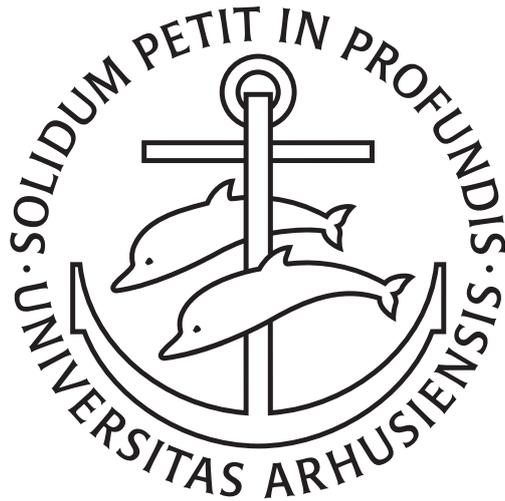


Homological algebra in subcategories

Nakayama functors, rank functions and differential modules



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PhD Dissertation

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Preface

This thesis is submitted in partial fulfilment of the requirements for the PhD degree. It consists of three papers written during the three-year course of my doctoral studies at Aarhus University.

- Chapter **I**, titled Nakayama functors on proper abelian subcategories, has been accepted for publication in *Publicacions Matemàtiques* [Nka24a]. It is available as a preprint on the arXiv ([arXiv:2312.07323v2](https://arxiv.org/abs/2312.07323v2)).
- Chapter **II**, titled Rank functions on $(d + 2)$ -angulated categories—a functorial approach, is available as a preprint on the arXiv ([arXiv:2405.19042v2](https://arxiv.org/abs/2405.19042v2), submitted in 2024) [Nka24b].
- Chapter **III**, titled Differential modules: a perspective on Bass’ question, is also available as a preprint on the arXiv ([arXiv:2504.15981](https://arxiv.org/abs/2504.15981), submitted in 2025) [Nka25].

Each paper is presented as a standalone chapter and has its own abstract and introduction. Collectively, they reflect a line of inquiry into homological structures arising in subcategorical structures.

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Declarations. This thesis was composed by myself and the work contained therein is my own, except where explicitly stated otherwise in the text. In accordance with GSNS rules, an old version of Chapter **I** of this thesis was also used in the progress report for the qualifying examination.

Acknowledgements. I would like to express my sincere gratitude to my supervisor, Peter Jørgensen, for his invaluable guidance and support throughout this work and the PhD. It was truly a privilege to have had you as a mathematical mentor.

I am grateful to the Extended Homological Algebra group at Aarhus University for providing a stimulating and supportive research environment.

Many thanks to Gustavo Jasso, who hosted me during a three-month research visit at the University of Cologne, and to the Algebra and Representation Theory group there for their warm welcome and insightful discussions.

I would also like to thank the friends with whom I studied representation theory in Bonn—your companionship and curiosity have made my experience in mathematics especially meaningful.

I am also grateful to Sira Gratz and Greg Stevenson, who guided me during my undergraduate studies, and to Jan Schröer, who supported me during my master's studies. Their encouragement and insight played an important role in shaping my mathematical journey.

Finally, I am deeply thankful to my family for their unwavering love and support.

Abstract

This thesis explores aspects of homological algebra within subcategories. It consists of three papers, each in its own chapter.

In Chapter **I**, we investigate the homological algebra of proper abelian subcategories within a triangulated category equipped with a Serre functor. By approximating the Serre functor, we construct Nakayama functors, which in turn enable the definition of the Auslander-Reiten translations. We show that suitable proper abelian subcategories are dualising \mathbf{k} -varieties and have enough projectives if and only if they have enough injectives. This framework yields a new proof for the existence of Auslander-Reiten sequences in the category of finite dimensional modules over a finite dimensional algebra.

Chapter **II** introduces a theory of rank functions on $(\mathbf{d} + 2)$ -angulated categories, generalising the notion of rank function on triangulated categories introduced by Chuang and Lazarev. We establish a bijective correspondence between object-defined and morphism-defined rank functions. Inspired by work of Conde, Gorsky, Marks and Zvonareva, we further demonstrate a bijective correspondence between rank functions on an Amiot-Lin $(\mathbf{d} + 2)$ -angulated categories and certain additive functions on its associated module category. This leads to a decomposition theorem: integral rank functions admit a factorisation into irreducible components in this setting.

Chapter **III** studies the homological theory of differential modules via the \mathbf{Q} -shaped derived category introduced by Holm and Jørgensen. We prove a differential module analogue of a classical result characterising when a finitely generated module over a local commutative noetherian ring has finite injective dimension. As an application, we provide a new characterisation of local Cohen-Macaulay rings using differential modules, offering an alternative perspective on a question originally posed by Bass.

Resumé

Denne afhandling udforsker aspekter af homologisk algebra i delkategorier. Den består af tre artikler, hver i sit eget kapitel.

I Kapitel **I** undersøger vi den homologiske algebra af ægte abelske delkategorier i en trianguleret kategori udstyret med en Serre-funktor. Ved at approksimere Serre-funktoren konstruerer vi Nakayama-funktorer, som igen muliggør definitionen af Auslander-Reiten-translationen. Vi viser, at passende ægte abelske delkategorier er dualiserende k -varieteter og har tilstrækkeligt mange projektive hvis og kun hvis de har tilstrækkeligt mange injektive. Disse rammer giver et nyt bevis for eksistensen af Auslander-Reiten-følger i kategorien af endeligdimensionale moduler over en endeligdimensional algebra.

Kapitel **II** introducerer en teori om rangfunktioner på $(d + 2)$ -angulerede kategorier, som generaliserer begrebet rangfunktion på triangulerede kategorier introduceret af Chuang og Lazarev. Vi etablerer en bijektiv korrespondance mellem objekt-definerede og morfi-definerede rangfunktioner. Inspireret af arbejde af Conde, Gorsky, Marks og Zvonareva demonstrerer vi yderligere en bijektiv korrespondance mellem rangfunktioner på en Amiot-Lin $(d+2)$ -anguleret kategori og visse additive funktioner på dens associerede modulkategori. Dette fører til en dekompositionssætning: heltallige rangfunktioner har en faktorisering i irreducible komponenter i disse rammer.

Kapitel **III** studerer den homologiske teori for differentialmoduler via den Q -formede afledte kategori introduceret af Holm og Jørgensen. Vi beviser en differentialmodul-analog af et klassisk resultat, der karakteriserer hvornår en endeligt frembragt modul over en lokal kommutativ noethersk ring har endelig injektiv dimension. Som en anvendelse giver vi en ny karakterisering af lokale Cohen-Macaulay-ringe ved brug af differentialmoduler, hvilket giver et alternativt perspektiv på et spørgsmål oprindeligt stillet af Bass.¹

¹Thanks to Christian, Marius, Mary and Peter for helping me with this translation.

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Introduction

Representation theory studies objects such as groups, rings and algebras, via their actions on structures such as sets, abelian groups and vector spaces. This procedure can be thought of as *making the abstract more concrete*². We call these acted-on structures *representations* or *modules*. To determine when two representations are truly distinct, we assign invariants to them that capture essential information. So the idea now is to study these invariants. Category theory teaches us that it is often fruitful not only to study objects intrinsically but also to examine their behaviour within a broader, natural setting. Therefore, homological algebra studies such invariants in naturally defined environments, leading us to the main objects of study in this thesis: abelian categories and triangulated categories.

Although the main topics in this thesis may initially appear unrelated, there is a common thread that weaves between them: homological algebra in subcategories—hence the thesis title. We now explain how each topic relates to this theme.

Chapter I is titled *Nakayama functors on proper abelian subcategories*. The explanation here is straightforward. We study abelian categories embedded in triangulated categories and prove intrinsic homological properties of the abelian subcategory by examining how well it sits within, and approximates, the ambient triangulated category.

Chapter II is titled *Rank functions on $(d + 2)$ -angulated categories—a functorial approach*. The original examples of such categories were defined as well-behaved subcategories of triangulated categories with strong approximation properties. To the best of the author’s knowledge, all known examples in the literature can be realised as subcategories of canonical triangulated categories, as further evidenced by recent work by Kvamme [Kva25]. Hence, the main object of study in this paper, at least for now, can be thought of as subcategories of triangulated categories.

Chapter III is titled *Differential modules: a perspective on Bass’ question*. The category of differential modules studied in this paper can be realised as a non-full subcategory of the category of complexes. As such, this paper explores how much homological algebra

²The following is a link to an artwork created by the author, an exploration of this slogan: <https://sites.google.com/view/makingtheabstractmoreconcrete>

can be done when we restrict attention to this subcategory and deliberately ‘forget’ some morphisms.

A journey through the main results

We now embark on a journey through the main results of this thesis. The aim here is to motivate and introduce these results, offering a glimpse into the author’s perspective and preferred ways of thinking about them. More detailed introductions and discussions can be found in the respective chapters.

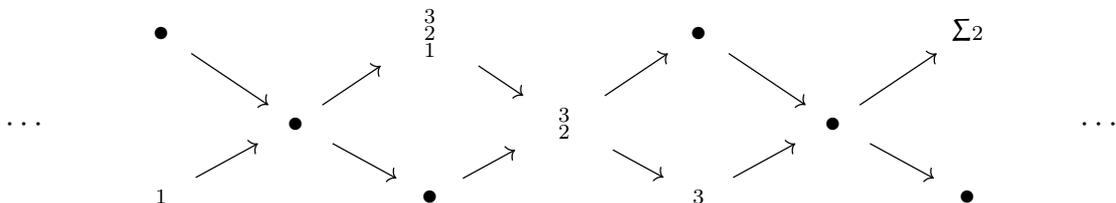
Chapter I

Chapter I investigates certain abelian subcategories of a triangulated category (\mathcal{T}, Σ) that come equipped with a Serre functor $\mathcal{T} \xrightarrow{\mathbb{S}} \mathcal{T}$. It begins with the following innocent question: *given a projective object P in a fixed abelian subcategory of \mathcal{T} , does the object $\mathbb{S}P$ lie in that same abelian subcategory?* Let us first examine a class of examples where the answer is *yes*.

Consider the heart \mathcal{H}_T of a t-structure on \mathcal{T} induced by a tilting object T . The heart \mathcal{H}_T is the full subcategory of \mathcal{T} consisting of the objects X in \mathcal{T} such that the Hom space $\mathcal{T}(T, \Sigma^i X)$ vanishes whenever the integer i is nonzero. Since T is a tilting object, it lies in \mathcal{H}_T by construction. The projective objects in \mathcal{H}_T are precisely coproducts of direct summands of T . It thus suffices to check whether $\mathbb{S}T$ lies in the heart \mathcal{H}_T , which follows from a straightforward computation using Serre duality: for i nonzero,

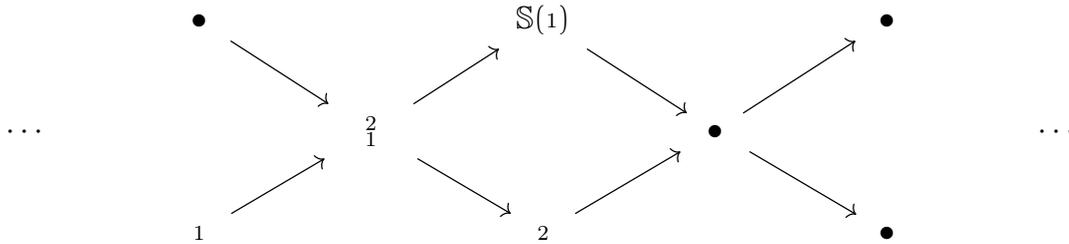
$$\mathcal{T}(T, \Sigma^i \mathbb{S}T) \cong D^2 \mathcal{T}(T, \Sigma^i \mathbb{S}T) \cong D \mathcal{T}(\Sigma^i \mathbb{S}T, T) \cong D \mathcal{T}(T, \Sigma^{-i} T) = 0.$$

A concrete example is as follows: let \mathcal{T} be the bounded derived category of the path algebra of the quiver $1 \leftarrow 2 \leftarrow 3$. As the path algebra has finite global dimension, \mathcal{T} admits a Serre functor given by the derived Nakayama functor. Representing modules by their composition factors, we let the tilting object T be the direct sum $T = 1 \oplus \frac{3}{1} \oplus 3$, where the summands correspond to the simple projective, the indecomposable projective-injective and the simple injective modules, considered as stalk complexes in degree zero. The associated heart is the additive closure $\mathcal{H}_T = \text{add}(\{T, \frac{3}{2}, \Sigma 2\})$. The action of the Serre functor on T is determined by the assignments $1 \mapsto \frac{3}{1} \mapsto 3 \mapsto \Sigma 2$. We include the Auslander-Reiten quiver of \mathcal{T} to help visualise this:



Now, let us examine an example where the answer to the innocent question is *no*.

Let \mathcal{T} be as above. The short exact sequence $0 \rightarrow 1 \rightarrow \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \rightarrow 2 \rightarrow 0$ in $\text{mod } kA_3$ induces an exact fully faithful embedding of module categories $\text{mod } kA_2 \rightarrow \text{mod } kA_3$ which sends the simple kA_2 -modules to simple kA_3 -modules. By viewing $\text{mod } kA_3$ as the heart of the canonical t-structure on \mathcal{T} , we may regard $\text{mod } kA_2$ as a full subcategory of \mathcal{T} . The action of the Serre functor on the simple projective kA_2 -module $P = 1$ does not land in $\text{mod } kA_2$, as shown in the Auslander-Reiten quiver of \mathcal{T} :



Although $\mathbb{S}P$ does not always lie in the considered abelian subcategory, it does not stray too far:

Theorem (Theorem I.5.4). *Let A be a finite dimensional algebra over a field k , let $\text{mod } A$ denote the category of finite dimensional A -modules, and let N be the standard Nakayama functor on $\text{mod } A$. If $\text{mod } A$ is a k -linear subcategory of \mathcal{T} , then for each projective object P in $\text{mod } A$, there is a strong $(\text{mod } A)$ -cover of the form $N(P) \rightarrow \mathbb{S}P$.*

In the previous example, the strong $(\text{mod } kA_2)$ -cover $N(P) \rightarrow \mathbb{S}P$ provided by the above theorem, is the irreducible morphism $\begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \rightarrow \mathbb{S}(1)$. This led us to formulate the following slogan, which captures the essence of Chapter I.

Slogan: *The Nakayama functor is a wannabe³ Serre functor.*

Guided by this slogan, we construct abstract Nakayama functors on suitable abelian categories \mathcal{A} via approximations of the Serre functor and deduce further strong intrinsic properties of \mathcal{A} . Our main results in Chapter I assume the following setup.

Setup. Let k be a field and \mathcal{T} be an idempotent complete Hom-finite k -linear triangulated category. Let \mathcal{A} be an extension closed k -linear proper abelian subcategory of \mathcal{T} and assume $\mathcal{T}(\mathcal{A}, \Sigma^{-1}\mathcal{A}) = 0$. Further, assume the following approximations exists:

- For each projective object P in \mathcal{A} there is an \mathcal{A} -precover in \mathcal{T} of the form $A_P \rightarrow \mathbb{S}P$.
- For each injective object I in \mathcal{A} there is an \mathcal{A} -preenvelope in \mathcal{T} of the form $\mathbb{S}^{-1}I \rightarrow B_I$.

³wannabe is a blend (or portmanteau) of the phrase ‘want to be’.

We summarise the main results of Chapter I in the following diagram, where an arrow

$$\mathbf{Setup} - \mathcal{A} \rightarrow \mathcal{P}$$

indicates that assuming the conditions in **Setup** and an additional assumption \mathcal{A} on \mathcal{A} , then \mathcal{A} enjoys property \mathcal{P} :

Theorem (Theorem I.3.4, Theorem I.3.8, Theorem I.3.10 and Theorem I.4.7).

$$\begin{array}{ccc}
 \text{Enough projectives} & \iff & \text{enough injectives} \\
 \uparrow & & \\
 \text{Finite length} & & \\
 \downarrow & & \\
 \mathbf{Setup} & \longrightarrow & \text{Proj } \mathcal{A} \simeq \text{Inj } \mathcal{A} \\
 \downarrow & & \\
 \text{Enough projectives and injectives} & & \\
 \downarrow & & \\
 \text{Dualising } \mathbf{k}\text{-variety and } AR\text{-sequences} & &
 \end{array}$$

The equivalence $\text{Proj } \mathcal{A} \simeq \text{Inj } \mathcal{A}$ between projective and injective objects of \mathcal{A} gives evidence for our slogan. That is, taking appropriate approximations of the Serre functor gives us a Nakayama functor. It is worth noting that the above theorems illustrate a phenomenon the author of this manuscript came across in [CPP22]: namely, that purely intrinsic properties of \mathcal{A} (i.e. \mathcal{P}) are deduced from how well \mathcal{A} sits inside of \mathcal{T} (i.e. **Setup** and \mathcal{A}).

As an application of the previous theory developed, we provide a new proof of the existence of Auslander-Reiten sequences in $\text{mod } \mathbf{A}$, for \mathbf{A} a finite dimensional algebra (see Theorem I.5.7). The idea is to find a suitable \mathcal{T} that $\text{mod } \mathbf{A}$ lies in, satisfying the considerations in **Setup**. We can take \mathcal{T} to be the stable category $\underline{\text{mod}} \widehat{\mathbf{A}}$ of the repetitive algebra of \mathbf{A} introduced in [Hap88].

Finally, these results can also serve as obstructions: they provide conditions that must be satisfied for an abelian category to admit such an embedding inside \mathcal{T} .

Chapter II

Chapter II proposes an extension of the classical notion of a *rank function* to the setting of higher homological algebra.

A rank function assigns a nonnegative numerical value to data in a triangulated category, subject to some axioms. One motivation for such a definition comes from mimicking the behaviour of dimension in the setting of finite dimensional vector spaces. Consider the exact sequence

$$\mathbf{U} \xrightarrow{f} \mathbf{V} \xrightarrow{g} \mathbf{W} \quad (1)$$

of finite dimensional vector spaces. The Rank-Nullity Theorem applied at \mathbf{V} yields the identity $\dim(\text{Ker } g) - \dim \mathbf{V} + \dim(\text{Im } g) = 0$. The exactness of (1) implies the equality

$\dim(\text{Ker } g) = \dim(\text{Im } f)$ and, noting that $\dim V = \dim(\text{Im } \text{id}_V)$, we may reformulate the above identity as

$$\dim(\text{Im } f) - \dim(\text{Im } \text{id}_V) + \dim(\text{Im } g) = 0. \quad (2)$$

This expression involves only the images of the linear maps appearing in the sequence (1).

To obtain a relation involving the objects themselves, we can apply the Rank-Nullity Theorem at U and V , and note that f need not be injective and g need not be surjective. This gives the inequality

$$\dim U - \dim V + \dim W \geq 0. \quad (3)$$

Now, consider a longer exact sequence of finite dimensional vector spaces

$$V_0 \rightarrow V_1 \rightarrow \cdots \rightarrow V_d \rightarrow V_{d+1}. \quad (4)$$

If d is a odd positive integer, then repeated application of the Rank-Nullity Theorem to this sequence yields the inequality

$$\sum_{i=0}^{d+1} (-1)^i \dim V_i \geq 0. \quad (5)$$

The inequality (5) does not hold in general for d not odd. This odd⁴ feature is unavoidable in Paper II and has been seen in other works concerning higher homological algebra (references to examples in the literature are given in the introduction to Paper II). The author does not know whether a ‘desirable’ higher analogue of (2) exists that involves *all* the linear maps in the sequence (4). However, the equation (2) does hold *locally* at V_i for $1 \leq i \leq d$.

In summary, given the exact sequence (4), assigning a linear map to the dimension of its image satisfies (2) locally at the inner V_i ’s, while assigning a vector space to its dimension satisfies the inequality (5). These two observations lead to the definitions of a rank function on morphisms and a rank function on objects in a $(d + 2)$ -angulated category (see Definition II.3.7 and Definition II.3).

Motivated by these considerations, we define and relate two notions of rank functions adapted to $(d + 2)$ -angulated categories:

Theorem (Theorem II.3.14). *Let d be an odd positive integer and let \mathcal{C} be an essentially small $(d + 2)$ -angulated category. There is an explicit bijection*

$$\{\text{Rank functions on objects in } \mathcal{C}\} \iff \{\text{Rank functions on morphisms in } \mathcal{C}\}.$$

There is a class of $(d + 2)$ -angulated categories originally defined by Amiot for $d = 1$ [Ami07] and extended by Lin to all $d \geq 1$ [Lin19]. With this class, rank functions admit the following additional correspondence:

⁴Pun intended.

Theorem (Theorem II.4.11). *Let \mathbf{d} be a positive odd integer, \mathbf{A} be a twisted $(\mathbf{d} + 2)$ -periodic finite dimensional algebra and endow $\mathbf{proj} \mathbf{A}$ with the Amiot-Lin $(\mathbf{d} + 2)$ -angulated category structure. There is an explicit bijection*

$$\{\text{Rank functions on morphisms in } \mathbf{proj} \mathbf{A}\} \iff \left\{ \begin{array}{l} \text{'Certain' additive functions} \\ \text{on } \mathbf{mod} \mathbf{A} \end{array} \right\}.$$

Moreover, this bijection respects a notion of integrality and irreducibility on both sides.

This characterisation enables a structural decomposition theorem for integral rank functions on $\mathbf{proj} \mathbf{A}$:

Theorem (Theorem II.4.14). *Let \mathbf{d} be a positive odd integer, \mathbf{A} be a twisted $(\mathbf{d} + 2)$ -periodic finite dimensional algebra and endow $\mathbf{proj} \mathbf{A}$ with the Amiot-Lin $(\mathbf{d} + 2)$ -angulated category structure. Then every integral rank function on morphisms in $\mathbf{proj} \mathbf{A}$ can be decomposed uniquely as a locally finite sum of irreducible rank functions on morphisms in $\mathbf{proj} \mathbf{A}$.*

Chapter III

Chapter III explores the homological algebra of differential modules—a structure obtained by removing the grading from complexes—through the lens of the Q-shaped derived category introduced by Holm and Jørgensen [HJ22].

Complexes lie at the heart of homological algebra, and with them, grading is nearly unavoidable. But what happens when we remove the grading entirely? This question motivates the study of differential modules, leading us to the following guiding philosophy:

Slogan: *Homological algebra without grading.*

To study this ungraded setting, we rely heavily on its connection to the graded world. Central to this connection is an adjoint triple:

$$\begin{array}{ccc} & \curvearrowright & \\ \{\text{Differential modules}\} & \longrightarrow & \{\text{Complexes}\}, \\ & \curvearrowleft & \\ & \pi_* & \end{array} \quad (6)$$

between the category of differential modules and that of complexes. The middle functor is a restriction functor, while π_* denotes the *cocompression functor*. Understanding which homological properties are preserved or reflected by these functors allows us to explore how the finiteness of the injective dimension of a finitely generated module over a local commutative noetherian ring can be detected in this ungraded world. As an immediate application, we show that this framework also detects when such a ring is Cohen–Macaulay:

Theorem (Theorem III.5.2 and Corollary III.5.3). *Let $(R, \mathfrak{m}, \mathfrak{k})$ be a local commutative noetherian ring and let M be a finitely generated R -module. Consider the differential R -module $(M, 0)$ with zero differential and choose a minimal semi-injective resolution $(M, 0) \rightarrow (I, d_1)$. Then the following statements are equivalent:*

1. M has finite injective dimension.
2. The number of direct summands of I isomorphic to the injective envelope of the residue field \mathfrak{k} is finite.

Furthermore, R is Cohen-Macaulay if and only if there exists a finitely generated R -module M for which these equivalent conditions hold.

This theorem offers a new perspective on a classical question posed by Bass [Bas63]. Its proof hinges on the fact that the functor π_* in (6) preserves semi-injectivity, prompting a deeper exploration into the structural properties of this functor.

Motivated by this, we further investigate when the cocompression functor π_* preserves *minimality* of semi-injectivity. We find three instances of an algebra A and a subclass of minimal semi-injective complexes

$$\mathcal{C}_A \subseteq \{\text{Minimal semi-injective complexes of } A\text{-modules}\},$$

where minimality is preserved under π_* . We summarise this in the following table:

Prop.	A	\mathcal{C}_A
III.6.2	Arbitrary	Complexes of length 2
III.6.3	Artinian	Bounded complexes
III.6.5	Local commutative noetherian of Krull dimension 1	Arbitrary

Foundational definitions with examples

We introduce several key notions that appear throughout Chapter I, Chapter II and Chapter III. Some of these were assumed to be known in the individual chapters but are included here for convenience.

Fix a hereditary commutative noetherian ring \mathbb{k} (for example, the ring of integers \mathbb{Z} or a field \mathbb{k}). A \mathbb{k} -category is a category enriched over the symmetric monoidal category of \mathbb{k} -modules. In particular, its Hom-sets are \mathbb{k} -modules and composition is \mathbb{k} -bilinear.

Triangulated categories (Verdier). A (\mathbb{k} -linear) *triangulated category* consists of the following data: a \mathbb{k} -category \mathcal{T} , a *suspension* \mathbb{k} -functor $\mathcal{T} \xrightarrow{\Sigma} \mathcal{T}$, which is an autoequivalence, and a collection of diagrams in \mathcal{T} of the form

$$x \rightarrow y \rightarrow z \rightarrow \Sigma x$$

called (*distinguished*) *triangles*. This data is subject to the following axioms:

T0. Consider a commutative diagram in \mathcal{T} of the form

$$\begin{array}{ccccccc} x & \longrightarrow & y & \longrightarrow & z & \longrightarrow & \Sigma x \\ \downarrow u & & \downarrow v & & \downarrow w & & \downarrow \Sigma u \\ x' & \longrightarrow & y' & \longrightarrow & z' & \longrightarrow & \Sigma x', \end{array}$$

where the morphisms u, v and w are isomorphisms. Then the top diagram is a triangle if and only if the bottom diagram is a triangle.

T1. For each object x in \mathcal{T} , the diagram $x \xrightarrow{\text{id}_x} x \rightarrow 0 \rightarrow \Sigma x$ is a triangle. For each morphism $x \xrightarrow{f} y$ in \mathcal{T} , there exists a triangle of the form $x \xrightarrow{f} y \rightarrow z \rightarrow \Sigma x$.

T2. The diagram $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} \Sigma x$ in \mathcal{T} is a triangle if and only if its rotation $y \xrightarrow{g} z \xrightarrow{h} \Sigma x \xrightarrow{-\Sigma f} \Sigma y$ is a triangle.

T3. Given a solid commutative diagram in \mathcal{T} of the form

$$\begin{array}{ccccccc} x & \longrightarrow & y & \longrightarrow & z & \longrightarrow & \Sigma x \\ \downarrow u & & \downarrow & & \downarrow & & \downarrow \Sigma u \\ x' & \longrightarrow & y' & \longrightarrow & z' & \longrightarrow & \Sigma x', \end{array}$$

where the rows are triangles, there exists a dashed morphism $z \rightarrow z'$ in \mathcal{T} rendering the entire diagram commutative.

T4. Consider a pair of composable morphisms $x \xrightarrow{f} y \xrightarrow{g} z$ fitting into some triangles

$$\begin{array}{l} x \xrightarrow{f} y \rightarrow \text{cone}(f) \rightarrow \Sigma x \\ y \xrightarrow{g} z \rightarrow \text{cone}(g) \rightarrow \Sigma y \\ x \xrightarrow{gf} z \rightarrow \text{cone}(gf) \rightarrow \Sigma x, \end{array}$$

in \mathcal{T} . Then there exists a triangle

$$\text{cone}(f) \rightarrow \text{cone}(gf) \rightarrow \text{cone}(g) \rightarrow \Sigma \text{cone}(f)$$

making the diagram

$$\begin{array}{ccccccc} x & \xrightarrow{f} & y & \longrightarrow & \text{cone}(f) & \longrightarrow & \Sigma x \\ \parallel & & \downarrow g & & \downarrow & & \parallel \\ x & \xrightarrow{gf} & z & \longrightarrow & \text{cone}(gf) & \longrightarrow & \Sigma x \\ \downarrow f & & \parallel & & \downarrow & & \downarrow \Sigma f \\ y & \xrightarrow{g} & z & \longrightarrow & \text{cone}(g) & \longrightarrow & \Sigma y \\ & & & & \downarrow & & \downarrow \Sigma u \\ & & & & \Sigma \text{cone}(f) & \longleftarrow & \Sigma \text{cone}(f) \end{array}$$

commutative.

A celebrated class of triangulated categories are derived categories of \mathbb{k} -algebras. Fix a \mathbb{k} -algebra A . The derived category $D(\text{Mod } A)$ of A is obtained from the abelian category of complexes of A -modules by formally inverting the quasi-isomorphisms. Since this procedure is difficult to work with directly, one often uses alternative models for the derived category, such as Frobenius exact categories or Quillen model categories.

Serre functor (Bondal and Kapranov). Let \mathcal{X} be a \mathbb{k} -category. A *Serre functor* is a \mathbb{k} -functor $\mathcal{X} \xrightarrow{\mathbb{S}} \mathcal{X}$, which is an autoequivalence, together with \mathbb{k} -module isomorphisms $\mathcal{X}(X, Y) \rightarrow \text{Hom}_{\mathbb{k}}(\mathcal{X}(Y, \mathbb{S}X), \mathbb{k})$, natural in objects X and Y in \mathcal{X} .

Suppose now that \mathbb{k} is an artinian ring and let A be an artin \mathbb{k} -algebra (for example, if \mathbb{k} is a field, A would be a finite dimensional \mathbb{k} -algebra). Recall that A is *Iwanaga-Gorenstein* if the regular left and right representations of A have finite injective dimension.

The perfect derived category $D^{\text{perf}}(A)$ of A is the triangulated subcategory of the derived category $D(\text{Mod } A)$ consisting of the compact objects. It admits a Serre functor if and only if A is Iwanaga-Gorenstein (see, for example, [Kra22, Thm. 6.4.6]).

Similarly, the bounded derived category $D^b(\text{mod } A)$ of A is a triangulated subcategory of $D(\text{Mod } A)$. It can be identified as the category obtained by inverting the quasi-isomorphisms in the category of bounded complexes of finitely generated A -modules. It admits a Serre functor if and only if A is Iwanaga-Gorenstein and each acyclic complex of finitely generated projective A -modules is contractible. This occurs if and only if A has finite global dimension (see, for example, [Kra22, Thm. 6.4.13]). In both cases, the Serre functor is given by the derived Nakayama functor.

Proper abelian subcategories (Jørgensen). A *proper abelian subcategory* of a triangulated category \mathcal{T} is a full \mathbb{k} -subcategory \mathcal{A} of \mathcal{T} that is closed under taking direct sums and direct summands and satisfies the following property: a given sequence

$$0 \rightarrow \mathbf{a} \xrightarrow{f} \mathbf{b} \xrightarrow{g} \mathbf{c} \rightarrow 0$$

in \mathcal{A} is exact if and only if there is a triangle

$$\mathbf{a} \xrightarrow{f} \mathbf{b} \xrightarrow{g} \mathbf{c} \rightarrow \Sigma \mathbf{a}$$

in \mathcal{T} .

Let us take \mathbb{k} to be a field. A well-studied class of proper abelian subcategories are the hearts of t -structures (see [BBD82, Théorème 1.3.6]). By construction, they enjoy the strong property that all negative extensions vanish—that is, the Hom bifunctor $\text{Hom}(-, \Sigma^{\leq -1}(-))$ (on the triangulated category) vanishes when restricted to the heart. However, this does not hold in general for arbitrary proper abelian subcategories.

Consider, for example, the bounded derived category $D^b(\mathbb{k}A_2)$ of the path algebra of the A_2 quiver $1 \leftarrow 2$, and let Σ be its suspension functor. Let P_1 be the unique

simple projective $\mathbb{k}\mathcal{A}_2$ -module and P_2 the unique indecomposable projective-injective $\mathbb{k}\mathcal{A}_2$ -module. Then the additive closure $\mathcal{A} = \text{add}(P_1 \oplus \Sigma P_2)$ is a semisimple abelian full subcategory of $D^b(\mathbb{k}\mathcal{A}_2)$. As $\mathcal{A}(P_1, P_2) \cong \mathcal{A}(P_1, \Sigma^{-1}\Sigma P_2)$, the radical inclusion $P_1 \rightarrow P_2$ represents a nontrivial negative extension between P_1 and ΣP_2 .

To see that \mathcal{A} is a proper abelian subcategory, observe that since \mathcal{A} is semisimple, all short exact sequences in \mathcal{A} are of the form

$$0 \rightarrow \mathfrak{a} \rightarrow \mathfrak{a} \oplus \mathfrak{c} \rightarrow \mathfrak{c} \rightarrow 0,$$

which correspond to the split triangles $\mathfrak{a} \rightarrow \mathfrak{a} \oplus \mathfrak{c} \rightarrow \mathfrak{c} \rightarrow \Sigma \mathfrak{a}$. This shows that \mathcal{A} is a *distinguished abelian subcategory* in the sense of Linckelmann [Lin24]. Moreover, since \mathcal{A} intersects trivially with $\Sigma \mathcal{A}$, it follows by [Lin24, Corollary 6.3] that \mathcal{A} is in fact a proper abelian subcategory.

We highlight here that it is easy to produce examples of distinguished abelian subcategories that are not proper. In the same setting as above, the additive closure $\text{add}(P_1 \oplus \Sigma P_1)$ is distinguished but not proper, for the same reasons outlined above. Explicitly, one can show that the definition fails because the triangle $P_1 \rightarrow 0 \rightarrow \Sigma P_1 \xrightarrow{-\text{id}} \Sigma P_1$ does not correspond to a short exact sequence in $\text{add}(P_1 \oplus \Sigma P_1)$.

Of course, many abelian subcategories are neither proper nor distinguished. For instance, take a non-wide abelian full subcategory \mathcal{B} of a module category $\text{mod } A$. The inclusion $\mathcal{B} \subseteq \text{mod } A \subseteq D^b(\text{mod } A)$, where $\text{mod } A \subseteq D^b(\text{mod } A)$ is the canonical heart inclusion, gives an example of such a category.

Finally, it is worth noting that proper abelian subcategories can also be found in triangulated categories that admit no nonzero hearts—such as in negative cluster categories. A method for constructing proper abelian subcategories in such settings was given by Jørgensen in [Jør22]. The key idea is to find a collection of simple-like objects (i.e. Hom-orthogonal bricks) that have few negative extensions between them and then take their extension closure.

($d+2$)-angulated categories (Geiss, Keller and Oppermann). Let d be a positive integer. A (\mathbb{k} -linear) $(d+2)$ -angulated category consists of the following data: a \mathbb{k} -category \mathcal{C} , a *suspension* \mathbb{k} -functor $\mathcal{C} \xrightarrow{\Sigma_d} \mathcal{C}$, which is an autoequivalence, and a collection of diagrams in \mathcal{C} of the form

$$\mathfrak{x}_0 \rightarrow \mathfrak{x}_1 \rightarrow \cdots \rightarrow \mathfrak{x}_{d+1} \rightarrow \Sigma_d \mathfrak{x}_0$$

called (*distinguished*) $(d+2)$ -angles. This data is subject to the following axioms:

- C0. The direct sum of two $(d+2)$ -angles is again a $(d+2)$ -angle. A direct summand of a $(d+2)$ -angle is again a $(d+2)$ -angle.

- C1. For each object x in \mathcal{C} , the diagram $x \xrightarrow{\text{id}_x} x \rightarrow 0 \rightarrow \cdots \rightarrow \Sigma x$ is a $(d+2)$ -angle. For each morphism $x_0 \xrightarrow{f_0} x_1$ in \mathcal{T} , there exists a $(d+2)$ -angle of the form $x_0 \xrightarrow{f_0} x_1 \rightarrow \cdots \rightarrow x_{d+1} \rightarrow \Sigma_d x_0$.
- C2. The diagram $x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} \cdots \rightarrow x_{d+1} \rightarrow \Sigma_d x_0$ in \mathcal{C} is a $(d+2)$ -angle if and only if its rotation $x_1 \xrightarrow{f_1} x_2 \rightarrow \cdots \rightarrow \Sigma_d x_0 \xrightarrow{(-1)^d \Sigma_d f_0} \Sigma_d x_1$ is a $(d+2)$ -angle.
- C3. Given a solid commutative diagram in \mathcal{C} of the form

$$\begin{array}{ccccccccccc} x_0 & \longrightarrow & x_1 & \longrightarrow & x_2 & \longrightarrow & \cdots & \longrightarrow & x_{d+1} & \longrightarrow & \Sigma_d x_0 \\ \downarrow u & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \Sigma_d u \\ y_0 & \longrightarrow & y_1 & \longrightarrow & y_2 & \longrightarrow & \cdots & \longrightarrow & y_{d+1} & \longrightarrow & \Sigma_d y_0, \end{array}$$

where the rows are $(d+2)$ -angles, there exist dashed morphisms $x_i \rightarrow y_i$ in \mathcal{C} , for $2 \leq i \leq d+1$, rendering the entire diagram commutative.

- T4. Consider a pair of composable morphisms $a \xrightarrow{\alpha} b \xrightarrow{\beta} c$ fitting into some $(d+2)$ -angles

$$\begin{aligned} a &\xrightarrow{\alpha} b \rightarrow x_2 \rightarrow \cdots \rightarrow x_{d+1} \rightarrow \Sigma_d a \\ b &\xrightarrow{\beta} c \rightarrow y_2 \rightarrow \cdots \rightarrow y_{d+1} \rightarrow \Sigma_d b \\ a &\xrightarrow{\beta\alpha} c \rightarrow z_2 \rightarrow \cdots \rightarrow z_{d+1} \rightarrow \Sigma_d a, \end{aligned}$$

in \mathcal{C} . Then there exist dashed morphisms fitting into a (noncommutative) diagram

$$\begin{array}{ccccccccccccccc} a & \xrightarrow{\alpha} & b & \longrightarrow & x_2 & \longrightarrow & x_3 & \longrightarrow & \cdots & \longrightarrow & x_d & \longrightarrow & x_{d+1} & \longrightarrow & \Sigma_d a \\ \parallel & & \downarrow \beta & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \parallel \\ a & \xrightarrow{\beta\alpha} & c & \longrightarrow & z_2 & \longrightarrow & z_3 & \longrightarrow & \cdots & \longrightarrow & z_d & \longrightarrow & z_{d+1} & \longrightarrow & \Sigma_d a \\ \downarrow \alpha & & \parallel & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \Sigma_d \alpha \\ b & \xrightarrow{\beta} & c & \longrightarrow & y_2 & \longrightarrow & y_3 & \longrightarrow & \cdots & \longrightarrow & y_d & \longrightarrow & y_{d+1} & \longrightarrow & \Sigma_d b \end{array}$$

such that the top squares commute, the bottom far right-most square commutes, and the induced totalisation (or mapping cone)

$$x_2 \rightarrow x_3 \oplus z_2 \rightarrow x_4 \oplus z_3 \oplus y_2 \rightarrow \cdots \rightarrow x_{d+1} \oplus y_d \oplus z_{d-1} \rightarrow y_{d+1} \oplus z_d \rightarrow z_{d+1}$$

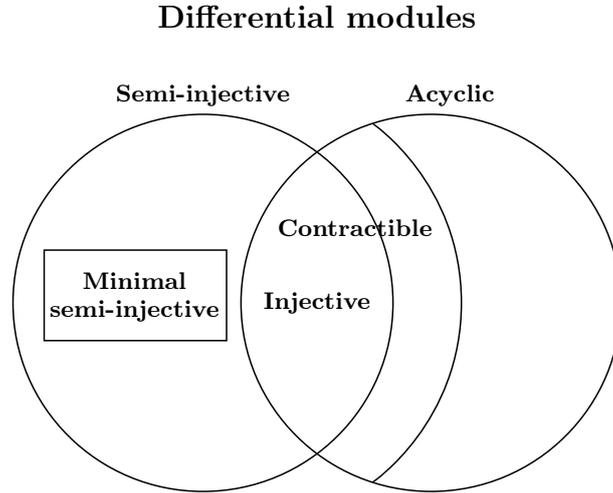
is a $(d+2)$ -angle.

We provide examples of $(d+2)$ -angulated categories in Chapter II. Rather than revisiting those here, we highlight a philosophical point that unifies such examples.

The original examples of $(d+2)$ -angulated categories arose as well-behaved, functorially finite subcategories of triangulated categories (see [GKO13, Thm. 1]). It was recently shown in [Kva25, Thm 7.5] that essentially every $(d+2)$ -angulated category arising in algebra is equivalent to these original examples. To the best of the author's knowledge, no example has yet been produced in the literature that is *not* equivalent to these original examples.

Differential module (Cartan and Eilenberg). Let A be a \mathbb{k} -algebra. A *differential A -module* is a pair (M, d_M) , where M is an A -module and $M \xrightarrow{d_M} M$ a square-zero A -module endomorphism. That is, $d_M \circ d_M = 0$.

We present the following Venn diagram representing various classes of homologically defined differential A -modules:



Rather than providing full definitions, we illustrate each region of the diagram with an example of a \mathbb{k} -algebra and a differential A -module that resides exclusively in that region, along with brief explanations. More detailed arguments can be developed using the theory developed in Chapter III.

- Differential module: for any A and for any nonzero A -module M , the differential A -module $(M, 0)$ is not acyclic and if M is not an injective A -module, it cannot be semi-injective.
- Acyclic: consider the truncated polynomial ring $A = \mathbb{k}[x]/(x^2)$. The differential module $(A, \cdot x)$ is acyclic but cannot be contractible nor semi-injective. One could see this since the 1-periodic complex of A -module given by $(A, \cdot x)$ is neither contractible nor semi-injective.
- Contractible: It is known, for any A , that all contractible differential A -modules are isomorphic to one of the form

$$\left(M \oplus M, \begin{pmatrix} 0 & 0 \\ \text{id}_M & 0 \end{pmatrix} \right),$$

for M an A -module.

- Injective: similarly to above, all differential A -modules are isomorphic to one of the form

$$\left(I \oplus I, \begin{pmatrix} 0 & 0 \\ \text{id}_I & 0 \end{pmatrix} \right),$$

for I an injective A -module.

- Semi-injective: consider the ring of integers \mathbb{Z} . We have the differential \mathbb{Z} -module

$$\left(\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}, \begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix} \right),$$

where $\mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z}$ is the canonical quotient. It is not injective as it is not of the above form. It is semi-injective as the \mathbb{Z} is hereditary and $\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$ is an injective \mathbb{Z} -module.

The structure of the thesis. The main body of the thesis is composed of three independent manuscripts: [Nka24a], [Nka24b] and [Nka25], corresponding to Chapter I, Chapter II and Chapter III, respectively. Each paper will begin with a title and an abstract, followed by an introduction, lemmas and then the main results. References are local to each chapter.

- Chapter I explores proper abelian subcategories of triangulated categories with a Serre functor. It is identical to the preprint [Nka24a] except that formatting and typesetting changes have been implemented to fit the thesis. This preprint has been accepted for publication in *Publicacions Matemàtiques*.
- Chapter II introduces the theory of rank functions into higher homological algebra. It is identical to the preprint [Nka24b] except that formatting and typesetting changes have been implemented to fit the thesis.
- Chapter III develops the homological algebra of differential modules. It is identical to the preprint [Nka25] except that formatting and typesetting changes have been implemented to fit the thesis.

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Nakayama functors on proper abelian subcategories

Abstract

We construct Nakayama functors on proper abelian subcategories of triangulated categories with a Serre functor using approximation theory. This, in turn, allows for the construction of Auslander-Reiten translates. As a result, we prove that suitable proper abelian subcategories are dualising k -varieties and have enough projectives if and only if they have enough injectives. As an application, we provide a new proof of the existence of Auslander-Reiten sequences in the category of finite dimensional modules over a finite dimensional algebra.

I.1 Introduction

Classical Auslander-Reiten theory was first introduced in [AR75] within the context of abelian categories. It was later extended into the triangulated realm in [Hap88]. Relative versions of Auslander-Reiten theory were also introduced in both the abelian [AS81] and triangulated [Jør09] cases. This paper aims to study Auslander-Reiten theory in a specific class of abelian subcategories of triangulated categories introduced in [Jør21]. The class known as the *proper abelian subcategories of triangulated categories* (see Definition I.1.4) generalises hearts of t -structures, and provides access to a theory concerning abelian subcategories, with possible nonzero negative extensions, of negative cluster categories.

Nakayama functors play an important role in the theory. In the abelian case, they provide an equivalence between the category of projective objects and the category of injective objects and allow for the construction of the Auslander-Reiten translates. Therefore, we begin by seeking a construction of Nakayama functors in proper abelian subcategories. Our results support a change of perspective from hearts of t -structures to proper abelian subcategories, as mentioned in [Jør21].

In this subsection, k is a field and \mathcal{T} is a Krull-Schmidt Hom-finite k -linear triangulated category with suspension functor Σ .

Definition I.1.1 ([Jør22, Definition 2.2 (i)]). We call a diagram $X \xrightarrow{x} Y \xrightarrow{y} Z$ in a triangulated category \mathcal{T} a *short triangle* if there exists a morphism $Z \xrightarrow{z} \Sigma X$ such that the augmented diagram $X \xrightarrow{x} Y \xrightarrow{y} Z \xrightarrow{z} \Sigma X$ is a triangle in \mathcal{T} .

Notation I.1.2. We say an exact sequence $X \xrightarrow{x} Y \xrightarrow{y} Z$ in an abelian category is left (right) exact if x is a monomorphism (y is an epimorphism). We say the sequence is short exact if it is both left exact and right exact. In particular, a projective presentation of an object X in an abelian category is a right exact sequence $P_1 \rightarrow P_0 \rightarrow X$, where P_0 and P_1 are projective objects in said abelian category. We use similar notation for injective copresentations.

Definition I.1.3. An *additive subcategory* of an additive category is a full subcategory closed under isomorphisms, direct sums and direct summands.

Definition I.1.4 ([Jør22, Definition 2.2 (ii)]). Let \mathcal{A} be an additive subcategory of a triangulated category \mathcal{T} . Then \mathcal{A} is a *proper abelian subcategory* of \mathcal{T} if it is abelian and if the following holds:

- The diagram $X \xrightarrow{x} Y \xrightarrow{y} Z$ is a short exact sequence in \mathcal{A} if and only if the diagram $X \xrightarrow{x} Y \xrightarrow{y} Z$ is a short triangle in \mathcal{T} whose terms X , Y and Z all lie in \mathcal{A} .

Remark I.1.5. Other generalisations of hearts of t-structures include *admissible abelian subcategories* [BBD82, Definition 1.2.5] and *distinguished abelian subcategories* [Lin24, Definition 1.1].

In the classical case, one may break down Auslander-Reiten theory into the following:

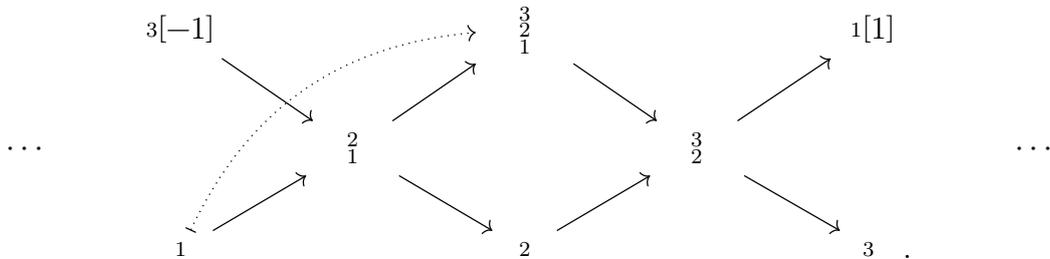
1. The study of the collection of non-projective objects and non-injective objects. These are intimately linked to the Auslander-Reiten translates and Auslander-Reiten sequences.
2. The study of the collection of projective objects and injective objects. These are intimately linked to Nakayama functors.

Using [Jør09, Theorem 3.1] (see [Fed19, Theorem B] for the $(d+2)$ -angulated case), *part 1* is well-understood for proper abelian subcategories: The Auslander-Reiten sequences in a proper abelian subcategory are controlled by approximation properties of the proper abelian subcategory with its ambient triangulated category. However, the aspects mentioned in *part 2* are not covered in [Jør09] and therefore, we aim to explore them in this paper. Other considerations about *part 1* can be found in [Kle97] and [Fed20] whereas other considerations about *part 2* can be found in [Fed22].

Definition I.1.6 ([BK89, page 519]). A *Serre functor* $\mathcal{X} \xrightarrow{\mathbb{S}} \mathcal{X}$ on a k -linear category \mathcal{X} is a k -linear autoequivalence together with isomorphisms $\mathcal{X}(X, Y) \xrightarrow{\eta_{X,Y}} D\mathcal{X}(Y, \mathbb{S}X)$ that are natural in objects X and Y in \mathcal{X} . Here $(\text{mod}_k)^{\text{op}} \xrightarrow{D} \text{mod}_k$ denotes the standard k -dual functor $\text{Hom}_k(-, k)$ where mod_k is the category of finite dimensional right modules over k . In this case, we say \mathcal{X} has *Serre duality*.

Let \mathcal{A} be either the category of finite dimensional (right) modules over a finite dimensional k -algebra of finite global dimension or a hereditary Ext-finite k -linear abelian category. Consider \mathcal{A} sitting inside its bounded derived category $\mathcal{D}^b(\mathcal{A})$ as the heart of the canonical t-structure. It was shown in [Hap88, I.4.6 and Theorem on page 37] and [RV02, Corollary I.3.4] that a Serre functor on $\mathcal{D}^b(\mathcal{A})$ restricts to an equivalence between projective objects in \mathcal{A} and injective objects in \mathcal{A} (see also [Hap91]). We would like to point out two problems that may occur in a more general setting. Firstly, the Serre functor might not send a projective object in \mathcal{A} to an object in \mathcal{A} , let alone an injective one (see Example I.1.7). Secondly, the Serre functor might not induce an equivalence. Under mild assumptions, Theorem A provides a remedy to these issues by constructing a functor via approximations of a Serre functor.

Example I.1.7 (Motivational example). The short exact sequence $0 \rightarrow 1 \rightarrow \frac{2}{1} \rightarrow 2 \rightarrow 0$ in $\text{mod } kA_3$ induces an exact fully faithful embedding of module categories $\text{mod}_{kA_2} \rightarrow \text{mod}_{kA_3}$. Therefore, considering mod_{kA_3} as the heart of the canonical t-structure on the bounded derived category $\mathcal{D}^b(\text{mod}_{kA_3})$, we can identify mod_{kA_2} as a full subcategory of $\mathcal{D}^b(\text{mod}_{kA_3})$. As the path algebra of the quiver A_3 is of finite global dimension, $\mathcal{D}^b(\text{mod}_{kA_3})$ has a Serre functor \mathbb{S} given by the left derived functor of $(-) \otimes_{kA_3} D(kA_3)$ (see [Hap88, Theorem on page 37]). One can check that \mathbb{S} sends the image of the simple projective kA_2 -module in $\mathcal{D}^b(\text{mod}_{kA_3})$ to an object that does not lie in mod_{kA_2} (identified as a full subcategory of $\mathcal{D}^b(\text{mod}_{kA_3})$). It is sent to the image of the indecomposable projective-injective object in mod_{kA_3} . If we consider the Auslander-Reiten quiver of $\mathcal{D}^b(\text{mod}_{kA_3})$, the above translates to the mapping $1 \xrightarrow{\mathbb{S}} \frac{3}{2}$ in the following diagram:



Although the object $\mathbb{S}(1) = \frac{3}{2}$ does not lie in mod_{kA_2} , we see that there is an irreducible morphism $\frac{2}{1} \xrightarrow{f} \frac{3}{2}$, where the object $\frac{2}{1}$ does indeed lie in mod_{kA_2} . Notice that $\nu(1) = \frac{2}{1}$, where ν is the Nakayama functor on mod_{kA_2} . We will see in Section I.5, that this

morphism $\nu(1) \xrightarrow{f} S(1)$ exhibits a strong approximation (see Definition I.1.8) of $S(1)$ inside of $\text{mod}_{\mathbf{k}\mathcal{A}_2}$.

We follow terminology due to [Eno81, Section 1] whereas the alternative terminology in the following is due to Auslander and Smalø, in which the concept was introduced in [AS80].

Definition I.1.8. Let \mathcal{Y} be an additive subcategory of an additive category \mathcal{X} and let X be an object in \mathcal{X} . For a morphism $Y \xrightarrow{y} X$ with Y in \mathcal{Y} , we have the following definitions:

- y is a \mathcal{Y} -precover of X if for each object Y' in \mathcal{Y} the map $\mathcal{X}(Y', Y) \xrightarrow{\mathcal{X}(Y', y)} \mathcal{X}(Y', X)$ is an epimorphism. Diagrammatically, this means for each morphism $Y' \rightarrow X$, there exists a morphism $Y' \rightarrow Y$ making the following diagram commute:

$$\begin{array}{ccc} & & Y' \\ & \swarrow \exists & \downarrow \\ Y & \xrightarrow{y} & X. \end{array}$$

Alternative terminology for y is a *right \mathcal{Y} -approximation of X* .

- y is *right minimal* if each endomorphism $Y \xrightarrow{y'} Y$ satisfying the equation $y = yy'$ is automatically an automorphism.
- y is a \mathcal{Y} -cover of X if it is both a \mathcal{Y} -precover of X and right minimal.
- y is a *strong \mathcal{Y} -cover of X* if for each object Y' in \mathcal{Y} the map $\mathcal{X}(Y', Y) \xrightarrow{\mathcal{X}(Y', y)} \mathcal{X}(Y', X)$ is an isomorphism. Diagrammatically, this means for each morphism $Y' \rightarrow X$, there exists a unique morphism $Y' \rightarrow Y$ making the following diagram commute:

$$\begin{array}{ccc} & & Y' \\ & \swarrow \exists! & \downarrow \\ Y & \xrightarrow{y} & X. \end{array}$$

\mathcal{Y} -preenvelopes (or alternatively, *left \mathcal{Y} -approximations*), *left minimality*, \mathcal{Y} -envelopes and *strong \mathcal{Y} -envelopes* are defined dually.

Definition I.1.9 ([AR74, page 307]). Let \mathcal{A} be an essentially small \mathbf{k} -linear category and let $\text{Mod}_{\mathcal{A}}$ denote the (abelian) category of \mathbf{k} -linear functors $\mathcal{A}^{\text{op}} \rightarrow \text{Mod}_{\mathbf{k}}$ (see [Aus74a, page 184]). We say that an object F in $\text{Mod}_{\mathcal{A}}$ is *finitely presented* (see [Aus74b, page 204]) if there exists a right exact sequence $\mathcal{A}(-, Y) \rightarrow \mathcal{A}(-, X) \rightarrow F$, with X and Y objects in \mathcal{A} (alternative terminology for such an F is *coherent*, see [Aus66, page 189]). Let $\text{mod}_{\mathcal{A}}$ denote the full subcategory of $\text{Mod}_{\mathcal{A}}$ consisting of the finitely presented objects in $\text{Mod}_{\mathcal{A}}$. The standard \mathbf{k} -dual functors induce exact functors

$$(\text{Mod}_{\mathcal{A}})^{\text{op}} \xrightarrow{\mathbb{D}} \text{Mod}_{\mathcal{A}^{\text{op}}} \quad \text{and} \quad \text{Mod}_{\mathcal{A}^{\text{op}}} \xrightarrow{\mathbb{D}} (\text{Mod}_{\mathcal{A}})^{\text{op}}. \quad (\text{I.1})$$

We say \mathcal{A} is a *dualising k-variety* if the functors in (I.1) restrict to functors

$$(\mathrm{mod}_{\mathcal{A}})^{\mathrm{op}} \xrightarrow{\mathbb{D}} \mathrm{mod}_{\mathcal{A}^{\mathrm{op}}} \quad \text{and} \quad \mathrm{mod}_{\mathcal{A}^{\mathrm{op}}} \xrightarrow{\mathbb{D}} (\mathrm{mod}_{\mathcal{A}})^{\mathrm{op}}.$$

Main results

In this subsection, k is a field and \mathcal{T} is a Krull-Schmidt Hom-finite k -linear triangulated category \mathcal{T} with suspension functor Σ and with a Serre functor $\mathcal{T} \xrightarrow{\mathbb{S}} \mathcal{T}$. We also fix a full abelian subcategory \mathcal{A} of \mathcal{T} . We let $\mathrm{Proj} \mathcal{A}$ denote the full subcategory of \mathcal{A} consisting of the projective objects and let $\mathrm{Inj} \mathcal{A}$ denote the full subcategory of \mathcal{A} consisting of the injective objects.

As witnessed in Example I.1.7, the assignment $P \mapsto \mathbb{S}P$ for P a projective object in \mathcal{A} only defines a functor $\mathrm{Proj} \mathcal{A} \rightarrow \mathcal{T}$. The idea of the next result is to correct this unwanted feature by requiring the existence of an \mathcal{A} -cover of $\mathbb{S}P$. It turns out, under mild assumptions (see Theorem I.3.4), this requirement ensures that the assignment $P \mapsto \mathbb{S}P \mapsto \nu P$, where $\nu P \rightarrow \mathbb{S}P$ is an \mathcal{A} -cover of $\mathbb{S}P$, is functorial and provides an equivalence $\mathrm{Proj} \mathcal{A} \simeq \mathrm{Inj} \mathcal{A}$ of additive categories.

Setup I.1.10. Let \mathcal{A} be an extension closed k -linear proper abelian subcategory of \mathcal{T} and assume $\mathcal{T}(\mathcal{A}, \Sigma^{-1}\mathcal{A}) = 0$. Further, assume the following:

- For each projective object P in \mathcal{A} there is an \mathcal{A} -cover in \mathcal{T} of the form $\nu P \xrightarrow{\alpha_P} \mathbb{S}P$.
- For each injective object I in \mathcal{A} there is an \mathcal{A} -envelope in \mathcal{T} of the form $\mathbb{S}^{-1}I \xrightarrow{\beta_I} \nu^{-}I$.

Theorem A (Theorem I.3.4). *Consider Setup I.1.10. Then the following hold:*

1. *The assignment $P \mapsto \nu P$ augments to an additive functor $\mathrm{Proj} \mathcal{A} \xrightarrow{\nu} \mathrm{Inj} \mathcal{A}$ such that*

$$\begin{array}{ccc} \nu P & \xrightarrow{\alpha_P} & \mathbb{S}P \\ \nu P \downarrow & & \downarrow \mathbb{S}P \\ \nu P' & \xrightarrow{\alpha_{P'}} & \mathbb{S}P' \end{array}$$

is commutative for each morphism $P \xrightarrow{P} P'$ in $\mathrm{Proj} \mathcal{A}$.

2. *The assignment $I \mapsto \nu^{-}I$ augments to an additive functor $\mathrm{Inj} \mathcal{A} \xrightarrow{\nu^{-}} \mathrm{Proj} \mathcal{A}$ such that*

$$\begin{array}{ccc} \mathbb{S}^{-1}I & \xrightarrow{\beta_I} & \nu^{-}I \\ \mathbb{S}^{-1}i \downarrow & & \downarrow \nu^{-}i \\ \mathbb{S}^{-1}I' & \xrightarrow{\beta_{I'}} & \nu^{-}I' \end{array}$$

is commutative for each morphism $I \xrightarrow{i} I'$ in $\mathrm{Inj} \mathcal{A}$.

Moreover, the functors ν and ν^{-} are mutual quasi-inverses.

Our next main theorem shows that \mathcal{A} enjoys a useful duality condition (see Definition I.1.9) and if we impose that \mathcal{A} is a length category, then the symmetry between the projective and injective objects in \mathcal{A} becomes stronger.

Theorem B (Theorem I.3.8 and Theorem I.3.10). *Consider Setup I.1.10. Then the following hold:*

1. *If \mathcal{A} is essentially small and has enough injectives and enough projectives, then \mathcal{A} is a dualising k -variety.*
2. *If every object in \mathcal{A} is of finite length, then \mathcal{A} has enough projectives if and only if \mathcal{A} has enough injectives.*

Having Theorem A at our disposal, we can construct Auslander-Reiten translates τ and τ^- in \mathcal{A} following the classical pedagogy (see the beginning of Section I.4 and Definition I.4.2). The fundamental properties of these mappings are given as our next main result.

Theorem C (Proposition I.4.3 and Proposition I.4.4). *Consider Setup I.1.10 and assume each object in \mathcal{A} has a projective cover and an injective envelope. Then there are Auslander-Reiten translates τ and τ^- on \mathcal{A} and they satisfy several standard properties.*

The next result is the existence of Auslander-Reiten sequences in \mathcal{A} .

Theorem D (Theorem I.4.7). *Consider Setup I.1.10 and assume each object in \mathcal{A} has a projective cover and an injective envelope. Then the following hold:*

1. *For each indecomposable non-projective object C in \mathcal{A} , there exists an Auslander-Reiten sequence in \mathcal{A} of the form*

$$0 \rightarrow \tau C \rightarrow E_C \rightarrow C \rightarrow 0.$$

2. *For each indecomposable non-injective object A in \mathcal{A} , there exists an Auslander-Reiten sequence in \mathcal{A} of the form*

$$0 \rightarrow A \rightarrow F_A \rightarrow \tau^- A \rightarrow 0.$$

When applying the methods presented in the main theorems above to the module category, we recover the standard Nakayama functors and Auslander-Reiten translates (see Theorem I.5.4 and Proposition I.5.6). As an application, we provide a new proof of the existence of Auslander-Reiten sequences in the finite dimensional module category of a finite dimensional algebra (see Theorem I.5.7).

Theorem A, Theorem B and Theorem D are instances of a fascinating and potentially powerful phenomenon that involves relating intrinsic properties of an abelian category

with its relationship to an ambient triangulated category (see the discussion in [CPP22, page 213]). In the same paper, it was shown that the heart of a bounded t-structure of a saturated Hom-finite Krull-Schmidt k -linear triangulated category is functorially finite in said triangulated if and only if the heart has enough projective and enough injective objects [CPP22, Corollary 2.8] (see [CPP22, Theorem 2.4] for a more general statement and compare with Theorem A and Theorem B, part 2).

Another such example of this phenomenon can be seen in [Jør22, Proposition 2.6], which says the following: Given an extension closed additive subcategory \mathcal{A} of a Hom-finite Krull-Schmidt k -linear triangulated category \mathcal{T} such that $\mathcal{T}(\mathcal{A}, \Sigma^{-1}\mathcal{A}) = 0$. If each object of \mathcal{A} has a $\Sigma\mathcal{A}$ -envelope, then there is an exact structure \mathcal{E} on \mathcal{A} such that the exact category $(\mathcal{A}, \mathcal{E})$ has enough projective objects.

Remark I.1.11. In [CPP22, Remark 2.15], the authors mention that it would be interesting to investigate whether their result [CPP22, Corollary 2.8] holds without the assumption of the triangulated category being saturated. The results in this paper assume the weaker condition that the triangulated category has a Serre functor (see [Kaw09, page 3]).

Some useful results

For the reader's convenience, we record some results that will be used multiple times throughout this paper. The first result is a useful criterion to use to show when a certain precover is actually a cover.

Lemma ([Kra00, Lemma 2.4]). *Consider a nonzero morphism $X \xrightarrow{f} Y$ in an additive category and suppose that $\text{End}(Y)$ is local. Then α is left minimal.*

The next result allows us to characterise the intrinsic property of an object being projective in a proper abelian subcategory in terms of the ambient triangulated category and its suspension functor.

Theorem ([Dye05, Theorem on page 1] (see also [Jør22, Theorem 3.5])). *Let \mathcal{T} be a triangulated category with suspension functor Σ and let \mathcal{A} be an additive full subcategory of \mathcal{T} that is closed under extensions and $\mathcal{T}(\mathcal{A}, \Sigma^{-1}\mathcal{A}) = 0$. Then an object P in \mathcal{A} is projective in \mathcal{A} if and only if $\mathcal{T}(P, \Sigma\mathcal{A}) = 0$.*

The final result will be used, sometimes in conjunction with Serre duality, to show that certain Hom spaces vanish.

Lemma (Triangulated Wakamatsu's Lemma [Jør09, Lemma 2.1]). *Let \mathcal{T} be a Krull-Schmidt Hom-finite k -linear triangulated category with suspension functor Σ and let \mathcal{A} be a full subcategory of \mathcal{T} that is closed under extensions and direct summands. Suppose that*

a morphism $A \xrightarrow{\alpha} T$ in \mathcal{T} is an \mathcal{A} -cover of an object T in \mathcal{T} and complete it to a triangle $A \xrightarrow{\alpha} T \rightarrow Z \rightarrow \Sigma A$. Then $\mathcal{T}(A, Z) = 0$.

Global setup

The following are taken throughout the paper:

- k is a field.
- The standard k -dual functor $(\mathbf{mod}_k)^{\text{op}} \xrightarrow{\text{Hom}_k(-, k)} \mathbf{mod}_k$, where \mathbf{mod}_k is the category of finite dimensional right modules over k , is denoted by D .
- All subcategories are assumed to be full subcategories closed under isomorphisms.
- k -linear categories are categories enriched over the category of k -vector spaces with finite direct sums.
- For an abelian category \mathcal{A} , the full subcategory of \mathcal{A} consisting of the projective objects is denoted by $\text{Proj } \mathcal{A}$ and the full subcategory of \mathcal{A} consisting of the injective objects is denoted by $\text{Inj } \mathcal{A}$.
- \mathcal{T} is a Krull-Schmidt Hom-finite k -linear triangulated category with suspension functor Σ and with a Serre functor $\mathcal{T} \xrightarrow{\mathbb{S}} \mathcal{T}$.

I.2 Lemmas

Lemmas on additive categories

Consider two objects X and X' in an additive category. We denote by $\text{Rad}(X, X')$ the radical morphisms between X and X' as seen in [Kel64, Lemma 6]. That is, a morphism $X \xrightarrow{x} X'$ in said additive category lies in $\text{Rad}(X, X')$ if for each morphism $X' \xrightarrow{x'} X$, we have that $1_X - x'x$ is an automorphism. The following lemma will be used in Proposition I.4.3.

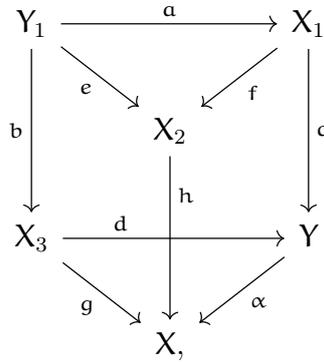
Lemma I.2.1. *Let \mathcal{X} and \mathcal{Y} be additive categories and let $\mathcal{X} \xrightarrow{F} \mathcal{Y}$ be a fully faithful additive functor. Then F induces an isomorphism $\text{Rad}(X, X') \rightarrow \text{Rad}(FX, FX')$ of abelian groups.*

Proof. Let $X \xrightarrow{x} X'$ be a radical morphism in \mathcal{X} . We will first show that $FX \xrightarrow{Fx} FX'$ is a radical morphism in \mathcal{Y} . To this end, let $FX' \xrightarrow{g} FX$ be a morphism in \mathcal{Y} . As F is fully faithful, g is of the form Fx' for some morphism $X' \xrightarrow{x'} X$ in \mathcal{X} . As x is a radical morphism in \mathcal{X} , we have that $1_X - x'x$ is an automorphism. Therefore, $1_{FX} - gF(x)$ is automorphism. We now have that the group isomorphism $\mathcal{X}(X, X') \rightarrow \mathcal{Y}(FX, FX')$ induced by F restricts to an injective group homomorphism $\text{Rad}(X, X') \rightarrow \text{Rad}(FX, FX')$. For surjectivity, we

let $FX \xrightarrow{f} FX'$ be a radical morphism in \mathcal{Y} . Again, we may take f to be of the form Fw for some morphism $X \xrightarrow{w} X'$ in \mathcal{X} . It suffices to show that w is a radical morphism in \mathcal{X} . Let $X' \xrightarrow{w'} X$ be a morphism in \mathcal{X} . Then $F(1_X - w'w) = 1_{FX} - F(w')f$ is an automorphism as f is a radical morphism in \mathcal{Y} . As F is fully faithful, it reflects isomorphisms. Therefore, $1_X - w'w$ is an automorphism and hence w is a radical morphism in \mathcal{X} . \square

The following Lemma will be a diagram trick involving a strong cover. It will be used in Lemma I.2.4 and Theorem I.3.4.

Lemma I.2.2. *Let \mathcal{X} be an additive category, let \mathcal{Y} be an additive subcategory of \mathcal{X} and consider the following diagram in \mathcal{X} :*



with the following properties:

- The objects Y and Y_1 both lie in \mathcal{Y} .
- All but the back-most square with vertices Y_1, X_1, X_3 and Y commute.

If α is a strong \mathcal{Y} -cover of X , then the back-most square commutes.

Proof. By the commutativity we have, the following equalities hold $\alpha db = gb = he = hfa = \alpha ca$. So if α is a strong \mathcal{Y} -cover, then we get $db = ca$ as required. \square

The following lemma will be used multiple times throughout this paper, where the functor $\mathcal{X} \xrightarrow{S} \mathcal{X}$ in the statement of the lemma will taken to be the Serre functor $\mathcal{T} \xrightarrow{S} \mathcal{T}$ mentioned in the Global setup. It will allow us to construct additive functors through the existence of approximations.

Lemma I.2.3. *Let \mathcal{X} be an additive category and let $\mathcal{X} \xrightarrow{S} \mathcal{X}$ be an additive endofunctor. Let \mathcal{Y} be an additive subcategory of \mathcal{X} and let \mathcal{Z} be an additive subcategory of \mathcal{Y} . Then the following hold:*

1. *If for each object Z in \mathcal{Z} there is a strong \mathcal{Y} -cover in \mathcal{X} of the form $C_Z \xrightarrow{c_Z} SZ$, then the assignment $Z \mapsto C_Z$ on objects augments to an additive functor $\mathcal{Z} \xrightarrow{C_{(-)}} \mathcal{Y}$ such*

that the induced diagram

$$\begin{array}{ccc} C_Z & \xrightarrow{c_Z} & SZ \\ C_Z \downarrow & & \downarrow S_Z \\ C_{Z'} & \xrightarrow{c_{Z'}} & SZ' \end{array}$$

is commutative for each morphism $Z \xrightarrow{z} Z'$ in \mathcal{Z} .

2. If for each object Z in \mathcal{Z} there is a strong \mathcal{Y} -envelope in \mathcal{X} of the form $S^{-1}Z \rightarrow E_Z$, then the assignment $Z \mapsto E_Z$ on objects augments to an additive functor $\mathcal{Z} \rightarrow \mathcal{Y}$ such that the induced diagram

$$\begin{array}{ccc} S^{-1}Z & \longrightarrow & E_Z \\ S^{-1}z \downarrow & & \downarrow E_z \\ S^{-1}Z' & \longrightarrow & E_{Z'} \end{array}$$

is commutative for each morphism $Z \xrightarrow{z} Z'$ in \mathcal{Z} .

Proof. Part 1. Given a morphism $Z \xrightarrow{z} Z'$ between objects in \mathcal{Z} , we have the composition $C_Z \xrightarrow{c_Z} SZ \xrightarrow{S_z} SZ'$. As $C_{Z'} \xrightarrow{c_{Z'}} SZ'$ is a strong \mathcal{Y} -cover in \mathcal{X} , there exists a unique morphism $C_Z \xrightarrow{c_z} C_{Z'}$ such that $S(z)c_Z = c_{Z'}C_Z$. The uniqueness of C_z in the assignment $z \mapsto C_z$ on morphisms ensures functoriality.

For additivity, let $Z \xrightarrow{z_1, z_2} Z'$ be morphisms between objects in \mathcal{Z} . We need to show $C_{(z_1+z_2)} = C_{z_1} + C_{z_2}$. By their respective definitions and as S is an additive functor, we have the following equalities:

$$c_{Z'}C_{(z_1+z_2)} = S(z_1 + z_2)c_Z = (Sz_1 + Sz_2)c_Z = c_{Z'}(C_{z_1} + C_{z_2}).$$

Additivity follows as $c_{Z'}$ is a strong \mathcal{Y} -cover.

Part 2. Dual to *part 1*. □

Lemma I.2.3 tells us that a global choice of strong \mathcal{Y} -covers of the objects SZ , for each object Z in \mathcal{Z} , give rise to an additive functor $\mathcal{Z} \rightarrow \mathcal{Y}$ fitting into the commutative diagram in said lemma. The next lemma, however, uses the uniqueness of the \mathcal{Y} -covers to ensure that any other global choice of \mathcal{Y} -covers of the objects SZ would result in a canonically naturally isomorphic functor. It will be used in Theorem I.5.4.

Lemma I.2.4. *Let \mathcal{X} be an additive category and let $\mathcal{X} \xrightarrow{S} \mathcal{X}$ be an additive endofunctor. Let \mathcal{Y} be an additive subcategory of \mathcal{X} and let \mathcal{Z} be an additive subcategory of \mathcal{Y} . Then the following hold:*

1. Assume for each object Z in \mathcal{Z} there are strong \mathcal{Y} -covers in \mathcal{X} of the form $C_Z \xrightarrow{c_Z} SZ$ and $D_Z \xrightarrow{d_Z} SZ$. The assignments $Z \mapsto C_Z$ and $Z \mapsto D_Z$ augment to two functors $\mathcal{Z} \rightarrow \mathcal{Y}$ by Lemma I.2.3. These functors are naturally isomorphic.

2. Assume for each object Z in \mathcal{Z} there are strong \mathcal{Y} -envelopes in \mathcal{X} of the form $S^{-1}Z \rightarrow E_Z$ and $S^{-1}Z \rightarrow F_Z$. The assignments $Z \mapsto E_Z$ and $Z \mapsto F_Z$ augment to two functors $\mathcal{Z} \rightarrow \mathcal{Y}$ by Lemma I.2.3. These functors are naturally isomorphic.

Proof. Assume for each object Z in \mathcal{Z} we have strong \mathcal{Y} -covers in \mathcal{X} of the form $C_Z \xrightarrow{c_Z} SZ$ and $D_Z \xrightarrow{d_Z} SZ$. By Lemma I.2.3, we have two functors $\mathcal{Z} \rightarrow \mathcal{Y}$, given by the assignments $Z \mapsto C_Z$ and $Z \mapsto D_Z$. As C_Z is an object in \mathcal{Y} and d_Z is a strong \mathcal{Y} -cover, there exists a unique isomorphism $C_Z \xrightarrow{\varphi_Z} D_Z$ such that $c_Z = d_Z \varphi_Z$. We now show that the collection $\{\varphi_Z \mid Z \text{ an object in } \mathcal{Z}\}$ of isomorphisms in \mathcal{Y} form the components of a natural isomorphism between the functors in question. Let $Z \xrightarrow{z} Z'$ be a morphism in \mathcal{Z} . We obtain the diagram:

$$\begin{array}{ccccc}
 C_Z & \xrightarrow{\varphi_Z} & D_Z & & \\
 \downarrow c_Z & \searrow c_Z & \swarrow d_Z & & \downarrow d_Z \\
 & & SZ & & \\
 & & \downarrow s_Z & & \\
 C_{Z'} & \xrightarrow{\varphi_{Z'}} & D_{Z'} & & \\
 \downarrow c_{Z'} & \searrow c_{Z'} & \swarrow d_{Z'} & & \downarrow d_{Z'} \\
 & & SZ' & &
 \end{array}$$

where all but the back-most square with vertices $C_Z, C_{Z'}, D_{Z'}$ and D_Z commute. By the application of Lemma I.2.2, we are done. \square

A lemma on k -linear abelian subcategories

The following lemma will be used in Proposition I.3.7.

Lemma I.2.5. *Let \mathcal{A} be a k -linear abelian subcategory of \mathcal{T} . Let X and P be objects in \mathcal{A} with P projective. Then the following two statements are equivalent:*

1. *There is a strong \mathcal{A} -cover in \mathcal{T} of the form $X \xrightarrow{\alpha_P} SP$.*
2. *There is a natural isomorphism of the form $D\mathcal{A}(P, -) \cong \mathcal{A}(-, X)$ of functors $\mathcal{A}^{\text{op}} \rightarrow \text{mod}_k$, i.e. the object X represents the functor $D\mathcal{A}(P, -)$.*

Dually, let Y and I be objects in \mathcal{A} with I injective. Then the following two statements are equivalent:

- 1'. *There is a strong \mathcal{A} -envelope in \mathcal{T} of the form $S^{-1}I \rightarrow Y$.*
- 2'. *There is a natural isomorphism of the form $D\mathcal{A}(-, I) \cong \mathcal{A}(Y, -)$ of functors $\mathcal{A} \rightarrow \text{mod}_k$, i.e. the object Y represents the functor $D\mathcal{A}(-, I)$.*

Proof. (1 \Rightarrow 2): Suppose there is a strong \mathcal{A} -cover of the form $X \xrightarrow{\alpha_P} \mathbb{S}P$. We have a natural isomorphism $\mathcal{A}(-, X) \xrightarrow{\mathcal{T}(-, \alpha_P)|_{\mathcal{A}}} \mathcal{T}(-, \mathbb{S}P)|_{\mathcal{A}} \cong D\mathcal{A}(P, -)$ of functors $\mathcal{A}^{\text{op}} \rightarrow \mathbf{mod}_k$, where the first whiskered composite natural transformation is a natural isomorphism as α_P is a strong \mathcal{A} -cover and the second natural isomorphism is given by Serre duality. Therefore, X represents the functor $D\mathcal{A}(P, -)$.

(2 \Rightarrow 1): Suppose the object X represents the functor $D\mathcal{A}(P, -)$. Then we have a natural isomorphism $\gamma: \mathcal{A}(-, X) \cong D\mathcal{A}(P, -) \cong \mathcal{T}(-, \mathbb{S}P)|_{\mathcal{A}}$ of functors $\mathcal{A}^{\text{op}} \rightarrow \mathbf{mod}_k$, where the second natural isomorphism is given by Serre duality. We will show that the morphism $X \xrightarrow{\gamma_X(1_X)} \mathbb{S}P$, where γ_X is the component of the natural isomorphism γ at X , is a strong \mathcal{A} -cover. Let $\alpha_P := \gamma_X(1_X)$. It suffices to show for each object A in \mathcal{A} , the component γ_A coincides with $\mathcal{A}(A, X) \xrightarrow{\mathcal{T}(A, \alpha_P)|_{\mathcal{A}}} \mathcal{T}(A, \mathbb{S}P)|_{\mathcal{A}}$. To this end, let $A \xrightarrow{a} X$ be a morphism in \mathcal{A} . By naturality, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{A}(X, X) & \xrightarrow{\gamma_X} & \mathcal{T}(X, \mathbb{S}P)|_{\mathcal{A}} \\ \mathcal{A}(a, X) \downarrow & & \downarrow \mathcal{T}(a, \mathbb{S}P)|_{\mathcal{A}} \\ \mathcal{A}(A, X) & \xrightarrow{\gamma_A} & \mathcal{T}(A, \mathbb{S}P)|_{\mathcal{A}}. \end{array}$$

Chasing the identity 1_X through this diagram yields the equality $\gamma_A(a) = \mathcal{T}(A, \alpha_P)|_{\mathcal{A}}(a)$.

Part 2. Dual to *part 1*. □

A lemma on abelian length categories

The following lemma will be used in Theorem [I.3.10](#).

Lemma I.2.6. *Let \mathcal{A} be an abelian category for which each object of \mathcal{A} is of finite length. Then the following hold:*

1. *If each simple object in \mathcal{A} admits a nonzero morphism to an injective object in \mathcal{A} , then \mathcal{A} has enough injectives.*
2. *If each simple object in \mathcal{A} admits a nonzero morphism from a projective object in \mathcal{A} , then \mathcal{A} has enough projectives.*

Proof. Part 1. Let A be an object in \mathcal{A} . As A is of finite length, there exists a finite composition series $0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_{n-1} \subseteq A_n = A$ of A . Denote the simple quotients A_{i+1}/A_i by S_{i+1} and note $A_1 = S_1$. There are short exact sequences of the form $A_i \rightarrow A_{i+1} \rightarrow S_{i+1}$ and by assumption, there are nonzero morphisms $S_{i+1} \rightarrow I_{i+1}$, where every I_{i+1} is an injective object in \mathcal{A} . In particular, these morphisms are monomorphisms in \mathcal{A} . By the dual of the Horseshoe Lemma [[EJ11](#), Lemma 8.2.1], we inductively get monomorphisms $A_{i+1} \rightarrow I_1 \oplus I_2 \oplus \cdots \oplus I_{i+1}$ in \mathcal{A} . In particular, we acquire a monomorphism $A = A_n \rightarrow I_1 \oplus I_2 \oplus \cdots \oplus I_n$ in \mathcal{A} , proving our statement.

Part 2. Dual to *part 1*. □

A lemma on proper abelian subcategories

The following lemma will be used in Proposition I.4.3, Theorem I.4.7 and implicitly in Proposition I.5.5.

Lemma I.2.7. *Let \mathcal{A} be an extension closed k -linear proper abelian subcategory of \mathcal{T} and assume $\mathcal{T}(\mathcal{A}, \Sigma^{-1}\mathcal{A}) = 0$. Further, assume the following:*

- For each projective object P in \mathcal{A} there is a strong \mathcal{A} -cover in \mathcal{T} of the form $C_P \xrightarrow{c_P} SP$.
- For each injective object I in \mathcal{A} there is a strong \mathcal{A} -envelope in \mathcal{T} of the form $S^{-1}I \rightarrow E_I$.

As $\text{Proj } \mathcal{A}$ is an additive subcategory of \mathcal{A} , Lemma I.2.3 gives rise to a functor $\text{Proj } \mathcal{A} \rightarrow \mathcal{A}$ given by the assignment $P \mapsto C_P$. The following then hold:

1. If $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} A$ is a projective presentation of A in \mathcal{A} , then there is an \mathcal{A} -precover of the form $X_A \xrightarrow{\theta} \Sigma^{-1}SA$, where X_A is the kernel of the induced morphism $C_{P_1} \xrightarrow{c_{P_1}} C_{P_0}$ in \mathcal{A} . Moreover, if A non-projective object in \mathcal{A} , then the \mathcal{A} -precover θ is necessarily nonzero.
2. If $B \rightarrow I^0 \rightarrow I^1$ is an injective copresentation of B in \mathcal{A} , then there is an \mathcal{A} -preenvelope of the form $\Sigma S^{-1}B \xrightarrow{\xi} Z_B$, where Z_B is the cokernel of the induced morphism $E_{I^0} \rightarrow E_{I^1}$ in \mathcal{A} . Moreover, if B non-injective object in \mathcal{A} , then the \mathcal{A} -preenvelope ξ is necessarily nonzero.

Proof. Part 1. Our assumption allows us to use Lemma I.2.3 to produce an additive functor $\text{Proj } \mathcal{A} \rightarrow \mathcal{A}$ given by $P \mapsto C_P$. The projective presentation $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} A$ of A induces a left exact sequence $X_A \xrightarrow{x_A} C_{P_1} \xrightarrow{c_{P_1}} C_{P_0}$, where x_A is the kernel of C_{P_1} . Complete the morphism $SP_1 \xrightarrow{sp_1} SP_0$ to a short triangle $Y \xrightarrow{y} SP_1 \xrightarrow{sp_1} SP_0$. As \mathcal{A} is a proper abelian subcategory, the short exact sequence $\Omega A \xrightarrow{\iota_1} P_0 \xrightarrow{p_0} A$, where ΩA is the syzygy of A , is a short triangle. By rotating, $\Sigma^{-1}P_0 \xrightarrow{-\Sigma p_0} \Sigma^{-1}A \rightarrow \Omega A$ is a short triangle. Therefore, there is a short triangle $\Sigma^{-1}SP_0 \xrightarrow{-\Sigma sp_0} \Sigma^{-1}SA \xrightarrow{w} \Sigma \Omega A$. Here we used that S is a triangulated functor (see [BK89, Proposition 3.3] for the classical reference and see [Boc08, Theorem A.4.4] for a proof which makes the natural transformation more evident). After rotating triangles, we get the following solid commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & X_A & \xrightarrow{x_A} & C_{P_1} & \xrightarrow{c_{P_1}} & C_{P_0} \\
 & & \downarrow \theta_1 & & \downarrow c_{P_1} & & \downarrow c_{P_0} \\
 \Sigma^{-1}SP_0 & \longrightarrow & Y & \xrightarrow{y} & SP_1 & \xrightarrow{sp_1} & SP_0 \\
 \parallel & & \downarrow \theta_2 & & \downarrow S\pi_1 & & \parallel \\
 \Sigma^{-1}SP_0 & \xrightarrow{-\Sigma sp_0} & \Sigma^{-1}SA & \xrightarrow{w} & \Sigma \Omega A & \xrightarrow{S\iota_1} & SP_0
 \end{array} \tag{I.2}$$

where the composition $\mathbb{S}P_1 \xrightarrow{\mathbb{S}\pi_1} \mathbb{S}\Omega A \xrightarrow{\mathbb{S}l_1} \mathbb{S}P_0$ is obtained by applying \mathbb{S} to the usual factorisation $p_1: P_1 \xrightarrow{\pi_1} \Omega A \xrightarrow{l_1} P_0$ of the morphism p_1 . Note that the bottom and the middle rows are both triangles. By an axiom of triangulated categories, there exists a morphism $Y \xrightarrow{\theta_2} \Sigma^{-1}\mathbb{S}A$. Since $\mathbb{S}(p_1)c_{P_1}x_A = 0$, there exists a morphism $X_A \xrightarrow{\theta_1} Y$ as y is a weak kernel of $\mathbb{S}p_1$. The morphisms θ_1 and θ_2 make the entire diagram I.2 commute. Taking $\theta = \theta_2\theta_1$, we show that θ is indeed an \mathcal{A} -precover. The next steps in our proof will consist of adding many new morphisms to diagram I.2, the reader is suggested to use diagram I.3 below to aid their reading (the dashed arrows in diagram I.3 will be introduced in the following argument).

$$\begin{array}{ccccccc}
0 & \longrightarrow & X_A & \xrightarrow{x_A} & C_{P_1} & \xrightarrow{C_{P_1}} & C_{P_0} \\
& & \downarrow \theta_1 & \swarrow \overline{b_1} & \nearrow b_1 & \downarrow c_{P_1} & \downarrow c_{P_0} \\
& & & B & & & \\
& & & \swarrow b & & & \\
\Sigma^{-1}\mathbb{S}P_0 & \longrightarrow & Y & \xrightarrow{y} & \mathbb{S}P_1 & \xrightarrow{\mathbb{S}p_1} & \mathbb{S}P_0 \\
& & \downarrow \theta_2 & \swarrow \overline{b_3} & \nearrow b_3 & \downarrow \mathbb{S}\pi_1 & \downarrow \\
& & & B & & & \\
& & & \swarrow b_2 & \searrow wb_2 & & \\
\Sigma^{-1}\mathbb{S}P_0 & \xrightarrow{-\Sigma\mathbb{S}p_0} & \Sigma^{-1}\mathbb{S}A & \xrightarrow{w} & \mathbb{S}\Omega A & \xrightarrow{\mathbb{S}l_1} & \mathbb{S}P_0.
\end{array} \tag{I.3}$$

First, we show that θ_1 is an \mathcal{A} -precover. Let $B \xrightarrow{b} Y$ be a morphism with the object B in \mathcal{A} . As c_{P_1} is a strong \mathcal{A} -precover, there exists a unique morphism $B \xrightarrow{b_1} C_{P_1}$ such that $yb = c_{P_1}b_1$. We have the following equalities: $c_{P_0}C_{P_1}b_1 = \mathbb{S}(p_1)c_{P_1}b_1 = \mathbb{S}(p_1)yb = 0$, where the last equality holds as consecutive morphisms in a triangle vanish. Then $C_{P_1}b_1 = 0$ as c_{P_0} is a strong \mathcal{A} -cover. As x_A is the kernel of C_{P_1} , there exists a unique morphism $B \xrightarrow{\overline{b_1}} X_A$ such that $b_1 = x_A\overline{b_1}$. We have the following equalities: $yb = c_{P_1}b_1 = c_{P_1}x_A\overline{b_1} = y\theta_1\overline{b_1}$, which gives $y(b - \theta_1\overline{b_1}) = 0$. As y is a weak kernel of $\mathbb{S}p_1$, the morphism $b - \theta_1\overline{b_1}$ factors through a morphism in the k -vector space $\mathcal{T}(B, \Sigma^{-1}\mathbb{S}P_0)$. But, $\mathcal{T}(B, \Sigma^{-1}\mathbb{S}P_0) \cong D\mathcal{T}(P_0, \Sigma B) = 0$ where the isomorphism is given by Serre duality and where the equality holds as P_0 is a projective object in \mathcal{A} (see [Dye05, Theorem on page 1]). Hence, we are done.

It now suffices to show that the map $\mathcal{T}(B, Y) \xrightarrow{\mathcal{T}(B, \theta_2)} \mathcal{T}(B, \Sigma^{-1}\mathbb{S}A)$ is an epimorphism for each object B in \mathcal{A} . To this end, let $B \xrightarrow{b_2} \Sigma^{-1}\mathbb{S}A$ be a morphism with B in \mathcal{A} . By Serre duality, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{T}(B, \mathbb{S}P_1) & \xrightarrow{\cong} & D\mathcal{T}(P_1, B) \\
\mathcal{T}(B, \mathbb{S}\pi_1) \downarrow & & \downarrow D\mathcal{T}(\pi_1, B) \\
\mathcal{T}(B, \mathbb{S}\Omega A) & \xrightarrow{\cong} & D\mathcal{T}(\Omega A, B).
\end{array}$$

As π_1 is an epimorphism, the map $D\mathcal{T}(\pi_1, B)$ is an epimorphism and hence, so is the map $\mathcal{T}(B, \mathbb{S}\pi_1)$. Therefore, there exists a morphism $B \xrightarrow{b_3} \mathbb{S}P_1$ such that $\omega b_2 = \mathbb{S}(\pi_1)b_3$. We have the following equalities: $\mathbb{S}(p_1)b_3 = \mathbb{S}(\iota_1)\mathbb{S}(\pi_1)b_3 = \mathbb{S}(\iota_1)\omega b_2 = 0$, where the last equality holds as consecutive morphisms in a triangle vanish. As y is a weak kernel of $\mathbb{S}p_1$, there exists a morphism $B \xrightarrow{\bar{b}_3} Y$ such that $b_3 = y\bar{b}_3$. We have the following equalities: $\omega b_2 = \mathbb{S}(\pi_1)b_3 = \mathbb{S}(\pi_1)y\bar{b}_3 = \omega\theta_2\bar{b}_3$, which gives $\omega(b_2 - \theta_2\bar{b}_3) = 0$. As ω is a weak kernel of $\mathbb{S}\iota_1$, the morphism $b_2 - \theta_2\bar{b}_3$ factors through a morphism in the k -vector space $\mathcal{T}(B, \Sigma^{-1}\mathbb{S}P_0)$. But, $\mathcal{T}(B, \Sigma^{-1}\mathbb{S}P_0) \cong D\mathcal{T}(P_0, \Sigma B) = 0$ where the isomorphism is given by Serre duality and where the equality holds as P_0 is a projective object in \mathcal{A} (see [Dye05, Theorem on page 1]). Hence, we are done.

Suppose that A is a non-projective object in \mathcal{A} . By [Dye05, Theorem on page 1], there exists an object T in \mathcal{A} with $\mathcal{T}(A, \Sigma T) \neq 0$. Then the k -vector space $D\mathcal{T}(A, \Sigma T)$ is also nonzero and we have $\mathcal{T}(T, \Sigma^{-1}\mathbb{S}A) \cong D\mathcal{T}(A, \Sigma T) \neq 0$, where the isomorphism holds by Serre duality. Therefore, we can pick a nonzero morphism $T \xrightarrow{t} \Sigma^{-1}\mathbb{S}A$. As θ is an \mathcal{A} -precover, the nonzero morphism t must factor through θ , resulting in θ necessarily being nonzero.

Part 2. Dual to part 1. □

I.3 Nakayama functors on proper abelian subcategories

The construction and fundamental properties

Setup I.3.1. We fix \mathcal{A} to be an extension closed k -linear proper abelian subcategory of \mathcal{T} and assume $\mathcal{T}(\mathcal{A}, \Sigma^{-1}\mathcal{A}) = 0$.

Proposition I.3.2. *Let \mathcal{A} be a category with Setup I.3.1. For P a projective object in \mathcal{A} and I an injective object in \mathcal{A} , the following hold:*

1. *Each \mathcal{A} -cover in \mathcal{T} of the form $C_P \xrightarrow{c_P} \mathbb{S}P$ is a strong \mathcal{A} -cover in \mathcal{T} .*
2. *Each \mathcal{A} -envelope in \mathcal{T} of the form $\mathbb{S}^{-1}I \rightarrow E_I$ is a strong \mathcal{A} -envelope in \mathcal{T} .*

Proof. Let $A \xrightarrow{a} C_P$ be a morphism in \mathcal{A} such that $c_P a = 0$. In order to show that c_P is a strong \mathcal{A} -cover in \mathcal{T} we need $a = 0$. Suppose a is a monomorphism in \mathcal{A} . As \mathcal{A} is a proper abelian subcategory, there is a short triangle $A \xrightarrow{a} C_P \xrightarrow{v} B$ in \mathcal{T} with B an object in \mathcal{A} . As $c_P a = 0$ and v is a weak cokernel of a , there exists a morphism $B \xrightarrow{b} \mathbb{S}P$ such that $c_P = bv$. As c_P is an \mathcal{A} -cover, there exists a morphism $B \xrightarrow{\bar{b}} C_P$ such that $b = c_P \bar{b}$.

The situation is depicted in the following diagram:

$$\begin{array}{ccccc}
 A & \xrightarrow{a} & C_P & \xrightarrow{v} & B \\
 & \searrow & \downarrow c_P & \swarrow & \uparrow \bar{v} \\
 & 0 & & & SP.
 \end{array}$$

We have the following equalities: $c_P = bv = c_P \bar{v}$. As c_P is right minimal, \bar{v} is an automorphism. As the composition of consecutive morphisms in a triangle vanishes, we have $\bar{v}va = 0$. It follows that $a = 0$.

Now suppose a is arbitrary. We choose a factorisation $a: A \xrightarrow{a_e} D \xrightarrow{a_m} C_P$ in \mathcal{A} where a_e is an epimorphism and a_m is a monomorphism. It suffices now to show $a_m = 0$. As \mathcal{A} is a proper abelian subcategory, there is a short triangle $A \xrightarrow{a_e} D \xrightarrow{d} \Sigma B$ with B an object in \mathcal{A} . As $c_P a_m a_e = c_P a = 0$ and d is a weak cokernel of a_e , there exists a morphism $\Sigma B \xrightarrow{b_1} SP$ such that $c_P a_m = b_1 d$. The situation is depicted in the following diagram:

$$\begin{array}{ccccc}
 A & \xrightarrow{a_e} & D & \xrightarrow{d} & \Sigma B \\
 & \searrow a & \downarrow a_m & \swarrow & \uparrow b_1 \\
 & 0 & C_P & \downarrow c_P & \\
 & & & & SP.
 \end{array}$$

The morphism b_1 lies in the k -vector space $\mathcal{T}(\Sigma B, SP)$. But, $\mathcal{T}(\Sigma B, SP) \cong D\mathcal{T}(P, \Sigma B) = 0$, where the isomorphism is given by Serre duality and where the equality holds as P is a projective object in \mathcal{A} (see [Dye05, Theorem on page 1]). Hence, $b_1 = 0$ and so $c_P a_m = 0$. As a_m is a monomorphism in \mathcal{A} , the previous argument tells us that $a_m = 0$, as required.

Part 2. Dual to part 1. □

Proposition I.3.3. *Let \mathcal{A} be a category with Setup I.3.1. Let X be an object in \mathcal{A} . Then the following hold:*

1. *If $C_X \rightarrow SX$ is an \mathcal{A} -cover in \mathcal{T} , then C_X is an injective object in \mathcal{A} .*
2. *If $S^{-1}X \rightarrow E_X$ is an \mathcal{A} -envelope in \mathcal{T} , then E_X is a projective object in \mathcal{A} .*

Proof. Part 1. Appealing to the dual of [Dye05, Theorem on page 1], we will show that $\mathcal{T}(A, \Sigma C_X) = 0$ for each object A in \mathcal{A} . To this end, let A be an object in \mathcal{A} and complete the morphism $C_X \rightarrow SX$ to a short triangle $C_X \rightarrow SX \rightarrow Z$. By properties of triangulated categories, the sequence of k -vector spaces $\mathcal{T}(A, Z) \rightarrow \mathcal{T}(A, \Sigma C_X) \rightarrow \mathcal{T}(A, \Sigma SX)$ is exact. The Triangulated Wakamatsu's Lemma [Jør09, Lemma 2.1] ensures $\mathcal{T}(A, Z) = 0$. By Serre duality we have $\mathcal{T}(A, \Sigma SX) \cong D\mathcal{T}(X, \Sigma^{-1}A) = 0$, where the latter k -vector space

vanishes since $\mathcal{T}(\mathcal{A}, \Sigma^{-1}\mathcal{A}) = 0$. It follows that $\mathcal{T}(\mathcal{A}, \Sigma C_X) = 0$ by the exactness of the aforementioned sequence.

Part 2. Dual to part 1. □

Theorem I.3.4. *Let \mathcal{A} be a category with Setup I.3.1. Then the following hold:*

1. *If for each projective object P in \mathcal{A} there is an \mathcal{A} -cover in \mathcal{T} of the form $\nu P \xrightarrow{\alpha_P} \mathbb{S}P$, then the assignment $P \mapsto \nu P$ augments to an additive functor $\text{Proj } \mathcal{A} \xrightarrow{\nu} \text{Inj } \mathcal{A}$ such that the induced diagram*

$$\begin{array}{ccc} \nu P & \xrightarrow{\alpha_P} & \mathbb{S}P \\ \nu p \downarrow & & \downarrow \mathbb{S}p \\ \nu P' & \xrightarrow{\alpha_{P'}} & \mathbb{S}P' \end{array}$$

is commutative for each morphism $P \xrightarrow{P} P'$ in $\text{Proj } \mathcal{A}$.

2. *If for each injective object I in \mathcal{A} there is an \mathcal{A} -envelope in \mathcal{T} of the form $\mathbb{S}^{-1}I \xrightarrow{\beta_I} \nu^{-1}I$, then the assignment $I \mapsto \nu^{-1}I$ augments to an additive functor $\text{Inj } \mathcal{A} \xrightarrow{\nu^{-1}} \text{Proj } \mathcal{A}$ such that the induced diagram*

$$\begin{array}{ccc} \mathbb{S}^{-1}I & \xrightarrow{\beta_I} & \nu^{-1}I \\ \mathbb{S}^{-1}i \downarrow & & \downarrow \nu^{-1}i \\ \mathbb{S}^{-1}I' & \xrightarrow{\beta_{I'}} & \nu^{-1}I' \end{array}$$

is commutative for each morphism $I \xrightarrow{i} I'$ in $\text{Inj } \mathcal{A}$.

Moreover, if the conditions from both part 1 and part 2 are satisfied then the functors ν and ν^{-1} are mutual quasi-inverses.

Proof. Part 1. The existence of a functor $\text{Proj } \mathcal{A} \rightarrow \mathcal{A}$ given by the assignment $P \mapsto \nu P$ is guaranteed by Lemma I.2.3 and Proposition I.3.2. Proposition I.3.3 assures that this assignment is a well-defined additive functor $\text{Proj } \mathcal{A} \xrightarrow{\nu} \text{Inj } \mathcal{A}$.

Part 2. Dual to part 1.

We now show that ν and ν^{-1} are mutual quasi-inverses. As both ν and ν^{-1} are additive functors, it suffices to first construct a collection of isomorphisms

$$\{I \rightarrow \nu P \mid \text{with } P = \nu^{-1}I \text{ and } I \text{ an indecomposable injective object in } \mathcal{A}\}$$

natural in I , and then to construct a collection of isomorphisms

$$\{\nu^{-1}J \rightarrow Q \mid \text{with } J = \nu Q \text{ and } Q \text{ an indecomposable projective object in } \mathcal{A}\}$$

natural in Q . To this end, let I be an indecomposable injective object of \mathcal{A} and set $P = \nu^{-1}I$. We have an \mathcal{A} -envelope $\mathbb{S}^{-1}I \xrightarrow{\beta_I} P$ which is automatically a strong \mathcal{A} -envelope

by Proposition I.3.2. As \mathbb{S} is an autoequivalence, we get a strong $\mathbb{S}\mathcal{A}$ -envelope $\gamma_I: I \cong \mathbb{S}\mathbb{S}^{-1}I \xrightarrow{\mathbb{S}\beta_I} \mathbb{S}P$. We first prove that γ_I is also a strong \mathcal{A} -cover.

Let A be an object in \mathcal{A} . By Serre duality, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{T}(\mathbb{S}P, \mathbb{S}A) & \xrightarrow{\cong} & D\mathcal{T}(A, \mathbb{S}P) \\ \mathcal{T}(\gamma_I, \mathbb{S}A) \downarrow & & \downarrow D\mathcal{T}(A, \gamma_I) \\ \mathcal{T}(I, \mathbb{S}A) & \xrightarrow{\cong} & D\mathcal{T}(A, I). \end{array} \quad (\text{I.4})$$

As γ_I is a strong $\mathbb{S}\mathcal{A}$ -envelope, the map $\mathcal{T}(\gamma_I, \mathbb{S}A)$ is an isomorphism. By the commutativity of diagram I.4, so too is $D\mathcal{T}(A, \gamma_I)$ an isomorphism and therefore γ_I is a strong \mathcal{A} -cover as required.

Notice that the object P is projective in \mathcal{A} using *part 2* and the definition of P . Furthermore, the \mathcal{A} -cover $\nu P \xrightarrow{\alpha_P} \mathbb{S}P$ is a strong \mathcal{A} -cover by Proposition I.3.2. We now construct the morphism $I \rightarrow \nu P$.

By the uniqueness of (strong) \mathcal{A} -covers, we get an (unique) isomorphism $I \xrightarrow{\bar{\gamma}_I} \nu P$ such that $\gamma_I = \alpha_P \bar{\gamma}_I$. We now show that $\bar{\gamma}_I$ is natural. Let $I \xrightarrow{i} J$ be a morphism between indecomposable injective objects of \mathcal{A} . We denote the morphism $\nu^{-}I \xrightarrow{\nu^{-}i} \nu^{-}J$ in \mathcal{A} by $P \xrightarrow{p} Q$. Then we get unique morphisms $\bar{\gamma}_I$ and $\bar{\gamma}_J$ such that $\gamma_I = \alpha_P \bar{\gamma}_I$ and $\gamma_J = \alpha_Q \bar{\gamma}_J$ hold. By definition of ν^{-} , we have $\beta_J \mathbb{S}^{-1}i = p\beta_I$ and therefore, after applying \mathbb{S} to this equation and using the natural isomorphism $\mathbb{1} \rightarrow \mathbb{S}\mathbb{S}^{-1}$, we get $\gamma_J i = \mathbb{S}(p)\gamma_I$. We obtain the diagram:

$$\begin{array}{ccccc} I & \xrightarrow{\bar{\gamma}_I} & \nu P & & \\ \downarrow i & \searrow \gamma_I & \swarrow \alpha_P & & \downarrow \nu p \\ & & \mathbb{S}P & & \\ & & \downarrow \mathbb{S}p & & \\ J & \xrightarrow{\bar{\gamma}_J} & \nu Q & & \\ \downarrow & \searrow \gamma_J & \swarrow \alpha_Q & & \\ & & \mathbb{S}Q & & \end{array}$$

As the right-facing square commutes by the definition of νP , we have that all but the back-most square with vertices $I, J, \nu Q$ and νP commute. By the application of Lemma I.2.2, we are done. Similarly, we construct the collection of natural isomorphisms $\nu^{-}J \rightarrow Q$. This proves that ν and ν^{-} are mutual quasi-inverses. \square

Remark I.3.5. Because we work in a Krull-Schmidt setting, every \mathcal{A} -precover can be made into an \mathcal{A} -cover by removing direct summands from the source object of the \mathcal{A} -precover (see [Jør09, Lemma 4.1]). As the dual situation is also true, the assumptions in Theorem I.3.4 may be relaxed.

Definition I.3.6. Let \mathcal{A} be a category with Setup I.3.1 such that the following hold:

- For each projective object P in \mathcal{A} there is an \mathcal{A} -cover in \mathcal{T} of the form $\nu P \xrightarrow{\alpha_P} \mathbb{S}P$.
- For each injective object I in \mathcal{A} there is an \mathcal{A} -envelope in \mathcal{T} of the form $\mathbb{S}^{-1}I \xrightarrow{\beta_I} \nu^{-1}I$.

Then we call the functors $\text{Proj } \mathcal{A} \xrightarrow{\nu} \text{Inj } \mathcal{A}$ and $\text{Inj } \mathcal{A} \xrightarrow{\nu^{-1}} \text{Proj } \mathcal{A}$ obtained from Theorem I.3.4 the *Nakayama functors on \mathcal{A} (obtained via approximations)* and we say that \mathcal{A} has a Nakayama functor.

Proposition I.3.7. *Let \mathcal{A} be a category with Setup I.3.1. Let X and P be objects in \mathcal{A} with P projective. Then the following three statements are equivalent:*

1. *There is an \mathcal{A} -cover in \mathcal{T} of the form $X \xrightarrow{\alpha_P} \mathbb{S}P$.*
2. *There is a strong \mathcal{A} -cover in \mathcal{T} of the form $X \xrightarrow{\alpha_P} \mathbb{S}P$.*
3. *There is a natural isomorphism of the form $D\mathcal{A}(P, -) \cong \mathcal{A}(-, X)$ of functors $\mathcal{A}^{\text{op}} \rightarrow \text{mod}_{\mathbb{k}}$, i.e. the object X represents the functor $D\mathcal{A}(P, -)$.*

Dually, let Y and I be objects in \mathcal{A} with I injective. Then the following three statements are equivalent:

- 1'. *There is an \mathcal{A} -envelope in \mathcal{T} of the form $\mathbb{S}^{-1}I \rightarrow Y$.*
- 2'. *There is a strong \mathcal{A} -envelope in \mathcal{T} of the form $\mathbb{S}^{-1}I \rightarrow Y$.*
- 3'. *There is a natural isomorphism of the form $D\mathcal{A}(-, I) \cong \mathcal{A}(Y, -)$ of functors $\mathcal{A} \rightarrow \text{mod}_{\mathbb{k}}$, i.e. the object Y represents the functor $D\mathcal{A}(-, I)$.*

Proof. Part 1. (1 \Leftrightarrow 2): This follows from Proposition I.3.2.

(2 \Leftrightarrow 3): This follows from Lemma I.2.5.

Part 2. Dual to part 1. □

Theorem I.3.8. *Let \mathcal{A} be an essentially small category that has a Nakayama functor (see Definition I.3.6). If \mathcal{A} has enough projectives and enough injectives, then \mathcal{A} is a dualising \mathbb{k} -variety.*

Proof. We first show that the induced functor $(\text{Mod}_{\mathcal{A}})^{\text{op}} \xrightarrow{\mathbb{D}} \text{Mod}_{\mathcal{A}^{\text{op}}}$ restricts to a functor $(\text{mod}_{\mathcal{A}})^{\text{op}} \rightarrow \text{mod}_{\mathcal{A}^{\text{op}}}$. To this end, let $\mathcal{A}^{\text{op}} \xrightarrow{F} \text{Mod}_{\mathbb{k}}$ be an object in $\text{mod}_{\mathcal{A}}$. Firstly, assume $F = \mathcal{A}(-, I)$ for I an injective object in \mathcal{A} . Then we have $\mathbb{D}(F) = \mathbb{D}\mathcal{A}(-, I) \cong \mathcal{T}(\mathbb{S}^{-1}I, -)|_{\mathcal{A}} \cong \mathcal{A}(\nu^{-1}I, -)$, where the first isomorphism holds since it holds pointwise by Serre duality and the second isomorphism holds as β_I is a strong \mathcal{A} -envelope (see Proposition I.3.2). Hence, $\mathbb{D}(F)$ lies in $\text{mod}_{\mathcal{A}^{\text{op}}}$.

Secondly, assume $F = \mathcal{A}(-, A)$ for A an object in \mathcal{A} . Then choose an injective copresentation $A \rightarrow I^0 \rightarrow I^1$ of A . As the Yoneda embedding $\mathcal{A} \rightarrow \text{Mod}_{\mathcal{A}}$ is left exact, the sequence $\mathcal{A}(-, A) \rightarrow \mathcal{A}(-, I^0) \rightarrow \mathcal{A}(-, I^1)$ is left exact. Using the exactness of \mathbb{D} ,

the sequence $\mathbb{D}\mathcal{A}(-, I^1) \rightarrow \mathbb{D}\mathcal{A}(-, I^0) \rightarrow \mathbb{D}\mathcal{A}(-, \mathbf{A}) = \mathbb{D}(\mathbf{F})$ is right exact. As $\mathbf{mod}_{\mathcal{A}^{\text{op}}}$ is closed under cokernels (see [Aus71, Proposition on page 41]) $\mathbb{D}(\mathbf{F})$ lies in $\mathbf{mod}_{\mathcal{A}^{\text{op}}}$.

Lastly, assume \mathbf{F} is arbitrary in $\mathbf{mod}_{\mathcal{A}}$. Then there is a right exact sequence of the form $\mathcal{A}(-, \mathbf{Y}) \rightarrow \mathcal{A}(-, \mathbf{X}) \rightarrow \mathbf{F}$. As \mathbb{D} is exact, the sequence $\mathbb{D}(\mathbf{F}) \rightarrow \mathbb{D}\mathcal{A}(-, \mathbf{X}) \rightarrow \mathbb{D}\mathcal{A}(-, \mathbf{Y})$ is left exact. As $\mathbf{mod}_{\mathcal{A}^{\text{op}}}$ is closed under kernels (see [Aus71, Proposition on page 41]) $\mathbb{D}(\mathbf{F})$ lies in $\mathbf{mod}_{\mathcal{A}^{\text{op}}}$. Moreover, the fact that the functor $\mathbf{Mod}_{\mathcal{A}^{\text{op}}} \xrightarrow{\mathbb{D}} (\mathbf{Mod}_{\mathcal{A}})^{\text{op}}$ restricts to a functor $\mathbf{mod}_{\mathcal{A}^{\text{op}}} \xrightarrow{\mathbb{D}} (\mathbf{mod}_{\mathcal{A}})^{\text{op}}$ follows by a dual argument to the one given above. \square

Remark I.3.9. The inclusion of points 3 and 3' in Proposition I.3.7 was inspired by ongoing work by Nan Gao, Julian Külshammer, Sondre Kvamme and Chrysostomos Psaroudakis presented at the *Homological Algebra and Representation Theory* conference (Karlovasi, Samos, 2023). They used the existence of the representations in both points 3 and 3' to define the notion of an abelian category admitting a Nakayama functor. This inspiration was instrumental in facilitating the proof of Theorem I.3.8.

Proper abelian length subcategories

Theorem I.3.10. *Let \mathcal{A} be a category that has a Nakayama functor (see Definition I.3.6) and assume each object of \mathcal{A} is of finite length. Then \mathcal{A} has enough projectives if and only if \mathcal{A} has enough injectives.*

Proof. ($\xRightarrow{\text{only if}}$): Assume \mathcal{A} has enough projectives and let \mathbf{A} be a simple object in \mathcal{A} . By assumption, there is a nonzero epimorphism $\mathbf{P} \xrightarrow{p} \mathbf{A}$ in \mathcal{A} with \mathbf{P} a projective object in \mathcal{A} . We first show that the composition $\nu\mathbf{P} \xrightarrow{\alpha_{\mathbf{P}}} \mathbb{S}\mathbf{P} \xrightarrow{\mathbb{S}p} \mathbb{S}\mathbf{A}$ is nonzero. Assume the contrary and complete $\alpha_{\mathbf{P}}$ to a short triangle $\nu\mathbf{P} \xrightarrow{\alpha_{\mathbf{P}}} \mathbb{S}\mathbf{P} \xrightarrow{g} \mathbf{Z}$. As $\mathbb{S}(p)\alpha_{\mathbf{P}} = 0$ and as g is a weak cokernel of $\alpha_{\mathbf{P}}$, we have that $\mathbb{S}p$ factors through a morphism in the \mathbf{k} -vector space $\mathcal{T}(\mathbf{Z}, \mathbb{S}\mathbf{A})$. But, $\mathcal{T}(\mathbf{Z}, \mathbb{S}\mathbf{A}) \cong D\mathcal{T}(\mathbf{A}, \mathbf{Z}) = 0$ where the isomorphism holds by Serre duality and the latter \mathbf{k} -vector space vanishes by the Triangulated Wakamatsu's Lemma [Jør09, Lemma 2.1]. Hence, we get $\mathbb{S}p = 0$ and therefore $p = 0$, a contradiction.

We now have that the composition $\nu\mathbf{P} \xrightarrow{\alpha_{\mathbf{P}}} \mathbb{S}\mathbf{P} \xrightarrow{\mathbb{S}p} \mathbb{S}\mathbf{A}$ is nonzero. By Serre duality, the existence of a nonzero morphism $\mathbf{A} \xrightarrow{a} \nu\mathbf{P}$ is ensured. Therefore, each simple object in \mathcal{A} admits a nonzero morphism to an injective object in \mathcal{A} (see Proposition I.3.3) and by Lemma I.2.6, \mathcal{A} has enough injectives.

($\xleftarrow{\text{if}}$): Dual to the previous argument. \square

I.4 Auslander-Reiten translates on proper abelian subcategories

The construction and fundamental properties

Setup I.4.1. Throughout this section, fix a category \mathcal{A} that has a Nakayama functor (see Definition I.3.6) and assume that each object in \mathcal{A} has a projective cover and an injective envelope.

For each indecomposable non-projective object A in \mathcal{A} and for each indecomposable non-injective object B in \mathcal{A} we fix

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} A \quad \text{and} \quad B \xrightarrow{i^0} I^0 \xrightarrow{i^1} I^1$$

to be a minimal projective presentation of A and a minimal injective copresentation of B , respectively. For each indecomposable projective object P in \mathcal{A} and for each indecomposable injective object I in \mathcal{A} , we fix

$$0 \rightarrow P \xrightarrow{1} P \quad \text{and} \quad I \xrightarrow{1} I \rightarrow 0$$

to be a minimal projective presentation of P and a minimal injective copresentation of I , respectively. Furthermore, we extend to general objects of \mathcal{A} by taking direct sums.

Definition I.4.2. Let C be an object in \mathcal{A} . We define the *Auslander-Reiten translate* τC of C as $\tau C = \text{Ker}(\nu Q_1 \xrightarrow{\nu q_1} \nu Q_0)$, where $Q_1 \xrightarrow{q_1} Q_0 \xrightarrow{q_0} C$ is the fixed minimal projective presentation of C . Similarly, we define the (*inverse*) *Auslander-Reiten translate* $\tau^- C$ of C as $\tau^- C = \text{Coker}(\nu^- J^0 \xrightarrow{\nu^- j^1} \nu^- J^1)$, where $C \xrightarrow{j^0} J^0 \xrightarrow{j^1} J^1$ is the minimal injective copresentation of C .

Proposition I.4.3. Let \mathcal{A} be a category with Setup I.4.1 and let A and A' be indecomposable objects in \mathcal{A} . Then the following hold:

1. $\tau(A \oplus A') \cong \tau A \oplus \tau A'$.
2. The object A is projective in $\mathcal{A} \iff \tau A = 0$.
3. If A is a non-projective object in \mathcal{A} , then τA is a non-injective object in \mathcal{A} .
4. If A is a non-projective object in \mathcal{A} , then the sequence $\tau A \xrightarrow{k_A} \nu P_1 \xrightarrow{\nu p_1} \nu P_0$ is a minimal injective copresentation of τA .
5. If A is a non-projective object in \mathcal{A} , then $\tau^- \tau A \cong A$.
6. If A is a non-projective object in \mathcal{A} , then τA is indecomposable.

7. If \mathbf{A} and \mathbf{A}' are non-projective objects in \mathcal{A} , then $\mathbf{A} \cong \mathbf{A}' \iff \tau\mathbf{A} \cong \tau\mathbf{A}'$.

Proof. Part 1. We have minimal projective presentations $\mathbf{P}_1 \xrightarrow{p_1} \mathbf{P}_0 \xrightarrow{p_0} \mathbf{A}$ and $\mathbf{P}'_1 \xrightarrow{p'_1} \mathbf{P}'_0 \xrightarrow{p'_0} \mathbf{A}'$ of \mathbf{A} and \mathbf{A}' respectively and therefore, $\mathbf{P}_1 \oplus \mathbf{P}'_1 \xrightarrow{p_1 \oplus p'_1} \mathbf{P}_0 \oplus \mathbf{P}'_0 \xrightarrow{p_0 \oplus p'_0} \mathbf{A} \oplus \mathbf{A}'$ is the fixed minimal projective presentation of $\mathbf{A} \oplus \mathbf{A}'$. By the application of ν , we arrive at the following solid commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tau(\mathbf{A} \oplus \mathbf{A}') & \longrightarrow & \nu(\mathbf{P}_1 \oplus \mathbf{P}'_1) & \xrightarrow{\nu(p_1 \oplus p'_1)} & \nu(\mathbf{P}_0 \oplus \mathbf{P}'_0) \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \tau\mathbf{A} \oplus \tau\mathbf{A}' & \longrightarrow & \nu\mathbf{P}_1 \oplus \nu\mathbf{P}'_1 & \xrightarrow{\nu p_1 \oplus \nu p'_1} & \nu\mathbf{P}_0 \oplus \nu\mathbf{P}'_0 \end{array}$$

where the vertical solid morphisms are the canonical isomorphisms induced by the additivity of ν (see Theorem I.3.4). The dashed morphism exists by the universal property of the kernel of $\nu p_1 \oplus \nu p'_1$ and the Five Lemma ensures it is an isomorphism.

Part 2. (\Rightarrow): Assume \mathbf{A} is a projective object in \mathcal{A} . Then $0 \rightarrow \mathbf{A} \xrightarrow{1} \mathbf{A}$ is the fixed minimal projective presentation of \mathbf{A} . As $0 = \nu 0 \rightarrow \nu\mathbf{A}$ is a monomorphism, it follows that $\tau\mathbf{A} = 0$.

(\Leftarrow): Assume $\tau\mathbf{A} = 0$. Then $\nu\mathbf{P}_1 \xrightarrow{\nu p_1} \nu\mathbf{P}_0$ is a monomorphism in \mathcal{A} with $\nu\mathbf{P}_1$ an injective object in \mathcal{A} . Therefore, νp_1 is a section. As ν is an equivalence, the morphism p_1 must also be a section. Hence, we have $\mathbf{P}_0 \cong \mathbf{P}_1 \oplus \mathbf{A}$ and therefore, \mathbf{A} is projective.

Part 3. Assume that \mathbf{A} is a non-projective object in \mathcal{A} . By Lemma I.2.7, there is a nonzero \mathcal{A} -precover of the form $\tau\mathbf{A} \xrightarrow{\theta} \Sigma^{-1}\mathbf{S}\mathbf{A}$ (notice that $\tau\mathbf{A}$ is $\mathbf{X}_{\mathbf{A}}$ in the statement of Lemma I.2.7). Showing that $\tau\mathbf{A}$ is not an injective object in \mathcal{A} amounts to finding an object \mathbf{C} in \mathcal{A} with $\mathcal{T}(\mathbf{C}, \Sigma\tau\mathbf{A}) \neq 0$ (see dual of [Dye05, Theorem on page 1]). Choosing $\mathbf{C} = \mathbf{A}$ and using Serre duality, we have $\mathcal{T}(\mathbf{A}, \Sigma\tau\mathbf{A}) \cong D\mathcal{T}(\tau\mathbf{A}, \Sigma^{-1}\mathbf{S}\mathbf{A})$, where the last k -vector space is non-vanishing since the nonzero morphism θ lies in $\mathcal{T}(\tau\mathbf{A}, \Sigma^{-1}\mathbf{S}\mathbf{A})$.

Part 4. It is clear the sequence is an injective copresentation of $\tau\mathbf{A}$. As ν is an additive equivalence (see Theorem I.3.4), Lemma I.2.1 tells us that νp_1 is a radical morphism as, by [Kra15, Proposition 3.10], p_1 is a radical morphism. The fact that $k_{\mathbf{A}}$ is an injective envelope follows from the dual of [Kra15, Proposition 3.10]. As $\nu(p_1)k_{\mathbf{A}} = 0$, the universal property of the cokernel of $k_{\mathbf{A}}$ ensures the existence of a unique morphism $\Omega^{-1}\tau\mathbf{A} \xrightarrow{\iota_1} \nu\mathbf{P}_0$ making the following diagram commutative:

$$\begin{array}{ccccc} 0 & \longrightarrow & \tau\mathbf{A} & \xrightarrow{k_{\mathbf{A}}} & \nu\mathbf{P}_1 & \xrightarrow{\nu p_1} & \nu\mathbf{P}_0 \\ & & & & \downarrow \pi_1 & \nearrow \iota_1 & \\ & & & & \Omega^{-1}\tau\mathbf{A} & & \end{array}$$

Here, $\Omega^{-1}\tau\mathbf{A}$ is the cosyzygy of $\tau\mathbf{A}$ and π_1 is the cokernel of $k_{\mathbf{A}}$. By diagram chasing, ι_1 is a monomorphism, so it suffices to show that ι_1 is an injective envelope.

To this end, we choose an injective envelope I of $\text{Coker } \nu p_1$ extending the injective copresentation of τA :

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \tau A & \xrightarrow{k_A} & \nu P_1 & \xrightarrow{\nu p_1} & \nu P_0 \xrightarrow{f} I \\
 & & & & \downarrow \pi_1 & \nearrow \iota_1 & \\
 & & & & \Omega^{-1}\tau A & &
 \end{array}$$

By the dual of [Kra15, Proposition 3.10], Lemma I.2.1 and the fact that ν and ν^- are mutual quasi-inverses (see Theorem I.3.4), it suffices to show that the morphism $g: P_0 \xrightarrow{\cong} \nu^- \nu P_0 \xrightarrow{\nu^- f} \nu^- I$ is a radical morphism. As $gp_1 = 0$ and p_0 is the cokernel of p_1 , there exists a unique morphism $A \xrightarrow{\alpha} \nu^- I$ such that the following diagram is commutative:

$$\begin{array}{ccccccc}
 P_1 & \xrightarrow{p_1} & P_0 & \xrightarrow{p_0} & A & \longrightarrow & 0 \\
 & & \searrow g & & \downarrow \alpha & & \\
 & & & & \nu^- I & &
 \end{array}$$

But, p_0 must be a radical morphism as A is a non-projective indecomposable object. Hence, $g = \alpha p_0$ is a radical morphism.

Part 5. Assume A is a non-projective object in \mathcal{A} . By *part 4*, $\tau A \rightarrow \nu P_1 \xrightarrow{\nu p_1} \nu P_0$ is a minimal injective copresentation of τA . Therefore, as minimal injective copresentations are unique up to isomorphism, we get the following commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \tau A & \longrightarrow & \nu P_1 & \xrightarrow{\nu p_1} & \nu P_0 \\
 & & & & \downarrow \cong & & \downarrow \cong \\
 0 & \longrightarrow & \tau A & \longrightarrow & I^0 & \xrightarrow{i^0} & I^1
 \end{array}$$

where the bottom row is our fixed injective copresentation of τA . Applying ν^- , we get the following solid commutative diagram:

$$\begin{array}{ccccccc}
 P_1 & \xrightarrow{p_1} & P_0 & \xrightarrow{p_0} & A & \longrightarrow & 0 \\
 \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
 \nu^- \nu P_1 & \xrightarrow{\nu^- \nu p_1} & \nu^- \nu P_0 & \xrightarrow{\nu^- \nu p_0} & C & \longrightarrow & 0 \\
 \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
 \nu^- I^0 & \xrightarrow{\nu^- i^1} & \nu^- I^1 & \longrightarrow & \tau^- \tau A & \longrightarrow & 0
 \end{array}$$

where C is the cokernel of $\nu^- \nu p_1$. Then the dashed morphisms exist by the universal property of the cokernel and the Five Lemma ensures that they are isomorphisms.

Part 6. Suppose A is a non-projective object in \mathcal{A} and assume, for a contradiction, we have $\tau A \cong X \oplus Y$, for nonzero objects X and Y in \mathcal{A} . We first show that both X and Y are non-injective objects in \mathcal{A} .

For a contradiction, assume Y is a nonzero injective object in \mathcal{A} . As $s: Y \rightarrow \tau A \xrightarrow{k_A} \nu P_1$ is a monomorphism, where $Y \rightarrow \tau A$ is the canonical inclusion morphism, s must be

a section. As $\nu(\mathfrak{p}_1)s = 0$, the object Y is isomorphic to a direct summand of νP_1 that lies in the kernel of $\nu \mathfrak{p}_1$. By applying ν^- , we get a section ν^-s with $\nu^-(\nu(\mathfrak{p}_1)s) = 0$. Similarly, the object ν^-Y is isomorphic to a direct summand of $\nu^- \nu P_1$ that lies in the kernel of $\nu^- \nu \mathfrak{p}_1$. As ν and ν^- are mutual quasi-inverses (see Theorem I.3.4), there is a nonzero direct summand of P_1 contained in the kernel of \mathfrak{p}_1 . Using that the projective cover $P_1 \rightarrow \Omega A$ is unique up to isomorphism, we get a contradiction to the dual of [KS98, Corollary 1.4].

Now, we have $A \cong \tau^- \tau A \cong \tau^- X \oplus \tau^- Y$ by *part 5* and the dual of *part 1*. By the dual of *part 2*, both objects $\tau^- X$ and $\tau^- Y$ are nonzero, a contradiction as A is indecomposable.

Part 7. (\Rightarrow): Assuming $A \cong A'$, the result follows by the uniqueness of minimal projective presentations, the universal property of kernels and the Five Lemma.

(\Leftarrow): Conversely, assume $\tau A \cong \tau A'$. By *part 5*, we have $A \cong \tau^- \tau A$ and $A' \cong \tau^- \tau A'$. Then by *part 6* and *part 3*, τA and $\tau A'$ are indecomposable non-injective objects in \mathcal{A} . Therefore, the dual argument of the “ \Rightarrow ” implication of *part 7* gives us $\tau^- \tau A \cong \tau^- \tau A'$. \square

We state the dual version of the above result for completeness.

Proposition I.4.4. *Let \mathcal{A} be a category with Setup I.4.1 and let B and B' be indecomposable objects in \mathcal{A} . Then the following hold:*

1. $\tau^-(B \oplus B') \cong \tau^-B \oplus \tau^-B'$.
2. The object B is injective in $\mathcal{A} \iff \tau^-B = 0$.
3. If B is a non-injective object in \mathcal{A} , then τ^-B is a non-projective object in \mathcal{A} .
4. If B is a non-injective object in \mathcal{A} , then the sequence $\nu^-I^0 \rightarrow \nu^-I^1 \rightarrow \tau^-B$ is a minimal projective presentation of τ^-B .
5. If B is a non-injective object in \mathcal{A} , then $\tau \tau^-B \cong B$.
6. If B is a non-injective object in \mathcal{A} , then τ^-B is indecomposable.
7. If B and B' are non-injective objects in \mathcal{A} , then $B \cong B' \iff \tau^-B \cong \tau^-B'$.

Corollary I.4.5. *Let \mathcal{A} be a category with Setup I.4.1. There is a bijective correspondence between the following:*

1. Isomorphism classes of indecomposable non-projective objects A in \mathcal{A} .
2. Isomorphism classes of indecomposable non-injective objects B in \mathcal{A} .

The bijective correspondence is given by the assignments $A \mapsto \tau A$ and $B \mapsto \tau^-B$.

The existence of Auslander-Reiten sequences in proper abelian subcategories

Remark I.4.6. In [Jør09, Theorem 3.1], the triangulated categories are assumed to be essentially small. This assumption can be dropped due to [INP24, Proposition 5.15].

Theorem I.4.7. *Let \mathcal{A} be a category with Setup I.4.1. Then the following hold:*

1. *For each indecomposable non-projective object C in \mathcal{A} , there exists an Auslander-Reiten sequence in \mathcal{A} of the form*

$$0 \rightarrow \tau C \rightarrow E_C \rightarrow C \rightarrow 0.$$

2. *For each indecomposable non-injective object A in \mathcal{A} , there exists an Auslander-Reiten sequence in \mathcal{A} of the form*

$$0 \rightarrow A \rightarrow F_A \rightarrow \tau^- A \rightarrow 0.$$

Proof. Part 1. By Lemma I.2.7, there exists a nonzero \mathcal{A} -precover of the form $\tau C \rightarrow \Sigma^{-1}SC$. But, τC is an indecomposable object in \mathcal{A} by Proposition I.4.3 and hence by the dual of [Kra00, Lemma 2.4], $\tau C \rightarrow \Sigma^{-1}SC$ is an \mathcal{A} -cover. The existence of the required Auslander-Reiten sequence is ensured by [Jør09, Theorem 3.1] (see also Remark I.4.6).

Part 2. Dual to part 1. □

I.5 Application to the module category of a finite dimensional algebra

In this section, we recover the standard Nakayama functors associated with the category of finite dimensional modules over a finite dimensional algebra and give a new proof of the existence of Auslander-Reiten sequences in such a category. The following setup applies throughout this section.

Setup I.5.1. Let A be a finite dimensional k -algebra and let mod_A denote the category of finite dimensional right modules over A . We also let proj_A denote the full subcategory of finitely generated projective A -modules and let inj_A denote the full subcategory of finitely generated injective A -modules.

Remark I.5.2. In [RV02, Theorem I.2.4], the ground field is assumed to be algebraically closed. This assumption can be dropped due to [INP24, Theorem 3.6].

By [RV02, Proposition I.1.4] and [RV02, Proposition I.2.3] (see also Remark I.5.2), we can associate to the Serre functor \mathbb{S} and the collection of natural isomorphisms

$$\{\mathcal{T}(X, Y) \xrightarrow{\eta_{X,Y}} D\mathcal{T}(Y, \mathbb{S}X) \mid \text{for } X \text{ and } Y \text{ objects in } \mathcal{T}\},$$

a collection of linear forms

$$\left\{ \mathcal{T}(Z, \mathbb{S}Z) \xrightarrow{\eta_Z} k \mid \text{for } Z \text{ an object in } \mathcal{T} \text{ and } \eta_Z := \eta_{Z,Z}(1_Z) \right\},$$

satisfying $\eta_Z(h_Z) \neq 0$ for each connecting homomorphism h_Z in an Auslander-Reiten triangle of the form $\Sigma^{-1}\mathbb{S}Z \rightarrow Y \rightarrow Z \xrightarrow{h_Z} \mathbb{S}Z$. Then for each morphism $X \xrightarrow{f} Y$ we have $\eta_{X,Y}(f)(-) = \eta_X(- \circ f)$ as linear forms on $\mathcal{T}(Y, \mathbb{S}X)$. It will be also useful to note that, for X and Y objects in \mathcal{T} , the morphism $\mathcal{T}(X, \mathbb{S}Y) \xrightarrow{\text{ev}} D^2\mathcal{T}(X, \mathbb{S}Y) \xrightarrow{D(\eta_{Y,X})} D\mathcal{T}(Y, X)$ is given by $g \mapsto \eta_Y(g \circ -)$, where ev is the evaluation map isomorphism.

Remark I.5.3. Similarly to Serre duality, the standard Nakayama functor $N(-) = - \otimes_{\mathbb{A}} D(\mathbb{A})$ on $\text{mod}_{\mathbb{A}}$ exhibits a duality as follows: For each finitely generated \mathbb{A} -module M and for each finitely generated projective \mathbb{A} -module P , we have the following composition of natural isomorphisms:

$$\text{Hom}_{\mathbb{A}}(P, M) \rightarrow D(P \otimes_{\mathbb{A}} D(M)) \rightarrow D(P \otimes_{\mathbb{A}} \text{Hom}_{\mathbb{A}}(M, D(\mathbb{A}))) \rightarrow D(\text{Hom}_{\mathbb{A}}(M, N(P))).$$

Here, the first isomorphism uses the evaluation isomorphism $M \cong D^2(M)$ and Tensor-Hom Adjunction, the second isomorphism uses the canonical isomorphism $M \cong M \otimes_{\mathbb{A}} \mathbb{A}$ and the Tensor-Hom Adjunction and the third isomorphism follows by the tensor evaluation isomorphism (a reference for these isomorphisms can be found in [CFH24, Section I.1]). We will denote this composition by

$$\text{Hom}_{\mathbb{A}}(P, M) \xrightarrow{\varepsilon_{P,M}} D\text{Hom}_{\mathbb{A}}(M, N(P)).$$

The isomorphism $\varepsilon_{P,M}$ is natural in both P and M . We will refer to this as *Nakayama duality*.

Recovering the standard Nakayama functors on the module category

Theorem I.5.4. *Let $\text{mod}_{\mathbb{A}}$ be a k -linear subcategory of \mathcal{T} . Let $N(-) = - \otimes_{\mathbb{A}} D(\mathbb{A})$ and let $N^{-}(-) = \text{Hom}_{\mathbb{A}}(D(\mathbb{A}), -)$ be the standard Nakayama functors on $\text{mod}_{\mathbb{A}}$. Then the following hold:*

1. *For each indecomposable finitely generated projective \mathbb{A} -module P , there is a strong $\text{mod}_{\mathbb{A}}$ -cover of the form $N(P) \xrightarrow{\alpha_P} \mathbb{S}P$ such that the diagram*

$$\begin{array}{ccc} N(P) & \xrightarrow{\alpha_P} & \mathbb{S}P \\ N(P) \downarrow & & \downarrow \mathbb{S}P \\ N(Q) & \xrightarrow{\alpha_Q} & \mathbb{S}Q \end{array}$$

is commutative for each \mathbf{A} -module homomorphism $P \xrightarrow{\mathbf{p}} Q$ between indecomposable finitely generated projective \mathbf{A} -modules. Moreover, the system of strong $\text{mod}_{\mathbf{A}}$ -covers $\{N(P) \xrightarrow{\alpha_P} \mathbb{S}P\}$ given above gives rise to a functor $\text{proj}_{\mathbf{A}} \xrightarrow{\mathbf{v}} \text{inj}_{\mathbf{A}}$ by Theorem I.3.4. The restricted Nakayama functor $\text{proj} \xrightarrow{\mathbf{N}} \text{inj}_{\mathbf{A}}$ is naturally isomorphic to \mathbf{v} .

2. For each indecomposable finitely generated injective \mathbf{A} -module I , there is a strong $\text{mod}_{\mathbf{A}}$ -envelope of the form $S^{-1}I \rightarrow N^{-}(I)$ such that the diagram

$$\begin{array}{ccc} S^{-1}I & \longrightarrow & N^{-}(I) \\ S^{-1}i \downarrow & & \downarrow N^{-}(i) \\ S^{-1}J & \longrightarrow & N^{-}(J) \end{array}$$

is commutative for each \mathbf{A} -module homomorphism $I \xrightarrow{i} J$ between indecomposable finitely generated injective \mathbf{A} -modules. Moreover, the system of strong $\text{mod}_{\mathbf{A}}$ -envelopes $\{S^{-1}I \rightarrow N^{-}(I)\}$ given above give rise to a functor $\text{inj}_{\mathbf{A}} \xrightarrow{\mathbf{v}^-} \text{proj}_{\mathbf{A}}$ by Theorem I.3.4. The restricted functor $\text{inj}_{\mathbf{A}} \xrightarrow{\mathbf{N}^-} \text{proj}_{\mathbf{A}}$ is naturally isomorphic to \mathbf{v}^- .

Proof. Part 1.

The construction of α_P . Using Remark I.5.3, we have a natural isomorphism:

$$\gamma_P^{(-)}: \text{Hom}_{\mathbf{A}}(-, N(P)) \xrightarrow{D(\varepsilon_P, -)^{\text{oev}}} D \text{Hom}_{\mathbf{A}}(P, -) \xrightarrow{(D(\eta_P, -)^{\text{oev}})^{-1}} \mathcal{T}(-, \mathbb{S}P).$$

Let $\alpha_P := \gamma_P^{N(P)}(1_{N(P)})$, where $\gamma_P^{N(P)}$ is the component of the natural isomorphism γ_P at $N(P)$.

α_P is a strong $\text{mod}_{\mathbf{A}}$ -cover. By definition, the morphism α_P is a strong $\text{mod}_{\mathbf{A}}$ -cover if and only if for each finitely generated \mathbf{A} -module M , the k -linear map

$$\text{Hom}_{\mathbf{A}}(M, N(P)) \xrightarrow{\mathcal{T}(M, \alpha_P)} \mathcal{T}(M, \mathbb{S}P)$$

is an isomorphism. As $\gamma_P^{(-)}$ is a natural isomorphism, it, therefore, suffices to show for each finitely generated \mathbf{A} -module M , the invertible component γ_P^M coincides with $\text{Hom}_{\mathbf{A}}(M, N(P)) \xrightarrow{\mathcal{T}(M, \alpha_P)} \mathcal{T}(M, \mathbb{S}P)$. To this end, let $M \xrightarrow{m} N(P)$ be an \mathbf{A} -module homomorphism. By naturality, the following diagram is commutative:

$$\begin{array}{ccc} \text{Hom}_{\mathbf{A}}(N(P), N(P)) & \xrightarrow{\gamma_P^{N(P)}} & \mathcal{T}(N(P), \mathbb{S}P) \\ \text{Hom}_{\mathbf{A}}(m, N(P)) \downarrow & & \downarrow \mathcal{T}(m, \mathbb{S}P) \\ \text{Hom}_{\mathbf{A}}(M, N(P)) & \xrightarrow{\gamma_P^M} & \mathcal{T}(M, \mathbb{S}P). \end{array}$$

Chasing the identity $1_{N(P)}$ through this diagram yields the equality

$$\gamma_P^M(m) = \mathcal{T}(M, \alpha_P)(m).$$

The commutativity of the induced diagram. Let $P \xrightarrow{\mathbf{p}} Q$ be a morphism between indecomposable finitely generated projective A -modules. We want to show that the diagram

$$\begin{array}{ccc} N(P) & \xrightarrow{\alpha_P} & \mathbb{S}P \\ N(\mathbf{p}) \downarrow & & \downarrow \mathbb{S}\mathbf{p} \\ N(Q) & \xrightarrow{\alpha_Q} & \mathbb{S}Q \end{array} \quad (\text{I.5})$$

is commutative. Using the naturality of γ in both components, the following diagram is commutative:

$$\begin{array}{ccccc} \text{Hom}_A(N(P), N(P)) & \xrightarrow{\cong} & D \text{Hom}_A(P, N(P)) & \xrightarrow{\cong} & \mathcal{T}(N(P), \mathbb{S}P) \\ \text{Hom}_A(N(P), N(\mathbf{p})) \downarrow & & \downarrow & & \downarrow \mathcal{T}(N(P), \mathbb{S}\mathbf{p}) \\ \text{Hom}_A(N(P), N(Q)) & \xrightarrow{\cong} & D \text{Hom}_A(Q, N(P)) & \xrightarrow{\cong} & \mathcal{T}(N(P), \mathbb{S}Q) \\ \text{Hom}_A(N(\mathbf{p}), N(Q)) \uparrow & & \uparrow & & \uparrow \mathcal{T}(N(\mathbf{p}), \mathbb{S}Q) \\ \text{Hom}_A(N(Q), N(Q)) & \xrightarrow{\cong} & D \text{Hom}_A(Q, N(Q)) & \xrightarrow{\cong} & \mathcal{T}(N(Q), \mathbb{S}Q). \end{array} \quad (\text{I.6})$$

Commutativity of (I.5) is equivalent to the validity of the equation

$$\mathcal{T}(N(P), \mathbb{S}\mathbf{p})(\alpha_P) = \mathcal{T}(N(\mathbf{p}), \mathbb{S}Q)(\alpha_Q)$$

and this, by commutativity of (I.6), is equivalent to the validity of the equation

$$\text{Hom}_A(N(P), N(\mathbf{p}))(1_{N(P)}) = \text{Hom}_A(N(\mathbf{p}), N(Q))(1_{N(Q)}),$$

which indeed holds.

The construction of ν is naturally isomorphic to N . As any finitely generated projective A -module P is a finite direct sum of indecomposable finitely generated projective A -modules and as the finite direct sum of strong covers is again a strong cover, we have strong mod_A -covers of the form $N(P) \rightarrow \mathbb{S}P$, where P is a finitely generated projective A -module. Theorem I.3.4 therefore, gives rise to a functor $\text{proj}_A \xrightarrow{\nu} \text{inj}_A$. The fact that the functors ν and the restricted Nakayama functor $\text{proj}_A \xrightarrow{N} \text{inj}_A$ are naturally isomorphic follows from Lemma I.2.4.

Part 2. This follows by a similar dual argument to *part 1*. □

Recovering the existence of Auslander-Reiten sequences in module categories

Proposition I.5.5. *Let mod_A be an extension closed proper abelian k -linear subcategory of \mathcal{T} and assume $\mathcal{T}(\text{mod}_A, \Sigma^{-1} \text{mod}_A) = 0$. Let $N(-) = - \otimes_A D(A)$ and let $N^-(-) = \text{Hom}_A(D(A), -)$ be the standard Nakayama functors on mod_A and let t and t^- denote the standard Auslander-Reiten translates in mod_A . Then the following hold:*

1. If $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M$ is a projective presentation of a non-projective indecomposable finitely generated A -module M , then there is a mod_A -cover of the form $t(M) \rightarrow \Sigma^{-1}\mathbb{S}M$ in \mathcal{T} , where $t(M) \rightarrow N(P_1)$ is the kernel of the induced A -module homomorphism $N(P_1) \rightarrow N(P_0)$ in mod_A .
2. If $L \rightarrow I^0 \rightarrow I^1$ is an injective copresentation of a non-injective indecomposable finitely generated A -module L , then there is a mod_A -envelope of the form $\Sigma\mathbb{S}^{-1}L \rightarrow t^-(L)$ in \mathcal{T} , where $N^-(I^1) \rightarrow t^-(L)$ is the cokernel of the induced A -module homomorphism $N(I^0) \rightarrow N(I^1)$ in mod_A .

Proof. Part 1. By Theorem I.5.4 and Lemma I.2.7, there is a nonzero mod_A -precover of the form $\tau M \xrightarrow{\theta} \Sigma^{-1}\mathbb{S}M$, where $\tau M \rightarrow \nu P_1$ is the kernel of the induced morphism $\nu P_1 \xrightarrow{\nu p_1} \nu P_0$ in mod_A . We have left exact sequences $\tau M \rightarrow \nu P_1 \xrightarrow{\nu p_1} \nu P_0$ and $t(M) \rightarrow N(P_1) \xrightarrow{N(p_1)} N(P_0)$. By the equivalence $\nu \cong N$ in Theorem I.5.4 and by the universal property of the kernel of $N(p_1)$, we get a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \tau M & \longrightarrow & \nu P_1 & \xrightarrow{\nu p_1} & \nu P_1 \\
& & \downarrow f & & \downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & t(M) & \longrightarrow & N(P_1) & \xrightarrow{N(p_1)} & N(P_0),
\end{array}$$

where the right two vertical A -module homomorphisms are isomorphisms. The Five Lemma ensures that the leftmost A -module homomorphism is also an isomorphism. Hence, $t(M) \xrightarrow{f^{-1}} \tau M \xrightarrow{\theta} \Sigma^{-1}\mathbb{S}M$ is a mod_A -precover. Finally, the minimality of the nonzero mod_A -precover θf^{-1} follows as $t(M)$ is indecomposable (see the dual of [Kra00, Lemma 2.4]).

Part 2. Dual to part 1. □

Proposition I.5.6. *Let mod_A be an extension closed proper abelian k -linear subcategory of \mathcal{T} and assume $\mathcal{T}(\text{mod}_A, \Sigma^{-1}\text{mod}_A) = 0$. Let t and t^- denote the standard Auslander-Reiten translates in mod_A . Then the following hold:*

1. For each indecomposable finitely generated non-projective A -module M , there is an Auslander-Reiten sequence in mod_A of the form

$$0 \rightarrow t(M) \rightarrow E_M \rightarrow M \rightarrow 0.$$

2. For each indecomposable finitely generated non-injective A -module L , there is an Auslander-Reiten sequence in mod_A of the form

$$0 \rightarrow L \rightarrow F_L \rightarrow t^-(L) \rightarrow 0.$$

Proof. Part 1. By Proposition I.5.5, there exists a mod_A -cover of the form $t(M) \rightarrow \Sigma^{-1}\mathbb{S}M$. The existence of the required Auslander-Reiten sequence is ensured by [Jør09, Theorem 3.1] (see Remark I.4.6).

Part 2. Dual to part 1. □

The following were defined in [HW83, page 351-352] and [Hap88, Chapter II.2.1] (see also [Hap87]): let \widehat{A} denote the repetitive algebra of A , let $\text{mod}_{\widehat{A}}$ denote the module category of finitely generated modules over \widehat{A} and let $\underline{\text{mod}}_{\widehat{A}}$ denote the stable module category of \widehat{A} . We now recover the existence of Auslander-Reiten sequences in mod_A .

Theorem I.5.7. *Let M be an indecomposable finitely generated non-projective A -module and let L an indecomposable finitely generated non-injective A -module. Let t and t^- denote the standard Auslander-Reiten translates in mod_A and let $N^-(-) = \text{Hom}_A(D(A), -)$ be the standard Nakayama functor on mod_A . Then the following hold:*

1. *There is an Auslander-Reiten sequence in mod_A of the form*

$$0 \rightarrow t(M) \rightarrow E_M \rightarrow M \rightarrow 0.$$

2. *There is an Auslander-Reiten sequence in mod_A of the form*

$$0 \rightarrow L \rightarrow F_L \rightarrow t^-(L) \rightarrow 0.$$

Proof. Part 1 and Part 2. By [Hap88, Theorem on page 16 and Lemma on page 62], the category $\underline{\text{mod}}_{\widehat{A}}$ is a triangulated category. We denote the suspension functor of $\underline{\text{mod}}_{\widehat{A}}$ by Σ . By [Hap88, Proposition on page 67], mod_A is the heart of a t-structure on $\underline{\text{mod}}_{\widehat{A}}$ and hence by [BBD82, Theorem 1.3.6 on page 31], mod_A is an extension closed k -linear proper abelian subcategory of $\underline{\text{mod}}_{\widehat{A}}$. By [Hap88, Lemma on page 63], we have isomorphisms $\underline{\text{Hom}}_{\widehat{A}}(M, \Sigma^i L) \cong \text{Ext}_A^i(M, L)$ for all i in \mathbb{Z} and therefore, $\underline{\text{Hom}}_{\widehat{A}}(\text{mod}_A, \Sigma^{-1} \text{mod}_A) = 0$. It was known in [HW83, Lemma on page 354], that $\text{mod}_{\widehat{A}}$ has Auslander-Reiten sequences. Later, it was stated in [Hap88], that $\underline{\text{mod}}_{\widehat{A}}$ has the so-called Auslander-Reiten triangles (one can use [INP24, Proposition 5.11], for example). Then by [RV02, Theorem I.2.4] (see also Remark I.5.2), the triangulated category $\underline{\text{mod}}_{\widehat{A}}$ has a Serre functor. As $\text{mod}_{\widehat{A}}$ is Hom-finite, so is $\underline{\text{mod}}_{\widehat{A}}$. To show that $\underline{\text{mod}}_{\widehat{A}}$ is Krull-Schmidt, it suffices to show that each idempotent in $\underline{\text{mod}}_{\widehat{A}}$ splits (see [Rin84, page 52], [Hap88, page 26], [CYZ08, Corollary A.2], [Kra15, Section 4] and [Sha23, Theorem 6.1]).

Let us first show that each idempotent in $\text{mod}_{\widehat{A}}$ splits. Following [Hap88, page 60], we let $M = (M_i, m_i)_{i \in \mathbb{Z}}$ be a finitely generated \widehat{A} -module. That is, M_i is a finitely generated A -module with finitely many M_i being nonzero and $M_i \xrightarrow{m_i} N^-(M_{i+1})$ is an A -module homomorphism such that $N^-(m_{i+1})m_i = 0$. Let $M \xrightarrow{e=(e_i)_{i \in \mathbb{Z}}} M$ be an \widehat{A} -module endomorphism. That is, $M_i \xrightarrow{e_i} M_i$ is an A -module homomorphism such that

$$\begin{array}{ccc} M_i & \xrightarrow{m_i} & N^-(M_{i+1}) \\ e_i \downarrow & & \downarrow N^-(e_{i+1}) \\ M_i & \xrightarrow{m_i} & N^-(M_{i+1}) \end{array}$$

commutes for each i in \mathbb{Z} . Assume that e is an idempotent in $\text{End}_{\widehat{\mathcal{A}}}(\mathcal{M})$. Then for each i in \mathbb{Z} , the \mathcal{A} -module homomorphism $M_i \xrightarrow{e_i} M_i$ is an idempotent in $\text{End}_{\mathcal{A}}(M_i)$. Hence, for each i in \mathbb{Z} , there exists a factorisation $e_i: M_i \xrightarrow{r_i} X_i \xrightarrow{s_i} M_i$ of e_i such that r_i and s_i are \mathcal{A} -module homomorphisms satisfying the equation $1_{X_i} = r_i s_i$. We define an \mathcal{A} -module homomorphism $X_i \xrightarrow{x_i} N^-(X_{i+1})$ as the following composition:

$$x_i: X_i \xrightarrow{s_i} M_i \xrightarrow{m_i} N^-(M_{i+1}) \xrightarrow{N^-(r_{i+1})} N^-(X_{i+1}).$$

It is not hard to check that the pair $(X_i, x_i)_{i \in \mathbb{Z}}$ defines a finitely generated $\widehat{\mathcal{A}}$ -module and that the sequences $r = (r_i)_{i \in \mathbb{Z}}$ and $s = (s_i)_{i \in \mathbb{Z}}$ define $\widehat{\mathcal{A}}$ -module homomorphisms such that $e = sr$ and $1_{\mathcal{M}} = rs$. Hence, e splits. By [CCS23, Proposition 5.9], the stable module category $\text{mod}_{\widehat{\mathcal{A}}}$ has split idempotents and therefore is Krull-Schmidt. Both parts of the theorem now follow from Proposition I.5.6. \square

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Rank functions on $(d + 2)$ -angulated categories—a functorial approach

Abstract

We introduce the notion of a rank function on a $(d + 2)$ -angulated category \mathcal{C} which generalises the notion of a rank function on a triangulated category. Inspired by work of Chuang and Lazarev, for d an odd positive integer, we prove that there is a bijective correspondence between rank functions defined on objects in \mathcal{C} and rank functions defined on morphisms in \mathcal{C} . Inspired by work of Conde, Gorsky, Marks and Zvonareva, for d an odd positive integer, we show there is a bijective correspondence between rank functions on $\text{proj } A$ and additive functions on $\text{mod}(\text{proj } A)$, where $\text{proj } A$ is endowed with the Amiot-Lin $(d + 2)$ -angulated category structure. This allows us to show that every integral rank function on $\text{proj } A$ can be decomposed into irreducible rank functions.

II.1 Introduction

Rank functions on triangulated categories were introduced in [CL21] to generalise Sylvester rank functions that were defined in [Sch85]. In this paper, we initiate the study of rank functions on $(d + 2)$ -angulated categories via the functorial approach investigated in [Con+24] and generalise [CL21, Proposition 2.4] and [Con+24, Theorem 3.11 and Theorem 4.2], the latter results to a class of $(d + 2)$ -angulated categories constructed in [Ami07; Lin19] (see Construction II.4.8). Note that by [JKM23, Theorem A], this class contains many $(d + 2)$ -angulated categories with finitely many indecomposable objects, see for example, [OT12, Theorem 5.2] and [JKM23, Proposition 2.29 and Theorem 2.2.20]. An example of a rank function on a $(d + 2)$ -angulated category is the total dimension of all the n -th cohomology groups, for n an integer multiple of d , of an object in the $(d + 2)$ -cluster tilting subcategory of the bounded derived category of a $(d + 2)$ -representation finite finite dimensional algebra (see Example II.3.3).

An odd feature that occurs in this paper when trying to pass from the triangulated world into the $(d + 2)$ -angulated world is that most of the main results are true only for d an odd positive integer. This phenomenon can also be observed in other works, such as [BT14, Theorem 4.6], [Rei20a, Theorem A, Corollary B and Theorem C] and [Rei20b, Theorem B].

The results in this paper concern the theory of higher homological algebra which originated in [Iya07a; Iya07b]. Since its origin, the theory has found many connections between other areas of mathematics such as in [HI11; IO11; OT12; DJL21; Wil22]. The literature concerning higher homological algebra is ever expanding and many authors have contributed to developing its general theory (see for example [GKO13; Jas16; Jør16; JK18; JJ19; Fed19; HJV20; Jør21; EN22; Kva22; Asa+22; HJS22; Kla23; Aug+25]).

Remark II.1.1. As a final remark in this introduction, we note that the ‘higher’ octahedral axiom [GKO13, Definition 1.1 (F4)] is not used in this paper. Consequently, all results established here apply to essentially small pre- $(d + 2)$ -angulated categories, despite them being presented in the context of essentially small $(d + 2)$ -angulated categories.

Global setup

- All categories are assumed to be locally small: for any ordered pair of objects in a category, the collection of morphisms between them forms a set.
- d will be a positive integer and sometimes we will require it to be an odd positive integer.
- For a finite dimensional algebra A over a field, we denote the category of finite dimensional right A -modules by $\text{mod } A$.

Main results (simplified)

See Remark II.1.1 whilst reading the main results of this paper.

The first main result in this paper is a direct generalisation of [CL21, Proposition 2.4] to $(d + 2)$ -angulated categories for d an odd positive integer. It says that rank functions can be defined either on the objects or on the morphisms in a $(d + 2)$ -angulated category.

Theorem E (Theorem II.3.14). *Let d be an odd positive integer and let \mathcal{C} be an essentially small $(d + 2)$ -angulated category. There is a (explicit) bijective correspondence between the following sets:*

- Rank functions on objects in \mathcal{C} .
- Rank functions on morphisms in \mathcal{C} .

The next main result generalises [Con+24, Theorem 3.11] to a class of $(d + 2)$ -angulated categories (see Construction II.4.8), for d an odd positive integer. It allows one to study rank functions on such $(d + 2)$ -angulated categories by studying additive functions on the abelian category consisting of additive contravariant functors from the $(d + 2)$ -angulated category into the category of abelian groups, and visa versa.

Theorem F (Theorem II.4.11). *Let d be a positive odd integer, A a suitable finite dimensional algebra and endow $\text{proj } A$ with the Amiot-Lin $(d + 2)$ -angulated category structure (see Construction II.4.8). There is a (explicit) bijective correspondence between the following:*

- Σ_d -invariant additive functions on $\text{mod}(\text{proj } A)$.
- Rank functions on morphisms in $\text{proj } A$.

The correspondence restricts to:

- Σ_d -invariant integral additive functions on $\text{mod}(\text{proj } A)$.
- Integral rank functions on morphisms in $\text{proj } A$.

The correspondence restricts further to:

- Σ_d -irreducible additive functions on $\text{mod}(\text{proj } A)$.
- Irreducible rank functions on morphisms in $\text{proj } A$.

The following theorem is a direct generalisation of the first part of [Con+24, Theorem 4.2] to $(d + 2)$ -angulated categories.

Theorem G (Theorem II.4.13). *Let \mathcal{C} be an essentially small $(d + 2)$ -angulated category. Then every Σ_d -invariant integral additive function on $\text{mod } \mathcal{C}$ can be decomposed uniquely as a locally finite sum of Σ_d -irreducible invariant additive functions on $\text{mod } \mathcal{C}$.*

Using the correspondence found in Theorem F and the decomposition in Theorem G, we can deduce the following decomposition theorem for integral rank functions.

Theorem H (Theorem II.4.14). *Let d be a positive odd integer, A a suitable finite dimensional algebra and endow $\text{proj } A$ with the Amiot-Lin $(d + 2)$ -angulated category structure (see Construction II.4.8). Then every integral rank function on morphisms in $\text{proj } A$ can be decomposed uniquely as a locally finite sum of irreducible rank functions on morphisms in $\text{proj } A$.*

Possible future directions. In [CL21, Remark 2.15], the idea is presented that rank functions could potentially replace stability conditions. It is noted that each stability condition corresponds to a specific rank function (see [CL21, Proposition 2.14]). As far as the Author knows, the concept of stability conditions in a $(d + 2)$ -angulated setting has not been generalised. Furthermore, the correspondence provides a continuous mapping from the space of stability conditions in a given triangulated category to the space of rank functions for that triangulated category. It was said that this could potentially establish a connection between rank functions and the compactification of a quotient of the space of stability conditions (see [CL21, Proposition 2.16] and the following sentence).

In [Con+24, Theorem C], a bijective correspondence is established between certain rank functions on a triangulated category and certain universal triangulated functors out of that same triangulated category and into sufficiently small triangulated categories. This correspondence relies on the need for a localisation theory and the theory addressed in [Kra05]. In contrast to stability conditions, the localisation theory of $(d + 2)$ -angulated categories has been addressed in [Lia23].

It would be intriguing to explore potential applications of the current rank functions defined on $(d + 2)$ -angulated categories, in a manner similar to the ones mentioned above.

Definitions and notation

This paper is concerned with $(d + 2)$ -angulated categories. Their definition and first properties, which we freely use (notably [GKO13, Proposition 2.5(a)]), can be found in [GKO13].

Notation II.1.2. We say an exact sequence $A' \xrightarrow{\alpha'} A \xrightarrow{\alpha''} A''$ in an abelian category is left (right) exact if α' is a monomorphism (α'' is an epimorphism). We say the sequence is short exact if it is both left exact and right exact.

Definition II.1.3 ([Aus66; Aus74a; Aus74b]). Let \mathcal{C} be a $(d + 2)$ -angulated category with suspension functor Σ_d .

- We denote the abelian category of additive functors $\mathcal{C}^{\text{op}} \rightarrow \text{Ab}$, where Ab denote the category of abelian groups, by $\text{Mod } \mathcal{C}$. The objects in $\text{Mod } \mathcal{C}$ are referred to as \mathcal{C} -modules and the morphisms in $\text{Mod } \mathcal{C}$ are referred to as \mathcal{C} -module homomorphisms.
- A \mathcal{C} -module M is *finitely presented* if there is a right exact sequence of \mathcal{C} -modules of the form $\mathcal{C}(-, X) \rightarrow \mathcal{C}(-, Y) \rightarrow M$, for some objects X and Y in \mathcal{C} (alternative terminology for such an M is *coherent*). We denote the full subcategory of $\text{Mod } \mathcal{C}$ consisting of the finitely presented \mathcal{C} -modules by $\text{mod } \mathcal{C}$. As \mathcal{C} has weak kernels, $\text{mod } \mathcal{C}$ is abelian (see [Aus71, Proposition on page 41]).

- The suspension functor Σ_d on \mathcal{C} induces an exact autoequivalence Σ_d on $\text{mod } \mathcal{C}$ given by the assignment $M \mapsto M\Sigma_d^{-1}$, for each \mathcal{C} -module M .

Definition II.1.4 ([Cra94a]). Let \mathcal{A} be an essentially small abelian category and let \mathcal{C} be an essentially small $(d+2)$ -angulated category with suspension functor Σ_d . Note that the category $\text{mod } \mathcal{C}$ is essentially small (this follows for example by [Cra94b, Theorem on page 1645]).

- An *additive function* α on \mathcal{A} is an assignment $A \mapsto \alpha(A)$, where A is an object in \mathcal{A} and $\alpha(A)$ is a nonnegative real number, that is constant on isomorphism classes and satisfies the equation $\alpha(A) = \alpha(A') + \alpha(A'')$ for each short exact sequence $A' \rightarrow A \rightarrow A''$ in \mathcal{A} . An additive function is *integral* if it takes values in the integers. An additive function is *zero* if it is given by the assignment $A \mapsto 0$, for each object A in \mathcal{A} and is *nonzero* otherwise.
- Consider additive functions α_1 and α_2 on \mathcal{A} . Their *sum* $\alpha_1 + \alpha_2$ is the additive function on \mathcal{A} given by the assignment $A \mapsto \alpha_1(A) + \alpha_2(A)$, for each object A in \mathcal{A} .
- Let I be a set and consider a set of additive functions $\{\alpha_i\}_{i \in I}$ on \mathcal{A} . If the set $\{i \in I \mid \alpha_i(A) \neq 0\}$ is finite for every object A in \mathcal{A} , then we define their *locally finite sum* $\sum_{i \in I} \alpha_i$ given by $A \mapsto \sum_{i \in I} \alpha_i(A)$, for each object A in \mathcal{A} .
- An additive function α on $\text{mod } \mathcal{C}$ is Σ_d -*invariant* if $\alpha(\Sigma_d M) = \alpha(M)$ for every finitely presented \mathcal{C} -module M .
- An additive function α on \mathcal{A} is *irreducible* if it is nonzero, integral and whenever there are integral additive functions α_1 and α_2 on \mathcal{A} with $\alpha = \alpha_1 + \alpha_2$, then α_1 is zero or α_2 is zero. An additive function α on $\text{mod } \mathcal{C}$ is Σ_d -*irreducible* if it is nonzero, integral, Σ_d -invariant and whenever there are integral Σ_d -invariant functions α_1 and α_2 on $\text{mod } \mathcal{C}$ with $\alpha = \alpha_1 + \alpha_2$, then α_1 is zero or α_2 is zero.

Definition II.1.5. Let $\mathcal{X} \xrightarrow{F} \mathcal{Y}$ be a functor between categories and let α be an assignment on \mathcal{Y} given by $Y \mapsto \alpha(Y)$, where Y is an object in \mathcal{Y} and $\alpha(Y)$ is a nonnegative real number. We define the *restriction* αF of α along F as the assignment on \mathcal{A} given by $A \mapsto \alpha(F(A))$, where A is an object in \mathcal{A} .

II.2 Lemmas

A lemma on additive functions

Lemma II.2.1. *Let \mathcal{A} and \mathcal{B} be abelian categories and let $\mathcal{A} \xrightarrow{F} \mathcal{B}$ be an equivalence. Let α be an assignment on \mathcal{B} given by $B \mapsto \alpha(B)$ such that $\alpha(B) = \alpha(B')$, whenever*

$B \cong B'$ in \mathcal{B} . If the restriction αF of α along F is an additive function on \mathcal{A} , then α is an additive function on \mathcal{B} .

Proof. Let $B' \rightarrow B \rightarrow B''$ be a short exact sequence in \mathcal{B} . Choose objects A, A' and A'' in \mathcal{A} such that $F(A) \cong B, F(A') \cong B'$ and $F(A'') \cong B''$. Then there is a short exact sequence in \mathcal{B} of the form $F(A') \xrightarrow{F(\alpha')} F(A) \xrightarrow{F(\alpha'')} F(A'')$, where $A' \xrightarrow{\alpha'} A$ and $A \xrightarrow{\alpha''} A''$ are morphisms in \mathcal{A} . But, F reflects short exact sequences (as it is in particular faithful and automatically additive and exact) and so $A' \xrightarrow{\alpha'} A \xrightarrow{\alpha''} A''$ is also a short exact sequence in \mathcal{A} . We have

$$\alpha(B) = \alpha(F(A)) = \alpha(F(A')) + \alpha(F(A'')) = \alpha(B') + \alpha(B''),$$

where the first and last equalities hold by assumption and the second equality holds as αF is an additive function on \mathcal{A} . \square

Schanuel's Lemma for $(d + 2)$ -angulated categories

The following subsection is a $(d + 2)$ -angulated version of [Kra16, Appendix A].

Definition II.2.2. Let \mathcal{C} be a $(d + 2)$ -angulated category. A $(d + 2)$ -angle

$$X := X_0 \xrightarrow{x_0} X_1 \rightarrow \cdots \rightarrow X_d \xrightarrow{x_d} X_{d+1} \rightarrow \Sigma_d X_0$$

in \mathcal{C} induces a presentation of a \mathcal{C} -module M if there is a right exact sequence of \mathcal{C} -modules

$$\mathcal{C}(-, X_d) \xrightarrow{\mathcal{C}(-, x_d)} \mathcal{C}(-, X_{d+1}) \rightarrow M.$$

Two $(d + 2)$ -angles are *homotopy equivalent* (this terminology follows [Kra16, Appendix A]) if they induce a presentation of the same \mathcal{C} -module. We say X induces a copresentation of N if there is a left exact sequence of \mathcal{C} -modules

$$N \rightarrow \mathcal{C}(-, X_0) \xrightarrow{\mathcal{C}(-, x_0)} \mathcal{C}(-, X_1).$$

Lemma II.2.3. Let \mathcal{C} be a $(d + 2)$ -angulated category, let

$$X := X_0 \xrightarrow{x_0} X_1 \rightarrow \cdots \rightarrow X_d \rightarrow X_{d+1} \xrightarrow{x_{d+1}} \Sigma_d X_0$$

be a $(d + 2)$ -angle in \mathcal{C} and let M be a \mathcal{C} -module. Then the following statements are equivalent:

1. X induces a presentation of M .
2. $M \cong \text{Im } \mathcal{C}(-, x_{d+1})$.
3. X induces a copresentation of $\Sigma_d^{-1} M$.

Proof. The $(d+2)$ -angle X induces the exact sequence of \mathcal{C} -modules

$$\delta: \mathcal{C}(-, \Sigma_d^{-1}X_{d+1}) \rightarrow \mathcal{C}(-, X_0) \rightarrow \mathcal{C}(-, X_1) \rightarrow \cdots \rightarrow \mathcal{C}(-, X_{d+1}) \rightarrow \mathcal{C}(-, \Sigma_d X_0).$$

(1 \Leftrightarrow 2): Both implications follow by considering the canonical image factorisation of $\mathcal{C}(-, x_{d+1})$:

$$\mathcal{C}(-, X_{d+1}) \rightarrow \text{Im } \mathcal{C}(-, x_{d+1}) \rightarrow \mathcal{C}(-, \Sigma_d X_0).$$

(2 \Rightarrow 3): We have $\Sigma_d^{-1}M \cong \Sigma_d^{-1} \text{Im } \mathcal{C}(-, x_{d+1}) \cong \text{Im } \mathcal{C}(-, \Sigma_d^{-1}(x_{d+1})) = \text{Ker } \mathcal{C}(-, x_0)$, where the first isomorphism holds by assumption, the second isomorphism holds as Σ_d^{-1} is an autoequivalence and the equality holds by the exactness of δ .

(3 \Rightarrow 2): We have $M \cong \Sigma_d \Sigma_d^{-1}M \cong \Sigma_d \text{Ker } \mathcal{C}(-, x_0) \cong \text{Ker } \mathcal{C}(-, \Sigma_d(x_0)) = \text{Im } \mathcal{C}(-, x_{d+1})$, where the second isomorphism follows by assumption, the third isomorphism follows as Σ_d is an autoequivalence and the equality holds by the exactness of δ . \square

Corollary II.2.4. *Let \mathcal{C} be a $(d+2)$ -angulated category and consider the $(d+2)$ -angles*

$$X := X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_{d+1} \xrightarrow{x_{d+1}} \Sigma_d X_0$$

and

$$Y := Y_0 \rightarrow Y_1 \rightarrow Y_2 \rightarrow \cdots \rightarrow Y_{d+1} \xrightarrow{y_{d+1}} \Sigma_d Y_0$$

in \mathcal{C} . Then X and Y are homotopy equivalent if and only if

$$\text{Im } \mathcal{C}(-, x_{d+1}) \cong \text{Im } \mathcal{C}(-, y_{d+1}).$$

Lemma II.2.5. (*$(d+2)$ -angulated Schanuel's Lemma*) *Let \mathcal{C} be a $(d+2)$ -angulated category. If*

$$X := X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_d \xrightarrow{x_d} X_{d+1} \rightarrow \Sigma_d X_0$$

and

$$Y := Y_0 \rightarrow Y_1 \rightarrow \cdots \rightarrow Y_d \xrightarrow{y_d} Y_{d+1} \rightarrow \Sigma_d Y_0$$

are homotopy equivalent $(d+2)$ -angles in \mathcal{C} , then

$$\bigoplus_{i \geq 0} (X_{2i} \oplus Y_{2i+1}) \cong \bigoplus_{i \geq 0} (X_{2i+1} \oplus Y_{2i}),$$

where we set $X_i = Y_i = 0$ for $i \geq d+2$.

Proof. As the $(d+2)$ -angles X and Y are homotopy equivalent, we get induced exact sequences of \mathcal{C} -modules

$$0 \rightarrow \Sigma_d^{-1}M \rightarrow \mathcal{C}(-, X_0) \rightarrow \mathcal{C}(-, X_1) \rightarrow \cdots \rightarrow \mathcal{C}(-, X_d) \xrightarrow{\mathcal{C}(-, x_d)} \mathcal{C}(-, X_{d+1}) \rightarrow M \rightarrow 0$$

and

$$0 \rightarrow \Sigma_d^{-1}M \rightarrow \mathcal{C}(-, Y_0) \rightarrow \mathcal{C}(-, Y_1) \rightarrow \cdots \rightarrow \mathcal{C}(-, Y_d) \xrightarrow{\mathcal{C}(-, y_d)} \mathcal{C}(-, Y_{d+1}) \rightarrow M \rightarrow 0.$$

We show that the above exact sequences represent the same class in $\text{Ext}_{\mathcal{C}}(M, \Sigma_d^{-1}M)$. By the Comparison Theorem (see [EM65, Proposition 3.2]) and the Yoneda Lemma, the identity \mathcal{C} -module homomorphism of M induces \mathcal{C} -module homomorphisms $\mathcal{C}(-, X_i) \xrightarrow{\mathcal{C}(-, f_i)} \mathcal{C}(-, Y_i)$ for $X_i \xrightarrow{f_i} Y_i$ a morphism in \mathcal{C} and $i = d, d + 1$, such that the diagram

$$\begin{array}{ccccc} \mathcal{C}(-, X_d) & \xrightarrow{\mathcal{C}(-, x_d)} & \mathcal{C}(-, X_{d+1}) & \longrightarrow & M \\ \mathcal{C}(-, f_d) \downarrow & & \downarrow \mathcal{C}(-, f_{d+1}) & & \parallel \\ \mathcal{C}(-, Y_d) & \xrightarrow{\mathcal{C}(-, y_d)} & \mathcal{C}(-, Y_{d+1}) & \longrightarrow & M \end{array}$$

commutes. As $f_{d+1}x_d = y_df_d$, an axiom of $(d + 2)$ -angulated categories ensures there exists a morphism

$$\begin{array}{ccccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots & \longrightarrow & X_d & \xrightarrow{x_d} & X_{d+1} & \longrightarrow & \Sigma_d X_0 \\ \downarrow f_0 & & \downarrow f_1 & & & & \downarrow f_d & & \downarrow f_{d+1} & & \downarrow \\ Y_0 & \longrightarrow & Y_1 & \longrightarrow & \cdots & \longrightarrow & Y_d & \xrightarrow{y_d} & Y_{d+1} & \longrightarrow & \Sigma_d Y_0 \end{array}$$

of $(d + 2)$ -angles between X and Y . This in turn induces the commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \Sigma_d^{-1}M & \longrightarrow & \mathcal{C}(-, X_0) & \longrightarrow & \cdots & \xrightarrow{\mathcal{C}(-, x_d)} & \mathcal{C}(-, X_{d+1}) & \longrightarrow & M & \longrightarrow & 0 \\ & & \parallel & & \downarrow \mathcal{C}(-, f_0) & & & & \downarrow \mathcal{C}(-, f_{d+1}) & & \parallel & & \\ 0 & \longrightarrow & \Sigma_d^{-1}M & \longrightarrow & \mathcal{C}(-, Y_0) & \longrightarrow & \cdots & \xrightarrow{\mathcal{C}(-, y_d)} & \mathcal{C}(-, Y_{d+1}) & \longrightarrow & M & \longrightarrow & 0, \end{array}$$

realising that the exact sequences mentioned at the start of this proof do indeed represent the same class in $\text{Ext}_{\mathcal{C}}(M, \Sigma_d^{-1}M)$. The result now follows by applying the variant of Schanuel's Lemma in [Kra16, Lemma A.2] and noting that the Yoneda Embedding reflects isomorphisms. \square

II.3 Rank functions on $(d + 2)$ -angulated categories

Rank functions defined on objects

We now use the objects in a $(d + 2)$ -angulated category to define a notion of a rank function and give some examples.

Definition II.3.1. Let \mathcal{C} be an essentially small $(d + 2)$ -angulated category. A *rank function on objects* ρ_o in \mathcal{C} is an assignment $X \mapsto \rho_o(X)$, where X is an object in \mathcal{C} and $\rho_o(X)$ is a nonnegative real number, that satisfies the following axioms:

RO0. ρ_o is constant on isomorphism classes of objects in \mathcal{C} . That is, for each pair of objects X and Y in \mathcal{C} such that $X \cong Y$, we have $\rho_o(X) = \rho_o(Y)$.

RO1. For each pair of objects X and Y in \mathcal{C} , we have $\rho_o(X \oplus Y) = \rho_o(X) + \rho_o(Y)$.

RO2. For each $(d + 2)$ -angle $X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_d \rightarrow X_{d+1} \rightarrow \Sigma_d X_0$ in \mathcal{C} , we have

$$\sum_{i=0}^{d+1} (-1)^i \rho_o(X_i) \geq 0.$$

RO3. For each object X in \mathcal{C} , we have $\rho_o(\Sigma_d X) = \rho_o(X)$.

Remark II.3.2. As \mathcal{C} is essentially small, axiom RO0 tells us that a rank function on objects in \mathcal{C} can be regarded as an honest function from the set of isomorphism classes of objects in \mathcal{C} to the nonnegative real numbers.

A standard example of a rank function on a triangulated category is the total dimension of the cohomology of an object in the bounded derived category of a finite dimensional algebra over a field (see [Con+24, Example 3.6]). The next example is a $(d + 2)$ -angulated version of this.

Example II.3.3. Let A be a finite dimensional algebra over a field k and consider the bounded derived category $\mathcal{T} := D^b(\text{mod } A)$ of A with suspension functor Σ . We further assume that A is d -representation finite (see [IO11, Definition 2.2]). Then there exists a d -cluster tilting subcategory \mathcal{C} of \mathcal{T} (that was constructed in [Iya11, Theorem 1.21]) such that the d -th power Σ^d of Σ endows \mathcal{C} with the structure of a $(d + 2)$ -angulated category (see [GKO13, Theorem 1 on page 109]). As A is d -representation finite, its global dimension is finite and hence, the canonical localisation functor $K^b(\text{proj } A) \rightarrow D^b(\text{mod } A)$, where $K^b(\text{proj } A)$ denotes the homotopy category of bounded complexes of finitely generated projective A -modules, is a triangle equivalence (this was stated in [Hap88], for a proof see for example [Kra22, Corollary 4.2.9]). Recall the definition of a compact object in a triangulated category with set-indexed coproducts (see [Nee92, Definition 1.1]).

For every object C in \mathcal{C} , we define an assignment $X \mapsto \rho_C(X)$, where X is an object in \mathcal{C} and $\rho_C(X)$ is given by the equation

$$\rho_C(X) = \sum_{j \in \mathbb{Z}} \dim_k \mathcal{C}(C, \Sigma^{dj} X). \quad (\text{II.1})$$

As a consequence of the equivalence $K^b(\text{proj } A) \rightarrow D^b(\text{mod } A)$, all objects in $D^b(\text{mod } A)$ are compact (see [Nee92, Lemma 2.2]). Therefore, we have

$$\sum_{j \in \mathbb{Z}} \dim_k \mathcal{C}(C, \Sigma^{dj} X) = \dim_k \left(\prod_{j \in \mathbb{Z}} \mathcal{C}(C, \Sigma^{dj} X) \right) = \dim_k \mathcal{C} \left(C, \prod_{j \in \mathbb{Z}} (\Sigma^{dj} X) \right),$$

where compactness is used for the second equality to hold. Hence, the sum in equation II.1 is finite. We show that this assignment defines a rank function $\rho_{\mathcal{C}}$ on objects in \mathcal{C} . It is clear that the value $\rho_{\mathcal{C}}(X)$ is a nonnegative real number for all objects X in \mathcal{C} .

RO0: This is clear as functors preserve isomorphisms and the k -dimension of a vector space is constant on isomorphism classes of vector spaces.

RO1: Follows by the additivity of the functor $\mathcal{C}(\mathcal{C}, \Sigma^{dj}(-))$ for each integer j .

RO2: Let $X := X_0 \xrightarrow{x_0} X_1 \xrightarrow{x_1} X_2 \rightarrow \cdots \rightarrow X_{d+1} \rightarrow \Sigma^d X_0$ be a $(d + 2)$ -angle in \mathcal{C} . For each integer j , the $(d + 2)$ -angle X induces the exact sequence of finite dimensional vector spaces

$$\mathcal{C}(\mathcal{C}, \Sigma^{dj}(X_0)) \rightarrow \mathcal{C}(\mathcal{C}, \Sigma^{dj}(X_1)) \rightarrow \cdots \rightarrow \mathcal{C}(\mathcal{C}, \Sigma^{dj}(X_{d+1})).$$

It follows that

$$\sum_{i=0}^{d+1} (-1)^i \dim_k \mathcal{C}(\mathcal{C}, \Sigma^{dj}(X_i)) \geq 0,$$

for each integer j . Adding all these quantities together, we have

$$\begin{aligned} 0 &\leq \sum_{j \in \mathbb{Z}} \sum_{i=0}^{d+1} (-1)^i \dim_k \mathcal{C}(\mathcal{C}, \Sigma^{dj}(X_i)) = \sum_{i=0}^{d+1} (-1)^i \sum_{j \in \mathbb{Z}} \dim_k \mathcal{C}(\mathcal{C}, \Sigma^{dj}(X_i)) \\ &= \sum_{i=0}^{d+1} (-1)^i \rho_{\mathcal{C}}(X_i), \end{aligned}$$

where the first equality holds as one of the summations is finite.

RO3: Follows by the definition of the suspension functor on \mathcal{C} and by the definition of $\rho_{\mathcal{C}}(X)$.

In particular, as the right regular representation A can be identified as a stalk complex in degree 0 in \mathcal{C} we may set $\mathcal{C} = A$ in equation II.1. In this case, for each object X in \mathcal{C} , the quantity $\rho_A(X)$ is the total dimension of the n -th cohomology groups of X , where n runs through the integer multiples of d .

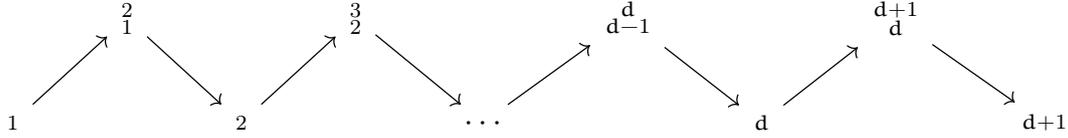
We will now calculate explicitly the rank function for a class of d -representation finite algebras.

Example II.3.4. We work over a field k . Let $A = A_2^d$ be the $(d - 1)$ -iterated higher-Auslander algebra of the path algebra of the quiver $\bullet \leftarrow \bullet$ (defined in [Iya11], but we use notation from [OT12, Theorem/Construction 3.3]). That is, A is the quotient of the path algebra of the quiver

$$1 \leftarrow 2 \leftarrow \cdots \leftarrow d + 1$$

by the two-sided ideal generated by all paths of length two. We let $P(i)$ denote the indecomposable projective A -module and let $I(i)$ denote the indecomposable projective A -module, both corresponding to the vertex i . Notice that $P(i + 1)$ is isomorphic to $I(i)$ as A -modules for $1 \leq i \leq d$, that $P(1)$ is the unique simple projective (non-injective)

A -module and that $I(d + 1)$ is the unique simple injective (non-projective) A -module. Denoting the indecomposable finite dimensional A -modules by their composition factors (for example, $P(2) = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$), the Auslander-Reiten quiver of $\text{mod } A$ is



where the composition of any morphism pointing to the top-right followed by the consecutive morphism pointing to the bottom-right is zero (for example, $1 \rightarrow \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \rightarrow 2$ is zero). Let $\mathcal{F} = \text{add}\{P(i), I(d + 1) \mid \text{for } 1 \leq i \leq d + 1\}$ be the additive closure of the indecomposable projective and indecomposable injective A -modules and let

$$\mathcal{C} = \text{add}\{\Sigma^{dj}F \mid \text{for } j \text{ an integer and for } F \text{ and object in } \mathcal{F}\}.$$

By [Jas16, Proposition 6.2] or [Vas21, Theorem 3] and [Iya11, Theorem 1.21], the algebra A is d -representation finite and \mathcal{C} is the d -cluster tilting subcategory of $D^b(\text{mod } A)$. By Example II.3.3, we have that the assignment

$$\rho_A(X) = \sum_{j \in \mathbb{Z}} \dim_k \mathcal{C}(A, \Sigma^{dj}X),$$

where X is an object in \mathcal{C} , is a rank function ρ_A on objects in \mathcal{C} . Up to isomorphism, the indecomposable objects in \mathcal{C} are $\Sigma^{dj}P(i)$ and $\Sigma^{dj}I(d + 1)$ for $1 \leq i \leq d + 1$ and j an integer and hence, by axiom RO0 and axiom RO1, it suffices to calculate ρ_A on representatives of the indecomposable objects in \mathcal{C} . We have

$$\mathcal{C}(A, \Sigma^{dj}X) \cong \text{Hom}_{\mathcal{K}^b(A)}(A, \Sigma^{dj}X) = H^0(\text{Hom}_A^\bullet(A, \Sigma^{dj}X)) \cong H^0(\Sigma^{dj}X) = H^{dj}(X),$$

where the first isomorphism holds as \mathcal{C} is a full subcategory of $D^b(\text{mod } A)$ and A is a semi-projective complex of A -modules (or see [Wei94, Corollary 10.47]). The last isomorphism holds by the enriched Yoneda Lemma and as H^0 is a functor. As the indecomposable objects are stalk complexes, for X an indecomposable object in \mathcal{C} , we have $\rho_A(X)$ is just the k -dimension of X when viewed as an A -module.

Using the class of $(d + 2)$ -angulated categories in Example II.3.4, it is easy to construct rank functions combinatorially. It boils down to a choice of finitely many nonnegative real numbers satisfying a finite number of inequalities.

Example II.3.5. Consider Example II.3.4 and let $d = 3$. The Auslander-Reiten quiver of \mathcal{C} is

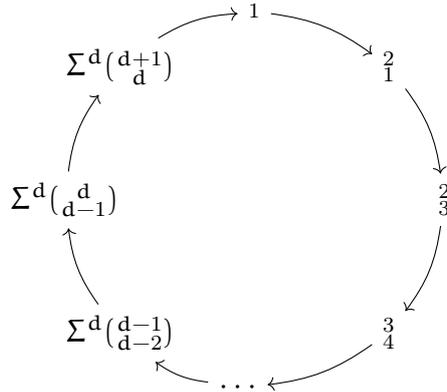
$$\dots \rightarrow \Sigma^{-3}(4) \rightarrow 1 \rightarrow \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \rightarrow \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \rightarrow \begin{smallmatrix} 4 \\ 3 \end{smallmatrix} \rightarrow 4 \rightarrow \Sigma^3(1) \rightarrow \Sigma^3\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right) \rightarrow \dots,$$

where the composition of any two consecutive morphisms is zero. The assignment

$$1 \mapsto 2, \quad \frac{2}{1} \mapsto 0, \quad \frac{3}{2} \mapsto 1, \quad \frac{4}{3} \mapsto 3 \quad \text{and} \quad 4 \mapsto 4,$$

uniquely define a rank function ρ_o on objects in \mathcal{C} .

Example II.3.6. Consider Example II.3.4. Let \mathcal{O}_A be the $(d + 2)$ -angulated cluster category of A (see [OT12, Definition 5.22]). From [OT12, Section 6 and Section 8], the Auslander-Reiten quiver of \mathcal{O}_A is



where there are $2d + 2$ indecomposable objects and the composition of any two consecutive morphisms is zero. We can use the Auslander-Reiten quiver to gain access to some typical $(d + 2)$ -angles in the following way: Start at an object in the Auslander-Reiten quiver and then follow the direction of the morphisms until you have met $d + 3$ objects to build a $(d + 2)$ -angle. For example, starting at the object 1, we get the following $(d + 2)$ -angle $1 \rightarrow \frac{2}{1} \rightarrow \cdots \rightarrow d + 1 \rightarrow \Sigma^d(1)$. Using axiom RO3, one can check that any rank function on objects in \mathcal{O}_A must be constant on indecomposable objects, namely, the objects in the Auslander-Reiten quiver of \mathcal{O}_A . Hence, the values of a given rank function on objects in \mathcal{O}_A will consist only of positive integer multiples of a specified nonnegative integer.

Rank functions defined on morphisms

We now use the morphisms in a $(d + 2)$ -angulated category to define a notion of a rank function and collect some needed properties.

Definition II.3.7. Let \mathcal{C} be an essentially small $(d + 2)$ -angulated category. A *rank function on morphisms* ρ_m in \mathcal{C} is an assignment $f \mapsto \rho_m(f)$, where f is a morphism in \mathcal{C} and $\rho_m(f)$ a nonnegative real number, that satisfies the following axioms:

RM0. ρ_m is constant on isomorphism classes of morphisms in \mathcal{C} . That is, for each pair of morphisms $X \xrightarrow{f} Y$ and $W \xrightarrow{g} Z$ in \mathcal{C} fitting into a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \psi \downarrow & & \downarrow \varphi \\ W & \xrightarrow{g} & Z, \end{array}$$

for ψ and φ isomorphisms in \mathcal{C} , we have $\rho_m(f) = \rho_m(g)$.

RM1. For each pair of morphisms f and g in \mathcal{C} , we have $\rho_m(f \oplus g) = \rho_m(f) + \rho_m(g)$.

RM2. For each consecutive pair of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in a $(d + 2)$ -angle in \mathcal{C} , we have $\rho_m(f) - \rho_m(1_Y) + \rho_m(g) = 0$.

RM3. For each morphism f in \mathcal{C} , we have $\rho_m(\Sigma_d f) = \rho_m(f)$.

Lemma II.3.8. *Let \mathcal{C} be an essentially small $(d + 2)$ -angulated category and let ρ be an assignment $f \mapsto \rho(f)$, where f is a morphism in \mathcal{C} and $\rho(f)$ is a nonnegative real number, satisfying axiom RM0. Consider a morphism $X \xrightarrow{f} Y$ in \mathcal{C} . Then the following statements hold:*

1. *If f is an isomorphism in \mathcal{C} , then $\rho(1_X) = \rho(f) = \rho(1_Y)$. In particular, $\rho(f) = \rho(f^{-1})$.*
2. *If ρ also satisfies axiom RM3, then $\rho(\Sigma_d^{-1} f) = \rho(f)$.*

Proof. Part 1: Suppose $X \xrightarrow{f} Y$ be an isomorphism in \mathcal{C} . The result follows by considering the following commutative diagram:

$$\begin{array}{ccc}
 X & \xrightarrow{1_X} & X \\
 1_X \downarrow & & \downarrow f \\
 X & \xrightarrow{f} & Y \\
 f \downarrow & & \downarrow 1_Y \\
 Y & \xrightarrow{1_Y} & Y.
 \end{array}$$

Part 2: Assume ρ also satisfies axiom RM3 and choose a natural isomorphism $\Sigma_d \Sigma_d^{-1} \xrightarrow{\varepsilon} 1_{\mathcal{C}}$, where $1_{\mathcal{C}}$ is the identity functor on \mathcal{C} . We then have that $\rho(\Sigma_d^{-1} f) = \rho(\Sigma_d \Sigma_d^{-1} f) = \rho(\varepsilon_Y^{-1} f \varepsilon_X) = \rho(f)$, where the first equality holds by axiom RM3, the second equality by naturality of ε and the third equality holds by axiom RM0 since ε_Y^{-1} and ε_X are isomorphisms. \square

Remark II.3.9. Given two morphisms f and g in \mathcal{C} . We write $f \sim g$ if there exist isomorphisms φ and ψ in \mathcal{C} such that $\varphi f = g \psi$ (whenever the composition makes sense). The relation \sim is an equivalence relation on the class $\text{Mor } \mathcal{C}$ and as \mathcal{C} is essentially small, there is a bijection between $\text{Mor } \mathcal{C} / \sim$ and the set of morphisms in a skeleton of \mathcal{C} . Hence, $\text{Mor } \mathcal{C} / \sim$ forms a set and by axiom RM0, a rank function on morphisms in \mathcal{C} can be regarded as an honest function from the set $\text{Mor } \mathcal{C} / \sim$ to the nonnegative real numbers.

Lemma II.3.10. *Let d be a positive odd integer, let \mathcal{C} be an essentially small $(d + 2)$ -angulated category and let ρ be an assignment $f \mapsto \rho(f)$, where f is a morphism in \mathcal{C}*

and $\rho(f)$ is a nonnegative real number, satisfying axioms RM0 and RM2. Consider the $(d + 2)$ -angles

$$X_0 \xrightarrow{x_0} X_1 \xrightarrow{x_1} X_2 \rightarrow \cdots \rightarrow X_{d+1} \xrightarrow{x_{d+1}} \Sigma_d X_0$$

and

$$Y_0 \xrightarrow{y_0} Y_1 \xrightarrow{y_1} Y_2 \rightarrow \cdots \rightarrow Y_{d+1} \xrightarrow{y_{d+1}} \Sigma_d Y_0$$

in \mathcal{C} . Then the following statements hold:

1. $\rho(x_0) + \rho(\Sigma_d x_0) = \sum_{i=0}^{d+1} (-1)^i \rho(1_{X_{i+1}})$, where we set $X_{d+2} := \Sigma_d X_0$.
2. If $\text{Im } \mathcal{C}(-, x_0) \cong \text{Im } \mathcal{C}(-, y_0)$, then $\rho(x_0) + \rho(\Sigma_d x_0) = \rho(y_0) + \rho(\Sigma_d y_0)$.

Proof. Part 1: Setting $X_{d+2} := \Sigma_d X_0$ and $x_{d+2} := \Sigma_d x_0$, we have

$$\rho(x_0) + \rho(\Sigma_d x_0) = \sum_{i=0}^{d+1} (-1)^i (\rho(x_i) + \rho(x_{i+1})) = \sum_{i=0}^{d+1} (-1)^i \rho(1_{X_{i+1}}),$$

where we added zeros for the first equality and the second equality holds by axiom RM2 and as $\rho(-\Sigma_d x_0) = \rho(\Sigma_d x_0)$ by axiom RM0 (notice the parity of d is used for both these equalities to hold).

Part 2: Again, setting $X_{d+2} := \Sigma_d X_0$ and using the above, we have

$$\rho(x_0) + \rho(\Sigma_d x_0) = \sum_{i=0}^{d+1} (-1)^i \rho(1_{X_{i+1}}) = \rho(1_{X_1 \oplus Y_2 \oplus X_3 \oplus \cdots \oplus X_d \oplus Y_{d+1} \oplus \Sigma_d X_0}) - \sum_{i=1}^{\frac{d+1}{2}} \rho(1_{X_{2i} \oplus Y_{2i}}),$$

where we added zeros for the second equality and used axiom RM1. Similarly, we have

$$\rho(y_0) + \rho(\Sigma_d y_0) = \rho(1_{Y_1 \oplus X_2 \oplus Y_3 \oplus \cdots \oplus Y_d \oplus X_{d+1} \oplus \Sigma_d Y_0}) - \sum_{i=1}^{\frac{d+1}{2}} \rho(1_{Y_{2i} \oplus X_{2i}}).$$

As $\text{Im } \mathcal{C}(-, x_0) \cong \text{Im } \mathcal{C}(-, y_0)$, then $\text{Im } \mathcal{C}(-, -\Sigma_d x_0) \cong \text{Im } \mathcal{C}(-, -\Sigma_d y_0)$. Therefore, the rotated $(d + 2)$ -angles

$$X_1 \xrightarrow{x_1} X_2 \rightarrow \cdots \rightarrow X_{d+1} \xrightarrow{x_{d+1}} \Sigma_d X_0 \xrightarrow{-\Sigma_d x_0} \Sigma_d X_1$$

and

$$Y_1 \xrightarrow{y_1} Y_2 \rightarrow \cdots \rightarrow Y_{d+1} \xrightarrow{y_{d+1}} \Sigma_d Y_0 \xrightarrow{-\Sigma_d y_0} \Sigma_d Y_1$$

are homotopy equivalent by Corollary II.2.4 and therefore, we have

$$X_1 \oplus Y_2 \oplus X_3 \oplus \cdots \oplus X_d \oplus Y_{d+1} \oplus \Sigma_d X_0 \cong Y_1 \oplus X_2 \oplus Y_3 \oplus \cdots \oplus Y_d \oplus X_{d+1} \oplus \Sigma_d Y_0$$

by Lemma II.2.5. The result follows as

$$\rho(1_{X_1 \oplus Y_2 \oplus X_3 \oplus \cdots \oplus X_d \oplus Y_{d+1} \oplus \Sigma_d X_0}) = \rho(1_{Y_1 \oplus X_2 \oplus Y_3 \oplus \cdots \oplus Y_d \oplus X_{d+1} \oplus \Sigma_d Y_0})$$

by Lemma II.3.8, part 1. □

A bijective correspondence between definitions of rank functions

We will now establish the connection between the previous two definitions of a rank function on a $(d + 2)$ -angulated category.

Setup II.3.11. Let \mathcal{C} be an essentially small $(d + 2)$ -angulated category. We define the following two sets:

- The rank functions on morphisms in \mathcal{C} which we denote by $\mathcal{R}_m^{\mathcal{C}}$.
- The rank functions on objects in \mathcal{C} which we denote by $\mathcal{R}_o^{\mathcal{C}}$.

We define two assignments:

1. Given a rank function on morphisms ρ_m in \mathcal{C} , we define an assignment $\Phi(\rho_m)$ on objects in \mathcal{C} to be given by $\Phi(\rho_m)(X) = \rho_m(1_X)$, for each object X in \mathcal{C} .
2. Given a rank function on objects ρ_o in \mathcal{C} , we define an assignment $\Psi(\rho_o)$ on morphisms in \mathcal{C} , given by

$$\Psi(\rho_o)(X_0 \xrightarrow{x_0} X_1) = \frac{1}{2} \left(\rho_o(X_0) + \sum_{i=1}^{d+1} (-1)^{i-1} \rho_o(X_i) \right),$$

for each morphism x_0 in \mathcal{C} with $X_0 \xrightarrow{x_0} X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_{d+1} \rightarrow \Sigma_d X_0$ a $(d + 2)$ -angle in \mathcal{C} . It will be shown that this definition is well defined in the proof of Proposition II.3.13.

Proposition II.3.12. *The assignment $\rho_m \mapsto \Phi(\rho_m)$ defined in Setup II.3.11, part 1, is a function $\mathcal{R}_m^{\mathcal{C}} \rightarrow \mathcal{R}_o^{\mathcal{C}}$.*

Proof. It is clear that the value $\Phi(\rho_m)(X)$ is a nonnegative real number for each object X in \mathcal{C} . We verify the axioms RO0, RO1, RO2 and RO3.

RO0: Let X and Y be isomorphic objects in \mathcal{C} . We have

$$\Phi(\rho_m)(X) = \rho_m(1_X) = \rho_m(1_Y) = \Phi(\rho_m)(Y),$$

where the second equality holds by Lemma II.3.8, part 1.

RO1: We have

$$\Phi(\rho_m)(X \oplus Y) = \rho_m(1_{X \oplus Y}) = \rho_m(1_X \oplus 1_Y) = \rho_m(1_X) + \rho_m(1_Y) = \Phi(\rho_m)(X) + \Phi(\rho_m)(Y),$$

where the second equality holds by functoriality and the third equality holds by axiom RM1.

RO2: Let $X_0 \xrightarrow{x_0} X_1 \xrightarrow{x_1} X_2 \xrightarrow{x_2} X_3 \xrightarrow{x_3} X_4 \rightarrow \cdots \rightarrow X_{d+1} \rightarrow \Sigma_d X_0$ be a $(d+2)$ -angle in \mathcal{C} . We have

$$\begin{aligned} \sum_{i=0}^{d+1} (-1)^i \Phi(\rho_m)(X_i) &= \sum_{i=0}^{d+1} (-1)^i \rho_m(1_{X_i}) = \rho_m(1_{X_0}) + \sum_{i=1}^{d+1} (-1)^i \rho_m(1_{X_i}) \\ &= (\rho_m(-\Sigma_d^{-1} x_{d+1}) + \rho_m(x_0)) \\ &\quad + \sum_{i=1}^{d+1} (-1)^i (\rho_m(x_{i-1}) + \rho_m(x_i)) \\ &= 2\rho_m(x_{d+1}) \geq 0, \end{aligned}$$

where the third equality holds by axiom RM2 and the fourth equality holds by axiom RM0, Lemma II.3.8, part 2 and axiom RM3.

RO3: We have $\Phi(\rho_m)(\Sigma_d X) = \rho_m(1_{\Sigma_d X}) = \rho_m(\Sigma_d 1_X) = \rho_m(1_X) = \Phi(\rho_m)(X)$, where the second equality holds by functoriality and the third equality holds by axiom RM3. \square

Proposition II.3.13. *Let d be an odd positive integer. Then the assignment $\rho_o \mapsto \Psi(\rho_o)$ defined in Setup II.3.11, part 2, is a well-defined function $\mathcal{R}_o^c \rightarrow \mathcal{R}_m^c$.*

Proof. We show that this definition is independent of the choice of $(d+2)$ -angle. Suppose x_0 can be completed to the two $(d+2)$ -angles $X_0 \xrightarrow{x_0} X_1 \rightarrow V_2 \rightarrow \cdots \rightarrow V_{d+1} \rightarrow \Sigma_d X_0$ and $X_0 \xrightarrow{x_0} X_1 \rightarrow W_2 \rightarrow \cdots \rightarrow W_{d+1} \rightarrow \Sigma_d X_0$. As both $(d+2)$ -angles start with the same morphism, the rotated $(d+2)$ -angles $X_1 \rightarrow V_2 \rightarrow \cdots \rightarrow V_{d+1} \rightarrow \Sigma_d X_0 \xrightarrow{-\Sigma_d x_0} \Sigma_d X_1$ and $X_1 \rightarrow W_2 \rightarrow \cdots \rightarrow W_{d+1} \rightarrow \Sigma_d X_0 \xrightarrow{-\Sigma_d x_0} \Sigma_d X_1$ are homotopy equivalent by Corollary II.2.4 and therefore, we have

$$X_1 \oplus W_2 \oplus V_3 \oplus \cdots \oplus V_d \oplus W_{d+1} \oplus \Sigma_d X_0 \cong X_1 \oplus V_2 \oplus W_3 \oplus \cdots \oplus W_d \oplus V_{d+1} \oplus \Sigma_d X_0$$

by Lemma II.2.5. By axiom RO0 we have

$$\rho_o(X_1 \oplus W_2 \oplus V_3 \oplus \cdots \oplus V_d \oplus W_{d+1} \oplus \Sigma_d X_0) = \rho_o(X_1 \oplus V_2 \oplus W_3 \oplus \cdots \oplus W_d \oplus V_{d+1} \oplus \Sigma_d X_0).$$

Then using axiom RO1, axiom RO3 and rearranging we get the following equation:

$$\rho_o(X_0) + \rho_o(X_1) + \sum_{i=2}^{d+1} (-1)^i \rho_o(V_i) = \rho_o(X_0) + \rho_o(X_1) + \sum_{i=2}^{d+1} (-1)^i \rho_o(W_i).$$

This proves that Φ is well-defined. It is clear from axiom RO2 that the value $\Psi(\rho_o)(f)$ is a nonnegative real number for each morphism f in \mathcal{C} . We now verify the axioms RM0, RM1, RM2 and RM3.

RM0: Let $X_0 \xrightarrow{x_0} X_1$ and $Y_0 \xrightarrow{y_0} Y_1$ be two morphisms in \mathcal{C} fitting into the following commutative diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{x_0} & X_1 \\ \psi \downarrow & & \downarrow \varphi \\ Y_0 & \xrightarrow{y_0} & Y_1, \end{array} \quad (\text{II.2})$$

for ψ and φ isomorphisms in \mathcal{C} . We complete x_0 and y_0 into the $(d + 2)$ -angles

$$X_0 \xrightarrow{x_0} X_1 \rightarrow V_2 \rightarrow \cdots \rightarrow V_{d+1} \rightarrow \Sigma_d X_0$$

and

$$Y_0 \xrightarrow{y_0} Y_1 \rightarrow W_2 \rightarrow \cdots \rightarrow W_{d+1} \rightarrow \Sigma_d X_0.$$

By the commutativity of II.2, we have $\text{Im } \mathcal{C}(-, x_0) \cong \text{Im } \mathcal{C}(-, y_0)$. Hence, the rotated $(d + 2)$ -angles

$$X_1 \rightarrow V_2 \rightarrow \cdots \rightarrow V_{d+1} \rightarrow \Sigma_d X_0 \xrightarrow{-\Sigma_d x_0} \Sigma_d X_1$$

and

$$Y_1 \rightarrow W_2 \rightarrow \cdots \rightarrow W_{d+1} \rightarrow \Sigma_d Y_0 \xrightarrow{-\Sigma_d y_0} \Sigma_d Y_1$$

are homotopy equivalent by Corollary II.2.4. Following the steps used in the argument above proving the well definedness of $\Psi(\rho_o)$ yields the required result.

RM1: Let $X_0 \xrightarrow{x_0} X_1$ and $Y_0 \xrightarrow{y_0} Y_1$ be morphisms in \mathcal{C} and complete them to the $(d + 2)$ -angles $X := X_0 \xrightarrow{x_0} X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_{d+1} \rightarrow \Sigma_d X_0$ and $Y := Y_0 \xrightarrow{y_0} Y_1 \rightarrow Y_2 \rightarrow \cdots \rightarrow Y_{d+1} \rightarrow \Sigma_d Y_0$ in \mathcal{C} . We have

$$\begin{aligned} \Psi(\rho_o)(x_0 \oplus y_0) &= \frac{1}{2} \left(\rho_o(X_0 \oplus Y_0) + \sum_{i=1}^{d+1} (-1)^{i-1} \rho_o(X_i \oplus Y_i) \right) \\ &= \frac{1}{2} \left(\rho_o(X_0) + \rho_o(Y_0) + \sum_{i=1}^{d+1} (-1)^{i-1} (\rho_o(X_i) + \rho_o(Y_i)) \right) \\ &= \frac{1}{2} \left(\rho_o(X_0) + \sum_{i=1}^{d+1} (-1)^{i-1} \rho_o(X_i) \right) + \frac{1}{2} \left(\rho_o(Y_0) + \sum_{i=1}^{d+1} (-1)^{i-1} \rho_o(Y_i) \right) \\ &= \Psi(\rho_o)(x_0) + \Psi(\rho_o)(y_0), \end{aligned}$$

where the first equality uses the $(d + 2)$ -angle $X \oplus Y$ defined as the direct sum of X and Y and the second equality holds by axiom RO1.

RM2: Let $X := X_0 \xrightarrow{x_0} X_1 \xrightarrow{x_1} X_2 \rightarrow \cdots \rightarrow X_{d+1} \rightarrow \Sigma_d X_0$ be a $(d + 2)$ -angle in \mathcal{C} . Then by definition we have

$$\Psi(\rho_o)(x_0) = \frac{1}{2} \left(\rho_o(X_0) + \sum_{i=1}^{d+1} (-1)^{i-1} \rho_o(X_i) \right).$$

By an axiom of $(d + 2)$ -angulated categories, the diagram $X_1 \xrightarrow{1_{X_1}} X_1 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow \Sigma_d X_1$ is a $(d + 2)$ -angle and so we have $\Psi(\rho_o)(1_{X_1}) = \frac{1}{2}(2\rho_o(X_1))$. The rotation

$$X_1 \xrightarrow{x_1} X_2 \rightarrow \cdots \rightarrow X_{d+1} \rightarrow \Sigma_d X_0 \xrightarrow{-\Sigma_d x_0} \Sigma_d X_1$$

of the $(d + 2)$ -angle X is also $(d + 2)$ -angle and so we have

$$\Psi(\rho_o)(x_1) = \frac{1}{2} \left(\rho_o(X_1) + \sum_{i=2}^{d+1} (-1)^i \rho_o(X_i) - \rho_o(\Sigma_d X_0) \right).$$

By rotating $(d + 2)$ -angles, to prove axiom RM2 holds it suffices to consider the pair of consecutive morphisms $X_0 \xrightarrow{x_0} X_1 \xrightarrow{x_1} X_2$. Using the above, we have

$$\begin{aligned}
\Psi(\rho_o)(x_0) - \Psi(\rho_o)(1_{X_1}) + \Psi(\rho_o)(x_1) &= \Psi(\rho_o)(x_0) + \Psi(\rho_o)(x_1) - \Psi(\rho_o)(1_{X_1}) \\
&= \frac{1}{2} \left(\rho_o(X_0) + \sum_{i=1}^{d+1} (-1)^{i-1} \rho_o(X_i) \right) \\
&\quad + \frac{1}{2} \left(\rho_o(X_1) + \sum_{i=2}^{d+1} (-1)^i \rho_o(X_i) - \rho_o(\Sigma_d X_0) \right) \\
&\quad - \frac{1}{2} (2\rho_o(X_1)) \\
&= \frac{1}{2} (\rho_o(X_0) + 2\rho_o(X_1) - \rho_o(\Sigma_d X_0)) - \rho_o(X_1) \\
&= 0,
\end{aligned}$$

where the third equality holds as $\sum_{i=1}^{d+1} (-1)^{i-1} \rho_o(X_i) + \sum_{i=2}^{d+1} (-1)^i \rho_o(X_i) = \rho_o(X_1)$ and the last equality follows by axiom RO3 (notice that the quantity would not vanish without the assumed parity of d).

RM3: Let $X_0 \xrightarrow{x_0} X_1$ be a morphism in \mathcal{C} and complete it to a $(d + 2)$ -angle

$$X_0 \xrightarrow{x_0} X_1 \xrightarrow{x_1} X_2 \rightarrow \cdots \rightarrow X_d \xrightarrow{x_d} X_{d+1} \xrightarrow{x_{d+1}} \Sigma_d X_0$$

in \mathcal{C} . There is a commutative diagram

$$\begin{array}{ccccccccccc}
\Sigma_d X_0 & \xrightarrow{-\Sigma_d x_0} & \Sigma_d X_1 & \xrightarrow{-\Sigma_d x_1} & \Sigma_d X_2 & \longrightarrow & \cdots & \longrightarrow & \Sigma_d X_d & \xrightarrow{-\Sigma_d x_d} & \Sigma_d X_{d+1} & \xrightarrow{-\Sigma_d x_{d+1}} & \Sigma_d^2 X_0 \\
\parallel & & \downarrow -1 & & \parallel & & & & \downarrow -1 & & \parallel & & \parallel \\
\Sigma_d X_0 & \xrightarrow{\Sigma_d x_0} & \Sigma_d X_1 & \xrightarrow{\Sigma_d x_1} & \Sigma_d X_2 & \longrightarrow & \cdots & \longrightarrow & \Sigma_d X_d & \xrightarrow{\Sigma_d x_d} & \Sigma_d X_{d+1} & \xrightarrow{-\Sigma_d x_{d+1}} & \Sigma_d^2 X_0
\end{array}$$

in \mathcal{C} . As the vertical morphisms are isomorphisms, the bottom row is a $(d + 2)$ -angle. Therefore, we use this $(d + 2)$ -angle to calculate

$$\begin{aligned}
\Psi(\rho_o)(\Sigma_d x_0) &= \frac{1}{2} \left(\rho_o(\Sigma_d X_0) + \sum_{i=1}^{d+1} (-1)^{i-1} \rho_o(\Sigma_d X_i) \right) \\
&= \frac{1}{2} \left(\rho_o(X_0) + \sum_{i=1}^{d+1} (-1)^{i-1} \rho_o(X_i) \right) \\
&= \Psi(\rho_o)(x_0),
\end{aligned}$$

where the second equality holds by axiom RO3. □

Theorem II.3.14. Consider Setup II.3.11 and suppose that d is an odd positive integer. Then there is a bijective correspondence between the following sets:

1. Rank functions on objects ρ_o in \mathcal{C} .

2. Rank functions on morphisms ρ_m in \mathcal{C} .

The bijective correspondence is given by the assignments $\rho_m \mapsto \Phi(\rho_m)$ and $\rho_o \mapsto \Psi(\rho_o)$.

Proof. By Proposition II.3.12 and Proposition II.3.13, there are well-defined functions $\mathcal{R}_m^{\mathcal{C}} \xrightarrow{\Phi} \mathcal{R}_o^{\mathcal{C}}$ and $\mathcal{R}_o^{\mathcal{C}} \xrightarrow{\Psi} \mathcal{R}_m^{\mathcal{C}}$. It is left to show that these are mutually inverse to each other.

Let ρ_m be a rank function on morphisms in \mathcal{C} . Let $X_0 \xrightarrow{x_0} X_1$ be a morphism in \mathcal{C} and complete it to a $(d+2)$ -angle $X_0 \xrightarrow{x_0} X_1 \xrightarrow{x_1} X_2 \xrightarrow{x_2} \dots \xrightarrow{x_d} X_{d+1} \xrightarrow{x_{d+1}} \Sigma_d X_0$ in \mathcal{C} . We need to show that $\Psi(\Phi(\rho_m)) = \rho_m$. We have

$$\begin{aligned} \Psi(\Phi(\rho_m))(x_0) &= \frac{1}{2} \left(\Phi(\rho_m)(X_0) + \sum_{i=1}^{d+1} (-1)^{i-1} \Phi(\rho_m)(X_i) \right) \\ &= \frac{1}{2} \left(\rho_m(1_{X_0}) + \sum_{i=1}^{d+1} (-1)^{i-1} \rho_m(1_{X_i}) \right) \\ &= \frac{1}{2} \left(\rho_m(-\Sigma_d^{-1} x_{d+1}) + \rho_m(x_0) + \sum_{i=1}^{d+1} (-1)^{i-1} (\rho_m(x_{i-1}) + \rho_m(x_i)) \right) \\ &= \frac{1}{2} (\rho_m(x_{d+1}) + 2\rho_m(x_0) - \rho_m(x_{d+1})) \\ &= \rho_m(x_0), \end{aligned}$$

where the third equality follows by axiom RM2, the fourth equality follows by axiom RM0, Lemma II.3.8, part 2 and axiom RM3.

Conversely, let ρ_o be a rank function on objects in \mathcal{C} . We need to show that $\Phi(\Psi(\rho_o)) = \rho_o$. We have $\Phi(\Psi(\rho_o))(X) = \Psi(\rho_o)(1_X) = \frac{1}{2}(2\rho_o(X)) = \rho_o(X)$. \square

Example II.3.15. Consider Example II.3.4 and let d be an odd integer. By Theorem II.3.14, we have the corresponding rank function $\Psi(\rho_{\mathcal{A}})$ on morphisms in \mathcal{C} . The Auslander-Reiten quiver of \mathcal{C} is

$$\dots \rightarrow \Sigma^{-d}(d+1) \rightarrow 1 \rightarrow \frac{2}{1} \rightarrow \dots \rightarrow d+1 \rightarrow \Sigma^d(1) \rightarrow \Sigma^d\left(\frac{2}{1}\right) \rightarrow \dots,$$

where the composition of any two consecutive morphisms is zero. The sequence

$$1 \rightarrow \frac{2}{1} \rightarrow \frac{3}{2} \rightarrow \dots \rightarrow d+1 \rightarrow \Sigma^d 1 \tag{II.3}$$

is a $(d+2)$ -angle in \mathcal{C} which allows us to calculate, for example, the value of the morphism $1 \rightarrow \frac{2}{1}$ under the rank function $\Psi(\rho_{\mathcal{A}})$: Noticing that d is an odd integer, we have

$$\begin{aligned} \Psi(\rho_{\mathcal{A}})\left(1 \rightarrow \frac{2}{1}\right) &= \frac{1}{2} \left(\rho_{\mathcal{A}}(1) + \sum_{i=1}^d (-1)^{i-1} \rho_{\mathcal{A}}\left(\frac{i+1}{i}\right) - \rho_{\mathcal{A}}(d+1) \right) \\ &= \frac{1}{2} \left(1 + \sum_{i=1}^d (-1)^{i-1} 2 - 1 \right) \\ &= 1, \end{aligned}$$

where the values for $\rho_{\mathcal{A}}$ in the second equality are given by the k -dimension of the objects. A similar calculation, using the rotation of the $(d + 2)$ -angle II.3, gives the values

$$\Psi(\rho_{\mathcal{A}})(i_{i-1} \rightarrow i_{i+1}) = \Psi(\rho_{\mathcal{A}})(d_{d+1} \rightarrow d+1) = 1 \quad \text{and} \quad \Psi(\rho_{\mathcal{A}})(4 \rightarrow 1) = 0,$$

where $2 \leq i \leq d - 1$. We can then use axiom RM3 to calculate the values assigned to the identity morphisms

$$\Psi(\rho_{\mathcal{A}})\left(1_{i_{i+1}}\right) = 2 \quad \text{and} \quad \Psi(\rho_{\mathcal{A}})(1_1) = \Psi(\rho_{\mathcal{A}})(1_{d+1}) = 1.$$

where $1 \leq i \leq d$. Notice that the values of $\Psi(\rho_{\mathcal{A}})$ on the identity morphisms do indeed coincide with the values of $\rho_{\mathcal{A}}$ on their respective objects. This agrees with the assignment in Setup II.3.11, part 1, that gives rise to one direction of the bijective correspondence in Theorem II.3.14.

II.4 A bijective correspondence between rank functions and additive functions

The bijective correspondence

We now connect the study of rank functions on $(d + 2)$ -angulated categories to the study of additive functions on associated abelian categories.

Definition II.4.1. Let \mathcal{C} be a $(d + 2)$ -angulated category.

- A rank function on morphisms in \mathcal{C} is *zero* if it is given by the assignment $f \mapsto 0$, for each morphism f in \mathcal{C} and is *nonzero* otherwise.
- Consider rank functions $\rho_{m,1}$ and $\rho_{m,2}$ on \mathcal{A} . Their *sum* $\rho_{m,1} + \rho_{m,2}$ is the rank function on morphisms in \mathcal{C} given by the assignment $f \mapsto \rho_{m,1}(f) + \rho_{m,2}(f)$, for each morphism f in \mathcal{C} .
- Let I be a set and consider a collection of rank functions $\{\rho_{m,i}\}_{i \in I}$ in \mathcal{C} . If the set $\{i \in I \mid \rho_{m,i}(f) \neq 0\}$ is finite for each morphism f in \mathcal{C} , then we define their *locally finite sum* $\sum_{i \in I} \rho_{m,i}$ given by $f \mapsto \sum_{i \in I} \rho_{m,i}(f)$, for each morphism f in \mathcal{C} .
- A rank function ρ_m on morphisms in \mathcal{C} is *irreducible* if it is nonzero, integral and whenever there are integral rank functions on morphisms $\rho_{m,1}$ and $\rho_{m,2}$ in \mathcal{C} with $\rho_m = \rho_{m,1} + \rho_{m,2}$, then $\rho_{m,1}$ is zero or $\rho_{m,2}$ is zero.

Setup II.4.2. Let \mathcal{C} be an essentially small $(d + 2)$ -angulated category. We define three collections:

- The Σ_d -invariant functions on $\text{mod } \mathcal{C}$ which we denote by $\mathcal{X}^{\mathcal{C}}$. That is, an assignment α that is given by $M \mapsto \alpha(M)$, where M is a finitely presented \mathcal{C} -module and $\alpha(M)$ is a nonnegative real number, lies in $\mathcal{X}^{\mathcal{C}}$ if $\alpha(\Sigma_d M) = \alpha(M)$ for each finitely presented \mathcal{C} -module M .
- The Σ_d -invariant additive functions on $\text{mod } \mathcal{C}$ which we denote by $\mathcal{A}^{\mathcal{C}}$.
- The rank functions on morphisms in \mathcal{C} which we denote by $\mathcal{R}_m^{\mathcal{C}}$.

We define two assignments:

1. Given a Σ_d -invariant additive function α on $\text{mod } \mathcal{C}$, we define $\varphi(\alpha)$ to be given by

$$\varphi(\alpha)(f) = \alpha(\text{Im } \mathcal{C}(-, f)),$$

for each morphism f in \mathcal{C} .

2. Given a rank function ρ_m on morphisms in \mathcal{C} , we define $\psi(\rho_m)$ to be given by

$$\psi(\rho_m)(M) = \rho_m(f),$$

for each finitely presented \mathcal{C} -module M such that $M \cong \text{Im } \mathcal{C}(-, f)$, for some morphism f in \mathcal{C} . Note that Lemma II.3.10 implies that $\psi(\rho_m)$ is well defined.

Proposition II.4.3. *The assignment $\alpha \mapsto \varphi(\alpha)$ defined in Setup II.4.2, part 1, is a well-defined mapping $\mathcal{A}^{\mathcal{C}} \xrightarrow{\varphi} \mathcal{R}_m^{\mathcal{C}}$.*

Proof. It is clear that the value $\varphi(\alpha)(f)$ is a nonnegative real number for each morphism f in \mathcal{C} . We verify the axioms in Definition II.3.7.

RM0: Let $X \xrightarrow{f} Y$ and $W \xrightarrow{g} Z$ be two morphisms in \mathcal{C} fitting into the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \psi \downarrow & & \downarrow \varphi \\ W & \xrightarrow{g} & Z, \end{array}$$

with ψ and φ isomorphisms in \mathcal{C} . Then by the commutativity of the above diagram, $\text{Im } \mathcal{C}(-, f) \cong \text{Im } \mathcal{C}(-, g)$ and so $\varphi(\alpha)(f) = \alpha(\text{Im } \mathcal{C}(-, f)) = \alpha(\text{Im } \mathcal{C}(-, g)) = \varphi(\alpha)(g)$, where the second equality follows as additive functions are constant on isomorphism classes.

RM1: Let $W \xrightarrow{f} X$ and $Y \xrightarrow{g} Z$ be morphisms in \mathcal{C} . Then there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{C}(-, W) & \longrightarrow & \mathcal{C}(-, W \oplus Y) & \longrightarrow & \mathcal{C}(-, Y) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Im } \mathcal{C}(-, f) & \longrightarrow & \text{Im } \mathcal{C}(-, f \oplus g) & \longrightarrow & \text{Im } \mathcal{C}(-, g) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{C}(-, X) & \longrightarrow & \mathcal{C}(-, X \oplus Z) & \longrightarrow & \mathcal{C}(-, Z) \longrightarrow 0, \end{array} \quad (\text{II.4})$$

of finitely presented \mathcal{C} -modules with exact rows. The vertical composition

$$\mathcal{C}(-, \mathbf{W}) \twoheadrightarrow \operatorname{Im} \mathcal{C}(-, f) \hookrightarrow \mathcal{C}(-, \mathbf{X})$$

is the canonical factorisation of the morphism $\mathcal{C}(-, \mathbf{W}) \xrightarrow{\mathcal{C}(-, f)} \mathcal{C}(-, \mathbf{X})$ and similarly for all other vertical compositions in II.4. As α is an additive function on $\operatorname{mod} \mathcal{C}$ and the middle row in II.4 is exact, we have $\varphi(\alpha)(f \oplus g) = \varphi(\alpha)(f) + \varphi(\alpha)(g)$.

RM2: Let $\mathbf{X} \xrightarrow{f} \mathbf{Y} \xrightarrow{g} \mathbf{Z}$ be a consecutive pair of morphisms in a $(d + 2)$ -angle in \mathcal{C} . Then there is an induced exact sequence $\mathcal{C}(-, \mathbf{X}) \xrightarrow{\mathcal{C}(-, f)} \mathcal{C}(-, \mathbf{Y}) \xrightarrow{\mathcal{C}(-, g)} \mathcal{C}(-, \mathbf{Z})$ of finitely presented \mathcal{C} -modules, which further gives a short exact sequence $\operatorname{Im} \mathcal{C}(-, f) \rightarrow \mathcal{C}(-, \mathbf{Y}) \rightarrow \operatorname{Im} \mathcal{C}(-, g)$ of finitely presented \mathcal{C} -modules. Noting that $\mathcal{C}(-, \mathbf{Y}) \cong \operatorname{Im} \mathcal{C}(-, 1_{\mathbf{Y}})$ and that α is an additive function on $\operatorname{mod} \mathcal{C}$ that is constant on isomorphic objects, we have $\varphi(\alpha)(1_{\mathbf{Y}}) = \varphi(\alpha)(f) + \varphi(\alpha)(g)$.

RM3: Let f be a morphism in \mathcal{C} . We have

$$\begin{aligned} \varphi(\alpha)(\Sigma_d f) &= \alpha(\operatorname{Im} \mathcal{C}(-, \Sigma_d f)) \\ &= \alpha(\operatorname{Im} \mathcal{C}(\Sigma_d^{-1}(-), f)) \\ &= \alpha(\Sigma_d \operatorname{Im} \mathcal{C}(-, f)) \\ &= \alpha(\operatorname{Im} \mathcal{C}(-, f)) \\ &= \varphi(\alpha)(f), \end{aligned}$$

where the second equality holds as $\operatorname{Im} \mathcal{C}(\Sigma_d^{-1}(-), f) \cong \Sigma_d \operatorname{Im} \mathcal{C}(-, f)$ and additive functions are equal on isomorphic objects and the fourth equality holds as α is Σ_d -invariant. \square

Proposition II.4.4. *The assignment $\rho_m \mapsto \psi(\rho_m)$ defined in Setup II.4.2, part 2, is a well-defined mapping $\mathcal{R}_m^{\mathcal{C}} \xrightarrow{\psi} \mathcal{X}^{\mathcal{C}}$.*

Proof. It is clear that the value $\psi(\rho_m)(M)$ is a nonnegative real number for each finitely presented \mathcal{C} -module M . Let M be a finitely presented \mathcal{C} -module such that $M \cong \operatorname{Im} \mathcal{C}(-, f)$, for some morphism f in \mathcal{C} . We have

$$\psi(\rho_m)(\Sigma_d M) = \rho_m(\Sigma_d(f)) = \rho_m(f) = \psi(M),$$

where the first equality holds as $\Sigma_d M \cong \Sigma_d \operatorname{Im} \mathcal{C}(-, f) \cong \operatorname{Im} \mathcal{C}(-, \Sigma_d(f))$ and the second equality holds by axiom RM3. \square

Proposition II.4.5. *The assignments defined in Setup II.4.2 satisfy the equality $\psi(\varphi(\alpha)) = \alpha$ for each additive function α on $\operatorname{mod} \mathcal{C}$.*

Proof. Let M be a finitely presented \mathcal{C} -module such that $M \cong \operatorname{Im} \mathcal{C}(-, f)$, for some morphism f in \mathcal{C} . We have $\psi(\varphi(\alpha))(M) = \varphi(\alpha)(f) = \alpha(\operatorname{Im} \mathcal{C}(-, f)) = \alpha(M)$. \square

Proposition II.4.6. *The assignment $\alpha \mapsto \varphi(\alpha)$ defined in Setup II.4.2, part 1, satisfies the equation $\varphi(\alpha_1 + \alpha_2) = \varphi(\alpha_1) + \varphi(\alpha_2)$, for additive functions α_1 and α_2 on $\text{mod } \mathcal{C}$.*

Proof. Let f be a morphism in \mathcal{C} . We have

$$\begin{aligned} \varphi(\alpha_1 + \alpha_2)(f) &= (\alpha_1 + \alpha_2)(\text{Im } \mathcal{C}(-, f)) \\ &= \alpha_1(\text{Im } \mathcal{C}(-, f)) + \alpha_2(\text{Im } \mathcal{C}(-, f)) \\ &= \varphi(\alpha_1)(f) + \varphi(\alpha_2)(f). \end{aligned} \quad \square$$

Proposition II.4.7. *The assignment $\rho_m \mapsto \psi(\rho_m)$ defined in Setup II.4.2, part 2, satisfies the equation $\psi(\rho_{m,1} + \rho_{m,2}) = \psi(\rho_{m,1}) + \psi(\rho_{m,2})$, for rank functions on morphisms $\rho_{m,1}$ and $\rho_{m,2}$ in \mathcal{C} .*

Proof. Let M be a finitely presented \mathcal{C} -module such that $M \cong \text{Im } \mathcal{C}(-, f)$, for some morphism f in \mathcal{C} . We have

$$\begin{aligned} \psi(\rho_{m,1} + \rho_{m,2})(M) &= (\rho_{m,1} + \rho_{m,2})(f) \\ &= \rho_{m,1}(f) + \rho_{m,2}(f) \\ &= \psi(\rho_{m,1})(M) + \psi(\rho_{m,2})(M). \end{aligned} \quad \square$$

We will now restrict our attention to a special class of $(d + 2)$ -angulated categories.

Construction II.4.8. The following is presented in [JKM23, Construction 2.2.13]: We work over a field k . Let d be a positive integer and let A be a basic finite dimensional algebra that is twisted $(d + 2)$ -periodic with respect to an algebra automorphism $A \xrightarrow{\sigma} A$. That is, the $(d + 2)$ -th syzygy of A considered as an A -bimodule is isomorphic to ${}_{\sigma^{-1}}A_1$, where ${}_{\sigma^{-1}}A_1$ is the A -bimodule twisted by σ^{-1} from the left. That is, ${}_{\sigma^{-1}}A_1$ denotes the A -bimodule whose underlying vector space is A and whose A -bimodule action is induced by the multiplication in A and is given by the left action $\mathbf{a}x \mapsto \sigma^{-1}(\mathbf{a})x$ and the right action $x\mathbf{b} \mapsto x\mathbf{b}$, for elements \mathbf{a}, \mathbf{b} and x in A . By [GSS03, Lemma 1.5] (and Remark before it), A is a selfinjective algebra. Therefore, there is an exact sequence of finite dimensional A -bimodules

$$\theta: 0 \rightarrow {}_{\sigma^{-1}}A_1 \rightarrow P_{d+1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0,$$

where each P_i is a projective-injective A -bimodule.

As σ is an algebra automorphism, the functors

$$\text{mod } A \xrightarrow{\Sigma_d} \text{mod } A$$

that is given by $M \mapsto M \otimes_A {}_{\sigma}A_1$ and

$$\text{mod } A \xrightarrow{\Sigma_d^{-1}} \text{mod } A$$

that is given by $M \mapsto M \otimes_{\mathcal{A}} \sigma^{-1} \mathcal{A}_1$ are exact and mutual quasi-inverses (see for example [Zim14, Lemma 1.10.10] and [SY11, Lemma 11.16]). Moreover, since for an element x in \mathcal{A} , there is an isomorphism of \mathcal{A} -modules $x\mathcal{A} \otimes_{\mathcal{A}} \sigma^{-1} \mathcal{A}_1 \cong \sigma(x)\mathcal{A}$ given by $x\mathbf{a} \otimes \mathbf{b} \mapsto \sigma(x)\mathbf{a}\mathbf{b}$, the functors Σ_d and Σ_d^{-1} restrict to mutual quasi-inverses on the subcategory $\text{proj } \mathcal{A}$.

We now define a class \diamond_{θ} of $(d + 2)$ -angles in $\text{proj } \mathcal{A}$ as follows: A sequence of finite dimensional projective \mathcal{A} -modules

$$\mathcal{X} := X_0 \xrightarrow{x_0} X_1 \xrightarrow{x_1} X_2 \rightarrow \cdots \rightarrow X_{d+1} \xrightarrow{x_{d+1}} \Sigma_d X_0$$

lies in \diamond_{θ} if the following conditions are satisfied:

C1. The augmented sequence

$$X_0 \xrightarrow{x_0} X_1 \rightarrow \cdots \rightarrow X_{d+1} \xrightarrow{x_{d+1}} \Sigma_d X_0 \xrightarrow{\Sigma_d x_0} \Sigma_d X_1$$

is exact in $\text{mod } \mathcal{A}$.

C2. As Σ_d is an equivalence, there is an exact sequence of finite dimensional \mathcal{A} -modules

$$0 \rightarrow \Sigma_d^{-1} C \xrightarrow{i} X_1 \xrightarrow{x_1} X_2 \rightarrow \cdots \rightarrow X_{d+1} \xrightarrow{x_{d+1}} \Sigma_d X_0 \xrightarrow{p} C \rightarrow 0, \quad (\text{II.5})$$

such that x_0 is equal to the composition $X_0 \cong \Sigma_d^{-1} \Sigma_d X_0 \xrightarrow{\Sigma_d^{-1} p} \Sigma_d^{-1} C \xrightarrow{i} X_1$ and where C denotes the cokernel of x_{d+1} . As the complex θ is contractible when considered as a complex of \mathcal{A}^{op} -modules (combine [SY11, Lemma 11.14] and [Jas16, dual of Proposition 2.6]), the complex

$$C \otimes_{\mathcal{A}} \theta \quad (\text{II.6})$$

of finite dimensional \mathcal{A} -modules is exact. We require the exact sequence II.5 and the exact sequence II.6 to be equivalent in $\text{Ext}_{\mathcal{A}}^{d+2}(C, \Sigma_d^{-1} C)$.

The collection of $(d + 2)$ -angles \diamond_{θ} defined above endows the pair $(\text{proj } \mathcal{A}, \Sigma_d)$ with the structure of a $(d + 2)$ -angulated category (see [Ami07, Theorem 8.1], [Lin19, Theorem 1.3] and [JKM23, Theorem 2.2.15]). Just as in [JKM23, Definition 2.2.17], we will refer to this $(d + 2)$ -angulated structure on $\text{proj } \mathcal{A}$ as the *Amiot-Lin $(d + 2)$ -angulated category structure*. The Amiot-Lin $(d + 2)$ -angulated categories cover a class of $(d + 2)$ -angulated categories that have been of interest, see for example, [OT12, Theorem 5.2] and [JKM23, Proposition 2.29 and Theorem 2.2.20].

Proposition II.4.9. *Let d be a positive odd integer. Consider Construction II.4.8 and endow $\text{proj } \mathcal{A}$ with the Amiot-Lin $(d + 2)$ -angulated category structure. Then the assignment $\rho_m \mapsto \psi(\rho_m)$ defined in Setup II.4.2, part 2, is a well-defined mapping $\mathcal{R}_m^{\text{proj } \mathcal{A}} \xrightarrow{\psi} \mathcal{A}^{\text{proj } \mathcal{A}}$.*

Proof. By Proposition II.4.4, it suffices to show that $\psi(\rho_m)$ is an additive function on $\text{mod}(\text{proj } \mathcal{A})$. By Lemma II.3.10 and axiom RM3, the assignment $\psi(\rho_m)$ is constant on isomorphism classes of objects in $\text{mod}(\text{proj } \mathcal{A})$. The Restricted Yoneda Embedding $Y_{\text{proj } \mathcal{A}}$ defined by $Y_{\text{proj } \mathcal{A}}(M) = \text{Hom}_{\mathcal{A}}(-, M)|_{\text{proj } \mathcal{A}}$ is an exact functor that also gives an equivalence $\text{mod } \mathcal{A} \xrightarrow{Y_{\text{proj } \mathcal{A}}} \text{mod}(\text{proj } \mathcal{A})$. Therefore, by Lemma II.2.1, it suffices to show that the restriction $\alpha := \psi(\rho_m)Y_{\text{proj } \mathcal{A}}$ is an additive function on $\text{mod } \mathcal{A}$. Given a finite dimensional \mathcal{A} -module M , we choose a morphism $P \xrightarrow{f} Q$ between finite dimensional projective \mathcal{A} -modules such that $M \cong \text{Im}(f)$ (this can be done as \mathcal{A} is selfinjective). For $P \xrightarrow{f} Q$ a morphism between finite dimensional projective \mathcal{A} -modules, we have the canonical image factorisation of f

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ & \searrow & \swarrow \\ & \text{Im}(f) & \end{array}$$

As the Restricted Yoneda Embedding $Y_{\text{proj } \mathcal{A}}$ is an exact functor, the diagram

$$\begin{array}{ccc} Y_{\text{proj } \mathcal{A}}(P) & \xrightarrow{Y_{\text{proj } \mathcal{A}}(f)} & Y_{\text{proj } \mathcal{A}}(Q) \\ & \searrow & \swarrow \\ & Y_{\text{proj } \mathcal{A}}(\text{Im}(f)) & \end{array}$$

is the canonical image factorisation of $Y_{\text{proj } \mathcal{A}}(f)$. Therefore, $Y_{\text{proj } \mathcal{A}}(\text{Im}(f)) \cong \text{Im}(Y_{\text{proj } \mathcal{A}}(f))$. This shows that the assignment α is given by

$$M \mapsto \rho_m(f),$$

for each finite dimensional \mathcal{A} -module M such that $M \cong \text{Im}(f)$, for f a morphism between finite dimensional projective \mathcal{A} -modules.

To this end, we let $M' \rightarrow M \rightarrow M''$ be a short exact sequence of finite dimensional \mathcal{A} -modules. As θ is contractible when viewed as a complex of \mathcal{A}^{op} -modules, then $M' \otimes_{\mathcal{A}} \theta \rightarrow M \otimes_{\mathcal{A}} \theta \rightarrow M'' \otimes_{\mathcal{A}} \theta$ is a short exact sequence of complexes of finite dimensional \mathcal{A} -modules which can be viewed as the following commutative diagram of finite dimensional \mathcal{A} -modules:

$$\begin{array}{cccccccccccc} 0 & \longrightarrow & \Sigma_d^{-1}M' & \xrightarrow{i'} & M' \otimes_{\mathcal{A}} P_{d+1} & \longrightarrow & \cdots & \longrightarrow & M' \otimes_{\mathcal{A}} P_1 & \xrightarrow{g'} & M' \otimes_{\mathcal{A}} P_0 & \xrightarrow{p'} & M' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Sigma_d^{-1}M & \xrightarrow{i} & M \otimes_{\mathcal{A}} P_{d+1} & \longrightarrow & \cdots & \longrightarrow & M \otimes_{\mathcal{A}} P_1 & \xrightarrow{g} & M \otimes_{\mathcal{A}} P_0 & \xrightarrow{p} & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Sigma_d^{-1}M'' & \xrightarrow{i''} & M'' \otimes_{\mathcal{A}} P_{d+1} & \longrightarrow & \cdots & \longrightarrow & M'' \otimes_{\mathcal{A}} P_1 & \xrightarrow{g''} & M'' \otimes_{\mathcal{A}} P_0 & \xrightarrow{p''} & M'' & \longrightarrow & 0. \end{array}$$

In particular, we get an exact sequence of finite dimensional \mathcal{A} -modules

$$M^* \otimes_{\mathcal{A}} \theta: 0 \rightarrow \Sigma_d^{-1} M^* \xrightarrow{i^*} M^* \otimes_{\mathcal{A}} P_{d+1} \rightarrow \cdots \rightarrow M^* \otimes_{\mathcal{A}} P_1 \xrightarrow{g^*} M^* \otimes_{\mathcal{A}} P_0 \xrightarrow{p^*} M^* \rightarrow 0,$$

where the symbol $*$ is a place holder for the symbol $'$, $''$ or no symbol (for example, M^* is either M' , M'' or M). We then construct the complexes of finite dimensional projective-injective \mathcal{A} -modules:

$$\delta^*: \Sigma_d^{-1}(M^* \otimes_{\mathcal{A}} P_0) \xrightarrow{f^*} M^* \otimes_{\mathcal{A}} P_{d+1} \rightarrow \cdots \rightarrow M^* \otimes_{\mathcal{A}} P_2 \rightarrow M^* \otimes_{\mathcal{A}} P_1 \xrightarrow{g^*} M^* \otimes_{\mathcal{A}} P_0,$$

where f^* is equal to the composition $\Sigma_d^{-1}(M^* \otimes_{\mathcal{A}} P_0) \xrightarrow{\Sigma_d^{-1}(p^*)} \Sigma_d^{-1}(M^*) \xrightarrow{i^*} M^* \otimes_{\mathcal{A}} P_{d+1}$. Consider the following diagram:

$$\begin{array}{ccc} M^* \otimes_{\mathcal{A}} P_0 & \xrightarrow{p^*} & M^* \\ \eta_{(M^* \otimes_{\mathcal{A}} P_0)} \downarrow & & \downarrow \eta_{M^*} \\ \Sigma_d \Sigma_d^{-1}(M^* \otimes_{\mathcal{A}} P_0) & \xrightarrow{\Sigma_d \Sigma_d^{-1} p^*} & \Sigma_d \Sigma_d^{-1}(M^*) \\ & \searrow \Sigma_d(f^*) & \downarrow \Sigma_d(i^*) \\ \Sigma_d \Sigma_d^{-1}(M^* \otimes_{\mathcal{A}} P_0) & \xrightarrow{\Sigma_d(f^*)} & \Sigma_d(M^* \otimes_{\mathcal{A}} P_{d+1}), \end{array} \quad (\text{II.7})$$

where $X \xrightarrow{\eta_X} \Sigma_d \Sigma_d^{-1} X$ denotes the X component of the natural isomorphism $\mathbb{1}_{\text{mod } \mathcal{A}} \xrightarrow{\eta} \Sigma_d \Sigma_d^{-1}$. Then the triangle in II.7 commutes by applying Σ_d to the composition of f^* mentioned above. The inner quadrilateral commutes as η is a natural isomorphism. Hence, the whole diagram II.7 is commutative. As the composition $p^* \eta_{(M^* \otimes_{\mathcal{A}} P_0)}^{-1}$ is an epimorphism and the composition $\Sigma_d(i^*) \eta_{M^*}$ is a monomorphism (note that Σ_d is exact), diagram II.7 gives us that $M^* \cong \text{Im} \left(\Sigma_d(i^*) \eta_{M^*} \circ p^* \eta_{(M^* \otimes_{\mathcal{A}} P_0)}^{-1} \right) = \text{Im}(\Sigma_d(f^*))$. By construction, the sequences δ^* satisfy conditions C1 and C2 in Construction II.4.8 and therefore, all lie in \diamond_{θ} . Each P_i is a projective-injective \mathcal{A} -bimodule and hence, each $M'' \otimes_{\mathcal{A}} P_i$ is a projective \mathcal{A} -module (see [SY11, Lemma 11.15]). In turn, each short exact sequence $M' \otimes_{\mathcal{A}} P_i \rightarrow M \otimes_{\mathcal{A}} P_i \rightarrow M'' \otimes_{\mathcal{A}} P_i$ is split as a sequence of \mathcal{A} -modules and hence, $M \otimes_{\mathcal{A}} P_i \cong (M' \otimes_{\mathcal{A}} P_i) \oplus (M'' \otimes_{\mathcal{A}} P_i)$. For ease of notation, we denote $M^* \otimes_{\mathcal{A}} P_i$ by X_i^* . As a result, we have

$$\begin{aligned} 2(\alpha(M') - \alpha(M) + \alpha(M'')) &= 2(\rho_m(\Sigma_d(f')) - \rho_m(\Sigma_d(f)) + \rho_m(\Sigma_d(f''))) \\ &= 2(\rho_m(f') - \rho_m(f) + \rho_m(f'')) \\ &= \sum_{i=0}^{d+1} (-1)^i \rho_m(1_{X_{d+1-i}^*}) - \sum_{i=0}^{d+1} (-1)^i \rho_m(1_{X_{d+1-i}}) \\ &\quad + \sum_{i=0}^{d+1} (-1)^i \rho_m(1_{X_{d+1-i}''}) \\ &= 0, \end{aligned}$$

where the second equality holds by axiom RM3, the third equality holds by Lemma II.3.10 and axiom RM3 and the last equality holds by axiom RM1, noting that $X_i \cong X'_i \oplus X''_i$. Hence, the assignment α is an additive function on $\text{mod } \mathcal{A}$ and therefore, $\psi(\rho_m)$ is an additive function on $\text{mod}(\text{proj } \mathcal{A})$. \square

Proposition II.4.10. *Consider Construction II.4.8 and endow $\text{proj } \mathcal{A}$ with the Amiot-Lin $(d+2)$ -angulated category structure. Then the assignments defined in Setup II.4.2 satisfy the equality $\varphi(\psi(\rho_m)) = \rho_m$, for each rank function ρ_m on morphisms in $\text{proj } \mathcal{A}$.*

Proof. Let f be a morphism in $\text{proj } \mathcal{A}$. We have

$$\varphi(\psi(\rho_m))(f) = \psi(\rho_m)(\text{Im Hom}_{\mathcal{A}}(-, f)|_{\text{proj } \mathcal{A}}) = \rho_m(f). \quad \square$$

Theorem II.4.11. *Let d be a positive odd integer. Consider Construction II.4.8 and endow $\text{proj } \mathcal{A}$ with the Amiot-Lin $(d+2)$ -angulated category structure. There is a bijective correspondence between the following:*

1. Σ_d -invariant additive functions α on $\text{mod}(\text{proj } \mathcal{A})$.
2. Rank functions ρ_m on morphisms in $\text{proj } \mathcal{A}$.

The following mutual inverses give the bijective correspondence:

- To a Σ_d -invariant additive function α on $\text{mod}(\text{proj } \mathcal{A})$, we assign $\varphi(\alpha)$ where

$$\varphi(\alpha)(f) = \alpha \left(\text{Im Hom}_{\mathcal{A}}(-, f)|_{\text{proj } \mathcal{A}} \right).$$

- To a rank function ρ_m on morphisms in $\text{proj } \mathcal{A}$, we assign $\psi(\rho_m)$ where

$$\psi(\rho_m)(M) = \rho_m(f)$$

for $M \cong \text{Im Hom}_{\mathcal{A}}(-, f)|_{\text{proj } \mathcal{A}}$.

Moreover, the bijective correspondence restricts to a bijection between the following:

- 1'. Σ_d -invariant integral additive functions α on $\text{mod}(\text{proj } \mathcal{A})$.
- 2'. Integral rank functions ρ_m on morphisms in $\text{proj } \mathcal{A}$.

Furthermore, the bijective correspondence restricts to a bijection between the following:

- 1''. Σ_d -irreducible additive functions α on $\text{mod}(\text{proj } \mathcal{A})$.
- 2''. Irreducible rank functions ρ_m on morphisms in $\text{proj } \mathcal{A}$.

Proof. Correspondence between 1 and 2: By Proposition II.4.3, Proposition II.4.9, Proposition II.4.10 and Proposition II.4.5, the mappings $\mathcal{A}^{\text{proj } \mathcal{A}} \xrightarrow{\varphi} \mathcal{R}_m^{\text{proj } \mathcal{A}}$ and $\mathcal{R}_m^{\text{proj } \mathcal{A}} \xrightarrow{\psi} \mathcal{A}^{\text{proj } \mathcal{A}}$ are mutual inverses.

Correspondence between 1' and 2': This is clear.

Correspondence between 1'' and 2'': Let α be a Σ_d -irreducible additive function on $\text{mod}(\text{proj } \mathcal{A})$ and assume that $\varphi(\alpha) = \rho_{m,1} + \rho_{m,2}$, where $\rho_{m,1}$ and $\rho_{m,2}$ are integral rank functions on morphisms in $\text{proj } \mathcal{A}$. Then $\alpha = \psi(\varphi(\alpha)) = \psi(\rho_{m,1}) + \psi(\rho_{m,2})$, where the first equality holds by Proposition II.4.5 and the second equality holds by Proposition II.4.7. As both $\psi(\rho_{m,1})$ and $\psi(\rho_{m,2})$ are integral (by the previous correspondence) and α is Σ_d -irreducible, then $\psi(\rho_{m,1}) = 0$ or $\psi(\rho_{m,2}) = 0$ and hence, $\rho_{m,1} = \varphi(\psi(\rho_{m,1})) = 0$ or $\rho_{m,2} = \varphi(\psi(\rho_{m,2})) = 0$, where we used Proposition II.4.10 and Proposition II.4.6. Therefore, $\varphi(\alpha)$ is an irreducible rank function.

Conversely, Let ρ_m be an irreducible rank function on morphisms in $\text{proj } \mathcal{A}$ and assume that $\psi(\rho_m) = \alpha_1 + \alpha_2$, where α_1 and α_2 are Σ_d -invariant integral additive functions on $\text{mod}(\text{proj } \mathcal{A})$. Similar to before, we have $\rho_m = \varphi(\psi(\rho_m)) = \varphi(\alpha_1) + \varphi(\alpha_2)$, where the first equality holds by Proposition II.4.10 and the second equality holds by Proposition II.4.6. As both $\varphi(\alpha_1)$ and $\varphi(\alpha_2)$ are Σ_d -invariant and integral (by the previous correspondence) and ρ_m is irreducible, then $\varphi(\alpha_1) = 0$ or $\varphi(\alpha_2) = 0$ and hence, $\alpha_1 = \psi(\varphi(\alpha_1)) = 0$ or $\alpha_2 = \psi(\varphi(\alpha_2)) = 0$, where we used Proposition II.4.5 and Proposition II.4.7. Therefore, $\psi(\rho_m)$ is a Σ_d -irreducible additive function. \square

Example II.4.12. Let d be a positive odd integer. Consider Construction II.4.8 and endow $\text{proj } \mathcal{A}$ with the Amiot-Lin $(d + 2)$ -angulated category structure. Assigning to a finite dimensional \mathcal{A} -module M its composition length $\mathfrak{l}(M)$ defines an additive function on $\text{mod } \mathcal{A} \simeq \text{mod}(\text{proj } \mathcal{A})$ with values in the integers. As an autoequivalence on $\text{mod } \mathcal{A}$ will preserve the composition length of a given finite dimensional \mathcal{A} -module, the assignment $M \mapsto \mathfrak{l}(M)$ is an Σ_d -invariant integral additive function. Passing this assignment under the correspondence in Theorem II.4.11, defines the integral rank function on morphisms in $\text{proj } \mathcal{A}$ given by $f \mapsto \mathfrak{l}(\text{Im}(f))$.

Two decomposition theorems

The proof of the following theorem follows the proof of [Con+24, Theorem 4.2].

Theorem II.4.13. *Let \mathcal{C} be an essentially small $(d + 2)$ -angulated category. Then every Σ_d -invariant integral additive function on $\text{mod } \mathcal{C}$ can be decomposed uniquely as a locally finite sum of Σ_d -irreducible invariant additive functions on $\text{mod } \mathcal{C}$.*

Proof. Let α be a Σ_d -invariant integral additive function on $\text{mod } \mathcal{C}$. As $\text{mod } \mathcal{C}$ is a locally finitely presented category (see [Cra94b, Theorem on page 1645]) therefore, by work of

[Cra94a; Cra94b] (see [Con+24, Theorem 2.8]), the additive function α decomposes into a locally finite sum

$$\alpha = \sum_{i \in I} \alpha_i, \quad (\text{II.8})$$

where each α_i is an irreducible integral additive function on $\text{mod } \mathcal{C}$. We have

$$\sum_{i \in I} \alpha_i = \alpha = \alpha_{\Sigma_d} = \sum_{i \in I} (\alpha_i \Sigma_d),$$

where the second equality follows as α is Σ_d -invariant. By [Con+24, Lemma 2.7], each additive function $\alpha_i \Sigma_d$ is irreducible since Σ_d^{-1} is in particular essentially surjective. By the uniqueness of decomposition II.8, there exists a bijective function $I \xrightarrow{f} I$, such that for each i in I we have $\alpha_i \Sigma_d = \alpha_{f(i)}$. Given an integral additive function β on $\text{mod } \mathcal{C}$, the Σ_d -orbit of β is defined to be the set $\text{orb}(\beta) = \{\beta \Sigma_d^n \mid \text{for } n \text{ an integer}\}$, where Σ_d^n denotes the n -fold composition of Σ_d when $n \geq 0$ and the n -fold composition of Σ_d^{-1} when $n < 0$. Fix an index j in I . As $\alpha_j \Sigma_d^n = \alpha_{f^n(j)}$ for all integers n , every element in $\text{orb}(\alpha_j)$ appears as a summand in the decomposition $\alpha = \sum_{i \in I} \alpha_i$ and therefore, we can write $\alpha = \beta_j + \beta$, where

$$\beta_j = \sum_{\gamma \in \text{orb}(\alpha_j)} \gamma$$

is a locally finite sum that is unique since II.8 was.

Appealing to the axiom of choice, there exists a subset K of I such that we have a disjoint union decomposition $\{\alpha_i\}_{i \in I} = \coprod_{k \in K} \text{orb}(\alpha_k)$. Noting there may exist indices i and j in I such that $\alpha_i = \alpha_j$, we can write

$$\alpha = \sum_{k \in K} \sum_{\substack{i \in I \\ \alpha_i = \alpha_k}} \beta_k.$$

By definition of $\text{orb}(\alpha_k)$ for each k , the additive function β_k is nonzero, integral and Σ_d -invariant. We show that β_k is in fact Σ_d -irreducible. Assume

$$\beta_k = \delta_1 + \delta_2, \quad (\text{II.9})$$

with δ_1 and δ_2 integral Σ_d -invariant functions on $\text{mod } \mathcal{C}$. We consider the decomposition of δ_1 and δ_2 into locally finite sums of irreducible integral additive functions on $\text{mod } \mathcal{C}$. By the uniqueness of the decomposition $\beta_k = \sum_{\gamma \in \text{orb}(\alpha_k)} \gamma$, the irreducible summands on the left-hand side and the right-hand side of II.9 must coincide. But by construction, all the irreducible summands on the left-hand side of II.9 are in the same Σ_d -orbit, which forces either δ_1 or δ_2 to be zero. \square

Theorem II.4.14. *Let d be a positive odd integer. Consider Construction II.4.8 and endow $\text{proj } \mathcal{A}$ with the Amiot-Lin $(d+2)$ -angulated category structure. Then every integral rank function on morphisms in $\text{proj } \mathcal{A}$ can be decomposed uniquely as a locally finite sum of irreducible rank functions on morphisms in $\text{proj } \mathcal{A}$.*

Proof. Combine Theorem II.4.11 and Theorem II.4.13. □

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Differential modules: a perspective on Bass' question

Abstract

Guided by the Q-shaped derived category framework introduced by Holm and Jørgensen, we provide a differential module analogue of a classical result that characterises when a finitely generated module over a local commutative noetherian ring has finite injective dimension. As an application, we characterise local Cohen–Macaulay rings using the homological algebra of differential modules.

III.1 Introduction

A differential module is a module equipped with a square-zero endomorphism called a differential. The notion first appeared in the foundational book by Cartan and Eilenberg [CE56, Chp. IV(1)], where differential modules served primarily as a stepping stone toward the study of complexes. A complex can be viewed as a differential module endowed with a grading such that the differential has degree one. Accordingly, Cartan and Eilenberg defined cocycles, boundaries, and cohomology for differential modules before specialising these notions to graded settings.

In recent years, differential modules have gained attention in commutative algebra and representation theory (see, for example, [ABI07; Rou06; Wei15; RZ17; Sta17; BE22; HJ24a]).

This paper is driven by the following three questions:

1. Is there a suitable theory of *homological algebra without grading*?
2. How would such a theory compare to the homological algebra of complexes?
3. Could this theory detect key properties of rings?

For question one: an affirmative answer is implicit in recent work by Holm and Jørgensen, who introduced the Q-shaped derived category [HJ22], a generalisation of the classical derived category. Their framework enables systematic homological constructions across various Q-shaped settings—one of which is the category of differential modules. We like to think of this as a theory of *homological algebra without grading*.

For example, (minimal) injective resolutions of modules appear naturally as complexes and motivate the notion of (minimal) semi-injective complexes. However, these resolutions do not directly suggest what a (minimal) semi-injective differential module should be. The Q-shaped framework developed in [HJ22; HJ24a] addresses this gap and provides access to such notions.

Moreover, differential modules admitting a projective flag—as considered in [ABI07, 2.1 and 2.2]—turn out to be a special case of the semi-projective differential modules arising naturally within the Q-shaped theory (see Example III.3.3). We therefore situate the category of differential modules within the Q-shaped framework.

For question two: we explore the adjunction triple $(\pi_!, \pi^*, \pi_*)$ (see (III.3)) between the category of differential modules and the category of complexes. The adjunction pair $(\pi_!, \pi^*)$ was previously considered in [ABI07, pg. 7]. It behaves well on projective-minded objects—reflecting the authors' focus on this adjunction pair and differential modules admitting a projective flag.

Recently in [HJ24a], a theory of minimal semi-injective Q-shaped objects was introduced. This motivates considering the remaining adjunction pair (π^*, π_*) into the picture, as it interacts nicely with injective-minded objects. We do so in this paper.

For question three: homological algebra underwent a major breakthrough when it was used to characterise when a local commutative noetherian ring is regular. This was done by Serre [Ser56] and independently by Auslander and Buchsbaum [AB56]. Later, Bass asked whether the existence of a finitely generated module of finite injective dimension suffices to characterise local Cohen–Macaulay rings (see [Bas63, Rmk. on pg. 14]). Using classical homological methods, this question was resolved—first in positive equicharacteristic by Peskine and Szpiro [PS73], and then in full generality by Roberts [Rob87; Rob98].

Inspired by these works, we offer a differential modules analogue of this classical result. The key insight is as follows:

Consider a finitely generated module over an arbitrary ring. Its injective dimension is fully captured by its minimal semi-injective resolution as a complex: this resolution has finite support (i.e. bounded or finite length) if and only if the module has finite injective dimension.

For differential modules, the situation is more subtle: since a differential module always has finite support, the standard finiteness condition does not suffice. However, in the setting of a local noetherian ring, we observe that a minimal semi-injective resolution has finite support if and only if the injective envelope of the residue field appears only finitely often. We prove that this characterisation remains valid in the category of differential modules.

The main theorem

Our main result gives a differential module analogue of a classical criterion for finitely generated modules of finite injective dimension. The result traces back to [Bas63, Lem. 2.7] (a complete modern reference is [CFH24, Thm. 16.4.8]). As a corollary, we provide a characterisation of local Cohen–Macaulay rings using differential modules.

Theorem I (Theorem III.5.2 and Corollary III.5.3). *Let $(R, \mathfrak{m}, \mathfrak{k})$ be a local commutative noetherian ring and let M be a finitely generated R -module. Consider the differential R -module $(M, 0)$ with zero differential and choose a minimal semi-injective resolution $(M, 0) \rightarrow (I, \mathfrak{d}_1)$. Then the following statements are equivalent:*

1. M has finite injective dimension.
2. The number of direct summands of I isomorphic to the injective envelope of the residue field \mathfrak{k} is finite.

Furthermore, R is Cohen–Macaulay if and only if there exists a finitely generated R -module M satisfying part 1 or part 2.

Structure of the paper and the global setup

Structure of the paper. In Section III.2, we prove a general lemma concerning the preservation of homologically defined objects under adjunctions arising in the Q -shaped framework. Section III.3 develops an adjunction triple between the category of differential modules and the category of complexes. We then apply the previous lemma to study the behaviour of this adjunction triple. In Section III.4, we define an ad hoc tensor product and internal Hom in the category of differential modules and study their interaction with the adjunctions. Section III.5 specialises to the local noetherian case and uses the preceding theory to prove our main theorem. As a corollary, we obtain a differential module version of a classical result in commutative algebra. Section III.6 establishes conditions under which the minimality of semi-injective objects is preserved under the functor π_* from Section III.3. Finally, in Section III.7, we provide examples illustrating the developed theory.

Global Setup. We work in the following setup:

- \mathbb{k} is a hereditary commutative noetherian ring (for example, the ring of integers \mathbb{Z} or a field \mathbb{k}). Hereafter, all tensor products with no subscripts are understood to be taken over this ground ring \mathbb{k} .
- A \mathbb{k} -category is a category enriched over the closed symmetric monoidal category of \mathbb{k} -modules.

For a \mathbb{k} -algebra A , we consider the following categories:

- $\text{Mod } A$ is the \mathbb{k} -category of right A -modules.
- $\text{mod } A$ is the full subcategory of $\text{Mod } A$ consisting of the finitely generated A -modules.
- $\text{Ch } A$ is the \mathbb{k} -category of (cohomologically graded) complexes of right A -modules.
- $\text{Dif } A$ is the \mathbb{k} -category of differential right A -modules. Its objects are pairs (X, d_X) consisting of an object X in $\text{Mod } A$ and an endomorphism $X \xrightarrow{d_X} X$ in $\text{Mod } A$ such that the equation $d_X d_X = 0$ holds. We refer to such a morphism d_X a *differential*. The Hom \mathbb{k} -module $\text{Hom}_{\text{Dif } A}((X, d_X), (X', d_{X'}))$ is the submodule of $\text{Hom}_{\text{Mod } A}(X, X')$ consisting of those morphisms f such that the equation $d_{X'} f = f d_X$ holds. In this case, we say f commutes or intertwines with the differentials. We sometimes represent a morphism $(X, d_X) \xrightarrow{f} (X', d_{X'})$ in $\text{Dif } A$ as a morphism between the corresponding differentials. That is, a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{d_X} & X \\ f \downarrow & & \downarrow f \\ X' & \xrightarrow{d_{X'}} & X' \end{array}$$

This paper uses the theory of semi-injective and semi-projective complexes. We suggest [CFH24, Sec. 5.2 and Sec. 5.3] as a reference.

III.2 A lemma

The following lemma concerns abelian \mathbb{k} -categories of the following type: given a \mathbb{k} -algebra A and a small \mathbb{k} -category \mathcal{C} , we have the abelian \mathbb{k} -category

$$\text{Mod}(\mathcal{C} \otimes A) := \text{Fun}(\mathcal{C}^{\text{op}}, \text{Mod } A)$$

of \mathbb{k} -functors from \mathcal{C}^{op} to $\text{Mod } A$ (the notation adopted in [HJ22] for the \mathbb{k} -category $\text{Mod}(\mathcal{C} \otimes A)$ is $\text{Mod}_{\mathcal{C}, A}$). It is a Grothendieck abelian category (see [HJ22, Prop. 3.12])

with all set indexed limits and set indexed colimits (see [Ste75, Ch. V, preamble and Cor. X.4.4]). It has enough projective objects (see [HJ22, Prop. 3.12(a)]), enough injective objects (see [Gro57, Thm. 1.10.1]) and a projective generator (see [HJ22, Prop. 3.12(a)]). Its Hom \mathbb{k} -functor is denoted by $\text{Hom}_{\mathcal{C} \otimes \mathcal{A}}$ and the right derived functor of $\text{Hom}_{\mathcal{C} \otimes \mathcal{A}}$ is denoted by $\text{Ext}_{\mathcal{C} \otimes \mathcal{A}}^1$. When \mathcal{C} satisfies the extra conditions given in [HJ24b, Setup 2.9], the abelian \mathbb{k} -category $\text{Mod}(\mathcal{C} \otimes \mathcal{A})$ permits the following notions:

- The notion of an ‘exact’ object (see [HJ22, Def. 4.1 and Thm. 7.1]). This generalises the acyclic (or exact) complexes.

The exact objects give rise to the following two classes of objects (which were also considered in [HJ22]):

- An object P in $\text{Mod}(\mathcal{C} \otimes \mathcal{A})$ is *semi-projective* if $\text{Ext}_{\mathcal{C} \otimes \mathcal{A}}^1(P, E) = 0$ for each exact object E in $\text{Mod}(\mathcal{C} \otimes \mathcal{A})$.
- An object I in $\text{Mod}(\mathcal{C} \otimes \mathcal{A})$ is *semi-injective* if $\text{Ext}_{\mathcal{C} \otimes \mathcal{A}}^1(E, I) = 0$ for each exact object E in $\text{Mod}(\mathcal{C} \otimes \mathcal{A})$.

To parse the following lemma, the reader is invited to replace the categories involved with the category of complexes where the exact objects are the acyclic complexes.

Lemma III.2.1. *Let \mathcal{A} and \mathcal{B} be \mathbb{k} -algebras and \mathcal{Q} and Γ be small \mathbb{k} -categories that satisfy [HJ24b, Setup 2.9]. Consider an adjunction pair (L, R) as follows:*

$$\text{Mod}(\mathcal{Q} \otimes \mathcal{A}) \xrightleftharpoons[R]{L} \text{Mod}(\Gamma \otimes \mathcal{B}),$$

and assume L and R are exact functors. The following statements hold:

1. If I is an injective object in $\text{Mod}(\Gamma \otimes \mathcal{B})$, then RI is an injective object in $\text{Mod}(\mathcal{Q} \otimes \mathcal{A})$.
2. If P is a projective object in $\text{Mod}(\mathcal{Q} \otimes \mathcal{A})$, then LP is a projective object in $\text{Mod}(\Gamma \otimes \mathcal{B})$.
3. Let X be an object in $\text{Mod}(\mathcal{Q} \otimes \mathcal{A})$ and let Y be an object in $\text{Mod}(\Gamma \otimes \mathcal{B})$. There is a natural isomorphism of \mathbb{k} -modules

$$\text{Ext}_{\Gamma \otimes \mathcal{B}}^1(LX, Y) \cong \text{Ext}_{\mathcal{Q} \otimes \mathcal{A}}^1(X, RY).$$

4. Suppose LE is an exact object in $\text{Mod}(\Gamma \otimes \mathcal{B})$ whenever E is an exact object in $\text{Mod}(\mathcal{Q} \otimes \mathcal{A})$. Then RI is a semi-injective object in $\text{Mod}(\mathcal{Q} \otimes \mathcal{A})$ whenever I is a semi-injective object in $\text{Mod}(\Gamma \otimes \mathcal{B})$.
5. Suppose RE is an exact object in $\text{Mod}(\mathcal{Q} \otimes \mathcal{A})$ whenever E is an exact object in $\text{Mod}(\Gamma \otimes \mathcal{B})$. Then LP is a semi-projective object in $\text{Mod}(\Gamma \otimes \mathcal{B})$ whenever P is a semi-projective object in $\text{Mod}(\mathcal{Q} \otimes \mathcal{A})$.

Proof. Part 1. The functor $\mathrm{Hom}_{\Gamma \otimes B}(\mathbf{L}(-), \mathbf{I})$ is exact as, by the assumptions, it is the composition of exact functors. The result follows by the adjunction isomorphism $\mathrm{Hom}_{\Gamma \otimes B}(\mathbf{L}(-), \mathbf{I}) \cong \mathrm{Hom}_{Q \otimes A}(-, \mathbf{R}\mathbf{I})$.

Part 2. Follows dually to part 1.

Part 3. Let $I^0 \rightarrow I^1 \rightarrow I^2$ be a truncated injective resolution of Y . By part 1 and as \mathbf{R} is an exact functor, the sequence $\mathbf{R}(I^0) \rightarrow \mathbf{R}(I^1) \rightarrow \mathbf{R}(I^2)$ is a truncated injective resolution of $\mathbf{R}Y$. We apply the functor $\mathrm{Hom}_{Q \otimes A}(X, -)$ to this shorted injective resolution and use the adjunction isomorphism of the adjunction pair (\mathbf{L}, \mathbf{R}) to get the following commutative diagram of \mathbb{k} -modules

$$\begin{array}{ccccc} \mathrm{Hom}_{\Gamma \otimes B}(\mathbf{L}X, I^0) & \longrightarrow & \mathrm{Hom}_{\Gamma \otimes B}(\mathbf{L}X, I^1) & \longrightarrow & \mathrm{Hom}_{\Gamma \otimes B}(\mathbf{L}X, I^2) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \mathrm{Hom}_{Q \otimes A}(X, \mathbf{R}(I^0)) & \longrightarrow & \mathrm{Hom}_{Q \otimes A}(X, \mathbf{R}(I^1)) & \longrightarrow & \mathrm{Hom}_{Q \otimes A}(X, \mathbf{R}(I^2)), \end{array}$$

where the vertical arrows are the adjunction isomorphisms. As any choice of injective resolution computes Ext , the claim follows by taking cohomology.

Part 4. Let I be a semi-injective object in $\mathrm{Mod}(\Gamma \otimes B)$. For each exact object E in $\mathrm{Mod}(Q \otimes A)$, the object $\mathbf{L}E$ is exact by assumption and therefore the \mathbb{k} -module $\mathrm{Ext}_{\Gamma \otimes B}^1(\mathbf{L}E, I)$ is zero. The result follows by the isomorphism found in part 3.

Part 5. Follows dually to part 4. □

III.3 The expansion and (co)compression functors

Consider a \mathbb{k} -algebra A . Let Γ be the \mathbb{k} -category of paths of the quiver

$$\cdots \longrightarrow 1 \longrightarrow 0 \longrightarrow -1 \longrightarrow \cdots \quad (\text{III.1})$$

modulo the relation that the concatenation of any two consecutive arrows is zero. The categories $\mathrm{Mod}(\Gamma \otimes A)$ and $\mathrm{Ch}A$ are isomorphic and we will henceforth identify them. Similarly, let Q be the \mathbb{k} -category of paths of the quiver

$$\begin{array}{c} \circlearrowleft \\ * \end{array} \quad (\text{III.2})$$

modulo the relation that the concatenation of the unique loop with itself is zero. The categories $\mathrm{Mod}(Q \otimes A)$ and $\mathrm{Dif}A$ are isomorphic and we will henceforth identify them. Let $\Gamma \xrightarrow{\pi} Q$ be the \mathbb{k} -functor induced by the unique morphism of quivers from the quiver (III.1) to the quiver (III.2). By the theory of Kan extensions (see [Mac98, Sec. X.3, Cor. 2 and its dual]), there is an adjunction triple $(\pi_!, \pi^*, \pi_*)$ as follows:

$$\begin{array}{ccc} & \xleftarrow{\pi_!} & \\ \mathrm{Dif}A & \xrightarrow{\pi^*} & \mathrm{Ch}A, \\ & \xleftarrow{\pi_*} & \end{array} \quad (\text{III.3})$$

where π^* is given by precomposition by π . Inspired by the terminology in [ABI07], where the top two functors appear, we call π^* the *expansion functor*, $\pi_!$ the *compression functor* and π_* the *cocompression functor*. For convenience, the explicit actions of these functors are given below.

Expansion functor. For an object (X, d_X) in DifA , the expansion $\pi^*(X, d_X)$ is the object in ChA given by the diagram

$$\dots \rightarrow X \xrightarrow{d_X} X \xrightarrow{d_X} X \rightarrow \dots$$

and for a morphism $(X, d_X) \xrightarrow{f} (X', d_{X'})$ in DifA , the expansion $\pi^*(X, d_X) \xrightarrow{\pi^*f} \pi^*(X', d_{X'})$ is the morphism in ChA given by the commutative diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & X & \xrightarrow{d_X} & X & \xrightarrow{d_X} & X \longrightarrow \dots \\ & & \downarrow f & & \downarrow f & & \downarrow f \\ \dots & \longrightarrow & X' & \xrightarrow{d_{X'}} & X' & \xrightarrow{d_{X'}} & X' \longrightarrow \dots \end{array}$$

Compression functor. For an object

$$Y = \dots \rightarrow Y^{i-1} \xrightarrow{d_Y^{i-1}} Y^i \xrightarrow{d_Y^i} Y^{i+1} \rightarrow \dots$$

in ChA , the compression $\pi_!Y$ is the object in DifA given by the pair

$$\left(\coprod_{i \in \mathbb{Z}} Y^i, \coprod_{i \in \mathbb{Z}} d_Y^i \right),$$

where $\coprod_{i \in \mathbb{Z}} d_Y^i$ is the canonical morphism induced by the compositions $Y^j \xrightarrow{d_Y^j} Y^{j+1} \rightarrow \coprod_{i \in \mathbb{Z}} Y^i$, for each integer j , where $Y^{j+1} \rightarrow \coprod_{i \in \mathbb{Z}} Y^i$ are the canonical inclusions. For a morphism $Y \xrightarrow{f} Z$:

$$\begin{array}{ccccccc} \dots & \longrightarrow & Y^{i-1} & \xrightarrow{d_Y^{i-1}} & Y^i & \xrightarrow{d_Y^i} & Y^{i+1} \longrightarrow \dots \\ & & \downarrow f^{i-1} & & \downarrow f^i & & \downarrow f^{i+1} \\ \dots & \longrightarrow & Z^{i-1} & \xrightarrow{d_Z^{i-1}} & Z^i & \xrightarrow{d_Z^i} & Z^{i+1} \longrightarrow \dots \end{array}$$

in ChA , the compression $\pi_!Y \xrightarrow{\pi_!f} \pi_!Z$ is the canonical morphism $\coprod_{i \in \mathbb{Z}} f^i$ in DifA , induced by the compositions $Y^j \xrightarrow{f^j} Z^j \rightarrow \coprod_{i \in \mathbb{Z}} Z^i$, where $Z^j \rightarrow \coprod_{i \in \mathbb{Z}} Z^i$ are the canonical inclusion.

Cocompression functor. For an object

$$Y = \dots \rightarrow Y^{i-1} \xrightarrow{d_Y^{i-1}} Y^i \xrightarrow{d_Y^i} Y^{i+1} \rightarrow \dots$$

in ChA , the cocompression π_*Y is the object in DifA given by the pair

$$\left(\prod_{i \in \mathbb{Z}} Y^i, \prod_{i \in \mathbb{Z}} d_Y^i \right),$$

where $\prod_{i \in \mathbb{Z}} d_Y^i$ is the canonical morphism induced by the compositions $\prod_{i \in \mathbb{Z}} Y^i \rightarrow Y^j \xrightarrow{d_Y^j} Y^{j+1}$, for each integer j , where $\prod_{i \in \mathbb{Z}} Y^i \rightarrow Y^j$ are the canonical projections. For a morphism $Y \xrightarrow{f} Z$:

$$\begin{array}{ccccccc} \dots & \longrightarrow & Y^{i-1} & \xrightarrow{d_Y^{i-1}} & Y^i & \xrightarrow{d_Y^i} & Y^{i+1} & \longrightarrow & \dots \\ & & \downarrow f^{i-1} & & \downarrow f^i & & \downarrow f^{i+1} & & \\ \dots & \longrightarrow & Z^{i-1} & \xrightarrow{d_Z^{i-1}} & Z^i & \xrightarrow{d_Z^i} & Z^{i+1} & \longrightarrow & \dots \end{array}$$

in ChA , the cocompression $\pi_*Y \xrightarrow{\pi_*f} \pi_*Z$ is the canonical morphism $\prod_{i \in \mathbb{Z}} f^i$ in DifA , induced by the morphism $\prod_{i \in \mathbb{Z}} Y^i \rightarrow Y^i \xrightarrow{f^i} Z^i$, where $\prod_{i \in \mathbb{Z}} Y^i \rightarrow Y^j$ are the canonical projections.

We now recall some additional functors that will be used later (see [HJ22, Cor. 3.9 and Prop. 7.15]).

Stalk functors. These are functors

$$\text{DifA} \xleftarrow{S_*} \text{Mod A} \xrightarrow{S_0} \text{ChA},$$

where $S_*M = (M, 0)$ and where S_0 identifies an A -module with its stalk complex in degree 0.

Evaluation functors and their left adjoint. There are functors

$$\text{DifA} \xrightarrow{E_*} \text{Mod A} \xleftarrow{E_i} \text{ChA},$$

where the index i ranges over the integers. The evaluations have object assignments given by $E_*(X, d_X) = X$ and $E_i(X^\bullet) = X^i$. The evaluations admit left adjoints

$$\text{DifA} \xleftarrow{F_*} \text{Mod A} \xrightarrow{F_i} \text{ChA},$$

with object assignments given by

$$F_*M = \left(M \oplus M, \begin{pmatrix} 0 & 0 \\ \text{id}_M & 0 \end{pmatrix} \right) \quad \text{and} \quad F_iM = \dots \rightarrow 0 \rightarrow M \xrightarrow{\text{id}_M} M \rightarrow 0 \rightarrow \dots,$$

where F_iM is concentrated in cohomological degrees i and $i + 1$.

Cohomology functors. These are functors

$$\text{Dif } \mathbf{A} \xrightarrow{H^*} \text{Mod } \mathbf{A} \xleftarrow{H^i} \text{Ch } \mathbf{A},$$

where the index i ranges over the integers. The cohomology functor H^* has object assignments given by $H^*(X, d_X) = \text{Ker } d_X / \text{Im } d_X$ and H^i is the usual i -th cohomology of an object in $\text{Ch } \mathbf{A}$.

We now address some definitions in the category $\text{Dif } \mathbf{A}$ (compare these to the analogous definitions in $\text{Ch } \mathbf{A}$).

Acyclic differential modules and quasi-isomorphisms. An object (X, d_X) in $\text{Dif } \mathbf{A}$ is *acyclic* if $H^*(X, d_X) = 0$. A morphism $(X, d_X) \xrightarrow{f} (Y, d_Y)$ in $\text{Dif } \mathbf{A}$ is a *quasi-isomorphism* if the obvious induced morphism $H^*(f)$ is an isomorphism in $\text{Mod } \mathbf{A}$.

Contractible differential modules. An object (X, d_X) in $\text{Dif } \mathbf{A}$ is *contractible* if there exists an endomorphism $X \xrightarrow{h} X$ in $\text{Mod } \mathbf{A}$ such that the equation $\text{id}_X = h d_X + d_X h$ holds. We call h a *contracting homotopy* of (X, d_X) .

Remark III.3.1. In [RZ17], the notion of a morphism in $\text{Dif } \mathbf{A}$ being homotopic to zero was considered. In their terminology, an object is contractible if its identity morphism is homotopic to zero. In [ABI07, 1.5 and Prop. 1.8], alternative definitions of a contractible object are considered. Proposition III.3.6 shows that all these definitions are equivalent.

Minimal semi-injective resolutions. An object (I, d_I) in $\text{Dif } \mathbf{A}$ is *semi-injective* if for each acyclic object (E, d_E) in $\text{Dif } \mathbf{A}$, we have $\text{Ext}_{\text{Dif } \mathbf{A}}^1((E, d_E), (I, d_I)) = 0$ (this is a special case of the semi-injective objects defined in section III.2). A *minimal semi-injective object* in $\text{Dif } \mathbf{A}$ is a semi-injective object whose only injective subobject is the zero object. A *semi-injective resolution* of an object (X, d_X) in $\text{Dif } \mathbf{A}$ is a quasi-isomorphism $(X, d_X) \rightarrow (I, d_I)$ with (I, d_I) a semi-injective object in $\text{Dif } \mathbf{A}$. If, in this case, (I, d_I) is a minimal semi-injective object, then we say $(X, d_X) \rightarrow (I, d_I)$ is a *minimal semi-injective resolution* of (X, d_X) in $\text{Dif } \mathbf{A}$.

Remark III.3.2. The above definitions are just special cases of the definitions found in [HJ24a, Def. A] applied to the category $\text{Dif } \mathbf{A}$ and hence by [HJ24a, Thm. 2.1], we have access to equivalent characterisations of a minimal semi-injective object in $\text{Dif } \mathbf{A}$.

Semi-projective objects. An object (P, d_P) in $\text{Dif } \mathbf{A}$ is *semi-projective* if for each acyclic object (E, d_E) in $\text{Dif } \mathbf{A}$, we have $\text{Ext}_{\text{Dif } \mathbf{A}}^1((P, d_P), (E, d_E)) = 0$ (this is a special case of the semi-injective objects defined in section III.2).

Example III.3.3 (Projective flags). In this example, we will suppress the differential of objects in $\text{Dif}A$. Let X be an object in $\text{Dif}A$. We say X admits a projective flag (see [ABI07, 2.1 and 2.2]) if there is a nested sequence

$$X^0 \subseteq X^1 \subseteq \dots \subseteq X^n \subseteq \dots$$

of subobjects of X in $\text{Dif}A$ with the property that $X^0 = 0$, $X = \bigcup_{i \geq 0} X^i$ and for every $i \geq 0$, the quotients X^{i+1}/X^i are isomorphic to S_*P , for P a projective object in $\text{Mod}A$. Every object in $\text{Dif}A$ that admits a projective flag lies in the class of objects in $\text{Dif}A$ that have a filtration by objects in $\{S_*P \mid P \text{ is a projective object in } \text{Mod}A\}$ (see [HJ19, p. 6.4]). Therefore, by [HJ24b, Prop. 3.2(a)], objects in $\text{Dif}A$ that admit a projective flag are also semi-projective. More generally, by the additivity of $\text{Ext}_{\text{Dif}A}^1$, if $X \rightarrow Y$ is a retraction in $\text{Dif}A$ with X admitting a projective flag, then Y is also semi-projective.

Remark III.3.4. Considering Example III.3.3 and noting that projective objects in $\text{Dif}A$ are in particular contractible (see [CE56, Prop. 2.5 and Prop. 2.3]), the result [ABI07, Thm. 2.3] is a special case of [HJ22, Thm. 4.4(a)].

We prove some basic properties of the expansion and (co)compression functors.

Proposition III.3.5. Consider the adjunction triple $(\pi_!, \pi^*, \pi_*)$ as follows:

$$\begin{array}{ccc} & \xleftarrow{\pi_!} & \\ \text{Dif}A & \xrightarrow{\pi^*} & \text{Ch}A \\ & \xleftarrow{\pi_*} & \end{array}$$

The following statements hold:

1. The expansion functor $\text{Dif}A \xrightarrow{\pi^*} \text{Ch}A$ is exact.
2. The compression functor $\text{Ch}A \xrightarrow{\pi_!} \text{Dif}A$ is exact.
3. The cocompression functor $\text{Ch}A \xrightarrow{\pi_*} \text{Dif}A$ is exact.

Proof. Part 1. This follows as π^* is both a left and a right adjoint.

Part 2. As $\pi_!$ is a left adjoint, it suffices to show that it is left exact. Let $X \xrightarrow{f} Y$ be a monomorphism in $\text{Ch}A$. Equivalently, this means that for each integer i , the morphism $E_i(f)$ in $\text{Mod}A$ is a monomorphism. We have to show that $\pi_!X \xrightarrow{\pi_!f} \pi_!Y$ is a monomorphism in $\text{Dif}A$ which occurs if and only if $E_*\pi_!X \xrightarrow{E_*\pi_!f} E_*\pi_!Y$ is a monomorphism in $\text{Mod}A$. But, $E_*\pi_!f$ is precisely $\coprod_{i \in \mathbb{Z}} E_i f$ which is a monomorphism as coproducts are exact in the AB4 category $\text{Mod}A$.

Part 3. This follows by a similar argument as seen in part 1, noting that products are exact in the AB4* category $\text{Mod}A$. \square

Hence, the adjunction pairs $(\pi_!, \pi^*)$ and (π^*, π_*) fit into Lemma III.2.1. We will now see that all working definitions of a contractible object in the literature are equivalent (see Remark III.3.1).

Proposition III.3.6. *Let (X, d_X) be an object in DifA . The following statements are equivalent:*

1. (X, d_X) is a contractible object in DifA .
2. $\pi^*(X, d_X)$ is a contractible object in ChA .
3. (X, d_X) is isomorphic to $F_* M$ for some object M in Mod A .
4. (X, d_X) is acyclic and the short exact sequence

$$0 \rightarrow \text{Ker } d_X \rightarrow X \rightarrow \text{Im } d_X \rightarrow 0$$

splits in Mod A .

Proof. The proof of statements 2, 3 and 4 being equivalent can be found in [ABI07, Prop. 1.8]. We now prove the equivalence of statements 1 and 2.

(1 \implies 2) : Let $X \xrightarrow{h} X$ be a contracting homotopy of (X, d_X) . Setting $h^i = h$ for each integer i , the sequence $(X \xrightarrow{h^i} X)_{i \in \mathbb{Z}}$ is a contracting homotopy for $\pi^*(X, d_X)$.

(2 \implies 1) : Let $(X \xrightarrow{h^i} X)_{i \in \mathbb{Z}}$ be a contracting homotopy of $\pi^*(X, d_X)$. Choose an integer i and set $h := h^{i+1} d_X h^i$. The following equation $\text{id}_X = h^{i+1} d_X + d_X h^i$ holds and implies both the equations $d_X = d_X h^i d_X$ and $d_X = d_X h^{i+1} d_X$. We have

$$\begin{aligned} \text{id}_X &= h^{i+1} d_X + d_X h^i = h^{i+1} (d_X h^i d_X) + (d_X h^{i+1} d_X) h^i \\ &= (h^{i+1} d_X h^i) d_X + d_X (h^{i+1} d_X h^i) = h d_X + d_X h. \quad \square \end{aligned}$$

The next result tells us how the cohomology functors interact with the expansion and (co)compression functors.

Proposition III.3.7. *Let (X, d_X) be an object in DifA and Y be an object in ChA . The following statements hold:*

1. For each integer i , we have an equality $H^i(\pi^*(X, d_X)) = H^*(X, d_X)$.
2. There is a natural isomorphism $H^*(\pi_! Y) \cong \coprod_{i \in \mathbb{Z}} H^i(Y)$ in Mod A .
3. There is a natural isomorphism $H^*(\pi_* Y) \cong \prod_{i \in \mathbb{Z}} H^i(Y)$ in Mod A .

Proof. Part 1. This follows from the explicit description of the expansion functor π^* .

Part 2. This is stated in [ABI07, pg. 7] and a proof is similar to part 3, which we give below.

Part 3. We have a composition of natural isomorphisms

$$\begin{aligned} H^*(\pi_*Y) &= \text{Ker} \left(\prod_{i \in \mathbb{Z}} d_Y^i \right) / \text{Im} \left(\prod_{i \in \mathbb{Z}} d_Y^i \right) \cong \prod_{i \in \mathbb{Z}} \text{Ker} d_Y^i / \prod_{i \in \mathbb{Z}} \text{Im} d_Y^i \\ &\cong \prod_{i \in \mathbb{Z}} (\text{Ker} d_Y^i / \text{Im} d_Y^i) = \prod_{i \in \mathbb{Z}} H^i(Y), \end{aligned}$$

that exist since products are exact in the AB4* category $\text{Mod } A$. \square

Corollary III.3.8. *The following statements hold:*

1. (X, d_X) is an acyclic object in $\text{Dif}A$ if and only if $\pi^*(X, d_X)$ is an acyclic object in $\text{Ch}A$.
2. Y is an acyclic object in $\text{Ch}A$ if and only if $\pi_!Y$ is an acyclic object in $\text{Dif}A$.
3. Y is an acyclic object in $\text{Ch}A$ if and only if π_*Y is an acyclic object in $\text{Dif}A$.
4. f is a quasi-isomorphism in $\text{Dif}A$ if and only if π^*f is a quasi-isomorphism in $\text{Ch}A$.
5. g is a quasi-isomorphism in $\text{Ch}A$ if and only if $\pi_!g$ is a quasi-isomorphism in $\text{Dif}A$.
6. g is a quasi-isomorphism in $\text{Ch}A$ if and only if π_*g is a quasi-isomorphism in $\text{Dif}A$.

Proof. The statements follow from Proposition III.3.7. \square

Our next aim is to study to which extent do the expansion and (co)compression functors interact with semi-injective and semi-projective objects.

Proposition III.3.9. *Let (I, d_I) be an object in $\text{Dif}A$ and J be an object in $\text{Ch}A$. The following statements hold:*

1. If J is an injective object in $\text{Ch}A$, then π_*J is an injective object in $\text{Dif}A$.
2. (I, d_I) is an injective object in $\text{Dif}A$ if and only if $\pi^*(I, d_I)$ is an injective object in $\text{Ch}A$.

Proof. Part 1. The statement follows from the adjunction pair (π^*, π_*) , Proposition III.3.5 and Lemma III.2.1 part 1.

Part 2. By [CE56, Prop. 2.5 and Prop. 2.3], (I, d_I) is an injective object in $\text{Dif}A$ if and only if (I, d_I) is contractible and I is an injective object in $\text{Mod } A$. By Proposition III.3.6 and the description of π^* , this holds if and only if $\pi^*(I, d_I)$ is contractible and for each integer i , the evaluation $E_i \pi^*(I, d_I)$ is an injective object in $\text{Mod } A$. By [CFH24, E 5.3.4], this holds if and only if $\pi^*(I, d_I)$ is an injective object in $\text{Ch}A$. \square

Proposition III.3.10. *Let (I, d_I) be an object in $\text{Dif}A$ and J be an object in $\text{Ch}A$. The following statements hold:*

1. If J is a semi-injective object in ChA , then $\pi_* J$ is a semi-injective object in DifA .
2. If (I, d_I) is a semi-injective object in DifA , then $\pi^*(I, d_I)$ is a semi-injective object in ChA .

Proof. Part 1. The statement follows from the adjunction pair (π^*, π_*) , Proposition III.3.5, Corollary III.3.8 part 1 and Lemma III.2.1 part 4.

Part 2. The statement follows from the adjunction pair $(\pi_!, \pi^*)$, Proposition III.3.5, Corollary III.3.8 part 2 and Lemma III.2.1 part 4. \square

Remark III.3.11. The dual statements of Proposition III.3.9 and Proposition III.3.10 also hold by their dual arguments. However, we will not use them here.

III.4 The differential tensor and Hom functors

Fix an object M in $\text{Mod}(A^{\text{op}} \otimes B)$.

Differential tensor product. For an object (X, d_X) in DifA , we define the *differential tensor product* of M and (X, d_X) as the object

$$(X, d_X) \boxtimes_A M = (X \otimes_A M, d_X \otimes_A M),$$

in DifB . This assignment augments to an additive functor $\text{DifA} \xrightarrow{-\boxtimes_A M} \text{DifB}$. This functor is related to the functor defined in [ABI07, Sec. 1.9].

Differential Hom. For an object (Y, d_Y) in DifB , we define the *differential Hom* of M and (Y, d_Y) as the object

$$\text{hom}_B(M, (Y, d_Y)) = (\text{Hom}_{\text{Mod } B}(M, Y), \text{Hom}_{\text{Mod } B}(M, d_Y))$$

in DifA . This assignment augments to an additive functor $\text{DifB} \xrightarrow{\text{hom}_B(M, -)} \text{DifA}$. These functors form an adjunction pair:

Proposition III.4.1. *Let M be an object in $\text{Mod}(A^{\text{op}} \otimes B)$. There is an adjunction isomorphism*

$$\text{Hom}_{\text{DifB}}((X, d_X) \boxtimes_A M, (Y, d_Y)) \cong \text{Hom}_{\text{DifA}}((X, d_X), \text{hom}_B(M, (Y, d_Y)))$$

of objects in $\text{Mod } k$, where (X, d_X) is an object in DifA and (Y, d_Y) an object in DifB . This adjunction isomorphism is given by the canonical currying operation

$$f \mapsto [x \mapsto f(x \otimes -)].$$

Proof. Let $X \otimes_A M \xrightarrow{f} Y$ be a morphism in $\text{Mod } B$. Then the following diagram on the left is commutative if and only if the following diagram on the right is commutative:

$$\begin{array}{ccc}
 X \otimes_A M & \xrightarrow{d_{X \otimes_A M}} & X \otimes_A M \\
 f \downarrow & & \downarrow f \\
 Y & \xrightarrow{d_Y} & Y
 \end{array}
 \quad
 \begin{array}{ccc}
 X & \xrightarrow{d_X} & X \\
 [x \mapsto f(x \otimes -)] \downarrow & & \downarrow [x \mapsto f(x \otimes -)] \\
 \text{Hom}_{\text{Mod } B}(M, Y) & \xrightarrow{\text{Hom}_{\text{Mod } B}(M, d_Y)} & \text{Hom}_{\text{Mod } B}(M, Y).
 \end{array}$$

This means that the Tensor-Hom Adjunction

$$\text{Hom}_{\text{Mod } B}(X \otimes_A M, Y) \cong \text{Hom}_{\text{Mod } A}(X, \text{Hom}_{\text{Mod } B}(M, Y))$$

restricts to the desired adjunction isomorphism. \square

The following result allows us to make new semi-injective objects from old semi-injective objects.

Proposition III.4.2. *Let M be an object in $\text{Mod}(A^{\text{op}} \otimes B)$ and (I, d_I) be an object in $\text{Dif}B$. Assume M is a projective object in $\text{Mod}(A^{\text{op}})$ and in $\text{Mod } B$. If (I, d_I) is semi-injective, then $\text{hom}_B(M, (I, d_I))$ is semi-injective in $\text{Dif}A$.*

Proof. The functors $\text{Ch}A \xrightarrow{- \otimes_A M} \text{Ch}B$ and $\text{Ch}B \xrightarrow{\text{Hom}_B(M, -)} \text{Ch}A$ are exact and preserve acyclic objects (see [CFH24, Prop. 5.2.10, Prop. 5.4.9 and Cor. 5.4.10]). Consequently, the functors $- \boxtimes_A M$ and $\text{hom}_B(M, -)$ are exact and preserve acyclic objects. The statement follows by applying Lemma III.2.1 to the adjunction pair $(- \boxtimes_A M, \text{hom}_B(M, -))$ seen in Proposition III.4.1. \square

Remark III.4.3. The dual statement of Proposition III.4.2 concerning the differential tensor product is also true by the dual argument.

For the next result, consider the tensor product functor \otimes_A and the Hom complex functor Hom_B on complexes.

Lemma III.4.4. *Let M be an object in $\text{Mod}(A^{\text{op}} \otimes B)$. The following statements hold:*

1. *For each object Z in $\text{Ch}B$, there is a natural isomorphism*

$$\text{hom}_B(M, \pi_* Z) \cong \pi_* \text{Hom}_B(S_0 M, Z).$$

2. *For each object Y in $\text{Ch}A$, there is a natural isomorphism*

$$\pi_! Y \boxtimes_A M \cong \pi_!(Y \otimes_A S_0 M).$$

Proof. Part 1: Let (W, \mathbf{d}_W) be an object in $\text{Dif}A$. We have the following composition of natural isomorphisms:

$$\begin{aligned} \text{Hom}_{\text{Dif}A}((W, \mathbf{d}_W), \pi_* \text{Hom}_B(S_0 M, Z)) &\cong \text{Hom}_{\text{Ch}A}(\pi^*(W, \mathbf{d}_W), \text{Hom}_B(S_0 M, Z)) \\ &\cong \text{Hom}_{\text{Ch}B}(\pi^*(W, \mathbf{d}_W) \otimes_A S_0 M, Z) \\ &\cong \text{Hom}_{\text{Ch}B}(\pi^*((W, \mathbf{d}_W) \boxtimes_A M), Z) \\ &\cong \text{Hom}_{\text{Dif}B}((W, \mathbf{d}_W) \boxtimes_A M, \pi_* Z) \\ &\cong \text{Hom}_{\text{Dif}A}((W, \mathbf{d}_W), \text{hom}_B(M, \pi_* Z)), \end{aligned}$$

where the first and fourth isomorphisms hold by the adjunction pair (π^*, π_*) , the second isomorphism holds by the Tensor-Hom adjunction, the third isomorphism holds as $\pi^*(W, \mathbf{d}_W) \otimes_A S_0 M$ and $\pi^*((W, \mathbf{d}_W) \boxtimes_A M)$ are canonically isomorphic and the last isomorphism holds by Proposition III.4.1. The result now follows from the Yoneda Lemma.

Part 2: This proof follows a similar argument as in part 1. \square

Proposition III.4.5. *Let J be a minimal semi-injective object in $\text{Ch}B$ and K be a semisimple object in $\text{Mod}B$. For any decomposition $\pi_* J = (I', \mathbf{d}_{I'}) \oplus (I'', \mathbf{d}_{I''})$ in $\text{Dif}B$, where $(I', \mathbf{d}_{I'})$ is a minimal semi-injective object and $(I'', \mathbf{d}_{I''})$ an injective object, the following statements hold:*

- *The object $\text{hom}_B(K, (I'', \mathbf{d}_{I''}))$ is zero.*
- *There is an isomorphism*

$$\text{hom}_B(K, (I', \mathbf{d}_{I'})) \cong \pi_* \text{Hom}_B(S_0 K, J)$$

of objects in $\text{Dif}A$.

In particular, there is an isomorphism

$$\text{Hom}_{\text{Mod}B}(K, I') \cong \prod_{i \in \mathbb{Z}} \text{Hom}_{\text{Mod}B}(K, E_i J)$$

of objects in $\text{Mod}A$.

Proof. Suppose we have such a decomposition of $\pi_* J$. Applying $\text{hom}_B(K, -)$ to it, we get

$$\text{hom}_B(K, (I', \mathbf{d}_{I'})) \oplus \text{hom}_B(K, (I'', \mathbf{d}_{I''})) \cong \text{hom}_B(K, \pi_* J) \cong \pi_* \text{Hom}_B(S_0 K, J),$$

where the first isomorphism holds by the additivity of the differential Hom functor and the second isomorphism holds by Lemma III.4.4 part 1. As J is a minimal semi-injective object and K is semisimple, $\pi_* \text{Hom}_B(S_0 K, J)$ has trivial differential (see [CFH24, Lem. 8.2.16]). In particular, $\text{hom}_B(K, (I'', \mathbf{d}_{I''}))$ has trivial differential. We will show $\text{hom}_B(K, (I'', \mathbf{d}_{I''}))$ is contractible, which then implies it must be zero.

As $(I'', \mathbf{d}_{I''})$ is an injective object in DifB , it is contractible (see [CE56, Prop. 2.5 and Prop. 2.3]). Then Proposition III.3.6, gives that the expansion $\pi^*(I'', \mathbf{d}_{I''})$ is also contractible and, therefore, so is $\text{Hom}_B(S_0 K, \pi^*(I'', \mathbf{d}_{I''}))$ (see [CFH24, Prop. 4.3.29]). By Proposition III.3.6 and as $\pi^* \text{hom}_B(K, (I'', \mathbf{d}_{I''}))$ is canonically isomorphic to $\text{Hom}_B(S_0 K, \pi^*(I'', \mathbf{d}_{I''}))$, the object $\text{hom}_B(K, (I'', \mathbf{d}_{I''}))$ is contractible. Therefore, $\text{hom}_B(K, (I'', \mathbf{d}_{I''}))$ is zero. This gives us the required isomorphism in DifA and the following isomorphism in Mod A is obtained by applying the evaluation functor E_* to this isomorphism. \square

Lemma III.4.6. *Let $(X, \mathbf{d}_X) \rightarrow (I, \mathbf{d}_I)$ and $(X, \mathbf{d}_X) \rightarrow (\tilde{I}, \mathbf{d}_{\tilde{I}})$ be semi-injective resolutions in DifA . If (I, \mathbf{d}_I) is minimal, then $(\tilde{I}, \mathbf{d}_{\tilde{I}})$ has the form $(\tilde{I}, \mathbf{d}_{\tilde{I}}) = (I', \mathbf{d}_{I'}) \oplus (I'', \mathbf{d}_{I''})$, where $(I'', \mathbf{d}_{I''})$ is an injective object in DifA and (I, \mathbf{d}_I) is isomorphic to $(I', \mathbf{d}_{I'})$ in DifA .*

Proof. A special consequence of [HJ24a, Thm. B(ii)] is that $(\tilde{I}, \mathbf{d}_{\tilde{I}})$ has the form $(\tilde{I}, \mathbf{d}_{\tilde{I}}) = (I', \mathbf{d}_{I'}) \oplus (I'', \mathbf{d}_{I''})$, with $(I', \mathbf{d}_{I'})$ a minimal semi-injective object in DifA and $(I'', \mathbf{d}_{I''})$ an injective object in DifA . As it is injective, the object $(I'', \mathbf{d}_{I''})$ is contractible and hence acyclic (see [CE56, Prop. 2.5 and Prop. 2.3]). Therefore, the morphism $(X, \mathbf{d}_X) \rightarrow (\tilde{I}, \mathbf{d}_{\tilde{I}}) \rightarrow (I', \mathbf{d}_{I'})$, where $(\tilde{I}, \mathbf{d}_{\tilde{I}}) \rightarrow (I', \mathbf{d}_{I'})$ is the canonical projection, is a minimal semi-injective resolution in DifA and hence (I, \mathbf{d}_I) and $(I', \mathbf{d}_{I'})$ are isomorphic in DifA (see [HJ24a, Thm. D]). \square

III.5 Bass numbers

Let \mathbf{R} be a commutative noetherian ring. Matlis' structure theorem [Mat58] (also see [CFH24, Thm. C.23]) for injective modules over \mathbf{R} states that every injective \mathbf{R} -module E decomposes into a coproduct of uniquely determined indecomposable injectives

$$E \cong \coprod_{\mathfrak{p} \in \text{Spec } \mathbf{R}} \left(\coprod_{\mu(E_{\mathfrak{p}})} E_{\mathbf{R}}(\mathbf{R}/\mathfrak{p}) \right),$$

where $\text{Spec } \mathbf{R}$ denotes the prime spectrum of the ring \mathbf{R} and the value $\mu(E_{\mathfrak{p}})$ is the dimension of $\text{Hom}_{\text{Mod } \mathbf{R}_{\mathfrak{p}}}(\kappa(\mathfrak{p}), E_{\mathfrak{p}})$ when viewed as a vector space over $\kappa(\mathfrak{p}) = \mathbf{R}_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$.

Further assume that $(\mathbf{R}, \mathfrak{m}, \mathbf{k})$ is a local commutative noetherian ring and let E be an \mathbf{R} -module. We set

$$\mu(E) = \dim_{\mathbf{k}} \text{Hom}_{\text{Mod } \mathbf{R}}(\mathbf{k}, E),$$

so that the quantity $\mu(E)$ is, by Matlis' structure theorem, the number of direct summand copies of $E_{\mathbf{R}}(\mathbf{k})$ in E . For a semi-injective object (I, \mathbf{d}_I) in DifR and for a semi-injective object J in ChR , we set

$$\mu_D(I, \mathbf{d}_I) := \mu(I) \quad \text{and} \quad \mu_C(J) := \sum_{i \in \mathbb{Z}} \mu(E_i J).$$

This makes sense by [HJ24b, Prop. 3.2(b)] for example. Let M be an object in $\text{mod } R$ and suppose $M \rightarrow J$ is a minimal semi-injective resolution in $\text{Ch}R$. In this case, for an integer i , the quantity $\mu(E_i J)$ is called the i -th Bass number of M . Since the Hom complex $\text{Hom}_R(k, J)$ has trivial differentials (see [CFH24, Lem. 8.2.16]), the i -th Bass number can be expressed as

$$\mu(E_i J) = \dim_k \text{Ext}_R^i(k, M),$$

which is finite (see [CFH24, Prop. 12.3.32]).

Proposition III.5.1. *Let (R, \mathfrak{m}, k) be a local commutative noetherian ring and J be a minimal semi-injective object in $\text{Ch}R$. For any decomposition $\pi_* J = (I', \mathfrak{d}_{I'}) \oplus (I'', \mathfrak{d}_{I''})$ in $\text{Dif}R$, where $(I', \mathfrak{d}_{I'})$ is a minimal semi-injective object and $(I'', \mathfrak{d}_{I''})$ an injective object, the following equations hold:*

- $\mu_D(I'', \mathfrak{d}_{I''}) = 0$.
- $\mu_D(I', \mathfrak{d}_{I'}) = \mu_C(J)$.

Proof. This follows by considering the (semi)simple object k in $\text{Mod}(k \otimes_{\mathbb{Z}} R)$ and applying Proposition III.4.5. □

The following is a differential module version of a classical commutative algebra result (see [Bas63, Lem. 2.7] for a classical reference and [CFH24, Thm. 16.4.8] for a modern reference).

Theorem III.5.2. *Let (R, \mathfrak{m}, k) be a local commutative noetherian ring and let M be an object in $\text{mod } R$ with a minimal semi-injective resolution $S_* M \rightarrow (I, \mathfrak{d}_I)$ in $\text{Dif}R$. Then M has finite injective dimension if and only if $\mu_D(I, \mathfrak{d}_I)$ is finite.*

Proof. Let $S_0 M \rightarrow J$ in $\text{Ch}R$ be a minimal semi-injective resolution in $\text{Ch}R$ and consider the compression $S_* M \cong \pi_* S_0 M \rightarrow \pi_* J$. By Corollary III.3.8 part 6 and Proposition III.3.10 part 1, $S_* M \cong \pi_* S_0 M \rightarrow \pi_* J$ is a semi-injective resolution in $\text{Dif}R$. Lemma III.4.6 gives us a decomposition $\pi_* J \cong (I, \mathfrak{d}_I) \oplus (I'', \mathfrak{d}_{I''})$, where $(I'', \mathfrak{d}_{I''})$ is an injective object in $\text{Dif}R$. Proposition III.5.1 gives us that $\mu_D(I, \mathfrak{d}_I) = \mu_C(J)$. Hence, $\mu_D(I, \mathfrak{d}_I)$ is finite if and only if $\mu_C(J)$ is finite. The equivalence follows since M has finite injective dimension if and only if $\mu_C(J)$ is finite (see [CFH24, Thm. 16.4.8]). □

Bass asked whether the existence of a finitely generated module of finite injective dimension suffices to characterise local Cohen–Macaulay rings (see [Bas63, Rmk. on pg. 14]). It turned out to be true. As a direct consequence of this fact, the homological algebra of differential modules can also characterise local Cohen–Macaulay rings.

Corollary III.5.3 (Bass' Question for differential modules). *A local commutative noetherian ring $(R, \mathfrak{m}, \mathfrak{k})$ is Cohen-Macaulay if and only if there exists a nonzero object M in $\text{mod } R$ whose minimal semi-injective resolution $S_* M \rightarrow (I, d_I)$ in $\text{Dif}R$ has that $\mu_D(I, d_I)$ is finite.*

Proof. The equivalence follows from [CFH24, Ex. 18.5.9], [CFH24, Cor. 18.5.8] and Theorem III.5.2. \square

III.6 Preservation of minimal semi-injectives

It is perhaps unsurprising that the expansion functor interacts well with the notion of minimality:

Proposition III.6.1. *Let (I, d_I) be a semi-injective object in $\text{Dif}A$. Then (I, d_I) is a minimal semi-injective object in $\text{Dif}A$ if and only if $\pi^*(I, d_I)$ is a minimal semi-injective object in $\text{Ch}A$.*

Proof. (\implies only if): By Proposition III.3.10 part 2, the expansion $\pi^*(I, d_I)$ is semi-injective object in $\text{Ch}A$. We use the characterisation found in [HJ24a, Thm. 2.1(vi)]. Let i be an integer, M an object in $\text{Mod } A$ and assume that the morphism

$$\begin{array}{ccccccccccc} F_i M & & \cdots & \longrightarrow & 0 & \longrightarrow & M & \xrightarrow{\text{id}_M} & M & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & & \downarrow & & \downarrow f_0 & & \downarrow f_1 & & & & \\ \downarrow \hat{f} = & & & & \downarrow & & \downarrow f_0 & & \downarrow f_1 & & & & \\ \pi^*(I, d_I) & & \cdots & \longrightarrow & I & \xrightarrow{d_I} & I & \xrightarrow{d_I} & I & \xrightarrow{d_I} & I & \longrightarrow & \cdots, \end{array}$$

is a monomorphism in $\text{Ch}A$. As \hat{f} intertwines the differentials, $f_1 = d_I f_0$. We are done if we show that M is zero. Consider the morphism

$$\begin{array}{ccc} F_* M & & M \oplus M \xrightarrow{\begin{pmatrix} 0 & 0 \\ \text{id}_M & 0 \end{pmatrix}} M \oplus M \\ \downarrow f^\dagger = & & \downarrow \begin{pmatrix} f_0 & d_I f_0 \end{pmatrix} \\ (I, d_I) & & I \xrightarrow{d_I} I \end{array}$$

in $\text{Dif}A$. As (I, d_I) is a minimal semi-injective object in $\text{Dif}A$, it suffices to show that f^\dagger is a monomorphism (see [HJ24a, Thm. 2.1(vi)]). To this end, let $x = (m, n)$ be an element in $M \oplus M$ and suppose $f^\dagger(x) = 0$. Applying the differential d_I , the equation $d_I f^\dagger(x) = 0$ simplifies to $d_I f_0(m) = 0$. As \hat{f} was assumed to be a monomorphism, $d_I f_0$ is also a monomorphism and we have that $m = 0$. But, then the equation $f^\dagger(x) = 0$ now simplifies to $d_I f_0(n) = 0$ and again, we get $n = 0$. Hence $x = (0, 0)$ and f^\dagger is a monomorphism. Therefore, M is zero.

(\Leftarrow^{if}) : We use the characterisation found in [HJ24a, Thm. 2.1(ii)]. Suppose (E, d_E) is an acyclic subobject of (I, d_I) . By Corollary III.3.8 part 1 and as π^* is an exact functor (see Proposition III.3.5 part 1), $\pi^*(E, d_E)$ is an acyclic subobject of $\pi^*(I, d_I)$. By minimality, $\pi^*(E, d_E)$ is zero (see [HJ24a, Thm. 2.1(ii)]). As π^* is faithful, (E, d_E) is also zero. \square

As the definition of a minimal semi-injective object involves homological and representation theoretic conditions, the question of whether the cocompression functor π_* preserves these objects is subtle. Later, we will see in Example III.7.3 that the minimality of semi-injective objects is not always preserved. Nevertheless, in this section we present some positive results that identify when it does.

Proposition III.6.2. *Let J be an object in ChA of the form*

$$J = \cdots \rightarrow 0 \rightarrow J^0 \xrightarrow{d_J^0} J^1 \rightarrow 0 \rightarrow \cdots .$$

*If J is a minimal semi-injective object, then π_*J is a minimal semi-injective object in DifA .*

Proof. We verify the characterisation of a minimal semi-injective object in DifA found in [HJ24a, Thm. 2.1(vi)]. By Proposition III.3.10 part 1, the cocompression π_*J is a semi-injective object in DifA , and takes the form

$$\pi_*J = \left(J^0 \oplus J^1, \begin{pmatrix} 0 & 0 \\ d_J^0 & 0 \end{pmatrix} \right).$$

Let M be an object in Mod A and assume that the morphism f^\dagger :

$$\begin{array}{ccc} F_* M & & M \oplus M \xrightarrow{\begin{pmatrix} 0 & 0 \\ \text{id}_M & 0 \end{pmatrix}} M \oplus M \\ \downarrow f^\dagger = & & \downarrow \begin{pmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{pmatrix} \\ \pi_* J & & J^0 \oplus J^1 \xrightarrow{\begin{pmatrix} 0 & 0 \\ d_J^0 & 0 \end{pmatrix}} J^0 \oplus J^1, \end{array}$$

is a monomorphism in DifA . As f^\dagger intertwines with the differentials, it must be a monomorphism of the form

$$M \oplus M \xrightarrow{\begin{pmatrix} f_{00} & 0 \\ f_{10} & d_J^0 f_{00} \end{pmatrix}} J^0 \oplus J^1.$$

In particular, $d_J^0 f_{00}: M \rightarrow J^1$ is a monomorphism. But then

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & M & \xrightarrow{\text{id}_M} & M & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow f_{00} & & \downarrow d_J^0 f_{00} & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & J^0 & \xrightarrow{d_J^0} & J^1 & \longrightarrow & 0 & \longrightarrow & \cdots, \end{array}$$

is a monomorphism in ChA and hence, by minimality of J , the object M is zero. \square

Let M be an object in $\text{Mod } A$. Let $\text{Soc } M$ denote the socle of an object M . By [AF92, Prop. 9.7], we know that $\text{Soc } M$ is equal to the intersection of all of the large subobjects of M . Consider a subobject L of M . We say L is *large* (also known as *essential*) if the zero object is the only subobject of M with a trivial intersection with L .

Proposition III.6.3. *Let A be a right Artinian \mathbb{k} -algebra and J be an object in ChA of the form*

$$J = \cdots \rightarrow J^0 \xrightarrow{d_J^0} J^1 \xrightarrow{d_J^1} J^2 \rightarrow \cdots.$$

Suppose J is bounded. If J is a minimal semi-injective object, then $\pi_ J$ is a minimal semi-injective object in $\text{Dif } A$.*

Proof. We verify the characterisation of a minimal semi-injective object in $\text{Dif } A$ found in [HJ24a, Thm. 2.1(vi)]. By Proposition III.3.10 part 1, the cocompression $\pi_* J$ is a semi-injective object in $\text{Dif } A$. As J is bounded, we can assume its cocompression is of the form

$$\pi_* J = \left(\bigoplus_{i=0}^n J^i, d_J := \bigoplus_{i=0}^n d_J^i \right),$$

for n some natural number. Let M be an object in $\text{Mod } A$ and assume that the morphism f^\dagger :

$$\begin{array}{ccc} F_* M & M \oplus M & \xrightarrow{\begin{pmatrix} 0 & 0 \\ \text{id}_M & 0 \end{pmatrix}} & M \oplus M \\ \downarrow f^\dagger = & \left(\begin{array}{c} f_0 \\ f_1 \end{array} \right) \downarrow & & \downarrow \left(\begin{array}{c} f_0 \\ f_1 \end{array} \right) \\ \pi_* J & \bigoplus_{i=0}^n J^i & \xrightarrow{d_J} & \bigoplus_{i=0}^n J^i \end{array}$$

is a monomorphism in $\text{Dif } A$. As f^\dagger intertwines with the differentials, it must be of the form

$$M \oplus M \xrightarrow{\begin{pmatrix} f_0 & d_J f_0 \end{pmatrix}} \bigoplus_{i=0}^n J^i.$$

In particular, the component $d_J f_0$ is also a monomorphism in $\text{Mod } A$. The socle of an object in $\text{Mod } A$ is equal to the intersection of all of its large subobjects (see [AF92, Prop. 9.7]) and as J is minimal, the subobject $\text{Ker } d_J^i$ is large in J_i (see [CFH24, Theorem B.21]) and therefore, the direct sum $\text{Ker } d_J \cong \bigoplus_{i=0}^n \text{Ker } d_J^i$ is large in $\bigoplus_{i=0}^n J_i$ (see [GW89, Prop.

5.6(d)). This implies that $\text{Soc}(\bigoplus_{i=0}^n J_i)$ is contained in $\text{Ker } d_J$. Now, the component f_0 maps $\text{Soc } M$ into $\text{Soc}(\bigoplus_{i=0}^n J_i)$ and therefore, the composition $d_J f_0$ maps $\text{Soc } M$ to zero. As $d_J f_0$ is a monomorphism, $\text{Soc } M$ is zero. But the only object in $\text{Mod } A$ with a trivial socle is the zero object (see [ARS97, Prop. 4.1(a)]). Note that the proof of [ARS97, Prop. 4.1(a)] remains true when the finitely generated assumption and the assumption that the ring is an algebra over a field is dropped (see [ARS97, Prop. 3.1(a)]). Hence, M is zero. \square

Remark III.6.4. Recall Proposition III.5.1. It tells us that over a local commutative noetherian ring, all the indecomposable injective summands $E(k)$ appearing in a minimal semi-injective complex J , when cocompressed by π_* , must land in the minimal semi-injective direct summand $(I', d_{I'})$ of the decomposition $\pi_* J = (I', d_{I'}) \oplus (I'', d_{I''})$. We will now use this fact to show another instance for when the cocompression functor π_* preserves minimal semi-injective objects.

Proposition III.6.5. *Let (R, \mathfrak{m}, k) be a local commutative noetherian ring of Krull dimension zero and let J be an object in $\text{Ch}R$. If J is a minimal semi-injective object, then $\pi_* J$ is a minimal semi-injective object in $\text{Dif}R$.*

Proof. By Proposition III.3.10 part 1, the cocompression $\pi_* J$ is a semi-injective object in $\text{Dif}R$. By [HJ24a, Thm. B(ii)], it is of the form $\pi_* J = (I', d_{I'}) \oplus (I'', d_{I''})$, where $(I', d_{I'})$ is minimal semi-injective and $(I'', d_{I''})$ injective. It suffices to show that $(I'', d_{I''})$ is zero. As R is of Krull dimension zero, $E(k)$ is the unique indecomposable injective object in $\text{Mod } R$. By Matlis' structure theorem [Mat58] (also see [CFH24, Thm. C.23]), each injective object E in $\text{Mod } R$ is a coproduct of $\mu(E)$ many summands isomorphic to $E(k)$. By Proposition III.5.1, the equation $\mu_D(I'', d_{I''}) = 0$ holds and thus, $(I'', d_{I''})$ must be zero. \square

III.7 Examples

Example III.7.1 (Hereditary rings). Considering Proposition III.6.2, we take A to be the ring of integers \mathbb{Z} and let I be the minimal semi-injective resolution of $S_0 \mathbb{Z}$ in $\text{Ch}\mathbb{Z}$:

$$I = \cdots \rightarrow 0 \rightarrow \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \rightarrow 0 \rightarrow \cdots .$$

Its cocompression

$$\pi_* I = \left(\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}, \begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix} \right),$$

is therefore a minimal semi-injective resolution of $S_*\mathbb{Z}$ in $\text{Dif}\mathbb{Z}$. By Proposition III.6.1, the expansion

$$\pi^*\pi_*I = \cdots \rightarrow \mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z} \xrightarrow{\begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix}} \mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z} \xrightarrow{\begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix}} \mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z} \xrightarrow{\begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix}} \mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z} \rightarrow \cdots$$

is a minimal semi-injective object in ChA .

Example III.7.2 (Infinite global dimension). Considering Proposition III.6.5, we take (R, \mathfrak{m}) to be $(\mathbb{Z}/4\mathbb{Z}, 2\mathbb{Z}/4\mathbb{Z})$ or the dual numbers $(k[x]/(x^2), (x)/(x^2))$ over a field k . Let I be the minimal semi-injective resolution of the residue field $S_0(R/\mathfrak{m})$ in ChR :

$$I = \cdots \rightarrow 0 \rightarrow R \xrightarrow{\delta} R \xrightarrow{\delta} R \rightarrow \cdots$$

Its cocompression is of the form

$$\pi_*I = \left(\prod_{i \in \mathbb{N}} R, \bar{\delta} \right),$$

where the differential $\bar{\delta}$ is given by $(t_0, t_1, t_2, \dots) \mapsto (0, \delta t_0, \delta t_1, \dots)$, and it is a minimal semi-injective resolution of $S_*(R/\mathfrak{m})$ in $\text{Dif}R$. Using Proposition III.6.1, the expansion

$$\pi^*\pi_*I = \cdots \rightarrow \prod_{i \in \mathbb{N}} R \xrightarrow{\bar{\delta}} \prod_{i \in \mathbb{N}} R \xrightarrow{\bar{\delta}} \prod_{i \in \mathbb{N}} R \rightarrow \cdots$$

is a minimal semi-injective object in ChR .

We will now see that the cocompression functor does not always preserve minimal semi-injective objects.

Example III.7.3 (Non-example hereditary rings). Let A be the ring of integers \mathbb{Z} and let J be the minimal semi-injective resolution of $S_0\mathbb{Z}$ as an object in $\text{Ch}\mathbb{Z}$:

$$J = \cdots \rightarrow 0 \rightarrow \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \rightarrow 0 \rightarrow \cdots$$

We consider the following object in ChA :

$$\prod_{i \geq 0} \Sigma^{-i}(J) = \cdots \rightarrow 0 \rightarrow \mathbb{Q} \xrightarrow{\begin{pmatrix} 0 \\ \pi \end{pmatrix}} \mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z} \xrightarrow{\begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix}} \mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z} \xrightarrow{\begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix}} \mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z} \rightarrow \cdots \quad (\text{III.4})$$

It is a semi-injective object in $\text{Ch}\mathbb{Z}$ as it is bounded on the left and consists of injective objects in $\text{Mod}A$ (see [CFH24, Ex. 5.3.12], but note that homological grading is used).

It is minimal as the kernel of each differential is large (see [HJ24a, Thm. 2.1(vii)]). By Proposition III.3.10 part 1, the cocompression

$$\pi_* \left(\prod_{i \geq 0} \Sigma^{-i}(J) \right) \quad (\text{III.5})$$

is a semi-injective object in $\text{Dif}\mathbb{Z}$. We will show that it is not minimal. By Proposition III.6.1, it is minimal if and only if its expansion

$$\pi^* \pi_* \left(\prod_{i \geq 0} \Sigma^{-i}(J) \right) \quad (\text{III.6})$$

is minimal. As (III.6) is periodic, minimality holds if and only if the kernel of the differential of (III.5) is large (see [HJ24a, Thm. 2.1(vii)]). The kernel of the differential of (III.5) is isomorphic to the product of the kernels of the differentials in (III.4). So, we need to show that the subobject

$$\text{Ker } \pi_* \left(\prod_{i \geq 0} \Sigma^{-i}(J) \right) \cong \mathbb{Z} \oplus \prod_{i \geq 1} (\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}) \subseteq \mathbb{Q} \oplus \prod_{i \geq 1} (\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}) \cong \pi_* \left(\prod_{i \geq 0} \Sigma^{-i}(J) \right)$$

is not large. To this end, let \mathcal{M} be the subobject of $\mathbb{Q} \oplus \prod_{i \geq 1} (\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z})$ generated by the element $\mathfrak{x} = (0, (\frac{1}{i}, 0)_{i \geq 1})$. As there is no integer \mathfrak{n} such that $\frac{\mathfrak{n}}{i}$ is an integer for all $i \geq 1$, there cannot exist a nonzero element of \mathcal{M} that also lies in $\mathbb{Z} \oplus \prod_{i \geq 1} (\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z})$. Hence, the kernel is not large.

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