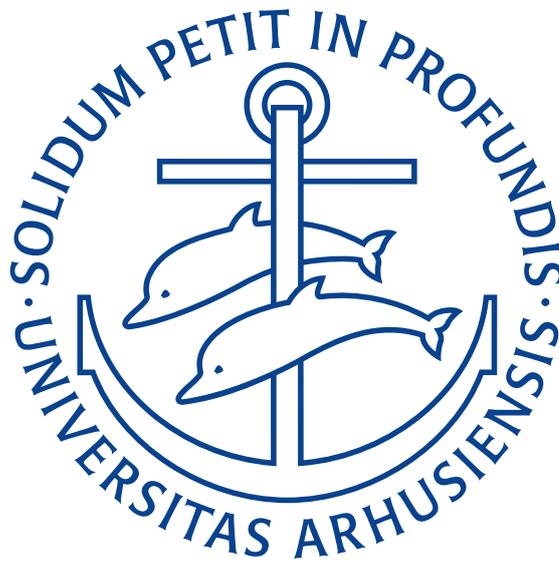# Crofton and Blaschke–Petkantschin Formulae: Applications in Stereology and Intersection Probabilities

PhD Dissertation

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Crofton and Blaschke–Petkantschin Formulae:
Applications in Stereology and Intersection Probabilities

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Preface

This dissertation summarizes the marathon that has been my PhD study at the Department of Mathematics, Aarhus University. The endeavor has been long. Starting in August 2021, my journey took me through the wonderful world of mathematics, valleys of despair, and up mountains where the whole world lay at your feet. This journey culminated in September 2025 with this dissertation.

Under the watchful eye of my main supervisor Markus Kiderlen, I set out on this journey with the aim of contributing to the fields of Convex Geometry and Integral Geometry. After a small detour, the top of the first mountain was in sight. The first result, *Crofton Formulae with a Fixed Subspace*, introduced new Crofton-type formulae unknown to the world. New mountain peaks were in sight. Through winding valleys and rugged mountain trails, I conquered another summit. *Stereologically adapted Crofton formulae for tensor valuations* were claimed, lifting previous results into the sky. On these winding roads, hidden treasures were recovered. The Blaschke–Petkantschin formulae in *A Blaschke–Petkantschin formula for linear and affine subspaces with application to intersection probabilities* connected the trail through integral geometry into intersection probabilities.

Now, as this journey comes to an end, I look back with gratitude at the challenges and the knowledge gained. It has been a privilege to explore these mathematical landscapes and contribute to their unfolding story.

With this, I wish to acknowledge the people who have guided and supported me on this path.

First and foremost, I would like to thank my supervisor Markus Kiderlen, for his support. Mathematics is a joy on its own, but what truly makes it special is the people you meet along the way. Markus exemplifies this as the *perfect supervisor* – always available, often with the answer, and, most importantly, deeply caring. His encouragement, whether in moments of success or struggle, made this PhD journey possible. For his years of mentorship and friendship, I am deeply grateful.

Special thanks to Christoph Thäle and his research group at Ruhr University Bochum for the opportunity to visit and collaborate. It was a memorable experience, one I hope to revisit someday.

I also wish to thank my colleagues at the Department of Mathematics for creating a warm, welcoming atmosphere. In particular, I am grateful to my office mates over the years – Ida Egendal, Ragnhild Laursen, Anton Tiepner, and Péter Juhász, for the camaraderie and pleasant breaks from work.

Finally, I owe my deepest gratitude to my family and friends, especially my fiancée, Maja Gry Nielsdatter. Throughout this journey, she kept me grounded. I thank her for her unwavering support, for celebrating the successes, and for the support when things did not go as planned. Her love reminds me that there is more to life than mathematics, and for that, I am forever thankful.

Emil Dare
Aarhus, September 2025

Abstract

This dissertation extends classical results in integral and stochastic geometry. We build on the classical Crofton formula. This formula connects intrinsic volumes of convex bodies in $\mathbb{R}^n$ to integrals over affine subspaces. We introduce new rotational Crofton formula where the integrating affine flats are required to contain a fixed subspace. Moreover, our approach naturally extends to the setting of vertical sections, a framework widely used in stereology to extract geometric information under less stringent randomness assumptions. In this vertical context, our results yield estimators that further broaden the applicability of Crofton-type methods in both theoretical analysis and practical applications.

We then extend these results into the setting of translation invariant Minkowski tensors. Classically, the invariantly integrated Minkowski tensor of the section profiles of a convex body K with flats is determined. According to Alesker's characterization theorem, this tensor can essentially be expressed as a linear combination of Minkowski tensors of K, and explicit but involved expressions for the corresponding coefficients have been found. Motivated by applications, we establish stereologically adapted versions of such Crofton formulae, where a predetermined Minkowski tensor of K is expressed as an invariant integral of a combination of Minkowski tensors of sections. We call such Crofton formulae *'stereologically adapted'* as they allow to assess spatial properties such as orientation or eccentricity of K from lower dimensional sections, and thus are useful in applications such as materials sciences and biological imaging.

Finally, we present a new version of a Blaschke–Petkantschin formula for intersections between linear and affine subspaces. This framework yields density functions for intersection probabilities that depend on the distance from the origin and also covers cases where the affine subspace is tangent to the unit sphere.

Together, these results bring new insights to integral geometry, close theoretical gaps, and open up for new applications in n-dimensional probability and stereology.

Resumé

Denne afhandling udvider klassiske resultater inden for integral- og stokastisk geometri. Vi bygger videre på den klassiske *Crofton-formel*. Denne formel forbinder indre volumener af konvekse legemer i $\mathbb{R}^n$ med integraler over affine underrum. Vi introducerer en ny rotationsinvariant Crofton-formel, hvor de integrerende affine underrum forudsættes at indeholde et fast underrum. Vores tilgang udvider desuden naturligt til vertikale snit, en ramme, der er udbredt i stereologi til at få geometrisk information under mindre strenge antagelser. I denne vertikale kontekst giver vores resultater estimatorer, der yderligere øger anvendeligheden af Crofton-lignende metoder i både teoretisk analyse og praktiske anvendelser.

Vi udvider derefter disse resultater til rammen af translationsinvariante Minkowski-tensorer. Klassisk bestemmes Minkowski-tensoren af snitprofiler af et konvekst legeme *K* som integraler over affine underrum. Ifølge Aleskers sætning kan sådanne tensorer hovedsageligt udtrykkes som lineære kombinationer af Minkowski-tensorer af *K*, og eksplicitte men komplekse udtryk for de tilsvarende koefficienter er kendt. Motiveret af anvendelser etablerer vi stereologisk tilpassede versioner af sådanne Crofton-formler, hvor en forudbestemt Minkowski-tensor af *K* udtrykkes som et invariant integral af en kombination af Minkowski-tensorer af snit. Vi kalder disse Crofton-formler *'stereologisk tilpassede'*, da de tillader estimering af rumlige egenskaber som orientering og excentricitet af *K* ud fra lavdimensionale snit, og er således anvendelige i eksempelvis materialevidenskab og biologi.

Afslutningsvis præsenterer vi en ny Blaschke–Petkantschin formel for skæringer mellem lineære og affine underrum. Dette giver tætheder for snit-sandsynligheder, som afhænger af afstanden til origo, og dækker også tilfælde, hvor det affine underrum er tangent til enhedssfæren.

Samlet set bringer disse resultater nye indsigter til integralgeometrien, lukker teoretiske huller og åbner nye anvendelsesmuligheder inden for sandsynlighed og stereologi.

Introduction

This chapter provides an overview of the topics and main results addressed in this dissertation. The central theme is *stochastic geometry*, with particular emphasis on the derivation and application of *Blaschke–Petkantschin formulae* and related techniques from *integral geometry*. The dissertation is based on the following three peer-reviewed and published research papers:

Paper A **Rotational Crofton formulae with a fixed subspace**
Published in *Advances in Applied Mathematics* (2024)
DOI: https://doi.org/10.1016/j.aam.2023.102611

Paper B **Stereologically adapted Crofton formulae for tensor valuations**
Published in *Advances in Applied Mathematics* (2024)
DOI: https://doi.org/10.1016/j.aam.2024.102754

Paper C **A Blaschke–Petkantschin formula for linear and affine subspaces with applications to intersection probabilities**
Published in *Nonlinear Analysis* (2025)
DOI: https://doi.org/10.1016/j.na.2024.113672

These three papers collectively form the core of this dissertation. Each paper is presented in its peer-reviewed form and conforms to its published versions. As different journals use different layouts, minor changes to layout, typesetting, and correction of typing errors have been made to present the papers in a uniform way. Paper A and C were joint projects, in which I contributed extensively at all stages, from research to writing and submission. Paper B lists me as the sole author, and thus I handled the whole process from start to submission.

Of noteworthy changes to the papers presented in this dissertation, we remark that a proper specification of a constant has been made in Paper B. For further information on this, see the end of Section 1.5. Finally, we note that the main introduction may resemble parts of the introductions given in [20, 22, 23], as it introduces concepts and notation from these three papers.

The overall structure of the dissertation is as follows. Chapter 1 contains preliminary remarks, notation, and a discussion of the three papers presented in this dissertation. The chapter begins with Section 1.1, which contains an introduction to the area of research and a motivation. Section 1.2 contains the notation and terminology used throughout this dissertation. At the end of this section, the concepts are supplemented with illustrative examples. Sections 1.3 through 1.5 contain discussions of Paper A through C, respectively. Each section ends with remarks on the applications of the results presented in these three papers. The final three chapters contain the three published papers.

1.1 MOTIVATION

Integral and stochastic geometry have grown over time as a mathematical framework for analyzing shapes and spatial structures. One of the fundamental tools in this field is the *Crofton formula*, which connects intrinsic volumes of convex bodies to integrals over affine subspaces. First introduced by Morgan Crofton in the 19th century ([14]) this result is now central in modern convex and integral geometry, see [75, Chap. 7]. Building on this foundation, recent research has sought to generalize Crofton-type formulae to more intricate settings involving valuations and tensorial quantities. To state a generic version of the problem considered in this dissertation, we will work with valuations (additive functionals) on the family $\mathcal{K}^n$ of convex bodies (non-empty compact convex subsets) in $\mathbb{R}^n$. For details on invariant measures, see Section 1.2 below.

Problem : Let Q be a valuation on $\mathcal{K}^n$ and fix a (sub)family F of all affine subspaces of a fixed dimension equipped with an invariant measure ν^*. Does there exist a measurement function φ such that

$$\int_F \varphi(K \cap E, E)\, \nu^*(dE) = Q(K) \tag{1.1}$$

holds for all $K \in \mathcal{K}^n$?

Seeking solutions to the above problem is motivated not only by theoretical interest, but also by practical applications. For instance, suppose ν^* is a probability measure on the family F. Then, given independent random subspaces $E_1, \ldots, E_n$ with distribution ν^*, the estimator

$$\hat{Q}(K) = \frac{1}{n} \sum_{i=1}^{n} \varphi(K \cap E_i, E_i) \tag{1.2}$$

is an unbiased estimator of $Q(K)$, provided that the identity (1.1) holds. In this way, solutions to the problem naturally give rise to sampling procedures, where the sampling space is the family F, and geometric data is extracted from K through intersections with randomly selected affine subspaces (flats).

A key advantage of such sampling schemes is that information about K is accessed only through its sections $K \cap E$, where $E \in F$. For this reason, throughout this dissertation we focus on cases where the integrand φ depends on K only through the section $K \cap E$, and possibly on E itself.

A classical example arises by taking F to be the space of affine q-flats in $\mathbb{R}^n$ for $q \in \{0, \ldots, n-1\}$, equipped with a motion-invariant measure. In this setting, the classical Crofton formula provides an explicit solution to (1.1) when $Q(K)$ is taken to be the volume, surface area, or another intrinsic volume of K, see (A.19) for the formula.

The use of solutions to (1.1) in practical applications began in the mid-20th century, where researchers combined geometric arguments with probabilistic methods to solve problems in microscopy and materials science. Methods for estimating particle characteristics, such as volume, surface area and mean width, became standard in biology and materials sciences. In the 1980s, this interplay between integral geometry and applied sciences further strengthened when Raymond E. Miles and Philip J. Davy introduced a design-based approach to sampling ([50, p. 12]).

In this decade another improvement was made by Adrian Baddeley and coworkers. They introducing design-based methods for vertical sections ([5, 6, 7]).

In certain applications, the estimator (1.2) is difficult or impossible to evaluate when F is the full family of affine subspaces of a fixed dimension. For instance, in layered materials visual or structural features of the intersection $E \cap K$ are most pronounced when $E \in F$ is perpendicular to the layers. This leads to the study of problem (1.1) with F being the family of affine subspaces, which are parallel to a fixed subspace of lower dimension. The solution, known as *vertical section formula* leads to a number of important sampling procedures such as the vertical rotator.

Originally, these estimators concerned basic real-valued quantities such as volume, surface area or mean width. In later years, solutions to (1.1) with more general choices of $Q(K)$ have been sought for, for instance letting $Q(K)$ be a fixed Minkowski tensor. With such a choice of $Q(K)$, solutions would allow for sampling schemes estimating spatial quantities such as the orientation of the object. For instance, [54] combines the techniques from vertical settings to estimate volume tensors in three dimensions.

Motivated by both the theoretical groundwork and practical applications arising from (1.1), this dissertation seeks to investigate new settings in which such Crofton-type formula can be formulated.

The first setting provides solutions to (1.1) where $Q(K)$ is a predetermined intrinsic volume. In this setting we consider two different classes of sampling families F.

We first consider the case where F consists of all subspaces of a fixed dimension that contain a predetermined linear subspace of lower dimension, hence we consider Crofton-type formulae with a fixed axis. In three dimensions, this relates to the sampling scheme in which the object is cut through a fixed point, a method already established in the literature, see for instance [48]. The second case studied, is the aforementioned setting of vertical sections, where F consists of all affine subspaces that are parallel to a fixed subspace of lower dimension.

Both cases are detailed in Paper A ([22]). The results in [22] can be generalized to tensor-valued valuations replacing the (real-valued) intrinsic volumes. These tensor-valued valuations are called Minkowski tensors and carry additional geometric information.

That is, we change the setting to solutions of (1.1) with $Q(K)$ being a translation-invariant Minkowski tensors, still with the two before-mentioned choices of sampling spaces F. Classically, the invariantly integrated Minkowski tensor of the section profiles of a convex body K with flats is determined. According to Alesker's characterization theorem (see [2]), this tensor can essentially be expressed as a linear combination of (products of powers of the metric tensor with) Minkowski tensors of K. The corresponding coefficients, although involved, have also been found. Such results are not our primary interest in our research context. We seek a measurement function such that the integral equals a predetermined translation-invariant Minkowski tensor. We achieve this for multiple examples and coin the term that the formulae is *stereologically adapted*. With these new formulae, we produce Crofton-type formulae with fixed axis for translation-invariant Minkowski tensors. Likewise, we produce vertical section formulae in the setting of translation-invariant Minkowski tensors, which contain new results even in the three-dimensional setting. These results are outlined in Paper B ([20]).

In the final contribution of this dissertation we deviate from the problem (1.1). Instead, we investigate intersection probabilities arising from intersections of affine subspaces and linear subspaces. Surprisingly, this investigation is closely connected to the two preceding papers, as all three rely heavily on Blaschke–Petkantschin formulae to calculate integrals of geometric

significance. In fact, we show that the intersection probabilities of interest can equivalently be seen as new Blaschke–Petkantschin formulae. In this final contribution we provide explicit densities for intersections of (affine)subspaces under various constraints. Notably, the results include cases where the affine subspaces are tangent to the unit sphere or intersect the unit ball, yielding new insights into intersection probabilities in arbitrary dimensions. The detailed findings are presented in Paper C ([23]). The theoretical results in this dissertation can, like the classical results, have multiple practical applications. For further information, we recommend [17], a newly published book on applications of these concepts. For historical notes, we recommend [50, 47], and for a newly published book on rotational integral geometry and its applications, we refer to [49].

1.2 BASIC NOTATION

We will now briefly introduce the overall notations used throughout this dissertation. As the literature and results in integral geometry are vast, we only state notation relevant for these three papers and refer to [74, 75] as more comprehensive references.

Throughout this dissertation we work in the Euclidean space $\mathbb{R}^n$, equipped with the usual inner product $\langle \cdot, \cdot \rangle$, associated norm $\| \cdot \|$ and orthogonal basis $e_1, \dots, e_n$. We let $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ and $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ be the unit ball and unit sphere, respectively. We also equip the space $\mathbb{R}^n$ with the d-dimensional Hausdorff measures $\mathcal{H}^d$ for $d = 0, \dots, n$ normalized in the usual way, for instance

$$\kappa_n = \mathcal{H}^n(B^n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(1 + \frac{n}{2})},$$

$$\omega_n = \mathcal{H}^{n-1}(S^{n-1}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})},$$

where $\Gamma(\cdot)$ is the gamma function. For a set $A \subseteq \mathbb{R}^n$, we let $\mathrm{span}(A)$, $\mathrm{aff}(A)$ and $\mathrm{conv}\, A = \mathrm{conv}(A)$ denote the linear, the affine and the convex hull of A, respectively. The set $A^{\perp} = \{x \in \mathbb{R}^n : \langle x, y \rangle = 0 \text{ for all } y \in A\}$ is the orthogonal complement of $\mathrm{span}\, A$. If B is another set, we let $A + B$ denote the Minkowski sum of these two sets. If $B = \{b\}$, we simply write $A + b$ for the translation of A by b. For $x \in \mathbb{R}^n$ we let $d(x, A) = \inf_{a \in A} \|x - a\|$ denote the distance from x to A. If K is a closed convex subset we let $p(x|K)$ denote the metric projection of x onto K. If L is a linear subspace we let $p(x|L)$ denote the orthogonal projection of x onto L, which also coincides with the metric projection of x onto L. For readability, we use the shorthand notation $x|K$ in Paper C.

1.2.1 *Subspaces and related notions*

The three papers presented in this dissertation work with particular families of flats (affine subspaces) of $\mathbb{R}^n$, which we briefly introduce following the notation from [75, Chap. 13]. We let $G(n, k)$ be the *Grassmannian* of k-dimensional *linear* subspaces of $\mathbb{R}^n$ and let $A(n, k)$ denote the family of *affine* k-dimensional flats of $\mathbb{R}^n$ for $k \in \{0, \dots, n\}$. Note that each $E \in A(n, k)$ can be written as $E = L + x$ for a $L \in G(n, k)$ and $x \in L^{\perp}$. We will denote the mapping that translates E into its linear component by $\mathrm{lin}(E)$. Formally, we define $\mathrm{lin}(E) = E - p(o|E)$ for $E \in A(n, k)$.

In all the papers we work with flats and subspaces incident with others. We therefore define, for a fixed $L \in G(n, p)$ with $p \in \{0, \ldots, n\}$, the spaces

$$G(L, k) = \begin{cases} \{L' \in G(n, k) \; : \; L' \subseteq L\}, & \text{if } k \leq p, \\ \{L' \in G(n, k) \; : \; L \subseteq L'\}, & \text{if } k > p, \end{cases}$$

and, similarly, for $E \in A(n, p)$

$$A(E, k) = \begin{cases} \{E' \in A(n, k) \; : \; E' \subseteq E\}, & \text{if } k \leq p, \\ \{E' \in A(n, k) \; : \; E \subseteq E'\}, & \text{if } k > p. \end{cases}$$

There exists a unique probability measure on $G(L, k)$ that is invariant under all rigid rotations that keep L fixed. We denote this measure by ν_k^L. Its construction is described in [75, p. 590]. If $L = \{0\}$ we let $\nu_k = \nu_k^{\{o\}}$ be the unique invariant probability measure on $G(n, k)$. We equip $A(E, k)$ with an invariant measure μ_k^E, which is invariant under all rigid rotations and translations fixing E. It is unique up to normalization, which we choose as in [75]. Furthermore, we let ν_k be an invariant measure on $A(n, k)$ normalized as in [75]. For an example of the construction of the measure μ_k we refer to Paper A or [75].

In Paper C, we need the group SO_n of proper rotations in $\mathbb{R}^n$, and we let ν denote the unique invariant probability (Haar) measure hereon, see [75, Chap. 13]. For $E \in A(n, q)$ we let λ_E denote the Lebesgue measure on E and if $q = n$ we abbreviate it to λ_n.

Throughout this dissertation we will encounter integral relations where the *subspace determinant* appears. Roughly speaking, this quantity describes the relative angle between two subspaces and is sometimes called the *'generalized sine function'*. If $L \in G(n, k)$ and $M \in G(n, r)$ with $k + r \leq n$, we define the subspace determinant as follows; Let $l_1, \ldots, l_k$ and $m_1, \ldots, m_r$ be orthogonal basis for L and M, respectively. Then, we define the subspace determinant

$$[L, M] = \lambda_{k+r} \left(\left\{ \sum_{j=1}^{k} \alpha_j l_j + \sum_{j=1}^{r} \beta_j m_j \; : \; \alpha_j \in [0, 1], \beta_j \in [0, 1] \right\} \right)$$

of L and M as the volume of the parallelepiped spanned by $l_1, \ldots, l_k, m_1, \ldots, m_r$. If $k = r = 1$, this parallelepiped simplifies to a parallelogram. Its area is equal to the sine function of the angle between the two spanning vectors l_1 and m_1 for L and M, respectively, see Figure 1.1.
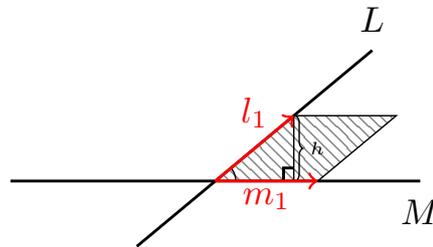

Figure 1.1: Two subspaces, L and M, spanned by the vectors l, m respectively. The area hatched equals $[M, L]$, which is also equal to the height h.

The subspace determinant will appear throughout this dissertation in multiple integral relations. For further remarks on this quantity, see the introduction of Paper A or [75].

1.2.2 *Convex sets and related notions*

We now turn our attention to notation regarding convex geometry, and in particular, convex bodies. A set $K \subset \mathbb{R}^n$ is called a *convex body* if it is nonempty, convex and compact. We denote the family of convex bodies in $\mathbb{R}^n$ by $\mathcal{K}^n$. We let $\mathcal{P}^n$ be the family of all convex polytopes (the convex hull of a finite set of points). The literature on these families is rich and we limit ourselves to only introduce notation relevant for this dissertation. For further information, we recommend the following, but definitely not exhaustive, list of references: [30, 43, 74, 75].

In Paper B, we use the notion of *the normal cone* of $K \in \mathcal{K}^n$ at $x \in K$ to construct a counterexample. We define the normal cone of $K \in \mathcal{K}^n$ at $x \in K$ according to [74, eq. (2.3)] as

$$N(K, x) = \{y \in \mathbb{R}^n \ : \ p(y|K) = x\} - x.$$

Thus, $N(K, x)$ consists of all outer normal vectors of K at x together with the zero vector. If F is a nonempty convex subset of K we further define $N(K, F)$ as $N(K, x)$ for x chosen in the relative interior of F. By [74, p. 83], this notion is independent of the choice of x. In Paper B we need $N(P, F)$ for $P \in \mathcal{P}^n$ and F a k-dimensional face (see [74, p. 16] for details on faces of a convex body). We let $\mathcal{F}_k(P)$ denote the collection of all faces of P of dimension k for $k = 0, \dots, n$.

Throughout the dissertation we will encounter *area measures* and *intrinsic volumes*, which we now will introduce in accordance with [74]. For a $K \in \mathcal{K}^n$ and $x \in \mathbb{R}^n \backslash K$ we define the outer unit normal vector

$$u_K(x) = \frac{x - p(x|K)}{\|x - p(x|K)\|}, \tag{1.3}$$

as the unit vector pointing from $p(x|K)$ to x. We define the *normal bundle*

$$\text{Nor}(K) = \{(p(x|K), u_K(x)) \ : \ x \in \mathbb{R}^n \backslash K\}$$

of K as the collection of points on the boundary of K with their respective outer normal (possible multiple). Thus, for each $x \in \mathbb{R}^n \backslash K$, we can associate a unique normal pair from $\text{Nor}(K)$. We have $\text{Nor}(K) \subseteq \Sigma = \mathbb{R}^n \times S^{n-1}$. If $\beta \subset \Sigma$ and $\epsilon > 0$ we can consider the *local parallel set*

$$\begin{aligned} M_\epsilon(K, \beta) &= \{x \in \mathbb{R}^n \ : \ 0 < d(x, K) \leq \epsilon, \ (p(x|K), u_K(x)) \in \beta\} \\ &= \{x \in (K + \epsilon B^n) \backslash K \ : \ (p(x|K), u_K(x)) \in \beta\}, \end{aligned}$$

which is the set of all vectors in $\mathbb{R}^n \backslash K$ with distance at most ϵ from K with associated normal pair in β, see Figure 1.2 for visualization.
If, furthermore, β is measurable, then the Lebesgue measure of $M_\epsilon(K, \beta)$ is a polynomial in $\epsilon > 0$ and the coefficients are measures in β. That is,

$$\lambda_n(M_\epsilon(K, \beta)) = \sum_{k=0}^{n-1} \epsilon^{n-k} \kappa_{n-k} \Lambda_k(K, \beta), \tag{1.4}$$

where $\Lambda_k(K, \cdot)$ are the *support measures* of K for $k = 0, \dots, n-1$, see [74, Thm. 4.2.1].

From [74, Thm. 4.2.1], it furthermore follows that the support measures are finite measures. Letting $\beta = \Sigma$ and adding $\lambda_n(K)$ to both sides of the above equation, we obtain the *Steiner formula*

$$\lambda_n(K + \epsilon B^n) = \sum_{k=0}^{n} \epsilon^{n-k} \kappa_{n-k} V_k(K),$$
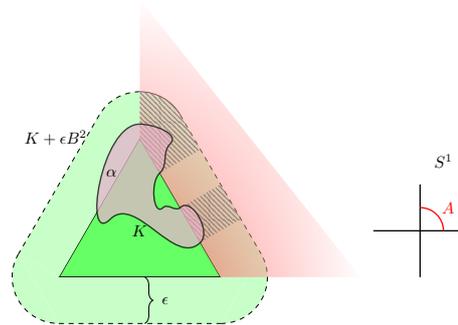
Figure 1.2: A triangle K with its parallel set $K + \epsilon B^2$. The purple set α is a subset of $\mathbb{R}^2$. The red cone marks all the points in $\mathbb{R}^2 \setminus K$ such that their outward normal vector is contained in $A \subset S^1$. The crosshatched area corresponds to $M_\epsilon(K, \alpha \times A)$.

if we define $V_k(K) = \Lambda_k(K, \Sigma)$ for $k = 0, \ldots, n-1$ and $V_n(K) = \lambda_n(K)$. The quantities V_k are the *intrinsic volumes*, which describe certain characteristics of a convex body, see [75]. For instance, $V_0(K) = 1$ is the Euler–Poincaré characteristic, $V_1(K)$ is proportional to the mean width of K and, if furthermore K is full-dimensional, then $V_{n-1}(K)$ equals one-half of the surface area of K. A visual representation of this decomposition can be found in Figure 1.3.
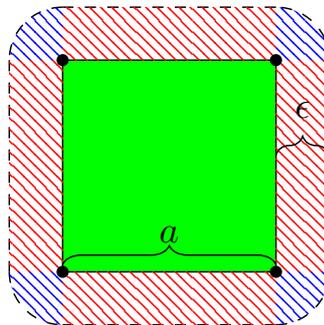
Figure 1.3: A cube K with side length a, where the parallel set $K + \epsilon B^2$ is hatched.

Here $K + \epsilon B^2$ is drawn for K being a cube with sidelenght a. The volume of $K + \epsilon B^2$ can be decomposed as

$$\lambda_2(K + \epsilon B^2) = \epsilon^2 \pi + \epsilon 4a + a^2, \tag{1.5}$$

in correspondence with the above mentioned interpretations.

This intuitive interpretation extends to convex polytopes. If $P \in \mathcal{P}^n$ and $k = 0, \ldots, n$ then $V_k(P)$ equals a weighted sum of the k-dimensional Hausdorff measures of all faces of P of dimension k, see [74, eq. (4.23)]. For a general convex body, this interpretation is no longer true. For instance, if $K = B^n$, then there are no faces of B^n of dimension k with $0 < k < n$. The intrinsic volumes can still be directly calculated. For example, expanding

$$\lambda(B^n + \epsilon B^n) = (1 + \epsilon)^n \kappa_n = \sum_{k=0}^{n} \binom{n}{k} \epsilon^{n-k} \kappa_n, \tag{1.6}$$

and comparing coefficients with the Steiner formula yields $V_k(B^n) = \frac{\kappa_n}{\kappa_{n-k}}\binom{n}{k}$. In particular, note that $V_{n-1}(B^n) = \frac{\omega_n}{2}$ equals half the surface area of B^n.

Besides the support measures, a set of marginal measures is of interest in Paper B. For $A \subset S^{n-1}$ measurable we define the *area measures*

$$S_k(K, A) = \frac{n\kappa_{n-k}}{\binom{n}{k}}\Lambda_k(K, \mathbb{R}^n \times A), \quad k = 0, \dots, n-1. \tag{1.7}$$

We adopt this normalization as when $K \in \mathcal{K}^n$ is full-dimensional then the *surface area measure* satisfies

$$S_{n-1}(K, A) = \mathcal{H}^{n-1}(\tau(K, A)),$$

where $\tau(K, \omega)$ is the set of boundary vectors of K with an associated outer normal in A. Thus, if $K \in \mathcal{K}^n$ is full-dimensional, then $S_{n-1}(K, A)$ can be interpreted as the surface area of those points on the boundary of K where the outer normals are elements of A, see [74, eq. (4.32)]. If $P \in \mathcal{P}^n$ is a polygon, then the area measures can be given more explicitly. By [74, eq. (4.3)] and [74, eq. (4.11)] we get

$$S_k(P, A) = \frac{1}{\binom{n-1}{k}}\sum_{F \in \mathcal{F}_k} \mathcal{H}^k(F)\mathcal{H}^{n-k-1}(N(P, F) \cap A), \tag{1.8}$$

for $k = 0, \dots, n-1$ and a measurable set $A \subset S^{n-1}$.

1.2.3 *Minkowski tensors and related notions*

In Paper B, these area measures play a central role as they serve as a foundation for defining specific tensors valuations of interest. The notation for tensors used in Paper B is consistent with the conventions in [10, 40, 44]. We will denote a multi-linear map from $(\mathbb{R}^n)^r$ to $\mathbb{R}$ as a *tensor of rank r* for $r \in \mathbb{N}_0$. If it is symmetric under all permutations of its r arguments, we call the tensor *symmetric*. We let $\mathbb{T}^r$ denote the set of symmetric tensors of rank r. We define the *symmetric tensor product* of $a_i \in \mathbb{T}^{r_i}$ for $r_i \in \mathbb{N}_0$ with $i = 1, 2$ as the tensor of rank $r = r_1 + r_2$ defined by

$$(a_1 \odot a_2)(x_1, \dots, x_r) = \frac{1}{r!}\sum_{\sigma \in S_r} a_1(x_{\sigma(1)}, \dots, x_{\sigma(r_1)})a_2(x_{\sigma(r_1+1)}, \dots, x_{\sigma(r)}),$$

for $x_1, \dots, x_r \in \mathbb{R}^n$, see [40, p. 29]. Here S_r denotes the group of permutations of the set $\{1, \dots, r\}$. We will use the abbreviations

$$a_1 a_2 = a_1 \odot a_2, \quad a_1^s = \underbrace{a_1 \odot a_1 \odot \cdots \odot a_1}_{s \text{ times}},$$

with $s \in \mathbb{N}$. We extend the notion letting $a_1^0 = 1$. A special tensor of rank 2 is the metric tensor, given by

$$Q(x, y) = \langle x, y \rangle, \quad x, y \in \mathbb{R}^n,$$

which will play a central role in formulae appearing in Paper B. We can now define the *Minkowski tensors* in accordance with [10, 25, 74]. They are tensor-valued valuations of rank $r + s$, $r, s \in \mathbb{N}_0$ given by

$$\Phi_k^{r,s}(K) = \frac{1}{r!s!}\frac{\omega_{n-k}}{\omega_{n-k+s}}\int_{\Sigma^n} x^r u^s \, \Lambda_k(K, d(x, u)), \quad k = 0, \dots, n-1, \tag{1.9}$$

$$\Phi_n^{r,0}(K) = \frac{1}{r!} \int_K x^r \, \lambda(dx).$$

for $K \in \mathcal{K}^n$. We understand integrals of tensor valuations pointwise. For instance, if $\mathbf{x} \in (\mathbb{R}^n)^{r+s}$ then

$$\Phi_k^{r,s}(K)(\mathbf{x}) = \frac{1}{r!s!} \frac{\omega_{n-k}}{\omega_{n-k+s}} \int_{\Sigma^n} (x^r u^s)(\mathbf{x}) \, \Lambda_k(K, d(x,u)), \quad k = 0, \dots, n-1.$$

Minkowski tensors with $r = 0$ are the main focus of Paper B. When $r = 0$, we note that (1.7) implies

$$\Phi_k^{0,s}(K) = \binom{n-1}{k} \frac{1}{\omega_{n-k+s}s!} \int_{S^{n-1}} u^s S_k(K, du), \quad k = 0, \dots, n-1, \tag{1.10}$$

corresponding to the definition presented in (B.6). As $\Phi_k^{0,s}(K)$ can be written as an integral with respect to an area measure it follows that $\Phi_k^{0,s}(K+t) = \Phi_k^{0,s}(K)$ for $t \in \mathbb{R}^n$, so Minkowski tensors with $r = 0$ are translation invariant.

For notational convenience, we define $\Phi_{k,s}(K) = \Phi_k^{0,s}(K)$ in Paper B. As tensors valuations are linear and symmetric we note that Minkowski tensors can be expressed in a coordinate representation. Letting $e_1, \dots, e_n$ denote the standard basis of $\mathbb{R}^n$ we can represent $\Phi_k^{r,s}(K)$ as an array of elements

$$\left(\Phi_k^{r,s}(K) \right)_{i_1, \dots, i_n} = \Phi_k^{r,s}(K)(\underbrace{e_1, \dots e_1}_{i_1}, \dots, \underbrace{e_n, \dots, e_n}_{i_n}), \tag{1.11}$$

with $i_1 + i_2 + \cdots + i_n = r + s$, see [25]. This representation illustrates how $\Phi_k^{r,s}(K)$ can be interpreted as a collection of specific cross-moments of $\Lambda_k(K, \cdot)$. In $\mathbb{R}^3$ estimation of $\Phi_k^{r,s}(K)$ is useful in studying the underlying convex body K, see for instance, [56, 79] for applications in brain imaging or [76] for an overview of their usefulness in materials science.

If $K \in \mathcal{K}^n$ is contained in a flat $E \in A(n,p)$ for $p \in \{1, \dots, n\}$, it can be seen as either a lower dimensional convex body in $\mathbb{R}^n$, or as a convex body in E. Hence, there are (at least) two possible ways to determine Minkowski tensors: the one just defined, and the other one, where this definition is applied with E as ambient space, still though as a tensor on $\mathbb{R}^n$. When we view E as the ambient space we define the *intrinsic support measures* $\Lambda_k^{(E)}(K, \cdot)$ of K for $k = 0, \dots, p-1$ relative to E in accordance with [52, p. 30] or [55] . If $L = \lin(E)$ is the linear subspace parallel to E, then $\Lambda_k^{(E)}(K, \cdot)$ is the image measure of $\Lambda_k(K, \cdot)$ under the almost everywhere defined mapping from $(\mathbb{R}^n \times S^{n-1})$ to $(\mathbb{R}^n \times (S^{n-1} \cap L))$ given by

$$(x, u) \mapsto \left(x, \frac{p(u|L)}{\|p(u|L)\|} \right).$$

With the intrinsic support measures we can define the intrinsic Minkowski tensors as

$$\Phi_{k,r,s}^{(E)}(K) = \frac{\omega_{p-k}}{r!s!\omega_{p-k+s}} \int_{E \times (S^{n-1} \cap \lin(E))} x^r u^s \, \Lambda_k^{(E)}(K, d(x,u)), \quad k = 0, \dots, p-1, \tag{1.12}$$

$$\Phi_{p,r,0}^{(E)} = \frac{1}{r!} \int_K x^r \lambda_E(dx),$$

where we still view them as tensors on $\mathbb{R}^n$.

The distinction between intrinsic Minkowski tensors and Minkowski tensors is relevant in Paper B. Here, *intrinsic translation invariant Minkowski tensors* $\Phi_{k,0,s}^{(E)}(K)$ appears using the adapted notation $\Phi_{k,s}^{(E)}(K)$. To see this distinction compare Theorem B.3 and Theorem B.4. For convex polytopes the distinction between (1.9) and (1.12) can be made more explicit. If $P \in \mathcal{P}^n$ is a polytope such that $P \subset E \in A(n, p)$, then

$$\Phi_{k,r,s}^{(E)}(P) = \frac{1}{r!s!\omega_{p-k+s}} \sum_{F \in \mathcal{F}_k(P)} \int_F x^r \lambda_F(dx) \int_{N(P,F)\cap S^{n-1}\cap\operatorname{lin}(E)} u^s\, \mathcal{H}^{p-k-1}(du), \qquad (1.13)$$

$$\Phi_k^{r,s}(P) = \frac{1}{r!s!\omega_{n-k+s}} \sum_{F \in \mathcal{F}_k(P)} \int_F x^r \lambda_F(dx) \int_{N(P,F)\cap S^{n-1}} u^s\, \mathcal{H}^{n-k-1}(du), \qquad (1.14)$$

for $k = 0, 1, \ldots, p - 1$, see [44, p. 116] (note that [44] uses a slightly different normalization, see (B.20) and (B.21)). Although both (1.13) and (1.14) describe tensors of rank $r + s$, we note that it is possible that $\Phi_{k,r,s}(P) \neq \Phi_{k,r,s}^{(E)}(P)$ for certain indices and $P \in \mathcal{P}^n$. For instance, if $P = [-1, 1] \times \{1\}$ is a line segment in $\mathbb{R}^2$ then $\Phi_{1,r,s}^{(e_2+\operatorname{span} e_1)}(P)(\mathbf{e_2}) = 0$ for all $r + s$ tuples $\mathbf{e_2} = (e_2, e_2, \ldots, e_2) \in (\mathbb{R}^n)^{r+s}$ with $r + s > 0$, whereas $\Phi_1^{r,s}(K)(\mathbf{e_2}) > 0$ if s is even.

1.2.4 *Examples*

We now present two examples that illustrate key aspects of the material introduced so far. These examples clarify the notation and demonstrate how to compute the relevant quantities using formulae presented in this dissertation. Additionally, one example provides a convex body for which the translation-invariant Minkowski tensor differs from the intrinsic translation-invariant Minkowski tensor. The considered examples are basic and well known. We include them here only for illustration. They outline the calculations of surface area measures and Minkowski tensors for selected convex bodies. If only the Minkowski tensors are of interest, a more direct derivation can be based on invariance arguments.

The two objects of interest are the three-dimensional centered unit cube $T = [-\frac{1}{2}, \frac{1}{2}]^3$, and the disc $B = B^3 \cap e_3^\perp$ embedded in $\mathbb{R}^3$.

Example 1.1 (**The centered cube** T).
We aim to calculate some translation-invariant Minkowski tensors of T using (1.10). We use (1.8) to calculate the area measures of T and start by describing the faces of T. If $a_1, \ldots, a_8$ are the corners of T arranged in increasing order, then it follows that

$$\mathcal{F}_0(T) = \Big\{ \{a_1\}, \{a_2\}, \ldots, \{a_8\} \Big\},$$

$$\mathcal{F}_1(T) = \Big\{ [a_1, a_2], [a_1, a_3], [a_1, a_5], [a_2, a_4], [a_2, a_6], [a_3, a_4], [a_3, a_7], [a_4, a_8],$$
$$[a_5, a_6], [a_5, a_7], [a_6, a_8], [a_7, a_8] \Big\},$$

$$\mathcal{F}_2(T) = \Big\{ \operatorname{conv}(a_1, a_2, a_3, a_4), \operatorname{conv}(a_5, a_6, a_7, a_8), \operatorname{conv}(a_3, a_4, a_7, a_8), \operatorname{conv}(a_1, a_2, a_5, a_6),$$
$$\operatorname{conv}(a_2, a_4, a_6, a_8), \operatorname{conv}(a_1, a_3, a_5, a_7) \Big\}$$

with $[a, b]$ being the line segment with endpoints $a, b \in \mathbb{R}^3$. For a visualization of this, see Figure 1.4.
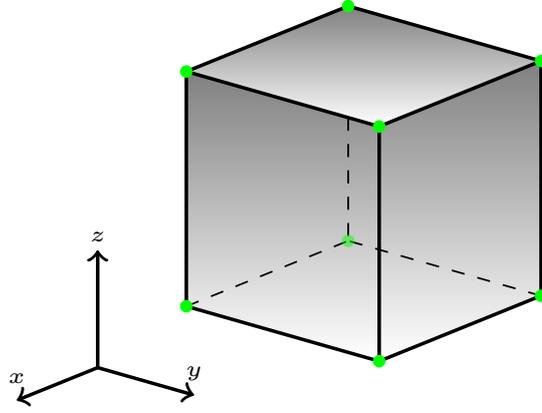
Figure 1.4: The cube T. The green vertices each correspond to a vertex $F \in \mathcal{F}_0(T)$. Each black line corresponds to an edge $F \in \mathcal{F}_1(T)$. The gray bounding squares each correspond to a facet $F \in \mathcal{F}_2(T)$.

Having described the faces of T, we now compute the corresponding normal cones. The normal cones associated with the vertices is given by

$$N(T, \{a_i\}) = \left\{ x \in \mathbb{R}^3 \setminus \{o\} \ : \ \text{sign}(\langle x, e_j \rangle) = a_i, \ j = 1, 2, 3 \right\} \cup \{o\} \tag{1.15}$$

for $i = 1, \ldots, 8$, which equals the octant of $\mathbb{R}^3$ containing a_i.
The normal cones associated with the edges are

$$N(T, F) = \left\{ x \in (F - p(o, F))^\perp \setminus \{o\} \ : \ \sphericalangle(x, p(o|F)) \in \left[\frac{-\pi}{4}, \frac{\pi}{4} \right] \right\} \cup \{o\}, \quad F \in \mathcal{F}_1(T). \tag{1.16}$$

The normal cones associated with the facets are the rays

$$N(T, F) = \{ t p(o|F) \ : \ t \geq 0 \}, \quad F \in \mathcal{F}_2(T). \tag{1.17}$$

We can now calculate the area measures. By [74, Thm. 4.2.3] it follows directly that $S_0(T, \cdot) = \mathcal{H}^2(\cdot)$. For $S_1(T, \cdot)$ and $S_2(T, \cdot)$ we apply (1.8) with (1.16) or (1.17), respectively. Thus, for $A \subset S^2$ a measurable set, we get

$$S_1(T, A) = \frac{1}{2} \sum_{i=1}^{3} \mathcal{H}^1(A \cap e_i^\perp), \tag{1.18}$$

$$S_2(T, A) = \sum_{i=1}^{3} \delta_{e_i}(A) + \delta_{-e_i}(A), \tag{1.19}$$

where δ_u is the Dirac measure supported by $u \in S^2$. Finally, applying (1.10) to the area measures yields

$$\Phi_1^{0,s}(T) = \frac{1}{\omega_{2+s} s!} \left(\int_{S^2 \cap e_1^\perp} u^s \, \mathcal{H}^1(du) + \int_{S^2 \cap e_2^\perp} u^s \, \mathcal{H}^1(du) \int_{S^2 \cap e_3^\perp} u^s \, \mathcal{H}^1(du) \right),$$

$$\Phi_2^{0,s}(T) = \begin{cases} \frac{2}{\omega_{1+s} s!} \sum_{i=1}^{3} e_i^s, & s \text{ even}, \\ 0, & s \text{ odd}. \end{cases}$$

Example 1.2. (**Unit disk in** $\mathbb{R}^3$) The unit disk $B = \{x \in B^3 : \langle x, e_3 \rangle = 0\}$ serves as an example where standard polytope-based formulas for area measures do not directly apply. Instead, we approximate B using inscribed polytopes, see [74, Thm. 4.1.1]. To this end, let B_n be a regular n-gon inscribed in B with vertices on the boundary of B, see Figure 1.5. Then $B_n \to B$ for $n \to \infty$ in the *Hausdorff metric* and $S_k(B_n, \cdot) \to S_k(B, \cdot)$ weakly, see [74, eq. (1.60) and Thm. 4.1.1].
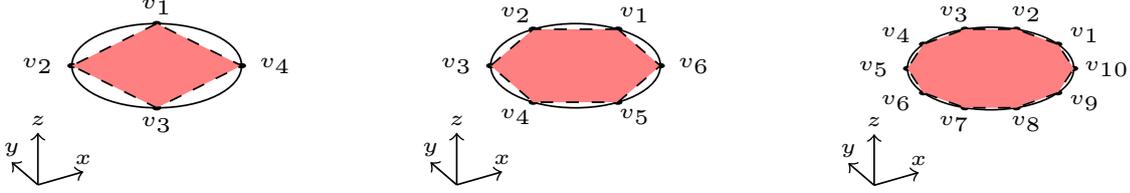
Figure 1.5: The disk B drawn in $\mathbb{R}^3$ with B_4, B_6 and B_{10} inscribed (in red), respectively from left to right.

We note that $\mathcal{F}_0(B_n)$ consists of the n vertices, $\mathcal{F}_1(B_n)$ consists of the n edges constituting the boundary of B_n and $\mathcal{F}_2(B_n) = B_n$. Since $S_0(B, \cdot) = \mathcal{H}^2(\cdot)$([74, Thm. 4.2.3]) we are primarily interested in $S_1(B_n, \cdot)$ and $S_2(B_n, \cdot)$. As these two measures are defined as a weak limit we will in the following let f be a continuous function and only specify $S_1(B, \cdot)$ and $S_2(B, \cdot)$ through integrals of such function.

As B_n is the only two-dimensional face of B_n (in $\mathbb{R}^3$) and has normal vectors e_3 and $-e_3$, applying (1.8) yields

$$\int_{S^{n-1}} f(u)\, S_2(B, du) = \lim_{n \to \infty} \mathcal{H}^2(B_n)\left(\int_{S^{n-1}} f(u)\, \delta_{e_3}(du) + \int_{S^{n-1}} f(u)\, \delta_{e_3}(du) \right) = \pi(f(e_3) + f(-e_3)).$$
(1.20)

Regarding $S_1(B, \cdot)$, we note that for F in $\mathcal{F}_1(B_n)$ the law of cosines yields

$$\mathcal{H}^1(F) = \sqrt{2\left(1 - \cos\left(\frac{2\pi}{n}\right)\right)}$$

and consequently,

$$\int_{S^{n-1}} f(u)\, S_1(B_n, du) = \frac{c_n}{2} \sum_{F \in \mathcal{F}_1(B_n)} \int_{S^2 \cap N(B_n, F)} f(u)\, \mathcal{H}^1(du)$$

with $c_n = \sqrt{2(1 - \cos(\frac{2\pi}{n}))}$. If u_F is the outer normal of $F \in \mathcal{F}_1(B_n)$ contained in $\mathrm{span}(e_1, e_2)$, then $S^2 \cap N(B_n, F)$ is the unit half-circle in $\mathrm{span}(u_F, e_3)$, see Figure 1.6.

As a result, by using [60, p. 1] or (A.5), we get

$$\int_{S^{n-1}} f(u)\, S_1(B, du) = \lim_{n \to \infty} \frac{1}{2} \int_{-1}^{1} (1 - t^2)^{-\frac{1}{2}} \left(c_n \sum_{F \in \mathcal{F}_1(B_n)} f(te_3 + \sqrt{1 - t^2}u_F) \right) \lambda_1(dt).$$

By dominated convergence, the limit can be moved under the integral, where

$$\lim_{n \to \infty} c_n \sum_{F \in \mathcal{F}_1(B_n)} f(te_3 + \sqrt{1 - t^2}u_F) = \int_{S^2 \cap \mathrm{span}(e_1, e_2)} f(te_3 + \sqrt{1 - t^2}u)\mathcal{H}^1(du),$$
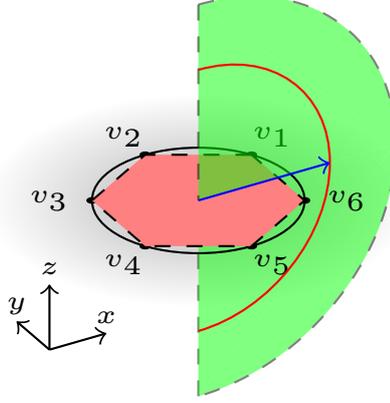
Figure 1.6: The disk B is drawn in $\mathbb{R}^3$ with B_6 inscribed in red shown from different directions. The vertices $v_1, \ldots, v_6$ are equidistantly placed on the unit circle in $\mathrm{span}(e_1, e_2)$. The green half-arc in the x-z plane corresponds to the normal cone of $F = [v_6, v_1] \in \mathcal{F}_1(B_6)$ with u_F marked in blue. The red half-arc marks the intersection of $N(B_6, F) \cap S^2$.

which can be seen by identification of $\mathrm{span}(e_1, e_2)$ with $\mathbb{R}^2$ and using that $B_n \to B^2$ under this identification. In summary, we conclude

$$\int_{S^{n-1}} f(u)\, S_1(B, du) = \frac{1}{2} \int_{-1}^{1} (1 - t^2)^{-\frac{1}{2}} \int_{S^2 \cap \mathrm{span}(e_1, e_2)} f(te_3 + \sqrt{1 - t^2}u)\mathcal{H}^1(du)\, dt$$

for any continuous function f. Thus, by (1.10) we get

$$\Phi_1^{0,s}(B) = \frac{1}{\omega_{2+s}s!} \int_{-1}^{1} (1 - t^2)^{-\frac{1}{2}} \int_{S^2 \cap \mathrm{span}(e_1, e_2)} \left(te_3 + \sqrt{1 - t^2}u\right)^s \mathcal{H}^1(du)\, dt.$$

An application of the binomial theorem yields

$$\Phi_1^{0,s}(B) = \frac{1}{\omega_{2+s}s!} \sum_{j=0}^{\frac{s}{2}} \binom{s}{2j} B\left(j + \frac{1}{2}, \frac{s+1}{2} - j\right) e_3^{2j} \int_{S^2 \cap \mathrm{span}(e_1, e_2)} u^{s-2j}\, \mathcal{H}^1(du)$$

where we have evaluated the appearing integrals over t using their relation to the beta integral (see (C.6)). Letting $L = \mathrm{span}(e_1, e_2)$ we can apply [49, eq. (2.47)] to evaluate the integral over the sphere in L yielding

$$\Phi_1^{0,s}(B) = \begin{cases} \sum_{j=0}^{\frac{s}{2}} C_{j,s} e_3^{2j} Q(L)^{\frac{s}{2} - j}, & s \text{ even}, \\ 0, & s \text{ odd}, \end{cases}$$

with $C_{j,s} = \dfrac{\Gamma(j + \frac{1}{2})\Gamma(\frac{s+1}{2} - j)^2}{\pi^{\frac{s+1}{2}} (s - 2j)!\,(2j)!\,\Gamma(\frac{s}{2} - j + 1)}$ and $Q(L)$ being the rank two tensor given by

$$Q(L)(x_1, x_2) = \langle x_1 | L, x_2 | L \rangle, \quad x_1, x_2 \in \mathbb{R}^n.$$

Finally, we note that

$$\Phi_2^{0,s}(B) = \begin{cases} \frac{2\pi}{\omega_{1+s}s!}e_3^2, & s \text{ even,} \\ 0, & s \text{ odd,} \end{cases}$$

by (1.20).

With these two examples, we end this section with some final notes on Minkowski tensors. For completeness, we also include the study of B^3, omitting the calculations. Here

$$S_k(B^n, \cdot) = \mathcal{H}^{n-1}(\cdot), \quad k = 0, \ldots, n-1, \tag{1.21}$$

which can be derived using rotational invariance arguments. The translation-invariant Minkowski tensors of rank two of the three convex bodies are given in Table 1.1. The tensor valuations are presented using a identification of rank two symmetric tensors with matrices. If $b \in \mathbb{T}^2$ then b can be identified with the matrix B with entries $B_{ij} = b(e_i, e_j)$.

Minkowski tensor	$k = 1$	$k = 2$
$\Phi_k^{0,2}(B^3)$	$\begin{pmatrix} \frac{2}{3\pi} & 0 & 0 \\ 0 & \frac{2}{3\pi} & 0 \\ 0 & 0 & \frac{2}{3\pi} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{6} \end{pmatrix}$
$\Phi_k^{0,2}(T)$	$\begin{pmatrix} \frac{1}{2\pi} & 0 & 0 \\ 0 & \frac{1}{2\pi} & 0 \\ 0 & 0 & \frac{1}{2\pi} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{4\pi} & 0 & 0 \\ 0 & \frac{1}{4\pi} & 0 \\ 0 & 0 & \frac{1}{4\pi} \end{pmatrix}$
$\Phi_k^{0,2}(B)$	$\begin{pmatrix} \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$

Table 1.1: Matrix representation of translation invariant Minkowski tensors for three different convex bodies in $\mathbb{R}^3$ Here, B^3 is the unit ball, T is the unit cube and B is the unit disk contained in $\mathrm{span}(e_1, e_2)$.

In Table 1.1, we see that some of these matrices are diagonal due to the symmetry of the convex bodies involved. As a result, as Minkowski tensors are homogeneous, we see that a single translation-invariant Minkowski tensor of rank two does not suffice to determine the underlying convex body, even up to translation. Nonetheless, determination of convex bodies given some Minkowski tensors can sometimes be done, for instance see [53].

In Example 1.2, we used $\mathbb{R}^3$ as the ambient space even though $B \subset \mathrm{span}(e_1, e_2)$. Using $\mathrm{span}(e_1, e_2)$ as the ambient space, we can calculate the intrinsic area measure of B. From (1.21), we conclude that $S_1^{\mathrm{span}(e_1,e_2)}(B, \cdot) = \mathcal{H}^1(\mathrm{span}(e_1, e_2) \cap \cdot)$ is the Hausdorff measure restricted to $\mathrm{span}(e_1, e_2)$ and

$$\Phi_{k,0,s}^{(\mathrm{span}(e_1,e_2))}(B) = \binom{2-1}{k} \frac{1}{\omega_{2-k+s}s!} \int_{S^2 \cap \mathrm{span}(e_1,e_2)} u^s \, \mathcal{H}^1(du), \quad k = 0, 1$$

is the intrinsic translation-invariant Minkowski tensors of rank two of B with $\mathrm{span}(e_1, e_2)$ as the ambient space. For comparison with Table 1.1 a matrix representation with $s = 2$ is

$$\Phi_{k,0,2}^{(\mathrm{span}(e_1,e_2))}(B) = \binom{2-1}{k} \frac{1}{\omega_{2-k+2}2!} \begin{pmatrix} \pi & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad k = 0, 1.$$

Comparing with Table 1.1, we see that $\Phi_{1,0,2}^{(\mathrm{span}(e_1,e_2))}(B) \neq \Phi_{1,s}(B)$. With this remark, we end the section on notation and examples.

1.3 Introduction to Paper A

Understanding the shape and size of objects from lower dimensional intersections is a fundamental problem in many sciences, from materials research to medical imaging. Mathematical tools, like the classical Crofton formula, allow to infer geometric characteristics from sections with random flats. In Paper A, we extend these ideas and provide a solution of (1.1) in two cases. The first case is letting the sampling space be $F = G(L_0, k)$ for a fixed $L_0 \in G(n, r)$ with $r < k$ and $Q(K) = V_m(K)$ for $m \in \{n + r - k, \ldots, n\}$. In this setting, we thus construct Crofton formulae with a fixed subspace. The proof combines a Blaschke–Petkantschin formula and the classical Crofton formula (see (A.19)) based on an idea similar to the one in [3], where the special case (put $r = 0$)

$$\int_{G(n,k)} \varphi_{L,q}^{\{o\}}(K \cap L)\, \nu_k(dL) = V_{n-j}(K), \qquad K \in \mathcal{K}^n, \tag{1.22}$$

with the measurement function

$$\varphi_{L,q}^{\{o\}}(K \cap L) = c \int_{A(L,q)} V_{q-j}(K \cap E)\, d(o, E)^{n-k}\, \mu_q^L(dE), \tag{1.23}$$

$n, k \in \mathbb{N}_0$, $1 \leq k \leq n$ was derived. The parameter q can be chosen in $\{j, \ldots, k-1\}$ and the constant c is explicitly known. The parameter q appears due to the use of a Blaschke–Petkantschin formula:

$$\int_{A(n,q)} f(E)\mu_q(dE) = \frac{\omega_{n-q}}{\omega_{k-q}} \int_{G(n,k)} \int_{A(L,q)} f(E)d(o, E)^{n-k}\, \mu_q^L(dE)\nu_k(dL),$$

valid for all $k \in \{1, \ldots, n-1\}$, $q \in \{0, \ldots, k-1\}$ and non-negative measurable functions f. This formula decomposes the motion-invariant integration over $A(n, q)$ into a rotation-invariant integration over linear subspaces $L \in G(n, k)$ and a subsequent integration over flats $E \in L$. To compensate for the fact that the choice of the motion-invariant measure of flats in L would give too strong preference to flats close to o, a weighting with a power of $d(o, E)$ must be included.

It should be emphasized that Blaschke–Petkantschin formulae play a central role not only in Paper A, but throughout the whole dissertation. They provide the framework for decomposing integrals and thereby enable many of the results presented in this dissertation. For an accessible and non-exclusive list of such formulae, we refer to [75].

The result (1.22) can now be derived by combining the classical Crofton formula with the above-displayed equation by letting f be a suitable intrinsic volume. As the classical Crofton

formula gives multiple formulae for a fixed intrinsic volume, this translates to variability in the parameter q in (1.22). Remarkably, the measurement function appearing in (1.23) is actually independent of this parameter q, which is a consequence of the classical Crofton formula, see [49, Thm. 4.8].

The first main result in Paper A is the construction of a measurement function $\varphi_{L,q}^{L_0}$ such that

$$\int_{G(L_0,k)} \varphi_{L,q}^{L_0}(K \cap L)\, \nu_k^{L_0}(dL) = V_{n-j}(K), \qquad K \in \mathcal{K}^n, \tag{1.24}$$

is valid for $n, r, k \in \mathbb{N}_0$ with $r + 1 \leq k \leq n$, where the parameter q can be chosen in $\{j, \ldots, k - (r+1)\}$ and $L_0 \in G(n,r)$ is to be considered fixed. The formula for the measurement function is given by

$$\varphi_{L,q}^{L_0}(K \cap L) = c_1 \int_{A(L,q)} V_{q-j}(K \cap E)\, D(E, L_0)^{n-k}\, \mu_q^L(dE). \tag{1.25}$$

Here, $D(E, L_0)$ is given by (B.31) and appears for the same reason as $d(o, E)$ appears in (1.23): it appears in the Blaschke–Petkantschin formula stated in Theorem A.3. As in (1.25), the apparent variability in the choice of measurement function $\varphi_{L,q}^{L_0}$ for varying q is due to the Blaschke–Petkantschin formula used. Even though q directly controls the dimension of the integrating flats $E \in A(L, q)$ we show in Theorem A.4, again by using the classical Crofton formula, that $\varphi_{L,q}^{L_0}$ is independent of q. These results cover the first half of Paper A and provide solutions to (1.1) in the setting of Crofton formulae with a fixed subspace.

The second part of the Paper concerns *vertical sections* and is contained in Section A.6. Appearing as an appendix in the published version, this part of the paper parallels the first half but now the sampling space F is the set of all affine flats that are parallel to a fixed lower-dimensional subspace L_0. Since such a flat E can be decomposed as the translation of a subspace L containing L_0 with a displacement vector x in $L^\perp$, we construct a measurement function $\tilde{\varphi}_{L+x,q}^{L_0}$ such that

$$\int_{G(L_0,k)} \int_{L^\perp} \tilde{\varphi}_{L+x,q}^{L_0}(K \cap (L + x))\, \lambda_{L^\perp}(dx)\, \nu_k^{L_0}(dL) = V_{n-j}(K), \quad K \in \mathcal{K}^n,$$

for suitable indices n, k, q, j and $L_0 \in G(n,r), r < k$, considered fixed. In $\mathbb{R}^3$, letting $L_0 = \text{span}(e_3)$, any flat $E = L + x$ with $L \in G(3,2)$, and $x \in L^\perp$ can be visualized as a vertical flat, thus giving rise to the name 'vertical sections'.

The structure of this part of the paper is as follows. The main result, Theorem A.7, provides the measurement function $\tilde{\varphi}_{L+x,q}^{L_0}$. Its construction is based, like that of $\varphi_{L,q}^{L_0}$, on a Blaschke–Petkantschin formula, which is stated in Theorem A.8. This formula introduces flexibility in the choice of the measurement function $\tilde{\varphi}_{L+x,q}^{L_0}$ through the parameter q. Furthermore, in Theorem A.9, we establish that the (apparently different) measurement functions appearing in Theorem A.7 actually coincide.

The results presented in this part of the paper have been shown previously in a slightly different setting. In 1984, Baddeley [6] derived a formula for $(\mathcal{H}^m, m)$-rectifiable sets. Specifically, if $L_0 \in G(n, n-p)$ is a fixed subspace and M is $(\mathcal{H}^m, m)$-rectifiable (see [28, 3.2.14] for definition), then

$$\mathcal{H}^m(M) = c \int_{G(L_0,n-p+r)} \int_{L^\perp} \int_{A(L+x,r)} J^r(L_0, T)\mathcal{H}^{m+r-n}(M \cap E)\, \nu_r^{L+x}(dE)\, \lambda_{L^\perp}(dx)\, \mu_{n-p+r}(dL), \tag{1.26}$$

for $0 < m < n \leq m + r$, $r < p \leq n - r$ and a known constant c, where the proof relies on the coarea formula. Here, $J(L_0, T)$ is a Jacobian term arising from the coarea formula.

If we assume that $K \in \mathcal{K}^n$ is full-dimensional, then (1.26) implies

$$V_n(K) = c \int_{G(L_0, n-p+r)} \int_{L^\perp} \int_{A(L+x, r)} J^r(L_0, T) V_r(K \cap E) \, \nu_r^{L+x}(dE) \, \lambda_{L^\perp}(dx) \, \mu_{n-p+r}(dL),$$

which coincides with Theorem A.9 for $r = n - p$, $j = 0$, $k = n - p + r$, and $q = k - r$. Similarly, (1.26) with $M = \partial K$ corresponds to a special case of Theorem A.9. Consequently, vertical section formulae have been known at least since 1984.

The results in Paper A contribute to the fields of integral geometry in multiple ways. We achieve new Crofton-type formulae, which expands the selection of Crofton-type formulae available. Also, the derived Blaschke–Petkantschin formulae can be applied in further work. For instance, it is used in the following Paper B to derive Crofton formula with a fixed axis for translation-invariant Minkowski tensors.

1.4 INTRODUCTION TO PAPER B

Intrinsic volumes describe some essential geometric features of convex bodies but fail to capture others, for instance, the orientation and elongation of a convex body. Translation-invariant Minkowski tensors (with $s > 0$), however, include additional directional and structural information. In Paper B, we focus on extending the classical Crofton-type formula to incorporate these tensorial shape descriptors. Specifically, we consider the problem posed in (1.1), now with the target quantity $Q(K)$ being a translation-invariant Minkowski tensor. Since Minkowski tensors of rank zero coincide with intrinsic volumes, the results in Paper B generalize those presented in Paper A. As we interpret integrals of tensors point-wise, we can reuse the Blaschke–Petkantschin formulae already derived in Paper A in order to, for instance, derive Crofton formula with a fixed axis valid for translation-invariant Minkowski tensors. The major obstacle in this is determining which type of classical Crofton formula (for tensors) to apply the different Blaschke–Petkantschin formulae to. As the classical Crofton formula is not stated in a stereologically adapted way for translation-invariant Minkowski tensors, the main objective of Paper B is to identify a suitable measurement tensor φ satisfying

$$\int_{A(n,q)} \varphi(K \cap E, E) \, \mu_q(dE) = \Phi_{n-j}^{0,s}(K), \quad K \in \mathcal{K}^n, \tag{1.27}$$

for appropriate indices n, q, j, s. Inspired by the classical Crofton formula for intrinsic volumes an initial guess would be that φ can be chosen proportional to a Minkowski tensor. This is not the case. Integrals of the form

$$\int_{A(n,q)} \Phi_k^{r,s}(K \cap E) \, \mu_q(dE), \quad K \in \mathcal{K}^n \tag{1.28}$$

has been calculated for most indices.

For instance, if $n, s, q, j \in \mathbb{N}$ with $1 \leq j < q < n$ and $K \in \mathcal{K}^n$ then

$$\int_{A(n,q)} \Phi_{q-j}^{0,s}(K \cap E)\, \mu_q(dE)$$

$$= \begin{bmatrix} n \\ n-q \end{bmatrix}^{-1} \binom{n-j}{q-j} \frac{(q-j)(n-q)}{2(n-j)} \frac{1}{\Gamma(\frac{n-j+s}{2})} \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \frac{\Gamma(\frac{n-q}{2}+z)\Gamma(\frac{q-j+s}{2}-z)}{(4\pi)^z z!} Q^z \Phi_{n-j}^{s-2z}(K),$$

(1.29)

with the flag coefficient $\begin{bmatrix} n \\ n-q \end{bmatrix}$ defined in (B.7), see [10, Thm. 4.12]. Since the constants involved are non-trivial, the expression (1.29) does not immediately yield a measurement tensor that solves (1.27). This underscores the central objective of Paper B. We aim to invert formulae of this type (if such an inversion is possible), thereby finding suitable measurement tensors $\varphi(E, K \cap E)$ that satisfy (1.27). In doing so, we consider a linear combination of tensors within the integral, explicitly isolating a single tensor as the integral of this linear combination.

A main result in Paper B is the explicit construction of measurement tensors φ solving (1.27), resulting in stereologically adapted Crofton formulae valid for $1 \leq j < q < n$ and $s \in \mathbb{N}$, as well as the special case $j = q$ provided that s is even. Having obtained these stereologically adapted Crofton formulae, we apply the Blaschke–Petkantschin formulae derived in Paper A to generalize the results from Paper A (Theorem A.2 and Theorem A.7) to versions valid for translation-invariant Minkowski tensors. Consequently, we establish a tensor Crofton formula with a fixed subspace and a tensor vertical section Crofton formula, each applicable to translation-invariant Minkowski tensors. Notably, the vertical section formulae for tensor valuations presented here provides genuinely new results even in three-dimensions.

We note, that we not only apply previously derived Blaschke–Petkantschin formulae, but also establish a new Blaschke–Petkantschin formula tailored for the case $j = q$ and s odd in the search for solutions to (1.27). This result is formulated in Theorem B.4, where the proof makes essential use of the Blaschke–Petkantschin formula (B.30) to create a framework in which [55, Thm. 3.6] can be applied. Since the formula was specifically used for this purpose, its role in Paper B is only marginal. For completeness, however, we now present and derive a slightly more general version of (B.30). The general lemma involves the constants $b_{n,q}$ for $1 \leq q \leq n$, as defined in [75, eq. (7.8)]. Its proof is postponed to after the main introduction to Paper B.

LEMMA 1.3. *Let $n, q, r \in \mathbb{N}$ with $q \in \{1, \ldots, n-1\}$ and $r \in \{n-q, \ldots, n\}$. Then, for any measurable function $f : A(n,q) \to [0, \infty)$, we have*

$$\int_{A(n,q)} f(E)\, \mu_q(dE)$$

$$= \frac{b_{r,n-q} b_{n-r+n-q,n-q}}{b_{n,n-q}} \int_{A(L_0, q+r-n)} \int_{A(M,q)} f(E) [\operatorname{lin}(E), L_0]^{q+r-n+1}\, \mu_q^M(dE)\, \mu_{q+r-n}^{L_0}(dM).$$

At first sight, the formula in Lemma 1.3 may resemble the Blaschke–Petkantschin formulae established in Paper A. This leads to the question of whether it opens up a new framework in which a solution to (1.1) might be obtained. The answer is negative. If $q \in \{1, \ldots, n-1\}$ and $j \in \{1, \ldots, q\}$, Lemma 1.3 can be combined with the classical Crofton formula (A.19) to yield

$$c \int_{A(L_0, q+r-n)} \int_{A(M,q)} V_{q-j}(K \cap E) [\operatorname{lin}(E), L_0]^{q+r-n+1}\, \mu_q^M(dE)\, \mu_{q+r-n}^{L_0}(dM) = V_{n-j}(K), \quad (1.30)$$

for $K \in \mathcal{K}^n$. Here, the constant $c = \frac{b_{r,n-q}b_{n-r+n-q,n-q}}{b_{n,n-q}} \cdot \left(c_{q-j,n}^{n-j,q}\right)^{-1}$ is defined using (A.20). Observe that (1.30) cannot be expressed in the form of (1.1), since the inner integral does not depend on K solely through $K \cap M$. This is no surprise, since in Lemma 1.3, one has $r + q - n \le \dim(L_0) = r$. Hence, heuristically, we can only observe geometrical features intrinsic to L_0. This is the reason why, as also mentioned in Paper A, when considering Crofton formulae with a fixed subspace L_0, we primarily consider the cases where the integrating flats are of strictly larger dimension than the fixed L_0.

Excluding the use of Blaschke–Petkantschin formulae a main strategy in proving the results in Paper B is as follows. We start by considering formulae such as (1.29), which are already established in the literature. To derive a sterologically adapted Crofton formula for translation-invariant Minkowski tensor of rank $s_0 > 0$, one considers (1.29) with n, q, j fixed and $s = 0, 1, \ldots, s_0$ as a system of linear equations. Solving this system for the Minkowski tensors on the right-hand side (if it is invertible) then yields a measurement function in (1.27) as a linear combination of products of Minkowski tensors of section profiles and powers of the metric tensor.

These arguments not only establish the existence of such a measurement function, but also allow explicit calculations and simplifications of the coefficients. To achieve this, we employ Zeilberger's algorithm. The algorithm is only briefly introduced around (B.38) in the appendix of Paper B. Therefore, we will now give a slightly more thorough introduction to illustrate its strengths and limitations. This introduction is based on [66, Chap. 6]. Suppose that $F : \mathbb{N}_0^2 \to \mathbb{R}$ is a hypergeometric function in both arguments, see [66] for a definition. We are interested in a sum of the form

$$f(n) = \sum_{k=0}^{K} F(n, k). \tag{1.31}$$

In many relevant applications, we may identify a new function $H(n, k)$, along with an integer $J \in \mathbb{N}_0$ and constants $\alpha_1, \ldots, \alpha_J$, which may depend on n, but not k, satisfying

$$\sum_{j=0}^{J} \alpha_j F(n + j, k) = H(n, k + 1) - H(n, k), \quad k = 0, \ldots, K. \tag{1.32}$$

This is a first step to solve the problem at hand, as summing over $k = 0, \ldots, K$ gives a telescoping sum and lead to

$$\sum_{j=0}^{J} \alpha_j f(n + j) = H(n, K + 1) - H(n, 0).$$

This recursive formula in $f(n)$ would then allow to calculate $f(n)$ for $n \ge J$ provided $f(0), \ldots, f(J - 1)$ can be calculated by other means. In particular, if $J = 1$ then only $f(0) = \sum_{k=0}^{K} F(0, k)$ is needed to determine $f(n)$ for all n.

To check if there exists $J \in \mathbb{N}_0$ such that the relation (1.32) can be established, Zeilberger's algorithm can be used, see [66, p. 102]. This algorithm checks if, for a fixed J, a solution to (1.32) exists, and if so it provides the constants $\alpha_1, \ldots, \alpha_J$ and the function $H(n, k)$. To provide the simplest possible relation, the algorithm is initialized with $J = 0$ and runs until it finds a $J \ge 0$ such that (1.32) holds true. The assumptions on F guarantee existence of some J such that (1.32) can be formulated and thus the algorithm terminates. For a fixed J, the algorithm

sets

$$t_k = \sum_{j=0}^{J} \alpha_j F(n+j, k),$$

which may depend on n for some temporary unknown parameters $\alpha_0, \ldots, \alpha_J$. Using the assumptions on F, the algorithm then executes a series of polynomial divisions on the fraction $\frac{t_{k+1}}{t_k}$. This leads to the decomposition

$$\frac{t_{k+1}}{t_k} = \frac{p(k+1)}{p(k)} \frac{p_2(k)}{p_3(k)},$$

where p, p_2, p_3 are polynomials (in k) and p_2 and p_3 are coprime and independent of $\alpha_0, \ldots, \alpha_J$. Only p depends on the temporary unknowns $\alpha_0, \ldots, \alpha_J$. To check if there exist $\alpha_0, \ldots, \alpha_J$ one defines another polynomial $b(k) = \sum_{l=0}^{\Delta} \beta_l k^l$ of degree Δ with unknown coefficients $\beta_0, \ldots, \beta_\Delta$ and considers the polynomial equation in $\Delta + J + 2$ unknowns

$$p_2(k)b(k+1) - p_3(k)b(k) = p(k). \tag{1.33}$$

This is the setup for *Gosper's algorithm*. This algorithm first checks if there exists a Δ such that a solution to (1.33) exists. If a solution exists for some Δ it furthermore provides an upper bound on Δ and constants $\alpha_0, \ldots, \alpha_J, \beta_0, \ldots, \beta_\Delta$ such that (1.33) holds true. Hence, we have two outcomes when we apply Gosper's algorithm. If there exists a solution to (1.33) then $H(n, k)$ takes the form $H(n, k) = \frac{p_3(k-1)}{p(k)} b(k) t_k$. If no solution exists, we start over with J replaced by $J + 1$. We do not delve into Gosper's Algorithm but instead refer to [66, Chap. 5], simply noting that it involves solving multiple linear equations.

These two steeps constitute the main algorithm, which can be structured as follows

- Fix $J = 0$.

- Do polynomial division on $\frac{t_{k+1}}{t_k}$ and find p, p_2, p_3.

- Use Gosper's algorithm to check if there exits a polynomial solution to

$$p_2(k)b(k+1) - p_3(k)b(k) = p(k)$$

  exists.

- If such $b(k)$ exists let $G(n, k) = \frac{p_3(k-1)}{p(k)} b(k) t_k$ else restart with $J = J + 1$.

A drawback of using Zeilberger's algorithm is that it is rarely traceable by hand due to the sheer number of polynomial divisions and linear equations involved. Consequently, the most straightforward way to verify the result is by directly checking whether (1.32) holds.

Another challenge lies in the value of J. To the author's knowledge, it appears to be coincidental that all applications of the algorithm in Paper B have resulted in $J \leq 1$. This makes checking the involved relations feasible. That the complexity rapidly increases with the value of J may limit applications of Zeilberger's algorithm in specific cases. We thus conclude this section with mentioning of an alternative method for computing sums. Although this method is not used in Paper B, we believe that it may be applicable to similar problems and thus merits a mention. In the recently published paper [70], the authors review and strongly advocate the use of *Egorychev's Method* for evaluation of sums. This method originates from Egorychev's book

[24] and is based on postponing summations by considering formal series. In [70] various ideas and techniques for employing this method are provided. We highly recommend consulting this paper for a comprehensive understanding and further insights into the method.

We conclude the main introduction to Paper B with the proof of Lemma 1.3, which parallels the beginning of the proof of Theorem B.4. For this reason, the arguments are presented only briefly and are best read after consulting the proof of Theorem B.4 in Paper B.

Proof of Lemma 1.3 . Let the setting be as in Lemma 1.3. By [75, Thm. 7.2.6] (with $s_1 = q$ and $s_0 = r$) we conclude

$$\int_{A(n,q)} f(E)\, \mu_q(dE) = c \int_{G(L_0,q+r-n)} \int_{G(M,q)} [L, L_0]^{q+r-n} \int_{L^\perp} f(L+x)\, \lambda_{L^\perp}(dx) \nu_q^M(dL) \nu_1^{L_0}(dM)$$

$$(1.34)$$

with $c = \frac{b_{r,n-q} b_{n-r+n-q,n-q}}{b_{n,n-q}}$. Using the definition of the subspace determinant (see [75, p. 598]) and [75, eq. (13.13)-(13.14)], we conclude

$$\int_{A(n,q)} f(E)\, \mu_q(dE) = \int_{A(L_0,q+r-n)} \int_{A(M,q)} f(E) J(E, M, L_0)\, \mu_q^M(dE)\, \mu_{q+r-n}^{L_0}(dM),$$

with

$$J(E, M, L_0) = c[\text{lin}(E), L_0]^{q+r-n} [\text{lin}(M)^\perp \cap L_0, \text{lin}(E)].$$

Thus, Lemma 1.3 holds true if we can simplify J. In this, note that $\text{lin}(M)^\perp \cap L_0$ and $\text{lin}(E)$ are a.e in general position with $\dim(\text{lin}(M)^\perp \cap L_0) + \dim(\text{lin}(E)) = n$ and thus

$$[\text{lin}(M)^\perp \cap L_0, \text{lin}(E)] = \left[\left(\text{lin}(M)^\perp \cap L_0\right)^\perp, \left(\text{lin}(E)\right)^\perp \right] = [\text{lin}(M) + L_0^\perp, \text{lin}(E)^\perp]$$

by [75, p. 598]. Now, as $\text{lin}(M)$ is perpendicular to $L_0^\perp$ and $\text{lin}(E)^\perp$, we can apply [75, eq. (7.9)] $q + r - n$ times resulting in

$$[\text{lin}(M) + L_0^\perp, \text{lin}(E)^\perp] = [L_0^\perp, \text{lin}(E)^\perp] = [L_0, \text{lin}(E)].$$

Recollecting the calculations ends the proof of Lemma 1.3. ∎

1.4.1 *Discrepancy of Theorem B.3 and the corresponding published version*

This subsection comments on the discrepancy of the constant appearing in Theorem B.3 and the corresponding result in the published version [20, Thm. 3]. In the published version a constant is missspecified and a supplementary corrigendum has been published, see [21]. To expand on this discrepancy we will use notation and results from Paper B, so this subsection is best read after reading Paper B.

The published version [20, Thm. 3] has a factor of $(2z)!$ in the denominator in the constant $d_{n,2s,q,q,z}$, which should not be present. In this dissertation [20, Thm. 3] is stated correctly in Theorem B.3.

To clarify why this factor should be omitted, we trace the proof of Theorem B.3 (or [20, Thm. 3]), which leads to the conclusion

$$\int_{A(n,q)} \sum_{z=0}^s h_{s,z} Q^{s-z} \tilde{\Phi}_{0,2z}^{(E)}(K \cap E)\, \mu_q(dE) = \tilde{\Phi}_{n-q,2s}(K), \quad K \in \mathcal{K}^n. \tag{1.35}$$

with

$$h_{s,z} = \frac{\sqrt{\pi}(2s-1)s!}{\Gamma(\frac{n+1-q}{2})\Gamma(\frac{q+1}{2})} \cdot (-1)^{z+1} \frac{4^z \Gamma(\frac{n+1}{2}+z)}{(s-z)!(2z)!}.$$

Here, $\tilde{\Phi}_{0,2z}^{(E)}(K \cap E)$ and $\tilde{\Phi}_{n-q,2s}(K)$ are translation-invariant Minkowski tensors with a different normalization then the one used in this dissertation. The transformations between these two normalizations are given by (B.20) and (B.21). Substituting these into (1.35), we get

$$\frac{\omega_{q+2z}(2z)!}{\omega_{q+2s}(2s)!} \int_{A(n,q)} \sum_{z=0}^{s} (-1)^{z+1} \frac{\sqrt{\pi}(2s-1)s!4^z \Gamma(\frac{n+1}{2}+z)}{\Gamma(\frac{n+1-q}{2})\Gamma(\frac{q+1}{2})(s-z)!(2z)!} Q^{s-z} \Phi_{0,2z}^{(E)}(K \cap E) \, \mu_q(dE)$$
$$= \Phi_{n-q,2s}(K), \quad K \in \mathcal{K}^n,$$

after collecting the constants on the left-hand side. Since

$$\frac{\omega_{q+2z}}{\omega_{q+2s}} = \pi^{z-q} \frac{\Gamma(\frac{q}{2}+s)}{\Gamma(\frac{q}{2}+z)},$$

we conclude

$$\frac{\omega_{q+2z}(2z)!}{\omega_{q+2s}(2s)!} (-1)^{z+1} \frac{\sqrt{\pi}(2s-1)s!4^z \Gamma(\frac{n+1}{2}+z)}{\Gamma(\frac{n+1-q}{2})\Gamma(\frac{q+1}{2})(s-z)!(2z)!} = \gamma_{n,2s,q,q} d_{n,2s,q,q,z}.$$

Here, $\gamma_{n,2s,q,q}$ and $d_{n,2s,q,q,z}$ are given in Theorem B.3. Of particular relevance for this discussion is the fact that the factor $(2z)!$ appearing in $h_{s,z}$ cancels when changing to the normalization of the Minkowski tensors used in this dissertation.

We end this subsection illustrating Theorem B.3 for $K = B^3$, $q = 1$ and $s = 1$. According to Theorem B.3 we need to confirm

$$\int_{A(3,1)} \frac{1}{2}\sqrt{\pi}\varphi_{1,1,2}(E, B^3 \cap E)\mu_q(dE) = \Phi_{2,2}(B^3), \tag{1.36}$$

where

$$\frac{1}{2}\sqrt{\pi}\varphi_{1,1,2}(E, B^3 \cap E) = \left(-\pi^{-\frac{3}{2}}Q\Phi_{0,0}^{(E)}(B^3 \cap E) + \frac{16}{\sqrt{\pi}}\Phi_{0,2}^{(E)}(B^3 \cap E) \right)$$
$$= 8\Phi_{0,2}^{(E)}(B^3 \cap E) - \frac{1}{2\pi}QV_0(B^3 \cap E), \quad E \in A(3,1).$$

Now fix a $E \in A(3,1)$. If $d(o, E) \leq 1$ then $B^3 \cap E$ is a line segment of lenght $2\sqrt{1-d(o,E)^2}$ contained in E. If $u_E \in S^2$ is such that $\mathrm{span}(u_E) = \mathrm{lin}(E)$ then (1.13) implies

$$\Phi_{0,2}^{(E)}(B^3 \cap E) = \frac{1}{2!\omega_3}(u_E^2 + (-u_E)^2) = \frac{1}{4\pi}u_E^2, \quad d(o, E) \leq 1.$$

For convenience let $u_E = 0$ if $d(o, E) > 1$. Then

$$\int_{A(3,1)} \Phi_{0,2}^{(E)}(B^3 \cap E) \, \mu_1(dE) = \frac{1}{\omega_3} \int_{S^2} \frac{1}{4\pi}u^2 \int_{u^\perp} 1_{\{x \leq 1\}} \, \lambda_{u^\perp}(dx) \, \mathcal{H}^2(du)$$

$$= \frac{\kappa_2}{4\pi\omega_3} \int_{S^2} u^2 \, \mathcal{H}^2(du).$$

Now [49, eq. (2.47)] implies

$$\int_{S^2} u^2 \, \mathcal{H}^2(du) = 2\frac{\omega_5}{\omega_3} Q = \frac{4\pi}{3} Q.$$

Hence, we conclude

$$\int_{A(3,1)} 8\Phi_{0,2}^{(E)}(K \cap E) - \frac{1}{2\pi} Q V_0(B^3 \cap E)\mu_1(dE) = 8\frac{1}{4\pi 4}\frac{4\pi}{3} Q - \frac{1}{2} Q = \frac{1}{6} Q,$$

where we evaluated the last part of the integral using (A.19). On the other hand it follows directly from Table 1.1 that

$$\Phi_{2,2}(B^3) = \frac{1}{6} Q.$$

Consequently, we have verified (1.36) for the ball $K = B^3$.

1.5 Introduction to Paper C

Having addressed Crofton-type formulae in Paper A and Paper B, the objective of Paper C can be treated independently of the framework related to (1.1). As a motivating example for the setup in Paper C, consider a sheet of paper on which a unit circle and its center (the origin) are drawn. One draws an isotropic random line through the origin, and then an affine line, chosen independently of the first one, with a distribution that only satisfies the condition that it is rotational invariant. These two lines intersect almost surely in a single point (see Figure C.1). The natural question arises: What is the distribution of this intersection point? In Paper C, we study such probabilities in arbitrary dimension. A central methodological tool is the family of Blaschke–Petkantschin-type formulae, which serve as a unifying framework linking the techniques used in Paper C with those in Papers A and B.

In geometry, the study of intersections between geometric objects is fundamental, with applications spanning probability, stereology, and integral geometry. For instance, the intersection of two affine flats (of suitable dimensions) is itself an affine flat, and is well understood, see [75, Thm. 7.2.9]. Similarly, the intersection of two subspaces (of suitable dimensions) is again a subspace, and integral formula is available, see [75, Thm. 7.2.4].

Surprisingly, the mixed case, in which the intersection involves a linear subspace and an affine flat, has received comparatively little attention in the literature. The aim of Paper C is to close this gap by characterizing the distribution of the intersection $E \cap L$, where $L \in G(n, q)$ is a linear subspace and $E \in A(n, n-q+\gamma)$ is an affine flat, for parameters $0 \leq \gamma < q < n$.

To this end, we consider integrals of the form

$$\int_{G(n,q)} \int_{A(n,n-q+\gamma)} f(E \cap L) \, H(E) \, \mu_{n-q+\gamma}(dE) \, \nu_q(dL), \tag{1.37}$$

where $H : A(n, n-q+\gamma) \to [0, \infty)$ is a measurable function. For instance, when $n = 2$, $q = 1$, $\gamma = 0$, and $H(E) = \mathbf{1}_{E \cap B^2 \neq \emptyset}$ for $E \in A(2, 1)$, the integral (1.37) directly corresponds to the

motivating example involving the intersection of a random line through the origin with an affine flat having a rotational invariant distribution.

Throughout Paper C, we assume that H is rotational invariant. This allows us, without loss of generality, to assume that f depends on the flat $E \cap L$ only through its distance from the origin. Equivalently, we may assume that f is also rotational invariant.

The main result of Paper C, stated as Theorem C.3, provides an explicit evaluation of the integral in (1.37) for measurable and positive functions f under a invariance assumption on H. The rotational invariance constraint can be transferred from H onto f using an averaging argument. Specifically, one has

$$\int_{G(n,q)} \int_{A(n,n-q+\gamma)} f(E \cap L) \, H(E) \, \mu_{n-q+\gamma}(dE) \, \nu_q(dL)$$
$$= \int_{G(n,q)} \int_{A(n,n-q+\gamma)} f(E \cap L) \left(\int_{SO(n)} H(\eta E) \, \nu(d\eta) \right) \mu_{n-q+\gamma}(dE) \, \nu_q(dL),$$

and consequently, Theorem C.3 can be applied to evaluate the right-hand side.

By employing a suitable Blaschke–Petkantschin-type formula, we express in Theorem C.3 the distribution of the random intersection $E \cap L$ in terms of explicit densities with respect to the measure μ_γ on $A(n, \gamma)$. Explicitly, we show that the integral (1.37) can be expressed in the form

$$D(n, q, \gamma) \int_{A(n,\gamma)} f(E) d(o, E)^{-(n-q)} J_H(d(o, E)) \, \mu_\gamma(dE) \tag{1.38}$$

for all measurable functions $f : A(n, \gamma) \to [0, \infty)$ and suitable indices. Here, J_H is a known function depending on H and $D(n, q, \gamma)$ a known constant, see Theorem C.3.

We then analyze the case where L follows the distribution ν_q and E is distributed according to the (normalized) invariant measure on all affine flats intersecting the closed ball of radius h, that is, we choose $H(E) = 1_{E \cap hB^n \neq \emptyset}$. As H only appears in (1.38) through J_H we obtain a simplified expression for the integral (1.37) in Corollary C.4 by simplifying J_H. From this, we derive an explicit density for the random variable $d(o, E \cap L)$, see Theorem C.8.

This result further allows us to analyze the case where E is tangent to the unit sphere. By differentiating with respect to h in Corollary C.4, we obtain a density for $d(o, E \cap L)$ under the constraint that E is tangent to the sphere of radius $h = 1$, see Theorem C.12.

Although geometric in nature, the main result of Paper C is connected to the theory of Blaschke–Petkantschin formulae. We show that Theorem C.3 is equivalent to a Blaschke–Petkantschin formula, given explicitly in Theorem C.7, thereby linking the intersection probabilities under study to classical tools of integral geometry.

This connection is further evident in the proof of Theorem C.3, where the probability of interest is expressed as a double integral. This integral is transformed using a known Blaschke–Petkantschin formula, reducing it to an integration over affine flats of dimension γ with an explicit Jacobian. By applying a sequence of such formulae to this Jacobian, the proof is completed.

In summary, we prove that intersection probabilities of the form considered in Paper C can not only be reformulated in terms of Blaschke–Petkantschin identities but also that these classical transformations serve as effective tools for their explicit evaluation.

A main assumption in Paper C is rotational invariance. Not contained in Theorem C.3 is the case where both f and H do not have this restraint. For completeness, we now present an

extension of Theorem C.3 that does not require rotational invariance of H. The proof of this result is a slight variation of the proof leading to Theorem C.3. To maintain a clear overview, the proof is postponed to the end of the section and is best read after consulting the proof of Theorem C.3, as many of the underlying ideas and steps carry over directly.

THEOREM 1.4. *Fix $n \geq 2$, $q \in \{1, \dots, n-1\}$ and $\gamma \in \{0, \dots, q-1\}$, and let $f : A(n, \gamma) \to [0, \infty)$ and $H : A(n, n - q + \gamma) \to [0, \infty)$ be measurable functions. Then*

$$I = \int_{G(n,q)} \int_{A(n,n-q+\gamma)} f(E \cap L) H(E) \, \mu_{n-q+\gamma}(dE) \, \nu_q(dL)$$

equals

$$\frac{D(n, q, \gamma)}{\omega_{n-q}} \int_{G(n,\gamma)} \int_0^\infty r^{q-\gamma-1} \int_{S^{n-1} \cap L_0^\perp} f(ru + L_0) I_3(u, r, L_0) \, \mathcal{H}^{n-\gamma-1}(du) \, \lambda_1(dr) \, \nu_\gamma(dL_0)$$

where

$$I_3(u, r, L_0) = \int_{G(L_0^\perp, n-q)} [u, M]^{\gamma+1} H\Big(L_0 + M + rp(u|M^\perp)\Big) \nu_{n-q}^{L_0^\perp}(dM). \tag{1.39}$$

and the constant $D(n, q, \gamma)$ is given by (C.14).

Consider the structure of Theorem 1.4. If $E \in A(n, n - q + \gamma)$ and $L \in G(n, q)$, then $E \cap L$ can almost everywhere be expressed in the form $E \cap L = L_0 + ru$, for some $L_0 \in G(n, \gamma)$, $u \in S^{n-1} \cap L_0^\perp$, and $r > 0$. Thus, the outer three integrals in Theorem 1.4 correspond to an integration over all possible intersections $E \cap L$ of dimension γ. The different intersections are weighted by $I_3(u, r, L_0)$ given in (1.39). Herein, the specified function H is evaluated at $L_0 + M + rp(u|M^\perp)$. If H is rotationally invariant, Theorem C.3 is recovered, as H then depends on M only through $p(u|M^\perp)$.

In particular, choosing $H(E) = 1_{\{B^n \cap E \neq \emptyset\}}$ for $E \in A(n, n-q+\gamma)$ yields $H(L_0+M+rp(u|M^\perp)) = 1_{\{r[u,M] \leq 1\}}$ and substituting this into Theorem 1.4, and simplifying leads, to Corollary C.4.

A natural application of Theorem 1.4 is the choice $H(E) = 1_{E \cap K \neq \emptyset}$ for $E \in A(n, n - q + \gamma)$ and $K \in \mathcal{K}^n$. Unfortunately, to the author's knowledge, this does not generally allow for clear simplifications. One reason for this is that if $\gamma < q - 1$, the involved flats can skew, complicating the geometric picture. On the other hand, if $\gamma = q - 1$, then

$$H(L_0 + M + rp(u|M^\perp)) = \begin{cases} 1, & r[u, M] \leq h_K\left(\frac{p(u|M^\perp)}{[u,M]}\right), \\ 0, & \text{else}, \end{cases} \tag{1.40}$$

where h_K is the support function of K, see [75] for the definition. This enables certain simplifications using [49, Prop. (3.15)]. However, these simplifications fall outside the scope of this dissertation and are therefore omitted, as they do not introduce any significant new contributions.

Before giving the proof of Theorem 1.4 we will briefly mention another use of Theorem C.3. It also allows for replacing the ambient space $\mathbb{R}^n$ with a hyperbolic space of constant negative curvature $K < 0$ as shown in [78]. In [78], they consider the setting where the ambient space $\mathbb{M}_K^n$ is a n-dimensional hyperbolic space of constant negative curvature $K < 0$. Herein, they fix a point $o \in \mathbb{M}_K^n$ referred to as the origin and equip $\mathbb{M}_K^n$ with a metric $d_K(\cdot, \cdot)$. They define $G_K(n, q)$

as the Grassmannian of q-dimensional totally geodesic submanifolds of $\mathbb{M}_K^n$ that pass through o and $A_k(n,q)$ as the space of all q-dimensional totally geodesic submanifolds of $\mathbb{M}_K^n$.

Then, by considering suitable probability measures, $\nu_{q,K}$ and $\mu_{q,K}^*$, on these two spaces, they consider the distribution of $L \cap E$ where L has the distribution $\nu_{q,K}$ and E has that of $\mu_{q,K}^*$. The latter is essentially the invariant measure on all hyperbolic flats intersecting the hyperbolic closed ball of radius 1, rescaled to form a probability measure. Hence, they consider a setting presented in Paper C in hyperbolic space. We refer to [73] for further details on hyperbolic spaces.

Using the main result from Paper C they derive statements for intersection probabilities in hyperbolic space. Here new phenomena appears when describing the law of $d_K(o, L \cap E)$. For instance, they show $\mathbb{P}(L \cap E \neq \emptyset) < 1$ for $K < 0$ thus in the hyperbolic setting the intersection can be empty. Likewise, for all $\alpha > 0$ they show

$$\mathbb{E}(d_K(o, E \cap L)^\alpha | d_K(o, E \cap L) < \infty) < \infty,$$

see [78, Cor. 3.5], which, as also mentioned in the article, should be seen in contrast to Corollary C.9, where finite moments in the Euclidian case require $\alpha < \gamma + 1$.

We not turn our attention to the proof of Theorem 1.4, which will be based on the following lemma. This lemma is a substitute for Lemma C.2 used in Paper C. Again, we recommend the reader to first consult the proof of Theorem C.3 in Paper C to better understand similarities and differences.

LEMMA 1.5. *Fix $n \geq 1$, $q \in \{1, \ldots, n-1\}$ and $\gamma \in \{0, \ldots, q-1\}$. Let $L_0 \in G(n, \gamma)$, $M \in G(L_0^\perp, n-q)$, and $f : S^{n-1} \cap L_0^\perp \to [0, \infty)$ be measurable. Then*

$$\int_{G(L_0^\perp, q-\gamma)} \int_{S^{n-1} \cap L} f(u)[M, L]^{\gamma+1} \, \mathcal{H}^{q-\gamma-1}(du) \, \nu_{q-\gamma}^{L_0^\perp}(dL)$$

$$= \frac{\omega_{q-\gamma}}{\omega_{n-\gamma}} c_1 \int_{S^{n-1} \cap L_0^\perp} [\operatorname{span} u, M]^{\gamma+1} f(u) \, \mathcal{H}^{n-\gamma-1}(du)$$

with $c_1 = \frac{\omega_{q-\gamma}}{\omega_{n-\gamma}} a(n-\gamma, n-q, q-\gamma, \gamma+1)$ where $a(n-\gamma, n-q, q-\gamma, \gamma+1)$ defined in Lemma C.2.

Proof. Let

$$I = \int_{G(L_0^\perp, q-\gamma)} \int_{S^{n-1} \cap L} f(u)[M, L]^{\gamma+1} \, \mathcal{H}^{q-\gamma-1}(du) \, \nu_{q-\gamma}^{L_0^\perp}(dL)$$

be the integral of interest. As this integral is intrinsic to $L_0^\perp$ we may identify $L_0^\perp$ with $\mathbb{R}^{n-\gamma}$ and conclude

$$I = \int_{G(n-\gamma, q-\gamma)} \int_{S^{n-\gamma} \cap L} f(\overrightarrow{u})[M, L]^{\gamma+1} \, \mathcal{H}^{q-\gamma-1}(du) \, \nu_{q-\gamma}^{L_0^\perp}(dL),$$

where $\overrightarrow{u}$ is understood as the embedding of $u \in \mathbb{R}^{n-\gamma}$ into $\mathbb{R}^n$. By [75, Thm. 7.1.1] we change the order of integration, yielding

$$I = \frac{\omega_{q-\gamma}}{\omega_{n-\gamma}} \int_{S^{n-\gamma}} f(\overrightarrow{u}) \int_{G(\operatorname{span}(u), q-\gamma)} [M, L]^{\gamma+1} \, \nu_{q-\gamma}^{\operatorname{span} u}(dL) \, \mathcal{H}^{n-\gamma-1}(du)$$

$$= \frac{\omega_{q-\gamma}}{\omega_{n-\gamma}} a(n-\gamma, n-q, q-\gamma, \gamma+1) \int_{S^{n-\gamma}} f(\overrightarrow{u})[\operatorname{span} u, M]^{\gamma+1} \, \mathcal{H}^{n-\gamma-1}(du),$$

using Lemma C.2. Reidentifying $\mathbb{R}^{n-\gamma}$ with $L_0^\perp$ concludes the proof. ∎

Proof of Theorem 1.4. For this proof, fix $n \geq 1$, $q \in \{1, \ldots, n-1\}$ and $\gamma \in \{0, \ldots, q-1\}$, and let $f : A(n, \gamma) \to [0, \infty)$ and $H : (n, n - q + \gamma) \to [0, \infty)$ be two measurable functions. The integral of interest is

$$I = \int_{G(n,q)} \int_{A(n,n-q+\gamma)} f(E \cap L) H(E) \, \mu_{n-q+\gamma}(dE) \, \nu_q(dL).$$

Proceeding as the first half of Theorem C.3 we get

$$I = \frac{\bar{b}}{\kappa_{n-q}} \int_{G(n,\gamma)} I_1(L_0, f) \, \nu_\gamma(dL_0) \tag{1.41}$$

with the constant $\bar{b}$ given by (C.21) and

$$I_1(L_0, f) = \int_{G(L_0^\perp, q-\gamma)} \int_{G(L_0^\perp, n-q)} \int_{L_0^\perp} f\Big(\big((M + t) \cap L\big) + L_0\Big) 1_{\{d(o, L+t) \leq 1\}} \tag{1.42}$$
$$\times H(M + L_0 + t) \, \lambda_{L_0^\perp}(dt) [M, L]^{\gamma+1} \nu_{n-q}^{L_0^\perp}(dM) \, \nu_{q-\gamma}^{L_0^\perp}(dL).$$

Now consider (1.42) for fixed $L_0 \in G(n, \gamma)$. Letting $f_0(t) = f(L_0 + t)$ and $H_0 = H(L_0 + t)$ for $t \in L_0^\perp$ we write

$$I_1(L_0, f) = \int_{G(L_0^\perp, q-\gamma)} \int_{G(L_0^\perp, n-q)} I_2(L, M) [M, L]^{\gamma+1} \nu_{n-q}^{L_0^\perp}(dM) \, \nu_{q-\gamma}^{L_0^\perp}(dL), \tag{1.43}$$

with

$$I_2(M, L) = \int_{L_0^\perp} f_0\Big((M + t) \cap L\Big) 1_{\{d(o, L+t) \leq 1\}} H_0(M + t) \, \lambda_{L_0^\perp}(dt)$$
$$= \int_{L_0^\perp \cap M} \int_{L_0^\perp \cap M^\perp} f_0\Big((M + x) \cap L\Big) H_0(M + x) 1_{\{d(o, L+x+y) \leq 1\}} \, \lambda_{L_0^\perp \cap M^\perp}(dx) \, \lambda_{L_0^\perp \cap M}(dy).$$

Applying (C.8) to the Lebesgue integral over $L_0^\perp \cap M^\perp = (L_0 + M)^\perp$ yields

$$I_2(M, L) = [M + L_0, L] \int_{L_0^\perp \cap M} \int_L f_0\Big((M + t) \cap L\Big) H_0\Big(M + t\Big) 1_{\{d(o, L+t|M^\perp+y) \leq 1\}} \, \lambda_L(dt) \, \lambda_{L_0^\perp \cap M}(dy)$$
$$= \int_L f_0\Big((M + t) \cap L\Big) H_0(M + t) I_3(M, L, t) \, \lambda_L(dt),$$

where we first used $t|(L_0^\perp \cap M^\perp) = t|M^\perp$ for any $t \in L$ as $L \subseteq L_0^\perp$, and then Tonelli's theorem. Here,

$$I_3(M, L, t) = [M + L_0, L] \int_{L_0^\perp \cap M} 1_{\{\|(t|M^\perp)|L^\perp + y|L^\perp\| \leq 1\}} \, \lambda_M(dy)$$
$$= [M, L] \int_M 1_{\{\|(t|M^\perp)|L^\perp + y|L^\perp\| \leq 1\}} \, \lambda_M(dy) = \kappa_{n-q}$$

using (C.8). Inserting into I_2 gives

$$I_2(M, L) = \kappa_{n-q} \int_L f_0\Big((M + t) \cap L\Big) H_0(M + t) \, \lambda_L(dt) = \kappa_{n-q} \int_L f_0(t) H_0(M + t) \, \lambda_L(dt),$$

using that for $t \in L$ we have $(M + t) \cap L = \{t\}$ almost everywhere. Applying (A.5) in L finally yields

$$I_2(M, L) = \kappa_{n-q} \int_0^\infty r^{q-\gamma-1} \int_{S^{n-1} \cap L} f_0(ru) H_0(M + ru) \, \mathcal{H}^{q-\gamma-1}(du) \, \lambda_1(dr).$$

Inserting this simplified expression into (1.43) we get

$$
\begin{aligned}
I_1(L_0, f) = {}& \kappa_{n-q} \int_{G(L_0^\perp, q-\gamma)} \int_{G(L_0^\perp, n-q)} \int_0^\infty r^{q-\gamma-1} \int_{S^{n-1} \cap L} \\
&\times f(ru) H_0(M + ru) \, \mathcal{H}^{q-\gamma-1}(du) \, dr \, [M, L]^{\gamma+1} \nu_{n-q}^{L_0^\perp}(dM) \, \nu_{q-\gamma}^{L_0^\perp}(dL) \\
= {}& \kappa_{n-q} \int_0^\infty r^{q-\gamma-1} \int_{G(L_0^\perp, n-q)} \int_{G(L_0^\perp, q-\gamma)} \int_{S^{n-1} \cap L} \\
&\times f(ru) H_0(M + ru) [M, L]^{\gamma+1} \, \mathcal{H}^{q-\gamma-1}(du) \, \nu_{q-\gamma}^{L_0^\perp}(dL) \nu_{n-q}^{L_0^\perp}(dM) \, \lambda_1(dr).
\end{aligned}
$$

Applying Lemma 1.5 to the two innermost integrals yields

$$
\begin{aligned}
I_1(L_0, f) = {}& \kappa_{n-q} c_1 \int_0^\infty r^{q-\gamma-1} \int_{G(L_0^\perp, n-q)} \int_{S^{n-1} \cap L_0^\perp} \\
&\times [\operatorname{span} u, M]^{\gamma+1} f_0(ru) H(M + ru) \, \mathcal{H}^{n-\gamma-1}(du) \nu_{n-q}^{L_0^\perp}(dM) \, dr \\
= {}& \kappa_{n-q} c_1 \int_0^\infty r^{q-\gamma-1} \int_{S^{n-1} \cap L_0^\perp} f_0(ru) I_3(u, r) \, \mathcal{H}^{n-\gamma-1}(du) \, dr,
\end{aligned}
$$

where the constant c_1 is given in Lemma 1.5, and

$$I_3(u, r) = \int_{G(L_0^\perp, n-q)} [\operatorname{span} u, M]^{\gamma+1} H_0(M + ru) \nu_{n-q}^{L_0^\perp}(dM).$$

Inserting in (1.41) completes the proof after a reduction of the involved constants, see the last part of the proof of Theorem C.3. ∎

Rotational Crofton Formulae with a Fixed Subspace

Emil Dare and Markus Kiderlen

Abstract

The classical Crofton formula explains how intrinsic volumes of a convex body K in n-dimensional Euclidean space can be obtained from integrating a measurement function at sections of K with invariantly moved affine flats. Motivated by stereological applications, we present variants of Crofton's formula, where the flats are constrained to contain a fixed linear subspace L_0, but are otherwise invariantly rotated. This main result generalizes a known rotational Crofton formula, which only covers the case $\dim L_0 = 0$. The proof combines a suitable Blaschke–Petkantschin formula with the classical Crofton formula. We also argue that our main result is best possible, in the sense that one cannot estimate intrinsic volumes of a set, based on lower-dimensional sections, other than those given by our result. Finally, we provide a proof for a well-established variant: an integral relation for vertical sections. Our formula is stated for intrinsic volumes of a given set, complementing the classical approach for Hausdorff measures.

A.1 Introduction

The classical Crofton formula expresses an intrinsic volume of a convex body K as invariant integral of another intrinsic volume of the intersection of K with affine subspaces. More specifically, for a set K in the family $\mathcal{K}^n$ of convex bodies (nonempty, compact, convex subsets) of $\mathbb{R}^n$, Crofton's intersection formula ([74, eq. (4.59)]) states

$$\int_{A(n,k)} \varphi(K \cap E)\, \mu_k(dE) = V_m(K), \tag{A.1}$$

for $k \in \{0, \dots, n\}$ and $m \in \{n - k, \dots, n\}$, where $A(n, k)$ is the family of k-dimensional affine subspaces (flats) in $\mathbb{R}^n$, μ_k is a motion invariant measure on that space, and $\varphi(K \cap E)$ is proportional to $V_{m+k-n}(K \cap E)$. The functionals $V_j : \mathcal{K}^n \to \mathbb{R}$ for $j = 0, \dots, n$ appearing in (A.1), are the *intrinsic volumes* usually defined as polynomial coefficients in *Steiner's formula* ([43, Thm. 3.10]), comprising $V_n(K)$, which is the ordinary volume, $V_{n-1}(K)$, which is proportional to the surface area and $V_1(K)$ being proportional to the mean width of K. We recommend the monograph [74] as an excellent reference for convex geometric notions and results.

The measure μ_k is not finite, but its restriction to the family $A_{K'}$ of flats hitting a compact reference set $K' \supset K$ is. Due to this fact, (A.1) can be used to obtain unbiased estimates of $V_m(K)$ from $K \cap E$, where $E \in A_{K'}$ is invariantly randomized. This has been used extensively in stereology, see [8] and the references therein.

However, in particular in biological applications, it is sometimes more convenient not to randomize over all flats in $A_{K'}$, but only over flats containing a fixed point, usually thought of as the origin. This led to the branch of *local stereology* (see [46]) and estimators such as the *nucleator* [37] and the *rotator* [48]. Although historically earlier, these estimators are

consequences of an underlying integral formula, the so-called *rotational Crofton formula*, derived independently in [3] and [36]. It is a variant of the classical Crofton formula and reads

$$\int_{G(n,k)} \varphi_L(K \cap L) \, \nu_k(dL) = V_m(K), \tag{A.2}$$

for $k \in \{1, \dots, n-1\}$ and $m \in \{n - k + 1, \dots, n\}$, where ν_k is the *rotation* invariant measure on the Grassmannian $G(n, k)$ of k-dimensional *linear* subspaces of $\mathbb{R}^n$. The measurement function $\varphi_L(K \cap L)$ is an explicitly known function of $K \cap L$ and L, but it is no longer proportional to an intrinsic volume of $K \cap L$.

Some stereological applications require even more constraints on the intersecting planes. For instance, in [48] the volume of a three-dimensional object is estimated from sections with ordinary planes, that all contain a given line L_0. The corresponding estimator is called *vertical rotator*. The practical implementation of this estimator for stereological applications can be found in [16] (not to be confused with [15], which gives early historical notes revolving around stereology). To state the underlying Crofton-type formula in $\mathbb{R}^n$ with a general fixed subspace $L_0 \in G(n, r)$, $r \in \{0, \dots, n-1\}$, and for general intrinsic volumes, is the main purpose of this paper. The result reads

$$\int_{G(L_0,k)} \varphi_L^{L_0}(K \cap L) \, \nu_k^{L_0}(dL) = V_m(K), \tag{A.3}$$

for $k \in \{r + 1, \dots, n\}$ and $m \in \{n + r - k + 1, \dots, n\}$. Here $G(L_0, k) \subset G(n, k)$ is the family of all k-dimensional linear subspaces containing L_0, and $\nu_k^{L_0}$ is the probability measure that is invariant under all rotations fixing L_0. The new measurement function $\varphi_L^{L_0}(K \cap L)$ is explicitly calculated and given in the main Theorem A.2 below. Clearly (A.3) reduces to the rotational Crofton formula (A.2) if $r = 0$, that is, if $L_0 = \{o\}$. It is therefore not surprising that the method of proof for (A.3) is a generalization of the one for (A.2). The idea of the latter can best be explained in stereological terms: A suitable Blaschke–Petkantschin formula ([75, p. 285]) allows generating a random q-dimensional flat E, $q = \{k - (n - m), \dots, k - 1\}$, in the isotropic subspace L in such a way that the distribution of E is motion invariant in $\mathbb{R}^n$. As $K \cap L \cap E = K \cap E$, the classical Crofton formula can thus be used to obtain explicit measurement functions in (A.2). As q may vary, one obtains $(k - 1) - (k - (n - m)) + 1 = n - m$ potentially different measurement functions, which, however, turn out to coincide when $L_0 = \{o\}$. Our proof of (A.3) proceeds along the same lines, but with a more general Blaschke–Petkantschin formula (Theorem A.3 below). We will again see that several potentially different measurement functions can be obtained depending on the parameter q. Extending the mentioned uniqueness for $L_0 = \{o\}$, Theorem A.4 states that they all coincide also in the general case. It should be noted that the special case $n = m = 3$ of (A.3) was already derived in [48] with a proof based on the Pappus-Guldinus theorem.

Note that (A.3) is different from the well-established stereological concept of *vertical sections*, where averages are taken over *affine planes* of given dimension that are *parallel* to a fixed subspace $L_0 \in G(n, r)$, $r \in \{1, \dots, n-1\}$. In our notation, the integral formula underlying this concept is

$$\int_{G(L_0,k)} \int_{L^\perp} \tilde{\varphi}_L^{L_0}\big(K \cap (L + x)\big) \, \lambda_{L^\perp}(dx) \, \nu_k^{L_0}(dL) = V_m(K), \tag{A.4}$$

for $k \in \{r + 1, \dots n\}$ and $m \in \{n + r - k, \dots, n\}$, where $\lambda_{L^\perp}$ is the Lebesgue-measure on the orthogonal complement $L^\perp$ of L. Baddeley [5] used this concept to estimate surface area of an

object in $\mathbb{R}^3$ based on vertical sections parallel to a fixed line ($r = 1$). A practical example involving a Paddington bear can be found in [7]. He extended this idea in [6] to arbitrary dimensions n and r for Hausdorff measures of rectifiable sets ([28, 3.2.14]) with a proof based on the coarea formula. For more information about vertical sections in stereology we recommend consulting the book [8] or the recent overview on vertical sections in [47]. To the best of our knowledge, the vertical section formula (A.4) has not been stated for intrinsic volumes in the literature. We therefore give the details of an independent proof based on a Blaschke–Petkantschin-type result in Section A.6, see Theorem A.7.

   This paper is organized as follows. In Section A.2 we recall preliminaries and basic notation. In Section A.3 we derive the main result, Theorem A.2, a rotational Crofton formula with a fixed subspace. This requires a suitable Blaschke–Petkantschin formula, which will be stated in Theorem A.3. Section A.4 collects basic properties and simplified representations of the measurement functions, including the aforementioned proof of independence from the parameter q. Section A.5 shows that no other intrinsic volumes than those described in Theorem A.2 can be obtained from section profiles under the given design. Hence, our results cannot be extended using different methods. The paper is supplied with an appendix discussing the vertical section formula (A.4).

A.2   Notation and preliminaries

For a set $A \subseteq \mathbb{R}^n$ we let $\mathrm{span}(A)$, $\mathrm{aff}(A)$ and $\mathrm{conv}\,A = \mathrm{conv}(A)$ denote the linear, the affine and the convex hull of A, respectively. The set $A^\perp = \{x \in \mathbb{R}^n : \langle x, y \rangle = 0 \text{ for all } y \in A\}$ is the orthogonal complement of $\mathrm{span}\,A$. We will write $\mathcal{H}^d$ for the d-dimensional Hausdorff measure on $\mathbb{R}^n$ for $d = 0, \ldots, n$, see e.g. [71]. We let $\kappa_n = \mathcal{H}^n(B^n) = \frac{\pi^{n/2}}{\Gamma(1+n/2)}$ be the volume of the Euclidean unit ball B^n in $\mathbb{R}^n$ and $\omega_n = \mathcal{H}^{n-1}(S^{n-1}) = n\kappa_n$ be the surface area of the Euclidean unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$. Throughout the paper we will use the following decomposition ([60, p. 1]) of the restriction of $\mathcal{H}^{n-1}$ to the Borel sets of the unit sphere, which can be thought of as cylindrical coordinates on the sphere:

$$\int_{S^{n-1}} f(u)\,\mathcal{H}^{n-1}(du) = \int_{S^{n-1} \cap v^\perp} \int_{-1}^{1} f(tv + \sqrt{1-t^2}\,w)(1-t^2)^{\frac{n-3}{2}}\,dt\,\mathcal{H}^{n-2}(dw). \tag{A.5}$$

Here, $v \in S^{n-1}$ is a fixed unit vector and $f : S^{n-1} \to [0, \infty)$ is measurable.

   We will follow the notation in [75, Chap. 13] for spaces of flats and subspaces in $\mathbb{R}^n$, $n \in \mathbb{N}$. For $k \in \{0, \ldots n\}$ we let $A(n, k)$ denote the family of *affine* k-dimensional flats of $\mathbb{R}^n$ and we let $G(n, k)$ be the *Grassmannian* of k-dimensional *linear* subspaces of $\mathbb{R}^n$. As we will work with flats and subspaces incident with others, we define for a fixed $L \in G(n, p)$, $p \in \{0, \ldots, n\}$, the spaces

$$G(L, k) = \begin{cases} \{L' \in G(n, k) : L' \subseteq L\}, & \text{if } k \leq p, \\ \{L' \in G(n, k) : L \subseteq L'\}, & \text{if } k > p, \end{cases}$$

and, similarly, for $E \in A(n, p)$

$$A(E, k) = \begin{cases} \{E' \in A(n, k) : E' \subseteq E\}, & \text{if } k \leq p, \\ \{E' \in A(n, k) : E \subseteq E'\}, & \text{if } k > p. \end{cases}$$

To avoid degenerate situations, we assume throughout this paper that the dimension n of the ambient space is at least 3. The unique invariant probability measure on $G(L, k)$ will be denoted by ν_k^L. Its construction is outlined in [75, p. 590]. Invariance is understood here with respect to all rigid rotations that keep L fixed. Similarly, there is a measure μ_k^E on $A(E, k)$, which is invariant under all rigid rotations and translations fixing E. It is unique up to normalization, which we choose as in [75]. More specifically, if $k \leq p$ and $E = L + x$ with $L \in G(n, p)$ and $x \in \mathbb{R}^n$, say, then

$$\int_{A(E,k)} f(E') \, \mu_k^E(dE') = \int_{G(L,k)} \int_{M^\perp \cap L} f(M + x + y) \, \lambda_{M^\perp \cap E}(dy) \, \nu_k^L(dM) \tag{A.6}$$

for any measurable function $f \geq 0$; cf. [75, eq. (13.13)]. Here, and later on, λ_F denotes the q-dimensional Lebesgue measure on a flat $F \in A(n, q)$. When the flat is clear from the context, we also use the notation $\lambda_q = \lambda_F$.

In the special case when $p = n$ we notice that $G(L, k) = G(n, k)$ and $A(E, k) = A(n, k)$. We will write ν_k for the invariant probability measure on $G(n, k)$ and μ_k for the invariant measure on $A(n, k)$. We will use the notation $p(x|E)$ for the orthogonal projection of $x \in \mathbb{R}^n$ on the flat $E \subset \mathbb{R}^n$.

To simplify notation, we use boldface letters to denote vectors of vectors. For instance, we write $\mathbf{x} = (x_1, \ldots, x_q) \in (\mathbb{R}^n)^q$ for the q-tuple of n-dimensional vectors, $q \in \{1, \ldots, n\}$.

Let

$$P(\mathbf{x}) = P(x_1, \ldots, x_q) = \left\{ \sum_{i=1}^q \alpha_i x_i \ : 0 \leq \alpha_i \leq 1, \ i = 1, \ldots, q \right\}$$

denote the parallelepiped spanned by $x_1, \ldots, x_q \in \mathbb{R}^n$. The q-dimensional volume of $P(\mathbf{x})$ is denoted by $\nabla_q(\mathbf{x})$.

For $\mathbf{x} \in (\mathbb{R}^n)^q$ the $(q-1)$-dimensional volume $\Delta_{q-1}(\mathbf{x})$ of $\text{conv}\{x_1, \ldots, x_q\}$ obeys

$$\Delta_{q-1}(x_1, \ldots, x_q) = \frac{1}{(q-1)!} \nabla_{q-1}(x_2 - x_1, \ldots, x_q - x_1), \tag{A.7}$$

see [75, eq. (7.6)]. We have

$$\nabla_{q+1}(x_1, \ldots, x_{q+1}) = \nabla_q(x_1, \ldots, x_q) d(x_{q+1}, L), \tag{A.8}$$

where $x_{q+1} \in \mathbb{R}^n$ and $L \in G(n, q)$ is a linear space containing $x_1, \ldots, x_q$, see [75, eq. (7.9)]. Extending these definitions, let

$$\nabla_{q,r}(\mathbf{x}, M) = \nabla_{q+r}(x_1, \ldots, x_q, u_1, \ldots u_r), \tag{A.9}$$

where $(u_1, \ldots, u_r)$ forms an orthonormal basis of $M \in G(n, r)$. If, in addition, $L \in G(n, k)$ with $r + k \leq n$ is given, the *subspace determinant* of L and M is given as $[L, M] = \nabla_{k,r}(\mathbf{v}, M)$, where $\mathbf{v} = (v_1, \ldots, v_k)$ forms an orthonormal basis of L. In the special cases $M = \{o\}$ or $L = \{o\}$, we define $[M, L] = 1$. If $\dim L + \dim M = n$ and $L \cap M = \{o\}$, the subspace determinant is equal to the factor by which the $(\dim M)$-dimensional Lebesgue measure is multiplied under the orthogonal projection from M to $L^\perp$, see [75, p. 598]. Hence, $\int_{\mathbb{R}^n} f(z) \, \lambda_n(dz)$ coincides with

$$\int_L \int_{L^\perp} f(x + y) \, \lambda_{L^\perp}(dx) \, \lambda_L(dy) = [L, M] \int_L \int_M f(x + y) \, \lambda_M(dx) \, \lambda_L(dy) \tag{A.10}$$

for any measurable function $f : \mathbb{R}^d \to [0, \infty)$.

For $q, r \in \{0, \ldots, n - 1\}$ with $q + r \leq n - 1$ let

$$D(E, L) = d(o, E) [\operatorname{span} E, L], \qquad E \in A(n, q), \ L \in G(n, r), \tag{A.11}$$

where $d(o, E)$ is the Euclidean distance of o to E. This quantity will play an important role later on, and we remark some of its properties in the following lemma.

LEMMA A.1. *Let $L \in G(n, r)$ and $E \in A(n, q)$ with $q + r \leq n - 1$ be given.*

(i) *If $E = \operatorname{aff}\{x_0, \ldots, x_q\} \in A(n, q)$ with $\mathbf{x} = (x_0, \ldots, x_q) \in (\mathbb{R}^n)^q$ then*

$$D(E, L) = \frac{1}{q!} \frac{\nabla_{q+1,r}(\mathbf{x}, L)}{\Delta_q(\mathbf{x})}. \tag{A.12}$$

(ii) *If $E = M + z$, $M \in G(n, q)$, then*

$$D(E, L) = d(E, L)[M, L], \tag{A.13}$$

where $d(E, L)$ is the Euclidean distance between E and L.

(iii) *If $E = M + z, M \in G(n, q), z \in M^{\perp}, q + r \leq n - 2$ and $u \in S^{n-1} \cap M^{\perp}$, then*

$$D(E + \operatorname{span}\{u\}, L) = D(E, L) \left\| p\big(u \big| (L + M + z)^{\perp}\big) \right\|. \tag{A.14}$$

Proof. We start with the proof of (i). The claim is trivial for $r = 0$, so we may assume that $\mathbf{v} = (v_1, \ldots, v_r)$ is an orthonormal basis of L. Define $M = \operatorname{span} E = \operatorname{span}\{x_0, \ldots, x_q\}$. If $M \cap L \neq \{o\}$, both sides of (A.12) vanish, so we may also assume $M \cap L = \{o\}$, implying $P(x_0, \ldots, x_q, v_1, \ldots, v_r) = P(\mathbf{x}) + P(\mathbf{v})$. This, the definition (A.9) and relation (A.10) applied in $M + L$ yield

$$
\begin{aligned}
\nabla_{q+1,r}(\mathbf{x}, L) &= \int_{\mathbb{R}^{q+1+r}} 1_{P(\mathbf{x})+P(\mathbf{v})}(z) \, \lambda_{q+1+r}(dz) \\
&= [L, M] \int_L \int_M 1_{P(\mathbf{x})+P(\mathbf{v})}(y + z) \, \lambda_M(dz) \, \lambda_L(dy) \\
&= [L, M] \, \lambda_r(P(\mathbf{v})) \lambda_{q+1}(P(\mathbf{x})) \\
&= [L, M] \nabla_{q+1}(\mathbf{x}).
\end{aligned}
$$

Now (A.7), the definition of Δ_q and the calculation of the volume of a $(q + 1)$-dimensional pyramid (using Tonelli's theorem) gives

$$
\begin{aligned}
\nabla_{q+1}(\mathbf{x}) &= (q + 1)! \Delta_{q+1}(o, x_1, \ldots, x_q) \\
&= q! \, \Delta_q(x_0, \ldots, x_q) d\big(o, \operatorname{aff}\{x_0, \ldots, x_q\}\big) \\
&= q! \, \Delta_q(\mathbf{x}) d(o, E).
\end{aligned}
$$

This shows assertion (i).

To show (ii) let $E = M + z$, $z \in M^\perp$, and assume that $u_1, \ldots, u_q$ is an orthonormal basis of M. As the affine hull of $z, z + u_1, \ldots, z + u_q$ is E, relation (A.12) and

$$(q!)\Delta_q(z, z + u_1, \ldots, z + u_q) = 1$$

(by (A.7)) show

$$D(E, L) = \nabla_{q+1,r}(z, z + u_1, \ldots, z + u_q, L)$$
$$= \nabla_{q+1,r}(z, u_1, \ldots, u_q, L),$$

where (A.8) was applied back and forth q times at the last step. Now (A.8) yields

$$D(E, L) = d(z, M + L)\nabla_{q,r}(u_1, \ldots, u_q, L) = d(E, L)[M, L],$$

as claimed.

We prove assertion (iii). If $z \in \operatorname{span}\{u\}$, the claim (A.14) is trivially true in view of the definition of $D(E, L)$, so we assume $z \notin \operatorname{span}\{u\}$ in the following. Using $z \in M^\perp$ and $u \in S^{n-1}$ we obtain

$$d(z, M + \operatorname{span}\{u\})^2 = \|z - \langle z, u\rangle u\|^2 = \|z\|^2 \, \|p(u|z^\perp)\|^2. \tag{A.15}$$

If $u_1, \ldots u_q$ is an orthonormal basis of M then $u_1, \ldots, u_q, \frac{z}{\|z\|}, \frac{p(u|z^\perp)}{\|p(u|z^\perp)\|}$ is an orthonormal basis of $M + \operatorname{span}\{z, u\}$. The definition of $D(E, L)$ and (A.15) give

$$D(E + \operatorname{span}\{u\}, L) = d(o, z + M + \operatorname{span}\{u\}) \left[M + \operatorname{span}\{z, u\}, L\right]$$
$$= d(z, M + \operatorname{span}\{u\}) \, \nabla_{q+2,r}\left(u_1, \ldots, u_q, \frac{z}{\|z\|}, \frac{p(u|z^\perp)}{\|p(u|z^\perp)\|}, L\right).$$
$$= \|z\| \, \nabla_{q+2,r}\left(u_1, \ldots, u_q, \frac{z}{\|z\|}, p(u|z^\perp), L\right).$$

Using that $(L + M + z)^\perp \subseteq z^\perp$ we conclude from (A.8), applied to the vector $\frac{p(u|z^\perp)}{\|p(u|z^\perp)\|}$, that

$$D(E + \operatorname{span}\{u\}, L) = \|z\| \, \nabla_{q+1,r}\left(u_1, \ldots u_q, \frac{z}{\|z\|}, L\right) \left\|p\left(u|(L + M + z)^\perp\right)\right\|.$$

As $\|z\| = d(o, E)$ and $u_1, \ldots u_q, \frac{z}{\|z\|}$ is an orthonormal basis of $M + \operatorname{span}\{z\}$, we have

$$\|z\| \, \nabla_{q+1,r}\left(u_1, \ldots u_q, \frac{z}{\|z\|}, L\right) = D(E, L)$$

and the assertion follows.

∎

The first statement in Lemma A.1 is a rather technical relation, only used in the proof of Theorem A.3. Lemma A.1.(ii) shows that $D(E, L)$ depends only on the distance between the two flats and the generalized angle between the corresponding linear subspaces. In particular, it shows that if $E \in A(n, 0)$ and $L \in G(n, r)$ then $D(E, L) = \|p(E|L^\perp)\|$. The last statement in Lemma A.1 is used in the proof of Theorem A.4.

We define the constants

$$b_{n,q} = \frac{\omega_{n-q+1} \cdots \omega_n}{\omega_1 \cdots \omega_q}, \quad 1 \le q \le n, \tag{A.16}$$

needed in certain Blaschke–Petkantschin formulae. Fix integers $k, q \ge 1$ and $r \ge 0$ such that $q + r \le k \le n$ and a subspace $L_0 \in G(n, r)$. Generically, the entries of a vector $\mathbf{x}$ consisting of q points in $\mathbb{R}^n$ and L_0 span a subspace $L \in G(L_0, r + q)$, which will be called a *pivot*. The following Blaschke–Petkantschin formula (with $k = q + r$) shows that invariant integration with respect to $\mathbf{x}$ can be achieved by invariantly integrating $\mathbf{x}$ in L, and then integrating the pivot:

$$\int_{(\mathbb{R}^n)^q} f(\mathbf{x}) \, \lambda^q(d\mathbf{x}) = \frac{b_{n-r,q}}{b_{k-r,q}} \int_{G(L_0,k)} \int_{L^q} f(\mathbf{x}) \nabla_{q,r}(\mathbf{x}, L_0)^{n-k} \lambda_L^q(d\mathbf{x}) \, \nu_k^{L_0}(dL) \tag{A.17}$$

holds for all measurable functions $f : (\mathbb{R}^n)^q \to [0, \infty)$, see [75, Thm. 7.2.2]. A proof of the Blaschke–Petkantschin formula (A.17), based on polar decomposition of the k-fold product of Lebesgue measure, can be found in [29, 59, 72]. A similar, affine Blaschke–Petkantschin formula ([75, Thm. 7.2.7]) states

$$\int_{(\mathbb{R}^n)^{q+1}} f(\mathbf{x}) \, \lambda^{q+1}(d\mathbf{x}) = b_{n,q}(q!)^{n-q} \int_{A(n,q)} \int_{E^{q+1}} f(\mathbf{x}) \Delta_q(\mathbf{x})^{n-q} \lambda_E^{q+1}(d\mathbf{x}) \, \mu_q(dE) \tag{A.18}$$

for all measurable functions $f : (\mathbb{R}^n)^{q+1} \to [0, \infty)$ and $q \in \{1, \ldots, n\}$. An overview of Blaschke–Petkantschin-type formulas with a sphere as pivot can be found in [61].

For later reference, we state the classical Crofton formula (see, e.g., [43, Thm. 5.10] or [74, eq. (4.59)]), which was already mentioned in the introduction in (A.1):

$$\int_{A(n,q)} V_{q-j}(K \cap E) \, \mu_q(dE) = c_{q-j,n}^{n-j,q} V_{n-j}(K), \quad K \in \mathcal{K}^n, \tag{A.19}$$

valid for $q \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, q\}$ with

$$c_{s_1,s_2}^{r_1,r_2} = \frac{r_1! \kappa_{r_1}}{s_1! \kappa_{s_1}} \frac{r_2! \kappa_{r_2}}{s_2! \kappa_{s_2}}. \tag{A.20}$$

Numerous variants and generalizations of (A.19) can be found in the literature, see the overview [39]–the following list exemplifies the variety without claiming completeness: the underlying Euclidean space can be replaced by a Finsler space [9, 64], the results for intrinsic volumes can be extended to support measures [33], the set class $\mathcal{K}^n$ can be generalized, for example to $\mathcal{U}_{PR}$-sets [68], a family of certain finite unions of sets with positive reach, and the invariant integration in (A.19) can be replaced by an invariant integration over translations of a fixed flat, leading to *translative Crofton formulae* [34, 68]. Our new result (A.3) can be considered as a generalization in the spirit of the last example, as we also replace integration over the full space $A(n, q)$ by integration over a geometrically meaningful subset.

A.3 Rotational Crofton Formulae with a Fixed Subspace

The main result of this paper is the following theorem. It uses the constant

$$\alpha_{n,k,q,r} = \frac{\omega_{k-q-r} \cdots \omega_{k-q}}{\omega_{n-q-r} \cdots \omega_{n-q}} \prod_{j=0}^{r-1} \frac{\omega_{n-j}}{\omega_{k-j}}. \tag{A.21}$$

For $r = 0$ the last product in (A.21) is set equal to 1 by convention.

THEOREM A.2 (Rotational Crofton formulae with a fixed subspace). *Let $n, r, k \in \mathbb{N}_0$ with $r + 1 \le k \le n$ be given and fix a subspace $L_0 \in G(n, r)$. Then, for $j = 0, \ldots, k - (r + 1)$,*

$$\int_{G(L_0,k)} \varphi_{L,q}^{L_0}(K \cap L) \, \nu_k^{L_0}(dL) = V_{n-j}(K), \qquad K \in \mathcal{K}^n, \tag{A.22}$$

holds with

$$\varphi_{L,q}^{L_0}(K \cap L) = \alpha_{n,k,q,r}^{-1} c_{n-j,q}^{q-j,n} \int_{A(L,q)} V_{q-j}(K \cap E) \, D(E, L_0)^{n-k} \, \mu_q^L(dE). \tag{A.23}$$

Here, $D(E, L_0)$ is given by (A.11)*, the leading constant is defined in* (A.20) *and* (A.21)*, and q can be chosen in $\{j, \ldots, k - (r + 1)\}$.*

The special case $r = 0$ of Theorem A.2 poses no constraints on the linear spaces in $G(L_0, k) = G(\{o\}, k) = G(n, k)$ and states that

$$\int_{G(n,k)} \varphi_{L,q}(K \cap L) \, \nu_k^{L_0}(dL) = V_{n-j}(K), \qquad K \in \mathcal{K}^n, \tag{A.24}$$

where

$$\varphi_{L,q}(K \cap L) = \frac{\omega_{n-q}}{\omega_{k-q}} c_{n-j,q}^{q-j,n} \int_{A(L,q)} V_{q-j}(K \cap E) d(o, E)^{n-k} \, \mu_q^L(dE) \tag{A.25}$$

for $q \in \{j, \ldots, k - 1\}$. This is the precise statement of the rotational Crofton formula in [3] already mentioned qualitatively in the introduction in (A.2). There thus appear to be $k - j$ different measurement functions. However, $\varphi_{L,j} = \cdots = \varphi_{L,k-1}$, so all these measurement functions coincide. This fact, although known, appears to be unpublished (see, however, the forthcoming publication [49]). It also follows from Theorem A.4 in Section A.4, which establishes the independence of the measurement functions of q for general $r \ge 0$.

The proof of Theorem A.2 will be given below. It is obtained by combining the classical Crofton formula (A.19) with the following Blaschke–Petkantschin result.

THEOREM A.3. *Let $n, q, r, k \in \mathbb{N}_0$ with $q + r + 1 \le k \le n$ be given and fix a subspace $L_0 \in G(n, r)$. Then, for any measurable function $f : A(n, q) \to [0, \infty)$ we have*

$$\int_{G(L_0,k)} \int_{A(L,q)} f(E) D(E, L_0)^{n-k} \, \mu_q^L(dE) \, \nu_k^{L_0}(dL)$$
$$= \alpha_{n,k,q,r} \int_{A(n,q)} f(E) \, \mu_q(dE), \tag{A.26}$$

with the constant (A.21) *and $D(E, L_0)$ given by* (A.11)*.*

Proof. We start with the special case $q = 0$. Equation (A.17) implies

$$\int_{A(n,0)} f(E) \, \mu_q(dE) = \int_{\mathbb{R}^n} f(\{x\}) \, \lambda(dx)$$
$$= \frac{b_{n-r,1}}{b_{k-r,1}} \int_{G(L_0,k)} \int_L f(\{x\}) \nabla_{1,r}(x, L_0)^{n-k} \, \lambda_L(dx) \, \nu_k^{L_0}(dL). \tag{A.27}$$

Since $\nabla_{1,r}(x, L_0) = D(\{x\}, L_0)$, μ_0^L is the image measure of λ_L under the identification $x \mapsto \{x\}$, and $\frac{b_{n-r,1}}{b_{k-r,1}} = \alpha_{n,k,0,r}^{-1}$ the relation (A.27) is equivalent to the claim when $q = 0$.

Now assume $q > 0$ and consider

$$g : (\mathbb{R}^n)^{q+1} \to \mathbb{R},$$
$$\mathbf{x} \mapsto f(\text{aff}(\mathbf{x}))\Delta_q(\mathbf{x})^{q-n}h(\mathbf{x}),$$

with $h(\mathbf{x}) = \prod_{i=1}^{q+1} 1_{B^n}(x_i - p(o|\,\text{aff}(\mathbf{x})))$, defined for all $\mathbf{x} \in (\mathbb{R}^n)^{q+1}$ such that $\text{aff}(x) \in A(n,k)$. Using (A.18) we get

$$\int_{(\mathbb{R}^n)^{q+1}} g(\mathbf{x})\, \lambda^{q+1}(d\mathbf{x})$$

$$= b_{nq}(q!)^{n-q} \int_{A(n,q)} \int_{E^{q+1}} f(\text{aff}(\mathbf{x}))\Delta_q(\mathbf{x})^{q-n}h(\mathbf{x})\Delta_q(\mathbf{x})^{n-q}\, \lambda_E^{q+1}(d\mathbf{x})\, \mu_q(dE)$$

$$= b_{nq}(q!)^{n-q} \int_{A(n,q)} f(E) \int_{E^{q+1}} h(\mathbf{x})\, \lambda_E^{q+1}(d\mathbf{x})\, \mu_q(dE)$$

$$= b_{nq}(q!)^{n-q}\kappa_q^{q+1} \int_{A(n,q)} f(E)\, \mu_q(dE). \tag{A.28}$$

On the other hand (A.17) with q replaced by $q + 1$ gives

$$\int_{(\mathbb{R}^n)^{q+1}} g(\mathbf{x})\, \lambda^{q+1}(d\mathbf{x}) = \frac{b_{n-r,q+1}}{b_{k-r,q+1}} \int_{G(L_0,k)} I(L)\, \nu_k^{L_0}(dL), \tag{A.29}$$

with

$$I(L) = \int_{L^{q+1}} g(\mathbf{x})\nabla_{q+1,r}(\mathbf{x}, L_0)^{n-k}\, \lambda_L^{q+1}(d\mathbf{x}).$$

Using $L_0 \subseteq L$, we can identify L with $\mathbb{R}^k$. An application of (A.18) in $\mathbb{R}^k$ then yields

$$I(L) = b_{k,q}(q!)^{k-q} \int_{A(k,q)} \int_{E^{q+1}} g(\mathbf{x})\Delta_q(\mathbf{x})^{k-q}\nabla_{q+1,r}(\mathbf{x}, L_0)^{n-k}\, \lambda_E^{q+1}(d\mathbf{x})\, \mu_q(dE).$$

Inserting the definition of g gives

$$I(L) = b_{k,q}(q!)^{n-q} \int_{A(k,q)} \int_{E^{q+1}} f\big(\text{aff}(\mathbf{x})\big)\left(\frac{1}{q!}\frac{\nabla_{q+1,r}(\mathbf{x}, L_0)}{\Delta_q(\mathbf{x})}\right)^{n-k}$$
$$\times h(\mathbf{x})\, \lambda_E^{q+1}(d\mathbf{x})\, \mu_q(dE)$$
$$= b_{k,q}(q!)^{n-q}\kappa_q^{q+1} \int_{A(k,q)} f(E)D(E, L_0)^{n-k}\, \mu_q(dE),$$

where Lemma A.1 was used in the last step.

Comparing (A.28) with (A.29), using the explicit form of $I(L)$ just derived, shows the claim (A.26). To simplify the constant, we used

$$\frac{b_{n,q}}{b_{k,q}}\frac{b_{k-r,q+1}}{b_{n-r,q+1}} = \frac{\omega_{n-q+1}\cdots\omega_n}{\omega_{n-r-q}\cdots\omega_{n-r}} \cdot \frac{\omega_{k-r-q}\cdots\omega_{k-r}}{\omega_{k-q+1}\cdots\omega_k} = \frac{\omega_{k-q-r}\cdots\omega_{k-q}}{\omega_{n-q-r}\cdots\omega_{n-q}}\prod_{j=0}^{r-1}\frac{\omega_{n-j}}{\omega_{k-j}},$$

which follows by induction in $r = 0, 1, \ldots k - q - 1$, if we define $\prod_{j=0}^{r-1}\frac{\omega_{n-j}}{\omega_{k-j}} = 1$ when $r = 0$. This concludes the proof. ∎

We can now give a proof of the main result.

Proof of Theorem A.2. With the assumptions of Theorem A.2, put $f(E) = V_{q-j}(K \cap E)$ in the Blaschke–Petkantschin formula (Theorem A.3) and observe that the right-hand side is a classical Crofton integral that can be evaluated using (A.19). ∎

A.4 THE MEASUREMENT FUNCTIONS

We collect properties and simplified representations of the measurement function defined in Theorem A.2. To avoid unnecessary repetitions, we put

$$c_0(q) = \alpha_{n,k,q,r}^{-1} c_{n-j,q}^{q-j,n} \tag{A.30}$$

for the leading constant in (A.23), thinking of all dimensions other than q as fixed.

The main result in this section is the independence of $\varphi_{L,q}^{L_0}$ from q given in Theorem A.4. With this independence in mind, we can simply choose one value of q and provide simplified expressions. This will be done in statements (ii) and (iii) of Proposition A.5 below. We start by stating the uniqueness result.

THEOREM A.4. *Let the setting be as in Theorem A.2. Then, $\varphi_{L,q}^{L_0}(L \cap K)$ is independent of $q \in \{j, \dots, k - (r+1)\}$.*

Proof. Let the assumptions of Theorem A.2 be satisfied. If $\{j, \dots, k - (r+1)\}$ is a singleton, the claim is trivial. Thus, we may assume $j < k - (r+1)$ and it is enough to show

$$\varphi_{L,q+1}^{L_0}(\cdot) = \varphi_{L,q}^{L_0}(\cdot) \tag{A.31}$$

for every fixed $q \geq j$ such that $q + 1 \leq k - (r+1)$.

Fix $E_1 \in A(L, q+1)$. Applying (A.19) to the convex body $K \cap E_1$ in E_1 gives

$$V_{(q+1)-j}(K \cap E_1) = c_{q-j+1,q}^{q-j,q+1} \int_{A(E_1,q)} V_{q-j}(K \cap E) \, \mu_q^{E_1}(dE).$$

Combining this with (A.23) yields

$$\varphi_{L,q+1}^{L_0}(K \cap L) = c_1 \int_{A(L,q+1)} \int_{A(E_1,q)} V_{q-j}(K \cap E) \, \mu_q^{E_1}(dE)$$
$$\times D(E_1, L_0)^{n-k} \, \mu_{q+1}^{L}(dE_1),$$

where $c_1 = c_0(q) \frac{\omega_{n-r-(q+1)}}{\omega_{k-r-(q+1)}} \frac{\omega_{k-q}}{\omega_{n-q}}$. Note that all flats involved are subsets of the k-dimensional subspace L, so we may intechange integrals due to [75, Thm. 7.1.2], applied in L. Identifying L with $\mathbb{R}^k$, we get

$$\varphi_{L,q+1}^{L_0}(K \cap L) = c_1 \int_{A(L,q)} V_{q-j}(K \cap E) h(E) \, \mu_q^{L}(dE), \tag{A.32}$$

where

$$h(E) = \int_{A(E,q+1)} D(E_1, L_0)^{n-k} \, \mu_q^E(dE_1).$$

is to be understood in $\mathbb{R}^k$. All orthogonal complements and unit spheres that will appear in the calculation of $h(E)$ will therefore also be understood relative to $\mathbb{R}^k$. Writing $E = M + z$ with $M \in G(k, q)$ and $z \in M^\perp$, [75, eq. (13.14)] gives

$$
\begin{aligned}
h(M + z) &= \int_{G(M,q+1)} D(L_1 + z, L_0)^{n-k} \, \nu_{q+1}^M(dL_1) \\
&= \omega_{k-q}^{-1} \int_{S^{k-1} \cap M^\perp} D(M + \mathrm{span}(u) + z, L_0)^{n-k} \, \mathcal{H}^{k-q-1}(du) \\
&= \omega_{k-q}^{-1} D(E, L)^{n-k} \int_{S^{k-1} \cap M^\perp} \|p(u | (L_0 + M + z)^\perp)\|^{n-k} \, \mathcal{H}^{k-q-1}(du),
\end{aligned}
$$

where we used (A.14) at the last equality sign.

We will now show that $z \neq o$ implies

$$
\begin{aligned}
I_z(M) &= \int_{S^{k-1} \cap M^\perp} \|p(u | (L_0 + M + z)^\perp)\|^{n-k} \, \mathcal{H}^{k-q-1}(du) \\
&= \frac{\omega_p}{\omega_{n-k+p}} \omega_{n-q},
\end{aligned}
\tag{A.33}
$$

with $p = \dim(L_0 + M + z)^\perp$.

To show relation (A.33) we apply (A.5) in the $(k-q)$-dimensional space $M^\perp$ with $v = z/\|z\|$ using that $p(v | (M + L + z)^\perp) = o$ to obtain

$$
I_z(M) = c' \int_{S^{k-1} \cap (M+z)^\perp} \|p(w | (L_0 + M + z)^\perp)\|^{n-k} \, \mathcal{H}^{k-q-2}(dw),
\tag{A.34}
$$

where

$$
c' = \int_{-1}^{1} (1 - t^2)^{\frac{n-q-3}{2}} \, dt = \frac{\omega_{n-q}}{\omega_{n-q-1}}
$$

has been evaluated by setting $f(u) = 1$ in (A.5).

For fixed $d, j, p \in \mathbb{N}$ with $p \leq d$, and $L' \in G(d, p)$ the result in [3, Lemma 1] gives

$$
\int_{S^{d-1}} \|p(u | L')\|^j \, du = \frac{\omega_{d-p} \omega_p}{2} \int_0^1 t^{\frac{j+p}{2}-1}(1-t)^{\frac{d-p}{2}-1} \, dt = \frac{\omega_{d+j}}{\omega_{p+j}} \omega_p.
\tag{A.35}
$$

Using this with $d = k - (q+1)$, $j = n - k$ and $p = \dim(L_0 + M + z)^\perp$ in (A.34) yields the claim (A.33).

This claim (A.31) follows by inserting (A.33) into $h(\cdot)$ and the result into (A.32) and observing that $p = k - r - (q+1)$ holds for almost every M. Indeed,

$$
\begin{aligned}
p = \dim(L_0 + M + z)^\perp &= k - \dim(L_0) - \dim(M + z) + \dim(L_0 \cap (M + z)) \\
&= k - r - (q+1) + 0
\end{aligned}
$$

a.e., as an invariance argument yields

$$
\int_{G(k,q)} \int_{M^\perp \cap RB^k} 1_{\dim(\mathrm{span}(M+z) \cap L_0) > 0} \, \lambda_{M^\perp}(dz) \, \nu_q(dM)
$$

$$= R^{k-q} \kappa_{k-q} \int_{G(k,q+1)} 1_{\dim(M' \cap L_0) > 0} \, \nu_{q+1}(dM') = 0$$

for all $R > 0$, where we used [74, Lemma 4.4.1] and $r + (q + 1) < k$. This concludes the proof and the theorem is shown. ∎

The following Proposition states properties of the measurement functions. Item (i) describes general characteristics. With Theorem A.4 in mind, we can focus on one value of $q \in \{j, \dots, k - (r+1)\}$ when providing reductions of $\varphi_{L,q}^{L_0}$. Item (ii) in Proposition A.5 below, chooses therefore the minimal possible q: it shows that the measurement function for $q = j$ can be interpreted as average of weighted projections onto j-dimensional linear subspaces that in turn are weighted according to their position relative to L_0. Item (iii) expresses simply the case $j = 0$, written for convenience. That the right-hand side of (A.37) yields a rotational Crofton formula with axis for the volume can also be seen directly from (A.17).

PROPOSITION A.5. *Let the assumptions of Theorem A.2 be satisfied and fix $q \in \{j, \dots, k - (r+1)\}$. When $L \in G(L_0, k)$ the following statements hold.*

(i) *$\varphi_{L,q}^{L_0}$ is an additive functional on the convex bodies in L. It is positive homogeneous of degree $n - j$ and invariant under all rotations fixing L and L_0.*

(ii) *We have*

$$\varphi_{L,j}^{L_0}(K') = c_0(j) \int_{G(L,j)} [M, L_0]^{n-k} \int_{K'|M^\perp} d(z, M + L_0)^{n-k} \, \lambda_{M^\perp}(dz) \, \nu_j^L(dM), \qquad \text{(A.36)}$$

for all convex bodies $K' \subset L$.

(iii) *If $j = 0$ (rotational integral for the volume) this simplifies to*

$$\varphi_{L,q}^{L_0}(K') = \frac{\omega_{n-r}}{\omega_{k-r}} \int_{K'} d(x, L_0)^{n-k} \, \lambda_L(dx), \qquad \text{(A.37)}$$

for all convex bodies $K' \subset L$.

Proof. The proof of (i) is straightforward. It uses additivity, homogeneity and motion invariance of the intrinsic volumes, see, e.g., [43, Section 3.3] and the fact that $\varphi_{L,q}^{L_0}$ is defined via an invariant integral.

To show (ii) we may assume $q = j$ due to Theorem A.4. Then (A.23), and an application of (A.6) and (A.13) yield

$$\varphi_{L,j}^{L_0}(K') = c_0(j) \int_{G(L,j)} [M, L_0]^{n-k} \int_{M^\perp \cap L} V_0(K' \cap (M + z)) d(z, M + L_0)^{n-k} \, \lambda_{M^\perp}(dz) \, \nu_j^L(dM).$$

The assertion now follows, observing that $V_0(K' \cap (M + z)) = \mathbf{1}_{K'|M^\perp}(z)$.

Finally, (iii) is a direct consequence of (A.36) with $j = 0$, observing that $G(L, 0)$ is a singleton containing only $\{o\}$. ∎

We remark that (A.36) can also be expressed in terms of the radial function of $K'|M^\perp$. The *radial function* of a convex body $K \subset \mathbb{R}^n$ with $o \in K$ is given by $\rho_K(u) = \sup\{t \geq 0 : t \cdot u \in K\}$ for $u \in S^{n-1}$. Introducing spherical coordinates one sees that

$$\int_K f(z) \, dz = \int_{S^{n-1}} \int_0^{\rho_K(u)} f(ru) r^{n+1} \, dr \, \mathcal{H}^{n-1}(du)$$

holds for every non-negative measurable function f. Applying this in $M^\perp$ with $f(z) = d(z, M + L_0)^{n-k}$ and $K = K'|M^\perp$ yields

$$\varphi_{L,j}^{L_0}(K') = \frac{c_0(j)}{n-j} \int_{G(L,j)} [M, L_0]^{n-k} \int_{S^{n-1} \cap M^\perp \cap L} d(u, M + L_0)^{n-k} \rho_{K'|M^\perp}^{n-j}(u) \, \mathcal{H}^{k-j-1}(du) \, \nu_j^L(dM)$$

provided that $o \in K'$. This gives an alternative representation of (A.37) in the case $j = 0$, namely,

$$\varphi_{L,0}^{L_0}(K') = \frac{\omega_{n-r}}{n\omega_{k-r}} \int_{S^{n-1} \cap L} d(u, L_0)^{n-k} \rho_{K'}^n(u) \, \mathcal{H}^{k-1}(du)$$

which may be advantageous when the radial function is easily accessible.

We will now briefly discuss which indices in Theorem A.2 lead to new formulae and which are already established in the literature. As already outlined directly after the statement of Theorem A.2, our main result reduces when $r = 0$ to the rotational Crofton formulae in [3] and [36]. If we further restrict considerations to $n = 3$ the resulting formulas date even further back as then the results are closely related to classical estimators: the *nucleator* [37] and the *rotator* [48], both already mentioned in the introduction. Of surface estimators in the three-dimensional space, we have the indices $n = 3, k = 2, r = 0, q = 1, j = 1$ where our formula is known as *integrated surfactor* or *flower estimator*.

When $r \neq 0$ in the three-dimensional setting, excluding the trivial case $k = 3$ only leaves us with the indices $r = 1, k = 2, q = 0$ and $j = 0$ yielding an integral relation for the volume from planar sections. The resulting estimator is well-known and goes under the name *vertical estimator* and to the authors' knowledge dates back to 1993, see [48].

In fact, all admissible estimators in the three-dimensional setting are already established in the rich literature in stereology and local stereology. When $n = 4$ the first new results appear. Due to (A.37) we get three different formulas for the volume of $K \in \mathbb{R}^4$ and one formula relating to the surface area of $K \in \mathbb{R}^n$. An example could be the choice of indices $n = 4, k = 3, r = q = j = 1$ resulting in

$$V_3(K) = c_0(1) \int_{G(L_0,3)} \int_{A(L,1)} V_0(K \cap E) d(o, E) \sin \sphericalangle(E, L_0) \, \mu_2^L(dE) \, \nu_2^{L_0}(dL),$$

with $\sphericalangle(E, L_0)$ being the minimal angle between the two lines E and L_0. This relates the surface area of a 4-dimensional convex body K to a double integration, where the inner integral is over lines passing through $K \cap L$ weighted by the relative angle to L_0 and their distance from the origin.

A.5 ROTATIONAL CROFTON FORMULAE WITH AXIS FOR OTHER INTRINSIC VOLUMES

We end this paper with a short discussion of the possibility to weaken the constraints on the indices in our main result. For a given $r \leq n - 1$ Theorem A.2 states rotational Crofton formulae

with axis for V_m, $m \in \{r + 1, \ldots, n\}$. Strictly speaking, the result for the homogeneity degree $m = r + 1$ is trivial, as it can only be achieved by choosing $k = n$, meaning that the left side of (A.22) depends on all of K instead of a lower dimensional section of K. Hence, only V_m, $m \in \{r + 2, \ldots, n\}$, can be obtained from lower dimensional sections in Theorem A.2. The question arises if this limitation is due to our method of proof or if it is a geometrical limitation that cannot be overcome with other methods either. The following proposition shows that the latter is the case. In this proposition, we exclude the case $k = n$, which just has been discussed.

PROPOSITION A.6. *Let $n, r, k \in \mathbb{N}_0$ with $r + 1 \leq k \leq n - 1$ be given. Fix a subspace $L_0 \in G(n, r)$ and $m \in \{0, \ldots, r + 1\}$. Then, there is no function φ satisfying*

$$\int_{G(L_0, k)} \varphi(K \cap L)\, \nu_k^{L_0}(dL) = V_m(K) \tag{A.38}$$

for all $K \in \mathcal{K}^n$.

Proof. To show that (A.38) is impossible, it is enough to find two convex bodies K_1, K_2 such that $K_1 \cap L = K_2 \cap L$ for almost all $L \in G(L_0, k)$ and $V_m(K_1) \neq V_m(K_2)$.

If $m = 0$ such an example is given by the singleton $K_1 = \{x\}$ with $x \notin L_0$ and the empty set $K_2 = \emptyset$. For $m > 0$ choose a linear subspace $L' \subset L_0$ of dimension $m - 1$ and a vector $u \in L_0^{\perp} \cap S^{n-1}$. Put

$$K_1 = (B^n \cap L') + \text{conv}\{o, u\},$$
$$K_2 = B^n \cap L'.$$

Then $K_1 \cap L = K_2 \cap L$ for almost all $L \in G(L_0, k)$ as $k \leq n - 1$. As $\dim K_1 = m$ and $\dim K_2 = m - 1$ we have $V_m(K_1) > 0$ and $V_m(K_2) = 0$. This yields the claim. ∎

Another possibility for rotational Crofton formulae of the type discussed in this paper is when integrating over $G(L_0, k)$ for a fixed axis $L_0 \in G(n, r)$ where $k \leq r$. This setting is not of practical interest, as any part of K outside L_0 cannot be observed by the sections considered. Hence, we only can hope to estimate intrinsic volumes of bodies $K \in \{K' \in \mathcal{K}^n : K' \subset L_0\}$. For such K one can of course identify $G(L_0, k)$ with $G(r, k)$ and just utilize the classical Crofton formulae with suitable parameters.

A.6   APPENDIX : VERTICAL SECTIONS FOR INTRINSIC VOLUMES

In this appendix we will give a proof of the vertical sections relation (A.4) based on a suitable Blaschke–Petkantschin formula. For a flat $E \in A(n, q)$ we let $\text{lin}(E) = E - p(o|E)$ denote the unique linear subspace of the same dimension which is parallel to E. It should be noted that $\text{lin}(E)$ and $\text{span}(E)$ typically differ. As $\text{span}(E) = \text{lin}(E) + \text{span}(\{x\})$ holds for arbitrary $x \in E$, they coincide if and only if E is a linear subspace. We will now prove the following result on vertical sections.

THEOREM A.7 (Vertical sections). *Let $n, r, k \in \mathbb{N}_0$ with $r + 1 \leq k \leq n$ be given and fix a subspace $L_0 \in G(n, r)$. Then, for $j = 0, \ldots, k - r$,*

$$\int_{G(L_0, k)} \int_{L^{\perp}} \tilde{\varphi}_{L+x, q}^{L_0}(K \cap (L + x))\, \lambda_{L^{\perp}}(dx)\, \nu_k^{L_0}(dL) = V_{n-j}(K), \qquad K \in \mathcal{K}^n,$$

holds with

$$\tilde{\varphi}_{L+x,q}^{L_0}(K \cap (L+x)) = d_{n,k,r,j}(q) \int_{A(L+x,q)} V_{q-j}(K \cap E) [\text{lin}(E), L_0]^{n-k} \mu_q^{L+x}(dE).$$

Here q can be chosen in $\{j, \ldots, k-r\}$. *The leading constant is*

$$d_{n,k,r,j}(q) = \frac{b_{n-r,q}}{b_{k-r,q}} \frac{b_{k,q}}{b_{n,q}} c_{n-j,q}^{q-j,n} = c_0(q) \frac{\omega_{k-r-q}}{\omega_{n-r-q}}$$

with $c_0(q)$ *given by* (A.30).

For $r = 0$, the classical Crofton formula (A.19), applied in the k-dimensional flat $L + x$, allows one to simplify the measurement function:

$$\tilde{\varphi}_{L+x,q}^{L_0}(K \cap (L+x)) = c_{n-j,k}^{k-j,n} V_{k-j}(K \cap (L+x)).$$

Hence, the special case $r = 0$ of Theorem A.7 reduces to the classical Crofton formula.

The vertical section formula in Theorem A.7 from k-dimensional sections allows for the estimation of V_m with $m \in \{n - k + r, \ldots, n\}$. For fixed r, the integral relation with vertical sections thus gives access to one more intrinsic volume of K, namely $V_{n-k+r}(K)$, as compared to the rotational Crofton formula with a fixed subspace (Theorem A.2). This is one of the reasons that the concept of vertical sections is so well-established in the applied literature.

We will now give a proof of Theorem A.7 using a suitable Blaschke–Petkantschin formula, which we will also prove. We mentioned already in the introduction that [6] showed vertical section formulae for Hausdorff measures of sets in $\mathbb{R}^n$. The rather contracted proof in that paper was based on the coarea formula. Interestingly enough, the result to come has the same structure as in [6, eq. (6)] if one replaces a Hausdorff measure for an appropriate intrinsic volume.

The proof of Theorem A.7 parallels the one in Section 3. To state the relevant Blaschke–Petkantschin formula, we will use the following relation:

$$\int_{L^\perp} \int_{A(L+x,q)} f(E) \mu_q^{L+x}(dE) \lambda_{L^\perp}(dx) = \int_{G(L,q)} \int_{M^\perp} f(M+x) \lambda_{M^\perp}(dx) \nu_q^L(dM), \tag{A.39}$$

valid for $L \in G(n,q)$ and any measurable function $f : A(n,q) \to [0, \infty)$, see [75, eq. (13.13)].

We now formulate an affine Blaschke–Petkantschin formula with a fixed subspace, the proof of which being inspired by [50, Lemma 2.2]. By "affine" we here mean that we want to choose an invariant flat parallel to the fixed subspace L_0. This can be done in a two-step procedure, where one first chooses a subspace L containing L_0 and afterward moves the subspace invariantly by a vector $x \in L^\perp$.

THEOREM A.8. *Let* $n, q, r, k \in \mathbb{N}_0$ *with* $q + r \le k \le n$ *be given and fix a subspace* $L_0 \in G(n,r)$. *Then, for any measurable function* $f : A(n,q) \to [0, \infty)$ *we have*

$$\int_{G(L_0,k)} \int_{L^\perp} \int_{A(L+x,q)} f(E) [\text{lin}(E), L_0]^{n-k} \mu_q^{L+x}(dE) \lambda_{L^\perp}(dx) \nu_k^{L_0}(dL)$$

$$= \frac{b_{k-r,q}}{b_{n-r,q}} \frac{b_{n,q}}{b_{k,q}} \int_{A(n,q)} f(E) \mu_q(dE).$$

Proof. For any measurable and positive function $g : G(n,q) \to [0, \infty)$ we have

$$\kappa_n^q \int_{G(n,q)} g(L)\, \nu_q(dL) = \int_{(B^n)^q} g(\text{span}\, \mathbf{x})\, \lambda_n^q(d\mathbf{x}), \tag{A.40}$$

as the image measure of the restriction of λ_n^q to $(B^n)^q$ under the mapping $\mathbf{x} \mapsto \text{span}\, \mathbf{x}$ is a rotationally invariant measure on $G(n,q)$ and thus a multiple of ν_q. Relation (A.17) with $f(\mathbf{x}) = g(\text{aff}\, \mathbf{x}) \mathbf{1}_{(B^n)^q}(\mathbf{x})$ gives

$$\kappa_n^q \int_{G(n,q)} g(L)\, \nu_q(dL) = \frac{b_{n-r,q}}{b_{k-r,q}} \int_{G(L_0,k)} I(L)\, \nu_k^{L_0}(dL), \tag{A.41}$$

where we used the abbreviation

$$I(L) = \int_{(B^n \cap L)^q} g(\text{span}\, \mathbf{x}) \nabla_{q,r}(\mathbf{x}, L_0)^{n-k} \lambda_L^q(d\mathbf{x}).$$

Now [75, Thm. 7.2.1], applied in L, allows us to write

$$I(L) = b_{k,q} \int_{G(L,q)} \int_{(B^n \cap M)^q} g(\text{span}\, \mathbf{x}) \nabla_{q,r}(\mathbf{x}, L_0)^{n-k} \nabla_q(\mathbf{x})^{k-q} \lambda_M^q(d\mathbf{x})\, \nu_q^L(dM)$$

$$= b_{k,q} \int_{G(L,q)} g(M) \int_{(B^n \cap M)^q} \nabla_{q,r}(\mathbf{x}, L_0)^{n-k} \nabla_q(\mathbf{x})^{k-q} \lambda_M^q(d\mathbf{x})\, \nu_q^L(dM),$$

as $\text{span}\, \mathbf{x} = M$ holds almost surely. The latter also implies

$$\nabla_{q,r}(\mathbf{x}, L_0) = [\text{span}\, \mathbf{x}, L_0] \nabla_q(\mathbf{x}) = [M, L_0] \nabla_q(\mathbf{x}),$$

almost surely. Hence, the inner integral coincides with

$$[M, L_0]^{n-k} \int_{(B^n \cap M)^q} \nabla_q(\mathbf{x})^{n-q} \lambda_M^q(d\mathbf{x})$$

$$= [M, L_0]^{n-k} \int_{G(n,q)} \int_{(B^n \cap M)^q} \nabla_q(\mathbf{x})^{n-q} \lambda_q^q(d\mathbf{x})\, \nu_k(dM)$$

$$= [M, L_0]^{n-k} \frac{1}{b_{n,q}} \int_{(B^n)^q} \lambda^q(d\mathbf{x})$$

$$= [M, L_0]^{n-k} \frac{\kappa_n^q}{b_{n,q}},$$

where we use rotational invariance in the first equality and (A.17) with $q = k$ and $r = 0$ to evaluate the double integral. Therefore,

$$I(L) = \frac{b_{k,q}}{b_{n,q}} \kappa_n^q \int_{G(L,q)} g(M)[M, L_0]^{n-k} \nu_q^L(dM),$$

can be inserted into (A.41) to arrive at

$$\int_{G(n,q)} g(L)\, \nu_q(dL) = \frac{b_{n-r,q}}{b_{k-r,q}} \frac{b_{k,q}}{b_{n,q}} \int_{G(L_0,k)} \int_{G(L,q)} g(M)[M, L_0]^{n-k} \nu_q^L(dM) \nu_k^{L_0}(dL).$$

Letting $g(L) = \int_{L^\perp} f(L + x)\,\lambda_L(dx)$ we arrive at

$$\frac{b_{k-r,q}}{b_{n-r,q}}\frac{b_{n,q}}{b_{k,q}} \int_{A(n,q)} f(E)\,dE = \int_{G(L_0,k)} \int_{G(L,q)} \int_{M^\perp} f(M + x)\,[M, L_0]^{n-k}$$
$$\times \lambda_{M^\perp}(dx)\,\nu_q^L(dM)\,\nu_k^{L_0}(dL).$$

The use of (A.39) concludes the proof. ∎

Notice how the proof proceeds along the same lines as the proof of Theorem A.3: We first transfer integration over subspaces to integration over tuples of vectors and then apply (A.17). It is also worth mentioning that we now allow for a larger range for the dimension q than in Theorem A.3, as $q = k - r$ is now allowed. The new normalizing constant closely resembles $\alpha_{n,k,q,r}$ and can be reduced into a similar structure. In fact, using $b_{a,q} = \frac{\omega_{q+1}}{\omega_{a-q}} b_{a,q+1}$ valid when $q + 1 \le a$ yields

$$\frac{b_{k-r,q}}{b_{n-r,q}}\frac{b_{n,q}}{b_{k,q}} = \alpha_{n,k,q,r}\frac{\omega_{n-r-q}}{\omega_{k-r-q}} \tag{A.42}$$

under the assumptions in Theorem A.8.

Proof of Theorem A.7. With the assumptions of Theorem A.7, put $f(E) = V_{q-j}(K \cap E)$ in the affine Blaschke–Petkantschin formula (Theorem A.8) and observe that the right-hand side is a classical Crofton integral that can be evaluated using (A.19). The constant is reduced using (A.42). ∎

We conclude this appendix with a result corresponding to Theorem A.4.

THEOREM A.9. *Let the setting be as in Theorem A.7. Then, $\tilde{\varphi}_{L,q}^{L_0}(L \cap K)$ is independent of $q \in \{j, \dots, k - r\}$.*

Proof. The proof of this theorem is a slight modification of the proof leading to Theorem A.4. We will therefore refer to this proof for expanding comments.

Let the assumptions of Theorem A.7 be satisfied. If $\{j, \dots, k - r\}$ is a singleton, the claim is trivial. Thus, we may assume $j < k - r$ and it is enough to show

$$\tilde{\varphi}_{L+x,q+1}^{L_0}(\cdot) = \tilde{\varphi}_{L+x,q}^{L_0}(\cdot) \tag{A.43}$$

for every fixed $q \ge j$ such that $q + 1 \le k - r$. Copying the arguments leading to (A.32) we conclude that

$$\tilde{\varphi}_{L+x,q+1}^{L_0}(K \cap (L + x)) = \tilde{c}_1 \int_{A(L+x,q)} V_{q-j}(K \cap E)\tilde{h}(E)\,\mu_q^L(dE), \tag{A.44}$$

with $\tilde{c}_1 = c_0(q)\frac{\omega_{k-q}}{\omega_{n-q}}$, where now

$$\tilde{h}(E) = \int_{A(E,q+1)} [\mathrm{lin}\, E_1, L_0]^{n-k}\,\mu_q^L(dE_1)$$

is to be understood in the k-dimensional affine space $L + x$. In analogy to the proof of Theorem A.4, the flat $L + x$ is identified with $\mathbb{R}^k$ to evaluate $\tilde{h}$. Writing $E = M + z$ with $M \in G(k, q)$ and $z \in M^\perp$, [75, eq. (13.14)] gives

$$
\begin{aligned}
\tilde{h}(M + z) &= \int_{G(M,q+1)} [\text{lin}(L_1 + z), L_0]^{n-k} \, \nu_{q+1}^M(dL_1) \\
&= \omega_{k-q}^{-1} \int_{S^{k-1} \cap M^\perp} [M + \text{span}\, u, L_0]^{n-k} \, \mathcal{H}^{k-q-1}(du) \\
&= \omega_{k-q}^{-1} [M, L_0]^{n-k} \int_{S^{k-1} \cap M^\perp} \|p(u|(L_0 + M)^\perp)\|^{n-k} \, \mathcal{H}^{k-q-1}(du),
\end{aligned}
$$

where we used

$$
[M + \text{span}\, u, L_0] = [M, L_0] \, \|p(u|(M + L_0)^\perp\| \tag{A.45}
$$

at the last equality sign. Equation (A.45) follows directly from (A.8), and is the main difference compared to the proof of Theorem A.4 where Lemma A.1 was used instead. As $[M, L_0] = [\text{lin}(E), L_0]$, we can conclude (A.43) and thereby end the proof by proving that

$$
\tilde{I}(M) = \int_{S^{k-1} \cap M^\perp} \|p(u|(L_0 + M)^\perp)\|^{n-k} \, \mathcal{H}^{k-q-1}(du) \tag{A.46}
$$

is constant except on a suitable null-set. Applying (A.35) now with $d = k - q$, $j = n - k$ and $p = k - r - q$ gives

$$
\tilde{I}(M) = \frac{\omega_{n-q}}{\omega_{n-r-q}} \omega_{k-r-q},
$$

except on the set where $\dim(M + L_0) < r + q$, which is the desired null-set. Collecting all the constants, we conclude

$$
\tilde{\varphi}_{L+x,q+1} = c_0(q) \frac{\omega_{k-q}}{\omega_{n-q}} \frac{1}{\omega_{k-q}} \frac{\omega_{n-q}}{\omega_{n-r-q}} \omega_{k-r-q} d(q)^{-1} \tilde{\varphi}_{L+x,q} = \tilde{\varphi}_{L+x,q},
$$

ending the proof. ∎

Stereologically Adapted Crofton Formulae for Tensor Valuations

B

By Emil Dare

Abstract

The classical Crofton formula explains how intrinsic volumes of a convex body Kin n-dimensional Euclidean space can be obtained from integrating a measurement function at sections of Kwith invariantly moved affine flats. We generalize this idea by constructing stereologically adapted Crofton formulae for translation invariant Minkowski tensors, expressing a prescribed tensor valuation as an invariant integral of a measurement function of section profiles with flats. The measurement functions are weighed sums of powers of the metric tensor times Minkowski valuations. The weights are determined explicitly from known Crofton formulae using Zeilberger's algorithm. The main result is an exhaustive set of measurement functions where the invariant integration is over flats. With the main result at hand, a Blaschke-Petkantschin formula allows us to establish new measurement functions valid when the invariant integration over flats is replaced by an invariant integration over subspaces containing a fixed subspace of lower dimension. Likewise, a stereologically adapted Crofton formula valid in the scheme of vertical sections is constructed. Only some special cases of this result have been stated explicitly before, with even the three dimensional case yielding a new stereological formula. Here, we obtain new vertical section formulae for the surface tensors of even rank.

B.1 Introduction

The classical Crofton formula expresses an intrinsic volume of a suitable set K as an invariant integral over affine subspaces. The integrand is another intrinsic volume of K intersected with the affine subspace. If K is an element of the family $\mathcal{K}^n$ of convex bodies (nonempty, compact, convex subsets) of $\mathbb{R}^n$, Crofton's intersection formula [74, eq. (4.59)] states

$$\int_{A(n,q)} \varphi(K \cap E)\mu_q(dE) = V_{n-j}(K), \tag{B.1}$$

for $q \in \{1, \ldots, n\}$ and $j \in \{0, \ldots, q\}$, where $A(n,q)$ is the family of q-dimensional affine subspaces (flats) in $\mathbb{R}^n$, μ_q is the motion invariant measure on that space normalized as in [75, Sec. 13.2] and $\varphi(K \cap E)$ is proportional to $V_{q-j}(K \cap E)$. The functionals $V_j : \mathcal{K}^n \to \mathbb{R}$ for $j = 0, \ldots, n$ appearing in (A.1), are the *intrinsic volumes*, usually defined as polynomial coefficients in Steiner's formula [43, Thm. 3.10], comprising $V_n(K)$, which is the ordinary volume, $V_{n-1}(K)$, which is proportional to the surface area and $V_1(K)$ being proportional to the mean width of K. We recommend the monograph [74] as an excellent reference for convex geometric notions and results.

In practice, estimation of intrinsic volumes is used to characterize certain properties of convex bodies in $\mathbb{R}^3$. For instance [5, 6, 7] concerns the estimation of the surface area (V_2) of a set.

The intrinsic volumes are also used in materials science to describe physical properties. In [62, p. 121–124], using the fact that V_1 is proportional to the integral of mean curvature,

the connectivity in copper sinter materials is described relating V_1 to a physical property of the material. Another example is found in [69] where intrinsic volumes are used to describe and model the geometry of fibre composites. Intrinsic volumes are also good characteristics in materials science, as they can be determined from spatial binary images using the algorithm described in [63, Chap. 5].

The intrinsic volumes are invariant under rotations (and translations) of the underlying object, which facilitates their calculation and interpretation. However, in applications where one wants to quantify 'orientations' of sets, such as the direction in which an object is most elongated, intrinsic volumes are insufficient. Translation invariant Minkowski tensors $\Phi_{k,s}(K)$, which are generalizations of intrinsic volumes to tensor-valued valuations, have been employed instead, as Minkowski tensors with rank $s > 1$ are not rotation invariant. That they contain far more information on the underlying convex body than intrinsic volumes is illustrated by the fact that a full-dimensional polytope P with at most $m \geq n + 1$ facets is determined in $\mathcal{K}^n$ up to translation by $\{\Phi_{n-1,s}(P) : s = 0, \ldots, m - n + 2\}$, see [53, Thm. 3.2]. It has been shown in [27, Thm. 2.1] that a centered ellipse $K \subset \mathbb{R}^2$ is uniquely determined by the tensor $\Phi_{1,2}(K)$, allowing for some illustrative set-valued test statistics for planar stationary particle processes, see [27, Fig. 1].

These tensors also appear in practical applications; for instance in [79] the authors estimate shape and orientation of pyramidal neurons using volume tensors based on optical microscopy. Here the available information is intersections of the neurons with centrally placed slices (thickened planes). In [56] volume tensors are estimated from vertical sections to study brain tissue. For other applications we refer to the introductions in [27, 79], where multiple references are given.

As the aim of the present paper is the construction of stereologically adapted Crofton formulae, we will now briefly note what has already been established in the literature and how our work differs. In [10, 41, 44, 77] Minkowski tensors of flat sections are integrated with respect to invariant measures. For instance [10, Thm. 4.12] states:

$$\int_{A(n,q)} \Phi_{q-j,s}(K \cap E) \, \mu_q(dE) = \sum_{z=0}^{\lfloor \frac{s}{2} \rfloor} \eta_{n,s,j,q,z} Q^z \Phi_{n-j,s-2z}(K), \quad K \in \mathcal{K}^n, \tag{B.2}$$

with $0 < j < q < n$. Here, Q is the metric tensor of rank 2 and Q^j is the j-fold symmetric tensor product of Q with itself. That the left-hand side can be written as a linear combination of tensors of the form $Q^z \Phi_{n-j}^{s-2z}$ for suitable integers is not surprising. The left-hand side of (B.2) is a tensor-valued translation invariant valuation in $K \in \mathcal{K}^n$, that satisfies the assumptions in [2, Lemma 3.1] and thus must be a linear combination of the tensors appearing on the right-hand side of (B.2).

The constants in (B.2) are known and given by

$$\eta_{n,s,j,q,z} = \begin{bmatrix} n \\ n-q \end{bmatrix}^{-1} \binom{n-j}{q-j} \frac{(q-j)(n-q)}{2(n-j)} \frac{1}{\Gamma(\frac{n-j+s}{2})} \frac{\Gamma(\frac{n-q}{2} + z)\Gamma(\frac{q-j+s}{2})}{(4\pi)^z z!},$$

with $\begin{bmatrix} n \\ n-q \end{bmatrix}$ defined through (B.7). We mention that more general Crofton formulae for not necessarily translation invariant Minkowski tensors are available. The early result [41, Thm. 2.3]

expresses the vary complicated constants in the analogue of (B.2) by a five-fold summation. The constants have been subsequently considerably simplified in [44, 45].

From a stereological point of view these results are not in a suitable form to be applied in practice. Here, one wants to construct a *measurement tensor* $\varphi_{q,j,s}(\cdot)$ which solves

$$\int_{A(n,q)} \varphi_{q,j,s}(E, K \cap E) \, \mu_q(dE) = \Phi_{n-j,s}(K), \quad K \in \mathcal{K}^n, \tag{B.3}$$

for $0 < j \leq q < n$ and $s \in \mathbb{N}$. Crofton formulae of the form (B.3) will be called *stereologically adapted*, as they explain how a given Minkowski tensor in $\mathbb{R}^n$ can be written as a Crofton integral with lower-dimensional flat sections. Once such formulae are established, they yield unbiased estimators in design-based stereology when the object is intersected by invariantly rotated planes, cf. the standard procedure for intrinsic volumes, for instance [50, eq. (2.6)]. When considering stereologically adapted Crofton formulae calculating explicit and simple expressions for the involved constants will be based on the use of Zeilberger's algorithm.

The main results of this paper can be summarized by the following theorem, which serves as an overview. In this we exclude the cases $s \in \{0, 1\}$ as when $s = 0$ then the statements reduce to the classical Crofton formula, and when $s = 1$ then a solution to (B.3) is trivial as $\Phi_{n-j,1}$ is the zero tensor. Thus, the results in this paper concern $s \geq 2$, where new results are proven.

THEOREM B.1. *Consider the functional equation* (B.3) *for* $n, s, j, q \in \mathbb{N}$ *with* $s \geq 2$.

(I) *If* $1 \leq j < q < n$ *then* (B.3) *holds with* $\varphi_{j,q,s}$ *given in Theorem* B.2.

(II) *If* $1 \leq j = q < n$ *and* s *is even then* (B.3) *holds with* $\varphi_{j,q,s}$ *given in Theorem* B.3.

(III) *If* $1 \leq j = q < n$ *and* s *is odd no solution to* (B.3) *can exist under mild conditions on the measurement tensor* $\varphi_{j,q,s}$, *see Theorem* B.4.

(IV) *For* $q < j$ *no solution to* (B.3) *can exist.*

The proof of (IV) is omitted in the present paper as arguments as in [22, Prop. 6] show that no solution valid for all $K \in \mathcal{K}^n$ can exist. The main idea is an identifiability argument. One can construct two different convex bodies K, K' such that $K \cap E = K' \cap E$ for almost all $E \in A(n, q)$ but $\Phi_{n-j,s}(K) \neq \Phi_{n-j,s}(K)$. This argument is also used in [53, Thm. 3.2].

In principle, Theorem B.1 (I) can be derived considering equation (B.2) for $s = 0, \ldots, s_0$ and solving this linear system of equations for $\Phi_{n-j,s_0}(K)$ when q and j are fixed. This approach shows that the measurement function for $\Phi_{n-j,s_0}(K)$ can be written as a weighted sum of Minkowski tensors of $K \cap E$. Theorem B.2 not only shows that the above mentioned linear system is invertible, but even yields explicit and relatively simple expressions for the corresponding weights. Concerning Theorem B.1 (II), the corresponding linear system can be based on [44, Cor. 5.6]. In Theorem B.3 it is shown that this system is only invertible for tensors of even rank (see (B.25)). Inverting these kinds of linear systems of equations has been applied in similar situations, see [42, eq. (7)–(10)] or [55, Thm. 3.4]. We also apply it in the present paper when proving Theorem B.3.

The Crofton formula in Theorem B.1 can, as mentioned earlier, be used to obtain stereological estimators of Minkowski tensors. In certain practical applications, however, additional constraints

must be imposed on the intersecting flats. We describe two such schemes in the second part of the paper.

In order to calculate the Crofton integral in (B.3) one has to be able to determine a tensor valuation of $K \cap E$ for μ_k-almost all $E \in A(n,k)$. In some applications, however, only intersections $K \cap E$ are available, where E is constrained to contain a fixed subspace $L_0 \in G(n,r)$, $0 \leq r < k$. Hence E must be an element of the space $G(L_0, k) = \{L \in G(n,k) \; : \; L_0 \subseteq L\}$. The invariant integration on this space is now understood with respect to the (unique) invariant probability measure $\nu_k^{L_0}$ on $G(L_0, k)$, see Section. B.4 for details. We will construct a translation invariant tensor valuation $\varphi_{L,q}^{L_0}$ satisfying

$$\int_{G(L_0,k)} \varphi_{L,q}^{L_0}(K \cap L) \, \nu_k^{L_0}(dL) = \Phi_{n-j,s}(K), \quad K \in \mathcal{K}. \tag{B.4}$$

This *rotational Crofton formula with a fixed axis for tensors* generalizes the main results in [22], which treated the case of intrinsic volumes ($s = 0$). When $r = 0$ we have $G(L_0, k) = G(n,k)$, so (B.4) becomes an (unconstrained) *rotational Crofton formula*, cf. [4] for the case $s = 0$.

The other scheme, not as restrictive as the first one, deals with the well-established concept of *vertical sections*. Here, the intersecting flat $E \in A(n,k)$ is only required to be *parallel* to a fixed subspace $L_0 \in G(n,r)$, $1 \leq r < k$

For real-valued measurement functions, this scheme is not new, see [5, 6, 7] for some, to the author's knowledge, of the earliest applications of this idea. We will briefly sketch the idea and refer to Section B.4, where this concept will formally be introduced. If L_0 is a fixed subspace of dimension r with $r < k$ we let $A_{Vert}(L_0, k)$ denote the set of flats of dimension k such that each $E \in A_{Vert}(L_0, k)$ contains L_0 when translated into the origin. Letting $\mu_{Vert,k}^{L_0}$ be an invariant measure on $A_{Vert}(L_0, k)$ we are asking for measurement tensors $\varphi_{E,q}^{L_0}$ such that

$$\int_{A_{Vert}(L_0,k)} \varphi_{E,q}^{L_0}(K \cap E) \, \mu_{Vert,k}^{L_0}(dE) = \Phi_{n-j,s}(K), \quad K \in \mathcal{K}^n, \tag{B.5}$$

for suitable indices. Solutions exist and are presented in Theorem B.7. To the author's knowledge, no such formulae are stated in the literature even for tensors of rank two. As Theorem B.7 allows for a solution for $n = 3$, it can be used in applications to estimate translation invariant Minkowski tensors of even rank based on sampling in the setting of vertical sections. For further information on this sampling procedure we again refer to [5, 6, 7] for some of the earliest implementations of this setting or [56] for a practical implementation of this technique for estimating volume tensors.

The structure of the paper is as follows. Section B.2 contains the three main results, Theorem B.2, Theorem B.3 and Theorem B.4 preceded by the required preliminaries. Section B.3 contains the proofs of Theorem B.2 and Theorem B.3, yielding claims (I) and (II) in Theorem B.1. Section B.4 deals with the proof of Theorem B.4. To prove this result, we introduce further notation from integral geometry. These notions are used throughout the second half of this paper. Section B.5 contains applications of the main results, resulting in Theorem B.6 and Theorem B.7, which provide measurement tensors solving (B.4) and (B.5), respectively. This section also contains a few examples. As all the main proofs rely on Zeilberger's algorithm, an appendix is provided where these calculations are collected.

B.2 Preliminaries and main results

We define, as in [74, eq. (5.108)], [55] or [10, p. 82], the translation invariant Minkowski tensors $\Phi_{k,s}$ as

$$\Phi_{k,s}(K) = \binom{n-1}{k} \frac{1}{\omega_{n-k+s}s!} \int_{S^{n-1}} u^s \, S_k(K, du), \quad k = 0, \ldots, n-1, \tag{B.6}$$

$$\Phi_{n,0}(K) = \lambda(K),$$

valid for $K \in \mathcal{K}^n$. Here $S_k(K, \cdot)$, denotes the *area measure* of K of order k for $k = 0, \ldots, n-1$, defined in [74, eq. (4.10)], $\omega_n = n\kappa_n$ is the surface area of the Euclidean unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ and $\kappa_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}$ is the volume of the Euclidean unit ball B^n in $\mathbb{R}^n$. We understand the integral of a tensor point wise.

We let λ denote the Lebesgue measure on $\mathbb{R}^n$ and write λ_E for the Lebesgue measure on a flat $E \subset \mathbb{R}^n$. It should be noted that the more general notation $\Phi_k^{r,s}$, which coincides with $\Phi_{k,s}$, when $r = 0$, is used in [44, 55, 74]. In the present paper, we will only work with translation invariant Minkowski tensors ($r = 0$) and therefore use this simplified notation. The area measures generalize the intrinsic volumes as the total mass of $S_k(K, \cdot)$ is proportional to $V_k(K)$. If K is a full-dimensional convex body then $S_{n-1}(K, A)$ is the $(n-1)$-dimensional Hausdorff measure of the set of points in the boundary of K with outer normals the Borel set $A \subset S^{n-1}$. For further information on area measures and their interpretation, we refer to [74, Thm. 4.2.1] and the subsequent discussion found therein. The intersection $K' = K \cap E$ of a convex body $K \in \mathcal{K}^n$ with a flat $E \in A(n, q)$ can be seen as a lower dimensional convex body in $\mathbb{R}^n$ or as a convex body in E. Hence, there are (at least) two possible ways to determine Minkowski tensors: the one just defined, and the other one, where this definition is applied with E as ambient space, still though as a tensor on $\mathbb{R}^n$. This distinction is used in Theorem B.3. Whenever we view E as the ambient space we will talk of the *intrinsic translation invariant Minkowski tensors* written as $\Phi_{k,s}^{(E)}(K')$. For an explicit definition we refer to the introduction in [55]. In contrast to intrinsic volumes, we do not in general have $\Phi_{k,s}^{(E)}(K') = \Phi_{k,s}(K')$ when $q < n$.

We will now state our first main result which requires the *flag coefficients*

$$\begin{bmatrix} n \\ l \end{bmatrix} = \binom{n}{l} \frac{\kappa_n}{\kappa_l \kappa_{n-l}}, \quad l \leq n, \tag{B.7}$$

defined as in [11]. In this, and results to come, we exclude $s = 1$ as $\Phi_{k,1}(K) = 0$ for all $K \in \mathcal{K}^n$.

Theorem B.2. *Let $n, s, q, j \in \mathbb{N}$ such that $1 \leq j < q < n$. For $s \geq 2$ we have*

$$\int_{A(n,q)} \varphi_{q,j,s}(E, K \cap E) \, \mu_q(dE) = \Phi_{n-j,s}(K), \quad K \in \mathcal{K}^n,$$

with

$$\varphi_{q,j,s}(E, \cdot) = \gamma_{n,s,j,q} \sum_{z=0}^{\lfloor s/2 \rfloor} d_{n,s,j,q,z} Q^z \Phi_{q-j,s-2z}.$$

Here

$$\gamma_{n,s,j,q} = \binom{n-j}{q-j}^{-1} \frac{2(n-j)}{(q-j)(n-q)} \begin{bmatrix} n \\ n-q \end{bmatrix} \frac{1}{\Gamma(\frac{s+q-j}{2})\Gamma(\frac{n-q}{2})},$$

$$d_{n,s,j,q,z} = \frac{(-1)^z}{(4\pi)^z z!} \Gamma\left(\frac{s+n-j}{2} - z\right) \prod_{i=0}^{z-1} \left(\frac{n-q}{2} - i\right).$$

Here, q controls the dimension of the flat $E \in A(n,q)$ that is used for sampling and $n-j$ denotes the homogeneity of the estimated tensor. For fixed q, the bound $j < q$ can, from a sampling perspective, be interpreted as sampling by flats of higher dimension offers more information from $E \cap K$ for $E \in A(n,q)$ about K allowing estimation of $\Phi_{n-j,s}(K)$ for more values of j. If $n-q$ is even, then $d_{n,s,j,q,z}$ vanishes for $z > \frac{n-q}{2}$, a fact which will be further explored in Section B.5.

Since Theorem B.2 requires $j < q$, it does not yield a Crofton formula for $\Phi_{n-q,s}(K)$. It is clear that $\varphi_{q,j,s}(E, \cdot)$ given in Theorem B.2 with $q = j$ can not be a measurement function for $\Phi_{n-q,s}(K)$ for $s \geq 2$, since

$$\int_{A(n,q)} \Phi_{0,s}(K \cap E)\, \mu_q(dE) = \begin{cases} c \cdot Q^{\frac{s}{2}}, & s \text{ even}, \\ 0, & s \text{ odd}, \end{cases}$$

where $c \neq 0$ is a constant given in [45, Thm. 4]. When $q = 1$ this calculation can also be found in [44, Thm. 5.12].

The border case $q = j$ is interesting as it, for instance, allows the estimation of $\Phi_{1,s}(K)$ by integration over $A(3,2)$. The reason this case is excluded reduces to an invertibility issue, as a coefficient in an integral relation equals 0 which results in division by 0 thus breaking a crucial part in the proof (see Lemma B.9). This raises the question if a formula can be constructed by other means. The answer involves the intrinsic translation invariant Minkowski tensors introduced earlier. The following complementary result deals with tensors of even rank.

THEOREM B.3. *Let $q \in \mathbb{N}$ such that $1 \leq q < n$. For $s \in \mathbb{N}$ we have*

$$\int_{A(n,q)} \varphi_{q,q,2s}(E, K \cap E)\, \mu_q(dE) = \Phi_{n-q,2s}(K), \quad K \in \mathcal{K}^n,$$

with

$$\varphi_{q,q,2s}(E, \cdot) = \gamma_{n,2s,q,q} \sum_{z=0}^{s} d_{n,2s,q,q,z} Q^{s-z} \Phi_{0,2z}^{(E)}.$$

Here

$$\gamma_{n,2s,q,q} = \frac{\sqrt{\pi}(2s-1)s!\Gamma(\frac{q}{2}+s)}{\Gamma(\frac{n+1-q}{2})\Gamma(\frac{q+1}{2})(2s)!},$$

$$d_{n,2s,q,q,z} = (-1)^{z+1} \frac{\pi^{z-s} 4^z \Gamma(\frac{n+1}{2}+z)}{(s-z)!\Gamma(\frac{q}{2}+z)}.$$

Combining Theorem B.2 and Theorem B.3 we have constructed measurement tensors for the Crofton formula (B.3) for $1 \leq j \leq q < n$ for $s \in \mathbb{N}\setminus\{1\}$, with the exception of the case where $q = j$ and s is odd. Remaining is therefore the question if there exists a measurement tensor $\varphi(E, K \cap E)$ such that

$$\int_{A(n,q)} \varphi(E, K \cap E)\, \mu_q(dE) = \Phi_{n-q,s}(K), \quad K \in \mathcal{K}^n, \tag{B.8}$$

is valid for odd s and $1 \leq q < n$. However, under sensible assumptions on the measurement tensor $\varphi(E, K \cap E)$ equation (B.8) cannot hold. Both measurement tensors in Theorem B.2 and Theorem B.3 depend on K only through its intersection with the flat E. Therefore, we only search for solutions to (B.8) among those measurement tensors that only depend on E and $E \cap K$ for $E \in A(n, q)$ valid for all $K \in \mathcal{K}^n$ satisfying some natural assumptions.

As we are searching for measurement functions with values in the space of tensors of a fixed rank we let $\mathbb{T}^s$ denote the space of symmetric tensors of rank s over $\mathbb{R}^n$. We endow this space with a norm $\| \cdot \|_{\mathbb{T}^s}$ which for instance can be chosen as in [42]. For a measurement function $\varphi : \mathcal{M} \to \mathbb{T}^s$, with $\mathcal{M} = \{(E, K) \in A(n, q) \times \mathcal{K}^n \mid K \subseteq E\}$ we introduce the following terminology.

- We say that φ is *integrable* if $E \mapsto \varphi(E, K \cap E)$ is measurable and

$$\int_{A(n,q)} \|\varphi(E, K \cap E)\|_{\mathbb{T}^s} \, \mu_q(dE) < \infty$$

  for all $K \in \mathcal{K}^n \cup \{\emptyset\}$.

- We say that φ is *simultaneously translation invariant* if

$$\varphi(E + x, (K \cap E) + x) = \varphi(E, K \cap E)$$

  for all $E \in A(n, q), K \in \mathcal{K}^n$ and $x \in \mathbb{R}^n$.

- We say that φ is *bounded* if

$$\sup\{\|\varphi(E, K \cap E)\|_{\mathbb{T}^s} \mid (E, K) \in A(n, q) \times \mathcal{K}^n\} < \infty.$$

- We say that φ is *uniformly regular* if for any convex body $K \subseteq B^n$ there exists a constant C_K such that

$$\int_{A(n,q)^*} \|\varphi(E + x, (K \cap E) + x)\|_{\mathbb{T}^s} \, \mu_q(dE) \leq C_K \tag{B.9}$$

  for all $x \in \mathbb{R}^n$ with $A(n, q)^* = \{E \in A(n, q) \mid E \cap B^n \neq \emptyset\}$ being the set of flats intersecting the unit ball.

If φ is simultaneously translation invariant and either integrable or bounded, then φ is uniformly regular. Using this, it follows that $\varphi_{j,q,s}$ from Theorem B.2 and Theorem B.3 is uniformly regular. The notion of being uniformly regular is, although without the naming, a direct generalization (put $q = 1$) of a uniform integrability condition presented in relation to [55, Thm. 3.6].

We now formulate a theorem covering the last case of interest when considering solutions to (B.3):

THEOREM B.4. *Let $n \geq 2$ and $s \neq 1$ be odd. For $1 \leq q < n$ there is no integrable and uniformly regular map $\varphi : \mathcal{M} \to \mathbb{T}^s$ such that*

$$\int_{A(n,q)} \varphi(E, K \cap E) \, \mu_q(dE) = \Phi_{n-q,s}(K) \tag{B.10}$$

hold true for all $K \in \mathcal{K}^n$.

This theorem is a direct generalization of [55, Thm. 3.6]. With the final main result stated, we now turn our attention to the proofs.

B.3 Proofs of Theorem B.2 and Theorem B.3

The structure of the proofs is as follows: To prove Theorem B.2 we rely on Zeilberger's algorithm in combination with a change of basis utilizing the Ψ-basis (see [11] or [44, Sec. 5.3.3]) to construct a solution to (B.3). This approach differs from the initial idea presented in the introduction as we avoid to invert a system of equations due to the involvement of the Ψ-basis. In the special case appearing in Theorem B.3 we cannot use this approach. Instead, we use the aforementioned approach of inversion by linear algebra. This approach is also used in [55, Thm. 3.4], where Theorem B.3 is derived in the special case $q = 1$, but with less explicit constants. Theorem B.4 in the special case $q = 1$ is covered by [55, Thm. 3.6]. All the calculations regarding Zeilberger's algorithm used in the proofs in this section are collected in the appendix. It should be noted that (B.6) and the main results of this paper are based on the normalization of Minkowski tensors used in [74, Sec. 5.4.2]. A different normalization of the Minkowski tensors was suggested in [44, Sec. 5.2] to simplify constants in integral formulae. Due to this, when proving Theorem B.1, we may use a different normalization of the Minkowski tensors to simplify the calculations and then transfer the results back into the normalization used in [74]. In general, it should be noted that we exclude the case $s = 0$ in all the proofs and theorems as then the statements reduce to special cases of the classical Crofton formula given in (B.1). This does not imply that no tensors of rank 0 is encountered in the proofs.

We will now formulate the foundation on which the proof of Theorem B.2 is based. It turns out that it is convenient to define a new set of so-called *trace-free* tensors $\Psi_{k,s}$. They are defined by the relation

$$\Psi_{k,s} = \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} c_{s,k,j} Q^j \Phi_{k,s-2j}, \tag{B.11}$$

for $0 \le k < n$ and $s \in \mathbb{N}_0$, where

$$c_{s,k,j} = \frac{(-1)^j \Gamma(\frac{n-k+s}{2}) \Gamma(\frac{n}{2} + s - 1 - j)}{(4\pi)^j j! \Gamma(\frac{n-k+s}{2} - j) \Gamma(\frac{n}{2} + s - 1)}. \tag{B.12}$$

According to [11, Prop. 4.16], we have

$$\Phi_{k,s} = \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} \beta_{s,k,j} Q^j \Psi_{k,s-2j}, \tag{B.13}$$

with

$$\beta_{s,k,j} = \frac{\Gamma(\frac{n-k+s}{2}) \Gamma(\frac{n}{2} + s - 2j)}{(4\pi)^j j! \Gamma(\frac{n-k+s}{2} - j) \Gamma(\frac{n}{2} + s - j)}. \tag{B.14}$$

In the special case $n = 2$ some of the constants $c_{s,k,j}$ and $\beta_{s,k,j}$ require a special interpretation, see the discussion in [11]. We will apply (B.11) and (B.13) in the proof of Theorem B.2, where $n \ge 3$ is assumed. The critical case $n = 2$ is therefore irrelevant in our application of (B.11) and (B.13). Note that (B.11) implies in particular $\Psi_{k,0} = \Phi_{k,0}$ and $\Psi_{k,1} = \Phi_{k,1} = 0$ for $0 \le k < n$.

The trace-free tensors are eigenfunctions of the Crofton operator (terminology adopted from [26, p. 2]). Indeed, if $k, l \in \mathbb{N}_0$ such that $k + l \leq n$ then

$$\int_{A(n,n-l)} \Psi_{k,s}(K \cap E) \, \mu_{n-l}(dE) = \alpha_{s,k,l} \Psi_{k+l,s}(K), \qquad (B.15)$$

where

$$\alpha_{s,k,l} = \frac{\omega_{s+k+l}}{\omega_{s+k} \omega_l} \binom{k+l}{k} \frac{kl}{k+l} \begin{bmatrix} n \\ l \end{bmatrix}^{-1}, \qquad (B.16)$$

see [11, Cor. 6.1]. Again, care should be taken when interpreting $\alpha_{s,k,l}$ in special cases. However, when $k > 0$ and $l > 0$ the constant $\alpha_{s,k,l}$ is well-defined by (B.16). We will apply (B.15) only in the proof of Theorem B.2, where k and l are both positive. Note that (B.15) is trivially true if $s = 1$, as both sides of (B.15) then coincide with the zero tensor (with $\alpha_{1,k,l}$ a nonzero constant).

We will now prove Theorem B.2.

Proof of Theorem B.2. To prove Theorem B.2 we will show the equivalent claim:

$$\int_{A(n,n-l)} \sum_{z=0}^{\lfloor s/2 \rfloor} f(z) Q^z \Phi_{k,s-2z}(K \cap E) \, \mu_{n-l}(dE) = \Phi_{k+l,s}(K), \quad K \in \mathcal{K}^n, \qquad (B.17)$$

with

$$f(z) = \left(\binom{k+l}{k}^{-1} \frac{2(k+l)}{kl} \begin{bmatrix} n \\ l \end{bmatrix} \right) \left(\frac{(-1)^z}{(4\pi)^z z!} \frac{\Gamma(\frac{s+k+l}{2} - z)}{\Gamma(\frac{s+k}{2})\Gamma(\frac{l}{2})} \prod_{i=0}^{z-1} \left(\frac{l}{2} - i \right) \right)$$

for all $k, l \in \mathbb{N}$ such that $k + l < n$, where we, for convenience, write $f(z)$ for a sequence of constants indexed by $z = 0, \ldots, \lfloor \frac{s}{2} \rfloor$. That this claim is equivalent to Theorem B.2 can be seen using the substitutions $q = n - l$ and $k + l = n - j$.

To show the claim, fix $K \in \mathcal{K}^n$ and integers $l, k > 0$ such that $l + k < n$ and $s \geq 2$. For abbreviation, we let $\tilde{s} = \lfloor \frac{s}{2} \rfloor$. From (B.13) we get

$$\Phi_{k+l,s}(K) = \sum_{j=0}^{\tilde{s}} \beta_{s,k+l,j} Q^j \Psi_{k+l,s-2j}(K).$$

Applying (B.15) to $Q^j \Psi_{k+l,s-2j}$ for $j \in \{0, \ldots, \tilde{s}\}$ yields

$$\Phi_{k+l,s}(K) = \int_{A(n,n-l)} \sum_{j=0}^{\tilde{s}} \beta_{s,k+l,j} (\alpha_{s-2j,k,l})^{-1} Q^j \Psi_{k,s-2j}(K \cap E) \, \mu_{n-l}(dE)$$

where $\alpha_{s-2j,k,l} \neq 0$ as $k, l > 0$. Finally use (B.11) to translate back to a linear combination of standard Minkowski tensors yielding

$$\Phi_{k+l,s}(K) = \int_{A(n,n-l)} \sum_{j=0}^{\tilde{s}} \beta_{s,k+l,j} Q^j (\alpha_{s-2j,k,l})^{-1}$$

$$\times \sum_{i=0}^{\tilde{s}-j} c_{s-2j,k,i} Q^i \Phi_{k,s-2j-2i}(K \cap E) \, \mu_{n-l}(dE)$$

$$= \int_{A(n,n-l)} \sum_{j=0}^{\tilde{s}} \sum_{i=0}^{\tilde{s}-j} \beta_{s,k+l,j}(\alpha_{s-2j,k,l})^{-1}$$
$$\times c_{s-2j,k,i} Q^{i+j} \Phi_{k,s-2(i+j)}(K \cap E) \, \mu_{n-l}(dE).$$

As, for any double indexed sequence $\{g_{i,j}\}$ with values in $\mathbb{R}$ it holds

$$\sum_{j=0}^{\tilde{s}} \sum_{i=0}^{\tilde{s}-j} g_{j,i} = \sum_{z=0}^{\tilde{s}} \sum_{j=0}^{z} g_{j,z-j},$$

we can change the order of summations yielding

$$\int_{A(n,n-l)} \sum_{z=0}^{\tilde{s}} \tilde{f}(z) Q^z \Phi_{k,s-2z}(K \cap E) \, \mu_{n-l}(dE) = \Phi_{k+l,s}(K),$$

where

$$\tilde{f}(z) = \sum_{j=0}^{z} \beta_{s,k+l,j}(\alpha_{s-2j,k,l})^{-1} c_{s-2j,k,z-j}. \tag{B.18}$$

A comparison with (B.17) shows that it remains to prove $\tilde{f}(z) = f(z)$ for $z = 0, \ldots, \lfloor \frac{s}{2} \rfloor$. To finalize the proof, we just need a suitable reduction of the constant (B.18). Letting $\tilde{f}(z,j)$ denote the jth summand in the representation (B.18) of $\tilde{f}$, $j = 0, \ldots, z$, this reduction is achieved using Zeilberger's algorithm. This algorithm constructs constants $G(z,j)$ for $j \in \{0, \ldots, z+1\}$ and two constants γ_1, γ_2, which are independent of j, but may depend on the other parameters, such that

$$\gamma_1 \tilde{f}(z,j) + \gamma_2 \tilde{f}(z+1,j) = G(z,j+1) - G(z,j) \tag{B.19}$$

with $\tilde{f}(z, z+1) = 0$. A summation over $j = 0, \ldots, z+1$ yields

$$\gamma_1 \tilde{f}(z) + \gamma_2 \tilde{f}(z+1) = G(z, z+2) - G(z, 0).$$

As seen in the proof of Lemma B.9, in the appendix the right-hand side equals 0. Hence, we get

$$\tilde{f}(z+1) = \tilde{f}(z) \frac{-\gamma_1}{\gamma_2}, \quad z = 0, \ldots, \left\lfloor \frac{s}{2} \right\rfloor - 1.$$

Now $\tilde{f}(z)$ can then be determined using this recursion. These calculations, involving the use of Zeilberger's algorithm, can be found in the appendix, see Lemma B.9. With the suitable simplification done, the proof of Theorem B.2 is completed.   ∎

With the proof of Theorem B.2 completed, we focus on the proof of Theorem B.3. This proof consists of two steps. First, we simplify a Crofton formula already established in the literature. Then we invert a system of linear equations. This step constitutes the main part of the proof of Theorem B.3. This approach is also used in [55]. Our starting point is [44, Cor. 5.6], which, however, uses a different normalization of the (intrinsic) translation invariant Minkowski tensors. In the present proof this normalization is adopted to allow easy comparison with the original results.

Therefore, define

$$\tilde{\Phi}_{k,s}^{(E)} = \omega_{q-k+s}s!\Phi_{k,s}^{(E)}, \tag{B.20}$$

$$\tilde{\Phi}_{k,s} = \omega_{n-k+s}s!\Phi_{k,s}, \tag{B.21}$$

for $E \in A(n,q)$. It should be noted that this normalization only appears in the proofs of Theorem B.3 and in an intermediate result. We will now state a lemma. This lemma is a special case of [44, Cor. 5.6] but the present version simplifies the constants. In this result the parameter d controls the parity of the rank of the tensors involved.

LEMMA B.5. *Let $q \in \mathbb{N}$ such that $q < n$ and $d \in \{0,1\}$. For $s \in \mathbb{N}$ we have*

$$\delta_{n,q,2s+d}\int_{A(n,q)} \tilde{\Phi}_{0,2s+d}^{(E)}(K \cap E)\, \mu_q(dE) = \begin{cases} \sum_{z=0}^{s} c_{n,s,q,z}Q^{s-z}\tilde{\Phi}_{n-q,2z}(K), & d = 0, \\ 0, & d = 1, \end{cases}$$

with

$$c_{n,s,q,z} = (-1)^{z+1}\frac{\sqrt{\pi}\Gamma(2s+1)\Gamma(\frac{n-q}{2}+s+1)}{4^s(2z-1)\Gamma(z+1)\Gamma(\frac{n+1}{2}+s)\Gamma(s-z+1)}, \quad z \le s,$$

and

$$\delta_{n,q,2s+d} = \frac{\pi\Gamma(\frac{n-q+2s+d}{2}+1)}{\Gamma(\frac{n+1-q}{2})\Gamma(\frac{q+1}{2})}. \tag{B.22}$$

Proof. The proof of this lemma consists essentially in a simplification of a constant appearing in an already known Crofton formula. For quick reference we will now state the formula which is a special case of [44, Cor. 5.6]:

$$\delta_{n,q,2s+d}\int_{A(n,q)} \tilde{\Phi}_{0,2s+d}^{(E)}(K \cap E)\, \mu_k(dE) = \sum_{z=0}^{s} \eta_{2s+d,z}Q^z\tilde{\Phi}_{n-q,2s+d-2z}(K), \tag{B.23}$$

for $K \in \mathcal{K}^n$ with $q, s \in \mathbb{N}_0$, $0 < q < n$ and $d \in \{0,1\}$. Here, δ_{2s+d} is given by (B.22) and

$$\eta_{2s+d,z} = \sum_{k=z}^{s}(-1)^{k-z}\binom{2s+d}{2k}\binom{k}{z}\Gamma(k+\frac{1}{2})\frac{\Gamma(s-k+1+\frac{d}{2})\Gamma(\frac{n-q}{2}+k)}{\Gamma(\frac{n+1}{2}+k)}$$

$$\times \left(\frac{n-q}{2}+k+\frac{(q-1)(k-z)}{2s-2z-1+d}\right), \tag{B.24}$$

where we define $\eta_{2s+d,s} = 0$ if $d = 1$. A reduction of $\eta_{2s+d,z}$ is possible resulting in

$$\eta_{2s+d,z} = \begin{cases} (-1)^{s+z+1}\frac{\sqrt{\pi}\Gamma(2s+1)\Gamma(\frac{n-q}{2}+s+1)}{4^s(2(s-z)-1)\Gamma(z+1)\Gamma(s-z+1)\Gamma(\frac{n+1}{2}+s)}, & d = 0, \\ 0, & d = 1. \end{cases} \tag{B.25}$$

Arguments for this proceed along the previous use of Zeilberger's algorithm and can be found in Lemma B.11. In particular, (B.49) contains the corresponding Zeilberger relation. From

(B.25) we immediately deduce that the right-hand side of (B.23) equals 0 whenever $d = 1$ corresponding to tensors of odd rank. Consequently, the linear system of equations mentioned in the introduction is not invertible for odd s. This is the reason why Theorem B.3 only considers even s. Using

$$\sum_{z=0}^{s} \eta_{2s,z} Q^z \Phi_{n-q,2s-2z}(K) = \sum_{z=0}^{s} \eta_{2s,s-z} Q^{s-z} \Phi_{n-q,2z}(K),$$

in combination with (B.23) and (B.25) concludes the proof of the lemma. ∎

We will now use the Crofton formula in Lemma B.5 to prove the stereologically adapted version in Theorem B.3.

Proof of Theorem B.3. The proof is essentially a generalization of [55, Thm. 3.4]. The main idea to represent $\tilde{\Phi}_{n-q,2s}$ as a Crofton integral, is to apply Lemma B.5 for all even ranks between 0 and $2s$ and then solve the corresponding linear system. In matrix notation we get

$$\begin{pmatrix} \delta_0 \int_{A(n,q)} \tilde{\Phi}_{0,0}^{(E)}(K \cap E) \, \mu_q(dE) \\ \delta_2 \int_{A(n,q)} \tilde{\Phi}_{0,2}^{(E)}(K \cap E) \, \mu_q(dE) \\ \vdots \\ \delta_{2s} \int_{A(n,q)} \tilde{\Phi}_{0,2s}^{(E)}(K \cap E) \, \mu_q(dE) \end{pmatrix} = C \begin{pmatrix} \tilde{\Phi}_{n-q,0}(K) \\ \tilde{\Phi}_{n-q,2}(K) \\ \vdots \\ \tilde{\Phi}_{n-q,2s}(K) \end{pmatrix}, \tag{B.26}$$

where we define $\delta_{2s} = \delta_{n,q,2s}$ to lighten the notation. Here, C is a lower triangular matrix with entries $c_{i,j}Q^{i-j}$, where $c_{i,j} = c_{n,i,q,j}$ for $j \leq i$. Define the lower triangular matrix D with entries $d_{i,j}Q^{i-j}$, where

$$d_{i,j} = (-1)^{j+1} \frac{(2i-1)\Gamma(i+1)4^j \Gamma(\frac{n+1}{2}+j)}{\Gamma(i-j+1)\Gamma(2j+1)\Gamma(\frac{n-q}{2}+j+1)\sqrt{\pi}}, \quad j \leq i. \tag{B.27}$$

We claim that DC is the identity matrix. Indeed, as DC is again a lower triangular matrix, this follows from two observations. Firstly,

$$(DC)_{i,i} = d_{i,i}c_{i,i} = 1,$$

which can be seen by direct calculation. Secondly, for $j < i$ we have $(DC)_{i,j} = Q^{i-j} \sum_{k=j}^{i} d_{i,k}c_{k,j}$. Inserting the definitions of $d_{i,k}$ and $c_{k,j}$ and removing all factors that are independent of the summation index k, the last sum turns out to be a multiple of

$$\sum_{k=j}^{i} (-1)^k \frac{1}{\Gamma(i-k+1)\Gamma(k-j+1)} = \frac{(-1)^j}{(i-j)!} \sum_{k=j}^{i} (-1)^{k-j} \binom{i-j}{k-j} = 0,$$

by the binomial theorem. Concluding, we have shown that DC is the identity matrix. Applying D to both sides of (B.26) thus gives

$$\int_{A(n,q)} \sum_{z=0}^{s} h_{s,z} Q^{s-z} \tilde{\Phi}_{0,2z}^{(E)}(K \cap E) \, \mu_q(dE) = \tilde{\Phi}_{n-q,2s}(K), \quad K \in \mathcal{K}^n,$$

with

$$h_{s,z} = \frac{\sqrt{\pi}(2s-1)s!}{\Gamma(\frac{n+1-q}{2})\Gamma(\frac{q+1}{2})} \cdot (-1)^{z+1} \frac{4^z \Gamma(\frac{n+1}{2}+z)}{(s-z)!(2z)!}.$$

Using (B.20) and (B.21) to translate back to normalized Minkowski tensors completes the proof of Theorem B.3. ∎

B.4 PROOF OF THEOREM B.4

Having proven the two theorems where a solution to (B.3) exists, we now study when a solution does not exist. For the proof, we need some notation, which will be from [75]. We introduce the *Grassmannian*, $G(n,k)$ as the family of k-dimensional *linear* subspaces of $\mathbb{R}^n$. We let ν_k denote the unique rotation invariant probability measure on $G(n,k)$. For a fixed $L \in G(n,p)$, $p \in \{0, \ldots, n\}$ we furthermore expand our notation by considering the flag spaces

$$G(L,k) = \begin{cases} \{L' \in G(n,k) \ : \ L' \subseteq L\}, & \text{if } k \le p, \\ \{L' \in G(n,k) \ : \ L \subseteq L'\}, & \text{if } k > p, \end{cases}$$

and, similarly, for $E \in A(n,p)$

$$A(E,k) = \begin{cases} \{E' \in A(n,k) \ : \ E' \subseteq E\}, & \text{if } k \le p, \\ \{E' \in A(n,k) \ : \ E \subseteq E'\}, & \text{if } k > p. \end{cases}$$

We equip $G(L,k)$ with the unique invariant probability measure ν_k^L. Here, invariance is understood with respect to all rigid rotations keeping L fixed. Likewise, we equip $A(E,k)$ with the invariant measure μ_k^E, which is invariant under all rigid rotations and translations fixing E. These definitions are directly adopted from [75, p. 590] where further comments can be found. If L is a subspace we let $L^\perp$ denote the orthogonal space to L. For a flat E we let $\text{lin}(E)$ denote the unique subspace parallel to E of the same dimension. If we let $p(o|E)$ denote the orthogonal projection of o onto E, then $\text{lin}(E) = E - p(o|E)$. We let $d(o,E) = \|p(o|E)\|$ and span E denote the linear hull of E. If M, L are two subspaces we, in accordance with [75, Sec. 14.1], define $[M, L]$ as the *subspace determinant* of M and L. If $\dim(M) = \dim(L) = 1$, then $[M,L]$ equals $\sin(\theta)$ where $\theta \in [0, \frac{\pi}{2}]$ is the angle between M and L. The proof of Theorem B.4 and Section B.5 requires the use of Blaschke–Petkantschin formulae. The common feature of these formulae is described in [75, p. 273]. An example of such a formula is [75, Thm. 7.2.6] (here in the special case $s_0 = n - q + 1$ and $s_1 = q$ for $1 < q < n$):

$$\int_{G(n,q)} f(L) \, \nu_q(dL) = \frac{\omega_{n-q+1}\omega_q}{\omega_1 \omega_n} \int_{G(L_0,1)} \int_{G(M,q)} f(L) [L, L_0] \, \nu_q^M(dL) \, \nu_1^{L_0}(dM), \tag{B.28}$$

with $L_0 \in G(n, n-q+1)$ and $f : G(n,q) \to \mathbb{R}_+$ measurable. Here invariant integration is decoupled by first fixing a line $M \in G(L_0, 1)$ (the pivot), integrating over all subspaces $L \in G(M,q)$ and then integrating over all $M \in G(L_0, 1)$. In this, a Jacobian may appear as a correction factor to account for this double integration. We will now prove Theorem B.4.

Proof of Theorem B.4. We start by describing the main idea behind the proof of Theorem B.4, which generalizes [55, Thm. 3.6]. To establish a contradiction, we suppose that there is an integrable and uniformly regular measurement function satisfying (B.10) for all convex bodies K. Then (B.10) also holds for all convex bodies in a fixed subspace L_0 of dimension $n - q + 1$. Establishing and using a suitable Blaschke–Petkantschin formula, this induces a Crofton formula with lines in L_0 for $\Phi_{n-q,s}(K)$. By [55, Thm. 3.6] this implies $\Phi_{n-q,s}(K) = 0$ for all $K \subset L_0$. However, we will see that there are $(n - q + 1)$-dimensional convex bodies with $\Phi_{n-q,s}(K) \neq 0$, yielding the desired contradiction.

The claim for $q = 1$ follows directly from [55, Thm. 3.6] so we may assume $q > 1$. Let $f : A(n, q) \to \mathbb{R}_+$ be measurable and fix $L_0 \in G(n, n - q + 1)$. Applying (B.28) to the function $L \mapsto \int_{L^\perp} f(L + x)\, \lambda_{L^\perp}(dx)$, $L \in G(n, q)$, yields

$$\int_{A(n,q)} f(E)\, \mu_q(dE) = \int_{G(n,q)} \int_{L^\perp} f(L + x)\, \lambda_{L^\perp}(dx)\, \nu_q(dL)$$
$$= c \int_{G(L_0,1)} \int_{G(M,q)} [L, L_0] \int_{L^\perp} f(L + x)\, \lambda_{L^\perp}(dx) \nu_q^M(dL) \nu_1^{L_0}(dM), \qquad (B.29)$$

with $c = \frac{\omega_{n-q+1}\omega_q}{\omega_1 \omega_n}$. As $L_0 \cap M^\perp$ and L are a.e. in general position and $\dim(L_0 \cap M^\perp) + \dim(L) = n$, the definition of the subspace determinant yields

$$\int_{L^\perp} f(L + x)\, \lambda_{L^\perp}(dx) = [M^\perp \cap L_0, L] \int_{M^\perp \cap L_0} f(L + x)\, \lambda_{M^\perp \cap L_0}(dx),$$

see [75, p. 598]. Inserting this in (B.29) and using [75, eq. (13.13)-(13.14)], we conclude the Blaschke–Petkantschin result

$$\int_{A(n,q)} f(E)\, \mu_q(dE) = \int_{A(L_0,1)} \int_{A(M,q)} f(E) J(E, M, L_0)\, \mu_q^M(dE)\, \mu_1^{L_0}(dM), \qquad (B.30)$$

with

$$J(E, M, L_0) = c\, [\mathrm{lin}(E), L_0]\, [\mathrm{lin}(M)^\perp \cap L_0, \mathrm{lin}(E)].$$

Using standard measure theoretic arguments, (B.30) also holds for all μ_q-integrable functions f.

Now suppose for contradiction that $\varphi : \mathcal{M} \to \mathbb{T}^s$ is integrable and uniformly regular satisfying (B.10) for all $K \in \mathcal{K}^n$. Fix $L_0 \in G(n, n - q + 1)$. For any convex body $K \subset L_0$ the components of $\varphi(\cdot, K \cap \cdot)$ are μ_q-integrable. Hence, using (B.10) and (B.30) applied to the components of φ implies

$$\Phi_{n-q,s}(K) = \int_{A(L_0,1)} \tilde{\varphi}(M, K \cap M)\, \mu_1^{L_0}(dM), \quad K \in \mathcal{K}^n, K \subseteq L_0$$

with

$$\tilde{\varphi}(M, K \cap M) = \int_{A(M,q)} \varphi(E, K \cap M) J(E, M, L_0)\, \mu_q(dE).$$

Here, we used $K \cap E = K \cap M$ for almost all $E \in G(M, q)$ whenever $M \in G(L_0, 1)$ since $K \subset L_0$. Inserting $1_{E \cap B^n \neq \emptyset}(E) \| \varphi(E, E \cap K) \|_{\mathbb{T}^s}$ in (B.30), we conclude that $\tilde{\varphi}(M, K \cap M)$, when viewing L_0 as the ambient space, satisfies the conditions in [55, Thm. 3.6]. The first part of the proof in [55, Thm. 3.6] thus implies that $\Phi_{n-q,s}$ is invariant under reflections of K at the origin: $\Phi_{n-q,s}(K) = \Phi_{n-q,s}(-K)$ for all $K \in \mathcal{K}^n$ with $K \subset L_0$. Due to the odd parity of s we conclude

$\Phi_{n-q,s}(K) = 0$ for all $K \in \mathcal{K}^n$ with $K \subset L_0$. This is a contradiction. For instance, $\Phi_{n-q,s}(K') \neq 0$ if K' is a right prism erected over a equilateral triangle as follows. As $\dim(L_0) = n - q + 1 \geq 2$, there exists a two-dimensional subspace $L' \subseteq L_0$. Let T be a nondegenerate equilateral triangle in L' with sides s_0, s_1, s_2 and corresponding outer normals $u_0, u_1, u_2 \in S^{n-1} \cap L'$. Let Q be the $(n - q - 1)$-dimensional unit cube in $L_0 \cap (L')^\perp$ and define the prism $K' = T + Q$. Using [74, eq. (4.3)] and [74, eq. (4.11)] we get

$$\binom{n-1}{n-q} \Phi_{n-q,s}(K') = \sum_{F \in \mathcal{F}_{n-q}(K')} \mathcal{H}^{n-q}(F) \int_{N(K',F) \cap S^{n-1}} u^s \, \mathcal{H}^{q-1}(du).$$

Here $\mathcal{F}_{n-q}(K')$ is the family of faces of K' of dimension $n - q$ and $N(K', F)$ the normal cone of the face $F \in \mathcal{F}_{n-q}(K')$. The measures $\mathcal{H}^{n-q}$ and $\mathcal{H}^{q-1}$ are the Hausdorff measures of dimension $n - q$ and $q - 1$, respectively. Any $F \in \mathcal{F}_{n-q}(K')$ is either of the form $F_i = s_i + Q$, $i = 0, 1, 2$ or $F = T + G$ with some $G \in \mathcal{F}_{n-q-2}(Q)$. Since $N(K', T + Q) \subset (L')^\perp$, evaluating $\Phi_{n-q,s}(K')$ at the s-tuple $\check{u} = (\check{u}, \check{u}, \dots, \check{u})$ with $\check{u} \in S^{n-1} \cap L'$ gives

$$\binom{n-1}{n-q} \Phi_{n-q,s}(K')(\check{u}) = \sum_{i=0}^{2} \mathcal{H}^{n-q}(s_i + Q) \int_{N(K',s_i+Q) \cap S^{n-1}} \langle u, \check{u} \rangle^s \mathcal{H}^{q-1}(du)$$

$$= \mathcal{H}^{n-q}(s_0 + Q) \omega_{q-1} \int_{-1}^{1} t^s (1 - t^2)^{\frac{q-3}{2}} \lambda(dt) \sum_{i=0}^{2} \langle u_i, \check{u} \rangle^s,$$

using spherical integration and the fact that $N(K', s_i + Q) = L_0^\perp + \{\alpha u_i : \alpha \geq 0\}$. Choosing $\check{u} = u_0$ and using $s > 1$, we have $\sum_{i=0}^{2} \langle u_i, \check{u} \rangle^s \neq 0$, so $\Phi_{n-q,s}(K') \neq 0$, as claimed.

This concludes the proof of Theorem B.4 and hence also the proof of Theorem B.1. ∎

B.5 Other stereologically adapted Crofton formulae

Having proved Theorem B.1 we now give applications and examples. So far, we have limited our study to formulae valid when the integration is over $A(n, q)$ for $0 < q < n$. To alter the integration domain, we will employ Blaschke–Petkantschin formulae. To this end define

$$D(E, L) = d(o, E) [\operatorname{span} E, L], \qquad E \in A(n, q), \ L \in G(n, r), \tag{B.31}$$

for $q, r \in \{0, \dots, n - 1\}$ with $q + r \leq n - 1$. This will serve as a Jacobian. Letting

$$\alpha_{n,k,q,r} = \frac{\omega_{k-q-r} \cdots \omega_{k-q}}{\omega_{n-q-r} \cdots \omega_{n-q}} \prod_{j=0}^{r-1} \frac{\omega_{n-j}}{\omega_{k-j}}, \tag{B.32}$$

we state a Blaschke–Petkantschin formula:

$$\alpha_{n,k,q,r} \int_{A(n,q)} f(E) \, \mu_q(dE) = \int_{G(L_0,k)} \int_{A(L,q)} f(E) D(E, L_0)^{n-k} \mu_q^L(dE) \nu_k^{L_0}(dL), \tag{B.33}$$

valid for $n, q, r, k \in \mathbb{N}_0$ with $q + r + 1 \leq k \leq n$, a fixed subspace $L_0 \in G(n, r)$ and any measurable function $f : A(n, q) \to [0, \infty)$, see [22, Thm. 3]. For insights on $D(E, L)$ and why it appears in this context, see the proof of [22, Thm. 3]. Here we see that invariant integration over flats

can be achieved by first picking a subspace (the pivot) L of dimension $k > q$ containing a fixed subspace L_0 and then integrating over all flats of dimension q in L weighting with the Jacobian $D(E, L_0)^{n-k}$. Viewing the inner integral of the right-hand side in (B.33) as a function $g(L, L_0)$ of L and L_0, a consequence of (B.33) is that we can shift from invariant integration over $A(n, q)$ to invariant integration over $G(L_0, k)$ by replacing $f(\cdot)$ with $g(\cdot, L_0)$. Combining (B.33) with Theorem B.2 or B.3 for suitable indices, we can construct measurement functions $\varphi_{L,q}^{L_0}$ solving (B.4) for specific value values of k, r, j and q.

THEOREM B.6 (Tensor Crofton formula with a fixed subspace). *Let $n, r, k, s \in \mathbb{N}$ with $r + 1 \le k < n$ and $s \ge 2$ be given and fix a subspace $L_0 \in G(n, r)$. Then, for $j = 1, \ldots, k - (r + 2)$,*

$$\int_{G(L_0,k)} \varphi_{L,q}^{L_0}(K \cap L)\, \nu_k^{L_0}(dL) = \Phi_{n-j,s}(K), \quad K \in \mathcal{K}^n,$$

holds with

$$\varphi_{L,q}^{L_0}(K \cap L) = \alpha_{n,k,q,r}^{-1} \int_{A(L,q)} \varphi_{q,j,s}(E, K \cap E) D(E, L_0)^{n-k}\, \mu_q^L(dE).$$

Here, $D(E, L_0)$ is given by (B.31), $\varphi_{q,j,s}(E, K \cap E)$ is as in Theorem B.2, the leading constant is defined in (B.32), and q can be chosen in $\{j+1, \ldots, k-(r+1)\}$. In addition, for $j = 1, \ldots, k-(r+1)$ and $q = j$ the formula still holds for s even with $\varphi_{q,q,s}(E, K \cap E)$ as in Theorem B.3.

The special case $L_0 = \{o\}$ of this result could also have been derived without the use of a Blaschke–Petkantschin formula using [77], where expressions for the integral over $G(n, q)$ of intrinsic translation invariant Minkowski tensors are provided, and then proceeding like in the proof of Theorem B.3.

When $L_0 = \{o\}$, the relation in Theorem B.6 is called a *rotational Crofton formula*. Rotational Crofton formulae for intrinsic volumes ($s = 0$) and for certain Minkowski tensors which are not translation invariant, have been shown in [3, Prop. 1] and [4, Cor. 4.4], respectively. For general $1 \le r \le n-1$ the Crofton formula with a fixed subspace was shown for intrinsic volumes ($s = 0$) in [22, Thm. 2]. For $s \ge 2$ the statement in Theorem B.6 is new. This theorem has some important implications from a stereological viewpoint. As $G(L_0, k)$ is a proper subset of $A(n, k)$, we should intuitively gain less information about $K \in \mathcal{K}^n$ when integrating over $G(L_0, k)$ compared to $A(n, k)$. This is indeed true when $j = q$: comparing Theorem B.3 with Theorem B.6 (s even), we see that $\Phi_{1,s}(K), \ldots, \Phi_{n-1,s}(K)$ can be written as Crofton integrals by Theorem B.3, whereas Theorem B.6 only allows for $\Phi_{r+2,s}(K), \ldots, \Phi_{n-1,s}(K)$ to be written as Crofton integrals. In particular, Theorem B.6 can only be applied in the three dimensional setting with $L_0 = \{o\}$ and $q = j = 1$ and no proper axis L_0 comes into play. To circumvent, this we consider the intermediate family of flats 'between' $A(n, k)$ and $G(L_0, k)$ being the family of all flats of dimension k parallel to $L_0 \in G(n, r)$ for $r < k$, i.e.,

$$A_{\text{Vert}}(L_0, k) = \{E \in A(n, k)\ :\ L_0 \subseteq \text{lin}(E)\}.$$

It should be noted that each $E \in A_{Vert}(L_0, k)$ can be written as $E = L + x$ for an $L \in G(L_0, k)$ and $x \in L^\perp$. With this identification, we can define a measure $\mu_{\text{Vert},k}^{L_0}$ by the defining relation

$$\mu_{\text{Vert},k}^{L_0}(A) = \int_{G(L_0,k)} \int_{L^\perp} 1_A(L + x)\, \lambda_L(dx)\, \mu_q^{L_0}(dL),$$

for a Borel set $A \subset A_{\mathrm{Vert}}(L_0, k)$. In three-dimensional space, L_0 is usually chosen to be one-dimensional and equal to the vertical direction, i.e., the z-axis. Therefore, the elements $E \in A_{\mathrm{Vert}}(k, L_0)$ are, also in general dimension, referred to as *vertical flats*.

It should be noted that Crofton-type integral over $A_{\mathrm{Vert}}(L_0, k)$, or *vertical section formulae*, are not new and date back to at least 1984, see [6]. Here, in $\mathbb{R}^3$, the sampling is over flats of dimension two parallel to the z-axis. This subbranch of stereology goes under the name *vertical sections*. With a Blaschke-Petkantschin formula translating invariant integration over $A(n, k)$ to $A_{\mathrm{Vert}}(L_0, k)$ by letting $E \in A_{\mathrm{Vert}}(L_0, k)$ be the pivot, we can construct a version of Theorem B.6 in the setting of vertical sections. The suitable Blaschke–Petkantschin formula is given in [22, Thm. 8], and requires the constants

$$b_{n,q} = \frac{\omega_{n-q+1} \cdots \omega_n}{\omega_1 \cdots \omega_q}, \quad 1 \leq q \leq n. \tag{B.34}$$

Combining Theorem B.2 or B.3 with [22, Thm. 8], we get a formula valid for translation invariant Minkowski tensors in the setting of vertical sections.

THEOREM B.7 (Vertical sections formulae for Minkowski tensors). *Let $n, r, k, s \in \mathbb{N}$ with $r + 1 \leq k < n$ and $s \geq 2$ be given, and fix a subspace $L_0 \in G(n, r)$. Then, for $j = 1, \ldots, k - (r+1)$,*

$$\int_{G(L_0, k)} \int_{L^\perp} \varphi_{L+x,q}^{L_0}(K \cap (L+x)) \, \lambda_{L^\perp}(dx) \, \nu_k^{L_0}(dL) = \Phi_{n-j,s}(K), \quad K \in \mathcal{K}^n,$$

holds with

$$\varphi_{L+x,q}^{L_0}(K \cap L) = \frac{b_{n-r,q}}{b_{k-r,q}} \frac{b_{k,q}}{b_{n,q}} \int_{A(L+x,q)} \varphi_{q,j,s}(E, K \cap E) \, [\mathrm{lin}(E), L_0]^{n-k} \, \mu_q^L(dE).$$

Here, $\varphi_{q,j,s}(E, K \cap E)$ is as in Theorem B.2 for $j < q$, the leading constant is defined in (B.34), and q can be chosen in $\{j+1, \ldots, k-r\}$. In addition, for $j = 1, \ldots, k-r$ and $q = j$ the formula still holds for s even with $\varphi_{q,q,s}(E, K \cap E)$ as in Theorem B.3.

Note that Theorem B.7 has a double integration compared to Theorem B.6. An implication of this, tracing back to the proof of the applied Blaschke-Petkantschin formula, is that j can be chosen equal to $k-r$. Therefore, Theorem B.7 gives a Crofton-type formula for $\Phi_{n-k+r,s}(K)$ when s is even, whereas Theorem B.6 does not. This has the practical implication that the surface area tensors of even rank can be written as Crofton-type integrals in $\mathbb{R}^3$ with a non-trivial subspace L_0 by setting $n = 3, k = 2$ and $q = j = r = 1$ in Theorem B.7.

Before giving a few examples of the adapted Crofton formulae presented in this paper, we return to the structure of $d_{n,s,j,q,z}$ in Theorem B.2. We already noted earlier that the measurement function in Theorem B.2 can be written as a sum of at most $\frac{n-q}{2} + 1$ non-vanishing terms if $n - q$ is even. Indeed $d_{n,s,j,q,z} = 0$ for $z > \frac{n-q}{2}$ in this case. The simplest case is when $n-q = 2, 1 \leq j < n-2$ and $s \geq 2$. Here,

$$\int_{A(n, n-2)} \varphi_{n-2,j,s}(E, K \cap E) \, \mu_{n-2}(dE) = \Phi_{n-j,s}(K), \quad K \in \mathcal{K}^n,$$

with

$$\varphi_{n-2,j,s}(E, \cdot) = \frac{n-1}{2\pi(n-j-1)(n-j-2)} \left(2\pi(n+s-j-2)\Phi_{n-2-j,s} - Q\Phi_{n-2-j,s-2} \right). \tag{B.35}$$

We define the trace of a symmetric tensor T of rank $s \geq 2$ by the defining relation

$$(T)(z_1, \ldots, z_{s-2}) = \sum_{j=1}^{s} T(z_1, \ldots, z_{s-2}, e_j, e_j),$$

for $z_1, \ldots, z_{s-2} \in \mathbb{R}^n$ and $e_1, \ldots, e_n$ a set of orthonormal vectors spanning $\mathbb{R}^n$. Direct calculations, cf. [54] gives

$$\Phi_{k,s-2} = \frac{2\pi s(s-1)}{n-k+s-2}(\Phi_{k,s}), \tag{B.36}$$

for $k \in \{0, \ldots, n-1\}$ and $s \geq 2$. A combination of this with (B.35) yields the following result.

COROLLARY B.8. *Let $j \in \mathbb{N}$ such that $0 < j < n-2$. For $s \geq 2$ we have*

$$\int_{A(n,n-2)} F(\Phi_{n-2-j,s}(K \cap E))\, \mu_q(dE) = \Phi_{n-j,s}(K), \quad K \in \mathcal{K}^n,$$

with

$$F(T) = \left(\frac{n-1}{(n-j-1)(n-j-2)}\right)\left((n+s-j-2)T - \frac{s(s-1)}{(s+j)}Q\, Tr(T)\right),$$

for a symmetric tensor T of rank s.

Here we see that $\{\Phi_{n-2-j,s}(K \cap E) \,:\, E \in A(n, n-2)\}$ is sufficient to determine $\Phi_{n-j,s}(K)$. Indeed, this is no surprise, as surface tensors of lower rank can be derived from a surface tensor of higher rank with same parity by repeated use of (B.36). This procedure of reapplying (B.36) would in principle work for all the adapted Crofton formulae presented in this paper. We opt out of presenting these formulae in this form as the measurement tensors become quite intractable, except for specific cases like the one in Corollary B.8.

We now conclude this paper with a special case of Theorem B.7, valid for $\mathbb{R}^3$. We only state the result for a tensor of rank two, but the Theorem allows estimation of all translation invariant tensors of even rank in $\mathbb{R}^3$. Specialized to $\mathbb{R}^3$, we obtain an apparently new section formula for the surface tensor of rank 2. If $K \in \mathcal{K}^3$ and $L_0 = \text{span}(u)$ with $u \in S^2$, then

$$\int_{G(L_0,2)} \int_{L^\perp} \varphi_{L+x,1}^{L_0}(K \cap (L+x))\, \lambda_{L^\perp}(dx)\, \nu_2^{L_0}(dL) = \Phi_{2,2}(K), \quad K \in \mathcal{K}^n, \tag{B.37}$$

holds with

$$\varphi_{L+x,1}^{L_0}(\cdot) = \frac{1}{2}\int_{A(L+x,1)}\left(8\pi\Phi_{0,2}^{(E)} - \frac{1}{2}Q\Phi_{0,0}^{(E)}\right)[\text{lin}(E), L_0]\, \mu_1^L(dE).$$

Relation (B.37) is a vertical section formula for the surface tensor of rank 2 in $\mathbb{R}^3$, generalizing [7, eq. (2)], where only the intrinsic volume V_2 (proportional to the surface area in $\mathbb{R}^3$) was considered.

B.6 Appendix

In this appendix we will show the claims related to the use of Zeilberger's algorithm appearing in the proofs of Theorem B.2 and Lemma B.5. All of the results to come are tailored to specific summations and provide little insight on their own. Nonetheless, the calculations provide insight into the machinery behind Zeilberger's algorithm and the origin of the different terms in the constants in Theorem B.2 and Theorem B.3. For the sake of readability, we will indicate the dependence of constants on indices using parentheses instead of subscripts. Thus, in the present appendix, we will write for instance $c(i, j)$ instead of $c_{i,j}$. Zeilberger's algorithm is a versatile tool to handle calculations of sums. In general, under suitable conditions on a function $h(z, j)$, it constructs constants $G(z, j)$ for $j \in \{0, \dots, z + 1\}$ and $\alpha_0, \dots, \alpha_k \in \mathbb{R}$ such that

$$\sum_{i=0}^{k} \alpha_i h(z + i, j) = G(z, j + 1) - G(z, j). \tag{B.38}$$

By summation over z this leads to a recursion formula with intricacy governed by the value of k. In this perspective the simplistic nature of (B.19) with $k = 1$ is a nice surprise and greatly simplifies the calculations. The principle behind the algorithm is essentially polynomial division combined with linear algebra. The algorithm can thus be implemented into software. In the present paper the implementation in MAPLE was used. As we insist in non-automatized proofs, we give formal derivations of the correctness of the output below. For further information about Zeilberger's algorithm consult [66, Chap. 6]. As in Theorem B.2, we also assume $n \geq 3$ in the following lemma. The assumption $n \geq 3$ implies that all terms and quotients in (B.40) are well-defined.

LEMMA B.9. *Let $k, l, n, s \in \mathbb{N}$ be such that $k + l < n$, $s \geq 2$ and $n \geq 3$. Consider the function $f : \mathbb{N}^2 \to \mathbb{R}$ given by*

$$f(z, j) = \begin{cases} \beta(s, k + l, j)\alpha(s - 2j, k, l)^{-1}c(s - 2j, k, z - j), & j \leq z \leq \lfloor \frac{s}{2} \rfloor, \\ 0, & \text{otherwise}, \end{cases}$$

defined through the use of (B.12), (B.14) and (B.16). Then

$$f(z) = \sum_{j=0}^{z} f(z, j) = \binom{k + l}{k}^{-1} \frac{2(k + l)}{kl} \begin{bmatrix} n \\ l \end{bmatrix} \frac{(-1)^z}{(4\pi)^z z!} \frac{\Gamma(\frac{s+k+l}{2} - z)}{\Gamma(\frac{s+k}{2})\Gamma(\frac{l}{2})} \prod_{i=0}^{z-1} \left(\frac{l}{2} - i \right)$$

whenever $z \leq \lfloor \frac{s}{2} \rfloor$.

Proof. Fix $z \leq \lfloor \frac{s}{2} \rfloor$ and $j \leq z$. Inserting the constants, rearranging and using

$$\frac{\omega_{s-2j+k}\omega_l}{\omega_{s-2j+k+l}} = \frac{2}{\Gamma(\frac{l}{2})} \cdot \frac{\Gamma(\frac{s+k+l}{2} - j)}{\Gamma(\frac{s+k}{2} - j)},$$

we conclude

$$f(z, j) = \alpha(z)g(z, j), \tag{B.39}$$

with

$$g(z, j) = (-1)^j \binom{z}{j} \frac{\Gamma(\frac{n-(k+l)+s}{2})\Gamma(\frac{n}{2}+s-2j)\Gamma(\frac{n-k+s}{2}-j)\Gamma(\frac{n}{2}+s-j-1-z)}{\Gamma(\frac{n-(k+l)+s}{2}-j)\Gamma(\frac{n}{2}+s-j)\Gamma(\frac{n-k+s}{2}-z)\Gamma(\frac{n}{2}+s-2j-1)}$$

$$\times \frac{\Gamma(\frac{s+k+l}{2}-j)}{\Gamma(\frac{s+k}{2}-j)\Gamma(\frac{l}{2})}, \tag{B.40}$$

$$\alpha(z) = \frac{(-1)^z}{(4\pi)^z}\binom{k+l}{k}^{-1}\frac{2(k+l)}{kl}\begin{bmatrix}n\\l\end{bmatrix}\frac{1}{z!}.$$

This reduces the problem to understanding $g(z) = \sum_{j=0}^z g(z, j)$, which will be calculated using Zeilberger's algorithm. Defining $g(z, z+1) = 0$ we are exactly in the setting of Lemma B.10 which yields

$$(l-2\beta)g(\beta) + (-k-l-s+2\beta+2)g(\beta+1) = 0, \quad \beta = 0, \dots, \left\lfloor \frac{s}{2} \right\rfloor - 1.$$

For $\beta < \lfloor \frac{s}{2} \rfloor$ we have $\frac{k+l+s}{2} - \beta - 1 \neq 0$, thus we get the recursion formula

$$g(\beta+1) = g(\beta)\frac{\frac{l}{2}-\beta}{\frac{k+l+s}{2}-\beta-1}, \quad \beta = 0, \dots, \left\lfloor \frac{s}{2} \right\rfloor - 1.$$

As

$$g(0) = \frac{\Gamma(\frac{s+k+l}{2})}{\Gamma(\frac{s+k}{2})\Gamma(\frac{l}{2})},$$

we conclude by induction that

$$g(z) = \frac{\Gamma(\frac{s+k+l}{2}-z)}{\Gamma(\frac{s+k}{2})\Gamma(\frac{l}{2})}\prod_{i=0}^{z-1}\left(\frac{l}{2}-i\right), \quad z = 0, \dots, \left\lfloor \frac{s}{2} \right\rfloor.$$

Inserting this representation of the function $g(z)$ into (B.39) yields the desired form of $f(z)$ and concludes the proof. ∎

To complete the proof of Lemma B.9, we now state and prove Lemma B.10. The linear relation herein for the function g has been established using the implementation of Zeilberger's algorithm in MAPLE. We give an elementary (but tedious) proof of this relation in the following lemma. To simplify calculations we naturally extend the definition of the binomial coefficient by defining $\binom{z}{j} = \frac{\Gamma(z+1)}{\Gamma(j+1)\Gamma(z-j+1)}$ for all reals j, z with $0 \leq j < z+1$. Likewise, we understand $(-1)^j$ as the complex number $e^{i\pi j}$ for $j \in \mathbb{R}$. Thus $g(z, j)$ appearing in the following is a complex valued function with $g(z, j) \in \mathbb{R}$ when $j \in \mathbb{N}$. This idea, by considering $g(z, j)$ as a complex valued function, is only used to simplify the proof of Lemma B.10. In this lemma we use limiting arguments to avoid division with zero as we first treat the case $j < z+1$ and then afterwards treat the case $j = z+1$ as a limit for $j \to z+1$. For instance $q(j)$ defined in the proof has $j - z - 1$ in the denominator, thus $q(z+1)$ is not well-defined. This limit argument has the advantage that all cases can be treated with one calculation.

Lemma B.10. *Let $k, l, n \in \mathbb{N}$ be such that $k + l < n$, $s \geq 2$ and $n \geq 3$. Fix $z \in \mathbb{N}$ such that $z \leq \lfloor \frac{s}{2} \rfloor - 1$. Consider the function $g : [0, z + 1] \to \mathbb{C}$ given by the extension of (B.40) when $j < z + 1$. For $j = z + 1$ we define $g(z, z + 1) = 0$. Then*

$$\sum_{j=0}^{z+1} (\gamma_1 g(z, j) + \gamma_2 g(z + 1, j)) = 0$$

with $\gamma_1 = l - 2z$ and $\gamma_2 = -k - l - s + 2z + 2$.

Proof. We will prove this lemma by showing

$$\gamma_1 g(z, j) + \gamma_2 g(z + 1, j) = G(z, j + 1) - G(z, j), \quad j \leq z, \tag{B.41}$$

$$\gamma_1 g(z, z + 1) + \gamma_2 g(z + 1, z + 1) = -G(z, z + 1), \tag{B.42}$$

holds with

$$G(z, j) := g(z, j) \frac{j(-n + k - s + 2j)(-s - k + 2j - l)}{(j - z - 1)(-n - 2s + 4j + 2)},$$

for $j < z + 1$ and

$$G(z, z + 1) = -2g(z, z) \frac{(\frac{n - (k+l)+s}{2} - z - 1)(\frac{n}{2} + s - z - 1)(\frac{s+k}{2} - z - 1)}{(\frac{n}{2} + s - 2z - 1)(\frac{n}{2} + s - 2z - 2)}.$$

Note that the conditions $z \leq \lfloor \frac{s}{2} \rfloor - 1$ and $n \geq 3$ combined with the bound $s - 2z \geq 2$ ensures that all fractions throughout this proof are well-defined. We will now show (B.41) and (B.42). To simplify notation we define

$$q(j) = \frac{j(-n + k - s + 2j)(-s - k + 2j - l)}{(j - z - 1)(-n - 2s + 4j + 2)}$$

for $j < z + 1$.

We start by showing that (B.41) holds true when $j < z$. Using $\Gamma(x)x = \Gamma(x + 1)$ extensively, it can be shown that

$$g(z + 1, j) = g(z, j) \cdot D_z(z, j), \tag{B.43}$$

with

$$D_z(z, j) = \frac{z + 1}{z + 1 - j} \frac{\frac{n - k + s}{2} - z - 1}{\frac{n}{2} + s - j - z - 2},$$

and

$$g(z, j + 1) = g(z, j) \cdot H_j(z, j), \tag{B.44}$$

with

$$H_j(z, j) := -\frac{(\frac{n - (k+l)+s}{2} - j - 1)(\frac{n}{2} + s - j - 1)(\frac{n}{2} + s - 2j - 1 - 2)(\frac{s+k}{2} - j - 1)}{(\frac{n - k + s}{2} - j - 1)(\frac{s+k+l}{2} - j - 1)(\frac{n}{2} + s - 2j - 1)}$$

$$\times \frac{z - j}{j + 1} \frac{1}{\frac{n}{2} + s - j - 1 - z - 1}.$$

Direct calculation shows

$$\gamma_1 + q(j) = H_j(z, j)q(j + 1) - \gamma_2 D_z(z, j).$$

Multiplying both sides by $g(z, j)$ yields

$$g(z, j)\Big(\gamma_1 + q(j)\Big) = g(z, j + 1)q(j + 1) - \gamma_2 g(z + 1, j), \tag{B.45}$$

where we used (B.44). Rearranging shows that this equals (B.41), clearing the case $j < z$.

We now consider $j = z$ in (B.41). Letting $j \to z$ along the positive real line from below in (B.45) we therefore conclude

$$g(z, z)\Big(\gamma_1 + q(z)\Big) = \lim_{j \to z} G(z, j + 1) - \gamma_2 g(z + 1, z). \tag{B.46}$$

When $j \in [0, z)$ then (B.44) yields

$$G(z, j + 1) = g(z, j)q(j + 1)H_j(z, j)$$

$$= -2g(z, j)\frac{(\frac{n-(k+l)+s}{2} - j - 1)(\frac{n}{2} + s - j - 1)(\frac{s+k}{2} - j - 1)}{(\frac{n}{2} + s - 2j - 1)}$$

$$\times \frac{1}{\frac{n}{2} + s - j - z - 2}.$$

From this we conclude

$$\lim_{j \to z} G(z, j + 1) = G(z, z + 1),$$

which, reinserted in (B.46) shows (B.41) when $j = z$.

We will now confirm (B.42). As $g(z, z + 1) = 0$ it is enough to show that

$$\gamma_2 g(z + 1, z + 1) = -G(z, z + 1).$$

To show this, let $j \in [0, z)$ and consider $g(z + 1, j + 1)$. First, using (B.44) to reduce j by one and then (B.43) to reduce z by one, yields

$$g(z + 1, j + 1) = g(z, j)H_j(z + 1, j)D_z(z, j).$$

Expanding this, we conclude

$$g(z + 1, j + 1) = g(z, j)\Bigg(-\frac{(\frac{n-(k+l)+s}{2} - j - 1)(\frac{n}{2} + s - j - 1)(\frac{n}{2} + s - 2j - 1 - 2)(\frac{s+k}{2} - j - 1)}{(\frac{n-k+s}{2} - j - 1)(\frac{s+k+l}{2} - j - 1)(\frac{n}{2} + s - 2j - 1)}$$

$$\times \frac{z + 1 - j}{j + 1}\frac{1}{\frac{n}{2} + s - j - 1 - z - 1 - 1}$$

$$\times \frac{z + 1}{z + 1 - j}\frac{\frac{n-k+s}{2} - z - 1}{\frac{n}{2} + s - j - 1 - z - 1}\Bigg).$$

Taking the limit $j \to z$ yields

$$g(z+1, z+1) = -g(z,z) \frac{(\frac{n-(k+l)+s}{2} - z - 1)(\frac{n}{2} + s - z - 1)(\frac{s+k}{2} - z - 1)}{(\frac{s+k+l}{2} - z - 1)(\frac{n}{2} + s - 2z - 1)} \frac{1}{z+1} \frac{z+1}{1} \frac{1}{\frac{n}{2} + s - 2z - 2}$$

$$= \frac{-G(z, z+1)}{\gamma_2},$$

showing (B.42). Using (B.41) and (B.42), we conclude

$$\sum_{j=0}^{z+1} \gamma_1 g(z, j) + \gamma_2 g(z+1, j) = -G(z, 0) = 0,$$

which completes the proof. ∎

We now turn our attention to the use of Zeilberger's algorithm in the proof of Lemma B.5. In Lemma B.10 we used complex-valued functions to simplify calculations. As the Zeilberger relation in Lemma B.11 is of a simpler nature, this is not needed in the next proof.

LEMMA B.11. *Let* $d \in \{0, 1\}$, $s, q, n \in \mathbb{N}$ *and* $z \in \mathbb{N}_0$ *be such that* $0 \leq z \leq s$, $0 < q < n$. *Fix* $k \in \{z, \dots, s\}$. *Then*

$$\eta(s, z, k, d) = (-1)^{k-z} \binom{2s+d}{2k} \binom{k}{z} \Gamma(k + \frac{1}{2}) \frac{\Gamma(s - k + 1 + \frac{d}{2})\Gamma(\frac{n-q}{2} + k)}{\Gamma(\frac{n+1}{2} + k)}$$

$$\times \left(\frac{n-q}{2} + k + \frac{(q-1)(k-z)}{2s - 2z - 1 + d} \right),$$

where we define $\eta(s, s, s, 1) = 0$, *satisfies*

$$\sum_{k=z}^{s} \eta(s, z, k, d) = \begin{cases} (-1)^{s+z+1} \frac{\sqrt{\pi}\Gamma(2s+1)\Gamma(\frac{n-q}{2} + s + 1)}{4^s (2(s-z)-1)\Gamma(z+1)\Gamma(s-z+1)\Gamma(\frac{n+1}{2} + s)}, & d = 0, \\ 0, & d = 1. \end{cases} \quad \text{(B.47)}$$

Proof. We start with the case $z = s$. If $d = 1$ then (B.47) holds by definition. If $d = 0$, Legendre's duplication formula, applied to $\eta(s, s, s, 0)$, yields the claim. Thus, we may assume $z < s$. Define

$$h_T(s, z, k, d) = (2n - 2 + 4k)(k - z),$$
$$h_B(s, z, k, d) = d(2k + n - q) + 2qk + 4ks - 4kz + 2ns - 2nz - 2qs - 4k - n + q + 2z,$$

for $k \in \{z, \dots, s\}$. As

$$h_B(s, z, k, d) \geq d(2k + n - q) + 4k(s - z - 1) + (n - q)(2(s - z) - 1) > 0,$$

we can define

$$G(s, z, k, d) = \eta(s, z, k, d) \frac{h_T(s, z, k, d)}{h_B(s, z, k, d)}, \quad k \in \{z, \dots, s\}. \quad \text{(B.48)}$$

We claim

$$\eta(s, z, k, d) = G(s, z, k, d) - G(s, z, k+1, d), \quad k \in \{z, \dots, s-1\}, \quad \text{(B.49)}$$

where $G(s, z, k, d)$ is defined in (B.48). To show this, fix a $k \in \{z, \ldots, s - 1\}$ and consider the following decomposition:

$$\eta(s, z, k, d) = (-1)^{k-z} a(k) p(k)$$

with

$$a(k) = \binom{2s + d}{2k} \binom{k}{z} \Gamma(k + \frac{1}{2}) \frac{\Gamma(s - k + 1 + \frac{d}{2}) \Gamma(\frac{n-q}{2} + k)}{\Gamma(\frac{n+1}{2} + k)},$$

$$p(k) = \left(\frac{n - q}{2} + k + \frac{(q - 1)(k - z)}{2s - 2z - 1 + d} \right).$$

Using $\Gamma(x + 1) = \Gamma(x)x$ direct calculations show

$$\frac{a(k + 1)}{a(k)} = \frac{(2s + d - 2k - 1)(2k + n - q)}{2(k + 1 - z)(n + 1 + 2k)}.$$

Likewise, we get

$$\frac{p(k + 1)}{p(k)} = \frac{p_T(k)}{p_B(k)},$$

with

$$p_T(k) = 2kd + nd - qd + 2kq + 2ks - 4kz + 2ns - 2nz - 2qs + 2d - 4k - n$$
$$+ 3q + 4s - 2z - 4,$$
$$p_B(k) = 2kd + nd - qd + 2qk + 4ks - 4kz + 2ns - 2nz - 2qs - 4k - n + q + 2z,$$

which simply follows by expanding all terms. By direct comparison we see that $p_T(k) = h_B(k+1)$ and $p_B(k) = h_B(k)$. Here, we use the short notation $h_B(k) = h_B(s, z, k, d)$. Now consider

$$\frac{a(k + 1)}{a(k)} \frac{p(k + 1)}{p(k)} \frac{h_T(k + 1)}{h_B(k + 1)} = \frac{a(k + 1)}{a(k)} \frac{h_T(k + 1)}{h_B(k)}. \tag{B.50}$$

As

$$\frac{a(k + 1)}{a(k)} h_T(k + 1) = (d - 2k + 2s - 1)(2k + n - q), \tag{B.51}$$

an expansion of the expressions involved shows

$$\frac{a(k + 1)}{a(k)} h_T(k + 1) + h_T(k) = h_B(k).$$

Inserting this in (B.50) we conclude

$$\frac{a(k + 1)}{a(k)} \frac{p(k + 1)}{p(k)} \frac{h_T(k + 1)}{h_B(k + 1)} + \frac{h_T(k)}{h_B(k)} = 1.$$

Reinserting the definition of $\eta(s, z, k, d)$ and $G(s, z, k, d)$ we conclude

$$\frac{(-1)^{k+1-z} G(s, z, k + 1, d)}{(-1)^{k-z} \eta(s, z, k, d)} + \frac{G(s, z, k, d)}{\eta(s, z, k, d)} = 1,$$

which, after a rearrangement shows (B.49). By (B.49) it now follows

$$\sum_{k=z}^{s} \eta(s, z, k, d) = \eta(s, z, s, d) - G(s, z, s, d) = \eta(s, z, s, d)\left(1 - \frac{h_T(s, z, s, d)}{h_B(s, z, s, d)}\right). \tag{B.52}$$

When $d = 1$ (B.47) follows as $\frac{h_T(s,z,s,1)}{h_B(s,z,s,1)} = 1$. For $d = 0$ we note

$$\left(1 - \frac{h_T(s, z, s, 0)}{h_B(s, z, s, 0)}\right) = -\frac{n - q + 2s}{2ns - 2nz + 4s^2 - 4sz - n + q - 4s + 2z},$$

$$\left(\frac{n - q}{2} + s + \frac{(q - 1)(s - z)}{2s - 2z - 1}\right) = \frac{2ns - 2nz + 4s^2 - 4sz - n + q - 4s + 2z}{2(2s - 2z - 1)}.$$

Thus, by using (B.52) and inserting $\eta(s, z, s, 0)$ we conclude

$$\sum_{k=z}^{s} \eta(s, z, k, 0) = (-1)^{s-z}\binom{2s}{2s}\binom{s}{z}\Gamma\left(s + \frac{1}{2}\right)\frac{\Gamma(1)\Gamma\left(\frac{n-q}{2} + s\right)}{\Gamma\left(\frac{n+1}{2} + s\right)} \times \left((-1)\frac{\frac{n-q}{2} + s}{2(s - z) - 1}\right)$$

showing (B.47) after rewriting this expression using Legendre duplication formula. This concludes the proof of Lemma B.11. ∎

A Blaschke–Petkantschin formula for linear and affine subspaces with application to intersection probabilities

By Emil Dare, Christoph Thäle and Markus Kiderlen

C

Abstract

Consider a uniformly distributed random linear subspace L and a stochastically independent random affine subspace E in $\mathbb{R}^n$, both of fixed dimension. For a natural class of distributions for E we show that the intersection $L \cap E$ admits a density with respect to the invariant measure. This density depends only on the distance $d(o, E \cap L)$ of $L \cap E$ to the origin and is derived explicitly. It can be written as the product of a power of $d(o, E \cap L)$ and a part involving an incomplete beta integral. Choosing E uniformly among all affine subspaces of fixed dimension hitting the unit ball, we derive an explicit density for the random variable $d(o, E \cap L)$ and study the behavior of the probability that $E \cap L$ hits the unit ball in high dimensions. Lastly, we show that our result can be extended to the setting where E is tangent to the unit sphere, in which case we again derive the density for $d(o, E \cap L)$. Our probabilistic results are derived by means of a new integral-geometric transformation formula of Blaschke–Petkantschin type.

C.1 Introduction and motivation

Fix dimension parameters $n \geq 2$, $q \in \{1, \dots, n-1\}$ and $\gamma \in \{0, \dots, q-1\}$. Let L_1 be a q-dimensional random linear subspace and L_2 be an $(n-q+\gamma)$-dimensional random linear subspace of $\mathbb{R}^n$. We assume that both subspaces are stochastically independent and that L_1 and L_2 are selected according to the uniform distribution on the Grassmannian $G(n, k)$ of all k-dimensional linear subspaces of $\mathbb{R}^n$, with $k = q$ and $k = n - q + \gamma$, respectively. In other words, we use the normalized rotation invariant measures ν_k, $k = q$ and $k = n - q + \gamma$, on these spaces as our underlying probability measures; these and further concepts will formally be introduced in Section C.2. The intersection $L_1 \cap L_2$ is almost surely a random subspace of $\mathbb{R}^n$ of dimension γ and its distribution is known to be the uniform distribution on the space $G(n, \gamma)$, see Figure C.1.

Let us now change the set-up and let E_1 be a q-dimensional random affine subspace and E_2 be another $(n-q+\gamma)$-dimensional random affine subspace of $\mathbb{R}^n$. Since the motion invariant measure μ_k on $A(n, k)$, $k \in \{0, \dots, n-1\}$, is not finite, we restrict attention to the set

$$[B^n]_k = \{E \in A(n, k) : E \cap B^n \neq \emptyset\}$$

of k-dimensional affine subspaces hitting the unit ball B^n of $\mathbb{R}^n$. We thus take E_1 and E_2 as random affine subspaces distributed according to the normalized measures μ_q and $\mu_{n-q+\gamma}$ restricted to $[B^n]_q$ and $[B^n]_{n-q+\gamma}$, respectively. Assuming that E_1 and E_2 are stochastically independent, the intersection $E_1 \cap E_2$ is almost surely a random affine subspace of $\mathbb{R}^n$ with dimension γ. However, the intersection of $E_1 \cap E_2$ with the unit ball B^n may or may not be empty with strictly positive probability, see Figure C.1.
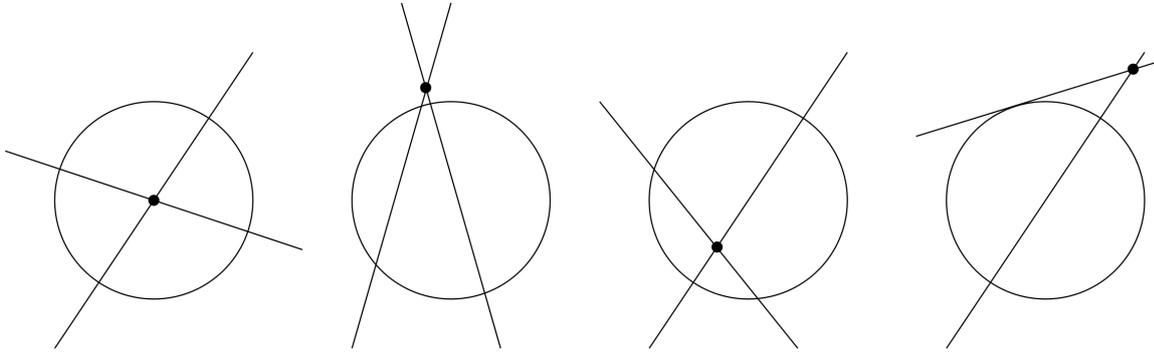
Figure C.1: Illustration in the case $n = 2$, $q = 1$, $\gamma = 0$. From left to right: Intersection of two linear subspaces; intersection of two affine subspaces hitting the unit ball; intersection of a linear with an affine subspace hitting the unit ball; intersection of a linear subspace with an affine subspace tangent to the unit sphere.

Already this basic observation shows that – in contrast to the case of random linear subspace discussed above – the distribution of $E_1 \cap E_2$ cannot coincide with the normalized motion-invariant measure on $[B^n]_\gamma$. More precisely, since the distance of the intersection of E_1 and E_2 to the origin $o \in \mathbb{R}^n$ can be arbitrarily large, the distribution of $E_1 \cap E_2$ cannot even be supported on a compact subset of $A(n, \gamma)$. In fact, it is known from [75, Thm. 7.2.8] that the distribution of $E_1 \cap E_2$ has a non-trivial density with respect to the invariant measure on $A(n, \gamma)$, which is proportional to

$$E \mapsto \int_{A(E,q) \cap [B^n]_q} \int_{A(E,n-q+\gamma) \cap [B^n]_{n-q+\gamma}} [E_1, E_2]^{\gamma+1} \, \mu_{n-q+\gamma}^E(dE_2) \mu_q^E(dE_1),$$

where $A(E, q)$ is the set of q-dimensional affine subspaces containing E and μ_q^E is the invariant measure on that space (similarly for $A(E, n - q + \gamma)$ and $\mu_{n-q+\gamma}^E$). Moreover, $[E_1, E_2]$ stands for the so-called subspace determinant, describing the relative position of E_1 and E_2, see below for a detailed definition. We remark that in the special case $n = 2$, $q = 1$ and $\gamma = 0$ this is a classical result of M. Crofton discussed in [12, Section 7], whereas the case $n = 3$, $q = 2$ and $\gamma = 1$ goes back to W. Blaschke [12, Section 33].

On a more abstract level, both problems just mentioned naturally lead to the study of what is known in the literature as integral-geometric transformation formulas of Blaschke–Petkantschin type. Such formulas go back to the pioneering works of W. Blaschke [12] in dimensions $n = 2$, and $n = 3$ and have more systematically been investigated by his student B. Petkantschin [65]. They have further been developed in [73] using the language of differential forms and in [75] by means of a measure-theoretic approach. Blaschke–Petkantschin formulas are fundamental devices in integral and stochastic geometry and have found various applications in convex geometry and geometric analysis [13, 18, 19, 38, 57, 58, 72] as well as in stereology [22, 31, 32, 46, 50, 51]. For the intersection of two linear subspaces a formula of this type is given by

$$\int_{G(n,q)} \int_{G(n,n-q+\gamma)} f(L_1, L_2) \, \nu_{n-q+\gamma}(dL_2) \nu_q(dL_1)$$

$$= c_1(n, q, \gamma) \int_{G(n,\gamma)} \int_{G(L,q)} \int_{G(L,n-q+\gamma)} f(L_1, L_2) [L_1, L_2]^\gamma \, \nu_{n-q+\gamma}^L(dL_2) \nu_q^L(dL_1) \nu_\gamma(dL), \tag{C.1}$$

where $f : G(n,q) \times G(n, n-q+\gamma) \to [0,\infty)$ is a measurable function and $G(L,q)$ is the relative Grassmannian of all q-dimensional linear subspaces of $\mathbb{R}^n$ containing L, whereas ν_q^L is the invariant probability measure on $G(L,q)$ which is invariant under all rotations of $\mathbb{R}^n$ that fix L (similarly for $G(L, n-q+\gamma)$ and $\nu_{n-q+\gamma}^L$). We refer to [75, Thm. 7.2.5] where also the value of the constant $c_1(n,q,\gamma)$ can be found, which only depends on the parameters in brackets. The corresponding formula for the intersection of two affine subspaces is a special case of [75, Thm. 7.2.8] and reads as follows:

$$
\begin{aligned}
&\int_{A(n,q)} \int_{A(n,n-q+\gamma)} f(E_1, E_2)\, \mu_{n-q+\gamma}(dE_2)\mu_q(dE_1) \\
&= c_2(n,q,\gamma) \int_{A(n,\gamma)} \int_{A(E,q)} \int_{A(E,n-q+\gamma)} f(E_1,E_2)[E_1,E_2]^{\gamma+1}\, \mu_{n-q+\gamma}^E(dE_2)\mu_q^E(dE_1)\mu_\gamma(dE)
\end{aligned}
\tag{C.2}
$$

for measurable functions $f : A(n,q) \times A(n, n-q+\gamma) \to [0,\infty)$. Here, $A(E,k)$ stands for the family of k-dimensional affine subspaces containing E and μ_k^E denotes the invariant measure on $A(E,k)$, $k = q$ and $k = n-q+\gamma$. The value of the constant $c_2(n,q,\gamma)$ only depends on the parameters in brackets and can be found in [75].

The present paper deals with a situation which in a sense is intermediate between (C.1) and (C.2), and combines the linear with the affine set-up. To the best of our knowledge, this has not found attention so far in the literature. More explicitly, let $L \in G(n,q)$ be a q-dimensional linear subspace of $\mathbb{R}^n$, $q \in \{1, \ldots, n-1\}$, and for $\gamma \in \{0, \ldots, q-1\}$ let $E \in G(n, n-q+\gamma)$ be an affine subspace of dimension $n-q+\gamma$. If E and L are in general position, their intersection $E \cap L$ is an affine subspace of dimension γ.

Our principal goal is the following. Find, for a given rotation invariant measure $\tilde{\mu}_{n-q+\gamma}$ on $A(n, n-q+\gamma)$ that is absolutely continuous with respect to $\mu_{n-q+\gamma}$, a measurable function $J : A(n, \gamma) \to [0,\infty)$ such that

$$
\int_{G(n,q)} \int_{A(n,n-q+\gamma)} f(E \cap L)\, \tilde{\mu}_{n-q+\gamma}(dE)\nu_q(dL) = \int_{A(n,\gamma)} f(E)\, J(E)\, \mu_\gamma(dE)
\tag{C.3}
$$

holds for all measurable functions $f : A(n, \gamma) \to [0,\infty)$. The main result of this paper provides an explicit description of $J(E)$ and its dependence on $\tilde{\mu}_{n-q+\gamma}$. In the important particular case $\tilde{\mu}_{n-q+\gamma} = \mu_{n-q+\gamma}$, it turns out that $J(E) = cd(o,E)^{-(n-q)}$, where c is a known constant and $d(o,E)$ stands for the distance of E to the origin o. Another interesting case arises when $\tilde{\mu}_{n-q+\gamma}$ is the restriction of $\mu_{n-q+\gamma}$ to $[hB^n]_{n-q+\gamma}$ for some fixed $h > 0$. Then, the left-hand side of (C.3) is

$$
\int_{G(n,q)} \int_{A(n,n-q+\gamma)} f(E \cap L)\, 1_{\{d(o,E) \leq h\}}\, \mu_{n-q+\gamma}(dE)\nu_q(dL),
\tag{C.4}
$$

and $J(E)$ involves, besides of $d(o,E)^{-(n-q)}$, an additional factor that can be expressed in terms of an incomplete beta function. In probabilistic terms, our new integral-geometric transformation formula (C.3) will allow us to determine the density with respect to the invariant measure μ_γ on $A(n, \gamma)$ of the intersection of a random linear subspace L of dimension q and a stochastically independent random subspace E of dimension $n-q+\gamma$ hitting the unit ball in $\mathbb{R}^n$, see Figure C.1. Moreover, we will also be able to determine the corresponding density with respect to μ_γ if the random affine subspace E is only tangent to the unit sphere, see again Figure C.1.

The remaining parts of this paper are structured as follows. In Section C.2 we set up the notation and gather some background material. Some preliminary considerations are contained in Section C.3 and in Section C.4 we formulate our main theorems, especially the new integral-geometric transformation formula of Blaschke–Petkantschin-type. In Sections C.5 and C.6 we present the two applications to intersection probabilities mentioned above. Finally, Section C.7 contains the proofs of our main results.

C.2 Notation and background material

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space for some fixed dimension $n \geq 1$. The *Euclidean norm* will always be denoted by $\| \cdot \|$ and by λ_n we indicate the *Lebesgue measure* on $\mathbb{R}^n$. The *Euclidean unit ball* and *sphere* are denoted by B^n and S^{n-1} and their volume and surface content are given by

$$\kappa_n = \lambda_n(B^n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \qquad \text{and} \qquad \omega_n = \mathcal{H}^{n-1}(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}, \tag{C.5}$$

respectively. Here, $\mathcal{H}^{n-1}$ is the $(n-1)$-*dimensional Hausdorff measure* in $\mathbb{R}^n$. Further, for a set $E \subset \mathbb{R}^n$ we define the distance $d(o, E) = \inf_{x \in E} \|x\|$ of E to the origin $o \in \mathbb{R}^n$. We will make use of the *incomplete beta function*

$$B(x; \alpha, \beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\, dt, \qquad 0 \leq x \leq 1,$$

with real parameters $\alpha, \beta > 0$. Notice that the *complete beta integral* satisfies

$$B\left(\tfrac{m}{2}, \tfrac{k}{2}\right) = B\left(1; \tfrac{m}{2}, \tfrac{k}{2}\right) = 2\frac{\omega_{m+k}}{\omega_m \omega_k}, \qquad m, k \in \mathbb{N}. \tag{C.6}$$

For $n \geq 1$ and $q \in \{0, \ldots, n\}$ we let $G(n, q)$ be the *Grassmannian* of all q-dimensional linear subspaces of $\mathbb{R}^n$ and write $A(n, q)$ for the family of q-dimensional affine subspaces of $\mathbb{R}^n$. We endow these spaces with the usual Borel σ-algebras and describe now shortly invariant measures on these families. Details can be found e.g. in [75], where also measurablility issues are discussed.

The group SO_n of rotations in $\mathbb{R}^n$ carries a unique invariant (or *Haar*) probability measure ν. This group acts naturally on the space $G(n, q)$ and we denote by ν_q the unique SO_n-invariant probability measure on $G(n, q)$. Both, SO_n and the group of translations act naturally on the *affine Grassmannian* $A(n, q)$. There exists a motion invariant measure on $A(n, q)$, and this measure is unique up to a multiplicative constant. We will use the motion invariant measure μ_q on the affine Grassmannian $A(n, q)$ given by

$$\mu_q(\,\cdot\,) = \int_{G(n,q)} \int_{L^\perp} 1_{\{L+x \in \,\cdot\,\}}\, \lambda_{L^\perp}(dx)\nu_q(dL), \tag{C.7}$$

where $\lambda_{L^\perp}$ denotes the Lebesgue measure on the orthogonal complement $L^\perp$ of $L \in G(n, q)$. Finally, for $M \in A(n, q)$ and $p \in \{0, \ldots, q\}$ we denote by $G(M, p)$ and $A(M, p)$ the *relative Grassmannian* of all p-dimensional linear and affine subspaces contained in M, respectively. If, on the other hand, $p \in \{q, \ldots, n\}$ then $G(M, p)$ and $A(M, p)$ are the sets of linear and affine

subspaces of dimension p that contain M. These spaces carry natural invariant measures ν_p^M and μ_p^M as described in [75, Sec. 7.1]. In particular, these measures satisfy

$$\int_{G(n,q)} \int_{G(M,p)} f(L)\, \nu_p^M(dL)\, \nu_q(dM) = \int_{G(n,p)} f(L)\, \nu_p(dL)$$

for all measurable functions $f : G(n,p) \to [0,\infty)$, and similarly

$$\int_{A(n,q)} \int_{A(F,p)} f(E)\, \mu_p^F(dE)\, \mu_q(dF) = \int_{A(n,p)} f(E)\, \mu_p(dE)$$

for all measurable functions $f : A(n,p) \to [0,\infty)$, see [75, Thm. 7.1.1 and Thm. 7.1.2].

Let $0 \le p, q \le n-1$ and fix $L \in G(n,p)$ and $M \in G(n,q)$. If $p+q \le n$ the *subspace determinant* $[L,M]$ is defined as the $(p+q)$-volume of a parallelepiped spanned by the union of an orthonormal basis in L and an orthonormal basis in M. If $p+q \ge n$ we define $[L,M] = [L^\perp, M^\perp]$. If $p+q = n$, both definitions coincide and $[L,M]$ is the factor by which the p-volume is multiplied under the orthogonal projection from L onto $M^\perp$ thus,

$$\int_{L^\perp} f(x)\, \lambda_{L^\perp}(dx) = [M,L] \int_M f(x|L^\perp)\, \lambda_M(dx), \qquad \text{(C.8)}$$

for measurable $f : L^\perp \to [0,\infty)$. Here, $x|L^\perp$ denotes the orthogonal projection of a point $x \in \mathbb{R}^n$ onto $L^\perp$. For further background on subspace determinants, we refer to [75, Sec. 14.1].

C.3 Preliminary considerations

Let $n \ge 1$ and $q \in \{1, \ldots, n-1\}$. The rotation group SO_n acts on the space of real-valued functions f on $A(n,q)$ by

$$(\eta f)(E) = f(\eta^{-1} E), \qquad E \in A(n,q),$$

for $\eta \in SO_n$. The function f is called *rotation invariant* if $\eta f = f$ for any rotation $\eta \in SO_n$. The *rotational mean* of f, given by

$$f_{\text{rot}}(E) = \int_{SO_n} (\eta f)(E)\, \nu(d\eta), \qquad E \in A(n,q),$$

is rotation invariant.

Assume that f is rotation invariant and $n \ge 2$. Then, we have $f(E) = f(E')$ for any two affine subspaces $E, E' \in A(n,q)$ with $d(o,E) = d(o,E')$, since there is a rotation $\eta \in SO_n$ with $\eta E = E'$. This implies that there is a function $f_I : [0,\infty) \to \mathbb{R}$, such that

$$f(E) = f_I(d(o,E)), \qquad E \in A(n,q). \qquad \text{(C.9)}$$

The following lemma shows that for our purposes, it is essentially enough to consider the class of rotation invariant functions.

LEMMA C.1. *Let $n \geq 2$, $q \in \{1, \ldots, n-1\}$, $\gamma \in \{0, \ldots, q-1\}$, a rotation invariant measure $\tilde{\mu}_{n-q+\gamma}$ on $A(n, n-q+\gamma)$ and a rotation invariant function $J : A(n, \gamma) \to [0, \infty)$ be given. Then*

$$\int_{G(n,q)} \int_{A(n,n-q+\gamma)} f(E \cap L) \, \tilde{\mu}_{n-q+\gamma}(dE) \, \nu(dL) = \int_{A(n,\gamma)} f(E) \, J(E) \, \mu_\gamma(dE) \qquad (C.10)$$

holds for all measurable functions $f : A(n, \gamma) \to [0, \infty)$, if it holds for those f that are in addition rotation invariant.

Proof. Suppose that (C.10) holds true for all non-negative, measurable and rotation invariant functions. Fix an arbitrary measurable function $f : A(n, q) \to [0, \infty)$. Using the rotation invariance of the measures $\tilde{\mu}_{n-q+\gamma}$ and ν_q and Tonelli's theorem, the left-hand side of (C.10) is

$$\int_{SO_n} \int_{G(n,q)} \int_{A(n,n-q+\gamma)} f(\eta^{-1}(E \cap L)) \, \tilde{\mu}_{n-q+\gamma}(dE) \, \nu_q(dL)\nu(d\eta)$$
$$= \int_{G(n,q)} \int_{A(n,n-q+\gamma)} f_{\mathrm{rot}}(E) \, \tilde{\mu}_{n-q+\gamma}(dE) \, \nu_q(dL).$$

Using the rotation invariance of μ_γ and J, a similar argument shows that the right-hand side of (C.10) is

$$\int_{A(n,\gamma)} f_{\mathrm{rot}}(E) J(E) \, \mu_\gamma(dE).$$

By assumption, (C.10) holds for the rotation invariant function f_{rot}, so the last two displayed expressions coincide and the assertion is shown. ∎

   In the proof of our main result, the following lemma will turn out to be of crucial importance. It can be seen a generalization of [41, Lem. 4.4]. In that result the authors prove that

$$A(n, k, r, \alpha) = \int_{G(n,k)} [F, L]^\alpha \, \nu_k(dL) = \prod_{i=0}^{n-r-1} \frac{\Gamma(\frac{n-i}{2})\Gamma(\frac{k-i+\alpha}{2})}{\Gamma(\frac{n-i+\alpha}{2})\Gamma(\frac{k-i}{2})}, \qquad (C.11)$$

for $\alpha \geq 0$, $r, k \in \{1, \ldots, n\}$ with $r + k \geq n$ and $F \in G(n, r)$, where the right-hand side is interpreted as 1 if $r = n$. The next lemma is a counterpart for the Grassmannian associated to a hyperplane. (We note that the definition of the subspace determinant in [41] is equivalent to our definition as $r + k \geq n$.)

LEMMA C.2. *Let $n \geq 2$, $\alpha \geq 0$ and $p, q \in \{1, \ldots, n-1\}$ with $p + q \leq n$ be given. Then, for any $u \in S^{n-1}$ and any fixed $M \in G(n, p)$ we have that*

$$\int_{G(\mathrm{span}\, u,q)} [L, M]^\alpha \, \nu_q^{\mathrm{span}\, u}(dL) = a(n, p, q, \alpha)[u, M]^\alpha \qquad (C.12)$$

with

$$a(n, p, q, \alpha) = \prod_{i=1}^{p} \frac{\Gamma(\frac{n-i}{2})\Gamma(\frac{n-q-i+\alpha+1}{2})}{\Gamma(\frac{n-i+\alpha}{2})\Gamma(\frac{n-q-i+1}{2})}.$$

Proof. If $u \in M$, then (C.12) holds trivially as both sides vanish. Hence, we may assume $u \notin M$. Suppose first that we also have $u \notin M^\perp$ and fix $L \in G(\operatorname{span} u, q)$. Since $\dim M^\perp + \dim L^\perp = 2n - (p+q) \geq n$, [67, Lem. 4.1] implies that

$$[L^\perp, M^\perp] = [L^\perp, M^\perp \cap u^\perp] \|u|M^\perp\|$$
$$= [L^\perp, M^\perp \cap u^\perp][u, M],$$

where we recall that $u|M^\perp$ is the projection of u onto $M^\perp$. Here, we used that fact that the subspace determinant $[L', M']$, as defined in [67], coincides with our definition whenever $\dim M' + \dim L' \geq n$, but differs otherwise, causing [67] to consider subspace determinants relative to $u^\perp$, which is not necessary using our definition. We thus obtain

$$[L, M] = [L^\perp, M^\perp] = [L^\perp, M^\perp \cap u^\perp][u, M],$$

a relation that is also true for $u \in M^\perp$, implying that the left-hand side of (C.12) coincides with

$$\int_{G(\operatorname{span} u, q)} [L^\perp, M^\perp \cap u^\perp]^a \, \nu_q^{\operatorname{span} u}(dL) \, [u, M]^\alpha = \int_{G(u^\perp, n-q)} [L, M^\perp \cap u^\perp]^a \, \nu_{q-1}^{u^\perp}(dL) \, [u, M]^\alpha.$$

The last integral is now of the form (C.11), but with $u^\perp$ instead of $\mathbb{R}^n$ as the ambient space. It is thus equal to $A(n-1, n-q, n-p-1, \alpha)$. This constant coincides with $a(n, p, q, \alpha)$, and the assertion is proven. ∎

C.4   Presentation of the main results

Having established the basic notions and concepts in Section C.3, we will now state our main result: a general reduction of integrals of the form (C.3). We remark that we imposed the dimensional constraints $0 \leq \gamma < q < n$ at the beginning of the introduction, since $\gamma = q$ or $q = n$, would imply that the left-hand side of (C.3) becomes trivial. These constraints imply the assumption $n \geq 2$, which we now adopt for the rest of the paper.

Recall that (C.3) involves a rotation invariant measure $\tilde{\mu}_{n-q+\gamma}$ on $A(n, n-q+\gamma)$ that is absolutely continuous with respect to $\mu_{n-q+\gamma}$. Hence, the Radon–Nikodym theorem guarantees the existence of a $\mu_{n-q+\gamma}$-density $\tilde{H} \geq 0$ for $\tilde{\mu}_{n-q+\gamma}$. The assumed rotation invariance implies that $H = \tilde{H}_{\operatorname{rot}}$ is also a $\mu_{n-q+\gamma}$-density for $\tilde{\mu}_{n-q+\gamma}$. We will state our results using this density, as the function $J(r) = J_H(r)$ can explicitly be expressed in terms of H. For specific choices of the density H, the function $J_H(r)$ in the statement of this theorem can be simplified and be made more explicit, as illustrated in Corollaries C.4 and C.6 below. The proof of Theorem C.3 is postponed to Section C.7 at the end of this paper.

THEOREM C.3. *Fix $n \geq 2$, $q \in \{1, \ldots, n-1\}$, $\gamma \in \{0, \ldots, q-1\}$, and let $H : A(n, n-q+\gamma) \rightarrow [0, \infty)$ be a measurable and rotation invariant function. Then*

$$\int_{G(n,q)} \int_{A(n,n-q+\gamma)} f(E \cap L) H(E) \, \mu_{n-q+\gamma}(dE) \, \nu_q(dL)$$
$$= D(n, q, \gamma) \int_{A(n,\gamma)} f(E) d(o, E)^{-(n-q)} J_H(d(o, E)) \, \mu_\gamma(dE)$$

for all measurable $f : A(n, \gamma) \to [0, \infty)$. Here,

$$J_H(r) = \int_0^1 H_I(rz)z^q(1 - z^2)^{\frac{n-q}{2}-1} \, dz, \tag{C.13}$$

with H_I defined through (C.9) *and the constant is given by*

$$D(n, q, \gamma) = \frac{\omega_{\gamma+1}\omega_{q-\gamma}\omega_{n-q}}{\omega_{n-(q-\gamma)+1}\omega_{n-\gamma}}. \tag{C.14}$$

We shall now discuss two special cases in which the function $J_H(r)$ and thus the integral relation in Theorem C.3 can be simplified. We start by considering the special case, where H is the function $H_h(E) = 1_{\{d(o,E)\le h\}}$ for some $h > 0$, corresponding to $\tilde{\mu}_{n-q+\gamma}$ in (C.3) being the restriction of $\mu_{n-q+\gamma}$ to $[hB^n]_{n-q+\gamma}$. Definition (C.13) and a substitution yield

$$J_{H_h}(r) = \frac{1}{2} \int_0^{(\min\{\frac{h}{r},1\})^2} z^{\frac{q+1}{2}-1}(1 - z)^{\frac{n-q}{2}-1} \, dz = \begin{cases} \frac{\omega_{n+1}}{\omega_{q+1}\omega_{n-q}} & : r \le h, \\ \frac{1}{2}B\big((\frac{h}{r})^2; \frac{q+1}{2}, \frac{n-q}{2}\big) & : r > h, \end{cases} \tag{C.15}$$

where (C.6) was used at the second equality sign. This implies the following result.

COROLLARY C.4. *Fix $n \ge 2$, $q \in \{1, \dots, n-1\}$, $\gamma \in \{0, \dots, q-1\}$, and $h > 0$. Then*

$$\int_{G(n,q)} \int_{[hB^n]_{n-q+\gamma}} f(E \cap L) \, \mu_{n-q+\gamma}(dE) \, \nu_q(dL)$$

$$= D(n, q, \gamma) \int_{A(n,\gamma)} f(E)d(o, E)^{-(n-q)} J(d(o, E)) \, \mu_\gamma(dE),$$

where $J = J_{H_h}$ is given by (C.15) *and where $D(n, q, \gamma)$ is the constant given in Theorem C.3.*

The incomplete beta integral, and hence the function J in Corollary C.4, can be expressed in terms of a hypergeometric function. Also, weight functions J_H associated to more general densities H can be expressed in terms of – possibly several – hypergeometric functions. For instance, if H_I is an even polynomial restricted to $[0, \infty)$, Euler's integral relation implies such a representation, see e.g. [1] for details.

Using the monotone convergence theorem when $h \to \infty$ in Corollary C.4, gives an explicit integral relation of the form (C.3) with $\tilde{\mu}_{n-q+\gamma} = \mu_{n-q+\gamma}$.

COROLLARY C.5. *Fix $n \ge 2$, $q \in \{1, \dots, n-1\}$ and $\gamma \in \{0, \dots, q-1\}$. Then*

$$\int_{G(n,q)} \int_{A(n,n-q+\gamma)} f(E \cap L) \, \mu_{n-q+\gamma}(dE) \, \nu_q(dL)$$

$$= \tilde{D}(n, q, \gamma) \int_{A(n,\gamma)} f(E)d(o, E)^{-(n-q)} \, \mu_\gamma(dE)$$

for all measurable functions $f : A(n, \gamma) \to [0, \infty)$. Here,

$$\tilde{D}(n, q, \gamma) = \frac{\omega_{n+1}\omega_{\gamma+1}\omega_{q-\gamma}}{\omega_{n-(q-\gamma)+1}\omega_{n-\gamma}\omega_{q+1}}.$$

Let us also report that Theorem C.3 allows an extension to multiple intersections in the following way. The proof is postponed to Section C.7.

COROLLARY C.6. *Fix $\ell, m, n \geq 1$, $q_1, \ldots, q_\ell \in \{0, \ldots, n\}$, put $q = q_1 + \cdots + q_\ell - (\ell - 1)n$. Suppose that $q \in \{1, \ldots, n-1\}$ and fix $\gamma \in \{0, \ldots, q-1\}$ and $p_1, \ldots, p_m \in \{0, \ldots, n\}$ such that $p_1 + \cdots + p_m - (m-1)n = n - q + \gamma$. Further, let $f : A(n, \gamma) \to [0, \infty)$ be a measurable function, and let $H : A(n, n-q+\gamma) \to [0, \infty)$ be a rotation invariant measurable function. Define*

$$I_{\ell,m} = \int_{G(n,q_1)} \cdots \int_{G(n,q_\ell)} \int_{A(n,p_1)} \cdots \int_{A(n,p_m)} f(E_1 \cap \ldots \cap E_m \cap L_1 \cap \ldots \cap L_\ell)$$
$$\times H(E_1 \cap \ldots \cap E_m) \, \mu_{p_m}(dE_m) \ldots \mu_{p_1}(dE_1) \nu_{q_\ell}(dL_\ell) \ldots \nu_{q_1}(dL_1).$$

Then

$$I_{\ell,m} = D(n, q, \gamma) \frac{\omega_{n-q+\gamma+1} \omega_{n+1}^m}{\omega_{n+1} \omega_{p_1} \cdots \omega_{p_m}} \int_{A(n,\gamma)} f(E) d(o, E)^{-(n-q)} J_H(d(o, E)) \, \mu_\gamma(dE),$$

where J_H is given by (C.13) and the leading constant by (C.14).

Finally, we state an alternative version of Theorem C.3, which appears to be more general, as the intersecting linear subspace now is fixed. Closer investigation shows, however, that the two statements are equivalent, if a suitable Blaschke–Petkantschin relation is applied. The proof of this equivalence will be given in Section C.7.

THEOREM C.7. *Fix $n \geq 2$, $q \in \{1, \ldots, n-1\}$, $\gamma \in \{0, \ldots, q-1\}$ and $L_0 \in G(n, q)$. Let $H : A(n, n-q+\gamma) \to [0, \infty)$ be a rotation invariant, measurable function. Then*

$$\int_{A(n,n-q+\gamma)} f(E \cap L_0) H(E) \, \mu_{n-q+\gamma}(dE) = \frac{\omega_{\gamma+1} \omega_{n-q}}{\omega_{n-(q-\gamma)+1}} \int_{A(L_0,\gamma)} f(E) J_H(d(o, E)) \, \mu_\gamma^{L_0}(dE) \quad \text{(C.16)}$$

for all measurable $f : A(L_0, \gamma) \to [0, \infty)$. Here, J_H is given in Theorem C.3.

Similar versions, that is, versions where the linear space is hold fixed, are possible also for Corollaries C.4-C.6.

C.5 INTERSECTION PROBABILITIES FOR LINEAR AND AFFINE SUBSPACES HITTING THE UNIT BALL

In this section we return to the problem from stochastic geometry already mentioned in the introduction. Namely, we consider a random linear subspace L of dimension $q \in \{1, \ldots, n-1\}$ in $\mathbb{R}^n$ with distribution ν_q and a random affine subspace E of dimension $n - q + \gamma$, where $\gamma \in \{0, \ldots, q-1\}$ is a fixed number. Recalling that

$$[B^n]_{n-q+\gamma} = \{E \in A(n, n-q+\gamma) : E \cap B^n \neq \emptyset\},$$

we use the restriction of $\kappa_{q-\gamma}^{-1} \mu_{n-q+\gamma}$ to $[B^n]_{n-q+\gamma}$ as distribution for E and suppose that L and E are stochastically independent. In fact, it follows from (C.7) that $\kappa_{q-\gamma}^{-1} \mu_{n-q+\gamma}$ is indeed a probability measure on $[B^n]_{n-q+\gamma}$. We are interested in the distribution of the random affine subspace $E \cap L$, which is almost surely of dimension γ. Since its distribution is invariant under

rotations of $\mathbb{R}^n$, all relevant information is contained in the distribution of the random variable $d(o, E \cap L)$ describing the distance of $E \cap L$ to the origin. This distribution turns out to have a density $f_{n,q,\gamma}$ with respect to Lebesgue measure on $[0, \infty)$. The next result shows that it is heavy tailed and asymptotically of Pareto type with shape parameter $\gamma + 2$.

THEOREM C.8. *Fix $n \geq 2$, $q \in \{1, \dots, n-1\}$ and $\gamma \in \{0, \dots, q-1\}$. Let L be a random linear subspace with distribution ν_q, and let E be a stochastically independent random affine subspace with distribution $\kappa_{q-\gamma}^{-1} \mu_{n-q-\gamma}$, restricted to $[B^n]_{n-q+\gamma}$, . Then, the probability density $f_{n,q,\gamma}(\delta)$ of the random variable $d(o, E \cap L)$ is given by*

$$f_{n,q,\gamma}(\delta) = (q-\gamma)\frac{\omega_{\gamma+1}\omega_{n-q}}{\omega_{n-(q-\gamma)+1}} \begin{cases} \frac{\omega_{n+1}}{\omega_{q+1}\omega_{n-q}}\delta^{q-\gamma-1} & : 0 \leq \delta \leq 1 \\ \frac{1}{2}\delta^{q-\gamma-1}B(\frac{1}{\delta^2}; \frac{q+1}{2}, \frac{n-q}{2}) & : \delta > 1. \end{cases}$$

Proof. Fix $\delta > 0$ and consider the probability of the event $d(o, E \cap L) \leq \delta$. Using Corollary C.4 with $h = 1$, we obtain

$$\mathbb{P}[d(o, E \cap L) \leq \delta] = \frac{1}{\kappa_{q-\gamma}}\int_{G(n,q)}\int_{A(n,n-q+\gamma)} 1_{\{d(o,E)\leq 1\}} 1_{\{d(o,E\cap L)\leq\delta\}} \mu_{n-q+\gamma}(dE)\nu_q(dL)$$

$$= \frac{D(n,q,\gamma)}{\kappa_{q-\gamma}}\int_{A(n,\gamma)} 1_{\{d(o,E)\leq\delta\}}d(o,E)^{-(n-q)}J_{H_1}(d(o,E))\,\mu_\gamma(dE),$$

with

$$J_{H_1}(r) = \begin{cases} \frac{\omega_{n+1}}{\omega_{q+1}\omega_{n-q}} & : r \leq 1 \\ \frac{1}{2}B(\frac{1}{r^2}; \frac{q+1}{2}, \frac{n-q}{2}) & : r > 1. \end{cases}$$

We now apply the decomposition (C.7), use $d(o, L + x) = \|x\|$ whenever $x \in L^\perp$, and then introduce spherical coordinates in $L^\perp$ to see that

$$\mathbb{P}[d(o, E \cap L) \leq \delta]$$
$$= \frac{D(n,q,\gamma)}{\kappa_{q-\gamma}}\int_{G(n,\gamma)}\int_{L^\perp} 1_{\{\|x\|\leq\delta\}}\|x\|^{-(n-q)}J_{H_1}(\|x\|)\,\lambda_{L^\perp}(dx)\nu_\gamma(dL)$$
$$= \frac{D(n,q,\gamma)\omega_{n-\gamma}}{\kappa_{q-\gamma}}\int_0^\delta r^{q-\gamma-1}J_{H_1}(r)\,dr.$$

Taking the derivative with respect to δ in the last expression, inserting the value of the constant $D(n, q, \gamma)$ of Theorem C.3 and then using (C.5) yields the result. ∎

Although the density $f_{n,q,\gamma}(\delta)$ is defined piecewise, we remark that it is continuous at the point $\delta = 1$. In fact, this follows from the continuity of the incomplete beta function in combination with (C.6). Let us also mention at this point that since we are working with the unit ball B^n as a reference set, the results of this section and also the next one continue to hold if the random linear subspace L is replaced by a deterministic linear subspace of the same dimension.

Since the density of $d(o, E \cap L)$ is asymptotically of Pareto type with shape parameter $\gamma + 2$, the moment properties of the random variable $d(o, E \cap L)$ depend on the intersection dimension γ. The next result delivers a precise description.

COROLLARY C.9. *Let $\alpha \in \mathbb{R}$. Under the assumptions of Theorem C.8, $\mathbb{E}d(o, E \cap L)^\alpha < \infty$ if and only if $\alpha \in (\gamma - q, \gamma + 1)$. In particular, if $\gamma = 0$ the random variable $d(o, E \cap L)$ has infinite expectation.*

Proof. We have to check under which conditions on α the product of δ^α with the probability density $f_{n,q,\gamma}(\delta)$ of Theorem C.8 is integrable at $\delta = 0$ and $\delta = \infty$. The function $\delta \mapsto \delta^{q-\gamma-1+\alpha}$ is integrable at $\delta = 0$, if and only if $\alpha > \gamma - q$. Moreover, since $B(\delta^{-2}; a, b) = \frac{\delta^{-2a}}{a} + O(\delta^{-2(a+1)})$ as $\delta \to \infty$, see [1, Section 6.6.8 and Section 15.7], the required integrability is satisfied, whenever the function $\delta^{\alpha+q-\gamma-1-(q+1)} = \delta^{\alpha-\gamma-2}$ is integrable at $\delta = \infty$. The latter holds if and only if $\alpha - \gamma - 2 < -1$, or equivalently, $\alpha < \gamma + 1$. This completes the proof. ∎

As anticipated above, the intersection of E and L may or may not hit the unit ball. By Theorem C.8 the probability for the first of these events is given by

$$\mathbb{P}[E \cap L \cap B^n \neq \emptyset] = (q - \gamma) \frac{\omega_{\gamma+1} \omega_{n-q}}{\omega_{n-(q-\gamma)+1}} \frac{\omega_{n+1}}{\omega_{q+1} \omega_{n-q}} \int_0^1 \delta^{q-\gamma-1} \, d\delta.$$

Simplification leads to the following result.

COROLLARY C.10. *Under the assumptions of Theorem C.8, we have*

$$p_{n,q,\gamma} = \mathbb{P}[E \cap L \cap B^n \neq \emptyset] = \frac{\omega_{\gamma+1} \omega_{n+1}}{\omega_{q+1} \omega_{n-(q-\gamma)+1}}.$$

This result allows quantifying the asymptotic behavior of the intersection probability $p_{n,q,\gamma}$ for fixed q and γ in high dimensions, that is, as $n \to \infty$. Using (C.5), we can rewrite $p_{n,q,\gamma}$ in terms of gamma functions as

$$p_{n,q,\gamma} = \frac{\Gamma(\frac{q+1}{2})\Gamma(\frac{n-(q-\gamma)+1}{2})}{\Gamma(\frac{\gamma+1}{2})\Gamma(\frac{n+1}{2})}.$$

Since q and γ are assumed to be fixed, we can apply Stirling's formula [1, Section 6.1.37] for the gamma function to see that

$$p_{n,q,\gamma} = \frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{\gamma+1}{2})}\left(\frac{2}{n}\right)^{\frac{q-\gamma}{2}}(1 + o_n(1)) = \frac{\omega_{\gamma+1}}{\omega_{q+1}}\left(\frac{2}{\pi n}\right)^{\frac{q-\gamma}{2}}(1 + o_n(1)),$$

as $n \to \infty$, where we write $o_n(1)$ for a sequence that converges to zero with n. Hence, in high dimensions, the intersection point of the random affine subspace E and the random linear subspace L will asymptotically almost surely be outside the unit ball B^n.

In the planar case $n = 2$ we can take $q = 1$ and $\gamma = 2$ and obtain

$$f_{2,1,0}(\delta) = \frac{2}{\pi} \begin{cases} 1 & : 0 \leq \delta \leq 1 \\ 1 - \sqrt{1 - \delta^{-2}} & : \delta > 1, \end{cases} \qquad p_{2,1,0} = \frac{2}{\pi}.$$

Moreover, if $n = 3$ we can consider the case $q = 2$ and $\gamma = 1$, which yields

$$f_{3,2,1}(\delta) = \begin{cases} \frac{\pi}{4} & : 0 \leq \delta \leq 1 \\ \frac{1}{2}(\text{arccsc}(\delta) - \sqrt{1 - \delta^{-2}}) & : \delta > 1, \end{cases} \qquad p_{3,2,1} = \frac{\pi}{4},$$

and the case $q = 1$ and $\gamma = 0$, where the probability density reduces to

$$f_{3,1,0}(\delta) = \frac{1}{2} \begin{cases} 1 & : 0 \leq \delta \leq 1 \\ \delta^{-2} & : \delta > 1, \end{cases} \qquad p_{3,1,0} = \frac{1}{2},$$
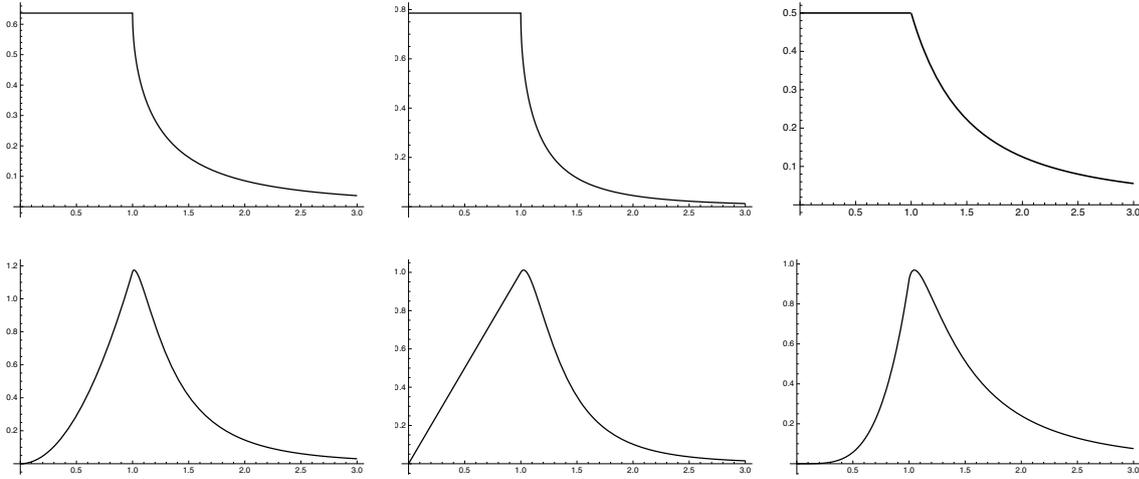
see also Figure C.2.

Figure C.2: First line: The probability densities $f_{2,1,0}$, $f_{3,2,1}$ and $f_{3,1,0}$ from left to right. Second line: The probability densities $f_{8,5,2}$, $f_{9,5,3}$ and $f_{9,6,1}$ from left to right.

In particular, if $n = 3$, $q = 2$ and $\gamma = 1$, we obtain

$$\mathbb{E}d(o, E \cap L) = \int_0^\infty \delta f_{3,2,1}(\delta)\, d\delta = \frac{\pi}{4},$$

whereas in the other two cases $\mathbb{E}d(o, E \cap L) = \infty$. To illustrate the potential complexity of the density functions $f_{n,q,\gamma}(\delta)$ we also record the following values:

n, q, γ	$f_{n,q,\gamma}$	$p_{n,q,\gamma}$	$\mathbb{E}d(o, E \cap L)$
$8, 5, 2$	$\begin{cases} \frac{128}{35\pi}\delta^2 & : 0 \le \delta \le 1 \\ \frac{48}{105\pi}\,\frac{8\delta^7 - (8\delta^6 + 4\delta^4 + 3\delta^2 - 15)\sqrt{\delta^2 - 1}}{\delta^5} & : \delta > 1, \end{cases}$	$\frac{128}{105\pi}$	$\frac{4}{\pi}$
$9, 5, 3$	$\begin{cases} \delta & : 0 \le \delta \le 1 \\ \frac{4\delta^2 - 3}{\delta^7} & : \delta > 1, \end{cases}$	$\frac{1}{2}$	$\frac{16}{15}$
$9, 6, 1$	$\begin{cases} \frac{75\pi}{265}\delta^4 & : 0 \le \delta \le 1 \\ \frac{75\delta^8 \arccsc(\delta) - 5(16\delta^6 + 10\delta^4 + 8\delta^2 - 48)\sqrt{\delta^2 - 1}}{128\delta^4} & : \delta > 1. \end{cases}$	$\frac{15\pi}{256}$	$\frac{5\pi}{8}$

We see that the complexity of the probability density $f_{n,q,\gamma}$ varies depending on n, q, γ. We notice in particular that $p_{9,5,3} = \frac{1}{2}$, which is the same as $p_{3,1,0}$ from before. In fact, for all odd n, there are (often multiple) ways we can choose q and γ such that $p_{n,q,\gamma} = \frac{1}{2}$. For instance, if we take $\gamma = \frac{n-1}{2} - 1$ and $q = \frac{n-1}{2} + 1$ or $q = n - 2$, then $p_{n,q,\gamma} = \frac{1}{2}$.

C.6   INTERSECTION PROBABILITIES FOR LINEAR AND AFFINE SUBSPACES TANGENT TO THE UNIT SPHERE

In this section, we consider another application of Theorem C.3 to intersection probabilities in stochastic geometry. As in the previous section, we consider a random linear subspace L of dimension $q \in \{1, \dots, n-1\}$ in $\mathbb{R}^n$ with distribution ν_q. In contrast, E is now a stochastically independent random affine subspace E in $\mathbb{R}^n$ of dimension $n - q + \gamma$ such that $d(o, E) = 1$. Thus,

E is tangent to the unit sphere S^{n-1} in $\mathbb{R}^n$. The rotation invariant probability measure on that space is given by

$$\sigma_{n-q+\gamma}(A) = \frac{1}{\omega_{q-\gamma}} \int_{G(n,n-q+\gamma)} \int_{S^{n-1}\cap M^\perp} 1_A(M+u) \, \mathcal{H}^{q-\gamma-1}(du) \, \nu_{n-q+\gamma}(dM) \tag{C.17}$$

for a Borel sets $A \subseteq \{E' \in A(n, n-q+\gamma) \ : \ d(o, E') = 1\}$. As in the previous section, we are interested in the distribution of $L \cap E$. Again, all relevant information is contained in the distribution of $d(o, E \cap L)$ due to rotation invariance. The next result is a stepping stone in the derivation of the probability density of the random variable $d(o, E \cap L)$.

LEMMA C.11. *Fix $n \geq 2$, $q \in \{1, \ldots, n-1\}$, $\gamma \in \{0, \ldots, q-1\}$ and let $f : A(n, \gamma) \to [0, \infty)$ be a measurable, rotation invariant and bounded function. Then*

$$\int_{G(n,q)} \int_{G(n,n-q+\gamma)} \int_{S^{n-1}\cap M^\perp} f((M+hu) \cap L) \, \mathcal{H}^{q-\gamma-1}(du) \, \nu_{n-q+\gamma}(dM) \, \nu_q(dL)$$

$$= D(n, q, \gamma)\omega_{n-\gamma}h^{\gamma+1} \int_h^\infty f_I(r)r^{-(\gamma+2)}\left(1 - \frac{h^2}{r^2}\right)^{\frac{n-q}{2}-1} dr$$

for almost every $h > 0$.

Proof. Let $f : A(n, \gamma) \to [0, \infty)$ satisfy the above assumptions and define the function $F : (0, \infty) \to [0, \infty)$ by

$$F(h) = D(n, q, \gamma) \int_{A(n,\gamma)} f(E)d(o, E)^{-(n-q)} J_{H_h}(d(o, E)) \, \mu_\gamma(dE)$$

$$= D(n, q, \gamma)\omega_{n-\gamma} \int_0^\infty f_I(r)r^{q-\gamma-1}J_{H_h}(r) \, dr,$$

with $D(n, q, \gamma)$ as in Theorem C.3 and J_{H_h} given by (C.15). Since

$$\frac{\partial}{\partial h}J_{H_h}(r) = \begin{cases} 0 & : 0 \leq r < h \\ r^{-q-1}h^q\left(1 - \frac{h^2}{r^2}\right)^{\frac{n-q}{2}-1} & : r > h, \end{cases}$$

we conclude that $F(h)$ is differentiable for almost all $h > 0$ with derivative

$$F'(h) = D(n, q, \gamma)\omega_{n-\gamma} \int_h^\infty f_I(r)r^{-\gamma-2}h^q\left(1 - \frac{h^2}{r^2}\right)^{\frac{n-q}{2}-1} dr.$$

Here, differentiation under the integral sign can be justified as follows. For fixed $r > 0$, the integrand $g(r, h) = f_I(r)r^{q-\gamma-1}J_{H_h}(r)$ is absolutely continuous as function of h. Since $(r, h) \mapsto \partial/(\partial h)g(r, h)$ is integrable on $(0, \infty) \times (a, b)$ for any $0 < a < b < \infty$, Fubini's theorem and Lebesgue differentiation theorem imply

$$F'(h) = \frac{d}{dh} \int_1^h \int_0^\infty \frac{\partial}{\partial h}g(r, t)drdt = \int_0^\infty \frac{\partial}{\partial h}g(r, h)dr$$

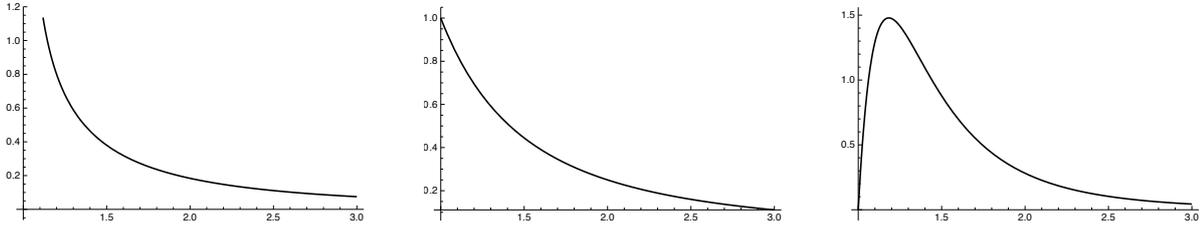for a.e. $h \in (a, b)$, yielding the intermediate assertion.

Figure C.3: From left to right: The probability densities $g_{2,1,0}$, $g_{3,1,0}$ and $g_{9,5,3}$.

On the other hand, Theorem C.3, relation (C.7) and spherical integration yield

$$F(h) = \int_{G(n,q)} \int_{G(n,n-q+\gamma)} \int_{S^{n-1} \cap M^\perp} \int_0^h f((M+ru) \cap L) r^{q-\gamma-1} \, dr \, \mathcal{H}^{q-\gamma-1}(du) \, \nu_{n-q+\gamma}(dM) \nu_q(dL),$$

and hence

$$F'(h) = \int_{G(n,q)} \int_{G(n,n-q+\gamma)} \int_{S^{n-1} \cap M^\perp} f((M+hu) \cap L) h^{q-\gamma-1} \, \mathcal{H}^{q-\gamma-1}(dy) \, \nu_{n-q+\gamma}(dM) \nu_q(dL)$$

for almost every $h > 0$ due to the Lebesgue differentiation theorem. This completes the proof of the lemma. ∎

THEOREM C.12. *Fix $n \geq 2$, $q \in \{1, \ldots, n-1\}$ and $\gamma \in \{0, \ldots, q-1\}$. Let L be a random linear subspace with distribution ν_q and let E be a stochastically independent random affine subspace, tangent to the unit sphere, with distribution $\sigma_{n-q+\gamma}$ given by (C.17). Then the random variable $d(o, E \cap L)^{-2}$ has a beta distribution with shape parameters*

$$a = \frac{\gamma+1}{2} \qquad and \qquad b = \frac{n-q}{2}.$$

More explicitly,

$$g_{n,q,\gamma}(r) = \frac{\omega_{\gamma+1} \omega_{n-q}}{\omega_{n-(q-\gamma)+1}} r^{-(\gamma+2)} \left(1 - \frac{1}{r^2}\right)^{\frac{n-q}{2}-1} 1_{\{r>1\}}$$

is a probability density for $d(o, E \cap L)$.

Figure C.3 illustrates the variety of density functions $g_{n,q,\gamma}$.

Proof. Let X be a beta-distributed random variable on $(0,1)$ with shape parameters $a, b > 0$. Then $X^{-1/2}$ has tail probabilities

$$1 - F_{X^{-1/2}}(t) = \mathbb{P}\left[X \leq \frac{1}{t^2}\right]$$

$$= \frac{1}{B(a,b)} \int_0^{1/t^2} x^{a-1}(1-x)^{b-1} \, dx$$

$$= \frac{2}{B(a,b)} \int_t^\infty r^{-2a-1} \left(1 - \frac{1}{r^2}\right)^{b-1} \, dr, \qquad t > 1. \tag{C.18}$$

On the other hand, applying Lemma C.11 with $f(E) = 1_{\{d(o,E) \geq t\}}$ yields

$$\frac{1}{\omega_{q-\gamma}} \int_{G(n,q)} \int_{G(n,n-q+\gamma)} \int_{S^{n-1} \cap M^\perp} 1_{\{d(o,(M+hu) \cap L) \geq t\}} \, \mathcal{H}^{q-\gamma-1}(du) \, \nu_{n-q+\gamma}(dM) \nu_q(dL)$$

$$= D(n,q,\gamma) \frac{\omega_{n-\gamma}}{\omega_{q-\gamma}} h^{\gamma+1} \int_h^\infty 1_{\{r \geq t\}} r^{-(\gamma+2)} \left(1 - \frac{h^2}{r^2}\right)^{\frac{n-q}{2}-1} dr \qquad (C.19)$$

for almost all $h > 0$. Using monotone convergence, one can show that both sides of (C.19) are continuous in h, so this relation extends to all $h > 0$. Letting $h = 1$, we see that the tail distributions of $d(o, E \cap L)$ are proportional to

$$\int_t^\infty r^{-(\gamma+2)} \left(1 - \frac{1}{r^2}\right)^{\frac{n-q}{2}-1} dr, \qquad t > 1.$$

A comparison with (C.18) shows the first claim. The explicit density $g_{n,q,\gamma}$ is obtained by differentiation of (C.19) when $h = 1$. ∎

Let us remark in this context that random affine subspaces with the property that their squared distance to the origin follows a beta distribution arise naturally in the theory of random beta polytopes. For example, let $X_0, \ldots, X_r$ with $r \in \{1, \ldots, n\}$ be stochastically independent random points in B^n with probability density proportional to $(1 - \|x\|^2)^{\frac{\nu-2}{2}}$ for some $\nu > 0$, and consider the random variable $d(o, M)^2$, where $M \in A(n, r)$ is the affine hull of $X_0, \ldots, X_r$. Then it follows from [35, Thm. 2.7] that this random variable has the beta distribution $\text{Beta}(\frac{n-r}{2}, \frac{\nu(r+1)+r(n-1)}{2})$.

From the explicit density of $d(o, E \cap L)$ in Theorem C.12, the existence of moments can directly be read off.

COROLLARY C.13. *Let $\alpha \in \mathbb{R}$. Under the assumptions of Theorem C.12, we have $\mathbb{E}d(o, E \cap L)^\alpha < \infty$ if and only if $\alpha < \gamma + 1$. In particular, if $\gamma = 0$ then the random variable $d(o, E \cap L)$ has infinite expectation. If $\gamma \geq 1$ we have*

$$\mathbb{E}d(o, E \cap L) = \frac{\omega_{\gamma+1} \omega_{n-q+\gamma}}{\omega_\gamma \omega_{n-(q-\gamma)+1}} = \frac{\omega_{n-q+\gamma}}{\omega_\gamma} (2\pi)^{-(n-q)}.$$

Applying Stirling's formula [1, Section 6.1.37] we conclude that for fixed $\gamma \geq 1$,

$$\frac{\mathbb{E}d(o, E \cap L)}{\sqrt{n}} \longrightarrow \frac{1}{\sqrt{2\pi}} \frac{\omega_{\gamma+1}}{\omega_\gamma},$$

as $n \to \infty$, independently of q.

C.7 PROOFS OF THEOREM C.3, COROLLARY C.6 AND THEOREM C.7

Proof of Theorem C.3. Given $E \in A(n,q)$, $q \in \{0, \ldots, n\}$, we will write $\text{lin}(E)$ for the linear subspace in $G(n,q)$ parallel to E. In other words, $\text{lin}(E) = E - x$ for all $x \in E$.

For the proof of Theorem C.3, fix $n \geq 1$, $q \in \{1, \ldots, n-1\}$ and $\gamma \in \{0, \ldots, q-1\}$, and let $f : A(n, \gamma) \to [0, \infty)$ be a measurable function. Due to Lemma C.1, we may assume without loss of generality that f is rotation invariant. The integral of interest is

$$I = \int_{G(n,q)} \int_{A(n,n-q+\gamma)} f(E \cap L) H(E) \, \mu_{n-q+\gamma}(dE) \, \nu_q(dL).$$

First note that I is indeed well-defined as for ν_q-almost all $L \in G(n, q)$ and $\mu_{n-q+\gamma}$-almost all $E \in A(n, n - q + \gamma)$ we have $\dim(E \cap L) = \gamma$. This follows from [75, Lem. 13.2.1], applied to L and $L' = \text{lin}(E)$.

Define the function $g : A(n, q) \times A(n, n - q + \gamma) \to \mathbb{R}$ by

$$g(E_1, E_2) = f\big(\text{lin}(E_1) \cap E_2\big) 1_{\{d(o, E_1) \leq 1\}} H(E_2),$$

where we let $f(\text{lin}(E_1) \cap E_2) = 0$ if $\dim(\text{lin}(E_1) \cap E_2) \neq \gamma$. Definition (C.7) implies

$$I = \frac{1}{\kappa_{n-q}} \int_{A(n,q)} \int_{A(n,n-q+\gamma)} g(E_1, E_2) \, \mu_{n-q+\gamma}(dE_2) \mu_q(dE_1).$$

Applying [75, Thm. 7.2.8] with $s_1 = q$ and $s_2 = n - q + \gamma$ yields

$$I = \frac{\bar{b}}{\kappa_{n-q}} \int_{A(n,\gamma)} \int_{A(E,q)} \int_{A(E,n-q+\gamma)} g(E_1, E_2) [\text{lin}(E_1), \text{lin}(E_2)]^{\gamma+1}$$
$$\times \mu_{n-q+\gamma}^E(dE_2) \, \mu_q^E(dE_1) \, \mu_\gamma(dE), \tag{C.20}$$

with the constant $\bar{b}$ given by

$$\bar{b} = b_{n,n-\gamma} \frac{b_{n-\gamma,n-q} b_{n-\gamma,q-\gamma}}{b_{n,n-q} b_{n,q-\gamma}}, \qquad b_{i,j} = \frac{\omega_{i-j+1} \cdots \omega_i}{\omega_1 \cdots \omega_j}, \quad i \in \mathbb{N}, j \in \{1, \ldots, i\}. \tag{C.21}$$

Expanding (C.20) by decomposing the measure μ_γ according to (C.7), applying [75, Eq. (13.14)] and Tonelli's theorem yields

$$I = \frac{\bar{b}}{\kappa_{n-q}} \int_{G(n,\gamma)} I_1(L_0, f) \, \nu_\gamma(dL_0) \tag{C.22}$$

with

$$I_1(L_0, f) = \int_{G(L_0,q)} \int_{G(L_0,n-q+\gamma)} \int_{L_0^\perp} f((M+t) \cap L) 1_{\{d(o, L+t) \leq 1\}} H(M+t)$$
$$\times \lambda_{L_0^\perp}(dt) [M, L]^{\gamma+1} \nu_{n-q+\gamma}^{L_0}(dM) \nu_q^{L_0}(dL).$$

Since

$$\int_{G(L_0,p)} h(L) \, \nu_p^{L_0}(dL) = \int_{G(L_0^\perp,p-\gamma)} h(L_0 + L) \, \nu_{p-\gamma}^{L_0^\perp}(dL)$$

for all $L_0 \in G(n, \gamma)$ and for any measurable function $h : G(L_0, p) \to [0, \infty)$ and integers $\gamma < p < n$, we conclude

$$I_1(L_0, f) = \int_{G(L_0^\perp,q-\gamma)} \int_{G(L_0^\perp,n-q)} \int_{L_0^\perp} f\Big(\big((M+t) \cap L\big) + L_0\Big) 1_{\{d(o, L+t) \leq 1\}}$$
$$\times H(M + L_0 + t) \lambda_{L_0^\perp}(dt) [M, L]^{\gamma+1} \nu_{n-q}^{L_0^\perp}(dM) \nu_{q-\gamma}^{L_0^\perp}(dL), \tag{C.23}$$

where we also have used $(M + L_0 + t) \cap (L + L_0) = \big((M + t) \cap L\big) + L_0$, $d(o, L + L_0 + t) = d(o, L + t)$ and $[M + L_0, L + L_0] = [M, L]$.

Now consider (C.23) for fixed $L_0 \in G(n, \gamma)$, $M \in G(L_0^\perp, n - q)$ and $L \in G(L_0^\perp, q - \gamma)$. As f and H are both rotation invariant, we may write

$$f\big(((M + t) \cap L) + L_0\big) = f_I\big(d(o, (M + t) \cap L)\big),$$
$$H(M + L_0 + t) = H_I(\|t|M^\perp\|),$$

with f_I and H_I satisfying (C.9). Thus, we may identify $L_0^\perp$ with $\mathbb{R}^{n-\gamma}$ and conclude that

$$I_1(L_0, f) = \int_{G(n-\gamma, q-\gamma)} \int_{G(n-\gamma, n-q)} I_2(L, M)[M, L]^{\gamma+1} \nu_{n-q}(dM) \, \nu_{q-\gamma}(dL), \tag{C.24}$$

with

$$I_2(M, L) = \int_{\mathbb{R}^{n-\gamma}} f_I(d(o, (M + t) \cap L)) 1_{\{d(o, L+t) \leq 1\}} H_I(\|t|M^\perp\|) \, \lambda_{n-\gamma}(dt)$$
$$= \int_M \int_{M^\perp} f_I(d(o, (M + x) \cap L)) H_I(\|x\|) 1_{\{d(o, L+x+y) \leq 1\}} \, \lambda_{M^\perp}(dx) \, \lambda_M(dy)$$

for $M \in G(n - \gamma, n - q)$ and $L \in G(n - \gamma, q - \gamma)$. Applying (C.8) to the Lebesgue integral over $M^\perp$ we get

$$I_2(M, L) = [M, L] \int_M \int_L f_I(d(o, (M + t) \cap L)) H_I(\|t|M^\perp\|) 1_{\{d(o, L+t|M^\perp+y) \leq 1\}} \, \lambda_L(dt) \, \lambda_M(dy)$$
$$= \int_L f_I(d(o, (M + t) \cap L)) H_I(\|t|M^\perp\|) I_3(M, L, t) \, \lambda_L(dt),$$

where we first used the fact that $M + t|M^\perp = M + t$ holds for any $t \in L$, and then Tonelli's theorem. Here,

$$I_3(M, L, t) = [M, L] \int_M 1_{\{\|(t|M^\perp)|L^\perp + y|L^\perp\| \leq 1\}} \, \lambda_M(dy).$$

Another application of (C.8) reveals that

$$I_3(M, L, t) = \int_{L^\perp} 1_{\{\|(t|M^\perp)|L^\perp + z\| \leq 1\}} \, \lambda_{L^\perp}(dz) = \kappa_{n-q}$$

is the volume of a unit ball in $L^\perp$, centered at $(t|M^\perp)|L^\perp$. Inserting this into I_2 gives

$$I_2(M, L) = \kappa_{n-q} \int_L f_I(d(o, (M + t) \cap L)) H_I(\|t|M^\perp\|) \, \lambda_L(dt).$$

Since $t \in L$ and $M \cap L = \{o\}$ for almost all L and M, we have

$$d(o, (M + t) \cap L) = d(o, \{t\}) = \|t\|.$$

This, and the use of spherical coordinates in L give

$$I_2(M, L) = \kappa_{n-q} \int_0^\infty f_I(r) r^{q-\gamma-1} \int_{S^{n-\gamma-1} \cap L} H_I(r[u, M]) \, \mathcal{H}^{q-\gamma-1}(du) \, dr.$$

Inserting this into (C.24) and the result into (C.22), we get after an application of Tonelli's theorem that

$$I = \bar{b} \int_0^\infty f_I(r) r^{q-\gamma-1} \int_{G(n-\gamma, n-q)} J_H(M, r) \, \nu_{n-q}(dM) \, dr, \tag{C.25}$$

with

$$J_H(M, r) = \int_{G(n-\gamma, q-\gamma)} \int_{S^{n-\gamma-1} \cap L} H_I(r[u, M]) \mathcal{H}^{q-\gamma-1}(du) [M, L]^{\gamma+1} \, \nu_{q-\gamma}(dL).$$

An invariance argument and [75, Thm. 7.1.1] imply

$$J_H(M, r) = \frac{\omega_{q-\gamma}}{\omega_{n-\gamma}} \int_{S^{n-\gamma-1}} H_I(r[u, M]) \int_{G(\mathrm{span}\, u, q-\gamma)} [M, L]^{\gamma+1} \, \nu_{q-\gamma}^{\mathrm{span}\, u}(dL) \, \mathcal{H}^{n-\gamma-1}(du).$$

Applying Lemma C.2 in $\mathbb{R}^{n-\gamma}$ to the innermost integral in $J_H(M, r)$ yields

$$J_H(M, r) = c_1 \int_{S^{n-\gamma-1}} H_I(r[u, M]) [u, M]^{\gamma+1} \, \mathcal{H}^{n-\gamma-1}(du)$$

with the constant $c_1 = a(n-\gamma, n-q, q-\gamma, \gamma+1) \frac{\omega_{q-\gamma}}{\omega_{n-\gamma}}$. To simplify $J_H(M, r)$ further, we use the fact that $[u, M] = \|u|M^\perp\|$ and apply [3, Lem. 1] with $B_p = M^\perp$, $p = q - \gamma$ and $d = n - \gamma$. This yields

$$J_H(M, r) = \frac{c_1}{2} \omega_{n-q} \omega_{q-\gamma} \int_0^1 H_I(rt^{\frac{1}{2}}) t^{\frac{q-1}{2}} (1-t)^{\frac{n-q}{2}-1} \, dt$$

$$= c_1 \omega_{n-q} \omega_{q-\gamma} \int_0^1 H_I(rz) z^q (1-z^2)^{\frac{n-q}{2}-1} \, dz,$$

using the substitution $z = \sqrt{t}$ in the last step. A comparison with (C.13) gives $J_H(M, r) = c_1 \omega_{n-q} \omega_{q-\gamma} J_H(r)$, so abbreviating

$$c_2 = \bar{b} c_1 \frac{\omega_{n-q} \omega_{q-\gamma}}{\omega_{n-\gamma}},$$

(C.25) becomes

$$I = c_2 \omega_{n-\gamma} \int_0^\infty f_I(r) r^{q-\gamma-1} J_H(r) \, dr$$

$$= c_2 \int_{A(n, \gamma)} f(E) d(o, E)^{-(n-q)} J_H(d(o, E)) \, \mu_\gamma(dE),$$

where we used (C.7) and spherical coordinates in $\mathrm{lin}(E)$. Hence, the theorem is shown once we have confirmed that

$$c_2 = D(n, q, \gamma). \tag{C.26}$$

We have

$$c_1 = \frac{\omega_{q-\gamma}}{\omega_{n-\gamma}} \prod_{i=1}^{n-q} \frac{\Gamma(\frac{n-\gamma-i}{2}) \Gamma(\frac{n-q-i+\gamma}{2}+1)}{\Gamma(\frac{n-i+1}{2}) \Gamma(\frac{n-q-i+1}{2})} = \frac{\omega_{q-\gamma}}{\omega_{n-\gamma}} \prod_{i=1}^{n-q} \frac{\omega_{n-i+1} \omega_{n-q-i+1}}{\omega_{n-\gamma-i} \omega_{n-q+\gamma+2-i}},$$

so direct insertion gives

$$\bar{b}c_1 = \left(\frac{\omega_{\gamma+1} \cdots \omega_n}{\omega_1 \cdots \omega_{n-\gamma}} \cdot \frac{\omega_{q-\gamma+1} \cdots \omega_{n-\gamma}}{\omega_{q+1} \cdots \omega_n} \cdot \frac{\omega_{n-q+1} \cdots \omega_{n-\gamma}}{\omega_{n-(q-\gamma)+1} \cdots \omega_n} \right)$$

$$\times \left(\frac{\omega_{q+1} \cdots \omega_n}{\omega_{q-\gamma} \cdots \omega_{n-\gamma-1}} \cdot \frac{\omega_1 \cdots \omega_{n-q}}{\omega_{\gamma+2} \cdots \omega_{n-(q-\gamma)+1}} \right) \frac{\omega_{q-\gamma}}{\omega_{n-\gamma}}$$

$$= \left(\frac{\omega_{q+1} \cdots \omega_n}{\omega_{q+1} \cdots \omega_n} \cdot \frac{\omega_{\gamma+1} \cdots \omega_n}{\omega_{\gamma+2} \cdots \omega_n} \cdot \frac{1}{\omega_{n-(q-\gamma)+1}} \right)$$

$$\times \left(\frac{\omega_{q-\gamma} \cdots \omega_{n-\gamma}}{\omega_{q-\gamma} \cdots \omega_{n-\gamma}} \cdot \frac{\omega_1 \cdots \omega_{n-\gamma}}{\omega_1 \cdots \omega_{n-\gamma}} \right)$$

$$= \frac{\omega_{\gamma+1}}{\omega_{n-(q-\gamma)+1}},$$

where the products were suitably sorted at the second equality sign, and simplified at the third equality sign. We thus get

$$c_2 = \frac{\omega_{\gamma+1} \omega_{q-\gamma} \omega_{n-q}}{\omega_{n-(q-\gamma)-1} \omega_{n-\gamma}} = D(n, q, \gamma).$$

This shows (C.26) and completes the proof. ∎

Proof of Corollary C.6. For a Borel set $B \subseteq G(n, q)$, define

$$\tilde{\nu}_q(B) = \int_{G(n,q_1)} \cdots \int_{G(n,q_\ell)} 1_B(L_1 \cap \ldots \cap L_\ell) \, \nu_{q_\ell}(dL_\ell) \ldots \nu_{q_1}(dL_1).$$

This gives rise to an invariant probability measure $\tilde{\nu}_q$ on $G(n, q)$. However, by [75, Thm. 13.2.11] there is only one such measure, which implies that $\tilde{\nu}_q = \nu_q$. This allows us in a first step to reduce the outer integral over $G(n, q_1), \ldots, G(n, q_m)$ in $I_{\ell,m}$ to a single integral over $G(n, q)$:

$$I_{\ell,m} = \int_{G(n,q)} \int_{A(n,p_1)} \cdots \int_{A(n,p_m)} f(E_1 \cap \ldots \cap E_m \cap L)$$

$$\times H(E_1 \cap \ldots \cap E_m) \, \mu_{p_m}(dE_m) \ldots \mu_{p_1}(dE_1) \nu_q(dL). \tag{C.27}$$

To also convert the inner integrals over $A(n, p_1), \ldots, A(n, p_m)$ to a single integral, let B be a Borel set in $A(n, n - q + \gamma)$ and define

$$\widehat{\mu}_{n-q+\gamma}(B) = \int_{A(n,p_1)} \cdots \int_{A(n,p_m)} 1_B(E_1 \cap \ldots \cap E_m) \, \mu_{p_m}(dE_m) \ldots \mu_{p_1}(dE_1).$$

The measure $\widehat{\mu}_{n-q+\gamma}$ is motion invariant on $A(n, n - q + \gamma)$. However, by [75, Thm. 13.1.3 and Thm. 13.2.12] all such measures are constant multiples of the invariant measure $\mu_{n-q+\gamma}$, so there exists some constant $c > 0$ such that $\widehat{\mu}_{n-q+\gamma} = c\mu_{n-q+\gamma}$. To determine the value of the constant c, we employ a special case of the Crofton formula [75, Thm. 5.1.1]. It says that

$$\int_{A(n,k)} \mathcal{H}^i(E \cap W) \, \mu_k(dE) = \frac{\omega_{n+1} \omega_{i+1}}{\omega_{k+1} \omega_{n-k+i+1}} \mathcal{H}^{n-k+i}(W)$$

for $0 \leq i \leq k \leq n-1$ and where $W \subset \mathbb{R}^n$ is a convex set of dimension $n-k+i$, that is, the affine hull of W has dimension $n-k+i$. Applying Crofton's formula with $i = k = n - q + \gamma$ and $W = B^n$ we obtain

$$\int_{A(n,n-q+\gamma)} \mathcal{H}^{n-q+\gamma}(E \cap B^n)\, \widehat{\mu}_{n-q+\gamma}(dE) = c \int_{A(n,n-q+\gamma)} \mathcal{H}^{n-q+\gamma}(E \cap B^n)\, \mu_{n-q+\gamma}(dE)$$
$$= c\mathcal{H}^n(B^n) = c\kappa_n.$$

On the other hand, by applying Crofton's formula repeatedly to each of the integrals in its definition, the integral on the left is

$$\int_{A(n,p_1)} \cdots \int_{A(n,p_m)} \mathcal{H}^{n-q+\gamma}(E_1 \cap \ldots \cap E_m \cap B^n)\, \mu_{p_m}(dE_m) \ldots \mu_{p_1}(dE_1)$$
$$= \frac{\omega_{n+1}\omega_{n-q+\gamma+1}}{\omega_{p_m+1}\omega_{2n-p_m-q+\gamma+1}} \int_{A(n,p_1)} \cdots \int_{A(n,p_{m-1})} \mathcal{H}^{2n-p_m-q+\gamma}(E_1 \cap \ldots \cap E_{m-1} \cap B^n)$$
$$\mu_{p_{m-1}}(dE_{m-1}) \ldots \mu_{p_1}(dE_1)$$

$$\vdots$$

$$= \frac{\omega_{n+1}\omega_{n-q+\gamma+1}}{\omega_{p_m+1}\omega_{2n-p_m-q+\gamma+1}} \frac{\omega_{n+1}\omega_{2n-p_m-q+\gamma}}{\omega_{p_{m-1}}\omega_{3n-p_m-p_{m-1}-q+\gamma+1}} \cdots$$
$$\cdots \frac{\omega_{n+1}\omega_{mn-p_m-\ldots-p_2-q+\gamma}}{\omega_{p_1+1}\omega_{n-p_1+mn-p_m-\ldots-p_2-q+\gamma+1}} \mathcal{H}^n(B^n)$$
$$= \frac{\omega_{n+1}^m \omega_{n-q+\gamma+1}}{\omega_{p_1} \cdots \omega_{p_m} \omega_{n+1}} \kappa_n.$$

Here, we simplified the telescopic product of the ω-terms and used our assumption $p_1 + \ldots + p_m - (m-1)n = n - 1 + \gamma$ in the last step. A comparison of these two expressions implies

$$\widehat{\mu}_{n-q+\gamma} = \frac{\omega_{n+1}^m \omega_{n-q+\gamma+1}}{\omega_{p_1} \cdots \omega_{p_m} \omega_{n+1}} \mu_{n-q+\gamma}.$$

As a consequence, we can reduce the inner integrals in (C.27) to a single integral over $A(n, n-q+\gamma)$:

$$I_{\ell,m} = \frac{\omega_{n+1}^m \omega_{n-q+\gamma+1}}{\omega_{p_1} \cdots \omega_{p_m} \omega_{n+1}} \int_{G(n,q)} \int_{A(n,n-q+\gamma)} f(E \cap L) H(E)\, \mu_{n-q+\gamma}(dE) \nu_q(dL).$$

The result of Corollary C.6 can now be concluded from Theorem C.3. ∎

The proof that the results of Theorems C.3 and C.7 are equivalent, is based on the following Blaschke–Petkantschin formula:

$$\int_{A(n,k)} g(E)\mu_k(dE) = \frac{\omega_{n-k}}{\omega_{r-k}} \int_{G(n,r)} \int_{A(L,k)} g(E)d(o,E)^{n-r} \mu_k^L(dE)\nu_r(dL), \qquad \text{(C.28)}$$

with integers $0 \leq k < r < n$ and measurable $g : A(n,k) \to [0, \infty)$, see e.g. [50, p. 33].

Proof of the equivalence of Theorems C.3 and C.7. That Theorem C.7 implies Theorem C.3 follows rather directly by invariant integration of (C.16) with respect to L_0 and an application of the Blaschke–Petkantschin formula (C.28) with $r = q$ and $k = \gamma$.

Assume now that Theorem C.3 holds, and let $f : A(L_0, \gamma) \to [0, \infty)$ be measurable. Assume first that f is rotation invariant under all rotations fixing L_0. The function $\tilde{f}(E) = f_I(d(o, E))$, $E \in A(n, \gamma)$, is the rotation invariant extension of f to $A(n, \gamma)$. Hence, the invariance of $\mu_{n-q+\gamma}$ gives

$$\int_{G(n,q)} \int_{A(n,n-q+\gamma)} \tilde{f}(E \cap L) H(E)\, \mu_{n-q+\gamma}(dE)\, \nu_q(dL)$$
$$= \int_{SO(n)} \int_{A(n,n-q+\gamma)} \tilde{f}(E \cap \vartheta L_0) H(E)\, \mu_{n-q+\gamma}(dE)\, \nu(d\vartheta)$$
$$= \int_{SO(n)} \int_{A(n,n-q+\gamma)} \tilde{f}\big(\vartheta(E \cap L_0)\big) H(\vartheta^{-1} E)\, \mu_{n-q+\gamma}(dE)\, \nu(d\vartheta)$$
$$= \int_{A(n,n-q+\gamma)} f(E \cap L_0) H(E)\, \mu_{n-q+\gamma}(dE).$$

On the other hand, the Blaschke–Petkantschin relation (C.28) and a similar reasoning shows

$$\int_{A(n,\gamma)} \tilde{f}(E) d(o, E)^{-(n-q)} J_H(d(o, E))\, \mu_\gamma(dE) = \frac{\omega_{n-\gamma}}{\omega_{q-\gamma}} \int_{A(L_0,\gamma)} f(E) J_H(d(o, E))\, \mu_\gamma^{L_0}(dE).$$

Theorem C.3 states that these two displayed expressions coincide up to multiplication with the constant $D(n, q, \gamma)$, and thus (C.16) holds for the function f chosen.

With arguments as in the proof of Lemma C.1, one can show that (C.16) holds for all measurable $f : A(L_0, \gamma) \to [0, \infty)$ if it holds for all such functions which are in addition invariant under all rotations fixing L_0. This proves that Theorem C.7 holds and concludes the proof of the equivalence. ∎

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