|z|^2)$ denotes a function decaying to zero at least quadratically.

The special coordinates are usually called *geodesic coordinates* or *normal coordinates*, and happen to be very useful for computations and gluing constructions.

In order to give another significant characterization, we shall denote with ∇^{ch} the Chern connection of associated to ω for (M, J). Then we can recall:

Proposition 1.1.3. *A metric ω is Kähler if and only if $\nabla^{ch} \equiv \nabla^{LC}$, where ∇^{LC} is the Levi-Civita connection of ω, i.e. if and only if ∇^{ch} has vanishing torsion tensor.*

This result essentially tells us that the hermitian and riemannian geometry of Kähler manifolds coincide, hence revealing a deeper interaction between the complex structure and the riemannian structure. In particular, it shows a compatibility of the riemannian curvature tensor with the complex structure, and hence tells us that it is still highly interesting in the Kähler setting to study the properties of said tensor the same way is done in riemannian geometry. One particular problem becomes then natural to be considered in Kähler geometry, that is the search for *Einstein metrics*, that in Kähler geometry are usually referred as *Kähler-Einstein metrics*. Kähler-Einstein metrics have been central in Kähler geometry, as they proved to be really a class of special metrics in the sense we discussed, thanks to the following celebrated results, which were originally proved in [A], [Y], [CDS], [Ti1].

Theorem 1.1.4 (Aubin-Yau, Calabi-Yau, Chen-Donaldson-Sun, Tian)**.** *Let M be a compact Kähler manifold and let $c_1(M)$ be its first Chern class. Then*

(i) if $c_1(M) < 0$ we can always find a Kähler-Einstein metric in $c_1(M)$;

(ii) if $c_1(M) = 0$ we can always find a Kähler Ricci-flat (which is Kähler-Einstein) metric in $c_1(M)$;

(iii) if $c_1(M) > 0$, M admits Kähler-Einstein metrics if and only if $(M, -K_M)$ is K-polystable.

For the proof of (i) and (ii), the main idea (given by Calabi, which he used to prove uniqueness of Kähler-Einstein metrics in each given Kähler class) is to rephrase the problem as a *complex Monge-Ampère equation*, and this can be achieved using a highly significant consequence of the existence of Kähler metrics, known as the $\partial\overline{\partial}$*-Lemma*.

Lemma 1.1.5. *On (M, ω) a compact Kähler manifold, every d-exact form is also $\partial\overline{\partial}$-exact.*

This Lemma ensures us that each Kähler class is parametrized by real valued smooth scalar functions, hence searching solutions to a differential equation inside a Kähler class can be reduced to a differential equation for scalar functions instead of tensors. As a final note, it is important to highlight the fact that the $\partial\overline{\partial}$-Lemma is not a characterization of the existence of Kähler metrics, and it was shown that there exist spaces on which the $\partial\overline{\partial}$-Lemma holds, but do not admit Kähler metrics (see [An], Section 2.1.3). Despite this fact, non-Kähler manifolds on which the $\partial\overline{\partial}$-Lemma holds are quite rare, and as of today their existence does not correspond to the existence of a class of metrics with some special property, hence when we will focus on studying certain classes of non-Kähler metrics, we will not have the Lemma available, hence we will have to deal with the complications dued to its absence.

Regarding instead part (iii), the problem was originally referred to as the *Yau-Tian-Donaldson conjecture*, which was initially suggested by Yau, who conjectured the existence of an algebro-geometric stability condition for Kähler-Eisntein metrics to exist in the case of Fano manifolds; as a response, Tian introduced the concept of *K-stability* (in [Ti]) based on Mabuchi's K-energy functional, which was later reformulated by Donaldson (in [D1]) in a purely algebro-geometric way. While this original statement of the conjecture has been solved (by Chen, Donaldson and Sun in [CDS] and independently by Tian in [Ti1]), the conjecture has been extended to the case of constant scalar curvature Kähler (cscK) metrics, and of today it represents one of the main research topics in Kähler geometry.

Conjecture 1.1.6 (Yau-Tian-Donaldson)**.** *A smooth polarised variety (M, L) admits cscK metrics in $c_1(L)$ if and only if it is K-polystable.*

It is signifcant to add that cscK metric are actually a particular case (the same as Kähler-Einstein metrics are a particular case of cscK metrics) of a larger family of metrics satisfying a curvature condition, known as *extremal Kähler metrics*, introduced by Calabi in [C], as critical points of the *Calabi functional*.

We will now move on to the non-Kähler setting, and we will see how the Kähler setting that we briefly presented above, guides the research for canonical metrics.

1.2 Special non-Kähler metrics

In the non-Kähler world, the first challenge encountered is to establish how to identify a metric as *special*. While there is no (apparently) natural choice for such metrics, if we look at the Kähler case we can make out some properties that special metrics should have;

in particular, we can conclude that a metric, in order to be a good candidate special metric, needs to satisfy

- a cohomological condition (e.g. the Kähler condition), and
- a curvature condition (e.g. the Einstein condition).

1.2.1 Cohomological conditions

If we focus first on the cohomological aspect, it is natural to search for a condition that is always satisfied by Kähler metrics, hence a generalization of the Kähler condition. In this direction, many notions have been introduced through the years, thus we shall recall a few of the more interesting ones.

Definition 1.2.1. A metric ω is called

- *balanced* if $d\omega^{n-1} = 0$ (introduced in [M]) ;
- *Gauduchon* if $\partial\overline{\partial}\omega^{n-1} = 0$ (introduced in [Ga2]);
- *pluriclosed* or *strong Kähler with torsion* (SKT) if $\partial\overline{\partial}\omega = 0$ (introduced in [Bi]);
- *astheno-Kähler* if $\partial\overline{\partial}\omega^{n-2} = 0$ (introduced in [JY]);
- *locally conformally Kähler* (LCK) if for all $p \in M$, it exists a neighborhood $U_p \subseteq M$ of p and $f : U_p \to \mathbb{R}$ smooth such that $\omega = e^f\eta$, with η a Kähler metric on U_p (introduced in [Lib1] and [Lib2]).

The main reason why so many notions popped out through the years, is that many of these generalize *independently* the Kähler condition, in the sense that combining this conditions on a metric (or on a manifold) might force the metric (or the manifold) to be Kähler, hence we shall spend some time to discuss what is known about the relations between the above conditions. First of all:

Remark 1.2.2. It is straighforward to notice that every balanced metric is Gauduchon. As a consequence, Gauduchon's *Théorème de l'excentricité nulle*, tells us that in each conformal class, if a balanced metric exists, it is unique up to homoteties.

Now, in order to further compare this metrics, we shall recall the following definition:

Definition 1.2.3. We call *torsion* 1*-form* of the metric ω the 1-form

$$\theta_\omega := \Lambda_\omega d\omega,$$

or equivalently, the 1-form satisfying the equation

$$d\omega^{n-1} = \theta_\omega \wedge \omega^{n-1}.$$

Remark 1.2.4. The torsion 1-form's definition immediately yields the following two explicit expressions
$$\theta = \Lambda_\omega d\omega = Jd^*\omega.$$

The torsion 1-form is useful for the following characterization, from which is once again clear the implication between the two.

Proposition 1.2.5. *A metric ω is*

- *Gauduchon if and only if θ_ω is co-closed;*
- *balanced if and only if θ_ω vanishes.*

This characterization is also useful to compare balanced metrics with LCK metrics, indeed the latter have a similar characterization that is

Proposition 1.2.6. *The metric ω is LCK if and only if it exists a closed 1-form θ such that*
$$d\omega = \theta \wedge \omega.$$
The form θ is called Lee form.

While it may seem confusing the choice of the letter θ to indicate also the Lee form, it is actually natural as the two objects coincide (up to a constant factor) when they both exist. Hence it easily follows that

Proposition 1.2.7. *If a metric ω is both balanced and LCK, then it is Kähler.*

There is also a very similar statement about balanced and SKT metrics, which instead has a more delicate proof, that is

Proposition 1.2.8 ([AI]). *Let ω be a hermitian metric on an n-dimensional complex manifold. Then it holds*

(i) $\langle i\partial\overline{\partial}\omega, \omega^2\rangle_\omega = |\theta|^2_\omega - |\partial\omega|^2_\omega + d^*\theta$;

(ii) $|\theta \wedge \omega|^2_\omega = (n-1)|\theta|^2_\omega$

In particular, it follows that

- *if ω is both balanced and pluriclosed, then ω is Kähler;*
- *if ω is both SKT and LCK, and $n \geq 3$, then ω is Kähler.*

Proof. (i) Since in our work in the following chapters we will be using the formula only in the balanced case, we shall prove it for ω balanced, making the computation much more straightforward (the following proof was done by Popovici in [Pop]). Indeed, it is easy to notice that

$$\langle i\partial\overline{\partial}\omega, \omega^2\rangle_\omega \frac{\omega^n}{n!} = -i\partial\omega \wedge \overline{\partial}\omega \wedge \frac{\omega^{n-3}}{(n-3)!}.$$

Moreover, when ω is balanced, Remark 1.2.4 and Proposition 1.2.5 tell us that $\partial\omega$ is a primitive form, hence the formula for the Hodge-star operator of primitive forms (see [Vo]) gives us that

$$*\partial\omega = i\partial\omega \wedge \frac{\omega^{n-3}}{(n-3)!},$$

from which combined with the previous formula gives

$$\langle i\partial\overline{\partial}\omega, \omega^2\rangle_\omega = -|\partial\omega|^2_\omega.$$

(ii) Using the representation of $\mathfrak{sl}(2,\mathbb{C})$ on the algebra of complex differential forms, and the fact that Λ_ω vanishes on 1-forms, we get

$$|\theta \wedge \omega|^2_\omega = \langle L\theta, L\theta\rangle_\omega = \langle \Lambda_\omega L\theta, \theta\rangle_\theta = (n-1)|\theta|^2.$$

$\square$

This type of results, and the many constructions of these metrics in the literature, naturally lead to extend the compatibility problem to the complex structure, that is: given a complex manifold (M, J) not admitting Kähler metrics, can we find two metrics - compatible with J - each one satisfying some special cohomological property? This type of problems have been central in non-Kähler geometry, as they could lead to a much better understanding of the non-Kähler world, leading to some sort of *orthogonal decomposition* of said world, hence they are extremely interesting. A very recent result in this sense involves a special subclass of LCK manifolds introduced in [V].

Definition 1.2.9. An LCK metric is called *Vaisman* if its Lee form is Levi-Civita parallel.

Then it holds:

Theorem 1.2.10 (Angella, Otiman [AO])**.** *A compact Vaisman manifold admitting SKT metrics or astheno-Kähler metrics or balanced metrics admits also Kähler metrics.*

This type of results however tend to be quite hard to achieve, and one of the most interesting compatibility problem is still open (originally proposed in [FV]), that is

Conjecture 1.2.11 (Fino, Vezzoni)**.** *A compact complex manifold admitting a balanced metric and a SKT metric admits also Kähler metrics.*

Many examples constructed in the literature have been proved to satisfy the conjecture (in Chapter 5 we will show that it holds true for the class of balanced manifolds we have constructed with Podestà), however the result is still far from being proved, even in restricted classes of compact complex manifolds.

As opposed to this results, it was also shown that some of the above conditions can coexist on a compact complex manifold, thus we shall summarize a few of them in the following remark.

Remark 1.2.12. It exists a compact complex manifold (M, J) without Kähler metrics, admitting

- SKT metrics and Gauduchon metrics, which can also be satisfied at the same time by only one metric, for example every SKT left-invariant metric is also Gauduchon;
- balanced metrics and astheno-Kähler metrics, such as in [FGV] and [LU];
- SKT metrics and LCK metrics, for example the Inoue-Bombieri surface; it is however expected (see Remark 3.2.1 in [O]) that in dimension at least three, the existence of this two type of metrics forces the existence of Kähler metrics.

It is thus clear how wild the non-Kähler world appears, and how difficult it is to establish which class of metrics might be the "best metrics".

1.2.2 Curvature conditions

As anticipated at the beginning of the section, the second class of conditions that can help identify a metric as special are curvature conditions. However, the non-Kähler setting presents itself immediately with a complication: there is no canonical choice of connection as in the Kähler case; indeed we can find a full line of connections compatible with both the metric and the complex structure, usually referred to as *canonical* 1*-parameter family of hermitian connections*, and said connections are sometimes called *Gauduchon connections*, as they were introduced by Gauduchon in [Ga4]. The definition is the following.

Definition 1.2.13. The canonical 1-parameter family of hermitian connections on (M, J, ω) is given by

$$\nabla^t := \nabla^{ch} + \frac{t-1}{4}\left(d^c\omega + \mathfrak{M}(d^c\omega)\right), \quad t \in \mathbb{R},$$

where $\mathfrak{M}$ is the involution $\mathfrak{M}(B)(X, Y, Z) := B(X, JY, JZ)$.

While it is clear that if ω is Kähler, then this line reduces to a point, consisting of the Levi-Civita connection, when the metric is not Kähler, all this connections are different choices that extend the Kähler case to the non-Kähler setting. Thus, a natural question to ask is whether there is a best choice of connection for an established cohmologically special class of metrics. As we will see, our focus will be centered on balanced metrics, and when it comes to these metrics, the connection that is usually considered is the Chern connection (corresponding to $t = 0$), and sometimes also the Strominger-Bismut connection (corresponding to $t = -1$).

We shall now spend a little bit of time recalling what is known about the curvature tensor of the Chern connection, in order to establish what conditions appear as effective to identify a canonical metric.

The first thing to recall is that the Chern connection is identified by the torsion being of type $(1,1)$, hence not necessarily zero, thus, when the metric is not Kähler we lose the symmetries that the vanishing of the torsion gives, and hence there are four possible ways to trace the Chern curvature tensor Θ. However, we will focus just on two out of the three, since they are the ones that actually have a significant geometric meaning. Following [ACS2], we recall

Definition 1.2.14. If $\Theta = \Theta_{i\bar{j}k\bar{l}}$ is the Chern curvature tensor of ω, we call

- *first Chern-Ricci form* (or just *Chern-Ricci form*) the trace taken on the third and fourth indices, i.e.
$$Ric^{ch}(\omega) \overset{\text{loc.}}{:=} g^{k\bar{l}}\Theta_{i\bar{j}k\bar{l}},$$
which extends globally;

- *second Chern-Ricci form* the trace taken on the first and second indices i.e.
$$S(\omega) \overset{\text{loc.}}{:=} g^{i\bar{j}}\Theta_{i\bar{j}k\bar{l}},$$
which also extends globally.

Remark 1.2.15. The first Chern-Ricci form is always closed and represents the first Bott-Chern class of the manifold, hence it presents itself as the "nearest" generalization of the Kähler-Ricci form to the non-Kähler setting, which is the reason why we just refer to it as *Chern-Ricci form*. However, we will see that also the second Chern-Ricci form carries important geometric information, but more related to the bundle geometry of the holomorphic tangent bundle, which we shall discuss in the next section.

This definitions lead naturally to a rephrasing, in terms of the Chern connection, of the Einstein metrics problem, hence we can define:

Definition 1.2.16. A metric ω is called

- Chern-Einstein if $Ric^{ch}(\omega) = \lambda\omega$ for some $\lambda \in \mathbb{R}$;
- Hermite-Einstein if $S(\omega) = \lambda\omega$ for some $\lambda \in \mathbb{R}$.

Remark 1.2.17. Since the Chern-Ricci form is closed, it is clear that a Chern-Einstein metric with $\lambda \neq 0$ is Kähler, thus the only Chern-Einstein metrics that are significant in non-Kähler geometry are the Chern-Ricci flat ones. A natural generalization could be to consider a weakened version of Chern-Einstein metrics, where λ is a real valued function, but Angella, Calamai and Spotti showed in [ACS2] that every *weak Chern-Einstein metric* with λ not identically zero is conformal to a Kähler metric, showing that even with this generalization, the only significant case remains the Chern-Ricci flat one.

The search for Chern-Ricci flat metrics has been quite active, and results in this direction where obtained for example by Tosatti and Weinkove in [TW], and by Székelyhidi, Tosatti and Weinkove in [STW], where they where able to obtain Chern-Ricci flat Gauduchon metrics on any compact complex manifold with vanishing first Bott-Chern class, i.e. metrics that satisfy both a cohomological condition and a curvature condition, hence metrics that present as "candidate" canonical non-Kähler metrics. As we will see in the next chapter, our interest will be centered on Chern-Ricci flat balanced metrics, which satisfy a stronger cohomological constraint, and hence give a better hope of having a finite dimensional moduli space (which Chern-Ricci flat Gauduchon metrics don't have), together with the relation to the Hull-Strominger system, which we shall discuss in the last section of this chapter.

As it happens in Kähler geometry, it is also interesting to consider a notion of *scalar curvature*, and luckily, there is a natural one since tracing both the Chern-Ricci forms lead to the same scalar function, hence

Definition 1.2.18. The *Chern scalar curvature* of a metric ω is

$$s^{ch}(\omega) \overset{\text{loc.}}{:=} g^{i\bar{j}} g^{k\bar{l}} \Theta_{i\bar{j}k\bar{l}},$$

and extends globally. It can also be written globally as

$$s^{ch}(\omega) = \frac{Ric^{ch}(\omega) \wedge \omega^{n-1}}{\omega^n}.$$

Thus, since csck metrics are central in the study of canonical metrics in Kähler geometry (as we recalled in the first section , we are interested in combining this curvature condition together with a cohomological one.

While the problem of prescribing the Chern scalar curvature has been studied with significant results in [ACS1] and [Fus], its combination with a cohomological condition does not have a very wide literature (a partial result in this sense was obtained by Shen in [Sh], with metrics satisfying a condition strenghtening the SKT one), hence in Chapter 3 and Chapter 5 we will present two constructions to obtain families of Chern scalar constant metrics which are also balanced.

1.3 Holomorphic vector bundles

As anticipated in the previous section, we will extend our discussion about canonical metrics to vector bundles, and our main reference will be [LT]. This case appears as interesting and natural when studying canonical metrics on manifolds, for example, as hinted at in the previous section, *second Chern-Einstein metrics* correspond actually to Hermite-Einstein metrics on the holomorphic tangent bundle. One further topic where metrics on bundles are crucial, is the study of the Hull-Strominger system, which - as we will see in the final section of this chapter - is a system of equations aiming to produce canonical metrics on *non-Kähler Calabi-Yau* manifolds, generalizing the Kähler Calabi-Yau setting. With this in mind, shall now recall some interesting facts about these bundles and their canonical metrics.

In this section, $(E, \overline{\partial}_E)$ will be a holomorphic vector bundle on a hermitian manifold (M, J, ω), endowed with a hermitian bundle metric h.

Definition 1.3.1. A metric h on E is said to be *Hermite-Einstein* with respect to ω if $\Lambda_\omega F_h = cId_E$, with $c \in \mathbb{R}$, where F_h is the curvature of the Chern connection of h.

Remark 1.3.2. It is clear that if we choose $E = TM$ with the standard holomorphic structure we get exactly the definition of second Chern-Einstein.

Remark 1.3.3. As for the Chern-Einstein problem, it is natural to consider a weaker version of the Hermite-Einstein equation, choosing c to be a real valued function. However, it is easily seen that if a *weak Hermite-Einstein* metric exists, then it is conformal to an actual Hermite-Einstein metric, thus it is only interesting to focus on the latter ones.

Remark 1.3.4. The constant c in the Hermite-Einstein equation, known as *Einstein constant*, is prescribed by the Hermite-Einstein equation itself, since it always holds

$$c = \frac{2\pi n \int_M \mathrm{tr}(iF_h) \wedge \omega^{n-1}}{\mathrm{rank}(E) \cdot Vol_\omega(M)}.$$

In particular,

- if ω is Gauduchon, c only depends on $c_1(E)$ and the metric ω;
- if ω is Kähler, c only depends on $c_1(E)$ and $[\omega]$, i.e. its purely cohomological.

The Hermite-Einstein equation can also be rephrased in a gauge-theoretical way, starting from the following definition.

Definition 1.3.5. A connection ∇ on E is called *Hermitian-Yang-Mills* if it satisfies

$$F_{\nabla}^{0,2} \equiv, 0$$
$$\Lambda_\omega F_\nabla = cId_E.$$

With this notion, we can produce a new equation for a given hermitian bundle $(E, h, \overline{\partial}_E)$, which can be shown to be equivalent to the Hermite-Einstein equation, that is

$$\Lambda_\omega F_{f\cdot\overline{\partial}_E} = cId_E,$$

for $f \in \mathcal{G}$, the complex gauge group of E, where $F_{f\cdot\overline{\partial}_E}$ denotes the curvature of the Chern connection of h with respect to the compatible complex structure $f \cdot \overline{\partial}_E := f^{-1} \circ \overline{\partial}_E \circ f$. To be more specific with the equivalence, what can be shown is that for a given hermitian bundle $(E, \overline{\partial}_E, h)$ we can find a Hermite-Einstein metric h' if and only if we can find $f \in \mathcal{G}$ such that the Chern connection of $(h, f \cdot \overline{\partial}_E)$ is Hermitan-Yang-Mills. It is thus common to mix the terminology and talk about *Hermite-Einstein connections* or *Hermite-Yang-Mills metrics*.

The existence of these metrics on holomorphic vector bundles has been thoroughly studied in the past decades, and was at the center of a conjecture known as *Hitchin-Kobayashi correspondence*, stated in the early '80s and ispired by the result for Riemann surfaces of Narasimhan-Seshadri (in [NaSe]) and proved after a few years by Donaldson (in [D]) and Uhlenbeck-Yau (in [UY]) in the Kähler case, and later extended to the general non-Kähler case by Buchdahl (in [Bu]) and Li-Yau (in [LY2]). Said correspondence, relates the existence of Hermite-Einstein metrics to an algebro-geometric property, called *slope stability*, introduced by Mumford in [Mu]. This notion was originally given only for bundles whose bases are Kähler manifolds (and the conjecture itself was stated originally for the Kähler case), but it was noted that it can be extended for any manifold endowed with a Gauduchon metric, which we can now recall.

Definition 1.3.6. For (M, ω) with ω Gauduchon, we define the $[\omega^{n-1}]_A$-*slope* of a torsion-free coherent sheaf $\mathcal{F}$ the quantity

$$\mu_{[\omega^{n-1}]_A}(\mathcal{F}) = \frac{c_1(E) \cdot [\omega^{n-1}]_A}{\text{rank}(\mathcal{F})}.$$

We then say that a bundle E on M is

- $[\omega^{n-1}]_A$-*slope semistable* if for every subsheaf $\mathcal{F} \subseteq E$ it holds
$$\mu_{[\omega^{n-1}]_A}(\mathcal{F}) \leq \mu_{[\omega^{n-1}]_A}(E);$$
- $[\omega^{n-1}]_A$-*slope stable* if for every subsheaf $\mathcal{F} \subseteq E$ it holds
$$\mu_{[\omega^{n-1}]_A}(\mathcal{F}) < \mu_{[\omega^{n-1}]_A}(E);$$
- $[\omega^{n-1}]_A$-*polystable* if
$$E = \bigoplus_i E_i,$$
with E_i stable and $\mu_{[\omega^{n-1}]_A}(E_i) = \mu_{[\omega^{n-1}]_A}(E_j)$ for all i, j.

This extended definition allowed to extend the conjecture to the non-Kähler setting. Moreover, it holds

Lemma 1.3.7 (Lemma 2.1.5 from [LT])**.** *If the bundle E on (M, ω) admits a Hermite-Einstein metric with respect to ω, then it admits a Hermite-Einstein metric with respect to any metric conformal to ω.*

Thus, thanks to the fact that every conformal class admits a Gauduchon metric, the Hitchin-Kobayashi correspondence makes sense for any compact complex manifold, which - as mentioned - was proved in this general setting by Li and Yau in [LY2].

Such a result is, in analogy with the existence theorems for Kähler-Einstein metrics, exactly the type of result highlighting the power that canonical metrics have when it comes to classification, as the Hitchin-Kobayashi correspondence has been a crucial tool in the study of moduli spaces of holomorphic vecotor bundles over Kähler manifolds. However, Hermite-Einstein metrics are not only significant in Kähler geometry, but are also central in problems for non-Kähler geometry, one of which is the Hull-Strominger system, which - as anticipated - we will discuss about in the final section of this chapter.

1.4 Balanced metrics

As previously stated, the class of cohomologically special metrics we are mostly interested in are balanced metrics, hence in this section we shall recall some important results related to this class of metrics, as well as aspects that are highly significant to the research work in this thesis. We shall also discuss some open problems involving balanced metrics, in order to motivate our interest towards this class of metrics and their importance in complex non-Kähler geometry.

The first property we will recall is the following characterization from [Ga1].

Proposition 1.4.1 (Gauduchon)**.** *Let (M, ω) be a compact hermitian manifold. Then ω is balanced if and only if on scalar functions holds*

$$\frac{1}{2}\Delta_\omega = \Box_\omega = \overline{\Box}_\omega,$$

where Δ_ω and $\Box_\omega$ are respectively the Hodge laplacian and the Dolbeault laplacian of ω.

This condition is extremely important, as it tells us that, despite the absence of a Kähler metric allowing us to have the perfect environment to perform analysis on the manifold, balanced metrics are still sufficiently special to allow us to do analysis with scalar functions exactly as in the Kähler case (in the following chapters, we will widely use this property). This fact in particular gave hope that many results from Kähler geometry achieved through geometric analysis techniques could be obtained also in the balanced setting. Among the others, it stands out Yau's Theorem and its extended version to Gauduchon metrics from Székelyhidi, Tosatti and Weinkove, which was explicitly proposed in [Tos] and it is still an open problem.

Conjecture 1.4.2 (Székelyhidi, Tosatti, Weinkove)**.** *Let (M, ω) be a compact balanced manifold and let $\Psi \in C_1^{BC}(M)$. Then, it exists a balanced metric $\omega_\Psi^{n-1} \in [\omega^{n-1}]_{BC}$ such that $Ric^{ch}(\omega_\Psi) = \Psi$.*

In the case where $c_1^{BC}(M)$ vanishes, this result goes in the direction of *geometrizing the balanced class* through finding a canonical metric in said class, identified by some curvature condition. However, it is still unknown if Chern-Ricci flat balanced metrics could be unique, or at least have finite dimensional moduli space, in a given balanced class. Hence the conjecture itself lies in a wider question that is

Question 1.4.3. *Can we find a condition on the Chern curvature identifying uniquely a canonical metric in a given balanced class?*

Once again, this problem is still far from being solved, and it is one of the reasons why balanced metrics appear as so interesting; in particular the $c_1^{BC}(M) = 0$ appears as particularly interesting in relation to the Hull-Strominger system.

Still regarding analytical aspects, there is also a characterization of the existence of balanced metrics in cohomological/geometric measure theory terms, from [M].

Theorem 1.4.4 (Michelsohn)**.** *A compact complex manifold M is balanced if and only if is* homologically balanced*, i.e. every non-zero d-closed $(n-1, n-1)$-current represents a non-zero class in $H_{2n-2}(M, \mathbb{R})$.*

Among the consequences of this theorem, the most significant one is the celebrated result by Alessandrini and Bassanelli in [AB1] and [AB2], that is:

Theorem 1.4.5 (Alessandrini, Bassanelli)**.** *The class of compact balanced manifolds is closed under proper modifications.*

In particular this shows that this weakening of the Kähler condition makes us gain this closure of the class, that instead does not hold for compact Kähler manifolds, thanks to a counterexample from Hironaka in [Hi].

If we now go back to the balanced condition in its original form, we can conclude this section seeing that, thanks to the properties of the Hodge-$*$ operator, it is easily seen that balanced condition for a metric ω is equivalent to

$$d^*\omega = 0$$

i.e. ω is *co-closed*. This suggests that the balanced condition might be - in some sense - *dual* to the Kähler condition. An example of this behaviour is in the following result from [M], showing that, while Kähler metrics are induced on submanifolds, balanced metrics are induced through submersions.

Proposition 1.4.6 (Michelsohn)**.** *Let M and N be compact complex manifolds.*

(i) If M and N are balanced, then $M \times N$ is also balanced.

(ii) If exists $f : M \to N$ a holomorphic submersion and M is balanced, then also N is balanced.

It is thus interesting to keep investigating balanced metrics in order to search for more results that could confirm further this duality relation with the Kähler condition.

1.5 The Hull-Strominger system

We will now conclude this preliminary chapter recalling the Hull-Strominger system with a brief discussion about its meaning when it comes to canonical metrics, and its role in motivating part of our research work. The main reference for this section are going to be the notes [GF] from Garcia-Fernandez.

The realm in which the system lives is the one of not necessarily Kähler Calabi-Yau manifolds (a relaxed version without this restriction was recently introduced by Gonzalez-Molina and Garcia-Fernandez in [GFGM1] as *coupled Hermitian-Einstein system*), hence we will first recall what we mean by this

Definition 1.5.1. A complex manifold M of dimension n is said to be *Calabi-Yau* if it admits a non-vanishing holomorphic global section Ω of the canonical bundle K_M. The $(n, 0)$-form Ω is called *holomorphic volume form.*

It is clear that, thanks to Yau's theorem, whenever the manifold is Kähler, this guarantees the existence of Kähler Ricci-flat metrics, and hence the classical notion of Kähler Calabi-Yau manifold.

Let us now give a remark about what immediate consequences we have from the existence of the holomorphic volume, showing us in particular that it is natural to study Chern-Ricci flat metrics on these spaces.

Remark 1.5.2. If (M, Ω) is a compact Calabi-Yau manifold, and ω is hermitian metric on M, it holds

$$||\Omega||^2_\omega \frac{\omega^n}{n!} = (-1)^{n(n-1)/2} i^n \Omega \wedge \overline{\Omega}.$$

This relation is extremely significant, as it implies that $\log ||\Omega||_\omega$ is a global $\partial\overline{\partial}$-potential for the Chern-Ricci form, from which it immediately follows that

- every Calabi-Yau manifold has $c_1^{BC}(M) = 0$;
- every hermitian metric ω is conformal to a Chern-Ricci flat metric, given by $\omega' := ||\Omega||^{2/n}_\omega \omega$.

We can now move on and recall the equations of the system.

Definition 1.5.3. Given a Calabi-Yau manifold (M, Ω) and a holomorphic vector bundle E on M, we say that the triple $(\omega,\ h,\ \overline{\partial}_T)$ is a solution of the *Hull-Strominger system* if it satisfies

$$\Lambda_\omega F_h = 0, \quad F_h^{0,2} = 0 \tag{1.1}$$

$$\Lambda_\omega R = 0, \tag{1.2}$$

$$d^*\omega - d^c \log ||\Omega||_\omega = 0, \tag{1.3}$$

$$dd^c\omega - \alpha(\mathrm{tr}R \wedge R - \mathrm{tr}F_h \wedge F_h) = 0, \tag{1.4}$$

where, α is a non-vanishing constant, ω is a hermitian metric on M, h is a hermitian metric along the fibers of E, $\overline{\partial}_T$ is a holomorphic structure on the tangent bundle of M, and R is the Chern curvature tensor of ω with respect to $\overline{\partial}_T$.

Remark 1.5.4. Notice that if we choose (E, h) to be the holomorphic tangent bundle with the metric ω, and take ω a Kähler Ricci-flat metric, we notice that this satisfies the system, thus being a solution of the Strominger system is a condition that generalizes being Kähler Calabi-Yau, making solutions to the system a promising candidate class of *canonical metrics* for non-Kähler Calabi-Yau manifolds.

In order to understand the link between the system and balanced metrics, we shall recall a result from Li and Yau in [LY1] (and also Gauntlett, Martelli and Waldram in [GMW]). In this result they show that Equation (1.5.3) is equivalent to a simpler equation involving balanced metrics, known as *conformally balanced equation*.

Proposition 1.5.5. *The dilatino equation is equivalent to the conformally balanced equation, i.e.*

$$d\left(||\Omega||_\omega \omega^{n-1}\right) = 0. \tag{1.5}$$

Thus a Calabi-Yau manifold that admits solutions of the dilatino equation has to be necessarily a balanced manifold, whence solutions to the Hull-Strominger system can be searched on the restricted class of balanced manifolds.

Remark 1.5.6. Actually Equation (1.5) allows us to show that every Calabi-Yau balanced manifold (M, Ω), $dim_{\mathbb{C}} M \geq 3$, always admits solutions to the dilatino equation; we can indeed show even more, that is: every balanced metric η is conformal to a solution of Equation (1.5), given by $\eta' = ||\Omega||_\eta^{-2/(n-2)}\eta$.

Remark 1.5.7. Combining Equation (1.5) with Remark 1.5.2, we immediately see that a balanced Chern-Ricci flat metric is always a solution of the dilatino equation. Hence, the system fuels the interest towards this class of metrics, presenting them as possible candidate canonical metrics.

Moreover, Equation (1.5) shows also that a solution ω to the dilatino equation gives a standard choice of a balanced class τ, given by

$$\tau := \left[||\Omega||_\omega \omega^{n-1}\right] \in H^{n-1,n-1}_{BC}(M, \mathbb{R}).$$

This, combined with Equation (1.2), in light of the Hitchin-Kobayashi correspondence, gives us also a necessary condition on the bundle E, that is the τ- slope polystability, together with $c_1(E) \cdot \tau = 0$. And necessary conditions are not over, since Equation (1.5.3), known as the *Bianchi identity*, or *anomaly cancellation equation*, implies that necessarily

$$ch_2(E) = ch_2(M) \in H^{2,2}_{BC}(M, \mathbb{R}), \tag{1.6}$$

where ch_2 denotes the second Chern character.

We shall remark that this necessary conditions were also conjectured to be sufficient, in the case of threefolds, by Yau in [Y1]:

Conjecture 1.5.8. *Let (M, Ω) be a compact Calabi-Yau threefold endowed with a balanced class τ, and let E be a holomorphic vector bundle satisfying $c_1(E) \cdot \tau = 0$ and (1.6). Then, if E is τ-stable then (M, Ω, E) admits a solution to the Hull-Strominger system.*

The conjecture is not even completely clear when M is Kähler; indeed, when τ is the square of a Kähler class, it was shown by Andreas and Garcia-Fernandez in [AGF] that the conjecture holds, and the result was recently strenghtened by Collins, Picard and Yau in [CPY2], where they showed that it holds while preserving the balanced class. However, Fu and Xiao in [FX] showed that a balanced class needs not to be the square of a Kähler class, hence the conjecture remains open in this case, as well as in the case of non-Kähler manifolds.

In the direction of this conjecture, in [GF] it was proposed an intermediate step:

Question 1.5.9 (Garcia-Fernandez)**.** *Given M a compact complex manifold, τ a balanced class and ρ a real $\partial\overline{\partial}$-exact $(2,2)$-form. Is there a balanced metric in τ such that $\partial\overline{\partial}\omega = \rho$?*

In light of Proposition 1.2.8, it is clear that we can not expect an affirmative answer in the general non-Kähler setting, however one can hope to identify a favorable condition that allows to answer positively to the question. Such an answer would be of great support for Yau's conjecture, and thus it keeps high the interest towards the study of balanced manifolds.

We will now end this section (and this preliminary chapter) by presenting an example of a non-Kähler solution of the system constructed by Fu and Yau in [FuY], which will be significant for our work in the next chapter. This solution forgets about Equation (1.5.3), and substitutes it by imposing the standard holomorphic structure on the tangent bundle, making the connection the standard Chern connection of the metric; this assumption clashes with the physical meaning of the system, but makes the problem more accessible, and hence a useful assumption to obtain preliminary (partial) solutions to the full system.

Example 1.5.10 (Fu-Yau)*.* The spaces on which this solutions are construct are the total spaces of a class of torus bundles over $K3$ surfaces, initially constructed by Goldstein and Prokushkin in [GP] as examples of threefolds not admitting Kähler metrics. We then start with (S, ω_{K3}) a $K3$ surfaces endowed with ω_{K3} a Kähler Calabi-Yau metric and Ω_{K3} a holomorphic volume form, and consider ω_1, ω_2 anti-slef-dual $(1,1)$-forms such that

$$[\omega_i/2\pi] \in H^2(S, \mathbb{Z}).$$

We then take X the total space of the fibered product of the $U(1)$ bundles identified by the cohomology classes of ω_1 and ω_2, and set θ a connection on X such that $iF_\theta = \omega_1 + \omega_2$. With this ingredients, we get that

$$\Omega := \Omega_{K3} \wedge \theta$$

defines a holomorphic volume for X, and the metric

$$\omega := p^*\omega_{K3} + \frac{i}{2}\theta \wedge \overline{\theta}$$

is a Chern-Ricci flat balanced metric. From here, we can obtain a family of balanced metrics with the rescaling

$$\omega_u := p^*(e^u \omega_{K3}) + \frac{i}{2}\theta \wedge \overline{\theta}, \tag{1.7}$$

where p is the fibration map. Now, if E is a degree-zero holomorphic vector bundle on S endowed with h_S a Hermite-Einstein metric with respect to ω_{K3}, then p^*E is again a degree-zero holomorphic vector bundle over X, endowed with p^*h_S a Hermite-Einstein metric with respect to ω_u. Hence, for what we have observed about the equations involved, the system reduces to just the Bianchi identity, which - when studied on the family ω_u - reduces to the following Monge-Ampère equation:

$$dd^c(e^u\omega_{K3} - \alpha e^{-u}\rho) + \frac{1}{2}dd^c u \wedge dd^c u = \mu\frac{\omega_{K3}^2}{2},$$

where ρ is a smooth real $(1,1)$-form on S, independent of u, such that

$$\mu\omega_{K3}^2 = (|\omega_1|^2 + |\omega_2|^2)\omega_{K3}^2 + \alpha(\mathrm{tr}F_h \wedge F_h - \mathrm{tr}R_{K3} \wedge R_{K3},$$

where R_{K3} is the Chern curvature of ω_{K3}. Solutions to Equation (1.7) (and hence to the system) are then given by the following result from [FuY]:

Theorem 1.5.11 (Fu-Yau). *Equation* (1.7) *has solutions for* $\alpha > 0$*, provided that*

$$0 = \int_S \mu\omega_{K3}^2 = \int_S (|\omega_1|^2 + |\omega_2|^2)\omega_{K3}^2 - 8\pi^2\alpha(24 - c_2(E)).$$

Chapter 2

Orbifolds and Chern-Ricci flat balanced metrics

In this Chapter we will discuss the paper *A Kümmer construction for Chern-Ricci flat balanced metrics* (see [GS]), from a joint work with Cristiano Spotti. The aim lying behind this work is to produce special non-Kähler metrics on spaces that are relevant for the Hull-Strominger system, obtaining partial solutions to said system. The main Theorem we will prove in this section is the following.

Theorem 2.0.1. *Let $(\tilde{M}, \tilde{\omega})$ be an n-dimensional non-Kähler Calabi-Yau orbifold with a finite family of isolated singularities, endowed with $\tilde{\omega}$ a singular Chern-Ricci flat balanced metric, and let M be a crepant resolution of $\tilde{M}$. Then M admits a Chern-Ricci flat balanced metric $\hat{\omega}$ such that*

$$[\hat{\omega}^{n-1}] = [\tilde{\omega}^{n-1}] + (-1)^{n-1}\varepsilon^{(2n-2)}(\sum_{i=1}^{k_j}\sum_{j=1}^{k} a_j^i PD[E_j^i])^{n-1},$$

where $PD[E_j^i]$ denotes the Poincaré dual of the class $[E_j^i]$.

2.1 The pre-gluing metric

Following several known gluing constructions from the literature (such as [AP], [BM], [J] and many others), our gluing process will be made of two main parts: the construction of a pre-gluing metric (which will be done in this section) obtained from a rough cut-off procedure providing an approximate solution to the problem, and a perturbative argument to obtain a genuine solution.

The goal of this section will be to prove the following:

Proposition 2.1.1. *Let $\tilde{M}$ be a Calabi-Yau orbifold with a finite family of isolated singularities, endowed with a Chern-Ricci flat balanced singular metric $\tilde{\omega}$, and suppose that it admits M a crepant resolution. Then M admits ω an approximately Chern-Ricci flat balanced metric.*

2.1.1 Chern-Ricci flat balanced orbifolds and their crepant resolutions

Before discussing the construction, we shall establish some notations for the reminder of the paper, and also use the occasion to briefly recall some known results from literature to understand better the framework we will be working in.

Throughout the paper we will denote with $\tilde{M}$ an n-dimensional non-Kähler Calabi-Yau orbifold, i.e. a complex orbifold endowed with a holomorphic volume form $\tilde{\Omega}$, with a finite family of isolated singularities, such that it admits a crepant resolution M.

Remark 2.1.2. A necessary condition for an orbifold to admit crepant resolutions is that the isotropy groups corresponding to the singularities are subgroups of $SL(n, \mathbb{C})$, and for $n = 3$ it is also sufficient (see [J]), making it a useful criterion to search for examples.

Remark 2.1.3. The exceptional set of a crepant resolution of an orbifold singularity is always divisiorial, i.e. in codimension 1. Indeed, it is known that orbifold singularities are "mild", meaning that (see for example [KM]) every orbifold is normal and $\mathbb{Q}$-factorial. But the existence of a (quasi-projective) small resolution would imply that the orbifold is not $\mathbb{Q}$-factorial, i.e. a contradiction.

We will also assume that $\tilde{M}$ is equipped with a singular balanced Chern-Ricci flat metric $\tilde{\omega}$, and thus it is worth giving examples of spaces that satisfy our assumptions, in order to ensure that we are working on an actually existing class of spaces.

Example 2.1.4. A first, trivial example is the one of quotients of tori with isolated orbifold singularities of the form $\mathbb{C}^3/\mathbb{Z}_3$. In these cases, we know that the quotient is equipped with a singular Kähler Calabi-Yau metric, and D. Joyce (in [J], for example) has shown that also their crepant resolutions admit Kähler Calabi-Yau metrics, which can be obtained via gluing construction in the same fashion as the one we are about to present. However, since every Kähler Ricci-flat metric is also balanced Chern-Ricci flat, we can still consider these spaces in our class, and - as we will se ahead - our construction does not ensure that the Chern-Ricci flat balanced metric obtained need to coincide with the Kähler Calabi-Yau, since the cohomology class preserved is going to be the balanced one, on which there are no known uniqueness results.

A possible variation on this argument could be to apply the (orbifold version of) the result of Tosatti and Weinkove in [TW1], which ensures us that we can find a Chern-Ricci flat balanced metric on the singular quotient of the torus, and thus provides a suitable metric for our construction.

Example 2.1.5. A more interesting example can be obtained on torus bundles on some algebraic K3 surfaces. Indeed, Goldstein and Prokushkin produced in [GP] a family of T^2 bundles on $K3$ surfaces that do not admit Kähler metrics; and they showed that these threefolds can be endowed with a balanced Chern-Ricci flat metric of the form

$$\eta = \pi^*\eta_{K3} + \frac{i}{2}\theta \wedge \overline{\theta},$$

where η_{K3} is the Calabi-Yau metric on the $K3$, and θ is a $(1,0)$-form arising from the duals of the horizontal lift of the coordinate vector fields on the $K3$. These bundles X inherit also a non-Kähler Calabi-Yau structure, i.e. a holomorphic volume form given by

$$\Omega = \Omega_{K3} \wedge \theta.$$

Now, while these are the building blocks of the Fu and Yau solutions for the Hull-Strominger system (see [FuY]), Becker, Tseng and Yau constructed (in [BTY], Section 6) a $\mathbb{Z}_3$ action on a subclass of the aforementioned torus bundles for some special choices of algebraic $K3$'s, of the form

$$\rho : (z_0, z_1, z_2, z_3, z_4, z) \longrightarrow (\zeta^2 z_0, \zeta^2 z_1, \zeta z_2, z_3, z_4, \zeta^2 z),$$

with ζ a cubic root of unity different from 1, and where the z_is are the homogeneous coordinates of the $\mathbb{P}^3$ in which the $K3$ lies, and z is the fiber coordinate. This action, despite not preserving the Calabi-Yau structures of the base and the fibres, it preserves Ω, together with the Chern-Ricci flat balanced metric η, producing an orbifold with 9 isolated singularities of the form $\mathbb{C}^3/\mathbb{Z}_3$, i.e. exactly from the family of orbifolds we are interested in working with.

Example 2.1.6. A further example comes from an action of $\mathbb{Z}_4$ on the Iwasawa manifold, constructed by Sferruzza and Tomassini in [ST]. In said paper they showed that the action of $Z_4 = \langle \sigma \rangle$ on $\mathbb{C}^3$, where

$$\sigma(z_1, z_2, z_3) := (iz_1, iz_2, -z_3),$$

descends to the quotient corresponding the (standard) Iwasawa manifold, producing 16 isolated singular points. Moreover, if we recall the standard coframe of invariant (with respect to the Heisenberg group operation) 1-forms

$$\varphi_1 := dz_1, \quad \varphi_2 := dz_2, \quad \varphi_3 := dz_3 - z_2 dz_1,$$

this can be used to construct a balanced metric

$$\omega := \frac{i}{2}(\varphi_1 \wedge \varphi_{\bar{1}} + \varphi_2 \wedge \varphi_{\bar{2}} + \varphi_3 \wedge \varphi_{\bar{3}}),$$

which descends to a Chern-Ricci flat balanced metric on the Iwasawa manifold, and is clearly invariant through σ, as well as the standard holomorphic volume of $\mathbb{C}^3$. Thus the quotient of the Iwasawa manifold through this action gives again an orbifold satisfying our hypotheses.

Our aim is to work on the crepant resolution M, and obtain via a gluing construction (using Joyce's ALE metrics on the bubble, see [J]) a family of Chern-Ricci flat balanced metrics from $(\tilde{M}, \tilde{\omega})$. In the following we will focus on the construction of the pre-gluing metric on M, that will be an *approximately* Chern-Ricci flat balanced metric. To make the presentation more clear, we will divide the process into three natural steps, and for simplicity assume that $\tilde{M}$ has just one singularity (the process obviously applies analogously to the case in which the singularities are any finite number). We are also going to compute explicitly a holomorphic volume form for M (starting from the one on $\tilde{M}$), since such form is a crucial ingredient for the deformation argument in the following section, as it can be used to obtain a global expression for the Chern-Ricci potential.

2.1.2 Pre-gluing - Step 1

We first glue together the metric $\tilde{\omega}$ with the flat metric ω_o centered at the singularity so that the resulting metric is balanced. This follows actually from the following remark, which holds for any balanced manifold and recovers a weaker version of the strategy used with normal coordinates in the Kähler case.

Lemma 2.1.7. *Given (X, η) an n-dimensional balanced orbifold with isolated singularities, for every $x \in X$ it exists a sufficiently small $\varepsilon > 0$, coordinates z centered at x and a balanced metric η_ε such that*

$$\eta_\varepsilon = \begin{cases} \omega_o & \text{if } |z| < \varepsilon \\ \eta & \text{if } |z| > 2\varepsilon \end{cases},$$

where ω_o is the flat metric around x, and such that $|\eta_\varepsilon|_{\omega_o} < c\varepsilon$ on $\{\varepsilon \leq |z| \leq 2\varepsilon\}$.

Proof. If (X, η) is an n-dimensional balanced orbifold and we fix any point $x \in M$, we can choose coordinates z around x such that, in a sufficiently small neighborhood of the point, it holds

$$\eta = \omega_o + O(|z|),$$

where ω_o is the flat metric in a neighborhood of x in the coordinates z. But now this means that if we take the $n-1$ power we obtain

$$\eta^{n-1} = \omega_o^{n-1} + \alpha,$$

where α is a closed $(n-1, n-1)$-form (thanks to the facts that η is balanced and ω_o is Kähler) such that $\alpha = O(|z|)$. Thus if we restrict to a simply connected neighborhood of x, it exists a form β such that

$$\alpha = d\beta,$$

and it can be chosen to be such that $\beta = O(|z|^2)$, since if we decompose $\beta = \beta_l + \beta_q$, where β_l is the component depending at most linearly on $|z|$ and β_q is the quadratic one, the fact that $\alpha = O(|z|)$ forces $d\beta_l = O(|z|)$ which holds if and only if $d\beta_l = 0$, thus we can always choose $\beta = \beta_q$.

Hence, if we introduce a cut-off function

$$\chi(y) := \begin{cases} 0 & \text{if } y \leq 1 \\ \text{non decreasing} & \text{if } 1 < y < 2 \\ 1 & \text{if } y \geq 2 \end{cases}$$

and call $r(z) := |z|$ the (flat) distance from x, we can take $\chi_\varepsilon(y) := \chi(y/\varepsilon)$ and define

$$\eta_\varepsilon^{n-1} := \omega_o^{n-1} + d(\chi_\varepsilon(r)\beta).$$

Here, the notation η_ε^{n-1} makes sense thanks to [M], since on the gluing region holds

$$|d(\chi_\varepsilon(r)\beta)| \leq |d\chi_\varepsilon||\beta| + |\chi_\varepsilon||d\beta| \leq c\varepsilon,$$

ensuring that $\eta_\varepsilon^{n-1} > 0$. Thus we have obtained a balanced metric η_ε on $X \setminus \{x\}$ which is exactly flat in a neighborhood of x. The same argument applies to the orbifold points after taking a cover chart. $\square$

We shall notice that, by proving the above lemma, we have showed *en passant* an Alessandrini-Bassanelli type of result, that is:

Proposition 2.1.8. *The class of balanced (not necessarily compact) manifolds is closed under blowups at finite families of isolated points.*

Thus we can start from our Chern-Ricci flat balanced metric $\tilde{\omega}$ on $\tilde{M}$ and obtain the corresponding cut-off metric $\tilde{\omega}_\varepsilon$ in a neighborhood of the orbifold singularity x by chosing coordinates z on the orbifold cover chart. For our construction, it will however be more convenient to slightly vary the cut-off function and, for $p > 0$, choose

$$\chi_{\varepsilon,p}(y) := \chi(y/\varepsilon^p)$$

so that the gluing region for $\tilde{\omega}_\varepsilon$ becomes $\{\varepsilon^p < r < 2\varepsilon^p\}$. Also, using again the results in [M], we can notice that, even though we are cutting at the level of $(n-1, n-1)$-forms, we have that on the gluing region the metric keeps being close to the flat metric, indeed:

Remark 2.1.9. Notice that we can choose a basis $\{e_j\}$ of 1-forms diagonalizing simultaneously ω_o (we can actually assume it to be the identity) and $\tilde{\omega}_\varepsilon$; this means that also ω_o^{n-1} and $\tilde{\omega}_\varepsilon^{n-1}$ are diagonal (in the sense of $(n-1, n-1)$-forms, implying that also the term $O(r)$ is necessarily diagonal with respect to this basis. Thus we can write

$$\tilde{\omega}_\varepsilon^{n-1} = \sum_{j=1}^{n} (1 + O(r))\widehat{e_j \wedge Je_j}$$

and applying Michelson's result with $\Lambda_j = 1 + O(r)$, we obtain $\tilde{\omega}_\varepsilon = \sum_{j=1}^{n} \lambda_j e_j \wedge Je_j$, with

$$\lambda_j = \frac{((1 + O(r)) \cdots (1 + O(r)))^{\frac{1}{n-1}}}{1 + O(r)} = 1 + O(r),$$

which implies, again thanks to Michelson's theorem

$$\omega = \sum_{j=1}^{n} (1 + O(r))\, e_j \wedge Je_j = \omega_o + O(r),$$

showing also that $d\omega$ has uniformly bounded norm.

2.1.3 Pre-gluing - Step 2

In this second step we instead perform the gluing between Joyce's Kähler-Ricci flat ALE metric ω_{ALE} and the flat metric ω_o of $\mathbb{C}^n$, on the crepant resolution $\hat{X}$ of the singular model $\mathbb{C}^n/G$, and we will actually be able to do it without losing the Kähler condition. To do this we recall that away from the singularity holds

$$\omega_{ALE} = \omega_o + Ai\partial\overline{\partial}(r^{2-2n} + o(r^{2-2n})),$$

where $A > 0$ is a constant and r is the (flat) distance from the singularity. This suggests introducing a large parameter R and a smooth cut-off function $\chi_R(x) := \chi_2(x/R)$ on

$[0, +\infty)$ such that

$$\chi_2(y) := \begin{cases} 1 & \text{if } y \leq \frac{1}{4}, \\ \text{Non increasing} & \text{if } \frac{1}{4} < y < \frac{1}{2}, \\ 0 & \text{if } y \geq \frac{1}{4}, \end{cases}$$

from which we introduce the family of closed $(1,1)$-forms

$$\omega_R = \omega_o + i\partial\overline{\partial}(\chi_R(r)(r^{2-2n} + o(r^{2-2n}))).$$

Once again, on the gluing region $G_R := \{\frac{R}{4} \leq r \leq \frac{R}{2}\}$ we have

$$|\omega_R - \omega_o|_{\omega_o} \leq |i\partial\overline{\partial}(\chi_R(r)(r^{2-2n} + o(r^{2-2n}))|_{\omega_o} \leq cR^{-2n} \leq cr^{-2n},$$

which clearly implies the positivity of ω_R also on G_R (as long as R is chosen to be sufficiently large) ensuring that ω_R is a Kähler metric on $\hat{X}$ which is exactly flat outside of a compact set.

2.1.4 Pre-gluing - Step 3

In this third and last step we want to glue together the metrics $\tilde{\omega}_\varepsilon$ from Step 1 with the metric ω_R from Step 2 by matching isometrically the exactly conical regions. In order to do this we are going to need to rescale by a constant $\lambda > 0$ the metric on $\hat{X}$, and we will now see that this constant is a geometric constant, since it is dictated by the geometries of the two metrics we are gluing together.

In what follows we will denote with z the coordinates on M_{reg} nearby the singularity and with ζ the coordinates on $\hat{X}$, both given by the identification with the singularity model $\mathbb{C}^n/G$. We then consider the regions

$$C_R := \{R/4 \leq r(\zeta) \leq 2R\} \subseteq \hat{X} \qquad \text{and} \qquad C_\varepsilon := \{\varepsilon^p/4 \leq r(z) \leq 2\varepsilon^p\} \subseteq M_{reg}$$

and define a biholomorphism between them by imposing

$$\zeta = \left(\frac{R}{\varepsilon^p}\right) z.$$

From this expression we have that on the identified region the following identity holds

$$r(\zeta) = r\left(\left(\frac{R}{\varepsilon^p}\right) z\right) = \frac{R}{\varepsilon^p} r(z)$$

which yields $\lambda = \lambda(\varepsilon, R) := \left(\frac{\varepsilon^p}{R}\right)^2$. From this follows $\lambda r^2(\zeta) = r^2(z)$, and thus on the identified conical regions $C'_R := \{R \leq r(\zeta) \leq 2R\} \simeq \{\varepsilon^p \leq r(z) \leq 2\varepsilon^p\} =: C'_\varepsilon$ holds

$$\lambda\omega_o(\zeta) = \omega_o(z), \qquad \text{and consequently} \qquad \lambda\omega_{\varepsilon,R} = \tilde{\omega}_\varepsilon.$$

Hence, λ is the needed rescaling factor, which allows us to define the glued family of balanced metrics on the crepant resolution M as

$$\omega_{\varepsilon,R} := \begin{cases} \lambda\omega_R & \text{on } r(\zeta) \leq R, \\ \omega_o & \text{on } \varepsilon^p \leq r(z) \leq 2\varepsilon^p, \\ \tilde{\omega}_\varepsilon & \text{on } r(z) \geq 2\varepsilon^p. \end{cases}$$

Remark 2.1.10. Notice that this first construction implies an Alessandrini-Bassanelli type result (see [AB1]) since it shows that any compact complex manifold bimeromorphic to a balanced orbifold with isolated singularities is also balanced.

In order to understand better the geometry of this new family of metrics, we shall obtain again some estimates on its distance from the flat metric on the gluing region, and since inside said region there is also an exactly flat part - whose geometry is also understood - which separates the two gluing regions from the first two steps, we can just estimate the distance separately on the two regions from the previous steps and then take the maximum.

Clearly, the metric is unaltered on the gluing region from Step 1, thus we still have on G_ε that

$$|\nabla^k_{\omega_o}(\omega - \omega_o)|_{\omega_o} \leq cr^{1-k},$$

for all $k \geq 0$.

On the other hand, since in this step we had to rescale the metric on $\hat{X}$, we have to check how it has affected the distance from the cone. To have clearer estimates, we will express also this one in terms of the small coordinates z, and we will relate the parameters R and ε by chosing $R = \varepsilon^{-q}$, with $q > 0$. We first notice that on G_R (actually the corresponding region through the biholomorphism) it holds

$$\begin{aligned} \langle \omega_{\varepsilon,R} - \omega_o, \omega_{\varepsilon,R} - \omega_o \rangle_{\omega_o}(z) &= \lambda^{-2} \langle \lambda(\omega_R - \omega_o), \lambda(\omega_R - \omega_o) \rangle_{\omega_o}(\zeta) \\ &= \langle \omega_R - \omega_o, \omega_R - \omega_o \rangle_{\omega_o}(\zeta) \end{aligned}$$

implying that $|\omega_{\varepsilon,R} - \omega_o|_{\omega_o}(z) = |\omega_R - \omega_o|_{\omega_o}(\zeta)$. From here, we can recall the estimate done in Step 2 and obtain

$$\begin{aligned} |\omega_{\varepsilon,R} - \omega_o|_{\omega_o}(z) \leq& |\omega_R - \omega_o|_{\omega_o}(\zeta) \\ &\leq cr^{-2n}(\zeta) = c\varepsilon^{2nq} \leq cr^{2nq/p}(z). \end{aligned}$$

which implies, on the whole gluing region, that for all $k \geq 0$ holds

$$|\nabla^k_{\omega_o}(\omega_{\varepsilon,R} - \omega_o)|_{\omega_o} \leq cr^{m-k},$$

where $m = \min\{1, 2nq/p\}$.

2.1.5 The Chern-Ricci potential

In order to use this description of the metrics to estimate the Chern-Ricci potential on the gluing region we are also going to need to understand how the holomorphic volume form of the resolution is related to the holomorphic volume of our background Calabi-Yau orbifold.

Before doing it we start by fixing some notation. Denote

- with $\tilde{\Omega}$ the holomorphic volume of M_{reg} such that
$$\tilde{\omega}^3 = i\tilde{\Omega} \wedge \overline{\tilde{\Omega}};$$
- with $\hat{\Omega}$ the rescaled holomorphic volume of the singularity model $\mathbb{C}^n/G$ (and its crepant resolution $\hat{X}$) in order to match the metric rescaling, i.e. $\hat{\Omega} := \lambda^{3/2}\Omega_o$ where
$$(\omega_{ALE})^3 = i\Omega_o \wedge \overline{\Omega}_o.$$

Now, in a neighborhood of the singularity it exists a holomorphic function h such that

$$\tilde{\Omega} = h\Omega_0.$$

On the other hand, under the rescaling biholomorphism that glues $\hat{X}$ to $M \setminus \{x\}$, we identify the Ω_o around the singularity with $\hat{\Omega}$, thus we can read h as a holomorphic function on the singularity model, and hence holomorphically extend it to a holomorphic function on the whole $\hat{X}$, and thus we can glue together $h\hat{\Omega}$ with $\tilde{\Omega}$ to obtain Ω a holomorphic volume for M.

We can also obtain information on h by noticing that, since $\tilde{\omega}$ is asymptotic to ω_o around the singularity, we obtain that around x it holds

$$(1 + O(|z|))\omega_o^3 = \tilde{\omega}^3 = i\tilde{\Omega} \wedge \overline{\tilde{\Omega}} = |h|^2 i\Omega_o \wedge \overline{\Omega}_o = |h|^2\omega_o^3$$

from which follows

$$|h| = 1 + O(r),$$

from which, by continuity, we have that $|h|^2 \equiv 1$ on the exceptional part.

Thus we can define a global Chern-Ricci potential as

$$f = f_{p,q,\varepsilon} := \log\left(\frac{i\hat{\Omega}\wedge\overline{\hat{\Omega}}}{\omega^3}\right).$$

and conclude this section by describing the behaviour of f in all the regions of M, to show that it is suitable to apply a deformation argument similar to the one done in [BM]. We have

- on $\{r(z) > 2\varepsilon^p\}$ hold $\omega = \tilde{\omega}$ and $\Omega = \tilde{\Omega}$, thus $f \equiv 0$;
- on $\{\varepsilon^p \leq r(z) \leq 2\varepsilon^p\}$ hold $\omega = \omega_o + O(r)$ and $\Omega\wedge\overline{\Omega} = \Omega_0\wedge\overline{\Omega_0} + O(r)$, from which we have
$$f = \log\left(\frac{\omega_o^3 + O(r)}{\Omega_0\wedge\overline{\Omega_0} + O(r)}\right) = \log(1+O(r)) = O(r);$$
- on $\{\frac{1}{2}\varepsilon^p \leq r(z) \leq \varepsilon^p\}$ hold $\omega = \omega_o$ and $\Omega\wedge\overline{\Omega} = i(1+O(r))\Omega_0\wedge\overline{\Omega_0}$, from which follows $f = O(r)$;
- on $\{\frac{1}{4}\varepsilon^p/2 \leq r(z) \leq \frac{1}{2}\varepsilon^p\}$ hold $\omega = \omega_o + O(r^{2nq/p})$ and $\Omega\wedge\overline{\Omega} = \Omega_0\wedge\overline{\Omega_0} + O(r)$, implying $f = O(r^m)$;
- on $\{r(z) < \varepsilon^p/2\}$ hold $\omega^3 = i\Omega_o\wedge\overline{\Omega}_o$ and $\Omega\wedge\overline{\Omega} = i(1+O(r))\Omega_0\wedge\overline{\Omega_0}$, giving once again $f = O(r)$.

Thus we can write globally (on M) that

$$|f| \leq cr^m,$$

ensuring that the metric ω is an *approximately Chern-Ricci flat* balanced metric (as wanted in Proposition 2.1.1), hence a suitable one to perform our gluing construction.

2.2 The deformation argument

In this section we will see that what was built in the previous section are exactly the ingredients we need to introduce a deformation argument in the same fashion as [BM], in order to obtain a balanced Chern-Ricci flat metric on our crepant resolution M. We will also analyze the cohomology class of the metric obtained and see why said metric is interested in the framework of the Hull-Strominger system.

2.2.1 The strategy

We will now set up the problem for this section. First of all we recall the deformation of the metric that preserves the balanced condition introduced in [FWW] (here taken with a particular ansatz):

$$\omega_\psi^{n-1} := \omega^{n-1} + i\partial\overline{\partial}(\psi\omega^{n-2}), \quad \psi \in C^\infty(M, \mathbb{R}) \text{ such that } \omega_\psi^{n-1} > 0.$$

Thus the problem we are interested in solving, following what was done in [BM], is the *balanced Monge-Ampère type* equation

$$\omega_\psi^n = e^f \omega^n \tag{2.1}$$

for $\psi \in C^\infty(M, \mathbb{R})$ such that $\omega_\psi^{n-1} > 0$.

Remark 2.2.1. The equation introduced above makes sense, because, as we've seen, $f = O(r^m)$, thus $e^f = 1 + O(r^m)$, meaning that $e^f\omega^n$ is nearby ω^n itself, hence it makes sense to try to obtain it as a small deformation of ω.

For practicality, it is useful to reformulate our equation as an operator on the space of smooth functions, thus we introduce $F : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ as

$$F(\psi) = F_\varepsilon(\psi) := \frac{\omega_\psi^n}{\omega^n} - e^f.$$

Our aim is then to solve the equation $F(\psi) = 0$ - which is equivalent to (2.1) - through a fixed point argument, hence the first step to take towards this argument is to compute the linearization at 0 of the operator F. To do this we shall introduce the notation $\omega_0' := \frac{d}{dt}_{|t=0}\omega_{tu}$, where ω_{tu} is the curve corresponding to the tangent vector $u \in C^\infty(M, \mathbb{R})$, and compute the derivative at zero of ω_{tu}^n in two different ways:

$$\frac{d}{dt}_{|t=0}\omega_{tu}^n = n\omega^{n-1} \wedge \omega_0';$$

$$\frac{d}{dt}_{|t=0}\omega_{tu}^n = i\partial\overline{\partial}(u\omega^{n-2}) \wedge \omega + \omega^{n-1} \wedge \omega_0'.$$

Even though none of these two expressions are explicit, we can put them together to obtain an explicit one for the linearization, that is

$$Lu := L_\varepsilon u = d_0F(u) = \frac{n}{n-1}\frac{i\partial\overline{\partial}(u\omega^{n-2}) \wedge \omega}{\omega^n}.$$

Here we can work through a few computations to get a clearer (an much more understandable) expression for the operator.

Lemma 2.2.2. *The linearized operator L can be written as*

$$Lu = \frac{1}{n-1}\left(\Delta_\omega u - \frac{1}{n-1}|\partial\omega|^2_\omega u\right) \tag{2.2}$$

for all $u \in C^\infty(M)$.

Proof. For all $n \geq 3$ it holds

$$\begin{aligned} i\partial\overline{\partial}(u\omega^{n-2}) =& i\partial(\overline{\partial}u \wedge \omega^{n-2} + (n-2)u\overline{\partial}\omega \wedge \omega^{n-3}) \\ =& i\partial\overline{\partial}u \wedge \omega^{n-2} - (n-2)i\overline{\partial}u \wedge \partial\omega \wedge \omega^{n-3} + (n-2)i\partial u \wedge \overline{\partial}\omega \wedge \omega^{n-3} \\ &+ (n-2)ui\partial\overline{\partial}\omega \wedge \omega^{n-3} - (n-2)(n-3)ui\overline{\partial}\omega \wedge \partial\omega \wedge \omega^{n-4}, \end{aligned}$$

and since the balanced condition $d\omega^{n-1} = 0$ implies $\partial\omega \wedge \omega^{n-2} = 0$, we get

$$\begin{aligned} &i\partial\overline{\partial}(u\omega^{n-2}) \wedge \omega \\ &= i\partial\overline{\partial}u \wedge \omega^{n-1} + (n-2)ui\partial\overline{\partial}\omega \wedge \omega^{n-2} - (n-2)(n-3)ui\overline{\partial}\omega \wedge \partial\omega \wedge \omega^{n-3} \\ &= \frac{1}{n}(\Delta_\omega u)\omega^n + (n-2)u(i\partial\overline{\partial}\omega \wedge \omega^{n-2} - (n-3)i\overline{\partial}\omega \wedge \partial\omega \wedge \omega^{n-3}). \end{aligned}$$

Now, applying the operator ∂ to the identity $\overline{\partial}\omega \wedge \omega^{n-2} = 0$, we get

$$0 = i\partial\overline{\partial}\omega \wedge \omega^{n-2} - (n-2)i\overline{\partial}\omega \wedge \partial\omega \wedge \omega^{n-3},$$

that is

$$i\partial\overline{\partial}\omega \wedge \omega^{n-2} = (n-2)i\overline{\partial}\omega \wedge \partial\omega \wedge \omega^{n-3},$$

giving us

$$i\partial\overline{\partial}(u\omega^{n-2}) \wedge \omega = \frac{1}{n}(\Delta_\omega u)\omega^n + (n-2)ui\partial\omega \wedge \overline{\partial}\omega \wedge \omega^{n-3}.$$

On the other hand, we can recall Proposition 1.2.8, which in the balanced case gives us

$$-|\partial\omega|^2_\omega \frac{\omega^n}{n!} = i\partial\omega \wedge \overline{\partial}\omega \wedge \frac{\omega^{n-3}}{(n-3)!},$$

from which we finally obtain the *linearized balanced Monge-Ampère type* operator

$$Lu = \frac{1}{n-1}\left(\Delta_\omega u - \frac{1}{n-1}|\partial\omega|^2_\omega u\right),$$

and we can clearly notice that it is bounded (using Remark 2.1.9) and L^2-self adjoint. □

Proposition 2.2.3. *The linear operator L introduced above has vanishing kernel on any n-dimensional ($n \geq 3$) compact balanced manifold (X, η), with η not Kähler.*

Proof. If $u \in \operatorname{Ker} L$, then also $uLu = 0$, which integrated on X gives us (using the balanced condition)

$$0 = \int_M \left(-u\Delta_\eta u + \frac{1}{n-1}|\partial\eta|^2_\eta u^2 \right) \eta^n = \int_M \left(|\nabla_\eta u|^2 + \frac{1}{n-1}|\partial\eta|^2_\eta u^2 \right) \eta^n,$$

from which necessarily

$$\begin{cases} |\nabla_\eta u| \equiv 0 \\ |\partial\eta|^2_\eta u^2 \equiv 0 \end{cases} \Leftrightarrow \begin{cases} u \equiv c \in \mathbb{R} \\ c^2 |\partial\eta|^2_\eta \equiv 0, \end{cases}$$

which implies, thanks to $\partial\eta \neq 0$, that $c = 0$, and hence $u \equiv 0$, i.e. L has vanishing kernel. □

Notice that the fact that the metric is not Kähler is crucial for the proof, since the non-vanishing of $\partial\eta$ ensures that the constants do not lie in the kernel of the operator.

2.2.2 Weighted analysis

Our aim is now to study the invertibility of the linear operator L, and we wish to do this in suitable weighted functional spaces. In order to introduce said spaces we shall start by introducing a weight function useful in our situation, and for simplicity we may assume that the neighbourhood of x on which the z coordinates are defined contains the region $\{r(z) \leq 1\}$ (this is true up to a rescaling). Define then

$$\rho = \rho_\varepsilon(z) := \begin{cases} \varepsilon^{p+q} & \text{on } r(z) \leq \varepsilon^{p+q}, \\ \text{non decreasing} & \text{on } \varepsilon^{p+q} \leq r(z) \leq 2\varepsilon^{p+q}, \\ r(z) & \text{on } 2\varepsilon^{p+q} \leq r(z) \leq 1/2, \\ \text{non decreasing} & \text{on } 1/2 \leq r(z) \leq 1, \\ 1 & \text{on } r(z) \geq 1, \end{cases}$$

Using this weight function we can introduce the weighted Hölder norm and its corresponding weighted Hölder spaces $C^{k,\alpha}_{\varepsilon,b}(M)$, where $k \geq 0$, $\alpha \in (0,1)$ is the Hölder constant, $b \in \mathbb{R}$ is the weight and ε indicates the dependence on the metric ω obtained by the gluing construction done above. We define

$$||u||_{C^{k,\alpha}_{\varepsilon,b}(M)} := \sum_{i=0}^{k} \sup_M |\rho^{b+i}\nabla^i_\varepsilon u|_\omega + \sup_{d_\varepsilon(x,y)<inj_\varepsilon} \left| \min\left(\rho^{b+k+\alpha}(x), \rho^{b+k+\alpha}(y)\right) \frac{\nabla^k_\varepsilon u(x) - \nabla^k_\varepsilon u(y)}{d_\varepsilon(x,y)^\alpha} \right|_\omega,$$

where inj_ε is the injectivity radius of the metric ω, and thus interpret F (and L) as operators defined as $F : C^{2,\alpha}_{\varepsilon,b}(M) \to C^{0,\alpha}_{\varepsilon,b+2}(M)$.

Following then the literature, we first wish to prove the following estimate.

Lemma 2.2.4. *With the same notations as above, for every $b \in (0, n-1)$ it exists $c > 0$ (independent of ε) such that for sufficiently small ε it holds*

$$||u||_{C^{2,\alpha}_{\varepsilon,b}} \leq c||Lu||_{C^{0,\alpha}_{\varepsilon,b+2}},$$

for all $u \in C^{2,\alpha}_{\varepsilon,b}$.

Proof. Suppose by contradiction that the above inequality does not hold. This means that for all $k \in \mathbb{N}$ we can find $\varepsilon_k > 0$ and $u_k \in C^{2,\alpha}_{\varepsilon_k,b}$ such that $\varepsilon_k \to 0$ as $k \to 0$, $||u_k||_{C^{2,\alpha}_{\varepsilon_k,b}} = 1$ and

$$||Lu_k||_{C^{0,\alpha}_{\varepsilon_k,b+2}} < \frac{1}{k}. \tag{2.3}$$

In the first place we analyze what happens on M_{reg}, i.e. away from the exceptional part. The properties of the sequence $\{u_k\}_{k\in\mathbb{N}}$ guarantee us that we can apply Arzela-Ascoli's Theorem, and hence up to subsequences we may assume $u_k \to u_\infty$ uniformly on compact subsets of M_{reg} in the sense of $C^{0,\alpha}_b$, with respect to $\tilde{\omega}$. Moreover, since for any compact set $K \subseteq M_{reg}$ there exists $n_K \in \mathbb{N}$ such that for all $k \geq n_K$ on K it holds $\omega = \tilde{\omega}$, and hence $\nabla_\omega = \nabla_{\tilde{\omega}}$, we actually have $C^{2,\alpha}_b$-convergence (again uniformly on compact subsets of M_{reg}). We shall then prove that u_∞ is necessarily identically zero on the whole M_{reg}. Indeed, take $\delta > 0$ and B_δ a ball of radius δ around the singularity, and notice that, calling $M_\delta := M \setminus B_\delta$, we get

$$0 = -\int_{M_\delta} u_\infty L_\infty u_\infty \tilde{\omega}^n = \int_{M_\delta} \left(-u_\infty \Delta_{\tilde{\omega}} u_\infty + \frac{1}{n-1}|d\tilde{\omega}|^2_{\tilde{\omega}} u_\infty^2\right)\tilde{\omega}^n, \tag{2.4}$$

and since $\tilde{\omega}$ is balanced it holds

$$d(i\overline{\partial}u_\infty \wedge (u_\infty\tilde{\omega}^{n-1})) = u_\infty i\partial\overline{\partial}u_\infty \wedge \tilde{\omega}^{n-1} + i\partial u_\infty \wedge \overline{\partial}u_\infty \wedge \tilde{\omega}^{n-1},$$

which combined with (2.4) gives

$$0 = \int_{\partial B_\delta} u_\infty i\overline{\partial} u_\infty \wedge \tilde{\omega}^{n-1} + \int_{M_\delta} \left(|\nabla_{\tilde{\omega}} u_\infty|^2 + \frac{1}{n-1} |d\tilde{\omega}|^2_{\tilde{\omega}} u_\infty^2 \right) \tilde{\omega}^n. \tag{2.5}$$

But if we call $d\hat{V}$ the volume form induced by the flat metric, we get

$$\left| \int_{\partial B_\delta} u_\infty i\overline{\partial} u_\infty \wedge \tilde{\omega}^{n-1} \right| \le c \int_{\partial B_\delta} |u_\infty|_\omega |\partial u_\infty|_\omega d\hat{V} \le c\delta^{2(n-1-b)},$$

thus choosing $b < n-1$ and taking the limit for $\delta \to 0$ in (2.5), we get $u \equiv 0$ on M_{reg} by repeating what was done in Remark 2.2.3.

Let now $M_c := \{r(z) \ge 1/2\} \subseteq M_{reg}$ be a compact set on which we know that $u_k \to 0$ uniformly in $C^{2,\alpha}_b$. To obtain a contradiction we want to prove that $\{u_k\}_{k\in\mathbb{N}}$ admits a subsequence uniformly convergent to zero in $C^{2,\alpha}_b$ also on $A := \{r(z) < 1/2\}$.

In order to work in this region, it is simpler to shift to the "large" coordinates ζ, i.e. the coordinates on the crepant resolution $\hat{X}$ away from the exceptional part. It is then useful to recall the relations

$$\zeta = \varepsilon^{-(p+q)} z \quad \text{and} \quad r(z) = \varepsilon^{p+q} r(\zeta),$$

from which we can write down the explicit identification

$$\left\{ r(z) < \frac{1}{2} \right\} = A \simeq \tilde{A} = \tilde{A}_\varepsilon = \left\{ r(\zeta) < \frac{1}{2} \varepsilon^{-(p+q)} \right\} \subseteq \hat{X};$$

this last set $\tilde{A}$ is the one we will be working on.

The first thing to do is rewrite the weight function in terms of this coordinates on $\tilde{A}$, resulting in

$$\rho = \begin{cases} \varepsilon^{p+q} & \text{on } r(\zeta) \le 1, \\ \text{non decreasing} & \text{on } 1 \le r(\zeta) \le 2, \\ \varepsilon^{p+q} r(\zeta) & \text{on } 2 \le r(\zeta) \le 1/2\varepsilon^{-(p+q)}. \end{cases}$$

Notice that the entire gluing region of the metric (from the previous step) is entirely contained inside the third region, i.e. $\{2 \le r(\zeta) \le 1/2\varepsilon^{-(p+q)}\}$.

We now go back to our sequence $\{u_k\}_{k\in\mathbb{N}}$. Since $||u_k||_{C^{2,\alpha}_{\varepsilon_k,b}} = 1$ for all $k \in \mathbb{N}$, we have in particular that on all $\tilde{A}_k := \tilde{A}_{\varepsilon_k}$ holds

$$|\rho^b u_k| \le c.$$

Introducing then the new sequence

$$U_k := \varepsilon_k^{b(p+q)} u_k,$$

the above weighted estimates for u_k imply the following ones for this new sequence:

$$\begin{cases} |U_k| \le c & \text{on } r(\zeta) \le 1, \\ |U_k| \le c & \text{on } 1 \le r(\zeta) \le 2, \\ |U_k| \le c r^{-b}(\zeta) & \text{on } 2 \le r(\zeta) \le 1/2\varepsilon_k^{-(p+q)}. \end{cases}$$

These estimates for U_k suggest us to introduce a new weight function $\tilde{\rho} = \tilde{\rho}_k$ on $\tilde{A}_k$ given by

$$\tilde{\rho}(\zeta) = \begin{cases} 1 & \text{on } r(\zeta) \le 1, \\ \text{non decreasing} & \text{on } 1 \le r(\zeta) \le 2, \\ r(\zeta) & \text{on } 2 \le r(\zeta) \le 1/2\varepsilon_k^{-(p+q)}, \end{cases},$$

with which we get that

$$|\tilde{\rho}^b U_k| \le c, \tag{2.6}$$

and analogous weighted estimates also for ∇U_k and $\nabla^2 U_k$, hence again by Ascoli-Arzelà theorem we have that $U_k \to U_\infty$ uniformly on compact sets of $\hat{X}$ (since $\tilde{A}_k \to \hat{X}$) in the sense of $\tilde{C}_b^{2,\alpha} = C_b^{2,\alpha}(\tilde{\rho})$, where this last space is the weighted Hölder space on $\hat{X}$ identified by the weight $\tilde{\rho}$ and the metric ω_{ALE}.

On the other hand, on any compact subset of $\hat{X}$, for sufficiently large k it holds

$$\rho^{b+2} L u_k = \tilde{\rho}^{b+2} \Delta_{\omega_{ALE}} U_k, \tag{2.7}$$

and since $\frac{1}{k} > ||\tilde{L} u_k||_{C^{0,\alpha}_{\varepsilon_k, b+2}}$, taking the limit in (2.7) we obtain that U_∞ is harmonic with respect to the ALE metric ω_{ALE}. Moreover, taking the limit in (2.6) ensures us that U_∞ decays at infinity, from which follows that $U_\infty \equiv 0$ on the whole $\hat{X}$, and thus $U_k \overset{\tilde{C}_b^{2,\alpha}}{\to} 0$ uniformly on compact sets of $\hat{X}$.

If we are now able to prove that U_k admits a subsequence converging uniformly to zero on the whole $\hat{X}$ in the sense $\tilde{C}_b^0$ we get our contradiction, and we are done. Indeed, if $U_k \overset{\tilde{C}_b^0}{\to} 0$ uniformly (up to subsequences) on $\hat{X}$, then scaled Schauder estimates imply that also $U_k \overset{\tilde{C}_b^{2,\alpha}}{\to} 0$ uniformly, which is the same as saying $u_k \overset{C^{2,\alpha}_{\varepsilon_k, b}}{\to} 0$ uniformly on $\{r(z) < 1/2\}$. Thus $\{u_k\}_{k \in \mathbb{N}}$ up to subsequences is uniformly convergent to zero on the whole manifold M, which is a contradiction with the fact that $||u_k||_{C^{2,\alpha}_{\varepsilon_k, b}} = 1$ for all $k \in \mathbb{N}$.

Now we will prove that the said uniformly convergent subsequence exists. If by contradiction this was not the case, since we have the uniform convergence on compact sets, we would be able to find $\delta > 0$ and $\{x_k\}_{k\in\mathbb{N}} \subseteq \hat{X}$, $x_k \in \tilde{A}_k$, such that $R_k := r(\zeta(x_k)) \to +\infty$ and $R_k^b U_k(x_k) \geq \delta$ for all $k \in \mathbb{N}$, and since $R_k \to +\infty$, we can actually assume $\tilde{\rho} \equiv r$ on the points of the sequence, from which we get that for all $k \in \mathbb{N}$ holds

$$R_k^b |U_k(x_k)| \geq \delta. \tag{2.8}$$

Naming then $r_k := r(z(x_k))$, recalling the relation between the two coordinates we have $\frac{1}{2} \geq r_k = \varepsilon_k^{p+q} R_k$, thus up to subsequences we can end up into two cases:

(i) if $r_k \to l > 0$, then $x_k \to x_\infty$, and since u_k is uniformly convergent on compact sets on M_{reg}, we get that $u_k(x_k)$ is bounded, giving

$$0 < \delta \leq R_k^b U_k(x_k) = (R_k \varepsilon_k^{p+q})^b u_k(x_k) = r_k^b u_k(x_k) \underset{k\to\infty}{\longrightarrow} 0,$$

which is a contradiction;

(ii) if $r_k \to 0$, let $X^* := \hat{X} \setminus E$ the singularity model and X' a copy of X^*, and we consider the biholomorphisms $\sigma_k : B_k \to A \setminus \{0\}$, given by

$$\sigma_k(z') := r_n z',$$

where $B_k := \{0 < r(z') < \frac{r_k^{-1}}{2}\} \subseteq X'$. Then, if we endow B_k with the metric

$$\theta_k := r_k^{-2} \sigma_k^* \omega,$$

it is easy to notice that the couple (B_k, θ_k) converges to (X', ω_{flat}), i.e. the standard singularity model. If we then introduce the functions

$$w_k := r_k^b \sigma_k^* u_k$$

on B_k, we notice that the pullback of the weight function ρ gives

$$\rho'(z') = \sigma_k^* \rho(z') = \begin{cases} \varepsilon_k^{p+q} & \text{on } r(z') < R_k^{-1}, \\ \text{non decreasing} & \text{on } R_k^{-1} \leq r(z') \leq 2R_k^{-1}, \\ r_k r(z') & \text{on } 2R_k^{-1} \leq r(z') < \frac{r_k^{-1}}{2}, \end{cases}$$

from which we get (pulling back the inequality $\rho^b |u_k| \leq 1$)

$$r^b(z') w_k(z') \leq 1 \tag{2.9}$$

on each $z' \in X$ (assuming k to be sufficiently large). Hence, this shows that for any compact $K \subseteq X'$, we can choose $k \in \mathbb{N}$ to be sufficiently large in order to have $K \subseteq B_k$ and $\rho'(z') = r_k r(z')$ on the whole K, and get that w_k is uniformly bounded on K; and since this works for any compact $K \subseteq X'$, we obtain that - up to subsequences - $\{w_k\}_{k\in\mathbb{N}}$ converges uniformly on compact sets of X' to a function w_∞, and from (2.9) we get that w_∞ is decaying at infinity. Moreover, recalling that $R_k^b U_k(x_k) \geq \delta$ for all $k \in \mathbb{N}$, if we introduce the sequence $y_k := \sigma_k^{-1}(x_k)$, it is straighforward to notice that from its definition follows that $w_k(y_k) \geq \delta$ and $||y_k||_{\theta_k} = 1$ for all $k \in \mathbb{N}$, thus implying that - up to subsequences - $y_k \to y_\infty \in X'$, and hence

$$w_\infty(y_\infty) > 0. \tag{2.10}$$

Now, if we recall the definition of the operator L and take the pullback with respect to σ_k of $\rho^{b+2} L u_k$, it is immediate to see that on every compact $K \subseteq X'$ we get

$$\begin{aligned} \sigma_k^* \left(\rho^{b+2} L u_k\right) &= \frac{n}{n-1} r^{b+2}(z') \left(\frac{i\partial\overline{\partial} w_k \wedge \theta_k^{n-1}}{\theta_k^n} + |d\theta_k|^2_{\theta_k} w_k \right) \\ &= \frac{n}{n-1} r^{b+2}(z') \Delta_{\theta_k} w_k + |d\theta_k|^2_{\theta_k} w_k, \end{aligned} \tag{2.11}$$

from which we have, taking the limit as $k \to +\infty$, that

$$\Delta_{\omega_{flat}} w_\infty \equiv 0 \quad \text{on } X',$$

i.e., w_∞ is harmonic on X' with respect to the flat metric. Thus, since it decays at infinity, we obtain $w_\infty \equiv 0$, which is a contradiction as (2.10) holds.

Thus the proof is complete. $\square$

As a direct consequence we get

Lemma 2.2.5. *The operator $L : C^{2,\alpha}_{\varepsilon,b}(M) \to C^{0,\alpha}_{\varepsilon,b+2}(M)$ defined above is a linear isomorphism for every $b \in (0, n-1)$.*

Proof. Notice that L is elliptic and shares its index with the laplacian, which is zero. Moreover, by Proposition 2.2.3 we have that L is injective, thus we automatically get that L is also surjective and - from 2.2.4 - has bounded inverse, thus L is a isomorphism. $\square$

With this result we can now show how to reformulate the original equation as a fixed point problem.

In order to do this we shall consider the expansion

$$F(\psi) = F(0) + L(\psi) + Q(\psi),$$

and thus rewrite the balanced Monge-Ampère type equation as

$$F(0) + L(\psi) + Q(\psi) = 0,$$

and using now Lemma 2.2.5, we get that our equation is therefore equivalent to

$$\psi = L^{-1}(-F(0) - Q(\psi)) =: N(\psi), \tag{2.12}$$

i.e. the search for a fixed point for the operator $N : C^{2,\alpha}_{\varepsilon,b}(M) \to C^{2,\alpha}_{\varepsilon,b}(M)$. To do this, we will have to identify the open set on which we wish to apply Banach's Lemma, and show that on said open set, the operator N can be restricted and gives rise to a contraction.

The first thing to do is the following remark.

Remark 2.2.6. If $C, \tau > 0$, and φ is a function on M such that $||\varphi||_{C^{2,\alpha}_{\varepsilon,-2}} \leq C\varepsilon^\tau$, thanks to Remark 2.2.2 it is straightforward to see that

$$||i\partial\overline{\partial}(\varphi\omega)||_{C^{0,\alpha}_{\varepsilon,0}} \leq ||\varphi||_{C^{2,\alpha}_{\varepsilon,-2}} \leq C\varepsilon^\tau,$$

thus we are guaranteed that, choosing ε to be sufficiently small, $\omega_\varphi^{n-1} > 0$, and thus its $(n-1)$ root ω_φ exists and is a balanced metric. Moreover, we can apply again the argument used in Remark 2.1.9, and obtain that if $||\varphi||_{C^{2,\alpha}_{\varepsilon,-2}} \leq C\varepsilon^\tau$, then

$$|\omega_\varphi - \omega|_\omega \leq c||\varphi||_{C^{2,\alpha}_{\varepsilon,-2}} \leq c\varepsilon^\tau,$$

which also implies that $\omega_\varphi \to \omega$, as $\varepsilon \to 0$.

Thanks to this remark, we have a suggestion on how to choose the open set on which apply Banach's Lemma, hence we introduce

$$U_\tau := \{\varphi \in C^{2,\alpha}_{\varepsilon,b} \mid ||\varphi||_{C^{2,\alpha}_{\varepsilon,b}} < \tilde{c}\varepsilon^{(p+q)(b+2)+\tau}\} \subseteq C^{2,\alpha}_{\varepsilon,b},$$

and we notice that for every $\varphi \in U_\tau$ it holds $||\varphi||_{C^{2,\alpha}_{\varepsilon,-2}} \leq C\varepsilon^\tau$, with C independent of φ and ε.

We will now prove that on U_τ, the operator N is a contraction. In particular, given $\varphi_1, \varphi_2 \in U_\tau$, we want to estimate

$$N(\varphi_1) - N(\varphi_2) = L^{-1}((\hat{Q}(\varphi_2) - \hat{Q}(\varphi_1))).$$

To do so, we notice that by the Mean Value Theorem we can find $t \in [0, 1]$ such that

$$Q(\varphi_1) - Q(\varphi_2) = dQ_\nu(\varphi_1 - \varphi_2) = (L_\nu - L)(\varphi_1 - \varphi_2),$$

where $\nu = t\varphi_1 + (1-t)\varphi_2 \in U_\tau$, and L_ν is the linearization of F at ν. With the same strategy used to compute L we can easily obtain an expression for L_ν, and thus get

$$(L_\nu - L)(\varphi_1 - \varphi_2) = \frac{n}{n-1}\frac{(\omega_\nu - \omega)\wedge i\partial\overline{\partial}((\varphi_1-\varphi_2)\omega^{n-2})}{\omega^n}.$$

From here, taking the norms with respect to ω, we can use the fact that $\nu \in U_\tau$ together with Remark 2.2.6, to obtain

$$|(L_\nu - L)(\varphi_1-\varphi_2)| \leq c|\omega_\nu - \omega|_\omega |i\partial\overline{\partial}((\varphi_1-\varphi_2)\omega^{n-2})|_\omega \leq c\varepsilon^\tau |i\partial\overline{\partial}((\varphi_1-\varphi_2)\omega^{n-2})|_\omega,$$

and thus, by multiplying the inequality with ρ^{b+2}, get

$$||Q(\varphi_1) - Q(\varphi_2)||_{C^{0,\alpha}_{b+2,\varepsilon}} \leq c\varepsilon^\tau ||\varphi_1 - \varphi_2||_{C^{2,\alpha}_{\varepsilon,b}}, \tag{2.13}$$

hence, choosing ε sufficiently small ensures us that N is a contraction on U_τ.

We are left with proving that $N(U_\tau) \subseteq U_\tau$. To do this we shall assume that $pm - q(b+2) > \tau > 0$ (which can easily be done), and see that for every $\varphi \in U_\tau$, thanks to estimate (2.13) and Lemma 2.2.4, we have

$$\begin{aligned} ||N(\varphi)|||_{C^{2,\alpha}_{\varepsilon,b}} \leq & ||N(\varphi) - N(0)|||_{C^{2,\alpha}_{\varepsilon,b}} + ||N(0)|||_{C^{2,\alpha}_{\varepsilon,b}} \\ \leq & c\varepsilon^\tau ||\varphi|||_{C^{2,\alpha}_{\varepsilon,b}} + ||L^{-1}(1-e^f)|||_{C^{2,\alpha}_{\varepsilon,b}} \\ \leq & c\varepsilon^\tau ||\varphi|||_{C^{2,\alpha}_{\varepsilon,b}} + ||f||_{C^{0,\alpha}_{\varepsilon,b+2}} \\ \leq & c\big(\varepsilon^{(p+q)(b+2)+2\tau} + \varepsilon^{p(b+2)+pm}\big) \\ \leq & c\varepsilon^{\min\{\tau, pm-q(b+2)-\tau\}}\varepsilon^{(p+q)(b+2)+\tau} \\ \leq & \tilde{c}\varepsilon^{(p+q)(b+2)+\tau}, \end{aligned}$$

implying that $N(U) \subseteq U$.

This shows that everything is into place to apply Banach's Lemma on the open set U and obtain $\hat{\omega}$ a Chern-Ricci flat balanced metric $\hat{\omega}$ on M, thus proving Theorem 2.0.1.

Remark 2.2.6 also implies:

Corollary 2.2.7. *The couple $(M, \hat{\omega})$ Gromov-Hausdorff converges to the singular Calabi-Yau metric on M_{reg} and, up to rescaling, to Joyce's ALE metrics nearby the exceptional curve.*

We conclude this part with a few remarks.

Remark 2.2.8. In light of Remark 2.1.3, Stokes' Theorem shows that - with the deformation given by the balanced Monge-Ampère type equation - the volume of the exceptional divisors remains the same as the one of the pre-gluing metric, i.e. the (scaled) volume of the ALE metric.

Remark 2.2.9. Thanks to what is known about Joyce's ALE metrics, if we have $k \in \mathbb{N}$ orbifold singularities and we call E^i_j, $i = 1, ..., k_j$ the exceptional divisors corresponding to the resolution of the j-th singularity, for $j = 1, ..., k$, from our construction we can conclude (in the same way as in [BM]) that

$$[\omega^{n-1}] = [\hat{\omega}^{n-1}] = [\tilde{\omega}^{n-1}] + (-1)^{n-1}\varepsilon^{(2n-2)}(\sum_{i=1}^{k_j}\sum_{j=1}^{k} a^i_j PD[E^i_j])^{n-1},$$

where $PD[E^i_j]$ denotes the Poincaré dual of the class $[E^i_j]$.

Thus completing the proof of Theorem 2.0.1.

Remark 2.2.10. It is known that for a manifold which is Calabi-Yau with holomorphic volume Ω, the existence of a Chern-Ricci flat balanced metric implies that Ω is parallel with respect to the Bismut connection associated to said metric. Among the other things, this implies that the restricted holonomy of the Bismut connection of Chern-Ricci flat balanced metrics is contained in $SU(n)$.

Remark 2.2.11. Even though this construction is done to address a non-Kähler situation, it can also be applied when $\tilde{M}$ is instead Kähler (Ricci flat). In this case we know from Joyce's theorem that M admits a Kähler Calabi-Yau metric ω_1, hence together with the balanced class induced by our Chern-Ricci flat balanced metric $\hat{\omega}$ we also have the one induced by ω_1. This two balanced classes need not be the same, however, even if they are to coincide, there is no uniqueness result that would guarantee that the two metrics have to be the same; moreover, the deformation we used in our construction does not cover the whole balanced class, hence in this case we are not even guaranteed that the two metrics are linked by our chosen deformation.

As a conclusion of this chapter, if we view the metric constructed in the system's scenario, we can make the following final remark in which we explain our ideas on how to expand our construction in this direction.

Remark 2.2.12. Given $\hat{\beta}$ a Chern-Ricci flat balanced metric on a Calabi-Yau threefold (Y, Ψ), it holds

$$||\Psi||_{\hat{\beta}} \equiv const.,$$

showing that our metric $\hat{\omega}$ gives a solution of the conformally balanced/dilatino equation on our crepant resolutions (M, Ω). Thus our construction gives us two solutions of the dilatino equation on (M, Ω), that are $\hat{\omega}$ and $\omega' := ||\hat{\Omega}||_{\omega}^{-2}\omega$, where this last one is the dilatino equation solution associated to the balanced metric ω obtained in the first part of the gluing construction. From here, thanks to the fact that this metrics are nearby a Kähler Ricci-flat metric, an idea could be to try and adapt stretegies as in [CPY1] or [DS] to construct a Hermite-Einstein metric on the tangent bundle with respect to the above metrics, and eventually from there try and extend it to a whole solution of the Hull-Strominger system, using - for example - some version of the approach of [AGF].

Other possible paths could instead be related to examples 2.1.5 and 2.1.6, on which it could be interesting to see if, again through a gluing process, if it is possible to construct new non-Kähler solutions to the Hull-Strominger system.

Chapter 3

Small resolutions of ordinary double points

As anticipated in the introduction, it is natural to ask weather or not the construction from the previous chapter can be adapted to the case of ordinary double points on threefolds, in order to fit our result in the conifold transition framework. Unfortunately issues show up, hence in the following we shall - after recalling the ingredients on Ordinary Double Points on threefolds - walk through our construction and see what continues to hold, see what fails and discuss ideas on how to eventually solve the issues. The partial result we obtained is the following.

Proposition 3.0.1. *Let $(\tilde{M}, \tilde{\omega})$ be a smoothable projective Kähler Calabi-Yau nodal threefold (with $\tilde{\omega}$ a singular Calabi-Yau metric), and let M be a compact (not necessarily Kähler) small resolution of $\tilde{M}$. Then M admits a balanced approximately Chern-Ricci flat metric ω such that*

$$[\omega^2] = [\tilde{\omega}^2] + \varepsilon^4[\mathbb{P}^1].$$

We will also discuss some ideas related to the slope stability of the holomorphic tangent bundle of this manifolds, in particular the idea of implementing Collins-Picard-Yau's method from [CPY1] to "reverse the arrow" from their work in order to construct Hermite-Einstein metrics on said bundle, and see that the same difficulty as for Chern-Ricci flat balanced metrics occurs. The following is joint work with Cristiano Spotti, and it is an extended version of the discussion in Section 4 of [GS].

3.1 Geometry and Topology of the small resolutions

The type of singularity adressed in this case is the one of Ordinary Double Points on threefolds (which are the most common kind of singularities), and are described by the local model

$$X := \{z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\} \subseteq \mathbb{C}^4,$$

which is known as the 3-dimensional *standard conifold*, whose only singular point is the origin. Then we have:

Definition 3.1.1. A singular point p in a singular threefold Y is called *ordinary double point* (ODP) if we can find a neighborhood $p \in U \subseteq M$ and a neighborhood $0 \in V \subseteq X$ such that U and V are biholomorphic through a map that sends p to 0.

These singularities arise naturally on threefolds when collapsing $(-1,-1)$-curves, i.e. rational curves biholomorphic to $\mathbb{P}^1$ whose normal bundle is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$, and actually this procedure to obtain ODPs covers all the possibilities on threefolds. Indeed, the standard conifold can be constructed in several ways, one of which is the following: consider the rank 2 bundle $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ on $\mathbb{P}^1$ and notice that the map

$$([X_1 : X_2], (w_1, w_2)) \mapsto (w_1X_1, w_1X_2, w_2X_1, w_2X_2)$$

maps $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ onto X - since X through a change of coordinates is biholomorphic to the set $\{W_1W_2 - W_3W_4 = 0\}$ - sending the zero section onto the origin. Moreover this map restricted to $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \setminus \mathbb{P}^1$ (where $\mathbb{P}^1$ is meant as the zero section) gives a biholomorphism with $X \setminus \{0\}$, proving our previous statement. This shows us that these singularities always admit small resolutions (with $\mathbb{P}^1$ as the exceptional curve) biholomorphic to $\hat{X} := \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$, and it can be shown that a singular threefold with n ordinary double points admits exactly 2^n small resolutions of this type (every singularity can be resolved with a curve in two distinct bimeromorphic ways).

Regarding instead the metric aspect of this singularities, the standard conifold X is naturally endowed with a conical structure. Indeed, we can introduce the function on $\mathbb{C}^4$

$$r(z) := ||z||^{\frac{2}{3}},$$

which restricted to X yields the conical distance to the singularity, and can be used to define the metric

$$\omega_{co,0} := \frac{3}{2} i\partial\overline{\partial} r^2,$$

on the smooth part of X, which is clearly Kähler. Moreover, it can be seen that $\omega_{co,0}$ is actually also Ricci flat, as well as a cone metric over the link $L := \{r = 1\} \subseteq X$ which can be written as

$$g_{co,0} = \frac{3}{2}(dr^2 + r^2 g_L),$$

with g_L a Sasaki-Einstein metric on the link L.

This metric structure of the standard conifold, with some further work, yields also a Kähler Calabi-Yau structure on the small resolution. In fact, Candelas and de la Ossa (see [CO]) constructed a family of metrics, depending on the parameter $a > 0$, of the form

$$\omega_{co,a} := i\partial\overline{\partial} f_a(r^3) + 4a^2 \pi^*_{\mathbb{P}^1}\omega_{FS},$$

where ω_{FS} is the Fubini-Study metric on $\mathbb{P}^1$, and f_a is smooth function satisfying the ODE

$$(xf_a'(x))^3 + 6a^2(xf_a'(x))^2 = x^2, \qquad f_a(x) \geq 0,$$

on $[0, +\infty)$, which immediately gives $f_a(x) = a^2 f_1(x/a^3)$. Here the function r is simply the conical distance from the singularity re-read on the resolution, hence portraying the conical distance from the exceptional curve. Moreover, this family of metrics is such that as $a \to 0$ the metrics $\omega_{co,a}$ converges, away from the exceptional curve, to the standard cone metric $\omega_{co,0}$, and it is also asymptotic (at infinity) to the cone metric $\omega_{co,0}$, and these facts can be seen explicitly with the following expansion from [CPY1].

Lemma 3.1.2. *For $x \gg 1$, the function $f_1(x)$ has a convergent expansion*

$$f_1(x) = \frac{3}{2}x^{\frac{2}{3}} - 2\log(x) + \sum_{n=0}^{+\infty} c_n x^{-\frac{2n}{3}}.$$

We shall now establish the notation and some assumptions useful to describe the gluing attempt, as well as show that the problem makes sense in the non-Kähler setting we are interested in.

First of all, we lay out the details of the setting and take $\tilde{M}$ a *smoothable* Kähler Calabi-Yau singular threefold obtained from the contraction of a finite family of disjoint $(-1,-1)$-curves in a compact complex threefold (thus the singular set of $\tilde{M}$ is made of a finite number of Ordinary Double Points) - hence with the regular part M_{reg} of $\tilde{M}$ equipped with $\tilde{\omega}$ a Kähler Calabi-Yau metric - and M a compact small resolution of $\tilde{M}$.

Remark 3.1.3. The reason why we have much stronger assumptions with respect to the orbifold case, is because for this type of singularities we are not aware of a version of Lemma 2.1.7, thus we need a condition to be able to smoothly cut-off the singular metric at the standard model around the singularity (given in this case by the standard cone metric $\omega_{co,0}$), and such condition is given exactly by the smoothability assumption, which allows us to apply the following result from Hein and Sun (Theorem 1.4 and Lemma A.1 from [HS]), which can be simplified for our purpose with the following statement (here written only for threefolds):

Theorem 3.1.4 (Hein-Sun). *Let $\tilde{M}$ be a smoothable singular threefold whose singular set is a finite family of ODPs endowed with a Kähler Calabi-Yau metric $\tilde{\omega}$ on its smooth part M_{reg}. Then for every singular point $p \in \tilde{M} \setminus M_{reg}$ there exist a constant $\lambda_0 > 0$, neighborhoods $p \in U_p \subseteq \tilde{M}$ and $0 \in V_p \subseteq X$, and a biholomorphism $P : V_p \setminus \{0\} \to U_p \setminus \{p\}$ such that*

$$P^*\tilde{\omega} - \omega_{co,0} = i\partial\overline{\partial}\varphi, \qquad \textit{for some } \varphi \in C^{\infty}_{2+\lambda_0},$$

where r is the conical distance from the singularities and $C^{\infty}_{2+\lambda_0}$ is the space of smooth functions with decay rate at zero of $2 + \lambda_0$ (i.e. an $f \in C^{\infty}_{2+\lambda_0}$ is a smooth function such that nearby zero it holds $|\nabla^k f| \leq cr^{2+\lambda_0-k}$ for all $k \geq 0$).

Anyway, what follows actually works if we replace the assumption above with: $\tilde{\omega}$ a singular Chern-Ricci flat balanced metric such that in a neighborhood of each singularity is asymptotic to the standard cone metric $\omega_{co,0}$.

Now, since our work aims to face the case of compact non-Kähler small resolutions of said $\tilde{M}$, before describing the gluing attempt it is significant to show that this kind of resolutions are actually a very common situation.

Remark 3.1.5. Thanks to a result from Cheltsov (see [Ch]) we know that a hypersurface $\tilde{M}$ in $\mathbb{P}^4$ of degree d with only isolated ODPs is factorial when $\tilde{M}$ has at most $(d-1)^2 - 1$ singularities, thus is in particular $\mathbb{Q}$-factorial. We can then apply the work from Namikawa and Steenbrink (see [NS]) to obtain that $\tilde{M}$ is smoothable, and hence, thanks to the results from Friedman (see [F]) we have that any small resolution M of $\tilde{M}$ with exceptional curves $C_1, ..., C_k$, $C_i \simeq \mathbb{P}^1$, satisfies necessarily a condition

$$\sum_{i=1}^{k} \lambda_i [C_i] = 0 \quad \text{in } H_2(M, \mathbb{R}), \quad \text{where each } \lambda_i \neq 0,$$

which immediately implies that if $\tilde{M}$ has only one ODP, then M can't be Kähler because it contains a homologically trivial curve (note that the generic quintic threefold has exactly one node, and is smoothable since it is a hypersurface in the projective space, hence satisfies this situation).

Moreover, Werner proved in [W] that M is projective if and only if all C_is are homologically non-trivial, and since M is Moishezon, projectivity is equivalent to Kählerness. Thus the class of examples above lies in a larger one, since every small resolution with at least a homologically trivial exceptional curve is non-Kähler.

Before discussing the construction, if we momentarily drop the curvature condition, it is straightforward from literature to conclude the existence of balanced metrics on the small resolution. Indeed:

Remark 3.1.6. Thanks to the results from Hironaka and Alessandrini-Bassanelli ([Hi] and [AB3]), we already know that such small resolutions admit balanced metrics, since blowing up the singularites produces a smooth Kähler threefold which is birational to the small resolution. This fact also shows that for the non-Kähler small resolutions we are considering, the Fino-Vezzoni conjecture (see [FV], Problem 3) holds true, since M is Moishezon, and thus we can apply Theorems B and C from [CRS] to obtain that M does not admit SKT metrics.

3.2 Gluing attempt and possible solutions

We will now present the gluing attempt. Since the proofs are essentially the same as the ones performed in Chapter 2, we will just avoid them and only state the results. Again for simplicity we will just work with one singularity.

The first thing to do is to produce a pre-gluing metric, and in the same fashion as we have done in Chapter 2, we do this in three steps.

(1) First, we glue the background singular metric $\tilde{\omega}$ to the standard cone metric around the singularity. To do so, we take a cut-off function χ_ε as in Step 1 from Chapter 2, and use Theorem 3.1.4. Indeed, if we take $p > 0$ and $\varepsilon > 0$ sufficiently small, so that on the region $\{0 < r \leq 2\varepsilon^p\} \subseteq X$ exists a constant $\lambda_0 > 0$ and is defined a function $\varphi \in C^\infty_{2+\lambda_0}$ such that

$$\tilde{\omega} = \omega_{co,0} + i\partial\overline{\partial}\varphi,$$

we can define the smooth real $(1,1)$-form

$$\tilde{\omega}_\varepsilon := \omega_{co,0} + i\partial\overline{\partial}(\chi_\varepsilon(r)\varphi),$$

which for ε sufficiently small defines a Kähler metric on M_{reg} which is exactly conical around the singularity.

(2) Now, we work on the small resolution of the conifold $\tilde{X}$ and glue the Candelas-de la Ossa metric $\omega_{co,a}$ to the standard cone metric, away from the exceptional curve, and since it's not possible to do it preserving the Kähler condition, we will do it maintaining the balanced one. This can be done thanks to the fact that the Candelas-de la Ossa metric is not exact at infinity, but its square is so, since it holds

$$\omega_{co,a}^2 = \left(i\partial\overline{\partial}\left(\frac{3}{2}r^2 + a^2\psi_a(r)\right)\right)^2 + 2a^2 i\partial\overline{\partial}\left(f_a(r^3) \wedge \pi^*\omega_{FS}\right).$$

Thus if we introduce a cut-off function χ_R as in Step 2 above, we can define the family of closed $(2,2)$-forms

$$\omega_{a,R}^2 = \left(i\partial\overline{\partial}\left(\frac{3}{2}r^2 + a^2\chi_R(r)\psi_a(r)\right)\right)^2 + 2a^2 i\partial\overline{\partial}\left(\chi_R(r)f_a(r^3)\right) \wedge \pi^*\omega_{FS},$$

which correspond to balanced metrics for sufficiently large $R > 0$.

(3) As in Step 3 from Chapter 2, we suitably rescale the metrics $\omega_{a,R}$ on the bubble with a geometric parameter λ and match the two pieces on their exactly conical regions, and hence define

$$\omega = \omega_{\varepsilon,R} := \begin{cases} \lambda\omega_{a,R} & \text{on } r(\zeta) \le R, \\ \omega_{co,0} & \text{on } \varepsilon^p \le r(z) \le 2\varepsilon^p, \\ \omega_\varepsilon & \text{on } r(z) \ge 2\varepsilon^p. \end{cases}$$

At this stage, as done above, we can just unify the parameters ε and R, and choose $R := \varepsilon^{-q}$, with $q > 0$, and using Remark 2.1.9 we can see that on the gluing region $\{\frac{1}{2}\varepsilon^p < r \le 2\varepsilon^p\}$ holds

$$\omega = \omega_{co,0} + O(r^{\lambda_0}) + O(r^{2q/p}\log r).$$

Moreover, we can also here match the holomorphic volumes of the singular threefold and of the small resolution to obtain an (almost) explicit holomorphic volume Ω for M, which can be used again to define the global Chern-Ricci potential

$$f = f_{p,q,\varepsilon} := \log\left(\frac{i\hat{\Omega}\wedge\overline{\hat{\Omega}}}{\omega^3}\right),$$

and obtain that globally on M holds

$$|f| = O(r^{\lambda_0}) + O(r^{2q/p}\log r),$$

i.e. a small Chern-Ricci potential.

Remark 3.2.1. As in Remark 2.2.12, the existence of this metric gives us immediately a solution to the dilatino equation, that is the metric $\omega' := ||\hat{\Omega}||_\omega^{-2}\omega$, which is still quite explicit, thus again a potentially interesting starting point for the construction of a solution to the Hull-Strominger system.

Let us now analyze then the cohomology class naturally associated to the metric ω just obtained, i.e. the $(2,2)$-class

$$[\omega^2] \in H^{2,2}_{dR}(M).$$

If we introduce two cut off functions $\theta_1, \theta_2 : [0, +\infty) \to [0, 1]$ defined as follows:

$$\theta_1(x) := \begin{cases} 1 & \text{if } x \leq \frac{1}{8}\varepsilon^{-q} \\ \text{non increasing} & \text{if } \frac{1}{8}\varepsilon^{-q} \leq x \leq \frac{1}{4}\varepsilon^{-q} \\ 0 & \text{if } x \geq \frac{1}{4}\varepsilon^{-q} \end{cases}$$

and

$$\theta_2(x) := \begin{cases} 0 & \text{if } x \leq 8\varepsilon^{-q} \\ \text{non decreasing} & \text{if } 8\varepsilon^{-q} \leq x \leq 16\varepsilon^{-q} \\ 1 & \text{if } x \geq 16\varepsilon^{-q}; \end{cases}$$

and since for sufficiently small ε we have that ω is exact on $K := \{\frac{1}{8}\varepsilon^{-q} \leq r(\zeta) \leq 8\varepsilon^{-q}\}$, it exists a 3-form β such that

$$\omega^2 = d\beta \quad \text{on } K.$$

Introduce then the form

$$\Omega_c := d(\theta_1(r(\zeta)) + \theta_2(r(\zeta)))\beta),$$

which is a smooth compactly supported form. Moreover, the form

$$\beta - (\theta_1(r) + \theta_2(r))\beta$$

can be extended as zero to the whole M, thanks to the definition of the cut-offs, and thus get that

$$[\omega^2] = [\Omega_c],$$

i.e. the class $[\omega^2]$ admits a compactly supported representative. In addition, the two cut-offs introduced also allow us to decompose $\Omega_c = \Omega_c' + \Omega_c''$, such that on K hold

$$\Omega_c' = d(\theta_1(r)\beta) \quad \text{and} \quad \Omega_c'' = d(\theta_2(r)\beta),$$

and both Ω_c' and Ω_c'' are compactly supported and closed; in particular said compact supports are respectively contained in $\hat{X}$ and M_{reg} (via the obvious identifications), and from their definition it is straightforward to see that

$$[\Omega_c'] = \varepsilon^{4(p+q)}[\omega_{co,a}^2] \in H_c^4(\hat{X})$$

and

$$[\Omega_c''] = [\tilde{\omega}^2] \in H_c^4(M_{reg}),$$

where H_c denotes the compactly supported cohomology group. Also, recalling that $\hat{X} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$, it is clear that $\hat{X}$ is homotopically equivalent to $\mathbb{P}^1$; hence applying Poincaré duality we get

$$H^4_c(\hat{X}) \simeq H^c_2(\hat{X}) \simeq H^c_2(\mathbb{P}^1) = H_2(\mathbb{P}^1) = \langle [\mathbb{P}^1] \rangle,$$

which means that the non-zero class $[\omega^2_{co,a}]$, up to multiplicative constants, is the Poincaré dual of the generator of $H_2(\mathbb{P}^1)$ (thus we can "confuse" them with each other), and thus we can write

$$[\omega^2] = [\tilde{\omega}^2] + \varepsilon^{4(p+q)}[\mathbb{P}^1] \quad \text{in } H^{2,2}_{dR}(M).$$

Finally, we also notice that

$$\int_{\mathbb{P}^1} \omega = \varepsilon^{2(p+q)} \int_{\mathbb{P}^1} \omega_{co,a} \xrightarrow[\varepsilon \to 0]{} 0,$$

hence the balanced class $[\omega^2]$, as $\varepsilon \to 0$, converges to a *nef class*, i.e. to the boundary of the balanced cone. This completes the proof of Proposition 3.0.1.

From what was proven above, the pre-gluing metric ω appears as suitable for a deformation argument, but unfortunately it is exactly here where the issue lies, and descends from the asymptotic behaviour of the Candelas-de la Ossa metrics.

Indeed, we can again consider the balanced Monge-Ampère type equation (2.1), obtained with our ansatz for the Fu-Wang-Wu balanced deformation, and obtain the corresponding operator F and its linearization at zero L (we use the same names of the operators used above since their expressions are unchanged). At this point, considering analogous weighted Hölder spaces as the ones used in the previous chapter, and a variation of F (following an argument of [Sz]) given by $\tilde{F}(\psi) := \frac{\omega^n_\psi}{\omega^n} - e^{f - ev_x \psi}$ we obtain, with essentially the same proof, the invertibility of the corresponding linearization $\tilde{L}$ and an estimate for its inverse (as in Lemma 2.2.4), i.e.

Lemma 3.2.2. *For every $b \in (0, 2)$ it exists $c > 0$ (independent of ε) such that for sufficiently small ε the operator $\tilde{L}$ is invertible and it holds*

$$||u||_{C^{2,\alpha}_{\varepsilon,b}} \leq c||\tilde{L}u||_{C^{0,\alpha}_{\varepsilon,b+2}},$$

for all $u \in C^{2,\alpha}_{\varepsilon,b}$.

From here, we see that we can again turn the equation $\tilde{F}(\psi) = 0$ (which still produces Chern-Ricci flat balanced metrics) into a fixed point problem. In order to do this we shall introduce the operators $\hat{F}, E, G : C^{2,\alpha}_{\varepsilon,b}(M) \to C^{0,\alpha}_{\varepsilon,b+2}(M)$ defined as

$$\hat{F}(\psi) := \frac{\omega^3_\psi}{\omega^3}, \quad E(\psi) := e^{f + ev_x(\psi)} \quad \text{and} \quad G(\psi) = e^f ev_x(\psi),$$

from which we can write

$$\tilde{F} = \hat{F} - E.$$

Now, we can consider the expansion

$$\hat{F}(\psi) = \hat{F}(0) + L(\psi) + \hat{Q}(\psi),$$

and thus rewrite $\tilde{F}(0) = 0$ as

$$\hat{F}(0) + L(\psi) + \hat{Q}(\psi) - E(\psi) = 0.$$

Here, we notice that $\tilde{L} = L - G$, thus we can rewrite $\tilde{F}(0) = 0$ once more and get

$$\hat{F}(0) + \tilde{L}(\psi) + \hat{Q}(\psi) + G(\psi) - E(\psi) = 0,$$

and using the above Lemma, we get that the balanced Monge-Ampère type equation is therefore equivalent to

$$\psi = \tilde{L}^{-1}(E(\psi) - G(\psi) - \hat{F}(0) - \hat{Q}(\psi)) =: N(\psi), \tag{3.1}$$

i.e. the search for a fixed point for the operator $N : C^{2,\alpha}_{\varepsilon,b}(M) \to C^{2,\alpha}_{\varepsilon,b}(M)$.

At this stage, analogously as above it is easy to check that on a suitable open set U_τ, with $\tau > 0$, given by

$$U_\tau := \{\varphi \in C^{2,\alpha}_{\varepsilon,b} \mid ||\varphi||_{C^{2,\alpha}_{\varepsilon,b}} < \tilde{c}\varepsilon^{(p+q)(b+2)+\tau}\} \subseteq C^{2,\alpha}_{\varepsilon,b},$$

it holds that N is a contraction operator. Unfortunately, it is impossible to consistently choose p, q and τ to repeat the above proof and obtain that

$$N(U_\tau) \subseteq U_\tau,$$

and this is caused by the asymptotic quadratic decay to the cone of the Candelas-de la Ossa metrics (unusual for Calabi-Yau metrics), making the norm $||F(0)||_{C^{0,\alpha}_{\varepsilon,b+2}}$ too large. Actually, what happens is that this quadratic decay is exactly the threshold for this argument to work, since if said decay was (arbitrarily) more than quadratic, the argument would have worked without issues.

Analyzing further the Candelas-de la Ossa metrics, one can see that if we just consider the cut-off metrics $\omega_{a,R}$ on the small resolution $\hat{X}$, these are exactly conical at infinity, thus a deformation argument as the one performed above could lead to Chern-Ricci flat balanced metrics with faster decay to the cone, but unfortunately the metric $\omega_{a,R}$ cannot be used to do this as the "initial error" given by the Chern-Ricci potential of said metric turns out to be blowing up with repsect to the weighted Hölder norm, suggesting that there might not be any Chern-Ricci flat balanced metrics in a neighborhood of the Candelas-de la Ossa metrics.

Hence, a possibility that we wish to explore in order to solve this issue, is to understand if it's possible to obtain Chern-Ricci flat balanced metrics on $\hat{X}$ which have fast decay but are not necessarily near to the Candelas-de la Ossa metrics, and the approach we think might be interesting to use is to try and obtain a balanced version of Conlon-Hein's result (see [CH]) starting from the metric $\omega_{a,R}$, which would immediately produce the missing ingredient to complete the above failed gluing construction.

Obviously such a problem comes with several challenges on the analytic side, as the balanced setting and the definition of the balanced Monge-Ampère type equation do not allow many of the tools typically to obtain Yau's estimates such as the Moser iteration technique, and the non-compact (even though weighted) setting makes it also hard to apply other inequalities that are typically used in non-Kähler settings such as the Cherrier inequality (see [TW1]).

Another possible interesting path to take could be to try and understand if the balanced class induced by the metric ω could be a polystable class for the holomorphic tangent bundle. This, thanks to the Hitchin-Kobayashi correspondence, would lead us to the existence of Hermite-Einstein metrics on said bundle, and thus add a block in the construction of a solution to the Hull-Strominger system.

3.3 Stability of the holomorphic tangent bundle

In the framework of the Hull-Strominger system, it is also a natural question to ask if the metric obtained in Proposition 2.0.1 from the previous section defines a balanced class making the holomorphic tangent bundle a polystable bundle. Following the strategy in [CPY1], one can attempt to obtain the polystability through a gluing procedure, thanks to the fact that both $\hat{X}$ and the singular manifold $\tilde{M}$ are endowed with Hermite-Einstein metrics with suitable asymptotic behaviour. We will however see that the issue appearing in the previous section is again preventing us from concluding the gluing procedure, thus in the following we will discuss the steps that work and why we are not able to conclude. This time, as a difference with the previous section, we shall provide the details of the proofs, as they differ in some aspects since we are working with endomorphisms of the tangent bundle and not just functions.

Let us then set up the problem as in [CPY1]. Given ω the pregluing metric on the small resolution M from Proposition 2.0.1, and indicating with H the metric induced by ω on the holomorphic tangent bundle $T^{1,0}M$, our aim is to deform H into a Hermite-Einstein metric, with respect to the balanced class $[\omega^2]$.

In order to achieve this, we shall follow the steps taken in Section 6 from [CPY1] and adapt them to our case (which will turn out to be slightly more simple).

Consider the Banach space

$$\mathcal{H} := \{u \in \Gamma(End\, T^{1,0}M) \mid u^\dagger = u\},$$

where $\dagger$ indicates the adjoint with respect to H, i.e. with respect to the L^2-product induced by (ω, H) that is

$$\langle u, v\rangle_{(\omega,H)} := \int_M (u, v)_H \omega^3,$$

where $(\_,\_)_H$ is the scalar product induced by H on the fibers of $T^{1,0}M$.

Consider then the operator $\mathcal{F} = \mathcal{F}_\varepsilon : \mathcal{H} \to \mathcal{H}$ given by

$$\mathcal{F}(u) := e^{u/2}\,(i\Lambda_\omega F_{H_u})\,e^{-u/2},$$

where $H_u := He^u$, for all $u \in \mathcal{H}$. Moreover, it is straightforward to notice that it holds the orthogonal decomposition

$$\mathcal{H} = \mathcal{W} \oplus \mathbb{C}Id,$$

where

$$\mathcal{W} = \left\{u \in \mathcal{H} \mid \int_M (\mathrm{Tr}u)\,\omega^3 = 0\right\},$$

and thanks to the facts that $c_1 = 0$ and $\mathcal{F}(u)^\dagger = \mathcal{F}(u)$, we get (as seen in [CPY1]) that $\mathcal{F}$ restricts to

$$\mathcal{F} : \mathcal{W} \to \mathcal{W}.$$

Now, our final aim is to find a Hermite-Einstein metric on $T^{1,0}M$ with respect to ω, and this can be achieved by finding $u \in \mathcal{W}$ such that $\mathcal{F}(u) = 0$. To approach this last problem as done by Collins-Picard-Yau, we notice that if we call $\mathcal{L}$ the linearization of $\mathcal{F}$ at 0 and $\mathcal{Q}$its corresponding quadratic part, we obtain the expansion

$$\mathcal{F}(u) = \mathcal{F}(0) + \mathcal{L}u + \mathcal{Q}(u). \tag{3.2}$$

In particular, if we are able to prove that $\mathcal{L}$ is an isomorphism, the equation $\mathcal{F}(u) = 0$, thanks to (3.1.2), becomes equivalent to solving the fixed point problem

$$\mathcal{M}(u) = u,$$

where

$$\mathcal{M}(u) := \mathcal{L}^{-1}\left(-\mathcal{F}(0) - \mathcal{Q}(u)\right). \tag{3.3}$$

Hence the plan to attack the problem is to first show that $\mathcal{L}$ is actually an isomorphism, and then conclude by showing that $\mathcal{M}$ is a contraction (in order to apply Banach's Lemma.

Before we start developing the two steps of the proof, it is significant to make the following remark to justify a technical assumption that we will be making in our proof, that is the *simplicity* of the holomorphic tangent bundle of M, that is the global holomorphic sections of the bundle $End(T^{1,0}M)$ are only scalar multiples of the identity.

Remark 3.3.1. It is known that a singular threefold $\tilde{M}$ of the type we are considering with n ordinary double points admits 2^n small resolutions, which are all bimeromorphic with each other through a birational transformation known as *Atiyah flop*. Moreover, in [CPY1] it was shown that whenever the small resolution is Kähler, than the bundle $T^{1,0}M$ is simple. We can however take this a little step further and show that simplicity is preserved through flops, meaning that if one of the small resolutions has simple holomorphic tangent bundle, then all the other small resolutions also have simple holomorphic tangent bundle. Indeed, if we take M, M' small resolutions of $\tilde{M}$ with M' such that $T^{1,0}M'$ simple, then if we take σ a holomorphic global section of $End(T^{1,0}M)$ and call $M_{reg} \subseteq \tilde{M}$ the regular part, we can consider the restriction

$$\tilde{\sigma} := \sigma_{|_{T^{1,0}M_{reg}}} : T^{1,0}M_{reg} \to T^{1,0}M_{reg},$$

which keeps being holomorphic. But we can now apply Hartogs' Theorem and extend $\tilde{\sigma}$ holomorphically to a global holomorphic section $\hat{\sigma}$ of $End(T^{1,0}M')$, and thus use the simplicity of $T^{1,0}M'$ to get that $\hat{\sigma} = cI$ for some $c \in \mathbb{C}$. Hence, the restriction of the holomorphic section σ to the open subset $T^{1,0}M_{reg}$ is identically of the form cI, thus necessarily also $\sigma = cI$, that is $T^{1,0}M$ is simple.

The above remark in particular shows that whenever our $\tilde{M}$ admits a Kähler small resolution, then all its small resolutions (Kähler and non-Kähler) have simple holomorphic tangent bundle, guaranteeing that (thanks also to Remark 3.1.5) assuming the simplicity of the holomorphic tangent bundle of M covers a very large class of cases (actually the most common ones).

The first thing to do is to compute the linearization of $\mathcal{F}$, and we can do it by reusing the computations on [CPY1] which give us

$$\begin{aligned} d\mathcal{F}_u(v) = &- e^{u/2}\left(g^{j\bar{k}}\partial_{\bar{k}}\nabla_j^{H_u}\left(e^u(d\exp)_u(v)\right)\right)e^{-u/2} \\ &+ (d\exp^{1/2})_u(v)(i\Lambda_\omega F_{H_u})e^{-u/2} + e^{u/2}(i\Lambda_\omega F_{H_u})(d\exp^{-1/2})_u(v), \end{aligned} \tag{3.4}$$

which evaluated in $u = 0$ gives

$$\mathcal{L}v = -g^{j\bar{k}}\partial_{\bar{k}}\nabla_j^H v - \frac{1}{2}[i\Lambda_\omega F_H, v].$$

In order to study the invertibility of this operator, we shall shift to work in weighted Hölder spaces (as done in [GS]), for which we recall the norm, for all $\alpha \in (0,1)$, is given by, for $u \in \Gamma^{k,\alpha}_{\varepsilon,b}(End(T^{1,0}M))$

$$||u||_{\Gamma^{k,\alpha}_{\varepsilon,b}(End(T^{1,0}M))} := \sum_{i=0}^{k} \sup_M |\rho^{b+i}\nabla^i_\varepsilon u|_\omega + \sup_{d_\varepsilon(x,y)<inj_\varepsilon} \left| \min\left(\rho^{b+k+\alpha}(x), \rho^{b+k+\alpha}(y)\right) \frac{\nabla^k_\varepsilon u(x) - \nabla^k_\varepsilon u(y)}{d_\varepsilon(x,y)^\alpha} \right|_\omega,$$

where inj_ε is the injectivity radius of the metric ω, ∇_ε denotes the Chern connection associated to H, and the weight function ρ is given by

$$\rho = \rho_\varepsilon(z) := \begin{cases} \varepsilon^{p+q} & \text{on } r(z) \leq \varepsilon^{p+q}, \\ \text{non decreasing} & \text{on } \varepsilon^{p+q} \leq r(z) \leq 2\varepsilon^{p+q}, \\ r(z) & \text{on } 2\varepsilon^{p+q} \leq r(z) \leq 1/2, \\ \text{non decreasing} & \text{on } 1/2 \leq r(z) \leq 1, \\ 1 & \text{on } r(z) \geq 1. \end{cases}$$

The functional space we will be working with is going to be $\Gamma^{k,\alpha}_{\varepsilon,b}(\mathcal{W}) := \Gamma^{k,\alpha}_{\varepsilon,b}(End(T^{1,0}M)) \cap \mathcal{W}$ (the subscript ε in the definition of the space is to enhance the dependence on the metrics ω and H, which both depend on the parameter ε).

Our aim is to prove the following.

Lemma 3.3.2. *If $T^{1,0}M$ is simple, there exists $c > 0$ such that for sufficiently small ε it holds*

$$||v||_{\Gamma^{2,\alpha}_{\varepsilon,b}(\mathcal{W})} \leq c||\mathcal{L}v||_{\Gamma^{0,\alpha}_{\varepsilon,b+2}(\mathcal{W})},$$

for all $v \in \Gamma^{2,\alpha}_{\varepsilon,b}(\mathcal{W})$.

Proof. Suppose by contradiction that such $c > 0$ does not exist, then we can find sequences $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}} \subseteq \Gamma^{k,\alpha}_{\varepsilon,b}(\mathcal{W})$ such that

$$||v_n||_{\Gamma^{2,\alpha}_{\varepsilon,b}(\mathcal{W})} = 1 \quad \text{and} \quad ||\mathcal{L}v_n||_{\Gamma^{0,\alpha}_{\varepsilon,b+2}(\mathcal{W})} \tag{3.5}$$

for all $n \in \mathbb{N}$, and $\varepsilon_n \to 0$ as $n \to +\infty$.

Consider then $K \subseteq M_{reg}$ compact and notice that for sufficiently large $n \in \mathbb{N}$, $\omega = \tilde{\omega}$ on K, that is ω is Kähler Ricci-flat, implying that $\Lambda_\omega F_H \equiv 0$, and hence (3.5) gives

$$||g^{j\bar{k}}\partial_{\bar{k}}\nabla^H_j v_n||_{\Gamma^{0,\alpha}_{\varepsilon,b+2}(\mathcal{W})} \underset{n\to+\infty}{\longrightarrow} 0. \tag{3.6}$$

Applying then Ascoli-Arzelà's Theorem, we get that - up to subsequences - our $\{v_n\}_{n\in\mathbb{N}}$ is uniformly convergent on compact sets of M_{reg} in the $C^{2,\alpha}_b$ sense. In particular, calling v_0 the limit, (3.6) implies

$$g^{j\bar{k}}\partial_{\bar{k}}\nabla^{\tilde{H}}_j v_0 = 0 \tag{3.7}$$

on every compact set of M_{reg} (here $\tilde{H}$ denotes the metric on the tangent bundle induced by $\tilde{\omega}$). Also, it is clear that $v_0^\dagger = v_0$ (here $\dagger$ denotes the adjoint with respect to $\tilde{H}$). Now, from (3.7) it follows (equally as seen in [CPY1])

$$\Delta_{\tilde{\omega}}|v_0|^2_{\tilde{H}} = |\overline{\nabla}v_0|^2_{\tilde{H}}. \tag{3.8}$$

In order to obtain a contradiction, our final aim is to show that $v_n \to 0$ on the whole M. Our first step towards this is to prove that $\overline{\nabla}v_0 \equiv 0$, i.e. v_0 is holomorphic on M_{reg}. To do this, for all $\delta > 0$ we introduce χ_δ a smooth cut-off function such that

$$\chi_\delta = \begin{cases} 0 & \text{on } r < \frac{\delta}{2} \\ \text{non decreasing} & \text{on } \frac{\delta}{2} < r < \delta \\ 1 & \text{on } r > \delta \end{cases}$$

and such that $|\Delta_{\tilde{\omega}}\chi_\delta| \leq c\delta^{-2}$.

Then for small δ, introducing $B_\delta := \{x \in M_{reg} \mid d(\mathbb{P}^1, x) < \delta\}$, $M_\delta := M_{reg} \setminus B_\delta$, we have

$$\begin{aligned} 2\int_{M_\delta} |\overline{\nabla}_{\tilde{H}}v_0|^2\tilde{\omega}^3 &\leq 2\int_{M_{reg}} \chi_\delta|\overline{\nabla}_{\tilde{H}}v_0|^2\tilde{\omega}^3 = \int_{M_{reg}} \chi_\delta\Delta|v_0|^2_{\tilde{H}}\tilde{\omega}^3 \\ &= \int_{M_{reg}} |v_0|^2_{\tilde{H}}\Delta_{\tilde{\omega}}\chi_\delta\tilde{\omega}^3 = \int_{\{\delta/2<r<\delta\}} |v_0|^2_{\tilde{H}}\Delta_{\tilde{\omega}}\chi_\delta\tilde{\omega}^3 \\ &\leq \int_{\{\delta/2<r<\delta\}} |v_0|^2_{\tilde{H}}|\Delta_{\tilde{\omega}}\chi_\delta|\tilde{\omega}^3 \leq c\delta^{-2}\int_{\{\delta/2<r<\delta\}} |v_0|^2_{\tilde{H}}\tilde{\omega}^3 \\ &\leq c\delta^{-2}\int_{\{\delta/2<r<\delta\}} |v_0|^2_{\tilde{H}}\omega^3_{co,0} \leq c\delta^{-2b-2}\int_{\{\delta/2<r<\delta\}} \omega^3_{co,0} \leq c\delta^{4-2b}. \end{aligned}$$

Thus, assuming $b < 2$ and taking the limit as $\delta \to 0$ we obtain

$$\int_{M_{reg}} |\overline{\nabla}v_0|^2_{\tilde{H}}\tilde{\omega}^3 = 0,$$

that is $\overline{\nabla}v_0 \equiv 0$ on M_{reg}, i.e. v_0 is holomorphic on M_{reg}. Moreover, by Hartogs' Theorem we have that v_0 extends holomorphically to the whole M.

On the other hand, by hypothesis it holds

$$\int_M (\mathrm{Tr} v_n)\omega^3 = 0$$

for all $n \in \mathbb{N}$. Hence, for sufficiently small $\gamma > 0$ (with M_γ defined as above) and large $n \in \mathbb{N}$, it holds

$$\int_{M_\gamma} (\mathrm{Tr} v_n)\omega^3 = \int_{M_\gamma} (\mathrm{Tr} v_n)\tilde{\omega}^3 \underset{n\to\infty}{\longrightarrow} \int_{M_\gamma} (\mathrm{Tr} v_0)\tilde{\omega}^3.$$

But we also have

$$\left| \int_{M_\gamma} (\mathrm{Tr} v_n)\omega^3 \right| = \left| \int_{B_\gamma} (\mathrm{Tr} v_n)\omega^3 \right| \leq \int_{B_\gamma} |\mathrm{Tr} v_n|\omega^3$$
$$\leq c\gamma^{-b} \int_{B_\gamma} \omega^3 \leq c\gamma^{6-b} \underset{\gamma\to 0}{\longrightarrow} 0.$$

Moreover

$$\left| \int_{M_\gamma} (\mathrm{Tr} v_0)\tilde{\omega}^3 \right| \underset{\gamma\to 0}{\longrightarrow} \left| \int_{M_{reg}} (\mathrm{Tr} v_0)\tilde{\omega}^3 \right|,$$

and hence

$$\int_{M_{reg}} (\mathrm{Tr} v_0)\tilde{\omega}^3 = 0.$$

But now this last information, combined with the holomorphicity of v_0 on the whole M and the hypothesis of simplicity for the tangent bundles implies necessarily that $v_0 \equiv 0$ on M. We will now work from here to get a contradiction.

From the weighted Schauder estimates, from (3.5) we get that for all $n \in \mathbb{N}$ holds

$$c(||v_n||_{\Gamma^{2,\alpha}_{\varepsilon,b}(\mathcal{W})} + \frac{1}{n}) \geq 1,$$

that is, it exists $\nu' > 0$ such that

$$||v_n||_{\Gamma^{2,\alpha}_{\varepsilon,b}(\mathcal{W})} \geq \nu'$$

for all $n \in \mathbb{N}$. It follows that it exists $\nu' > \nu > 0$ and a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq M$ such that

$$\rho^b(x_n)|v_n(x_n)|_H \geq \nu \tag{3.9}$$

for all $n \in \mathbb{N}$. We then have two possibilities.

If

$$\liminf_{n\to+\infty} r_n > 0,$$

where $r_n := r(z(x_n))$, then - up to subsequences - we can assume $x_n \to x_\infty$ together with $r(z(x_\infty)) > 0$, in particular it follows that for sufficiently large $n \in \mathbb{N}$ we get $\rho(x_n) = r(x_n)$, from which

$$|v_0(x_\infty)|_{\tilde{H}} \geq \nu r(z(x_\infty))^{-b} > 0,$$

i.e. a contradiction, since $v_0 \equiv 0$.

If instead it holds

$$\liminf_{n\to+\infty} r_n = 0,$$

again up to subsequences we can assume $r_n \to 0$. Re-reading this scenario in the ζ coordinates leads to two further subcases.

(i) If $r(\zeta(x_n))$ is bounded, up to subsequences we can assume $r(\zeta(x_n)) \to \tilde{r} \geq 0$. In this case we can repeat the strategy we adopted in [GS] (also previously adopted in [BM]), and notice that reading ρ, $\{v_n\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}}$ in the ζ coordinates on $\hat{X}$, gives us that $x_n \to x_\infty \in \{r(\zeta) \geq L\} \subseteq \hat{X}$, for some $L > 0$. Moreover, the bound on the weighted norms of the v_ns imply that the sequence $V_n := \varepsilon_n^b v_n$ is - once again up to subsequences - uniformly convergent on compact sets of $\hat{X}$ (thanks to Ascoli-Arzelà's Theorem) to some V_∞, from which we get

$$0 < \nu \leq |\rho^b(x_n)v_n(x_n)|_{H_{co,1}} \leq |r^b(\zeta(x_n))V_n(x_n)|_{H_{co,1}},$$

where $H_{co,1}$ is the metric induced on the tangent bundle by the Candelas-de la Ossa metric $\omega_{co,1}$. Taking then the limit as $n \to +\infty$ in this last inequality gives us

$$0 < \nu \leq r^b(\zeta(x_\infty))|V_\infty(x_\infty)|_{H_{co,1}}. \tag{3.10}$$

But if we recall the definition of $\mathcal{L}$, and the fact that

$$\rho^{b+2}\mathcal{L}v_n \underset{n\to+\infty}{\longrightarrow} 0,$$

we have

$$\rho^{b+2}\mathcal{L}v_n \to \tilde{\rho}^{b+2} g^{j\bar{k}}\partial_{\bar{k}}\nabla_j^{H_{co,1}} V_\infty,$$

where $\tilde{\rho}$ is the weight function on $\hat{X}$ given by

$$\tilde{\rho}(\zeta) = \begin{cases} 1 & \text{on } r(\zeta) \leq 1, \\ \text{non decreasing} & \text{on } 1 \leq r(\zeta) \leq 2, \\ r(\zeta) & \text{on } 2 \leq r(\zeta) \leq 1/2\varepsilon_n^{-1}, \end{cases} .$$

This gives again that $\Delta_{\omega_{co,1}}|V_\infty|^2_{H_{co,1}} = |\overline{\nabla} V_\infty|^2_{H_{co,1}}$, implying in particular that $|V_\infty|^2_{H_{co,1}}$ is subharmonic and obviously non-negative. This last two facts, combined with the facts that $|V_\infty|^2_{H_{co,1}} \leq cr^{-2b}$ and $\omega_{co,1}$ is asymptotically conical, imply - applying an asymptotically conical version of Lemma 6.9 from [CPY1] - that $V_\infty \equiv 0$, hence a contradiction since (2.13) holds.

(ii) If instead $r(\zeta(x_n))$ is unbounded, we can assume $r(\zeta(x_n)) \to +\infty$, which in particular implies $\rho(x_n) = r(x_n)$, and thus from (3.9) we get

$$r^b(x_n)|v_n(x_n)|_H \geq \nu \quad \text{for all } n \in \mathbb{N}. \tag{3.11}$$

Consider then the region $A^* := \{0 < r(z) < \frac{1}{2}\}$, and the family of biholomorphisms

$$\sigma_n : C_n \to A^*$$

given by $\sigma_n(z') := r_n^{3/2} z'$, where $C_n := \{0 < r(z') < \frac{r_n^{-1}}{2}\} \subseteq \hat{X}$. We can then introduce on C_n the metrics $\eta_n := r_n^{-2}\sigma_n^*\omega$ and the sequence $y_n := \sigma_n^{-1}(x_n) \in C_n$, and notice that

$$r(y_n) = 1 \quad \text{for all } n \in \mathbb{N},$$

giving us in particular that $y_n \to y_\infty \in \hat{X}$ up to subsequences.

On the other hand we can introduce the sequence of bundle endomorphisms

$$W_n := r_n^b \sigma_n^* v_n$$

and notice that (3.11) implies that

$$|w_n(y_n)|_{\eta_n} = r^b(z'(y_n))|w_n(y_n)|_{\eta_n} \geq \nu \quad \text{for all } n \in \mathbb{N}. \tag{3.12}$$

Moreover, using again Ascoli-Arzelà's Theorem we can assume $W_n \to W_\infty$ uniformly on compact sets of X (since $C_n \to \hat{X}$). Thus, noticing that $\eta_n \to \omega_{co,0}$ as $n \to \infty$, taking the limit in (3.12) gives

$$|W_\infty(y_\infty))|_{\omega_co,0} \geq \nu > 0. \tag{3.13}$$

On the other hand, taking the pullback

$$\sigma_n^*\left(\rho^{b+2}\mathcal{L}v_n\right)$$

we obtain (on any $K \subseteq \hat{X}$ compact subset)

$$\begin{aligned}\sigma_n^*\left(\rho^{b+2}\mathcal{L}v_n\right) &= \sigma_n^*(\rho^{b+2})\left(\sigma_n^*(g^{j\bar{k}})\partial_{\bar{k}}\nabla_j^{\sigma_n^*H}(\sigma_n^*v_n) - \frac{1}{2}[\Lambda_{\sigma_n^*\omega}F_{\sigma_n^*H}, \sigma_n^*v_n]\right)\\ &= r^{b+2}\left(\eta_n^{j\bar{k}}\partial_{\bar{k}}\nabla_j^{\sigma_n^*H}W_n - \frac{1}{2}[\Lambda_{\eta_n}F_{r_n^{-2}\sigma_n^*H}, W_n]\right),\end{aligned}$$

which after taking the limit as $n \to +\infty$, implies once again

$$\Delta_{\omega_{co,0}} |W_\infty|^2_{H_{co,0}} = |\overline{\nabla} W_\infty|^2_{H_{co,0}}.$$

In particular we have that $|W_\infty|^2_{H_{co,0}}$ is subharmonic and non-negative, and thanks to the weighted estimates we have in the hypothesis, it also satisfies

$$|W_\infty|^2_{H_{co,0}} \leq cr^{-2b},$$

hence applying Lemma 6.9 from [CPY1] we get $W_\infty \equiv 0$, which is a contradiction since (3.13) holds.

Thus the estimate holds necessarily. □

From this we can easily obtain the invertibility of $\mathcal{L}$.

Theorem 3.3.3. *The linear operator $\mathcal{L} : \Gamma^{2,\alpha}_{\varepsilon,b}(\mathcal{W}) \to \Gamma^{0,\alpha}_{\varepsilon,b+2}(\mathcal{W})$ defined above is an isomorphism.*

Proof. As seen in Lemma 3.3.2 we have that $\mathcal{L}$ is injective. Moreover, it is straightforward to notice that $\mathcal{L}$ is elliptic and shares its principal symbol with the laplacian, thus it is also surjective. Hence it admits an inverse $\mathcal{L}^{-1}$ which is still continuous thanks to the estimate from Lemma 3.3.2. □

At this stage, the natural choiche of a neighborhood of zero to study $\mathcal{M}$ on is resembling the choice done in the previous section, that is

$$U_\tau := \left\{ u \in \Gamma^{2,\alpha}_{\varepsilon,b}(\mathcal{W}) \mid ||u||_{\Gamma^{2,\alpha}_{\varepsilon,b}(\mathcal{W})} \leq \tilde{c}\varepsilon^{b+2+\tau} \right\} \subseteq \Gamma^{2,\alpha}_{\varepsilon,b}(\mathcal{W}).$$

As it happened previously, we are once again able to show that $\mathcal{M}$ contracts distances on this neighborhood, but we cannot show that U_τ is preserved by $\mathcal{M}$, and the problem is again related to the "inital error" of the pregluing metric, as it happens that

$$||\mathcal{N}(0)||_{\Gamma^{2,\alpha}_{\varepsilon,b}(\mathcal{W})} \leq c||\mathcal{F}(0)||_{\Gamma^{0,\alpha}_{\varepsilon,b+2}(\mathcal{W})} \leq c\varepsilon^b(\varepsilon^{\lambda_0)} + \varepsilon^{2q}\log\varepsilon),$$

which is not enough to conclude that $\mathcal{M}(U_\tau) \subseteq U_\tau$.

It however remains of central interest to try and understand if $[\omega^2]$ makes the holomorphic tangent bundle into a slope stable bundle, as it is the balanced class naturally associated to a solution of the dilatino equation, hence we plan to explore alternative approaches to try and obtain this property for the bundle.

Chapter 4

Blowing up Chern-Ricci flat balanced manifolds

In this chapter I will be presenting a result obtained with Elia Fusi. It consists of another gluing construction in the balanced case, this time aimed to the construction of Chern-scalar constant balanced metrics, in an attempt of extending the result from Arezzo and Pacard in [AP] to the balanced case. The statement follows.

Theorem 4.0.1. *Let M be a compact complex manifold of dimension n, endowed with $\tilde{\omega}$ a Chern-Ricci flat balanced metric. Then, the blowup $\hat{M}$ at a point $x \in M$ admits Chern-scalar constant balanced metric $\hat{\omega}$ such that*

$$[\omega^{n-1}]_{BC} = [\tilde{\omega}^{n-1}]_{BC} + (-1)^{n-1}\varepsilon^{(2n-2)}[\mathbb{P}^{n-1}].$$

As we will see, at the moment the construction only works assuming that the base manifold is Chern-Ricci flat, but we are currently working towards extending it to the general Chern-scalar constant balanced case.

4.1 The approximate solution

Let M be a compact complex manifold of dimension $n \geq 3$, endowed with $\tilde{\omega}$ a Chern-scalar constant balanced metric, and let $\hat{M}$ be the blowup at a point $x \in M$. Our aim is to obtain a Chern-scalar constant balanced metric on $\hat{M}$, and the first step towards this will be to construct an *approximate solution*. In order to do this, we shall implement a cut-off argument (as in [GS]) followed by a description of the behaviour of the newly obtained metric in the gluing region, to ensure that the metric is indeed an approximate solution.

4.1.1 Cutting off

In order to produce the metric we want, we shall first establish which are the ingredients we will use. One of the main components of our argument is the Burns-Simanca metric ω_{BS}, introduced by Burns and Simanca (see [LeB] and [Sim]), which is a scalar-flat, asymptotically flat Kähler metric on $Bl_0\mathbb{C}^n$, already used successfully in gluing constructions of cscK metrics on blow-ups by Arezzo and Pacard in [AP] (see also [Sz]). Hence, we want to glue together the background metric $\tilde{\omega}$ with ω_{BS} on a flat region, and in order to do this, our second main ingredient will be the balanced property. Let us then start to describe this gluing process by seeing how the balanced property intervenes, and this can be done recalling Lemma 2.1.7. Thus, starting from $\tilde{\omega}$, we can obtain the corresponding $\tilde{\omega}_\varepsilon$, which is exactly flat in a neighborhood of x.

On the other hand, we can consider coordinates ζ on $\hat{X} := Bl_x(\mathbb{C}^n) \setminus \mathbb{CP}^{n-1} = \mathbb{C}^n \setminus \{0\} =: X \setminus \{0\}$, and a cut-off function

$$\psi(y) := \begin{cases} 1 & \text{if } y \leq \frac{1}{4}, \\ \text{Non increasing} & \text{if } \frac{1}{4} < y < \frac{1}{2}, \\ 0 & \text{if } y \geq \frac{1}{4}, \end{cases}$$

which, for all $q > 0$, can be rescaled to

$$\psi_\varepsilon(y) := \psi(\varepsilon^q y).$$

This rescaling, makes the cut-off happen *far away* from the exceptional divisor, hence in the asymptotically flat part. Hence, thanks to the fact that the Burns-Simanca metric, away from the exceptional divisor, has the following expansion

$$\omega_{BS} = i\partial\overline{\partial}(|\zeta|^2 + \gamma(|\zeta|)),$$

with $\gamma(|\zeta|) = O(|\zeta|^{4-2n})$, we can introduce the family of closed $(1,1)$-forms

$$\omega_{BS,\varepsilon} := i\partial\overline{\partial}(|\zeta|^2 + \psi_\varepsilon(|\zeta|)\gamma(|\zeta|)),$$

and easily see that, on the cut-off region $\{\frac{1}{4}\varepsilon^{-q} \leq |\zeta| \leq \frac{1}{2}\varepsilon^{-q}\}$, it holds

$$\omega_{BS,\varepsilon} = \omega_o + O(|\zeta|^{2-2n}), \tag{4.1}$$

where now ω_o denotes the flat metric on $\mathbb{C}^n \setminus \{0\}$ induced by the coordinates ζ, from which it follows that, for sufficiently small ε, $\omega_{BS,\varepsilon}$ is an asymptotically exactly flat Kähler metric on $\hat{X}$.

If we then consider the biholomorphism

$$z = \varepsilon^{p+q}\zeta,$$

this gives the identification

$$\left\{\frac{1}{4}\varepsilon^{-q} \leq |\zeta| \leq 2\varepsilon^{-q}\right\} \equiv \left\{\frac{1}{4}\varepsilon^{p} \leq |z| \leq 2\varepsilon^{p}\right\}$$

with we wich we can topologically realize $\hat{M}$, and also obtain that

$$|z|^2 = \varepsilon^{2(p+q)}|\zeta|^2,$$

which tells that on $\hat{M}$, the metrics $\varepsilon^{2(p+q)}\omega_{BS,\varepsilon}$ and ω'_ε coincide (with the flat metric), on the region

$$\left\{\frac{1}{2}\varepsilon^{-q} \leq |\zeta| \leq \varepsilon^{-q}\right\} \equiv \left\{\frac{1}{2}\varepsilon^{p} \leq |z| \leq \varepsilon^{p}\right\},$$

hence allowing us to glue $\tilde{\omega}_\varepsilon$ and $\varepsilon^{2(p+q)}\omega_{BS,\varepsilon}$ to a global balanced metric ω on $\hat{M}$.

4.1.2 Behaviour of the new metric

We will now describe the behaviour of ω, and make sure that it is the approximate solution we were searching for.

First of all, it is clear that the metric is unaltered on $\{\varepsilon^p \leq |z| \leq 2\varepsilon^p\}$, on which we still have

$$|\nabla^k_{\omega_o}(\omega - \omega_o)|_{\omega_o} \leq c|z|^{1-k},$$

for all $k \geq 0$.

On the other hand, since to obtain ω we had to rescale the metric $\omega_{BS,\varepsilon}$ on $\hat{X}$, we have to check how it has affected the distance from the flat metric. To have clearer estimates, we will express also this one in terms of the "small" coordinates z. The main thing to observe, is that on $\{\frac{1}{4}\varepsilon^{-q} \leq |\zeta| \leq \frac{1}{2}\varepsilon^{-q}\}$ it holds

$$\begin{aligned}\langle \omega - \omega_o, \omega - \omega_o\rangle_{\omega_o}(z) &= \varepsilon^{-4(p+q)}\langle \varepsilon^{2(p+q)}(\omega_{BS,\varepsilon} - \omega_o), \varepsilon^{2(p+q)}(\omega_{BS,\varepsilon} - \omega_o)\rangle_{\omega_o}(\zeta)\\ &= \langle \omega_{BS,\varepsilon} - \omega_o, \omega_{BS,\varepsilon} - \omega_o\rangle_{\omega_o}(\zeta)\end{aligned}$$

implying that $|\omega - \omega_o|_{\omega_o}(z) = |\omega_{BS,\varepsilon} - \omega_o|_{\omega_o}(\zeta)$. From here, we can recall the expansion (4.1) and obtain

$$|\omega - \omega_o|_{\omega_o}(z) \leq |\omega_{BS,\varepsilon} - \omega_o|_{\omega_o}(\zeta) \leq c|\zeta|^{2-2n} \leq c\varepsilon^{(2n-2)q} \leq c|z|^{(2n-2)q/p},$$

which implies, on the whole gluing region, that, for all $k \geq 0$, holds

$$|\nabla^k_{\omega_o}(\omega - \omega_o)|_{\omega_o} \leq cr^{m-k},$$

where $m = \min\{1, (2n-2)q/p\}$, showing again that ω is indeed a metric on $\hat{M} := Bl_x(M)$. Moreover, the closeness between the metric ω and the flat metric ω shows us that ω is suitable to perform analysis with, and hence we can try to search for a Chern-scalar constant balanced metric through a deformation argument.

As a final note, we can see that as in the Kähler case, we have information on the scalar curvature of the metrics we wish to construct. Also, for simplicity, from now on we shall assume $p + q = 1$.

Remark 4.1.1. As for the first Chern class of Kähler manifolds, we have that blowing up a point on a non Kähler metric yields

$$c_1^{BC}(\hat{M}) = c_1^{BC}(M) + (n-1)[\mathbb{P}^{n-1}],$$

where here we denoted with $[\mathbb{P}^{n-1}]$ the Poincarè dual of the $(2n-2)$-homology class defined by the exceptional divisor.

On the other hand, the construction of the metric ω explained above, gives that

$$[\omega^{n-1}]_{BC} = [\tilde{\omega}^{n-1}]_{BC} + [\varepsilon^2 \omega_{BS}]^{n-1},$$

and since $[\omega_{BS}] = -[\mathbb{P}^{n-1}]$, we get

$$[\omega^{n-1}]_{BC} = [\tilde{\omega}^{n-1}]_{BC} + (-1)^{n-1}\varepsilon^{2n-2}[\mathbb{P}^{n-1}].$$

As highligthed in [ACS1, Proposition 2.6], the value of the Chern scalar curvature of ω must be equal to the *Gauduchon degree* of the conformal class of ω, $\Gamma(\{\omega\})$, introduced in [Ga3, I.17]. Thus, as it holds

$$s^{ch}(\omega) = \Gamma(\{\omega\}) = \int_{\hat{M}} nRic^{ch}(\omega) \wedge \omega^{n-1} = c_1^{BC}(\hat{M})[\omega^{n-1}]_{BC}, \tag{4.2}$$

it follows that

$$\begin{aligned} s^{ch}(\omega) =& n(c_1^{BC}(M) + (n-1)[\mathbb{P}^{n-1}])([\tilde{\omega}^{n-1}]_{BC} + (-1)^{n-1}\varepsilon^{2n-2}[\mathbb{P}^{n-1}]) \\ =& s^{ch}(\tilde{\omega}) - n(n-1)\varepsilon^{2n-2}. \end{aligned}$$

4.2 Setting up the equation

We now wish to obtain a Chern-scalar constant balanced metric starting from the approximate solution, and as done in [BM], [AP], [Sz] and many others, we plan to do it through a deformation argument. In particular, since we wish to work inside the balanced class of ω, we will consider the balanced deformation introduced by [FWW], with the ansatz considered in [GS], that is

$$\omega_\varphi^{n-1} := \omega^{n-1} + i\partial\overline{\partial}(\varphi\omega^{n-2}), \quad \varphi \in C^\infty(M, \mathbb{R}) \text{ such that } \omega_\varphi^{n-1} > 0.$$

Thus the problem we are interested in solving, following what was done in [Sz], is the equation

$$s^{ch}(\omega_\varphi) = const. \tag{4.3}$$

for $\varphi \in C^\infty(M, \mathbb{R})$ such that $\omega_\varphi^{n-1} > 0$. Now, as observed in Remark 4.1.1, we can expect the solution to have a scalar Chern curvature near to the one of the background metric, thus we can rephrase equation (4.3) with

$$\mathcal{S}(\varphi) := s^{ch}(\omega_\varphi) - s^{ch}(\tilde{\omega}) = c \tag{4.4}$$

for $\varphi \in C^\infty(M, \mathbb{R})$ and $c \in \mathbb{R}$. Moreover, we can get rid of the unknown constant by rewriting the equation as

$$\tilde{\mathcal{S}}(\psi) := s^{ch}(\omega_\varphi) - s^{ch}(\tilde{\omega}) - \int_{\hat{M}} \varphi\omega^n = 0. \tag{4.5}$$

This last version of the equation encodes the unknown constant from equation (4.4), and we will see that it will help us in obtaining the invertibility of the linearized operator of $\tilde{\mathcal{S}}$. The interest in this linearized operator, is that we wish to solve the problem of equation (4.5) with Banach's Lemma in a neighborhood of zero, hence our next step will be to obtain the linearization at 0 of $\tilde{\mathcal{S}}$.

4.2.1 Computation of the linearized operator

We thus want to obtain an explicit expression for the operator $\mathcal{L}(u) := d_0 s^{ch}(u) = \frac{d}{dt}|_{t=0} s^{ch}(\omega_{t,u})$, where $\omega_{t,u}$ is an arbitrary curve of Hermitian metrics, lying in $[\omega^{n-1}]_{BC}$, such that $\omega_{0,u} = \omega$ and $\omega'_{t,u}(0) = u$. In order to do this, first of all, we recall that equation (4.2), gives us

$$s^{ch}(\omega)\omega^n = n\mathrm{Ric}^{ch}(\omega) \wedge \omega^{n-1}\,. \tag{4.6}$$

Then, using $\omega_{t,u}^{n-1} = \omega^{n-1} + ti\partial\bar\partial(u\omega^{n-2})$ and differentiating at $t = 0$ the relation above, we have that

$$\mathcal{L}(u)\omega^n + s^{ch}(\omega)\frac{d}{dt}|_{t=0}\omega_{t,u}^n = n\frac{d}{dt}|_{t=0}Ric^{ch}(\omega_{t,u}) \wedge \omega^{n-1} + nRic^{ch}(\omega) \wedge i\partial\bar\partial(u\omega^{n-2})\,. \tag{4.7}$$

Following then the computations and notations in Chapter 2 (i.e. the ones in [GS]), we have that

$$\begin{aligned} \frac{d}{dt}|_{t=0}\omega_{t,u}^n &= L(u)\omega^n = \frac{n}{n-1}i\partial\bar\partial(u\omega^{n-2}) \wedge \omega\,, \\ \frac{d}{dt}|_{t=0}Ric^{ch}(\omega_{t,u}) &= -\, i\partial\bar\partial L(u)\,. \end{aligned} \tag{4.8}$$

Then, using (4.8) in (4.7), we obtain

$$\mathcal{L}(u) = -\Delta_\omega L(u) + n\frac{i\partial\bar\partial(u\omega^{n-2}) \wedge \left(Ric^{ch}(\omega) - \frac{1}{n-1}s^{ch}(\omega)\omega\right)}{\omega^n}\,. \tag{4.9}$$

Remark 4.2.1. If we restrict $\mathcal{L}$ to $M_x := M \setminus \{x\}$ and assume $\tilde\omega$ to be balanced with constant Chern-scalar curvature case, if $u \in \ker \mathcal{L}$ then,

$$\int_{M_x} s^{ch}(\omega)|d\omega|^2 u\frac{\omega^n}{n!} = 0\,.$$

Indeed, as long as $b < 2n - 4$ we have that

$$\int_{M_x} \Delta_\omega Lu\frac{\omega^n}{n!} = 0\,, \quad \int_{M_x} i\partial\bar\partial(u\omega^{n-2} \wedge Ric^{ch}(\omega)) = 0\,,$$

while

$$\begin{aligned} \frac{n}{n-1}\int_{M_x} s^{ch}(\omega)i\partial\bar\partial(u\omega^{n-2}) \wedge \omega &= \int_{M_x} s^{ch}(\omega)L(u)\frac{\omega^n}{n!} \\ &= \frac{s^{ch}(\omega)}{n-1}\int_{M_x} |d\omega|^2 u\frac{\omega^n}{n!} \end{aligned}$$

Then,

$$0 = \int_{M_x} \mathcal{L}u\frac{\omega^n}{n!} = \frac{1}{n-1}\int_{M_x} s^{ch}(\omega)|d\omega|^2 u\frac{\omega^n}{n!}\,,$$

giving us the claim.

The next step will be to understand more about the linearization at 0 of $\tilde S$, which clearly is given by

$$\tilde{\mathcal{L}}u := d_0\tilde{\mathcal{S}}u = \mathcal{L}u + \int_{\hat M} u\omega^n.$$

4.2.2 Inverting the linearized operator

We will now focus on obtaining the invertibility of the operator $\tilde{\mathcal{L}}$. However, we need this invertibility to be uniform with respect to ε, so that, once we reformulate the equation as a fixed point problem, we will be able to solve it. Hence, in order to do this, we shall introduce suitable weighted spaces, once again following [BM], as done in the previous chapters. We then define (we can always assume up to rescaling that the open set on which the z coordinates are defined contains the region $\{|z| \leq 1\}$)

$$\rho = \rho_\varepsilon(z) := \begin{cases} \varepsilon & \text{on } r(z) \leq \varepsilon, \\ \text{non decreasing} & \text{on } \varepsilon \leq r(z) \leq 2\varepsilon, \\ r(z) & \text{on } 2\varepsilon \leq r(z) \leq 1/2, \\ \text{non decreasing} & \text{on } 1/2 \leq r(z) \leq 1, \\ 1 & \text{on } r(z) \geq 1, \end{cases}$$

We then introduce, for all $b \in \mathbb{R}$, the *weighted Hölder norm* as

$$\begin{aligned} ||u||_{C^{k,\alpha}_{\varepsilon,b}(M)} :=& \sum_{i=0}^{k} \sup_M |\rho^{b+i} \nabla^i_\varepsilon u|_\omega \\ &+ \sup_{d_\varepsilon(x,y)<inj_\varepsilon} \left| \min\left(\rho^{b+k+\alpha}(x), \rho^{b+k+\alpha}(y)\right) \frac{\nabla^k_\varepsilon u(x) - \nabla^k_\varepsilon u(y)}{d_\varepsilon(x,y)^\alpha} \right|_\omega, \end{aligned}$$

where inj_ε is the injectivity radius of the metric ω; and consequently define the corresponding *weighted Hölder spaces* $C^{k,\alpha}_{\varepsilon,b}(M)$, where $k \geq 0$, $\alpha \in (0,1)$ is the Hölder constant, and ε indicates the dependence on the pre-gluing metric ω obtained above. Hence we can interpret $\tilde{\mathcal{S}}$ as

$$\tilde{\mathcal{S}} : C^{4,\alpha}_{\varepsilon,b}(\hat{M}) \to C^{0,\alpha}_{\varepsilon,b+4}(\hat{M}),$$

and obtain the following result (we will keep following the strategy in [BM]), which will imply the uniform invertibility of the linearized operator. In order to be able to complete the proof, we will assume from now on that the background metric $\tilde{\omega}$ is Chern-Ricci flat.

Theorem 4.2.2. *For any $b \in (0, n-1)$, there exists $C > 0$ such that, for all $u \in C^{4,\alpha}_{b,\varepsilon}(\hat{M})$, we have*

$$\|u\|_{C^{4,\alpha}_{b,\varepsilon}(\hat{M})} \leq C \|\tilde{\mathcal{L}} u\|_{C^{0,\alpha}_{b+4,\varepsilon}(\hat{M})}.$$

Proof. Suppose by contradiction that statement does not hold. Hence, we can find sequences $\{\varepsilon_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}_{>0}$ and $\{u_k\}_{k\in\mathbb{N}} \subseteq C^{4,\alpha}_{\varepsilon_k,b}(\hat{M})$ such that

$$\varepsilon_k \underset{k\to+\infty}{\longrightarrow} 0\,, \quad ||u_k||_{C^{4,\alpha}_{\varepsilon_k,b}} = 1\,, \quad \forall\, k \in \mathbb{N}\,, \tag{4.10}$$

and

$$||\tilde{\mathcal{L}}u_k||_{C^{0,\alpha}_{\varepsilon_k,b+4}} < \frac{1}{k} \quad \forall\, k \in \mathbb{N}. \tag{4.11}$$

We will focus first on M_x. By applying Ascoli-Arzelà's Theorem, we have that $u_k \to u_\infty$ uniformly on compact subsets of M_x in the sense of $C_b^{4,\alpha}$, with respect to the background metric $\tilde{\omega}$. This implies in particular that on any compact subset of M_x, thanks to the fact that $\tilde{\omega}$ is a Chern-Ricci flat balanced metric, it holds

$$\mathcal{L}u_k \to \Delta_{\tilde{\omega}}(L_{\tilde{\omega}}u_\infty), \tag{4.12}$$

i.e. $\mathcal{L}u_k$ converges uniformly on compact sets to a continuous function on M_x. If we then fix a point $y \in M_x$, in the region where $\rho \equiv 1$, condition (4.11) implies that

$$\tilde{\mathcal{L}}u_k(y) \to 0,$$

which, combined with equation (4.12), implies that the real sequence $\int_{\hat{M}} u_k\omega^n$ has finite limit, hence by Lebesgue's Theorem we get

$$\int_{\hat{M}} u_k\omega^n \to \int_{M_x} u_\infty\tilde{\omega}^n. \tag{4.13}$$

If we now integrate $\tilde{\mathcal{L}}u_k$ on M_x, using equations (4.12) and (4.13) and assuming $b < 2n-4$, we obtain

$$0 = \int_{M_x} \tilde{\mathcal{L}}u_\infty\tilde{\omega}^n = \int_{M_x} \Delta_{\tilde{\omega}}(L_{\tilde{\omega}}u_\infty)\tilde{\omega}^n + Vol_{\tilde{\omega}}(M)\int_{M_x} u_\infty\tilde{\omega}^n = Vol_{\tilde{\omega}}(M)\int_{M_x} u_\infty\tilde{\omega}^n, \tag{4.14}$$

hence $\int_{M_x} u_\infty\tilde{\omega}^n = 0$, and thus

$$\Delta_{\tilde{\omega}}(L_{\tilde{\omega}}u_\infty) = 0,$$

from which follows that u_∞ is such that $Lu_\infty \equiv c \in \mathbb{R}$. Following then [GS], if we integrate on M_x the equation $0 = u_\infty L_{\tilde{\omega}}u_\infty = cu_\infty$, and assume $b < n-1$, we get

$$\begin{aligned} 0 &= \int_{M_x} \left(-u_\infty\Delta_{\tilde{\omega}}u_\infty + \frac{1}{n-1}|d\tilde{\omega}|^2_{\tilde{\omega}}u^2_\infty\right)\tilde{\omega}^n \\ &= \int_{M_x} \left(|\nabla_{\tilde{\omega}}u_\infty|^2 + \frac{1}{n-1}|d\tilde{\omega}|^2_{\tilde{\omega}}u^2_\infty\right)\tilde{\omega}^n = c\int_{M_x} u_\infty\tilde{\omega}^n = 0, \end{aligned} \tag{4.15}$$

implying that u_∞ is constant, which paired with $\int_{M_x} u_\infty\tilde{\omega}^n = 0$, gives us that $u_\infty \equiv 0$ on M_x.

We thus fix the compact set $M_c := M \setminus \{|z| < 1/2\}$, and focus on $A := \{|z| < 1/2\}$, on which we wish to obtain uniform convergence to zero. For convenience, we shall shift to the "large" coordinates ζ, i.e. the coordinates on the blow-up $\hat{X}$ defined outside the exceptional divisor. Recalling then that

$$\zeta = \varepsilon^{-1} z \quad \text{and} \quad |z| = \varepsilon|\zeta|,$$

we have the identification

$$A \simeq \tilde{A} = \tilde{A}_\varepsilon := \left\{ |\zeta| < \frac{1}{2}\varepsilon^{-1} \right\} \subseteq \hat{X},$$

and the last description will be the one we will use.

First of all, we shall rewrite ρ with respect to ζ on $\tilde{A}$, giving

$$\rho = \begin{cases} \varepsilon & \text{on } |\zeta| \leq 1, \\ \text{non decreasing} & \text{on } 1 \leq |\zeta| \leq 2, \\ \varepsilon|\zeta| & \text{on } 2 \leq |\zeta| \leq 1/2\varepsilon^{-1}. \end{cases}$$

It follows that, if we go back to $\{u_k\}_{k\in\mathbb{N}}$ and recall (4.10), we have in particular that on all $\tilde{A}_k := \tilde{A}_{\varepsilon_k}$ it holds

$$|\rho^b u_k| \leq c.$$

This suggests us to introduce the new sequence

$$U_k := \varepsilon_k^b u_k,$$

and using again (4.10), we obtain

$$\begin{cases} |U_k| \leq c & \text{on } |\zeta| \leq 1, \\ |U_k| \leq c & \text{on } 1 \leq |\zeta| \leq 2, \\ |U_k| \leq c|z|^{-b}(\zeta) & \text{on } 2 \leq |\zeta| \leq 1/2\varepsilon_k^{-1}, \end{cases}$$

and the same for its derivatives up to the fourth degree. These estimates for U_k bring us to consider a new weight function $\tilde{\rho} = \tilde{\rho}_k$ on $\tilde{A}_k$ given by

$$\tilde{\rho}(\zeta) = \begin{cases} 1 & \text{on } |\zeta| \leq 1, \\ \text{non decreasing} & \text{on } 1 \leq |\zeta| \leq 2, \\ |\zeta| & \text{on } 2 \leq |\zeta| \leq 1/2\varepsilon_k^{-1}, \end{cases} ,$$

with which we get that

$$|\tilde{\rho}^b U_k| \leq c, \tag{4.16}$$

and estimates also for $\nabla^m U_k$, $m = 1, ..., 4$, hence again by Ascoli-Arzelà's Theorem we have that $U_k \to U_\infty$ uniformly on compact sets of $\hat{X}$ (since $\tilde{A}_k \to \hat{X}$) in the sense of $\tilde{C}_b^{4,\alpha} = C_b^{4,\alpha}(\tilde{\rho})$, where this last space is the weighted Hölder space on $\hat{X}$ given by the weight $\tilde{\rho}$ and the metric ω_{BS}.

On the other hand, on any compact subset of $\hat{X}$, for sufficiently large k it holds

$$\rho^{b+4}\mathcal{L}u_k = -\frac{1}{n-1}\tilde{\rho}^{b+4}\mathcal{D}^*\mathcal{D}U_k, \tag{4.17}$$

where $\mathcal{D}^*\mathcal{D}$ is the Lichnerowitz operator corresponding to ω_{BS}. Thus, since it holds $\frac{1}{k} > ||\tilde{\mathcal{L}}u_k||_{C^{0,\alpha}_{\varepsilon_k,b+4}}$, taking the limit in (4.17) we obtain that U_∞ is in the kernel of $\mathcal{D}^*\mathcal{D}$ with respect to the Burns-Simanca metric ω_{BS}, and thus, applying Proposition 8.10 from [Sz], we get that U_∞ is necessarily constant, which needs to be zero as U_∞ decays at infinity (from inequality (4.16)), hence $U_k \xrightarrow{\tilde{C}_b^{4,\alpha}} 0$ uniformly on compact sets of $\hat{X}$.

In order to conclude, we will show that U_k admits a subsequence uniformly convergent to zero on the whole $\hat{X}$ in the sense $\tilde{C}_b^0$. This, combined with the scaled Schauder estimates, will imply that also $U_k \xrightarrow{\tilde{C}_b^{4,\alpha}} 0$ uniformly. On the other hand, this is equivalent to $u_k \to 0$ uniformly on $\{|z| < 1/2\}$ in $C^{4,\alpha}_{\varepsilon,b}$. Together with the fact that u_k converges uniformly to zero on M_c, it gives a contradiction with the fact that $||u_k||_{C^{4,\alpha}_{\varepsilon,b}} = 1$ for all $k \in \mathbb{N}$.

The final step of the proof will be to show that such subsequence necessarily exists. Indeed, if we assume by contradiction that such subsequence does not exist, then we can find a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \hat{X}$ and a $\delta > 0$ such that

$$R_k := |\zeta(x_k)| \to +\infty \tag{4.18}$$

and

$$\rho^b(x_k)|U_k(x_k)| \geq \delta \quad \forall k \geq 0, \tag{4.19}$$

where this last condition can be rewritten (up to choosing sufficiently large k) as

$$R_k^b|U_k(x_k)| \geq \delta \quad \forall k \geq 0. \tag{4.20}$$

If we then define $r_k := |z(x_k)|$, we have that $r_k = \varepsilon_k R_k$ for all $k \in \mathbb{N}$, from which (up to subsequences) we see that we can only fall into two cases:

- if $\lim_{k\to+\infty} r_k = r > 0$, then it means that we can assume $x_k \to x_\infty$, which combined with the uniform convergence to zero on compact sets (of M_x) of the sequence $\{u_k\}_{k\in\mathbb{N}}$ gives
$$0 < \delta \leq R_k^b|U_k(x_k)| = r_k^b u_k(x_k) \to 0,$$
i.e. a contradiction;

- if instead $\lim_{k\to+\infty} r_k = 0$, we take X' a copy of X, and for all $k \geq 0$ we introduce the holomorphic maps
$$\sigma_k : B_k \to A^* := A \setminus \{0\},$$
given by $\sigma_k(z') = r_k z'$, where $B_k := \{0 < |z'| < r_k/2\} \subseteq X'$, over which we can define the metrics
$$\theta_k := r_k^{-2}\sigma_k^*\omega,$$
and easily observe that $(B_k, \theta_k) \to (X', \omega_o)$, where ω_o here denotes the flat metric induced by the coordinates z'. Then it is natural to consider the functions on each B_k given by
$$W_k := r_k^b \sigma_k^* u_k\,, \quad \forall k \in \mathbb{N},$$
and the pullback weight function
$$\rho'(z') = \sigma_k^*\rho(z') = \begin{cases} \varepsilon_k & \text{on } |z'| \leq R_k^{-1} \\ \text{non increasing} & \text{on } R_k^{-1} < |z'| < 2R_k^{-1} \\ r_k|z'| & \text{on } 2R_k^{-1} \leq |z'| < \frac{1}{2}r_k^{-1}. \end{cases} \tag{4.21}$$
Now, if we pullback (4.10) with σ_k, we immediately obtain that $\{W_k\}_{k\in\mathbb{N}}$ is a sequence that is uniformly bounded on compact sets in the $C_b^{4,\alpha}$ sense, thus by Ascoli-Arzela's Theorem we can assume that $W_k \to W_\infty$, and still from pulling back (4.10) obtain that W_∞ is a $C^{4,\alpha}$-function on X' decaying to infinity. Moreover, analyzing the pieces of the pullback
$$\sigma_k^*(\mathcal{L}u_k) = \sigma_k^*(\Delta_\omega(Lu_k) + \frac{ni\partial\overline{\partial}(u_k\omega^{n-2}) \wedge (\mathrm{Ric}^{ch}(\omega) - \frac{1}{n-1}s^{ch}(\omega)}{\omega^n})$$
we can see that
  - $\sigma_k^*\Delta_\omega(Lu_k) = r_k^{-(b+4)}\Delta_{\theta_k}L_{\theta_k}W_k$, where L_{θ_k} is the operator L with θ_k substituting ω;
  - $\sigma_k^*(\mathrm{Ric}^{ch}(\omega) - \frac{1}{n-1}s^{ch}(\omega)) = \mathrm{Ric}^{ch}(\theta_k) - \frac{1}{n-1}s^{ch}(\theta_k)$;
  - $\sigma_k^*(i\partial\overline{\partial}(u_k\omega^{n-2}) = r_k^{-(b+4)}i\partial\overline{\partial}(W_k\theta_k^{n-2})$,

hence, pulling back with σ_k inequality (4.11) and taking the limit in k, we obtain that W_∞ is biharmonic on X'; and pulling back (4.10) and recalling (4.21), we obtain that W_∞ decays at infinity, implying necessarily that $W_\infty \equiv 0$ on X'. On the other hand, if we define the sequence $y_k := \sigma_k^{-1}(x_k) \in X'$, it is straighforward to see that $|y_k| = 1$ for all $k \in \mathbb{N}$, hence it can be assumed to be convergent to some y_∞, which combined with the limit of pullback through σ_k of (4.20), implies $W_\infty(y_\infty) > 0$, i.e. a contradicition with the fact that $W_\infty \equiv 0$.

Hence the thesis is proven. □

As a consequence we easily obtain the uniform invertibility.

Lemma 4.2.3. *The operator*

$$\tilde{\mathcal{L}}\colon C^{4,\alpha}_{\varepsilon,b}(\hat{M}) \to C^{0,\alpha}_{\varepsilon,b+4}(\hat{M})$$

is an isomorphism for $b \in (0, n-1)$.

Proof. Thanks to Theorem (4.2.2), we have that $\tilde{\mathcal{L}}$ is injective. Moreover, $\tilde{\mathcal{L}}$ is clearly elliptic and it has the same index of Δ^2_ω which is 0. This automatically guarantees the claim. □

Remark 4.2.4. If we compare this result with the Kähler case, we notice that our proof of the invertibility of (the limit of) $\tilde{\mathcal{L}}$ on M_x, imposes the condition $b < n-1$, which is stronger then $b < 2n-4$ which appears in the Kähler case (see Proposition 8.10 in [Sz]). Our additional restriction comes from the integration by parts needed to obtain the identity (4.15), showing that this stronger condition is consequence of the non-Kähler nature of the problem.

4.3 The fixed point problem

We can now reformulate (4.5) as a fixed point problem. In order to do so, we consider the expansion

$$s^{ch}(\omega_u) = s^{ch}(\omega) + \mathcal{L}u + Q(u)\,,$$

where Q is the quadratic part of $s^{ch}(\omega_u)$. Then, (4.5) can be rewritten as

$$s^{ch}(\omega) + \tilde{\mathcal{L}}u + Q(u) = 0\,.$$

Now, using (4.2.3), we obtain that

$$\mathcal{N}(u) := -\tilde{\mathcal{L}}^{-1}(s^{ch}(\omega) + Q(u)) = u\,. \tag{4.22}$$

So, in order to find a solution to (4.5), we need to show that

$$\mathcal{N}\colon C^{4,\alpha}_{b,\varepsilon}(\hat{M}) \to C^{4,\alpha}_{b,\varepsilon}(\hat{M})$$

is a contraction on a suitable open neighborhood of zero in $C^{4,\alpha}_{b,\varepsilon}$.

4.3.1 Determining the open set

In order to determine this open set, we observe that, if $\|\varphi\|_{C^{4,\alpha}_{-2,\varepsilon}} \leq C\varepsilon^{\tau}$ for some $C, \tau > 0$, then,

$$\|i\partial\bar{\partial}(\varphi\omega^{n-2})\|_{C^{2,\alpha}_{0,\varepsilon}} \leq C\|\varphi\omega^{n-2}\|_{C^{4,\alpha}_{-2,\varepsilon}} \leq C\|\varphi\|_{C^{4,\alpha}_{-2,\varepsilon}} \leq C\varepsilon^{\tau}\,, \tag{4.23}$$

where the second inequality is due to the fact that $\|\omega^{n-2}\|_{C^{4,\alpha}_{0,\varepsilon}} \leq C$. Up to choosing ε sufficiently small, this guarantees that $\omega_{\varphi}^{n-1} > 0$, hence provides a balanced metric, thanks to [M], as well as

$$\|\omega_{\varphi}^{n-1} - \omega^{n-1}\|_{C^{2,\alpha}_{0,\varepsilon}} = \|i\partial\bar{\partial}(\varphi\omega^{n-2})\|_{C^{2,\alpha}_{0,\varepsilon}} \leq C\varepsilon^{\tau}\,. \tag{4.24}$$

Moreover, arguing as in Remark 2.8 in [GS], we can fix a point $y \in \hat{M}$ and consider holomorphic coordinates so that, in y, ω is the identity and ω_{φ} is diagonal with eigenvalues λ_i. On the other hand, ω^{n-1} will be again the "identity" and ω_{φ}^{n-1} will have eigeinvalues Λ_i. But, thanks to (4.24), we know that

$$\Lambda_i = 1 + O(\varepsilon^{\tau})\,,$$

which implies that $\lambda_i = \left(\prod_{j\neq i}\Lambda_j\right)^{\frac{1}{n-1}} = 1 + O(\varepsilon^{\tau})$. This last fact readily guarantees that

$$\|\omega_{\varphi} - \omega\|_{C^{2,\alpha}_{0,\varepsilon}} \leq C\varepsilon^{\tau}\,, \tag{4.25}$$

which in particular gives that $\omega_{\varphi} \to \omega$ as $\varepsilon \to 0$. As in [GS], we then consider the open set

$$U_{\tau} := \{\varphi \in C^{4,\alpha}_{b,\varepsilon}(\hat{M}) \quad | \quad \|\varphi\|_{C^{4,\alpha}_{b,\varepsilon}} \leq C\varepsilon^{b+2+\tau}\}\,,$$

and we note that, if $\varphi \in U_{\tau}$, then

$$\|\varphi\|_{C^{4,\alpha}_{-2,\varepsilon}} \leq \varepsilon^{-(b+2)}\|\varphi\|_{C^{4,\alpha}_{b,\varepsilon}} \leq C\varepsilon^{\tau}\,, \tag{4.26}$$

where the first inequality is due to the fact that $\|\varphi\|_{C^{k,\alpha}_{a,\varepsilon}} \leq \varepsilon^{-b+a}\|\varphi\|_{C^{k,\alpha}_{b,\varepsilon}}$, for any $k \geq 0$, $a \leq b$, thanks to the definition of our weight. This inequality guarantees also that every $\psi \in U_{\tau}$, is not only small in the weighted sense, but it is so also in the standard sense, ensuring that our setting for the problem makes sense in this set.

We are thus left with the estimates to obtain that $\mathcal{N}$ preserves U_{τ} and is a contraction on it.

4.3.2 Estimates

We first show that $\mathcal{N}$ contracts distances on U_τ, which thanks to Theorem 4.2.3 reduces to showing that Q contracts distances. Thus, fixed $\varphi_1, \varphi_2 \in U_\tau$, the Mean value Theorem guarantees that there exists $t \in [0,1]$ such that, defined $\chi := t\varphi_1 + (1-t)\varphi_2$ (which clearly is also contained in U_τ), we have

$$Q(\varphi_1) - Q(\varphi_2) = d_\chi Q(\varphi_1 - \varphi_2) = (\mathcal{L}_\chi - \mathcal{L})(\varphi_1 - \varphi_2)\,.$$

We now need to compute $\mathcal{L}_\chi$, i.e. the differential $d_\chi \mathcal{S}$. As done before, we define $\omega_{\chi,u}(s) := \omega_\chi + si\partial\bar{\partial}(u\omega^{n-2})$, then,

$$\mathcal{L}_\chi(u) = \frac{d}{ds}|_{s=0} s^{ch}(\omega_{\chi,u}(s))\,.$$

But, differentiating again (4.6), we obtain that

$$\begin{aligned}\mathcal{L}_\chi(u)\omega_\chi^n =& n\frac{d}{ds}|_{s=0} Ric^{ch}(\omega_{\chi,u}(s)) \wedge \omega_\chi^{n-1} \\ &+ nRic^{ch}(\omega_\chi) \wedge \frac{d}{ds}|_{s=0}\omega_\chi^{n-1}(s) - s^{ch}(\omega_\chi)\frac{d}{ds}|_{s=0}\omega_{\chi,u}(s)^n\,.\end{aligned}$$

As done before, we have

$$\begin{aligned}\frac{d}{ds}|_{s=0}\omega_{\chi,u}(s)^{n-1} =& i\partial\bar{\partial}(u\omega^{n-1})\,, \\ \frac{d}{ds}|_{s=0}\omega_{\chi,u}(s)^{n} =& \frac{n}{n-1} i\partial\bar{\partial}(u\omega^{n-2}) \wedge \omega_\chi\,, \\ \frac{d}{ds}|_{s=0} Ric^{ch}(\omega_{\chi,u}(s)) =& -i\partial\bar{\partial}\frac{d}{ds}|_{s=0}\log(\omega_{\chi,u}(s)^n) = -i\partial\bar{\partial}\left(\frac{\frac{d}{ds}|_{s=0}\omega_{\chi,u}(s)^n}{\omega_\chi^n}\right) \\ =& -i\partial\bar{\partial}\left(\frac{i\partial\bar{\partial}(u\omega^{n-2}) \wedge \omega_\chi}{\omega_\chi^n}\right)\,.\end{aligned} \tag{4.27}$$

Then, defining

$$L_\chi(u) = \frac{n}{n-1}\frac{i\partial\bar{\partial}(u\omega^{n-2}) \wedge \omega_\chi}{\omega_\chi^n}$$

we have that

$$\mathcal{L}_\chi(u) = -\Delta_{\omega_\chi} L_\chi(u) + \frac{ni\partial\bar{\partial}(u\omega^{n-2}) \wedge (Ric^{ch}(\omega_\chi) - \frac{1}{n-1}s^{ch}(\omega_\chi)\omega_\chi)}{\omega_\chi^n}$$

Before going through the estimates, we need to explore the relation between the differential operators we are working with. First of all, we define the function

$$g(\chi) = \frac{\omega^n}{\omega_\chi^n},$$

then, for any function v at least C^2, we have

$$\Delta_{\omega_\chi} v = \frac{ni\partial\bar{\partial}v \wedge \omega_\chi^n - 1}{\omega_\chi^n} = g(\chi)\left(\Delta_\omega v + \frac{ni\partial\bar{\partial}v \wedge i\partial\bar{\partial}(\chi\omega^{n-2})}{\omega^n}\right), \tag{4.28}$$

which gives us that

$$\Delta_{\omega_\chi} v - \Delta_\omega v = (g(\chi) - 1)\Delta_\omega v + g(\chi)\left(\frac{ni\partial\bar{\partial}v \wedge i\partial\bar{\partial}(\chi\omega^{n-2})}{\omega^n}\right).$$

For the sake of simplicity, we will denote

$$E(v) := g(\chi)\left(\frac{ni\partial\bar{\partial}v \wedge i\partial\bar{\partial}(\chi\omega^{n-2})}{\omega^n}\right)$$

so that (4.28) can be rewritten as

$$\Delta_{\omega_\chi} v = g(\chi)\Delta_\omega v + E(v). \tag{4.29}$$

Moreover, we define

$$\begin{aligned} G(v) :=& \frac{i\partial\bar{\partial}(v\omega^{n-2}) \wedge (Ric^{ch}(\omega_\chi) - \frac{1}{n-1}s^{ch}(\omega_\chi)\omega_\chi)}{\omega_\chi^n} \\ &- \frac{i\partial\bar{\partial}(v\omega^{n-2}) \wedge (Ric^{ch}(\omega) - \frac{1}{n-1}s^{ch}(\omega)\omega)}{\omega^n} \\ =& \frac{i\partial\bar{\partial}(v\omega^{n-2}) \wedge \big(g(\chi)Ric^{ch}(\omega_\chi) - Ric^{ch}(\omega) - \frac{1}{n-1}(g(\chi)s^{ch}(\omega_\chi)\omega_\chi - s^{ch}(\omega)\omega)\big)}{\omega^n}. \end{aligned}$$

Then, using (4.29) and these new notations, we have

$$\begin{aligned} (\mathcal{L}_\chi - \mathcal{L})(v) =& -(g(\chi)\Delta_\omega L_\chi(v) - \Delta_\omega Lu) + E(L_\chi v) + G(v) \\ =& -g(\chi)\Delta_\omega(L_\chi - L)v - (g(\chi) - 1)\Delta_\omega Lv + E(L_\chi v) + G(v). \end{aligned} \tag{4.30}$$

We will then breakdown the estimates in a series of smaller lemmas which will be used to conclude.

Lemma 4.3.1. *Considering χ as above,*

$$\|g(\chi) - 1\|_{C^{2,\alpha}_{0,\varepsilon}} \leq C\varepsilon^\tau\,, \quad \|g(\chi)\|_{C^{2,\alpha}_{0,\varepsilon}} \leq 1 + C\varepsilon^\tau\,. \tag{4.31}$$

Proof. Obviously, it is sufficient to prove the first inequality, since the second one can be recovered by that one using the triangle inequality and the fact that $\|1\|_{C^{2,\alpha}_{0,\varepsilon}} = 1$. In order to prove the first one, we observe that

$$\begin{aligned}\omega^n - \omega_\chi^n =& \omega_\chi^{n-1} \wedge (\omega - \omega_\chi) + \omega \wedge (\omega^{n-1} - \omega_\chi^{n-1})\\ =& \omega^{n-1} \wedge (\omega - \omega_\chi) + i\partial\bar\partial(\chi\omega^{n-2}) \wedge (\omega - \omega_\chi) + \omega \wedge (\omega^{n-1} - \omega_\chi^{n-1})\,.\end{aligned}$$

Now, from this, we have

$$\begin{aligned}\|\omega^n - \omega_\chi^n\|_{C^{2,\alpha}_{0,\varepsilon}} \leq& C\|\omega - \omega_\chi\|_{C^{2,\alpha}_{0,\varepsilon}} + C\|i\partial\bar\partial(\chi\omega^{n-2})\|_{C^{2,\alpha}_{0,\varepsilon}}\|\omega - \omega_\chi\|_{C^{2,\alpha}_{0,\varepsilon}}\\ &+ C\|\omega^{n-1} - \omega_\chi^{n-1}\|_{C^{2,\alpha}_{0,\varepsilon}}\\ \leq& C\varepsilon^\tau\end{aligned}$$

where the last inequality is due to the (4.23), (4.24), (4.25). This last inequality readily implies that

$$\omega^n = \omega_\chi^n + O(\varepsilon^\tau)\,,$$

which gives us the claim. □

Lemma 4.3.2. *For ε sufficiently small, we have that, for any $v \in C^{2,\alpha}_{b,\varepsilon}$,*

$$\|E(v)\|_{C^{0,\alpha}_{b+2,\varepsilon}} \leq C\varepsilon^\tau\|v\|_{C^{2,\alpha}_{b,\varepsilon}}\,.$$

Proof. Recalling the definition of E, we can conclude that

$$\begin{aligned}\|E(v)\|_{C^{0,\alpha}_{b+2,\varepsilon}} \leq& C\|g(\chi)\|_{C^{0,\alpha}_{0,\varepsilon}}\|i\partial\bar\partial v\|_{C^{0,\alpha}_{b+2,\varepsilon}}\|i\partial\bar\partial(\chi\omega^{n-2})\|_{C^{0,\alpha}_{0,\varepsilon}}\\ \leq& C\|g(\chi)\|_{C^{0,\alpha}_{0,\varepsilon}}\|i\partial\bar\partial v\|_{C^{0,\alpha}_{b+2,\varepsilon}}\|i\partial\bar\partial(\chi\omega^{n-2})\|_{C^{2,\alpha}_{0,\varepsilon}}\,.\end{aligned}$$

But, now, using (4.31) and (4.23), we have that

$$\|E(v)\|_{C^{0,\alpha}_{b+2,\varepsilon}} \leq C(1+\varepsilon^\tau)\varepsilon^\tau\|v\|_{C^{2,\alpha}_{b,\varepsilon}} \leq C\varepsilon^\tau\|v\|_{C^{2,\alpha}_{b,\varepsilon}}\,.$$

□

Before showing the next Lemma, we notice that for any v at least C^2, it holds

$$\begin{aligned}
L_\chi v - Lv &= \frac{n}{n-1}\left(\frac{i\partial\bar{\partial}(v\omega^{n-2})\wedge\omega_\chi}{\omega_\chi^n} - \frac{i\partial\bar{\partial}(v\omega^{n-2})\wedge\omega}{\omega^n}\right) \\
&= \frac{n}{n-1}\left(g(\chi)\left(\frac{i\partial\bar{\partial}(v\omega^{n-2})\wedge\omega_\chi}{\omega^n}\right) - \frac{i\partial\bar{\partial}(v\omega^{n-2})\wedge\omega}{\omega^n}\right) \\
&= \frac{n}{n-1}\left(g(\chi)\left(\frac{i\partial\bar{\partial}(v\omega^{n-2})\wedge\omega_\chi}{\omega^n}\right) + (g(\chi)-1)Lv\right).
\end{aligned} \tag{4.32}$$

Lemma 4.3.3. *For ε sufficiently small, for any $v \in C^{4,\alpha}_{b,\varepsilon}$, we have that*

$$\|L_\chi v - Lv\|_{C^{2,\alpha}_{b+2,\varepsilon}} \leq C\varepsilon^\tau \|v\|_{C^{4,\alpha}_{b,\varepsilon}}.$$

Proof. Thanks to (4.32), we can obtain that

$$\begin{aligned}
\|L_\chi v - Lv\|_{C^{2,\alpha}_{b+2,\varepsilon}} \leq& C\Big(\|g(\chi)\|_{C^{2,\alpha}_{0,\varepsilon}}\|\omega_\chi-\omega\|_{C^{2,\alpha}_{0,\varepsilon}}\|i\partial\bar{\partial}(v\omega^{n-2})\|_{C^{2,\alpha}_{b+2,\varepsilon}} \\
&+\|g(\chi)-1\|_{C^{2,\alpha}_{0,\varepsilon}}\|Lv\|_{C^{2,\alpha}_{0,\varepsilon}}\Big) \\
\leq& C\left(\|g(\chi)\|_{C^{2,\alpha}_{0,\varepsilon}}\|\omega_\chi-\omega\|_{C^{2,\alpha}_{0,\varepsilon}}\|v\|_{C^{4,\alpha}_{b,\varepsilon}} + \|g(\chi)-1\|_{C^{2,\alpha}_{0,\varepsilon}}\|Lv\|_{C^{2,\alpha}_{0,\varepsilon}}\right).
\end{aligned}$$

Now, we can use (4.31), (4.25) and the continuity of $L\colon C^{4,\alpha}_{b,\varepsilon} \to C^{2,\alpha}_{b+2,\varepsilon}$ to obtain that

$$\|L_\chi v - Lv\|_{C^{2,\alpha}_{b+2,\varepsilon}} \leq C((1+\varepsilon^\tau)\varepsilon^\tau)\|v\|_{C^{4,\alpha}_{b,\varepsilon}} + \varepsilon^\tau\|v\|_{C^{4,\alpha}_{b,\varepsilon}}) \leq C\varepsilon^\tau\|v\|_{C^{4,\alpha}_{b,\varepsilon}},$$

concluding the proof. □

It remains to analize $G(v)$. In order to do so, we need two more estimates.

Lemma 4.3.4. *For ε sufficiently small,*

$$\|g(\chi)Ric^{ch}(\omega_\chi) - Ric^{ch}(\omega)\|_{C^{0,\alpha}_{2,\varepsilon}} \leq C\varepsilon^\tau, \quad \|g(\chi)s^{ch}(\omega_\chi)\omega_\chi - s^{ch}(\omega)\omega\|_{C^{0,\alpha}_{2,\varepsilon}} \leq C\varepsilon^\tau.$$

Proof. We have that

$$\begin{aligned}
g(\chi)Ric^{ch}(\omega_\chi) - Ric^{ch}(\omega) &= g(\chi)(Ric^{ch}(\omega_\chi) - Ric^{ch}(\omega)) + (g(\chi)-1)Ric^{ch}(\omega) \\
&= g(\chi)i\partial\bar{\partial}\log(g(\chi)) + (g(\chi)-1)Ric^{ch}(\omega)
\end{aligned}$$

On the other hand, we have

$$\|Ric^{ch}(\omega)\|_{C^{0,\alpha}_{2,\varepsilon}} \leq C, \quad \|s^{ch}(\omega)\|_{C^{0,\alpha}_{2,\varepsilon}} \leq C. \tag{4.33}$$

Indeed, we know that $\omega = \omega_0 + O(|z|^m)$ and then $\omega^n = \omega_0^n + O(|z|^m)$ which implies that

$$Ric^{ch}(\omega) = O(|z|^{m-2}), \quad s^{ch}(\omega) = O(|z|^{m-2})$$

implying that

$$\rho^2 Ric^{ch}(\omega) = O(|z|^m), \quad \rho^2 s^{ch}(\omega) = O(|z|^m)$$

giving us the claim. Now,

$$\begin{aligned} \|g(\chi)Ric^{ch}(\omega_\chi) - Ric^{ch}(\omega)\|_{C^{0,\alpha}_{2,\varepsilon}} \leq & \|g(\chi)\|_{C^{0,\alpha}_{0,\varepsilon}} \|i\partial\bar\partial \log g(\chi)\|_{C^{0,\alpha}_{2,\varepsilon}} \\ & + \|g(\chi) - 1\|_{C^{0,\alpha}_{0,\varepsilon}} \|Ric^{ch}\|_{C^{0,\alpha}_{2,\varepsilon}} \\ \leq & C(1+\varepsilon^\tau)\|\log g(\chi)\|_{C^{2,\alpha}_{0,\varepsilon}} + C\varepsilon^\tau . \end{aligned} \tag{4.34}$$

But if we now recall inequalities (4.31), we can use the Taylor expansion and obtain from (4.34) the first claim. As for the second one, we observe that

$$\begin{aligned} g(\chi)s^{ch}(\omega_\chi)\omega_\chi - s^{ch}(\omega)\omega =& g(\chi)(s^{ch}(\omega_\chi) - s^{ch}(\omega))\omega_\chi + g(\chi)s^{ch}(\omega)(\omega_\chi - \omega) \\ & + (g(\chi)-1)s^{ch}(\omega)\omega. \end{aligned}$$

Moreover, using (4.31), (4.25) and (4.33), we have that

$$\begin{aligned} \|g(\chi)s^{ch}(\omega_\chi)\omega_\chi - s^{ch}(\omega)\omega\|_{C^{0,\alpha}_{2,\varepsilon}} \leq & C(1+\varepsilon^\tau)\|(s^{ch}(\omega_\chi) - s^{ch}(\omega))\omega_\chi\|_{C^{0,\alpha}_{2,\varepsilon}} \\ & + C(1+\varepsilon^\tau)\varepsilon^\tau + C\varepsilon^\tau \\ \leq & C\varepsilon^\tau + C(1+\varepsilon^\tau)\|s^{ch}(\omega_\chi) - s^{ch}(\omega)\|_{C^{0,\alpha}_{2,\varepsilon}} \|\omega_\chi\|_{C^{0,\alpha}_{0,\varepsilon}} . \end{aligned} \tag{4.35}$$

Again, using (4.25), we have that

$$\|\omega_\chi\|_{C^{0,\alpha}_{0,\varepsilon}} \leq \|\omega\|_{C^{0,\alpha}_{0,\varepsilon}} + \|\omega_\chi - \omega\|_{C^{0,\alpha}_{0,\varepsilon}} \leq C(1+\varepsilon^\tau),$$

which put into (4.35) gives that

$$\|g(\chi)s^{ch}(\omega_\chi)\omega_\chi - s^{ch}(\omega)\omega\|_{C^{0,\alpha}_{2,\varepsilon}} \leq C\varepsilon^\tau + C(1+\varepsilon^\tau)^2\|s^{ch}(\omega_\chi) - s^{ch}(\omega)\|_{C^{0,\alpha}_{2,\varepsilon}} \tag{4.36}$$

On the other hand, we have

$$
\begin{aligned}
s^{ch}(\omega_\chi) - s^{ch}(\omega) =& \frac{nRic^{ch}(\omega_\chi)\wedge\omega_\chi^{n-1}}{\omega_\chi^n} - s^{ch}(\omega)\\
=& \frac{nRic^{ch}(\omega_\chi)\wedge\omega^{n-1}}{\omega_\chi^n} + \frac{nRic^{ch}(\omega_\chi)\wedge i\partial\bar\partial(\chi\omega^{n-2})}{\omega_\chi^n} - s^{ch}(\omega)\\
=& (g(\chi)-1)s^{ch}(\omega) + g(\chi)\frac{ni\partial\bar\partial\log g(\chi)\wedge\omega^{n-1}}{\omega^n}\\
&+ g(\chi)\frac{nRic^{ch}(\omega_\chi)\wedge i\partial\bar\partial(\chi\omega^{n-2})}{\omega^n}\\
=& (g(\chi)-1)s^{ch}(\omega) + g(\chi)\Delta_\omega\log(g(\chi))\\
&+ g(\chi)\frac{nRic^{ch}(\omega_\chi)\wedge i\partial\bar\partial(\chi\omega^{n-2})}{\omega^n}\,,
\end{aligned}
$$

and then, using again (4.31) and (4.33)

$$
\begin{aligned}
\|s^{ch}(\omega_\chi)-s^{ch}(\omega)\|_{C^{0,\alpha}_{2,\varepsilon}} \le& C\varepsilon^\tau + C(1+\varepsilon^\tau)\varepsilon^\tau\\
&+ C(1+\varepsilon^\tau)\|Ric^{ch}(\omega_\chi)\wedge i\partial\bar\partial(\chi\omega^{n-2}))\|_{C^{0,\alpha}_{2,\varepsilon}}\\
\le& C\varepsilon^\tau + C(1+\varepsilon^\tau)\|Ric^{ch}(\omega_\chi)\|_{C^{0,\alpha}_{2,\varepsilon}}\|i\partial\bar\partial(\chi\omega^{n-2})\|_{C^{0,\alpha}_{0,\varepsilon}}\,.
\end{aligned}
$$

But, we have

$$
\begin{aligned}
\|Ric^{ch}(\omega_\chi)\|_{C^{0,\alpha}_{2,\varepsilon}} \le \|Ric^{ch}(\omega)\|_{C^{0,\alpha}_{2,\varepsilon}} + \|i\partial\bar\partial\log g(\chi)\|_{C^{0,\alpha}_{2,\varepsilon}} \le C(1+\varepsilon^\tau)\\
\|i\partial\bar\partial(\chi\omega^{n-2})\|_{C^{0,\alpha}_{0,\varepsilon}} \le \|\chi\omega^{n-2}\|_{C^{2,\alpha}_{-2,\varepsilon}} \le C\|\chi\|_{C^{4,\alpha}_{-2,\varepsilon}} \le C\varepsilon^\tau
\end{aligned}
\tag{4.37}
$$

where the last inequality is due to (4.26). Putting (4.37) into (4.36), we have the claim. □

Thus, using Lemma (4.3.4), we can conclude that

$$
\begin{aligned}
&\|G(v)\|_{C^{0,\alpha}_{b+4,\varepsilon}}\\
\le\ & C\|\partial\bar\partial(v\omega^{n-2})\|_{C^{0,\alpha}_{b+2}}\left\|g(\chi)Ric^{ch}(\omega_\chi) - Ric^{ch}(\omega) - \frac{g(\chi)s^{ch}(\omega_\chi)\omega_\chi - s^{ch}(\omega)\omega}{n-1}\right\|_{C^{0,\alpha}_{2,\alpha}}\\
\le\ & C\varepsilon^\tau\|v\|_{C^{4,\alpha}_{b,\varepsilon}}
\end{aligned}
\tag{4.38}
$$

We are finally ready to prove that $\mathcal{N}$ is a contraction operator on U_τ.

Proposition 4.3.5. *For ε sufficiently small and $b < n-1$, the operator $\mathcal{N}$ is a contraction and $\mathcal{N}(U_\tau) \subseteq U_\tau$.*

Proof. Consider $v = \varphi_1 - \varphi_2$ as above,

$$\|\mathcal{N}(\varphi_1) - \mathcal{N}(\varphi_2)\|_{C^{4,\alpha}_{b,\varepsilon}} \leq C\|(\mathcal{L}_\chi - \mathcal{L})v\|_{C^{0,\alpha}_{b+4,\varepsilon}} .$$

Using (4.30), (4.31), Lemma (4.3.3), Lemma (4.3.2) and (4.38) and the continuity of $\Delta_\omega \colon C^{4,\alpha}_{b,\varepsilon} \to C^{2,\alpha}_{b+2}$ and that of $L \colon C^{2,\alpha}_{b+2,\varepsilon} \to C^{0,\alpha}_{b+4,\varepsilon}$, we have

$$\|(\mathcal{L}_\chi - \mathcal{L})v\|_{C^{0,\alpha}_{b+4,\varepsilon}} \leq C\varepsilon^\tau \|v\|_{C^{4,\alpha}_{b,\varepsilon}}$$

which, after choosing ε sufficiently small, guarantees that $\mathcal{N}$ is a contraction. Now fix $\varphi \in U_\tau$, we have that

$$\begin{aligned}
\|\mathcal{N}(\varphi)\|_{C^{4,\alpha}_{b,\varepsilon}} &\leq \|\mathcal{N}(0)\|_{C^{4,\alpha}_{b,\varepsilon}} + \|\mathcal{N}(\varphi) - \mathcal{N}(0)\|_{C^{4,\alpha}_{b,\varepsilon}} \leq \|\mathcal{N}(0)\|_{C^{4,\alpha}_{b,\varepsilon}} + C\varepsilon^\tau\|\varphi\|_{C^{4,\alpha}_{b,\varepsilon}} \\
&\leq \|\mathcal{N}(0)\|_{C^{4,\alpha}_{b,\varepsilon}} + C\varepsilon^{2\tau+b+2} \leq \|\tilde{\mathcal{L}}^{-1}(s^{ch}(\omega))\|_{C^{4,\alpha}_{b,\varepsilon}} + C\varepsilon^{2\tau+b+2} \\
&\leq C\|s^{ch}(\omega)\|_{C^{0,\alpha}_{b+4,\varepsilon}} + C\varepsilon^{2\tau+b+2} .
\end{aligned}$$

On the other hand,

$$\|s^{ch}(\omega)\|_{C^{0,\alpha}_{b+4,\varepsilon}} \leq C\varepsilon^{p(m+b+2)} ,$$

from which follows

$$\|\mathcal{N}(\varphi)\|_{C^{4,\alpha}_{b,\varepsilon}} \leq C\varepsilon^{p(m+b+2)} + C\varepsilon^{2\tau+b+2} \leq C\varepsilon^{\min\{\tau, pm-q(b+2)-\tau\}}\varepsilon^{\tau+b+2} .$$

It is then sufficient to notice that τ can be chosen such that $pm - (1-p)(b+2) > \tau > 0$, giving us the claim the claim.

□

Hence Theorem 4.0.1 is proven.

As anticipated, our plan is to extend the proof to the whole Chern-scalar constant balanced case, and we are currently in the process to overcome the analyitical difficulties arising from the non-Kähler setting.

Chapter 5

Real semisimple Lie groups and balanced metrics

Despite our work has been mostly focusing on gluing constructions, another very common technique to search for special metrics is the use of symmetries. The work in this last chapter is joint work with Fabio Podestà and it can be found in [GiPo]. The paper lies in the realm of Lie groups and homogeneous spaces, and consists of a construction of a class of spaces admitting balanced metrics, and satisfying some curvature and topological constraints. The main statements are the following.

Theorem 5.0.1. *Every non-compact simple Lie group* $\mathrm{G_o}$ *of even dimension and of inner type admits an invariant complex structure* J *and* ω *an invariant balanced* J*-Hermitian metric. Moreover, if* Γ *is a cocompact lattice, the quotient* $M = \Gamma\backslash G_o$ *inherits the balanced structure.*

Proposition 5.0.2. *Let* $\mathrm{G_o}$ *be a non-compact simple group of even dimension and of inner type together with a co-compact lattice* $\Gamma \subset \mathrm{G_o}$*. If* $\mathcal{M} = \Gamma\backslash\mathrm{G_o}$ *is endowed with a standard complex structure and a Hermitian balanced metric* h*, then the Chern Ricci form* ρ *of* h *never vanishes and the Kodaira dimension* $\kappa(M) = -\infty$*.*

By the end, we will notice that the spaces obtained form a quite wide class of complex manifolds admitting balanced metrics, which have vanishing first Chern class but non-vanishing first Bott-Chern class, showing again how the absence of the $\partial\overline{\partial}$-lemma in the non-Kähler world makes things a lot more unpredictable.

We will also see that the class of spaces constructed do not admit any SKT metric, providing a new class of spaces in which the Fino-Vezzoni conjecture holds. In particular we obtained:

Theorem 5.0.3. *Let* $\mathrm{G_o}$ *be a non-compact simple even-dimensional Lie group of inner type endowed with the invariant complex structure* J *as in Theorem 5.0.1. If* Γ *is a co-compact lattice of* $\mathrm{G_o}$*, then the complex manifold* $(\mathcal{M}, J)$ *with* $\mathcal{M} = \Gamma\backslash\mathrm{G_o}$ *does not carry any pluriclosed metric.*

5.1 Preliminaries

Let $\mathfrak{g}_o$ be a real simple $2n$-dimensional Lie algebra and let $\mathrm{G_o}$ be a connected Lie group with Lie algebra $\mathfrak{g}_o$. It is well known that either the complexification $\mathfrak{g}_o^c$ is a complex simple Lie algebra (and in this case $\mathfrak{g}_o$ is called *absolutely simple*) or $\mathfrak{g}_o$ is the realification $\mathfrak{g}_\mathbb{R}$ of a complex simple Lie algebra $\mathfrak{g}$ (see e.g. [He]).

When $\mathfrak{g}_o$ is even dimensional, it is known ([Mo], see also [Sas]) that $\mathfrak{g}_o$ admits an invariant complex structure, namely an endomorphism $\mathrm{J} \in \mathrm{End}\,(\mathfrak{g}_\mathrm{o})$ with $\mathrm{J}^2 = -\mathrm{Id}$ which extends by left translation to an almost complex structure on $\mathrm{G_o}$ with vanishing Nijenhuis tensor. This last condition can be written at the level of the Lie algebra $\mathfrak{g}_o^c$ as

$$\mathfrak{g}_o^c = \mathfrak{g}_o^{1,0} \oplus \mathfrak{g}_o^{0,1}, \qquad [\mathfrak{g}_o^{1,0}, \mathfrak{g}_o^{1,0}] \subseteq \mathfrak{g}_o^{1,0},$$

where $\mathfrak{g}_o^{1,0}, \mathfrak{g}_o^{0,1}$ are the $+i, -i$-eigenspace of J on $\mathfrak{g}_o^c$ respectively.

When $\mathrm{G_o}$ is non-compact, a result due to Borel ([Bo]), guarantees the existence of a discrete, torsion-free cocompact lattice $\Gamma \subset \mathrm{G_o}$ so that $\mathcal{M} := \Gamma\backslash\mathrm{G_o}$ is compact and the left-invariant complex structure J on $\mathrm{G_o}$ descends to a complex structure J on $\mathcal{M}$.

When $\mathrm{G_o}$ is compact and even-dimensional, i.e. $\mathfrak{g}_o$ is of compact type, we recall that the existence of an invariant complex structure was already established by Samelson ([Sam]), while in [Pi] it was shown that every invariant complex structure on $\mathrm{G_o}$ is obtained by means of Samelson's construction.

If we now consider an even-dimensional simple Lie group $\mathrm{G_o}$ and a compact quotient $\mathcal{M}$ endowed with an invariant complex structure J, we are interested in the existence of special Hermitian metrics h. The following proposition states a known fact, namely the non-existence of (invariant) Kähler structures.

Proposition 5.1.1. *Let* $\mathrm{G_o}$ *be a semisimple Lie group endowed with a left invariant complex structure* J *and let* $\Gamma \subset \mathrm{G_o}$ *be a cocompact lattice so that* $\mathcal{M} = \Gamma\backslash\mathrm{G_o}$ *is compact. Then the group* $\mathrm{G_o}$ *does not admit any invariant Kähler metric and* $\mathcal{M}$ *is not Kähler.*

Proof. The first assertion is contained in [Chu], but we give here an elementary proof. If ω is an invariant symplectic form on $\mathfrak{g}_o$, then the closedness condition $d\omega = 0$ can be written as follows for $x, y, z \in \mathfrak{g}_o$

$$\omega([x, y], z) + \omega([z, x], y) + \omega([y, z], x) = 0. \tag{5.1}$$

If B denotes the non-degenerate Cartan-Killing form of $\mathfrak{g}_o$, then we can define the endomorphism $F \in \mathrm{End}\,(\mathfrak{g}_o)$ by $B(Fx, y) = \omega(x, y)$ $(x, y \in \mathfrak{s})$ so that using the biinvariance of B, (5.1) can be written as

$$B(F([x, y]), z) - B([Fx, y], z) - B([x, Fy], z) = 0,$$

hence F turns out to be a derivation of $\mathfrak{g}_o$. As $\mathfrak{g}_o$ is semisimple, there exists a unique $u \in \mathfrak{g}_o$ with $F = \mathrm{ad}(u)$, so that for $x, y \in \mathfrak{g}_o$, $\omega(x, y) = B([u, x], y)$ and therefore $u \in \ker \omega$, a contradiction.

We now suppose that the compact complex manifold $\mathcal{M}$ has a Kähler metric with Kähler form ω. Using ω and a symmetrization procedure that goes back to [Bel], we now construct an *invariant* Kähler form on $\mathrm{G_o}$, obtaining a contradiction. We fix a basis $x_1, ..., x_{2n}$ of $\mathfrak{g}_o$ and we extend each vector as a left invariant vector fields on $\mathrm{G_o}$; these vector fields can be projected down to M as vector fields $x_1^*, \ldots, x_{2n}^*$ that span the tangent space TM at each point. As $\mathrm{G_o}$ is semisimple, we can find a biinvariant volume form $d\mu$, that also descends to a volume form on $\mathcal{M}$. We now define a left-invariant non-degenerate 2-form ϕ on $\mathrm{G_o}$ by setting

$$\phi_e(x_i, x_j) := \int_M \omega(x_i^*, x_j^*)\, d\mu.$$

As ϕ is left invariant and ω is closed, we have for $i, j, k = 1, \ldots, 2n$

$$\begin{aligned} 3\, d\phi(x_i, x_j, x_k) &= -\sum_{\text{cyclic } (i,j,k)} \phi([x_i, x_j], x_k) = -\int_M \sum_{\text{cyclic } (i,j,k)} \omega([x_i^*, x_j^*], x_k^*)\, d\mu \\ &= -\int_M \sum_{\text{cyclic } (i,j,k)} x_i^* \omega(x_j^*, x_k^*)\, d\mu. \end{aligned}$$

As $\mathcal{L}_{x_i^*} d\mu = 0$ for every i, we have

$$\int_M x_i^* \omega(x_j^*, x_k^*)\, d\mu = \int_M \mathcal{L}_{x_i^*}(\omega(x_j^*, x_k^*)\, d\mu) = 0$$

by Stokes' theorem and therefore we obtain that $d\phi = 0$, hence ϕ is invariant and symplectic, a contradiction. $\square$

Therefore we are interested in the existence of special Hermitian metrics on the complex manifold $(\mathcal{M}, \mathrm{J})$, in particular balanced and pluriclosed metrics, when the group $\mathrm{G_o}$ is of non-compact type.

The case of a simple Lie algebra $\mathfrak{g}_o$ which is the realification of a complex simple Lie algebra $\mathfrak{g}$ can be easily treated and will be dealt with in subsection 5.1.3.

We will now focus on some subclasses of simple real algebras, namely those which are absolutely simple and of inner type.

5.1.1 Simple Lie algebras of inner type

Let $\mathfrak{g}_o$ be an absolutely simple real algebra (i.e. $\mathfrak{g}_o^c$ is a simple Lie algebra) of non-compact type. It is well-known that $\mathfrak{g}_o$ admits a Cartan decompositon

$$\mathfrak{g}_o = \mathfrak{k} + \mathfrak{p},$$

where $\mathfrak{k}$ is a maximal compactly embedded subalgebra and

$$[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k},$$

so that $(\mathfrak{g}_o, \mathfrak{k})$ is a symmetric pair. Moreover the algebra $\mathfrak{g}_o$ is said to be *of inner type* when the symmetric pair $(\mathfrak{g}_o, \mathfrak{k})$ is of inner type, i.e. when a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{g}_o$ or equivalently its complexification $\mathfrak{t}^c$ is a Cartan subalgebra of $\mathfrak{g}_o^c$. Using the notation as in [He], p. 126, we obtain the list of all inner symmetric pairs $(\mathfrak{g}_o, \mathfrak{k})$ of non-compact type with $\mathfrak{g}_o$ simple and even dimensional (Table 1).

Type	$\mathfrak{g}$	$\mathfrak{k}$	conditions
A	$\mathfrak{su}(p,q)$	$\mathfrak{su}(p)+\mathfrak{su}(q)+\mathbb{R}$	$p \geq q \geq 1,\ p+q$ odd
B	$\mathfrak{so}(2p+1,2q)$	$\mathfrak{so}(2p+1)+\mathfrak{so}(2q)$	$p \geq 0, q \geq 1,\ p+q$ even
C	$\mathfrak{sp}(2n,\mathbb{R})$	$\mathfrak{su}(2n)+\mathbb{R}$	$n \geq 1$
C	$\mathfrak{sp}(p,q)$	$\mathfrak{sp}(p)+\mathfrak{sp}(q)$	$p,q \geq 1,\ p+q$ even
D	$\mathfrak{so}(4n)^*$	$\mathfrak{su}(2n)+\mathbb{R}$	$n \geq 2$
D	$\mathfrak{so}(2p,2q)$	$\mathfrak{so}(2p)+\mathfrak{so}(2q)$	$p,q \geq 1, p+q$ even ≥ 4
G	$\mathfrak{g}_{2(2)}$	$\mathfrak{su}(2)+\mathfrak{su}(2)$	
F	$\mathfrak{f}_{4(-20)}$	$\mathfrak{so}(9)$	
F	$\mathfrak{f}_{4(4)}$	$\mathfrak{su}(2)+\mathfrak{sp}(3)$	
E	$\mathfrak{e}_{6(2)}$	$\mathfrak{su}(2)+\mathfrak{su}(6)$	
E	$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(10)+\mathbb{R}$	
E	$\mathfrak{e}_{8(8)}$	$\mathfrak{so}(16)$	
E	$\mathfrak{e}_{8(-24)}$	$\mathfrak{su}(2)+\mathfrak{e}_7$	

Table 5.1: Inner symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ of non-compact type with $\mathfrak{g}$ simple and even dimensional.

5.1.2 Invariant complex structures

In this section we will describe how to construct invariant complex structures on even-dimensional absolutely simple non-compact Lie algebras $\mathfrak{g}_o$ of inner type.

We fix a maximal abelian subalgebra $\mathfrak{t} \subseteq \mathfrak{k}$, so that $\mathfrak{h} := \mathfrak{t}^c$ is a Cartan subalgebra of $\mathfrak{g} := \mathfrak{g}_o^c$. Note that if $\mathfrak{g}_o$ is even dimensional , the same holds for $\mathfrak{t}$. The corresponding root system is denoted by R and we have the following decompositions

$$\mathfrak{k}^c = \mathfrak{t}^c \oplus \bigoplus_{\alpha \in R_{\mathfrak{k}}} \mathfrak{g}_\alpha, \quad \mathfrak{p}^c = \bigoplus_{\alpha \in R_{\mathfrak{p}}} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ denotes the root space relative to $\alpha \in R$. A root α will be called *compact* (resp. *non-compact*), when $\mathfrak{g}_\alpha \subseteq \mathfrak{k}^c$ (resp. $\mathfrak{g}_\alpha \subseteq \mathfrak{p}^c$) and the set of all compact (resp. non-compact) roots is denoted by $R_{\mathfrak{k}}$ (resp. $R_{\mathfrak{p}}$). It is a standard fact that $\mathfrak{u} := \mathfrak{k} + i\mathfrak{p} \subseteq \mathfrak{g}$ is a compact real form of $\mathfrak{g}$ and that we can choose the standard Weyl basis $\{E_\alpha\}_{\alpha \in R}$ of root spaces so that

$$\tau(E_\alpha) = -E_{-\alpha}, \qquad B(E_\alpha, E_{-\alpha}) = 1, \qquad [E_\alpha, E_{-\alpha}] = H_\alpha$$

where τ denotes the anticomplex involution defining $\mathfrak{u}$, B is the Cartan Killing form of $\mathfrak{g}$ and H_α is the B-dual of α (see e.g. [He]). If σ is the involutive anticomplex map defining $\mathfrak{g}_o$, we then have that

$$\begin{aligned} \sigma(E_\alpha) &= -E_{-\alpha}, \quad \alpha \in R_{\mathfrak{k}}, \\ \sigma(E_\alpha) &= E_{-\alpha}, \quad \alpha \in R_{\mathfrak{p}}. \end{aligned}$$

If we fix an ordering , namely a splitting $R = R^+ \cup R^-$ with $R^- = -R^+$ and $(R^+ + R^+) \cap R \subseteq R^+$, we can define a subalgebra

$$\mathfrak{q} := \mathfrak{h}_1 \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha,$$

where $\mathfrak{h}_1 \subset \mathfrak{h}$ is a subspace so that $\mathfrak{h}_1 \oplus \sigma(\mathfrak{h}_1) = \mathfrak{h}$. The subalgebra $\mathfrak{q} \subset \mathfrak{g}$ defined in this way satisfies

$$\mathfrak{g} = \mathfrak{q} \oplus \sigma(\mathfrak{q})$$

and therefore it defines a complex structure J on $\mathfrak{g}_o$ with the property that $\mathfrak{q} = \mathfrak{g}_o^{10}$. This complex structure depends on the arbitrary choice of $\mathfrak{h}_1$, i.e. on the arbitrary choice of a complex structure on $\mathfrak{t}$.

We remark that the complex structure J enjoys the further property of being $\mathrm{ad}(\mathfrak{t})$-invariant, namely

$$[\mathrm{ad}(x), \mathrm{J}] = 0, \quad \mathrm{x} \in \mathfrak{t}.$$

Therefore if $\mathrm{G_o}$ is a Lie group with Lie algebra $\mathfrak{g}_o$, then J extends to a left-invariant complex structure on $\mathrm{G_o}$ and it will be also right-invariant with respect to right translations by elements $h \in \mathrm{T} := \exp(\mathfrak{t})$ (note that T might be non-compact, unless $\mathrm{G_o}$ has finite center).

We will call such an invariant complex structure *standard*.

Remark 5.1.2. In [Sn] the class of (simple) real Lie algebras of inner type is called "Class I" and it is then proved that *every* invariant complex structure in these algebras are standard, with respect to a suitable choice of a Cartan subalgebra (such complex structures are called regular in [Sn]).

5.1.3 Invariant metrics and the balanced condition

Let $\mathcal{M}$ be a compact complex manifold of the form $\Gamma\backslash\mathrm{G_o}$, endowed with a complex structure J which is induced by a standard invariant complex structure J on $\mathrm{G_o}$, as in the previous section. It is clear that any left invariant J-Hermitian metric h on $\mathrm{G_o}$ induces an Hermitian metric $\bar{h}$ on $\mathcal{M}$ and $\bar{h}$ is balanced or pluriclosed if and only if h is so. For the converse, we prove the following

Proposition 5.1.3. *If* $(\mathcal{M}, \mathrm{J})$ *admits a balanced (pluriclosed) Hermitian metric, there exists a left invariant and right* T*-invariant Hermitian metric on* $\mathrm{G_o}$ *which is balanced (pluriclosed resp.).*

Proof. Suppose we have a balanced metric h on M with associated fundamental form ω. Then using the same notation and arguments as in the proof of Prop.5.1.1, we define a left-invariant positive $(n-1, n-1)$-form ϕ on G_o as follows

$$\phi_e(x_{i_1}, \ldots, x_{i_{2n-2}}) := \int_M \omega^{n-1}(x^*_{i_1}, \ldots, x^*_{i_{2n-2}})\, d\mu.$$

As $d\omega^{n-1} = 0$, we obtain that also $d\phi = 0$. Therefore, we can find an unique $(1,1)$-form $\hat{\omega}$ so that $\hat{\omega}^{n-1} = \phi$ (see [M]) and the metric given by $\hat{\omega}$ is balanced. As ϕ is left invariant, so is $\hat{\omega}$ by uniqueness. Now, the group $\mathrm{Ad}(\mathrm{T})$ is compact and using a standard avaraging process we can make ϕ_e also $\mathrm{Ad}(\mathrm{T})$-invariant. This means that ϕ is also invariant under right T-translations. Again, by the uniqueness, the same will hold true for $\hat{\omega}$.

As for the pluriclosed condition, the lifted metric from $\mathcal{M}$ to $\mathrm{G_o}$ is clearly pluriclosed and can be made T-invariant by a standard averaging. □

Remark 5.1.4. We can now deal with the case when $\mathfrak{g}_o$ is the realification of a simple Lie algebra $\mathfrak{g}$. In this case the complex structure J commutes with $\mathrm{ad}(\mathfrak{g}_o)$ and $\mathfrak{g}_o = \mathfrak{u} + i\mathfrak{u}$ is a Cartan decomposition, where $\mathfrak{u}$ is a compact real form of $\mathfrak{g}$. Let $\mathrm{G_o}$ be a real group with algebra $\mathfrak{g}_o$ and let $\mathcal{U}$ be the compact subgroup with algebra $\mathfrak{u}$. Then the metric h which coincides with $-B$ on $\mathfrak{u}$, with B on $i\mathfrak{u}$ and such that $h(\mathfrak{u}, i\mathfrak{u}) = 0$ is a Hermitian metric which is balanced. Indeed, h is $\mathrm{Ad}(\mathcal{U})$-invariant and therefore the corresponding $\delta\omega$ is $\mathrm{Ad}(\mathcal{U})$-invariant 1-form, hence it vanishes identically. This is consistent with the fact that complex parallelizable manifolds carry balanced metrics as they carry Chern-flat metrics, as noted in [Ga1], p. 121 (see also [AG], [Gr]).

On the other hand, $\mathrm{G_o}$ admits no invariant pluriclosed metric. Indeed, any such metric h can be avaraged to produce an $\mathrm{Ad}(\mathcal{U})$-invariant pluriclosed metric, which would be balanced by the previous argument. This is not possible, as a metric which is balanced and pluriclosed at the same time has to be Kähler (see e.g. [AI]), contrary to Prop 5.1.1.

We now focus on the case where $\mathfrak{g}_o$ is absolutely simple of inner type, endowed with an invariant standard complex structure. We fix a Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{k}$ with corresponding root system $R = R_{\mathfrak{k}} \cup R_{\mathfrak{p}}$ as in section 5.1.1 and we consider an ordering $R = R^+ \cup R^-$ giving an invariant complex structure $\mathrm{J_o}$ on $\mathfrak{g}_o/\mathfrak{t}$. We extend $\mathrm{J_o}$ to an invariant complex structure J on $\mathfrak{g}_o$.

We also fix a basis of a complement of $\mathfrak{t}$ in $\mathfrak{g}_o$

$$v_\alpha := \frac{1}{\sqrt{2}}(E_\alpha - E_{-\alpha}),\ w_\alpha := \frac{i}{\sqrt{2}}(E_\alpha + E_{-\alpha}),\ \alpha \in R^+_{\mathfrak{k}},$$

$$v_\alpha := \frac{1}{\sqrt{2}}(E_\alpha + E_{-\alpha}),\ w_\alpha := \frac{i}{\sqrt{2}}(E_\alpha - E_{-\alpha}),\ \alpha \in R^+_{\mathfrak{p}},$$

so that $v_\alpha, w_\alpha \in \mathfrak{g}_o$ for every $\alpha \in R^+$ and moreover

$$\begin{aligned} Jv_\alpha &= w_\alpha,\ Jw_\alpha = -v_\alpha, \\ [H, v_\alpha] &= -i\alpha(H)w_\alpha,\ H \in \mathfrak{h}, \\ [v_\alpha, w_\alpha] &= iH_\alpha,\ \alpha \in R^+_{\mathfrak{k}}, \\ [v_\alpha, w_\alpha] &= -iH_\alpha,\ \alpha \in R^+_{\mathfrak{p}}. \end{aligned}$$

We now construct invariant Hermitian metrics h on $\mathfrak{g}_o$. First, we define h on $\mathfrak{t}$ by choosing a J-Hermitian metric $h_{\mathfrak{t}}$ on $\mathfrak{t}$. If we set $\mathfrak{m}_\alpha := \mathrm{Span}\{v_\alpha, w_\alpha\}_{\alpha \in R^+}$, we define for $\alpha \neq \beta \in R^+$

$$\begin{gathered} h(\mathfrak{t}, \mathfrak{m}_\alpha) = 0, \quad h(\mathfrak{m}_\alpha, \mathfrak{m}_\beta) = 0, \\ h(v_\alpha, v_\alpha) = h(w_\alpha, w_\alpha) = h_\alpha^2, \quad h(v_\alpha, w_\alpha) = 0 \end{gathered}$$

for $h_\alpha \in \mathbb{R}^+$.

In particular we are interested in constructing balanced Hermitian metrics, namely Hermitian metrics whose associated $(1,1)$-form $\omega = h(\cdot, \mathrm{J}\cdot)$ satisfies $d\omega^{n-1} = 0$ or equivalently $\delta\omega = 0$, where δ denotes the codifferential.

We use the standard expression of the codifferential in terms of the Levi Civita connection ∇ of h (see e.g. [Bes], p.34)

$$\delta\omega(x) = -\mathrm{Tr}\nabla.\omega(\cdot, x) = -\sum_{i=1}^{2n} \nabla_{e_i}\omega(e_i, x) =$$

$$= \sum_{i=1}^{2n} \left(\omega(\nabla_{e_i} e_i, x) + \omega(e_i, \nabla_{e_i} x) \right),$$

where $\{e_i\}$ is an orthonormal basis of $\mathfrak{g}_o$ w.r.t. h. Note that both h and J are $\mathrm{ad}(\mathfrak{t})$-invariant and therefore $\delta\omega$ is $\mathrm{ad}(\mathfrak{t})$-invariant too. This last fact implies that $\delta\omega$ vanishes identically if and only if $\delta\omega(x) = 0$ for every $x \in \mathfrak{t}$.

We have the following expression for the Levi Civita connection, namely for $x, y, z \in \mathfrak{g}_o$

$$2h(\nabla_x y, z) = h([x, y], z) + h([z, x], y) + h([z, y], x).$$

Then for every $x \in \mathfrak{t}$, $y \in \mathfrak{g}_o$

$$h(\nabla_y y, x) = h([x, y], y) = 0.$$

Therefore for $x \in \mathfrak{t}$ we have

$$\delta\omega(x) = \sum_i \omega(e_i, \nabla_{e_i} x) = -\sum_i h(Je_i, \nabla_{e_i} x) = \tag{5.2}$$

$$= -\frac{1}{2} \left(h([e_i, x], Je_i) + h([Je_i, e_i], x) + h([Je_i, x], e_i) \right).$$

We now observe that J is $\mathrm{ad}(\mathfrak{t})$-invariant and hence $h([Je_i, x], e_i) = -h([e_i, x], Je_i)$ for every $i = 1, \ldots, 2n$, so that (5.2) can be written as

$$-\delta\omega(x) = \frac{1}{2} \sum_i h([Je_i, e_i], x) =$$

$$= \frac{1}{2} \cdot 2 \left(\sum_{\alpha \in R^+_{\mathfrak{k}}} \frac{1}{h^2_\alpha} h([w_\alpha, v_\alpha], x) + \sum_{\alpha \in R^+_{\mathfrak{p}}} \frac{1}{h^2_\alpha} h([w_\alpha, v_\alpha], x) \right) =$$

$$= \sum_{\alpha \in R^+_{\mathfrak{k}}} \frac{1}{h^2_\alpha} h(-iH_\alpha, x) + \sum_{\alpha \in R^+_{\mathfrak{p}}} \frac{1}{h^2_\alpha} h(iH_\alpha, x),$$

whence $\delta\omega|_{\mathfrak{t}} = 0$ if and only if

$$-\sum_{\alpha \in R^+_{\mathfrak{k}}} \frac{1}{h^2_\alpha} H_\alpha + \sum_{\alpha \in R^+_{\mathfrak{p}}} \frac{1}{h^2_\alpha} H_\alpha = 0.$$

Summing up, the metric h is balanced when the following equation is satisfied

$$\sum_{\alpha \in R^+_{\mathfrak{k}}} \frac{1}{h^2_\alpha} \alpha = \sum_{\alpha \in R^+_{\mathfrak{p}}} \frac{1}{h^2_\alpha} \alpha. \tag{5.3}$$

Note that this does *not* depend on the choice of the metric along the toral part $\mathfrak{t}$.

5.2 Proof of the main result

In this section we will prove our main result Theorem 5.0.1.

We keep the same notation as in the previous sections and we start noting that equation (5.3) involves the unknowns $\{h_\alpha\}_{\alpha\in R^+}$ and also a choice of positive roots, i.e. an ordering or equivalenty a complex structure on $\mathfrak{g}_o$. We will always fix a complex structure on $\mathfrak{t}$ once for all. It is known that giving an ordering on the root system R is equivalent to the choice of a system of simple roots Π and that two systems of simple roots are conjugate under the action of the Weyl group W. We may fix a system of simple roots $\Pi = \{\alpha_1,\dots,\alpha_r\}$ and put $\Pi = \Pi_c \cup \Pi_{nc}$, where $\Pi_{c/nc}$ denotes the set of simple roots which are compact or noncompact. We set $\Pi_c = \{\phi_1,\dots,\phi_k\}$, $\Pi_{nc} = \{\psi_1,\dots,\psi_l\}$, $k+l = r = \mathrm{rank}(\mathfrak{g}_o)$. Each root $\alpha \in R^+$ can be written as

$$\alpha = \sum_{i=1}^{k} n_i(\alpha)\phi_i + \sum_{j=1}^{l} m_j(\alpha)\psi_j$$

for $n_i(\alpha), m_j(\alpha) \in \mathbb{N}$ nonnegative integers. If we set $g_\alpha := \frac{1}{h_\alpha^2}$ and $g_j := g_{\phi_j}, h_j := g_{\psi_j}$, equation (5.3) can be written as

$$\sum_{\alpha\in R^+_{\mathfrak{k}},\alpha\notin\Pi} g_\alpha \left(\sum_{j=1}^{k} n_j(\alpha)\phi_j + \sum_{j=1}^{l} m_j(\alpha)\psi_j\right) + \sum_{j=1}^{k} g_j\phi_j =$$

$$= \sum_{\alpha\in R^+_{\mathfrak{p}},\alpha\notin\Pi} g_\alpha \left(\sum_{j=1}^{k} n_j(\alpha)\phi_j + \sum_{j=1}^{l} m_j(\alpha)\psi_j\right) + \sum_{j=1}^{l} h_j\psi_j,$$

and therefore

$$\begin{cases} g_j = \displaystyle\sum_{\alpha\in R^+_{\mathfrak{p}},\ \alpha\notin\Pi} g_\alpha n_j(\alpha) - \sum_{\alpha\in R^+_{\mathfrak{k}},\ \alpha\notin\Pi} g_\alpha n_j(\alpha), \quad j = 1,\dots,k, \\ h_j = \displaystyle\sum_{\alpha\in R^+_{\mathfrak{k}},\ \alpha\notin\Pi} g_\alpha m_j(\alpha) - \sum_{\alpha\in R^+_{\mathfrak{p}},\ \alpha\notin\Pi} g_\alpha m_j(\alpha), \quad j = 1,\dots,l. \end{cases} \tag{5.4}$$

Remark 5.2.1. If we consider for instance the case $\mathfrak{g}_o = \mathfrak{su}(p,q)$ ($p+q$ even, $p,q \geq 2$) and the standard system of simple roots $\Pi = \{\epsilon_1-\epsilon_2, \epsilon_2-\epsilon_3,\dots,\epsilon_{p-1}-\epsilon_p, \epsilon_p - \epsilon_{p+1},\dots,\epsilon_{p+q-1}-\epsilon_{p+q}\}$ of $\mathfrak{sl}(p+q,\mathbb{C})$, then $\Pi_{nc} = \{\epsilon_p-\epsilon_{p+1}\}$ and Π_c gives a system of simple roots for the semisimple part $\mathfrak{k}_{ss}$ of $\mathfrak{k}$. This means that every root $\alpha \in R^+_{\mathfrak{k}}, \alpha \notin \Pi$ is a linear combination of roots in Π_c and therefore the right hand side of the last equation in (5.4) is nonpositive, so that (5.4) has no solution. This shows that the choice of the invariant complex structure might not be straightforward.

The following lemma provides key tools in our argument.

Lemma 5.2.2. *For each symmetric pair* $(\mathfrak{g}_o, \mathfrak{k})$ *as in Table 1,* $(\mathfrak{g}_o, \mathfrak{k}) \ncong (\mathfrak{so}(1, 2n), \mathfrak{so}(2n))$ *and given a Cartan subalgebra* $\mathfrak{t} \subseteq \mathfrak{k}$ *with corresponding root system* R*, there exists an ordering of the roots, hence a system of simple roots* Π*, such that*

$$\forall \psi \in \Pi_{nc}\, \exists \psi' \in \Pi_{nc} \text{ with } \psi + \psi' \in R. \tag{5.5}$$

This implies that, if $\Pi_{nc} = \{\psi_1, \ldots, \psi_l\}$, then for every $\psi_j \in \Pi_{nc}$ there exists $\alpha \in R^+_{\mathfrak{k}}$ with $m_j(\alpha) \neq 0$ and $\alpha \in \mathrm{Span}\{\Pi_{nc}\}$.

Remark Note that $\mathfrak{sp}(1, 1) \cong \mathfrak{so}(1, 4)$ is also not admissible in the above Lemma. In general, for $\mathfrak{g}_o = \mathfrak{so}(1, 2n)$ we have the standard system $\Pi = \{\epsilon_i - \epsilon_{i+1}, \epsilon_n,\ i = 1, \ldots, n-1\}$ with $\Pi_{nc} = \{\epsilon_n\}$. As R_c consists precisely of all the short roots, it is clear that for any element σ of the Weyl group $W \cong \mathbb{Z}_2^n \ltimes \mathcal{S}_n$ we have that $\sigma(\Pi)_{nc}$ consists of one element. We will deal with this case later on.

Proof. We first deal with the classical case. We start with the standard system of simple roots Π, following the notation as in [He]. It is immediate to check that in this case Π_{nc} consists of a single root ψ.

Assume first to be in the case where ψ is a short root. Let Λ be the set of all simple roots which are connected to ψ in the Dynkin diagram relative to Π. If $s \in W$ denotes the reflection around ψ, then s leaves every element $\Pi \setminus \Lambda$ pointwise fixed. We observe that Λ consists of either at most three short roots or it contains a long root. In the first case, $s(\Lambda) = \{\psi + \lambda |\ \lambda \in \Lambda\} \subseteq R_{\mathfrak{p}}$ so that $s(\Pi)_{nc} = \{-\psi, s(\Lambda)\}$ and therefore the system of simple roots $s(\Pi)$ satisfies (5.5). If Λ contains a long root, then it also contains a short root, unless $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{so}(2, 3), \mathbb{R} + \mathfrak{so}(3))$, that is isomorphic to $(\mathfrak{sp}(2), \mathfrak{u}(2))$; this case will be dealt with in the second part of the proof. Therefore $\Lambda = \{\phi_1, \phi_2\}$ with ϕ_1 short and ϕ_2 long. Again the reflection s around ψ gives $s(\phi_1) = \psi + \phi_1$ and $s(\phi_2) = \phi_2 + 2\psi \in R_{\mathfrak{k}}$ or $s(\phi_2) = \psi + \phi_2 \in R_{\mathfrak{p}}$. This implies that the system of simple roots $s(\Pi)$ has $s(\Pi)_{nc} = \{-\psi, \psi + \phi_1\}$ or $\{-\psi, \psi + \phi_1, \psi + \phi_2\}$ and in both cases it satisfies (5.5).

We are then left with the case where ψ is a long root, namely the case where $\mathfrak{g}_o = \mathfrak{sp}(2n, \mathbb{R})$ and $\mathfrak{k} = \mathfrak{u}(2n)$. A standard system of simple roots is given by $\Pi = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{2n-1} - \epsilon_{2n}, 2\epsilon_{2n}\}$ and $\Pi_{nc} = \{\psi = 2\epsilon_{2n}\}$. Again using s_β, we see that $s_\beta(\Pi)_{nc} = \{-2\epsilon_{2n}, \epsilon_{2n-1} + \epsilon_{2n}\}$ so that condition (5.5) is satisfied.

We may now deal with the exceptional cases. Starting with the standard system of simple roots Π, we list the set Π_{nc}, that turns out to consist of a single root β. For each case, using the symmetry s_β we obtain the system of simple roots $\Pi' := s_\beta(\Pi)$ that satisfies condition (5.5).

(1) $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{g}_2, \mathfrak{su}(2) + \mathfrak{su}(2))$. Here $\Pi = \{\alpha, \beta\}$, with β long. We have $\Pi_{nc} = \{\beta\}$ and $\Pi' = \{-\beta, \alpha + \beta\}$.

(2) $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{f}_{4(-20)}, \mathfrak{so}(9))$. According to [He], the standard system of simple roots is $\Pi = \{\alpha_1 = \epsilon_2 - \epsilon_3, \alpha_2 = \epsilon_3 - \epsilon_4, \alpha_3 = \epsilon_4, \alpha_4 = \frac{1}{2}(e_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)\}$ so that $\Pi_{nc} = \{\alpha_4\}$ and therefore $\Pi'_{nc} = \{-\alpha_4, \alpha_4 + \alpha_3\}$.
(3) $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{f}_{4(4)}, \mathfrak{su}(2) + \mathfrak{sp}(3))$. In this case $\Pi_{nc} = \{\alpha_1\}$ and therefore $\Pi'_{nc} = \{-\alpha_1, \alpha_1 + \alpha_2\}$.
(4) $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{e}_{8(8)}, \mathfrak{so}(16))$. For $\mathfrak{e}_8$ we have the standard system of simple roots

$$\alpha_1 = \frac{1}{2}(\epsilon_1 + \epsilon_8) - \frac{1}{2}(\epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7), \alpha_2 = \epsilon_1 + \epsilon_2,$$

$$\alpha_j = \epsilon_{j-1} - \epsilon_{j-2},\ j = 3, \dots, 8.$$

Then $\Pi_{nc} = \{\alpha_1\}$ and $\Pi'_{nc} = \{-\alpha_1, \alpha_1 + \alpha_3\}$.
(5) $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{e}_{8(-24)}, \mathfrak{su}(2) + \mathfrak{e}_7)$. Keeping the same notation for simple roots as above, we have $\Pi_{nc} = \{\alpha_8\}$ and $\Pi'_{nc} = \{-\alpha_8, \alpha_8 + \alpha_7\}$.
(6) $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{e}_{6(2)}, \mathfrak{su}(2) + \mathfrak{su}(6))$. As the system root system Π can be taken to be composed of the simple roots $\{\alpha_1, \dots, \alpha_6\}$ of $\mathfrak{e}_8$, we have $\Pi_{nc} = \{\alpha_2\}$ and $\Pi'_{nc} = \{-\alpha_2, \alpha_2 + \alpha_4\}$.
(7) $(\mathfrak{g}_o, \mathfrak{k}) = (\mathfrak{e}_{6(-14)}, \mathbb{R} + \mathfrak{so}(10))$. We have $\Pi_{nc} = \{\alpha_1\}$ and $\Pi'_{nc} = \{-\alpha_1, \alpha_1 + \alpha_3\}$. □

Lemma 5.2.3. *For every system of simple roots $\Pi = \Pi_c \cup \Pi_{nc}$ with $\Pi_c = \{\phi_1, \dots, \phi_k\}$ we have*

$$\forall\, j = 1, \dots, k,\ \exists\, \alpha \in R^+_{\mathfrak{p}},\ \alpha \notin \Pi :\ n_j(\alpha) \neq 0,$$

where $n_j(\alpha)$ denotes the coordinate of α along the root ϕ_j.

Proof. We start noting that the centralizer $C_{\mathfrak{k}^c}(\mathfrak{p}^c) = C_{\mathfrak{k}}(\mathfrak{p})^c = \{0\}$. It then follows that $[E_{\phi_j}, \mathfrak{p}^c] \neq \{0\}$, hence there exists $\gamma \in R_{\mathfrak{p}}$ with $[E_{\phi_j}, E_\gamma] \neq 0$, i.e. $\phi_j + \gamma \in R_{\mathfrak{p}}$. Now, if $\gamma > 0$, then $\alpha := \phi_j + \gamma \in R^+_{\mathfrak{p}} \setminus \Pi$ and $n_j(\alpha) \geq 1$. Suppose now $\gamma < 0$. We write $\gamma = c_j\phi_j + \sum_{\theta \in \Pi \setminus \phi_j} c_\theta \theta$ for some nonpositive integers c_j, c_θ. As $\gamma \neq -\phi_j$, there exists at least one negative coefficient $c_\theta < 0$, for some $\theta \in \Pi, \theta \neq \phi_j$. Therefore the root $\gamma + \phi_j$ must be negative and $1 + c_j \leq 0$, i.e. $\alpha := -\gamma \in R^+_{\mathfrak{p}} \setminus \Pi$ and $n_j(\alpha) = -c_j \geq 1$. □

We now fix a system of simple roots Π as in Lemma 5.2.2. In order to solve the corresponding system of equations (5.4) for the positive unknowns $\{g_i, h_j, g_\alpha\}$, we will show how to choose the positive values $\{g_\alpha\}_{\alpha \in R^+ \setminus \Pi}$ in such a way to guarantee that the constants $\{g_i, h_j\}$, defined to satisfy (5.4), are positive.

We set

$$\Sigma_{\mathfrak{k}} := \{\alpha \in R^+_{\mathfrak{k}} \,|\, \alpha \notin \Pi, \alpha \in \mathrm{Span}\{\Pi_{nc}\}\}, \qquad A_{\mathfrak{k}} = (R^+_{\mathfrak{k}} \setminus \Pi_c) \setminus \Sigma_{\mathfrak{k}}.$$

Then the system of equations (5.4) can be written as

$$\begin{cases} g_j = \sum_{\alpha\in R^+_{\mathfrak{p}},\ \alpha\notin\Pi} g_\alpha n_j(\alpha) - \sum_{\alpha\in A_{\mathfrak{k}}} g_\alpha n_j(\alpha), & j=1,\dots,k,\ (1) \\ h_j = \sum_{\alpha\in R^+_{\mathfrak{k}},\ \alpha\notin\Pi} g_\alpha m_j(\alpha) - \sum_{\alpha\in R^+_{\mathfrak{p}},\ \alpha\notin\Pi} g_\alpha m_j(\alpha), & j=1,\dots,l.\ (2) \end{cases} \tag{5.6}$$

We start assigning $g_\alpha = 1$ for every $\alpha\in A_{\mathfrak{k}}$.

Then, for every $j=1,\dots,k$, we use Lemma 5.2.3 selecting a root $\alpha\in R^+_{\mathfrak{p}}$ with $n_j(\alpha)\neq 0$, $\alpha\notin\Pi$. This root α, which depends on j, contributes to the first sum in the right hand side of equation (1) in (5.6) and the value g_α can be chosen big enough so that g_j is strictly positive. Summing up, we can assign values $\{g_\alpha\}_{\alpha\in R^+_{\mathfrak{p}}\setminus\Pi_{nc}}$ so that all g_j, $j=1,\dots,k$ can be defined as in (5.6), (1), and are strictly positive.

We now turn to equation (5.6)-(2), which can now be written as

$$h_j = \sum_{\alpha\in\Sigma_{\mathfrak{k}}} g_\alpha m_j(\alpha) + \sum_{\alpha\in A_{\mathfrak{k}}} m_j(\alpha) - \sum_{\alpha\in R^+_{\mathfrak{p}},\ \alpha\notin\Pi} g_\alpha m_j(\alpha), \tag{5.7}$$

where in the right hand side the last two sums have a fixed value. Now, by Lemma 5.2.2, we know that for every $j=1,\dots,l$, we can find $\alpha\in\Sigma_{\mathfrak{k}}$ with $m_j(\alpha)\neq 0$. These roots can be used to choose the coefficients $\{g_\beta\}_{\beta\in\Sigma_{\mathfrak{k}}}$ big enough to guarantee that h_j, when defined to satisfy (5.7), is strictly positive.

In order to complete the proof of our main result Theorem 5.0.1, we are left with the case $(\mathfrak{g}_o,\mathfrak{k}) = (\mathfrak{so}(1,2n),\mathfrak{so}(2n))$ with standard system of simple roots $\Pi=\{\epsilon_i-\epsilon_{i+1},\epsilon_n,\ i=1,\dots,n-1\}$, $\Pi_{nc}=\{\epsilon_n\}$. We see that

$$R^+_{\mathfrak{k}} = \{\epsilon_i\pm\epsilon_j,\ i<j\},\quad R_{\mathfrak{p}} = \{\epsilon_1,\dots,\epsilon_n\}.$$

Now, we use equation (5.3) and search for positive real numbers $\{x,y,z_i,\ i=1,\dots,n\}$ so that

$$x\cdot\sum_{i<y}\epsilon_i-\epsilon_j + y\cdot\sum_{i<j}\epsilon_i+e_j = \sum_{i=1}^n z_i\epsilon_i,$$

i.e.

$$\sum_{i=1}^n[(x+y)(n-i)+(x-y)(i-1)]\epsilon_i = \sum_{i=1}^n z_i\epsilon_i.$$

It is clear that the above equation has positive solutions by simply choosing $x>y>0$. This concludes the proof of Theorem 5.0.1. $\square$

Remark 5.2.4. We can consider the metric h_o which coincides with $-B$ on the compact part $\mathfrak{k}$, with B on $\mathfrak{p}$ and such that $h_o(\mathfrak{k}, \mathfrak{p}) = 0$. This metric is easily seen to depend only on $\mathfrak{g}_o$ and *not* on the Cartan decomposition $\mathfrak{g}_o = \mathfrak{k} + \mathfrak{p}$. We could then ask whether there exists a suitable complex structure such that the metric h_o turns out to be balanced. The resulting equation has been already treated in [APo] and has a solution if and only if $\mathfrak{g}_o = \mathfrak{su}(p, p+1) \cong \mathfrak{su}(p+1, p)$ for $p \geq 1$.

5.3 Non-existence of pluriclosed metrics

In this section we prove our result Theorem 5.0.3 concerning the non-existence of pluriclosed metrics on the compact quotients $\mathcal{M}$ we have constructed in the previous sections and we will keep the same notation used above.

Suppose now that h is a pluriclosed metric on $\mathcal{M} = \Gamma\backslash\mathrm{G_o}$. Then we can obtain a pluriclosed invariant metric h on $\mathrm{G_o}$ which is also invariant under right T-translations. It follows that on $\mathfrak{g}$ we have

$$h(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0 \quad \text{if} \quad \beta \neq -\alpha.$$

In order to write down the condition $dd^c\omega = 0$, where ω is the fundamental form of h, we recall the standard formula for the differential of invariant forms (see e.g. [He], p.136). If ϕ is any invariant k-form on $\mathrm{G_o}$ or equivalently on $\mathfrak{g}_o$, then for every $v_o, \ldots, v_k$ in $\mathfrak{g}_o$

$$(k+1) \cdot d\phi(v_o, \ldots, v_k) = \sum_{i<j} (-1)^{i+j} \phi([v_i, v_j], v_o, \ldots, \widehat{v_i}, \ldots, \widehat{v_j} \ldots, v_k).$$

We set $\phi := d^c\omega$ and compute $d\phi(E_\alpha, E_{-\alpha}, E_\beta, E_{-\beta})$ for $\alpha, \beta \in R^+$. We have

$$4\, d\phi(E_\alpha, E_{-\alpha}, E_\beta, E_{-\beta}) = -\phi(H_\alpha, E_\beta, E_{-\beta}) + \phi(N_{\alpha\beta} E_{\alpha+\beta}, E_{-\alpha}, E_{-\beta})$$
$$-\phi(N_{\alpha,-\beta} E_{\alpha-\beta}, E_{-\alpha}, E_\beta) - \phi(N_{-\alpha,\beta} E_{\beta-\alpha}, E_\alpha, E_{-\beta}) + \phi(N_{-\alpha,-\beta} E_{-\alpha-\beta} E_\alpha, E_\beta)$$
$$-\phi(H_\beta, E_\alpha, E_{-\alpha}),$$

where we use the standard notation $[E_\gamma, E_\epsilon] = N_{\gamma,\epsilon} E_{\gamma+\epsilon}$ for every $\gamma, \epsilon \in R$. Using the known identities for the Weyl basis (see [He], p. 172,176), we can write that

$$4\, d\phi(E_\alpha, E_{-\alpha}, E_\beta, E_{-\beta}) = -\phi(H_\alpha, E_\beta, E_{-\beta}) - \phi(H_\beta, E_\alpha, E_{-\alpha})$$
$$+2\phi(N_{\alpha\beta} E_{\alpha+\beta}, E_{-\alpha}, E_{-\beta}) - 2\phi(N_{\alpha,-\beta} E_{\alpha-\beta}, E_{-\alpha}, E_\beta).$$

We also introduce the notation $JE_\gamma = i\epsilon_\gamma E_\gamma$ for every $\gamma \in R$, where $\epsilon_\gamma = \pm 1$ according to $\gamma \in R^\pm$. Then

$$4\, dd^c\omega(E_\alpha, E_{-\alpha}, E_\beta, E_{-\beta}) = -d\omega(JH_\alpha, E_\beta, E_{-\beta}) - d\omega(JH_\beta, E_\alpha, E_{-\alpha})$$
$$-2iN_{\alpha,\beta} d\omega(E_{\alpha+\beta}, E_{-\alpha}, E_{-\beta}) - 2iN_{\alpha,-\beta}\epsilon_{\alpha-\beta} d\omega(E_{\alpha-\beta}, E_{-\alpha}, E_\beta).$$

Now we easily compute

$$3\, d\omega(JH_\alpha, E_\beta, E_{-\beta}) = -\omega(H_\beta, JH_\alpha)$$

and

$$3\, d\omega(E_{\alpha+\beta}, E_{-\alpha}, E_{-\beta}) = N_{\alpha,\beta}(\omega(E_\alpha, E_{-\alpha}) + \omega(E_\beta, E_{-\beta}) - \omega(E_{\alpha+\beta}, E_{-\alpha-\beta})),$$

where we have used the fact that $N_{\alpha,\beta} = N_{\alpha+\beta,-\beta} = -N_{\alpha+\beta,-\alpha}$ (see [He], p. 172). Similarly,

$$3\, d\omega(E_{\alpha-\beta}, E_{-\alpha}, E_\beta) = N_{\alpha,-\beta}(-\omega(E_\beta, E_{-\beta}) + \omega(E_\alpha, E_{-\alpha}) - \omega(E_{\alpha-\beta}, E_{\beta-\alpha})).$$

Summing up we have

$$\begin{aligned} 12\, dd^c\omega(E_\alpha, E_{-\alpha}, E_\beta, E_{-\beta}) &= -2\omega(JH_\alpha, H_\beta) \\ -2iN_{\alpha,\beta}^2(\omega(E_\alpha, E_{-\alpha}) + \omega(E_\beta, E_{-\beta}) &- \omega(E_{\alpha+\beta}, E_{-\alpha-\beta})) \\ -2iN_{\alpha,-\beta}^2\epsilon_{\alpha-\beta}(-\omega(E_\beta, E_{-\beta}) + \omega(E_\alpha, E_{-\alpha}) &- \omega(E_{\alpha-\beta}, E_{\beta-\alpha})). \end{aligned} \tag{5.8}$$

If we now set $a_\alpha := h(E_\alpha, E_{-\alpha})$, the pluriclosed condition and (5.8) imply that

$$0 = -h(H_\alpha, H_\beta) - iN_{\alpha,\beta}^2(-ia_\alpha - ia_\beta + ia_{\alpha+\beta})$$
$$-iN_{\alpha,-\beta}^2\epsilon_{\alpha-\beta}(-i\epsilon_{\beta-\alpha}a_{\alpha-\beta} + ia_\beta - ia_\alpha),$$

hence

$$h(H_\alpha, H_\beta) = N_{\alpha,\beta}^2(a_{\alpha+\beta} - a_\alpha - a_\beta) + N_{\alpha,-\beta}^2\epsilon_{\alpha-\beta}(\epsilon_{\alpha-\beta}a_{\alpha-\beta} + a_\beta - a_\alpha). \tag{5.9}$$

We recall that

$$\begin{aligned} a_\alpha = h(E_\alpha, E_{-\alpha}) = -h(v_\alpha, v_\alpha) < 0, &\qquad \alpha \in R_{\mathfrak{k}}^+, \\ a_\alpha = h(E_\alpha, E_{-\alpha}) = h(v_\alpha, v_\alpha) > 0, &\qquad \alpha \in R_{\mathfrak{p}}^+, \\ h(H_\alpha, H_\beta) = -h(iH_\alpha, iH_\beta) \in \mathbb{R}, &\qquad h(H_\alpha, H_\alpha) < 0. \end{aligned}$$

Now, we note that the existence of the complex structure J, which we constructed in section 5.2, relies on Lemma 5.2.2. In particular, when $\mathfrak{g}_o \neq \mathfrak{so}(1, 2n)$, we have the existence of two simple roots $\psi_1, \psi_2 \in \Pi_{nc}$ with $\psi_1 + \psi_2 = \phi \in R_{\mathfrak{k}}$. The following lemma is elementary.

Lemma 5.3.1. *Either $\psi_1 + 2\psi_2 \notin R$ or $\psi_2 + 2\psi_1 \notin R$.*

Proof. As ψ_1, ψ_2 are simple, we have $\pm(\psi_1 - \psi_2) \notin R$. Now, $\psi_i + n\psi_j \in R$ if and only if $0 \leq n \leq q_j$ with $q_j = -2\frac{\langle \psi_1, \psi_2 \rangle}{||\psi_j||^2} \in \mathbb{N}$ for $i \neq j$. It is then clear that $q_1, q_2 \geq 2$ is impossible, as $\psi_1 \neq \psi_2$ implies $q_1 \cdot q_2 < 4$. □

Suppose then that $\phi + \psi_1 = \psi_2 + 2\psi_1 \notin R$. We now apply (5.9) with two possible choices for α, β, namely:

(1) $\alpha = \psi_1, \beta = \psi_2$. Then

$$h(H_{\psi_1}, H_{\psi_2}) = N^2_{\psi_1,\psi_2}(a_\phi - a_{\psi_1} - a_{\psi_2}).$$

(2) $\alpha = \phi, \beta = \psi_1$. Then

$$h(H_\phi, H_{\psi_2}) = N^2_{\phi,-\psi_1}(a_{\psi_2} + a_{\psi_1} - a_\phi).$$

Subtracting (1) from (2) we get

$$h(H_{\psi_2}, H_{\psi_2}) = \left(N^2_{\phi,-\psi_1} + N^2_{\psi_1,\psi_2}\right)(a_{\psi_2} + a_{\psi_1} - a_\phi).$$

This is a contradiction, as $h(H_{\psi_2}, H_{\psi_2}) < 0$, while $a_{\psi_i} > 0$ for $i = 1, 2$ and $a_\phi < 0$.

We are left with the case $\mathfrak{g}_o = \mathfrak{so}(1, 2n)$, that we have dealt with separately in section 5.2. In this case the complex structure J is defined by the standard system of positive roots, namely $R^+ = \{\epsilon_i \pm \epsilon_j, \epsilon_i,\ 1 \leq i \neq j \leq n\}$. In particular $R^+_\mathfrak{k} = \{\epsilon_i \pm \epsilon_j\}_{i \neq j}$ and $R^+_\mathfrak{p} = \{\epsilon_i\}_{i=1,\dots,n}$. We now consider $\psi_i = \epsilon_i$, $i = 1, 2$, $\phi_1 = \psi_1 + \psi_2 \in R^+_\mathfrak{k}$ and $\phi_2 = \psi_1 - \psi_2 \in R^+_\mathfrak{k}$. We apply (5.9) in two different ways:

(1) $\alpha = \psi_1, \beta = \psi_2$. Then

$$h(H_{\psi_1}, H_{\psi_2}) = N^2_{\psi_1,\psi_2}(a_\phi - a_{\psi_1} - a_{\psi_2}) + N^2_{\psi_1,-\psi_2}(a_{\phi_2} + a_{\psi_2} - a_{\psi_1}).$$

(2) $\alpha = \phi_1, \beta = \psi_2$. Note that $\phi_1 + \psi_1 \notin R$. Then

$$h(H_{\phi_1}, H_{\psi_1}) = N^2_{\phi_1,-\psi_1}(a_{\psi_2} + a_{\psi_1} - a_{\phi_1}).$$

Therefore

$$h(H_{\psi_1}, H_{\psi_1}) = (N^2_{\phi_1,-\psi_1} + N^2_{\psi_1,\psi_2})(a_{\psi_2} + a_{\psi_1} - a_{\phi_1}) + N^2_{\psi_1,-\psi_2}(a_{\psi_1} - a_{\phi_2} - a_{\psi_2})$$

We now recall that, if $\gamma, \delta \in R$, then $N^2_{\gamma,\delta} = \frac{q(1-p)}{2}||\gamma||^2$, where $\delta + n\gamma$, $p \leq n \leq q$, is the γ-series containing δ (see [He], p.176). We then immediately see that $N^2_{\psi_1,\psi_2} = N^2_{\psi_1,-\psi_2}$ and noting furthermore that $N^2_{\phi_1,-\psi_1} = N^2_{\psi_1,\psi_2}$, we can write

$$h(H_{\psi_1}, H_{\psi_1}) = N^2_{\psi_1,\psi_2}(a_{\psi_2} + 3a_{\psi_1} - 2a_{\phi_1} - a_{\phi_2}),$$

hence the contradiction $h(H_{\psi_1}, H_{\psi_1}) > 0$. This concludes the proof of Theorem 5.0.3.

5.4 Geometric properties

In this last section, we collect some properties of the complex Hermitian manifolds we have constructed in the previous section, to obtain a the proof of Proposition 5.0.2.

Proof. We consider a standard complex structure J on a manifold $\mathcal{M} = \Gamma\backslash \mathrm{G_o}$. We denote by D the Chern connection relative to a Hermitian metric h which is induced by an invariant metric on $\mathrm{G_o}$, again denoted by h. We can moreover suppose that h is invariant by the right T-translations.

If $x \in \mathfrak{g}_o$ we define the endomorphism $D_x \in \mathrm{End}\,(\mathfrak{g}_o)$ as follows: given $y \in \mathfrak{g}_o$, we extend x, y as left invariant vector fields x^*, y^* on $\mathrm{G_o}$ and we put $D_x y := D_{x^*}y^*|_e$. Clearly $D_x \in \mathfrak{so}(\mathfrak{g}_o, h)$ and $[D_x, \mathrm{J}] = 0$. Moreover

$$D_x y = [x, y]^{10}, \qquad \forall x \in \mathfrak{g}_o^{01},\ y \in \mathfrak{g}_o^{10}, \tag{5.10}$$

that follows from the fact that $T^{1,1} = 0$, where T is torsion of D.

If R denote the curvature of the Chern connection, where $R_{xy} = [D_x, D_y] - D_{[x,y]}$, we are interested in the first Ricci form ρ given by

$$\rho(x, y) = -\frac{1}{2}\mathrm{Tr}(J \circ R_{xy}).$$

As the complex structure and the metric are both invariant under the adjoint action of the group $T = \exp(\mathfrak{t})$, we see that

$$\rho(\mathfrak{t}, E_\alpha) = 0,\ \forall \alpha \in R,$$

$$\rho(E_\alpha, E_\beta) \neq 0 \text{ implies } \beta = -\alpha,\ \alpha, \beta \in R.$$

Therefore we can compute

$$\rho(E_\alpha, E_{-\alpha}) = \frac{1}{2}\mathrm{Tr}(JD_{H_\alpha}).$$

Lemma 5.4.1. *For every $x \in \mathfrak{h}$*

$$D_x = \mathrm{ad}(x).$$

Proof. We use similar arguments as in [Po]. It will suffice to consider the case where $x \in \mathfrak{h}^{10}$; then for every $\alpha \in R^+$ we have

$$D_x E_{-\alpha} = [x, E_{-\alpha}]^{01} = [x, E_{-\alpha}], \quad D_x \mathfrak{h}^{01} = 0$$

by (5.10). Then if $\beta \in R^+$ we have

$$h(D_x E_\alpha, E_{-\beta}) = -h(E_\alpha, D_x E_{-\beta}) = -\beta(x) h(E_\alpha, E_{-\beta}) = 0 \qquad \text{if } \alpha \neq \beta,$$

so that $D_x E_\alpha = \alpha(x) E_\alpha = [x, E_\alpha] \pmod{\mathfrak{h}}$. As $h(D_x E_\alpha, \mathfrak{h}^{01}) = -h(E_\alpha, D_x \mathfrak{h}^{01}) = 0$, we conclude that

$$D_x E_\alpha = [x, E_\alpha].$$

Finally, $h(D_x \mathfrak{h}^{10}, \mathfrak{h}^{01}) = 0$ and $h(D_x \mathfrak{h}^{10}, E_{-\alpha}) = -h(\mathfrak{h}^{10}, [x, E_{-\alpha}]) = 0$, so that $D_x \mathfrak{h} = 0 = [x, \mathfrak{h}]$. □

It follows that

$$\rho(\mathfrak{h}, \mathfrak{h}) = 0$$

and

$$\rho(E_\alpha, E_{-\alpha}) = \frac{1}{2}\left(2 \sum_{\beta \in R^+} i\beta(H_\alpha)\right) = B(H_\alpha, \delta), \tag{5.11}$$

where

$$\delta = \sum_{\beta \in R^+} iH_\beta \in \mathfrak{t} \neq 0,$$

hence ρ never vanishes.

We now show that the tensor powers $K_{\mathcal{M}}^{\otimes m}$ are holomorphically non trivial for every $m \geq 1$. Indeed, the metric h induces a Hermitian metric on the line bundles $K_{\mathcal{M}}^{\otimes m}$ with curvature form $m\rho$. If Ω is a nowhere vanishing holomorphic section of $K_{\mathcal{M}}^{\otimes m}$, then $m\rho = -i\partial\overline{\partial}\log(||\Omega||^2)$. If we denote by $\widehat{\ }$ the result of the symmetrization process, which commutes with the operators ∂ and $\overline{\partial}$, we obtain on $\mathrm{G_o}$ that $\widehat{\rho} = -i\partial\overline{\partial}\widehat{\log(||\Omega||^2)} = 0$. As ρ is invariant, $\widehat{\rho} = \rho = 0$ and we get a contradiction as $\delta \neq 0$.

The claim $\kappa(\mathcal{M}) = -\infty$ now follows from Thm. 1.4 in [Ya]. □

Remark 5.4.2. We note that the manifold $\mathcal{M}$ is parallelizable and therefore $c_1(M) = 0$, hence the Chern Ricci form ρ is exact. Moreover the Chern scalar curvature s^{Ch} vanishes identically, as it can be deduced from the expression (5.11) of ρ or in a simpler way* since $d\omega^{n-1} = 0$ and

$$0 = \int_M \rho \wedge \omega^{n-1} = \frac{1}{n}\int_M s^{Ch}\omega^n = s^{Ch}\int_M \omega^n.$$

We also remark here that the balanced condition implies that the two scalar curvatures that one can obtain tracing the Chern curvature tensor coincide (see [Ga3], p. 501).

We finally note that also for a compact group K endowed with an invariant complex structure we have $h^{n,0}(\mathrm{K}) = 0$, see [Pi], Prop. 3.7.

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