Irrationality, transcendence, and other subjects in number theory

Mathias Løkkegaard Laursen

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Supervised by Simon Kristensen

Department of Mathematics
Aarhus University
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Abstract

In English

The main part of the thesis provides new criteria ensuring irrationality, transcendence, linear independence over a field, or algebraic independence for numbers expressed as infinite series $\sum_{n=1}^{\infty} 1/a_n$ that are generated by sequences $\{a_n\}_{n=1}^{\infty}$ of algebraic numbers containing a subsequence of sufficiently rapidly increasing modulus. Using similar methods, the thesis also provides new linear independence criteria for numbers expressed as continued fractions $[0; a_1, a_2, \ldots]$ and new irrationality and transcendence criteria for numbers expressed as infinite products $\prod_{n=1}^{\infty}(1 + 1/a_n)$ or as infinite products of infinite series $\prod_{m=1}^{\infty}\left(1 + \sum_{n=1}^{\infty} 1/a_{m,n}\right)$, again generated by sequences $\{a_n\}_{n=1}^{\infty}$ or $\{a_{m,n}\}_{n=1}^{\infty}$ of algebraic numbers containing a subsequence of sufficiently rapidly increasing modulus. All proofs apply a method originally developed by Erdős. The results are compared to related notions of irrationality, transcendence, linear independence over a field, and algebraic independence of sequences rather than numbers.

The thesis also contains two smaller chapters related to different subfields of number theory. The first of these settles two conjectures regarding which values are possible as the measure of certain p-adic sets. In the first of these conjectures, the sets originate as a p-adic variant to those originally considered in the famous Duffin–Schaeffer Conjecture, while the latter conjecture considers related sets based on a slightly different Diophantine inequality.

The final chapter provides an asymptotic equivalence for the number of integer partitions over the Fibonacci numbers or over some other strictly increasing linearly recurrent sequence where the associated characteristic polynomial satisfies certain mild conditions.

In Danish

I hovedparten af denne afhandling gives nye kriterier, der sikrer irrationalitet, transcendens, lineær uafhængighed over et legeme eller algebraisk uafhængighed for tal skrevet som uendelige rækker $\sum_{n=1}^{\infty} 1/a_n$, der er genereret af følger $\{a_n\}_{n=1}^{\infty}$ af algebraiske tal indeholdende en delfølge af tilpas hurtigt voksende modulus. Gennem lignende metoder giver afhandlingen også nye kriterier for lineær uafhængighed af tal skrevet som kædebrøker $[0; a_1, a_2, \ldots]$ samt irrationalitets- og transcendenskriterier for tal skrevet som uendelige produkter $\prod_{n=1}^{\infty}(1 + 1/a_n)$ eller som uendelige produkter af uendelige rækker $\prod_{m=1}^{\infty} \left(1 + \sum_{n=1}^{\infty} 1/a_{m,n}\right)$, igen genereret af følger $\{a_n\}_{n=1}^{\infty}$ eller $\{a_{m,n}\}_{n=1}^{\infty}$ af algebraiske tal indeholdende en delfølge af tilpas hurtigt voksende modulus. Beviserne anvender alle en metode, som oprindeligt er udviklet af Erdős. Resultaterne sammenholdes med relaterede begreber om irrationalitet, transcendens, lineær uafhængighed og algebraisk uafhængighed af følger frem for tal.

Afhandlingen indeholder også to mindre kapitler om andre emner inden for talteori. I det første besvares formodninger om hvilke værdier, der er mulige som målet på visse p-adiske mængder. I det første af disse formodninger opstår disse mængder som p-adiske pendanter til mængderne fra den berømte Duffin–Schaefferformodning, mens den anden formodning omhandler relaterede mængder, hvor den underliggende diofantiske ulighed er delvist ændret.

Det sidste kapitel giver en asymptotisk ækvivalens for antallet af heltalspartitioner over fibonaccitallene eller over en anden strengt voksende lineært rekursiv følge, hvis karakteristiske polynomium overholder visse milde betingelser.

Preface

This PhD thesis is split into three chapters. While all chapters are entirely within the field of number theory, they belong to different branches that have little to do with each other. For this reason, they may be read independently and in any order, with each chapter having a separate introduction. **The only exceptions are the notions of irrationality and transcendence, which are used to motivate Chapter 2, and of conjugates, which are used in Chapter 3, relying on definitions and – in the motivation of Chapter 2 – classical results introduced early in Chapter 1.**

ALT 0 The only exceptions are the notions of irrationality and transcendence, which are used to motivate Chapter 2, and of conjugates, which are used in Chapter 3. These notions rely on definitions and – in the motivation of Chapter 2 – classical results introduced early in Chapter 1.

ALT 1 The only exceptions are the notions of irrationality and transcendence, which are used to motivate Chapter 2, and of conjugates, which are used in Chapter 3, with each of these notions being introduced in Chapter 1.

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ALT 3 The only exceptions are the notions of irrationality, transcendence, and conjugates, which are introduced in Chapter 1, with Chapter 2 using classical results on irrationality and transcendence as part of its motivation and Chapter 3 briefly using conjugates to help formulate the assumptions for one of its main results.

ALT 4 The only exceptions are the notions of irrationality, transcendence, and conjugates, which are introduced in Chapter 1; Chapter 2 uses classical irrationality and transcendence criteria as part of its motivation, and Chapter 3 uses the notion of conjugates to help formulate the assumptions for one of its main results.

Each chapter presents a number of new results by the current author. Some of these results were proven in collaboration with others [9, 23, 24, 35], while some results were not [36–39]. The preprints are all freely accessible on the distribution service arXiv.org, henceforth referred to simply as arXiv. While arXiv does not perform peer-review, all papers have been submitted to scientific journals that do, and the preprints have been updated to reflect any significant changes. At the time of writing, the three papers [36–38] have been published while the other five are currently under review.

While the papers are aimed at experts within their respective fields, this author has strived toward making the rest of the thesis accessible to as broad an audience as possible. This is done towards the ideal that scientific work should be as easily accessible to the public as possible. Given that most mathematical research requires a fairly high amount of specialization, the possible availability can be rather limited, but it is the hope of this author that the present thesis will be understandable to most graduates in mathematics. Particular efforts have been made to make the introductions of Chapters 1 and 3 approachable. Consequently, these parts of the thesis are estimated to require the least mathematical background, and it is the hope that at least the first page of these chapters will make sense even to non-mathematicians.

Chapter 1 takes up the far majority of the thesis and concerns itself with irrationality, transcendence, linear independence, and – to some degree – algebraic independence of numbers. The chapter contains six out of the eight papers mentioned above. The results are also compared to corresponding but less broadly known notions of irrationality, transcendence, linear independence over a field, and algebraic independence for sequences. All six papers use an analytical method originally developed by Erdős [11], which is named the *Erdős Jump* by the current author. In each paper, the method has been generalized from former versions to fit the problem at hand, though the paper [39] entirely reuses the Erdős Jump from [38]. The chapter is divided into the following seven sections.

- Section 1.1 introduces the reader to the field of study.
- Section 1.2 then introduces the Erdős Jump in greater detail and presents a conjecture that aims to capture all generalizations present in the papers [23, 24, 35, 36, 38, 39].
- Section 1.3 introduces some algebraic number theory that was deemed too involved for the introduction but that is relevant for the subsequent sections.
- Section 1.4 introduces and compares the papers [36] and [35], which provide irrationality criteria for infinite series and infinite products, respectively, of algebraic numbers. The methods of proof in the two papers are highly similar, though the method is improved in the latter paper, [35], which thereby achieves a stronger result. We then use this improvement to strengthen the main result of [36] to a result that has not been published prior to this thesis.

- In continuation hereof, Section 1.5 presents the paper [24], which provides criteria for linear independence of continued fractions generated by sequences of algebraic numbers but using a different algebraic method from that in the preceding section.
- Section 1.6 then presents the papers [38] and [39], which provide transcendence criteria for infinite series and infinite products, respectively, of algebraic numbers. Again, the methods of proof are highly similar and were improved in the latter paper, leading to more relaxed and slightly simpler criteria.
- Finally, Section 1.7 presents the paper [23], which provides criteria for algebraic independence. The paper also introduces new notions for irrationality and transcendence of sequences, which we generalize to notions of linear independence over a field and algebraic independence.

Chapter 2 is concerned with p-adic variants of the famous theorem formerly known as the Duffin–Schaeffer Conjecture and presents the paper [37] by the current author. This chapter assumes a basic understanding of measure theory and analytical completions, both being fundamental to the field and too involved to introduce in the present text. The chapter is divided into the following two sections.

- Section 2.1 introduces the reader to the field as well as two newer conjectures regarding which measure values are attainable for certain p-adic variants of the sets considered in the Duffin–Schaeffer Conjecture. One of these new conjectures was introduced in a joint work by Simon Kristensen and the current author [34], which was submitted for publication before this author started his PhD studies. For this reason, this paper is not presented in its entirety and is only to be considered as context for the subsequent section.
- Section 2.2 introduces the paper [37], which settles the two new conjectures mentioned above.

Chapter 3 is concerned with partition functions with a particular interest in the partitions over the Fibonacci numbers. It is divided into the following two sections.

- Section 3.1 introduces the reader to the field of study.
- Section 3.2 introduces the paper [9], which provides asymptotic equivalences for the partition functions over the Fibonacci sequence and a broad family of linearly recurrent sequences.

Unless otherwise stated, this thesis uses standard mathematical notation. Since some notation is used differently among different mathematicians, let us briefly cover some of the most basic notation that is subject to various interpretations. In this text, $\mathbb{N}$ is used to denote the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is used to denote the non-negative integers. Secondly, we will use (a, b) only to denote open intervals or pairs of mathematical objects (such as integers), denoting the greatest common divisor of two integers by $\gcd(a, b)$.[1] Finally, log denotes the natural logarithm while $\log_2$ is used for the logarithm to base 2.

[1]In the literature, many authors write (a, b) for the greatest common divisor. This likely originates from the study of ideals in ring theory, where the ideal generated by a and b is commonly denoted (a, b) and is equal to the ideal generated by $\gcd(a, b)$.

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Contents

Chapter 1

Irrational and Transcendental Sequences

1.1 Introduction

As the story goes, the ancient Greeks long believed that any conceivable number could be written as the fraction of two positive integers. It was therefore not received well when it was proven that no such fraction exists for something as simple as the diagonal of the square with side length 1, also known as $\sqrt{2}$, making this the first irrational number. Legend has it that the gods in their fury summoned a storm while the culprit behind the proof was travelling by sea, thus capsizing the ship and drowning him. Fortunately, it seems, the gods have calmed down and no longer punish mathematicians who dabble in the dark arts of irrationality. Not even working with transcendence – a special kind of irrationality, which we will define below – appears to be of any great risk, even though the Greek gods must have found this even more despicable than mere irrationality. The first proof of the existence of transcendental numbers was made in 1844 by Joseph Liouville [40], while Charles Hermite [28] was the first to prove an already famous number to be transcendental when he did so for Euler's number $e = \sum_{n=0}^{\infty} 1/n!$ in 1873.[1] Even worse was it when Georg Cantor [7] in 1874 proved that nearly all real numbers have to be transcendental. Given that all three of them lived several decades after publishing their respective results and they each reached an age of more than 70 years, it should be of no particular risk for the reader to continue reading the present chapter about irrationality and transcendence.

Before moving ahead, let us first take a step back and consider the definitions of rational, irrational, algebraic, and transcendental numbers. Recall that a real number a is *rational* if it there exists a positive integer q so that the resulting

[1] Hermit's proof was later modified by Lindemann to prove that also π is transcendental.

number $p = aq$ is again an integer, and we say that a is *irrational* if this cannot be done. In other words, a is rational if it solves an equation of the form $qx - p = 0$ where p and q are integers with $q > 0$. Seeing how $qx - p$ is a polynomial of degree 1, a natural step for broadening the set of rational numbers is to consider more general polynomial equations, such as

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_d x^d = 0, \tag{1.1}$$

where d is a positive integer, and the coefficients $c_0, c_1, \ldots, c_d$ are integers with $c_d > 0$. We then say that a real or complex number a is *algebraic* if it solves such an equation, and we then define the *degree* of a, denoted $\deg a$, as the smallest d where this is possible. If a solves equation (1.1) with $c_d = 1$, then we also say that a is an *algebraic integer*. We use the symbol $\overline{\mathbb{Q}}$ to denote the set of all algebraic numbers. $\sqrt{2}$, which solves the equation $x^2 - 2 = 0$, is an example of an algebraic integer that is also irrational. If a real or complex number is not algebraic, then we say that it is *transcendental* and that its degree is ∞. This makes the transcendental numbers are a special kind of irrational numbers.

It is a well-known fact in number theory that if a is a rational number, then there is a constant $C > 0$ that depends only on a so that

$$\left| a - \frac{p}{q} \right| \geq \frac{C}{q}, \tag{1.2}$$

for all integers p and q with $q > 0$ and $p/q \neq a$. Hence, a number is automatically irrational if no such C exists. Another way to determine irrationality is in the form of continued fractions. Let $\{a_n\}_{n=0}^{\infty}$ a sequence of be real or complex numbers with $a_n \neq 0$ for all $n \geq 1$, and let N be a positive integer. We then define the *finite continued fraction* generated by $a_0, a_1, \ldots, a_N$ as

$$[a_0; a_1, a_2, \ldots, a_N] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_N}}}}$$

when $[a_N;] \neq 0, [a_{N-1}; a_N] \neq 0, \ldots, [a_1; a_2, \ldots, a_N] \neq 0$, writing $[a_N;] = a_N$. In the case that $[a_0; a_1, a_2, \ldots, a_N]$ is well-defined for all large $N \in \mathbb{N}$ and converges for $N \to \infty$, we define the *infinite continued fraction* generated by $\{a_n\}_{n=0}^{\infty}$ as

$$[a_0; a_1, a_2, \ldots,] := \lim_{N\to\infty} [a_0; a_1, a_2, \ldots, a_N] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}},$$

If the a_n are all integers with $a_n > 0$ when $n > 0$, then the finite continued fractions are all well-defined and converge to a number $a = [a_0; a_1, a_2, \ldots]$. Furthermore, this a does not satisfy inequality (1.2) for any choice of C and is therefore irrational. In fact, the p, q used to disprove inequality (1.2) for a may be chosen among the finite continued fractions $[a_0; a_1, \ldots, a_N]$. See, e.g., [6] for a proof.

Let us now turn our attention to criteria that ensure transcendence instead. Liouville's construction of the 'first' transcendental numbers was done through a generalized version of inequality (1.2), which he proved in the same paper [40].

Theorem 1.1 (Liouville, 1844)**.** *Let a be an algebraic number of degree at most d. Then there is a constant $C > 0$, depending only on a, so that*

$$\left| a - \frac{p}{q} \right| \geq \frac{C}{q^d}$$

is true for all integers p and q with $q > 0$ and $p/q \neq a$.

From this result, it is easy to prove that the Liouville constant

$$L = \sum_{n=1}^{\infty} 10^{-n!} = \frac{1}{10} + \frac{1}{100} + \frac{1}{1,000,000} + \frac{1}{(1,000,000)^4} + \cdots$$

cannot be algebraic, by picking

$$\frac{p}{q} = \sum_{n=1}^{N} 10^{-n!} = \frac{1}{10} + \frac{1}{100} + \frac{1}{1,000,000} + \frac{1}{(1,000,000)^4} + \cdots + \frac{1}{10^{-N!}}$$

for various values of N and noticing

$$\left| L - \frac{p}{q} \right| < \frac{2}{q^{N+1}}.$$

In the time after Liouville presented this theorem, several improvements were made to the implied transcendence criterion. This culminated 111 years later with the following theorem due to Roth [46].

Theorem 1.2 (Roth, 1955)**.** *Let a be an irrational real algebraic number, and let $\delta > 0$. Then there is a constant $C > 0$, depending only on a and δ, so that*

$$\left| a - \frac{p}{q} \right| \geq \frac{C}{q^{2+\delta}}$$

is true for all integers p and q with $q > 0$.

One of the most remarkable properties of transcendental numbers is how incredibly difficult it can be to prove that a specific one of them is even irrational. A famous example of such difficulties is concerned with the Riemann zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad s > 1. \tag{1.3}$$

When $s > 1$ is an even integer, we know that $\zeta(s)$ is transcendental, and we have no reason to believe that this should not also be the case when $s > 1$ is an odd integer. However, while Apéry was able to prove that $\zeta(3)$ is irrational in 1979 [3], it remains an open question if it is transcendental. Even worse, irrationality (or rationality) has not yet been shown for any $\zeta(s)$ when $s > 3$ is a fixed odd integer.

The main focus of this chapter is on irrationality and transcendence criteria inspired by the below theorem due to Erdős from 1975. Specifically, we are interested in such results for infinite series $\sum_{n=1}^{\infty} 1/a_n$, infinite products $\prod_{n=1}^{\infty}(1+1/a_n)$, and continued fractions $[0; a_1, a_2, \ldots]$ where the numbers a_n are algebraic. We will also present criteria related to infinite product of infinite series $\prod_{m=1}^{\infty}(1+\sum_{n=1}^{\infty} 1/a_{n,m})$. While these criteria by no means ensure irrationality or transcendence for numbers such as $\zeta(s)$, they have the neat property that small alterations to the sequence of a_n are less likely to affect whether the criteria are satisfied. All of these results are inspired by and generalize the below theorem by Erdős [11].

Theorem 1.3 (Erdős, 1975)**.** *Let $\varepsilon > 0$, and let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence of integers such that $a_n \geq n^{1+\varepsilon}$ for all n. Suppose*

$$\limsup_{n\to\infty} a_n^{2^{-n}} = \infty.$$

Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of positive integers. Then the number $\sum_{n=1}^{\infty} 1/(a_n c_n)$ is irrational.

Like many other results on irrationality and transcendence of numbers, Erdős' proof may be split into two parts; an algebraic part and an analytical one. The algebraic part is rather simple. By a quick argument, it is sufficient to prove the theorem for $c_n = 1$. Then, pretending that $\sum_{n=1}^{\infty} 1/a_n$ is rational, one writes $q = \prod_{n=1}^{N} a_n$ and $p = q\sum_{n=1}^{N} /a_n$, so that inequality (1.2) implies

$$\left(\prod_{n=1}^{N} a_n\right) \sum_{n=N+1}^{\infty} \frac{1}{a_n} = \left(\prod_{n=1}^{N} a_n\right) \left| \sum_{n=1}^{\infty} \frac{1}{a_n} - \sum_{n=1}^{N} \frac{1}{a_n} \right| \leq C \tag{1.4}$$

for a constant $C > 0$ that depends only on $\sum_{n=1}^{\infty} /a_n$. In the analytical part of the proof, which is much more involved, Erdős proves that this inequality cannot be satisfied for all N under the given assumptions on a_n. We will give a more detailed sketch of this part of the proof in Section 1.2 as it plays a vital role for not only the theorems given below but also for the new results by the current author and his co-authors, as presented in Sections 1.4–1.7 .

Before considering further results, we will need a few more definitions. We start with the below definition of expressible sets, a term that appears to originally have been used in the below sense by Hančl, Nair, and Šustek [25] for infinite series. Later, Kolouch and Novotný [32] introduced a variant of the term for infinite products. The new notion for continued fractions follow this pattern.

Definition 1.4 (Expressible sets)**.** Let $(\{a_{1,n}\}_{n=1}^{\infty}, \ldots, \{a_{K,n}\}_{n=1}^{\infty})$ be a list of sequences of non-zero real or complex numbers. We then define the associated *expressible set of series* as

$$E_{\Sigma}\Big(\{a_{i,n}\}_{n=1}^{\infty}\Big)_{i=1}^{K} := \left\{ \left(\sum_{n=1}^{\infty} \frac{1}{a_{i,n}c_n}\right)_{i=1}^{K} : c_n \in \mathbb{N}\ \forall n \in \mathbb{N} \right\},$$

the associated *expressible set of products* as

$$E_{\Pi}\Big(\{a_{i,n}\}_{n=1}^{\infty}\Big)_{i=1}^{K} := \left\{ \left(\prod_{n=1}^{\infty} \left(1 + \frac{1}{a_{i,n}c_n}\right)\right)_{i=1}^{K} : c_n \in \mathbb{N}\ \forall n \in \mathbb{N} \right\},$$

and the associated *expressible set of continued fractions* as

$$E_{\mathrm{CF}}\Big(\{a_{i,n}\}_{n=1}^{\infty}\Big)_{i=1}^{K} := \Big\{ \big([0; a_{i,1}c_1, a_{i,2}c_2, \ldots]\big)_{i=1}^{K} : c_n \in \mathbb{N}\ \forall n \in \mathbb{N} \Big\}.$$

Our main use of the above definitions of expressible sets will be as a means to better introduce the following notions of irrationality, transcendence, and linear independence of sequences. These terms of irrationality and transcendence are slightly older than that of expressible sets and were coined by Hančl in the papers [15] and [16]. The term 'linear independence' was first used in the below sense in the paper [22] by Hančl, Korčeková, and Novotný and was called 'linear unrelatedness' in previous papers, starting with the paper [17] by Hančl. The below notion of $X_{\mathbb{K}}$-irrationality for arbitrary fields $\mathbb{K}$ is more recent and was introduced by the current author in [39] to get a more finely incremented terminology.

Definition 1.5. Let $\mathbb{K}$ be a field containing $\mathbb{Q}$, and let X be a placeholder for either one of the labels Σ, Π, and CF. We then say that a sequence $\{a_n\}_{n=1}^{\infty}$ of real or complex numbers is

- $X_{\mathbb{K}}$-*irrational* if $E_X\{a_n\}_{n=1}^{\infty}$ does not contain any element from $\mathbb{K}$,

- X-*irrational* if it is $X_{\mathbb{Q}}$-irrational, or
- X-*transcendental* if it is $X_{\overline{\mathbb{Q}}}$-irrational.

We further say that sequences $\{a_{1,n}\}_{n=1}^{\infty}, \dots, \{a_{K,n}\}_{n=1}^{\infty}$ of real or complex numbers are X-*linearly independent over* $\mathbb{K}$ if the numbers $1, \xi_1, \dots, \xi_K$ are linearly independent over $\mathbb{K}$ for all lists $(\xi_1, \dots, \xi_K)$ in $E_X(\{a_{1,n}\}_{n=1}^{\infty}, \dots, \{a_{K,n}\}_{n=1}^{\infty})$.

Remark 1.6. While we could make corresponding definitions for infinite products of infinite series, the current author is unaware of any results that could meaningfully be described as providing proper '$\Pi\Sigma$-irrationality'.

In this language, the conclusion of Theorem 1.3 may now be restated as $\{a_n\}_{n=1}^{\infty}$ being Σ-irrational. The rest of this chapter covers a selection of generalizations and variants of that theorem. While many results were not originally proven with the above definitions in mind, we will either reformulate such results correspondingly or extract corollaries that use Definition 1.5.

The first generalization of Theorem 1.3 that we will consider here is one by Hančl and Sobková [26]. They provided a criterion for linear independence of sequences when the *limsup* criterion of Theorem 1.3 is sufficiently strengthened in terms of the number of sequences considered. Soon after this, Hančl [19] proved that the conclusion remains true when the *limsup* criterion is replaced by the condition that the corresponding *liminf* and *limsup* values are different. Combined, we get the following theorem. In the theorem and for the rest of this chapter, we write $\log_2^\alpha x$ as shorthand for $(\log_2 x)^\alpha$.

Theorem 1.7 (Hančl–Sobková, 2003–2004)**.** *Let* $\alpha \in (0,1)$, $\varepsilon > 0$, *and* $K \in \mathbb{N}$. *For* $i = 1, \dots, K$, *let* $\{a_{i,n}\}_{n=1}^{\infty}$ *and* $\{b_{i,n}\}_{n=1}^{\infty}$ *be sequences of positive integers such that*

$$n^{1+\varepsilon} < a_{1,n} \le a_{1,n+1}, \quad b_n < 2^{\log_2^\alpha a_{1,n}}, \quad 2^{-\log_2^\alpha a_{1,n}} a_{i,n} \le a_{1,n} \le 2^{\log_2^\alpha a_{1,n}} a_{i,n},$$

and such that the sequence $\left\{a_{1,n}^{(1+K)^{-n}}\right\}_{n=1}^{\infty}$ *diverges in* $\mathbb{R}$. *Then the sequences* $\{a_{1,n}/b_{1,n}\}_{n=1}^{\infty}, \dots, \{a_{K,n}/b_{K,n}\}_{n=1}^{\infty}$ *are linearly independent over* $\mathbb{Q}$.

Taking $K = 1$ in this theorem, we replace the *limsup* condition in Erdős' Theoerem 1.3 by the slightly more relaxed requirement that the sequence $\{a_n^{2^{-n}}\}_{n=1}^{\infty}$ is divergent in $\mathbb{R}$.

Our next result is due to Andersen and Kristensen [1] and generalizes Theorem 1.3 in a different direction, favouring what turns out to be $\Sigma_{\mathbb{K}}$-irrationality rather than Σ-linear independence over $\mathbb{Q}$ and allowing a_n to potentially be irrational algebraic numbers, provided $|a_n|$ satisfies a sufficiently strict *limsup* criterion. To better phrase their theorem, we need to increase our vocabulary a bit.

Definition 1.8. Let a be an algebraic number. We then say that b is a *conjugate* of a if it solves inequality (1.1) with $c_0, \dots, c_d \in \mathbb{Z}$ whenever a does. The maximum absolute value of a or any of its conjugates is called the *house* of a and is denoted $\overline{|a|}$.

Remark 1.9. Including itself, any algebraic number a has exactly $\deg a$ conjugates.

Definition 1.10. A field $\mathbb{K}$ is said to be a *number field* if there is an algebraic number a such that $\mathbb{K} = \mathbb{Q}(a)$, which is to say that each element of $\mathbb{K}$ may be written as $c_0 + c_1 a + \cdots c_{d-1} a^{d-1}$ where $d = \deg a$, and $c_0, c_1, \dots, c_{d-1}$ may be any rational numbers. In this case, we also say that $\mathbb{K}$ is of *degree* d.

We will also use $\Re(x)$ and $\Im(x)$ to denote the real and imaginary values, respectively, of a given complex number x.

Theorem 1.11 (Andersen–Kristensen, 2019)**.** *Let $d, D \in \mathbb{N}$ be positive integers, and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of algebraic integers of degree $\deg a_n \le d$ such that*

$$n^{1+\varepsilon} \le \overline{|a_n|} = |a_n| \le |a_{n+1}|$$

such and that either $\Re(a_n) > 0$ for all n or $\Im(a_n) > 0$ for all n. Suppose

$$\limsup_{n\to\infty} |a_n|^{D^n \prod_{i=1}^{n-1}(d^i+d)^{-1}} = \infty.$$

Then $\{a_n\}_{n=1}^{\infty}$ is $(\Sigma, \mathbb{K})$-irrational for any number field $\mathbb{K}$ of degree at most D.

Remark 1.12. In the original phrasing of this theorem, the conclusion was only that $\deg \sum_{n=1}^{\infty} 1/a_n > D$, while the present formulation is equivalent to the stronger statement that $\deg \sum_{n=1}^{\infty} 1/(a_n c_n) > D$ for all sequences $\{c_n\}_{n=1}^{\infty}$ of positive integers. The stronger statement follows quite easily from the original, however (see the proof of Corollary 1.27 in Section 1.4).

In the paper [36], which will be presented in Section 1.4, the current author combines the arguments of [19] and Theorem 1.11 into a result from which a criterion for Σ-linear independence over $\mathbb{K}$ may be extracted through a *liminf*$<$*limsup*$<\infty$ condition. Also in Section 1.4, we will consider two variations of Theorem 1.11 that consider infinite products and infinite products of infinite series, respectively. These results were proven by Kristensen and the current author in [35] and generalize existing irrationality results by Hančl and Kolouch [20,21] where a_n is assumed to be rational.

In Section 1.5, we then consider a result by Hančl, Leta, and the current author [24], which provides a criterion for Σ-linear independence over $\mathbb{K}$ of continued fractions for a fixed number field $\mathbb{K}$, taking inspiration from a continued fractions variant of Theorem 1.11 by Andersen and Kristensen [2].

It may be noted that Theorem 1.11 also provides a criterion for Σ-transcendence when the *limsup* condition is satisfied for all $D \in \mathbb{N}$. Relying on a generalized version of Theorem 1.1, this is, unfortunately, rather restrictive. This becomes particularly clear when comparing with the following result by Hančl [18], which uses Theorem 1.2 to achieve Σ-transcendence of sequences.

Theorem 1.13 (Hančl, 2001)**.** *Let $\delta, \varepsilon > 0$, let $\alpha \in \left(\frac{\log(3+2\varepsilon)}{\log(3+2\varepsilon+\delta)}, 1\right)$, and let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of positive integers such that*

$$n^{1+\varepsilon} < a_n \leq a_{n+1}, \quad b_n < a_n^{\varepsilon/(1+\varepsilon)} 2^{-\log_2^{\alpha} a_n} \quad \textit{and} \quad \limsup_{n\to\infty} a_n^{(3+2\varepsilon+\delta)^{-n}} = \infty.$$

Then the number $\{a_n/b_n\}_{n=1}^{\infty}$ is Σ-transcendental.

In Section 1.6, we will consider the papers [38, 39], in which the current author generalizes the above theorem to provide criteria for Σ- and Π-transcendence, though with slightly less lenient bounds of b_n. This is, in part, done through an application of Schmidt's Subspace Theorem [47]. The theorem will be presented in that section and is a generalization of Roth's Theorem.

Finally, in Section 1.7, we consider alternative versions of Definitions 1.4 and 1.5, replacing the condition $c_n \in \mathbb{N}$ with $c_n \in \mathbb{Z}$ and $p \nmid c_n$ for one or multiple prime numbers p. We then relate these definitions to results from a recent paper by Hančl, Kristensen, and the current author [23]. Inspired by Theorem 1.11, the paper provides new criteria for irrationality, transcendence, linear independence, and algebraic independence of numbers expressed as infinite series. We say that a list of complex numbers $(a_1, \ldots, a_K)$ is *algebraically independent* if $P(a_1, \ldots, a_K) \neq 0$ for all polynomials P in K variables and with integer coefficients.

1.2 The Erdős Jump

Recall how the proof of Theorem 1.3 may be split into an algebraic and an analytical part, with the algebraic part showing that if $\sum_{n=1}^{\infty} 1/a_n$ is rational, then there is a fixed $C > 0$ such that inequality (1.4), which reads

$$\left(\prod_{n=1}^{N} a_n\right) \sum_{n=N+1}^{\infty} \frac{1}{a_n} \geq C,$$

is satisfied for all positive integers N. In this section, we will consider the analytical part of the proof, in which this inequality is proven to be impossible.

Due to the assumption $\limsup_{n\to\infty} a_n^{2^{-n}} = \infty$, there will be times when a_{N+1} is particularly large compared to the values of previous a_n (if a_n accelerates fast enough, this may happen for all N). We will then say that the sequence $\{a_n\}_{n=1}^{\infty}$ makes a *jump* at time N.

The main idea of the proof is to find N such that a_n makes a large jump at time N and then remains sufficiently large in terms of n for long enough thereafter to ensure that $\sum_{n=N+1}^{\infty} 1/a_n$ becomes so small compared to $\prod_{n=1}^{N} a_n$ that inequality (1.4) cannot be satisfied. As it turns out, "sufficiently large" can be taken to mean $a_n \geq 2^n$. If this is true for all n, we just need a lower bound for the largest jumps by a_n, which turns out to not be too difficult. More care has to be taken when the sequence $\{a_n\}_{n=1}^{\infty}$ stagnates for periods of time long enough that we will get $a_n < 2^n$ infinitely often. In this case, it may very well be that the next stagnation comes right after our jump, which would allow values of $\sum_{n=N+1}^{\infty} 1/a_n$ that fail to contradict inequality (1.4). Since $\limsup_{n\to\infty} a_n^{2^{-n}} = \infty$, there will, for all positive numbers A, be infinitely many k such that $a_k > A^{2^k}$. Combined with the assumption $a_n \geq n^{1+\varepsilon}$, this gives some control of $\sum_{n=k}^{\infty} 1/a_n$. However, since each such k may potentially be followed by a long stagnation, it may not be enough to disprove inequality 1.4 on its own. Erdős solved this problem by first fixing such a k, then identifying the most recent stagnation, and finally timing the choice of N to be at the very first jump after this stagnation. This guarantees that no stagnation happens between N and k, which gives us a decent upper bound on the infinite series $\sum_{n=N+1}^{\infty} 1/a_n$, while the preceding stagnation itself ensures that the product $\prod_{n=1}^{N} a_n$ cannot be too large, either. By picking the A mentioned earlier to be sufficiently large, it follows that the resulting number $\left(\prod_{n=1}^{N} a_n\right) \sum_{n=N+1}^{\infty} 1/a_n$ can be made arbitrarily small.

Because of how central the jump at time N and the timing hereof are to the proof, the current author names this method the *Erdős Jump* (or *Jump* for short), given that no one else seems to have named it in the literature.

When the Erdős Jump is used to prove a new theorem, it is often the case that the entire argument or large parts of it are presented again since different forms of generalization are required (see, e.g., [1,2,17–21,25,26]). The same pattern can be seen in the papers by the current author that use an Erdős Jump [23,24,35,36,38, 39], where only the paper [39] does not make a new Jump but rather refers back to the one made in [38]. However different, all these generalizations have certain similarities and appear to be special cases of a more general Erdős Jump. In writing the present thesis, this led the current author to the following conjectures. The first seed toward these conjectures was planted by the paper [25], which uses a more flexibly phrased Jump than many other papers. [25] also inspired the Jumps made in the papers [23] and [38].

Conjecture 1.14. *Let $\varepsilon, K > 0$, and let $\{m_n\}_{n=1}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$ be sequences of real numbers such that $m_n \geq 1$ and $\mu_n > 0$. Write $M_n = \prod_{i=1}^{n} m_i$. Let $\{a_n\}_{n=1}^{\infty}$,*

$\{b_n\}_{n=1}^{\infty}$, *and* $\{E_n\}_{n=1}^{\infty}$ *be sequence of positive numbers such that*

$$a_{n+1} \geq a_n \geq n^{1+\varepsilon}, \quad \limsup_{n\to\infty} \frac{\log b_n}{\log a_n} \leq 0,$$

and

$$E_n \leq \max\left\{ b_n^{n^K \max_{1\leq i\leq n} \mu_i}, 2^{Kn^{-2}(\log\log n)^{-1}\prod_{i=1}^{n-1}(m_i+\mu_i M_i)} \right\}.$$

If the sequence $\left\{ a_n^{1/\prod_{i=1}^{n-1}(m_i+\mu_i M_i)} \right\}_{n=1}^{\infty}$ *diverges in* $\mathbb{R}$, *then*

$$\liminf_{N\to\infty} \left(\left(E_N \prod_{n=1}^{N} a_n^{\mu_n} \right)^{M_N} \sum_{n=N+1}^{\infty} \frac{b_n}{a_n} \right) = 0.$$

Assuming this conjecture to be true, it is easily generalized to the following statement, which allows values of b_n that are much closer to a_n, in exchange for a harsher divergence criterion.

Conjecture 1.15. *Let* ε, K, $\{m_n\}_{n=1}^{\infty}$, $\{\mu_n\}_{n=1}^{\infty}$, $\{M_n\}_{n=1}^{\infty}$, *and* $\{a_n\}_{n=1}^{\infty}$ *be given as in Conjecture 1.14. Let* $\beta \in [0, \varepsilon/(1+\varepsilon))$, *and let* $\{b_n\}_{n=1}^{\infty}$ *and* $\{E_n\}_{n=1}^{\infty}$ *be sequences of positive numbers such that* $\limsup_{n\to\infty}(\log b_n/\log a_n) \leq \beta$ *and*

$$E_n \leq \max\left\{ \left(\frac{b_n}{a_n^{\beta}}\right)^{n^K \max_{1\leq i\leq n}\mu_i}, 2^{Kn^{-2}(\log\log n)^{-1}\prod_{i=1}^{n-1}\left(m_i+\frac{\mu_i M_i}{1-\beta}\right)} \right\}.$$

If the sequence $\left\{ a_n^{1/\prod_{i=1}^{n-1}\left(m_i+\frac{\mu_i M_i}{1-\beta}\right)} \right\}_{n=1}^{\infty}$ *diverges in* $\mathbb{R}$, *then*

$$\liminf_{N\to\infty} \left(\left(E_N \prod_{n=1}^{N} a_n^{\mu_n} \right)^{M_N} \sum_{n=N+1}^{\infty} \frac{b_n}{a_n} \right) = 0.$$

Proof (assuming Conjecture 1.14). Write

$$a_n' = a_n^{1-\beta}, \quad \varepsilon' = (1+\varepsilon)(1-\beta) - 1, \quad b_n' = \frac{b_n}{a_n^{\beta}} \quad \text{and} \quad \mu_n' = \frac{\mu_n}{1-\beta}.$$

From this follows that $\varepsilon' > 0$,

$$a_{n+1}' \geq a_n' \geq \left(n^{1+\varepsilon}\right)^{1-\beta} = n^{1+\varepsilon'},$$

and

$$\limsup_{n\to\infty} \frac{\log b_n'}{\log a_n'} = \frac{1}{1-\beta} \limsup_{n\to\infty} \frac{\log b_n - \beta\log a_n}{\log a_n} \leq 0.$$

Then the sequence

$$\left\{ {a'_n}^{1/\prod_{i=1}^{n-1}(m_i+\mu'_i M_i)} \right\}_{n=1}^{\infty} = \left\{ \left(a_n^{1/\prod_{i=1}^{n-1}\left(m_i+\frac{\mu_i M_i}{1-\beta}\right)} \right)^{1-\beta} \right\}_{n=1}^{\infty}$$

clearly diverges, while

$$\begin{aligned}\left(2^{N^K} \prod_{n=1}^{N} \left(a_n^{1-\beta} b_n\right)^{\mu_n} \right)^{M_N} \sum_{n=N+1}^{\infty} \frac{b_n}{a_n} &= \left(2^{N^K} \prod_{n=1}^{N} \left({a'_n}^{\frac{1}{1-\beta}} b'_n\right)^{\mu_n} \right)^{M_N} \sum_{n=N+1}^{\infty} \frac{b'_n}{a'_n} \\ &\le \left(2^{N^K} \prod_{n=1}^{N} (a'_n b'_n)^{\mu'_n} \right)^{M_N} \sum_{n=N+1}^{\infty} \frac{b'_n}{a'_n}.\end{aligned}$$

The statement now follows immediately from Conjecture 1.14. □

Combining the Erdős Jumps from [23, 35], we get the following lemma. The paper [35] is presented in Section 1.4, while [23] is presented in Section 1.7.

Lemma 1.16. *Conjecture 1.14 is true in the following cases.*

1. *$m_i, \mu_i \in \mathbb{N}$ with fixed $\mu_i = \mu$, $b_n \le 2^{\log_2^\alpha a_n}$ for some fixed constant $\alpha \in (0,1)$, and $E_n = b_n^{n^2}$.*
2. *$m_i = 1$, $\mu_i = \mu$ is fixed, $b_n \le 2^{\log_2^\alpha a_n}$ for some fixed constant $\alpha \in (0,1)$, $E_n = \max\{b_n^{n^2}, n^{-3} 2^{(1+\mu)^{n+1}}\}$, and $\limsup_{n\to\infty} a_n^{(1+\mu)^{-n}} = \infty$.*
3. *$m_i \in \mathbb{N}$, $\mu_i = i + 1 - M_{i-1}^{-1}$, $b_n \le a_n^{(\log\log a_n)^{-3-\varepsilon}}$, $E_n = 2^{n^2} \prod_{i=1}^{n} b_i^{\mu_i}$, and $\limsup_{n\to\infty} a_n^{1/(n! \prod_{i=1}^{n-1} M_i)} = \infty$.*

Furthermore, Conjecture 1.15 is true given the assumptions of case 2 where the bound on b_n weakened to $b_n \le |a_n|^\beta 2^{\log_2^\alpha a_n}$ and the limsup *condition strengthened to $\limsup_{n\to\infty} a_n^{(1+\mu/(1-\beta))^{-n}} = \infty$.*

Remark 1.17. In case 3, $n! \prod_{i=1}^{n-1} M_i$ is equal to $\prod_{i=1}^{n-1} (m_i + \mu_i M_i)$ due to the choice of μ_i. Similarly, $(1+\mu)^{n-1} = \prod_{n=1}^{n-1}(m_i + \mu_i M_i)$ in case 2.

Proof. Case 1. This follows from [35, Lemma 16] with $D = 1$, $d_1 = \mu m_1$, and $d_{n+1} = m_{n+1}$.

Case 2. This follows from the proof of Lemma 11 in [23] with $M = \mu - 1$. While the lemma assumes $\mu \ge 1$, this additional assumption is easily removed (see subsection 1.7.2).

Case 3. This follows from [35, Lemma 19] where $D = 1$ and $D_n = M_n$.

The statement regarding Conjecture 1.15 follows from case 2 and the conditional proof of the conjecture. This completes the present proof. □

1.3 Some additional algebra

In this section, we introduce further notions and theorems that will play a central role in phrasing and proving results that will be presented in the next few sections. Unless otherwise stated, the introduced terminology and results are standard knowledge within the field of algebraic number theory.

We start by expanding our vocabulary regarding fields and field extensions.

Definition 1.18. Let $\mathbb{K}$ and $\mathbb{L}$ be fields. We then make the following definitions.

- If $\mathbb{L} \supseteq \mathbb{K}$, then we say that $\mathbb{L}$ is a *field extension* of $\mathbb{K}$, and we define the *degree* of this field extension, denoted $[\mathbb{L} : \mathbb{K}]$, as the dimension of $\mathbb{L}$ when viewed as a $\mathbb{K}$-vector space.
- Let a lie in a field extension of $\mathbb{K}$. If there exist $d \in \mathbb{N}$ such that

$$c_0 + c_1 a + \ldots + c_d a^d = 0,$$

  for the right choices of $c_0, \ldots, c_d \in \mathbb{K}$ with $c_d \neq 0$, then we define the *degree* of a over $\mathbb{K}$, denoted $\deg_{\mathbb{K}} a$, as the smallest such d. If no such d exists, we write $\deg_{\mathbb{K}} a = \infty$.
- If $\mathbb{K} \subseteq \overline{\mathbb{Q}}$, we say that a map $\sigma : \mathbb{K} \to \overline{\mathbb{Q}}$ is an *embedding* into $\overline{\mathbb{Q}}$ if it is a ring homomorphism over $\mathbb{K}$, i.e., if $\sigma(a + bc) = \sigma(a) + \sigma(b)\sigma(c)$ for all $a, b, c \in \mathbb{K}$.

Notation. Let $\mathbb{K}$ be a field, and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of numbers from a field extension of $\mathbb{K}$. Then we have the following notation.

- If $\deg_{\mathbb{K}} a_n = d < \infty$, we write $\mathbb{K}(a_n) = \left\{\sum_{i=0}^{d-1} c_i a_n^i : c_i \in \mathbb{K}\right\}$.
- If $\deg_{\mathbb{K}} a_n = \infty$, we write $\mathbb{K}(a_n) = \left\{\sum_{i=-K}^{K} c_i a_n^i : c_i \in \mathbb{K}, K \in \mathbb{N}\right\}$.
- We write $\mathbb{K}(a_m, a_n) = (\mathbb{K}(a_m))(a_n)$, i.e., $\mathbb{K}(a_m, a_n) = \mathbb{K}'(a_n)$ if $\mathbb{K}' = \mathbb{K}(a_m)$.
- We write $\mathbb{K}(a_1, a_2, \ldots) = \bigcup_{n=1}^{\infty} \mathbb{K}(a_1, a_2, \ldots, a_n)$.
- If $\mathbb{K} \subseteq \overline{\mathbb{Q}}$, we write $\mathcal{O}_{\mathbb{K}}$ for the set of algebraic integers in $\mathbb{K}$.

Remark 1.19. The following facts will be considered common knowledge when using the above notation and definitions.

- $\mathbb{K}(a)$ always defines a field, and so does $\mathbb{K}(a_1, a_2, \ldots)$.
- The above definitions of degree generalize the notions introduced in Section 1.1 since it easily follows that $[\mathbb{K}(a) : \mathbb{K}] = \deg_{\mathbb{K}} a$ and $\deg_{\mathbb{Q}} a = \deg a$.

- By a simple argument from linear algebra involving bases, it follows that $[\mathbb{K}(a,b):\mathbb{K}] = [\mathbb{K}(a,b):\mathbb{K}(a)][\mathbb{K}(a):\mathbb{K}]$.
- If $\mathbb{K} \subseteq \overline{\mathbb{Q}}$, then any embedding $\sigma : \mathbb{K} \to \overline{\mathbb{Q}}$ is automatically injective, and the number of such embeddings is equal to $[\mathbb{K} : \mathbb{Q}]$.

We will now introduce ways of estimating the complexity of algebraic numbers.

Definition 1.20. Let a be an algebraic number of degree d. We then make the following definitions.

- The *minimal polynomial* of a is the unique polynomial $P_a = c_0 + c_1X + \ldots + c_dX^d$ such that $P_a(a) = 0$, $c_0, \ldots, c_{d-1} \in \mathbb{Z}$, $c_d \in \mathbb{N}$, and $\gcd(c_0, \ldots, c_d) = 1$.
- The leading coefficient c_d of P_a is called the *denominator* of a, denoted $\operatorname{den} a$.
- Factoring P_a as $P_a = c_d(X - a^{(1)}) \cdots (X - a^{(d)})$, *Mahler measure* of a is defined as
$$M(a) := c_d \prod_{i=1}^{d} \max\left\{1, \left|a^{(i)}\right|\right\}.$$

Notice that the denominator of an algebraic number a is also the smallest positive integer c such that ac is an algebraic integer.

One of the advantages of the Mahler measure is that it provides a way to measure the complexity of an algebraic number. Another and particularly useful way to do the same is with the Weil height, defined below. This definition is more advanced and relies on the notions of places and their associated local fields. Since places and local fields take a while to introduce and are not used elsewhere in this chapter, the interested reader is instead referred to [48, Section 3.1], which provides a neat and brief introduction.[2] Alternatively, the reader may understand the Weil height in terms of the Mahler measure through the remarkable Theorem 1.22 below.

Definition 1.21. Let a be an algebraic number. We then define the *Weil height* of a as
$$H(a) := \prod_{\nu \in M_{\mathbb{K}}} \max\{1, |a|_\nu\}^{[\mathbb{K}_\nu:\mathbb{Q}_\nu]/[\mathbb{K}:\mathbb{Q}]},$$
where $\mathbb{K}$ is an arbitrary number field containing a, $M_{\mathbb{K}}$ denotes the set of places for $\mathbb{K}$, and $\mathbb{K}_\nu$ is the local field of $\mathbb{K}$ at ν.

[2] In [48], *normalized absolute values* are used instead of *places*, but this makes no difference since places are simply the equivalence classes of absolute values, of which the normalized ones are canonical representatives.

This definition does not depend on the choice of $\mathbb{K}$ (see [48] for a proof).

Recalling Definition 1.8, we may compare the Weil height, the Mahler measure, and the house using the following classical theorem. The equality is proven in [48] while the inequality is a trivial consequence of Definitions 1.8 and 1.20.

Theorem 1.22. *Let a be an algebraic number of degree d. Then*

$$H(a)^d = M(\alpha) \leq (\operatorname{den} a) \max\{1, \overline{|a|}^d\}$$

The Weil height appears in the rather useful Theorem 1.25, which is presented in the below Section 1.4, but it also behaves rather nicely with respect to addition and multiplication. This is seen in the following lemma, a proof of which is found in [48].

Lemma 1.23. *Let $a_1, \ldots, a_n \in \overline{\mathbb{Q}}$ with $a_1 \neq 0$. Then*

$$H(a_1 + \cdots + a_n) \leq nH(a_1) \cdots H(a_n), \quad H(a_1 a_2) \leq H(a_1)H(b_1),$$
$$H(1/a_1) = H(a_1).$$

The following lemma has also been used for bounding the house of a sum of algebraic numbers. Its two statements are proven in [38] and [39], respectively, but are most likely also proven elsewhere in the literature.

Lemma 1.24. *Let $x_1, \ldots, x_d$ be algebraic numbers. Then there is a constant $C_1 > 0$, depending only on $x_1, \ldots, x_d$, so that for any $(c_1, \ldots, c_d) \in \mathbb{Q}^d$,*

$$\overline{|c_1 x_1 + c_2 x_2 + \cdots + c_d x_d|} \leq C_1 \max_{1 \leq i \leq d} |c_i|.$$

If furthermore $x_1, \ldots, x_d$ are linearly independent over $\mathbb{Q}$, then there is another constant $C_2 > 0$, depending only on $x_1, \ldots, x_d$, so that for any $(c_1, \ldots, c_d) \in \mathbb{Q}^d$,

$$\overline{|c_1 x_1 + c_2 x_2 + \cdots + c_d x_d|} \geq C_2 \max_{1 \leq i \leq d} |c_i|.$$

1.4 Irrationality of series and products

In this section, we will consider criteria for $\mathbb{K}$-irrationality of infinite series, infinite products, and infinite products of infinite series. These results are based on the papers [36] and [35] by the current author, the latter paper being a joint work with Simon Kristensen. The preprints of the papers are printed in their entirety in subsections 1.4.3 and 1.4.4, respectively. In subsection 1.4.1, we will consider a number of examples.

Common for all these results is that they rely on the below generalization of Theorem 1.1, a proof of which may be extracted from [6, Theorem A.1]. H is the Weil height as defined in Definition 1.21.

Theorem 1.25. *Let a, b be non-conjugate algebraic numbers. Then*

$$|a-b| \geq \big(2H(a)H(b)\big)^{-\deg(a)\deg(b)}.$$

We are now ready to state the below theorem, which is the main result of the paper [36]. It generalizes Theorem 1.7 and extends Theorem 1.11, combining arguments from both theorems.

Theorem 1.26 (Laursen, 2023)**.** *Let $D, K \in \mathbb{N}$, $\alpha \in [0,1)$ $\varepsilon > 0$, and $\zeta \in \mathbb{C}$ with $|\zeta| = 1$. Let $\{d_n\}_{n\in\mathbb{N}}$ be a sequence of natural numbers, and write $D_n = \prod_{i=1}^{n} d_i$. For $i = 1, \ldots, K$, let $\{a_{i,n}\}_{n\in\mathbb{N}}$ and $\{b_{i,n}\}_{n\in\mathbb{N}}$ be sequences of algebraic and positive integers, respectively, so that*

$$\begin{aligned} n^{1+\varepsilon} \leq |a_{1,n}| \leq |a_{1,n+1}|, \qquad & [\mathbb{Q}\left(a_{1,n}, \ldots, a_{K,n}\right) : \mathbb{Q}] \leq d_n, \\ b_{i,n}\overline{\lceil a_{i,n} \rceil} \leq 2^{\log_2^\alpha |a_{1,n}|}|a_{i,n}|, \qquad & |a_{i,n}|2^{-\log_2^\alpha |a_{1,n}|} < |a_{1,n}| < |a_{i,n}|2^{\log_2^\alpha |a_{1,n}|}, \\ \Re\left(\zeta \frac{b_{i,n}}{a_{i,n}}\right) > 0, \qquad & \lim_{n\to\infty} \frac{\Re(\zeta b_{i,n}/a_{i,n})}{\Re(\zeta b_{i+1,n}/a_{i+1,n})} = 0, \end{aligned}$$

writing $b_{K+1,n}, a_{K+1,n} = 1$, and

$$\liminf_{n\to\infty} |a_{1,n}|^{1/\left(D^n \prod_{i=1}^{n-1}(d_i + KD_i)\right)} < \limsup_{n\to\infty} |a_{1,n}|^{1/\left(D^n \prod_{i=1}^{n-1}(d_i + KD_i)\right)} < \infty. \tag{1.5}$$

Then the numbers $1, \sum_{n=1}^{\infty} b_{1,n}/a_{1,n}, \ldots, \sum_{n=1}^{\infty} b_{K,n}/a_{K,n}$ are linearly independent over $\mathbb{K}$ for all number fields $\mathbb{K}$ of degree at most D.

In [36], it was assumed that $|a_{1,n}| < |a_{1,n+1}|$, but making the inequality soft does not affect the proof in any meaningful way, and it allows us to extract the following corollary regarding $\Sigma_{\mathbb{K}}$-irrationality.

Corollary 1.27. *Let $d \in \mathbb{N}$. Suppose all assumptions of Theorem 1.26 are satisfied with $d_n = d$ for all n. Then the sequences $\{a_{1,n}/b_{1,n}\}_{n=1}^{\infty}, \ldots, \{a_{K,n}/b_{K,n}\}_{n=1}^{\infty}$ are Σ-linearly independent over $\mathbb{K}$ for all number fields $\mathbb{K}$ of degree at most D.*

Proof. Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of positive integers. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection such that $A_{1,n} := a_{1,\sigma(n)}c_{\sigma(n)}$ is of non-decreasing modulus, and set $A_{i,n} := a_{i,\sigma(n)}c_{\sigma(n)}$, $B_{i,n} := b_{i,\sigma(n)}c_{\sigma(n)}$. Then the sequences $\{A_{i,n}\}_{n=1}^{\infty}$ and $\{B_{i,n}\}_{n=1}^{\infty}$, with $i = 1, \ldots, K$, satisfy the assumptions of Theorem 1.26 with $d_n = d$. This completes the proof. □

Remark 1.28. By this corollary, the *limsup* condition of Theorem 1.11 may be replaced with the more general assumption that $|a_n|^{1/\left(D^n \prod_{i=1}^{n-1}(d_i + KD_i)\right)}$ is divergent in $\mathbb{R}$.

In the above corollary, we fixed $d_n = d$ due to the necessary re-ordering of $|a_{1,n}c_n|$. Suppose for a moment that we did not make this assumption. Depending on $\{c_n\}_{n=1}^\infty$, the reordering may move $(a_{1,n}, \dots, a_{K,n})$ arbitrarily around. For a sufficiently devious $\{c_n\}_{n=1}^\infty$, we may therefore be in the case that this reordering postpones those indices n that have small values of d_n increasingly far while bringing the indices n with large values of d_n correspondingly forward. If done correctly, this will increase the resulting D_n significantly for all values of n, which in turn makes the *liminf<limsup* criterion much harder to satisfy as it now appears plausible to end up with $\limsup_{n\to\infty} |A_{1,n}|^{1/\left(D^n \prod_{i=1}^{n-1}(d_i'+KD_i')\right)} = 1$, where $d_n' = d_{\sigma(n)}$ and $D_n' = \prod_{i=1}^{n-1} d_i'$. Of course, this may be avoided if c_n is sufficiently large for the right n, but it is doubtable that this will always be the case.

In the paper [35], Kristensen and the current author found that the products versions of Theorems 1.11 and 1.26 follow from almost the exact same proofs. By tweaking the algebraic part of the proofs, the following somewhat stronger theorem was shown.

Theorem 1.29 (Kristensen–Laursen, 2025 on arXiv)**.** *Let $D \in \mathbb{N}$, $\varepsilon > 0$, $\alpha \in (0,1)$, and $e \in \{-1,1\}$. Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be sequences of algebraic and positive integers, respectively. Write $\mathbb{K}_0 = \mathbb{Q}$, $\mathbb{K}_n = \mathbb{K}_{n-1}(a_n)$, $d_n = [\mathbb{K}_n : \mathbb{K}_{n-1}]$, and $D_n = \prod_{i=1}^n d_i$. Let $\mathbb{K}$ denote the field $\bigcup_{n=1}^\infty \mathbb{K}_n$, and suppose for all n and some fixed $e \in \{-1,1\}$ that*

$$n^{1+\varepsilon} \le |a_n| \le |a_{n+1}|, \qquad \overline{|a_n|}\, b_n \le |a_n| 2^{(\log_2|a_n|)^\alpha}, \qquad e(\Re(a_n/b_n) + 1/2) \ge 0,$$

with $(\Re(a_n/b_n)+1/2)e > 0$ infinitely often, and that $|a_n|^{1/(D^n\prod_{i=1}^{n-1}(d_i+D_i))}$ diverges in $\mathbb{R}$ as $n \to \infty$. Then $\deg_{\mathbb{K}} \prod_{n=1}^\infty (1 + b_n/a_n) > D$.

Restricted to a statement on $\Pi_{\mathbb{K}}$-irrationality, we get the following corollary. Notice that the bound of real values is strengthened for $e = 1$ in order to make it resilient against significant scalings of a_n. The same is observed by the current author in [39].

Corollary 1.30. *Let $d, D \in \mathbb{N}$, $\varepsilon > 0$, $\alpha \in (0,1)$, $e \in \{-1,1\}$, and let $\mathbb{K}$ be a number field. Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be sequences of algebraic integers with $\deg_{\mathbb{K}} a_n \le d$ and $b_n \in \mathbb{N}$. Suppose for all n that*

$$n^{1+\varepsilon} \le |a_n| \le |a_{n+1}|, \qquad \overline{|a_n|}\, b_n \le |a_n| 2^{(\log_2|a_n|)^\alpha},$$

$$e\Re\left(\frac{a_n}{b_n}\right) \ge \begin{cases} 0 & \text{if } e = 1, \\ 1/2 & \text{if } e = -1, \end{cases}$$

with $\Re(a_n/b_n) \ne -1/2$ infinitely often, and that $|a_n|^{1/(D^n\prod_{i=1}^{n-1}(d+d^i[\mathbb{K}:\mathbb{Q}]))}$ diverges in $\mathbb{R}$ as $n \to \infty$. Then $\{a_n/b_n\}_{n=1}^\infty$ is $(\Sigma, \mathbb{L})$-irrational for all field extensions $\mathbb{L} \supseteq \mathbb{K}(a_1, a_2, \dots)$ of degree at most D.

The improvement of the proof that leads to the stronger statements of both theorem and corollary compared to the series setting is surprisingly simple and relies on how Theorem 1.25 is applied. In the algebraic arguments of Theorems 1.11 and 1.26 with $K = 1$, we write $\gamma = \sum_{n=1}^{\infty} \frac{b_n}{a_n}$ and $\gamma_N = \sum_{n=1}^{N} \frac{b_n}{a_n}$. For Theorem 1.29, we have $\gamma = \prod_{n=1}^{\infty}(1 + \frac{b_n}{a_n})$ and $\gamma_N = \prod_{n=1}^{N}(1 + \frac{b_n}{a_n})$ instead, but this is of no significance in the following. Then we assume, towards contradiction, that $\deg \gamma \le D$ (for Theorem 1.29, we only assume $\deg_{\mathbb{K}} \gamma \le D$).

After bounding $H(\gamma_N)$ and ensuring that γ_N eventually differs from γ, we apply Theorem 1.25. In the proofs of Theorems 1.11 and 1.26, we take $a = \gamma$ and $b = \gamma_N$, which leads to

$$|\gamma - \gamma_N| \ge \big(2H(\gamma)H(\gamma_N)\big)^{-(\deg \gamma)\deg \gamma_N} \ge \big(2H(\gamma)H(\gamma_N)\big)^{-D_N \deg \gamma}, \tag{1.6}$$

using that $\deg \gamma_N \le \deg a_1 \cdots \deg a_N$. Notice that the moment after we apply Theorem 1.25 in this way, we can no longer take advantage of any algebraic connection between γ and γ_N. In the proof of Theorem 1.29, we instead take $a = \gamma - \gamma_N$ and $b = 0$. Since $H(0) = \deg 0 = 1$, this now leads to

$$|\gamma - \gamma_N| \ge \big(2H(\gamma - \gamma_N)\big)^{-\deg(\gamma-\gamma_N)} \ge \big(4H(\gamma)H(\gamma_N)\big)^{-\deg(\gamma-\gamma_N)},$$

using Lemma 1.23 for the second inequality. This now becomes a much stronger bound if $\deg(\gamma - \gamma_N)$ is significantly smaller than $D_N \deg \gamma$, which is easily the case if γ has a large degree and lies in a field extension of $\mathbb{K}$.

Doing these improvements to Theorem 1.11, Theorem 1.26, and Corollary 1.27, we find the following theorem on linear independence of numbers and sequences, which has not previously been published. Drawing inspiration from the papers [38, 39], the theorem also contains an additional improvement by replacing the exponent $D^n \prod_{i=1}^{n-1}(d_i + D_i K)$ with the slightly smaller $\prod_{i=1}^{n-1}(d_i + DD_i K)$.

Theorem 1.31. *Let $D, K \in \mathbb{N}$, $\alpha \in [0, 1)$ $\varepsilon > 0$, and $\zeta \in \mathbb{C}$ with $|\zeta| = 1$. Let $\{\mathbb{K}_n\}_{n=1}^{\infty}$ be a sequence of number fields satisfying $\mathbb{K}_n \subseteq \mathbb{K}_{n+1}$, and let $\mathbb{K}$ denote the field $\bigcup_{n=1}^{\infty} \mathbb{K}_n$. Write $d_1 = [\mathbb{K}_1 : \mathbb{Q}]$, $d_{n+1} = [\mathbb{K}_{n+1} : \mathbb{K}_n]$, and $D_n = \prod_{i=1}^{n} d_i$. For $i = 1, \dots, K$, let $\{a_{i,n}\}_{n\in\mathbb{N}}$ and $\{b_{i,n}\}_{n\in\mathbb{N}}$ be sequences of algebraic integers with $a_{i,n} \in \mathcal{O}_{\mathbb{K}_n}$ and $b_{i,n} \in \mathbb{N}$, respectively, so that*

$$n^{1+\varepsilon} \le |a_{1,n}| \le |a_{1,n+1}|, \qquad b_{i,n}\overline{|a_{i,n}|} \le 2^{(\log_2 |a_{1,n}|)^{\alpha}} |a_{i,n}|,$$

$$|a_{i,n}|2^{-(\log_2 |a_{1,n}|)^{\alpha}} < |a_{1,n}| < |a_{i,n}|2^{(\log_2 |a_{1,n}|)^{\alpha}}, \qquad \Re\left(\zeta \frac{b_{1,n}}{a_{1,n}}\right) > 0,$$

and, when $i < K$,

$$\lim_{n\to\infty} \frac{|b_{i,n}/a_{i,n}|}{|b_{i+1,n}/a_{i+1,n}|} = 0. \tag{1.7}$$

If the sequence $\left\{|a_{1,n}|^{1/\prod_{i=1}^{n-1}(d_i+DD_iK)}\right\}_{n=1}^{\infty}$ *diverges in* $\mathbb{R}$*, then the numbers* 1, $\sum_{n=1}^{\infty}\frac{b_{1,n}}{a_{1,n}},\ldots,\sum_{n=1}^{\infty}\frac{b_{K,n}}{a_{K,n}}$ *are linearly independent over* $\mathbb{L}$ *for all field extensions* $\mathbb{L}\supseteq\mathbb{K}$ *of degree at most* D.

Let $d\in\mathbb{N}$*, and let* $\mathbb{K}_0$ *be a number field. Suppose* $[\mathbb{K}_0(a_{1,n},\ldots,a_{K,n}):\mathbb{K}_0]\leq d$ *for all* $n\in\mathbb{N}$ *and that the sequence* $\left\{|a_{1,n}|^{1/\prod_{i=1}^{n-1}(d+d^iD[\mathbb{K}_0:\mathbb{Q}])}\right\}_{n=1}^{\infty}$ *diverges in* $\mathbb{R}$*. Then the sequences* $\{a_{1,n}/b_{1,n}\}_{n=1}^{\infty},\ldots,\{a_{K,n}/b_{K,n}\}_{n=1}^{\infty}$ *are* Σ*-linearly independent over* $\mathbb{L}$ *for all field extensions* $\mathbb{L}\supseteq\mathbb{K}_0(a_{1,n},\ldots,a_{K,n}:n\in\mathbb{N})$ *of degree at most* D.

If we replace equation (1.7) *with*

$$\Re\left(\zeta\frac{b_{i+1,n}}{a_{i+1,n}}\right)>0\quad\text{and}\quad\lim_{n\to\infty}\frac{\Re(\zeta b_{i,n}/a_{i,n})}{\Re(\zeta b_{i+1,n}/a_{i+1,n})}=0,\tag{1.8}$$

then the above statements still hold but where '[...] independent over $\mathbb{L}$*' is replaced with '[...] independent over* $\mathbb{L}\cap\mathbb{R}$*'.*

We prove this in subsection 1.4.2. Considering the matters of Σ- and Π-irrationality settled, let us move on to an irrationality result for infinite products of infinite series that Kristensen and the current author dealt with in the second half of [35]. In this setting, we have to be more restrictive regarding the real and imaginary values of a_n/b_n than in the series and products cases. For that reason, we introduce the following definition.

Definition 1.32. Let $(a_{n,m})_{m,n\in\mathbb{N}}$ and $(b_{n,m})_{m,n\in\mathbb{N}}$ be infinite arrays of algebraic and positive integers, respectively. We then say that $(a_{n,m})_{m,n\in\mathbb{N}}$ and $(b_{n,m})_{m,n\in\mathbb{N}}$ form a ΠΣ-*neat pair* if at least one of the following conditions is satisfied.

1. $\Re(a_{n,m})\geq 0$ and $e\Im a_{n,m}\geq 0$ for all $m,n\in\mathbb{N}$, where $e\in\{-1,1\}$ is fixed.
2. $\Re(a_{n,m}/b_{n,m})\geq-\frac{1}{2}$ for all sufficiently large $m+n$ with $>$ infinitely often, and $e\Im(a_{n,m})\geq|\Re(a_{n,m})|$ for all $m,n\in\mathbb{N}$, where $e\in\{-1,1\}$ is fixed.
3. $|\Im(a_{n,m})|\leq\Re(a_{n,m})$ for all $m,n\in\mathbb{N}$.
4. $X<1$, $\Re(a_{n,m}/b_{n,m})\leq 0$, and $|\Im(a_{n,m})|\leq R|\Re(a_{n,m})|$ for all $m,n\in\mathbb{N}$, where $X=\sup_{m\in\mathbb{N}}\{\sum_{n=1}^{\infty}\frac{b_{n,m}}{|a_{n,m}|}\}$ and $R\in(0,1/X)$ are fixed.

We then get the following theorem.

Theorem 1.33 (Kristensen–Laursen, 2025 on arXiv)**.** *Let* $D\in\mathbb{N}$*, let* $\varepsilon>0$*. Let* $(a_{n,m})_{m,n\in\mathbb{N}}$ *and* $(b_{n,m})_{m,n\in\mathbb{N}}$ *be infinite arrays of algebraic and positive integers, respectively, that form a* ΠΣ*-neat pair. Suppose that the sequence* $\{|a_{n,1}|\}_{n=1}^{\infty}$ *is non-decreasing and that for* n *sufficiently large,*

$$n^{1+\varepsilon}\leq|a_{n,1}|,\qquad\sum_{j=1}^{n}\left|\frac{b_{n-j+1,j}}{a_{n-j+1,j}}\right|\leq|a_{n,1}|^{-1+(\log\log|\alpha|)^{-3-\varepsilon}},$$

and

$$\prod_{j=1}^{n} \overline{|a_{n-j+1,j}|} \leq |a_{n,1}|^{n+(\log\log|\alpha|)^{-3-\varepsilon}}.$$

Write $\mathbb{K}_1 = \mathbb{Q}(a_{1,1})$, $\mathbb{K}_n = \mathbb{K}_{n-1}(a_{1,n+1}, a_{2,n}, \ldots, a_{n+1,1})$, *and* $D_n = [\mathbb{K}_n : \mathbb{Q}]$. *Let* $\mathbb{K}$ *denote the field* $\bigcup_{n=1}^{\infty} \mathbb{K}_n$. *Then* $\deg_{\mathbb{K}} \prod_{m=1}^{\infty} (1 + \sum_{n=1}^{\infty} b_{n,m}/a_{n,m}) > D$ *if*

$$\limsup_{n\to\infty} |a_{n,1}|^{1/(D^n n! \prod_{i=1}^{n-1} D_i)} = \infty.$$

The proof combines ideas from the proofs of Theorem 1.11, Theorem 1.29, and the paper [21] by Hančl and Kolouch.

It would now seem natural to extract an irrationality statement like those in Corollaries 1.27 and 1.30. Unfortunately, this is not as easily done for the following reason. In the series and product settings, the number of $a_{i,j}$ and $b_{i,j}$ considered for fixed n did not depend on n itself. Meanwhile, for Theorem 1.33, we have increasingly many $a_{i,j}$ and $b_{i,j}$ to consider for each n, namely those with $i + j - 1 = n$. These $a_{i,j}$ and $b_{i,j}$ are furthermore compared to each other without any immediate connection to those considered for other values of n except for the fact that $\{|a_{n,1}|\}_{n=1}^{\infty}$ is non-decreasing and satisfies a specific *limsup* condition. If we now try to reorder the terms, we will be forced to mix collections of $a_{i,j}$ and $b_{i,j}$ from different values of n and compare them. How to handle this appears to be a study in itself, though one this author fears would be of too little reward. For now, the closest we get to a notion of '$\Sigma\Pi$-irrationality' is the below trivial corollary to Theorem 1.33.

Corollary 1.34. *Keeping the assumptions and notation of Theorem 1.33, let* $\{c_n\}_{n=2}^{\infty}$ *be a sequence of positive integers such that* $\{|a_n|c_{n-1}\}_{n=1}^{\infty}$ *is non-decreasing. Then* $\deg_{\mathbb{K}} \prod_{m=1}^{\infty} (1 + \sum_{n=1}^{\infty} b_{n,m}/(a_{n,m}c_{m+n})) > D$ *if*

$$\limsup_{n\to\infty} |a_{n,1}|^{1/(D^n n! \prod_{i=1}^{n-1} D_i)} = \infty.$$

Notice that the assumptions of Theorem 1.33 imply more lenient bounds of b_n in terms of $a_{n,1}$ than would be expected from Theorems 1.26 and 1.29, thus allowing $b_{n,1}\overline{|a_{n,1}|}$ as large as $|a_{n,1}|^{1+1/\log^{3+\varepsilon}\log|a_{n,1}|}$. This is due to a more effective Erdős Jump (see Section 1.2). This Jump has, however, not yet been made in the cases of *liminf*<*limsup*<∞.

1.4.1 Examples

In this subsection, we will present a list of examples to the theorems presented above. Most of these examples are inspired by [23, Remark 4]. Recall the Riemann zeta function, defined by equation (1.3). A direct consequence of Theorem 1.3 is that the number $\sum_{n=1}^{\infty} 1/(n^s a_n)$ is irrational when $s > 1$ is an integer and $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive integers such that $\limsup_{n\to\infty} a_n^{2^{-n}} = \infty$. Our first example provides a similar statement when $s > 1$ is a rational number but not necessarily an integer. This is then generalized in the subsequent example to a statement of linear independence of multiple series. In all the below examples, π denotes the prime counting function, i.e., $\pi(n)$ is the number of prime numbers $p \le n$.

Example 1.35. Let $p, q \in \mathbb{N}$ with $p > q$, and let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive integers such that $a_n^{1/\prod_{i=1}^{n-1}(1+q^{\pi(i)})}$ diverges in $\mathbb{R}$. Then the number $\sum_{n=1}^{\infty} 1/(n^{p/q} a_n)$ is irrational by Theorems 1.11 and 1.26. Instead applying Theorem 1.31, we get the stronger result that $\sum_{n=1}^{\infty} 1/(n^{p/q} a_n) \notin \mathbb{Q}(\sqrt[q]{2}, \sqrt[q]{3}, \ldots)$.

Example 1.36. Let $s_1 > \ldots > s_K > 1$ be distinct rational numbers, let Q be the least common multiple of the denominators of $s_1, \ldots, s_K$, and let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive integers with $a_n \ge n^{\log n}$ such that the sequence $\left\{a_n^{1/\prod_{i=1}^{n-1}(1+Q^{\pi(i)})}\right\}_{n=1}^{\infty}$ diverges in $\mathbb{R}$. Then the numbers $1, \sum_{n=1}^{\infty} 1/(n^{s_1} a_n), \ldots, \sum_{n=1}^{\infty} 1/(n^{s_K} a_n)$ are linearly independent over $\mathbb{Q}(\sqrt[Q]{2}, \sqrt[Q]{3}, \ldots)$ and, in particular, over $\mathbb{Q}$. This follows from Theorem 1.31 with $\alpha = 2/3$ and $a_{i,n} = n^{s_i} a_n$.

We now consider similar examples for infinite products and for infinite products of infinite series.

Example 1.37. Let $p, q \in \mathbb{N}$ with $p > q$, and let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive integers such that $a_n^{1/\prod_{i=1}^{n-1}(1+q^{\pi(i)})}$ diverges in $\mathbb{R}$. By Theorem 1.29, the number $\prod_{n=1}^{\infty}\big(1 + 1/(n^{p/q} a_n)\big)$ is irrational and, in fact, not contained in the field $\mathbb{Q}(\sqrt[q]{2}, \sqrt[q]{3}, \ldots)$.

Example 1.38. Let $q, m_0 \in \mathbb{N}$, and let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive integers with $a_n \ge n^{\log n}$ such that $\limsup_{n\to\infty} a_n^{1/(n!\prod_{i=1}^{n-1} q^{\pi(i)})} = \infty$. Write $a_{m,1} = 1$ and $a_{m,n} = a_{n+m-m_0-1}$ for $n > 1$. Then Theorem 1.33 with $\alpha = 2/3$ ensures that the number

$$\prod_{m=m_0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{m/q} a_{m,n}} = \prod_{m=m_0}^{\infty} \left(1 + \sum_{n=2}^{\infty} \frac{1}{n^{m/q} a_{m,n}}\right)$$

is irrational and, in fact, not contained in the field $\mathbb{Q}(\sqrt[q]{2}, \sqrt[q]{3}, \ldots)$.

Remark 1.39. The assumption $a_n \ge n^{\log n}$ that is present in Examples 1.36 and 1.38 is most likely unnecessary for the examples to be true. However, removing this assumption requires improvements to the Erdős Jump, similar to those made for case 2 of Lemma 1.16. While this should not be too difficult, it will require more time than this author has available at the time of writing.

Our final example is a special case of [39, Example 3.1].

Example 1.40. Let x be an algebraic integer with $x = \overline{|x|} > 1$. By Corollary 1.30 and Theorem 1.31, the sequence $\{x^{h_n}\}_{n=1}^{\infty}$ is $\Sigma_{\mathbb{Q}(x)}$- and $\Pi_{\mathbb{Q}(x)}$-irrational if $h_n \in \mathbb{N}$ and $h_n \geq n^2$ for all n and $h_n \geq 3^n \log n$ infinitely often. For instance, we may take $h_n = 3^n n$ or $h_n = 3^{2^{\lfloor \log_2 n \rfloor}} + n$, where $\lfloor \log_2 n \rfloor$ denotes the largest integer no greater that $\log_2 n$.

1.4.2 Proof of an improved theorem

We will now prove Theorem 1.31. The analytical part of the proof is an application of Lemma 1.16, while the algebraic part is covered by the following two lemmas. The first lemma combines ideas from [36, Lemma 11] and [35, Lemma 15], while the latter generalizes a statement used in the proof of [36, Lemma 11].

Lemma 1.41. *Let $D, K \in \mathbb{N}$, let $\{\mathbb{K}_n\}_{n=1}^{\infty}$ be a sequence of number fields satisfying $\mathbb{K}_n \subseteq \mathbb{K}_{n+1}$, and let $\mathbb{K}$ denote the field $\bigcup_{n=1}^{\infty} \mathbb{K}_n$. Write $\mathbb{K}_0 = \mathbb{Q}$, $d_n = [\mathbb{K}_n : \mathbb{K}_{n-1}]$, and $D_n = \prod_{i=1}^{n} d_i$. For $i = 1, \ldots, K$, let $\{a_{i,n}\}_{n\in\mathbb{N}}$ and $\{b_{i,n}\}_{n\in\mathbb{N}}$ be sequences of algebraic integers such that $a_{i,n} \in \mathcal{O}_{\mathbb{K}_n}$, $b_{i,n} \in \mathbb{N}$, $b_{i,n} \leq \max_{1\leq i\leq n} a_{i,n}$, and $\sum_{n=1}^{\infty} |b_{i,n}/a_{i,n}|$ converges for all i. Let $\mathbb{L}$ be a field extension of $\mathbb{K}$ of degree at most D, let $\beta_1, \ldots, \beta_K \in \mathbb{L}$, and write*

$$\gamma_N = \sum_{i=1}^{K} \beta_i \sum_{n=1}^{N} \frac{b_{i,n}}{a_{i,n}} \quad \textit{and} \quad \gamma = \sum_{i=1}^{K} \beta_i \sum_{n=1}^{\infty} \frac{b_{i,n}}{a_{i,n}}$$

Suppose that $\{\gamma_N\}_{N=1}^{\infty}$ does not contain any constant subsequence and that $\gamma \in \mathbb{L}$. Then

$$\lim_{N\to\infty} \left(\left(2^{N^2} \prod_{n=1}^{N} \max_{1\leq i\leq K} \overline{|a_{i,n}|}^K \right)^{DD_N} \sum_{n=N+1}^{\infty} \max_{1\leq i\leq K} \frac{b_{i,n}}{|a_{i,n}|} \right) = \infty.$$

Proof. Because γ_N does not attain the same value infinitely often, we must have $\gamma_N \neq \gamma$ for all large enough N. We then have from Theorem 1.25 that

$$|\gamma - \gamma_N| \geq \big(2H(\gamma - \gamma_N)\big)^{-\deg(\gamma-\gamma_N)}. \tag{1.9}$$

Pick $\xi \in \mathbb{L}$ such that $\mathbb{L} = \mathbb{K}(\xi)$, and write $\mathbb{L}_n = \mathbb{K}_n(\xi)$. Since $\deg_{\mathbb{K}} \xi = [\mathbb{L} : \mathbb{K}] \leq D$, there is a polynomial $P \in \mathbb{K}[X]$ of $\deg P \leq D$ and $P(\xi) = 0$. Because $\mathbb{K} = \bigcup_{n=1}^{\infty} \mathbb{K}_n$ and $\mathbb{K}_n \subseteq \mathbb{K}_{n+1}$, it follows that the coefficients of P must be contained in $\mathbb{K}_n$ for all sufficiently large n. Hence, $[\mathbb{L}_n : \mathbb{K}_n] \leq D$ when n is large enough. Clearly, we also have $\mathbb{L} = \bigcup_{n=1}^{\infty} \mathbb{L}_n$, which means that we eventually have $\beta_1, \beta_2, \ldots, \beta_K, \gamma \in \mathbb{L}_N$ and, consequently, $\gamma - \gamma_N \in \mathbb{L}_N$. The upshot is that when N is sufficiently large,

$$\deg(\gamma - \gamma_N) \leq [\mathbb{L}_N : \mathbb{Q}] = [\mathbb{L}_N : \mathbb{K}_N] \prod_{n=1}^{N} [\mathbb{K}_n : \mathbb{K}_{n-1}] \leq DD_N.$$

Combined with inequality (1.9), we have

$$|\gamma - \gamma_N| \geq \big(2H(\gamma - \gamma_N)\big)^{-DD_N} \tag{1.10}$$

By Lemma 1.23 and Theorem 1.22,

$$\begin{aligned} H(\gamma - \gamma_N) &\leq (K+2)H(\gamma) \prod_{i=1}^{K} \left(H(\beta_i)(N+1) \prod_{n=1}^{N} H\left(\frac{b_{i,n}}{a_{i,n}}\right)\right) \\ &\leq 2^{N/D-1} \prod_{i=1}^{K}\prod_{n=1}^{N} H\left(\frac{a_{i,n}}{b_{i,n}}\right) \leq 2^{N/D-1} \prod_{n=1}^{N}\prod_{i=1}^{K} \max\{\lceil |a_{i,n}| \rceil, b_{i,n}\}. \end{aligned}$$

for all sufficiently large N. Since $b_{i,n} \leq \max_{1\leq i\leq n} |a_{i,n}|$, inequality (1.10) now yields

$$|\gamma - \gamma_N| \geq \left(2^N \prod_{n=1}^{N} \max_{1\leq i\leq n} \lceil |a_{i,n}| \rceil^{DK}\right)^{-D_N}$$

From this and the triangle inequality follows that

$$\begin{aligned} \left(2^N \prod_{n=1}^{N} \max_{1\leq i\leq n} \lceil |a_{i,n}| \rceil^{DK}\right)^{-D_N} &\leq \left| \sum_{i=1}^{K} \beta_i \sum_{n=N+1}^{\infty} \frac{b_{i,n}}{a_{i,n}} \right| \leq \sum_{i=1}^{K} |\beta_i| \sum_{n=N+1}^{\infty} \left| \frac{b_{i,n}}{a_{i,n}} \right| \\ &\leq N \sum_{n=N+1}^{\infty} \max_{1\leq i\leq K} \left| \frac{b_{i,n}}{a_{i,n}} \right|, \end{aligned}$$

for all large enough N, which completes the proof as it implies

$$\left(2^{N^2} \prod_{n=1}^{N} \max_{1\leq i\leq n} \lceil |a_{i,n}| \rceil^{DK}\right)^{D_N} \sum_{n=N+1}^{\infty} \max_{1\leq i\leq K} \left| \frac{b_{i,n}}{a_{i,n}} \right| \geq \frac{2^{N(N-1)D_N}}{N}. \qquad \square$$

Lemma 1.42. *Let $\zeta \in \mathbb{C}$ with $|\zeta| = 1$. For $i = 1, \ldots, K$, let $\{a_{i,n}\}_{n\in\mathbb{N}}$ and $\{b_{i,n}\}_{n\in\mathbb{N}}$ be sequences of non-zero complex numbers with $b_{i,n} = |b_{i,n}|$ so that*

$$\Re\left(\zeta b_{1,n}/a_{1,n}\right) > 0.$$

Let $\beta_1, \ldots, \beta_K$ be complex numbers that are not all 0, and write

$$\gamma_N = \sum_{i=1}^{K} \beta_i \sum_{n=1}^{N} \frac{b_{i,n}}{a_{i,n}}$$

If equation (1.7) *is true whenever $1 \leq i < K$, or if $\beta_1, \ldots, \beta_K \in \mathbb{R}$ and equation* (1.8) *is true whenever $1 \leq i < K$, then the sequence $\{\gamma_N\}_{N=1}^{\infty}$ does not contain any constant subsequence.*

Proof. Let $M, N \in \mathbb{N}$ with $M > N$, and let $R \leq K$ be the maximal index such that $\beta_R \neq 0$. If $R = 1$, then

$$\Re\left(\zeta\frac{\gamma_N}{\beta}\right) = \Re\left(\sum_{n=1}^{N}\zeta\frac{b_{1,n}}{a_{1,n}}\right) = \sum_{n=1}^{N}\Re\left(\zeta\frac{b_{1,n}}{a_{1,n}}\right)$$

is strictly increasing, and the proof is complete. We then suppose $R > 2$. Notice that it will be sufficient to prove that $\gamma_M \neq \gamma_N$ when N is sufficiently large.

Suppose equation (1.7) is true for $i = 1, \ldots, R-1$. By the converse triangle inequality,

$$|\gamma_M - \gamma_N| = \left|\sum_{i=1}^{R}\beta_i\sum_{n=N+1}^{M}\frac{b_{i,n}}{a_{i,n}}\right| \geq |\beta_R|\left(\sum_{n=N+1}^{M}\left|\frac{b_{R,n}}{a_{R,n}}\right| - \sum_{i=1}^{R-1}\left|\frac{\beta_i}{\beta_R}\right|\left|\frac{b_{i,n}}{a_{i,n}}\right|\right).$$

Writing $\beta = \max_{1\leq i<R}|\beta_i|$, it now follows from equation (1.7) that when N is sufficiently large,

$$|\gamma_M - \gamma_N| \geq |\beta_R|\sum_{n=N+1}^{M}\left(\left|\frac{b_{i,n}}{a_{i,n}}\right| - \frac{(R-1)\beta}{|\beta_R|}\left|\frac{b_{R-1,n}}{a_{R-1,n}}\right|\right) > 0.$$

This completes the proof in the present case.

Suppose now that $\beta_1, \ldots, \beta_K \in \mathbb{R}$ and that assumption (1.8) is true whenever $1 \leq i < K$. The proof is then identical to that of the claim in the proof of [36, Lemma 11]. We repeat the argument here for clarity, making only a minimal amount of changes to notation and wording.

Assume without loss of generality that $\beta_R > 0$ (otherwise replace each β_i by $-\beta_i$ for all i). Using that the the real value map is linear over $\mathbb{R}$, together with the first part of assumption 1.8, we find for each $n \in \mathbb{N}$ that

$$\begin{aligned}\Re\left(\zeta\sum_{j=1}^{K}\beta_j\frac{b_{j,n}}{\alpha_{j,n}}\right) &= \sum_{j=1}^{K}\Re\left(\zeta\beta_j\frac{b_{j,n}}{\alpha_{j,n}}\right)\\ &= \Re\left(\zeta\beta_R\frac{b_{R,n}}{\alpha_{R,n}}\right)\left(1 + \sum_{j=1}^{R-1}\frac{\Re(\zeta\beta_j b_{j,n}/\alpha_{j,n})}{\Re(\zeta\beta_R b_{R,n}/\alpha_{R,n})}\right).\end{aligned}$$

The second part of assumption (1.8) then implies for n sufficiently large that

$$\left|\sum_{j=1}^{R-1}\beta_j\frac{\Re(\zeta b_{j,n}/\alpha_{j,n})}{\Re(\zeta b_{R,n}/\alpha_{R,n})}\right| < \beta_R,$$

and thus $\Re\left(\zeta \sum_{j=1}^{K} \beta_j \frac{b_{j,n}}{\alpha_{j,n}}\right) > 0$, by the assumption that $\Re(\zeta b_{j,n}/a_{j,n}) > 0$. For N sufficiently large, it hence follows that

$$\Re(\zeta\gamma_N) = \sum_{n=1}^{N} \Re\left(\zeta \sum_{j=1}^{K} \beta_j \frac{b_{j,n}}{\alpha_{j,n}}\right) < \sum_{n=1}^{M} \Re\left(\zeta \sum_{j=1}^{K} \beta_j \frac{b_{j,n}}{\alpha_{j,n}}\right) = \Re(\zeta\gamma_M),$$

which completes the proof. □

We are now ready to properly prove Theorem 1.31.

Proof of Theorem 1.31. We first prove the statements of linear independence of numbers, so we may assume that the sequence $\left\{|a_{1,n}|^{1/\prod_{i=1}^{n-1}(d_i+DD_iK)}\right\}_{n=1}^{\infty}$ diverges in $\mathbb{R}$. If assumption (1.8) with $\beta_1, \ldots, \beta_K \in \mathbb{R}$ is used rather than equation (1.7), then replace $\mathbb{L}$ by $\mathbb{L} \cap \mathbb{R}$ in the following. Let $(\beta_1, \ldots, \beta_K) \in \mathbb{L}^K \setminus \{0\}^K$, and let γ_N and γ be defined as in Lemma 1.41. If we had $\gamma \in \mathbb{L}$, then Lemmas 1.42 and 1.41 would imply that

$$\lim_{N\to\infty} \left(\left(2^{N^2} \prod_{n=1}^{N} \max_{1\le i\le K} \overline{|a_{i,n}|}^{DK}\right)^{D_N} \sum_{n=N+1}^{\infty} \max_{1\le i\le K} \frac{b_{i,n}}{|a_{i,n}|}\right) = \infty.$$

Applying the bounds of $b_{i,n}$ and $\overline{|a_{i,n}|}$, we get

$$\begin{aligned}\left(2^{N^2} \prod_{n=1}^{N} |a_{1,n}|^{DK} 2^{DK \log_2^{\alpha} |a_{1,n}|}\right)^{D_N} \sum_{n=N+1}^{\infty} \frac{2^{2\log_2^{\alpha}|a_{1,n}|}}{|a_{1,n}|} \\ \ge \left(2^{N^2} \prod_{n=1}^{N} \max_{1\le i\le K} \overline{|a_{i,n}|}^{DK}\right)^{D_N} \sum_{n=N+1}^{\infty} \max_{1\le i\le K} \frac{b_{i,n}}{|a_{i,n}|} \ge 1,\end{aligned}$$

which contradicts case 1 of Lemma 1.16 with $m_n = d_n$ and $\mu = DK$. Hence, $\gamma \notin \mathbb{L}$. Since$(\beta_1, \ldots, \beta_K) \in \mathbb{L}^K \setminus \{0\}^K$ where chosen arbitrarily, this proves that the numbers $1, \sum_{n=1}^{\infty} \frac{a_{1,n}}{b_{1,n}}, \ldots, \sum_{n=1}^{\infty} \frac{a_{K,n}}{b_{K,n}}$ are linearly independent over $\mathbb{L}$.

The argument for linear independence of sequences then follows in complete parallel to Corollary 1.27. □

1.4.3 Paper 1: Algebraic degree of reciprocal algebraic integers

Below, the reader will find the paper [36], which has the current author as its sole author. The paper was published in *Rocky Mountains Journal of Mathematics* in April 2022 and is available through the link `https://doi.org/10.1007/s40993-024-00553-2`. While the published version of the paper is not available without a subscription, the preprint is freely available on arXiv through the link `https://arxiv.org/abs/2203.11786` or via the arXiv identifier 2203.11786.

Though the published version is nicer to look at than the preprint, the differences between the two versions are of no real significance. These differences entail a different formatting, corrections of minor typing errors, and slight changes to wording and notation. Finally, the theorems and lemmas are numbered differently in the two versions; where the preprint writes Theorem 1, Theorem 2, ..., Theorem 4, Lemma 1, Lemma 2, ..., Lemma 8, and Theorem 5, the published version instead writes Theorem 1, Theorem 2, ..., Theorem 4, Lemma 5, Lemma 6, ..., Lemma 12, and Theorem 13. The lemmas and theorems are presented in the same order in the two versions, however.

The journal preferred that the published version would not be included in this thesis. Respecting this, we will instead see the preprint as it is available on arXiv but with the numbering of its lemmas and theorems changed to match that of the published paper. The presented version has a length of 15 pages, numbered 1 through 15.

ALGEBRAIC DEGREE OF SERIES OF RECIPROCAL ALGEBRAIC INTEGERS

MATHIAS LØKKEGAARD LAURSEN

Abstract. In this paper, I give sufficient conditions for any linear combination in $\mathbb{Q}$ of numbers $\sum_{n=1}^{\infty} \frac{b_{1,n}}{\alpha_{1,n}}, \ldots, \sum_{n=1}^{\infty} \frac{b_{K,n}}{\alpha_{K,n}}$ to have algebraic degree greater than an arbitrary fixed integer D when the numbers $\alpha_{i,n}$ are algebraic integers of sufficiently rapidly increasing modulus and the $b_{i,n}$ are positive integers that are not too large.

1. Introduction

In 1975, Erdős [3] gave a sufficient condition for an increasing series of reciprocal integers to be irrational, as stated in the below theorem.

Theorem 1 (Erdős)**.** *Let $\{a_n\}_{n\in\mathbb{N}}$ be an increasing sequence of natural numbers such that $a_n > n^{1+\varepsilon}$ for some $\varepsilon > 0$ for all n sufficiently large. If $\limsup_{n\to\infty} a_n^{1/2^n} = \infty$, then $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is irrational.*

Later, in [4], Hančl extended this theorem to also cover the case of

$$1 \leq \limsup_{n\to\infty} a_n^{1/2^n} < \limsup_{n\to\infty} a_n^{1/2^n} < \infty,$$

while also providing a related condition for finitely many series of fractions $\sum_{n=1}^{\infty} \frac{b_{i,n}}{a_{i,n}}$ $(i = 1, \ldots, K)$ with sufficiently small and positive $b_{i,n}$ to be irrational and linear independent over $\mathbb{Q}$.

Theorem 2 (Hančl)**.** *Let $K \in \mathbb{N}$, and let $A_1, A_2, a, \varepsilon > 0$ be real numbers such that $a < 1 \leq A_1 < A_2$. For $i = 1, \ldots, K$, let $\{a_{i,n}\}_{n\in\mathbb{N}}$ and $\{b_{i,n}\}_{n\in\mathbb{N}}$ each be sequences of natural numbers. Suppose that*

$$\begin{aligned}
\forall n \in \mathbb{N} : \quad & n^{1+\varepsilon} \leq a_{1,n} < a_{1,n+1}, \\
& \limsup_{n\to\infty} {a_{1,n}}^{1/(K+1)^n} = A_2, \\
& \liminf_{n\to\infty} {a_{1,n}}^{1/(K+1)^n} = A_1, \\
\forall n \in \mathbb{N}\ \forall 1 \leq i \leq K : \quad & b_{i,n} < 2^{(\log_2 a_{1,n})^a}, \\
\forall 1 \leq i < j \leq K : \quad & \lim_{n\to\infty} \frac{b_{i,n} a_{j,n}}{a_{i,n} b_{j,n}} = 0, \\
\forall 1 < i \leq K : \quad & a_{i,n}\ 2^{-(\log_2 a_{1,n})^a} < a_{1,n} < a_{i,n}\ 2^{(\log_2 a_{1,n})^a}.
\end{aligned}$$

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Then $\sum_{n=1}^{\infty} \frac{b_{1,n}}{a_{1,n}}, \ldots, \sum_{n=1}^{\infty} \frac{b_{K,n}}{a_{K,n}}$ *are irrational and linearly independent over* $\mathbb{Q}$*.*

In 2017, Hančl and Nair [5] showed that integer sequences of the form $a_{n+1} = a_n^2 - a_n + 1$ with $a_1 \geq 2$ will satisfy both $\lim_{n\to\infty} a_n \in \mathbb{Q}$ and

$$1 < \sqrt[4]{a_1^2 - a_1} \leq \liminf_{n\to\infty} a_n^{1/2^n} = \limsup_{n\to\infty} a_n^{1/2^n} < \infty,$$

which exemplifies that the requirement $A_1 < A_2$ cannot in general be omitted from Theorem 2. The main result of [5] was a variant of Theorem 1 where the a_n may be square roots of positive integers when $\lim_{n\to\infty} a_n^{1/2^{n^2/2}} = \infty$. Two years later, this result was generalised to the below theorem by Andersen and Kristensen [1], which gives a sufficient condition for $\sum_{n=1}^{\infty} 1/\alpha_n$ to have large algebraic degree when α_n are algebraic integers of bounded degree.

Theorem 3 (Andersen and Kristensen)**.** *Let* $d, D \in \mathbb{N}$*,* $\varepsilon > 0$*, and let* $\{\alpha_n\}_{n\in\mathbb{N}}$ *be a series of algebraic integers of maximal degree* d *such that*

$$\begin{aligned} \forall n \in \mathbb{N}: \quad & n^{1+\varepsilon} \leq |\alpha_n| < |\alpha_{n+1}|, \\ & \limsup_{n\to\infty} |\alpha_n|^{\overline{\frac{1}{D^n \prod_{i=1}^{n-1}(d+d^i)}}} = \infty, \\ \forall n \in \mathbb{N}: \quad & \overline{|\alpha_n|} = |\alpha_n|. \end{aligned}$$

Suppose that $\Re(\alpha_n) > 0$ *holds for all* $n \in \mathbb{N}$ *or that* $\Im(\alpha_n) > 0$ *holds for all* $n \in \mathbb{N}$*. Then* $\deg \sum_{n=1}^{\infty} \frac{1}{\alpha_n}$ *is strictly greater than* D*.*

Both in the above theorem and for the remainder of this paper, $\overline{|\alpha|}$ denotes the house of an algebraic number α, which is defined as the maximum modulus among the conjugates of α.

As Andersen and Kristensen note in their paper, their proof only really needs $\overline{|\alpha_n|}$ to be bounded by $C|\alpha_n|$ for some uniform constant $C > 0$. Similarly, the restriction on the sign of the real (or imaginary) value of α_n is only to enforce that all α_n are contained in an open half plane not containing 0, which ensures that the partial sums $\sum_{n=1}^{N} \frac{1}{\alpha_n}$ are non-zero and non-conjugate to $\sum_{n=1}^{\infty} \frac{1}{\alpha_n}$ for sufficiently large N.

The main result of this paper is a generalisation of Theorem 2 in the spirit of Theorem 3, and the proof will combine the arguments used by the respective papers. For the sake of clarity, we will, however, be slightly more explicit with the open half-plane containing all α_n, as compared to Andersen's and Kristensen's proof of Theorem 3. For this purpose, I introduce notation $\Re_\zeta(z)$ to denote $\Re(\overline{\zeta} z)$ for $\zeta \in \mathbb{C}\setminus\{0\}$ and $z \in \mathbb{C}$, as $\Re_\zeta(z) > 0$ is then equivalent to z lying in the open half-plane with 0 on its border and moving in direction of ζ.

Theorem 4. *Let $D, K \in \mathbb{N}$, let $A_1, A_2, a, \varepsilon > 0$ be real numbers such that $a < 1 \leq A_1 < A_2$, and let $\zeta \in \mathbb{C}$ with $|\zeta| = 1$. Let $\{d_n\}_{n\in\mathbb{N}}$ be a sequence of natural numbers, and write $D_n = \prod_{i=1}^{n} d_i$. For $i = 1, \ldots, K$, let $\{b_{i,n}\}_{n\in\mathbb{N}}$ be a sequence of natural numbers, and let $\{\alpha_{i,n}\}_{n\in\mathbb{N}}$ be sequences of algebraic integers such that*

$$\forall n \in \mathbb{N} : |\alpha_{1,n}| < |\alpha_{1,n+1}| \tag{1}$$

$$\forall n \in \mathbb{N} : |\alpha_{1,n}| \geq n^{1+\varepsilon} \tag{2}$$

$$\forall n \in \mathbb{N} : \left[\mathbb{Q}\left(\alpha_{1,n}, \ldots, \alpha_{K,n}\right) : \mathbb{Q}\right] \leq d_n \tag{3}$$

$$\liminf_{n\to\infty} |\alpha_{1,n}|^{\frac{1}{D^n \prod_{i=1}^{n-1}(KD_i+d_i)}} = A_1 \tag{4}$$

$$\limsup_{n\to\infty} |\alpha_{1,n}|^{\frac{1}{D^n \prod_{i=1}^{n-1}(KD_i+d_i)}} = A_2 \tag{5}$$

$$\forall n \in \mathbb{N}\ \forall 1 < i \leq K : 2^{-(\log_2|\alpha_{1,n}|)^a} < \frac{|\alpha_{1,n}|}{|\alpha_{i,n}|} < 2^{(\log_2|\alpha_{1,n}|)^a} \tag{6}$$

$$\forall n \in \mathbb{N}\ \forall 1 \leq i \leq K : b_{i,n}\overline{|\alpha_{i,n}|} \leq 2^{(\log_2|\alpha_{1,n}|)^a}|\alpha_{i,n}| \tag{7}$$

$$\forall n \in \mathbb{N}\ \forall 1 \leq i \leq K : \Re_\zeta\left(\frac{b_{i,n}}{\alpha_{i,n}}\right) > 0, \tag{8}$$

$$\forall 1 \leq i < j \leq K : \lim_{n\to\infty}\left(\Re_\zeta\left(\frac{b_{i,n}}{\alpha_{i,n}}\right)/\Re_\zeta\left(\frac{b_{j,n}}{\alpha_{j,n}}\right)\right) = 0. \tag{9}$$

Then $\deg\gamma > D$ when γ is any non-trivial linear combination over $\mathbb{Q}$ of the numbers $\sum_{n=1}^{\infty}\frac{b_{1,n}}{\alpha_{1,n}}, \ldots, \sum_{n=1}^{\infty}\frac{b_{K,n}}{\alpha_{K,n}}$.

2. Auxiliary Results

The proof of Theorem 4 will be split into two parts, the first of which will be based around the Weil height and the Mahler measure. We recall the definitions below.

For K being some finite field extension of $\mathbb{Q}$ of degree d, we define for $\alpha \in K$ the Weil height of α as the number

$$H(\alpha) := \prod_{\nu\in M_K} \max\{1, |\alpha|_\nu\}^{d_\nu/d},$$

where M_k denotes the set of places of K, and $d_\nu = [K_\nu : \mathbb{Q}_\nu]$ denotes the degree of the completion of K with respect to place ν as an extension of the completion of $\mathbb{Q}$ with respect to ν. With the normalisation in the exponent d_ν/d, the definition is independent of the field K containing α. We define the Mahler measure of α as

$$M(\alpha) := |a_d| \prod_{i=1}^{n} \max\{1, |\alpha_i|\},$$

where a_d here denotes leading coefficient of the minimal polynomial in $\mathbb{Z}[X]$ of α, and $\alpha_1, \ldots, \alpha_d$ denote the conjugates of α.

The proof will furthermore use the following lemmas, the first of which relates Weil height, Mahler measure, and house of algebraic integers. The main part of the statement, $H(\alpha) = M(\alpha)^{1/d}$, is a classical result, which is presented in [8]. The rest is essentially a trivial consideration, see [1].

Lemma 5. *Let α be an algebraic number of degree d. Then*

$$H(\alpha) = M(\alpha)^{1/d} \leq \overline{|\alpha|} \leq M(\alpha) = H(\alpha)^d$$

The second lemma is a list of further classical results regarding the Weil height, see [8].

Lemma 6. *Let α, β be algebraic numbers. Then*

$$H(1/\alpha) = H(\alpha) \text{ if } \alpha \neq 0, \quad H(\alpha+\beta) \leq 2H(\alpha)H(\beta),$$
$$H(\alpha\beta) \leq H(\alpha)H(\beta)$$

Similar results are likewise true for the degree function, as seen by the below lemma.

Lemma 7. *Let α, β be algebraic numbers. Then*

$$\deg(1/\alpha) = \deg(\alpha) \text{ if } \alpha \neq 0,$$
$$\deg(\alpha+\beta) \leq \deg(\alpha)\deg(\beta), \quad \deg(\alpha\beta) \leq \deg(\alpha)\deg(\beta)$$

This is essentially trivial: Following the spirit of [6], the inequalities come from noting that $\alpha+\beta$ and $\alpha\beta$ both lie in the field extension $\mathbb{Q}(\alpha,\beta)$, which clearly has degree at most $\deg(\alpha)\deg(\beta)$ over $\mathbb{Q}$. Noting $1/\alpha \in \mathbb{Q}(\alpha)$ and $\alpha \in \mathbb{Q}(1/\alpha)$ for $\alpha \neq 0$, it is likewise obvious that $\deg(1/\alpha) = \deg \alpha$.

The below lemma is central for the first part of the proof of Theorem 4, and seems to originally be from [7]. A proof may also be extracted from the proof of Theorem A.1 in Appendix A of [2].

Lemma 8. *Let α, β be non-conjugate algebraic numbers. Then*

$$|\alpha - \beta| \geq \frac{1}{2^{\deg(\alpha)\deg(\beta)} M(\alpha)^{\deg(\beta)} M(\beta)^{\deg(\alpha)}}$$

In the second part of the proof of Theorem 4, we will occasionally need the below simple estimate related to the exponent of the limes superior and limes inferior.

Lemma 9. *Let $D, K, N \in \mathbb{N}$ be natural numbers, and let $\{d_n\}_{n\in\mathbb{N}}$ be a sequence of natural numbers. Writing $D_n = \prod_{i=1}^n d_i$, we have*

$$D^{N+1} \prod_{i=1}^{N} (KD_i + d_i) \geq KDD_n \sum_{n=1}^{N} D^n \prod_{i=1}^{n-1} (KD_i + d_i).$$

Proof. The first statement proven by induction in N. Note that the statement clearly holds for $N = 1$. Suppose it holds for $N - 1$, for some $N > 1$. Then

$$\begin{aligned} D^{N+1}\prod_{i=1}^{N}(KD_i + d_i) &= D(KD_N + d_N)\left(D^N\prod_{i=1}^{N-1}(KD_i + d_i)\right) \\ &\geq KDD_N\left(D^N\prod_{i=1}^{N-1}(KD_i + d_i)\right) \\ &\quad + d_N\left(KDD_{N-1}\sum_{n=1}^{N-1}D^n\prod_{i=1}^{n-1}(KD_i + d_i)\right) \\ &= KDD_N\sum_{n=1}^{N}D^n\prod_{i=1}^{n-1}(KD_i + d_i). \end{aligned}$$

□

Near the end of the proof, we will use a generalised version of a lemma from [3], which Erdős used for proving Theorem 1. The current version is presented and proven in [1].

Lemma 10. *Let $\varepsilon > 0$, and let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence of real numbers satisfying $a_n > n^{1+\varepsilon}$ for all $n \in \mathbb{N}$. Then*

$$\sum_{n=N}\frac{1}{a_n} < \frac{2 + 1/\varepsilon}{a_N^{\varepsilon/(1+\varepsilon)}}.$$

3. Proof of Main Result

The proof of Theorem 4 will be split into two lemmas:

Lemma 11. *Let $D, K \in \mathbb{N}$, let $\zeta \in \mathbb{C}$ with $|\zeta| = 1$, and let $a, c, A_2 > 0$ such that $c < a < 1 < A_2$. Let $\{d_n\}_{n\in\mathbb{N}}$ be a sequence of natural numbers, and write $D_n = \prod_{i=1}^{n} d_i$. For $i = 1, \dots, K$, let $\{\alpha_{i,n}\}_{n\in\mathbb{N}}$ be a sequence of algebraic integers, and $\{b_{i,n}\}_{n\in\mathbb{N}}$ be a sequence of natural numbers. Suppose that equations* (3), (5), (6), (7), (8), (9) *are satisfied, let $\beta_1, \dots, \beta_K \in \mathbb{Z}$ be integers that are not all 0, and write*

$$\gamma = \sum_{j=1}^{K}\beta_j\sum_{n=1}^{\infty}\frac{b_{j,n}}{\alpha_{j,n}}, \qquad \gamma(N) = \sum_{j=1}^{K}\beta_j\sum_{n=N+1}^{\infty}\frac{b_{j,n}}{\alpha_{j,n}}$$

If $\deg\gamma \leq D$ and $c \in (a, 1)$, then

$$|\gamma(N)|\left(2^{D^{cN}\prod_{i=1}^{N-1}(KD_i+d_i)^c}\prod_{n=1}^{N}|\alpha_{1,n}|^K\right)^{DD_N} \geq 1$$

holds for all sufficiently large N.

Lemma 12. *Let $D, K \in \mathbb{N}$, and let $A_1, A_2, a, \varepsilon > 0$ be real numbers such that $a < 1 \leq A_1 < A_2$. Let $\{d_n\}_{n\in\mathbb{N}}$ be a sequence of natural numbers, and write $D_n = \prod_{i=1}^n d_i$. For $i = 1, \dots, K$, let $\{\alpha_{i,n}\}_{n\in\mathbb{N}}$ and $\{b_{i,n}\}_{n\in\mathbb{N}}$ be sequences of complex numbers. Suppose that equations* (1), (2), (4), (5), (6) *hold and that*

$$\forall n \in \mathbb{N}\ \forall 1 \leq i \leq K : \quad |b_{i,n}| \leq 2^{(\log_2 |\alpha_{1,n}|)^a}. \tag{10}$$

Let $\beta_1, \dots, \beta_K \in \mathbb{Z}$ be integers that are not all 0, and write

$$\gamma(N) = \sum_{j=1}^{K} \beta_j \sum_{n=N+1}^{\infty} \frac{b_{j,n}}{\alpha_{j,n}}$$

Let $c \in (a, 1)$. Then

$$\liminf_{N\to\infty} |\gamma(N)| \left(2^{D^{cN} \prod_{i=1}^{N-1}(KD_i+d_i)^c} \prod_{n=1}^{N} |\alpha_{1,n}|^K \right)^{DD_N} = 0.$$

One minor result that will be briefly used for proving both lemmas is that equation (5) implies

$$|\alpha_{1,n}| \leq (2A_2)^{D^n \prod_{i=1}^{n-1}(KD_i+d_i)} \tag{11}$$

for n sufficiently large

We now prove Lemma 11:

Proof(Lemma 11). We introduce further notation

$$\gamma_N := \sum_{i=1}^{K} \beta_i \sum_{n=1}^{N} \frac{b_{i,n}}{\alpha_{i,n}}, \qquad \beta := \prod_{i=1}^{K} H(\beta_i).$$

By Lemma 7 and equation (3), we quickly find

$$\deg \gamma_N \leq \prod_{n=1}^{N} \deg \left(\sum_{i=1}^{K} \beta \frac{b_{i,n}}{\alpha_{i,n}} \right) \leq D_N.$$

Applying Lemma 5 and Lemma 6 followed by equation (7), we then get

$$\begin{aligned}
M(\gamma_N) = H(\gamma_N)^{\deg \gamma_N} &\leq \left(2^{NK} \prod_{i=1}^{K} H(\beta_i) \prod_{n=1}^{N} H(b_{i,n}) H\left(\frac{1}{\alpha_{i,n}}\right) \right)^{D_N} \\
&= \left(2^{KN} \beta \prod_{i=1}^{K} \prod_{n=1}^{N} H(\alpha_{i,n}) H(b_{i,n}) \right)^{D_N} \\
&\leq \left(2^{KN} \beta \prod_{i=1}^{K} \prod_{n=1}^{N} \overline{|\alpha_{i,n}|}\, \overline{|b_{i,n}|} \right)^{D_N} \\
&\leq \left(\beta 2^{KN + KN(\log_2 |\alpha_{1,N}|)^a} \prod_{i=1}^{K} \prod_{n=1}^{N} |\alpha_{i,n}| \right)^{D_N},
\end{aligned}$$

using that $\alpha_{1,n}$ is non-decreasing and that $\lceil b_{i,n} \rceil = b_{i,n}$ as $b_{i,n} \in \mathbb{N}$. From equations (6) and (11), we then have for N sufficiently large that

$$M(\gamma_N) < \left(2^{2KN\left(\log_2(2A_2)D^N \prod_{i=1}^{N-1}(KD_i+d_i)\right)^a} \prod_{n=1}^{N} |\alpha_{1,n}|^K \right)^{D_N}$$

$$\leq \left(\frac{2^{D^{cN} \prod_{i=1}^{N-1}(KD_i+d_i)^c}}{2H(\gamma)} \prod_{n=1}^{N} |\alpha_{1,n}|^K \right)^{D_N}. \tag{12}$$

We now wish to apply Lemma 10 to get an estimate on $|\gamma(N)|$. To do so, we need $\gamma \neq \gamma_N$, which is ensured for sufficiently large N if the γ_N are mutually distinct from a some point.

Claim ($\gamma_M \neq \gamma_N$ for $M > N$ sufficiently large)**.** To see this, let R be the maximal value of i such that $\beta_i \neq 0$ and assume without loss of generality that $\beta_R > 0$ (otherwise replace each β_i by $-\beta_i$ for all i). Using that $\Re_\zeta$ is clearly linear in $\mathbb{R}$, we find for each $n \in \mathbb{N}$ that

$$\begin{aligned} \Re_\zeta \left(\sum_{j=1}^{K} \beta_j \frac{b_{j,n}}{\alpha_{j,n}} \right) &= \sum_{j=1}^{K} \beta_j \Re_\zeta \left(\frac{b_{j,n}}{\alpha_{j,n}} \right) \\ &= \Re_\zeta \left(\frac{b_{R,n}}{\alpha_{R,n}} \right) \left(\beta_R + \sum_{j=1}^{R-1} \beta_j \frac{\Re_\zeta(b_{j,n}/\alpha_{j,n})}{\Re_\zeta(b_{R,n}/\alpha_{R,n})} \right) \end{aligned}$$

Equation (9) then implies that for n sufficiently large, we have

$$\left| \sum_{j=1}^{R-1} \beta_j \frac{\Re_\zeta(b_{j,n}/\alpha_{j,n})}{\Re_\zeta(b_{R,n}/\alpha_{R,n})} \right| < \beta_R,$$

and thus $\Re_\zeta \left(\sum_{j=1}^{K} \beta_j \frac{b_{j,n}}{\alpha_{j,n}} \right) > 0$, by equation (8). For N sufficiently large, it hence follows that

$$\Re_\zeta \gamma_N = \sum_{n=1}^{N} \Re_\zeta \sum_{j=1}^{K} \beta_j \frac{b_{j,n}}{\alpha_{j,n}} < \sum_{n=1}^{M} \Re_\zeta \sum_{j=1}^{K} \beta_j \frac{b_{j,n}}{\alpha_{j,n}} = \Re_\zeta \gamma_M,$$

which implies the claim.

Since γ can have at most D conjugates, it then follows that γ and γ_N must be non-conjugate for N sufficiently large, and we may apply Lemma 8, Lemma 5, and equation (12) (in that order) to find

$$\begin{aligned} |\gamma(N)| = |\gamma - \gamma_N| &\geq \frac{1}{2^{\deg(\gamma)\deg(\gamma_N)} M(\gamma)^{\deg(\gamma_N)} M(\gamma_N)^{\deg(\gamma)}} \\ &\geq \frac{1}{2^{DD_N} H(\gamma)^{DD_N} M(\gamma_N)^D} \\ &\geq \frac{1}{\left(2^{D^{cN} \prod_{i=1}^{N-1}(KD_i+d_i)^c} \prod_{n=1}^{N} |\alpha_{1,n}|^K \right)^{DD_N}}. \end{aligned}$$

This proves the lemma. □

Proof (Lemma 12). Applying equations (6) and (10) followed by (1), we find

$$|\gamma(N)| = \left| \sum_{n=N+1}^{\infty} \sum_{j=1}^{K} \frac{b_{j,n}}{\alpha_{j,n}} \right| \leq \sum_{n=N+1}^{\infty} \sum_{j=1}^{K} \frac{|b_{j,n}|}{|\alpha_{j,n}|}$$

$$\leq \sum_{n=N+1}^{\infty} \frac{K 2^{(\log_2 |\alpha_{1,n}|)^a}}{|\alpha_{1,n}| 2^{-(\log_2 |\alpha_{1,n}|)^a}} \leq \sum_{n=N+1}^{\infty} \frac{2^{(\log_2 |\alpha_{1,n}|)^c}}{|\alpha_{1,n}|}, \tag{13}$$

for N sufficiently large.

We now split into two cases, both using the notation

$$a_n := |\alpha_{1,n}|, \qquad S_n := a_n^{\frac{1}{D^n \prod_{i=n}^{n-1}(KD_i + d_i)}}.$$

Case 1 ($a_n \geq 2^n$ for all n sufficiently large). We continue on equation (13) and use that the function $x^{(\log_2 x)^c}/x$ is decreasing for $x > 1$ to find

$$|\gamma(N)| \leq \sum_{N<n\leq \log a_{N+1}}^{\infty} \frac{2^{(\log_2 a_n)^c}}{a_n} + \sum_{n>\log a_{N+1}}^{\infty} \frac{2^{(\log_2 a_n)^c}}{a_n}$$

$$\leq \frac{2^{2(\log_2 a_{N+1})^c}}{a_{N+1}} + \sum_{n>\log a_{N+1}}^{\infty} \frac{2^{(\log_2 2^n)^c}}{2^n}$$

$$= \frac{2^{2(\log_2 a_{N+1})^c}}{a_{N+1}} + \sum_{n>\log a_{N+1}}^{\infty} \frac{1}{2^{n-cn}}$$

$$\leq \frac{2^{2(\log_2 a_{N+1})^c}}{a_{N+1}} + C \frac{1}{2^{\log_2 a_{N+1} - (\log_2 a_{N+1})^c}} \leq \frac{2^{(\log_2 a_{N+1})^\omega}}{a_{N+1}}, \tag{14}$$

for sufficiently large N, where $C > 0$ and $\omega \in (c, 1)$ do not depend on N. The above equation is (safe for notational differences) a direct transcription of equation (14) of [4], which is repeated here for clarity.

Next, we will make a choice of N that will later show the conclusion of Lemma 12. Let $\delta > 0$ be a "sufficiently" small number (we will later make uniform assumptions on its size). By equations (5) and (4), there exist $s_0 \in N$ such that

$$\max\{1, A_1 - \delta\} < S_n < A_2 + \delta \tag{15}$$

holds for all $n \geq s_0$. For each such s_0, pick $s_1 \in \mathbb{N}$ minimal such that

$$s_1 > D^{s_0} \prod_{i=1}^{s_0-1} (KD_i + d_i), \quad \max\{1, A_1 - \delta\} < S_{s_1} < A_1 + \delta, \tag{16}$$

and pick then $s_2 \in \mathbb{N}$ minimal such that

$$s_2 > s_1, \quad A_2 - \delta < S_{s_2} < A_2 + \delta. \tag{17}$$

For sufficiently large s_0, pick $N = N(s_0) \in \mathbb{N}$ minimal such that

$$s_1 \leq N < s_2, \quad S_{N+1} > \left(1 + \frac{1}{(N+1)^2}\right) \max_{s_1 \leq j \leq N} \{S_j, A_2 - 2\delta\}. \tag{18}$$

This is doable as the contrary would imply

$$\begin{aligned} A_2 - \delta < S_{s_2} &\leq \left(1 + \frac{1}{s_2^2}\right) \max_{s_1 \leq j < s_2} \{S_j, A_2 - 2\delta\} \\ &\leq \ldots \leq \max\{S_{s_1}, A_2 - 2\delta\} \prod_{j=s_1+1}^{s_2} \left(1 + \frac{1}{j^2}\right) \\ &\leq (A_2 - 2\delta) \prod_{j=s_1+1}^{\infty} \left(1 + \frac{1}{j^2}\right), \end{aligned}$$

which would be a contradiction for large enough s_0 (and thus s_1), regardless of δ.

We then apply equation (18) along with Lemma 9 to find

$$\begin{aligned} a_{N+1} &= S_{N+1}^{D^{N+1} \prod_{i=1}^{N} (KD_i + d_i)} \\ &> \left(1 + \frac{1}{(N+1)^2}\right)^{D^{N+1} \prod_{i=1}^{N} (KD_i + d_i)} \\ &\quad \max_{s_1 \leq j \leq N} \{S_j, A_2 - 2\delta\}^{D^{N+1} \prod_{i=1}^{N} (KD_i + d_i)} \\ &\geq \left(1 + \frac{1}{(N+1)^2}\right)^{D^{N+1} \prod_{i=1}^{N} (KD_i + d_i)} \\ &\quad \left(\max_{s_1 \leq j \leq N} \{S_j, A_2 - 2\delta\}^{\sum_{n=1}^{N} D^n \prod_{i=1}^{n-1} (KD_i + d_i)}\right)^{KDD_N} \\ &\geq \left(1 + \frac{1}{(N+1)^2}\right)^{D^{N+1} \prod_{i=1}^{N} (KD_i + d_i)} \left(\prod_{n=s_1+1}^{N} a_n\right)^{KDD_N} \\ &\quad \left(\prod_{n=1}^{s_1} (A_2 - 2\delta)^{D^n \prod_{i=1}^{n-1} (KD_i + d_i)}\right)^{KDD_N} \\ &\geq \left(1 + \frac{1}{(N+1)^2}\right)^{D^{N+1} \prod_{i=1}^{N} (KD_i + d_i)} \left(\prod_{n=1}^{N} a_n\right)^{KDD_N} \\ &\quad \left(\frac{1}{\prod_{n=1}^{s_0-1} a_n} \prod_{n=s_0}^{s_1} \frac{(A_2 - 2\delta)^{D^n \prod_{i=1}^{n-1} (KD_i + d_i)}}{a_n}\right)^{KDD_N}, \end{aligned} \tag{19}$$

for a small enough choice of δ. For sufficiently large s_0, equation (11) gives

$$\frac{1}{\prod_{n=1}^{s_0-1} a_n} \geq \frac{1}{\prod_{n=1}^{s_0-1}(3A_2)^{D^n \prod_{i=1}^{n-1}(KD_i+d_i)}} \geq (3A_2)^{-N}, \tag{20}$$

by choice of s_1 and N. Meanwhile, equations (15) and (16) followed by Lemma 9 give

$$\prod_{n=s_0}^{s_1} \frac{(A_2-2\delta)^{D^n \prod_{i=1}^{n-1}(KD_i+d_i)}}{a_n}$$
$$\geq \left(\prod_{n=s_0}^{s_1-1} \frac{(A_2-2\delta)^{D^n \prod_{i=1}^{n-1}(KD_i+d_i)}}{(A_2+\delta)^{D^n \prod_{i=1}^{n-1}(KD_i+d_i)}}\right) \frac{(A_2-2\delta)^{D^{s_1} \prod_{i=1}^{s_1-1}(KD_i+d_i)}}{(A_1+\delta)^{D^{s_1} \prod_{i=1}^{s_1-1}(KD_i+d_i)}}$$
$$\geq \prod_{n=s_0}^{s_1-1} \left(\frac{(A_2-2\delta)^2}{(A_2+\delta)(A_1+\delta)}\right)^{D^n \prod_{i=1}^{n-1}(KD_i+d_i)} \geq 1, \tag{21}$$

by choosing $\delta > 0$ small enough that $(A_2-2\delta)^2 > (A_2+\delta)(A_1+\delta)$. Notice that since d_i and K are all positive integers, we must have $KD_i + d_i \geq 2$, which ensures

$$\prod_{i=1}^{N}(KD_i+d_i) \geq \frac{\log 2}{\log\left(1+\frac{1}{(N+1)^2}\right)} N^3 D_N \prod_{i=1}^{N-1}(KD_i+d_i)^{\omega},$$

for large enough N (recall $c < \omega < 1$), using that $1/\log\left(1+\frac{1}{(N+1)^2}\right)$ is dominated by the polynomial $(N+1)^2$. Thus

$$\left(1+\frac{1}{(N+1)^2}\right)^{D^{N+1}\prod_{i=1}^{N}(KD_i+d_i)} \geq 2^{N^3 D^{N+1} D_N \prod_{i=1}^{N-1}(KD_i+d_i)^{\omega}}. \tag{22}$$

Applying this as well as equations (20) and (21) to equation (19), we have

$$a_{N+1} \geq \left(\prod_{n=1}^{N} a_n\right)^{KDD_N} 2^{N^3 D^{N+1} D_N \prod_{i=1}^{N-1}(KD_i+d_i)^{\omega}} (3A_2)^{-KNDD_N}$$
$$\geq \left(\prod_{n=1}^{N} a_n\right)^{KDD_N} 2^{N^2 D^{N+1} \prod_{i=1}^{N}(KD_i+d_i)^{\omega}},$$

for $N(s_0)$ large enough, using that $(KD_N+d_N)/D_N \leq K+1$, and K is constant. Recalling equations (14) and (11), we now have

$$|\gamma(N)| \leq \frac{2^{(\log_2 a_{N+1})^{\omega}}}{a_{N+1}} \leq \left(\prod_{n=1}^{N} a_n\right)^{-KDD_N} \frac{2^{\left(\log_2(2A_2) D^{N+1}\prod_{i=1}^{N}(KD_i+d_i)\right)^{\omega}}}{2^{N^2 D^{N+1}\prod_{i=1}^{N}(KD_i+d_i)^{\omega}}}$$
$$\leq \left(\prod_{n=1}^{N} a_n\right)^{-KDD_N} 2^{-ND^{N+1}\prod_{i=1}^{N}(KD_i+d_i)^{\omega}},$$

and so

$$|\gamma(N)|\left(2^{\left(D^N\prod_{i=1}^{N-1}(KD_i+d_i)\right)^c}\prod_{n=1}^{N}\overline{|a_{1,n}|}^K\right)^{DD_N}$$
$$\leq \frac{2^{D^{cN}\prod_{i=1}^{N-1}(KD_i+d_i)^c}}{2^{ND^N\prod_{i=1}^{N}(KD_i+d_i)^\omega}} \leq 2^{-(K+1)^{\omega N}},$$

for all sufficiently large $N(s_0)$. As this becomes arbitrarily small as s_0 tends to infinity, the lemma follows.

Case 2 ($a_n < 2^n$ infinitely often). Put $A = (1 + A_2)/2$. By equation (5), we may pick arbitrarily large $k \in \mathbb{N}$ such that

$$S_k > A. \tag{23}$$

For each such k, pick $k_0 \in \mathbb{N}$ maximal such that

$$k_0 \leq k, \qquad a_{k_0} < 2^{k_0}. \tag{24}$$

Notice that the case assumption implies

$$k_0 \underset{k\to\infty}{\longrightarrow} \infty. \tag{25}$$

As clearly $k_0 < k$ for just slightly large k, pick $N \in \mathbb{N}$ minimal such that

$$k_0 \leq N < k, \qquad S_{N+1} > \left(1 + \frac{1}{(N+1)^2}\right)\max_{k_0\leq j\leq N} S_j. \tag{26}$$

Such N must exist as the contrary would imply

$$A < S_k \leq \left(1+\frac{1}{k^2}\right)\max_{k_0\leq j<k} S_j \leq \cdots \leq S_{k_0}\prod_{j=k_0}^{k}\left(1+\frac{1}{j^2}\right)$$
$$< S_{k_0}\prod_{j=k_0}^{\infty}\left(1+\frac{1}{j^2}\right)$$

for large enough k, as the number

$$C_k := S_{k_0}\prod_{j=k_0}^{\infty}\left(1+\frac{1}{j^2}\right)$$

tends to 1 as k (and thus k_0, by (25)) tends to infinity. Following the same argument, we may also conclude that $S_n < C_k$ for all $k_0 \leq n \leq N$ when k is sufficiently large. That leads to

$$\prod_{n=1}^{N} a_n = \left(\prod_{n=1}^{k_0} a_n\right)\prod_{n=k_0+1}^{N} a_n < \left(\prod_{n=1}^{k_0} 2^k\right)\prod_{n=k_0+1}^{N} C_k^{D^n\prod_{i=1}^{n-1}(KD_i+d_i)}$$
$$\leq 2^{k_0^2} C_k^{D^N\prod_{i=1}^{N-1}(KD_i+d_i)}\prod_{n=k_0+1}^{N-1} C_k^{D^n\prod_{i=1}^{n-1}(KD_i+d_i)},$$

by using the choice of k_0 and equation (1). Applying Lemma 9 and the lower bound on N, we reach

$$\prod_{n=1}^{N} a_n < 2^{N^2} \left(C_k^{D^N \prod_{i=1}^{N-1}(KD_i+d_i)} \right)^2 = 2^{N^2} \left(C_k^2 \right)^{D^N \prod_{i=1}^{N-1}(KD_i+d_i)} . \tag{27}$$

Aiming for a lower bound on a_{N+1}, we use equation (26) and Lemma 9 to find

$$\begin{aligned} a_{N+1} &= S_{N+1}^{D^{N+1} \prod_{i=1}^{N}(KD_i+d_i)} \\ &> \left(1 + \frac{1}{(N+1)^2}\right)^{D^{N+1} \prod_{i=1}^{N}(KD_i+d_i)} \\ &\quad \left(\max_{k_0 \le j \le N} S_j\right)^{D^{N+1} \prod_{i=1}^{N}(KD_i+d_i)} \\ &\ge \left(1 + \frac{1}{(N+1)^2}\right)^{D^{N+1} \prod_{i=1}^{N}(KD_i+d_i)} \\ &\quad \left(\max_{k_0 \le j \le N} S_j\right)^{KDD_N \sum_{n=1}^{N} D^n \prod_{i=1}^{n-1}(KD_i+d_i)} \\ &\ge \left(1 + \frac{1}{(N+1)^2}\right)^{D^{N+1} \prod_{i=1}^{N}(KD_i+d_i)} \left(\prod_{n=1}^{N} a_n\right)^{KDD_N} \\ &\quad \left(\prod_{n=1}^{k_0-1} \frac{1}{a_n}\right)^{KDD_N} . \end{aligned}$$

By equation (1) and the choice of k_0, we have

$$\left(\prod_{n=1}^{k_0-1} \frac{1}{a_n}\right)^{KDD_N} \ge \left(\prod_{n=1}^{k_0-1} 2^{-k_0}\right)^{KDD_N} \ge 2^{-KN^2DD_N}.$$

Recalling equation (22) (which uses neither case assumption or choice of N), we get for sufficiently large N that

$$\begin{aligned} a_{N+1} &> 2^{N^3 D^{N+1} D_N \prod_{i=1}^{N-1}(KD_i+d_i)^{\omega}} \left(\prod_{n=1}^{N} a_n\right)^{KDD_N} 2^{-KN^2DD_N} \\ &\ge 2^{N^2 D^{N+1} \prod_{i=1}^{N}(KD_i+d_i)^{\omega}} \left(\prod_{n=1}^{N} a_n\right)^{KDD_N} . \end{aligned} \tag{28}$$

Repeating equation (20) of [4], we use that the function $2^{(\log x)^c}/x$ is decreasing combined with equation (2) to see that

$$\begin{aligned}\sum_{n=k}^{\infty}\frac{2^{(\log_2 a_n)^c}}{a_n} &= \sum_{k\le n\le a_k^a}\frac{2^{(\log_2 a_n)^c}}{a_n}+\sum_{n>a_k^a}^{\infty}\frac{2^{(\log_2 a_n)^c}}{a_n}\\ &\le a_k^a\frac{2^{(\log_2 a_k)^c}}{a_k}+\sum_{n>a_k^a}\frac{2^{(\log_2 n^{1+\varepsilon})^c}}{n^{1+\varepsilon}}\\ &\le a_k^{(a-1)/2}+\sum_{n>a_k^a}\frac{1}{n^{1+\varepsilon/2}}\le a_k^{(a-1)/2}+B_0\frac{1}{(a_k^a)^{1+\varepsilon/2}}\\ &\le a_k^{(a-1)/2}+a_k^{-a\varepsilon/3}\le a_k^{-B},\end{aligned}$$

for k sufficiently large, for some $0<B<1<B_0$ not depending on k. By equations (13), (23) and (28), we then have

$$|\gamma(N)|\le\sum_{n=N+1}^{k-1}\frac{2^{(\log_2 a_n)^c}}{a_n}+\sum_{n=k}^{\infty}\frac{2^{(\log_2 a_n)^c}}{a_n}\le\frac{2^{(\log_2 a_{N+1})^\omega}}{a_{N+1}}+a_k^{-B}$$

Thus

$$\begin{aligned}&|\gamma(N)|\left(2^{D^{cN}\prod_{i=1}^{N-1}(KD_i+d_i)^c}\prod_{n=1}^{N}a_n^K\right)^{DD_N}\\ &\quad\le\left(\frac{2^{(\log_2 a_{N+1})^\omega}}{a_{N+1}}+a_k^{-B}\right)\left(2^{D^{cN}\prod_{i=1}^{N-1}(KD_i+d_i)^c}\prod_{n=1}^{N}a_n^K\right)^{DD_N}\end{aligned}$$

It follows by $c<\omega$ and equations (11) and (28) that

$$\begin{aligned}&\frac{2^{(\log_2 a_{N+1})^\omega}}{a_{N+1}}\left(2^{D^{cN}\prod_{i=1}^{N-1}(KD_i+d_i)^c}\prod_{n=1}^{N}a_n^K\right)^{DD_N}\\ &\qquad<\frac{2^{(\log_2(2A_2)+1)D^{\omega(N+1)}\prod_{i=1}^{N}(KD_i+d_i)^\omega}}{2^{N^2D^{N+1}\prod_{i=1}^{N}(KD_i+d_i)^\omega}}<2^{-(K+1)^{\omega N}},\end{aligned}$$

for sufficiently large N. Meanwhile, equations (23) and (27) imply that

$$\begin{aligned}&a_k^{-B}\left(2^{((K+1)DD_{N/2})^{cN}}\prod_{n=1}^{N}a_n^K\right)^{DD_N}\\ &\quad<\left(2^{D^{cN}\prod_{i=1}^{N-1}(KD_i+d_i)^c}\right)^{DD_N}\frac{\left(2^{N^2}\,(C_k^2)^{D^N\prod_{i=1}^{N-1}(KD_i+d_i)}\right)^{DD_N}}{A^{BD^k\prod_{n=1}^{k-1}(KD_i+d_i)}}\\ &\quad=\left(2^{N^2+D^{cN}\prod_{i=1}^{N-1}(KD_i+d_i)^c}\right)^{DD_N}\frac{(C_k^2)^{D^{N+1}D_N\prod_{i=1}^{N-1}(KD_i+d_i)}}{(A^B)^{D^{N+1}\prod_{n=1}^{N}(KD_i+d_i)}}\\ &\quad<2^{D^{cN}\prod_{i=1}^{N}(KD_i+d_i)^c}\Big/(A^{B/2})^{D^{N+1}\prod_{n=1}^{N}(KD_i+d_i)}\le 2^{-(K+1)^{cN}},\end{aligned}$$

using that $C_k^2 < A^{B/2}$ for k (and thus N) sufficiently large. For k sufficiently large, we conlcule

$$|\gamma(N)| \left(2^{((K+1)DD_{N/2})^{cN}} \prod_{n=1}^{N} a_n^K \right)^{DD_N} < 2^{-(K+1)^{\omega N}} + 2^{-(K+1)^{cN}},$$

which clearly tends to 0 as k (and thus N) grows large, and the lemma follows. □

Proof (Theorem 4). It is clear that the entire hypothesis of Lemma 12 is implied by the hypothesis of Theorem 4, as equation (7) implies equation (10). It is likewise clear that the only part of the hypothesis of Lemma 11 that is not also used in the hypothesis of Theorem 4 is the assumption that $\deg\gamma \leq D$. As the conclusions of the two lemmas are mutually exclusive, we conclude $\deg\gamma > D$. □

Concluding Remarks

As in the case of Theorem 3, the requirements using $\Re_\zeta$ (i.e. equations (8) and (9)) are used solely to ensure that γ_N is non-zero and non-conjugate to γ for all sufficiently large N. Consequently, these requirements may be replaced any other set of conditions ensuring that property. Note, however, that the property is required as one might otherwise construct a sequence converging to a rational number while satisfying all other parts of the hypothesis.

In the case of $K = 1$, Theorem 4 implies

Theorem 13. *Let $D \in \mathbb{N}$ be a natural number, let $\zeta \in \mathbb{C}$ with $|\zeta| = 1$, and let $a, \varepsilon > 0$ be real numbers. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of algebraic integers, and let $\{b_n\}_{n\in\mathbb{N}}$ be a sequences of rational integers. For $n \in \mathbb{N}$, write $d_n = \deg\alpha_n$ and $D_n = \prod_{i=1}^n d_i$. Suppose that*

$$\forall n \in \mathbb{N}: \quad n^{1+\varepsilon} \leq |\alpha_n| < |\alpha_{n+1}|,$$

$$1 \leq \liminf_{n\to\infty} |\alpha_n|^{\frac{1}{D^n \prod_{i=1}^{n-1}(D_i+d_i)}} < \limsup_{n\to\infty} |\alpha_n|^{\frac{1}{D^n \prod_{i=1}^{n-1}(D_i+d_i)}} < \infty,$$

$$\forall n \in \mathbb{N}: \quad b_n\overline{|\alpha_n|} \leq 2^{(\log_2|\alpha_n|)^a}|\alpha_n|,$$

$$\forall n \in \mathbb{N}: \quad \Re_\zeta(\alpha_n) > 0.$$

Then $\sum_{n=1}^\infty \frac{1}{\alpha_n}$ has algebraic degree strictly greater than D.

By doing the right modifications to the proof of Theorem 3, it may be improved so that the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ only needs to satisfy the hypothesis of Theorem 13 where the requirement

$$1 \leq \liminf_{n\to\infty} |\alpha_n|^{\frac{1}{D^n \prod_{i=1}^{n-1}(D_i+d_i)}} < \limsup_{n\to\infty} |\alpha_n|^{\frac{1}{D^n \prod_{i=1}^{n-1}(D_i+d_i)}} < \infty$$

is replaced by $\limsup_{n\to\infty} |\alpha_n|^{\frac{1}{D^n \prod_{i=1}^{n-1}(D_i+d_i)}} = \infty$. This will in particular remove the restriction that the α_n must be of bounded algebraic degree while also slacking the upper bound on $\overline{|\alpha_n|}$.

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M. L. Laursen, Department of Mathematics, Aarhus University, Ny Munkegade 118, DK-8000 Aarhus C, Denmark
Email address: mll@math.au.dk

1.4.4 Paper 2: Infinite products with algebraic numbers

Below, the reader will find the most recent preprint of the paper [35], which is joint work between Simon Kristensen and the current author. The paper is currently under review but has not yet been accepted for publication. The preprint is available on arXiv through the link https://arXiv.org/abs/2502.03154v1 or via the arXiv identifier 2502.03154. It has a length of 27 pages, numbered 1 through 27.

INFINITE PRODUCTS WITH ALGEBRAIC NUMBERS

SIMON KRISTENSEN AND MATHIAS LØKKEGAARD LAURSEN

Abstract. We obtain general criteria for giving a lower bound on the degree of numbers of the form $\prod_{n=1}^{\infty}\left(1+\frac{b_n}{\alpha_n}\right)$ or of the form $\prod_{m=1}^{\infty}\left(1+\sum_{n=1}^{\infty}\frac{b_{n,m}}{\alpha_{n,m}}\right)$, where the α_n and $\alpha_{n,m}$ are assumed to be algebraic integers, and the b_n and $b_{n,m}$ are natural numbers. In each case, we give a lower bound of the degree over the smallest extension of $\mathbb{Q}$ containing all algebraic numbers in the expression. The criteria obtained depend on growth conditions on the involved quantities.

1. Introduction

Proving that a comcrete number is irrational can be a difficult task. Proving transcencence results can be even more difficult. In the present paper, we are concerned with general criteria showing that a number represented in a certain way has lower bounded degree. The criteria are on parameters of the representation, and so the representation of the number will reveal arithmetical properties of the number itself. This study has a long history, and we begin by giving some relevant highlights.

In [4], Erdős proved that if $\varepsilon > 0$ is fixed and $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence of positive integers satisfying $a_n \geq n^{1+\varepsilon}$ and

$$\limsup_{n\to\infty} a_n^{1/2^n} = \infty,$$

then the number $\sum_{n=1}^{\infty}\frac{1}{a_n c_n}$ is irrational for all sequences of positive integers $\{c_n\}_{n=1}^{\infty}$. This result has since seen many generalizations, including criteria for irrationality of infinite products and continued fractions (see [6] for an overview). Later, Andersen, Kristensen and Laursen [1, 2, 7] have provided criteria for getting a lower bound on the algebraic degree of series of reciprocals of algebraic integers as well as continued fractions with algebraic integers as partial coefficients.

This leaves the case of infinite products, which we deal with in this note. In the assumptions for our theorems and in their proofs, we let $\overline{|\alpha|}$ denote the *house* of an algebraic number α, i.e., the maximum modulus among α and its algebraic conjugates.

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Theorem 1. *Let $D \in \mathbb{N}$, $\varepsilon > 0$, $a \in (0,1)$, $e \in \{-1,1\}$, let $\{b_n\}_{n=1}^{\infty}$ be a sequence of positive integers, and let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of algebraic integers, such that $\overline{|\alpha_n|}\, b_n \leq |\alpha_n| 2^{(\log_2 |\alpha_n|)^a}$. Suppose that $|\alpha_n|$ increases, and that $|\alpha_n| > n^{1+\varepsilon}$ for n sufficiently large. Furthermore, we suppose $(\Re(\alpha_n/b_n) + 1/2)e \geq 0$ for all $n \in \mathbb{N}$ with strict inequality for infinitely many $n \in \mathbb{N}$. Write $\mathbb{K}_0 = \mathbb{Q}$, $\mathbb{K}_{n+1} = \mathbb{K}_n(\alpha_{n+1})$, $d_n = \deg_{\mathbb{K}_{n-1}} \alpha_n$ and $D_n = \prod_{i=1}^{n} d_i$. Finally, suppose that $|\alpha_n|^{1/(D^n \prod_{i=1}^{n-1}(D_i + d_i))}$ diverges in $\mathbb{R}$ as $n \to \infty$. Write $\mathbb{K} = \mathbb{Q}(\alpha_1, \alpha_2, \ldots)$. Then*

$$\deg_{\mathbb{K}} \left(\prod_{n=1}^{\infty} \left(1 + \frac{b_n}{\alpha_n} \right) \right) > D.$$

This theorem generalizes a result by Hančl and Kolouch [4], which restricted α_n to be positive integers and only covered the case $D = 1$ and $|\alpha_n|^{1/(D^n \prod_{i=1}^{n-1}(D_i + d_i))} = \infty$. [4] does, however, give a more lenient bound for b_n. In our concluding remark we point to how one would get similarly relaxed bounds on b_n for the present paper.

We also provide a proof for the below theorem regarding infinite products of infinite series, which generalizes another theorem by Hančl and Kolouch [5], with their version having $\alpha_n \in \mathbb{N}$ and $D = 1$.

Theorem 2. *Let $D \in \mathbb{N}$, let $\varepsilon > 0$, let $(b_{n,m})_{m.n \in \mathbb{N}}$ be an infinite array of positive integers, and let $(\alpha_{n,m})_{m,n \in \mathbb{N}}$ be an infinite array of algebraic integers. Suppose that $|\alpha_{n,1}|$ increases, and that for n sufficiently large,*

$$n^{1+\varepsilon} \leq |\alpha_{n,1}|, \tag{1}$$

$$\sum_{j=1}^{n} \left| \frac{b_{n-j+1,j}}{\alpha_{n-j+1,j}} \right| \leq |\alpha_{n,1}|^{-1+(\log\log|\alpha|)^{-3-\varepsilon}}, \tag{2}$$

$$\prod_{j=1}^{n} \overline{|\alpha_{n-j+1,j}|} \leq |\alpha_{n,1}|^{n+(\log\log|\alpha|)^{-3-\varepsilon}}. \tag{3}$$

Furthermore, we suppose that $\Re(\alpha_{n,m}) \geq 0$ and $e\Im\alpha_{n,m} \geq 0$ for all pairs (m,n), where $e \in \{-1,1\}$ is fixed. Write $\mathbb{K}_0 = \mathbb{Q}$, $\mathbb{K}_{n+1} = \mathbb{K}_n(\alpha_{1,n+1}, \alpha_{2,n}, \ldots, \alpha_{n+1,1})$, and $D_n = [\mathbb{K}_n : \mathbb{Q}]$. Finally, suppose that

$$\limsup_{N \to \infty} |\alpha_{N,1}|^{\overline{\frac{1}{D^N N! \prod_{n=1}^{N-1} D_n}}} = \infty. \tag{4}$$

Let $\mathbb{K} = \mathbb{Q}(\alpha_{m,n} : m,n \in \mathbb{N})$. Then

$$\deg_{\mathbb{K}} \left(\prod_{m=1}^{\infty} \left(1 + \sum_{n=1}^{\infty} \frac{b_{n,m}}{\alpha_{n,m}} \right) \right) > D.$$

Remark. As will be evident from the proof, the restrictions on real and imaginary values of $\alpha_{n,m}/b_{n,m}$ are only there to ensure that the sequence

$\left\{\prod_{m=1}^{N}\left(1+\sum_{n=1}^{N-m+1}\frac{b_{n,m}}{\alpha_{n,m}}\right)\right\}_{N=1}^{\infty}$ does not take the same value infinitely often and that the terms $\left(1+\sum_{n=1}^{\infty}\frac{b_{n,m}}{\alpha_{n,m}}\right)$ are non-zero. In fact, either of the following assumptions would also have been sufficient. We will prove this together with the theorem.

- $\Re(\frac{\alpha_{n,m}}{b_{n,m}}) \geq -\frac{1}{2}$ for all sufficiently large $m+n$ with $>$ infinitely often, and $e\Im\alpha_{n,m} \geq |\Re(\alpha_{n,m})|$ for all m, n, where $e \in \{-1, 1\}$ is fixed.
- $|\Im(\alpha_{n,m})| \leq \Re(\alpha_{n,m})$ for all m, n.
- $X < 1$, $\Re(\frac{\alpha_{n,m}}{b_{n,m}}) \leq 0$, and $|\Im(\alpha_{n,m})| \leq R|\Re(\alpha_{n,m})|$ for all m, n, where $X = \sup_{m\in\mathbb{N}}\{\sum_{n=1}^{\infty}\frac{b_{n,m}}{|\alpha_{n,m}|}\}$ and $R \in (0, 1/X)$ are fixed.

2. Auxiliary results

We will make heavy use of Weil heights and Mahler measures of algebraic numbers. We recall the definitions.

Let α be an algebraic number, let K be a number field containing α and let M_K denote the set of places of K. Then, the (Weil) height of α is defined as

$$H(\alpha) = \prod_{\nu\in M_K} \max\{1, |\alpha|_\nu\}^{d_\nu/d},$$

where $d = [K : \mathbb{Q}]$ and $d_\nu = [K_\nu : \mathbb{Q}_\nu]$, and where K_ν and $\mathbb{Q}_\nu$ denote the completions of the fields at the place ν. With the normalisation in the exponent, the height becomes independent of the field K.

We will also need to define the Mahler measure of α. For this purpose, suppose that α is algebraic of degree d and let $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ denote the conjugates of α. Finally, let a_d denote the leading coefficient of the minimal polynomial of α defined over $\mathbb{Z}$. The Mahler measure of α is defined as

$$M(\alpha) = |a_d| \prod_{i=1}^{d} \max\{1, |\alpha_i|\}.$$

Here, the only place playing a role is the usual Archimedean one, i.e. the modulus in the complex plane.

The following wonderful result is classical, see e.g. [11].

Theorem 3. *For an algebraic number α of degree d,*

$$H(\alpha) = M(\alpha)^{1/d}.$$

The following lemma from [1] relates heights and houses.

Lemma 4. *Let α be an algebraic integer of degree d. Then,*

$$H(\alpha) = M(\alpha)^{1/d} \leq \overline{|\alpha|} \leq M(\alpha) = H(\alpha)^d.$$

The inequalities are best possible.

We will need to know that the height remains unchanged on taking the reciprocal. This is also classical, see [11].

Lemma 5. *Let α be a non-zero algebraic number. Then, $H(\alpha) = H(1/\alpha)$.*

[11] also provides bounds of the Weil height of sums and products of algebraic numbers.

Lemma 6. *Let $n \in \mathbb{N}$, and let $\beta_1, \ldots, \beta_n$ be algebraic numbers. Then,*

$$H\left(\sum_{i=1}^{n} \beta_i\right) \leq 2^n \prod_{i=1}^{n} H(\beta_i), \quad \text{and } H\left(\prod_{i=1}^{n} \beta_i\right) \leq \prod_{i=1}^{n} H(\beta_i).$$

Our proof depends critically on the Liouville–Mignotte inequality [9, 10], which is the following.

Lemma 7. *Let α and β be non-conjugate algebraic numbers. Then,*

$$|\alpha - \beta| \geq (2H(\alpha)H(\beta))^{-\deg(\alpha)\deg(\beta)}.$$

A nice proof can be found in [3]. The following two lemmas are found in [1].

Lemma 8. *Let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence of real numbers such that $a_n > n^{1+\varepsilon}$ for some $\varepsilon > 0$ and all $n \in \mathbb{N}$. Then, for all $N \in \mathbb{N}$,*

$$\sum_{n=N}^{\infty} \frac{1}{a_n} < \frac{2 + \frac{1}{\varepsilon}}{a_N^{\varepsilon/(1+\varepsilon)}}.$$

Lemma 9. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that*

$$\limsup_{n \to \infty} a_n = \infty$$

Then for infinitely many $N \in \mathbb{N}$,

$$a_{N+1} > \left(1 + \frac{1}{k^2}\right) \max_{1 \leq n \leq N} a_n.$$

The following three lemmas are taken from [5]. While the first two of the below lemmas assumed $\alpha_{n,1}$ to be integers in their original form, this property is never used in the proofs, so they remain valid in the present formulation. The third lemma has been generalized slightly from [5], but the proof is the same.

Lemma 10. *Let ε and $\alpha_{n,1}$ be given as in Theorem 2. Then, for N sufficiently large,*

$$\sum_{n=N}^{\infty} |\alpha_{n,1}|^{-1 + \frac{1}{\log^{3+\varepsilon} \log|\alpha_{n,1}|}} < |\alpha_{N,1}|^{-\frac{\varepsilon}{2(1+\varepsilon)}}$$

Lemma 11. *Let* ε *and* $\alpha_{n,1}$ *be given as in Theorem 2 such that*

$$2^n < |\alpha_{n,1}| \tag{5}$$

Then, for N *sufficiently large,*

$$\sum_{n=N}^{\infty} |\alpha_{n,1}|^{-1+\frac{1}{\log^{3+\varepsilon}\log|\alpha_{n,1}|}} < |\alpha_{N,1}|^{-1+\frac{1}{\log^{3+\varepsilon/2}\log|\alpha_{N,1}|}}$$

Lemma 12. *Let* $\delta \in [0,1)$*, and let* $D \in \mathbb{N}$*, let* $(D_n)_{n=1}^{\infty}$ *be a sequence of natural numbers. Suppose* $(a_n)_{n=1}^{\infty}$ *is a non-decreasing sequence of positive real numbers such that*

$$\limsup_{n\to\infty} a_n^{\frac{1}{D^n(n+\delta)!\prod_{i=1}^{n-1} D_i}} = \infty. \tag{6}$$

Then, for infinitely many N*,*

$$a_{N+1}^{\frac{1}{D^{N+1}(N+1+\delta)!\prod_{i=1}^{N} D_i}} > \left(1+\frac{1}{N^2}\right)\max_{1\le n\le N} a_n^{\frac{1}{D^n(n+\delta)!\prod_{i=1}^{n-1} D_i}} \tag{7}$$

and

$$a_{N+1} > \left(\left(1+\frac{1}{N^2}\right)^{D^N(N+1+\delta)!\prod_{i=1}^{N-1} D_i} \prod_{n=1}^{N} a_n^{n+\delta}\right)^{DD_N}. \tag{8}$$

As some applications of Lemma 11 are a little opaque, we will state a consequence of it that is more easily applied. It follows immediately by adding infinitely many terms to the finite sum of the corollary and subsequently applying Lemma 11.

Corollary 13. *Let* ε *and* $\alpha_{n,1}$ *be given as in Theorem 2 such that*

$$2^n < |\alpha_{n,1}|,$$

for $n \in [t,k]$ *for infinitly many disjoint intervals* $[t,k]$*. Then, for* t *sufficiently large,*

$$\sum_{n=t}^{k} |\alpha_{n,1}|^{-1+\frac{1}{\log^{3+\varepsilon}\log|\alpha_{n,1}|}} < |\alpha_{t,1}|^{-1+\frac{1}{\log^{3+\varepsilon/2}\log|\alpha_{t,1}|}}$$

Finally, we present another lemma that will be useful for proving Theorem 2.

Lemma 14. *Let* $(a_n)_{n=1}^{\infty}$ *be a sequence of complex numbers such that* $\prod_{n=1}^{\infty}(1+a_n)$ *is absolutely convergent. Write*

$$C = \sup_{K\in\mathbb{N}} \prod_{n=1}^{K-1} |1+a_n|.$$

Then

$$\left|1 - \prod_{n=1}^{\infty}(1+a_n)\right| \le C\sum_{n=1}^{\infty}|a_n|.$$

Proof. Let $K \in \mathbb{N}$. We will then show that

$$\left|1 - \prod_{n=1}^{K}(1 + a_n)\right| \leq C \sum_{n=1}^{K} |a_n|$$

If $K = 1$, this is trivial. If $K > 1$, it follows by induction upon noting

$$\begin{aligned}\left|1 - \prod_{n=1}^{K}(1 + a_n)\right| &\leq \left|1 - \prod_{n=1}^{K-1}(1 + a_n)\right| + |a_K| \prod_{n=1}^{K-1} |1 + a_n| \\ &\leq \left|1 - \prod_{n=1}^{K-1}(1 + a_n)\right| + C|a_K|.\end{aligned}$$

The lemma then follows by letting K tend to infinity. □

3. Proof of Theorem 1

The theorem follows from the following two lemmas

Lemma 15. *Let D, d_n, D_n, a, ε, α_n, and b_n be given as in Theorem 1, except that $|\alpha_n|^{\frac{1}{D^n \prod_{n=1}^{N-1}(d_i+D_i)}}$ need not diverge. Suppose $\prod_{n=1}^{\infty}\left(1 + \frac{b_n}{\alpha_n}\right)$ has degree at most D over $\mathbb{K}$. Then*

$$\liminf_{N\to\infty} \left(2^{N^2 \log_2^a |\alpha_N|} \prod_{n=1}^{N} |\alpha_n|\right)^{DD_N} \sum_{n=N+1}^{\infty} \left|\frac{b_n}{\alpha_n}\right| = \infty. \tag{9}$$

Proof. For $N \in \mathbb{N}$, let

$$x = \prod_{n=1}^{\infty}\left(1 + \frac{1}{\alpha_n}\right) \quad \text{and} \quad x_N = \prod_{n=1}^{N}\left(1 + \frac{1}{\alpha_n}\right).$$

By Lemmas 6 and 5,

$$\begin{aligned}H(x - x_N) &\leq 2H(x) \prod_{n=1}^{N} 2H(\alpha_n)H(1/b_n) \\ &= 2^{N+1}H(x) \prod_{n=1}^{N} H(\alpha_n)H(b_n).\end{aligned}$$

Appealing to Lemma 4, we then have

$$H(x - x_N) \leq 2^{N+1}H(x) \prod_{n=1}^{N} \overline{|\alpha_n|}\, b_n. \tag{10}$$

A simple calculation shows that $|1 + b_n/\alpha_n| - 1$ is negative, 0, or positive when $\Re(\alpha_n/b_n) + 1/2$ is negative, 0, or positive, respectively, while the bounds $|\alpha_n| \geq |\alpha_1| > 1$ and $b_n \leq 2^{\log_2^a |\alpha_n|}$ ensure that $|b_n/\alpha_n| < 1$ and thereby $x_N \neq 0$. Hence, the restriction on $\Re(\alpha_n/b_n)$ implies that $\{|x_N|\}_{N=1}^{\infty}$ is monotonous but not constant, so that $x_N \neq x$. Since

$x - x_N$ must be algebraic due to $\deg_{\mathbb{K}_N} x = D < \infty$, we get from Lemma 7 with $\alpha = x - x_N$ and $\beta = 0$ that

$$|x - x_N| \geq \frac{1}{(2H(x - x_N))^{\deg(x - x_N)}}.$$

Since clearly $\mathbb{K} = \bigcup_{n=1}^{\infty} \mathbb{K}_n$, $\deg_{\mathbb{K}} x = \deg_{\mathbb{K}_N} x$ for all sufficiently large N. Then $x - x_N \in \mathbb{K}_N(x)$, and so

$$\deg(x - x_N) \leq [\mathbb{K}_N : \mathbb{Q}] \deg_{\mathbb{K}_N} x \leq D \prod_{n=1}^{N} [\mathbb{K}_N : \mathbb{K}_{N-1}] = DD_N.$$

Recalling inequality (10), we continue the lower bound of $|x - x_N|$,

$$|x - x_N| \geq \left(2^{N+2} H(x) \prod_{n=1}^{N} \overline{|\alpha_n|}\, b_n \right)^{-DD_N}.$$

Then applying the assumed upper bound of $\overline{|a_n|}\, b_n$, we have

$$|x - x_N| \geq \left(2^{N+1} H(x) \prod_{n=1}^{N} 2^{2 \log_2^a |\alpha_n|} |\alpha_n| \right)^{-DD_N}$$

(11) $$\geq \left(2^{2N \log_2^a |\alpha_N|} \prod_{n=1}^{N} |\alpha_n| \right)^{-DD_N},$$

for all sufficiently large N

To get an upper bound on $|x - x_N|$, let $K \geq N$. Then

$$\left| 1 - \frac{x_{K+1}}{x_N} \right| \leq \left| 1 - \frac{x_K}{x_N} \right| + \left| \frac{b_{K+1}}{\alpha_{K+1}} \right| \left| \frac{x_K}{x_N} \right|$$

Recalling that $|x_N|$ is monotonous and taking induction in K, we have

$$\left| 1 - \frac{x_{K+1}}{x_N} \right| \leq \sum_{n=N+1}^{K+1} \left| \frac{x_K}{x_N} \right| \left| \frac{b_n}{\alpha_n} \right| \leq \max\left\{ 1, \left| \frac{x_K}{x_N} \right| \right\} \sum_{n=N+1}^{K+1} \left| \frac{x_K}{x_N} \right| \left| \frac{b_n}{\alpha_n} \right|$$
$$\leq \max\left\{ \frac{1}{|x_N|}, \left| \frac{x_K}{x_N} \right| \right\} \sum_{n=N+1}^{K+1} \left| \frac{x_K}{x_N} \right| \left| \frac{b_n}{\alpha_n} \right|.$$

Letting K tend to infinity, we then get

$$|x - x_N| = |x_N| \left| 1 - \frac{x}{x_N} \right| \leq \max\{1, |x|\} \sum_{n=N+1}^{\infty} \left| \frac{b_n}{\alpha_n} \right|.$$

Combining this with inequality (11), we conclude

$$\begin{aligned}\left(2^{N^2\log_2^a|\alpha_n|}\prod_{n=1}^N|\alpha_n|\right)^{DD_N}&\sum_{n=N+1}^\infty\left|\frac{b_n}{\alpha_n}\right|\\ &\geq\left(2^{N^2\log_2^a|\alpha_n|}\prod_{n=1}^N|\alpha_n|\right)^{DD_N}\sum_{n=N+1}^\infty\left|\frac{b_n}{\alpha_n}\right|\\ &\geq\frac{2^{DD_N(N^2-2N)\log_2^a|\alpha_n|}}{\max\{1,|x|\}}\underset{N\to\infty}{\longrightarrow}\infty,\end{aligned}$$

and the proof is complete. □

Lemma 16. *Let D, d_n, D_n, a, ε, and α_n be given as in Theorem 1, and let $c\in(a,1)$. Then*

$$\liminf_{N\to\infty}\left(2^{N^2\log_2^\alpha|\alpha_N|}\prod_{n=1}^N|\alpha_n|\right)^{DD_N}\left|\sum_{n=N+1}^\infty\frac{b_n}{\alpha_n}\right|=0. \tag{12}$$

Proof. Write

$$a_n=|\alpha_n|,\quad\text{and}\quad H_n=a_n^{\frac{1}{D^n\prod_{i=1}^{n-1}(D_i+d_i)}}.$$

The case of

$$\liminf_{n\to\infty}H_n<\limsup_{n\to\infty}H_n<\infty,$$

is merely a special case of Lemma 12 from [7], by noting that for all $c\in(a,1)$,

$$2^{N^2\log_2^a a_N}=2^{N^2D^{aN}\prod_{i=1}^{N-1}(D_i+d_i)^a\log_2^a H_n}<2^{D^{cN}\prod_{i=1}^{N-1}(D_i+d_i)^c}.$$

Henceforth, we assume $\limsup_{n\to\infty}|\alpha_n|^{\frac{1}{D^n\prod_{i=1}^{n-1}(D_i+d_i)}}=\infty$. Recall the definitions of a_n and H_n above. Write further

$$D_{n,\delta}=D_n+\delta\quad\text{and}\quad H_{n,\delta}=a_n^{\frac{1}{D^n\prod_{i=1}^{n-1}D_{i,\delta}+d_i}},$$

for any $\delta\geq 0$. Note that

$$\begin{aligned}\left(\max_{1\leq n\leq N}H_{n,\delta}\right)^{D^{N+1}\prod_{i=1}^N D_{i,\delta}+d_i}&\geq a_N^{DD_{N,\delta}}\left(\max_{1\leq n\leq N}H_{n,\delta}\right)^{D^{N+1}d_N\prod_{i=1}^{N-1}D_{i,\delta}+d_i}\\ &\geq a_N^{DD_{N,\delta}}\left(\max_{1\leq n\leq N}H_{n,\delta}\right)^{D^{N+1}D_{N,\delta}\prod_{i=1}^{N-2}D_{i,\delta}+d_i}\\ &\quad\left(\max_{1\leq n\leq N}H_{n,\delta}\right)^{D^{N+1}d_Nd_{N+1}\prod_{i=1}^{N-2}D_{i,\delta}+d_i}\\ &\geq\cdots\geq\left(\prod_{n=1}^N a_n^{D^{N-n}}\right)^{DD_{N,\delta}}\geq\prod_{n=1}^N a_n^{DD_{N,\delta}}.\end{aligned}\tag{13}$$

Let $c \in (a,1)$ and notice that

$$\log_2\left(1+\frac{1}{N^2}\right)D^{N+1}\prod_{i=1}^{N} D_{i,\delta}+d_i > D^{N+1}N^3 D_{N,\delta}\prod_{i=1}^{N-1}(D_i+d_i)^c,$$

for all sufficiently large N. Combined with inequality (13), Lemma 9 now implies that if $\limsup_{n\to\infty} H_{n,\delta}$, then

$$a_{N+1} \geq \left(1+\frac{1}{N^2}\right)^{D^{N+1}\prod_{i=1}^{N} D_{i,\delta}+d_i}\left(\max_{1\leq n\leq N} H_{n,\delta}\right)^{D^{N+1}\prod_{i=1}^{N} D_{i,\delta}+d_i}$$

$$(14)\qquad > \left(2^{D^N N^3 \prod_{i=1}^{N-1}(D_i+d_i)^c}\prod_{n=1}^{N} a_n\right)^{DD_{N,\delta}},$$

for infinitely many N.

We will split the proof into several cases, but before doing so, notice that if $a_n \geq 2^n$ for all sufficiently large n, then

$$\sum_{n=N+1}^{\infty}\frac{b_n}{a_n} \leq \sum_{n=N+1}^{\lfloor \log_2 a_{N+1}+1\rfloor}\frac{2^{\log_2^a a_n}}{a_n} + \sum_{n>\log_2 a_{N+1}+1}\frac{2^{\log_2^a a_n}}{a_n}$$

$$\leq \frac{\log_2 a_{N+1}+1}{a_{N+1}}2^{\log_2^a a_{N+1}} + \sum_{n>\log_2 a_{N+1}+1}^{\infty}\frac{2^{n^a}}{2^n}$$

$$(15)\qquad \leq \frac{\log_2 a_{N+1}+1}{a_{N+1}}2^{\log_2^a a_{N+1}} + C\frac{2^{\log_2^a a_{N+1}}}{a_{N+1}} \leq \frac{2^{2\log_2^a a_{N+1}}}{a_{N+1}},$$

for a suitably fixed $C>0$ and all sufficiently large N.

Case 1. $a_n \geq 2^n$ for all sufficiently large n, and $\limsup_{n\to\infty} H_{n,\delta}=\infty$ for some $\delta>0$. Fix such a δ. Combining inequalities (14) and (13), there are infinitely many N satisfying

$$\sum_{n=N+1}^{\infty}\frac{b_n}{a_n} \leq \left(2^{D^N N^3\prod_{i=1}^{N-1}(D_i+d_i)^c}\prod_{n=1}^{N} a_n\right)^{-DD_{N,\delta}}$$

$$\cdot 2^{2\log_2^\alpha\left(2^{D^N N^3\prod_{i=1}^{N-1}(D_i+d_i)^c}\prod_{n=1}^{N} a_n\right)^{DD_{N,\delta}}}$$

$$\leq \prod_{n=1}^{N} a_n^{-DD_{N,\delta/2}},$$

so that

$$\left(2^{N^2\log_2^a a_N}\prod_{n=1}^{N} a_n\right)^{DD_N}\sum_{n=N+1}^{\infty}\frac{b_n}{a_n} \leq \prod_{n=1}^{N} a_n^{DD_{N,\delta/3}}\prod_{n=1}^{N} a_n^{-DD_{N,\delta/2}}$$

$$\leq \prod_{n=1}^{N} a_n^{-DD_{N,\delta/6}},$$

which becomes arbitrarily small as N increases.

Case 2. $a_n \geq 2^n$ for all sufficiently large n, but $\limsup_{n\to\infty} H_{n,\delta} < \infty$ for all $\delta > 0$. Recall $a < c < 1$, and pick $\delta > 0$ such that

$$m + \delta \leq m^{c/(2a)}$$

for all $m \geq 2$. Then there must be some $C > 0$ such that

$$\log_2 a_N \leq CD^N \prod_{i=1}^{N-1} D_{i,\delta} + d_i \leq \prod_{i=1}^{N-1} (D_i + d_i)^{c/a},$$

for all sufficiently large N, and so

$$2^{\log_2^a a_N} \leq 2^{\prod_{i=1}^{N-1}(D_i+d_i)^c} \tag{16}$$

Inserting this and inequality (14) into inequality (13) now yields

$$\sum_{n=N+1}^{\infty} \frac{b_n}{a_n} \leq \frac{2^{2\log_2^a a_{N+1}}}{a_{N+1}} \leq \frac{2^{\log_2^a\left(\left(2^{D^N N^3 \prod_{i=1}^{N-1}(D_i+d_i)^c}\prod_{n=1}^{N} a_n\right)^{DD_N}\right)}}{\left(2^{D^N N^3 \prod_{i=1}^{N-1}(D_i+d_i)^c}\prod_{n=1}^{N} a_n\right)^{DD_N}}$$

for infinitely many N. By inequality (16),

$$\begin{aligned}
&\log_2^a\left(\left(2^{D^N N^3 \prod_{i=1}^{N-1}(D_i+d_i)^c}\prod_{n=1}^{N} a_n\right)^{DD_N}\right) \\
&\quad\leq (DD_N)^a\left(D^N N^3 \prod_{i=1}^{N-1}(D_i+d_i)^c + 2\sum_{n=1}^{N}\prod_{i=1}^{N-1}(D_i+d_i)^{c/a}\right)^a \\
&\quad\leq (DD_N)^a\left(2D^N N^3 \prod_{i=1}^{N-1}(D_i+d_i)^{c/a}\right)^a \\
&\quad= 2^a D^{a(N+1)} D_N^a N^{3a} \prod_{i=1}^{N-1}(D_i+d_i)^c,
\end{aligned}$$

Continuing our bound on $\sum_{n=N+1}^{\infty} \frac{b_n}{a_n}$, we now have

$$\begin{aligned}
\sum_{n=N+1}^{\infty} \frac{b_n}{a_n} &\leq \frac{2^{2^a D^{a(N+1)} D_N^a N^{3a} \prod_{i=1}^{N-1}(D_i+d_i)^c}}{\left(2^{D^N N^3 \prod_{i=1}^{N-1}(D_i+d_i)^c}\prod_{n=1}^{N} a_n\right)^{DD_N}} \\
&\leq \left(2^{2D^N N^2 \prod_{i=1}^{N-1}(D_i+d_i)^c}\prod_{n=1}^{N} a_n\right)^{-DD_N}
\end{aligned}$$

Using this and inequality (16), we conclude

$$\begin{aligned}
\left(2^{N^2\log_2^a a_N}\prod_{n=1}^{N} a_n\right)^{DD_N} \sum_{n=N+1}^{\infty} \frac{b_n}{a_n} &\leq \frac{2^{N^2\prod_{i=1}^{N-1}(D_i+d_i)^c}}{2^{2D^N N^2\prod_{i=1}^{N-1}(D_i+d_i)^c}} \\
&\leq \frac{1}{2^{D^N N^2\prod_{i=1}^{N-1}(D_i+d_i)^c}},
\end{aligned}$$

Case 3. $a_n < 2^n$ infinitely often.

Let $A > 1$ be a large number, and pick $k_1 \in \mathbb{N}$ such that

$$H_{k_1} > 2^A. \tag{17}$$

Then pick $k_2 \leq k_1$ maximal such that

$$a_{k_2} < 2^{k_2}. \tag{18}$$

Notice that k_1 grows large when A does since $\limsup_{n\to\infty} H_n = \infty$. The case assumption then implies

$$k_2 \underset{A\to\infty}{\longrightarrow} \infty. \tag{19}$$

By Lemma 9, we may now pick $N \geq k_2$ minimal such that

$$H_{N+1} > \left(1 + \frac{1}{(N+1)^2}\right) \max_{k_2 \leq j \leq N} H_j$$

Since $\prod_{n=1}^{\infty}\left(1+\frac{1}{(n+1)^2}\right) < \infty$, it follows by the choices of k_1 and k_2 that $N < k_1$ when A is large enough. Notice that N satisfies inequality (14) with $\delta = 0$.

Since a_n is increasing, it follows by the choices of k_2 and N that

$$\begin{aligned}\prod_{n=1}^{N} a_n &\leq a_{k_2}^{k_2} \prod_{n=k_2+1}^{N}\left(1+\frac{1}{N^2}\right)^{n-k_2} a_k \leq a_{k_2}^{N} \prod_{n=k_2+1}^{N}\left(1+\frac{1}{N^2}\right)^{N} \\ &\leq 2^{N^2} \prod_{n=k_2+1}^{\infty}\left(1+\frac{1}{N^2}\right)^{N}\end{aligned}$$

Since $\prod_{n=1}^{\infty}\left(1+\frac{1}{N^2}\right) < \infty$, it follows for large enough values of A (and thereby k_2) that

$$\prod_{n=1}^{N} a_n \leq 2^{N^3}. \tag{20}$$

We now turn to estimating the infinite series. By maximality of k_2, we have that $a_n \geq 2^n$ for each $n \in [N+1, k_1)$. Hence, by inequalities (15) and (14),

$$\begin{aligned}\sum_{n=N+1}^{k_1-1} \frac{b_n}{a_n} \leq \sum_{n=N+1}^{k_1-1} \frac{b_n}{a_n} + \sum_{n=k_1}^{\infty} \frac{1}{2^n} &\leq \frac{2^{2\log_2^a a_{N+1}}}{a_{N+1}} \\ &\leq \frac{2^{2\log_2^a\left(2^{D^N N^3 \prod_{i=1}^{N-1}(D_i+d_i)^c} \prod_{n=1}^{N} a_n\right)^{DD_N}}}{\left(2^{D^N N^3 \prod_{i=1}^{N-1}(D_i+d_i)^c} \prod_{n=1}^{N} a_n\right)^{DD_N}}\end{aligned}$$

By applying inequality (20) in the exponent of the numerator, we find

$$\log_2\left(\left(2^{D^N N^3 \prod_{i=1}^{N-1}(D_i+d_i)^c}\prod_{n=1}^{N} a_n\right)^{DD_N}\right) \leq DD_N\left(D^N N^3 \prod_{i=1}^{N-1}(D_i+d_i)^c + N^3\right) \leq 2D^{N+1}D_N N^3 \prod_{i=1}^{N-1}(D_i+d_i)^c,$$

and so

$$\sum_{n=N+1}^{k_1-1}\frac{b_n}{a_n} \leq \frac{2^{4D^{a(N+1)}D_N^a N^{3a}\prod_{i=1}^{N-1}(D_i+d_i)^{ac}}}{\left(2^{D^N N^3\prod_{i=1}^{N-1}(D_i+d_i)^c}\prod_{n=1}^{N} a_n\right)^{DD_N}} \leq \frac{1}{\left(2^{D^N N^2\prod_{i=1}^{N-1}(D_i+d_i)^c}\prod_{n=1}^{N} a_n\right)^{DD_N}}. \tag{21}$$

Noticing $a_n/b_n \geq n^{1+\varepsilon/2}$, Lemma 8 and the bound on b_n yield

$$\sum_{n=k_1}^{\infty}\frac{b_n}{a_n} \leq \frac{2+\frac{2}{\varepsilon}}{(a_{k_1}/b_{k_1})^{\varepsilon/(2+\varepsilon)}} \leq \frac{2(1+\frac{1}{\varepsilon})2^{\frac{\varepsilon}{2+\varepsilon}\log_2^a a_{k_1}}}{a_{k_1}^{\varepsilon/(2+\varepsilon)}}$$

By choice of k_1 and since $N < k_1$, we then have

$$\sum_{n=k_1}^{\infty}\frac{b_n}{a_n} \leq \frac{2(1+\frac{1}{\varepsilon})2^{\frac{\varepsilon}{2+\varepsilon}(AD^{k_1}\prod_{i=1}^{k_1-1}(D_i+d_i))^a}}{2^{\frac{\varepsilon}{2+\varepsilon}AD^{k_1}\prod_{i=1}^{k_1-1}(D_i+d_i)}} \leq \frac{1}{2^{\frac{\varepsilon}{3+\varepsilon}AD^{N+1}\prod_{i=1}^{N}(D_i+d_i)}}$$

Together with inequality (21), we then have

$$\sum_{n=N+1}^{\infty}\frac{b_n}{a_n} = \sum_{n=N+1}^{k_1-1}\frac{b_n}{a_n} + \sum_{n=k_1}^{\infty}\frac{b_n}{a_n} \leq \frac{1}{\left(2^{D^N N^2\prod_{i=1}^{N-1}(D_i+d_i)^c}\prod_{n=1}^{N} a_n\right)^{DD_N}} + \frac{1}{2^{\frac{\varepsilon}{3+\varepsilon}AD^{N+1}\prod_{i=1}^{N}(D_i+d_i)}}.$$

Hence,

$$\left(2^{N^2\log_2^a a_N}\prod_{n=1}^N a_n\right)^{DD_N}\sum_{n=N+1}^\infty \frac{b_n}{a_n} \le \frac{2^{N^2\log_2^a a_N}}{2^{D^{N+1}D_N N^2\prod_{i=1}^{N-1}(D_i+d_i)^c}} + \frac{2^{N^2\log_2^a a_N}\prod_{n=1}^N a_n}{2^{\frac{\varepsilon}{3+\varepsilon}AD^{N+1}\prod_{i=1}^N(D_i+d_i)}}.$$

The first summand clearly tends to 0 as A (and thereby N) grows large. The other summand is estimated through inequality (20),

$$\frac{2^{N^2\log_2^a a_N}\prod_{n=1}^N a_n}{2^{\frac{\varepsilon}{3+\varepsilon}AD^{N+1}\prod_{i=1}^N(D_i+d_i)}} \le \frac{2^{N^{2+3a}+N^3}}{2^{\frac{\varepsilon}{3+\varepsilon}AD^{N+1}\prod_{i=1}^N(D_i+d_i)}},$$

which shows that also this summand tends to 0. Thereby,

$$\left(2^{N^2\log_2^a a_N}\prod_{n=1}^N a_n\right)^{DD_N}\sum_{n=N+1}^\infty \frac{b_n}{a_n}$$

can be made arbitrarily small.

Since we have now covered all possible cases, the proof is complete. □

Proof Theorem 1. Since the conclusions of Lemmas 15 and 16 are quite clearly mutually exclusive, the theorem follows by comparing hypotheses. □

4. Proof of Theorem 2

For this section, we will write

$$x = \prod_{m=1}^\infty\left(1+\sum_{n=1}^\infty \frac{b_{n,m}}{\alpha_{n,m}}\right), \qquad x_N = \prod_{m=1}^N\left(1+\sum_{n=1}^{N-m+1} \frac{b_{n,m}}{\alpha_{n,m}}\right).$$

The proof of Theorem 2 and the remark following it will be split into the following three lemmas.

Lemma 17. *Let $\alpha_{m,n}$ and $b_{m,n}$ be given as in Theorem 2, except for the restrictions on real and imaginary values, and suppose one of the following statements holds.*

- *I. $\Re(\frac{\alpha_{n,m}}{b_{n,m}}) \ge 0$ and $e\Im\alpha_{n,m} \ge 0$ for all m, n.*
- *II. $\Re(\frac{\alpha_{n,m}}{b_{n,m}}) \ge -\frac{1}{2}$ for all sufficiently large $m+n$ with $>$ infinitely often and that $e\Im\alpha_{n,m} \ge |\Re(\alpha_{n,m})|$ for all m, n.*
- *III. $\Re(\alpha_{n,m}) \ge |\Im(\alpha_{n,m})|$ for all m, n.*
- *IV. $X < 1$, $\Re(\frac{\alpha_{n,m}}{b_{n,m}}) \le \frac{-1}{2(1-XR)}$, and $|\Im(\alpha_{n,m})| \le R|\Re(\alpha_{n,m})|$ for all pairs (m, n), where $X = \sup_{m\in\mathbb{N}}\{\sum_{n=1}^\infty \frac{b_{n,m}}{|\alpha_{n,m}|}\}$ and $R \in [1, 1/X)$ are fixed.*

Then $|1+\sum_{n=1}^{\infty} b_{n,m}/\alpha_{n,m}| \geq C_0$ *for a fixed number* $C_0 \in (0,1)$ *that does not depend on* m*, and the sequence* $\{x_N\}_{N=1}^{\infty}$ *does not contain the same number infinitely often.*

Proof. We first consider the estimate $|1+\sum_{n=1}^{\infty} b_{n,m}/\alpha_{n,m}| \geq C_0$ In statements I and III, $\Re(\alpha_{n,m}/b_{n,m}) \geq 0$, which implies $\Re(b_{n,m}/\alpha_{n,m}) \geq 0$, and so

$$\left|1+\sum_{n=1}^{\infty}\frac{b_{n,m}}{\alpha_{n,m}}\right| \geq \Re\left(1+\sum_{n=1}^{\infty}\frac{b_{n,m}}{\alpha_{n,m}}\right) = 1+\sum_{n=1}^{\infty}\Re\left(\frac{b_{n,m}}{\alpha_{n,m}}\right) \geq 1.$$

As for statement II, the bound on the imaginary value implies

$$\left|\Im\left(\frac{b_{n,m}}{\alpha_{n,m}}\right)\right| = \frac{b_{n,m}|\Im(\alpha_{n,m})|}{|\alpha_{n,m}|^2} \geq \frac{b_{n,m}\Re(\alpha_{n,m})}{|\alpha_{n,m}|^2} = \Re\left(\frac{b_{n,m}}{\alpha_{n,m}}\right).$$

From this and the converse triangle inequality follows

$$\begin{aligned}\left|1+\sum_{n=1}^{\infty}\frac{b_{n,m}}{\alpha_{n,m}}\right| &\geq \max\left\{\left|1+\sum_{n=1}^{\infty}\Re\left(\frac{b_{n,m}}{\alpha_{n,m}}\right)\right|, \left|\sum_{n=1}^{\infty}\Im\left(\frac{b_{n,m}}{\alpha_{n,m}}\right)\right|\right\}\\ &\geq \max\left\{1-\sum_{n=1}^{\infty}\left|\Re\left(\frac{b_{n,m}}{\alpha_{n,m}}\right)\right|, \sum_{n=1}^{\infty}\left|\Re\left(\frac{b_{n,m}}{\alpha_{n,m}}\right)\right|\right\}\\ &\geq \frac{1}{2}.\end{aligned}$$

For statement IV, we again use the converse triangle inequality,

$$\left|1+\sum_{n=1}^{\infty}\frac{b_{n,m}}{\alpha_{n,m}}\right| \geq 1-X > 0.$$

We now turn our attention to the sequence $\{x_N\}_{N=1}^{\infty}$. Consider

$$\begin{aligned}\frac{|x_N|}{|x_{N-1}|} &= \prod_{m=1}^{N}\left|1+\frac{\frac{b_{N-m+1,m}}{\alpha_{N-m+1,m}}}{1+\sum_{n=1}^{N-m}\frac{b_{n,m}}{\alpha_{n,m}}}\right|\\ &= \prod_{m=1}^{N}\left|1+\frac{1}{\frac{\alpha_{N-m+1,m}}{b_{N-m+1,m}}\left(1+\sum_{n=1}^{N-m}\frac{b_{n,m}}{\alpha_{n,m}}\right)}\right|. \qquad (22)\end{aligned}$$

We will focus on the numbers

$$\xi_{N,m} = \frac{\alpha_{N-m+1,m}}{b_{N-m+1,m}}\left(1+\sum_{n=1}^{N-m}\frac{b_{n,m}}{\alpha_{n,m}}\right).$$

Since

$$\begin{aligned}|1+\xi_{N,m}^{-1}|^2 &= \frac{(|\xi_{N,m}|^2+\Re(\xi_{N,m}))^2+\Im(\xi_{N,m})^2}{|\xi_{N,m}|^4}\\ &= 1+\frac{1+2\Re(\xi_{N,m})}{|\xi_{N,m}|^2},\end{aligned}$$

it follows that the number $|1+1/\xi_{N,m}|-1$ will be 0, negative, or positive exactly when the number $\Re(\xi_{N,m}) + 1/2$ is 0, negative, or positive, respectively. Hence, the proof will follow from equation (22) if we can show that $\Re(\xi_{N,m}) + 1/2$ is either always non-negative or always non-positive and that it is non-zero infinitely often.

We calculate

$$\begin{aligned}\Re(\xi_{N,m}) &= \Re\left(\frac{\alpha_{N-m+1,m}}{b_{N-m+1,m}}\left(1+\sum_{n=1}^{N-m}\frac{b_{n,m}}{\alpha_{n,m}}\right)\right)\\ &= \Re\left(\frac{\alpha_{N-m+1,m}}{b_{N-m+1,m}}\right)+\sum_{n=1}^{N-m}\frac{\Re\left(\frac{\alpha_{N-m+1,m}}{b_{N-m+1,m}}\right)\Re\left(\alpha_{n,m}\right)b_{n,m}}{\left|\alpha_{n,m}\right|^2}\\ &\quad+\sum_{n=1}^{N-m}\frac{\Im\left(\frac{\alpha_{N-m+1,m}}{b_{N-m+1,m}}\right)\Im\left(\alpha_{n,m}\right)b_{n,m}}{\left|\alpha_{n,m}\right|^2} \qquad (23)\end{aligned}$$

If statement I holds, then certainly $\xi_{N,m} \geq 0 > -1/2$ for all N, m, and we are done.

If Statement II holds, then

$$\begin{aligned}\Im\left(\frac{\alpha_{N-m+1,m}}{b_{N-m+1,m}}\right)\Im\left(\alpha_{n,m}\right) &= \left|\Im\left(\frac{\alpha_{N-m+1,m}}{b_{N-m+1,m}}\right)\right|\left|\Im\left(\alpha_{n,m}\right)\right|\\ &\geq \left|\Re\left(\frac{\alpha_{N-m+1,m}}{b_{N-m+1,m}}\right)\right|\left|\Re\left(\alpha_{n,m}\right)\right|,\end{aligned}$$

so that equation (23) yields

$$\xi_{N,m} \geq \Re\left(\frac{\alpha_{N-m+1,m}}{b_{N-m+1,m}}\right) \geq -\frac{1}{2},$$

with strict inequality for infinitely many pairs of indices (N, m).

If statement III holds, we find $\xi_{N,m} \geq 0 > -1/2$, using parallel arguments to those used for statement II.

Finally, suppose that statement IV holds. Then equation (23) implies

$$\begin{aligned}
\Re(\xi_{N,m}) &= -\left|\Re\left(\frac{\alpha_{N-m+1,m}}{b_{N-m+1,m}}\right)\right| \\
&\quad + \sum_{n=1}^{N-m} b_{n,m} \frac{\left|\Re\left(\frac{\alpha_{N-m+1,m}}{b_{N-m+1,m}}\right)\right| \left|\Re\left(\alpha_{n,m}\right)\right| + \Im\left(\frac{\alpha_{N-m+1,m}}{b_{N-m+1,m}}\right)\Im\left(\alpha_{n,m}\right)}{\left|\alpha_{n,m}\right|^2} \\
&\leq -\left|\Re\left(\frac{\alpha_{N-m+1,m}}{b_{N-m+1,m}}\right)\right| \\
&\quad + \max\left\{\left|\Re\left(\frac{\alpha_{N-m+1,m}}{b_{N-m+1,m}}\right)\right|, \left|\Im\left(\frac{\alpha_{N-m+1,m}}{b_{N-m+1,m}}\right)\right|\right\} \sum_{n=1}^{N-m} \frac{b_{n,m}}{|\alpha_{n,m}|} \\
&\leq -\left|\Re\left(\frac{\alpha_{N-m+1,m}}{b_{N-m+1,m}}\right)\right| + R\left|\Re\left(\frac{\alpha_{N-m+1,m}}{b_{N-m+1,m}}\right)\right| \sum_{n=1}^{N-m} \frac{b_{n,m}}{|\alpha_{n,m}|} \\
&< -\left|\Re\left(\frac{\alpha_{N-m+1,m}}{b_{N-m+1,m}}\right)\right| (1 - RX) \leq -\frac{1}{2},
\end{aligned}$$

for all pairs (N, m). Since $\left|\Im\left(\frac{\alpha_{N-m+1,m}}{b_{N-m+1,m}}\right)\right| \leq R\left|\Re\left(\frac{\alpha_{N-m+1,m}}{b_{N-m+1,m}}\right)\right|$, it follows that $\left|\Re\left(\frac{\alpha_{N-m+1,m}}{b_{N-m+1,m}}\right)\right| \geq \sqrt{1+R^2}^{-1}\left|\frac{\alpha_{N-m+1,m}}{b_{N-m+1,m}}\right|$. By inequalities (1) and (2), this converges uniformly to ∞ as $N \to \infty$. Hence, $\Re(\xi_{N,m}) < -1/2$ for all m when N is sufficiently large. This completes the proof. □

Lemma 18. *Let D, $d_{n,m}$, D_n, ε, $b_{m,n}$, and $\alpha_{m,n}$ be given as in Theorem 2, except that we do not assume equation* (4) *and that the restriction on real and imaginary values may be replaced by either of the three alternative restrictions posed in the remark following the theorem. Suppose* $\deg_{\mathbb{K}} x \leq D$*. Then*

$$\lim_{N\to\infty} \left(2^{N^2} \prod_{n=1}^{N} |\alpha_{n,1}|^{n+\frac{n+2}{\log^{3+\varepsilon}\log|\alpha_{n,1}|}}\right)^{DD_N} \sum_{n=N+1}^{\infty} \frac{|\alpha_{n,1}|^{\frac{1}{\log^{3+\varepsilon}\log|\alpha_{n,1}|}}}{|\alpha_{n,1}|} = \infty.$$

Proof. Suppose that $\deg_{\mathbb{K}} x \leq d$, which ensure that x is algebraic.

By Lemmas 6 and 5, we have

$$\begin{aligned}
H(x_N) &\leq \prod_{m=1}^{N}\left(2^{N-m+1}\prod_{n=1}^{N-m+1} b_{n,m} H\left(\frac{1}{\alpha_{n,m}}\right)\right) \\
&= 2^{N(N+1)/2} \prod_{n=1}^{N}\prod_{j=1}^{n} b_{n-j+1,j} H(\alpha_{n-j+1,j})
\end{aligned}$$

We then apply Lemma 4 as well as inequalities (2) and (3) to find

$$
\begin{aligned}
H(x_N) &\le 2^{N(N+1)/2} \prod_{n=1}^{N} \prod_{j=1}^{n} |\alpha_{n,1}|^{-1+\frac{1}{\log^{3+\varepsilon}\log|\alpha_{n,1}|}} \left\lceil\overline{\alpha_{n-j+1,j}}\right\rceil^2 \\
&\le 2^{N(N+1)/2} \prod_{n=1}^{N} |\alpha_{n,1}|^{-n+n\frac{1}{\log^{3+\varepsilon}\log|\alpha_{n,1}|}} |\alpha_{n,1}|^{2n+2\frac{1}{\log^{3+\varepsilon}\log|\alpha_{n,1}|}} \\
&= 2^{N(N+1)/2} \prod_{n=1}^{N} |\alpha_{n,1}|^{n+\frac{n+2}{\log^{3+\varepsilon}\log|\alpha_{n,1}|}}. \qquad (24)
\end{aligned}
$$

By Lemma 17, it follows that $x_N \neq x$ for all sufficiently large N, which means that we may apply Lemma 7 with $\alpha = x - x_N$ and $\beta = 0$, leading to

$$|x - x_N| \ge (2H(x - x_N))^{-\deg(x - x_N)}.$$

Since clearly $\mathbb{K} = \bigcup_{n=1}^{\infty} \mathbb{K}_n$, $\deg_{\mathbb{K}} x = \deg_{\mathbb{K}_N} x$ for all sufficiently large N. Then $x - x_N \in \mathbb{K}_N(x)$, and so

$$\deg(x - x_N) \le [\mathbb{K}_N : \mathbb{Q}] \deg_{\mathbb{K}_N} x \le D \prod_{n=1}^{N} [\mathbb{K}_N : \mathbb{K}_{N-1}] = DD_N.$$

Using this and inequality (24), we continue the above estimate on $|x - x_n|$ and find

$$|x - x_N| \ge \left(2^{\frac{2}{3}N^2} \prod_{n=1}^{N} |\alpha_{n,1}|^{n+\frac{n+2}{\log^{3+\varepsilon}\log|\alpha_{n,1}|}} \right)^{-DD_N}.$$

To also get an upper bound of $|x - x_N|$, notice that the number

$$C = \sup_{K,N\in\mathbb{N}_0} \left\{ \prod_{m=1}^{N} \left| 1 + \frac{\sum_{n=N-m+2}^{\infty} \frac{b_{m,n}}{a_{m,n}}}{1 + \sum_{n=1}^{N-m+1} \frac{b_{m,n}}{a_{m,n}}} \right| \prod_{m=N+1}^{N+K} \left| 1 + \sum_{n=1}^{\infty} \frac{b_{m,n}}{a_{m,n}} \right| \right\}$$

is a finite, positive number. Lemma 14 now yields

$$
\begin{aligned}
\frac{|x - x_N|}{|x_N|} &= \left| 1 - \prod_{m=1}^{N} \left(1 + \frac{\sum_{n=N-m+2}^{\infty} \frac{b_{m,n}}{a_{m,n}}}{1 + \sum_{n=1}^{N-m+1} \frac{b_{m,n}}{a_{m,n}}} \right) \prod_{m=N+1}^{\infty} \left(1 + \sum_{n=1}^{\infty} \frac{b_{m,n}}{a_{m,n}} \right) \right| \\
&\le C \left(\sum_{m=1}^{N} \left| \frac{\sum_{n=N-m+2}^{\infty} \frac{b_{m,n}}{a_{m,n}}}{1 + \sum_{n=1}^{N-m+1} \frac{b_{m,n}}{a_{m,n}}} \right| + \sum_{m=N+1}^{\infty} \left| \sum_{n=1}^{\infty} \frac{b_{m,n}}{a_{m,n}} \right| \right).
\end{aligned}
$$

By Lemma 17, the numbers $\left|1 - \sum_{n=1}^{\infty} \frac{b_{m,n}}{a_{m,n}}\right|$ have a uniform lower bound, C_0, while inequalities (1) and (2) ensure that each $\lim_{m\to\infty} \sum_{n=1}^{\infty} \left|\frac{b_{m,n}}{a_{m,n}}\right| = 0$. Hence, the numbers $\left|1 + \sum_{n=1}^{N-m+1} \frac{b_{m,n}}{a_{m,n}}\right|$ are bounded by $C_0/2$ for all sufficiently large N, and so

$$\begin{aligned}\frac{|x-x_N|}{|x_N|} &\leq C\Bigg(\frac{2}{C_0}\sum_{m=1}^{N}\Bigg|\sum_{n=N-m+2}^{\infty}\frac{b_{m,n}}{a_{m,n}}\Bigg| + \sum_{m=N+1}^{\infty}\Bigg|\sum_{n=1}^{\infty}\frac{b_{m,n}}{a_{m,n}}\Bigg|\Bigg)\\ &\leq \frac{2C}{C_0}\Bigg(\sum_{m=1}^{N}\sum_{n=N-m+2}^{\infty}\Bigg|\frac{b_{m,n}}{a_{m,n}}\Bigg| + \sum_{m=N+1}^{\infty}\Bigg|\sum_{n=1}^{\infty}\frac{b_{m,n}}{a_{m,n}}\Bigg|\Bigg)\\ &= \frac{2C}{C_0}\sum_{n+m\geq N+2}^{\infty}\Bigg|\frac{b_{m,n}}{a_{m,n}}\Bigg| = \frac{2C}{C_0}\sum_{n=N+1}^{\infty}\sum_{j=1}^{n}\Bigg|\frac{b_{n-j+1,j}}{a_{n-j+1,j}}\Bigg|.\end{aligned}$$

by also applying the triangle inequality in the second estimate. By inequality (2), we now have

$$\begin{aligned}|x-x_N| &\leq \frac{2C}{C_0}|x_N|\sum_{n=N+1}^{\infty}|\alpha_{n,1}|^{-1+\frac{1}{\log^{3+\varepsilon}\log|\alpha_{n,1}|}}\\ &\leq C'\sum_{n=N+1}^{\infty}|\alpha_{n,1}|^{-1+\frac{1}{\log^{3+\varepsilon}\log|\alpha_{n,1}|}},\end{aligned}$$

for a suitable constant $C'>0$ that does not depend on N. Recalling the lower bound on $|x-x_N|$ found above, we conclude

$$\begin{aligned}&\left(2^{N^2}\prod_{n=1}^{N}|\alpha_{n,1}|^{n+\frac{n+2}{\log^{3+\varepsilon}\log|\alpha_{n,1}|}}\right)^{DD_N}\sum_{n=N+1}^{\infty}\frac{|\alpha_{n,1}|^{\frac{1}{\log^{3+\varepsilon}\log|\alpha_{n,1}|}}}{|\alpha_{n,1}|}\\ &\qquad\geq\frac{2^{DD_N\frac{N^2}{3}}}{C'}\underset{N\to\infty}{\longrightarrow}\infty.\end{aligned}$$ □

Lemma 19. *Let D, $d_{n,m}$, D_N, ε, and $\alpha_{n,1}$ be given as in Theorem 2. Then*

$$\liminf_{N\to\infty}\left(2^{N^2}\prod_{n=1}^{N}|\alpha_{n,1}|^{n+\frac{n+2}{\log^{3+\varepsilon}\log|\alpha_{n,1}|}}\right)^{DD_N}\sum_{n=N+1}^{\infty}\frac{|\alpha_{n,1}|^{\frac{1}{\log^{3+\varepsilon}\log|\alpha_{n,1}|}}}{|\alpha_{n,1}|}=0.$$

Proof (original). To simplify notation, we introduce $a_n=|\alpha_{n,1}|$ and

$$Z_N=\left(2^{N^2}\prod_{n=1}^{N}a_n^{n+\frac{n+2}{\log^{3+\varepsilon}\log a_n}}\right)^{DD_N}\sum_{n=N+1}^{\infty}\frac{a_n^{\frac{1}{\log^{3+\varepsilon}\log a_n}}}{a_n},$$

so that our aim is to prove that Z_N has no positive lower bound. We now split into four cases.

Case 1. Assume that equation (5) holds for all sufficiently large N and that there is a real number $0<\delta<1$ such that a_n and δ satisfy equation (6). By Lemma 12 and equation (4), we then have infinitely many N so that

$$a_{N+1} > \left(\left(1+\frac{1}{N^2}\right)^{D^N(N+1+\delta)!\prod_{i=1}^{N-1}D_i}\prod_{n=1}^{N}a_n^{n+\delta}\right)^{DD_N}. \tag{25}$$

Then

$$a_{N+1} > \left(1 + \frac{1}{N^2}\right)^{D^{N+1}(N+1+\delta)!\prod_{i=1}^{N} D_i} \geq 2^{N^{5+\varepsilon} DD_N} \tag{26}$$

and

$$\begin{aligned}\log\log a_{N+1} &\geq \log\left(DD_N \log\left((1+N^{-2})^{D^N(N+1+\delta)!\prod_{i=1}^{N-1} D_i}\right)\right)\\ &\geq \log\left(\frac{(N+1+\delta)!}{2N^2} D^{N+1}\prod_{i=1}^{N} D_i\right) > \log((N-1+\delta)!)\\ &\geq \frac{(N-1+\delta)\log(N-1+\delta)}{2} \geq \frac{N\log N}{3} + \log 2.\end{aligned}$$

Using the latter after applying Lemma 11 and inequality (25), we have

$$\begin{aligned}&\sum_{n=N+1}^{\infty} \frac{a_n^{\frac{1}{\log^{3+\varepsilon}\log a_n}}}{a_n}\left(\prod_{n=1}^{N} a_n^{\frac{n+2}{\log^{3+\varepsilon}\log a_n}}\right)^{DD_N}\\ &\quad< a_{N+1}^{-1+\frac{1}{\log^{3+\varepsilon/2}\log a_{N+1}}+\frac{2}{\log^{3+\varepsilon}\log a_{N+1}}} < a_{N+1}^{-1+\frac{2}{\log^{3+\varepsilon/2}\log a_{N+1}}}\\ &\quad< a_{N+1}^{-1+\left(\frac{N\log N}{3}\right)^{-3-\varepsilon/2}}.\end{aligned}$$

By equation (26), we have

$$a_{N+1}^{\left(\frac{N\log N}{3}\right)^{-3-\varepsilon/2}-N^{-3-\varepsilon/2}} < a_{N+1}^{N^{-3-\varepsilon}} < 2^{-N^2 DD_N},$$

and so

$$\begin{aligned}\left(2^{N^2}\prod_{n=1}^{N} a_n^{\frac{n+2}{\log^{3+\varepsilon}\log a_n}}\right)^{DD_N} \sum_{n=N+1}^{\infty} \frac{a_n^{\frac{1}{\log^{3+\varepsilon}\log a_n}}}{a_n} &< a_{N+1}^{-1+N^{-3-\varepsilon/2}}\\ &\leq \left(\prod_{n=1}^{N} a_n^{n+\delta}\right)^{DD_N(-1+N^{-3-\varepsilon/2})},\end{aligned} \tag{27}$$

Thus,

$$\begin{aligned}Z_N &= \left(2^{N^2}\prod_{n=1}^{N} a_n^{n+\frac{n+2}{\log^{3+\varepsilon}\log a_n}}\right)^{DD_N} \sum_{n=N+1}^{\infty} \frac{a_n^{\frac{1}{\log^{3+\varepsilon}\log a_n}}}{a_n}\\ &< \left(\prod_{n=1}^{N} a_n^{-\delta+(n+\delta)N^{-3-\varepsilon/2}}\right)^{DD_N}\\ &\leq \left(\prod_{n=1}^{N} a_n^{-\delta/2}\right)^{DD_N}.\end{aligned}$$

As there are infinitely many such N, this completes the proof in the present case.

Case 2. Assume that equation (5) holds for all sufficiently large N and that there is no real number $0 < \delta < 1$ such that a_n and δ satisfy equation (6). For all $0 < \delta < 1$, we then have

$$a_n < 2^{D^n (n+\delta)! \prod_{i=1}^{n-1} D_i}, \tag{28}$$

for all sufficiently large n.

By Lemma 12 and equation (4), we have infinitely many N so that

$$a_{N+1} > \left(\left(1 + \frac{1}{N^2} \right)^{D^N (N+1)! \prod_{i=1}^{N-1} D_i} \prod_{n=1}^{N} a_n^n \right)^{DD_N}. \tag{29}$$

Notice that all arguments leading to (27) in Case 1 remain valid when we replace δ by 0. Hence, equation (29) implies

$$\begin{aligned} \left(2^{N^2} \prod_{n=1}^{N} a_n^{\frac{n+2}{\log^{3+\varepsilon} \log a_n}} \right)^{DD_N} \sum_{n=N+1}^{\infty} \frac{a_n^{\frac{1}{\log^{3+\varepsilon} \log a_n}}}{a_n} &< a_{N+1}^{-1+N^{-3-\varepsilon/2}} \\ &< \left(\left(1 + \frac{1}{N^2} \right)^{(N+1)!} \prod_{n=1}^{N} a_n^n \right)^{DD_N \left(-1 + N^{-3-\varepsilon/2} \right)}. \end{aligned}$$

Let $\delta > 0$ be sufficiently small. When the above N grow sufficiently large, equation (28) and the fact that $(1 + 1/N)^N > 2$ imply

$$\begin{aligned} Z_N &= \left(2^{N^2} \prod_{n=1}^{N} a_n^{n + \frac{n+2}{\log^{3+\varepsilon} \log a_n}} \right)^{DD_N} \sum_{n=N+1}^{\infty} \frac{a_n^{\frac{1}{\log^{3+\varepsilon} \log a_n}}}{a_n} \\ &< \left(\left(1 + \frac{1}{N^2} \right)^{(N+1)! \left(-1 + N^{-3-\varepsilon/2} \right)} \prod_{n=1}^{N} a_n^{nN^{-3-\varepsilon/2}} \right)^{DD_N} \\ &\leq \left(2^{N! \left(-1 + N^{-3-\varepsilon/2} \right)} \prod_{n=1}^{N} 2^{(n+\delta)! \frac{n}{N^{3+\varepsilon/2}}} \right)^{DD_N} \\ &\leq \left(2^{DD_N} \right)^{\frac{N^2 (N+\delta)! + N!}{N^{3+\varepsilon/2}} - N!} < 2^{-N}, \end{aligned}$$

and so the proof is complete in this case.

Case 3. Assume that

$$a_n \leq 2^n \tag{30}$$

holds for infinitely many N and that there is a real number $0 < \delta < 1$ such that a_n and δ satisfy equation (6).

We fix an $A > 0$. By (6), there is an $n \in \mathbb{N}$ such that

$$a_n^{\frac{1}{D^n (n+\delta)! \prod_{i=1}^{n-1} D_i}} > A,$$

so we may pick k_1 minimal with this property. We then choose $k_2 < k_1$ maximal such that $a_{k_2} \leq 2^{k_2}$ by (30). If no such k_2 exists, we increase

A until it does. This is possible since k_1 tends to infinity with A and since (30) is satisfied for infinitely many indices.

Now,

$$a_{k_1} > A^{D^n(n+\delta)!\prod_{i=1}^{n-1} D_i} = 2^{\log_2(A)D^n(n+\delta)!\prod_{i=1}^{n-1} D_i}, \tag{31}$$

so there is an $n < k_1$ with

$$a_{n+1} > 2^{D^{n+1}(n+1+\delta)!\prod_{i=1}^{n} D_i}. \tag{32}$$

We pick $N \in [k_2, k_1)$ minimal such that the latter holds. Such an index exists since

$$a_{k_2} \le 2^{k_2} \le 2^{D^{k_2}(k_2+\delta)!\prod_{i=1}^{k_2-1} D_i},$$

since a_n is increasing, and since a_{k_1} satisfies (31). Note that as A increases, both k_1 and k_2 will increase. Hence, N will tend to infinity as A tends to infinity.

Consider first the product,

$$\prod_{n=1}^{N} a_n^{n+\frac{n+2}{\log^{3+\varepsilon}\log a_n}} = \left(\prod_{n=1}^{k_2} a_n^{n+\frac{n+2}{\log^{3+\varepsilon}\log a_n}}\right)\left(\prod_{n=k_2+1}^{N} a_n^{n+\frac{n+2}{\log^{3+\varepsilon}\log a_n}}\right) = M_1 \cdot M_2.$$

By choice of k_2, since a_n is an increasing sequence,

$$M_1 \le \prod_{n=1}^{k_2} 2^{n^2+n\frac{n+2}{\log^{3+\varepsilon}\log 2^n}} \le \prod_{n=1}^{k_2} 2^{n\left(k_2+\frac{k_2+2}{\log^{3+\varepsilon}\log 2^{k_2}}\right)} \le 2^{k_2^3},$$

by carrying out the multipliction and noticing the triangular number in the exponent.

Now, for M_2 we have by (32)

$$\begin{aligned} M_2 &\le \prod_{n=k_2+1}^{N} \left(2^{D^n(n+\delta)!\prod_{i=1}^{n-1} D_i}\right)^{n+\frac{n+2}{\log^{3+\varepsilon}\log a_n}} \\ &\le \prod_{n=k_2+1}^{N} 2^{\left(D^n(n+\delta)!\prod_{i=1}^{n-1} D_i\right)\left(n+\frac{n+2}{\log^{3+\varepsilon}(D^{n+1}(n+1+\delta)!(\prod_{i=1}^{n} D_i)\log 2)}\right)} \\ &\le 2^{\left(D^N(N+\delta)!\prod_{i=1}^{N-1} D_i\right)\left(N+\frac{1}{N^2}\right)} \prod_{n=k_2+1}^{N-1} 2^{\left(D^n(n+\delta)!\prod_{i=1}^{n-1} D_i\right)\left(n+\frac{1}{n^2}\right)} \\ &\le \left(2^{(N+\delta)!\left(N+\frac{1}{N^2}\right)} \prod_{n=k_2+1}^{N-1} 2^{(n+\delta)!\left(n+\frac{1}{n^2}\right)}\right)^{D^N\prod_{i=1}^{N-1} D_i}, \end{aligned}$$

where we have bounded $\frac{n+2}{\log^{3+\varepsilon} D^{n+1}(n+1+\delta)!\prod_{i=1}^{n} D_i \log 2}$ rather brutally by $1/n^2$. To continue,

$$2^{(N+\delta)!\left(N+\frac{1}{N^2}\right)} = 2^{(N+1+\delta)!-\left(1+\delta-\frac{1}{N^2}\right)(N+\delta)!}.$$

Thus, for N large enough, which we can ensure by increasing A, the term

$$2^{-\left(1+\delta-\frac{1}{N^2}\right)(N+\delta)!}$$

will cancel out the product over the remaining n's as well as the term coming from M_1. Thus,

$$M_1 M_2 \leq 2^{\left((N+1+\delta)!-\frac{\delta}{2}(N+\delta)!\right)D^N \prod_{i=1}^{N-1} D_i}.$$

Now, consider the sum

$$\sum_{n=N+1}^{\infty} \frac{a_n^{\frac{1}{\log^{3+\varepsilon}\log a_n}}}{a_n} = \sum_{n=N+1}^{k_1-1} \frac{a_n^{\frac{1}{\log^{3+\varepsilon}\log a_n}}}{a_n} + \sum_{n=k_1}^{\infty} \frac{a_n^{\frac{1}{\log^{3+\varepsilon}\log a_n}}}{a_n} = S_1 + S_2.$$

By Corollary 13, choice of N, and (32),

$$S_1 \leq a_{N+1}^{\frac{1}{\log^{3+\varepsilon/2}\log a_{N+1}}-1} \leq 2^{D^{N+1}(N+1+\delta)!(\frac{1}{(N+1)^3}-1)\prod_{i=1}^{N+1} D_i},$$

by a brutal estimate in the exponent. For S_2, Lemma 10 together with (31) gives us that

$$S_2 \leq a_{k_1}^{-\frac{\varepsilon}{2(1-\varepsilon)}} \leq 2^{-(\log_2 A)\frac{\varepsilon}{2(1-\varepsilon)}D^{N+1}(N+1+\delta)!\prod_{i=1}^{N} D_i}.$$

The upshot is that

$$Z_N \leq \left(2^{N^2} M_1 M_2\right)^{DD_N} (S_1 + S_2).$$

But this evidently tends to 0 as A – and hence N – increases, by inserting all the above estimates. This completes the proof in this case.

Case 4. Assume that equation (30) holds for infinitely many N and that there is no real number $0 < \delta < 1$ such that a_n and δ satisfy equation (6). For all $0 < \delta < 1$, equation (28) holds for all sufficiently large n. Note that by the *limsup* assumption in Theorem 2, we instead have equation (6) if we let $\delta = 0$.

Let $\delta > 0$ be fixed, and let A be sufficiently large. By inequality (28),

$$a_n \leq 2^{D^n(n+\delta)!\prod_{i=1}^{n-1} D_i} \tag{33}$$

holds for all sufficiently large $n \in \mathbb{N}$. Pick $k_1 \in \mathbb{N}$ to be minimal with the property that

$$a_{k_1} > A^{D^{k_1}k_1!\prod_{i=1}^{k_1-1} D_i}.$$

Now, choose $k_2 < k_1$ to be maximal so that $a_{k_2} \leq 2^{k_2}$. Should no such k_2 exist, then choose A larger. Note also that for A increasing, both k_1 and k_2 must increase.

As before, we now choose $N \in [k_2, k_1)$ such that a large jump must occur at this place. Concretely, we use Lemma 9 and inequality (6) to let $N \geq k_2$ be minimal with

$$a_{N+1}^{\frac{1}{D^{N+1}(N+1)!\prod_{i=1}^{N} D_i}} > \left(1 + \frac{1}{(N+1)^{1+\varepsilon/4}}\right) \max_{j=k_2,\dots,N} a_j^{\frac{1}{D^j j! \prod_{i=1}^{j-1} D_i}}.$$

Since $\prod_{n=1}^{\infty}\left(1+\frac{1}{(N+1)^{1+\varepsilon/4}}\right) < \infty$, we must have $N < k_1$ when k_1 is sufficiently large.

We claim that Z_N tends to zero with N, which again suffices, as both k_1 and k_2 tend to infinity with A, so that a subsequence of left hand sides in the lemma will tend to zero.

Again, as in the preceding case, we will let

$$\prod_{n=1}^{N} a_n^{n+\frac{n+2}{\log^{3+\varepsilon}\log a_n}} = \left(\prod_{n=1}^{k_2} a_n^{n+\frac{n+2}{\log^{3+\varepsilon}\log a_n}}\right)\left(\prod_{n=k_2+1}^{N} a_n^{n+\frac{n+2}{\log^{3+\varepsilon}\log a_n}}\right) = M_1 \cdot M_2.$$

For $r < k_2$, $a_r \leq a_{k_2} \leq 2^{k_2}$, so as before

$$M_1 \leq \prod_{n=1}^{k_2} 2^{n^2+n\frac{n+2}{\log^{3+\varepsilon}\log 2^n}} \leq \prod_{n=1}^{k_2} 2^{n\left(k_2+\frac{k_2+2}{\log^{3+\varepsilon}\log 2^{k_2}}\right)} \leq 2^{k_2^3}.$$

Now, by minimality of N, it follows for each $r \in (k_2, N]$ that

$$a_r^{\frac{1}{D^n n! \prod_{i=1}^{n-1} D_i}} \leq \left(1+\frac{1}{n^{1+\varepsilon/4}}\right) \max_{j=k_2,\dots,n-1} a_j^{\frac{1}{D^j j! \prod_{i=1}^{j-1} D_i}}.$$

Using this successively, we find that

$$a_n^{\frac{1}{D^n n! \prod_{i=1}^{n-1} D_i}} \leq \left(1+\frac{1}{n^{1+\varepsilon/4}}\right)\left(1+\frac{1}{(n-1)^{1+\varepsilon/4}}\right) \max_{j=k_2,\dots,n-2} a_j^{\frac{1}{D^j j! \prod_{i=1}^{j-1} D_i}}$$
$$\leq \cdots \leq a_{k_2}^{\frac{1}{D^{k_2} k_2! \prod_{i=1}^{k_2-1} D_i}} \prod_{j=k_2+1}^{n}\left(1+\frac{1}{j^{1+\varepsilon/4}}\right)$$

The latter is a partial product of a convergent infinite product, and so can be bounded by a constant depending only on ε. Since $a_{k_2} < 2^{k_2}$, the first factor is also bounded by $\sqrt{2}$, say. The upshot is that for some B depending only on ε,

$$a_n \leq B^{D^n n! \prod_{i=1}^{n-1} D_i},$$

so that

$$M_2 \leq \prod_{n=k_2+1}^{N} B^{\left(D^n n! \prod_{i=1}^{n-1} D_i\right)\left(n+\frac{n+2}{\log^{3+\varepsilon}\log a_n}\right)} \leq \prod_{n=k_2+1}^{N} B^{\left(D^n n! \prod_{i=1}^{n-1} D_i\right)\left(n+\frac{n+2}{\log^{3+\varepsilon}(n\log 2)}\right)},$$

since $a_n > 2^n$ by maximality of k_2.

Again as in the preceding case, consider now the sum

$$\sum_{n=t}^{\infty} \frac{a_n^{\frac{1}{\log^{3+\varepsilon}\log a_n}}}{a_n} = \sum_{n=t}^{k_1-1} \frac{a_n^{\frac{1}{\log^{3+\varepsilon}\log a_n}}}{a_n} + \sum_{n=k_1}^{\infty} \frac{a_n^{\frac{1}{\log^{3+\varepsilon}\log a_n}}}{a_n} = S_1 + S_2.$$

By Corollary 13 as before,

$$S_1 \leq a_{N+1}^{\frac{1}{\log^{3+\varepsilon/2}\log a_{N+1}}-1} \tag{34}$$

Since

$$a_{N+1}^{\frac{1}{D^{N+1}(N+1)!\prod_{i=1}^{N} D_i}} > \left(1+\frac{1}{(N+1)^{1+\varepsilon/4}}\right)\max_{j=s,\dots,N} a_j^{\frac{1}{D^j j!\prod_{i=1}^{j-1} D_i}},$$

certainly,

$$\begin{aligned} a_{N+1} &> \left(\left(1+\frac{1}{(N+1)^{1+\varepsilon/4}}\right)\max_{j=s,\dots,N} a_j^{\frac{1}{D^j j!\prod_{i=1}^{j-1} D_i}}\right)^{D^{N+1}(N+1)!\prod_{i=1}^{N} D_i} \\ &\geq \left(1+\frac{1}{N^{1+\varepsilon/4}}\right)^{D^{N+1}(N+1)!\prod_{i=1}^{N} D_i}\prod_{r=1}^{N} a_r. \end{aligned} \tag{35}$$

Furthermore, by minimality of t, for each $n \in (k_2, N]$,

$$\begin{aligned} a_n^{\frac{1}{D^n n!\prod_{i=1}^{n-1} D_i}} &\leq \left(1+\frac{1}{n^{1+\varepsilon/4}}\right)\max_{j=k_2,\dots,n-1} a_j^{\frac{1}{D^j j!\prod_{i=1}^{j-1} D_i}} \\ &\leq \left(1+\frac{1}{n^{1+\varepsilon/4}}\right)\left(1+\frac{1}{(n-1)^{1+\varepsilon/4}}\right)\max_{j=k_2,\dots,n-2} a_j^{\frac{1}{D^j j!\prod_{i=1}^{j-1} D_i}} \\ &\leq \dots \\ &\leq \prod_{j=k_2+1}^{n}\left(1+\frac{1}{j^{1+\varepsilon/4}}\right) a_{k_2}^{\frac{1}{D^j j!\prod_{i=1}^{j-1} D_i}}. \end{aligned}$$

Since $a_{k_2} \leq 2^{k_2}$, this is bounded by a constant, K say, on estimating the product by its infinite counterpart. Consequenly, since a_n is an increasing sequence by assumption,

$$\prod_{r=1}^{N} a_r = \prod_{r=1}^{k_2} a_r \prod_{r=k_2+1}^{N} a_r \leq 2^{k_2^2}\prod_{r=k_2+1}^{N} K^{D^r r!\prod_{i=1}^{r-1} D_i} \tag{36}$$

We insert (35) into (34) to obtain

$$S_1 \leq \left(\left(1+\frac{1}{N^{1+\varepsilon/4}}\right)^{D^{N+1}(N+1)!\prod_{i=1}^{N} D_i}\prod_{r=1}^{N} a_r\right)^{\frac{1}{\log^{3+\varepsilon/2}\log a_{N+1}}-1}.$$

Using (33),

$$S_1 \leq \left(\left(1+\frac{1}{N^{1+\varepsilon/4}}\right)^{D^t t!\prod_{i=1}^{N} D_i}\prod_{r=1}^{N} a_r\right)^{\frac{1}{t^{3+\varepsilon/4}}-1}.$$

Using finally (36),

$$S_1 \leq \frac{\left(1+\frac{1}{N^{1+\varepsilon/4}}\right)^{\frac{1}{D^{N+1}(N+1)!\prod_{i=1}^{N} D_i}\left(\frac{1}{N^{3+\varepsilon/4}}-1\right)} \left(2^{k_2^2}\prod_{r=k_2+1}^{N} K^{D^r r!\prod_{i=1}^{r-1} D_i}\right)^{\frac{1}{N^{3+\varepsilon/4}}}}{\prod_{r=1}^{N} a_r}$$

The second summand S_2 is estmated by Lemma 10, so that

$$S_2 = \sum_{n=k_1}^{\infty} \frac{a_n^{\frac{1}{\log^{3+\varepsilon}\log a_n}}}{a_n} < |a_{k_1}|^{-\frac{\varepsilon}{2(1+\varepsilon)}} < A^{-\varepsilon(D^{k_1}k_1!\prod_{i=1}^{k_1-1} D_i)/2(1+\varepsilon)}.$$

In other words, since

$$Z_N = (2^{N^2} M_1 M_2)^{DD_N}(S_1 + S_2),$$

by inserting the estimates above, Z_N can be made arbitrarily small by increasing A, which in turn corresponds to increasing t, so that in this case the *liminf* of the Lemma is also equal to zero. The four cases exhaust the possibilities of satisfying the conditions of the lemma, and so the proof is complete. □

5. Concluding Remarks

In the light of Theorem 1, we expect that Theorem 2 will remain true if the *limsup* criterion (4) is weakened to the assumption that $|\alpha_{N,1}|^{\frac{1}{D^N N!\prod_{n=1}^{N-1} D_n}}$ diverges in $\mathbb{R}$, though this will require additional arguments in lemma 19, most likely in the form of two additional cases to be considered. We deemed this question out of scope for the current paper, however.

Similarly, the proof of Theorem 2 may be modified so that the bound on $b_n\overline{|a_n|}$ can be loosened to $b_n \leq |a_n|^{(\log\log|a_n|)^{-3-\varepsilon}}$ and $\overline{|a_n|} \leq |a_n|^{1+(\log\log|a_n|)^{-3-\varepsilon}}$, thus presenting the same lenient bound on b_n as found in [4]. Note however that in order to accomplish this, we would then have to strengthen the divergence assumption to

$$\limsup_{n\to\infty} |a_n|^{1/D^n\prod_{i=1}^{n-1}(D_i+d_i)} = \infty,$$

at least until the case of

$$\liminf_{n\to\infty} |a_n|^{1/D^n\prod_{i=1}^{n-1}(D_i+d_i)} < \limsup_{n\to\infty} |a_n|^{1/D^n\prod_{i=1}^{n-1}(D_i+d_i)} < \infty$$

has been handled.

Furthermore, in [8], an analogue of Theorem 1 for series $\sum_{n=1}^{\infty}\frac{b_n}{\alpha_n}$ with $D_n = d$ constant was proven (Proposition 4.3 of the paper) and where the divergence criterion is replaced by *limsup* criterion. Compared to the present paper and [1, 7], the exponent in the *limsup* expression is not

$$\limsup_{n\to\infty} |\alpha_n|^{D^{-n}(1+d)^{-n}} = \infty,$$

as would be expected, but rather

$$\limsup_{n\to\infty} |\alpha_n|^{(1+dD)^{-n}} = \infty.$$

We therefore suspect that Theorem 1 may be improved so that the exponent in the divergence criterion may be replaced with

$$\frac{1}{\prod_{i=1}^{n-1}(d_i + DD_i)}.$$

This is less strict when $D > 1$. We further suspect tha the exponent in the *limsup* expression of Theorem 2 may be replaced with

$$\frac{1}{\prod_{i=1}^{N-1}(d_i + iDD_i)},$$

which is easily checked to be more lenient when $DD_i > 1$.

It is also likely that the restrictions on real and imaginary values in Theorem 2 – including the alternative restrictions presented in the subsequent remark – may be weakened to some extent.

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S. Kristensen, Department of Mathematics, Aarhus University, Ny Munkegade 118, DK-8000 Aarhus C, Denmark
Email address: sik@math.au.dk

M. L. Laursen, Department of Mathematics, Aarhus University, Ny Munkegade 118, DK-8000 Aarhus C, Denmark
Email address: mll@math.au.dk

1.5 Linear independence of continued fractions

In this section, we consider linear independence in the continued fractions setting, based on the joint paper [24] by Jaroslav Hančl, Jitu Berhanu Leta, and the current author. Recall the notation

$$[0; a_1, a_2, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}.$$

from Section 1.1. The main result of the paper is the below theorem, which generalizes the paper [2].

Theorem 1.43 (Hančl–Laursen–Leta, 2024 on arXiv)**.** *Let $K \in \mathbb{N}$, let $\mathbb{K}$ be a number field of degree d, write $D = \max(2, dK - 1)$, and let $\alpha \in (0,1)$. For each $i = 1, \ldots, K$, let $\{a_{n,i}\}_{n=1}^{\infty}$ be a sequence of positive real numbers from $\mathbb{K}$ of the form $a_{n,i} = (S_{n,i}b_{n,i} + c_{n,i})/d_{i,n}$, where $S_{n,i} \in \mathbb{Z}$ and $b_{n,i}, c_{n,i}, d_{n,i} \in \mathcal{O}_{\mathbb{K}}$, and suppose that for all $n \in \mathbb{N}$,*

$$\overline{|b_{i,n}|}, \overline{|c_{i,n}|}, \overline{|d_{i,n}|} \leq \max(2^{\log_2^{\alpha} a_{i,n}}, 2^{D^{\alpha n}}), \quad a_{1,n} < \max(a_{K,n} 2^{\log_2^{\alpha} a_{K,n}}, 2^{D^{\alpha n}}),$$

and

$$\overline{|a_{i,n}^{-1}|} \leq \frac{1}{2}. \tag{1.11}$$

For $i = 1, \ldots, K-1$, assume furthermore that

$$\liminf_{n\to\infty} \sqrt{n}\left(\frac{a_{n,i}}{a_{n,i+1}} - 1\right) > 0. \tag{1.12}$$

Suppose that

$$\limsup_{n\to\infty} a_n^{D^{-n}} = \infty.$$

Then the sequences $\{a_{1,n}\}_{n=1}^{\infty}, \ldots, \{a_{1,n}\}_{n=1}^{\infty}$ are CF*-linearly independent over $\mathbb{K}$.*

This result is also true if instead of (1.11)*,*

$$e_{i,\sigma}\Re(\sigma a_{i,n}) \geq \min\{2^{-\log_2^{\alpha} a_{n,1}}, 2^{-D^{\alpha n}}\} \tag{1.13}$$

holds for all $i = 1, \ldots, K$, all embeddings σ of $\mathbb{K}$ into $\overline{\mathbb{Q}}$, and all $n \in \mathbb{N}$, where $e_{j,\sigma} \in \{-1, 1\}$ does not depend on n.

Remark 1.44. In the original phrasing of the theorem, the conclusion is only that the numbers $1, [0; a_{1,1}, a_{1,2}, \ldots], \ldots, [0; a_{K,1}, a_{K,2}, \ldots]$ are linearly independent over $\mathbb{K}$. However, none of the assumptions are affected by replacing $a_{i,n}$ and $S_{i,n}$ by $c_n a_{i,n}$ and $c_n S_{i,n}$, respectively, when all c_n are positive integers. Hence, the present formulation is equally true.

From this theorem, we may extract the following corollary. Notice that CF-linear independence over $\overline{\mathbb{Q}}$ is a generalization of CF-transcendence.

Corollary 1.45. *Suppose all assumptions of Theorem 1.43 are satisfied. If furthermore*

$$\limsup_{n\to\infty} a_{i,n}^{A^{-n}} = \infty,$$

is true for for all $A \in \mathbb{N}$, then the sequences $\{a_{1,n}\}_{n=1}^{\infty}, \dots, \{a_{K,n}\}_{n=1}^{\infty}$ are $\mathrm{CF}_{\overline{\mathbb{Q}}}$-linearly independent over $\mathbb{K}$.

It is a well-known fact from the theory of continued fractions that

$$[0; a_1, a_2, \dots, a_N] = \frac{p_N}{q_N},$$

where

$$\begin{aligned} p_0 &= 0, & p_1 &= 1, & p_{n+2} &= a_{n+2}p_{n+1} + p_n, \\ q_0 &= 1, & q_1 &= a_1, & q_{n+2} &= a_{n+2}q_{n+1} + q_n. \end{aligned} \tag{1.14}$$

When a_n is an integer for all $n \in \mathbb{N}$ and positive for $n \geq 1$, then q_N and p_N are always coprime integers with q_N positive. If the a_n are not assumed to be integers, then this is no longer guaranteed, but the above formula remains true. Notice that if a_n is a positive real number for all $n \in \mathbb{N}$, then so are p_n and q_n.

As part of proving Theorem 1.43, the authors also prove the below lemma about comparing the sequences q_N generated from two different continued fractions.

Lemma 1.46 (Hančl–Laursen–Leta, 2024 on arXiv)**.** *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of real numbers with $a_n, b_n \geq 1$ such that inequality* (1.12) *is satisfied. For each $n \in \mathbb{N}$, let $q_{n,a}$ and $q_{n,b}$ be defined by formula* (1.14) *using the finite continued fractions $[0; a_1, a_2, \dots, a_n]$ and $[0; b_1, b_2, \dots, b_n]$, respectively. Then*

$$\lim_{n\to\infty} \frac{q_{n,a}}{q_{n,b}} = \infty.$$

1.5.1 Examples

The paper [24] contains various applications of Theorem 1.43. While some are presented as examples, others are given the status of corollary or even theorem. We will here give special attention to the two applications that are considered proper theorems. The reader may find the remaining ones in [24, Section 3] as presented in subsection 1.5.2 together with a number of unsolved questions to inspire future work.

We start by presenting [24, Theorem 1], which is an immediate consequence of Theorem 1.43. In the language of irrational sequences, it reads as follows.

Theorem 1.47 (Hančl–Laursen–Leta, 2024 on arXiv). *Let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive integers such that $\limsup_{n\to\infty} a_n^{3^{-n}} = \infty$, and let $\{p_n\}_{n=1}^{\infty}$ be the sequence of all prime numbers. Then the sequences $\{a_n(1+p_n/n)\sqrt{2}\}_{n=1}^{\infty}$ and $\{a_n(p_n/n)\sqrt{2}\}_{n=1}^{\infty}$ are* CF*-linearly independent over over $\mathbb{Q}(\sqrt{2})$. In particular, the numbers*

$$\Big[0; a_1\Big(1+\frac{p_1}{1}\Big)\sqrt{2}, \dots, a_n\Big(1+\frac{p_n}{n}\Big)\sqrt{2}, \dots\Big], \quad \Big[0; a_1\frac{p_1}{1}\sqrt{2}, \dots, a_n\frac{p_n}{n}\sqrt{2}, \dots\Big],$$

and 1 are linearly independent over $\mathbb{Q}(\sqrt{2})$.

Remark 1.48. In the original version of this theorem, the first of these sequences, $\{a_n(1+p_n/n)\sqrt{2}\}_{n=1}^{\infty}$, was presented as $\{(1+p_n/n)\sqrt{2}\}_{n=1}^{\infty}$ instead; this was a typing error.

As in subsection 1.4.1, we let π denote the prime counting function, i.e., $\pi(K)$ is the number of prime numbers less than or equal to K. By the Prime Number Theorem, we have $\pi(K) \sim K/\log K$, which is to say that $\lim_{n\to\infty} \pi(K)/(K/\log K) = 1$. This implies that $p_n \sim n \log n$, which is part of the reason why inequality (1.12) is satisfied in Theorem 1.47.

The prime counting function is also used in [24, Theorem 3] as presented below. This theorem is proven by slightly modifying the proof of Theorem 1.43. It is again trivial to generalize the original statement of linear independence of numbers to a statement of CF-linear independence of sequences.

Theorem 1.49 (Hančl–Laursen–Leta, 2024 on arXiv). *Let $K \in \mathbb{N}$ with $K \geq 2$, write $\mathbb{K} = \mathbb{Q}(\sqrt{2}, \dots, \sqrt{K})$, and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of integers such that $a_n \geq K$ and $\limsup_{n\to\infty} a_n^{(K2^{\pi(K)})^{-n}} = \infty$. Then the sequences $\{\sqrt{1}a_n\}_{n=1}^{\infty}, \dots, \{\sqrt{K}a_n\}_{n=1}^{\infty}$ are* CF*-linearly independent over $\mathbb{K}$.*

1.5.2 Paper 3: Linear independence of continued fractions with algebraic terms

Below, the reader will find the most recent preprint of the paper [24], which is joint work between Jaroslav Hančl, Jitu Berhanu Leta, and the current author. The paper is currently under review but has not yet been accepted for publication. The preprint is available on arXiv through the link https://arxiv.org/abs/2406.19047v1 or by using the arXiv identifier 2406.19047. It has a length of 26 pages, numbered 1 through 26.

Linear independence of continued fractions with algebraic terms

Jaroslav Hančl, Mathias L. Laursen, and Jitu Berhanu Leta

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Abstract

We give conditions on sequences of positive algebraic numbers $\{a_{n,j}\}_{n=1}^{\infty}$, $j = 1, \ldots, M$ and number field $\mathbb{K}$ to ensure that the numbers defined by the continued fractions $[0; a_{1,j}, a_{2,j}, \ldots]$, $j = 1, \ldots, M$ and 1 are linearly independent over $\mathbb{K}$.

1 Introduction

Following Erdős [4], Davenport and Roth [3] we prove:

Theorem 1. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a non-decreasing sequence of positive integers such that* $\lim_{n\to\infty} a_n^{\frac{1}{3^n}} = \infty$ *and* $\{p_n\}_{n=1}^{\infty}$ *be the increasing sequence of all primes. Then the continued fractions*

$$\left[0; \left(1+\frac{p_1}{1}\right)\sqrt{2}, \dots, \left(1+\frac{p_n}{n}\right)\sqrt{2}, \dots\right], \quad \left[0; a_1\frac{p_1}{1}\sqrt{2}, \dots, a_n\frac{p_n}{n}\sqrt{2}, \dots\right]$$

and number 1 *are linearly independent over* $\mathbb{Q}(\sqrt{2})$ *particularly over* $\mathbb{Q}$ *.*

This is an immediate consequence of Theorem 2, which will be introduced in the chapter Main Results. The results presented in this paper have some history. Forty years ago Davenport and Roth in [3] proved that the continued fraction $[a_1; a_2, \dots]$, where $a_1, a_2, \dots$ are positive integers satisfying $\limsup_{n\to\infty}\left((\log\log a_n)\frac{\sqrt{\log n}}{n}\right) = \infty$, is a transcedental number. Hančl [8] found some criteria for continued fractions to be linearly independent. Andersen and Kristensen [1] come with special conditions on continued fractions consisting of algebraic integers to be irrational or transcedental numbers. The generalization of transcendence is algebraic independence and there are several results concerning the algebraic independence of continued fractions, see, for instance, [2], [7] or [13]. In 1975, Erdős [4] proved that if $\{a_n\}_{n=1}^{\infty}$ is a non-decreasing sequence of positive integers such that $\lim_{n\to\infty} a_n^{\frac{1}{2^n}} = \infty$ then the number $\sum_{n=1}^{\infty}\frac{1}{a_n}$ is irrational. Later, in 1991, Hančl [6] proved that if $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive real numbers such that $a_n \le 2^{\frac{1}{n^2}2^n}$ holds for any positive integer n, then there exists a sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers such that the number $\sum_{n=1}^{\infty}\frac{1}{c_n a_n}$ is rational. Rucki [20] established a criterion for the sums of reciprocals of a sequence of natural numbers to be irrational. Genčev [5] obtained some irrationality results with the help of special transformations. Then Hančl and Sobková [9] established the linear independence of the sums of certain infinite series. Using Padé approximation Matala-aho and Zudilin [17] obtained some interesting results in irrationality of infinite series. Recently, Hančl and Kolouch [11] gave a criterion for infinite products of rational numbers to be irrational. A nice review of these results can be found in [14] and [15].

Our results are of a quite general character and written in the spirit of Erdös. This method was later develop by Hančl and Sobková [10], see also [12]. We do not for instance require that the elements of $\{a_n\}_{n=1}^{\infty}$ be approximable by the elements of a finite union of power sequences or be associated with any differential equation as in the method of K. Mahler, for which the reader is referred to K. Nishioka's book [19]. Let us mention also [18].

The main result of this paper is Theorem 2. Many consequences and examples of this theorem can be found in the chapter Main Results. Lemma 14 deals with the conditions which guarantee that the ratio of two linear recurrences tends to infinity.

2 Notations

We use the standard notation: $\mathbb{N}$ and $\mathbb{Z}$ the set of non-negative integers and integers, respectively. For a positive real number x the expression $\log x$, $\log_2 x$ and $\pi(x)$ denotes the natural logarithm of x, the logarithm base 2 of x and number of primes less than or equal to x, respectively. If x is positive integer then $d(x)$ denotes the number of divisors of x.

Let $\frac{p_n}{q_n} = [a_0; a_1, a_2, \cdots, a_n]$ be the n-th partial fraction of the real number

$a = [a_0; a_1, a_2, \cdots]$. We have

$$p_0 = a_0, \quad q_0 = 1, \quad p_1 = a_1 a_0 + 1, \quad q_1 = a_1, \quad p_{n+2} = a_{n+2} p_{n+1} + p_n,$$

$$q_{n+2} = a_{n+2} q_{n+1} + q_n, \quad q_{n+1} p_n - p_{n+1} q_n = (-1)^{n+1},$$

$$\begin{aligned} a = [a_0; a_1, a_2, \cdots] &= \left[a_0; a_1, a_2, \cdots, a_n, [a_{n+1}; a_{n+2}, a_{n+3}, \cdots]\right] \\ &= \frac{p_n[a_{n+1}; a_{n+2}, a_{n+3}, \cdots] + p_{n-1}}{q_n[a_{n+1}; a_{n+2}, a_{n+3}, \cdots] + q_{n-1}}, \end{aligned}$$

$$\begin{aligned} a - \frac{p_n}{q_n} &= \frac{(-1)^n}{q_n^2([a_{n+1}; a_{n+2}, \dots] + [0; a_n, \dots, a_1])} \\ &= \frac{(-1)^n}{q_n^2([a_{n+1}; a_{n+2}, \dots] + \frac{q_{n-1}}{q_n})}, \end{aligned} \tag{1}$$

$$\frac{p_n}{q_n} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{q_k q_{k-1}}, \tag{2}$$

and

$$q_n = a_n q_{n-1} + q_{n-2} > a_n q_{n-1} > \cdots > \prod_{k=1}^{n} a_k, \ a_k > 0, \ k = 1, \dots, n \tag{3}$$

for all $n \in \mathbb{N}$. If $a = [a_0; a_1, a_2, \cdots, a_k]$ is finite and $k \geq 1$, then we suppose that $a_k \neq 1$. All of this can be found in the book of Schmidt [21] pages 7-10. If x is algebraic number and $x_1 = x, x_2, \dots, x_k$ all are its different conjugates, then the house of x is the maximal modulus among the conjugates of x i.e. $\overline{|x|} = \max_{1 \leq j \leq k} |x_j|$.

3 Main Results

Theorem 2. *Let D and M be positive integers, and let $\gamma \in (0, 1)$. Let $\mathbb{K}$ be an algebraic field such that $\deg \mathbb{K} = D$. For every $j = 1, \dots, M$ let $\{S_{n,j}\}_{n=1}^{\infty}$, $\{a_{n,j}\}_{n=1}^{\infty}$, $\{b_{n,j}\}_{n=1}^{\infty}$, $\{c_{n,j}\}_{n=1}^{\infty}$, $\{d_{n,j}\}_{n=1}^{\infty}$ be the sequences and*

$\alpha_j = [a_{0,j}; a_{1,j}, \ldots]$ be continued fractions such that for all $n \in \mathbb{N}$, we have $S_{n,j} \in \mathbb{Z}$, $b_{n,j}, c_{n,j}, d_{n,j} \in \mathbb{K}$ are algebraic integers,

$$a_{n,j} = \frac{S_{n,j} b_{n,j} + c_{n,j}}{d_{n,j}} \geq 1, \tag{4}$$

$$\overline{\left| \frac{1}{a_{n,j}} \right|} \geq \frac{1}{2}, \tag{5}$$

$$a_{n,1} < \max(a_{n,M} 2^{(\log_2 a_{n,M})^\gamma}, 2^{d^{n\gamma}}), \tag{6}$$

and

$$\overline{|b_{n,j}|}, \overline{|c_{n,j}|}, \overline{|d_{n,j}|} \leq \max(2^{(\log_2 a_{n,j})^\gamma}, 2^{d^{n\gamma}}). \tag{7}$$

Assume that

$$\liminf_{n\to\infty} \sqrt{n} \Big(\frac{a_{n,j}}{a_{n,j+1}} - 1 \Big) > 0 \tag{8}$$

for all $j = 1, \ldots, M-1$. Set $d = \max(2, DM - 1)$. Suppose that

$$\limsup_{n\to\infty} a_{n,j}^{\frac{1}{d^n}} = \infty. \tag{9}$$

Then the numbers $\alpha_1, \ldots, \alpha_M$ and 1 are linearly independent over $\mathbb{K}$.

This result is also true if instead of (5),

$$(-1)^{e_{j,\sigma}} \Re(\sigma a_{n,j}) \geq \max(2^{(\log_2 a_{n,1})^\gamma}, 2^{d^{n\gamma}})^{-1} \tag{10}$$

holds for all $j = 1, \ldots, M$ and all embeddings σ of $\mathbb{K}$ into $\overline{\mathbb{Q}}$, where $e_{j,\sigma} \in \{0, 1\}$ does not depend on n.

Theorem 3. *Let $K \geq 2$ be an integer and let $\{a_n\}_{n=1}^\infty$ be a sequence of integers greater or equal to K and such that $\limsup_{n\to\infty} a_n^{\frac{1}{(K2^{\pi(K)})^n}} = \infty$. Then the continued fractions $[0; \sqrt{j}a_1, \sqrt{j}a_2, \ldots]$, $j = 1, 2, \ldots, K$ and the number 1 are linearly independent over $\mathbb{Q}(\sqrt{1}, \sqrt{2}, \ldots, \sqrt{K})$ particularly over $\mathbb{Q}$.*

Corollary 4. *Let K be a positive integer and let $\{a_n\}_{n=1}^\infty$ be a sequence of positive integers greater or equal to K and such that $\limsup_{n\to\infty} a_n^{\frac{1}{d^n}} = \infty$ where $d = \max(2, K-1)$. Then the continued fractions $[0; \frac{a_1}{j}, \frac{a_2}{j}, \ldots]$, $j = 1, 2, \ldots, K$ and the number 1 are linearly independent over $\mathbb{Q}$.*

Example 5. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of positive integers greater or equal to 3 and such that* $\limsup_{n\to\infty} a_n^{\frac{1}{2^n}} = \infty$. *Then the continued fractions* $[0; a_1, a_2, \ldots]$, $[0; \frac{a_1}{2}, \frac{a_3}{2}, \ldots]$, $[0; \frac{a_3}{3}, \frac{a_3}{3}, \ldots]$ *and number* 1 *are linearly independent over* $\mathbb{Q}$.

Corollary 6. *Let* $K \geq 2$ *be an integer and* $P(x)$ *be a polynomial with integer coefficients and* $\deg P = K$. *Assume that all roots* $\alpha_1, \alpha_2, \ldots, \alpha_K$ *of* $P(x)$ *are real, different and greater than 1. Let* $\{a_n\}_{n=1}^{\infty}$ *be the sequence of positive integers such that* $\limsup_{n\to\infty} a_n^{\frac{1}{(K^2-1)^n}} = \infty$. *Then the continued fractions* $[0; \alpha_j a_1, \alpha_j a_2, \ldots], j = 1, \ldots, K$ *and the number* 1 *are linearly independent over* $\mathbb{Q}(\alpha_1, \ldots, \alpha_K)$ *particularly over* $\mathbb{Q}$.

Example 7. *Let* $\{a_n\}_{n=1}^{\infty}$ *be the sequence of positive integeres such that* $\limsup_{n\to\infty} a_n^{\frac{1}{3^n}} = \infty$. *Then the continued fractions*

$$\left[0; \left(4+\sqrt{2}\right)a_1, \left(4+\sqrt{2}\right)a_2, \ldots\right], \ \left[0; \left(4-\sqrt{2}\right)a_1, \left(4-\sqrt{2}\right)a_2, \ldots\right]$$

and the number 1 *are linearly independent over* $\mathbb{Q}(\sqrt{2})$ *particularly they are linearly independent over* $\mathbb{Q}$.

Example 8. *Let* K *be a positive integer, let* $\varphi = (\sqrt{5}+1)/2$ *be the golden ratio, and let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of positive integers such that* $\limsup_{n\to\infty} a_n^{\frac{1}{d^n}} = \infty$ *where* $d = \max\{2, 2K-1\}$. *Then the continued fractions*

$$\left[0; \varphi a_1, \varphi^3 a_2, \varphi^5 a_3, \ldots\right], \ \left[0; \varphi^2 a_1, \varphi^4 a_2, \varphi^6 a_3 \ldots\right], \ \ldots,$$
$$\left[0; \varphi^K a_1, \varphi^{K+2} a_2, \varphi^{K+4} a_3, \ldots\right],$$

and the number 1 *are linearly independent over* $\mathbb{Q}(\alpha)$.

Corollary 9. *Let* K *be an integer and let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of positive integers such that* $\limsup_{n\to\infty} a_n^{\frac{1}{d^n}} = \infty$ *where* $d = \max(2, K-1)$. *Then the continued fractions* $[0; a_1(j+\sum_{k=1}^{1}\frac{1}{k}), \ldots, a_n(j+\sum_{k=1}^{n}\frac{1}{k}), \ldots]$, $j = 1, \ldots, K$ *and number* 1 *are linearly independent over* $\mathbb{Q}$.

Corollary 10. *Let K be an integer and $\{a_n\}_{n=0}^{\infty}$ be a sequence of positive integers such that* $\limsup_{n\to\infty} a_n^{\frac{1}{d^n}} = \infty$ *where* $d = \max(2, K-1)$. *Then the continued fractions* $[0; a_1(1+\frac{\pi(1)}{1})^j, \dots, a_n(1+\frac{\pi(n)}{n})^j, \dots]$, $j = 0, 1, \dots, K-1$ *and number* 1 *are linearly independent over* $\mathbb{Q}$.

Example 11. *Let $\{a_n\}_{n=1}^{\infty}$ be the sequence of positive integer such that* $\limsup_{n\to\infty} a_n^{\frac{1}{3^n}} = \infty$. *Then the continued fractions*

$$\big[0; a_1 d(1)\sqrt{2}, \dots, a_n d(n)\sqrt{2}, \dots\big],\ \big[0; a_1(1+d(1))\sqrt{2}, \dots, a_n(1+d(n))\sqrt{2}, \dots\big],$$

and the number 1 *are linearly independent over* $\mathbb{Q}(\sqrt{2})$ *particularly over* $\mathbb{Q}$.

Question 12. *Does every sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers such that* $\limsup_{n\to\infty} a_n^{\frac{1}{2^n}} = \infty$, *then the continued fractions* $[0; a_1+\sqrt{2}, a_2+\sqrt{2}, \dots]$ *,and* $[0; a_1+\sqrt{3}, a_2+\sqrt{3}, \dots]$ *are linearly independent over* $\mathbb{Q}$*?*

Question 13. *Check if there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers such that the continued fractions* $[0; a_1, a_2, \dots]$ *and* $[0; a_1+1, a_2+2, \dots]$ *are linearly dependent over* $\mathbb{Q}$*?*

Lemma 14. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of real numbers greater or equal to 1 and such that*

$$\liminf_{n\to\infty} \sqrt{n}\Big(\frac{a_n}{b_n} - 1\Big) > 0. \tag{11}$$

Let $q_{n,a}$ and $q_{n,b}$ be denominator of n-th partial of the continued fraction $a = [0; a_1, a_2, \dots]$ *and* $b = [0; b_1, b_2, \dots]$, *respectively. Then*

$$\lim_{n\to\infty} \frac{q_{n,a}}{q_{n,b}} = \infty. \tag{12}$$

Question 15. *Is that possible to substitute condition (11) by the condition* $\limsup_{n\to\infty} n(\frac{a_n}{b_n} - 1) > 0$*?*

Remark 16. *We cannot substitute condition (11) by the weaker condition* $\limsup_{n\to\infty} n^2(\frac{a_n}{b_n} - 1) > 0$. *For example set* $a_n = n^2 + 1$ *and* $b_n = n^2$ *for all* $n \in \mathbb{N}$. *Then*

$$\frac{q_{n,a}}{q_{n,b}} = \frac{(n^2+1)q_{n-1,a} + q_{n-2,a}}{n^2 q_{n-1,b} + q_{n-2,b}} = \frac{n^2+1}{n^2}\frac{q_{n-1,a}}{q_{n-1,b}}\left(\frac{1 + \frac{q_{n-2,a}}{(n^2+1)q_{n-2,a}}}{1 + \frac{q_{n-2,b}}{n^2 q_{n-2,b}}}\right)$$
$$< \left(1 + \frac{1}{n^2}\right)^2 \frac{q_{n-1,a}}{q_{n-2,b}} < \cdots < \prod_{n=1}^{\infty}\left(1 + \frac{1}{n^2}\right)^2 = const < \infty.$$

Lemma 17. *Let* $z_0, z_1, z_2, \ldots, z_n$ *be complex numbers such that for every* $k = 0, 1, 2, \ldots, n$ *we have*

$$|z_k| \geq 2. \tag{13}$$

Then

$$\big|[z_0; z_1, z_2, \ldots, z_n]\big| \geq 1. \tag{14}$$

Lemma 18. *Let* $z_0, z_1, z_2, \ldots, z_n$ *be complex numbers with* $\Re(z_k) > 0$ *for every* $k = 0, 1, 2, \ldots, n$. *Then*

$$|\Re\big([-z_0; -z_1, -z_2, \ldots, -z_n]\big)| = \Re\big([z_0; z_1, z_2, \ldots, z_n]\big) \geq \Re(z_0).$$

4 Proofs

Proof of Lemma 17 . Lemma 17 can be proved by mathematical induction using the inequality $|a + b| \geq ||a| - |b||$, which holds for all complex numbers a and b. □

Proof of Lemma 18 . Lemma 18 can be proved by mathematical induction using that $\Re(a+b^{-1}) = \Re(a) + \Re(b)/|b|^2$, which holds for all non-zero complex numbers a and b. □

Proof of Theorem 2. Let $\sigma_1, \ldots, \sigma_D$ be the set of embeddings of $\mathbb{K}$ into $\overline{\mathbb{Q}}$, where σ_1 is the identity. Then the number $\prod_{i=1}^{D} \sigma_i(x)$ is a rational number for every $x \in \mathbb{K}$.

If $b_{n,j} \neq 0$, then we have that $|\prod_{i=1}^{D} \sigma_i(b_{n,j})| = A$ where A is a positive integer. From this and (7) we obtain that for every $i = 1, \ldots, D$

$$|\sigma_i(b_{n,j})| = \frac{A}{|\prod_{I=1, I\neq i}^{D} \sigma_I(b_{n,j})|} \geq \frac{1}{\max(2^{(D-1)(\log_2 a_{n,j})^\gamma}, 2^{(D-1)d^{n\gamma}})} \qquad (15)$$

and similarly

$$|\sigma_i(d_{n,j})| \geq \frac{1}{\max(2^{(D-1)(\log_2 a_{n,j})^\gamma}, 2^{(D-1)d^{n\gamma}})}. \qquad (16)$$

Inequalities (4), (7) and (15) yield

$$|S_{n,j}| = \left|\frac{d_{n,j}a_{n,j} - c_{n,j}}{b_{n,j}}\right| \leq 2a_{n,j} \max(2^{(D \log_2 a_{n,j})^\gamma}, 2^{Dd^{n\gamma}}). \qquad (17)$$

Suppose that the numbers $\alpha_1, \alpha_2, \ldots, \alpha_M$ and 1 are linearly dependent over $\mathbb{K}$. Then there exist $A_1, A_2, \ldots, A_M \in \mathbb{K}$, not all equal to zero such that $\sum_{j=1}^{M} A_j\alpha_j \in \mathbb{K}$. Let us write $\sum_{j=1}^{M} A_j\alpha_j = y$ where $y \in \mathbb{K}$. Therefore $\sum_{j=1}^{M} A_j\alpha_j - y = 0$. Let n_0 and n be sufficiently large and such that $n \geq n_0$. Then we have

$$\sum_{j=1}^{M} A_j\alpha_j - y = \sum_{j=1}^{M} A_j\Big(\alpha_j - \frac{p_{n,j}}{q_{n,j}}\Big) + \sum_{j=1}^{M} A_j\frac{p_{n,j}}{q_{n,j}} - y = 0.$$

Hence

$$\sum_{j=1}^{M} A_j\frac{p_{n,j}}{q_{n,j}} - y = -\sum_{j=1}^{M} A_j\Big(\alpha_j - \frac{p_{n,j}}{q_{n,j}}\Big). \qquad (18)$$

Now we prove that for all large n we have

$$\left|\sum_{j=1}^{M} A_j\frac{p_{n,j}}{q_{n,j}} - y\right| > 0. \qquad (19)$$

Without loss of generality let M^* be an integer such that $A_M = \cdots = A_{M^*+1} = 0$ and $A_{M^*} \neq 0$. From this and (1), we obtain that

$$\sum_{j=1}^{M} A_j(\alpha_j - \frac{p_{n,j}}{q_{n,j}}) = \sum_{j=1}^{M^*} A_j(\alpha_j - \frac{p_{n,j}}{q_{n,j}})$$
$$= \sum_{j=1}^{M^*} \frac{A_j(-1)^n}{q_{n,j}^2(a_{n+1,j} + [0; a_{n+2,j}, a_{n+3,j}, \ldots] + [0; a_{n,j}, \ldots, a_{1,j}])}.$$

This and Lemma 14 yield

$$\begin{aligned}
&\Big|\sum_{j=1}^{M} A_j\Big(\alpha_j - \frac{p_{n,j}}{q_{n,j}}\Big)\Big| \\
&\geq \frac{|A_{M^*}|}{q_{n,M^*}^2(a_{n+1,M^*}+[0;a_{n+2,j},a_{n+3,j},\dots]+[0;a_{n,j},\dots,a_{1,j}])} - \sum_{j=1}^{M^*-1}\frac{|A_j|}{q_{n,j}^2 a_{n+1,j}} \\
&\geq \frac{|A_{M^*}|}{3q_{n,M^*}^2 a_{n+1,M^*}} - \sum_{j=1}^{M^*-1}\frac{|A_j|}{q_{n,j}^2 a_{n+1,j}} \\
&= \frac{|A_{M^*}|}{3q_{n,M^*}^2 a_{n+1,M^*}}\Bigg(1 - \sum_{j=1}^{M^*-1}\frac{\frac{3|A_j|}{|A_{M^*}|}}{(\frac{q_{n,j}}{q_{n,M^*}})^2\frac{a_{n+1,j}}{a_{n+1,M^*}}}\Bigg) > 0.
\end{aligned}$$

This and (18) yield (19).

Using (1) yields that

$$\begin{aligned}
\Big|\sum_{j=1}^{M} A_j(\alpha_j - \frac{p_{n,j}}{q_{n,j}})\Big| &= \Big|\sum_{j=1}^{M^*} A_j(\alpha_j - \frac{p_{n,j}}{q_{n,j}})\Big| \\
&= \Big|\sum_{j=1}^{M^*}\frac{A_j(-1)^n}{q_{n,j}^2(a_{n+1,j}+[0;a_{n+2,j},a_{n+3,j},\dots]+[0;a_{n,j},\dots,a_{1,j}])}\Big| \\
&\leq \sum_{j=1}^{M^*}\frac{\max_{j=1,2,\dots,M^*}(|A_j|)}{q_{n,j}^2 a_{n+1,j}}. \qquad (20)
\end{aligned}$$

From this, Lemma 14 and (8) we obtain that

$$\begin{aligned}
\Big|\sum_{j=1}^{M} A_j(\alpha_j - \frac{p_{n,j}}{q_{n,j}^2})\Big| &\leq \max_{j=1,,\dots,M}(|A_j|)\sum_{j=1}^{M}\frac{1}{q_{n,j}^2 a_{n+1,j}} \\
&= \frac{\max_{j=1,2,\dots,M}(|A_j|)}{q_{n,M}^2 a_{n+1,M}}\Bigg(1 + \sum_{j=1}^{M-1}(\frac{q_{n,M}}{q_{n,j}})^2\frac{a_{n+1,M}}{a_{n+1,j}}\Bigg) \\
&\leq \frac{M\max_{j=1,2,\dots,M}|A_j|}{q_{n,M}^2 a_{n+1,M}} = \frac{c}{q_{n,M}^2 a_{n+1,M}}
\end{aligned}$$

where $c = M\max_{j=1,2,\dots,M}|A_j|$ is a constant which does not depend on n. This, (18), and (19) yield

$$0 < \Big|\sum_{j=1}^{M} A_j\frac{p_{n,j}}{q_{n,j}} - y\Big| < \frac{c}{q_{n,M}^2 a_{n+1,M}}.$$

It implies that

$$\prod_{i=1}^{D}\left|\sigma_i\Big(\sum_{j=1}^{M}A_j\frac{p_{n,j}}{q_{n,j}}-y\Big)\right|\le\prod_{i=2}^{D}\left|\sigma_i\Big(\sum_{j=1}^{M}A_j\frac{p_{n,j}}{q_{n,j}}-y\Big)\frac{c}{q_{n,M}^2a_{n+1,M}}\right|.$$

Hence,

$$\frac{c\,\big|\prod_{i=2}^{D}\sum_{j=1}^{M}(\sigma_i(A_j\frac{p_{n,j}}{q_{n,j}})-\sigma_i(y))\big|}{q_{n,M}^2a_{n+1,M}}\ge\left|\prod_{i=1}^{D}\sum_{j=1}^{M}\Big(\sigma_i\Big(A_j\frac{p_{n,j}}{q_{n,j}}\Big)-\sigma_i(y)\Big)\right|. \quad (21)$$

From (19) and by Galois theory the number on the right is a rational number $\frac{a}{b}, (a,b)=1, a,b\in\mathbb{Z}^+$ such that there exist a constant C which does not depend on n and such that

$$\begin{aligned}0<b&\le C\prod_{i=1}^{D}\prod_{j=1}^{M}\left|\sigma_i\Big(\Big(\prod_{k=1}^{n}d_{k,j}\Big)q_{n,j}\Big)\right|\\&=C\prod_{i=1}^{D}\prod_{j=1}^{M}|\sigma_i(q_{n,j})|\prod_{k=1}^{n}|\sigma_i(d_{k,j})|.\end{aligned}$$

This and (7) yield

$$0<b\le C\prod_{i=1}^{D}\prod_{j=1}^{M}|\sigma_i(q_{n,j})|\prod_{k=1}^{n}\max(2^{(\log_2 a_{k,j})^\gamma},2^{d^{k\gamma}}). \quad (22)$$

Write $R_{k,j}=\max(2^{(\log_2 a_{k,j})^\gamma},2^{d^{k\gamma}})$ and notice that $\overline{|1/a_{k,j}|}\le R_{k,j}$ regardless of whether we assume (5) or (10). Combining this with inequality (4), we find

$$\begin{aligned}|\sigma_i(q_{n,j})|&=|\sigma_i(a_{n,j}q_{n-1,j}+q_{n-2,j})|\\&\le|\sigma_i(a_{n,j})||\sigma_i(q_{n-1,j})|+|\sigma_i(q_{n-2,j})|\\&\le|\sigma_i(q_{n-2,j})|\big(|\sigma_i(a_{n,j})||\sigma_i(a_{n-1,j})|+1\big)+|\sigma_i(a_{n,j})||\sigma_i(q_{n-3,j})|\\&<(1+R_{n,j}R_{n-1,j})|\sigma_i(a_{n,j})|\big(|\sigma_i(a_{n-1,j})||\sigma_i(q_{n-2,j})|+|\sigma_i(q_{n-3,j})|\big)\\&<\cdots<\prod_{k=1}^{n}(1+R_{k,j}R_{k-1,j})|\sigma_i(a_{k,j})|<2^nR_{n,j}\prod_{k=1}^{n-1}R_{k,j}^2|\sigma_i(a_{k,j})|\\&<\frac{\prod_{k=1}^{n}R_{k,j}^2|\sigma_i(a_{k,j})|}{C^{1/DM}}=\frac{\prod_{k=1}^{n}\max(2^{(\log_2 a_{k,j})^\gamma},2^{d^{k\gamma}})|\sigma_i(a_{k,j})|}{C^{1/DM}}.\end{aligned}$$

This and (22) yield

$$0 < b \le C \prod_{i=1}^{D} \prod_{j=1}^{M} |\sigma_i(q_{n,j})| \prod_{k=1}^{n} \max(2^{(\log_2 a_{k,j})^\gamma}, 2^{d^{k\gamma}})$$
$$< \prod_{i=1}^{D} \prod_{j=1}^{M} \prod_{k=1}^{n} |\sigma_i(a_{k,j})| \max(2^{(\log_2 a_{k,j})^\gamma}, 2^{d^{k\gamma}})^3.$$

From this, (4) and (16) we obtain that

$$0 < b \le \prod_{i=1}^{D} \prod_{j=1}^{M} \prod_{k=1}^{n} \frac{|\sigma_i(S_{k,j} b_{k,j} + c_{k,j})|}{|\sigma_i(d_{k,j})|} \max(2^{(\log_2 a_{k,j})^\gamma}, 2^{d^{k\gamma}})^3$$
$$\le \prod_{i=1}^{D} \prod_{j=1}^{M} \prod_{k=1}^{n} (|S_{k,j}||\sigma_i(b_{k,j})| + |\sigma_i(c_{k,j})|) \max(2^{(\log_2 a_{k,j})^\gamma}, 2^{d^{k\gamma}})^{D+2}.$$

This and (7) imply that

$$0 < b < \prod_{i=1}^{D} \prod_{j=1}^{M} \prod_{k=1}^{n} (|S_{k,j}| + 1) \max(2^{(\log_2 a_{k,j})^\gamma}, 2^{d^{k\gamma}})^{D+3}.$$

This and (17) yield

$$0 < b < \prod_{i=1}^{D} \prod_{j=1}^{M} \prod_{k=1}^{n} \Big(2a_{k,j} \max(2^{(D \log_2 a_{k,j})^\gamma}, 2^{Dd^{k\gamma}}) + 1\Big) \times$$
$$\max(2^{(\log_2 a_{k,j})^\gamma}, 2^{d^{k\gamma}})^{D+3}$$
$$\le \prod_{i=1}^{D} \prod_{j=1}^{M} \prod_{k=1}^{n} \Big(a_{k,j} \max(2^{(\log_2 a_{k,j})^\gamma}, 2^{d^{k\gamma}})^{5D}\Big)$$

This and (8) imply that

$$0 < b < \prod_{i=1}^{D} \prod_{j=1}^{M} \prod_{k=1}^{n} \Big(a_{k,1} \max(2^{(\log_2 a_{k,1})^\gamma}, 2^{d^{k\gamma}})^{5D}\Big)$$
$$= \prod_{k=1}^{n} \Big(a_{k,1}^{DM} \max(2^{(\log_2 a_{k,1})^\gamma}, 2^{d^{k\gamma}})^{5D^2 M}\Big) \qquad (23)$$

We now calculate

$$
\begin{aligned}
|\sigma_i(q_{n,j})| =& |\sigma_i(a_{n,j})\sigma_i(q_{n-1,j}) + \sigma_i(q_{n-2,j})| \\
=& |\sigma_i(q_{n-1,j})| \left| \sigma_i(a_{n,j}) + \frac{\sigma_i(q_{n-2,j})}{\sigma_i(q_{n-1,j})} \right| \\
=& |\sigma_i(q_{n-1,j})| |\sigma_i(a_{n,j}) + [0; \sigma_i(a_{n-1,j}), \sigma_i(a_{n-2,j}), \ldots, \sigma_i(a_{1,j})]| \\
=& |\sigma_i(q_{n-1,j})| |[\sigma_i(a_{n,j}); \sigma_i(a_{n-1,j}), \sigma_i(a_{n-2,j}), \ldots, \sigma_i(a_{1,j})]|. \qquad (24)
\end{aligned}
$$

If (5) is satisfied, we then apply Lemma 17 to (24) and find

$$
|\sigma_i(q_{n,j})| \geq |\sigma_i(q_{n-1,j})| \geq |\sigma_i(q_{n-2,j})| \geq \cdots \geq |\sigma_i(q_{0,j})| = 1,
$$

while (10) would allow us to apply Lemma 18 to (24) and obtain

$$
\begin{aligned}
|\sigma_i(q_{n,j})| &\geq |\sigma_i(q_{n-1,j})| |\Re([\sigma_i(a_{n,j}); \sigma_i(a_{n-1,j}), \sigma_i(a_{n-2,j}), \ldots, \sigma_i(a_{1,j})])| \\
&\geq |\sigma_i(q_{n-1,j})| |\Re\sigma_i(a_{n,j})| \geq \frac{|\sigma_i(q_{n-1,j})|}{\max(2^{(\log_2 a_{k,1})^\gamma}, 2^{d^{k\gamma}})} \\
&\geq \cdots \geq \prod_{k=1}^{n} \max(2^{(\log_2 a_{k,j})^\gamma}, 2^{d^{k\gamma}})^{-1}
\end{aligned}
$$

Whether (5) or (10) is true, this and (2) then yield

$$
\begin{aligned}
&\left| \prod_{i=2}^{D} \left(\sum_{j=1}^{M} (\sigma_i(A_j \frac{p_{n,j}}{q_{n,j}}) - \sigma_i(y)) \right) \right| \\
&= \left| \prod_{i=2}^{D} \left(\sum_{j=1}^{M} (\sigma_i(A_j \sum_{k=1}^{n} \frac{(-1)^{k+1}}{q_{k,j} q_{k-1,j}} - \sigma_i(y))) \right) \right| \\
&\leq \prod_{i=2}^{D} \left(\sum_{j=1}^{M} |\sigma_i(A_j)| \sum_{k=1}^{n} \frac{1}{|\sigma_i(q_{k,j})| |\sigma_i(q_{k-1,j})|} \right) \\
&\leq \prod_{i=2}^{D} \left(\sum_{j=1}^{M} |\sigma_i(A_j)| \sum_{k=1}^{n} \max(2^{(\log_2 a_{k,j})^\gamma}, 2^{d^{k\gamma}}) \prod_{l=1}^{k-1} \max(2^{(\log_2 a_{l,j})^\gamma}, 2^{d^{l\gamma}})^2 \right) \\
&\leq \prod_{k=1}^{n} \max(2^{(\log_2 a_{k,1})^\gamma}, 2^{d^{k\gamma}})^{2D}, \qquad (25)
\end{aligned}
$$

for all sufficiently large n.

From (21),(23), and (25), we obtain that for all sufficiently large n,

$$
\begin{aligned}
q_{n,M}^2 a_{n+1,M} &< \frac{1}{c}\prod_{k=1}^{n}\left(a_{n,1}^{DM}\max(2^{(\log_2 a_{k,1})^\gamma}, 2^{d^{k\gamma}})^{7D^2M}\right)\\
&\le \left(\prod_{k=1}^{n} a_{n,1}^{DM}\right)\left(\prod_{k=1}^{n} 2^{7D^2M(\log_2 a_{k,1})^\gamma}\right)\left(\prod_{k=1}^{n} 2^{7D^2Md^{k\gamma}}\right)\\
&= 2^{7D^2M\frac{d^{\gamma(n+1)}}{d^\gamma-1}}\left(\prod_{k=1}^{n} a_{n,1}^{DM}\right)\left(\prod_{k=1}^{n} 2^{7D^2M(\log_2 a_{k,1})^\gamma}\right)\\
&\le 2^{7D^2M\frac{d^{\gamma(n+1)}}{d^\gamma-1}}\left(\prod_{k=1}^{n} a_{n,1}^{DM}\right)\left(\prod_{k=1}^{n} 2^{7D^2M(\log_2 a_{k,1})^\gamma}\right)
\end{aligned}
$$

From this and (6) we obtain that

$$
\begin{aligned}
q_{n,M}^2 a_{n+1,M} < 2^{7D^2M\frac{d^{\gamma(n+1)}}{d^\gamma-1}}&\left(\prod_{k=1}^{n}\max(a_{k,M}2^{(\log_2 a_{k,M})^\gamma}, 2^{d^{k\gamma}})^{DM}\right)\times\\
&\left(\prod_{k=1}^{n} 2^{7D^2M(\log_2 \max(a_{k,M}2^{(\log_2 a_{k,M})^\gamma}, 2^{d^{k\gamma}}))^\gamma}\right).
\end{aligned}
$$

Set $b_k = \max\{a_{k,M}, 2^{d^{k\gamma}}\}$. From this and (23) we obtain for sufficiently large n that

$$
\begin{aligned}
q_{n,M}^2 a_{n+1,M} &< 2^{7D^2M\frac{d^{\gamma(n+1)}}{d^\gamma-1}}\left(\prod_{k=1}^{n} b_k 2^{(\log_2 b_k)^\gamma}\right)^{DM}\left(\prod_{k=1}^{n} 2^{8D^2M(\log_2 b_k)^\gamma}\right)\\
&\le 2^{7D^2M\frac{d^{\gamma(n+1)}}{d^\gamma-1}}\left(\prod_{k=1}^{n} b_k\right)^{DM}\left(\prod_{k=1}^{n} 2^{8D^2M(\log_2 b_k)^\gamma}\right). \qquad (26)
\end{aligned}
$$

Inequality (3) implies that

$$
q_{n,M} > \prod_{k=1}^{n} a_{k,M} \ge \prod_{k=1}^{n} b_k \prod_{k=1}^{n} 2^{-d^{k\gamma}} \ge 2^{-\frac{d^{\gamma(n+1)}}{d^\gamma-1}}\prod_{k=1}^{n} b_k.
$$

This and inequality (26) imply

$$
a_{n+1,M} < 2^{9D^2M\frac{d^{\gamma(n+1)}}{d^\gamma-1}}\left(\prod_{k=1}^{n} b_k\right)^{d-1}\left(\prod_{k=1}^{n} 2^{8D^2M(\log_2 b_k)^\gamma}\right) \qquad (27)
$$

Now the proof falls into two cases:

Case 1

Assume that there is $\delta > 0$ such that

$$\limsup_{n\to\infty} b_n^{\frac{1}{(d+\delta)^n}} = \infty. \tag{28}$$

From this and Borel's theorem we obtain that there exist infinitely many N such that

$$b_{N+1}^{\frac{1}{(d+\delta)^{N+1}}} > \left(1+\frac{1}{N^2}\right) \max_{k=1,\dots,N} b_k^{\frac{1}{(d+\delta)^k}}.$$

Therefore

$$b_{N+1} > \left(1+\frac{1}{N^2}\right)^{(d+\delta)^{N+1}} \left(\max_{k=1,\dots,N} b_k^{\frac{1}{(d+\delta)^k}}\right)^{(d+\delta)^{N+1}} > 2^{d^{N+1}} \tag{29}$$

and

$$\begin{aligned} b_{N+1} &> \left(1+\frac{1}{N^2}\right)^{(d+\delta)^{N+1}} \left(\max_{k=1,\dots,N} b_k^{\frac{1}{(d+\delta)^k}}\right)^{(d+\delta)^{N+1}} \\ &> \left(1+\frac{1}{N^2}\right)^{(d+\delta)^{N+1}} \left(\max_{k=1,\dots,N} b_k^{\frac{1}{(d+\delta)^k}}\right)^{(d+\delta-1)((d+\delta)^N+(d+\delta)^{N-1}+\cdots+1)} \\ &> 2^{\frac{1}{N^3}(d+\delta)^{N+1}} \left(\prod_{k=1}^{N} b_k\right)^{d+\delta-1}. \end{aligned} \tag{30}$$

hold for infinitely many N. From inequatity (29) and the fact that $b_{n+1} = \max(a_{N+1,M}, 2^{d\gamma(N+1)})$ we obtain that $b_{N+1} = a_{N+1,M}$. This and (30) imply that

$$a_{N+1,M} > \left(\prod_{k=1}^{N} b_k\right)^{d-1} \left(\prod_{k=1}^{N} b_k\right)^{\delta} 2^{\frac{1}{N^3}(d+\delta)^{N+1}}$$

a contradiction with (27).

Case 2

Suppose that there is no $\delta > 0$ such that (28) holds. Then for every $\delta > 0$, there exist n_1 such that for every $n > n_1$, we have

$$b_n < 2^{(d+\delta/2)^n}. \tag{31}$$

This implies that for all sufficiently large n,

$$\prod_{k=1}^{n} 2^{8D^2M(\log_2 b_k)^\gamma} \le \prod_{k=1}^{n} 2^{8D^2M(\log_2 2^{(d+\delta)^k})^\gamma} = \prod_{k=1}^{n} 2^{8D^2M(d+\delta)^{k\gamma}}$$
$$= 2^{8D^2M\frac{(d+\delta)^{\gamma(n+1)}-(d+\delta)^\gamma}{(d+\delta)^\gamma-1}} \le 2^{8D^2M\frac{(d+\delta)^{\gamma(n+1)}}{(d+\delta)^\gamma-1}}. \qquad (32)$$

Borel's theorem and (9) yield that for infinitely many N,

$$b_{N+1}^{\frac{1}{d^{N+1}}} > \left(1+\frac{1}{N^2}\right) \max_{k=1,2,\dots,N} b_k^{\frac{1}{d^k}}$$

holds. It implies that

$$b_{N+1} > \left(1+\frac{1}{N^2}\right)^{d^{N+1}} \left(\max_{k=1,2,\dots,N} b_k^{\frac{1}{d^k}}\right)^{d^{N+1}}$$
$$> \left(1+\frac{1}{N^2}\right)^{d^{N+1}} \left(\max_{k=1,2,\dots,N} b_k^{\frac{1}{d^k}}\right)^{(d-1)(d^N+d^{N-1}+\dots+1)}$$
$$> \left(1+\frac{1}{N^2}\right)^{d^{N+1}} \left(\prod_{k=1}^{N} b_k\right)^{d-1}.$$

Therefore

$$b_{N+1} > 2^{\frac{1}{N^3}d^{N+1}} \left(\prod_{k=1}^{N} b_k\right)^{d-1}.$$

From this and the fact that $b_{N+1} = \max(a_{N+1,M}, 2^{d^{(N+1)\gamma}})$ we obtain that $b_{N+1} = a_{N+1,M}$. Hence

$$a_{N+1,M} > 2^{\frac{1}{N^3}d^{N+1}} \left(\prod_{k=1}^{N} b_k\right)^{d-1}.$$

This and (32) yield

$$a_{N+1,M} > 2^{\frac{1}{N^3}d^{N+1}} \left(\prod_{k=1}^{N} b_k\right)^{d-1} \left(\prod_{k=1}^{N} 2^{7D^2M(\log_2 b_k)^\gamma}\right) \left(\prod_{k=1}^{N} 2^{7D^2M(\log_2 b_k)^\gamma}\right)^{-1}$$
$$\ge 2^{\frac{1}{N^3}d^{N+1}-8D^2M\frac{(d+\delta)^{\gamma(N+1)}}{(d+\delta)^\gamma-1}} \left(\prod_{k=1}^{N} b_k\right)^{d-1} \left(\prod_{k=1}^{N} 2^{8D^2M(\log_2 b_k)^\gamma}\right),$$

which contradicts (27) for a sufficiently small choice of δ. □

Proof of Theorem 3. We follow the proof of Theorem 2 and the exception will be only the lower estimation of partial denominaters for the continued fractions $\alpha_2 = [0; -\sqrt{2}a_1, -\sqrt{2}a_2, \ldots]$ and $\alpha_3 = [0; -\sqrt{3}a_1, -\sqrt{3}a_2, \ldots]$. For α_2 we have

$$\begin{aligned} q_{n+1,\alpha_2} &= -\sqrt{2}a_{n+1,\alpha_2}q_{n,\alpha_2} + q_{n-1,\alpha_2} \\ &= -\sqrt{2}a_{n+1,\alpha_2}q_{n,\alpha_2}\left(1 + \frac{q_{n-1,\alpha_2}}{-\sqrt{2}a_{n+1,\alpha_2}q_{n,\alpha_2}}\right) \\ &= -\sqrt{2}a_{n+1,\alpha_2}q_{n,\alpha_2}\left(1 + \frac{1}{-\sqrt{2}a_{n+1,\alpha_2}}[0; -\sqrt{2}a_n, -\sqrt{2}a_{n-1}, \ldots, -\sqrt{2}a_1]\right). \end{aligned}$$

Hence

$$\begin{aligned} |q_{n+1,\alpha_2}| =& \sqrt{2}a_{n+1,\alpha_2}|q_{n,\alpha_2}|\left(1 + \frac{1}{\sqrt{2}a_{n+1,\alpha_2}}[0; \sqrt{2}a_n, \sqrt{2}a_{n-1}, \ldots, \sqrt{2}a_1]\right) \\ \geq& |q_{n,\alpha_2}| \geq \cdots \geq |q_{1,\alpha_2}| = 1 \end{aligned}$$

and (25) follows. Similarly for α_3. □

Proof of Corollary 4. Corollary 4 is an immediate consequence of Theorem 2 when we set $D = 1$ and $M = K$. □

Proof of Example 5. Example 5 is the immediate consequence of Corollary 4 if we set $K = 3$. □

Proof of Corollary 6. Corollary 6 is the immediate consequence of Theorem 2 if we set $D = M = K$. □

Proof of Example 7. Example 7 is the immediate consequence of Corollary 6 when we set $K = 2$ and $P(x) = x^2 - 8x + 14$. □

Proof of Example 8. This is an immediate consequence of Theorem 2 if we set $D = \deg \alpha$, since $\varphi^{2n}a_{n+j}$ clearly satisfies (10). □

Proof of Corollary 9. This is the immediate consequence of Theorem 2 if we set $D = 1$, $K = M$ and the fact that $c = \lim_{n\to\infty}(-\log n + \sum_{j=1}^{n} \frac{1}{j})$ is the Euler–Mascheroni constant. $\square$

Proof of Corollary 10. This is the immediate consequence of Theorem 2 if we set $D = 1$, $K = M$ and the fact that $\lim_{n\to\infty} \frac{\pi(n)}{\frac{n}{\log n}} = 1$. $\square$

Proof of Example 11. This is the immediate consequence of Theorem 2 if we set $D = M = 2$ and the well-known facts that $\liminf_{n\to\infty} d(n) = 2$ and $\limsup_{n\to\infty} \frac{\log d(n) \log\log n}{\log n} = \log 2$. $\square$

Proof of Lemma 14. From (11) we obtain that there exists positive real number ε and positive integer n_0 such that for every positive integer $n \geq n_0 - 4$ we have

$$a_n \geq \left(1 + \frac{\varepsilon}{\sqrt{n}}\right) b_n, \tag{33}$$

$$\frac{\varepsilon^2 - \frac{\varepsilon}{\sqrt{n}+\sqrt{n+1}}}{\sqrt{n+1}(\sqrt{n}+\varepsilon)} - \frac{2}{(n+1)\ln(n+1)} > 0. \tag{34}$$

and the function

$$f(x) = \frac{\varepsilon^2 - \frac{\varepsilon}{\sqrt{x}+\sqrt{x+1}}}{\sqrt{x+1}(\sqrt{x}+\varepsilon)}$$

is decreasing for $x > n_0$.

Set $c = [a_0, a_1, a_2, \ldots, a_{n_0-5}, (1 + \frac{\varepsilon}{\sqrt{n_0-4}})b_{n_0-4}, (1 + \frac{\varepsilon}{\sqrt{n_0-3}})b_{n_0-3}, \ldots]$ and $q_{n,c}$ the denominator of its n-th partial. Then for all positive integers n we have

$$\frac{q_{n,a}}{q_{n,b}} \geq \frac{q_{n,c}}{q_{n,b}}. \tag{35}$$

To prove Lemma 14 we prove that for every large n we have

$$\frac{q_{n+1,c}}{q_{n+1,b}} \geq \left(1 + \frac{1}{(n+1)\ln(n+1)}\right) \frac{q_{n,c}}{q_{n,b}}. \tag{36}$$

Then this, (35) and the fact that $\prod_{j=n_0+1}^{\infty}(1+\frac{1}{j\ln j})=\infty$ imply that

$$\lim_{n\to\infty}\frac{q_{n,a}}{q_{n,b}} \geq \lim_{n\to\infty}\frac{q_{n,c}}{q_{n,b}} \geq \lim_{n\to\infty}\left(1+\frac{1}{n\ln n}\right)\frac{q_{n-1,c}}{q_{n-1,b}}$$
$$\geq \cdots \geq \lim_{n\to\infty}\left(\prod_{j=n_0+1}^{n+1}(1+\frac{1}{j\ln j})\right)\frac{q_{n_0,c}}{q_{n_0,b}}$$
$$= \frac{q_{n_0,c}}{q_{n_0,b}}\prod_{j=n_0+1}^{\infty}\left(1+\frac{1}{j\ln j}\right)=\infty$$

and (12) follows.

To prove (36) let us set $x_n = 1+\frac{\varepsilon}{\sqrt{n}}$ and $y_n = 1+\frac{1}{n\ln n}$. Then we have

$$\frac{q_{n+1,c}}{q_{n+1,b}} = \frac{x_{n+1}b_{n+1}q_{n,c}+q_{n-1,c}}{b_{n+1}q_{n,b}+q_{n-1,b}}$$
$$= \frac{(x_{n+1}-y_{n+1}+y_{n+1})b_{n+1}q_{n,c}+q_{n-1,c}}{b_{n+1}q_{n,b}+q_{n-1,b}}$$
$$= \frac{y_{n+1}b_{n+1}q_{n,c}\left(\frac{x_{n+1}-y_{n+1}}{y_{n+1}}+1+\frac{1}{b_{n+1}y_{n+1}\frac{q_{n,c}}{q_{n-1,c}}}\right)}{b_{n+1}q_{n,b}\left(1+\frac{1}{b_{n+1}\frac{q_{n,b}}{q_{n-1,b}}}\right)}.$$

This yields that it is enough to prove that

$$\frac{x_{n+1}-y_{n+1}}{y_{n+1}}+1+\frac{1}{b_{n+1}y_{n+1}\frac{q_{n,c}}{q_{n-1,c}}} \geq 1+\frac{1}{b_{n+1}\frac{q_{n,b}}{q_{n-1,b}}}$$

and this is equivalent to

$$A = \frac{x_{n+1}-y_{n+1}}{y_{n+1}}+\frac{1}{b_{n+1}}\left(\frac{1}{y_{n+1}(x_nb_n+\frac{q_{n-2,c}}{q_{n-1,c}})}-\frac{1}{b_n+\frac{q_{n-2,b}}{q_{n-1,b}}}\right)\geq 0. \quad (37)$$

Now we have

$$\frac{1}{y_{n+1}(x_nb_n+\frac{q_{n-2,c}}{q_{n-1,c}})}-\frac{1}{b_n+\frac{q_{n-2,b}}{q_{n-1,b}}} = \frac{b_n+\frac{q_{n-2,b}}{q_{n-1,b}}-y_{n+1}x_nb_n-\frac{q_{n-2,c}}{q_{n-1,c}}y_{n+1}}{(y_{n+1}x_nb_n+\frac{q_{n-2,c}}{q_{n-1,c}}y_{n+1})(b_n+\frac{q_{n-2,b}}{q_{n-1,b}})}$$

$$= \frac{1-y_{n+1}x_n+\frac{1}{b_n}(\frac{q_{n-2,b}}{q_{n-1,b}}-\frac{q_{n-2,c}}{q_{n-1,c}}y_{n+1})}{(y_{n+1}x_n+\frac{1}{b_n}\frac{q_{n-2,c}}{q_{n-1,c}}y_{n+1})(b_n+\frac{q_{n-2,b}}{q_{n-1,b}})}. \quad (38)$$

This, the fact that $1 - y_{n+1}x_n < 0$ and $b_{n+1} \geq 1$ imply that

$$\frac{1}{y_{n+1}(x_n b_n + \frac{q_{n-2,c}}{q_{n-1,c}})} - \frac{1}{b_n + \frac{q_{n-2,b}}{q_{n-1,b}}} = \frac{1 - y_{n+1}x_n + \frac{1}{b_n}(\frac{q_{n-2,b}}{q_{n-1,b}} - \frac{q_{n-2,c}}{q_{n-1,c}}y_{n+1})}{(y_{n+1}x_n + \frac{1}{b_n}\frac{q_{n-2,c}}{q_{n-1,c}}y_{n+1})(b_n + \frac{q_{n-2,b}}{q_{n-1,b}})}$$

$$\geq \frac{1 - y_{n+1}x_n}{y_{n+1}x_n} + \frac{\frac{1}{b_n}(\frac{q_{n-2,b}}{q_{n-1,b}} - \frac{q_{n-2,c}}{q_{n-1,c}}y_{n+1})}{(y_{n+1}x_n + \frac{1}{b_n}\frac{q_{n-2,c}}{q_{n-1,c}}y_{n+1})(b_n + \frac{q_{n-2,b}}{q_{n-1,b}})}. \tag{39}$$

Set $E_n(y_{n+1}) = \frac{1}{b_n}(\frac{q_{n-2,b}}{q_{n-1,b}} - \frac{q_{n-2,c}}{q_{n-1,c}}y_{n+1})$, $D = (y_{n+1}x_n + \frac{1}{b_n}\frac{q_{n-2,c}}{q_{n-1,c}}y_{n+1})(b_n + \frac{q_{n-2,b}}{q_{n-1,b}})$ and $F = \frac{x_{n+1}-y_{n+1}}{y_{n+1}} + \frac{1-y_{n+1}x_n}{y_{n+1}x_n}$. This, (38) and (39) yield that to prove (37), it is enough to prove that

$$F + \frac{1}{b_{n+1}}\frac{E_n(y_{n+1})}{D} \geq 0. \tag{40}$$

Set $E^*_{n-1}(y_{n+1}) = \frac{1}{b_{n-1}}(\frac{q_{n-3,c}}{q_{n-2,c}} - \frac{q_{n-3,b}}{q_{n-2,b}}y_{n+1})$ and $D_{n-1} = (x_{n-1} + \frac{1}{b_{n-1}}\frac{q_{n-3,c}}{q_{n-2,c}})(b_{n-1} + \frac{q_{n-3,b}}{q_{n-2,b}})$. Now we have

$$\begin{aligned} E_n(y_{n+1}) =& \frac{1}{b_n}(\frac{q_{n-2,b}}{q_{n-1,b}} - \frac{q_{n-2,c}}{q_{n-1,c}}y_{n+1}) \\ =& \frac{1}{b_n}\left(\frac{1}{(b_{n-1} + \frac{q_{n-3,b}}{q_{n-2,b}})} - \frac{y_{n+1}}{b_{n-1}x_{n-1} + \frac{q_{n-3,c}}{q_{n-2,c}}}\right) \\ =& \frac{1}{b_n}\frac{x_{n-1} - y_{n+1} + E^*_{n-1}(y_{n+1})}{D_{n-1}} \end{aligned} \tag{41}$$

and

$$\begin{aligned} E^*_{n-1}(y_{n+1}) =& \frac{1}{b_{n-1}}\left(\frac{q_{n-3,c}}{q_{n-2,c}} - \frac{q_{n-3,b}}{q_{n-2,b}}y_{n+1}\right) \\ =& \frac{1}{b_{n-1}}\left(\frac{1}{(b_{n-2}x_{n-2} + \frac{q_{n-4,c}}{q_{n-3,c}})} - \frac{y_{n+1}}{b_{n-2} + \frac{q_{n-4,b}}{q_{n-3,b}}}\right) \\ =& \frac{1}{b_{n-1}}\frac{1 - x_{n-2}y_{n+1} + E_{n-2}(y_{n+1})}{D_{n-2}}. \end{aligned}$$

From this and the fact that $1 - x_{n-2}y_{n+1} < 0$ we obtain that

$$\begin{aligned} E^*_{n-1}(y_{n+1}) =& \frac{1}{b_{n-1}}\frac{1 - x_{n-2}y_{n+1} + E_{n-2}(y_{n+1})}{D_{n-2}} \\ >& \frac{1 - x_{n-2}y_{n+1}}{x_{n-2}} + \frac{1}{b_{n-1}}\frac{E_{n-2}(y_{n+1})}{D_{n-2}}. \end{aligned}$$

This and (41) imply that

$$E_n(y_{n+1}) = \frac{1}{b_n}\frac{x_{n-1} - y_{n+1} + E^*_{n-1}(y_{n+1})}{D_{n-1}}$$
$$> \frac{1}{b_n}\frac{x_{n-1} - y_{n+1} + \frac{1-x_{n-2}y_{n+1}}{x_{n-2}} + \frac{1}{b_{n-1}}\frac{E_{n-2}(y_{n+1})}{D_{n-2}}}{D_{n-1}}. \tag{42}$$

For every $j = n_0, \ldots, n$ we have

$$x_{j+1} - y_{n+1} + \frac{1 - x_j y_{n+1}}{x_j}$$
$$= 1 + \frac{\varepsilon}{\sqrt{(j+1)}} - 1 - \frac{1}{(n+1)\ln(n+1)} + \frac{1 - (1 + \frac{\varepsilon}{\sqrt{j}})(1 + \frac{1}{(n+1)\ln(n+1)})}{1 + \frac{\varepsilon}{\sqrt{j}}}$$
$$= \frac{\varepsilon}{\sqrt{(j+1)}} - \frac{1}{(n+1)\ln(n+1)} + \frac{1}{1 + \frac{\varepsilon}{\sqrt{j}}} - 1 - \frac{1}{(n+1)\ln(n+1)}$$
$$= \frac{\varepsilon}{\sqrt{(j+1)}} - \frac{2}{(n+1)\ln(n+1)} - \frac{\varepsilon}{\sqrt{j} + \varepsilon} = \frac{\varepsilon^2 - \frac{\varepsilon}{\sqrt{j}+\sqrt{j+1}}}{\sqrt{(j+1)}(\sqrt{j} + \varepsilon)} - \frac{2}{(n+1)\ln(n+1)}.$$

This and the fact that the function $f(x) = \frac{\varepsilon^2 - \frac{\varepsilon}{\sqrt{x}+\sqrt{x+1}}}{\sqrt{x+1}(\sqrt{x}+\varepsilon)}$ is decreasing for $x > n_0$ we obtain that

$$x_{j+1} - y_{n+1} + \frac{1 - x_j y_{n+1}}{x_j} = \frac{\varepsilon^2 - \frac{\varepsilon}{\sqrt{j}+\sqrt{j+1}}}{\sqrt{(j+1)}(\sqrt{j} + \varepsilon)} - \frac{2}{(n+1)\ln(n+1)}$$
$$\geq \frac{\varepsilon^2 - \frac{\varepsilon}{\sqrt{n}+\sqrt{n+1}}}{\sqrt{(n+1)}(\sqrt{n} + \varepsilon)} - \frac{2}{(n+1)\ln(n+1)}. \tag{43}$$

From this and (34) we obtain that $x_{j+1} - y_{n+1} + \frac{1-x_j y_{n+1}}{x_j} > 0$. Hence $x_{n-1} - y_{n+1} + \frac{1-x_{n-2}y_{n+1}}{x_{n-2}} > 0$. This and (42) yield that

$$E_n(y_{n+1}) > \frac{1}{b_n}\frac{x_{n-1} - y_{n+1} + \frac{1-x_{n-2}y_{n+1}}{x_{n-2}} + \frac{1}{b_{n-1}}\frac{E_{n-2}(y_{n+1})}{D_{n-2}}}{D_{n-1}}$$
$$> \frac{1}{b_n}\frac{\frac{1}{b_{n-1}}\frac{E_{n-2}(y_{n+1})}{D_{n-2}}}{D_{n-1}}.$$

If we repeat this procedure then we obtain

$$\begin{aligned} E_n(y_{n+1}) >& \frac{1}{b_n}\frac{\frac{1}{b_{n-1}}(\frac{E_{n-2}(y_{n+1})}{D_{n-2}})}{D_{n-1}} > \cdots > \frac{E_{n-2[\frac{n}{2}]+2n_0-2}(y_{n+1})}{\prod_{j=n}^{n-2[\frac{n}{2}]+2n_0} b_j D_{j-1}} \\ >& \frac{-|E_{n-2[\frac{n}{2}]+2n_0-2}(y_{n+1})|}{\prod_{j=n}^{n-2[\frac{n}{2}]+2n_0} x_{j-1}} = \frac{-|E_{n-2[\frac{n}{2}]+2n_0-2}(y_{n+1})|}{\prod_{j=n}^{n-2[\frac{n}{2}]+2n_0}(1+\frac{\varepsilon}{\sqrt{j-1}})} \\ =& \frac{-|E_{n-2[\frac{n}{2}]+2n_0-2}(y_{n+1})|}{e^{\sum_{j=n}^{n-2[\frac{n}{2}]+2n_0}\log(1+\frac{\varepsilon}{\sqrt{j-1}})}} > \frac{-|E_{n-2[\frac{n}{2}]+2n_0-2}(y_{n+1})|}{e^{\sqrt{2}\varepsilon(\sqrt{n-2}-\sqrt{2n_0+2})}}. \end{aligned}$$

This implies that to prove (40) it is enough to prove that

$$F - \frac{|E_{n-2[\frac{n}{2}]+2n_0-2}(y_{n+1})|}{e^{\sqrt{2}\varepsilon(\sqrt{n-2}-\sqrt{2n_0+2})}} \geq 0. \tag{44}$$

From (43) we obtain that

$$\begin{aligned} F =& \frac{1+\frac{\varepsilon}{\sqrt{(n+1)}} - 1 - \frac{1}{(n+1)\ln(n+1)}}{y_{n+1}} + \frac{1-(1+\frac{\varepsilon}{\sqrt{n}})(1+\frac{1}{(n+1)\ln(n+1)})}{y_{n+1}x_n} \\ =& \frac{1}{y_{n+1}}(\frac{\varepsilon^2 - \frac{\varepsilon}{\sqrt{n}+\sqrt{n+1}}}{\sqrt{(n+1)}(\sqrt{n}+\varepsilon)} - \frac{2}{(n+1)\ln(n+1)}). \end{aligned}$$

This implies that inequality (44) has the form

$$\frac{1}{y_{n+1}}(\frac{\varepsilon^2 - \frac{\varepsilon}{\sqrt{n}+\sqrt{n+1}}}{\sqrt{(n+1)}(\sqrt{n}+\varepsilon)} - \frac{2}{(n+1)\ln(n+1)}) - \frac{|E_{n-2[\frac{n}{2}]+2n_0-2}(y_{n+1})|}{e^{\sqrt{2}\varepsilon(\sqrt{n-2}-\sqrt{2n_0+2})}} \geq 0$$

which holds for all sufficiently large n. The proof of Lemma 14 is complete.

□

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J. Hančl and J. Leta

Department of Mathematics, Faculty of Science, University of Ostrava, 30. dubna 22, 701 03 Ostrava 1, Czech Republic.

e-mail: hancl@osu.cz, jiiberhanu@gmail.com

M. Laursen

Department of Mathematics, Aarhus University, Ny Munkegade 118, 8000 Aarhus C, Denmark.

e-mail: mll@math.au.dk

1.6 Transcendence of series and products

In this section, we will consider criteria for Σ- and Π-transcendence of sequences whose elements all belong to a fixed number field $\mathbb{K}$, based on the papers [38, 39] by the current author. In this section, $\mathcal{N} : \overline{\mathbb{Q}} \to \mathbb{Q}$ denotes the map that sends algebraic numbers to the product of their conjugates. This function is related to the notion of field norms. If $\mathbb{K}$ is a number field, and $\sigma_1, \ldots, \sigma_d$ are its distinct embeddings into $\overline{\mathbb{Q}}$, then we define *the field norm* for the extension $\mathbb{K} \supseteq \mathbb{Q}$ is as the map $\mathcal{N}_{\mathbb{K}} : \mathbb{K} \to \mathbb{Q}$ given by $\mathcal{N}_{\mathbb{K}}(a) = \sigma_1(a) \cdots \sigma_d(a)$. When $a \in \mathbb{K}$, we have in particular that $\mathcal{N}_{\mathbb{K}}(a) = \mathcal{N}(a)^{\deg_{\mathbb{K}} a}$.

The first two results govern Σ-transcendence and were proven in [38]. As was shown in the same paper, each theorem admits sequences $\{a_n/b_n\}_{n=1}^{\infty}$ the other does not.

Theorem 1.50 (Laursen, 2024)**.** *Let $\mathbb{K}$ be a number field of degree $d \geq 2$, consider real numbers $\alpha \in (0,1)$, $\varepsilon > 0$, $\beta \in [0, \varepsilon/(1+\varepsilon))$, and $y \geq 1$, and let $\zeta \in \mathbb{C}$ with $\zeta \neq 0$. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of non-zero numbers from $\mathbb{N}$ and $\mathcal{O}_{\mathbb{K}}$, respectively, with $n^{1+\varepsilon} \leq a_n \leq a_{n+1}$. Suppose that for sufficiently large n,*

$$|b_n| \leq a_n^{\beta} 2^{\log_2^{\alpha} a_n}, \qquad \overline{|b_n|} \leq a_n^{y} 2^{\log_2^{\alpha} a_n}, \quad \text{and} \quad \Re(\zeta b_n) > 0. \tag{1.15}$$

Then the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ is Σ-irrational if

$$\limsup_{n\to\infty} a_n^{\left(\frac{dy}{1-\beta}+1\right)^{-n}} = \infty,$$

and it is Σ-transcendental if

$$\limsup_{n\to\infty} a_n^{\left(\frac{d^2y}{1-\beta}+1\right)^{-n}} = \infty.$$

Theorem 1.51 (Laursen, 2024)**.** *Let $\mathbb{K}$ be a number field of degree d, consider real numbers $\alpha \in (0,1)$, $\delta, \varepsilon > 0$, $\beta \in [0, \varepsilon/(1+\varepsilon))$, $y_1 \geq 1$, $y_2 \geq \beta$, $\eta_1 \in [0, (d-1)y_1 + y_2]$, and $\eta_2 \geq 1$, and let $\zeta \in \mathbb{C}$ with $\zeta \neq 0$. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of non-zero numbers from $\mathcal{O}_{\mathbb{K}}$ with , and let $\{r_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that $r_n \mid a_n$ in $\mathcal{O}_{\mathbb{K}}$. Suppose that for all sufficiently large n,*

$$n^{1+\varepsilon} \leq a_n \leq a_{n+1}, \qquad |b_n| \leq |a_n|^{\beta} 2^{\log_2^{\alpha} |a_n|}, \qquad \overline{|b_n|} \leq |a_n|^{y_2} 2^{\log_2^{\alpha} |a_n|},$$

$$\overline{|a_n|} \leq |a_n|^{y_1} 2^{\log_2^{\alpha} |a_n|}, \qquad |\mathcal{N}_{\mathbb{K}}(a_n)| \geq |a_n|^{\eta_1} 2^{-\log_2^{\alpha} |a_n|},$$

$$r_n \left| \mathcal{N}\left(\frac{a_n}{r_n}\right) \right| \leq |a_n|^{\eta_2} 2^{\log_2^{\alpha} |a_n|}, \quad \text{and} \quad \Re\left(\zeta \frac{a_n}{b_n}\right) > 0.$$

are satisfied. Then the sequence $\{a_n/b_n\}_{n=1}^\infty$ *is* Σ*-irrational if*

$$\limsup_{n\to\infty} |a_n|^{\left(\frac{d(y_1+y_2)}{1-\beta}+1\right)^{-n}} = \infty,$$

and it is Σ*-transcendental if*

$$\limsup_{n\to\infty} |a_n|^{\left(\frac{d((d-1)y_1+y_2-\eta_1+\eta_2)+\eta_2+\delta}{1-\beta}+1\right)^{-n}} = \limsup_{n\to\infty} |a_n|^{\left(\frac{d^2(y_1+y_2)}{1-\beta}+1\right)^{-n}} = \infty.$$

Remark 1.52. In the original phrasings of these theorems, a_n and b_n were written in a $\mathbb{Q}$-linear basis of $\mathbb{K}$, bounding the corresponding max norm of a_n and b_n rather than $\overline{|a_n|}$ and $\overline{|b_n|}$. As follows from Lemma 1.24, however, this affects the respective bounds by at most a fixed constant $C > 0$, and then we just replace α by $\alpha' \in (\alpha, 1)$ if necessary.

The two theorems have almost identical proofs and split into two halves with each their algebraic argument and Erdős jump. The first half considered in the paper provides a criterion that ensures that $\sum_{n=1}^\infty b_n/a_n$ is either transcendental or an element of $\mathbb{K}$, relying heavily on an application of Schmidt's Subspace Theorem [47]. This theorem, presented below, uses the notion of *linear forms*, which are polynomials of the form $L = c_1X_1 + \ldots + c_dX_d$ where $X_1, \ldots, X_d$ are free variables, and $c_1, \ldots, c_d$ are the corresponding coefficients. While it may not be clear from first glance, the theorem is a generalization of Theorem 1.2

Theorem 1.53 (Schmidt, 1980)**.** *Let* $L_1, \ldots, L_d$ *be* $\mathbb{Q}$*-linearly independent linear forms in d variables with algebraic coefficients. For any* $\delta > 0$*, there exists a finite collection of proper subspaces* $T_1, \ldots, T_w \subsetneq \mathbb{Q}^d$ *such that any* $x \in \mathbb{Z}^d$ *with*

$$|L_1(x) \cdots L_d(x)| < |x|^{-\delta}$$

is contained in at least one subspace T_i*.*

In this chapter, we specifically have $L_i = X_i$ for $i < d$ and $L_d = c_1X_1 + \ldots + c_dX_d$ with $c_d \neq 0$. Compared to Theorem 1.2, the coefficient c_d then plays the role of a, X_d plays the role of q, and $c_1X_1 + \ldots + c_{d-1}X_{d-1}$ plays the role of $-p$. Using a few technical arguments as well, this leads to the following lemma from [38], in which d has been replaced by $d + 1$, the coefficients $c_1, \ldots, c_{d+1}$ have been renamed to $-x_1, \ldots, -x_d, s$, and the variables $X_1, \ldots, X_{d+1}$ have been renamed to $p_1, \ldots, p_d, q$.

Lemma 1.54 (Laursen, 2024)**.** *Let* $x_1, \ldots, x_d, s$ *be algebraic numbers such that s is* $\mathbb{Q}$*-linearly independent of* $x_1, \ldots, x_d$*, and let* $C, \delta > 0$*. Then the inequality*

$$\left| qs - \sum_{i=1}^d p_i x_i \right| \prod_{i=1}^d \max\{1, |p_i|\} < q^{-\delta}.$$

has only finitely many solutions $(p_1, \ldots, p_d, q) \in \mathbb{Z}^d \times \mathbb{N}$ *with* $|p_i| \leq q^C$*.*

Remark 1.55. In the original phrasing in [38] of the above lemma, the current author forgot to add the needed condition $|p_i| \leq q^C$, which is what allows us to replace $|(p_1, \ldots, p_d, q)|^{-\delta}$ by $q^{-\delta}$. However, since this assumption is satisfied whenever the lemma is applied, the proofs of [38] remain valid.

Picking $s = \sum_{n=1}^{\infty} \frac{b_n}{a_n c_n}$ and $p_{1,N}, \ldots, p_{d,N}, q_N$ with sufficient care, including so that $(p_{d,N}x_1 + p_{2,N}x_2 + \cdots + p_{d,N}x_d)/q = \sum_{n=1}^{N} b_n/(a_n c_n)$, Lemmas 1.16 and 1.54 then lead to the conclusion that $\sum_{n=1}^{\infty} b_n/(a_n c_n)$ is either transcendental or contained in $\mathbb{K}$. These choices of $p_{1,N}, \ldots, p_{d,N}, q_N$ are rather simple when $a_n \in \mathbb{Z}$. In that case, we simply take $q_N = \prod_{n=1}^{N} a_n$ and pick $p_{i,N}$ accordingly with respect to a chosen basis $x_1, \ldots, x_d$ of $\mathbb{K}$. If we are not guaranteed $a_n \in \mathbb{Z}$, we have to be more careful with the choice of q_N (and, by extension, $p_{1,N}, \ldots, p_{d,N}$). This is the main reason why Theorem 1.51 has more assumptions to check than Theorem 1.50 and why it has the more intricate *limsup* condition

$$\limsup_{n\to\infty} |a_n|^{\left(\frac{d((d-1)y_1+y_2+\eta_2-\eta_1)+\delta}{1-\beta}+1\right)^{-n}} = \infty.$$

The remaining half of the proofs of Theorems 1.50 and 1.51 uses the methods behind Theorems 1.11 and 1.26 to provide the Σ-irrationality statement and ensure that $\sum_{n=1}^{\infty} b_n/(a_n c_n) \notin \mathbb{K}$.

Similar to what we experienced in Section 1.4, it is not too difficult to translate the proofs of Theorems 1.50 and 1.51 into criteria for Π-irrationality and Π-transcendence. By making modifications to the arguments corresponding to those made in Section 1.4 as well as further improvements, the current author found the following results in [39]. While presenting these theorems, we will elaborate on brief remarks made in the same paper on how to improve Theorems 1.50 and 1.51.

Theorem 1.56. *Let $\mathbb{K}$ be a number field of degree d, and consider real numbers $\alpha \in (0,1)$, $\varepsilon > 0$, $\beta \in [0, \varepsilon/(1+\varepsilon))$, $y_1 \geq 1$, $y_2 \geq \beta$, $z_1 \geq -y_2$, $z_2 \geq 0$, and $e \in \{0,1\}$. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of non-zero numbers in $\mathcal{O}_{\mathbb{K}}$, and let $\{r_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that $r_n \mid a_n$ in $\mathcal{O}_{\mathbb{K}}$. Let d_0 be a positive integer, and suppose that for all sufficiently large n,*

$$n^{1+\varepsilon} \leq |a_n| \leq |a_{n+1}|, \qquad |b_n| < |a_n|^{\beta} 2^{\log_2^{\alpha}|a_n|}, \qquad \overline{|b_n|} \leq |a_n|^{y_2} 2^{\log_2^{\alpha}|a_n|},$$

$$\overline{|a_n|} \leq |a_n|^{y_1} 2^{\log_2^{\alpha}|a_n|}, \qquad \overline{|a_n^{-1}|} \leq |a_n|^{z_1} 2^{\log_2^{\alpha}|a_n|}, \qquad \deg(a_n/b_n) \geq d_0,$$

$$r_n \left| \mathcal{N}\left(\frac{a_n}{r_n}\right) \right| \leq |a_n|^{z_2} 2^{\log_2^{\alpha}|a_n|}, \quad \text{and} \quad e\Re\left(\frac{a_n}{b_n}\right) \geq \begin{cases} 0 & \text{if } e = 1, \\ 1/2 & \text{if } e = -1, \end{cases}$$

with $\Re(a_n/b_n) \neq -1/2$ infinitely often. Then the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ is $\Pi_{\mathbb{K}}$-irrational if

$$\limsup_{n\to\infty} |a_n|^{\left(\frac{d(y_2+z_1+z_2/d_0)}{1-\beta}+1\right)^{-n}} = \infty, \tag{1.16}$$

$$\limsup_{n\to\infty} |a_n|^{\left(\frac{d(y_1+y_2)}{1-\beta}+1\right)^{-n}} = \infty, \tag{1.17}$$

or, in the case that all n satisfy $a_n \in \mathbb{Z}$ or $b_n \in \mathbb{Z}$, if

$$\limsup_{n\to\infty} |a_n|^{\left(\frac{d\max\{y_1,y_2\}}{1-\beta}+1\right)^{-n}} = \infty. \tag{1.18}$$

Remark 1.57. The reason why we reach $\Pi_{\mathbb{K}}$-irrationality rather than mere Π-irrationality is the exact same as it was for Theorem 1.29 in section 1.4 and is equally applicable in the infinite series setting. Hence, we may improve the statements of Σ-irrationality in Theorems 1.50 and 1.51 to statements of $\Sigma_{\mathbb{K}}$-irrationality without changing any assumptions.

Remark 1.58. As can be seen in the proof in [39], the difference between using condition 1.16 compared to condition 1.17 or 1.18 comes solely from the difference between bounding $H(b_n/a_n)$ via Theorem 1.22 as

$$\begin{aligned} H\left(\frac{b_n}{a_n}\right) &\leq \operatorname{den}\left(\frac{b_n}{a_n}\right)^{1/\deg(b_n/a_n)} \max\left\{1, \boxed{\frac{b_n}{a_n}}\right\} \\ &\leq \left(r_n \mathcal{N}\left(\frac{a_n}{r_n}\right)^{1/d_0}\right) \max\left\{1, \boxed{b_n}\boxed{a_n^{-1}}\right\} \\ &\leq |a_n|^{z_2/d_0+y_2+z_1} 2^{3\log_2 \alpha |a_n|}, \end{aligned}$$

or via Lemma 1.23 and Theorem 1.22 as

$$H\left(\frac{b_n}{a_n}\right) \leq H(b_n)H(a_n) \leq \boxed{b_n}\boxed{a_n} \leq |a_n|^{y_2+y_1} 2^{2\log_2 \alpha |a_n|}$$

or, when $a_n \in \mathbb{Z}$ or $b_n \in \mathbb{Z}$,

$$\begin{aligned} H\left(\frac{b_n}{a_n}\right) = H\left(\frac{a_n}{n_n}\right) &\leq \begin{cases} |a_n| \max\{1, \boxed{b_n/a_n}\} & \text{if } a_n \in \mathbb{Z}, \\ |b_n| \max\{1, \boxed{a_n/b_n}\} & \text{if } b_n \in \mathbb{Z} \end{cases} \\ &\leq |a_n|^{\max\{y_1,y_2\}} 2^{\log_2 \alpha |a_n|}. \end{aligned}$$

Hence, the *limsup* condition for irrationality in Theorem 1.51 may be replaced by equation (1.16) or, when all n satisfy $a_n \in \mathbb{Z}$ or $b_n \in \mathbb{Z}$, by equation (1.18). This does not affect the validity of Remark 1.57 above.

Theorem 1.59 (Laursen, 2025 on arXiv)**.** *Let $\mathbb{K}$ be a number field of degree $d \in \mathbb{N}$, and consider real numbers $\delta, \varepsilon > 0$, $\alpha \in (0,1)$, $\beta \in [0, \varepsilon/(1+\varepsilon))$, $e \in \{-1, 1\}$, and $y \geq 1$. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of non-zero numbers from $\mathbb{N}$ and $\mathcal{O}_{\mathbb{K}}$,* *respectively, and let* $\{r_n\}_{n=1}^{\infty}$ *be a sequence of positive integers such that* $r_n \mid a_n$ *in* $\mathcal{O}_{\mathbb{K}}$. *Suppose that for all sufficiently large* n,

$$n^{1+\varepsilon} \leq |a_n| \leq |a_{n+1}|, \qquad |b_n| < |a_n|^{\beta} 2^{\log_2^{\alpha} |a_n|}, \qquad \boxed{b_n} \leq |a_n|^{y} 2^{\log_2^{\alpha} |a_n|},$$

and

$$e\Re\left(\frac{a_n}{b_n}\right) \geq \begin{cases} 0 & \textit{if } e = 1, \\ 1/2 & \textit{if } e = -1, \end{cases}$$

with $\Re(a_n/b_n) \neq -1/2$ *infinitely often. Then the sequence* $\{a_n/b_n\}_{n=1}^{\infty}$ *is* Π-*transcendental if*

$$\limsup_{n\to\infty} a_n^{\left(\frac{dy+1+\delta}{1-\beta}+1\right)^{-n}} = \infty.$$

Theorem 1.60. *Let* $\mathbb{K}$ *be a number field of degree* $d \in \mathbb{N}$, *and consider real numbers* $\alpha \in (0,1)$, $\delta, \varepsilon > 0$, $\beta \in [0, \varepsilon/(1+\varepsilon))$, $e \in \{-1, 1\}$, $y \geq \beta$, $z_1 \geq -y$, *and* $z_2 \geq 0$. *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be sequences of non-zero numbers in* $\mathcal{O}_{\mathbb{K}}$, *and let* $\{r_n\}_{n=1}^{\infty}$ *be a sequence of positive integers such that* $r_n \mid a_n$ *in* $\mathcal{O}_{\mathbb{K}}$. *Suppose that for all sufficiently large* n,

$$n^{1+\varepsilon} \leq |a_n| \leq |a_{n+1}|, \qquad |b_n| < |a_n|^{\beta} 2^{\log_2^{\alpha} |a_n|}, \qquad \boxed{b_n} \leq |a_n|^{y} 2^{\log_2^{\alpha} |a_n|},$$

$$\boxed{a_n^{-1}} \leq |a_n|^{z_1} 2^{\log_2^{\alpha} |a_n|}, \qquad r_n \left| \mathcal{N}\left(\frac{a_n}{r_n}\right) \right| \leq |a_n|^{z_2} 2^{\log_2^{\alpha} |a_n|},$$

and

$$e\Re\left(\frac{a_n}{b_n}\right) \geq \begin{cases} 0 & \textit{if } e = 1, \\ 1/2 & \textit{if } e = -1, \end{cases}$$

with $\Re(a_n/b_n) \neq -1/2$ *infinitely often. Then the sequence* $\{a_n/b_n\}_{n=1}^{\infty}$ *is* Π-*transcendental if*

$$\limsup_{n\to\infty} |a_n|^{\left(\frac{d(y+z_1+z_2)+z_2+\delta}{1-\beta}+1\right)^{-n}} = \infty.$$

Remark 1.61. The more slacked *limsup* criteria of Theorems 1.59 and 1.60 compared those of Theorems 1.50 and 1.51 are due to changes in argumentation that are equally valid if the products $\prod(1 + b_n/a_n)$ are replaced by the series $\sum b_n/a_n$. Hence, we may replace the *limsup* criteria for transcendence in Theorems 1.50 and 1.51 with those of Theorems 1.59 and 1.60. In fact, this improvement to Theorem 1.50 is an immediate consequence of Remark 1.57.

The main change in the proof of Theorem 1.60 compared to Theorem 1.51, other than using a stronger result to ensure $\mathbb{K}$-irrationality, relied on how the $\mathbb{Q}$-linear coefficients $p_{i,N}/q_N$ used for Lemma 1.54 were bounded. By immediately using Lemma 1.24 to bound these coefficients by a scalar times $\left|\sum_{i=1}^{d} p_{i,N} x_i / q_N\right|$, the subsequent estimates became more simple to handle, and it became easy to spot a simpler and more efficient bound, which then led to the improved *limsup* criterion.

As we also experienced in Section 1.4, we may slacken the assumptions to some degree if we only seek irrationality or transcendence for the specific series or product generated by the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ rather than the sequence itself. This leads to the below theorem from [39].

Theorem 1.62 (Laursen, 2025 on arXiv)**.** *In Theorems 1.56 and 1.60, suppose we weaken the assumption that $z_2 \geq 1$ and the bounds on $\Re(a_n/b_n)$ to $z_2 \geq 0$ and $e\Re\left(a_n/b_n + 1/2\right) \geq 0$ with strict inequality infinitely often. Then the statements of the theorems still hold but with the statements of $\Pi_{\mathbb{K}}$-irrationality and Π-transcendence weakened to $\prod_{n=1}^{\infty}(1 + b_n/a_n) \notin \mathbb{K}$ and $\prod_{n=1}^{\infty}(1 + b_n/a_n) \notin \overline{\mathbb{Q}}$, respectively.*

Remark 1.63. After applying Remarks 1.57–1.61 to Theorem 1.51, we may similarly replace $z_2 \geq 1$ by $z_2 \geq 0$ if we also replace $\Sigma_{\mathbb{K}}$-irrationality and Σ-transcendence by $\sum_{n=1}^{\infty} b_n/a_n \notin \mathbb{K}$ and $\sum_{n=1}^{\infty} b_n/a_n \notin \overline{\mathbb{Q}}$, respectively.

In allowing z_2 below 1, this theorem provides the greatest surprise of the paper, at least in the eyes of the current author. Suddenly, we may use the Erdős Jump to prove irrationality for a number $\sum_{n=1}^{\infty} 1/a_n$ with $\limsup_{n\to\infty} |a_n|^{2^{-n}} = 1$, something that this author has not seen anywhere in the literature. Admittedly, a_n has to have some quite restrictive arithmetic properties in order to have $y_2 + z_1 + z_2 < 1$, but it is possible, as is evident from Example 1.64 below and [39, Example 3.2].

1.6.1 Examples

In this subsection, it is assumed that Theorems 1.50 and 1.51 have been improved through Remarks 1.57, 1.58, and 1.61 so that they match Theorems 1.56, 1.59, and 1.60 in strength.

For the purpose of the following examples, let $\varphi = (1 + \sqrt{5})/2$ be the golden ratio, and let $F_1, F_2, \ldots$ be the Fibonacci numbers, defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_n + F_{n+1}$. It follows that $\varphi^n = F_n\varphi + F_{n-1}$ and, thereby, $F_n \approx \varphi^{n-1}$. The reasons for using φ to construct examples are as follows.

- With a minimal polynomial of $X^2 - X - 1$, φ is fairly simple and, at least to this author, easier to relate to than more obscure numbers.
- φ is an algebraic unit, meaning that it is an algebraic integer whose multiplicative inverse is again an algebraic integer or, equivalently, $\mathcal{N}(\varphi) = 1$, which makes it easier to construct examples with specific values of z_1 and z_2.

- Being a real number greater than 1 means that all powers of φ remain in the positive half plane and contribute to the increase in modulus of the resulting sequence $\{a_n\}_{n=1}^{\infty}$.

We will also consider another real algebraic unit greater than one, namely the *supergolden ratio* ψ, which is the real root of the polynomial $X^3 - X^2 - 1$. In addition to sharing the above properties with φ, ψ also satisfies $\lceil 1/\psi \rceil = \sqrt{\psi}$, thus allowing a smaller z_1 for a sequence like $\{\psi^{h_n}\}_{n=1}^{\infty}$. Defining $\hat{F}_1 = \hat{F}_2 = \hat{F}_3 = 1$ and $\hat{F}_{n+3} = \hat{F}_n + \hat{F}_{n+2}$, we have $\psi^n = \hat{F}_n\psi + \hat{F}_{n-2}$ and $\hat{F}_n \approx \psi^{n-1}$.

Recall the Riemann zeta function ζ from equation (1.3). Inspired by [23, Remark 4], we notice that Theorems 1.3 and 1.13 imply

$$\sum_{n=1}^{\infty} \frac{1}{n^s a_n} \notin \mathbb{Q} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^s A_n} \notin \overline{\mathbb{Q}},$$

respectively, when $\{a_n\}_{n=1}^{\infty}$ and $\{A_n\}_{n=1}^{\infty}$ are non-decreasing sequences of integers with $\limsup_{n\to\infty} a_n^{2^{-n}} = \infty$ and $\limsup_{n\to\infty} A_n^{(3+\delta)^{-n}} = \infty$ for some fixed $\delta > 0$. Inspired by [23, Remark 3], we may modify [39, Examples 3.2 and 3.5], to weaken these *limsup* restrictions on a_n and A_n if we multiply with suitable sequences of algebraic units with comparable growth. In the below examples, we use $\lfloor a \rfloor$ to denote the largest rational integer less than or equal to a given real number a.

Example 1.64. Let $s > 1$ be a positive integer. For $i > 1$, let $\{a_{i,n}\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive integers with $\limsup_{n\to\infty} a_{i,n}^{(1+1/i)^{-n}} = \infty$. Then, by Remark 1.63 and Theorem 1.51 with $d_0 = d$, $z_1 = y_2 = 0$, and $z_2 = 1/i$,

$$\sum_{n=1}^{\infty} \frac{1}{n^s a_{2,n} \varphi^{\lfloor \log(n^s a_{2,n})/\log\varphi \rfloor}} \notin \mathbb{Q}(\varphi) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^s a_{3,n} \psi^{2\lfloor \log(n^s a_{3,n})/\log\varphi \rfloor}} \notin \mathbb{Q}(\psi).$$

More generally, if $x > 1$ is an algebraic unit with $\lceil 1/x \rceil = x^{1/(i-1)}$, then

$$\sum_{n=1}^{\infty} \frac{1}{n^s a_{i,n} x^{(i-1)\lfloor \log(n^s a_{i,n})/\log x \rfloor}} \notin \mathbb{Q}(x).$$

Remark 1.65. Notice that the *limsup* condition required for the general part of the example becomes more lenient the larger values of i that have a matching x in the above sense. Seeing how $\lceil 1/x \rceil = x^{1/(\deg x - 1)}$ for $x = \varphi, \psi$, it is this author's hope that the same is true for x of arbitrarily large degree.

Example 1.66. Let $s > 1$ be a positive integer. For $j > 0$, let $\{A_{i,n}\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive integers with $\limsup_{n\to\infty} A_{i,n}^{(2+1/i+\delta)^{-n}} = \infty$ for any fixed $\delta > 0$. Then the numbers

$$\sum_{n=1}^{\infty} \frac{1}{n^s A_{2,n} \varphi^{\lfloor \log_\varphi(n^s A_{2,n}) \rfloor}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^s A_{3,n} \psi^{2\lfloor \log_\psi(n^s A_{3,n}) \rfloor}}$$

are transcendental. More generally, suppose $x > 1$ is an algebraic unit of degree d with $\overline{|1/x|} = x^{1/(i-1)}$. Writing $j = (d+1)/i - 1$, the number

$$\sum_{n=1}^{\infty} \frac{1}{n^s A_{1/j,n} x^{(i-1)\lfloor \log_x(n^s A_{1/j,n}) \rfloor}}$$

is transcendental.

Remark 1.67. Notice that with the right algebraic units x, assuming such x exist, the *limsup*, condition can be made arbitrary close to that of Theorem 1.3. Compared to Example 1.64, we have to be more careful with these x, however, as i has to grow sufficiently fast in terms of d.

Notice that each of the above examples would be equally true in the setting of infinite products, though the connection to $\zeta(s)$ might be less clear.

The remaining examples, which all deal with irrationality and transcendence of sequences, are further special cases of those provided in [39]. We start by combining Examples 3.2 and 3.5 of the paper. The reader may notice that the considered sequences have the same arithmetic properties as in the previous examples except for much stricter *limsup* conditions, which are a consequence of the restriction $z_2 \geq 1$.

Example 1.68. For $i \in \mathbb{N}$, let $\{h_{i,n}\}_{n=1}^{\infty}$ be a strictly increasing sequence of integers with $h_{i,n} \geq (i + 1/i)^n$ and $h_{i,n}^{(j)} \geq (1 + j - 1/i)^n$ infinitely often. Writing $h_n = h_{1,n}$, Theorems 1.51 and 1.56 ensure that the sequence $\{F_{h_n} \varphi^{h_n}\}_{n=1}^{\infty}$ is both $\Sigma_{\mathbb{Q}(\varphi)}$- and $\Pi_{\mathbb{Q}(\varphi)}$-irrational and that the sequence $\{\hat{F}_{h_n} \psi^{2h_n}\}_{n=1}^{\infty}$ is $\Sigma_{\mathbb{Q}(\psi)}$- and $\Pi_{\mathbb{Q}(\psi)}$-irrational. By Theorems 1.51 and 1.60, the sequences $\{F_{h_{4,n}} \varphi^{h_{4,n}}\}_{n=1}^{\infty}$ and $\{\hat{F}_{h_{5,n}} \psi^{2h_{5,n}}\}_{n=1}^{\infty}$ are both Σ- and Π-transcendental.

As an application of Theorem 1.50 and its product counterparts, we now combine Examples 3.1 and 3.4 from [39] as follows.

Example 1.69. Let $\{h_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of integers with $h_n \geq 3^n n$ infinitely often. By Theorems 1.50 and 1.56, the sequence $\{F_{h_n}/(1+\varphi^{-h_n})\}_{n=1}^{\infty}$ is both $\Sigma_{\mathbb{Q}(\varphi)}$-irrational and $\Pi_{\mathbb{Q}(\varphi)}$-irrational. Using Theorems 1.50 and 1.59, $\{F_{h_n}/(1 + \varphi^{-h_n})\}_{n=1}^{\infty}$ is furthermore Σ- and Π-transcendental if $h_n \geq (4 + 1/4)^n$ infinitely often.

In the last irrationality example of this section, which is based on [39, Example 3.3], we consider a situation where *limsup* condition (1.17) is our best option.

Example 1.70. Let $\{h_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of integers with $h_n \geq 7^n \log n$ infinitely often. Then the sequence $\left\{\left(2^{h_n} + \sqrt[3]{2}^n\right)/\left(1 + \left(\sqrt[3]{2} - 1\right)^{h_n}\right)\right\}_{n=1}^{\infty}$ is $\Sigma_{\mathbb{Q}(\sqrt[3]{2})}$- and $\Pi_{\mathbb{Q}(\sqrt[3]{2})}$-irrational, by following Theorems 1.51 and 1.56.

Finally, in the below special case of [39, Example 3.6], we consider transcendence for a sequence constructed solely by powers of ψ.

Example 1.71. Let $\{h_n\}_{n=1}^{\infty}$ and $\{h'_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of integers with $h_n \geq (7 - 1/3)^n$ and $h'_n \geq (3 - 1/3)^n$ infinitely often. By Theorems 1.51 and 1.60, $\{\psi^{h_n}\}_{n=1}^{\infty}$ is Π-transcendental, and the number $\prod_{n=1}^{\infty}(1 + \psi^{-h'_n})$ is transcendental.

1.6.2 Paper 4: Transcendence of certain infinite series

Below, the reader will find the paper [38], which has the current author as its sole author. The paper is published in *Research in Number Theory* in July 2024 as an open access article and is available through the link https://doi.org/10.1007/s40993-024-00553-2. It has a length of 25 pages, numbered 1 through 25.

When reading Lemma 3.3 of the paper as well as its proof, the reader may replace the assumption $M \geq 1$ by $M > 0$, which does not affect the validity of the proof later in the paper. The only point where it is actively used that $M \geq 1$ rather than $M > 0$ is in a single line on page 22 of the paper, which reads

$$a_{k_1}^{\left(\frac{M}{1-\beta}+1\right)^{-k_1}} \leq 2^{k_1 2^{-k_1}} \leq \frac{1}{2}.$$

However, this is not an issue When $\mu \in (0, 1)$, this should be replaced by

$$a_{k_1}^{\left(\frac{M}{1-\beta}+1\right)^{-k_1}} \leq 2^{k_1\left(\frac{M}{1-\beta}+1\right)^{-k_1}} \leq \frac{1}{2},$$

which is valid even for $M \in (0, 1)$ when k_1 is sufficiently large, This is a valid replacement since it is argued earlier on the same page that k_1 is unbounded and only large values of k_1 are important to the proof.

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RESEARCH

Transcendence of certain sequences of algebraic numbers

Mathias L. Laursen

*Correspondence:
mll@math.au.dk
Department of Mathematics, Aarhus University, Ny Munkegade 118, 8000 Aarhus C, Denmark

Abstract

Using Schmidt's Subspace Theorem, this paper improves and extends an existing transcendence result for sequences of algebraic numbers. The theorems thus produced correspond to a central theorem on the irrationality of sequences due to Erdős.

Keywords: Transcendence, Irrationality, Schmidt's Subspace Theorem, Series of algebraic numbers, Algebraic degree

Mathematics Subject Classification: 11J81, 11J87

1 Introduction and main results

Proving whether a given real number is algebraic, or even rational, can be a quite frustrating endeavour. While more than a century and a half have passed since Hermite proved that the number $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ is transcendental in early 1873, it remains unsolved if the number $\sum_{n=0}^{\infty} \frac{1}{n!+1}$ is even irrational, despite what may appear as a much similar construction. Similarly, it is well-known that the Riemann ζ function defined as $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $\Re(s) > 1$ is transcendental when s is a positive even integer while the question of irrationality remains open when $s \geq 5$ is any fixed odd integer. In other words, we have a multitude of interesting numbers that we know to be transcendental but where a small perturbation to the infinite series used to describe them renders even the question of irrationality exceedingly hard to settle. Aiming away from frustrations of this kind, this paper studies irrationality and transcendence criteria that are less sensitive to such perturbations.

Following the notions of Erdős and Graham [1] (respectively Hančl [2]), we say that a sequence $\{a_n\}_{n=1}^{\infty}$ of real or complex numbers is irrational (respectively transcendental) if the sum of the series $\sum_{n=1}^{\infty} \frac{1}{a_n c_n}$ is irrational (respectively transcendental) for any sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers. An early and central result on irrational sequences was proven in 1975 by Erdős [3].

Theorem 1.1 (Erdős) *Let $\varepsilon > 0$, and let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence of integers such that $a_n \geq n^{1+\varepsilon}$ for all n. Suppose*

$$\limsup_{n\to\infty} a_n^{2^{-n}} = \infty.$$

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Then the sequence $\{a_n\}_{n=1}^{\infty}$ *is irrational.*

As noted in [3], the number 2 in the theorem is best possible, in the sense that there exist sequences $\{a_n\}_{n=1}^{\infty}$ of positive integers that satisfy $\limsup_{n\to\infty} a_n^{A^{-n}} = \infty$ for all $A < 2$ while the sum of the series $\sum_{n=1}^{\infty} 1/a_n$ is rational. Still, much effort has been applied to extend this result (see [4] for a broader overview). One such result is the below theorem by Hančl [5], which gives a corresponding condition for a sequence of (not necessarily integral) rational numbers to be transcendental.

Theorem 1.2 (Hančl) *Let* $\gamma > 2\varepsilon > 0$ *and* $1 > \alpha > \frac{\log(3+2\varepsilon)}{\log(3+\gamma)}$, *and let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be sequences of positive integers such that*

$$n^{1+\varepsilon} < a_n \le a_{n+1}, \quad \limsup_{n\to\infty} a_n^{(3+\gamma)^{-n}} = \infty,$$

and

$$b_n < a_n^{\varepsilon/(1+\varepsilon)} 2^{-\log_2^{\alpha} a_n}. \tag{1}$$

Then the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ *is transcendental.*

One way to extend on this result is to broaden the family of numbers that may be contained by the sequence to a greater class of algebraic numbers. The best known result in this direction is due to Andersen and Kristensen [6], which gives sufficient conditions for bounding the algebraic degree from below. In their theorem, they use the notion of algebraic integers, which are defined as the algebraic numbers that have a monic minimal polynomial over the integers. Given an algebraic extension K of $\mathbb{Q}$, we will use $\mathcal{O}_K$ to denote the set of algebraic integers contained in K. Recall that $\mathcal{O}_K$ forms a subring of K. They also use the notion of a *house*, written as $\overline{|a|} := \max_{1\le j\le d} |a^{(j)}|$, where $a^{(1)}, \ldots, a^{(d)}$ denotes the conjugates of an algebraic number a, i.e., the roots of its minimal polynomial over $\mathbb{Z}$. In terms of irrational and transcendental sequences, the theorem reduces to the below result.

Theorem 1.3 (Andersen–Kristensen) *Let* $d \in \mathbb{N}$ *be a positive integer, and let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of algebraic integers of degree* $\deg a_n \le d$ *such that*

$$n^{1+\varepsilon} \le \overline{|a_n|} = |a_n| \le |a_{n+1}|.$$

Suppose $\Re(a_n) > 0$ *for all n or that* $\Im(a_n) > 0$ *for all n. If*

$$\limsup_{n\to\infty} |a_n|^{\prod_{i=1}^{n-1}(d^i+d)^{-1}} = \infty,$$

then $\{a_n\}_{n=1}^{\infty}$ *is irrational. Furthermore, if for all* $D \in \mathbb{N}$,

$$\limsup_{n\to\infty} |a_n|^{D^{-n}\prod_{i=1}^{n-1}(d^i+d)^{-1}} = \infty,$$

then $\{a_n\}_{n=1}^{\infty}$ *is transcendental.*

Note that the restriction on $\Re(a_n)$ or $\Im(a_n)$ corresponds to the restriction that a_n be positive in Theorem 1.1. Furthermore, as can be seen in the proof, the somewhat extensive assumptions on the divergence of the limsup of a_n is in part due to the fact that each successive a_N may potentially increase the algebraic degree of $\sum_{n=1}^{N} 1/a_n$ by a factor of d. By assuming the a_n to come from a fixed number field K, the limsup conditions would thus be much weakened, replacing the product $\prod_{i=1}^{n-1}(d^i + d)^{-1}$ with $(2d)^{-n}$. The main results of this paper are improvements to this result when all a_n are contained in the same number field, in terms of allowing sequences with non-integral elements, replacing the restriction $\overline{|a_n|} = |a_n|$ with much weaker conditions, and weakening the limsup criteria. In our first result, below, we assume a_n to be rational but allow b_n to attain certain algebraic irrational values. This will be proven in Sect. 4.

Theorem 1.4 *Let K be a number field of dimension $d \geq 2$, and let $x_1, \ldots, x_D \in K$. Consider real numbers $\varepsilon > 0$, $0 < \alpha < 1 \leq \gamma$, and $\beta \in [0, \frac{\varepsilon}{1+\varepsilon})$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that*

$$n^{1+\varepsilon} \leq a_n \leq a_{n+1}, \tag{2}$$

and let $\{b_n\}_{n=1}^{\infty}$ be a sequence of non-zero numbers so that $b_n = \sum_{i=1}^{D} b_{i,n} x_i$ for suitable $b_{i,n} \in \mathbb{Z}$. Suppose for n sufficiently large, $i = 1, \ldots, D$, and some fixed $\zeta \in \mathbb{C}$,

$$|b_n| \leq a_n^{\beta} 2^{\log_2^{\alpha} a_n}, \quad |b_{i,n}| \leq a_n^{\gamma} 2^{\log_2^{\alpha} a_n}, \tag{3}$$

and

$$\Re(\zeta b_n) > 0. \tag{4}$$

Then the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ is irrational if

$$\limsup_{n\to\infty} a_n^{\left(\frac{d\gamma}{1-\beta}+1\right)^{-n}} = \infty,$$

and it is transcendental if

$$\limsup_{n\to\infty} a_n^{\left(\frac{d^2\gamma}{1-\beta}+1\right)^{-n}} = \infty. \tag{5}$$

Remark 1.5 In the proof of the theorem, assumption (4) will only be used once and only to ensure that the partial sums $\sum_{n=1}^{N} \frac{b_n}{a_n}$ do not take the same value infinitely often (see the proof of Proposition 4.3 later in this paper). Therefore, assumption (4) can be replaced by any other assumption that preserves this property.

The main novelty of this result is the improved transcendence criterion, which relies on Schmidt's Subspace Theorem along with ideas from [7] to exclude near all algebraic numbers as possible values for the sum of $\sum_{n=1}^{\infty} \frac{1}{a_n c_n}$, leaving only a finite field of potential algebraic values to be dealt with in the spirit of [6].

Since the above theorem assumes a_n to be rational, much of the arithmetic information regarding the number a_n/b_n – such as its algebraic degree – is carried solely by b_n. This is in some contrast to Theorems 1.1 and 1.3, which can be viewed as having b_n constantly 1, so that all arithmetic information is stored in a_n alone. By modifying the proof of Theorem 1.4 in order to get a result where we again have most of the arithmetic information carried by a_n, we reach the below result. Unfortunately, this version of the theorem is a bit more complicated to read, which is in part due to the method of proof as it requires $\sum_{n=1}^{N} \frac{b_n}{a_n c_n}$ to be written as a $\mathbb{Q}$-linear combination of the x_i – something that is more easily and neatly done when the a_n are guaranteed to be rational.

In the theorem and for the rest of this paper, $\mathcal{N} : \overline{\mathbb{Q}} \to \mathbb{Q}$ denotes the map that sends each algebraic number to the product of its algebraic conjugates, and $\mathcal{N}_K : K \to \mathbb{Q}$

denotes the field norm for the finite extension $K \subseteq \mathbb{Q}$. Notice that $\mathcal{N}_K(a) = \mathcal{N}(a)^{d/\deg a}$ for all $a \in K$, where d denotes the degree of the extension $K \subseteq \mathbb{Q}$.

Theorem 1.6 *Let K be a number field of dimension d, and let $x_1, \dots, x_D \in K$. Consider real numbers $\alpha, \delta, \varepsilon > 0$, $\beta, \eta_1 \geq 0$, and $\eta_2, y \geq 1$ such that $\alpha < 1$, $\beta < \varepsilon/(1+\varepsilon)$, and $\eta_1 \leq (d-1)y+\beta$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-zero numbers given by $a_n = \sum_{i=1}^{D} a_{i,n}x_i$ with $a_{i,n} \in \mathbb{Z}$ such that for all sufficiently large n,*

$$n^{1+\varepsilon} \leq |a_n| \leq |a_{n+1}|, \tag{6}$$
$$|\mathcal{N}_K(a_n)| \geq |a_n|^{\eta_1} 2^{-\log_2^{\alpha} |a_n|}, \tag{7}$$

and

$$r_n \big|\mathcal{N}(a_n/r_n)\big| \leq |a_n|^{\eta_2} 2^{\log_2^{\alpha} |a_n|}, \tag{8}$$

where $r_n := \gcd(a_{1,n}, \dots, a_{D,n})$. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that for some fixed $\zeta \in \mathbb{C}$, each $i = 1, \dots, D$, and all sufficiently large n,

$$b_n \leq |a_n|^{\beta} 2^{\log_2^{\alpha} |a_n|}, \quad |a_{i,n}| \leq |a_n|^{y} 2^{\log_2^{\alpha} |a_n|},$$

and $\Re(\zeta a_n) > 0$. Then the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ is irrational if

$$\limsup_{n\to\infty} |a_n|^{\left(\frac{d(y+\beta)}{1-\beta}+1\right)^{-n}} = \infty,$$

and it is transcendental if

$$\limsup_{n\to\infty} |a_n|^{\left(\frac{\eta_2+d((d-1)y+\beta+\eta_2-\eta_1)+\delta}{1-\beta}+1\right)^{-n}} = \limsup_{n\to\infty} |a_n|^{\left(\frac{d^2(y+\beta)}{1-\beta}+1\right)^{-n}} = \infty.$$

In the proof of Theorem 1.6, it makes little difference if we also allow b_n to be irrational. Doing so leads to the below generalization, which we will prove in Sect. 5.

Theorem 1.7 *Let K be a number field of dimension d, and let $x_1, \dots, x_D \in K$. Consider real numbers $\alpha, \delta, \varepsilon > 0$, $\beta, \eta_1 \geq 0$, $\eta_2, y_1 \geq 1$, and $y_2 \geq \beta$ such that $\alpha < 1$, $\beta < \varepsilon/(1+\varepsilon)$, and $\eta_1 \leq (d-1)y_1 + y_2$. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of non-zero numbers given by $a_n = \sum_{i=1}^{D} a_{i,n}x_i$ and $b_n = \sum_{i=1}^{D} b_{i,n}x_i$ with $a_{i,n}, b_{i,n} \in \mathbb{Z}$ such that inequalities (6), (7), and (8) are satisfied for n sufficiently large. For each $i = 1, \dots, D$, and some fixed $\zeta \in \mathbb{C}$, suppose additionally that*

$$|a_{i,n}| \leq |a_n|^{y_1} 2^{\log_2^{\alpha} |a_n|}, \quad |b_{i,n}| \leq |a_n|^{y_2} 2^{\log_2^{\alpha} |a_n|}, \tag{9}$$
$$|b_n| \leq |a_n|^{\beta} 2^{\log_2^{\alpha} |a_n|}, \tag{10}$$

and

$$\Re(\zeta a_n/b_n) > 0, \tag{11}$$

when n is sufficiently large. Then the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ is irrational if

$$\limsup_{n\to\infty} |a_n|^{\left(\frac{d(y_1+y_2)}{1-\beta}+1\right)^{-n}} = \infty,$$

and it is transcendental if

$$\limsup_{n\to\infty} |a_n|^{\left(\frac{\eta_2+d((d-1)y_1+y_2+\eta_2-\eta_1)+\delta}{1-\beta}+1\right)^{-n}} = \limsup_{n\to\infty} |a_n|^{\left(\frac{d^2(y_1+y_2)}{1-\beta}+1\right)^{-n}} = \infty. \tag{12}$$

Remark 1.8 Similarly to Theorem 1.4, the assumption (11) can be replaced with any other assumption that ensures that the partial sums $\sum_{n=1}^{N} \frac{b_n}{a_n}$ do not have the same value infinitely often (see the proof of Proposition 5.3 later in this paper).

Remark 1.9 Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ satisfy the assumptions of either of Theorems 1.4, 1.6, and 1.7 for some choice of $x_1, \ldots, x_D$. If $x'_1, \ldots, x'_{D'} \in K$ such that a_n and b_n lie in the $\mathbb{Q}$-linear span of these numbers, then $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ satisfy the assumptions of the same theorem with $x'_1/Q, \ldots, x'_{D'}/Q$ instead of $x_1, \ldots, x_D$, where Q is a positive integer that depends only on $x_1, \ldots, x_D$ and $x'_1, \ldots, x'_{D'}$.

To see that this is indeed the case, pick one of Theorems 1.4, 1.6, and 1.7, use the notation from that theorem, and assume the conditions are satisfied. Let d' denote the dimension of the $\mathbb{Q}$-linear span of $x'_1, \ldots, x_{D'}$. By renumbering if necessary, we may assume that $x'_1, \ldots, x'_{d'}$ are linearly independent. Let $\tilde{x}_1, \ldots \tilde{x}_d$ be a $\mathbb{Q}$-linear basis of K so that $\tilde{x}_j = x'_j$ for $1 \le j \le d'$, and write $x_i = \sum_{j=1}^{d} \frac{p_{i,j}}{q_{i,j}} \tilde{x}_j$, for suitable choices of $p_{i,j} \in \mathbb{Z}$ and $q_{i,j} \in \mathbb{N}$. Pick $Q = \prod_{i=1}^{D} \prod_{j=1}^{d} q_{i,j}$. Let ξ denote either letter a or b so that ξ_n is not assumed to be a positive integer by the chosen theorem. Seeing that

$$\xi_n = \sum_{i=1}^{D} \xi_{i,n} x_i = \sum_{j=1}^{d} \sum_{i=1}^{D} \frac{Q}{q_{i,j}} p_{i,j} a_{i,n} \frac{\tilde{x}_j}{Q},$$

write $\tilde{\xi}_{j,n} = \sum_{i=1}^{D} \frac{Q}{q_{i,j}} p_{i,j} \xi_{i,n}$, and set $\xi'_{j,n} = \tilde{\xi}_{j,n}$ for $1 \le j \le d'$ and $\xi'_{j,n} = 0$ for $d' < j \le D'$. Since each ξ_n is contained in the span of $x'_1, \ldots, x'_{D'}$ (and so in the span of $\tilde{x}_1 = x'_1, \ldots, \tilde{x}_{d'} = x'_{d'}$), it follows that $\tilde{\xi}_{j,n} = 0$ for $d' < j \le d$, and so

$$\xi_n = \sum_{j=1}^{d} \tilde{\xi}_{j,n} \frac{\tilde{x}_j}{Q} = \sum_{j=1}^{D'} \xi'_{j,n} \frac{x'_j}{Q},$$

while

$$\max_{1\le j\le D'} |\xi'_{j,n}| = \max_{1\le j\le d} |\tilde{\xi}_{j,n}| \le \sum_{i=1}^{D} \frac{Q}{q_{i,j}} |p_{i,j}||\xi_{i,n}| \le DQ \max_{i,j} |p_{i,j}| \max_{1\le i\le D} |\xi_{i,n}|.$$

By replacing α with $(1+\alpha)/2$, it follows that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ satisfy the assumptions of the chosen theorem with $x'_1/Q, \ldots, x'_{D'}$ instead of $x_1, \ldots, x_D$.

Notice that the sum $y_1 + y_2$ in Theorem 1.7 corresponds to $y + \beta$ in Theorem 1.4. As such, one should not expect to be able to derive Theorem 1.4 as a corollary to Theorem 1.7. This is further underlined by Example 2.3 in section 2. Similarly, as will be seen from Example 2.7, there are cases where Theorem 1.7 is applicable while the other two theorems are not.

2 Examples

We will now go through a few applications of the main theorems in order to better understand the strengths and differences of applicability between them. For this purpose, we will say that a theorem is immediately applicable to a sequence $\{x_n\}_{n=1}^{\infty}$ if there are sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ that satisfy $x_n = a_n/b_n$ and the assumptions of the theorem.

For these examples, we make use of the Fibonacci sequence F_n, defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$, along with the golden ratio $\varphi = (1+\sqrt{5})/2$ and its conjugate $\bar{\varphi} = (1-\sqrt{5})/2 = -\varphi^{-1}$. Recall that $\varphi^n = F_n\varphi + F_{n-1}$ and $\bar{\varphi}^n = F_n\bar{\varphi} + F_{n-1}$ for each $n \in \mathbb{N}$.

The first example, below, shows the strengths of Theorem 1.4 in terms of providing transcendence of K-linear combinations of multiple series of rational numbers when K is a suitable number field.

Example 2.1 Let x be any algebraic number of degree at most 2, and let $\{c_n\}_{n=1}^{\infty}$ be a sequence of positive integers. Then

$$x\sum_{n=1}^{\infty}\frac{1}{F_{9^n n}c_n} + \sum_{n=1}^{\infty}\frac{1}{F_{9^n n+1}c_n}$$

is a transcendental number. To see this, write

$$x\sum_{n=1}^{\infty}\frac{1}{F_{9^n n}c_n} + \sum_{n=1}^{\infty}\frac{1}{F_{9^n n+1}c_n} = \sum_{n=1}^{\infty}\frac{F_{9^n n+1}x + F_{9^n n}}{F_{9^n n}F_{9^n n+1}c_n}.$$

Aiming to use Theorem 1.4, pick $x_2 = x$, $\beta = 1/2$, $y = 1$, and any $0 < \alpha < 1 < \varepsilon$. Suppose $x \neq \bar{\varphi}$. The transcendence follows if we can find $\zeta \in \mathbb{C}$ such that $\Re(\zeta(F_{9^n n+1}x + F_{9^n n})) > 0$ for all sufficiently large n. If $\Im(x) \neq 0$, pick $\zeta = -i\Im(x)$. Otherwise, pick $\zeta = x - \bar{\varphi}$, as then

$$\zeta(F_{9^n n+1}x + F_{9^n n}) = F_{9^n n}(x-\bar{\varphi})\left(x + \frac{F_{9^n n-1}}{F_{9^n n}}\right) \tag{13}$$

and $\lim_{n\to\infty} F_{9^n n-1}/F_{9^n n} = 1/\varphi = -\bar{\varphi}$ ensure that $\zeta(F_{9^n n+1}x + F_{9^n n})$ is a positive real number when n is sufficiently large, and we are done.

This leaves us with the case of $x = \bar{\varphi}$, where we have

$$F_{9^n n+1}\bar{\varphi} + F_{9^n n} = \bar{\varphi}^{9^n n+1}.$$

While we have no hope of getting $\Re(\zeta\bar{\varphi}^{9^n n+1}) > 0$ for all large n, Remark 1.5 allows us to ignore this if we can show that

$$s_N := \sum_{n=1}^{N-1}\frac{\bar{\varphi}^{9^n n+1}}{F_{9^n n}F_{9^n n+1}}.$$

does not take the same value for infinitely many N. To see this, we let $M > N$ and use the converse triangle inequality to find

$$|s_N - s_M| = \left|\sum_{n=N}^{M}\frac{\bar{\varphi}^{9^n n+1}}{F_{9^n n}F_{9^n n+1}}\right| > 0,$$

for all sufficiently large N, and the example is complete.

The next example shows Theorem 1.4 is not easily replaced by Theorem 1.7 in the above example. For this purpose, we will need a simple lemma, which will be proven in Sect. 3, right after Lemma 3.6.

Lemma 2.2 *Let x be a fixed non-zero algebraic number, and let $a, b \in \mathbb{Z}$. Then there is a constant $C > 0$, depending only on x, so that $|a + bx| \geq C \max\{|a|, |b|, 1\}^{-2\deg x}$ when $a + bx \neq 0$.*

Example 2.3 In Example 2.1, Theorem 1.7 would not have been immediately applicable on the sequences that appears when $x \neq \bar{\varphi}$ is quadratic irrational. To see this, notice that we must have

$$a_n = F_{9^n n} F_{9^n n+1} \tilde{a}_n \quad \text{and} \quad b_n = (F_{9^n n+1} x + F_{9^n n}) \tilde{a}_n,$$

for some suitable sequence of $\tilde{a}_n \in \overline{\mathbb{Q}}$ such that κa_n and κb_n are all algebraic integers for some fixed $\kappa \in \mathbb{N}$. If $\tilde{a}_n \notin \mathbb{Q}(x)$ for some n, then we get $d \geq 4$ and so

$$\frac{d^2(y_1 + y_2)}{1 - \beta} + 1 \geq 17 > 9,$$

making Theorem 1.7 inapplicable, so we assume $\tilde{a}_n \in \mathbb{Q}(x)$ for all n. Due to Eq. (13), it follows that

$$y_2 \geq \beta \geq \frac{\limsup_{n\to\infty} \log|b_n|}{\limsup_{n\to\infty} \log|a_n|} = \frac{1+c}{2+c}, \quad \text{where } c = \limsup_{n\to\infty} \frac{\log|\tilde{a}_n|}{\log F_{9^n n}}.$$

Note that we need $c > -2$ in order to have a y_1 that satisfies the bound on $a_{i,n}$ for all n. By Remark 1.9, we may assume $D = 2$, $x_1 = 1$, and $x_2 = x$. Writing $\tilde{a}_n = \tilde{a}_{1,n} + \tilde{a}_{2,n} x$ with $\tilde{a}_{1,n}, \tilde{a}_{2,n} \in \mathbb{Q}$, we obtain from Lemma 2.2 that

$$|\tilde{a}_n| \geq C \max\{|\tilde{a}_{1,n}|, |\tilde{a}_{2,n}|\}^{-4},$$

where $C > 0$ is a constant that depends only on x, and so

$$y_1 \geq \limsup_{n\to\infty} \frac{\log(F_{9^n n} F_{9^n n+1} \max\{|\tilde{a}_{1,n}|, |\tilde{a}_{2,n}|\})}{\log(F_{9^n n} F_{9^n n+1} |\tilde{a}_n|)} \geq \frac{2 - \frac{c}{4}}{2+c}.$$

If $c \leq -1$, then

$$\frac{d^2(y_1 + y_2)}{1 - \beta} + 1 \geq \frac{4\left(\frac{2-c/4}{2+c} + 0\right)}{1 - 0} + 1 = \frac{8 - c}{2 + c} + 1 \geq 9 + 1 > 9.$$

while assuming $c > -1$ yields

$$\frac{d^2(y_1 + y_2)}{1 - \beta} + 1 \geq \frac{4\left(\frac{2-c/4}{2+c} + \frac{1+c}{2+c}\right)}{1 - \frac{1+c}{2+c}} + 1 = 12 + 3c + 1 > 9.$$

This shows that Theorem 1.7 is not immediately applicable and concludes the example.

In the remaining examples, we fix $K = \mathbb{Q}(\sqrt{5}) = \mathbb{Q}(\varphi)$, $D = d = 2$, and $x_1 = 1$, while the value of x_2 will be chosen as either φ or $\bar{\varphi}$. The aim of these examples is to show some more simple applications of the theorems while also highlighting the differences in applicability.

Example 2.4 The sequence $\{n^{5^n}\varphi^n\}_{n=1}^{\infty}$ is transcendental. This follows from Theorem 1.6 with $x_2 = \varphi$, $\beta = 0$, $y = \eta_1 = \eta_2 = 1$, and arbitrary $\alpha, \delta, \varepsilon \in (0, 1)$. Alternatively, one could apply Theorem 1.4 with $x_2 = \bar{\varphi}$, $\beta = 0$, $y = 1$, and any $\alpha, \varepsilon \in (0, 1)$ upon rewriting to

$$n^{5^n}\varphi^n = \frac{n^{5^n}}{(-\bar{\varphi})^n} = \frac{n^{5^n}}{(-1)^n(F_n\bar{\varphi} + F_{n-1})}.$$

Example 2.5 The sequence $\{\varphi^{7^n}\}_{n=1}^{\infty}$ is transcendental. This follows from Theorem 1.6 with $x_2 = \varphi$, $\beta = \eta_1 = 0$, and $\eta_2 = y = 1$. Note that if we wished to immediately apply Theorem 1.4, we would be left with

$$b_n = a_n\varphi^{-7^n} = a_n(F_{7^n-1} - F_{7^n}\varphi^{-1}).$$

Now, if $a_n \leq F_{7^n}$, we have $\beta = 0$ but must take $y \geq 2$, which means that the limsup criterion is not satisfied since $d^2y/(1-\beta) + 1 \geq 9 > 7$. If $a_n \geq F_{7^n}$ is large enough, we may achieve $y = 2 - \gamma$ for some $0 \leq \gamma < 1$, but then it is easily shown that we get $\beta \geq \gamma$, so that

$$\frac{d^2y}{1-\beta} + 1 \geq 9 > 7,$$

and the divergence criterion remains unsatisfied.

Example 2.6 The sequence $\{F_{9^n n+1}\varphi^{9^n n}/F_{9^n n}\}_{n=1}^{\infty}$ is transcendental. This follows by rewriting into

$$\frac{F_{9^n n+1}\varphi^{9^n n}c_n}{F_{9^n n}} = \frac{F_{9^n n+1}}{F_{9^n n}(-\bar{\varphi})^{9^n n}} = \frac{F_{9^n n+1}}{F_{9^n n}(-1)^n(F_{9^n n-1} + F_{9^n n}\bar{\varphi})},$$

and then applying Theorem 1.4 with $x_2 = \bar{\varphi}$, $\beta = 0$, and $y = 2$ or, alternatively, Theorem 1.7 with $x_2 = \bar{\varphi}$, $\beta = 0$, $\eta_1 = \eta_2 = y_1 = 1$, and $y_2 = 2$. On the other hand, Theorem 1.6 is not immediately applicable, as can be seen through similar arguments to those in Example 2.5.

In our final example, below, we consider a sequence where only Theorem 1.7 is applicable. This will also serve as an example that we may encounter

$$|a_n|^{\left(\frac{\eta_2 + d((d-1)y_1 + y_2 + \eta_2 - \eta_1) + \delta}{1-\beta} + 1\right)^{-n}} > |a_n|^{\left(\frac{d^2(y_1+y_2)}{1-\beta} + 1\right)^{-n}}$$

for some sequences, though it should be mentioned that there appears to be no connection between this inequality and the applicability of the other theorems.

Example 2.7 The sequence $\{\varphi^{2\cdot 14^n}/(F_{14^n} + \varphi)\}_{n=1}^{\infty}$ is transcendental. This follows from Theorem 1.7 by taking $\eta_1 = 0$, $\beta = y_1 = 1/2$, and $\eta_2 = y_2 = 1$. Here, neither one of Theorems 1.4 and 1.6 is immediately applicable, as seen through similar arguments to those in Example 2.5.

3 Preliminaries

A central tool for proving the main results is Schmidt's Subspace Theorem [8], below.

Theorem 3.1 (Schmidt) *Let $L_1, \ldots, L_d$ be $\mathbb{Q}$-linearly independent linear forms in d variables with algebraic coefficients. For any $\delta > 0$, there exists a finite collection of proper subspaces $T_1, \ldots, T_w \subsetneq \mathbb{Q}^d$ such that any $x \in \mathbb{Z}^d$ with*

$$|L_1(x)\cdots L_n(x)| < |x|^{-\delta}$$

is contained in $\bigcup_{i=1}^{w} T_i$.

This theorem will be used together with the following lemma, which is found in a paper by Hančl, Nair, and Šustek [7]. The M used in the present version of the lemma equals 1 plus the M used in the original. [7] also had additional assumptions for the lemma (such as the a_n being integers and M having a greater lower bound), but those assumptions were never used in the proof and were only there for the sake of the main theorem of that paper.

Lemma 3.2 *Consider real numbers $\varepsilon > 0$, $0 < \alpha < 1 \leq M$, and $\beta \in [0, \frac{1+\varepsilon}{\varepsilon})$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that*

$$n^{1+\varepsilon} \leq a_n \leq a_{n+1} \quad \text{and} \quad \limsup_{n\to\infty} a_n^{\left(\frac{M}{1-\beta}+1\right)^{-n}} = \infty.$$

Let $x_1, \ldots, x_d \in \mathbb{C}$, let $y_1, \ldots, y_d \geq 1$, and let $\{b_n\}_{n=1}^{\infty}$ be a sequence of complex numbers with $b_n = \sum_{i=1}^{d} x_i b_{i,n}$ for some $b_{1,n}, \ldots, b_{d,n} \in \mathbb{Z}$, such that for all sufficiently large n and each $i = 1, \ldots, d$,

$$b_n \leq a_n^{\beta} 2^{\log_2^{\alpha} a_n}$$

and

$$|b_{i,n}| \leq |a_n|^{y_i} 2^{\log_2^{\alpha} |a_n|}.$$

Finally, let $\{c_n\}_{n=1}^{\infty}$ be a sequence of positive integers. Then there is a positive real number $E > 0$ such that the inequality

$$\left| \sum_{n=1}^{\infty} \frac{b_n}{a_n c_n} - \frac{\sum_{i=1}^{d} p_i x_i}{q} \right| < \frac{1}{(\log_2^2 q) 2^{d \log_2^{(1+2\alpha)/3} q} q^M} \tag{14}$$

has infinitely many solutions $(p_1, \ldots, p_d, q) \in \mathbb{Z}^d \times \mathbb{N}$ satisfying

$$|p_i| \leq E 2^{\log_2^{(1+2\alpha)/3} q} q^{y_i}, \quad \text{for all } i = 1, \ldots, d. \tag{15}$$

Another central tool for the proofs is the following lemma, which is a slight strengthening of another lemma from [7] and follows from a much similar proof. For clarity, we will go through the proof in Sect. 6.

Lemma 3.3 *Let $\varepsilon > 0$, $0 < \alpha < 1 \leq M$, and $\beta \in [0, \frac{1+\varepsilon}{\varepsilon})$. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of positive real numbers such that for all sufficiently large n,*

$$n^{1+\varepsilon} \leq a_n \leq a_{n+1}, \qquad b_n \leq a_n^{\beta} 2^{\log_2^{\alpha} a_n},$$

and

$$\limsup_{n\to\infty} a_n^{\left(\frac{M}{1-\beta}+1\right)^{-n}} = \infty. \tag{16}$$

Let $0 < c < 1$ *be fixed. Then*

$$\liminf_{N\to\infty} \left(2^{N^2 \log_2^c a_{N-1}} \left(\prod_{n=1}^{N-1} a_n^M \right) \sum_{n=N}^{\infty} \frac{b_n}{a_n} \right) = 0.$$

We now present a few notions from algebraic number theory that will be relevant in the proofs of the main theorems. Let a be an algebraic number with minimal polynomial $\sum_{i=0}^{d} c_i X^i$ over the integers ($c_d > 0$). The leading coefficient, c_d, is also called the *denominator* of a, since $c_d a$ is an algebraic integer while $c'a$ is not for any rational integer $0 < c' < c_d$. By rewriting the minimal polynomial of a as $c_d \prod_{i=1}^{d}(X - a_i)$ instead, the Mahler measure of a is defined as

$$M(a) := c_d \prod_{i=1}^{d} \max\{1, |a_i|\}.$$

Surprisingly closely related to this is the Weil height, which we define as

$$H(\alpha) := \prod_{v\in M_K} \max\{1, |a|_v\}^{[K_v:\mathbb{Q}_v]/[K:\mathbb{Q}]},$$

where K is any number field containing a, M_K denotes the set of places of K, K_v is the local field of K at v, and $[K : L]$ denotes the degree of a field extension $K \supseteq L$. This does not depend on the choice of K (see [9] for a proof). We will compare and estimate the house, Mahler measure, and Weil height using the following classical results.

Lemma 3.4 *Let a be an algebraic number with denominator* c_d. *Then*

$$H(a)^d = M(a) \le |c_d| \max\{ \overline{|a|}^{\,d}, 1\}.$$

Proof The inequality is a trivial consequence of the definitions. For the equality, see [9, Lemma 3.10]. □

Lemma 3.5 *Let* $a, b \in \overline{\mathbb{Q}}$ *with* $a \neq 0$. *Then*

$$H(a+b) \le 2H(a)H(b), \quad H(ab) \le H(a)H(b),$$
$$H(1/a) = H(a).$$

Proof See [9]. □

Lemma 3.6 (Liouville Inequality) *Let a, b be non-conjugate algebraic numbers. Then*

$$|a-b| \ge \left(2H(a)H(b)\right)^{-\deg(a)\deg(b)}.$$

Proof This can be extracted from [10, Theorem A.1]. □

Proof of Lemma 2.2 Pick $C = \min\{(2H(x))^{-\deg x}, |x|\}$. If a or b is 0, the statement is trivially true. If $ab \neq 0$, then we may apply Lemmas 3.6 and 3.5 to conclude

$$\begin{aligned}|a+bx| &\geq (2H(a)H(-bx))^{-\deg(a)\deg(-bx)} \geq (2H(a)H(b)H(x))^{-\deg(x)}\\ &= (2H(x))^{-\deg(x)}|ab|^{-\deg x} \geq C\max\{|a|,|b|\}^{-\deg x}.\end{aligned}$$

□

In order to prove Theorem 1.7, we will need a different version of Lemma 3.2, the proof of which will use some elementary of Galois theory. Recall that a field extension $K \supseteq \mathbb{Q}$ is called a Galois extension if all irreducible polynomials over $\mathbb{Q}$ that have a root in K split into linear factors over K. Note that this is equivalent to K being closed under conjugation. We use $\mathrm{Gal}(K)$ to denote the associated Galois group, i.e., the field automorphisms on K that preserve $\mathbb{Q}$. Recall that for any finite field extension $L \supseteq \mathbb{Q}$, L has a unique finite field extension $K \supseteq L$ of minimal degree such that $K \supseteq \mathbb{Q}$ is Galois (see, e.g., Theorem 11.6 of [11]). This also implies the below lemma, which will be relevant for the proofs of both Theorems 1.4 and 1.7.

Lemma 3.7 *Let $a_1,\dots,a_d \in \overline{\mathbb{Q}}$. Then there is a constant C, depending only on $a_1,\dots,a_d$, so that for any $(c_1,\dots,c_d) \in \mathbb{Q}^d$,*

$$\boxed{c_1a_1 + c_2a_2 + \cdots + c_da_d} \leq C \max_{1\leq i\leq d} |c_i|.$$

Proof Let $K \supseteq \mathbb{Q}$ be the smallest Galois extension of $\mathbb{Q}$ containing $a_1,\dots,a_d$. Since conjugation is a field automorphism on $\overline{\mathbb{Q}}$, and K is closed under conjugation, we find

$$\boxed{\sum_{i=1}^{d} c_ia_i} \leq \max_{\psi\in\mathrm{Gal}(K)} \left|\psi\left(\sum_{i=1}^{d} c_ia_i\right)\right| = \max_{\psi\in\mathrm{Gal}(K)} \left|\sum_{i=1}^{d} c_i\psi(a_i)\right|.$$

The proof is then completed by an application of the triangle inequality,

$$\boxed{\sum_{i=1}^{d} c_ia_i} \leq \left(d \max_{\psi\in\mathrm{Gal}(K)} \max_{0\leq i\leq d} |\psi(a_i)|\right) \max_{0\leq i\leq d} |c_i|.$$

□

4 Proof of Theorem 1.4

We will first prove the below result, which is inspired by the main theorem of [7].

Theorem 4.1 *Let $d \in \mathbb{N}$ be a positive integer, and consider real numbers $\delta, \varepsilon > 0$, $0 < \alpha < 1 \leq y_1,\dots,y_n$, and $\beta \in [0, \frac{\varepsilon}{1+\varepsilon})$. Let furthermore $x_1,\dots,x_d$ be algebraic numbers, and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers that satisfy inequality (2) and*

$$\limsup_{n\to\infty} a_n^{\left(\frac{1+\sum_{i=1}^{d} y_i+\delta}{1-\beta}+1\right)^{-n}} = \infty.$$

Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of non-zero numbers given by $b_n = \sum_{i=1}^{d} b_{i,n}x_i$ where $b_{i,n} \in \mathbb{Z}$ and such that the inequalities of inequality (3) are satisfied for n sufficiently large and each $i = 1,\dots,d$. Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of positive integers. Then the number $\sum_{n=1}^{\infty} \frac{b_n}{a_nc_n}$ is either transcendental or a $\mathbb{Q}$-linear combination of $x_1,\dots,x_d$.

The main difference between the proof of this theorem and that of the corresponding one in [7] lies in the below application of Theorem 3.1, which replaces [7, Lemma 7].

Lemma 4.2 *Let $x_1, \dots, x_d, s$ be algebraic numbers such that s is $\mathbb{Q}$-linearly independent of $x_1, \dots, x_d$, and let $\delta > 0$. Then the inequality,*

$$\left| qs - \sum_{i=1}^{d} p_i x_i \right| \prod_{i=1}^{d} \max\{1, |p_i|\} < q^{-\delta}, \tag{17}$$

has only finitely many solutions $(p_1, \dots, p_d, q) \in \mathbb{Z}^d \times \mathbb{N}$.

Proof This will be proven by induction, using the convention that linear independence of the empty set is equivalent to being non-zero. Let S denote the set of solutions $(p_1, \dots, p_d, q) \in (\mathbb{Z} \setminus \{0\})^d \times \mathbb{N}$ to inequality (17). For $d = 0$ or $S = \emptyset$, S is clearly finite, so suppose $d > 0$, $S \neq \emptyset$, and that the lemma is true for $d' = d - 1$.

Note that all elements of S satisfy

$$\left| qs - \sum_{i=1}^{d} p_i x_i \right| \prod_{i=1}^{d} |p_i| < q^{-\delta}.$$

By Theorem 3.1, there is a finite collection of proper subspaces $T_1, \dots, T_w \subsetneq \mathbb{Q}^{d+1}$ such that $S \subseteq \bigcup_{l=1}^{w} T_l$. Write $S_l = S \cap T_l$ for $l = 1, \dots, w$, and let $1 \leq l \leq w$ such that $S_l \neq \emptyset$. Then T_l contains an element with a non-zero q-entry. Since $\dim T_l \leq d$, it follows that there is a j such that the p_j-entry is given as a fixed linear combination of the remaining entries for all elements in T_l. By renumbering if necessary, we may assume that $j = d$ and then pick $r_1, \dots, r_d \in \mathbb{Q}$ such that $p_d = r_1 p_1 + \cdots + r_{d-1} p_{d-1} + r_d q$ for all $(p_1, \dots, p_d, q) \in T_l$. For elements of S_l, inequality (17) now reduces to

$$\begin{aligned} q^{-\delta} &> \left| q(s - x_d r_d) - \sum_{i=1}^{d-1} p_i (x_i + x_d r_i) \right| \prod_{i=1}^{d} \max\{1, |p_i|\} \\ &\geq \left| q(s - x_1 r_1) - \sum_{i=1}^{d-1} p_i (x_i + x_1 r_i) \right| \prod_{i=1}^{d-1} \max\{1, |p_i|\}. \end{aligned}$$

Hence, S_l is finite by induction. Since $S = S_1 \cup \cdots \cup S_w$, this completes the proof. □

Proof of Theorem 4.1 Put $M = 1 + \sum_{i=1}^{d} y_i + \delta$. By Lemma 3.2, there are infinitely many $(p_1, \dots, p_d, q) \in \mathbb{Z}^d \times \mathbb{N}$ satisfying both inequalities (14) and (15), where we may take E to be rational. Rewriting inequality (14) using the above choice of M, we find

$$\left| q \sum_{n=1}^{\infty} \frac{b_n}{a_n c_n} - \sum_{i=1}^{d} p_i x_i \right| \prod_{i=1}^{d} \left(q^{y_i} 2^{\log_2^{(1+2\alpha)/3} q} \right) < q^{-\delta},$$

and it follows from inequality (15) that

$$\left| q E^{-d} \sum_{n=1}^{\infty} \frac{b_n}{a_n c_n} - \sum_{i=1}^{d} p_i E^{-d} x_i \right| \prod_{i=1}^{d} \max\{1, |p_i|\} < q^{-\delta}.$$

Lemma 4.2 then implies that $E^{-d} \sum_{n=1}^{\infty} \frac{b_n}{a_n c_n}$ cannot both be algebraic and $\mathbb{Q}$-linearly independent of $E^{-d} x_1, \dots, E^{-d} x_k$. Since E is rational, this completes the proof. □

To finish the proof of Theorem 1.4, we will use the below proposition to ensure that $\sum_{n=1}^{\infty} \frac{b_n}{a_n c_n}$ is indeed $\mathbb{Q}$-linearly independent of $x_1, \ldots, x_d$.

Proposition 4.3 *Let $d, \tilde{d} \in \mathbb{N}$, let K be a number field of degree d, let $x_1, \ldots, x_D \in K$, and consider real numbers $\varepsilon > 0$, $0 < \alpha < 1$, $\beta \in [0, \frac{\varepsilon}{1+\varepsilon})$, and $y \geq 1$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers that satisfy inequality (2) and*

$$\limsup_{n\to\infty} a_n^{\left(\frac{d\tilde{d}y}{1-\beta}+1\right)^{-n}} = \infty.$$

Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of non-zero numbers given by $b_n = \sum_{i=1}^{d} b_{i,n} x_i$ where $b_{i,n} \in \mathbb{Z}$ and such that inequalities (3) and (4) are satisfied for all sufficiently large n. Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of positive integers. Then the number $\sum_{n=1}^{\infty} \frac{b_n}{a_n c_n}$ has degree strictly greater than $\tilde{d}$.

Proof Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection such that $A_n = a_{\sigma(n)} c_{\sigma(n)}$ is increasing, and put $B_n = b_{\sigma(n)}$ and $B_{i,n} = B_{i,\sigma(n)}$. Since clearly $A_n \geq a_n$ for all n, we get that $\{A_n\}_{n=1}^{\infty}$, $\{B_{i,n}\}_{n=1}^{\infty}$, and $\{B_n\}_{n=1}^{\infty}$ satisfy all assumptions of the proposition. For the remainder of the proof, we may therefore assume that $c_n = 1$ for all n.

Assume towards contradiction that $\deg\left(\sum_{n=1}^{\infty} \frac{b_n}{a_n}\right) \leq \tilde{d}$, and write

$$s = \sum_{n=1}^{\infty} \frac{b_n}{a_n}, \qquad s_N = \sum_{n=1}^{N-1} \frac{b_n}{a_n}.$$

Note that $\deg(s_N) \leq d$, and let c denote the least common multiple of the denominators of $x_1, \ldots, x_D$. Then the denominator of s_N is at most $c \prod_{n=1}^{N-1} a_n$. By Lemmas 3.4 and 3.7,

$$H(s_N) \leq c\left(\prod_{n=1}^{N-1} a_n\right) \overline{|s_N|} \leq C_1 \left(\prod_{n=1}^{N-1} a_n\right) \max_{1\leq i\leq D} \left|\sum_{n=1}^{N-1} \frac{b_{i,n}}{a_n}\right|, \tag{18}$$

where $C_1 > 0$ is some sufficiently large constant depending only on $x_1, \ldots, x_D$. Since $y \geq 1$, the triangle inequality and inequality (3) then imply

$$\left|\sum_{n=1}^{N-1} \frac{b_{i,n}}{a_n}\right| \leq \sum_{n=1}^{N-1} a_n^{y-1} 2^{\log_2^\alpha a_n} \leq N 2^{\log_2^\alpha a_n} a_{N-1}^{y-1},$$

for all sufficiently large N. Applying this to inequality (18) and once again using that $y \geq 1$, we obtain that when N is sufficiently large,

$$H(s_N) \leq C_1 \left(\prod_{n=1}^{N-1} a_n\right) N a_{N-1}^{y-1} 2^{\log_2^\alpha a_{N-1}} \leq 2^{(3d\tilde{d})^{-1} N^2 \log_2^\alpha a_{N-1}} \prod_{n=1}^{N-1} a_n^y. \tag{19}$$

When N grows large, inequality (4) makes $\Re(\zeta s_N)$ strictly increasing. Since s has only finitely many conjugates, s and s_N can thus only be conjugate numbers for finitely many N. When N is sufficiently large, it therefore follows from the triangle inequality and Lemma 3.6 that

$$\sum_{n=N}^{\infty} \left|\frac{b_n}{a_n}\right| \geq |s - s_N| \geq (2H(s)H(s_N))^{-\deg(s)\deg(s_n)},$$

and so, recalling that $\deg s \leq \tilde{d}$ and $\deg s_n \leq d$ while applying inequality (19),

$$\sum_{n=N}^{\infty} \left|\frac{b_n}{a_n}\right| \geq \left(2H(s) 2^{(3d\tilde{d})^{-1} N^2 \log_2 a_{N-1}} \prod_{n=1}^{N-1} a_n^y\right)^{-d\tilde{d}} > \sqrt{2}^{N^2 \log_2^\alpha a_{N-1}} \prod_{n=1}^{N-1} a_n^{-d\tilde{d}y}$$

We conclude that for all large enough N,

$$2^{N^2 \log_2^\alpha a_{N-1}} \left(\prod_{i=1}^{N-1} a_n^{d\tilde{d}y} \right) \sum_{n=N}^{\infty} \left| \frac{b_n}{a_n} \right| > \sqrt{2}^{N^2 \log_2^\alpha a_{N-1}},$$

which contradicts Lemma 3.3 and thus completes the proof. □

Proof of Theorem 1.4 By Remark 1.9, we may assume that $D = d$ and that $x_1, \ldots, x_d$ forms a $\mathbb{Q}$-linear basis of K. The irrationality statement is then simply Proposition 4.3 with $\tilde{d} = 1$, while the transcendence statement follows from Theorem 4.1 and Proposition 4.3 with $\tilde{d} = d$. □

5 Proof of Theorem 1.7

Our first step in proving Theorem 1.7 will be to prove the below parallel result to Theorem 4.1.

Theorem 5.1 *Let K be a number field with $\mathbb{Q}$-linear basis $x_1, \ldots, x_d$, and consider real numbers $\alpha, \varepsilon > 0$, $\beta, \eta_1, y_1 \geq 0$, and $\eta_2, y_2 \geq 1$ such that $\alpha < 1$, $\beta < \varepsilon/(1+\varepsilon)$, and $\eta_1 \leq (d-1)y_1 + y_2$. Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be non-zero sequences in K given by $a_n = \sum_{i=1}^d a_{i,n} x_i$ and $b_n = \sum_{i=1}^d b_{i,n} x_i$ where $a_{i,n}, b_{i,n} \in \mathbb{Z}$ such that*

$$\limsup_{n\to\infty} |a_n|^{\left(\frac{\eta_2 + d((d-1)y_1 + y_2 - \eta_1 + \eta_2) + \delta}{1-\beta} + 1 \right)^{-n}} = \infty$$

and inequalities (6), (7), (8), (9), and (10) are satisfied for each $i = 1, \ldots, d$ and all sufficiently large n. Then the number $\sum_{n=1}^\infty \frac{b_n}{a_n c_n}$ is either transcendental or a $\mathbb{Q}$-linear combination of $x_1, \ldots, x_d$.

This is not quite as neat as Theorem 4.1, and the reason for this is to be found in Lemma 3.2. As part of its proof in [7], the authors write b_n/a_n as a $\mathbb{Q}$-linear combination of $x_1, \ldots, x_d$, which is fairly elegantly done when a_n is rational and less so when a_n may be irrational. Using the Galois theory mentioned by the end of Sect. 3 to make the corresponding modifications, we reach the below lemma.

Lemma 5.2 *Using the notation and assumptions of Theorem 5.1, the inequality*

$$\left| \sum_{n=1}^{\infty} \frac{b_n}{a_n c_n} - \frac{\sum_{i=1}^d p_i x_i}{q} \right| < \frac{1}{2^{d \log_2^{(1+2\alpha)/3} q}\, q^{d+1+d\frac{(d-1)y_1+y_2-\eta_1}{\eta_2}}} \tag{20}$$

has infinitely many solutions $(p_1, \ldots, p_d, q) \in \mathbb{Z}^d \times \mathbb{N}$ satisfying

$$|p_i| \leq 2^{\log_2^{(1+2\alpha)/3} q}\, q^{1+\frac{(d-1)y_1+y_2-\eta_1}{\eta_2}}, \qquad \text{for all } i = 1, \ldots, d. \tag{21}$$

Proof Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection such that the sequence $\{A_n\}_{n=1}^\infty$, given by $A_n = a_{\sigma(n)} c_{\sigma(n)}$, is of increasing modulus. Put $B_{i,n} = b_{i,\sigma(n)}$, $B_n = b_{\sigma(n)}$, and $R_n = r_{\sigma(n)}$. Since these new sequences satisfy the hypothesis of the lemma, we may assume without loss of generality that $c_n = 1$ for all n.

Let $\tilde{K} \supseteq \mathbb{Q}$ be the smallest Galois extension of $\mathbb{Q}$ with $K \subseteq \tilde{K}$. Pick $x_{d+1}, \ldots, x_D$ such that $x_1, \ldots, x_D$ is a $\mathbb{Q}$-linear basis of $\tilde{K}$, and let π_i denote the i'th coordinate map in this basis. Note that $\pi_i(a_n) = a_{i,n}$ and $\pi_i(b_n) = b_{i,n}$ when $1 \leq i \leq d$. Pick $c > 0$ and $\kappa \in \mathbb{N}$ so that

$$c \geq \left|\pi_i\left(\prod_{k=1}^{d'} g_k(x_{j_k})\right)\right| \qquad \text{and} \qquad \kappa\pi_i\left(\prod_{k=1}^{d'} g_k(x_{j_k})\right) \in \mathbb{Z},$$

for all $d' \mid d$, all $i, j_1, \ldots, j_{d'} \in \{1, \ldots, d\}$, and all $g_1, \ldots, g_{d'} \in \mathrm{Gal}(\tilde{K})$.

For each $n \in \mathbb{N}$, pick $\tilde{a}_n \in \mathbb{Z}$ of minimal modulus such that

$$\tilde{a}_n r_n \mathcal{N}(a_n/r_n) \geq |a_n|^{\eta_2},$$

and note by inequality (8) that then

$$\tilde{a}_n r_n \mathcal{N}(a_n/r_n) \leq 2|a_n|^{\eta_2} 2^{\log_2^{\alpha}|a_n|}. \tag{22}$$

Write $d_n = \deg a_n$ and pick $g_1, \ldots, g_{d_n} \in \mathrm{Gal}(\tilde{K})$ such that $(g_k(a_n))_{k=1}^{d_n}$ runs through all d_n conjugates of a_n, with $g_1(a_n) = a_n$. It follows that

$$\begin{aligned}\pi_i\left(\kappa r_n \mathcal{N}\left(\frac{a_n}{r_n}\right)\frac{b_n}{a_n}\right) &= \kappa\pi_i\left(\left(\sum_{j=1}^{d} b_{j,n}x_j\right)\prod_{k=2}^{d_n}\sum_{j=1}^{d}\frac{a_{j,n}}{r_n}g_k(x_j)\right)\\ &= \sum_{j_1=1}^{d} b_{j_1,n}\sum_{j_2=1}^{d}\frac{a_{j_2,n}}{r_n}\cdots\sum_{j_{d_n}=1}^{d}\frac{a_{j_{d_n},n}}{r_n}\kappa\pi_i\left(\prod_{k=1}^{d_n} g_k(x_{j_k})\right)\end{aligned}$$

must be an integer by choice of κ since $d_n \mid d$. Define

$$q_N := \kappa\prod_{n=1}^{N-1}\tilde{a}_n r_n \mathcal{N}(a_n/r_n) \quad \text{and} \quad p_{i,N} := \pi_i\left(q_N\sum_{n=1}^{N-1}\frac{b_n}{a_n}\right),$$

and note that $q_N \in \mathbb{N}$ and $p_{i,N} \in \mathbb{Z}$ by the above considerations. Set $M = \eta_2 + d((d-1)y_1 + y_2 - \eta_1 + \eta_2) + \delta$. The choice of $p_{i,N}$, the triangle inequality, and Lemma 3.3 show that for infinitely many N,

$$\begin{aligned}\left|\sum_{n=1}^{\infty}\frac{b_n}{a_n} - \frac{\sum_{i=1}^{d} p_{i,N}x_i}{q_N}\right| &= \left|\sum_{n=1}^{\infty}\frac{b_n}{a_n} - \sum_{n=1}^{N-1}\frac{b_n}{a_n}\right| \leq \sum_{n=N}^{\infty}\left|\frac{b_n}{a_n}\right|\\ &< 2^{-\log_2^{(1+\alpha)/2}\prod_{n=1}^{N-1}|a_n|}\prod_{n=1}^{N-1}|a_n|^{-M}.\end{aligned}$$

Inequality (22), with the choices of q_N and M, then implies

$$\begin{aligned}\left|\sum_{n=1}^{\infty}\frac{b_n}{a_n} - \frac{\sum_{i=1}^{d} p_{i,N}x_i}{q_N}\right| &< 2^{-\log_2^{(1+\alpha)/2}\prod_{n=1}^{N-1}|a_n|}\prod_{n=1}^{N-1}(\tilde{a}_n r_n \mathcal{N}(a_n/r_n))^{-M/\eta_2}\\ &\leq 2^{-\log_2^{(1+\alpha)/2} q_N} q_N^{-M/\eta_2}\\ &= 2^{-\log_2^{(1+\alpha)/2} q_N} q_N^{-1-d-d\frac{(d-1)y_1+y_2-\eta_1}{\eta_2}}.\end{aligned}$$

Since $\log_2^{(1+\alpha)/2} q_N \geq d\log_2^{(1+2\alpha)/3} q_N$ when N (and thereby q_N) is sufficiently large, we conclude that inequality (20) is satisfied for $q = q_N$ and $p_i = p_{i,N}$, for infinitely many choices of N.

We now just need to check that inequality (21) is also satisfied for $q = q_N$ and $p_i = p_{i,N}$. We start by noting

$$\frac{b_n}{a_n} = \frac{1}{\mathcal{N}(a_n)^{d/d_n}} \left(\sum_{j=1}^{d} b_{j,n} x_j \right) \left(\sum_{j=1}^{d} a_{j,n} x_j \right)^{d/d_n - 1} \prod_{k=2}^{d_n} \left(\sum_{j=1}^{d} a_{j,n} g_k(x_j) \right)^{d/d_n}$$
$$= \frac{1}{\mathcal{N}_K(a_n)} \sum_{j_1=1}^{d} \cdots \sum_{j_d=1}^{d} b_{j_1,n} a_{j_2,n} \cdots a_{j_d,n} \prod_{k=1}^{d_n} \prod_{l=1}^{d/d_n} g_k(x_{j_{kd_n+l}}).$$

It then follows from the triangle inequality and the choice of c that

$$\left| \pi_i \left(\frac{b_n}{a_n} \right) \right| \leq d^d c \frac{1}{\mathcal{N}(a_n)} \max_{1 \leq j \leq d} |b_{j,n}| \max_{1 \leq j \leq d} |a_{j,n}|^{d-1}.$$

Hence, by inequalities (7) and (9),

$$\left| \pi_i \left(\frac{b_n}{a_n} \right) \right| \leq d^d c \, |a_n|^{-\eta_1 + y_2 + y_1(d-1)} 2^{(d+1) \log_2^\alpha |a_n|}.$$

Recalling the choice of $p_{i,N}$, we now find

$$\left| \frac{p_{i,N}}{q_N} \right| \leq d^d c \sum_{n=1}^{N-1} |a_n|^{y_1(d-1) + y_2 - \eta_1} 2^{(d+1) \log_2^\alpha |a_n|}$$
$$\leq N |a_{N-1}|^{y_1(d-1) + y_2 - \eta_1} 2^{(d+2) \log_2^\alpha |a_N|}.$$

For all sufficiently large N, the choice of q_N and inequality (6) lead to

$$\log_2^\alpha q_N \geq \log_2^\alpha \left((N-2)! |a_{N-1}| \right) \geq \frac{1}{d+3} \max\{\log_2 N, \log_2^\alpha |a_{N-1}|\}.$$

Since $\eta_1 \leq y_1(d-1) + y_2$, this means that

$$\left| \frac{p_{i,N}}{q_N} \right| \leq |a_{N-1}|^{y_1(d-1) + y_2 - \eta_1} 2^{\log_2^\alpha |q_N|} \leq |q_N|^{\frac{y_1(d-1) + y_2 - \eta_1}{\eta_2}} 2^{\log_2^\alpha |q_N|},$$

by choice of $\tilde{a}_N$ and q_N, and the proof is complete. □

Proof of Theorem 5.1 This follows in full parallel to the proof of Theorem 4.1, with $M = d(y_1(d-1) + y_2 + \eta_2 - \eta_1) + \delta$ and using Lemma 5.2 in place of Lemma 3.2. In the application of Lemma 4.2, replace δ with δ/η_2. □

We now just need a variant of Proposition 4.3 where we allow the a_n to be irrational.

Proposition 5.3 *Let $d, \tilde{d} \in \mathbb{N}$, let K be a number field of degree d, let $x_1, \ldots, x_D \in K$, and consider real numbers $\varepsilon > 0$, $y_1 \geq 1 > \alpha > 0$, $\beta \in [0, \frac{\varepsilon}{1+\varepsilon})$, and $y_2 \geq \beta$. Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be sequences of non-zero numbers given by $a_n = \sum_{i=1}^D a_{i,n} x_i$ and $b_n = \sum_{i=1}^D b_{i,n} x_i$ where $a_{i,n}, b_{i,n} \in \mathbb{Z}$ and satisfy inequality (6),*

$$n^{1+\varepsilon} \leq |a_n| \leq |a_{n+1}|,$$

and

$$\limsup_{n\to\infty} |a_n|^{\left(\frac{d\tilde{d}(y_1+y_2)}{1-\beta}+1\right)^{-n}} = \infty.$$

Suppose for all sufficiently large n, each $i = 1\dots, D$, and some fixed $\zeta \in \mathbb{C}$ that inequalities (9)*, (10), and (11) are satisfied. Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of positive integers. Then the number $\sum_{n=1}^{\infty} \frac{b_n}{a_n c_n}$ has algebraic degree strictly greater than $\tilde{d}$.*

The main change from the proof of Proposition 4.3 is that we now use Lemma 3.5 a few times before using Lemmas 3.4 and 3.7 in the estimate of $H(s_N)$. This makes the proof closer to that in [6].

Proof of Proposition 5.3 The proof is essentially the same as that of Proposition 4.3, except that the calculation starting with inequality (18) and ending with inequality (19) is replaced by

$$\begin{aligned}
H(s_N) &\le 2^{N-2}\prod_{i=1}^{N-1} H(a_n)H(b_n) \le 2^{N-2}\prod_{i=1}^{N-1} c^{2/d}\, \overline{|a_n|}\ \overline{|b_n|} \\
&\le C^N \prod_{i=1}^{N-1} \max_{1\le i\le D}|a_{i,n}| \max_{1\le i\le D}|b_{i,n}| \le C^N \prod_{i=1}^{N-1} |a_n|^{y_1+y_2} 2^{2\log_2^{\alpha}|a_n|}, \\
&\le 2^{(3d\tilde{d})^{-1}N^2\log_2^{\alpha}|a_{N-1}|}\prod_{i=1}^{N-1}|a_n|^{y_1+y_2},
\end{aligned}$$

using Lemmas 3.5, 3.4, 3.7 and the inequalities of inequality (9), where $C > 0$ is some sufficiently large constant that depends only on $x_1, \dots, x_D$. The rest of the proof follows the exact same arguments as those for the proof of Proposition 4.3 but with y replaced with $y_1 + y_2$.

□

Proof of Theorem 1.7 By Remark 1.9, we may assume that $D = d$ and that $x_1, \dots, x_d$ form a $\mathbb{Q}$-linear basis of K. Then the irrationality statement is identical to Proposition 5.3 with $\tilde{d} = 1$, while the transcendence statement follows from Theorem 5.1 and Proposition 5.3 with $\tilde{d} = d$. □

6 Proof of Lemma 3.3

The proof of Lemma 3.3 closely follows the proof of [7, Lemma 5]. Similarly to Lemma 3.2, the below lemmas are taken from [7], in which the first three of them appear with additional assumptions that are never used in their proofs.

Lemma 6.1 *$\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ satisfy the assumptions of Lemma 3.3. Then there is a fixed number $0 < \gamma < 1$ such that for all sufficiently large N,*

$$\sum_{n=N}^{\infty} \frac{b_n}{a_n} \le \frac{1}{a_N^{\gamma}}.$$

Lemma 6.2 *Let β, $\{a_n\}_{n=1}^{\infty}$, and $\{b_n\}_{n=1}^{\infty}$ satisfy the assumptions of Lemma 3.3. Suppose that $a_n \ge 2^n$ for all sufficiently large n. Then there is a fixed number $0 < \Gamma < 1$ such that for all sufficiently large N,*

$$\sum_{n=N}^{\infty} \frac{b_n}{a_n} \le \frac{2^{\log_2^{\Gamma} a_N}}{a_N^{1-\beta}}.$$

Lemma 6.3 *Let β, $\{a_n\}_{n=1}^{\infty}$, and $\{b_n\}_{n=1}^{\infty}$ satisfy the assumptions of Lemma 3.3. Then there is a fixed number $0 < \Gamma < 1$ so that if $N \le Q$ are sufficiently large and $a_n \ge 2^n$ for $n = N, \dots, Q$, then*

$$\sum_{n=N}^{Q} \frac{b_n}{a_n} \le \frac{2^{\log_2^{\Gamma} a_N}}{a_N^{1-\beta}}.$$

Lemma 6.4 *Let $\{y_n\}_{n=1}^{\infty}$ be an unbounded sequence of positive real numbers. Then there are infinitely many positive integers N such that*

$$y_N > \left(1 + \frac{1}{N^2}\right) \max_{1 \le n < N} y_n.$$

By a simple induction argument, we notice for $k < N$ and $\delta \ge 0$ that

$$\begin{aligned}(M+1+\delta)^N &= (M+1+\delta)^{N-1} + (M+\delta)(M+1+\delta)^{N-1} = \cdots \\ &= (M+1+\delta)^k + (M+\delta)\sum_{n=k}^{N-1}(M+1+\delta)^n. \end{aligned} \tag{23}$$

This will be used both in the proof of Lemma 3.3 and for proving the below lemma, which is to be used together with Lemma 6.4 and Eq. (16).

Lemma 6.5 *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers, and let k be a positive integer. Then for all $N > k$,*

$$\left(\max_{k \le n < N} a_n^{\left(\frac{M}{1-\beta}+1\right)^{-n}}\right)^{\left(\frac{M}{1-\beta}+1\right)^N} > \prod_{n=k}^{N-1} a_n^{\frac{M}{1-\beta}}.$$

Proof We use equation (23) with $\delta = 0$ to find

$$\begin{aligned}\left(\max_{k \le n < N} a_n^{\left(\frac{M}{1-\beta}+1\right)^{-n}}\right)^{\left(\frac{M}{1-\beta}+1\right)^N} &> \left(\max_{k \le n < N} a_n^{\left(\frac{M}{1-\beta}+1\right)^{-n}}\right)^{\frac{M}{1-\beta}\sum_{n=k}^{N-1}\left(\frac{M}{1-\beta}+1\right)^n} \\ &\ge \prod_{n=k}^{N-1}\left(a_n^{\left(\frac{M}{1-\beta}+1\right)^{-n}}\right)^{\frac{M}{1-\beta}\left(\frac{M}{1-\beta}+1\right)^n} = \prod_{n=k}^{N-1} a_n^{\frac{M}{1-\beta}}.\end{aligned}$$

□

Proof of Lemma 3.3 We will split into three cases, depending on whether

$$\limsup_{n\to\infty} a_n^{\left(\frac{M}{1-\beta}+1+\delta\right)^{-n}} = \infty \tag{24}$$

holds for some fixed $\delta > 0$ and whether $a_n < 2^n$ infinitely often. To shorten notation, write

$$Z_N = 2^{N^2 \log_2^c a_{N-1}} \left(\prod_{n=1}^{N-1} a_n^M \right) \sum_{n=N}^{k_2} \frac{b_n}{a_n}.$$

Case 1 (inequality (24) **is satisfied for some** $\delta > 0$**)** Pick $0 < \gamma < 1$ as in Lemma 6.1, and let $z > 2$ be some sufficiently large number. Pick $k_1, k_2, N \in \mathbb{N}$ as follows. Let k_2 be the smallest integer such that

$$a_{k_2}^{\left(\frac{M}{1-\beta}+1+\delta\right)^{-k_2}} > z^{1/\gamma}, \tag{25}$$

let k_1 be the largest integer such that $k_1 < k_2$ and

$$a_{k_1} \le z^{k_1}, \tag{26}$$

and let N be the smallest number such that $N > k_1$ and

$$a_N^{\left(\frac{M}{1-\beta}+1+\delta\right)^{-N}} \ge z. \tag{27}$$

Note that $k_2 \ge N > k_1$ and that $k_1 \to \infty$ as $z \to \infty$. From the above choices of k_1 and N, it follows that $a_n < z^{(\frac{M}{1-\beta}+1+\delta)^n}$ when $k_1 \le n < N$. Hence, by also applying Eq. (23)

$$\prod_{n=k_1}^{N-1} a_n < z^{\sum_{n=k_1}^{N-1} (\frac{M}{1-\beta}+1+\delta)^n} < z^{(\frac{M}{1-\beta}+\delta)^{-1} (\frac{M}{1-\beta}+1+\delta)^N},$$

while inequality (26) implies

$$\prod_{n=1}^{k_1-1} a_n \le a_{k_1-1}^{k_1-1} < a_{k_1}^{k_1} \le z^{k_1^2} < z^{N^2},$$

since a_n is increasing and $k_1 < N$. Thus

$$\prod_{n=1}^{N-1} |a_n| < z^{N^2 + \left(\frac{M}{1-\beta}+\delta\right)^{-1} \left(\frac{M}{1-\beta}+1+\delta\right)^N}. \tag{28}$$

Since γ was chosen as in Lemma 6.1, we have for each sufficiently large z (and thereby k_2) that

$$\sum_{n=k_2}^{\infty} \frac{b_n}{a_n} \le \frac{1}{a_{k_2}^{\gamma}}.$$

Combining this with inequality (25) and the fact that $N \le k_2$, we find that

$$\sum_{n=k_2}^{\infty} \frac{b_n}{a_n} \le \frac{1}{z^{\left(\frac{M}{1-\beta}+1+\delta\right)^{k_2}}} \le \frac{1}{z^{\left(\frac{M}{1-\beta}+1+\delta\right)^{N}}} \tag{29}$$

when z is sufficiently large. Since

$$1 - \beta = \frac{M}{\left(\frac{M}{1-\beta}\right)} > \frac{M}{\frac{M}{1-\beta} + \delta},$$

we may pick a fixed number ζ such that

$$\frac{M}{M/(1-\beta)+\delta} < \zeta < 1-\beta. \tag{30}$$

Now pick $0 < \Gamma < 1$ as in Lemma 6.3. Since k_1 is the largest number less than k_2 satisfying inequality (26) and $k_1 < N \le k_2$, it follows that

$$\sum_{n=N}^{k_2-1} \frac{b_n}{a_n} \le \frac{2^{\log_2^\Gamma a_N}}{a_N^{1-\beta}}$$

when z (and thereby N) is sufficiently large. Applying inequalities (30) and (27), this yields that for all sufficiently large z,

$$\sum_{n=N}^{k_2-1} \frac{b_n}{a_n} < \frac{1}{a_N^{\zeta}} \le \frac{1}{z^{\zeta\left(\frac{M}{1-\beta}+1+\delta\right)^N}}.$$

Combined with inequality (29), we conclude

$$\sum_{n=N}^{\infty} \frac{b_n}{a_n} < \frac{1}{z^{\zeta\left(\frac{M}{1-\beta}+1+\delta\right)^N}} + \frac{1}{z^{\left(\frac{M}{1-\beta}+1+\delta\right)^N}} < \frac{2}{z^{\zeta\left(\frac{M}{1-\beta}+1+\delta\right)}}, \tag{31}$$

when z is sufficiently large.

Since N is the smallest number strictly greater than k_1 satisfying inequality (27), and since k_1 cannot satisfy inequality (27), due to inequality (26), we get in particular that $N-1$ does not satisfy inequality (27). Thus,

$$2^{N^2 \log_2^c a_{N-1}} \le 2^{N^2\left(\frac{M}{1-\beta}+1+\delta\right)^{(N-1)c} \log_2^c z} < z^{N^2\left(\frac{M}{1-\beta}+1+\delta\right)^{Nc}}. \tag{32}$$

When z is sufficiently large, we obtain from inequalities (28), (31), and (32) that

$$\begin{aligned} Z_N &= 2^{N^2 \log_2^c a_{N-1}} \left(\prod_{n=1}^{N-1} a_n^M\right) \sum_{n=N}^{k_2} \frac{b_n}{a_n} \\ &< z^{N^2\left(\frac{M}{1-\beta}+1+\delta\right)^{Nc} + MN^2 + \left(\frac{M}{M/(1-\beta)+\delta} - \zeta\right)\left(\frac{M}{1-\beta}+1+\delta\right)^N}. \end{aligned}$$

To simplify notation, write $\zeta' = \zeta - \frac{M}{M/(1-\beta)+\delta}$, which is positive due to inequality (30). We then continue our calculation to find that

$$Z_N < z^{N^2\left(\frac{M}{1-\beta}+1+\delta\right)^{Nc} + MN^2 - \zeta'\left(\frac{M}{1-\beta}+1+\delta\right)^N} < z^{-\frac{\zeta'}{2}\left(\frac{M}{1-\beta}+1+\delta\right)^N},$$

when z is sufficiently large. As the right-hand side clearly tends to 0 as z tends to infinity, we get the desired result.

Case 2 (inequality (24) is not satisfied for any fixed $\delta > 0$) This case is a bit more involved than the other one and will need to be split into two subcases, depending on whether $a_n \le 2^n$ infinitely often. However, both cases will need an estimate of the expression

$$\frac{2^{n^2 \log_2^c a_{n-1} + \log_2^\Gamma a_n}}{(1+(n-1)^{-2})^{(1-\beta)\left(\frac{M}{1-\beta}+1\right)^n}}, \tag{33}$$

where $\Gamma \in (0,1)$ is a fixed number to be chosen in each subcase. Set $\Gamma_0 = \max\{\Gamma, c\}$, and pick $\delta > 0$ so small that

$$\left(\frac{M}{1-\beta}+1+\delta\right)^{(1+\Gamma_0)/2} < \left(\frac{M}{1-\beta}+1\right)^{(2+\Gamma_0)/3}. \tag{34}$$

When n is sufficiently large, the case assumption will ensure that $a_n \le 2^{\left(\frac{M}{1-\beta}+1+\delta\right)^n}$, and so it follows that

$$\begin{aligned} 2^{n^2 \log_2^c a_{n-1} + \log_2^\Gamma |a_n|} &\le 2^{n^2\left(\frac{M}{1-\beta}+1+\delta\right)^{cn} + \left(\frac{M}{1-\beta}+1+\delta\right)^{\Gamma n}} \le 2^{\left(\frac{M}{1-\beta}+1+\delta\right)^{n(1+\Gamma_0)/2}} \\ &< 2^{\left(\frac{M}{1-\beta}+1\right)^{n(2+\Gamma_0)/3}}, \end{aligned}$$

by applying inequality (34) in the last inequality. As for the denominator of expression. (33), the Taylor expansion of $\log_2(1+x)$ implies that $\log_2(1+n^{-2}) \ge n^{-5/2}$. Therefore,

$$\frac{2^{n^2 \log_2^c a_{n-1} + \log_2^\Gamma a_n}}{(1+n^{-2})^{(1-\beta)\left(\frac{M}{1-\beta}+1\right)^n}} < \frac{2^{\left(\frac{M}{1-\beta}+1\right)^{n(2+\Gamma_0)/3}}}{2^{n^{-5/2}(1-\beta)\left(\frac{M}{1-\beta}+1\right)^n}} \le 2^{-n^{-3}\left(\frac{M}{1-\beta}+1\right)^n}, \tag{35}$$

for all sufficiently large n.

Case 2.a ($a_n \ge 2^n$ **for all but finitely many** n) By picking Γ as in Lemma 6.2, we get for all sufficiently large $N \in \mathbb{N}$ that

$$\sum_{n=N}^{\infty} \left|\frac{b_n}{a_n}\right| \le \frac{2^{\log_2^\Gamma |a_N|}}{|a_N|^{1-\beta}}.$$

At the same time, it follows from Lemma 6.5 and Lemma 6.4 with Eq. (16) that there are infinitely many $N \in \mathbb{N}$ such that

$$\begin{aligned} \prod_{n=1}^{N-1} |a_n|^M &< \frac{\left(\max_{k\le n<N} a_n^{\left(\frac{M}{1-\beta}+1\right)^{-n}}\right)^{(1-\beta)\left(\frac{M}{1-\beta}+1\right)^N}}{(1+N^{-2})^{(1-\beta)\left(\frac{M}{1-\beta}+1\right)^N}} \\ &< \frac{|a_N|^{1-\beta}}{(1+N^{-2})^{(1-\beta)\left(\frac{M}{1-\beta}+1\right)^N}}. \end{aligned}$$

Hence, for these infinitely many N,

$$Z_N = 2^{N^2 \log_2^c a_{N-1}} \left(\prod_{n=1}^{N-1} a_n^M\right) \sum_{n=N}^{k_2} \frac{b_n}{a_n} < \frac{2^{N^2 \log_2^c a_{N-1} + \log_2^\Gamma |a_N|}}{(1+N^{-2})^{(1-\beta)\left(\frac{M}{1-\beta}+1\right)^N}}.$$

From this and inequality (35), we obtain that for infinitely many N,

$$Z_N < 2^{-N^{-3}\left(\frac{M}{1-\beta}+1\right)^N},$$

and we are done.

Case 2.b ($a_n < 2^n$ **infinitely often**) Let $z > 0$ be sufficiently large, and pick $k_1, k_2, N \in \mathbb{N}$ as follows. Let k_2 be the smallest integer such that

$$a_{k_2}^{\left(\frac{M}{1-\beta}+1\right)^{-k_2}} > z, \tag{36}$$

and let k_1 be the largest integer such that $k_1 < k_2$ and

$$a_{k_1} < 2^{k_2}. \tag{37}$$

Due to the assumption that $a_n < 2^n$ infinitely often and the fact that k_2 is clearly unbounded, k_1 is also unbounded. Applying Lemma 6.4 with $k = k_1$ to Eq. (16), we pick N to be the smallest integer such that $N > k_1$ and

$$a_N^{\left(\frac{M}{1-\beta}+1\right)^{-N}} > \left(1+N^{-2}\right) \max_{k_1 \le n < N} a_n^{\left(\frac{M}{1-\beta}+1\right)^{-n}}. \tag{38}$$

Whenever $k_1 < n < N$, we then find by induction that

$$\begin{aligned} a_n^{\left(\frac{M}{1-\beta}+1\right)^{-n}} &\le \left(1+n^{-2}\right) \max_{k_1 \le m < n} a_m^{\left(\frac{M}{1-\beta}+1\right)^{-m}} \le \cdots \\ &\le \left(\prod_{m=k_1+1}^{n} \left(1+m^{-2}\right)\right) a_{k_1}^{\left(\frac{M}{1-\beta}+1\right)^{-k_1}} \\ &\le a_{k_1}^{\left(\frac{M}{1-\beta}+1\right)^{-k_1}} \prod_{m=1}^{\infty} \left(1+m^{-2}\right). \end{aligned} \tag{39}$$

Since $\log(1+x) \le x$, we find that

$$\prod_{m=1}^{\infty} \left(1+m^{-2}\right) = \exp\left(\sum_{m=1}^{\infty} \log\left(1+m^{-2}\right)\right) \le \exp\left(\sum_{m=1}^{\infty} m^{-2}\right) = \exp\left(\frac{\pi^2}{6}\right) < 6.$$

Similarly, inequality (37) and the fact that $2^{n/2^n} \le 1/2$ allow us to deduce

$$a_{k_1}^{\left(\frac{M}{1-\beta}+1\right)^{-k_1}} \le 2^{k_1 2^{-k_1}} \le \frac{1}{2}.$$

Recalling inequality (39), it follows that

$$a_n^{\left(\frac{M}{1-\beta}+1\right)^{-n}} < \frac{1}{2} \cdot 6 = 3, \tag{40}$$

for each $k_1 \le n < N$, and so, due to Eq. (23) with $\delta = 0$, it follows that

$$\prod_{n=k_1}^{N-1} a_n^M < 3^{M \sum_{n=k_1}^{N-1} \left(\frac{M}{1-\beta}+1\right)^n} < 3^{\left(\frac{M}{1-\beta}+1\right)^N}$$

when z (and thereby N) is sufficiently large. Using inequality (37) and the fact that a_n is non-decreasing to estimate

$$\prod_{n=1}^{k_1-1} a_n^M \le a_{k_1}^{Mk_1} \le 2^{Mk_1^2} < 2^{MN^2}, \tag{41}$$

we may then conclude that

$$\prod_{n=1}^{N-1} a_n = \left(\prod_{n=1}^{k_1-1} a_n\right) \prod_{n=k_1}^{N-1} a_n < 2^{MN^2} 3^{\left(\frac{M}{1-\beta}+1\right)^N} \le 4^{\left(\frac{M}{1-\beta}+1\right)^N}, \tag{42}$$

for all sufficiently large values of z. On the other hand, we might also estimate $\prod_{n=k_1}^{N-1} a_n^M$, using Lemma 6.5 and inequality (38) instead, which leads to

$$\prod_{n=k_1}^{N-1} a_n^M \leq \frac{a_N^{1-\beta}}{(1+N^{-2})^{(1-\beta)\left(\frac{M}{1-\beta}+1\right)^N}},$$

so that we, by means of inequality (41), reach

$$\prod_{n=1}^{N-1} a_n^M \leq 2^{MN^2} \frac{a_N^{1-\beta}}{(1+N^{-2})^{(1-\beta)\left(\frac{M}{1-\beta}+1\right)^N}}, \tag{43}$$

for all sufficiently large z (and thereby N).

Let $0 < \gamma < 1$ be given as in Lemma 6.1. From this, inequality (36), and the fact that $N \leq k_2$ when z is large, we then obtain

$$\sum_{n=k_2}^{\infty} \frac{b_n}{a_n} \leq a_{k_2}^{-\gamma} < z^{-\gamma\left(\frac{M}{1-\beta}+1\right)^{k_2}} \leq z^{-\gamma\left(\frac{M}{1-\beta}+1\right)^N}, \tag{44}$$

when z is sufficiently large. Let similarly $0 < \Gamma < 1$ be given as in Lemma 6.3. Since k_1 is the largest number less than k_2 that satisfies inequality (37) and $k_1 < N \leq k_2$, we have

$$\sum_{n=N}^{k_2-1} \frac{b_n}{a_n} \leq \frac{2^{\log_2^{\Gamma} a_N}}{a_N^{1-\beta}}.$$

Together with inequality (44), this leads to

$$\sum_{n=N}^{\infty} \frac{b_n}{a_n} = \sum_{n=N}^{k_2-1} \frac{b_n}{a_n} + \sum_{n=k_2}^{\infty} \frac{b_n}{a_n} \leq \frac{2^{\log_2^{\Gamma} a_N}}{a_N^{1-\beta}} + z^{-\gamma\left(\frac{M}{1-\beta}+1\right)^N}. \tag{45}$$

Combining inequalities (42), (43), and (45), we obtain

$$\begin{aligned} Z_N &= 2^{N^2 \log_2^c a_{N-1}} \left(\prod_{n=1}^{N-1} a_n^M\right) \sum_{n=N}^{k_2} \frac{b_n}{a_n} \\ &< 2^{N^2 \log_2^c a_{N-1}} \min\left\{4^{\left(\frac{M}{1-\beta}+1\right)^N}, 2^{MN^2} \frac{a_N^{1-\beta}}{(1+N^{-2})^{(1-\beta)\left(\frac{M}{1-\beta}+1\right)^N}}\right\} \\ &\quad \cdot \left(\frac{2^{\log_2^{\Gamma} a_N}}{a_N^{1-\beta}} + z^{-\gamma\left(\frac{M}{1-\beta}+1\right)^N}\right) \\ &\leq \frac{2^{N^2 \log_2^c a_{N-1} + MN^2 + \log_2^{\Gamma} a_N}}{(1+N^{-2})^{(1-\beta)\left(\frac{M}{1-\beta}+1\right)^N}} + \frac{2^{N^2 \log_2^c a_{N-1}} 4^{\left(\frac{M}{1-\beta}+1\right)^N}}{z^{\gamma\left(\frac{M}{1-\beta}+1\right)^N}}, \end{aligned} \tag{46}$$

for all sufficiently large z. From inequality (35), it follows that

$$\frac{2^{N^2 \log_2^c a_{N-1} + MN^2 + \log_2^{\Gamma} a_N}}{(1+N^{-2})^{(1-\beta)\left(\frac{M}{1-\beta}+1\right)^N}} < \frac{2^{MN^2}}{2^{N^{-3}\left(\frac{M}{1-\beta}+1\right)^N}},$$

which clearly tends to 0 as z (and thereby N) tends to infinity. We are thus left to show that the remaining term of the right-hand side of inequality (46) also approaches 0 when z grows large. Fortunately, this immediately follows from the calculation that

$$\frac{2^{N^2 \log_2^c a_{N-1}} 4^{\left(\frac{M}{1-\beta}+1\right)^N}}{z^{\gamma\left(\frac{M}{1-\beta}+1\right)^N}} \leq \frac{5^{\left(\frac{M}{1-\beta}+1\right)^N}}{z^{\gamma\left(\frac{M}{1-\beta}+1\right)^N}} \leq z^{-\frac{\gamma}{2}\left(\frac{M}{1-\beta}+1\right)^N},$$

for all sufficiently large z. This completes the proof. □

7 Concluding remarks

In [12], the present author proves a variant of the main theorem of [6]. In particular, this implies that one may replace $\limsup_{n\to\infty} |a_n|^{\prod_{i=1}^{n-1}(d^i+d)^{-1}} = \infty$ with

$$\liminf_{n\to\infty} |a_n|^{\prod_{i=1}^{n-1}(d^i+d)^{-1}} < \limsup_{n\to\infty} |a_n|^{\prod_{i=1}^{n-1}(d^i+d)^{-1}} < \infty$$

in Theorem 1.3 and still get irrationality. Certainly, a corresponding result can be proven for Propositions 4.3 and 5.3 as well, leading to alternative versions of Theorems 1.4, 1.6, and 1.7, though we will not do that here.

We will now compare the main theorems to Theorem 1.2. If we set $d = 1$ in Theorems 1.6 or Theorem 1.7, we get the below corollary, which also appears if we assume $d = 1$ in the proof of Theorem 1.4.

Corollary 7.1 *Let $\alpha, \delta, \varepsilon > 0$ be positive real numbers with $\alpha < 1$, and let $\beta \in [0, \frac{\varepsilon}{1+\varepsilon})$. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of positive integers so that*

$$n^{1+\varepsilon} \leq a_n \leq a_{n+1}, \quad \limsup_{n\to\infty} a_n^{\left(\frac{2+\delta}{1-\beta}+1\right)^{-n}} = \infty,$$

and for all sufficiently large n,

$$b_n \leq a_n^{\beta} 2^{\log_2^{\alpha} a_n}. \tag{47}$$

Then the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ is transcendental.

This corollary also follows from Theorem 1.2 by replacing ε with $\beta/(1-\beta) + \delta/3$ and putting $\gamma = \frac{2\beta+\delta}{1-\beta}$, while noting that $a_n^{\beta} 2^{\log_2^{\alpha_1} a_n} < a_n^{\varepsilon/(1+\varepsilon)}/2^{\log_2^{\alpha_2} a_n}$ for any fixed values of $\alpha_1, \alpha_2 \in (0, 1)$ and all sufficiently large n. On the other hand, Theorem 1.2 is slightly stronger than Corollary 7.1 since inequality (1) allows $\log|b_n|/\log|a_n|$ to approach $\varepsilon/(1+\varepsilon)$ as $n \to \infty$, which is prevented by inequality (47). Quite naturally, this raises the following question.

Question 7.2 *Suppose we replaced β by $\varepsilon/(1+\varepsilon)$, let α be sufficiently close to 1, and replaced the assumption $|b_n| \leq |a_n|^{\beta} 2^{\log_2^{\alpha} a_n}$ by $|b_n| < |a_n|^{\varepsilon/(1+\varepsilon)} 2^{-\log_2^{\alpha}|a_n|}$ for all sufficiently large n. Would Theorems 1.4, 1.6, and 1.7 then remain true?*

While the above comparison between Corollary 7.1 and Theorem 1.2 would suggest an affirmative answer, this question is not so easily answered. To see why, we start by taking a brief look at Hančl's proof of Theorem 1.2 as presented in [5]. Part of the proof follows an argument much similar to Lemma 3.3. The most significant difference is that Hančl takes advantage of the fact that γ can always be replaced by a smaller value $\gamma' > 2\varepsilon$ without affecting whether any assumption of the theorem is satisfied. This means that he only ever has to consider what corresponds to case 1 from the proof of Lemma 3.3. In the proofs of Theorems 4.1 and 5.1, we can make the same trick by replacing δ with a smaller positive number, so the proofs of these theorems should be easily modified to allow the changes proposed by the above question. The problem arises with Propositions 4.3 and 5.3 – here none of the parameters present in the limsup conditions can be reduced without strengthening some other assumption, and so we cannot apply Hančl's trick and must deal with some version of case 2 from the proof of Lemma 3.3. When imposing the changes from Question 7.2, this becomes a much more difficult task, and there appears to be no immediate way of modifying the proof of Lemma 3.3 so that all of case 2 is covered.

Therefore, at least until Question 7.2 is answered in the affirmative for at least one of the theorems, we cannot both get the large values of $|b_n|$ that Theorem 1.2 suggests without making the limsup conditions more strict.

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Publisher's Note

1.6.3 Paper 5: Irrationality and transcendence of infinite products

Below, the reader will find the most recent preprint of the paper [39], which has the current author as its sole author. The paper is currently under review but has not yet been accepted for publication. The preprint is available on arXiv through the link https://arxiv.org/abs/2503.01575v1 or by using the arXiv identifier 2503.01575. It has a length of 14 pages, numbered 1 through 14.

In order to prove the irrationality part of Theorem 1.62 presented earlier in this section, the assumption $M \geq 1$ has to be weakened to $M > 0$ in Lemma 4.1 of the below paper. The first page of subsection 1.6.2 explains why the lemma remains true with this slight change of assumption.

TRANSCENDENCE CRITERIA FOR INFINITE PRODUCTS OF ALGEBRAIC NUMBERS

MATHIAS L. LAURSEN

Abstract. Using an application of Schmidt's Subspace Theorem, this paper gives new transcendence criteria for rapidly converging infinite products of algebraic numbers. The paper also improves existing criteria for irrationality of products and criteria for irrationality and transcendence of infinite series. These results generalize a classical theorem on the irrationality of infinite series due to Erdős.

1. Introduction

Proving whether a given real number is algebraic, or even rational, can be a quite frustrating endeavour. While more than a century and a half have passed since Hermite proved that the number $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ is transcendental in early 1873, it remains unsolved if the number $\sum_{n=0}^{\infty} \frac{1}{n!+1}$ is even irrational, despite what may appear as a much similar construction. Similarly, it is well-known that the Riemann ζ function defined as $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $\Re(s) > 1$ is transcendental when s is a positive even integer while the question of irrationality remains open when $s \geq 5$ is any fixed odd integer. In other words, we have a multitude of interesting numbers that we know to be transcendental but where a small perturbation to the infinite series used to describe them renders even the question of irrationality exceedingly hard to settle. Aiming away from frustrations of this kind, this paper studies irrationality and transcendence criteria that are less sensitive to such perturbations.

Following the notions of Hančl [3, 4], we say that a sequence $\{a_n\}_{n=1}^{\infty}$ of real or complex numbers is Σ-irrational (respectively Σ-transcendental) if the sum of the series $\sum_{n=1}^{\infty} \frac{1}{a_n c_n}$ is irrational (respectively transcendental) for any sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers. Inspired by this, we say that $\{a_n\}_{n=1}^{\infty}$ is (Π, K)-irrational if the number $\prod_{n=1}^{\infty} \big(1 + \frac{1}{a_n c_n}\big)$ lies outside of a given field K for all sequences $\{c_n\}_{n=1}^{\infty}$ of positive integers. We then say that $\{a_n\}_{n=1}^{\infty}$ is Π-irrational if it is $\Pi_{\mathbb{Q}}$-irrational and that it is Π-transcendental if it is $\Pi_{\overline{\mathbb{Q}}}$-irrational.

The first result on Σ-irrationality was proven in 1975 by Erdős [2].

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Theorem 1.1 (Erdős)**.** *Let $\varepsilon > 0$, and let $\{a_n\}_{n=1}^{\infty}$ be an increasing sequence of integers such that $a_n \geq n^{1+\varepsilon}$ for all n. Suppose*

$$\limsup_{n\to\infty} a_n^{2^{-n}} = \infty.$$

Then the sequence $\{a_n\}_{n=1}^{\infty}$ is Σ-irrational.

Since then, more Σ-related results have come to light, such as criteria for Σ-transcendence (starting with [4] in 1996) or $\mathbb{Q}$-linear independence of the numbers $1, \sum_{n=1}^{\infty} \frac{b_{1,n}}{a_{1,n}c_n}, \dots, \sum_{n=1}^{\infty} \frac{b_{K,n}}{a_{K,n}c_n}$ (starting with [5] in 1999).

Meanwhile, the first result on Π-irrationality was not published until 2011, where Hančl and Kolouch [6] proved that sequences $\{a_n\}_{n=1}^{\infty}$ satisfying the assumptions of Theorem 1.1 will also be Π-irrational. This result was extended by Kristensen and the current author in [7] where a_n are allowed to be algebraic integers from a broader family of algebraic numbers and where a lower bound on algebraic degree of the numbers $\prod_{n=1}^{\infty}(1 + a_n^{-1})$ is given. Recall that an algebraic integer is an algebraic number whose primitive polynomial over $\mathbb{Z}$ is monic. We use $\mathcal{O}_{\mathbb{K}}$ to denote the ring of algebraic integers contained in a given field $\mathbb{K}$. When only considering the questions of Π-irrationality and Π-transcendence, this result specializes to the below theorem. To the current author's knowledge, this is so far the only available result regarding Π-transcendence. The notation $\overline{|a_n|}$ denotes the maximum modulus amongst the conjugates of a_n.

Theorem 1.2 (Kristensen and Laursen)**.** *Let $\mathbb{K}$ be a number field of degree d, let $\tilde{d}, D \in \mathbb{N}$ be positive integers, and consider $\varepsilon > 0$ and $\alpha \in (0,1)$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of algebraic integers such that $n^{1+\varepsilon} < |a_n| \leq |a_{n+1}|$. Write $\tilde{\mathbb{K}} = \mathbb{Q}(a_n : n \in \mathbb{N})$, let $\mathbb{K}'$ be a field extension of degree D of $\tilde{\mathbb{K}}$, and let $\{b_n\}_{n=1}^{\infty}$ be a sequence of positive integers. Suppose $\overline{|a_n|} b_n \leq |a_n| 2^{\log_2^{\alpha}|a_n|}$, $\deg_{\mathbb{K}} a_n \leq \tilde{d}$, and $e\Re(a_n/b_n - 1/4) \geq 1/4$ for all n with $e \in \{-1, 1\}$ fixed and $\Re(a_n/b_n) \neq -1/2$ infinitely often. Then $\{a_n\}_{n=1}^{\infty}$ is $\Pi_{\mathbb{K}'}$-irrational if $|a_n|^{D^{-n}\prod_{i=1}^{n-1}(\tilde{d}^i d + \tilde{d})^{-1}} = \infty$ diverges in $\mathbb{R}$, and $\{a_n\}_{n=1}^{\infty}$ is Π-transcendental if for all $A > 0$,*

$$\limsup_{n\to\infty} |a_n|^{A^{-n}\prod_{i=1}^{n-1}(d^i + d)^{-1}} = \infty.$$

In the present paper, we will improve this result in the case $\tilde{d} = 1$, i.e., when $a_n \in \mathbb{K}$ for all n, while providing conditions for when $\overline{|a_n|}|b_n|$ is large and when b_n is picked in $\mathbb{K}$ rather than $\mathbb{Q}$. The most significant improvement lies in the transcendence criterion, where the new version allows one to stop at a finite A. Results of this nature was proven for infinite series in [8] by the current author. In that paper, the most simple criteria were found when a_n is assumed to be rational while b_n carries the algebraic degree of the number. Revisiting these ideas, we will not only achieve stronger results for infinite products but also improve the theorems from [8].

2. Main results

As in [8], our main results will be variations of each other. When restricting our attention to rational numbers, we get the following irrationality and transcendence criteria.

Theorem 2.1. *Let* $\varepsilon > 0$, $0 < \alpha < 1$, *and* $\beta \in [0, \frac{\varepsilon}{1+\varepsilon})$. *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be sequences of positive integers such that*

$$n^{1+\varepsilon} < a_n \leq a_{n+1} \quad \text{and} \quad b_n \leq a_n^{\beta} 2^{\log_2^{\alpha} a_n}.$$

Then the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ *is* Π*-irrational if*

$$\limsup_{n\to\infty} a_n^{(\frac{1}{1-\beta}+1)^{-n}} = \infty,$$

and it is Π*-transcendental if*

$$\limsup_{n\to\infty} a_n^{(\frac{2+\delta}{1-\beta}+1)^{-n}} = \infty.$$

Moving on to algebraic numbers, we need to be more careful and to take into account the arithmetic properties of a_n and b_n. We will first consider the question of irrationality, which is also the easiest to prove. We provide three different *limsup* criteria, the latter two of which correspond to the irrationality conditions given in [8], while the first condition is new and will be used for proving the transcendence criteria of the subsequent theorems. While the theorem has a good number of assumptions, some of them may be skipped, depending on which *limsup* condition one means to imply; thus inequality (3) may be skipped when using condition (8), while inequalities (5) and (6) may be skipped when using condition (9) or (10). In this theorem, and for the remainder of the current paper, $\mathcal{N} : \overline{\mathbb{Q}} \to \mathbb{Q}$ denotes the map that sends an algebraic number to the product of its (algebraic) conjugates. A reader familiar with field norms will notice that $\mathcal{N}(a)$ is exactly the field norm associated with $\mathbb{Q}(a)$, evaluated at a.

Theorem 2.2. *Let* $\mathbb{K}$ *be a number field with of degree* $d \in \mathbb{N}$, *and consider real numbers* $\alpha \in (0,1)$, $\varepsilon > 0$, $\beta \in [0, \varepsilon/(1+\varepsilon))$, $y_1 \geq 1$, $y_2 \geq \beta$, $z_1 \geq -y_2$, $z_2 \geq 0$, *and* $e \in \{-1, 1\}$. *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be sequences of non-zero numbers in* $\mathcal{O}_{\mathbb{K}}$ *such that*

$$n^{1+\varepsilon} \leq |a_n| \leq |a_{n+1}|, \tag{1}$$

$$|b_n| < |a_n|^{\beta} 2^{\log_2^{\alpha} |a_n|}, \tag{2}$$

$$\overline{|a_n|} \leq |a_n|^{y_1} 2^{\log_2^{\alpha} |a_n|}, \tag{3}$$

$$\overline{|b_n|} \leq |a_n|^{y_2} 2^{\log_2^{\alpha} |a_n|}, \tag{4}$$

$$\overline{\left|a_n^{-1}\right|} \leq |a_n|^{z_1} 2^{\log_2^{\alpha}|a_n|}, \tag{5}$$

$$r_n \left|\mathcal{N}(a_n/r_n)\right| \leq |a_n|^{z_2} 2^{\log_2^{\alpha}|a_n|}, \tag{6}$$

and

$$\Re\left(\frac{a_n}{b_n}\right) \begin{cases} \geq 0 & \text{if } e = 1, \\ \leq -1/2 & \text{if } e = -1, \end{cases} \tag{7}$$

where each r_n is a positive integer dividing a_n, and $\Re(a_n/b_n) \neq -1/2$ infinitely often. Let $d_0 \in \mathbb{N}$ and suppose $\deg(a_n/b_n) \geq d_0$ for all large enough n. Then the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ is $\Pi_{\mathbb{K}}$-irrational if

$$\limsup_{n\to\infty} |a_n|^{\left(\frac{d(y_2+z_1+z_2/d_0)}{1-\beta}+1\right)^{-n}} = \infty, \tag{8}$$

$$\limsup_{n\to\infty} |a_n|^{\left(\frac{d(y_1+y_2)}{1-\beta}+1\right)^{-n}} = \infty, \tag{9}$$

or, in the case that $a_n \in \mathbb{Z}$ or $b_n \in \mathbb{Z}$ for each n, if

$$\limsup_{n\to\infty} |a_n|^{\left(\frac{d\max\{y_1,y_2\}}{1-\beta}+1\right)^{-n}} = \infty. \tag{10}$$

The main novelty of this theorem lies in *limsup* condition (8) since it grants Π-irrationality for sequences $\{a_n\}_{n=1}^{\infty}$ satisfying both $\deg a_n > 1$ and

$$\lim_{n\to\infty} |a_n|^{1/(2+\delta)^n} = 1,$$

for all $\delta > 0$, while granting a somewhat weaker result when the same is true for $\delta = 0$, as seen in Example 3.2. Meanwhile, all former Σ- or Π-irrationality statements in the literature require at least

$$\lim_{n\to\infty} |a_n|^{(d+1)^{-n}} = \infty.$$

when $\mathbb{Q}(a_1, a_2, \ldots)$ is a finite field of degree d.

Theorem 2.2 also provides a stronger irrationality statement than those in [8], where only $\Sigma_{\mathbb{Q}}$-irrationality was proven. This improvement, however, does not rely on any difference between products and series, and so the irrationality statements of [8] may easily be strengthened to match Theorem 2.2 by modifying the proofs accordingly. An important consequence of the improved irrationality statement is that we may slack the transcendence criterion when $d > 1$, giving us the below theorem. In parallel to [8], transcendence is proven through an application of Schmidt's Subspace Theorem, which will give us that each product $\prod_{n=1}^{\infty}(1 + \frac{b_n}{c_n a_n})$ is either transcendental or contained in $\mathbb{K}$, then dealing with $\mathbb{K}$ through Theorem 2.2. As with Theorem 2.2, the corresponding theorem in [8] may be improved to match this result.

Theorem 2.3. *Let $\mathbb{K}$ be a number field of degree $d \in \mathbb{N}$, and consider real numbers $\delta, \varepsilon > 0$, $\alpha \in (0,1)$, $\beta \in [0, \frac{\varepsilon}{1+\varepsilon})$, $e \in \{-1, 1\}$, and $y \geq 1$. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be a sequences of non-zero numbers from $\mathbb{Z}$ and $\mathcal{O}_{\mathbb{K}}$, respectively, such that inequalities* (1), (2), (4), *and* (12) *are satisfied with $y_2 = y$. Suppose that $\Re(a_n/b_n) \neq -1/2$ infinitely often and that*

$$\limsup_{n\to\infty} a_n^{\left(\frac{dy+1+\delta}{1-\beta}+1\right)^{-n}} = \infty.$$

Then the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ is Π-transcendental.

In order to use the above mentioned application of Schmidt's Subspace Theorem, we need to write the approximants $\prod_{n=1}^{N}(1 + \frac{b_n}{a_n c_n})$ as a $\mathbb{Q}$-linear combination of some basis $x_1, \ldots, x_d$ of $\mathbb{K}$. If we allow $a_n \in \mathcal{O}_{\mathbb{K}}$, we then need to also consider the coordinates of a_n^{-1} and the associated least common denominator, which makes both theorem and proof a bit more involved but allows us to conclude the following theorem. Again, we get an improvement compared to [8], and updating the proof in that paper accordingly will give a matching result for Σ-irrationality.

Theorem 2.4. *Let $\mathbb{K}$ be a number field with of degree $d \in \mathbb{N}$, and consider real numbers $\alpha \in (0,1)$, $\delta, \varepsilon > 0$, $\beta \in [0, \varepsilon/(1+\varepsilon))$, $e \in \{-1, 1\}$, $y \geq \beta$, $z_1 \geq -y$, and $z_2 \geq 0$. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of non-zero numbers in $\mathcal{O}_{\mathbb{K}}$ such that inequalities* (1), (2), (4)–(6), *and* (12) *are satisfied with $y_2 = y$, $r_n \in \mathbb{Z}$, and $r_n \mid a_n$. Suppose that $\Re(a_n/b_n) \neq -1/2$ infinitely often and that*

$$\limsup_{n\to\infty} |a_n|^{\left(\frac{d(y+z_1+z_2)+z_2+\delta}{1-\beta}+1\right)^{-n}} = \infty. \tag{11}$$

Then the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ is Π-transcendental.

When specializing to $a_n \in \mathbb{N}$, we retrieve Theorem 2.3, which has the advantage of being more easily checked. Further specializing to $b_n \in \mathbb{N}$, we reach Theorem 2.1. However, when allowing $a_n \in \mathcal{O}_{\mathbb{K}}$ and specializing to $b_n \in \mathbb{N}$, we do not get the same simplification as is the case for Theorem 2.3, and we will for that reason not state it as a separate theorem.

Theorem 2.5. *Replace the assumptions $z_2 \geq 1$ and* (7) *with the weaker assumptions $z_2 \geq 0$ and*

$$e\Re\left(\frac{a_n}{b_n} + \frac{1}{2}\right) \geq 0. \tag{12}$$

Then Theorems 2.2, 2.3, and 2.4 remain valid if we replace the statements of $\Pi_{\mathbb{K}}$-irrationality and $\Pi_{\mathbb{K}}$-transcendence with $\xi \notin \mathbb{K}$ and ξ being transcendental, respectively, where ξ is the number $\prod_{n=1}^{\infty}(1 + b_n/a_n)$.

3. Applications

In the following examples, φ is the golden ratio (the positive root of x^2-x-1) and ψ is the supergolden ratio (the real root of x^3-x^2-1). Let F_n and $\hat{F}_n$ be the corresponding linear recurrences, i.e., $F_1 = F_2 = 1$, $F_{n+2} = F_n + F_{n+1}$, $\hat{F}_1 = \hat{F}_2 = \hat{F}_3 = 1$, and $\hat{F}_{n+3} = \hat{F}_n + \hat{F}_{n+2}$. Notice that φ and ψ are units with $\overline{|\varphi^{-1}|} = \varphi$ and $\overline{|\psi^{-1}|} = \psi^{1/2}$. We will also use the notation of $\lceil a \rceil$ to denote the smallest integer $k \geq a$ when $a \in \mathbb{R}$.

If for a given example below, some of the assumptions (1) through (12) are not be satisfied for all of the first finitely many n, apply first the relevant theorem first on $\{a_{n+N}/b_{n+N}\}_{n=1}^{\infty}$ and then realize that the conclusion then also holds for $\{a_n/b_n\}_{n=1}^{\infty}$.

Example 3.1. Let $\{h_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of integers with $h_n \geq 3^n n$ infinitely often. Relying on condition (10) of Theorem 2.2, the sequences $\{\varphi^{h_n}\}_{n=1}^{\infty}$, $\{\varphi^{h_n} + b_n\}_{n=1}^{\infty}$, and $\{F_{h_n}/b_n\}_{n=1}^{\infty}$ are all $\Pi_{\mathbb{Q}(\varphi)}$-irrational when $b_n \in \mathbb{Q}(\varphi)$ with $0 < b_n \leq 2^{h_n^{\alpha}}$ and $\overline{|b_n|} \leq F_{h_n}$. The same is true if φ is replaced by any other quadratic irrational number x with $\overline{|x|} = x > 1$.

The next example shows how relaxed condition (8) becomes when the numbers a_n have the right arithmetic properties.

Example 3.2. For $i = 1, 2, 3$, let $\{h_{i,n}\}_{n=1}^{\infty}$ be a strictly increasing sequence of integers with $h_{i,n} \geq (1 + 1/i)^n \log n$ infinitely often. Using condition (8), Theorem 2.2 ensures that the sequences $\{F_{h_{1,n}}\varphi^{h_{1,n}}\}_{n=1}^{\infty}$ and $\{\hat{F}_{\lceil h_{1,n}/2 \rceil}\psi^{h_{1,n}}\}_{n=1}^{\infty}$ are $\Pi_{\mathbb{Q}(\varphi)}$-irrational and $\Pi_{\mathbb{Q}(\psi)}$-irrational, respectively, while we may use Theorem 2.5 to further get $\prod_{n=1}^{\infty}(1 + F_{h_{2,n}}^{-1}\varphi^{-h_{2,n}}) \notin \mathbb{Q}(\varphi)$ and $\prod_{n=1}^{\infty} 1 + \hat{F}_{\lceil h_{3,n}/2 \rceil}^{-1}\psi^{-h_{3,n}} \notin \mathbb{Q}(\psi)$.

More generally, if $x > 1$ is an algebraic unit with $\overline{|x^{-1}|} = x^z$, then $\{\lceil x^{zh_{1,n}} \rceil x^{h_{1,n}}\}_{n=1}^{\infty}$ is $\Pi_{\mathbb{Q}(x)}$-irrational while $\prod_{n=1}^{\infty}(1 + \frac{1}{\lceil x^{-zh_n} \rceil^{-1} x^{-h_n}}) \notin \mathbb{Q}(x)$ when $\{h_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of integers with $h_n \geq (1 + Z)^n \log n$ infinitely often where $Z = z/(1 + z)$.

We then provide an example where condition (9) is preferred over condition (8) and where we may have $a_n, b_n \notin \mathbb{Z}$ infinitely often.

Example 3.3. Let $\{h_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of integers with $h_n \geq 7^n \log n$ infinitely often. Then condition (8) of Theorem 2.2 ensures that the sequence $\{(2^{h_n} + \sqrt[3]{2}^n)/b_n\}_{n=1}^{\infty}$ is $(\Pi, \mathbb{Q}(\sqrt[3]{2}))$-irrational if $b_n \in \mathcal{O}_{\mathbb{Q}(\sqrt[3]{2})}$ with $0 < b_n < 2^{h_n^{\alpha}}$ and $\overline{|b_n|} \leq 2^{h_n}$ for all n. The same is true when $\sqrt[3]{2}$ is replaced for any other d'th root of a positive integer and $h_n \geq (2d + 1)^n \log n$ infinitely often.

We will now give examples of Π-transcendental sequences. The first one shows how simple it is to apply Theorem 2.3 while the following two show how lenient equation (11) of Theorem 2.4 can be when the right sequences are considered.

Example 3.4. For each $i \in \mathbb{N}$, let $\{h_{i,n}\}_{n=1}^{\infty}$ be a strictly increasing sequence of integers with $h_{i,n} \geq (i+1/i)^n$ infinitely often. By Theorem 2.3, $\{F_{h_{4,n}}/(1+\varphi^{-h_n})\}$ is Π-transcendental by picking $\beta = 0$ and $y = 1$. More generally, if $x > 1$ is an algebraic unit of degree d, then the sequence $\{\lceil x^{h_{d+2,n}} \rceil/(1 + x^{-h_{d+2,n}})\}_{n=1}^{\infty}$ is Π-transcendental.

Example 3.5. For $i \in \mathbb{N}$, let $\{h_{i,n}\}_{n=1}^{\infty}$ and $\{h'_{i,n}\}_{n=1}^{\infty}$ be strictly increasing sequences of integers with $h_{i,n} \geq (i + 1/i)^n$ and $h'_{i,n} \geq (3 - 1/i)^n$ infinitely often. Then Theorem 2.4 implies that the sequences $\{F_{h_{4,n}}\varphi^{h_{4,n}}\}_{n=1}^{\infty}$ and $\{\hat{F}_{\lceil h_{5,n}/2\rceil}\psi^{h_{3,n}}\}_{n=1}^{\infty}$ are Π-transcendental while Theorem 2.5 ensures that also the numbers $\prod_{n=1}^{\infty}(1 + F_{h'_{3,n}}^{-1}\varphi^{-h'_{3,n}})$ and $\prod_{n=1}^{\infty} 1 + \hat{F}_{\lceil h'_{2,n}/2\rceil}^{-1}\psi^{-h'_{2,n}}$ are transcendental.

More generally, if $x > 1$ is an algebraic unit with $\overline{|x^{-1}|} = x^z$ and $\deg x = d$, then $\{\lceil x^{z h_{d+2,n}}\rceil x^{h_{d+2,n}}\}_{n=1}^{\infty}$ is Π-transcendental while we further have $\prod_{n=1}^{\infty}(1 + \frac{1}{\lceil x^{-z h_n}\rceil^{-1} x^{-h_n}}) \notin \mathbb{Q}(x)$ when $\{h_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of integers with $h_n \geq (2 + dZ)^n$ infinitely often where $Z = z/(1 + z)$.

Example 3.6. Let $\{h_n\}_{n=1}^{\infty}$ and $\{h'_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of integers with $h_n \geq (7 - 1/3)^n$ and $h'_n \geq (3 - 1/3)^n$ infinitely often. By Theorems 2.4 and 2.5, $\{\psi^{h_n}\}$ is Π-transcendental, and the number $\prod_{n=1}^{\infty}(1 + \psi^{-h'_n})$ is transcendental. A similar argument can be made for any other real algebraic unit x with $1 < \overline{|x^{-1}|} < x^{2/d}$ and assuming $h_n \geq (3 - \delta)^n$ for some sufficiently small $\delta > 0$.

4. Preliminaries

In this section, we provide useful definitions and lemmas that are either proven elsewhere or that have particularly short proofs.

We start by introducing two lemmas that are proven in [8]. The first lemma uses the method of proof introduced in [2], while the other one is an application of Schmidt's Subspace Theorem [9].

Lemma 4.1. *Let $\varepsilon > 0$, $0 < \alpha < 1 \leq M$, and $\beta \in [0, \frac{1+\varepsilon}{\varepsilon})$. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of numbers in $\mathbb{C}$ that satisfy inequalities* (1) *and* (2). *If*

$$\limsup_{n\to\infty} |a_n|^{\left(\frac{M}{1-\beta}+1\right)^{-n}} = \infty,$$

then for all fixed $0 < c < 1$,

$$\liminf_{N\to\infty} \sum_{n=N+1}^{\infty} \left|\frac{b_n}{a_n}\right| \left(\prod_{n=1}^{N} |a_n|^M\right) 2^{N^2 \log_2^c |a_{N-1}|} = 0.$$

Lemma 4.2. *Let $x_1, \dots, x_d, s$ be algebraic numbers such that s is $\mathbb{Q}$-linearly independent of $x_1, \dots, x_d$, and let $C, \delta > 0$. Then the inequality*

$$\left| qs - \sum_{i=1}^{d} p_i x_i \right| \prod_{i=1}^{d} \max\{1, |p_i|\} < q^{-\delta} \tag{13}$$

has only finitely many solutions $(p_1, \dots, p_d, q) \in \mathbb{Z}^d \times \mathbb{N}$ with $|p_i| \le q^C$.

The below lemma can be extracted from the proof of Lemma 15 in [7] and replaces the triangle inequality from the series setting.

Lemma 4.3. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that the infinite product $\prod_{n=1}^{\infty} |1 + a_n|$ converges monotonously. Then, for all N,*

$$\left| \prod_{n=1}^{\infty} (1 + a_n) - \prod_{n=1}^{N} (1 + a_n) \right| \le \max\left\{1, \prod_{n=1}^{\infty} |1 + a_n| \right\} \sum_{n=N+1}^{\infty} |a_n|.$$

We now present certain notions to further describe algebraic numbers. Let a be an algebraic number with minimal polynomial $\sum_{i=0}^{d} c_i X^i$ over the integers, with $c_d > 0$. The leading coefficient, c_d, is also called the *denominator* of a, since $c_d a$ is an algebraic integer while $c' a$ is not for any rational integer $0 < c' < c_d$.

By rewriting the minimal polynomial of a as $c_d \prod_{i=1}^{d} (X - a_i)$ instead, we define the Mahler measure as

$$M(a) := c_d \prod_{i=1}^{d} \max\{1, |a_i|\}.$$

Surprisingly closely related to this is the Weil height, which we define as

$$H(\alpha) := \prod_{\nu \in M_{\mathbb{K}}} \max\{1, |a|_\nu\}^{[\mathbb{K}_\nu : \mathbb{Q}_\nu]/[\mathbb{K}:\mathbb{Q}]},$$

where $\mathbb{K}$ is any number field containing a, $M_{\mathbb{K}}$ denotes the set of places of $\mathbb{K}$, $\mathbb{K}_\nu$ is the local field of $\mathbb{K}$ at ν, and $[\mathbb{K} : \mathbb{K}']$ denotes the degree of a field extension $\mathbb{K} \supseteq \mathbb{K}'$. This does not depend on the choice of $\mathbb{K}$ (see [10] for a proof). We will compare and estimate the house, Mahler measure, and Weil height using the following classical results.

Lemma 4.4. *Let a be an algebraic number with denominator c_d. Then*

$$H(\alpha)^d = M(\alpha) \le |c_d| \max\{1, \overline{|\alpha|}^d\}.$$

Proof. The inequality is a trivial consequence of the definitions. For the equality, see Lemma 3.10 of [10]. □

Lemma 4.5. *Let $a_1, \dots, a_n \in \overline{\mathbb{Q}}$ with $a_1 \neq 0$. Then*

$$H(a_1 + \dots + a_n) \le nH(a_1) \cdots H(a_n), \quad H(a_1 a_2) \le H(a_1)H(b_1),$$
$$H\left(1/a_1\right) = H(a_1).$$

Proof. See [10]. □

Lemma 4.6 (Liouville Inequality)**.** *Let α be a non-zero algebraic number. Then*

$$|\alpha| \geq \left(2H(\alpha)\right)^{-\deg(\alpha)}.$$

Proof. This can be extracted from [1, Theorem A.1]. □

Finally, we will be using the below lemma to compare the house with a given max norm on a finite field $\mathbb{K}$ seen as a $\mathbb{Q}$-vector space.

Lemma 4.7. *Let $a_1, \dots, a_d \in \overline{\mathbb{Q}}$. Then there is a constant $C_1 > 0$, depending only on $a_1, \dots, a_d$, so that for any $(c_1, \dots, c_d) \in \mathbb{Q}^d$,*

$$\overline{|c_1a_1 + c_2a_2 + \cdots + c_da_d|} \leq C_1 \max_{1\leq i\leq d} |c_i|.$$

If $a_1, \dots a_d$ are linearly independent over $\mathbb{Q}$, then there also is a constant $C_2 > 0$ such that

$$\overline{|c_1a_1 + c_2a_2 + \cdots + c_da_d|} \geq C_2 \max_{1\leq i\leq d} |c_i|.$$

Proof. The first statement is proven in [8], so we limit our attention to the second one. We make the proof by induction. The lemma is trivial for $d = 1$, so suppose $d > 1$ and that the statement holds for $d' = d-1$. If $c_i = 0$ for any i, the induction is trivial, so suppose not. Write

$$\alpha = \frac{c_1a_1 + c_2a_2 + \cdots + c_da_d}{a_d} = c_1\frac{a_1}{a_d} + c_2\frac{a_2}{a_d} + \cdots + c_{d-1}\frac{a_{d-1}}{a_d} + c_d.$$

Since $a_1, \dots, a_d$ are linearly independent over $\mathbb{Q}$, $c_1, \dots, c_d \neq 0$, and $d > 1$, then α must be irrational. Let $\sigma_1, \dots, \sigma_D$ be the distinct embeddings of $\mathbb{Q}(a_1, \dots, a_n)$ into $\overline{\mathbb{Q}}$.

We then have by the induction assumption that

$$\max_{1\leq i\leq D} \overline{|\alpha - \sigma_i(\alpha)|} = \max_{1\leq i\leq D} \overline{\left|\sum_{j=1}^{d-1} c_j\left(\frac{a_j}{a_d} - \sigma_i\left(\frac{a_j}{a_d}\right)\right)\right|} \geq C_2' \max_{1\leq i<d}\{|c_i|\},$$

for some $C_2' > 0$ that depends only on $a_1/a_d, \dots, a_{d-1}/a_d$ and the set $\{\sigma_1, \dots, \sigma_D\}$, which in turn only depend on $a_1, \dots, a_d$. Pick j such that $\overline{|\alpha - \sigma_j\alpha|} = \max_{1\leq i\leq D} \overline{|\alpha - \sigma_i(\alpha)|}$. Then

$$\max_{1\leq i\leq D} \overline{|\alpha - \sigma_i(\alpha)|} = |\sigma_k(\sigma_j(\alpha)) - \sigma_k(\alpha)|,$$

for a suitable k. Noticing that $1/\overline{|1/a_d|}$ is the modulus of the smallest conjugate of a_d, we then have

$$\overline{\left|\sum_{i=1}^{d} c_ia_i\right|} = \overline{|a_d\alpha|} \geq \frac{\overline{|\alpha|}}{\overline{|1/a_d|}} \geq \frac{\max\{|\sigma_k(\sigma_j(\alpha))|, |\sigma_k(\alpha)|\}}{\overline{|1/a_d|}}$$

$$\geq \frac{|\sigma_k(\sigma_j(\alpha)) - \sigma_k(\alpha)|}{2\overline{|1/a_d|}} \geq \frac{C_2'}{2\overline{|1/a_d|}} \max_{1\leq i<d}\{|c_i|\},$$

and the proof is complete. □

5. Proof of Main Theorems

We will first prove the irrationality result of Theorem 2.5, which we state below as a separate result for future reference.

Proposition 5.1. *Use the notation of Theorem 2.2 except that we assume $z_2 \geq 0$ rather than $z_2 \geq 1$. Suppose assumptions* (1)-(6) *and* (12) *are satisfied. Then the number $\prod_{n=1}^{\infty}(1+b_n/a_n)$ is not contained in $\mathbb{K}$ if either condition* (8) *or* (9) *is satisfied. If, for all $n \in \mathbb{N}$, $b_n \in \mathbb{Z}$ or $a_n \in \mathbb{Z}$, then we may replace condition* (9) *with condition* (10).

Theorem 2.2, combining ideas from [7] and [8].

Proof of Theorem 2.2. Write

$$\gamma = \prod_{n=1}^{\infty}\left(1+\frac{b_n}{a_n}\right) \quad \text{and} \quad \gamma_N = \prod_{n=1}^{N-1}\left(1+\frac{b_n}{a_n}\right),$$

and assume towards contradiction that $\gamma \in \mathbb{K}$.

We start by boudning $H(a_n/b_n)$. Applying Lemmas 4.5 and 4.4 followed by inequalities (3) and (4), we find

$$H\left(\frac{b_n}{a_n}\right) = H\left(\frac{a_n}{b_n}\right) \leq \begin{cases} \max\{|a_n|, \overline{|b_n|}\} & \text{if } a_n \in \mathbb{Z}, \\ \max\{|b_n|, \overline{|a_n|}\} & \text{if } b_n \in \mathbb{Z}, \\ H(a_n)H(b_n) \leq \overline{|a_n|}\,\overline{|b_n|} & \text{regardless} \end{cases}$$
$$\leq \begin{cases} |a_n|^{\max\{y_1,y_2\}} 2^{\log_2^{\alpha} a_n} & \text{if } \forall\, m \in \mathbb{N} : a_m \in \mathbb{Z} \text{ or } b_m \in \mathbb{Z}, \\ |a_n|^{y_1+y_2} 2^{2\log_2^{\alpha} a_n} & \text{otherwise.} \end{cases}$$

Instead using only Lemma 4.4, we have

$$H\left(\frac{b_n}{a_n}\right) \leq \left| r_n \mathcal{N}\left(\frac{a_n}{r_n}\right)\right|^{1/\deg(b_n/a_n)} \max\left\{1, \overline{\left|\frac{b_n}{a_n}\right|}\right\}$$
$$\leq \left| r_n \mathcal{N}\left(\frac{a_n}{r_n}\right)\right|^{1/d_0} \max\left\{1, \overline{|a_n^{-1}|}\,\overline{|b_n|}\right\},$$

so that inequalities (4), (5), and (6) yield

$$H\left(\frac{b_n}{a_n}\right) \leq |a_n|^{y_2+z_1+z_2/d_0} 2^{3\log_2^{\alpha}|a_n|}.$$

Writing $y = \min\{y', y_2 + z_1 + z_2/d_0\}$ where

$$y' = \begin{cases} \max\{y_1, y_2\} & \text{if for all } n \in \mathbb{N} : a_n \in \mathbb{Z} \text{ or } b_n \in \mathbb{Z}, \\ y_1 + y_2 & \text{otherwise,} \end{cases}$$

the above bounds on $H(a_n/b_n)$ combine to

$$H\left(\frac{b_n}{a_n}\right) \leq |a_n|^{y} 2^{3\log_2^{\alpha}|a_n|}.$$

Then, by Lemma 4.5,

$$H(\gamma-\gamma_N) \le 2H(\gamma)\prod_{n=1}^{N-1}\left(2H\left(\frac{a_n}{b_n}\right)\right) \le 2^{2N\log_2^\alpha|a_{N-1}|}\prod_{n=1}^{N-1}|a_n|^y, \tag{14}$$

for large values of N.

We now wish to apply Lemma 4.6. By inequalities (1) and (2), $|1+b_n/a_n|>0$ for all n. Hence, inequality (12) and the fact that $\Re(a_n/b_n)\neq -1/2$ infinitely often makes $|\gamma_N|$ monotonous with $|\gamma_N|\neq|\gamma|$, being non-decreasing for $e=1$ and non-increasing for $e=-1$. Hence $\gamma-\gamma_N\neq 0$ are different for all large enough N, and we may thus apply the lemma and then inequality (14) to find

$$|\gamma-\gamma_N| \ge (2H(\gamma-\gamma_N))^{-\deg(\gamma-\gamma_N)} \ge \left(2^{3N\log_2^\alpha|a_{N-1}|}\prod_{n=1}^{N-1}|a_n|^y\right)^{-d}.$$

Hence, by Lemma 4.3,

$$\prod_{n=1}^{N-1}|a_n|^{-dy} \le 2^{3dN\log_2^\alpha|a_{N-1}|}|\gamma-\gamma_N| \le 2^{4dN\log_2^\alpha|a_{N-1}|}\sum_{n=N}^{\infty}\left|\frac{b_n}{a_n}\right|,$$

and thereby

$$\lim_{N\to\infty}\sum_{n=N+1}^{\infty}\left|\frac{b_n}{a_n}\right|\left(\prod_{n=1}^{N}|a_n|^{dy}\right)2^{N^2\log_2^\alpha|a_{N-1}|}=\infty.$$

Recalling the choice of y along with the relevant assumption (8), (9), or (10), this contradicts Lemma 4.1. This completes the proof. □

We are now left to prove Theorem 2.4. This will follow from Theorem 2.2 and the below result, the proof of which takes inspiration from [8].

Theorem 5.2. *Using the notation of Theorem 2.4, suppose all assumptions of that theorem, except that inequality* (12) *does not have to be satisfied and the assumption $z_2\ge 1$ is replaced with $z_2>0$. Then the number $\prod_{n=1}^{\infty}(1+b_n/a_n)$ is either transcendental or contained in $\mathbb{K}$.*

To prove this theorem, we will need the below lemma.

Lemma 5.3. *Use the notation and assumptions of Theorem 5.2. Write $M=d(y+z_1+z_2)+z_2+\delta$, and let $x_1,\dots,x_d$ span $\mathcal{O}_{\mathbb{K}}$ as a $\mathbb{Z}$-module. Then there exist sequences $\{p_{1,N}\}_{N=1}^{\infty}$, …, $\{p_{d,N}\}_{N=1}^{\infty}$, and $\{q_N\}_{N=1}^{\infty}$ with $p_{i,N}\in\mathbb{Z}$ and $q_N\in\mathbb{N}$ with $q_N>2^N$ such that the inequalities*

$$|p_{i,N}| \le 2^{N\log_2^\alpha q_N} q_N^{1+\frac{y+z_1}{z_2}} \tag{15}$$

are satisfied for each $i=1,\dots,d$ and all sufficiently large N, while

$$\left|\sum_{n=1}^{\infty}\frac{b_n}{a_n}-\frac{\sum_{i=1}^{d}p_{i,N}x_i}{q_N}\right| < \frac{1}{2^{dN\log_2^\alpha q_N}q^{M/z_2}} \tag{16}$$

is satisfied for infinitely many $N\in\mathbb{N}$.

Proof. Clearly, $(x_1, \dots, x_d)$ forms a $\mathbb{Q}$-linear basis of $\mathbb{K}$. Let $\pi_1, \dots, \pi_d : \mathbb{K} \to \mathbb{Q}$ denote the associated coordinate maps. Notice that π_i maps integers to integers $\pi_i(\alpha) \in \mathbb{Z}$ for all $\alpha \in \mathcal{O}_{\mathbb{K}}$.

For each $n \in \mathbb{N}$, pick $\tilde{a}_n \in \mathbb{Z}$ of minimal modulus such that

$$\tilde{a}_{n+1} r_{n+1} \mathcal{N}(a_{n+1}/r_{n+1}) \geq \tilde{a}_n r_n \mathcal{N}(a_n/r_n) \geq |a_n|^{z_2},$$

noting by inequality (6) that then

$$\tilde{a}_n r_n \mathcal{N}(a_n/r_n) \leq 2|a_n|^{z_2} 2^{\log_2^\alpha |a_n|}. \tag{17}$$

Then choose

$$q_N = \prod_{n=1}^{N-1} \left(\tilde{a}_n r_n \mathcal{N}\left(\frac{a_n}{r_n}\right)\right) \quad \text{and} \quad p_{i,N} = \pi_i \left(q_N \prod_{n=1}^{N-1} \left(1 + \frac{b_n}{a_n}\right)\right).$$

Clearly, $q_N \in \mathbb{N}$ with $q_N > 2^N$ for large values of N, while we have $p_{i,N} \in \mathbb{Z}$ since

$$\left(q_N \prod_{n=1}^{N-1} \left(1 + \frac{b_n}{a_n}\right)\right) = \prod_{n=1}^{N-1} \left(\tilde{a}_n (a_n + b_n) \frac{\mathcal{N}(a_n/r_n)}{a_n/r_n}\right) \in \mathcal{O}_{\mathbb{K}}.$$

Noticing

$$\prod_{n=1}^{N} \left(1 + \frac{b_n}{a_n}\right) = \sum_{\mathcal{S} \subseteq \{1,\dots,N\}} \prod_{n \in \mathcal{S}} \frac{b_n}{a_n},$$

it follows from linearity of π_i and the triangle inequality

$$\left|\frac{p_{i,N}}{q_N}\right| = \left| \pi_i \left(\prod_{n=1}^{N-1} \left(1 + \frac{b_n}{a_n}\right)\right)\right| \leq \sum_{\mathcal{S} \subseteq \{1,\dots,N\}} \left| \pi_i \left(\prod_{n \in \mathcal{S}} \frac{b_n}{a_n}\right)\right|.$$

By Lemma 4.7, there then is a $C \geq 1$ such that

$$\left|\frac{p_{i,N}}{q_N}\right| \leq \sum_{\mathcal{S} \subseteq \{1,\dots,N\}} C \boxed{\prod_{n \in \mathcal{S}} \frac{b_n}{a_n}} \leq 2^N C \prod_{n=1}^{N-1} \max\left\{1, \boxed{b_n}\boxed{a_n^{-1}}\right\}.$$

Hence, by inequalities (4) and (5) along with the fact that $z_1 \geq -y$,

$$\left|\frac{p_{i,N}}{q_N}\right| \leq 2^N C \prod_{n=1}^{N-1} |a_n|^{y+z_1} 2^{2\log_2^\alpha |a_n|} \leq 2^{z_2^\alpha N \log_2^\alpha |a_{N-1}|} \prod_{n=1}^{N-1} |a_n|^{y+z_1}$$

for all sufficiently large N. By inequality (17) and the choice of q_N, we conclude inequality (15), since

$$\begin{aligned} \left|\frac{p_{i,N}}{q_N}\right| &< 2^{N \log_2^\alpha(\tilde{a}_n r_n \mathcal{N}(a_n/r_n))} \prod_{n=1}^{N-1} (\tilde{a}_n r_n \mathcal{N}(a_n/r_n))^{\frac{y+z_1}{z_2}} \\ &\leq 2^{N \log_2^\alpha q_N} q_N^{\frac{y+z_1}{z_2}}. \end{aligned}$$

We now just have to ensure that inequality (16) holds for infinitely many N. The choice of $p_{i,N}$ and Lemma 4.3 show that

$$\left|\prod_{n=1}^{\infty}\left(1+\frac{b_n}{a_n}\right)-\frac{\sum_{i=1}^{d}p_{i,N}x_i}{q_N}\right| = \left|\prod_{n=1}^{\infty}\left(1+\frac{b_n}{a_n}\right)-\prod_{n=1}^{N-1}\left(1+\frac{b_n}{a_n}\right)\right| \le C'\sum_{n=N}^{\infty}\left|\frac{b_n}{a_n}\right|$$

holds for all sufficiently large N and a suitable $C' \ge 1$. By inequality (11) and the choice of M, Lemma 4.1 then implies that

$$\left|\prod_{n=1}^{\infty}\left(1+\frac{b_n}{a_n}\right)-\frac{\sum_{i=1}^{d}p_{i,N}x_i}{q_N}\right| < \left(\prod_{n=1}^{N}|a_n|^{-M}\right)2^{-N^2\log_2^\alpha|a_{N-1}|}$$

for infinitely many N. For these N, the choices of $\tilde{a}_n$ and q_N now allow us to conclude inequation (16) by calculating

$$\left|\prod_{n=1}^{\infty}\left(1+\frac{b_n}{a_n}\right)-\frac{\sum_{i=1}^{d}p_{i,N}x_i}{q_N}\right| < \frac{q_N^{-M/z_2}}{2^{2dN^{2-\alpha}\log_2^\alpha((r_{N-1}\tilde{a}_{N-1}\mathcal{N}(a_n/r_n))^N)/z_1^\alpha}} \le \frac{1}{2^{dN\log_2^\alpha q_N}q_N^{M/z_2}}.$$

This completes the proof. □

Proof of Theorem 5.2. By Lemma 5.3, infinitely many $(p_1,\ldots,p_d,q)\in\mathbb{Z}^d\times\mathbb{N}$ satisfy both inequalities (15) and (16) with $|p_i|<q^C$ for some fixed $C>0$ that does not depend on $(p_1,\ldots,p_d,q)$. We first rewrite inequality (16) by recalling $M=d(y+z_1+z_2)+z_2+\delta$,

$$\left|q\prod_{n=1}^{\infty}\left(1+\frac{b_n}{a_n}\right)-\sum_{i=1}^{d}p_ix_i\right|\prod_{i=1}^{d}\left(q^{1+\frac{d(y+z_1+z_2)}{z_2}}2^{N\log_2^\alpha q}\right) < q^{-\delta/z_2}.$$

It then follows from the inequalities of (15) that

$$\left|q\prod_{n=1}^{\infty}\left(1+\frac{b_n}{a_n}\right)-\sum_{i=1}^{d}p_ix_i\right|\prod_{i=1}^{d}\max\{1,|p_i|\} < q^{-\delta/z_2}.$$

Lemma 4.2 now implies that $\prod_{n=1}^{\infty}\left(1+b_n/a_n\right)$ cannot both be algebraic and $\mathbb{Q}$-linearly independent of $x_1,\ldots,x_d$. In other words, the number $\prod_{n=1}^{\infty}\left(1+b_n/a_n\right)$ is either contained in $\mathbb{K}$ or transcendental, and the proof is complete. □

Proof of Theorem 2.5. Replace δ and z_2 with $\delta'=\delta/(d+2)$ and $z_2'=z_2+\delta'$, respectively. Then the statement follows from Proposition 5.1 and Theorem 5.2. □

Proof of Theorems 2.2 and 2.4. Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of positive integers. Replacing a_n and r_n by a_nc_n and c_nr_n, respectively, then invalidates neither $a_n\ge n^{1+\varepsilon}$ nor any of the inequalities (2)-(7) since we clearly have $\varepsilon, \beta, y_2 \geq 0$, $y_1, z_2 \geq 1$, and $z_1 \geq -1$. Rearranging the terms of $\{a_n c_n/b_n\}_{n=1}^{\infty}$ so that $|a_n c_n|$ becomes non-decreasing then allows us to apply Theorem 2.5 on the number $\prod_{n=1}^{\infty}\left(1+\frac{b_n}{a_n c_n}\right)$. This completes the proof. □

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M. L. Laursen, Department of Mathematics, Aarhus University, Ny Munkegade 118, DK-8000 Aarhus C, Denmark

Email address: mathiaslaursen@hotmail.com

1.7 Algebraic independence of infinite series

This section presents the recent paper [23] by Hančl, Kristensen, and the current author. The paper contains new criteria for linear and algebraic independence of finite sets real numbers expressed as infinite series of rational numbers as well as new notions for irrationality and transcendence of sequences.

Recall from the end of Section 1.1 that a list of complex numbers $(a_1, \ldots, a_K)$ is algebraically independent if $P(a_1, \ldots, a_K) \neq 0$ for all integer polynomials $P \in \mathbb{Z}[X_1, \ldots, X_K]$ in K variables. When $K = 1$, this is just a repetition of the definition of transcendence. When $K \geq 2$, however, we have a property that is strictly stronger than being linearly independent over $\overline{\mathbb{Q}}$, which is the best we can get from any of the papers presented in the preceding sections.

The new notions of irrationality and transcendence are as follows. Notice that the main difference from Definitions 1.4 and 1.5 is that we assume $p \nmid c_n$ rather than assuming $c_n > 0$.

Definition 1.72. Let $\{a_{1,n}\}_{n=1}^{\infty}, \ldots, \{a_{K,n}\}_{n=1}^{\infty}$ be sequences of real or complex numbers, and let $\mathcal{P}$ be a non-empty set of prime numbers.

- The $(\Sigma, \mathcal{P})$-*expressible* set of $\{a_{1,n}\}_{n=1}^{\infty}, \ldots, \{a_{K,n}\}_{n=1}^{\infty}$ is defined as

$$E_{\Sigma,\mathcal{P}}\big(\{a_{i,n}\}_{n=1}^{\infty}\big)_{i=1}^{K} := \left\{ \left(\sum_{n=1}^{\infty} \frac{1}{a_{i,n} c_n} \right)_{i=1}^{K} : c_n \in \mathbb{Z} \setminus \bigcup_{p \in \mathcal{P}} p\mathbb{Z} \text{ for all } n \in \mathbb{N} \right\}.$$

- The sequences $\{a_{1,n}\}_{n=1}^{\infty}, \ldots, \{a_{K,n}\}_{n=1}^{\infty}$ are said to be $(\Sigma, \mathcal{P})$-*linearly independent over* $\mathbb{Q}$ if the numbers $1, \xi_1, \ldots, \xi_K$ are linearly independent over $\mathbb{Q}$ for all lists $(\xi_1, \ldots, \xi_d)$ in $E_{\Sigma,\mathcal{P}}(\{a_{1,n}\}_{n=1}^{\infty}, \ldots, \{a_{K,n}\}_{n=1}^{\infty})$

- The sequences $\{a_{1,n}\}_{n=1}^{\infty}, \ldots, \{a_{K,n}\}_{n=1}^{\infty}$ are said to be $(\Sigma, \mathcal{P})$-*algebraically independent* if each list of numbers in $E_{\Sigma,\mathcal{P}}(\{a_{1,n}\}_{n=1}^{\infty}, \ldots, \{a_{K,n}\}_{n=1}^{\infty})$ is algebraically independent.

- We say that a single sequence $\{a_n\}_{n=1}^{\infty}$ is $(\Sigma, \mathcal{P})$-*irrational* (respectively $(\Sigma, \mathcal{P})$-*transcendental*) if all elements of $E_{\Sigma,\mathcal{P}}\{a_n\}_{n=1}^{\infty}$ are irrational (respectively transcendental).

When $\mathcal{P} = \{p\}$, we may write (Σ, p) in place of $(\Sigma, \{p\})$ for each of the above definitions.

Inspired by [15,16], the definitions of $(\Sigma, \mathcal{P})$-irrationality and $(\Sigma, \mathcal{P})$-transcendence (including when $\mathcal{P}$ is replaced by p) were introduced in [23], while the notions of $(\Sigma, \mathcal{P})$-linear independence and $(\Sigma, \mathcal{P})$-algebraic independence do not appear to have been used before the present thesis.

In parallel to Definition 1.5, one may generalize to notions of $(X_{\mathbb{K}}, \mathcal{P})$-irrationality, $(X, \mathcal{P})$-transcendence, $(X, \mathcal{P})$-linear independence over $\mathbb{K}$, and $(X, \mathcal{P})$-algebraic independence when X is any one of the labels Σ, Π, and CF. We will, however, only need the notions presented in Definition 1.72.

Notice that in all generalizations of Erdős' Theorem 1.3 presented so far in this chapter, there has been one or more bounds related to the real value of a_n (or to a complex number times a_n). When dealing with the p- and $\mathcal{P}$-variants of irrationality, transcendence, and so on, we instead introduce certain bounds on the p-adic valuation of a_n. The *p-adic valuation*, denoted ν_p, is defined so that $\nu_p(0) = \infty$ and $\nu_p(n) \in \mathbb{N}_0$ with $p^{\nu_p(n)} \mid n$ and $p^{\nu_p(n)+1} \nmid n$ when n is a non-zero integer. We then get the following alternative version of Theorem 1.3, which combines Theorems 5 and 10 from [23].

Theorem 1.73 (Hančl–Kristensen–Laursen, 2025 on arXiv)**.** *Let $\alpha \in (0,1)$, let $\varepsilon > 0$, let p be a prime number, and let $C_p > 0$. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of non-zero integers with $p \nmid \gcd(a_n, b_n)$ such that for each $n \in \mathbb{N}$,*

$$\nu_p(a_n) \le C_p \text{ or } \nu_p(a_n) \ne \nu_p(a_1), \ldots, \nu_p(a_{n-1}) \tag{1.19}$$

and

$$\sup_{n\in\mathbb{N}} \nu_p(a_n) = \infty. \tag{1.20}$$

Suppose that

$$|a_{n+1}| \ge |a_n| \ge n^{1+\varepsilon}, \quad |b_n| \le 2^{\log_2^{\alpha} |a_n|}, \quad \text{and} \quad \limsup_{n\to\infty} |a_n|^{2^{-n}} = \infty. \tag{1.21}$$

Then the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ is (Σ, p)-irrational.

Suppose we replace equation (1.20) by $\sup_{n\in\mathbb{N}} \nu_p(a_n) > C_p$. If the assumptions are now all satisfied for all p in an infinite set of prime numbers $\mathcal{P}$, then $\{a_n/b_n\}_{n=1}^{\infty}$ is $(\Sigma, \mathcal{P})$-irrational.

If furthermore, $\limsup_{n\to\infty} |a_n|^{d^{-n}} = \infty$ for all $d \in \mathbb{N}$, then the above statements of p- and $\mathcal{P}$-irrationality may be replaced by statements of p- and $\mathcal{P}$-transcendence, respectively.

Notice that the *limsup* condition required for transcendence is rather strict and no more lenient than that of Theorem 1.11. Having seen Theorems 1.13 and 1.50, it would only be natural for the reader to expect a weakening of this assumption. Not wanting to disappoint, let us consider the below result, which is a special case of [38, Theorem 5.1] by the current author.

Theorem 1.74 (Laursen, 2024)**.** *Let $\alpha \in (0,1)$ and $\delta, \varepsilon > 0$. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of non-zero integers that satisfy assumption (1.21) and*

$$\limsup_{n\to\infty} a_n^{(3+\delta)^{-n}} = \infty.$$

Let $\{c_n\}_{n=1}^{\infty}$ *be a sequence of positive integers. Then the number* $\sum_{n=1}^{\infty} b_n/(a_n c_n)$ *is either transcendental or rational.*

Notice that the above theorem makes no assumptions on valuation or sign of a_n other than $a_n \neq 0$. Combined with the irrationality statement of Theorem 1.73, we reach the following theorem.

Theorem 1.75. *Let* $\alpha \in (0,1)$, *let* $\delta, \varepsilon > 0$, *let* p *be a prime number, and let* $C_p > 0$. *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be sequences of non-zero integers with* $p \nmid \gcd(a_n, b_n)$ *such that assumptions* (1.19)–(1.21) *are satisfied for all* $n \in \mathbb{N}$. *If*

$$\limsup_{n\to\infty} a_n^{(3+\delta)^{-n}} = \infty,$$

Then the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ *is* (Σ, p)*-transcendental.*

Suppose we replace equation (1.20) *by* $\sup_{n\in\mathbb{N}} \nu_p(a_n) > C_p$. *If the assumptions are now all satisfied for all* p *in an infinite set of prime numbers* $\mathcal{P}$, *then* $\{a_n/b_n\}_{n=1}^{\infty}$ *is* $(\Sigma, \mathcal{P})$*-transcendental.*

Proof. Let $\mathcal{P}$ be either the set containing the single prime number p or the infinite set of prime numbers from the theorem, depending on which assumptions are made. Let $\{c_n\}_{n=1}^{\infty}$ be a sequence of integers with $p \nmid c_n$ for all $p \in \mathcal{P}$. We are then to show that the number $\sum_{n=1}^{\infty} b_n/(a_n c_n)$ is transcendental. We already know that it is irrational from Theorem 1.73. Transcendence now follows from Theorem 1.74 by rewriting

$$\sum_{n=1}^{\infty} \frac{b_n}{a_n c_n} = \sum_{n=1}^{\infty} \frac{b_n}{(a_n c_n/|c_n|)|c_n|}. \qquad \square$$

Like most other results from [23], Theorem 1.73 is proven as a corollary to Theorems 1 and 6 of the paper. Combined, these theorems read as follows.

Theorem 1.76 (Hančl–Kristensen–Laursen, 2025 on arXiv)**.** *Let* K *and* d *be positive integers, let* $\alpha \in (0,1)$, *let* $\varepsilon > 0$, *and let* p *be a prime number. For* $i = 1, \ldots, K$, *let* $\{a_{i,n}\}_{n=1}^{\infty}$ *and* $\{b_{i,n}\}_{n=1}^{\infty}$ *be sequences of non-zero integers with* $p \nmid \gcd(a_{i,n}, b_{k,n})$ *such that for each sufficiently large* $N \in \mathbb{N}$,

$$\nu_p(a_{i,n}) = \max_{1\le m\le N} \nu_p(a_{i,m})$$

for exactly one $n \le N$ *and*

$$\lim_{N\to\infty} \Big(\max_{1\le n\le N} \nu_p(a_{i,n}) - d \max_{1\le n\le N} \nu_p(a_{i-1,n}) \Big) = \infty, \tag{1.22}$$

writing $a_{0,n} = 1$ *for all* n. *Suppose there is a sequence* $\{a_n\}_{n=1}^{\infty}$ *of integers such that for every* $i = 1, \ldots, K$ *and each sufficiently large* $n \in \mathbb{N}$,

$$n^{1+\varepsilon} \le a_n \le a_{n+1}, \qquad |b_{i,n}| \le 2^{(\log_2 a_n)^{\alpha}},$$

$$a_n 2^{-(\log_2 a_n)^\alpha} \le |a_{i,n}| \le \max\left\{a_n 2^{(\log_2 a_n)^\alpha}, 2^{n^{-3}(Kd+1)^n}\right\},$$

and

$$\limsup_{n\to\infty} a_n^{(Kd+1)^{-n}} = \infty.$$

Then the list of numbers $\left(\sum_{n=1}^\infty b_{1,n}/a_{1,n}, \ldots, \sum_{n=1}^\infty b_{K,n}/a_{K,n}\right)$ *is not the root of any non-zero polynomial of* K *variables with integer coefficients and degree less than or equal to* d.

The same is true if we replace equation (1.22) *by*

$$\max_{n=1,\ldots,N} \nu_p(a_{i,n}) > d \max_{n=1,\ldots,N} \nu_p(a_{i-1,n}), \tag{1.23}$$

and assume that all assumptions are satisfied for not one but infinitely many distinct prime numbers p.

Suppose $K = 2$ *and that* d *is replaced by 3 in equation* (1.22) *and inequality* (1.23). *If all assumptions are otherwise satisfied, then the pair of numbers* $\left(\sum_{n=1}^\infty b_{1,n}/a_{1,n}, \sum_{n=1}^\infty b_{2,n}/a_{2,n}\right)$ *is non-degenerately independent of order* d.

The term *non-degenerately independent of order* d is introduced in [23] and means that the two numbers considered are not the root of any $P \in \mathbb{Z}[X_1, X_2]$ of degree at most d unless that P belongs to a rather sparse family of polynomials. To be precise, this family consists of exactly those polynomials of degree 4 or greater for which the set of solutions to $P(x_1, x_2) = 0$ over $\mathbb{C}$ defines a manifold of geometric genus at most 1. A pair of numbers is called *non-degenerately independent* if it is non-degenerately independent of order d for all $d \in \mathbb{N}$.

The remaining results of [23] regard algebraic independence of numbers [23, Theorems 2 and 7] and non-degenerate independence [23, Theorems 3 and 8] (see subsection 1.7.2).

We now extract the following criteria for linear and algebraic independence of the (Σ, p) and $(\Sigma, \mathcal{P})$ types, which are not explicitly stated in the paper.

Theorem 1.77. *Let* K *and* d *be positive integers, let* $\alpha \in (0,1)$, *let* $\varepsilon > 0$, *and let* p *be a prime number. For* $i = 1, \ldots, K$, *let* $\{a_{i,n}\}_{n=1}^\infty$ *and* $\{b_{i,n}\}_{n=1}^\infty$ *be sequences of non-zero integers with* $p \nmid \gcd(a_{i,n}, b_{k,n})$ *such that for all* $n \in \mathbb{N}$,

$$\Big[\nu_p(a_{i,n}) \le C \quad or \quad \nu_p(a_{i,n}) \ne \nu_p(a_1), \ldots, \nu_p(a_{n-1})\Big], \quad \nu_p(a_{i,n}) \ge \nu_p(a_{i-1,n}), \tag{1.24}$$

and

$$\limsup_{n\to\infty} \big(\nu_p(a_{i,n}) - \nu_p(a_{i-1,n})\big) = \infty, \tag{1.25}$$

writing $a_{0,n} = 1$ *for all* n. *Suppose there is a sequence* $\{a_n\}_{n=1}^\infty$ *of integers such that for every* $i = 1, \ldots, K$ *and each sufficiently large* $n \in \mathbb{N}$,

$$n^{1+\varepsilon} \le a_n \le a_{n+1}, \quad |b_{i,n}| \le 2^{(\log_2 a_n)^\alpha},$$

$$a_n 2^{-(\log_2 a_n)^\alpha} \le |a_{i,n}| \le \max\left\{a_n 2^{(\log_2 a_n)^\alpha}, 2^{n^{-3}(Kd+1)^n}\right\},$$

and

$$\limsup_{n\to\infty} a_n^{(K+1)^{-n}} = \infty. \tag{1.26}$$

Then the sequences $\{a_{1,n}/b_{1,n}\}_{n=1}^\infty, \dots, \{a_{K,n}/b_{K,n}\}_{n=1}^\infty$ *are* (Σ, p)*-linearly independent over* $\mathbb{Q}$*. If furthermore, for all* $A \in \mathbb{N}$,

$$\limsup_{n\to\infty} \frac{\nu_p(a_{i,n})}{1+\nu_p(a_{i-1,n})} = \infty \quad \textit{and} \quad \limsup_{n\to\infty} a_n^{A^{-n}} = \infty, \tag{1.27}$$

then the sequences $\{a_{1,n}/b_{1,n}\}_{n=1}^\infty, \dots, \{a_{K,n}/b_{K,n}\}_{n=1}^\infty$ *are* (Σ, p)*-algebraically independent.*

Let $\mathcal{P}$ *be an infinite set of prime numbers. Replace inequality* (1.25) *by*

$$\limsup_{n\to\infty} \big(\nu_p(a_{i,n}) - \max\{C, \nu_p(a_{i-1,n})\}\big) > 0, \tag{1.28}$$

and assumption (1.27) *by*

$$\limsup_{n\to\infty} \big(\nu_p(a_{i,n}) - \max\{C, A\nu_p(a_{i-1,n})\}\big) > 0 \quad \textit{and} \quad \limsup_{n\to\infty} a_n^{A^{-n}} = \infty. \tag{1.29}$$

Suppose that any assumption made for p *is made for all* $p \in \mathcal{P}$*. Then the above statements remain true but with '*(Σ, p)*-linearly independent' and '*(Σ, p)*-algebraically independent' replaced by '*$(\Sigma, \mathcal{P})$*-linearly independent' and '*$(\Sigma, \mathcal{P})$*-algebraically independent', respectively.*

Notice that assumption (1.29) implies (1.27) for $i > 1$ but not for $i = 1$. The proof of this theorem is essentially the same as that of Theorem 1.73, but we will repeat it here for clarity, modifying the details to fit the current statement.

Proof. Let $\mathcal{P}$ be either the set containing the single prime number p or the infinite set of prime numbers from the theorem, depending on which assumptions are made. Let $\{c_n\}_{n=1}^\infty$ be an infinite set of integers with $p \nmid c_n$ for all $p \in \mathcal{P}$, and let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection so that $\{A_n\}_{n=1}^\infty$ is non-decreasing where $A_n = a_{\sigma(n)}|c_{\sigma(n)}|$. Put $A_{i,n} = a_{i,\sigma(n)}c_{\sigma(n)}$ and $B_{i,n} = b_{i,\sigma(n)}$. Notice that all assumptions of Theorem 1.76 not involving gcd or ν_p are then clearly satisfied. Let $p \in \mathcal{P}$. Since $p \nmid c_n$, we have $\nu_p(A_{i,n}) = a_{i,\sigma(n)}$ and $\gcd(A_{i,n}, B_{i,n}) = \gcd(a_{i,\sigma(n)}, b_{i,\sigma(n)})$. Then notice that assumption (1.24) combined with either equation (1.25) or (1.28) implies the remaining assumptions of Theorem 1.76 with $d = 1$. If we instead combine assumption (1.24) with either equation (1.27) or (1.29), then the remaining assumptions of Theorem 1.76 are satisfied for all $d \in \mathbb{N}$. This completes the proof. □

Compared to the transcendence criterion in Theorems 1.75 and 1.77, the *limsup* criterion $\limsup_{n\to\infty} a_n^{A^{-n}} = \infty$ required for algebraic independence appears rather strict. Unfortunately, this author has not discovered any way to prove a more lenient assumption.

1.7.1 Examples

Recall the Riemann zeta function ζ as defined in equation (1.3). Related to this, the following example may then be extracted from Remark 4 of [23].

Example 1.78 (Hančl–Kristensen–Laursen, 2025 on arXiv)**.** Let $s > 1$ be a positive integer, and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of odd integers with $|a_{n+1}| \geq |a_n|$ and $\limsup_{n\to\infty} |a_n|^{2^{-n}} = \infty$. Then the number $\sum_{n=1}^{\infty} 1/(n^s a_n)$ is irrational. This follows from Theorem 1.76 with $d = 1$ and $p = 2$.

Remark 1.79. This example only works for the prime $p = 2$. The reason is as follows. For any given k, the smallest positive integer n with $\nu_p(n) \geq k$ is the number p^k, and the next one is $2p^k$. When $p = 2$, we have $\nu_2(2^k) < \nu_2(2\cdot 2^k)$, which is well in line with the assumptions. However, when $p > 2$, then $\nu_p(p^k) = \nu_p(2p^k)$, which breaks one of the first assumptions of Theorem 1.76.

This example is easily modified to one regarding transcendence.

Example 1.80. Let $s > 1$ be a positive integer, let $\delta > 0$, and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of odd integers with $|a_{n+1}| \geq |a_n|$ and $\limsup_{n\to\infty} |a_n|^{(3+\delta)^{-n}} = \infty$. Then the number $\sum_{n=1}^{\infty} 1/(n^s a_n)$ is transcendental. This follows by combining the above example with Theorem 1.74.

In comparing the new Theorem 1.75 with Theorem 1.73, we may also improve Example 6 of [23] to the following.

Example 1.81. Let $z \in \mathbb{N}$ with $z \geq 2$, let $\delta > 0$, and let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive integers with $\gcd(a_n, z) = 1$ and $\limsup_{n\to\infty} a_n^{(3+\delta)^{-n}} = \infty$. Let $\{r_n\}_{n=1}^{\infty}$ be a sequence of pairwise different non-negative integers. Then the sequence $\{z^{r_n} a_n\}_{n=1}^{\infty}$ is (Σ, p)-transcendental for all prime numbers p dividing z.

By combining the ideas from Examples 1, 2, and 4 of [23], we get new examples on (Σ, p)-algebraic independence and (Σ, p)-linear independence.

Example 1.82. Let z be a non-zero integer, and let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive integers with $p \nmid a_n$ such that $\limsup_{n\to\infty} a_n^{A^{-n}} = \infty$ for every positive integer A. Then the sequences $\{a_n z^n\}_{n=1}^{\infty}, \{a_n z^{n^2}\}_{n=1}^{\infty}, \ldots, \{a_n z^{n^K}\}_{n=1}^{\infty}$ are (Σ, p)-algebraically independent for all prime numbers p dividing z.

Example 1.83. Let $K, z \in \mathbb{N}$, and let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of integers with $p \nmid a_n$ such that $\limsup_{n\to\infty} a_n^{(K+1)^{-n}} = \infty$ for all $A \in \mathbb{N}$. Then the sequences $\{a_n z^n\}_{n=1}^{\infty}, \{a_n z^{2n}\}_{n=1}^{\infty}, \ldots, \{a_n z^{Kn}\}_{n=1}^{\infty}$ are (Σ, p)-linearly independent over $\mathbb{Q}$ for all prime numbers p dividing z.

Meanwhile, examples of $(\Sigma, \mathcal{P})$-linear independence and $(\Sigma, \mathcal{P})$-algebraic independence with infinite $\mathcal{P}$ may be extracted straight from Examples 7 and 8 of [23].

Example 1.84. Let K be a positive integer, and let $\{r_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of positive integers such that $\limsup_{n\to\infty} r_n/(K+1)^n = \infty$. Let $\{p_n\}_{n=1}^{\infty}$ be an unbounded and non-decreasing sequence of prime numbers, and let $\mathcal{P}$ be an unbounded subset of these p_n. Then the sequences $\{p_n^{1+r_n}\}_{n=1}^{\infty}, \dots, \{p_n^{K+r_n}\}_{n=1}^{\infty}$ are $(\Sigma, \mathcal{P})$-linearly independent over $\mathbb{Q}$.

Example 1.85. Let $\{p_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of odd prime numbers, and let $\{r_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of integers such that $r_n \geq p_n$ and $\limsup_{n\to\infty}(r_n/A^n) = \infty$ for all $A \in \mathbb{N}$. Let $\mathcal{P}$ be a set of infinitely many of these p_n, and write $a_{1,n} = 2^{r_n} p_n$ and $a_{k+1,n} = 2^{r_n}(p_1 \cdots p_n)^{n^{k-1}}$ for all $k, n \in \mathbb{N}$. Then the sequences $\{a_{1,n}\}_{n=1}^{\infty}, \dots, \{a_{1,n}\}_{n=1}^{\infty}$ are $(\Sigma, \mathcal{P})$-algebraically independent.

The *Examples and applications* section on pages 8–11 of the below paper contain the original versions of the examples presented above as well as additional examples, including some for (Σ, p)- and $(\Sigma, \mathcal{P})$-irrationality.

1.7.2 Paper 6: Algebraic independence of infinite series

Below, the reader will find the most recent preprint of the paper [23], which is joint work between Jaroslav Hančl, Simon Kristensen, and the current author. The paper is currently under review but has not yet been accepted for publication. A preprint is available on arXiv through the link `https://arxiv.org/abs/2502.19079v1` or by using the arXiv identifier 2502.19079.

The day before the deadline of this thesis, the current author found a small technical mistake in the preprint on arXiv, which is corrected in the below version. The mistake was found in the proof of [23, Lemma 22] and required an improvement to [23, Lemma 11] and its proof.

This mistake has been corrected in the below version of the paper, which has a total length of 30 pages, including front page and abstract, with the remaining 28 pages being numbered 1 through 28.

Similar to the case of [38], which is presented in subsection 1.6.2, the Erdős Jump (i.e., Lemma 11) of the below paper can be further improved. Specifically, the assumption $M \geq 0$ of the lemma can be replaced by $M > -1$ without affecting the proof in any meaningful way. This is not important to the paper itself, but it may be for a future paper, as it happened in [39].

Algebraic independence of infinite series

Jaroslav Hančl, Simon Kristensen, and Mathias L. Laursen

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Abstract

We give conditions on a finite set of series of rational numbers to ensure that they are algebraically independent. Specialising our results to polynomials of lower degree, we also obtain new results on irrationality and $\mathbb{Q}$-linear independence of such series.

1 Introduction

The study of irrationality and transcendence of numbers date back to antiquity, where an unfortunate Pythagorean lost his life in proving the irrationality of $\sqrt{2}$. Much later, in the late nineteenth century, the existence of transcendental numbers was proven, explicit examples were provided, and some famous numbers such as π and e were shown to be transcendental. Later on, concepts such as linear independence over $\mathbb{Q}$, over $\overline{\mathbb{Q}}$ and algebraic independence of numbers have been important. In the first two cases, one thinks of the set of real numbers $\mathbb{R}$ as a vector space defined over the rationals, $\mathbb{Q}$, or the algebraic numbers, $\overline{\mathbb{Q}}$, and investigate whether a set of numbers is linearly independent over the base field as vectors. For algebraic independence, one investigates whether a set of K real numbers has the property that no non-zero integer polynomial in K variables vanishes at this set. In this case, the numbers are said to be algebraically independent. Of course, if one considers only polynomials of degree 1, the notion of linear independence over $\mathbb{Q}$ is rediscovered. If furthermore $K = 1$, the question of irrationality is rediscovered.

The present paper is concerned with algebraic independence of a finite set of real numbers given in terms of series of rationals. More precisely, we will derive conditions on such a set of series which ensure that the numbers are algebraically independent. An important result in this respect is the 1975 result of Erdős [4], where it is shown that if $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence of positive integers with $a_n > n^{1+\epsilon}$ for some $\epsilon > 0$ and

$$\limsup_{n\to\infty} a_n^{1/2^n} = \infty,$$

then the number $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is irrational. The result has been extended numerous times by various authors to encompass a wider range of series [7] as well as series with more general denominators [10, 1, 16].

Our objective here is to obtain criteria ensuring algebraic independence of a set of such series. We will allow for numerators as well as denominators, although our series will remain defined over the rationals. One novelty of our approach is that we are able to remove the restriction of positivity in Erdős' result at some expense. The conditions obtained will look somewhat cumbersome and technical. In order to demonstrate their applicability, we will provide a large number of examples of concrete series, which we show to be irrational, linearly independent over $\mathbb{Q}$ or even algebraically independent.

2 Main results

We will now state our main theorem. We will be considering series of the form

$$\alpha_k = \sum_{n=1}^{\infty} \frac{b_{k,n}}{a_{k,n}},$$

where for $k = 1, 2, \ldots, K$, $\{a_{k,n}\}_{n=1}^{\infty}$ and $\{b_{k,n}\}_{n=1}^{\infty}$ are sequences of non-zero integers. Our results depend heavily on the joint divisibility properties of the denominators. In our main result, we express the required properties in terms of the p-adic valuation. As usual, for a prime p and an integer n, we will let $\nu_p(n)$ denote the p-adic valuation of n, i.e., $\nu_p(0) = \infty$ and, for $n \neq 0$, $\nu_p(n)$ is the unique non-negative integer such that $p^{\nu_p(n)}|n$ but $p^{\nu_p(n)+1} \nmid n$. With this in mind, we now state our main result on the algebraic independence of a finite set of series of the above form. In some of the theorems, we give conditions to ensure that non-zero polynomials $P \in \mathbb{Z}[x_1, \ldots, x_K]$ of a degree bounded by a fixed integer d will all satisfy $P(\alpha_1, \ldots, \alpha_K) \neq 0$. We here take the degree of a polynomial to be the maximum degree of its monomial terms, and we define $\deg(x_1^{i_1} \cdots x_K^{i_K}) = i_1 + \cdots + i_K$.

When only two series are considered, so that $K = 2$, we are able to obtain much weaker criteria at the cost of arriving at a somewhat weaker conclusion. To fix ideas, we define a weaker notion here. For a polynomial $P(x_1, x_2)$, let $\tilde{P}(x_1, x_2, x_3)$ be the projective version of P, i.e., the homogeneous polynomial obtained from P by multiplying each monomial with x_3 sufficiently many times that the resulting polynomial is homogeneous of the same degree as the original one. This polynomial defines a plane, projective curve, $\mathcal{C}_P \in \mathbb{P}^2$. If this curve is smooth, its (geometric) genus $g(\mathcal{C}_P)$ is given by the genus–degree formula,

$$g(\mathcal{C}_P) = \frac{(\deg P - 1)(\deg P - 2)}{2}.$$

If the curve is singular, the genus decreases. An ordinary singularity of multiplicity r decreases the genus by $r(r-1)/2$. Non-ordinary singularities need to be examined individually. We will say that two numbers α_1 and α_2 are *non-degenerately independent of order* d if no integer polynomial P in two variables of degree at most d such that $\deg P \leq 3$ or $g(\mathcal{C}_P) \geq 2$ vanishes at the point (α_1, α_2). If two numbers α_1 and α_2 are non-degenerately independent of order d for any d, we will say that α_1 and α_2 are *non-degenerately algebraically independent*.

For polynomials P such that the resulting plane, projective curve is smooth, the genus condition is satisfied as soon as $\deg P > 3$. For higher degrees, we are removing certain potential algebraic dependencies with this definition, arising from the decrease in genus from the singularities. One should stress that most polynomials give rise to a smooth curve, and that if the degree is large, most non-smooth curves will have genus ≥ 2. In work in progress [15], F. Pazuki and the second named author obtain a counting estimate for polynomials of fixed degree d resulting in a curve of fixed genus g.

Theorem 1. *Let K and d be positive integers, let ε and κ be positive real numbers with $\kappa < 1$, and let p be a prime number. For $k = 1, \ldots, K$, let $\{a_{k,n}\}_{n=1}^{\infty}$ and $\{b_{k,n}\}_{n=1}^{\infty}$ be sequences of non-zero integers with $p \nmid \gcd(a_{k,n}, b_{k,n})$ such that for each sufficiently large $N \in \mathbb{N}$,*

$$\nu_p(a_{k,n}) = \max_{1 \leq m \leq N} \nu_p(a_{k,m}) \tag{1}$$

for exactly one $n \leq N$ and

$$\lim_{N \to \infty} \left(\max_{1 \leq n \leq N} \nu_p(a_{k,n}) - d \max_{1 \leq n \leq N} \nu_p(a_{k-1,n}) \right) = \infty, \tag{2}$$

writing $a_{0,n} = 1$ for all n. Suppose there is a sequence $\{a_n\}_{n=1}^{\infty}$ of integers such that for every $k = 1, \ldots, K$ and each sufficiently large $n \in \mathbb{N}$,

$$n^{1+\varepsilon} \leq a_n \leq a_{n+1}, \tag{3}$$

$$a_n 2^{-(\log_2 a_n)^{\kappa}} \leq |a_{k,n}| \leq \max\left\{a_n 2^{(\log_2 a_n)^{\kappa}}, 2^{n^{-3}(Kd+1)^n}\right\}, \tag{4}$$

$$|b_{k,n}| \leq 2^{(\log_2 a_n)^{\kappa}}, \tag{5}$$

and

$$\limsup_{n \to \infty} a_n^{\frac{1}{(Kd+1)^n}} = \infty. \tag{6}$$

For $k = 1, \ldots, K$, set $\alpha_k = \sum_{n=1}^{\infty} \frac{b_{k,n}}{a_{k,n}}$. Then $(\alpha_1, \ldots, \alpha_K)$ is not the root of any non-zero polynomial of K variables with integer coefficients and degree less than or equal to d.

If $K = 2$, this is also true when d is replaced by 3 in equation (2) and the non-vanishing conclusion is replaced by non-degenerately independent of order d.

It is worth noting that in the last case of the theorem, if one replaces 3 with any higher number d', say, (2) automatically rules out the existence of polynomials of degree $\leq d'$ vanishing at the point (α_1, α_2). This cuts down on the number of degenerate polynomials potentially obstructing full independence.

The conditions of Theorem 1 may seem unwieldy to check, and their origin may not be terribly clear. Let us take a moment to digest them. Condition (3), (4), (5) and (6) are reminiscent of the conditions in Erdős' paper [4], except that no numerators were present in that paper, and it had $K = 1$ and $a_n = a_{1,n}$. Nonetheless, the role played by these conditions is similar and will be used to show that the theorem follows from a result about the partial sums $\sum_{n=N}^{\infty} b_{k,n}/a_{k,n}$ when N is large. Assumptions (4) and (5) arise from considering several numbers and from having a non-fixed denominators, respectively, and essentially allow us to replace the sequences $\{a_{k,n}\}_{n=1}^{\infty}$ and $\{b_{k,n}\}_{n=1}^{\infty}$ with the single sequence $\{a_n\}_{n=1}^{\infty}$ for a central part of the proof.

The novelty of our result lies in the origins of assumptions (1) and (2). As we shall see, their role is to ensure that no integer polynomial in K variables will vanish at infinitely many of the partial sums of the numbers α_k.

From the conditions in Theorem 1, we could easily formulate conditions under which (2) and (6) hold for every value of $d \in \mathbb{N}$. This immediately leads us to a criterion for algebraic independence of the α_k in the following way.

Theorem 2. *Let K be a positive integer, let p be a prime number, and let ε and κ be positive real numbers with $\kappa < 1$. For $k = 1, \ldots, K$, let $\{a_{k,n}\}_{n=1}^{\infty}$ and $\{b_{k,n}\}_{n=1}^{\infty}$ be sequences of non-zero integers with $p \nmid \gcd(a_{k,n}, b_{k,n})$ so that assumption (1) is satisfied for exactly one $n \leq N$ when N is sufficiently large, while*

$$\lim_{N\to\infty} \frac{\max_{n=1,\ldots,N} \nu_p(a_{k,n})}{1 + \max_{n=1,\ldots,N} \nu_p(a_{k-1,n})} = \infty, \tag{7}$$

where $a_{0,n} = 1$ for each $n \in \mathbb{N}$. Suppose there is a non-decreasing sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers that, for each $k = 1, \ldots, K$ and all sufficiently large $n, A \in \mathbb{N}$, satisfies assumption (3), (5),

$$a_n 2^{-(\log_2 a_n)^{\kappa}} \leq |a_{k,n}| \leq \max\left\{a_n 2^{(\log_2 a_n)^{\kappa}}, 2^{A^n}\right\}, \tag{8}$$

and

$$\limsup_{n\to\infty} a_n^{\frac{1}{A^n}} = \infty. \tag{9}$$

For $k = 1, \ldots, K$, set $\alpha_k = \sum_{n=1}^{\infty} \frac{b_{k,n}}{a_{k,n}}$. Then $\alpha_1, \ldots, \alpha_K$ are algebraically independent over $\mathbb{Q}$.

As stated in the results, when $K = 2$, most of the conditions needed for our two previous results become simpler at the cost of removing potential degenerate cases of algebraic dependence. Indeed, it is straightforward to check in this case that the following holds.

Theorem 3. *Let $\{a_{k,n}\}_{n=1}^{\infty}$, $\{b_{k,n}\}_{n=1}^{\infty}$, $\{a_n\}_{n=1}^{\infty}$, and α_k be given as in Theorem 1 with $K = 2$, $d = 3$, and the additional assumption that for every positive integer A,*

$$\limsup_{n\to\infty} a_n^{\frac{1}{A^n}} = \infty.$$

Then α_1 and α_2 are non-degenerately algebraically independent over $\mathbb{Q}$.

If we focus on $K = 1$, we also get new irrationality and transcendence criteria, which we will phrase in terms of the following definitions of p-irrationality and p-transcendence.

Definition 4. Let p be a prime number. We say that a sequence of non-zero numbers $\{a_n\}_{n=1}^{\infty}$ is p-irrational or p-transcendental, if for every sequence $\{c_n\}_{n=1}^{\infty}$ of integers c_n with $p \nmid c_n$, the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{a_n c_n}$$

converges to a number that is irrational or transcendental, respectively.

This generalises the notions of irrational and transcendental sequences introduced by Erdős [4] and Hančl [6], respectively, where it is assumed that $c_n > 0$ rather than $p \nmid c_n$. From Theorem 1, we obtain the below result.

Theorem 5. *Let p be a prime number, let C be a positive integer, and let ε and κ be positive real numbers with $\kappa < 1$. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of non-zero integers with $p \nmid \gcd(a_n, b_n)$ such that $\limsup_{n\to\infty} \nu_p(a_n) = \infty$ and either $\nu_p(a_m) \neq \nu_p(a_n)$ or $\nu_p(a_m) = \nu_p(a_n) \le C$ when $m \neq n$. Suppose that $a_{n+1} \ge a_n \ge n^{1+\varepsilon}$ and $|b_n| \le 2^{(\log_2 |a_n|)^{\kappa}}$ when n is sufficiently large. Then the sequence $\{a_n/b_n\}$ is p-irrational if*

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{2^n}} = \infty, \tag{10}$$

and it is p-transcendental if equation (9) *is satisfied for every positive integer A.*

Proof. Let $c_n \in \mathbb{Z}$ with $p \nmid c_n$, and let $\sigma : \mathbb{N} \to \mathbb{N}$ be the bijection so that $|A_n|$ is non-decreasing where $A_n = a_{\sigma(n)} c_{\sigma(n)}$, and put $B_n = b_{\sigma(n)}$. If equation (9) is satisfied, let $d \in \mathbb{N}$ be arbitrary. If not, assume equation (10), and put $d = 1$. Since certainly $\sum_{n=1}^{\infty} \frac{b_n}{a_n c_n} = \sum_{n=1}^{\infty} \frac{B_n}{A_n}$, the theorem follows if we can show that $\{A_n\}_{n=1}$ and $\{B_n\}_{n=1}^{\infty}$ satisfy the assumptions of Theorem 1. We immediately notice that inequalities (2), (4), (5), and (6) are satisfied. Since A_n is non-decreasing and $|a_n c_n| \ge a_n \ge n^{1+\varepsilon}$, inequality (3) must also be satisfied. As $p \nmid c_n, \gcd(a_n, b_n)$, it follows that $p \nmid \gcd(A_n, B_n)$, while the facts that $p \nmid c_n$, $\limsup_{n\to\infty} \nu_p(n) = \infty$, and $\nu_p(n) \neq \nu_p(m)$ when $\nu_p(n)$ and $m \neq n$ are large ensure that equation (1) can be satisfied by at most one $n \leq N$ when N is large. Thus, all assumptions of Theorem 1 are satisfied. $\square$

Remark 1. When $\{c_n\}_{n=1}^{\infty}$ is a sequence of non-zero integers, as is the case when all c_n are non-divisible by any given prime, then $\alpha = \sum_{n=1}^{\infty} \frac{b_n}{a_n c_n}$ is absolutely convergent, and so the resulting number is the same if we reorder the terms. Hence, if we have $\nu_p(a_m) \neq \nu_p(a_n)$ for all $m \neq n$, we can freely reorder the terms to ensure that $\nu_p(a_n)$ is strictly increasing. Then $a_n \geq p^{\nu_p(a_n)} \geq p^{n-1}$, and so we automatically get the assumptions $\limsup \nu_p(a_n) = \infty$ and $a_n \geq n^{1+\varepsilon}$ when n is sufficiently large, thus reducing the number of assumptions one needs to check when applying Theorem 5 to such a sequence.

In Theorem 1, we might also consider what happens if we keep track of multiple prime numbers p rather than just one. This leads to the below theorem, the proof of which is almost identical to that of Theorem 1. As such, these two theorems will be proven in unison.

Theorem 6. *Let K and d be positive integers, let ε and κ be positive real numbers with $\kappa < 1$ and let $\mathcal{P}$ be a set of infinitely many prime numbers. For $k = 1, \ldots, K$, let $\{a_{k,n}\}_{n=1}^{\infty}$ and $\{b_{k,n}\}_{n=1}^{\infty}$ be sequences of non-zero integers with $p \nmid \gcd(a_{k,n}, b_{k,n})$ for each fixed $p \in \mathcal{P}$. Suppose that for each sufficiently large $N \in \mathbb{N}$, exactly one $n \leq N$ satisfies equation* (1) *while*

$$\max_{n=1,\ldots,N} \nu_p(a_{k,n}) > d \max_{n=1,\ldots,N} \nu_p(a_{k-1,n}), \tag{11}$$

where $a_{0,n} = 1$. Suppose that there is a sequence $\{a_n\}_{n=1}^{\infty}$ that satisfies assumptions (3), (4), (5), *and* (6) *for every $k = 1, \ldots, K$ and $n \in \mathbb{N}$. For $k = 1, \ldots, K$, set $\alpha_k = \sum_{n=1}^{\infty} \frac{b_{k,n}}{a_{k,n}}$. Then $(\alpha_1, \ldots, \alpha_K)$ is not the root of any non-zero polynomial of K variables with integer coefficients and degree less than or equal to d.*

If $K = 2$, this is also true when d is replaced by 3 in equation (11) *and the conclusion is replaced by non-degenerate independence of order d.*

As with Theorem 1, one could replace 3 in the last part by a number $d' > 3$ to cut down on the number of potentially 'bad' polynomials

This theorem has corollaries in complete parallel to Theorems 2 and 3.

Theorem 7. *Let K be a positive integer, let $\mathcal{P}$ be an infinite set of a prime numbers, and let ε and κ be positive real numbers with $\kappa < 1$. For $k = 1, \ldots, K$, let $\{a_{k,n}\}_{n=1}^{\infty}$ and $\{b_{k,n}\}_{n=1}^{\infty}$ be sequences of non-zero integers such* *that, for all* $p \in \mathcal{P}$, $p \nmid \gcd(a_{k,n}, b_{k,n})$, *assumption* (1) *is satisfied for exactly one* $n \leq N$ *when* N *is sufficiently large, and if* $k > 1$, *then equation* (7) *is satisfied. Suppose there is a non-decreasing sequence* $\{a_n\}_{n=1}^{\infty}$ *of positive integers that, for each* $k = 1, \ldots, K$ *and all sufficiently large* $A, n \in \mathbb{N}$, *satisfies assumptions* (3), (5), (8), *and* (9). *For* $k = 1, \ldots, K$, *set* $\alpha_k = \sum_{n=1}^{\infty} \frac{b_{k,n}}{a_{k,n}}$. *Then* $\alpha_1, \ldots, \alpha_K$ *are algebraically independent over* $\mathbb{Q}$.

Theorem 8. *Let* $\{a_{k,n}\}_{n=1}^{\infty}$, $\{b_{k,n}\}_{n=1}^{\infty}$, $\{a_n\}_{n=1}^{\infty}$, *and* α_k *be given as in Theorem 6 with* $K = 2$, $d = 3$, *and the additional assumption that for every positive integer* A,

$$\limsup_{n\to\infty} a_n^{\frac{1}{A^n}} = \infty.$$

Then α_1 *and* α_2 *are non-degenerately algebraically independent over* $\mathbb{Q}$.

Remark 2. Theorems 2 and 7 are very similar in their assumptions. In fact, the only differences are that Theorem 7 does not assume equation (7) for $K = 1$ and that it makes its assumptions for infinitely many primes p. This is a consequence of how an unbounded d makes $\nu_p(a_{k,n})$ unbounded when $k > 1$, which in turn makes the differences between inequality (11) and equation (2) small when d grows large. As such, Theorem 7 should only be used when $\nu_p(a_{n,1})$ is bounded for all fixed $p \in \mathcal{P}$ (or when it is unknown if this is the case); otherwise, Theorem 2 would be sufficient by just picking any one p from $\mathcal{P}$ with unbounded $\nu_p(a_{1,n})$.

Aiming for a result corresponding to Theorem 5, we introduce notions of $\mathcal{P}$-irrationality and $\mathcal{P}$-transcendence when $\mathcal{P}$ is a set of prime numbers.

Definition 9. Let $\mathcal{P}$ be a set of prime numbers. We say that a sequence $\{a_n\}_{n=1}^{\infty}$ is $\mathcal{P}$-irrational or $\mathcal{P}$-transcendental if the series $\sum_{n=1}^{\infty} \frac{1}{a_n c_n}$ converges to an irrational or transcendental number, respectively, for all sequences of integers $\{c_n\}_{n=1}^{\infty}$ with $p \nmid c_n$ for all $p \in \mathcal{P}$.

By considerations similar to those in the last part of Remark 2, it is clear that when $K = 1$, Theorem 6 is only interesting when $\nu_p(a_{1,n})$ is bounded for all fixed p, since one could otherwise apply Theorem 1 with much less effort instead. For that reason, we formulate the result corresponding to Theorem 5 as follows.

Theorem 10. *Let* $\mathcal{P}$ *be a set of infinitely many prime numbers, and let* ε *and* κ *be positive real numbers with* $\kappa < 1$. *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be sequences of non-zero integers such that, for each fixed* $p \in \mathcal{P}$, $p \nmid \gcd(a_n, b_n)$ *for all* n, *while* $\nu_p(a_n)$ *is bounded and attains its maximum value exactly once. Suppose* *that* $a_{n+1} \geq a_n \geq n^{1+\varepsilon}$ *and* $|b_n| \leq 2^{(\log_2 |a_n|)^\kappa}$. *Then the sequence* $\{a_n/b_n\}$ *is* $\mathcal{P}$*-irrational if equation* (10) *is satisfied, and it is* $\mathcal{P}$*-transcendental if equation* (9) *is satisfied for every positive integer* A.

Proof. The proof is essentially the same as that of Theorem 5, except that we apply Theorem 6 rather than Theorem 1. As such, our interest in equation (2) is replaced with inequality (11). □

3 Examples and applications

In this section, we give a number of examples of applications of our results. It is of interest that the terms of our series have varying signs since this was not allowed in previous papers, such as [1, 4, 6, 7, 16]. Let $f_1, \ldots, f_K$ denote arbitrary functions from $\mathbb{N}$ into $\mathbb{Z}$. This way, we can express the varying signs as $(-1)^{f_k(n)}$. Possible choices for $f_k(n)$ could, for instance, be the product kn, the number of divisors of n, or the Euler totient function of n, i.e., the number of $m = 1, \ldots, n$ such that $\gcd(m, n) = 1$.

We start by giving some simple examples to highlight the immediate applications of our results in terms of algebraic independence. For these examples, use Theorem 2.

Example 1. Let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive odd integers such that for every positive integer A,

$$\limsup_{n\to\infty} a_n^{\frac{1}{A^n}} = \infty.$$

Set $\alpha_k = \sum_{n=1}^{\infty} \frac{(-1)^{f_k(n)}}{a_n 2^{n^k}}$, $k = 1, \ldots, K$. Then the numbers $\alpha_1, \ldots, \alpha_K$ are algebraically independent over $\mathbb{Q}$.

Example 2. Let K and z be positive integers with $z \geq 2$, and let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of integers with $\gcd(a_n, z) = 1$, $a_n \geq n^{1+\varepsilon}$, and

$$\limsup_{n\to\infty} a_n^{\frac{1}{A^n}} = \infty,$$

for every positive integer A. Set $\alpha_k = \sum_{n=1}^{\infty} \frac{(-1)^{f_k(n)}}{a_n z^{\nu_2(n)^k}}$, $k = 1, \ldots, K$. Then the numbers $\alpha_1, \ldots, \alpha_K$ are algebraically independent over $\mathbb{Q}$.

As is seen from the theorems, we have non-degenerate algebraic independence with weaker restrictions when we know that $K = 2$, which gives us the below example by applying Theorem 3.

Example 3. Let $z \geq 2$ be a positive integer, and let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive integers with $\gcd(z, a_n) = 1$ and, for all positive integers A, $\limsup a_n^{1/A^n} = \infty$. For $k = 1, 2, \ldots$, set $\alpha_k = \sum_{n=1}^{\infty} \frac{(-1)^{f_k(n)}}{a_n z^{(3^k-1)n}}$. If $k \neq l$, then α_k and α_l are non-degenerately algebraically independent.

Meanwhile, by using the more general Theorem 1, we are also able to prove irrationality and linear independence of series where we are not currently able to determine algebraic independence, as seen in the next example.

Example 4. Let K and z be positive integers with $z \geq 2$, and let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive integers such that $\gcd(a_n, z) = 1$ and $\limsup a_n^{1/(K+1)^n} = \infty$. For $k = 1, \ldots, K$, set $\alpha_k = \sum_{n=1}^{\infty} \frac{(-1)^{f_k(n)}}{a_n z^{k\,\nu_p(n)}}$. Then $1, \alpha_1, \ldots, \alpha_K$ are linearly independent over $\mathbb{Q}$.

Remark 3. We are not able to prove that the number $\zeta(5) = \sum_{n=1}^{\infty} \frac{1}{n^5}$ is irrational. Erdős [4] proved that if $\{a_n\}_{n=1}^{\infty}$ is a non-decreasing sequence of positive integers such that $\limsup_{n\to\infty} a_n^{\frac{1}{2^n}} = \infty$, then $\alpha = \sum_{n=1}^{\infty} \frac{1}{a_n n^5}$ is an irrational number. From Theorem 1 with $p = 2$, we now obtain that if all a_n are also odd, then the number $\sum_{n=1}^{\infty} \frac{(-1)^{f(n)}}{a_n n^5}$ is irrational for all functions $f : \mathbb{N} \to \mathbb{Z}$.

Remark 4. For integer functions $f : \mathbb{N} \to \mathbb{Z}$ and $g(n) : \mathbb{N} \to \mathbb{N}$, write $\zeta_{f,g}(k) = \sum_{n=1}^{\infty} \frac{(-1)^{f(n)}}{g(n)n^k}$. Then $\zeta_{0,1}$ is Riemann's zeta function, where 0 and 1 denote the functions that are constantly 0 and 1, respectively. We get from Remark 3 that $\zeta_{f,g}(5)$ is irrational when $\{g(n)\}_{n=1}^{\infty}$ defines a non-decreasing sequence of positive odd integers with $\limsup_{n\to\infty} g(n)^{1/2^n} = \infty$, regardless of f. In fact, $\zeta_{f,g}(k)$ is irrational for all such g and all integers $k \geq 2$. However, if we remove the conditions that $g(n)$ is non-decreasing and that $\limsup_{n\to\infty} g(n) = \infty$ and fix $f(n)$ and $k \geq 2$, then we can pick g so that $\zeta_{f,g}(k)$ is rational. If $f(n)$ is both infinitely often even and infinitely often odd, we can take a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$$\zeta_{f,g}(k) = \sum_{n=1}^{\infty} \frac{(-1)^{f(n)}}{n^k g(n)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sigma(n)^k g(\sigma(n))}.$$

Then picking g so that $g(\sigma(2m)) = \sigma(2m-1)^k$ and $g(\sigma(2m-1)) = \sigma(2m)^k$, we get $\zeta_{f,g}(k) = 0$, which is rational. If $f(n)$ is even only finitely often, then $\zeta_{f,g}(k)$ is a rational number minus the number $\sum_{n=N}^{\infty} \frac{1}{n^k g(n)}$. By a result due to Hančl [5], we can now choose g so that this number is rational. If $f(n)$ is odd only finitely often, we just note that $\zeta_{f,g}(k) = -\zeta_{f+1,g}(k)$. Then $1 + f(n)$ is even only finitely often, and we can pick g so that $\zeta_{f,g}(k)$ is rational by the previous consideration.

Open Problem 5. While it is well-known that $\zeta_{0,1}(2) = \zeta(2) = \pi^2/6$ is transcendental, and Apéry [2] showed that $\zeta_{0,1}(3) = \zeta(3)$ is irrational, the authors do not know if $\zeta_{f,1}(k) = \sum_{n=1}^{\infty} \frac{(-1)^{f(n)}}{n^k}$ is irrational in general for $k \geq 2$.

We now give examples of p-irrationality and p-transcendence of a sequence, using Theorem 5 and Remark 1.

Example 5. Let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive odd integers with $\limsup_{n\to\infty} a_n^{1/2^n} = \infty$. Let $\{r_n\}_{n=1}^{\infty}$ be a sequence of pairwise different non-negative integers. Then the sequence $\{2^{r_n} a_n\}_{n=1}^{\infty}$ is 2-irrational. Likewise, for any integer $z \geq 2$, if we assume $\gcd(a_n, z) = 1$ in place of a_n being odd, then the sequence $\{z^{r_n} a_n\}_{n=1}^{\infty}$ is p-irrational for all primes p dividing z.

Example 6. Let $z \geq 2$ be a positive integer, and let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive integers coprime with z such that, for all $A \in \mathbb{N}$, $\limsup_{n\to\infty} a_n^{1/A^n} = \infty$. Let $\{r_n\}_{n=1}^{\infty}$ be a sequence of pairwise different non-negative integers. Then the sequence $\{z^{r_n} a_n\}_{n=1}^{\infty}$ is p-transcendental for all primes p dividing z.

Finally, we present the below four examples of Theorems 6, 7, 8, and 10, respectively. Comparing these with Examples 4, 1, 3, and 5, respectively, we see some of the differences in applicability between considering a single prime or infinitely many.

Example 7. Let K be a positive integer, and let $\{r_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of positive integers such that

$$\limsup_{n\to\infty} \frac{r_n}{(K+1)^n} > 0.$$

Let $\{p_n\}_{n=1}^{\infty}$ be an unbounded and non-decreasing sequence of prime numbers. For $k = 1, \ldots, K$, set $\alpha_k = \sum_{n=1}^{\infty} \frac{(-1)^{f_k(n)}}{p_n^{k+r_n}}$. Then the numbers $1, \alpha_1, \ldots, \alpha_K$ are linearly independent over $\mathbb{Q}$.

Example 8. Let $\{p_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of odd prime numbers, and let $\{r_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of integers such that $r_n \geq p_n$ and $\limsup_{n\to\infty}(r_n/A^n) = \infty$ for all $A \in \mathbb{N}$. Set $\alpha_1 = \sum_{n=1}^{\infty} \frac{(-1)^{f_1(n)}}{2^{r_n} p_n}$ and $\alpha_{k+1} = \sum_{n=1}^{\infty} \frac{(-1)^{f_k(n)}}{2^{r_n}(p_1 \cdots p_n)^{n^{k-1}}}$ for each $k \in \mathbb{N}$ with $k > 1$. Then $\alpha_1, \ldots, \alpha_K$ are algebraically independent for all $K \in \mathbb{N}$.

For this example, it may not be clear that we can actually pick a sequence $\{a_n\}_{n=1}^{\infty}$ of integers that satisfies (8) for each k and all large values of A and

n. Write $a_{1,n} = 2^{r_n} p_n$ and $a_{k,n} = 2^{r_n}(p_1 \cdots p_n)^{n^{k-1}}$ for $k > 1$, fix K, and pick $a_n = 2^{r_n}$ and $\kappa = 1/2$. Then the only assumption that is not trivially satisfied is the upper bound of (8) when $k > 1$. Since $a_{1,n} < \cdots < a_{K,n}$, it suffices to prove this when $A = 2$, $k = K > 1$ and $a_{K,n} > 2^{2^n}$. By using $\log_2$, the upper bound of (8) is equivalent to

$$r_n + \log_2\left((p_1 \cdots p_n)^{n^k}\right) = \log_2 a_{1,n} \leq \log_2\left(a_n 2^{(\log_2 a_n)^\kappa}\right) = r_n + r_n^{1/2},$$

i.e.,

$$r_n^{1/2} \geq \log_2\left((p_1 \cdots p_n)^{n^k}\right) = n^k \sum_{m=1}^{n} \log_2 p_m.$$

If $r_n \geq n^{4K}$, then this is indeed satisfied, recalling that $r_n \geq p_n$ and calculating

$$r_n^{1/2} = r_n^{1/4} r_n^{1/4} \geq (n^{4K})^{1/4} p_n^{1/4} > n^{K-1} n \log_2 p_n \geq n^{K-1} \sum_{m=1}^{n} \log_2 p_m.$$

If $p_n \geq n^{4K}$, we are therefore done, so consider the case $p_n < n^{4K}$. Since $a_{K,n} > 2^{2^n}$, we then have

$$2^{2^n} < a_{K,n} = 2^{r_n}(p_1 \cdots p_n)^{n^{K-1}} < 2^{r_n} p_n^{n^K} < 2^{r_n} n^{4Kn^K}.$$

As this clearly ensures that $r_n \geq n^{4K}$ for all large values of n, we are done.

Example 9. Let $\{p_n\}_{n=1}^\infty$ be a strictly increasing sequence of odd prime numbers, and let $\{r_n\}_{n=1}^\infty$ be a non-decreasing sequence of integers such that $r_n \geq p_n$ and $\limsup_{n\to\infty}(r_n/A^n) = \infty$ for all positive integers A. For $k = 1, 2, \ldots$, set $\alpha_k = \sum_{n=1}^\infty \frac{(-1)^{f_k(n)}}{2^{r_n} p_n^{3^k - 1}}$. If $k \neq l$, then α_k and α_l are non-degenerately algebraically independent.

Example 10. Let $\{p_n\}_{n=1}^\infty$ be an increasing sequence of prime numbers, let $\mathcal{P}$ be an infinite subset of $\{p_n : n \in \mathbb{N}\}$, and let $\{a_n\}_{n=1}^\infty$ be an increasing sequence of positive integers with $\limsup_{n\to\infty}(a_n p_n)^{1/2^n} = \infty$ and $p \nmid a_n$ for all $p \in \mathcal{P}$. Then the sequences $\{a_n p_n\}_{n=1}^\infty$ and $\{a_n p_n^2 p_{n-1} \cdots p_1\}_{n=1}^\infty$ are each $\mathcal{P}$-irrational. If furthermore, $\limsup_{n\to\infty}(a_n p_n)^{1/A^n} = \infty$ for all positive integers A, then the sequences $\{a_n p_n\}_{n=1}^\infty$ and $\{a_n p_n^2 p_{n-1} \cdots p_1\}_{n=1}^\infty$ are each $\mathcal{P}$-transcendental.

4 Preliminaries

In this section, we give some auxiliary results needed for the proofs of the main theorems. The first one is a slight strengthening of Lemma 5 of [10] when specialized to a single sequence $\{a_n/b_n\}$ with $b_n \leq 2^{(\log_2 a_n)^\kappa}$.

Lemma 11. *Let $\varepsilon > 0$, $0 < \kappa < 1$, and $M \geq 0$. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of positive integers such that a_n is non-decreasing and*

$$\limsup_{n\to\infty} a_n^{(M+2)^{-n}} = \infty. \tag{12}$$

Suppose that for all sufficiently large n that

$$a_n \geq n^{1+\varepsilon}$$

and

$$b_n \leq 2^{(\log a_n)^{\kappa}}.$$

Then for any fixed $0 < c < 1$,

$$\liminf_{N\to\infty} 2^{\max\{N^2(\log_2 a_N)^c,\, N^{-3}(M+2)^{N+1}\}} \left(\prod_{n=1}^{N-1} a_n\right)^{M+1} \sum_{n=N}^{\infty} \left|\frac{b_n}{a_n}\right| = 0.$$

Not surprisingly, the proof will closely follow that of Lemma 5 in [10] and use the same preliminary lemmas, below, which are proven in [10]. In their original form, the lemmas had additional assumptions, but these were never used in their proofs and are thus omitted here. They may also be extracted from the proof in [4].

Lemma 12. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ satisfy the assumptions of Lemma 11. Then there exists a number $\gamma > 0$ that does not depend on N and such that for all sufficiently large N,*

$$\sum_{n=N}^{\infty} \frac{b_n}{a_n} \leq \frac{1}{a_N^{\gamma}}.$$

Lemma 13. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ satisfy the assumptions of Lemma 11. Suppose that $a_n \geq 2^n$ for all sufficiently large n. Then there is a fixed $0 < \Gamma < 1$ such that for all sufficiently large N,*

$$\sum_{n=N}^{\infty} \frac{b_n}{a_n} \leq \frac{2^{(\log_2 a_N)^{\Gamma}}}{a_N}.$$

Lemma 14. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ satisfy the assumptions of Lemma 11. Then there is a fixed number $0 < \Gamma < 1$ so that if N and Q are sufficiently large and $a_n \geq 2^n$ for $n = N, \ldots, Q$, then*

$$\sum_{n=N}^{Q} \frac{b_n}{a_n} \leq \frac{2^{(\log_2 a_N)^{\Gamma}}}{a_N}.$$

Lemma 15. *Let $\{y_n\}_{n=1}^{\infty}$ be an unbounded sequence of positive real numbers. Then there are infinitely many N such that*

$$y_N > \left(1 + \frac{1}{N^2}\right) \max_{1 \le n < N} y_n.$$

By a simple induction argument, we notice that for $k < N$ and $\delta \ge 0$,

$$\begin{aligned}(M+2+\delta)^N &= (M+2+\delta)^{N-1} + (M+1+\delta)(M+2+\delta)^{N-1} = \cdots \\ &= (M+2+\delta)^k + (M+1+\delta)\sum_{n=k}^{N-1}(M+2+\delta)^n, \qquad (13)\end{aligned}$$

which will be used for the proof of Lemma 11. Equation (13) also helps prove the below corollary to Lemma 15.

Corollary 16. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ satisfy the assumptions of Lemma 11. Let k be a positive integer. Then for infinitely many $N > k$,*

$$a_N > \left(1 + \frac{1}{N^2}\right)^{(M+2)^N} \left(\max_{k \le n < N} a_n^{(M+2)^{-n}}\right)^{(M+2)^N}.$$

For such N, we further have

$$a_N > \left(1 + \frac{1}{N^2}\right)^{(M+2)^N} \prod_{n=k}^{N-1} a_n^{M+1}.$$

Proof. The first inequality follows immediately from Lemma 15 by taking $y_n = a_n^{(M+2)^{-n}}$ for $n \ge k$ and $y_n = y_k$ for $n < k$. We then use (13) to conclude

$$\begin{aligned}\left(\max_{k \le n < N} a_n^{(M+2)^{-n}}\right)^{(M+2)^N} &\ge \left(\max_{k \le n < N} a_n^{(M+2)^{-n}}\right)^{(M+1)\sum_{n=k}^{N-1}(M+2)^n} \\ &\ge \prod_{n=k}^{N-1} a_n^{M+1}. \qquad \square\end{aligned}$$

Proof of Lemma 11. To shorten notation, write

$$E_n = \max\{n^2(\log_2 a_n)^{\kappa}, n^{-3}(M+2)^{n+1}\}$$

and

$$Z_N = 2^{E_N}\left(\prod_{n=1}^{N-1} a_n\right)^{M+1} \sum_{n=N}^{k_2} \frac{b_n}{a_n}$$

We here split the proof into two cases depending on whether

$$\limsup_{n\to\infty} a_n^{(M+2+\delta)^{-n}} = \infty \tag{14}$$

is true some fixed $\delta > 0$.

Case 1 (equation (14) **holds for some** $\delta > 0$**).** Pick $0 < \gamma < 1$ as in Lemma 12, and let $z > 2$ be some sufficiently large number. Pick $k_1, k_2, N \in \mathbb{N}$ as follows. Let k_2 be the smallest integer such that

$$a_{k_2}^{(M+2+\delta)^{-k_2}} > z^{1/\gamma}, \tag{15}$$

let k_1 be the largest integer such that $k_1 < k_2$ and

$$a_{k_1} \le z^{k_1}, \tag{16}$$

and let N be the smallest number such that $N > k_1$ and

$$a_N^{(M+2+\delta)^{-N}} \ge z. \tag{17}$$

Note that $N \le k_2$ and $N \to \infty$ as $z \to \infty$. By inequalities (16) and (17), $a_n < z^{(M+2+\delta)^n}$ when $k_1 \le n < N$. Hence,

$$2^{N^2(\log_2 a_n)^\kappa} \le 2^{N(M+2+\delta)^{(N-1)c}(\log_2 z)^c} < z^{N^2(M+2+\delta)^{Nc}}, \tag{18}$$

while also

$$\prod_{n=k_1}^{N-1} a_n < \prod_{n=k_1+1}^{N-1} z^{(M+2+\delta)^n} = z^{\sum_{n=k_1+1}^{N-1}(M+2+\delta)^n}.$$

From this and equation (13), we obtain

$$\prod_{n=k_1}^{N-1} a_n < z^{\frac{(M+2+\delta)^N}{M+1+\delta}},$$

while inequality (16) together with the facts that $N > k_1$ and a_n is non-decreasing yields

$$\prod_{n=1}^{k_1-1} a_n \le a_{k_1}^{k_1-1} < z^{N^2}.$$

Thus,

$$\prod_{n=1}^{N-1} a_n^{M+1} < z^{(M+1)N^2+(M+1)\frac{(M+2+\delta)^N}{M+1+\delta}}. \tag{19}$$

Having a bound for the product in Z_N, we move on to bounding the infinite series. Let $0 < \Gamma < 1$ be given as in Lemma 14. Let $\zeta \in \left(\frac{M+1}{M+1+\delta}, 1\right)$ be a fixed number that does not depend on z. Since k_1 is the largest number less than k_2 satisfying (16), we have $a_n > z^n > 2^n$ for $N \le n \le k_2$. By Lemma 14, this means that when z (and thereby N) is sufficiently large, then

$$\sum_{n=N}^{k_2} \frac{b_n}{a_n} \le \frac{2^{(\log_2 a_N)^\Gamma}}{a_N} \le \frac{1}{a_N^\zeta}.$$

This and inequality (17) imply

$$\sum_{n=N}^{k_2} \frac{b_n}{a_n} \le z^{-(M+2+\delta)^N \zeta}.$$

From Lemma 12, inequality (15), and the fact that a_n is non-decreasing, it follows for all sufficiently large z that

$$\sum_{n=k_2+1}^{\infty} \frac{b_n}{a_n} \le a_{k_2+1}^{-\gamma} \le a_{k_2}^{-\gamma} \le z^{-(M+2+\delta)^{k_2}},$$

and so we have

$$\sum_{n=N}^{\infty} \frac{b_n}{a_n} \le z^{-(M+2+\delta)^N \zeta} + z^{-(M+2+\delta)^{k_2}} \le \frac{2}{z^{(M+2+\delta)^N \zeta}}. \tag{20}$$

Write $\zeta' = \zeta - \frac{M+1}{M+1+\delta}$ and note that $\zeta' > 0$. By inequalities (18), (19), and (20), we find that for sufficiently large z (and thus N),

$$\begin{aligned} Z_N &= 2^{E_N} \left(\prod_{n=1}^{N-1} a_n^M \right) \sum_{n=N}^{k_2} \frac{b_n}{a_n} \\ &< z^{N^{-3}(M+2+\delta)^N + (M+1)N^2 + (M+2+\delta)^N \left(\frac{M+1}{M+1+\delta} - \zeta\right)} \\ &= z^{N^{-3}(M+2+\delta)^N + MN^2 - \zeta'(M+2+\delta)^N} < z^{-\frac{\zeta'}{2}(M+2+\delta)^N}. \end{aligned}$$

Hence, Z_N tends to 0 when z grows toward infinity, and this case is complete.

Case 2 (equation (14) **does not hold for any** $\delta > 0$**).** This case will be split into further 2 subcases, depending on whether $a_n < 2^n$ infinitely often. Before we do that, we make an observation to be used for both cases.

Given any fixed number Γ, set $\Gamma_0 = \max\{c, \Gamma\}$ and $\tilde{\Gamma} = (1+2\Gamma_0)/(2+\Gamma_0)$. Note that $\Gamma_0 < \tilde{\Gamma} < 1$. By the assumption that equation (14) holds for no

$\delta > 0$, we may pick a small number $\delta_\Gamma > 0$ that does not depend on n, such that

$$(M + 2 + \delta_\Gamma)^{(2+\Gamma_0)/3} < M + 2$$

and $a_n < 2^{(M+2+\delta_\Gamma)^n}$ for all sufficiently large n. Due to this and the fact that a_n is non-decreasing,

$$\begin{aligned} n^2(\log_2 a_n)^c + (\log_2 a_n)^\Gamma &< 2n^2(\log_2 a_n)^{\tilde{\Gamma}} < 2n^2\big(M + 2 + \delta_\Gamma\big)^{n\Gamma_0} \\ &< 2n^2(M + 2)^{n\frac{3\Gamma_0}{2+\Gamma_0}} < (M + 2)^{n\frac{1+2\Gamma_0}{2+\Gamma_0}} \\ &= (M + 2)^{\tilde{\Gamma}n} < n^{-3}(M + 2)^n \end{aligned} \tag{21}$$

Recalling $(1 + n^{-2}) = 2^{\log_2(1+n^{-2})}$ and using the Taylor expansion of $\log_2 x$ around 1, we have that $(1 + n^{-2}) > 2^{n^{-2}}$ when n is large. From this and inequality (21), we obtain for all large enough n that

$$\frac{2^{E_n+(\log_2 a_n)^\Gamma}}{(1 + n^{-2})^{(M+2)^n}} < \frac{2^{2n^{-3}(M+2)^{n+1}}}{2^{n^{-2}(M+2)^n}} < 2^{-n^{-3}(M+2)^n}. \tag{22}$$

Case 2a ($a_n < 2^n$ for at most finitely many n). By Lemma 13, we may pick $0 < \Gamma < 1$ such that for all sufficiently large N,

$$\sum_{n=N}^{\infty} \frac{b_n}{a_n} \leq \frac{2^{(\log_2 a_N)^\Gamma}}{a_N}.$$

This and Corollary 16 imply that for infinitely many N,

$$\begin{aligned} Z_N &< 2^{E_N} \frac{a_N}{(1 + N^{-2})^{(M+2)^N}} \frac{2^{(\log_2 a_N)^\Gamma}}{a_N} \\ &= \frac{2^{E_N+(\log_2 a_N)^\Gamma}}{2^{(M+2)^N \log_2(1+N^{-2})}}. \end{aligned}$$

From this and inequality (22), we then have

$$Z_N < 2^{-N^{-3}(M+2)^N}$$

for infinitely many N, and we are done.

Case 2b ($a_n < 2^n$ infinitely often). Let $C > 0$ be sufficiently large. Pick $k_1, k_2, N \in \mathbb{N}$ that depend on C as follows. Let k_2 be the smallest integer such that

$$a_{k_2}^{(M+2)^{-k_2}} > C, \tag{23}$$

and let k_1 be the largest integer such that $k_1 < k_2$ and

$$a_{k_1} < 2^{k_1}. \tag{24}$$

Since $k_2 \to \infty$ as $C \to \infty$, the assumption that $a_n < 2^n$ infinitely often implies that also $k_1 \to \infty$ as $C \to \infty$. Using Corollary 16, pick $N > k_1$ to be the smallest integer such that

$$a_N > \left(1 + \frac{1}{N^2}\right)^{(M+2)^N} \left(\max_{k_1 \le n < N} a_n^{(M+2)^{-n}}\right)^{(M+2)^N}. \tag{25}$$

Consequently, we get by induction that if $k_1 < n < N$, then

$$\begin{aligned} a_n^{(M+2)^{-n}} &\le \left(1 + \frac{1}{n^2}\right) \max_{k_1 \le i < n} a_i^{(M+2)^{-i}} \\ &\le \cdots \le a_{k_1}^{(M+2)^{-k_1}} \prod_{m=k_1+1}^{n} \left(1 + \frac{1}{m^2}\right). \end{aligned}$$

Therefore, each n with $k_1 \le n < N$ must satisfy

$$a_n < 8^{(M+2)^n} = 2^{3(M+2)^n}, \tag{26}$$

using that $\prod_{m=1}^{\infty}(1 + m^{-2}) < 4$ and that $a_{k_1}^{(M+2)^{-k_1}} < 2$ by inequality (24). Note that inequalities (23) and (26) ensure $N \le k_2$ when C is large.

Using inequality (24) along with the facts that a_n is non-decreasing and $N > k_1$, we find

$$\prod_{n=1}^{k_1 - 1} a_n \le a_{k_1-1}^{k_1-1} < a_{k_1}^{k_1} < 2^{k_1^2} < 2^{N^2}. \tag{27}$$

From inequalities (26) and (27), we get

$$\prod_{n=1}^{N-1} a_n^{M+1} = \left(\prod_{n=1}^{k_1-1} a_n^{M+1}\right) \prod_{n=k_1}^{N-1} a_n^{M+1} < 2^{N^2(M+1) + 3\sum_{n=k_1}^{N-1}(M+2)^n}.$$

Due to this and equation (13), all large enough C (and thus N) must satisfy

$$\begin{aligned} \prod_{n=1}^{N-1} a_n^{M+1} &< 2^{N^2(M+1) + 3\sum_{n=k_1}^{N-1}(M+2)^n} < 2^{N^2(M+1) + 3(M+2)^N} \\ &< 2^{4(M+2)^N}. \end{aligned} \tag{28}$$

Meanwhile, inequality (25) allows us to apply Corollary 16 and find

$$\prod_{n=k_1}^{N-1} a_n^{M+1} \le \frac{a_N}{(1+N^{-2})^{(M+2)^N}},$$

which together with inequality (27) yields

$$\begin{aligned}\prod_{n=1}^{N-1} a_n^{M+1} &= \left(\prod_{n=1}^{k_1-1} a_n^{M+1}\right)\prod_{n=k_1}^{N-1} a_n^{M+1} \\ &< 2^{N^2(M+1)}\frac{a_N}{(1+N^{-2})^{(M+2)^N}}. \end{aligned} \tag{29}$$

Having two upper bounds of $\prod_{n=1}^{N-1} a_n^{M+1}$, we move on to also bounding $\sum_{n=N}^{\infty} b_n/a_n$. Pick $\Gamma \in (0,1)$ as in Lemma 14. Since $k_1 < N \le k_2$, k_1 is the greatest integer less than k_2 that satisfies inequality (24), and $a_{k_2} > 2^{k_2}$ due to inequality (23), we obtain from Lemma 14 that

$$\sum_{n=N}^{k_2} \frac{b_n}{a_n} \le \frac{2^{(\log_2 a_N)^\Gamma}}{a_N}. \tag{30}$$

Similarly, pick $\gamma \in (0,1)$ as in Lemma 12. Then

$$\sum_{n=k_2+1}^{\infty} \frac{b_n}{a_n} \le \frac{1}{a_{k_2+1}^{\gamma}} \le \frac{1}{a_{k_2}^{\gamma}},$$

since a_n is non-decreasing. Estimating this further by applying inequality (23) together with the fact that $k_2 \ge N$, we get

$$\sum_{n=k_2+1}^{\infty} \frac{b_n}{a_n} \le \frac{1}{a_{k_2}^{\gamma}} < \frac{1}{C^{\gamma(M+2)^{k_2}}} \le \frac{1}{C^{\gamma(M+2)^N}}.$$

This and inequality (30) imply

$$\sum_{n=N}^{\infty} \frac{b_n}{a_n} = \sum_{n=N}^{k_2} \frac{b_n}{a_n} + \sum_{n=k_2+1}^{\infty} \frac{b_n}{a_n} \le \frac{2^{(\log_2 a_N)^\Gamma}}{a_N} + \frac{1}{C^{\gamma(M+2)^N}}. \tag{31}$$

From inequalities (28), (29), and (31) follows that

$$
\begin{aligned}
Z_N &= 2^{E_N} \left(\prod_{n=1}^{N-1} a_n \right)^{M+1} \sum_{n=N}^{k_2} \frac{b_n}{a_n} \\
&< 2^{E_N} \min \left\{ 2^{4(M+2)^N}, \frac{2^{N^2(M+1)} a_N}{(1+N^{-2})^{(M+2)^N}} \right\} \\
&\cdot \left(\frac{2^{(\log_2 a_N)^\Gamma}}{a_N} + \frac{1}{C^{\gamma(M+2)^N}} \right) \\
&\leq \frac{2^{E_N + N^2(M+1) + (\log_2 a_N)^\Gamma}}{(1+N^{-2})^{(M+2)^N}} + \frac{2^{E_N + 4(M+2)^N}}{C^{\gamma(M+2)^N}}.
\end{aligned}
$$

By inequalities (21) and (22), we then have

$$
Z_N < 2^{-N^{-3}(M+2)^N} + \frac{2^{5(M+2)^N}}{C^{\gamma(M+2)^N}},
$$

which tends to 0 as C grows large, due to the facts that $N > k_1$, $k_1 \to \infty$ as $C \to \infty$, and $\tilde{\Gamma} < 1$. This completes the proof. □

Lemma 11 is to be used in connection with the below elementary result.

Lemma 17. *Let $R \geq 1$, let $P \in \mathbb{Z}[x_1, \ldots, x_K]$ be a polynomial, and let $\alpha_1, \ldots, \alpha_K, \beta_1, \ldots, \beta_K \in \mathbb{C}$ with $|\alpha_k|, |\beta_k| \leq R$ for each k. Then there is a constant $C > 0$, depending only on P and R, such that*

$$
|P(\alpha_1, \ldots, \alpha_K) - P(\beta_1, \ldots, \beta_K)| \leq C \max_{1 \leq k \leq K} |\alpha_k - \beta_k|
$$

Proof. For $i_k \in \mathbb{N}$, note that

$$
\begin{aligned}
\prod_{k=1}^{K} \alpha_k^{i_k} - \prod_{k=1}^{K} \beta_k^{i_k} &= \sum_{k=1}^{K} \left(\prod_{j=1}^{k-1} \alpha_l^{i_l} \right) (\alpha_k^{i_k} - \beta_k^{i_k}) \prod_{j=k+1}^{K} \beta_l^{i_l} \\
&= \sum_{k=1}^{K} \left((\alpha_k - \beta_k) \sum_{j=1}^{i_k} \alpha_k^{j-1} \beta_k^{k-j} \right) \left(\prod_{j=1}^{k-1} \alpha_l^{i_l} \right) \prod_{j=k+1}^{K} \beta_l^{i_l}. \quad (32)
\end{aligned}
$$

Write $P(x_1, \ldots, x_K) = \sum_{i_1, \ldots, i_k} c_{i_1, \ldots, i_K} x_1^{i_1} \cdots x_K^{i_K}$, $d = \deg P$, $\alpha = (\alpha_1, \ldots, \alpha_K)$, and $\beta = (\beta_1, \ldots, \beta_K)$. Then the triangle inequality, equation (32), and the facts that $\deg P = d$ and $|\alpha_k|, |\beta_k| \leq R$ let us conclude

$$\begin{aligned}|P(\alpha_1, \ldots, \alpha_K) - P(\beta_1, \ldots, \beta_K)| &= \left| \sum_{i_1,\ldots,i_k} c_{i_1,\ldots,i_K} \left(\prod_{k=1}^{K} \alpha_k^{i_k} - \prod_{k=1}^{K} \beta_k^{i_k} \right) \right| \\ &\leq \sum_{i_1,\ldots,i_k} |c_{i_1,\ldots,i_K}| \sum_{k=1}^{K} |\alpha_k - \beta_k| d R^{d-1} \\ &\leq \left(dKR^{d-1} \sum_{i_1,\ldots,i_k} |c_{i_1,\ldots,i_K}| \right) \max_{1 \leq k \leq K} |\alpha_k - \beta_k|. \quad \square\end{aligned}$$

In order to handle the case of $K = 2$ and $d > 3$ in Theorems 1 and 6, we will need a few results from algebraic geometry. Let $\mathbb{P}^K$ denote the K-dimensional complex projective space, i.e., $\mathcal{P} = (\mathbb{C}^{K+1} \setminus 0)/\!\sim$ where $\sim$ denotes the equivalence relation that $x \sim y$ if $x = \alpha y$ for some $\alpha \in \mathbb{C} \setminus 0$. For $i = 1, \ldots, n$, let $P_i \in \mathbb{C}[x_1, \ldots, x_K]$ be a polynomial, and let $\tilde{P}_i \in \mathbb{C}[x_1, \ldots, x_{K+1}]$ denote the unique homogeneous polynomial of minimal degree such that $P_i(x_1, \ldots, x_K) = \tilde{P}_i(x_1 \ldots, x_K, 1)$. We then say that the set

$$V(P_1, \ldots, P_n) = \{[x] \in \mathbb{P}^K : \tilde{P}_1(x) = \cdots = \tilde{P}_n(x) = 0\}$$

is a *projective variety*. If the coefficients of $P_1, \ldots, P_n$ are all rational numbers, we say that $V(P_1, \ldots, P_n)$ is a projective variety over $\mathbb{Q}$.

Let V_1 and V_2 be two non-empty irreducible projective varieties that are proper subvarieties of $\mathbb{P}^K$. We then say that $f : V_1 \dashrightarrow V_2$ is a *rational map* if there is a proper subvariety $B_1 \subsetneq V_1$ (which may be reducible) such that f restricted to $V_1 \setminus B_1$ is a well-defined rational function, i.e., each coordinate map is given as the quotient of two polynomials. If there is another rational map $g : V_2 \dashrightarrow V_1$ so that $g \circ f$ coincides with the identity where it is defined, then we say that f is a *birational map* and that V_1 and V_2 are *birationally equivalent*. We furthermore say that f is a rational (resp. birational) map over $\mathbb{Q}$ if the implied rational function can be chosen so that the coefficients of its coordinate maps are contained in $\mathbb{Q}$, and we say that V_1 and V_2 are birationally $\mathbb{Q}$-equivalent if there is a birational map $V_1 \dashrightarrow V_2$ over $\mathbb{Q}$. We will need the below three theorems, of which the first is known as Fatings's Theorem, and the second is a consequence of the degree–genus formula. Genus here refers to the geometric genus. All three theorems can be found in [13].

Theorem 18 (Faltings)**.** *Let A be a non-singular irreducible projective variety over $\mathbb{Q}$ of genus $g > 1$ and dimension 1. Then A has only finitely many rational points.*

Theorem 19. *Let $V \subseteq \mathbb{P}^2$ be a non-singular irreducible projective variety of degree d and dimension 1. Then the genus of V equals $(d-1)(d-2)/2$.*

Theorem 20. *Let $V \subseteq \mathbb{P}^2$ be an irreducible singular projective variety over $\mathbb{Q}$ of dimension 1. Then V is birationally $\mathbb{Q}$-equivalent to a smooth irreducible projective variety over $\mathbb{Q}$.*

Finally, we will also need the below simple result.

Lemma 21. *Let V_1, V_2 be birationally equivalent irreducible projective varieties of dimension 1. Then the implied rational map $f : V_1 \dashrightarrow V_2$ is defined for all but finitely many elements of V_1.*

Proof. Because V_1 is irreducible, any non-empty proper subvariety B of V_1 must be of a lower dimension than V_1. Since V_1 is of dimension 1, this makes B finite. □

5 Proof of Theorems 1 and 6

For this section, let d and K be positive integers, and let $\{a_{k,n}\}_{n=1}^{\infty}$ and $\{b_{k,n}\}_{n=1}^{\infty}$ be sequences of non-zero integers such that each of the sequences $\{\alpha_{1,N}\}_{N=1}^{\infty}, \dots, \{\alpha_{K,N}\}_{N=1}^{\infty}$ defined by

$$\alpha_{k,N} = \sum_{n=1}^{N} \frac{b_{k,n}}{a_{k,n}}$$

converge, and we write $\alpha_k = \lim_{N\to\infty} \alpha_{k,N}$ for the corresponding limit.

Theorems 1 and 6 have almost identical proofs. For that reason, we will prove them simultaneously. Roughly speaking, their proofs can be divided into an analytical part, which is covered by the below lemma, and an algebraic part, which is covered by the subsequent two lemmas.

Lemma 22. *Let ε and κ be positive real numbers with $\kappa < 1$. Suppose for all $k = 1, \dots, K$ and $n \in \mathbb{N}$ that equations (3), (4), (5), and (6) are satisfied. Let $P \in \mathbb{Z}[x_1, \dots, x_K]$ be a polynomial of degree at most d. If $P(\alpha_{1,N}, \dots \alpha_{K,N}) \neq 0$ for all sufficiently large N, then $P(\alpha_1, \dots \alpha_K) \neq 0$.*

Proof. Let N be sufficiently large. Then $P(\alpha_{1,N}, \dots, \alpha_{K,N}) \neq 0$. Therefore, since $\deg P \leq d$ and $\alpha_{k,N} \prod_{n=1}^{N} |a_{k,n}|$ must be integral, we get

$$|P(\alpha_{1,N}, \dots, \alpha_{K,N})| \geq \left(\prod_{n=1}^{N} \prod_{k=1}^{K} |a_{k,n}| \right)^{-d}$$

and so, by inequality (4),

$$
\begin{aligned}
|P(\alpha_{1,N},\ldots,\alpha_{K,N})| &\geq \prod_{n=1}^{N}\left(a_n^K 2^{(K-1)\max\{(\log_2 a_n)^\kappa,\, n^{-3}(Kd+1)^n\}}\right)^{-d}\\
&\geq 2^{-\max\{N^2(\log_2 a_N)^\kappa,\, N^{-3}(Kd+1)^{N+1}\}}\prod_{n=1}^{N} a_n^{-dK}. \qquad (33)
\end{aligned}
$$

Meanwhile, Lemma 17 implies that there is a $C>0$ depending only on P and on $\sup_{k,N}|\alpha_{k,N}|$ such that

$$
\begin{aligned}
|P(\alpha_1,\ldots,\alpha_K)-P(\alpha_{1,N},\ldots,\alpha_{K,N})| &\leq C\max_{1\leq k\leq K}|\alpha_k-\alpha_{k,N}|\\
&= C\max_{1\leq k\leq K}\left|\sum_{n=N+1}^{\infty}\frac{b_{k,n}}{a_{k,n}}\right|.
\end{aligned}
$$

Therefore, due to the triangle inequality followed by inequalities (4) and (5),

$$
\begin{aligned}
|P(\alpha_1,\ldots,\alpha_K)-P(\alpha_{1,N},\ldots,\alpha_{K,N})| &\leq C\max_{1\leq k\leq K}\sum_{n=N+1}^{\infty}\frac{|b_{k,n}|}{|a_{k,n}|}\\
&\leq C\sum_{n=N+1}^{\infty}\frac{2^{2(\log_2 a_n)^\kappa}}{a_n} = C\sum_{n=N+1}^{\infty}\frac{4^{(\log_2 a_n)^\kappa}}{a_n}.
\end{aligned}
$$

This, the triangle inequality, and estimate (33) imply that

$$
\begin{aligned}
|P(\alpha_1,\ldots,\alpha_K)| \geq{}& |P(\alpha_{1,N},\ldots,\alpha_{K,N})| - |P(\alpha_1,\ldots,\alpha_K)\\
&- P(\alpha_{1,N},\ldots,\alpha_{K,N})|\\
\geq{}& 2^{-\max\{N^2(\log_2 a_N)^\kappa,\, N^{-3}(Kd+1)^{N+1}\}}\prod_{n=1}^{N} a_n^{-dK}\\
&- C\sum_{n=N+1}^{\infty}\frac{4^{(\log_2 a_n)^\kappa}}{a_n}.
\end{aligned}
$$

Hence, to prove the lemma, it is enough to show that there are infinitely many N such that

$$
2^{-\max\{N^2(\log_2 a_N)^\kappa,\, N^{-3}(Kd+1)^{N+1}\}}\prod_{n=1}^{N} a_n^{-dK} > C\sum_{n=N+1}^{\infty}\frac{4^{(\log_2 a_n)^\kappa}}{a_n}
$$

or, equivalently,

$$
2^{\max\{N^2(\log_2 a_N)^\kappa,\, N^{-3}(Kd+1)^{N+1}\}}\left(\prod_{n=1}^{N} a_n^{-dK}\right)\sum_{n=N+1}^{\infty}\frac{4^{(\log_2 a_n)^\kappa}}{a_n} < C^{-1}. \qquad (34)
$$

Note that

$$2^{\max\{N^2(\log_2 a_N)^\kappa,\, N^{-3}(Kd+1)^{N+1}\}}\left(\prod_{n=1}^{N} a_n^{-dK}\right)\sum_{n=N+1}^{\infty}\frac{4^{(\log_2 a_n)^\kappa}}{a_n}$$
$$\leq 2^{N^2(\log_2 a_N)^\kappa}\left(\prod_{n=1}^{N} a_n\right)^{dK}\sum_{n=N+1}^{\infty}\frac{\lceil 4^{(\log_2 a_n)^\kappa}\rceil}{a_n}, \tag{35}$$

for all large enough values of N. Taking $\kappa' \in (\kappa, 1)$, we have $\lceil 4^{(\log_2 a_n)^\kappa}\rceil \leq 2^{(\log_2 a_n)^{\kappa'}}$ when N is sufficiently large. Using this together with assumptions (3) and (6), we may apply Lemma 11 with $M = dK - 1$. Therefore, we can pick infinitely many N such that the right-hand-side of (35) is smaller than C^{-1}, and so inequality (34) follows, and the proof is complete. □

For the remaining two lemmas, let $\mathcal{P}$ be a set of either a single prime number or infinitely many prime numbers, and assume that $p \nmid \gcd(b_{k,n}, a_{k,n})$ for all $k = 1, \ldots, K$ and $n \in \mathbb{N}$. Recall the elementary facts that

$$\nu_p(ab) = \nu_p(a) + \nu_p(b), \qquad \nu_p(a+b)\begin{cases} = \nu_p(a), & \text{if } \nu_p(a) < \nu_p(b), \\ \geq \nu_p(a), & \text{if } \nu_p(a) = \nu_p(b). \end{cases} \tag{36}$$

From the first of these two facts, ν_p extends to a function $\mathbb{Q} \to \mathbb{Z}$ by $\nu_p(0) = \infty$ and $\nu_p(q) = \nu_p(a) - \nu_p(b)$ when $q = a/b$; note that $\nu_p(q)$ does not depend on the choice of representative (a, b).

In the proof of the below lemma, we will order tuples of indices colexicographically, i.e., we will say that

$$(i_1, \ldots, i_{n+1}) < (j_1, \ldots, j_{n+1})$$

if $i_{n+1} < j_{n+1}$, or if both $i_{n+1} = j_{n+1}$ and $(i_1, \ldots, i_n) < (j_1, \ldots, j_n)$.

Lemma 23. *Suppose for each fixed $k = 1, \ldots, K$ and $p \in \mathcal{P}$ that equation* (1) *holds for all $N \in \mathbb{N}$ and that inequality* (11) *holds for all sufficiently large N. If $\mathcal{P} = \{p\}$, assume additionally that equation* (2) *holds for each $k = 1, \ldots, K$. Let $P \in \mathbb{Z}[x_1, \ldots, x_K]$ be a non-zero polynomial of degree at most d. Then $P(\alpha_{1,N}, \ldots, \alpha_{K,N}) \neq 0$ for all sufficiently large N.*

Proof. Write

$$P(x_1, \ldots, x_K) = \sum_{i_1,\ldots,i_K} c_{i_1,\ldots,i_K} x_1^{i_1} \cdots x_K^{i_K},$$

and let $c_{j_1,\dots,j_K} x_1^{j_1} \cdots x_K^{j_K}$ be its leading term in colexicographic order. We will now prove that for a suitably chosen $p \in \mathcal{P}$ and each sufficiently large N, the term $c_{j_1,\dots,j_K} \alpha_{1,N}^{j_1} \cdots \alpha_{K,N}^{j_K}$ has strictly lower p-adic valuation than any of the other terms of $P(\alpha_{1,N}, \dots, \alpha_{K,N})$. By (36), this will then imply the lemma. To simplify notation, write

$$C_{i_1,\dots,i_K} = \frac{c_{i_1,\dots,i_K}}{c_{j_1,\dots,j_K}} \prod_{k=1}^{K} \alpha_{k,N}^{i_k - j_k}. \tag{37}$$

If $|\mathcal{P}| = 1$, let p be the unique prime in $\mathcal{P}$. Otherwise, $|\mathcal{P}|$ is infinite, and we instead pick p such that

$$\nu_p(c_{j_1,\dots,j_K}) = 0. \tag{38}$$

Since clearly $C_{j_1,\dots,j_K} = 1$ by equation (37), the lemma follows if we can prove for all $(i_1, \dots, i_K) \neq (j_1, \dots, j_K)$ with $c_{i_1,\dots,i_K} \neq 0$ that

$$\nu_p(C_{i_1,\dots,i_K}) > 0 \tag{39}$$

when N is sufficiently large.

Suppose $(i_1, \dots, i_K) \neq (j_1, \dots, j_K)$ with $c_{i_1,\dots,i_K} \neq 0$, and let m be the largest index such that $i_k \neq j_k$. Note that this implies $i_m < j_m$ since $c_{j_1,\dots,j_K} x^{j_1} \cdots x^{j_K}$ is the leading term of P. When N is sufficiently large, we have from inequality (11) that $\max_{1 \le n \le N} \nu_p(a_{k,n}) > 0$ for each $k = 1, \dots, K$, which together with (1), (36), and the fact that $p \nmid \gcd(a_{k,n}, b_{k,n})$ implies that $\nu_p(\alpha_{k,N}) = -\max_{1 \le n \le N} \nu_p(a_{k,n})$. Using this and equation (37), we find that when N is sufficiently large,

$$\begin{aligned} \nu_p(C_{i_1,\dots,i_K}) &= \nu_p(c_{i_1,\dots,i_K}) - \nu_p(c_{j_1,\dots,j_K}) + \sum_{k=1}^{K} (i_k - j_k) \nu_p(\alpha_{k,N}) \\ &= \nu_p(c_{i_1,\dots,i_K}) - \nu_p(c_{j_1,\dots,j_K}) + \sum_{k=1}^{m} (j_k - i_k) \max_{1 \le n \le N} \nu_p(a_{k,n}). \end{aligned}$$

Hence, using the facts that $i_m < j_m$, $c_{i_1,\dots,i_K} \in \mathbb{Z}$, and $a_{k,n} \in \mathbb{Z}$,

$$\nu_p(C_{i_1,\dots,i_K}) \ge -\nu_p(c_{j_1,\dots,j_K}) + \max_{1 \le n \le N} \nu_p(a_{m,n}) - \sum_{k=1}^{m-1} i_k \max_{1 \le n \le N} \nu_p(a_{k,n}).$$

This, the fact that $\deg P \le d$, and inequality (11) yield

$$\nu_p(C_{i_1,\dots,i_K}) \ge -\nu_p(c_{j_1,\dots,j_K}) + \max_{1 \le n \le N} \nu_p(a_{m,n}) - d \max_{1 \le n \le N} \nu_p(a_{m-1,n}). \tag{40}$$

If $|\mathcal{P}| = 1$, then inequality (39) follows from inequality (40) and equation (2) when N is sufficiently large. Similarly, if $|\mathcal{P}| = \infty$, then (39) is verified by applying inequality (38) and equation (11) to inequality (40) when N is sufficiently large. This completes the proof. □

The final lemma of this paper provides an improvement to Lemma 23 when $K = 2$ and the degree of P is greater than 3.

Lemma 24. *Suppose the assumptions of Lemma 23 are satisfied for $d = 3$ and $K = 2$. Let $P \in \mathbb{Z}[x_1, x_2]$ be a non-zero polynomial of arbitrary degree such that P defines a curve of genus ≥ 2. Then $P(\alpha_{1,N}, \alpha_{2,N}) \neq 0$ for all sufficiently large N.*

As before, we remark that choosing a different value d' in place of 3 will rule out polynomials of degree $\leq d'$. This will make the genus requirement less restrictive. Indeed, there are fewer polynomials resulting in low genus if the degree becomes higher.

Proof of Theorem 1. Since equation (2) implies inequality (11), the theorem follows by combining Lemma 22 with Lemmas 23 and 24. □

Proof of Theorem 6. This follows by combining Lemma 22 with Lemmas 23 and 24. □

Proof of Lemma 24. Without loss of generality, P is irreducible; if not, then we may write P as the product of irreducible polynomials $P_1, \ldots, P_L \in \mathbb{Z}[x_1, x_2]$, and we would then just have to prove the lemma for each of these P_l. If $\deg P \leq 3$, the statement follows from Lemma 23, so we may assume $\deg P > 3$.

Since $p \nmid \gcd(b_{k,n}, a_{k,n})$ for all $p \in \mathcal{P}$, equations (1) and (36) together with either (11) (if $|\mathcal{P}| = \infty$) or (2) (if $\mathcal{P} = \{p\}$) implies that the minimal denominator of $\alpha_{k,N}$ tends to infinity as N tends to infinity. Hence, $\alpha_{k,N}$ does not take the same value infinitely often.

Introducing an extra variable X_3, let $\tilde{P} \in \mathbb{Z}[X_1, X_2, X_3]$ be the homogenised version of P, and consider the projective curve $\mathcal{C}_P = \{[x] \in \mathbb{P}^2 : \tilde{P}(x) = 0\}$. Since $P(\alpha_{1,N}, \alpha_{2,N}) = \tilde{P}(\alpha_{1,N}, \alpha_{2,N}, 1)$ and since $\alpha_{k,N}$ is always rational but does not take the same value infinitely often, it suffices to prove that $\mathcal{C}_P$ contains only finitely many rational points.

Suppose first that $\mathcal{C}_P$ is non-singular. By the degree–genus formula [13, Theorem A.4.2.6.], the genus of $\mathcal{C}_P$ is at least 2 since P is irreducible with $\deg P > 3$. By Faltings's Theorem, such a variety contains at most finitely many rational points, and the lemma is proven.

Next, suppose that $\mathcal{C}_P$ is singular. In this case, we may normalise V_P, by appealing to e.g. [13, Theorem A4.1.4.], to obtain a smooth projective curve birationally equivalent to $\mathcal{C}_P$. As mentioned in the introduction, this may result in a curve of lower genus than predicted by the degree–genus formula. Ordinary singularities result in a decrease of $\frac{1}{2}r(r-1)$, where r is the multiplicity of the singularity. Non-ordinary singularities decrease the genus by a quantity which must be calculated on a case-by-case basis, but such polynomials are very rare indeed. At any rate, the assumptions of the lemma is that the resulting curve has genus at least 2, so by Falting's Theorem there are only finitely many rational points on $\mathcal{C}_P$. For the original polynomial (before normalisation), we observe that desingularising again would not increase the number of rational points, and we are done. $\square$

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J. Hančl
Department of Mathematics, Faculty of Science, University of Ostrava, 30. dubna 22, 701 03 Ostrava 1, Czech Republic.
e-mail: hancl@osu.cz

S. Kristensen
Department of Mathematics, Aarhus University, Ny Munkegade 118, 8000 Aarhus C, Denmark.
e-mail: sik@math.au.dk

M. L. Laursen
Department of Mathematics, Aarhus University, Ny Munkegade 118, 8000 Aarhus C, Denmark.
e-mail: mll@math.au.dk

Chapter 2

p-adic Duffin–Schaeffer Sets

2.1 Introduction

In the study of irrationality and transcendence of numbers as introduced in the first few pages of Chapter 1, one often encounters various Diophantine inequalities, i.e., inequalities with integer coefficients where one is interested in finding or counting the set of integer solutions. One such inequality is the following, where α is a fixed real number, ψ is a map from $\mathbb{N}$ to $\mathbb{R}_{\geq 0}$,

$$\left|\alpha - \frac{a}{n}\right| \leq \frac{\psi(n)}{n}. \tag{2.1}$$

Depending on how ψ is chosen, various arithmetic properties of α may be deduced from the size of the solution set of inequality (2.1). We may, for instance, reformulate Theorem 1.1 with $d = 1$ and Theorem 1.2 from Chapter 1 into the following statement.

Theorem 2.1. *Let $\alpha \in \mathbb{R}$. Then α is irrational if inequality* (2.1) *has infinitely many solutions with $a/n \neq \alpha$ and $\psi(n) = 1$. If inequality* (2.1) *also has infinitely many solutions with $a/n \neq \alpha$ and $\psi(n) = n^{-1-\delta}$ for some $\delta > 0$, then α is transcendental.*

Let $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be fixed, and let $m \in \mathbb{N}$. Notice that if (a, n) is a solution to inequality (2.1) for some α, then $(a + mn, n)$ is a solution for $\alpha + m$. Hence, the number of solutions for fixed n is unaffected by adding or subtracting integers from α. Because of this, it is sufficient to consider α between 0 and 1 to gain information about the real numbers in general, at least when it comes to inequality (2.1).

We will now consider the set of α from the unit interval with infinitely many solutions (a, n), formally defined as

$$\mathcal{K}(\psi) := \{\alpha \in [0, 1] : (2.1) \text{ is true for infinitely many } (a, n) \in \mathbb{Z} \times \mathbb{N}\}.$$

An immediate benefit from studying subsets of $[0,1]$ rather than of $\mathbb{R}$ is that the Lebesgue measure, which we denote as λ, is a probability measure on $[0,1]$. This gives us access to a number of tools from probability theory that are less easy to apply on $\mathbb{R}$. In 1924, Alexandre Khintchine [31] used some of these tools to prove the following theorem. To the current author's knowledge, this theorem started the field now known as *metric Diophantine approximation*, to which all theorems of this chapter belong. Named after him, we will also refer to $\mathcal{K}(\psi)$ as the *Khintchine set* associated to ψ.

Theorem 2.2 (Khintchine, 1924)**.** *Let $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$, and suppose that $n\psi(n)$ is monotonous. Then*

$$\lambda(\mathcal{K}(\psi)) = \begin{cases} 1 & \text{if } \sum_{n=1}^{\infty} \psi(n) = \infty, \\ 0 & \text{if } \sum_{n=1}^{\infty} \psi(n) < \infty. \end{cases}$$

The main limitation of Theorem 2.2 is the restriction that $n\psi(n)$ must be monotonous. While it might seem possible Khintchine just lacked the arguments to prove it more generally, Duffin and Schaeffer [10] showed in 1941 that at least some restriction has to be put on ψ for the statement to be true. Specifically, they constructed an approximation function $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ that contradicts it but with $n\psi(n)$ being far from monotonous. They then suggested a related but slightly different set to $\mathcal{K}(\psi)$, which we will call the *Duffin–Schaeffer set* and define as

$$\mathcal{A}^{\infty}(\psi) := \{\alpha \in [0,1] : (2.1) \text{ is true for infinitely many } a/n \in \mathbb{Q}\}, \tag{2.2}$$

where each rational number $a/n \in \mathbb{Q}$ is in reduced form, i.e., $a \in \mathbb{Z}$ and $n \in \mathbb{N}$ with $\gcd(a,n) = 1$. By weighting $\psi(n)$ based on the number of rational numbers in the interval $[0,1)$ with denominator n, they proposed the following formula for deciding the Lebesgue measure of $\mathcal{A}^{\infty}(\psi)$. Here and for the rest of this chapter, φ denotes the Euler totient function, i.e., $\varphi(n)$ is the number of integers $1, 2, \ldots, n$ that are coprime with n.

Conjecture 2.3 (Duffin–Schaeffer, 1941)**.** *Let $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$. Then*

$$\lambda(\mathcal{A}^{\infty}(\psi)) = \begin{cases} 1 & \text{if } \sum_{n=1}^{\infty} \psi(n)\varphi(n)/n = \infty, \\ 0 & \text{if } \sum_{n=1}^{\infty} \psi(n)\varphi(n)/n < \infty. \end{cases}$$

While it may seem similar to Theorem 2.2, it remained unproven for the better part of a century. It even took two decades before Gallagher [14] in 1961 managed to prove that $\lambda(\mathcal{A}^{\infty}(\psi))$ cannot take other values than 0 or 1.

Theorem 2.4 (Gallagher, 1961)**.** $\lambda(\mathcal{A}^{\infty}(\psi)) \in \{0,1\}$ *for all* $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$.

After yet another 29 years, Pollington and Vaughan [45] came close to a proof, leaving a relatively small family of ψ to be considered. They also fully proved a higher dimensional variant of the conjecture for dimension 2 and greater. In 2020, after 79 years, the conjecture was finally proven by Koukoulopoulos and Maynard [33].

Theorem 2.5 (Koukoulopoulos–Maynard, 2020)**.** *Conjecture 2.3 is true.*

In the meantime, Haynes [27] had introduced a p-adic variant of $\mathcal{A}^\infty(\psi)$, hoping that it might be easier to prove the theorem within the realm of p-adic numbers (defined below) and then move the result back into the real setting.

Definition 2.6. Let p be a prime number. We then have the following definitions.

- The *p-adic valuation*, denoted ν_p, is defined on $\mathbb{Q}$ as follows.
  - $\nu_p(0) = \infty$.
  - If $n \in \mathbb{Z}$ with $n \neq 0$, then $\nu_p(n)$ is the largest non-negative integer such that $p^{\nu_p(n)} \mid n$ and $p^{\nu_p(n)+1} \nmid n$.
  - If $(m, n) \in \mathbb{Z} \times \mathbb{N}$, then $\nu_p(m/n) = \nu_p(m) - \nu_p(n)$.
- Writing $p^{-\infty} = 0$, the *p-adic absolute value* on $\mathbb{Q}$ is given by $|x|_p = p^{-\nu_p(x)}$.
- The metric $(x, y) \mapsto |x - y|_p$ induced by the p-adic absolute value is called the *p-adic metric*.
- The *set of p-adic numbers*, denoted $\mathbb{Q}_p$, is defined as the analytical completion of $\mathbb{Q}$ with respect to the p-adic metric.
- The *set of p-adic integers*, denoted $\mathbb{Z}_p$, is defined as the analytical completion of $\mathbb{Z}$ with respect to the p-adic metric or, equivalently, as the set of p-adic numbers x with $|x|_p \leq 1$.
- The set of *p-adic units*, denoted $\mathbb{Z}_p^\times$, is defined as the set of p-adic integers that have a multiplicative inverse in $\mathbb{Z}_p$ or, equivalently, as the set of p-adic numbers x with $|x|_p = 1$.
- We will use μ_p to denote the *p-adic Haar measure* normalized on $\mathbb{Z}_p$, i.e., the unique translation invariant Borel measure on $\mathbb{Q}_p$ with $\mu_p(\mathbb{Z}_p) = 1$.

Just like the unit interval is equal to the set of real numbers $\sum_{n=1}^{\infty} a_n p^{-n}$ in the usual metric with $a_n \in \{0, \ldots, p-1\}$ for all n, $\mathbb{Z}_p$ is equal to the set of p-adic numbers of the form $\sum_{n=1}^{\infty} a_n p^{n-1}$ in the p-adic metric with $a_n \in \{0, \ldots, p-1\}$ for all n. For this reason, it makes sense to think of $\mathbb{Z}_p$ as a p-adic parallel to the unit interval, which we will do for the remainder of this chapter.

The p-adic numbers differ from the real numbers in many ways. Since $|a|_p$ can be arbitrarily small among the integers, we are suddenly in the situation that inequality (2.1) always has infinitely many solutions for each value of n when $\psi(n) > 0$, $\alpha \in \mathbb{Z}_p$, and $|\cdot|$ is replaced with $|\cdot|_p$. Since this makes the inequality redundant, we modify it in the spirit of Jarník [30] and Lutz [41],

$$\left|\alpha - \frac{a}{b}\right|_p \leq \frac{\psi(n)}{n}, \qquad n = \max\{|a|, b\}. \tag{2.3}$$

Readers familiar with height functions will notice that $\max\{|a|, b\}$ is exactly the classical height function on the number a/b when a and b are coprime. Inserting these notions into equation (2.2), we get a p-adic parallel to the Duffin–Schaeffer set,

$$\mathcal{B}^p(\psi) := \{\alpha \in \mathbb{Z}_p : (2.3) \text{ is true for infinitely many } a/b \in \mathbb{Q}\}.$$

Again, a/b is assumed to be in reduced form. This is not the only way to define a p-adic variant of the Duffin–Schaeffer set, however. When studying $\mathcal{A}^\infty(\psi)$, it is convenient to first express it as a *limsup set*, which is to say, write it as

$$\mathcal{A}^\infty(\psi) = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \mathcal{A}_n^\infty(\psi) \tag{2.4}$$

for a suitable sequence of sets $\{\mathcal{A}_n^\infty(\psi)\}_{n=1}^\infty$. From the definition of $\mathcal{A}^\infty(\psi)$, it is easy to see that this is achieved by taking

$$\mathcal{A}_n^\infty(\psi) := \bigcup_{\substack{0 \leq a \leq n \\ \gcd(a,n)=1}} B_{[0,1]}\left(\frac{a}{n}, \frac{\psi(n)}{n}\right),$$

where $B_X(x, r)$ denotes the closed ball in the metric space X with radius r around the point x. In particular,

$$B_{[0,1]}\left(\frac{a}{n}, \frac{\psi(n)}{n}\right) = \left[\frac{a - \psi(n)}{n}, \frac{a + \psi(n)}{n}\right] \cap [0, 1],$$

which has a p-adic variant in the form of

$$B_{\mathbb{Z}_p}\left(\frac{a}{n}, \frac{\psi(n)}{n}\right) = \left\{\alpha \in \mathbb{Z}_p : \left|\alpha - \frac{a}{n}\right|_p \leq \frac{\psi(n)}{n}\right\}.$$

Based on this, Haynes [27] defined a different p-adic Duffin–Schaeffer set to $\mathcal{B}^p(\psi)$,

$$\mathcal{A}^p(\psi) := \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \mathcal{A}_n^p(\psi),$$

where

$$\mathcal{A}_n^p(\psi) := \bigcup_{\substack{-n \le a \le n \\ \gcd(a,n)=1}} B_{\mathbb{Z}_p}\left(\frac{a}{n}, \frac{\psi(n)}{n}\right).$$

He preferred this variant since it – unlike $\mathcal{B}^p(\psi)$ – always has a measure of 0 or 1 (see [27] for a proof), and he proposed the following conjecture.

Conjecture 2.7 (Haynes, 2010)**.** *Let p be a prime number, and let $\psi : \mathbb{N} \to \mathbb{R}_{\ge 0}$. Then*

$$\mu_p(\mathcal{A}^p(\psi)) = \begin{cases} 1 & \text{if } \sum_{n=1}^{\infty} \mu_p(\mathcal{A}_n^p(\psi)) = \infty, \\ 0 & \text{if } \sum_{n=1}^{\infty} \mu_p(\mathcal{A}_n^p(\psi)) < \infty. \end{cases}$$

Among other interesting results in [27], Haynes proved that a certain strengthening of Conjecture 2.3 would imply Conjecture 2.7. In 2023, Kristensen and the current author [34] combined this with a modification of the proof of Theorem 2.5 by Koukoulopoulos and Maynard to settle the conjecture.

Theorem 2.8 (Kristensen–Laursen, 2023)**.** *Conjecture 2.7 is true.*

The same paper also provided a few theorems regarding $\mu_p(\mathcal{B}^p(\psi))$, one of these being the below result, which considers the spectrum of values that $\mu_p(\mathcal{B}^p(\psi))$ attains when we allow $\psi : \mathbb{N} \to \mathbb{R}_{\ge 0}$ to vary. To ease notation, let us write

$$\mathcal{S}_{\mathcal{B}^p} := \bigcup_{\psi:\mathbb{N}\to\mathbb{R}_{\ge 0}} \{\mu_p(\mathcal{B}^p(\psi))\}.$$

Going forward, this definition will also be used where $\mathcal{B}^p$ is replaced by other labels X such that $X(\psi)$ denotes a set inspired by the Khintchine set or the Duffin–Schaeffer set, writing $\mu_\infty = \lambda$ for $p = \infty$. This way, Theorem 2.4 may be rephrased as $\mathcal{S}_{\mathcal{A}^\infty} = \{0, 1\}$. In the following theorem and for the rest of this chapter, we use the notation of $a + bX = \{a + bx : x \in X\}$ when $a, b \in Y$ and $X \subseteq Y$ for some set of numbers Y. The theorem also uses the following definition of Cantor sets, which is taken from [43].

Definition 2.9. Let $x \in (0, 1/2)$, write $I_{0,1} = [0, 1]$, and consider the following iterative process. At step n for $n \in \mathbb{N}_0$, we will have a list of intervals $I_{n,1}, \ldots, I_{n,2^n}$, each of which is of length x^n. We then remove the middle of each $I_{n,j}$, leaving behind two closed intervals $I_{n+1,2j-1}(x)$ and $I_{n+1,2j}(x)$ of length $x\lambda(I_{n,j}) = x^{n+1}$ to be used in the next iterative step. Noticing that $\bigcup_{j=1}^{2^{n+1}} I_{n+1}(x) \subseteq \bigcup_{j=1}^{2^n} I_n(x)$, we define the *Cantor set* with parameter x as the limit of this sequence of joined sets,

$$C(x) := \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{2^n} I_{n,j}(x) = \left\{ \sum_{n=0}^{\infty} c_n x^n : c_n \in \{0, 1 - x\} \text{ for all } n \in \mathbb{N}_0 \right\}.$$

This process can, to some extent, also be done for $x = 0$ or $x = 1/2$, leading to the sets $C(0) = \{0, 1\}$ and $C(1/2) = [0, 1]$.

Theorem 2.10 (Kristensen–Laursen, 2023)**.** *Let p be a prime number. Then*

$$p^{-1}C(p^{-1}) \cup \{1\} \subseteq \mathcal{S}_{\mathcal{B}^p} \subseteq [0, p^{-1}] \cup \{1\}.$$

In particular, $\mathcal{S}_{\mathcal{B}^2} = [0, 1/2] \cup \{1\}$.

Remark 2.11. If we were to replace p with ∞ and write $\mathcal{B}^\infty = \mathcal{A}^\infty$ in the conclusion of this theorem (with the convention $\infty^{-1} = 0$), we retrieve the statement of Theorem 2.4. Hence, the above theorem may be said to hold for $p = \infty$ as well.

This theorem inspired the following conjecture, based on an optimism towards having an easily described spectrum and an expectation that when $[0, 1] \setminus \mathcal{S}_{\mathcal{B}^p}$ is an open interval for $p = 2$ and $p = \infty$ (writing $\mathcal{B}^\infty = \mathcal{A}^\infty$), then it is probably also true for the remaining values of p. Besides, it would be quite the coincidence if $\mathcal{S}_{\mathcal{B}^p}$ were to be a mildly modified Cantor set.

Conjecture 2.12 (Kristensen–Laursen, 2023)**.** $\mathcal{S}_{\mathcal{B}^p} = [0, 1/p] \cup \{1\}$ *for all prime numbers p.*

In Section 2.2, we settle this conjecture by presenting the paper [37] by the current author, in which $\mathcal{S}_{\mathcal{B}^p}$ was in fact shown to be a mildly modified Cantor set.

When writing inequality (2.1), we implicitly use the denominator to quantify the complexity of a rational number, whereas inequality (2.3) uses the maximum of numerator and denominator to do the same. While each of these ways of measuring the complexity is meaningful in its respective setting, there are cases where a third way is preferable. This is, for instance, the case in [4], where Badziahin and Bugeaud replaced inequalities (2.1) and (2.3) with the multiplicative inequality

$$\left|\alpha - \frac{a}{b}\right|_p \leq \frac{\psi(n)}{n}, \qquad n = |ab|. \tag{2.5}$$

Based on this, they construct what we will call the *multiplicative p-adic Duffin–Schaeffer set*,

$$\mathfrak{A}^p(\psi) = \left\{\alpha \in \mathbb{Z}_p : \text{(2.5) is true for infinitely many } \frac{a}{b} \in \mathbb{Q}\right\}, \tag{2.6}$$

where we assume a/b to be reduced fractions, same as we did in the definitions of $\mathcal{A}^\infty(\psi)$ and $\mathcal{B}^p(\psi)$. Badziahin and Bugeaud then proved a theorem similar to Conjectures 2.3 and 2.7 but with certain technical restrictions on ψ. As a brief remark, they suggested that these restrictions might not be needed, likely inspired by the apparent similarities between $\mathfrak{A}^p(\psi)$ and either of $\mathcal{A}^\infty(\psi)$ and $\mathcal{A}^p(\psi)$. If this suggestion were true, it would imply the following milder conjecture, which would then correspond to Theorem 2.4.

Conjecture 2.13 (Badziahin–Bugeaud, 2022). *Let p be a prime number. Then $\mathcal{S}_{\mathfrak{A}^p} = \{0, 1\}$.*

This conjecture will also be settled in Section 2.2, where another result from [37] states that $\mathcal{S}_{\mathfrak{A}^p}$ equals the unit interval.

2.2 Attainable values for $\mu_p(\mathcal{B}_p)$ and $\mu_p(\mathfrak{A}_p)$

In this section, we will consider the main results of the paper [37] by the current author. Recall Conjecture 2.12 from Section 2.1. It follows from the definition of $\mathcal{B}^p$ that

$$\mathcal{B}^p(\psi) = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} (\mathcal{A}_n^p(\psi) \cup \mathcal{C}_n^p(\psi)),$$

where

$$\mathcal{C}_n^p(\psi) = \bigcup_{\substack{0<|a|<n \\ \gcd(a,n)=1}} B_{\mathbb{Z}_p}\left(\frac{n}{a}, \frac{\psi(n)}{n}\right).$$

Defining

$$\mathcal{C}^p(\psi) := \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \mathcal{C}_n^p(\psi),$$

the current author settles Conjecture 2.12 in the paper [37] in the form of the following theorem, which may be considered a shell-wise $\mathcal{C}^p$-variant of Theorem 2.4.

Theorem 2.14 (Laursen, 2023). *Let p be a prime number, and let $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$. Then*

$$\mu_p(\mathcal{C}^p \cap p^k \mathbb{Z}^\times) \in \{0, (p-1)/p^{k+1}\}.$$

In particular, $\mathcal{S}_{\mathcal{C}^p} = C(p^{-1})$ and $\mathcal{S}_{\mathcal{B}^p} = p^{-1}C(p^{-1}) \cup \{1\}$.

Remark 2.15. A reader familiar with the Hausdorff dimension, denoted $\dim_{\mathcal{H}}$, will notice that $\dim_{\mathcal{H}} \mathcal{S}_{\mathcal{B}^p} = \dim_{\mathcal{H}} \mathcal{S}_{\mathcal{C}^p} = \log 2/\log p$, which fits how $\dim_{\mathcal{H}} \mathcal{S}_{\mathcal{A}^\infty} = 0$. See [43] for a definition of $\dim_{\mathcal{H}}$ and a proof that $\dim_{\mathcal{H}} C(x) = \log 2/\log(x^{-1})$.

The proof of Theorem 2.14 is inspired by that of [27, Lemma 1] and uses many of the same ideas. Part of this involves p-adic variants of classical results regarding the Lebesgue measure.

Expecting that the remaining results of [27] may also be altered to work for $\mathcal{C}^p(\psi) \cap p^k\mathbb{Z}^\times$ instead of $\mathcal{A}^p(\psi)$, the current author proposes a conjecture equivalent to the following at the end of [37]. In the conjecture and for the rest of this section, we write

$$\kappa(\psi) = \inf\left\{k \in \mathbb{N}_0 : \psi(n) \geq pn \text{ for infinitely many } n \in p^k\mathbb{Z}_p\right\},$$

with the convention that $\inf \emptyset = \infty$.

Conjecture 2.16 (Laursen, 2023)**.** *Let p be a prime number, let $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$, and let $k \in \mathbb{N}_0$. If $k < \kappa(\psi)$, then*

$$\mu_p(\mathcal{C}^p(\psi) \cap p^k\mathbb{Z}_p^\times) = \begin{cases} (p-1)/p^{k+1} & \text{if } \sum_{\nu_p(n)=k} \psi(n)\varphi(n)/n = \infty, \\ 0 & \text{if } \sum_{\nu_p(n)=k} \psi(n)\varphi(n)/n < \infty. \end{cases}$$

This conjecture is to be seen in the context of the below theorem, which follows immediately from Theorem 2.8 and the fact that $\mathcal{B}^p(\psi) = \mathcal{A}^p(\psi) \cup \mathcal{C}^p(\psi)$. Together, the conjecture (if proven true) and the theorem provide a formula for deciding $\mu_p(\mathcal{C}^p(\psi))$ and $\mu_p(\mathcal{B}^p(\psi))$ for any $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$.

Theorem 2.17 (Laursen, 2023)**.** *Let p be a prime number, and let $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$. Then*

$$\mu_p(\mathcal{C}^p(\psi)) = p^{-\kappa(\psi)} + \sum_{n=0}^{\kappa(\psi)-1} \mu_p(\mathcal{C}^p(\psi) \cap p^n\mathbb{Z}_p^\times)$$

and

$$\mu_p(\mathcal{B}^p(\psi)) = \begin{cases} 1 & \text{if } \sum_{\nu_p(n)=0} \psi(n)\varphi(n)/n = \infty, \\ \mu_p(\mathcal{C}^p(\psi)) & \text{if } \sum_{\nu_p(n)=0} \psi(n)\varphi(n)/n < \infty. \end{cases}$$

Now recall Conjecture 2.13 from Section 2.1. A similar conjecture could likewise be posed for the set

$$\mathfrak{K}^p(\psi) := \{\alpha \in \mathbb{Z}_p : (2.5) \text{ holds for infinitely many } (a,b) \in \mathbb{Z} \times \mathbb{N}\}, \tag{2.7}$$

which is inspired by the Khintchine set. By writing $|\cdot|_\infty = |\cdot|$, $\mathbb{Z}_\infty = [0,1]$, and $\mu_\infty = \lambda$, equations (2.6) and (2.7) give notions of multiplicative Duffin–Schaeffer sets and multiplicative Khintchine sets, respectively, over the real numbers.

Conjecture 2.13 is then disproven for both $\mathfrak{A}^p(\psi)$ and $\mathfrak{K}^p(\psi)$, whether p is a prime number or ∞, by the following result from [37].

Theorem 2.18 (Laursen, 2023)**.** *Let p be a prime number or ∞, and let $x \in [0,1]$. Then an approximation function $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ can be constructed such that $\mu_p(\mathfrak{K}^p(\psi)) = \mu_p(\mathfrak{A}^p(\psi)) = x$. In particular, $\mathcal{S}_{\mathfrak{A}^p} = \mathcal{S}_{\mathfrak{K}^p} = [0,1]$.*

Remark 2.19. There are also $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ with $\mu_p(\mathfrak{K}^p(\psi)) \neq \mu_p(\mathfrak{A}^p(\psi))$. When p is a prime, this happens for the function $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ that Haynes [27] used to prove $\mathcal{S}_{\mathcal{B}^p} \neq \{0, 1\}$. This ψ is defined by $\psi(n) = n/p$ when $p \mid n$ and $\psi(n) = 0$ when $p \nmid n$. It then follows that $\mathfrak{K}^p(\psi) = \mathbb{Z}_p$ and $\mathfrak{A}^p(\psi) = p\mathbb{Z}_p$, so that

$$\mu_p(\mathfrak{K}^p(\psi)) = 1 > 1/p = \mu_p(\mathfrak{A}^p(\psi)).$$

While this theorem is relatively easy to prove for $p = \infty$, it is much more difficult when p is a prime number. In the presented proof, the prime numbers are divided into 5 families: The primes $p > 5$ congruent to 1 modulo 4, the primes $p > 5$ congruent to 3 modulo 4, the prime 2, the prime 3, and finally the prime 5. Given a number $x \in [0, 1]$, $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ is constructed through a method that draws inspiration from the proof of Theorem 2.10 in [34] but is much more involved and depends on which of the 5 families the considered prime number belongs to. Of the three prime numbers that requires individual attention, the prime 2 was the easiest to handle while the prime 5 was the most difficult.

2.2.1 Paper 7: Attainable measures for certain types of p-adic Duffin–Schaeffer sets

Below, the reader will find the paper [37], which has the current author as its sole author. It was published in *Mathematica Scandinavica* in 2023 and is available through the link https://doi.org/10.7146/math.scand.a-139832. It has a length of 29 pages, numbered 452 through 480.

While the published version of the paper is not currently available on the webpage of *Mathematica Scandinavica* without a subscription, it will be so in the year 2028, 5 years after its publication. The preprint is also available on arXiv through the link https://arxiv.org/abs/2212.03619 or by using its arXiv identifyier 2212.03619, though without the formatting of *Mathematica Scandinavica* and with some minor typing errors.

Below, the reader will find the published version of the paper, which the journal has kindly allowed the current author to include in this thesis.

MATH. SCAND. 129 (2023), 452–480

ATTAINABLE MEASURES FOR CERTAIN TYPES OF p-ADIC DUFFIN-SCHAEFFER SETS

MATHIAS L. LAURSEN

Abstract

This paper settles recent conjectures concerning the p-adic Haar measure applied to a family of sets defined in terms of Diophantine approximation. This is done by determining the spectrum of measure values for each family and seeing that this contradicts the corresponding conjectures.

1. Introduction

In Diophantine approximation, we are often interested in determining when a given number $\alpha \in [0,1]$ has infinitely many solutions to inequalities of the form

$$\left|\alpha - \frac{a}{n}\right| < \frac{\psi(n)}{n}, \tag{1.1}$$

for some chosen function $\psi\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$. In 1924, Khintchine [7] gave one of the first metric results regarding this inequality, as stated below.

THEOREM (Khintchine). *Let* $\psi\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ *and write*

$$\mathscr{K}(\psi) = \big\{\alpha \in [0,1] : (1.1) \textit{ holds for infinitely many } (a,n) \in \mathbb{Z} \times \mathbb{N}\big\}.$$

If $\psi(n)$ *is monotonic, then* $\mathscr{K}(\psi)$ *has Lebesgue measure* 1 *if* $\sum_{n=1}^{\infty} \psi(n) = \infty$ *and has Lebesgue measure* 0 *otherwise.*

Later, Duffin and Schaeffer [3] tried to remove the monotonicity condition and found a counterexample, which lead them to suppose that it would be more natural to consider the set

$$\mathscr{A}(\psi) = \big\{\alpha \in [0,1] : (1.1) \text{ holds for infinitely many } \tfrac{a}{n} \in \mathbb{Q}\big\}, \tag{1.2}$$

where the fractions a/n are assumed to be reduced, i.e. $\gcd(a,n) = 1$. They then famously conjectured the below theorem, which was only proven a couple of years ago by Koukoulopoulos and Maynard [8].

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THEOREM (Duffin-Schaeffer). *Let* $\psi\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$. *Then*

$$|\mathscr{A}(\psi)| = \begin{cases} 0 & \text{if } \displaystyle\sum_{n=1}^{\infty} \frac{\psi(n)\phi(n)}{n} < \infty, \\ 1 & \text{if } \displaystyle\sum_{n=1}^{\infty} \frac{\psi(n)\phi(n)}{n} = \infty. \end{cases}$$

Here and throughout this paper, ϕ denotes the Euler totient function. Named after the conjecture, we will refer to $\mathscr{A}(\psi)$ as the (real) Duffin-Schaeffer set. In handling the conjecture, one usually writes it as a lim-sup set $\mathscr{A}(\psi) = \limsup_{n\to\infty} \mathscr{A}_n(\psi)$ where

$$\mathscr{A}_n(\psi) = \bigcup_{\substack{1\leq a\leq n \\ \gcd(a,n)=1}} B_{\mathbb{R}}\left(\frac{a}{n}, \frac{\psi(n)}{n}\right) \cap [0, 1].$$

We use $B_X(x, r)$ to denote the closed ball of radius r around the point x in metric space X when $r > 0$, and the singleton $\{x\}$ when $r = 0$.

Before the conjecture was settled, Haynes [5] proposed a p-adic variant of the conjecture in terms of the p-adic Haar measure μ_p with $\mu_p(\mathbb{Z}_p) = 1$ and a p-adic Duffin-Schaeffer set, $\mathscr{A}^p$. To this end, he first translated $\mathscr{A}_n$ into

$$\mathscr{A}_n^p(\psi) := \bigcup_{\substack{|a|\leq n \\ \gcd(a,n)=1}} B_{\mathbb{Q}_p}\left(\frac{a}{n}, \frac{\psi(n)}{n}\right) \cap \mathbb{Z}_p$$

and then defined $\mathscr{A}^p$ as the limsup set of $\mathscr{A}_n^p(\psi)$. From this, Haynes phrased a p-adic Duffin-Schaeffer conjecture for the set $\mathscr{A}^p(\psi)$. One of the main results of Haynes' paper was that if a certain 'quasi-independentness on average' criterion, closely related to the real Duffin-Schaeffer conjecture, were to hold, then his conjecture would follow. In [9], Kristensen and Laursen modified arguments from [8] to prove this 'quasi-independentness on average' for relevant ψ, thus settling the conjecture in the affirmative, so we may now state it as a theorem.

THEOREM (p-adic Duffin-Schaeffer). *We have*

$$\mu_p(\mathscr{A}^p(\psi)) = \begin{cases} 0 & \text{if } \displaystyle\sum_{n=1}^{\infty} \mu_p(\mathscr{A}_n^p(\psi)) < \infty \\ 1 & \text{if } \displaystyle\sum_{n=1}^{\infty} \mu_p(\mathscr{A}_n^p(\psi)) = \infty. \end{cases}$$

However, the set $\mathscr{A}^p(\psi)$ is not the only natural choice for a p-adic variant of $\mathscr{A}(\psi)$, as one might instead start from equation (1.2) when translating the question to a p-adic context. For this, we first need a translation of inequality (1.1). Following Jarník [6] and Lutz [10], this should be

$$\left|\alpha - \frac{a}{n}\right|_p \leq \frac{\psi(\max\{|a|, |n|\})}{\max\{|a|, |n|\}}. \tag{1.3}$$

By fixing $\max\{|a|, |n|\}$, we limit both the numerator and the denominator. If we, as is done in the real case, merely compared based on n, the inequality would trivially have infinitely many answers $a \in \mathbb{Z}$ for any $x \in \mathbb{Z}_p$ whenever $\psi(n) > 0$, as the fractions a/n would be dense in the ball of radius $|n|_p^{-1}$ around the origin, by virtue of $\mathbb{Z}$ being dense in $\mathbb{Z}_p$. If we impose the condition that a/n be a reduced fraction, the argument becomes slightly more obscure, but the fractions would still be dense in $\mathbb{Z}^p$ when $\gcd(a, n) = 1$ – just add the proper multiple of a sufficiently large power of p to a when $\gcd(a, n) > 1$. We then define the alternative p-adic Duffin-Schaeffer set as

$$\mathscr{B}^p(\psi) = \left\{\alpha \in \mathbb{Z}_p : (1.3) \text{ holds for infinitely many } \tfrac{a}{n} \in \mathbb{Q}\right\}.$$

This set was also briefly considered by Haynes [5], who showed that this set allowed for $\mu_p(\mathscr{B}^p(\psi)) \neq 0, 1$, by explicitly constructing a ψ such that the measure was p^{-1}. Kristensen and Laursen [9] revisited the set and found an uncountable collection of possible values for $\mu_p(\mathscr{B}^p(\psi))$ when ψ varies. For $p = 2$, they found that the possible values were exactly those in the set $[0, 1/2] \cup \{1\}$ and conjectured that the set of possible values would be $[0, 1/p] \cup \{1\}$ for $p > 2$. The first main result of this paper rejects this conjecture by proving that the spectrum of possible values of $\mu_p(\mathscr{B}^p(\psi))$ is in fact limited to those already found in [9]. As noted in that paper, this means that the spectrum becomes a Lebesgue-null set of Hausdorff dimension $\log 2/\log p$.

Furthermore, a new paper by Badziahin and Bugeaud [1] introduces another Duffin-Schaeffer-like set of p-adic numbers, which we will denote by $\mathfrak{A}'^p(\psi)$. The definition of this set is related to that of $\mathscr{B}^p(\psi)$ where inequality (1.3) is replaced by

$$\left|\alpha - \frac{a}{b}\right|_p < \frac{\psi(|ab|)}{|ab|}, \tag{1.4}$$

so that $\mathfrak{A}'^p(\psi)$ becomes

$$\mathfrak{A}'^p(\psi) = \left\{\alpha \in \mathbb{Z}_p : (1.4) \text{ holds for infinitely many } \tfrac{a}{b} \in \mathbb{Q}\right\}.$$

This set is not quite a generalisation of the original Duffin-Schaeffer set, but it appears to be of a similar nature. In their paper, Badziahin and Bugeaud prove a theorem inspired by the above theorem due to Khintchine and by the Duffin-Schaeffer Theorem. Akin to Khintchine, they achieve their result by imposing certain restrictions on the approximation function ψ, including some weak growth restrictions. After stating the theorem, they suggest that it may hold with these restrictions weakened or perhaps even removed. We will in this paper show that said restrictions cannot be removed entirely as the spectrum of values attainable by $\mu_p(\mathfrak{A}'^p(\psi))$ is the entire unit interval when ψ is allowed to be any function onto $\mathbb{R}_{\geq 0}$.

2. Main results

For ease of notation, we will suppress ψ when talking about the various Duffin-Schaeffer-inspired sets in the cases where there is no ambiguity towards the underlying ψ.

The most obvious reason for the difference between $\mathscr{A}^p$ and $\mathscr{B}^p$ is that $\mathscr{A}^p$ only allows fractions where the numerator is bounded by the denominator while $\mathscr{B}^p$ allows numerators of any size relative to the denominator. As such, it appears natural to consider what happens with the remaining fractions, where the numerator is greater than the denominator. This leads to a third p-adic Duffin-Schaeffer set, $\mathscr{C}^p$, which we define as the limsup set of sets $\mathscr{C}_n^p$ given by

$$\mathscr{C}_n^p = \bigcup_{\substack{|a|<n \\ \gcd(a,n)=1}} B_{\mathbb{Q}_p}\left(\frac{n}{a}, \frac{\psi(n)}{n}\right) \cap \mathbb{Z}_p,$$

for $n > 1$, and $\mathscr{C}_1^p = \emptyset$ for $n = 1$. It is then easy to see that $\mathscr{B}^p$ is the limsup set of $\mathscr{B}_n^p := \mathscr{A}_n^p \cup \mathscr{C}_n^p$. By Dirichlet's pigeonhole principle, we then get

$$\mathscr{B}^p = \mathscr{A}^p \cup \mathscr{C}^p. \tag{2.1}$$

If $p \nmid n$ and $p \mid a$, then $|n/a|_p > 1$, so that $B(n/a, \psi(n)/n) \cap \mathbb{Z}_p = \emptyset$ unless $\psi(n)/n \geq |a|_p^{-1} > 1$, in which case $\mathscr{C}_n^p = \mathbb{Z}_p = B(n/1, \psi(n)/n) \cap \mathbb{Z}_p$. By this realization, we may instead write

$$\mathscr{C}_n^p(\psi) = \bigcup_{\substack{|a|<n \\ \gcd(a,pn)=1}} B_{\mathbb{Q}_p}\left(\frac{n}{a}, \frac{\psi(n)}{n}\right) \cap \mathbb{Z}_p,$$

Note that this is also applicable for $n = 1$, as the union is empty in that case. We will therefore use this as the *de facto* definition of $\mathscr{C}_n^p(\psi)$ going forward.

We are now ready to state the first main result of this paper, which determines the spectra of values for $\mu_p(\mathscr{C}^p)$ and $\mu_p(\mathscr{B}^p)$, respectively.

THEOREM 2.1. *Let $x \in [0, 1]$. There exists a function $\psi\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that $\mu_p(\mathscr{C}^p(\psi)) = x$ if, and only if, x is of the form*

$$x = \sum_{k=0}^{\infty} x_k (p-1) p^{-k-1}, \quad x_k \in \{0, 1\}\ \forall k \in \mathbb{N}_0.$$

There exists a function $\psi\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that $\mu_p(\mathscr{B}^p(\psi)) = x$ if and only if, $x = 1$ or x is of the form

$$x = \sum_{k=1}^{\infty} x_k (p-1) p^{-k-1}, \quad x_k \in \{0, 1\}\ \forall k \in \mathbb{N}.$$

We now turn our attention towards the set $\mathfrak{A}'^p$ from [1]. In order to have notation more in line with the Duffin-Schaeffer sets $\mathscr{A}$ and $\mathscr{B}^p$, we alter inequality (1.4) slightly,

$$\left|\alpha - \frac{a}{b}\right|_p \leq \frac{\psi(|ab|)}{|ab|}, \tag{2.2}$$

which leads to a set

$$\mathfrak{A}^p(\psi) = \left\{\alpha \in \mathbb{Z}_p : \text{(2.2) holds for infinitely many } \tfrac{a}{b} \in \mathbb{Q}\right\}.$$

To see that this set will have the same spectrum of measures as $\mathfrak{A}'^p$, define ψ' by

$$\psi'(n) = \begin{cases} p\psi(n) & \text{if } \psi(n) = n/p^k \text{ for some } k \in \mathbb{Z}, \\ \psi(n) & \text{otherwise.} \end{cases}$$

Note that

$$\left|\alpha - \frac{a}{n}\right|_p \leq 0$$

has at most 1 solution $a/n \in \mathbb{Q}$ for any given α, and so the collection of n with $\psi(n) = 0$ contributes to neither $\mathfrak{A}^p(\psi)$ nor $\mathfrak{A}'^p(\psi')$. For $\psi(n) > 0$, inequality (1.4) with ψ' is equivalent to inequality (2.2) with ψ. It thus follows that $\mathfrak{A}^p(\psi) = \mathfrak{A}'^p(\psi')$. Since there is an obvious bijection between ψ and ψ', $\mathfrak{A}^p$ and $\mathfrak{A}'^p$ have the same spectrum of measures.

We will also consider a Khintchine-like variant of the set, defined as

$$\mathfrak{K}^p(\psi) = \left\{\alpha \in \mathbb{Z}_p : \text{(2.2) holds for infinitely many } (a, b) \in (\mathbb{Z} \setminus \{0\}) \times \mathbb{N}\right\}.$$

By taking $\mathbb{Z}_\infty$, $\mathbb{Q}_\infty$, $|\cdot|_\infty$, and μ_∞ to mean $[0, 1]$, $\mathbb{R}$, $|\cdot|$, and the Lebesque measure, respectively, we define related sets over the real numbers when allowing $p = \infty$.

As is the case for the 'proper' Duffin-Schaeffer sets, we will want to write $\mathfrak{A}^p$ and $\mathfrak{R}^p$ as the limsup set of sets $\mathfrak{A}_n^p$ and $\mathfrak{R}_n^p$, respectively. Write $B_{\mathbb{Q}_p}(\pm a, r) = B_{\mathbb{Q}_p}(a, r) \cup B_{\mathbb{Q}_p}(-a, r)$ and $n = \prod_{i=1}^{\omega(n)} p_i^{v_{p_i}(n)}$, where $\omega(n)$ denotes the number of distinct prime divisors of n. By defining

$$\mathfrak{R}_n^p(\psi) = \bigcup_{a|n} B_{\mathbb{Q}_p}\left(\pm\frac{a}{n/a}, \frac{\psi(n)}{n}\right) \cap \mathbb{Z}_p,$$

$$\mathfrak{A}_n^p(\psi) = \bigcup_{a_1,\ldots,a_{\omega(n)} \in \{0,1\}} B_{\mathbb{Q}_p}\left(\pm\frac{\prod_{a_i=1} p_i^{v_{p_i}(n)}}{\prod_{a_i=0} p_i^{v_{p_i}(n)}}, \frac{\psi(n)}{n}\right) \cap \mathbb{Z}_p,$$

we immediately achieve

$$\mathfrak{R}^p = \limsup_{n\to\infty} \mathfrak{R}_n^p, \qquad \mathfrak{A}^p = \limsup_{n\to\infty} \mathfrak{A}_n^p.$$

It so happens that the spectra of values for $|\mathfrak{A}^\infty|$ and $|\mathfrak{R}^\infty|$ are surprisingly easy to settle while the spectra of values for $\mu_p(\mathfrak{A}^p)$ and $\mu_p(\mathfrak{R}^p)$ when $p < \infty$ require significantly more care. However, the spectra are nonetheless independent of p as they always take up the entire unit interval as seen by the below theorem. We will prove the easy case of $p = \infty$ immediately and save $p < \infty$ for Section 5. That section will be independent of Sections 3 and 4.

THEOREM 2.2. *Let p be a prime or $p = \infty$, and let $x \in [0, 1]$. Then there exists a $\psi\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that $\mu_p(\mathfrak{A}^p(\psi)) = \mu_p(\mathfrak{R}^p(\psi)) = x$.*

PROOF FOR $p = \infty$. Let $x \in [0, 1]$ and define $\psi\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ by

$$\psi(n) = \begin{cases} nx & \text{if } n \text{ is a prime,} \\ 0 & \text{otherwise.} \end{cases}$$

Since ψ is supported on the primes, we immediately have that $\mathfrak{A}^\infty = \mathfrak{R}^\infty$. We are thus left with showing that $|\mathfrak{A}^\infty| = x$. Upon applying the Borel-Cantelli Lemma, this follows by a brief calculation, where we use q to denote primes:

$$\begin{aligned} |\mathfrak{A}^\infty| &= \left|\limsup_{q\to\infty} \bigcup_{e_1,e_2\in\{-1,1\}} B_{\mathbb{R}}(e_1 q^{e_2}, x) \cap [0, 1]\right| \\ &= \left|\limsup_{q\to\infty} [0, x + q^{-1}]\right| = x. \end{aligned}$$

This completes the proof

Note that $\mathfrak{A}_n^p(\psi) \subseteq \mathfrak{R}_n^p(\psi)$ will hold with strict inclusion for all small enough ψ, so we should not expect equality between $\mu_p(\mathfrak{A}^p)$ and $\mu_p(\mathfrak{R}^p)$ in general. In fact, if we in the above proof had taken

$$\psi(n) = \begin{cases} nx & \text{if } n \text{ is the square of a prime,} \\ 0 & \text{otherwise,} \end{cases}$$

we would still find $|\mathfrak{A}^\infty| = x$ while $|\mathfrak{K}^\infty| = \min\{2x, 1\}$ as $\mathfrak{K}^p$ now also accepts infinitely many copies of $1 = q/q$ as approximants and thereby includes the interval $[1 - x, 1]$ in $\mathfrak{K}^\infty$.

3. Some *p*-adic measure theory

In this section, we will present a series of general measure theoretical results that will be used for the proof of Theorem 2.1. Except for Lemma 3.5, these results all appear to have been applied to some extent in [5], even though only Lemma 3.2 was formally stated. All results are to be applied in proving Proposition 4.2, which will be introduced in Section 4 and plays a central role in proving the 'only if' parts of Theorem 2.1.

Note that each non-negative real number $x \in \mathbb{R}_{\geq 0}$ has a canonical base p expansion of the form

$$x = \sum_{n=N}^{\infty} a_n p^{-n},$$

where $a_n \in \{0, \dots, p-1\}$, $\liminf a_n < p - 1$, and $N \in \mathbb{Z}_{\leq 0}$ is maximal possible. Throughout this paper, the function $\iota_p \colon \mathbb{R}_{\geq 0} \to \mathbb{Q}_p$ denotes the associated map

$$x = \sum_{n=N}^{\infty} a_n p^{-n} \mapsto \iota_p(x) = \sum_{n=N}^{\infty} a_n p^n.$$

This function is measure preserving in that $\lambda(A) = p\mu_p(\iota_p(A))$ for any measurable subset $A \subseteq \mathbb{R}$.

LEMMA 3.1. *The preimage map ι_p^{-1} maps balls $B \subseteq \mathbb{Q}_p$ to half-open intervals in $\mathbb{R}$ of length $p\mu_p(B)$. In particular, ι_p is measurable, and the associated push-forward measure on the Lebesgue measure, $\iota_{p\#}$, is equal to $p\mu_p$.*

PROOF. Notice that

$$\begin{aligned} \iota_p^{-1}(\mathbb{Z}_p) &= \left\{ \sum_{m=0}^{\infty} b_m p^{-m} \in \mathbb{R} : b_m \in \{0, \dots, p-1\},\ \liminf_{m\to\infty} b_m < p - 1 \right\} \\ &= [0, p). \end{aligned}$$

Let B be a ball in $\mathbb{Z}_p$ and write $B = \sum_{m=0}^{M-1} b_m p^m + p^M \mathbb{Z}_p$ for some $M \in \mathbb{N}_0$ and $b_0, \ldots, b_{M-1} \in \{0, \ldots, p-1\}$. We then have

$$\iota_p^{-1}(B) = \iota_p^{-1}\left(\sum_{m=0}^{M-1} b_m p^m\right) + p^{-M}\iota_p^{-1}(\mathbb{Z}_p) = \sum_{m=0}^{M-1} b_m p^{-m} + [0, p^{1-M}),$$

which proves the first part of the lemma. Since the Borel algebra of $\mathbb{Z}_p$ is generated by the collection of balls in $\mathbb{Z}_p$, and the intervals are contained in the Borel algebra of $\mathbb{R}_{\geq 0}$, ι_p is a measurable function, and the push-forward $\iota_{p\#}$ is a Borel-measure on $\mathbb{Q}_p$. From the above equation, we notice in particular that for all balls B in $\mathbb{Q}_p$,

$$\iota_{p\#}(B) = p\mu_p(B).$$

Since all balls have finite measure, and the collection of the empty set and all balls in $\mathbb{Q}_p$ is preserved under pairwise intersection and generates the Borel algebra of $\mathbb{Q}_p$, this implies that $\iota_{p\#} = p\mu_p$, by the Uniqueness of Measures Theorem, and the proof is complete.

The alternative definition of μ_p provided by the above lemma gives a tool to translate measure theoretic results regarding $\mathbb{R}$ into measure theoretic results regarding $\mathbb{Q}_p$. It appears that Haynes may have used this alternative definition in [5], though it was not explicitly stated. One result that very easily translates using ι_p is the below lemma from [4], which is a modification of a lemma from [2].

LEMMA 3.2. *Let* $\{I_n\}_{n\in\mathbb{N}}$ *be a sequence of real intervals with* $\lambda(I_n) \underset{n\to\infty}{\longrightarrow} 0$, *and let* $\{U_n\}_{n\in\mathbb{N}}$ *be a sequence of measurable sets such that*

$$U_n \subseteq I_n, \quad |U_n| \geq \varepsilon|I_n| \qquad \forall n \in \mathbb{N},$$

for some fixed $0 < \varepsilon < 1$. *Then* $|\limsup_{n\to\infty} U_n| = |\limsup_{n\to\infty} I_n|$.

COROLLARY 3.3. *Let* $\{B_n\}_{n\in\mathbb{N}}$ *be a sequence of p-adic balls with the property* $\mu_p(B_n) \underset{n\to\infty}{\longrightarrow} 0$. *Suppose* $\{U_n\}_{n\in\mathbb{N}}$ *is a sequence of measurable sets such that, for some*

$$U_n \subseteq B_n, \quad \mu_p(U_n) \geq \varepsilon\mu_p(B_n) \qquad \forall n \in \mathbb{N},$$

for some fixed $0 < \varepsilon < 1$. *Then* $\mu_p(\limsup_{n\to\infty} U_n) = \mu_p(\limsup_{n\to\infty} B_n)$.

PROOF. Applying $p\mu_p = \iota_{p\#}$ by Lemma 3.1, we have

$$p\mu_p\big(\limsup_{n\to\infty} U_n\big) = \left|\iota_p^{-1}\left(\bigcap_{N=1}^{\infty}\bigcup_{n\geq N} U_n\right)\right| = \lim_{N\to\infty}\left|\iota_p^{-1}\left(\bigcup_{n\geq N} U_n\right)\right|$$

$$= \left|\limsup_{n\to\infty} \iota_p^{-1}(U_n)\right|.$$

By repeating the process for B_n, the statement follows by Lemma 3.2, and the proof is complete.

Another relevant result that may be derived using ι_p is a p-adic version of the Lebesgue Density Theorem, as presented below.

LEMMA 3.4 (Lebesgue Density Theorem for $\mathbb{Q}_p$). *Let A be a measurable subset of $\mathbb{Q}_p$ such that $\mu_p(A) > 0$. Then A contains a point of density 1, which is to say that there exists $a \in A$ such that if $\{B_M\}_{M\in\mathbb{N}}$ is a sequence of balls $B_M \ni a$ of radius $r(B_M) \underset{M\to\infty}{\longrightarrow} 0$, then*

$$\frac{\mu_p(A \cap B_M)}{\mu_p(B_M)} \underset{M\to\infty}{\longrightarrow} 1.$$

PROOF. By Lemma 3.1, we have $p\mu_p = \iota_{p\#}$. Thus $\tilde{A} = \iota_p^{-1}(A)$ has positive Lebesgue measure, which means that it must contain a point $x \in \tilde{A}$ of density 1, by the Lebesgue Density Theorem. Put $a = \iota_p(x)$, and let $\{B_M\}$ be a collection of balls around a in $\mathbb{Q}_p$ of radius $r(B_M) \underset{M\to\infty}{\longrightarrow} 0$.

Let $\varepsilon > 0$, and put $\tilde{B}_M = \iota_p^{-1}(B_M)$. Since ι_p^{-1} maps balls to half-open intervals of length $p\mu_p(B_M)$ by Lemma 3.1, and we must have $\tilde{B}_M \ni x$, it follows that $|\tilde{A} \cap \tilde{B}_M| > 0$, as x is of density 1 in $\tilde{A}$. Using once more that $\tilde{B}$ is an interval, this allows us to pick open intervals $I_M \supseteq \tilde{B}_M$ such that $|\tilde{A} \cap I_M| \le p^{\varepsilon}|\tilde{A} \cap \tilde{B}_M|$ and $|I_M| \le 2|\tilde{B}_M|$. Hence,

$$\frac{\mu_p(A \cap B_M)}{\mu_p(B_M)} = \frac{|\tilde{A} \cap \tilde{B}_M|}{|\tilde{B}_M|} \ge \frac{p^{-\varepsilon}|\tilde{A} \cap I_M|}{|I_M|}. \tag{3.1}$$

As $I_M \ni x$, write $I_M = (x - s_M, x + t_M)$ with $s_M, t_M > 0$ and put $u_M = \max\{s_M, t_M\}$. Then

$$\begin{aligned} 1 \ge \frac{|\tilde{A} \cap I_M|}{|I_M|} &\ge \frac{|\tilde{A} \cap (x - u_M, x + u_M)| - (u_M - s_M) - (u_M - t_M)}{s_M + t_M} \\ &= \frac{2u_M}{s_M + t_M} \frac{|\tilde{A} \cap (x - u_M, x + u_M)|}{|(x - u_M, x + u_M)|} - \frac{2u_M}{s_M + t_M} + 1. \end{aligned}$$

Since x is of density 1, and we have

$$u_M \le 2|I_M| \le 4|\tilde{B}_M| = 4p\,\mu_p(B_M) \underset{M\to\infty}{\longrightarrow} 0,$$

the squeezing lemma then implies that $|\tilde{A} \cap I_M|/|I_M| \to 1$ as $M \to \infty$.

Combined with equation (3.1), this means that

$$\liminf_{M\to\infty} \frac{\mu_p(A\cap B_M)}{\mu_p(B_M)} \geq p^{-\varepsilon},$$

and the lemma follows by letting ε tend to 0.

We will also use the below lemma, which relies on elementary algebra and then the Uniqueness of Measures Theorem. This does not use ι_p.

LEMMA 3.5. *Let $f\colon \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ denote the map $x \mapsto 1/x$. Then f maps balls in $\mathbb{Z}_p^\times$ to balls of the same measure in $\mathbb{Z}_p^\times$. In particular, f preserves μ_p restricted to $\mathbb{Z}_p^\times$.*

PROOF. This is a simple calculation.

4. Proof of Theorem 2.1

To prove Theorem 2.1, we will use two propositions. The first one puts $\mu_p(\mathscr{B}^p)$ equal to $\mu_p(\mathscr{A}^p)$ or $\mu_p(\mathscr{C}^p)$, depending on a divergence criterion, and splits $\mathscr{C}^p$ into smaller pieces to be dealt with individually.

In the proof and for the rest of this paper, we will use $\sqcup$ to denote the disjoint union of sets.

PROPOSITION 4.1. *Let l be the minimal $k \in \mathbb{N}_0$ such that $\psi(n)/n \geq p^{-k}$ for infinitely many $n \in p^k\mathbb{N}$, if such a k exists. Otherwise, put $l = \infty$ and write $p^{-\infty} = 0$. Then*

$$\mu_p(\mathscr{B}^p) = \begin{cases} \mu_p(\mathscr{A}^p) = 1 & \text{if } \displaystyle\sum_{p\nmid n} \frac{\phi(n)\psi(n)}{n} = \infty, \\ \mu_p(\mathscr{C}^p) & \text{if } \displaystyle\sum_{p\nmid n} \frac{\phi(n)\psi(n)}{n} < \infty, \end{cases} \tag{4.1}$$

$$\mu_p(\mathscr{C}^p) = p^{-l} + \sum_{0\leq k<l} \mu_p\big(\mathscr{C}^p \cap p^k\mathbb{Z}_p^\times\big). \tag{4.2}$$

Furthermore, if $k < l$, then $\mathscr{C}^p \cap p^k\mathbb{Z}_p^\times = \limsup_{\nu_p(n)=k} \mathscr{C}_n^p$.

PROOF. We start by proving equation (4.1). If $\sum_{p\nmid n} \frac{\phi(n)\psi(n)}{n} = \infty$, this is [9, Theorem 2], so suppose not. Then $\sum_{p\nmid n} \mu_p(\mathscr{A}_n^p) < \infty$. If $\sum_{p|n} \mu_p(\mathscr{A}_n^p) = \infty$, then we must have $\psi(n) \geq pn$ infinitely often, and so the definitions imply $\mathscr{B}_n^p \supseteq \mathscr{C}_n^p \supseteq B_{\mathbb{Q}_p}(n/1, p) \cap \mathbb{Z}_p = \mathbb{Z}_p$ for these n when $n > 1$. Hence, $\mathscr{B}^p = \mathscr{C}^p = \mathbb{Z}_p$. If $\sum_{p|n} \mu_p(\mathscr{A}_n^p) < \infty$, then the Borel-Cantelli Lemma implies $\mu_p(\mathscr{A}^p) = 0$, and the statement follows by equation (2.1).

Moving on to equation (4.2), notice that

$$\mathscr{C}^p = (\mathscr{C}^p \cap p^l \mathbb{Z}_p) \sqcup \bigsqcup_{0 \le k < l} \mathscr{C}^p \cap p^k \mathbb{Z}_p^\times,$$

with the convention of $p^\infty = 0$ (as an element of $\mathbb{Z}_p$) in the case of $l = \infty$. Equation (4.2) is then equivalent to $\mu_p(\mathscr{C}^p \cap p^l \mathbb{Z}_p) = p^l$. If $l = \infty$, this is trivial, so suppose $l < \infty$. Let $\alpha \in p^l \mathbb{Z}_p$. By definition of l, there are infinitely many $n \in p^l \mathbb{N}$ with $\psi(n) \ge p^{-l} n$. For these n, $|\alpha - n/1|_p \le p^{-l} \le \psi(n)/n$, and so $\alpha \in \mathscr{C}_n^p$. Hence, $\mathscr{C}^p \supseteq p^l \mathbb{Z}_p$, implying $\mu_p(\mathscr{C}^p \cap p^l \mathbb{Z}_p) = p^l$ as claimed.

As for the final part of the statement, this is vacuous if $l = 0$, so suppose $l > 0$ and let $0 \le j \le k < l$. Then $\psi(n)/n < p^{-j}$ for all but finitely many $n \in p^j \mathbb{N}$. If $\nu_p(n) = j$, then $\alpha \in \mathscr{C}_n^p$ would imply

$$|\alpha|_p = \left| \alpha - \frac{n}{a} + \frac{n}{a} \right|_p = p^{-j},$$

for some $|a| < n$ with $\gcd(a, pn) = 1$ when n is sufficiently large, so that

$$\limsup_{\nu_p(n) = j} \mathscr{C}_n^p \subseteq p^j \mathbb{Z}_p^\times.$$

If $\nu_p(n) > k$, we would instead find

$$|\alpha|_p \le \max\left\{ \left| \alpha - \frac{n}{a} \right|_p, \left| \frac{n}{a} \right|_p \right\} < p^{-k},$$

for some $|a| < n$ with $\gcd(a, pn) = 1$ when n is sufficiently large, so that

$$\limsup_{\nu_p(n) > k} \mathscr{C}_n^p \subseteq p^{k+1} \mathbb{Z}_p.$$

By the pigeon-hole principle, the proposition is proven upon calculating

$$\begin{aligned} \mathscr{C}^p \cap p^k \mathbb{Z}_p^\times &= \Big(\limsup_{\nu_p(n) > k} \mathscr{C}_n^p \cap p^k \mathbb{Z}_p^\times \Big) \cup \bigcup_{j=0}^{k} \limsup_{\nu_p(n) = j} \mathscr{C}_n^p \cap p^k \mathbb{Z}_p^\times \\ &= \limsup_{\nu_p(n) = k} \mathscr{C}_n^p. \end{aligned}$$

The other proposition leading to Theorem 2.1 is a bit more involved and may be thought of as a shell-wise zero-full law for $\mathscr{C}^p$. This relates to, and is inspired by, the zero-one laws $|\mathscr{A}| \in \{0, 1\}$ [4] and $\mu_p(\mathscr{A}^p) \in \{0, 1\}$ [5].

PROPOSITION 4.2. *Let $k \in \mathbb{N}_0$. Then $\mu_p(\mathscr{C}^p \cap p^k \mathbb{Z}_p^\times) \in \{0, (p-1)/p^{k+1}\}$.*

The proof of this proposition follows the same overall structure as the proof of the 0-1 law in [5], with some modifications. In that light, it is perhaps not surprising that we will need the below lemma, which corresponds to the less trivial part of [5, Lemma 2]. We will here continue to use $\omega(n)$ to denote the number of prime divisors of n. The proof of the lemma will apply the M+Âbius function μ, which is defined as $\mu(d) = (-1)^{\omega(d)}$ when d is square free, and $\mu(d) = 0$ otherwise. In this context, the following three facts, which can be found in [11], will be applied without proof.

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } d = 1, \\ 0 & \text{if } d \neq 1, \end{cases} \tag{4.3}$$

$$\sum_{d|n} \mu(d) \frac{n}{d} = \phi(n), \tag{4.4}$$

$$\frac{\phi(n)}{n} = \prod_{q|n} (1 - q^{-1}). \tag{4.5}$$

LEMMA 4.3. *If* $n > 1$ *and* $\psi(n) > 4^{\omega(n)}$, *then* $\mathscr{C}_n^p \supseteq p^{\nu_p(n)} \mathbb{Z}_p^\times$.

PROOF. Put $k = \nu_p(n)$, and let $\alpha \in p^k \mathbb{Z}_p^\times$. Then $n/\alpha \in \mathbb{Z}_p^\times$, and we write

$$\frac{n}{\alpha} = \sum_{m=0}^{\infty} b_m p^m, \quad b_m \in \{0, \dots, p-1\}\ \forall m \in \mathbb{N}_0, b_0 \neq 0.$$

If $\psi(n) \geq n/p^k$, then clearly $\alpha \in \mathscr{C}_n^p$ since $|n|_p = |\alpha|_p = p^{-k}$ implies $\left|\alpha - \frac{n}{1}\right|_p \leq p^{-k}$, so suppose not. Pick $N \in \mathbb{Z}$ such that $\psi(n)/n \in [p^{-N}, p^{-N+1})$. Note that $N > k$. Our job of proving $\alpha \in \mathscr{C}_n^p$ then reduces to finding an a with $|a| < n$ and $\gcd(a, pn) = 1$ such that $\nu_p(\alpha - n/a) \geq N$. As $\nu_p(\alpha) = k$ and $p \nmid a$, the latter part is equivalent to $\nu_p(a - n/\alpha) \geq N - k$. The proof is then complete if we find an a with $|a| < n$ and $\gcd(a, pn) = 1$ such that

$$a = \frac{n}{\alpha} + \sum_{m=N-k}^{\infty} c_m p^m = \sum_{m=0}^{N-k-1} b_m p^m + \sum_{m=N-k}^{\infty} (b_m + c_m) p^m,$$

with $c_m \in \{0, \dots p-1\}$ for all m. Write $b = \sum_{m=0}^{N-k-1} b_m p^m$. Since $p \nmid b$ as $b_0 \neq 0$ and $N > k$, all elements of the set

$$A = \left\{a \in \mathbb{Z} : |a| < n, \gcd(a, n) = 1, a \equiv b \bmod p^{N-k}\right\}$$

satisfy these criteria, and so we are done if we can show that $\#A > 0$. By equation (4.3), we have

$$\begin{aligned}\#A &= \sum_{\substack{|a|<n \\ a\equiv b \bmod p^{N-k}}} \sum_{d|\gcd(a,n)} \mu(d) = \sum_{\substack{d|n \\ p\nmid d}} \sum_{\substack{|a|<n,\ d|a \\ a\equiv b \bmod p^{N-k}}} \mu(d) \\ &= \sum_{d|n/p^k} \mu(d) \sum_{\substack{|l|<n/d \\ l\equiv bd^{-1} \bmod p^{N-k}}} 1.\end{aligned}$$

To simplify notation, let $\tilde{n} = n/p^k$. For each $d \mid \tilde{n}$, pick $x_d \in \mathbb{Z}$ such that $x_d \equiv bd^{-1} \bmod p^{N-k}$, and let k_d denote the difference $\#((-n/d, n/d) \cap (x_d + p^{N-k}\mathbb{Z})) - 2n/(p^{N-k}d)$. Then

$$\begin{aligned}\#A &= \sum_{d|\tilde{n}} \mu(d) \sum_{\substack{|l|<n/d \\ l\in x_d+p^{N-k}\mathbb{Z}}} 1 = \sum_{d|\tilde{n}} \mu(d)\left(\frac{2n}{p^{N-k}d} + k_d\right) \\ &= \frac{2p^k}{p^{N-k}} \sum_{d|\tilde{n}} \mu(d)\frac{\tilde{n}}{d} + \sum_{d|\tilde{n}} \mu(d)k_d = 2p^{-N}p^{2k}\phi(\tilde{n}) + \sum_{d|\tilde{n}} \mu(d)k_d \\ &\geq 2p^k\frac{\psi(n)}{n}\phi(n) + \sum_{d|\tilde{n}} \mu(d)k_d,\end{aligned}$$

where the final equality and the inequality follow from equation (4.4) and the choices of N and k, respectively. Notice that $|k_d| \leq 1$, which combined with the assumption that $\psi(n) > 4^{\omega(n)}$ implies

$$\#A > 2p^k\frac{4^{\omega(n)}}{n}\phi(n) - \sum_{d|\tilde{n}} |\mu(d)| = 2p^k\frac{\phi(n)}{n}4^{\omega(n)} - 2^{\omega(\tilde{n})}.$$

By equation (4.5), we have

$$2\frac{\phi(n)}{n}4^{\omega(n)} = 2\prod_{\substack{q|n \\ q \text{ prime}}} 4(1-q^{-1}) \geq 2\prod_{\substack{q|n \\ q \text{ prime}}} 2 = 2^{\omega(n)+1},$$

and so $\#A > 2^{\omega(n)} > 0$. We conclude $\alpha \in \mathscr{C}_n^p$ and thus $\mathscr{C}_n^p \supseteq p^k\mathbb{Z}_p^\times$, and the proof is complete.

We will also need the below lemma, which is essentially a specialisation of Corollary 3.3 in the context of $\mathscr{C}^p$.

LEMMA 4.4. *Let* $k \in \mathbb{N}_0$, *and suppose* $\operatorname{supp}(\psi) \subseteq p^k\mathbb{N} \setminus p^{k+1}\mathbb{N}$. *Then* $\mu_p\big(\mathscr{C}^p(x\psi) \cap p^k\mathbb{Z}_p^\times\big) = \mu_p\big(\mathscr{C}^p(\psi) \cap p^k\mathbb{Z}_p^\times\big)$ *for all* $x > 0$.

PROOF. There is nothing to prove if $x = 1$ or if $\psi(n) > 0$ for only finitely many n, so suppose neither is the case. By replacing ψ with $\psi' = x\psi$ and x by $x' = 1/x$ if necessary, we may assume without loss of generality that $x \in (0, 1)$. If $x\psi(n) \geq 4^{\omega(n)}$ infinitely often, then Lemma 4.3 implies that $\mathscr{C}^p(\psi), \mathscr{C}^p(x\psi) \supseteq p^k\mathbb{Z}_p^\times$, so suppose not. Then, for n sufficiently large,

$$\frac{\psi(n)}{n} \leq x^{-1}\frac{4^{\omega(n)}}{n} \leq 16x^{-1}\frac{4^{\log_5(n)}}{n} \underset{n\to\infty}{\longrightarrow} 0.$$

Since $\mu_p(\mathscr{C}_n^p(\psi)) = 0$ when $\psi(n) = 0$, we have by the Borel-Cantelli Lemma that

$$\mu_p\big(\mathscr{C}^p(\psi)\big) = \mu_p\Big(\limsup_{\psi(n)>0} \mathscr{C}_n^p(\psi)\Big),$$

$$\mu_p\big(\mathscr{C}^p(x\psi)\big) = \mu_p\Big(\limsup_{\psi(n)>0} \mathscr{C}_n^p(x\psi)\Big).$$

Notice that for $\psi(n) > 0$, $\mathscr{C}_n^p(\psi)$ is a finite union of proper balls $B_{i_n}, \dots, B_{i_{n+1}-1}$ of radius $\psi(n)/n \underset{n\to\infty}{\longrightarrow} 0$, and that each ball B_{i_n+j} is matched one-to-one by a ball U_{i_n+j} from $\mathscr{C}_n^p(x\psi)$ satisfying

$$U_{i_n+j} \subseteq B_{i_n+j}, \quad \mu_p(U_{i_n+j}) \geq \frac{x}{p}\mu_p(B_{i_n+j}).$$

From Corollary 3.3, it follows that $\mu_p(\mathscr{C}^p(x\psi)) = \mu_p(\mathscr{C}^p(\psi))$. This completes the proof since clearly $\mathscr{C}^p(x\psi) \subseteq \mathscr{C}^p(\psi)$.

We are now ready to complete the proof of the shell-wise p-adic zero-full law. The remaining part of the proof is where it differs the most from the proof of the 0-1 law in [5], though it still follows the same overall idea.

PROOF OF PROPOSITION 4.2. Let l be defined as in Proposition 4.1. If $k \geq l$, then $l < \infty$, and it follows from the proof of that proposition that $\mathscr{C}^p(\psi) \supseteq p^l\mathbb{Z}_p \supseteq p^k\mathbb{Z}_p^\times$, and we are done, so suppose $k > l$. We then have that $\mathscr{C}^p(\psi) \cap p^k\mathbb{Z}_p^\times = \limsup_{\nu_p(n)=k} \mathscr{C}_n^p(\psi)$, and we may hence assume, without loss of generality, that $\operatorname{supp}(\psi) \subseteq p^k\mathbb{N} \setminus p^{k+1}\mathbb{N}$. Following the arguments in the proof of Lemma 4.4, $\limsup_{n\to\infty} \psi(n)/n > 0$ would likewise imply $\mathscr{C}_n^p(\psi) \supseteq p^k\mathbb{Z}_p^\times$, so suppose $\limsup_{n\to\infty} \psi(n)/n = 0$. We then have, for any fixed j, that

$$\frac{\psi(n)}{n} < p^{-j} \quad \text{for all } n \geq N_j, \text{ for some } N_j \in \mathbb{N}.$$

Let $\tau_1\colon \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ and $\tau_2\colon p^k\mathbb{Z}_p^\times \to p^k\mathbb{Z}_p^\times$ be given by

$$\tau_1(b) = \begin{cases} \sum_{m=0}^{\infty} b_{m+1}p^m & \text{if } b_1 \neq 0, \\ 1 + \sum_{m=0}^{\infty} b_{m+1}p^m & \text{if } b_1 = 0, \end{cases}$$

$$\tau_2(p^k b) = p^k/\tau_1(b),$$

for $b = \sum_{m=0}^{\infty} b_m p^m \in \mathbb{Z}_p^\times$. For $K \geq 2$ and $b = \sum_{m=0}^{K-1} b_m p^m \in \mathbb{Z}_p^\times$, note that

$$\tau_1\big(b + p^K\mathbb{Z}_p\big) = \tau_1(b_1) + \sum_{m=1}^{K-2} b_{m+1}p^m + p^{K-1}\mathbb{Z}_p. \tag{4.6}$$

Thus, τ_1 maps balls of centre b and radius p^{-K} to balls of centre $\tau_1(b)$ and radius p^{1-K} when $K \geq 2$. This makes any restriction of τ_1 to a ball in $\mathbb{Z}_p^\times$ of radius at most p^{-2} into a homeomorphism onto its image as it is clearly a bijection under such a restriction. By Lemma 3.5, these properties extend to a restriction of τ_2 when replacing K by $M \geq k+2$. Let B be a ball of radius at most p^{-k-2}. The inverse of τ_2 restricted to B, $\tau_2|_B^{-1}$ is thus measurable and has a push-forward measure, $(\tau_2|_B^{-1})_{\#\mu}$, satisfying

$$(\tau_2|_B^{-1})_{\#\mu}(\tilde{B}) = \mu_p(\tau_2(\tilde{B})) = p\mu_p(\tilde{B}),$$

for all balls $\tilde{B} \subseteq \tau_2(B)$. By the proof of Lemma 3.1, this means that $(\tau_2|_B^{-1})_{\#\mu} = p\mu_p$, i.e.,

$$\mu_p(\tau_2(A)) = p\mu_p(A), \tag{4.7}$$

for all Borel subsets $A \subseteq B$.

Let $\alpha \in \mathscr{C}^p(\psi) \cap p^k\mathbb{Z}_p^\times$ and write

$$\alpha' := \alpha/p^k = \sum_{m=0}^{\infty} a_m p^m \in \mathbb{Z}_p^\times, \quad a_i \in \{0, \ldots, p-1\}.$$

Let $n \in \mathbb{N}$ such that $\alpha \in \mathscr{C}_n^p(\psi)$. Since $\lim_{n\to\infty} \psi(n)/n = 0$, we have $\psi(n)/n \leq p^{-k-2}$ for n large enough. By the assumption on the support of ψ, we may write $n = p^k n'$ for some $n' \in \mathbb{N} \setminus p\mathbb{N}$ and pick some $|a| < |n|$ with $\gcd(a, pn) = 1$ such that

$$|n'\alpha' - a|_p = \left|\frac{n}{\alpha} - a\right|_p = p^k\left|\alpha - \frac{n}{a}\right|_p \leq p^k\frac{\psi(n)}{n}.$$

If $a_1 \neq 0$, put $a' = (a - a_0 n')/p$. Then

$$\left|\frac{n}{\tau_2(\alpha)} - a'\right|_p = \left|n'\tau_1(\alpha') - \frac{a - a_0 n'}{p}\right|_p = \left|\frac{n'(\alpha' - a_0) - (a - a_0 n')}{p}\right|_p$$
$$= p|n'\alpha' - a|_p \leq p^{k+1}\frac{\psi(n)}{n},$$
$$|a'| = \left|\frac{a - a_0 n'}{p}\right| \leq \frac{|a| + (p-1)n}{p} < n,$$
$$a' = \frac{a - a_0 n'}{p} = \frac{a - (a_0 + pa_1)n'}{p} + a_1 n'.$$

Since $p^2 \mid a - \alpha' n'$, we have $p \mid (a - (a_0 + pa_1)n')/p \in \mathbb{Z}$. From the last equation, we can thus deduce that $\gcd(a', pn') = \gcd(a', pn) = 1$, as $p \nmid a_1 n'$ and $\gcd(a, n') = 1$. Combined with the other two equations, we conclude that $\tau_2(\alpha) \in \mathscr{C}_n^p(p\psi)$. If $a_1 = 0$, we put $a' = (a + (p - a_0)n')/p$ and reach the same conclusion, based on similar calculations. Hence, $\tau_2(\mathscr{C}^p(\psi)) \subseteq \mathscr{C}^p(p\psi)$. By induction, we then have

$$\tau_2^j(\mathscr{C}^p(\psi)) \subseteq \mathscr{C}^p(p^j\psi), \tag{4.8}$$

for all $j \in \mathbb{N}$, where τ_2^j denotes the composition of j copies of τ_2.

If $\mu_p(\mathscr{C}^p(\psi)) = 0$, we are done, so suppose that $\mu_p(\mathscr{C}^p(\psi)) > 0$. Then Lemma 3.4 implies that $\mathscr{C}^p(\psi)$ contains a point α of density 1, i.e.,

$$\frac{\mu_p(\mathscr{C}^p(\psi) \cap B_M(\alpha))}{\mu_p(B_M(\alpha))} \underset{M\to\infty}{\longrightarrow} 1,$$

where we use $B_M(\alpha)$ as short-hand notation of the ball $B_{\mathbb{Q}_p}(\alpha, p^{-M}) = \alpha + p^M\mathbb{Z}_p$. Let $\varepsilon > 0$ and pick $M \geq k + 2$ such that

$$\mu_p(\mathscr{C}^p(\psi) \cap B_M(\alpha)) \geq (1 - \varepsilon)\mu_p(B_M(\alpha)) = (1 - \varepsilon)p^{-M}.$$

Pick $x = p^k \sum_{m=0}^{M-k-1} x_m p^m \in \mathbb{N}$ such that $B_M(x) = B_M(\alpha)$. Put $A = \tau_2^{M-k-1}(\mathscr{C}^p \cap B_M(x))$. By Lemma 4.4, equation (4.6), and inclusion (4.8), we have

$$\mu_p\big(\mathscr{C}^p(\psi) \cap p^k\mathbb{Z}_p^\times\big) = \mu_p\big(\mathscr{C}^p(p^{M-k}\psi) \cap p^k\mathbb{Z}_p^\times\big)$$
$$\geq \mu_p\big(\tau_2^{M-k}(\mathscr{C}^p(\psi) \cap B_M(x))\big) = \mu_p(\tau_2(A)). \tag{4.9}$$

Let $\tilde{x} = \tau_2^{M-k-1}(x)$ and note that $\tilde{x} = p^k\tilde{x}_0$ for some $\tilde{x}_0 \in \{1, \ldots, p-1\}$.

Then

$$A \subseteq B_{k+1}(\tilde{x}) = p^k \bigsqcup_{i=0}^{p-1} \big(\tilde{x}_0 + ip + p^2\mathbb{Z}_p\big).$$

By applying equation (4.7) and then later the above inclusion, we find

$$\begin{aligned}
\mu_p(\tau_2(A)) &\geq \sum_{i=1}^{p-1} \mu_p(\tau_2(A \cap B_{k+2}(\tilde{x} + ip))) \\
&= p \sum_{i=1}^{p-1} \mu_p(A \cap B_{k+2}(\tilde{x} + ip)) \\
&= p\mu_p(A \setminus B_{k+2}(\tilde{x})) \geq p(\mu_p(A) - p^{-k-2}) \\
&\geq (p-1)\mu_p(A)
\end{aligned}$$

Meanwhile, an iterative application of equations (4.6) and (4.7) yield

$$\mu_p(A) = p^{M-k-1}\mu_p(\mathscr{C}^p \cap B_M(x)) \geq \frac{1-\varepsilon}{p^{k+1}},$$

so that $\mu_p\big(\mathscr{C}^p(\psi) \cap p^k\mathbb{Z}_p^\times\big) \geq \frac{p-1}{p^{k+1}}(1-\varepsilon)$, by inequality (4.9). This completes the proof.

Proof of Theorem 2.1. We start by the ‘only if’ parts. For $\mathscr{C}^p$, it follows by Proposition 4.2 upon noting

$$\mathscr{C}^p = (\mathscr{C}^p \cap \{0\}) \sqcup \bigsqcup_{k=0}^{\infty} \big(\mathscr{C}^p \cap p^k\mathbb{Z}_p^\times\big).$$

As for $\mathscr{B}^p$, suppose $\mu_p(\mathscr{B}^p) < 1$. By Proposition 4.1, $\sum_{p\nmid n} \phi(n)\psi(n)/n < \infty$, $\mu_p(\mathscr{B}^p) = \mu_p(\mathscr{C}^p)$, and $\mathscr{C}^p \cap \mathbb{Z}_p^\times = \limsup_{p\nmid n} \mathscr{C}_n^p$. Since

$$\sum_{p\nmid n} \mu_p(\mathscr{C}_n^p) \leq \sum_{p\nmid n} \frac{2\phi(n)\psi(n)}{n} < \infty,$$

the Borel-Cantelli Lemma implies $\mu_p(\mathscr{C}^p \cap \mathbb{Z}_p^\times) = 0$, and we are done by the above consideration of $\mathscr{C}^p$.

The ‘if’ parts of the theorem are already dealt with in the proof of Theorem 3 of [9] (at least for $\mathscr{B}^p$), but we will repeat the argument here for clarity, shortened by means of Proposition 4.1. Let $x_k \in \{0, 1\}$ for $k \in \mathbb{N}_0$ and define $\psi\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ by

$$\psi(n) = x_{\nu_p(n)} \frac{n}{p^{\nu_p(n)+1}}.$$

Let $k \in \mathbb{N}_0$, and let $q > p$ be some large prime. If $x_k = 0$, we clearly have $\limsup_{\nu_p(n)=k} \mathscr{C}_n^p = \emptyset$. If, on the other hand, $x_k = 1$, then $\psi(p^k q)/(p^k q) = p^{-k-1}$, and so

$$\mathscr{C}^p_{p^k q} = \bigcup_{\substack{|a|<p^k q \\ \gcd(a, p^{k+1}q)=1}} B_{\mathbb{Q}_p}\left(\frac{p^k q}{a}, p^{-k-1}\right)$$
$$\supseteq \bigcup_{a=1}^{p-1} p^k \frac{q}{a} + p^{k+1}\mathbb{Z}_p = p^k \bigcup_{a=1}^{p-1} \frac{q}{a} + p\mathbb{Z}_p$$

Since q is a unit modulo p, and since inversion and multiplication by units only permute the set of units modulo p, this implies that $\mathscr{C}^p_{p^k q} \supseteq p^k \mathbb{Z}_p^\times$. As there are infinitely many such q, we get $\mathscr{C}^p \supseteq p^k \mathbb{Z}_p^\times$. Note that if $x_0 = 1$, then $\sum_{p\nmid n} \phi(n)\psi(n)/n = \infty$. By Proposition 4.1, this means that

$$\mu_p(\mathscr{C}^p) = \sum_{x_k=1} \mu_p(p^k \mathbb{Z}_p^\times) = \sum_{k=0}^{\infty} x_k (p-1) p^{-k-1},$$
$$\mu_p(\mathscr{B}^p) = \begin{cases} 1 & \text{if } x_0 = 1, \\ \displaystyle\sum_{k=1}^{\infty} x_k (p-1) p^{-k-1} & \text{if } x_0 = 0, \end{cases}$$

as the value l from the proposition is clearly infinite. This completes the proof of Theorem 2.1.

5. Proof of Theorem 2.2 for $p < \infty$

The proof of Theorem 2.2 follows the same main idea as the 'if' part of Theorem 2.1, though some details are different. The main difference is that additional care is needed for the choice of the support. This will rely heavily on the below theorem due to Dirichlet, which may be found in [11].

DIRICHLET'S THEOREM ON PRIMES. *Let $a, b \in \mathbb{N}$ such that* $\gcd(a, b) = 1$. *Then there are infinitely many primes $q \equiv a \mod b$.*

To simplify the notation, the symbol $\pm$ will be used to implicitly denote the union of the cases of $+$ and $-$, respectively, in place of $\pm$. We thus write

$$B(\pm a^{\pm 1}, r) = B(a^{\pm 1}, r) \cup B(-a^{\pm 1}, r)$$
$$= B(a, r) \cup B(a^{-1}, r) \cup B(-a, r) \cup B(-a^{-1}, r).$$

Furthermore, we continue using $\sqcup$ to denote the disjoint union of sets. Recall that we already proved Theorem 2.2 for $p = \infty$ in Section 2.

PROOF OF THEOREM 2.2 FOR $p < \infty$. Let $x \in [0, 1]$. If $x = 1$, the statement is trivial, so suppose not. We then write $x = \sum_{k=0}^{\infty} x_k p^{-k-1}$ where the x_k are chosen such that $\liminf_{k\to\infty} x_k < p-1$. Pick $K = \min\{k \in \mathbb{N}_0 : x_k < p-1\}$, and let $g \in \mathbb{N}$ be such that $g + p\mathbb{Z}_p$ generates the multiplicative group $\mathbb{Z}_p/p\mathbb{Z}_p$. As the rest of the construction will depend on the prime p, we split into four cases. In the first two cases, which construct ψ for primes $p > 5$ according to their congruency classes modulo 4, we do not put any further restrictions on the choice of g. In the other two cases, which deal with $p = 2$ and $p = 3, 5$, respectively, we will need some further restrictions on g and therefore fix a specific value, depending on p.

Case 1: $(5 < p \equiv 1 \mod 4)$. For $a \in \{0, \dots, p-1\}$ and $k \in \mathbb{N}_0$, define

$$I_a := \begin{cases} \emptyset & \text{if } a < 4, \\ \left\{1, g^{\frac{p-1}{4}}\right\} \cup \{g^i : 2 \le i \le a/4\} & \text{if } 4 \le a < p-1, \\ \{1, \dots, p-1\} & \text{if } a = p-1, \end{cases}$$

$$r_k := \begin{cases} x_k - 4\left\lfloor \dfrac{x_k}{4} \right\rfloor + \displaystyle\sum_{l=k+1}^{\infty} x_l p^{k-l} & \text{if } x_k < p-1, \\ 0 & \text{if } x_k = p-1. \end{cases}$$

Since clearly $r_k \in [0, 4]$, we may write

$$\frac{r_k}{4} = \sum_{i=1}^{\infty} b_{k,i} p^{-i}, \qquad b_{k,i} \in \{0, 1, \dots, p-1\}.$$

Based on this, we construct ψ as

$$\psi(n) := \begin{cases} f_k(q) & \text{if } n = p^k q, \text{ where } k \le K, \\ 0 & \text{otherwise,} \end{cases}$$

where we use q to denote primes other than p, and

$$f_k(q) = \begin{cases} q/p & \text{if } q \equiv m \mod p, \text{ where } m \in I_{x_k}, \\ q/p^{i+1} & \text{if } q \equiv g + b' p^i \mod p^{i+1}, \\ & \quad \text{where } 1 \le b' \le b_{k,i},\ i \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $k \le K$. Then

$$\begin{aligned}\mathfrak{A}^p_{p^k q} &= \begin{cases} B_{\mathbb{Q}_p}(\pm q^{\pm 1} p^k, p^{-k-1}) & \text{if } q \equiv m \bmod p, \text{ where } m \in I_{x_k}, \\ B_{\mathbb{Q}_p}(\pm q^{\pm 1} p^k, p^{-k-i-1}) & \text{if } q \equiv g + b' p^i \bmod p^{i+1}, \\ & \text{where } 1 \le b' \le b_{k,i},\, i \in \mathbb{N}, \\ B_{\mathbb{Q}_p}(\pm q^{\pm 1} p^k, 0) & \text{otherwise} \end{cases} \\ &= p^k \begin{cases} \pm q^{\pm 1} + p\mathbb{Z}_p & \text{if } q \equiv g^i \bmod p, \text{ where } g^i \in I_{x_k}, \\ \pm q^{\pm 1} + p^{i+1}\mathbb{Z}_p & \text{if } q \equiv g + b' p^i \bmod p^{i+1}, \\ & \text{where } 1 \le b' \le b_{k,i},\, i \in \mathbb{N}, \\ \{\pm q^{\pm 1}\}, & \text{otherwise,} \end{cases} \qquad (5.1) \\ &\subseteq p^k \mathbb{Z}_p^\times. \end{aligned}$$

If $k < K$, then $I_{x_k} = \{1, \dots, p-1\}$, and so we are in the first case for all q, implying that $\limsup_{q\to\infty} \mathfrak{A}^p_{p^k q} = p^k \mathbb{Z}_p^\times$, by Dirichlet's Theorem on primes. By Dirichlet's pigeonhole principle, this means that

$$\mathfrak{R}^p \supseteq \mathfrak{A}^p \supseteq \limsup_{\substack{n\to\infty \\ n = p^k q,\ k<K}} \mathfrak{A}^p_n = \bigcup_{k=0}^{K-1} \limsup_{q\to\infty} \mathfrak{A}^p_{p^k q} = \mathbb{Z}_p \setminus p^K \mathbb{Z}_p. \qquad (5.2)$$

We are thus left to consider $\mathfrak{R}^p \cap p^K \mathbb{Z}_p$ and $\mathfrak{A}^p \cap p^K \mathbb{Z}_p$. Since q is prime, and $\psi(p^k q)/(p^k q) \le p^{-k-1}$, we note, for arbitrary $k \in \mathbb{N}_0$,

$$\mathfrak{R}^p_{p^k q} \subseteq \mathfrak{A}^p_{p^k q} \cup \bigcup_{0<j\le k/2} p^{k-2j} \mathbb{Z}_p^\times. \qquad (5.3)$$

Since $\mathfrak{R}^p_n$ is finite when $\psi(n) = 0$, and $\mathfrak{R}^p_{qp^k} \subseteq \mathbb{Z}_p \setminus p^{k+1}\mathbb{Z}_p$ by equation (5.1) and inclusion (5.3), we have

$$\mu_p(\mathfrak{R}^p \cap p^K \mathbb{Z}_p) = \mu_p\Big(\limsup_{n\to\infty} \mathfrak{R}^p_n \cap p^K \mathbb{Z}_p\Big) = \mu_p\Big(\limsup_{q\to\infty} \mathfrak{R}^p_{p^K q} \cap p^K \mathbb{Z}_p\Big).$$

Applying inclusion (5.3) and then equation (5.1) once more, we find

$$\begin{aligned}\mu_p(\mathfrak{R}^p \cap p^K \mathbb{Z}_p) &= \mu_p\Big(\limsup_{q\to\infty} \mathfrak{A}^p_{p^K q} \cap p^K \mathbb{Z}_p\Big) \\ &= \mu_p\Big(\limsup_{q\to\infty} \mathfrak{A}^p_{p^K q} \mathbb{Z}_p\Big) = \mu_p\big(\mathfrak{A}^p \cap p^K \mathbb{Z}_p\big).\end{aligned}$$

Since $x_k = p-1$ for $0 \le k < K$, inclusion (5.2) implies

$$\mu_p(\mathfrak{R}^p) = \mu_p(\mathfrak{A}^p) = \mu_p(\mathbb{Z}_p \setminus p^K\mathbb{Z}_p) + \mu_p\Big(\limsup_{q\to\infty} \mathfrak{A}^p_{p^K q}\Big)$$
$$= \sum_{k=0}^{K-1} x_k p^{-k-1} + \mu_p\Big(\limsup_{q\to\infty} \mathfrak{A}^p_{p^K q}\Big).$$

The theorem hence follows for the current case if we can show that

$$\mu_p\Big(\limsup_{q\to\infty} \mathfrak{A}^p_{p^K q}\Big) = \sum_{k=K}^{\infty} x_k p^{-k-1}. \tag{5.4}$$

Applying Dirichlet's Theorem on primes on equation (5.1) for $k = K$, we find

$$\begin{aligned}\limsup_{q\to\infty} \mathfrak{A}^p_{p^K q} &= \bigcup_{m\in I_{x_K}} (\pm m^{\pm 1} p^K + p^{K+1}\mathbb{Z}_p) \\ &\quad \cup \bigcup_{i=1}^{\infty} \bigcup_{b'=1}^{b_{K,i}} (\pm(g+b'p^i)^{\pm 1} p^K + p^{K+i+1}\mathbb{Z}_p) \\ &= \bigsqcup_{m\in I_{x_K}} (\pm m^{\pm 1} p^K + p^{K+1}\mathbb{Z}_p) \\ &\quad \sqcup \bigcup_{i=1}^{\infty} \bigcup_{b'=1}^{b_{K,i}} (\pm(g+b'p^i)^{\pm 1} p^K + p^{K+i+1}\mathbb{Z}_p),\end{aligned} \tag{5.5}$$

using that $(\pm m_1^{\pm 1} p^K + p^{K+1}\mathbb{Z}_p) \cap (\pm m_2^{\pm 1} p^K + p^{K+1}\mathbb{Z}_p) = \emptyset$ for distinct $m_1, m_2 \in \{g^j : 0 \le j \le (p-1)/4\}$. We clearly also have

$$(\pm(g+b_1'p^i)^{\pm 1} p^K + p^{K+i+1}\mathbb{Z}_p) \cap (\pm(g+b_2'p^j)^{\pm 1} p^K + p^{K+j+1}\mathbb{Z}_p) = \emptyset,$$

for $i \ne j$ or $b_1' \ne b_2'$. Applying this to the above calculation, we find

$$\begin{aligned}\mu_p\Big(\limsup_{q\to\infty} \mathfrak{A}^p_{p^K q}\Big) &= \sum_{m\in I_{x_K}} \mu_p(\pm m^{\pm 1} p^K + p^{K+1}\mathbb{Z}_p) \\ &\quad + \sum_{l=1}^{\infty} \sum_{b'=1}^{b_{K,l}} \mu_p(\pm(g+b'p^l)^{\pm 1} p^K + p^{K+l+1}\mathbb{Z}_p) \\ &= p^{-K}\Bigg(\sum_{m\in I_{x_K}} \mu_p(\pm m^{\pm 1} + p\mathbb{Z}_p) \\ &\quad + \sum_{l=1}^{\infty} \sum_{b'=1}^{b_{K,l}} \mu_p(\pm(g+b'p^l)^{\pm 1} + p^{l+1}\mathbb{Z}_p)\Bigg).\end{aligned} \tag{5.6}$$

In order to determine the first sum in equation (5.6), note that

$$-g^i \equiv g^{\frac{p-1}{2}+i}, \quad g^{-i} \equiv g^{p-1-i}, \quad -g^{-i} \equiv g^{\frac{p-1}{2}-i}$$

modulo p. For $1 \leq i < (p-1)/4$, we have

$$\frac{p-1}{4} < \frac{p-1}{2} - i < \frac{p-1}{2} < \frac{p-1}{2} + i$$
$$< 3\frac{p-1}{4} < p-1-i < p-1,$$

implying that $g^i, g^{-i}, -g, -g^{-i}$ are not congruent modulo p^l for $l \geq 1$. A similar argument yields $1 = 1^{-1} \not\equiv -1 = -1^{-1}$ and

$$g^{(p-1)/4} \equiv -g^{-(p-1)/4} \not\equiv -g^{(p-1)/4} \equiv g^{-(p-1)/4}$$

modulo p. Hence, for integers $1 \leq i < (p-1)/4$, $0 \leq b' \leq p-1$, $l \geq 1$, and $j \in \{0, (p-1)/4\}$,

$$\begin{aligned} \mu_p(\pm(g^i + b'p^l)^{\pm 1} + p^{l+1}\mathbb{Z}_p) &= 4/p^{l+1}, \\ \mu_p(\pm g^{\pm j} + p\mathbb{Z}_p) &= 2/p. \end{aligned} \tag{5.7}$$

We are now ready to handle the first sum of (5.6). For $x_K \geq 4$, we find

$$\begin{aligned} \sum_{m \in I_{x_K}} \mu_p(\pm m^{\pm 1} + p\mathbb{Z}_p) &= \sum_{i=2}^{\lfloor x_K/4 \rfloor} \mu_p(\pm g^{\pm(p-1)/4} + p\mathbb{Z}_p) \\ &\quad + \mu_p(\pm 1^{\pm 1} + p\mathbb{Z}_p) + \mu_p(\pm g^{\pm(p-1)/4} + p\mathbb{Z}_p) \\ &= \left(\sum_{i=2}^{\lfloor x_K/4 \rfloor} \frac{4}{p}\right) + \frac{2}{p} + \frac{2}{p} = \frac{4}{p}\left\lfloor \frac{x_k}{4} \right\rfloor. \end{aligned}$$

For $x_K < 4$, $I_{x_K} = \emptyset$, and so we reach the same conclusion in that case.

Equation (5.6) and the definitions of $b_{a,i}$, r_a, and x_k then allow us to conclude equation (5.4) as we see

$$\begin{aligned} &\mu_p\big(\limsup_{q\to\infty} \mathfrak{A}^p_{p^K q}\big) \\ &\quad = p^{-K}\left(\frac{4}{p}\left\lfloor \frac{x_K}{4} \right\rfloor + \sum_{i=1}^{\infty} \sum_{b'=1}^{b_{K,i}} \frac{4}{p^{i+1}}\right) = p^{-K-1}4\left(\left\lfloor \frac{x_K}{4} \right\rfloor + \sum_{i=1}^{\infty} b_{k,i} p^{-i}\right) \\ &\quad = p^{-K-1}4\left(\left\lfloor \frac{x_K}{4} \right\rfloor + \frac{r_K}{4}\right) = p^{-K-1} \sum_{l=K}^{\infty} p^{K-l} x_l = \sum_{l=K}^{\infty} x_l p^{-l-1}. \end{aligned}$$

Case 2: ($5 < p \equiv 3 \bmod 4$). This case closely follows the structure of Case 1. The main difference is a modification to the definitions of I_a and r_k, so that

$$I_a := \begin{cases} \emptyset & \text{if } a < 2, \\ \{1\} \cup \{g^i : 2 \le i \le (a+2)/4\} & \text{if } 2 \le a < p-1, \\ \{1, \dots, p-1\} & \text{if } a = p-1, \end{cases}$$

$$r_k := \begin{cases} \sum_{l=k}^{\infty} x_l p^{k-l} & \text{if } x_k < 2, \\ \sum_{l=k}^{\infty} x_l p^{k-l} - 4\left\lfloor \frac{x_k - 2}{4} \right\rfloor - 2 & \text{if } 2 \le x_k < p-1, \\ 0 & \text{if } x_k = p-1. \end{cases}$$

Note that all arguments until and including equation (5.7) remain valid, except that we no longer have an integer $j = (p-1)/4$. For $x_K < 2$, the remaining arguments are unchanged, so suppose $x_K \ge 2$. Then

$$\begin{aligned} \sum_{m \in I_{x_K}} \mu_p\big(\pm m^{\pm 1} + p\mathbb{Z}_p\big) &= \sum_{i=2}^{\lfloor (x_K+2)/4 \rfloor} \mu_p(\pm g^{\pm (p-1)/4} + p\mathbb{Z}_p) \\ &\quad + \mu_p(\pm 1^{\pm 1} + p\mathbb{Z}_p) \\ &= \left(\sum_{i=2}^{\lfloor (x_K+2)/4 \rfloor} \frac{4}{p} \right) + \frac{2}{p} = \frac{4}{p} \left\lfloor \frac{x_k - 2}{4} \right\rfloor + \frac{2}{p}. \end{aligned}$$

Applying this and equation (5.7) to equation (5.4), we conclude

$$\begin{aligned} \mu_p\Big(\limsup_{q \to \infty} \mathfrak{A}^p_{p^K q}\Big) &= p^{-K} \left(\frac{4}{p} \left\lfloor \frac{x_k - 2}{4} \right\rfloor + \frac{2}{p} + \sum_{i=1}^{\infty} \sum_{b'=1}^{b_{K,i}} \frac{4}{p^{i+1}} \right) \\ &= p^{-K-1} 4 \left(\left\lfloor \frac{x_k - 2}{4} \right\rfloor + \frac{1}{2} + \sum_{i=1}^{\infty} b_{k,i} p^{-i} \right) \\ &= p^{-K-1} 4 \left(\left\lfloor \frac{x_k - 2}{4} \right\rfloor + \frac{1}{2} + \frac{r_K}{4} \right) \\ &= p^{-K-1} \sum_{l=K}^{\infty} p^{K-l} x_l = \sum_{l=K}^{\infty} x_l p^{-l-1}. \end{aligned}$$

Case 3: ($p = 2$). In this case, we will use the same construction as in Case 1, except that we fix $g = 1$ and change $b_{K,i}$ such that

$$\frac{r_K}{2} = \sum_{i=1}^{\infty} b_{K,i} 2^{-i}.$$

In terms of I_a, the case $a = p - 1$ is more important than the case $a < 4$, so we read the construction as $I_1 = \{1\}$ for $p = 2$. Note that all arguments of the proof for Case 1 until equation (5.6) remain valid. Since $(1 + 2^i)^{-1} \equiv 1 + 2^i \bmod 2^{i+1}$, $b_{K,i} \in \{0, 1\}$, and $x_K = 0$, we find

$$\begin{aligned}
\mu_2\big(\limsup_{q\to\infty} \mathfrak{A}^2_{2^K q}\big) &= 2^{-K} \sum_{i=1}^{\infty} b_{K,i} \mu_2\big(\pm(1 + 2^i) + 2^{i+1}\mathbb{Z}_2\big) \\
&= 2^{-K} \sum_{i=1}^{\infty} b_{K,i} 2 \cdot 2^{-i-1} = 2^{-K} \frac{r_K}{2}, \\
&= 2^{-K-1} \sum_{l=K+1}^{\infty} x_l 2^{K-l} = \sum_{l=K}^{\infty} x_l 2^{-l-1},
\end{aligned}$$

which completes the proof in this case.

Case 4: ($p = 3, 5$). We use the same construction as in Case 2, except that we further change r_k and ψ so that

$$r_k = \begin{cases} x_K - 2\left\lfloor \frac{x_K}{2} \right\rfloor + \sum\limits_{l=K+2}^{\infty} x_l p^{K-l} & \text{if } k = K, \\ x_k - 2\left\lfloor \frac{x_k}{2} \right\rfloor & \text{if } k \neq K, \end{cases}$$

$$\psi(n) := \begin{cases} f_k(q) & \text{if } n = p^k q, k \leq K + 1, \\ 0 & \text{otherwise.} \end{cases}$$

The value of g will also be fixed, depending on p. Note that the arguments of Case 2 (which follows the arguments of Case 1) remain valid all the way to inclusion (5.3) and that equation (5.1) now also holds for $k = K + 1$. By following the same argument as in Case 1, we find

$$\mu_p\big(\mathfrak{R}^p \cap p^K\mathbb{Z}_p\big) = \mu_p\Big(\limsup_{q\to\infty}\big(\mathfrak{R}^p_{p^K q} \cup \mathfrak{R}^p_{p^{K+1} q}\big) \cap p^K\mathbb{Z}_p\Big)$$

By inclusion (5.3), it then follows that

$$\begin{aligned}\mu_p\big(\mathfrak{R}^p \cap p^K\mathbb{Z}_p\big) &= \mu_p\Big(\limsup_{q\to\infty}\big(\mathfrak{A}^p_{p^Kq} \cup \mathfrak{A}^p_{p^{K+1}q}\big)\Big)\\ &= \mu_p\Big(\limsup_{q\to\infty}\mathfrak{A}^p_{p^Kq}\Big) + \mu_p\Big(\limsup_{q\to\infty}\mathfrak{A}^p_{p^{K+1}q}\Big)\\ &= \mu_p\big(\mathfrak{A}^p \cap p^K\mathbb{Z}_p\big),\end{aligned}$$

Recalling $x_k = p-1$ for $k < K$, inclusion (5.2) implies

$$\begin{aligned}\mu_p(\mathfrak{R}^p) = \mu_p(\mathfrak{A}^p) = \sum_{k=0}^{K-1} x_k p^{-k-1} &+ \mu_p\Big(\limsup_{q\to\infty}\mathfrak{A}^p_{p^Kq}\Big)\\ &+ \mu_p\Big(\limsup_{q\to\infty}\mathfrak{A}^p_{p^{K+1}q}\Big),\end{aligned}$$

so that we are left to prove

$$\mu_p\Big(\limsup_{q\to\infty}\mathfrak{A}^p_{p^Kq}\Big) + \mu_p\Big(\limsup_{q\to\infty}\mathfrak{A}^p_{p^{K+1}q}\Big) = \sum_{k=K}^{\infty} x_k p^{-k-1}.$$

Note that equation (5.5) is preserved and that it remains valid for K replaced by $K+1$. Let $k \in \{K, K+1\}$. Then

$$\bigcup_{m\in I_{x_k}} \pm mp^k + p^{k+1}\mathbb{Z}_p = p^k \begin{cases} \mathbb{Z}_p^\times & \text{if } x_k = p-1,\\ (\pm 1 + p\mathbb{Z}_p) & \text{if } 2 \le x_k < p-1,\\ \emptyset & \text{otherwise.}\end{cases}$$

From this follows

$$\begin{aligned}\mu_p\Big(\limsup_{q\to\infty}\mathfrak{A}^p_{p^kq}\Big) = p^{-k-1}2\Big\lfloor\frac{x_k}{2}\Big\rfloor&\\ + p^{-k}\sum_{i=1}^{\infty}\mu_p&\Big(\bigcup_{b'=1}^{b_{k,i}}(\pm(g+b'p^i)^{\pm 1} + p^{i+1}\mathbb{Z}_p)\Big).\end{aligned}$$

We are now done if we, for each of $p = 3, 5$, can show

$$\sum_{i=1}^{\infty}\mu_p\Big(\bigcup_{b'=1}^{b_{k,i}}(\pm(g+b'p^i)^{\pm 1} + p^{i+1}\mathbb{Z}_p)\Big) = r_k, \tag{5.8}$$

as this would imply

$$\mu_p\Big(\limsup_{q\to\infty}\mathfrak{A}^p_{p^k q}\Big) = p^{-k-1}2\Big\lfloor\frac{x_k}{2}\Big\rfloor + p^{-k}r_k$$
$$= \begin{cases} x_K p^{-K} + \displaystyle\sum_{l=K+2}^{\infty} x_l p^{-l} & \text{if } k = K, \\ x_{K+1}p^{-(K+1)} & \text{if } k = K+1. \end{cases}$$

For $p = 3$, fix $g = 2$. Then $\liminf_{l\to\infty} x_l < 2$, and we have

$$\frac{r_k}{4} \le \frac{1}{4}\Big(1 + \sum_{l=k+2}^{\infty} x_l 3^{k-l}\Big) < \frac{1}{4}\Big(1 + \frac{1}{3}\Big) = \frac{1}{3},$$

i.e., $b_{1,k} = 0$. For $i \ge 2$ and $1 \le b' \le b_{k,i}$, notice that

$$\begin{aligned} (2 + b'3^i) + 3^{i+1}\mathbb{Z}_3 &\subseteq 2 + 3^2\mathbb{Z}_3, \\ -(2 + b'3^i) + 3^{i+1}\mathbb{Z}_3 &\subseteq 7 + 3^2\mathbb{Z}_3, \\ (2 + b'3^i)^{-1} + 3^{i+1}\mathbb{Z}_3 &\subseteq 5 + 3^2\mathbb{Z}_3, \\ -(2 + b'3^i)^{-1} + 3^{i+1}\mathbb{Z}_3 &\subseteq 4 + 3^2\mathbb{Z}_3. \end{aligned}$$

Since all balls on the right-hand-side of the above four inclusions are disjoint, the same holds for the four balls defining $(\pm 2^{\pm 1} + b'3^i + 3^{i+1}\mathbb{Z}_3)$. When we then vary i, we note that the sets

$$(\pm 2 + b'3^i)^{\pm 1} + 3^{i+1}\mathbb{Z}_3 \subseteq \pm 2^{\pm 1} + 3^i\mathbb{Z}_3^\times$$

are also disjoint. We conclude equation (5.8) by calculating

$$\sum_{i=1}^{\infty}\mu_3\Big(\bigcup_{b'=1}^{b_{k,i}}(\pm(2 + b'3^i)^{\pm 1} + 3^{i+1}\mathbb{Z}_3)\Big)$$
$$= \sum_{i=1}^{\infty}\sum_{b'=1}^{b_{k,i}} 4\mu_3(3^{i+1}\mathbb{Z}_3) = 4\sum_{i=1}^{\infty} b_{k,i}3^{-i-1} = r_k.$$

For $p = 5$, fix $g = 3$ and estimate

$$\frac{r_k}{4} \le \frac{1}{4}\Big(1 + \sum_{l=K+2}^{\infty} x_l 5^{x_K - l}\Big) < \frac{1}{4}\Big(1 + \frac{1}{5}\Big) < \frac{2}{5},$$

so that $b_{k,1} \in \{0, 1\}$. For $i \geq 1$ and $1 \leq b' \leq b_{k,i}$, let $a_i \in \{0, 1\}$ such that $a_1 = b'$ and $a_i = 0$ for $i > 1$. We then find

$$\begin{aligned}(3 + b'5^i) + 5^{i+1}\mathbb{Z}_5 &\subseteq 3 + a_i 5 + 5^2\mathbb{Z}_5,\\ -(3 + b'5^i) + 5^{i+1}\mathbb{Z}_5 &\subseteq 2 + (4 - a_i)5 + 5^2\mathbb{Z}_5,\\ (3 + b'5^i)^{-1} + 5^{i+1}\mathbb{Z}_5 &\subseteq 2 + (3 + a_i)5 + 5^2\mathbb{Z}_5,\\ -(3 + b'5^i)^{-1} + 5^{i+1}\mathbb{Z}_5 &\subseteq 3 + (1 - a_i)5 + 5^2\mathbb{Z}_5.\end{aligned}$$

We are again left with four disjoint balls on the right-hand-side, for i fixed. We then apply arguments in parallel to those for $p = 3$ and conclude equation (5.8). This completes the proof.

In cases 3 and 4 of the above proof, note that the choice of generator g matters; for $p = 2$, $g > 1$ would lead to $\pm g^{\pm 1}$ representing four unique values modulo 2^i for sufficiently large i, where we want exactly 2 unique values. For $p = 3$, $g \equiv -1 \bmod 9$ would on the other hand lead to $\pm g^{\pm 1}$ representing only two unique values modulo 9, where we want exactly 4. The same issue arises modulo 25 for $p = 5$ when $g + a_1 5 = g + 5 \equiv \pm 7 \bmod 25$.

Note also that the alterations introduced in Case 4 would not be sufficient for $p = 5$ if they were to be applied to Case 1, even though $5 \equiv 1 \bmod 4$, since it would allow any $b_{k,1}$ between 0 and 4, which would lead to $g + a_1 5 \equiv \pm 7 \bmod 25$ for some a_1, regardless of g.

Finally, note that the construction in Case 4 would also work for $p = 2$ by putting $g = 3$ and $r_k = 0$ for $k \neq K$, though that would not actually shorten the proof as we would then have to give $p = 2$ the same amount of special attention as we gave each of $p = 3$ and $p = 5$. This suggests that we are in a peculiar case of $p = 2$ *not* being the most troublesome prime, as that title appears to go to $p = 5$, with $p = 3$ as a close second.

6. Concluding remarks

Considering how Proposition 4.2 acts as a shell-wise $\mathscr{C}^p$-variant of the 0-1 law on $\mathscr{A}^p$ from [5], it appears rather plausible that the p-adic Duffin-Schaeffer theorem should also have a shell-wise $\mathscr{C}^p$-variant, as formally stated below.

Conjecture. *Let $\psi\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$, and let p be a prime. Suppose* supp $\psi \subseteq p^k\mathbb{N} \setminus p^{k+1}\mathbb{N}$ *for some $k \in \mathbb{N}_0$. Then*

$$\mu_p\big(\mathscr{C}^p \cap p^k\mathbb{Z}_p^\times\big) = \begin{cases} (p-1)/p^{k+1} & \text{if } \sum_{n=1}^{\infty} \mu_p(\mathscr{C}_n^p) = \infty,\\ 0 & \text{if } \sum_{n=1}^{\infty} \mu_p(\mathscr{C}_n^p) < \infty.\end{cases}$$

As with the original and p-adic Duffin-Schaeffer Theorems, the Borel-Cantelli Lemma directly implies $\mu_p(\mathscr{C}^p) = 0$ when the series converges. If the conjecture holds true, it combines with Proposition 4.1 to provide an explicit formula for determining the p-adic measures of $\mathscr{B}^p$ and $\mathscr{C}^p$. It is expected that the conjecture will follow from a modification of the proof of [5, Theorem 2] combined with [8, Proposition 5.4], following the structure presented in [9]. In the light of [9, Theorem 2], it appears only natural if the measures $\mu_p(\mathscr{C}_n^p)$ in the divergence criterion may be replaced by the fractions $\phi(n)\psi(n)/n$ from the real Duffin-Schaeffer Theorem, by following a similar argument.

As for the set $\mathfrak{A}^p$, one might try to modify the construction with the aim of decreasing the spectrum of possible measure values, similarly to what Haynes [5] achieved in constructing $\mathscr{A}^p$ instead of $\mathscr{B}^p$. In his construction, Haynes effectively removed the sets $\mathscr{B}_n^p$ that were restricted to specific shells $p^k\mathbb{Z}_p^\times$ from consideration as he indirectly forced the sets $\mathscr{A}_n^p$ with $p \mid n$ to be either empty or full [5, Lemma 2]. Trying to get a similar modification of $\mathfrak{A}^p$, we might consider the set

$$\limsup_{n\to\infty} \bigcup_{\substack{d\mid n\\ \gcd(d,pn/d)=1}} B_{\mathbb{Q}_p}\left(\frac{d}{n/d}, \frac{\psi(n)}{n}\right).$$

However, for $p \neq 3, 5$ and $x \in [0, (p-1)/p] \cup \{1\}$, the ψ constructed in the proof of Theorem 2.2 will still produce measure x. For $p = 3, 5$, ψ only works for $x \in \{1, (p-1)/p\} \cup \bigcup_{x_0=0}^{p-2} \frac{x_0}{p} + [0, p^{-2})$, but it seems reasonable that there should also exist a ψ for the remaining $x \in [0, (p-1)/p] \cup \{1\}$; perhaps a hybrid between the constructions from cases 3 and 4 would do the trick. Note that this attack would work identically if the above modification were carried out on $\mathfrak{K}^p$ instead. As such, there does not seem to be any immediate 'correction' to the set $\mathfrak{A}^p$ that would make it satisfy a 0-1 law in general.

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DEPARTMENT OF MATHEMATICS
AARHUS UNIVERSITY
8000 AARHUS C
DENMARK
E-mail: mll@math.au.dk

Chapter 3

Partition functions and the Fibonacci numbers

3.1 Introduction

In his book *Liber Abaci* from 1202, Leonardo Pisano (posthumously known as Fibonacci) introduced the now widely used Hindu-Arabic numeral system to the Europeans. To motivate it, he presented a collection of arithmetic problems as evidence of its strength compared to the Roman numerals in terms of calculating with and expressing large numbers. In the perhaps most famous of these problems, one imagines living in a world where rabbits reach maturity after a month, produce a new pair of rabbits once a month thereafter, and live on indefinitely. Starting with a single pair of newborn rabbits, one is then tasked with keeping track of the population. Named after Pisano, the number of rabbits at the start of the n'th month in this scenario is called the n'th Fibonacci number and denoted F_n. It follows that the first two Fibonacci numbers are both equal to 1 and that each subsequent Fibonacci number is given as $F_{n+2} = F_{n+1} + F_n$, so that we have

$$F_1 = 1, \quad F_2 = 1, \quad F_3 = 2, \quad F_4 = 3, \quad F_5 = 5, \quad F_6 = 8, \quad F_7 = 13,$$

and so on. The sequence also has close ties to the *golden ratio* (also called the *golden mean* in the literature), denoted by the Greek letter φ. It is classically defined as the unique positive number with the property that if two line segments are given where one is φ times longer than the other, then their combined length is φ times greater than the length of the longest one of them. In more modern terms, we say that φ is the positive solution to the equation $X + 1 = X^2$. By solving this second order equation, one finds that

$$\varphi = \frac{1 + \sqrt{5}}{2}.$$

It also follows from the equation $X + 1 = X^2$ that $\varphi^{n+2} = \varphi^{n+1} + \varphi^n$, which is strikingly similar to the recurrence $F_{n+2} = F_{n+1} + F_n$. More connections between the Fibonacci numbers and the golden ratio exist, and the two may in fact be thought of as different sides of the same coin.

Given the simple and yet significant nature of the Fibonacci sequence, it is only natural to start asking which other properties the sequence has and then how these properties may generalize to related sequences. In this chapter, we will investigate one such question, which was handled by Coons, Kristensen, and the current author in the paper [9]. More precisely, we will be interested in describing the asymptotic behaviour of the number of *partitions*, defined below, of positive integers n over the sequence $\{F_k\}_{k=2}^{\infty}$. We exclude F_1 from the sequence to ensure that the same number does not occur twice.

Definition 3.1. Let $\{P_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of positive integers, and let $n \in \mathbb{N}$. We then say that a sequence $\{a_k\}_{n=1}^{\infty}$ with $a_k \in \mathbb{N}_0$ is a *partition* of n over $\{P_k\}_{k=1}^{\infty}$ if it satisfies

$$n = a_1 P_1 + a_2 P_2 + a_3 P_3 + \cdots , \tag{3.1}$$

We use $p_P(n)$ and $p_F(n)$ to denote the number of partitions of n over $\{P_k\}_{k=1}^{\infty}$ and $\{F_k\}_{k=2}^{\infty}$, respectively. The map $p_P : \mathbb{N} \to \mathbb{N}_0$ is also called the *partition function* over $\{P_k\}_{k=1}^{\infty}$.

With this definition in hand, our interest in $p_F(n)$ is a specific instance of the following question. To ensure $p_P(n) \geq 1$ for all n, we will enforce $P_1 = 1$.

Question 3.2. *Given a strictly increasing sequence $\{P_k\}_{k=1}^{\infty}$ of positive integers with $P_1 = 1$, what is the asymptotic behaviour of $p_P(n)$ as $n \to \infty$?*

When dealing with this question, we will use the following standard asymptotic notation.

Notation. Let $f, g, h : \mathbb{N} \to \mathbb{R}$. We then write as follows.

- $f(n) = g(n) + O(h(n))$ if $|f(n) - g(n)| \leq C|h(n)|$ for all $n \in \mathbb{N}$ and some fixed constant $C > 0$.
- $f(n) \asymp g(n)$ if $f(n) = O(g(n))$ and $g(n) = O(f(n))$.
- $f(n) = g(n) + o(h(n))$ if $f(n) - g(n) = O(h(n))$ but $h(n) \neq O(f(n) - g(n))$. If $h(n) \neq 0$ for all n, this is equivalent to $\lim_{n\to\infty} |f(n) - g(n)|/|h(n)| = 0$.
- $f \sim g$ if $f(n) = g(n) + o(f(n))$ or, equivalently, $f(n) = g(n)(1 + o(1))$. In this case, we also say that f and g are *asymptotically equivalent*.

Question 3.2 is not new. Already in 1918, Hardy and Ramanujan gave an answer for the sequence $\{k\}_{k=1}^{\infty}$ of all positive integers, writing p for the corresponding partition function.

Theorem 3.3 (Hardy–Ramanujan, 1918)**.** *Let p be the partition function for the sequence $\{k\}_{k=1}^{\infty}$. Then*

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}.$$

While the endeavour might be tempting, answering Question 3.2 in full generality does not appear feasible. For that reason, we will limit our attention to linearly recurrent sequences as defined below.

Definition 3.4. We say that a sequence $\{P_k\}_{k=1}^{\infty}$ of positive integers is *linearly recurrent* if there is a fixed positive integer d and fixed integers $c_0, \dots, c_{d-1}$ such that

$$P_{k+d} = c_0 P_k + c_1 P_{k+1} + \cdots + c_{k-1} P_{k+d-1}.$$

In that case, the smallest such d is called the *degree* of $\{P_k\}_{k=1}$. Writing $d = \deg\{P_k\}_{k=1}^{\infty}$, the *characteristic polynomial* of $\{P_k\}_{k=1}$ is (uniquely) defined as

$$\chi_P(X) := X^d - c_{d-1}X^{d-1} - \cdots - c_2X^2 - c_1X - c_0.$$

Writing $\chi_P(X) = (X - a_1)\cdots(X - a_d)$, we say that a_i is a dominant root of χ_P if $|a_i| \geq |a_j|$ for all j.

Remark 3.5. The sequence $P_k = k$ considered by Hardy and Ramanujan satisfies $P_{k+2} = 2P_{k+1} - P_k$ and is thereby a linearly recurrent sequence of degree 2 as no c satisfies $k + 1 = ck$ for all k. This also shows that the characteristic polynomial is not necessarily irreducible since we get $\chi_P = (X^2 - 2X + 1) = (X - 1)^2$.

For a proof that χ_P is indeed well defined, see [29].

With the requirements $P_1 = 1$ and $P_{n+1} > P_n$, the linearly recurrent sequences of degree 1 are exactly the sequences $\{r^k\}_{k=0}^{\infty}$ with $r \geq 2$. The number of partitions $p_r(n)$ of such sequences was estimated by Mahler in 1940 [42]. In 1948, this was improved by de Bruijn [5], who found and specified an oscillation in the leading term. His result may be phrased as follows.

Theorem 3.6 (de Bruijn, 1948)**.** *Let $r \geq 2$ be a fixed integer. Then there is an explicit positive 1-periodic function $\psi_r(x)$ such that*

$$p_r(rn) \sim \psi_r\left(\frac{\log n - \log\log n}{\log r}\right) n^{B_r(n)} (\log n)^{C_r(n)},$$

where $B_r(n)$ and $C_r(n)$ are given by

$$B_r(n) = \frac{\log n - 2\log\log n + \log r + 2\log\log r + 2}{2\log r}$$

and

$$C_r(n) = \frac{\log\log n - 2\log r - 2\log\log r}{2\log r}.$$

Remark 3.7. As noted by Mahler [42], $p_r(rn+m) = p_r(rn)$ for $m = 0, 1, \ldots, r-1$. Hence, if we were to write out the asymptotics of $p_r(n)$ rather than $p_r(rn)$, we would have to handle an error term in $B_r(n/r)$ and $C_r(n/r)$ in order to have them stagnate for n between rn' and $rn' + r - 1$.

Having an answer for linear recurrences of degree 1, the natural next step in answering Question 3.2 is to consider the linear recurrences of degree 2 or greater, of which the Fibonacci sequence is a central example, especially among those with an irreducible characteristic polynomial. While this appears not to have been done prior to the paper [9] by Coons, Kristensen, and the current author, similar answers have been handled when additional restrictions are put on the sequence $\{a_k\}_{k=1}^{\infty}$ from Definition 3.1. An early such result was made by Zeckendorf in 1972, where he proved that there for each $n \in \mathbb{N}$ is exactly one partition $\{z_k\}_{k=2}^{\infty}$ over $\{F_k\}_{k=2}^{\infty}$ with $z_k \in \{0,1\}$ and $\min\{z_k, z_{k+1}\} = 0$. We will call this partition the *Zeckendorf representation* of n. This result was later generalized by Fraenkel in 1985 [13] to a theorem that provides natural restrictions that permit exactly one partition $\{a_k\}_{k=1}^{\infty}$ over $\{P_k\}_{k=1}^{\infty}$ for each n when $\{P_k\}_{k=1}^{\infty}$ is a strictly increasing sequence of integers with $P_1 = 1$. It should be noted that the algorithm for at least the recurrent sequences of degree 2 had been known long before, though it does not appear to have been used to actually generate a representation of positive integers; Ostrowski applied a version of it already in 1921 for a study of continued fractions [44].

Slacking the requirements of the Zeckendorf representation, we may instead consider the number $q_F(n)$ of *distinct partitions* of integers n over $\{F_k\}_{k=2}^{\infty}$, which is to say, the number of partitions with $a_k \in \{0,1\}$ for all k. Recently, in 2021, Chow and Slattery [8] gave an explicit formula for $q_F(n)$ as a function of the list of indices k with $z_k = 1$ in the Zeckendorf representation of n. Studying q_F further, they found that $\sum_{n=1}^{N} q_F(n) \asymp N^{\log 2/\log\varphi}$ and showed that there are oscillations in the main terms. This author is not aware of any generalizations of these results to a broader family of sequences $\{P_k\}_{k=1}^{\infty}$. Notice, however, that such a family will be rather small if we want to have $q_P(n) \geq 1$ for all n since this requires $P_{k+1} \leq 1 + P_1 + \cdots P_k$.

In the below section, we consider the main results of the paper [9], which answer Question 3.2 for the Fibonacci sequence and a broad family of related linearly recurrent sequences.

3.2 Partitions over the Fibonacci sequence and other linear recurrences

The main theorem of this chapter is the following result from the paper [9] by Coons, Kristensen and the current author. In this result and going forward, γ denotes the *Euler Mascheroni constant.* The theorem is of roughly the same form as Theorem 3.6 but with greater control of the error term.

Theorem 3.8. *Let $p_F(n)$ be the number of partitions of n over non-distinct Fibonacci numbers. Then, for $n > 1$,*

$$p_F(n) = \psi_F\left(\frac{\log n}{\log\varphi}\right) n^{B_F(n)} (\log n)^{C_F(n)} \left(1 + O\left(\frac{(\log\log n)^2}{\log n}\right)\right),$$

where

$$B_F(n) := \frac{1}{2\log\varphi}\left(\log n - 2\log\log n + \psi_{0,F}\left(\frac{\log n}{\log\varphi}\right)\right),$$
$$C_F(n) := \frac{1}{2\log\varphi}\left(\log\log n - \psi_{0,F}\left(\frac{\log n}{\log\varphi}\right) - 8\gamma - 2\log 5 + 8\log\varphi + 2\right),$$

and where the functions $\psi_F(x)$ and $\psi_{0,F}(x)$ are 1-periodic and explicitly computable with $\psi_F(x) > 0$.

Remark 3.9. While different, the 1-periodic functions $\psi_F(x)$ and $\psi_{0,F}(x)$ are easily computed from those in [9, Theorem 1]. Then same is true for the 1-periodic functions in Theorem 3.10 below.

After this theorem is proven, it is immediately generalized to cover the family of linearly recurrent sequences that satisfy the following conditions.

- $\{P_k\}_{k=1}^{\infty}$ is strictly increasing with $P_1 = 1$.
- The characteristic polynomial, χ_P, is irreducible and of degree at least 2.
- χ_P has a single dominant root, β.

Since χ_P is irreducible, it follows from a theorem in [12] that there are constants $\lambda, \lambda_2, \ldots, \lambda_r \in \mathbb{C}$ such that

$$P_k = \lambda\beta^k + \lambda_2^k\beta_1 + \cdots + \lambda_r^k\beta_r, \tag{3.2}$$

where $\beta_2, \ldots, \beta_r$ are the conjugates of β other than β itself. The notion of conjugates is defined in Definition 1.8 in Section 1.1. By another theorem in [12], we also have $|P_k| \asymp |\beta|^k$ since β is a dominant root. Since P_k is positive and increasing, and β is the only dominant root, it now follows from equation (3.2) that β and λ must both be positive real numbers.

Theorem 3.10. *Let $\{P_k\}_{k=1}^{\infty}$ be a strictly increasing linearly recurrent sequence of integers with $P_1 = 1$. Suppose the associated characteristic polynomial is irreducible and has a unique dominant root β, and let λ be defined by (3.2). Then, for $n > 1$,*

$$p_P(n) = \psi_P\left(\frac{\log n}{\log \beta}\right) n^{B_P(n)} (\log n)^{C_P(n)} \left(1 + O\left(\frac{(\log\log n)^2}{\log n}\right)\right),$$

where

$$B_P(n) := \frac{1}{2\log\beta}\left(\log n - 2\log\log n + \psi_{0,P}\left(\frac{\log n}{\log\beta}\right)\right),$$
$$C_P(n) := \frac{1}{2\log\beta}\left(\log\log n + \psi_{0,P}\left(\frac{\log n}{\log\beta}\right) - 8\gamma + 4\log\lambda + 4\log\beta + 2\right),$$

and where the functions $\psi_P(x)$ and $\psi_{0,P}(x)$ are 1-periodic and explicitly computable with $\psi_P(x) > 0$.

Remark 3.11. Theorem 3.8 is a special case of this theorem. Write $P_n = F_{n+1}$. The roots of $\chi_P = X^2 - X - 1$ are then φ and its conjugate $\bar{\varphi} = -1/\varphi = (1-\sqrt{5})/2$. It follows from simple induction that $\varphi^n = F_n\varphi + F_{n-1}$ and $\bar{\varphi}^n = F_n\bar{\varphi} + F_{n-1}$. Hence,

$$P_n = F_{n+1} = \frac{\varphi^{n+1} - \bar{\varphi}^{n+1}}{\varphi - \bar{\varphi}} = \frac{\varphi}{\sqrt{5}}\varphi^n + \frac{1}{\varphi\sqrt{5}}\bar{\varphi}^n.$$

In particular, $\beta = \varphi$, and $\lambda = \varphi/\sqrt{5}$, and we rediscover Theorem 3.8 from Theorem 3.10.

Focusing on the Fibonacci numbers, the paper [9] proofs Theorem 3.8 before generalizing the method to prove Theorem 3.10. Since the argument is slightly simpler for the Fibonacci numbers, this also provides the reader with an easier and perhaps more intuitive proof than for the general setting. Both proofs are inspired by the arguments used by de Bruijn [5].

3.2.1 Paper 8: Asymptotics for partitions over the Fibonacci numbers and related sequences

Below, the reader will find the most recent preprint of the paper [9], which is joint work between Michael Coons, Simon Kristensen and the current author. The paper is currently under review and is to appear in *Combinatorics and Number Theory*. The preprint is available on arXiv through the link `https://arxiv.org/abs/2312.07404v3` or by using the arXiv identifier 2312.07404. It has a length of 20 pages, numbered 1 through 20.

ASYMPTOTICS FOR PARTITIONS OVER THE FIBONACCI NUMBERS AND RELATED SEQUENCES

MICHAEL COONS, SIMON KRISTENSEN, AND MATHIAS L. LAURSEN

ABSTRACT. In this paper, harkening back to ideas of Hardy and Ramanujan, Mahler and de Bruijn, with the addition of more recent results on the Fibonacci Dirichlet series, we determine the asymptotic number of ways $p_F(n)$ to write an integer as the sum of non-distinct Fibonacci numbers. This appears to be the first such asymptotic result concerning non-distinct partitions over Fibonacci numbers. As well, under weak conditions, we prove analogous results for a general linear recurrences.

1. INTRODUCTION

We consider the number $p_F(n)$ of non-distinct partitions of n over the Fibonacci sequence, F_k. Specifically, for a positive integer n, $p_F(n)$ is the number of solutions of

$$n = a_2F_2 + a_3F_3 + \cdots + a_kF_k + \cdots, \tag{1}$$

in nonnegative integers a_k. In recent work, Chow and Slattery [2] gave results on the number distinct partitions, $q_F(n)$ over the Fibonacci sequence, and noted that their work shows that neither $q_F(n)$, nor its partial sums, have a 'nice' asymptotic formula. In particular, they showed that there is some oscillation in the partial sums of $q_F(n)$, and they gave bounds on these oscillations. In a similar vein, very recently, Sana [14] showed that there are also oscillations in the partial sums of the powers $(q_F(n))^m$ for each m. In this paper, with a related motivation, we determine asymptotic formulas for $p_F(n)$ and describe the oscillations that occur.

The asymptotic theory of partitions goes back to the celebrated result Hardy and Ramanujan who, in 1918, showed that the number of ways $p(n)$ to write a positive integer as the sum of positive integers satisfies $p(n) \sim (4n\sqrt{3})^{-1}e^{\pi\sqrt{2n/3}}$, as $n \to \infty$. Here, $p(n)$ has an asymptotic with a non-oscillating main term. Mahler [8] and de Bruijn [1] encountered a partition asymptotic with an oscillatory main term—they considered the number of ways $p_r(n)$ of writing n as the sum of non-distinct rth powers, for a positive integer $r \geqslant 2$. Here, we contribute the following result.

Theorem 1. *Let $p_F(n)$ be the number of partitions of n over non-distinct Fibonacci numbers. Then, as $n \to \infty$,*

$$\log p_F(n) \sim \frac{(\log n)^2}{2\log\varphi}.$$

In particular,

$$p_F(n) = A_F\left(\frac{\log n}{\log\varphi}\right) n^{B_F(n)}(\log n)^{C_F(n)}\left(1 + O\left(\frac{(\log\log n)^2}{\log n}\right)\right),$$

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where

$$A_F(x) := \sqrt{\frac{1}{2\pi \log \varphi}} \cdot \exp\left(\frac{(\log\log\varphi)^2}{2\log\varphi} + \log\log\varphi\left(\frac{c_3}{\log\varphi} - 1\right) + c_2 + 2\gamma + \psi_1(x)\right),$$

$$B_F(n) := \frac{1}{2\log\varphi}\left(\log n - 2\log\log n - \frac{4\psi_0\left(\frac{\log n}{\log\varphi}\right)}{\log\varphi} + 2\log\log\varphi + 2c_3 - 4\log\varphi + 2\right),$$

$$C_F(n) := \frac{1}{2\log\varphi}\left(\log\log n - \frac{4\psi_0\left(\frac{\log n}{\log\varphi}\right)}{\log\varphi} + 2\log\log\varphi - 2c_3 + 3\log\varphi + 4\right),$$

$$c_2 := \frac{3\gamma^2}{2\log\varphi} - \frac{\gamma_1}{\log\varphi} + \frac{\pi^2}{12\log\varphi} + 2\gamma\left(\frac{\log 5 - \log\varphi}{2\log\varphi}\right) + \sum_{k\geqslant 1}\frac{(-1)^k}{k(\varphi^{2k} + (-1)^{k+1})}, \text{ and}$$

$$c_3 := \frac{1}{2}\log 5 - \frac{1}{2}\log\varphi + 2\gamma,$$

where φ is the golden mean, γ is the Euler–Mascheroni constant, γ_1 is the first Stieltjes constant, and $\psi_0(x)$ and $\psi_1(x)$ are explicitly computable 1-periodic functions.

Theorem 1 seems to be the first asymptotic result chracterising non-distinct partitions over Fibonacci numbers. Distinct partitions over Fibonacci numbers have received considerable attention; see [2] and the references therein for details. In very recent work, addressing a question of Chow and Slattery [2], Kempton [7] has shown that $n^{-\frac{\log 2}{\log\varphi}}\sum_{m\leqslant n} q_F(m)$ is log-periodic. Our proof of Theorem 1 gives a way to describe the related log-periodic function for $p_F(n)$.

Our main result on Fibonacci partitions followa from a result of Coons and Kirsten [3], which uses a saddle-point method, which itself was inspired by the work of Nanda [9] and Richmond [12, 13]. In particular, Richmond [12] was not only able to give a new proof of Hardy and Ramanujan's above-mentioned result, he gave asymptotics for all of the moments of $p(n)$. The method therein, and herein, relies on the existence of invertible asymptotics for the related saddle point. In our situation, the lead order asymptotic of the saddle point is monotonic, so we have invertibility, but, additionally, we are able compute the second-order term, which is oscillatory. These terms are asymptotically close enough, so that they both contribute to the outcome of Theorem 1. We note that our results are heavily related to analytic properties of the Fibonacci zeta function (defined and discussed in more detail below); in particular, the Fibonacci zeta function is defined by a Dirichlet series which converges in the positive right half-plane. This presents added difficulties compared to the more fully-examine case of partitions related to Dirichlet series whose abscissa of convergence is strictly positive—for an interesting study on the asymptotics of partitions related to to Dirichlet series whose abscissa of convergence is strictly positive, see Debruyne and Tenenbaum [4].

This paper is organised as follows. In Sections 2 and 3, we focus the proof of Theorem 1. In particular, inspired by ideas of de Bruijn [1], we prove an exact formula for the generating function of $p_F(n)$ in Section 2. We then use a saddle-point method in Section 3 to give the asymptotic result for $p_F(n)$. Finally, in Section 4, we give the complete extension of Theorem 1 to the case of a linear recurrence with dominant root. In particular, suppose that P_k is a strictly increasing linearly recurrent sequence of positive integers of degree at least 2 with $P_1 = 1$, such that the characteristic polynomial $\chi_P(x)$ of P_k has a single dominant root $\beta > 0$ and $P_k = \lambda\beta^k + \lambda_2\beta_2^k + \cdots + \lambda_r\beta_r^k$, where $\lambda, \lambda_2, \ldots, \lambda_r$ are constants and $\beta_2, \ldots, \beta_r$ are the algebraic conjugates of β, then we have

Theorem 2. *Let $p_P(n)$ be the number of partitions of n over non-distinct elements of the sequence P_k. Then, as $n \to \infty$,*

$$p_P(n) = A_P\left(\frac{\log n}{\log \beta}\right) n^{B_P(n)} (\log n)^{C_P(n)} \left(1 + O\left(\frac{(\log\log n)^2}{\log n}\right)\right),$$

where $A_P(x)$ is the a positive 1-periodic function satisfying

$$A_P(x) := \sqrt{\frac{1}{2\pi \log \beta}} \cdot \exp\left(\frac{(\log\log\beta)^2}{2\log\beta} + (2\gamma - \log\lambda)\frac{\log\log\beta}{\log\beta} - \frac{1}{2}\log\log\beta + C_2 + \psi_4(x)\right),$$

$$B_P(n) := \frac{1}{2\log\beta}\left(\log n - 2\log\log n - \frac{4\psi_3\left(\frac{\log n}{\log\beta}\right)}{\log\beta} + 2\log\log\beta + 4\gamma - 2\log\lambda - 2\log\beta + 2\right),$$

$$C_P(n) := \frac{1}{2\log\beta}\left(\log\log n + 2 - \frac{4\psi_3\left(\frac{\log n}{\log\beta}\right)}{\log\beta} + 2\log\log\beta - 4\gamma + 2\log\lambda + 2\log\beta + 2\right),$$

where $\psi_3(x)$ and $\psi_4(x)$ are explicitly computable 1-periodic functions, and C_2 is the constant defined in Proposition 9.

2. An exact formula for for the generating series of $p_F(n)$

To prove Theorem 1, we study the asymptotics of the generating series

$$F_2(x) := \sum_{n\geqslant 0} p_F(n)x^n = \prod_{k\geqslant 2}\left(1 - x^{F_k}\right)^{-1}$$

as $x \to 1^-$. Note that we are starting with $F_2 = 1$, and not with $F_1 = 1$, since we wish to avoid having two representations of 1 in our partitions. We will necessarily need to consider a Fibonacci Dirichlet series. Navas [10] determined most of the properties we need by considering the analytic continuation of the series $\zeta_F(z) = \sum_{k\geqslant 1} F_k^{-z}$, but here with 1 doubly represented. It turns out this is not much of a problem. To deal with this, we merely consider the product

$$F(x) = (1-x)^{-1}F_2(x) = \prod_{k\geqslant 1}\left(1 - x^{F_k}\right)^{-1}, \tag{2}$$

then translate the results back to $F_2(x)$. It is convenient to change variables, setting $x = e^{-s}$, so that we are considering the function $F(e^{-s})$, with particular interest in the asymptotics as $s \to 0^+$.

Taking the logarithm and using the Taylor series of the logarithm near 1,

$$\log F(e^{-s}) = -\sum_{k=1}^{\infty}\log\left(1 - e^{-sF_k}\right) = \sum_{k=0}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m}e^{-sF_k m}. \tag{3}$$

Mellin's formula for the exponential function states that for $a > 0$ and $w > 0$,

$$e^{-w} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Gamma(z)w^{-z}dz.$$

Inserting this into (3) and interchanging integration and summation, as we may by absolute convergence, and finally rearranging the sum,

$$\log F(e^{-s}) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\sum_{k=0}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m}\Gamma(z)\left(sF_k m\right)^{-z}dz = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}s^{-z}\Gamma(z)\zeta(1+z)\zeta_F(z)dz, \tag{4}$$

where $a > 0$, $\zeta(z+1)$ denotes the Riemann ζ-function at $z+1$, and, as above $\zeta_F(z) := \sum_{k\geqslant 1} F_k^{-z}$. Note that $\zeta_F(z)$ is absolutely convergent for $\Re(z) > 0$ and continuable to a meromorphic function on all of $\mathbb{C}$; this is discussed more below—see Navas [10].

We would like to estimate the final integral in (4) using Cauchy's formula; that is, we would like to move the vertical contour towards $-\infty$. Applying the functional equation of the Riemann ζ-function, we find that

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_{-n+\frac{1}{2}-i\infty}^{-n+\frac{1}{2}+i\infty} s^{-z}\Gamma(z)\zeta(1+z)\zeta_F(z)dz = 0,$$

where n runs over the positive integers, and, as we shall see in what follows, the vertical line avoids poles of the integrand. Also, contributions from horizontal paths of integration do not contribute in the limit, as they are moved up or down respectively. Thus, the integral in (4) is nothing but the sum over the residues at the poles of the integrand. In what follows, we prove

Theorem 3. *The function $F_2(z)$ defined above satisfies, as $s \to 0^+$,*

$$\log F_2(e^{-s}) = \frac{(\log s)^2}{2\log\varphi} - (\log s)\left(\frac{c_3}{\log\varphi} - 1\right) + c_2 + 2\gamma + f(s) + O(s^2).$$

where $f(s) = f(\varphi s) + O(s^2)$, $c_3 := \frac{1}{2}\log 5 - \frac{1}{2}\log\varphi + 2\gamma$, and

$$c_2 := \frac{3\gamma^2}{2\log\varphi} - \frac{\gamma_1}{\log\varphi} + \frac{\pi^2}{12\log\varphi} + 2\gamma\left(\frac{\log 5 - \log\varphi}{2\log\varphi}\right) + \sum_{k\geqslant 1} \frac{(-1)^k}{k(\varphi^{2k} + (-1)^{k+1})}.$$

The proof of Theorem 3 will come as a direct application of five propositions, each having to do with contributions coming from certain singularities of $s^{-z}\Gamma(z)\zeta(1+z)\zeta_F(z)$, and one lemma. To this end, note that for $s > 0$, the function $z \mapsto s^{-z}$ can be expanded in a Taylor series as

$$s^{-z} = 1 - z\log s + \tfrac{1}{2}(\log s)^2 z^2 + O(z^3), \tag{5}$$

which converges for all $z \in \mathbb{C}$. The Γ-function has simple poles at non-positive integers, where the residue at $-n$ is

$$\operatorname*{Res}_{s=-n}\{\Gamma(s)\} = \frac{(-1)^n}{n!}.$$

The function $\zeta(z+1)$ has a simple pole with residue 1 at $z = 0$, and the trivial zeros of $\zeta(z+1)$ at all points in $-2\mathbb{N} - 1$ will cancel the corresponding poles of Γ.

The function $\zeta_F(z)$ is the most mysterious of the functions we will consider here, but much is known due to work of Navas [10]. We will use several of these properties, which we have gathered into the following proposition.

Proposition 1 (Navas, 2001)**.** *The Dirichlet series $\zeta_F(z)$ can be continued analytically to a meromorphic function on all of $\mathbb{C}$, still called $\zeta_F(z)$, whose singularities are simple poles at $s = s(n,k) = -2k + \frac{\pi i(2n+k)}{\log\varphi}$, for $n, k \in \mathbb{Z}$ and $k \geqslant 0$, with*

$$\operatorname*{Res}_{z=s}\{\zeta_F(z)\} = \frac{(-1)^k 5^{s/2}\binom{-s}{k}}{\log\varphi}.$$

Moreover, we have that $\zeta_F(-(4m+2)) = 0$ for each $m \in \mathbb{N}_0$, and $\zeta_F(-1) = -1$.

The poles of $\zeta_F(z)$ collude with the poles of $s^{-z}\Gamma(z)\zeta(1+z)$ to become poles of higher order. Combining our knowledge of the functions s^{-z}, $\Gamma(z)$ and $\zeta(1+z)$ with Proposition 1, the integrand $s^{-z}\Gamma(z)\zeta(1+z)\zeta_F(z)$ has

- a triple pole at $z = 0$,
- a simple pole coming from $\Gamma(z)$ at $z = -1$,
- double poles at $z \in -4\mathbb{N}$, and
- simple poles off the real line at $z = s(n,k)$, for $k \geqslant 0$ and $k \neq -2n$.

Note that the poles of $\Gamma(z)$ at $z=-(4m+2)$ for $m\in\mathbb{N}_0$ are cancelled by the corresponding zero of $\zeta_F(z)$. The higher order poles require more terms in the expansions of $\Gamma(z)$, $\zeta(1+z)$ and $\zeta_F(z)$. We will consider these in the order above.

To calculate the contribution of the triple pole at $z=0$, we need the first three terms of the Laurent expansions of the contributing functions, i.e., of $\Gamma(z)$, $\zeta(z+1)$ and $\zeta_F(z)$. The former two are well known,

$$\Gamma(z)=\frac{1}{z}+\gamma+\tfrac{1}{2}\left(\gamma^2+\frac{\pi^2}{6}\right)z+O(z^2), \tag{6}$$

where γ is the Euler–Mascheroni constant, and

$$\zeta(1+z)=\frac{1}{z}+\gamma-\gamma_1 z+O(z^2), \tag{7}$$

where γ_1 is the first Stieltjes constant,

$$\gamma_1=\lim_{m\to\infty}\left(\sum_{k=1}^{m}\frac{\log k}{k}-\tfrac{1}{2}(\log m)^2\right).$$

For $\zeta_F(z)$, results beyond the residue are not available in literature, so we present them here.

Lemma 1. *Near $z=0$, we have*

$$\zeta_F(z)=\frac{1}{\log\varphi}\cdot\frac{1}{z}+\frac{\log 5-\log\varphi}{2\log\varphi}+\left(\frac{\log\varphi-3\log 5}{12}+\frac{(\log 5)^2}{8\log\varphi}+c_1\right)z+O(z^2),$$

where $c_1:=\sum_{k\geqslant 1}\frac{(-1)^k}{k(\varphi^{2k}+(-1)^{k+1})}\approx -0.20436188.$

Proof. The first two terms were found in [10]. For clarity, we repeat the same process here. As a first step, noting $\bar\varphi=-1/\varphi$ and $F_n=(\varphi^n-\bar\varphi^n)/\sqrt{5}$, we have

$$\zeta_F(z)=\sum_{n\geqslant 1}\frac{1}{F_n^z}=5^{z/2}\sum_{n\geqslant 1}\frac{1}{(\varphi^n-\bar\varphi^n)^z}. \tag{8}$$

We recover the Taylor expansion of $5^{z/2}$ from that of the exponential function,

$$5^{z/2}=e^{\log(5)z/2}=\sum_{k\geqslant 0}\frac{(\log 5)^k}{2^k k!}z^k=1+\frac{\log 5}{2}z+\frac{(\log 5)^2}{8}z^2+O(z^3). \tag{9}$$

For the series, we start with an application of the binomial theorem to give

$$\sum_{n\geqslant 1}\frac{1}{(\varphi^n-\bar\varphi^n)^z}=\sum_{n\geqslant 1}\frac{1}{\varphi^{nz}}\left(1+(-1)^{n+1}\varphi^{-2n}\right)^{-z}=\sum_{n\geqslant 1}\frac{1}{\varphi^{nz}}\sum_{k\geqslant 0}\binom{-z}{k}(-1)^{k(n+1)}\varphi^{-2nk}.$$

Since the double sum is absolutely convergent, as argued in [10], we swap the order of summation and recognise a geometric series to give

$$\begin{aligned}\sum_{n\geqslant 1}\frac{1}{(\varphi^n-\bar\varphi^n)^z}&=\sum_{k\geqslant 0}\binom{-z}{k}(-1)^k\sum_{n\geqslant 1}\left((-1)^k\varphi^{-(z+2k)}\right)^n\\&=\sum_{k\geqslant 0}\binom{z+k-1}{k}\frac{(-1)^k\varphi^{-(z+2k)}}{1-(-1)^k\varphi^{-(z+2k)}}\\&=\sum_{k\geqslant 0}\frac{\Gamma(z+k)}{\Gamma(z)\Gamma(k+1)}\cdot\frac{(-1)^k}{\varphi^{z+2k}+(-1)^{k+1}}\\&=\frac{1}{\varphi^z-1}+\sum_{k\geqslant 1}\frac{\Gamma(z+k)}{\Gamma(z)k!}\cdot\frac{(-1)^k}{\varphi^{z+2k}+(-1)^{k+1}}.\end{aligned}$$

The first term is particularly cumbersome, but a few applications of L'Hôpital's rule gives

$$\frac{1}{\varphi^z - 1} = \frac{1}{\log\varphi}\cdot\frac{1}{z} - \frac{1}{2} + \frac{\log\varphi}{12}z + O(z^2).$$

For the remaining summands, by $\frac{1}{\Gamma(z)} = z + O(z^2)$, $\Gamma(z+k) = \Gamma(k) + O(z) = (k-1)! + O(z)$, and

$$\frac{(-1)^k}{\varphi^{z+2k} + (-1)^{k+1}} = \frac{(-1)^k}{\varphi^{2k} + (-1)^{k+1}} + O(z),$$

we have that

$$\sum_{n\geqslant 1}\frac{1}{(\varphi^n - \bar{\varphi}^n)^z} = \frac{1}{\log\varphi}\cdot\frac{1}{z} - \frac{1}{2} + \left(\frac{\log\varphi}{12} + c_1\right)z + O(z^2),$$

where c_1 is defined as in the statement of the lemma. Thus, using (9),

$$\begin{aligned}\zeta_F(z) &= 5^{z/2}\left(\frac{1}{\log\varphi}\cdot\frac{1}{z} - \frac{1}{2} + \left(\frac{\log\varphi}{12} + c_1\right)z + O(z^2)\right)\\ &= \frac{1}{\log\varphi}\cdot\frac{1}{z} + \left(-\frac{1}{2} + \frac{\log 5}{2\log\varphi}\right) + \left(\frac{\log\varphi}{12} + c_1 - \frac{\log 5}{4} + \frac{(\log 5)^2}{8\log\varphi}\right)z + O(z^2)\\ &= \frac{1}{\log\varphi}\cdot\frac{1}{z} + \frac{\log 5 - \log\varphi}{2\log\varphi} + \left(\frac{\log\varphi - 3\log 5}{12} + \frac{(\log 5)^2}{8\log\varphi} + c_1\right)z + O(z^2),\end{aligned}$$

which is the desired result. □

Proposition 2. *We have*

$$\operatorname*{Res}_{z=0}\left\{s^{-z}\Gamma(z)\zeta(1+z)\zeta_F(z)\right\} = \frac{(\log s)^2}{2\log\varphi} - \left(\frac{1}{2}\log 5 - \frac{1}{2}\log\varphi + 2\gamma\right)\frac{\log s}{\log\varphi} + c_2,$$

where, as in Theorem 1,

$$c_2 := \frac{3\gamma^2}{2\log\varphi} - \frac{\gamma_1}{\log\varphi} + \frac{\pi^2}{12\log\varphi} + 2\gamma\left(\frac{\log 5 - \log\varphi}{2\log\varphi}\right) + \sum_{k\geqslant 1}\frac{(-1)^k}{k(\varphi^{2k} + (-1)^{k+1})}.$$

Proof. Around $z = 0$, using (5), (6) and (7), we have

$$s^{-z}\Gamma(z)\zeta(1+z) = \frac{1}{z^2} + \frac{2\gamma - \log s}{z} + \left(\frac{(\gamma - \log s)(3\gamma - \log s)}{2} - \gamma_1 + \frac{\pi^2}{12}\right) + O(z).$$

Thus, by Lemma 1, we have

$$\begin{aligned}\operatorname*{Res}_{z=0}\left\{s^{-z}\Gamma(z)\zeta(1+z)\zeta_F(z)\right\} &= \frac{1}{\log\varphi}\left(\frac{(\gamma - \log s)(3\gamma - \log s)}{2} - \gamma_1 + \frac{\pi^2}{12}\right)\\ &\quad + \frac{\log 5 - \log\varphi}{2\log\varphi}(2\gamma - \log s) + \left(\frac{\log\varphi - 3\log 5}{12} + \frac{(\log 5)^2}{8\log\varphi} + c_1\right).\end{aligned}$$

Gathering powers of $\log s$ finishes the proof. □

The simple pole at $z = -1$ is a straightforward calculation, using known values.

Proposition 3. *We have*

$$\operatorname*{Res}_{z=-1}\left\{s^{-z}\Gamma(z)\zeta(1+z)\zeta_F(z)\right\} = -\frac{s}{2}.$$

Proof. We calculate, using that $\operatorname*{Res}_{z=-1}\{\Gamma(z)\} = -1$, $\zeta(0) = -1/2$, and $\zeta_F(-1) = -1$ to give

$$\operatorname*{Res}_{z=-1}\left\{s^{-z}\Gamma(z)\zeta(1+z)\zeta_F(z)\right\} = s\,\zeta(0)\,\zeta_F(-1)\cdot\operatorname*{Res}_{z=-1}\{\Gamma(z)\} = s\left(\frac{-1}{2}\right)(-1)(-1) = -\frac{s}{2}.$$

For the calculation of $\zeta_F(-1)$ see Navas [10, Eq. 9] □

For the double poles at $z \in -4\mathbb{N}$, we require the first two terms of the Laurent expansions of $\Gamma(z)$ and $\zeta_F(z)$ respectively around these points. For $\Gamma(z)$, the first term of the expansion is well known to be $\frac{1}{(4n)!} \cdot \frac{1}{z+4n}$. The constant term is surprisingly difficult to find in literature, but it can be easily calculated as the derivative of $(z+4n)\Gamma(z)$, evaluated at $z = -4n$. In order to accomplish this, we repeatedly apply the functional equation $z\Gamma(z) = \Gamma(z+1)$ to find that

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \cdots = \frac{\Gamma(z+4n+1)}{z(z+1)\cdots(z+4n)},$$

so that we only need to evaluate the derivative

$$\frac{d}{dz}\left\{\frac{\Gamma(z+4n+1)}{z(z+1)\cdots(z+4n-1)}\right\}$$

at $z = -4n$. Recalling that $\Gamma(1) = 1$ and $\Gamma'(1) = -\gamma$, we find that the constant term of the Laurent series of $\Gamma(z)$ around $z = -4n$ is

$$\frac{d}{dz}\left\{\frac{\Gamma(z+4n+1)}{z(z+1)\cdots(z+4n-1)}\right\}\Big|_{z=-4n} = \frac{1}{(4n)!}\left(\sum_{k=1}^{4n}\frac{1}{k} - \gamma\right),$$

so that near $z = -4n$,

$$\Gamma(z) = \frac{1}{z+4n} \cdot \frac{1}{(4n)!} + \frac{1}{(4n)!}\left(\sum_{k=1}^{4n}\frac{1}{k} - \gamma\right) + O(z+4n). \tag{10}$$

We also require the constant term of the Laurent series of $\zeta_F(z)$ around $z = -4n$. Following Navas [10],

$$\begin{aligned}\zeta_F(z) &= 5^{z/2}\sum_{k=0}^{\infty}\binom{-z}{k}\frac{(-1)^k}{\varphi^{z+2k} + (-1)^{k+1}} = 5^{z/2}\sum_{k=0}^{\infty}\frac{\Gamma(1-z)}{\Gamma(1-z-k)k!} \cdot \frac{(-1)^k}{\varphi^{z+2k} + (-1)^{k+1}} \\ &= 5^{z/2}\frac{\Gamma(1-z)}{\Gamma(1-z-2n)(2n)!} \cdot \frac{1}{\varphi^{z+4n} - 1} + 5^{z/2}\sum_{\substack{k=0 \\ k\neq 2n}}^{\infty}\frac{\Gamma(1-z)}{\Gamma(1-z-k)k!} \cdot \frac{(-1)^k}{\varphi^{z+2k} + (-1)^{k+1}}.\end{aligned}$$

The second term is holomorphic at $z = -4n$ and contributes to the constant term with its value, which we denote by c_{-4n}. For the first term, we note that

$$\lim_{z\to -4n} 5^{z/2} = 5^{-2n}, \qquad \text{and} \qquad \lim_{z\to -4n}\frac{\Gamma(1-z)}{\Gamma(1-z-2n)(2n)!} = \frac{(4n)!}{((2n)!)^2}.$$

Thus, it remains to note that

$$\lim_{z\to -4n}\frac{d}{dz}\left\{\frac{z+4n}{\varphi^{z+4n} - 1}\right\} = -\frac{1}{2},$$

which is easily shown by differentiating and applying L'Hôpital's rule twice. Proposition 1 gives that the residue of $\zeta_F(z)$ at $z = -4n$ is

$$b_{-4n} := \operatorname*{Res}_{z=-4n}\{\zeta_F(z)\} = \frac{5^{-2n}}{\log\varphi}\binom{4n}{2n} = \frac{5^{-2n}}{\log\varphi} \cdot \frac{(4n)!}{((2n)!)^2},$$

so that near $z = -4n$, we have

$$\zeta_F(z) = \frac{b_{-4n}}{z+4n} + \left(c_{-4n} - \frac{5^{-2n}}{2} \cdot \frac{(4n)!}{((2n)!)^2}\right) + O(z+4n). \tag{11}$$

Proposition 4. *We have*

$$g(s) := \sum_{n=1}^{\infty}\operatorname*{Res}_{z=-4n}\left\{s^{-z}\Gamma(z)\zeta(1+z)\zeta_F(z)\right\} = \sum_{n=1}^{\infty}\alpha_n\, s^{4n} - \log s\sum_{n=1}^{\infty}\beta_n s^{4n},$$

where $\beta_n := b_{-4n}\zeta(1-4n)/(4n)!$,

$$\alpha_n := \frac{B_{2n}}{4n\cdot(4n)!}\left(b_{-4n}\left(\sum_{k=1}^{4n}\frac{1}{k}-\gamma\right)+c_{-4n}-\frac{5^{-2n}}{2}\frac{(4n)!}{((2n)!)^2}\right)+\frac{b_{-4n}}{(4n)!}\cdot\zeta'(1-4n),$$

and B_{2n} *is the* $2n$*-th Bernoulli number. Moreover, as* $s\to 0^+$*, we have* $g(s)=O(s^2)$*.*

Proof. The first part of the proposition follows from (10), (11), and the expansions, around $z=-4n$,

$$s^{-z}=s^{4n}s^{-(z+4n)}=s^{4n}-(z+4n)\log s+O((z+4n)^2),$$

and $\zeta(1+z)=\zeta(1-4n)+\zeta'(1-4n)(z+4n)+O((z+4n)^2)$ along with the fact that $\zeta(1-4n)=B_{2n}/4n$; see Titchmarch [16, p. 19].

For the second part, we start by using that

$$\zeta'(1-4n)=\frac{(-1)^{k+1}(4n)!}{2n(2\pi)^{2n}}\zeta'(4n)+\frac{B_{4n}}{4n}\left(\sum_{k=1}^{4n-1}\frac{1}{k}-\gamma-\log(2\pi)\right).$$

Using the facts that $\zeta'(4n)\sim -2^{-4n}\log 2$ and $|B_{4n}|\sim\frac{2(4n)!}{(2\pi)^{4n}}$ with Stirling's approximation of the factorial, we have that

$$\frac{b_{-4n}}{(4n)!}\cdot\zeta'(1-4n)=O\left(\frac{(4n)!}{(40\pi)^{2n}n((2n)!)^2}\right)=O\left(\frac{1}{(10\pi)^{2n}n^{3/2}}\right)=O(1).$$

Now, we have at hand asymptotic information about all of the quantities in α_n except for c_{-4n}. It turns out that c_{-4n} is uniformly bounded; more precisely $|c_{-4n}|<3$. To see this, note that for all $n\geqslant 1$, we have

$$|c_{-4n}|=\left|\frac{1}{5^{2n}}\sum_{\substack{k=0\\k\neq 2n}}^{\infty}\binom{4n}{k}\cdot\frac{(-1)^k}{\varphi^{-4n+2k}+(-1)^{k+1}}\right|=\frac{1}{5^{2n}}\sum_{\substack{k=0\\k\neq 2n}}^{4n}\binom{4n}{k}\cdot\varphi^{4n-2k}\left|\frac{1}{1-\varphi^{4n-2k}}\right|$$

since $\binom{4n}{k}=0$ for $k>4n$. Now the value of $\left|\frac{1}{1-\varphi^{4n-2k}}\right|$ is maximal when $k=2n-1$ (recall, $k\neq 2n$), and there, it is approximately 2.6180, which yields

$$|c_{-4n}|<\frac{3}{5^{2n}}\sum_{\substack{k=0\\k\neq 2n}}^{4n}\binom{4n}{k}\cdot\varphi^{4n-2k}\leqslant\frac{3}{5^{2n}}\sum_{k=0}^{4n}\binom{4n}{k}\cdot\varphi^{4n-k}\left(\frac{1}{\varphi}\right)^k=\frac{3}{5^{2n}}\left(\varphi+\frac{1}{\varphi}\right)^{4n}=3,$$

which shows that c_{-4n} is uniformly bounded. With this in hand, we will now use the fact that $|B_{2n}|=\frac{2(2n)!}{(2\pi)^{2n}}\zeta(2n)$ and $\zeta(2n)\sim 1$ as $n\to\infty$, to finish our proof. To this end, using the definition of b_{-4n} and Stirling's approximation, we have

$$\begin{aligned}
|\alpha_n| &< \frac{|B_{2n}|}{4n\cdot(4n)!}\left(\frac{5^{-2n}}{\log\varphi}\cdot\frac{(4n)!}{((2n)!)^2}\left(\sum_{k=1}^{4n}\frac{1}{k}-\gamma\right)+3+\frac{5^{-2n}}{2}\frac{(4n)!}{((2n)!)^2}\right)+O(1)\\
&=\zeta(2n)\frac{2(2n)!}{4n\cdot(2\pi)^{2n}\cdot(4n)!}\left(\frac{5^{-2n}}{\log\varphi}\cdot\frac{(4n)!}{((2n)!)^2}\left(\sum_{k=1}^{4n}\frac{1}{k}-\gamma\right)+3+\frac{5^{-2n}}{2}\frac{(4n)!}{((2n)!)^2}\right)+O(1)\\
&\sim\frac{2(2n)!}{4n\cdot(2\pi)^{2n}\cdot(4n)!}\left(\frac{5^{-2n}}{\log\varphi}\cdot\frac{(4n)!}{((2n)!)^2}\log(4n)+3+\frac{5^{-2n}}{2}\frac{(4n)!}{((2n)!)^2}\right)+O(1)\\
&=\frac{2}{4n\cdot(2\pi)^{2n}}\left(\frac{5^{-2n}}{\log\varphi}\cdot\frac{\log(4n)}{(2n)!}+\frac{3\cdot(2n)!}{(4n)!}+\frac{5^{-2n}}{2(2n)!}\right)+O(1)\\
&\sim\frac{2}{4n\cdot(2\pi)^{2n}}\cdot\frac{5^{-2n}}{\log\varphi}\cdot\frac{\log(4n)}{(2n)!}+O(1)\sim\frac{1}{4\sqrt{\pi}\log\varphi}\cdot\frac{\log n}{\sqrt{n}}\cdot\left(\frac{e}{20\pi n}\right)^{2n}+O(1)=O(1).
\end{aligned}$$

Similarly, we have that

$$|\beta_n| \sim \frac{1}{4\sqrt{\pi}\log\varphi} \cdot \frac{1}{\sqrt{n}} \cdot \left(\frac{e}{20\pi n}\right)^{2n} = O(1).$$

To see $g(s) = O(s^2)$ as $s \to 0^+$, we note that, using L'Hôpital's rule, we have $\lim_{s\to 0^+} s^2 \log s = 0$. Using this, along with an application of the first part of the proposition,

$$g(s) = \left(\alpha_1 s^4 - \beta_1 s^4 \log s\right)(1 + O(s^4)) = O(s^2). \qquad \square$$

Finally, at the simple poles of $\zeta_F(z)$ off the real line, which occur at $z = s(n,k)$ with $k \geqslant 0$ and $n \neq -k/2$, we note that $s^{-z}\Gamma(z)\zeta(1+z)$ is analytic at these points. Concerning the contributions from these singularities, we have the following result.

Proposition 5. *We have*

$$\begin{aligned} h(s) &:= \sum_{k=0}^{\infty} \sum_{\substack{n=-\infty \\ n\neq -k/2}}^{\infty} \operatorname*{Res}_{z=s(n,k)} \left\{s^{-z}\Gamma(z)\zeta(1+z)\zeta_F(z)\right\} \\ &= \frac{1}{\log\varphi} \sum_{k=0}^{\infty} \sum_{\substack{n=-\infty \\ n\neq -k/2}}^{\infty} (-1)^k \binom{-s(n,k)}{k} \left(\frac{s}{\sqrt{5}}\right)^{-s(n,k)} \Gamma(s(n,k))\,\zeta(1+s(n,k)). \end{aligned}$$

Moreover, $h(s) = h(\varphi s) + O(s^2)$ *as* $s \to 0^+$.

Proof. The form $h(s)$ of these contributions is immediate using Proposition 1. Also, since the terms in each sum over n is symmetric about the real axis, $h(s)$ is real. It remains to examine the analytic properties of $h(s)$ as a function of s.

We first consider the $k = 0$ term of $h(s)$, which we denote by $[k = 0]h(s)$. Noting that

$$\begin{aligned} \left(\frac{\varphi s}{\sqrt{5}}\right)^{\frac{i\pi(2n)}{\log\varphi}} &= \cos\left(\frac{2\pi n}{\log\varphi}\log\left(\frac{\varphi\alpha}{\sqrt{5}}\right)\right) + i\sin\left(\frac{2\pi n}{\log\varphi}\log\left(\frac{\varphi\alpha}{\sqrt{5}}\right)\right) \\ &= \cos\left(2\pi n + \frac{2\pi n}{\log\varphi}\log\left(\frac{\alpha}{\sqrt{5}}\right)\right) + i\sin\left(2\pi n + \frac{2\pi n}{\log\varphi}\log\left(\frac{\alpha}{\sqrt{5}}\right)\right) \\ &= \cos\left(\frac{2\pi n}{\log\varphi}\log\left(\frac{\alpha}{\sqrt{5}}\right)\right) + i\sin\left(\frac{2\pi n}{\log\varphi}\log\left(\frac{\alpha}{\sqrt{5}}\right)\right) = \left(\frac{s}{\sqrt{5}}\right)^{\frac{i\pi(2n)}{\log\varphi}}, \end{aligned}$$

we have that $[k = 0]h(\varphi s) = [k = 0]h(s)$.

For $k \neq 0$, using the functional equations for ζ and Γ, since we are examining complex values, there is a positive constant $d_{n,k}$ that is uniformly bounded such that

$$\begin{aligned} \left|\Gamma(s(n,k))\zeta(1+s(n,k))\right| &= \left|\frac{\zeta(-s(n,k))2^{s(n,k)}\pi^{s(n,k)+1}}{s(n,k)\sin\left(-\frac{s(n,k)\pi}{2}\right)}\right| \\ &= \left|\frac{\pi\cdot\zeta\left(2k - i\frac{\pi(2n+k)}{\log\varphi}\right)}{\left(-2k + i\frac{\pi(2n+k)}{\log\varphi}\right)(2\pi)^{2k}\sin\left(i\frac{\pi^2(2n+k)}{2\log\varphi}\right)}\right| \sim d_{n,k}\cdot\frac{e^{-\frac{\pi^2 k}{2\log\varphi}}}{(2\pi)^{2k}}\cdot e^{-\frac{\pi^2|n|}{\log\varphi}}, \end{aligned}$$

where we have used that $|\sin(z)|$ is π-periodic in the $\Re(z)$, that $|e^{i\theta}| = 1$ for all θ, and that, as k or $|n|$ (or both) grows, $\left|\sin\left(i\frac{\pi^2(2n+k)}{2\log\varphi}\right)\right| \sim e^{\frac{\pi^2 k}{2\log\varphi}}e^{\frac{\pi^2|n|}{\log\varphi}}/2$.

It remains to deal with the factor $\binom{-s(n,k)}{k}$. To this end, note that

$$\left|\binom{-s(n,k)}{k}\right| = \left|\frac{\Gamma(1-s(n,k))}{k!\,\Gamma(1-s(n,k)-k)}\right| = \frac{1}{k!}\left(\prod_{j=k+1}^{2k}\left(j^2+\left(\tfrac{\pi(2n+k)}{\log\varphi}\right)^2\right)\right)^{1/2}$$

$$= \frac{(kn)^k}{k!}\left(\prod_{j=k+1}^{2k}\left(\left(\frac{j}{nk}\right)^2+\left(\frac{\pi\left(\frac{2}{k}+\frac{1}{n}\right)}{\log\varphi}\right)^2\right)\right)^{1/2} \leqslant \frac{(kn)^k}{k!}\left(\left(\frac{2}{n}\right)^2+\left(\frac{\pi\left(\frac{2}{k}+\frac{1}{n}\right)}{\log\varphi}\right)^2\right)^{k/2},$$

and, independent of k,

$$\left(\left(\frac{2}{n}\right)^2+\left(\frac{\pi\left(\frac{2}{k}+\frac{1}{n}\right)}{\log\varphi}\right)^2\right)^{k/2} \xrightarrow{n\to\infty} \left(\frac{2\pi}{k\log\varphi}\right)^k.$$

Thus, there is a $d>0$, independent from n and k, such that any term of $h(s)$ with $k\neq 0$, satisfies,

$$\left|[k\neq 0]h(s)\right| \leqslant d\sum_{\substack{n=-\infty\\ n\neq -k/2}}^{\infty}\left(\frac{s}{\sqrt{5}}\right)^{2k}\frac{(kn)^k}{k!}\left(\frac{2\pi}{k\log\varphi}\right)^k\cdot d_{n,k}\cdot\frac{e^{-\frac{\pi^2 k}{2\log\varphi}}}{(2\pi)^{2k}}\cdot e^{-\frac{\pi^2|n|}{\log\varphi}}$$

$$= d\left(\frac{s}{\sqrt{5}}\right)^{2k}\frac{1}{k!}\left(\frac{1}{\log\varphi}\right)^k\frac{e^{-\frac{\pi^2 k}{2\log\varphi}}}{(2\pi)^k}\sum_{\substack{n=-\infty\\ n\neq -k/2}}^{\infty} d_{n,k}\cdot n^k\cdot e^{-\frac{\pi^2|n|}{\log\varphi}},$$

where $d_{n,k}$ here is different from above, but still a uniformly bounded positive constant. So, there is a positive constant d, also different from above, but still independent of n and k, such that

$$\left|[k\neq 0]h(s)\right| \leqslant d\left(\frac{s^2}{10\pi\cdot\log\varphi\cdot e^{\frac{\pi^2}{2\log\varphi}}}\right)^k\frac{1}{k!}\sum_{n=0}^{\infty} n^k e^{-\frac{\pi^2|n|}{\log\varphi}}$$

$$= d\left(\frac{s^2}{10\pi\cdot\log\varphi\cdot e^{\frac{\pi^2}{2\log\varphi}}}\right)^k\frac{1}{k!}\cdot\frac{e^{-\frac{\pi^2}{2\log\varphi}}A_k\left(e^{-\frac{\pi^2}{2\log\varphi}}\right)}{\left(1-e^{-\frac{\pi^2}{2\log\varphi}}\right)^{k+1}}$$

$$= d\left(\frac{s^2}{10\pi\cdot\log\varphi\cdot\left(e^{\frac{\pi^2}{2\log\varphi}}-1\right)}\right)^k\frac{1}{k!}\cdot\frac{A_k\left(e^{-\frac{\pi^2}{2\log\varphi}}\right)}{\left(e^{\frac{\pi^2}{2\log\varphi}}-1\right)}$$

$$< \frac{d}{\left(e^{\frac{\pi^2}{2\log\varphi}}-1\right)}\left(\frac{s^2}{10\pi\cdot\log\varphi\cdot\left(e^{\frac{\pi^2}{2\log\varphi}}-1\right)}\right)^k,$$

where, for $k\geqslant 1$, we have used that $\sum_{n\geqslant 0} n^k x^n = x\,A_k(x)(1-x)^{-(k+1)}$, where $A_k(x)\in\mathbb{Z}_{>0}[x]$ is the k-th Eulerian Polynomial, which satisfies $\deg A_k(x)=k-1$ and $A_k(1)=k!$. Hence, the terms $[k\neq 0]h(s)$ contribute $O(s^2)$ collectively, and the lemma follows. □

Lemma 2. *As $s\to 0^+$, we have*

$$\log(1-e^{-s}) = \log s - 2\gamma - \frac{s}{2} + O(s^2).$$

Proof. We follow the method above, writing

$$-\log(1-e^{-s}) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} s^{-z}\Gamma(z)\zeta(1+z)dz,$$

where, for now, $a>0$. We use the expansions of $\Gamma(z)$ and $\zeta(1+z)$ around $z=0$ from above, with the fact that the integrand has a simple pole at $z=-1$ coming from $\Gamma(z)$ to get that, as $s\to 0^+$,

$$-\log(1-e^{-s}) = -\log s + 2\gamma - s\zeta(0) + O(s^2)$$

which, since $\zeta(0) = -1/2$, when multiplied by -1, yields the desired result. □

Using the relationship in (2) and combining this lemma with the previous four propositions proves Theorem 3.

3. Fibonacci partitions $p_F(n)$ via the saddle point method

In this section, we prove our main result using a saddle point method. To achieve this, we must determine the behaviour as $s \to 0^+$ of each of the pieces in the expansion of $\log F(e^{-s})$.

We begin in the same way as Hardy and Ramanujan [6], using Cauchy's integral formula, but diverge from their argument almost immediately. We have

$$p_F(n) = \frac{1}{2\pi i}\int_{C_\varepsilon} \frac{F_2(z)}{z^{m+1}} dz = \frac{1}{2\pi i}\int_{C_\varepsilon} \frac{e^{n(-\log z + \frac{1}{n}\log F_2(z))}}{z} dz = \frac{1}{2\pi i}\int_{s.p.} e^{n(s+\frac{1}{n}\log F_2(e^{-s}))} ds,$$

where C_ε indicates a positively oriented circle of radius $\varepsilon \in (0,1)$ and *s.p.* indicates a path that goes through the saddle point $s = \alpha$ of the integrand, that is, the point $s = \alpha$ which is the solution of the equation

$$\frac{d}{ds}\left(s + \frac{1}{n}\log F_2(e^{-s})\right) = 0. \tag{12}$$

Our main result on Fibonacci partitions will follow from a result of Coons and Kirsten [3], which itself was inspired by the work of Nanda [9] and Richmond [12, 13].

Theorem 4 (Coons and Kirsten, 2009)**.** *If $\Lambda(x) = \prod_{k\geqslant 1}\left(1 - x^{\lambda_k}\right)^{-1}$ generates a sequence $p_\lambda(n)$, and $s = \alpha$ is the solution of (12) with $F_2(e^{-s})$ replaced by $\Lambda(e^{-s})$, then, as n tends to infinity,*

$$p_\lambda(n) = \frac{e^{n\alpha}\Lambda(e^{-\alpha})}{\sqrt{2\pi}}\left(\sqrt{\frac{1}{-\left.\frac{dn}{ds}\right|_{s=\alpha}}} + O\left(\frac{1}{n^{3/2}}\right)\right).$$

Here, α must be thought of as being replaced by its large-n asymptotic expansion so that the asymptotic of $p_\lambda(n)$ represents a large-n asymptotic.

Remark 1. The proof of Theorem 4 was accomplished by iteratively applying an asymptotic result on exponential integrals, which can be found in the book of Olver [11, p. 127, Theorem 7.1]. Note that Coons and Kirsten [3] use the notation $t^0_\Lambda(n)$ for our definition of $p_\lambda(n)$ above. ♢

Remark 2. The statement "α must be thought of as being replaced by its large-n asymptotic expansion" may seem a bit cumbersome, but, here, the point is that the solution of (12) gives n as a function of the saddle point α as an (asymptotically) monotonic function, so it is invertible; that is, there is a well-defined asymptotic for the saddle point α in terms n—this is precisely the method we employ.

To apply this Theorem 4, we must first determine the saddle point for large values of n. From (12), we have that

$$n = -\left.\frac{d}{ds}\log F_2(e^{-s})\right|_{s=\alpha}. \tag{13}$$

We will use this combined with Theorem 3 to prove the following result.

Lemma 3. *There exists a function $h_0(s)$ satisfying $h_0(s) = h_0(\varphi s)$ such that for sufficiently large n, or, equivalently, for sufficiently small $\alpha > 0$,*

$$n = \frac{-\log(\alpha)}{\alpha\log\varphi} + \frac{h_0(\alpha)}{\alpha} + O(\alpha).$$

Proof. Let $f(s) := g(s) + h(s)$, where $g(s)$ and $h(s)$ are as in Propositions 4 and 5, respectively. As $s \to 0^+$, using Proposition 4 and one application of L'Hôpital's rule gives $g(s) = O(s^2)$ (in fact, one gets that $g(s) = O(s^{4-\varepsilon})$ for any fixed small positive ε, but only $O(s^2)$ is necessary after later comparison with the asymptotics for $h(s)$). Now, collectively, the sum of all of the terms with $k > 0$ in the formula for $h(s)$ in Proposition 5 go to zero as $s \to 0^+$, since $\Re(-s(n,k)) = 2k > 0$. Thus, as $s = \alpha \to 0^+$, separating out the $k = 0$ term of the sum, denoting it $[k = 0]h(s)$, and noting that $s(n, 0) = \frac{2\pi i n}{\log\varphi}$, we have,

$$f(\alpha) = [k = 0]h(\alpha) + O(\alpha^2) = \frac{1}{\log\varphi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{\alpha}{\sqrt{5}}\right)^{-\frac{2\pi i n}{\log\varphi}} \Gamma(\tfrac{2\pi i n}{\log\varphi})\,\zeta(1 + \tfrac{2\pi i n}{\log\varphi}) + O(\alpha^2),$$

so that

$$f'(\alpha) = \frac{\sqrt{5}}{\alpha} \cdot [k = 0]h(\alpha) + O(\alpha).$$

The form of $[k = 0]h(s)$ immediately implies that $[k = 0]h(s) = [k = 0]h(\varphi s)$.

Now, we calculate

$$\begin{aligned} n &= -\frac{d}{ds} \log F_2(e^{-s})\Big|_{s=\alpha} \\ &= -\frac{d}{ds}\left(\frac{(\log s)^2}{2\log\varphi} - (\log s)\left(\frac{c_3}{\log\varphi} - 1\right) + c_2 + 2\gamma + f(s) + O(s^2)\right)\Bigg|_{s=\alpha} \\ &= -\frac{\log\alpha}{\alpha\log\varphi} - \frac{1}{\alpha}\left(\frac{c_3}{\log\varphi} - 1 - \sqrt{5}\cdot[k = 0]h(\alpha)\right) + O(\alpha). \qquad (14) \end{aligned}$$

setting $h_0(s) := \sqrt{5} \cdot [k = 0]h(s) + 1 - c_3/\log\varphi$ gives the result. ☐

Remark 3. Lemma 3 provides the leading two terms for n in terms of the saddle point α. Here, the first term shows that the relationship is asymptotically monotonic, so that it can be inverted. The second term is oscillatory. If one follows this method and tries to apply it to the distinct Fibonacci partitions function $q_F(n)$ the resulting asymptotic is not monotonic—the leading term is oscillatory. This is precisely why this method doesn't immediately generalise to distinct Fibonacci partitions.

While the above lemma gives n as a function of α, we necessarily need α as a function of n to apply the saddle point method. We achieve this via the Lambert W-function.

Proposition 6. *There is a continuous* 1*-periodic function* $\psi_0(x)$*, such that for sufficiently small* $\alpha > 0$*, or, equivalently, for sufficiently large* n*,*

$$\alpha = \frac{W\left(e^{\psi_0\left(\frac{\log n}{\log\varphi}\right)/\log\varphi} n/\log\varphi\right)}{n\log\varphi}\left(1 + O\left(\frac{\log n}{n^2}\right)\right),$$

where $W(x)$ *denotes Lambert's* W*-function.*

Proof. Note that the previous lemma gives that

$$n = \left(\frac{-\log(\alpha)}{\alpha\log\varphi} + \frac{h_0(\alpha)}{\alpha}\right)\left(1 + O\left(\frac{\alpha^2}{\log\alpha}\right)\right),$$

where $h_0(s)$ is fixed along any sequence $\{x\varphi^m\}_{m\geqslant 0}$. We use this property to invert the above relationship between α and n. Note that the relationship is invertible because the lead asymptotics are strictly monotonic. Now, when one inverts $n \sim A\frac{1}{\alpha}\log\frac{1}{\alpha} + B\frac{1}{\alpha}$, one gets $\alpha \sim A \cdot W\left(e^{B/A}\frac{n}{A}\right)/n \sim A\log n/n$, where $W(x)$ is Lambert's W-function. Doing this along the sequences $\{x\varphi^m\}_{m\geqslant 0}$ to ensure a constant $B = B(x)$, we then reconstruct, using a fundamental interval, say $x \in [\varphi, \varphi^2]$, to get a continuous 1-periodic function $\psi_0(x)$ such that for large n,

$$\alpha = \frac{W\left(e^{\psi_0\left(\frac{\log n}{\log \varphi}\right)/\log\varphi} n/\log\varphi\right)}{n\log\varphi}\left(1+O\left(\frac{\log n}{n^2}\right)\right).$$

Here, we have used the original relationship to find that $O(\alpha/\log\alpha) = O(1/n)$, and then the inverse, noting that $W(n) \sim \log n$. □

In what follows, we will use Proposition 6 to give asymptotics for several functions of α, including $n\alpha$, $\frac{dn}{ds}\big|_{s=\alpha}$, $\log\alpha$ and $(\log\alpha)^2$. To give our end result, we will need varying orders of precision for the asymptotics for each of these terms. In particular, first order asymptotics of the Lambert W-function will not be enough to deal with $(\log\alpha)^2$, though they will be enough for some terms so we record them below. For convenience, we note here that the Lambert W-function satisfies,

$$W(x) = \log x - \log\log x + \frac{\log\log x}{\log x} + O\left(\frac{\log\log x}{(\log x)^2}\right), \tag{15}$$

as $x \to \infty$; see, e.g., Corless et al. [5].

Corollary 1. *For sufficiently small $\alpha > 0$, or, equivalently, for sufficiently large n,*

$$\alpha = \frac{\log n}{n\log\varphi}\left(1+O\left(\frac{\log\log n}{\log n}\right)\right),$$

and

$$\log\alpha = -\log n + \log\log n - \log\log\varphi + O\left(\frac{\log\log n}{\log n}\right).$$

Proof. The first result follows directly from the fact that

$$W(x) = \log x(1 + O(\log\log x/\log x))$$

for all x sufficiently large. The second follows immediately from the first. □

Corollary 2. *For sufficiently small $\alpha > 0$, or, equivalently, for sufficiently large n,*

$$\left.\frac{dn}{ds}\right|_{s=\alpha} = \frac{-n^2\log\varphi}{\log n}\left(1+O\left(\frac{\log\log n}{\log n}\right)\right).$$

Proof. We start with Lemma 3 in the form $n = -\log(\alpha)/(\alpha\log\varphi) + O(1/\alpha)$, and take a derivative, then apply both parts of Corollary 1 to get

$$\left.\frac{dn}{ds}\right|_{s=\alpha} = \frac{\log\alpha}{\alpha^2\log\varphi} + O\left(\frac{1}{\alpha^2}\right) = \frac{\log\alpha}{\alpha^2\log\varphi}\left(1+O\left(\frac{1}{\log\alpha}\right)\right) = \frac{-n^2\log\varphi}{\log n}\left(1+O\left(\frac{\log\log n}{\log n}\right)\right),$$

which finishes the proof. □

The final term necessary is $(\log\alpha)^2$. Here, in addition to using the full asymptotic in (15), we will use the fact that for any y, $\log W(y) = \log y - W(y)$.

Corollary 3. *For sufficiently small $\alpha > 0$, or, equivalently, for sufficiently large n,*

$$(\log\alpha)^2 = \log n\left(\log n - 2\log\log n - \frac{4\psi_0\left(\frac{\log n}{\log\varphi}\right)}{\log\varphi} + 2\log\log\varphi\right)$$
$$+\log\log n\left(\log\log n + 2 - \frac{4\psi_0\left(\frac{\log n}{\log\varphi}\right)}{\log\varphi} + 2\log\log\varphi\right)$$
$$+(\log\log\varphi)^2 - \frac{2\psi_0\left(\frac{\log n}{\log\varphi}\right)}{\log\varphi} + 2\log\log\varphi + O\left(\frac{(\log\log n)^2}{\log n}\right).$$

Proof. We start with Proposition 6 and take the natural logarithm of both sides to obtain

$$(\log\alpha)^2 = \left(\log\left(\frac{W\left(e^{\psi_0\left(\frac{\log n}{\log\varphi}\right)/\log\varphi}n/\log\varphi\right)}{n\log\varphi}\right)\right)^2 + O\left(\frac{(\log n)^2}{n^2}\right), \tag{16}$$

since

$$\log\left(\frac{W\left(e^{\psi_0\left(\frac{\log n}{\log\varphi}\right)/\log\varphi}n/\log\varphi\right)}{n\log\varphi}\right) = O(\log n).$$

We now use the full force of (15) to give an asymptotic for the first term in (16), first noting that

$$\log\left(\frac{W\left(e^{\psi_0\left(\frac{\log n}{\log\varphi}\right)/\log\varphi}n/\log\varphi\right)}{n\log\varphi}\right) = \log\left(e^{\psi_0\left(\frac{\log n}{\log\varphi}\right)/\log\varphi}n/\log\varphi\right)$$
$$-\log(n\log\varphi) - W\left(e^{\psi_0\left(\frac{\log n}{\log\varphi}\right)/\log\varphi}n/\log\varphi\right)$$
$$= \frac{\psi_0\left(\frac{\log n}{\log\varphi}\right)}{\log\varphi} - W\left(e^{\psi_0\left(\frac{\log n}{\log\varphi}\right)/\log\varphi}n/\log\varphi\right).$$

Thus,

$$\left(\log\left(\frac{W\left(e^{\psi_0\left(\frac{\log n}{\log\varphi}\right)/\log\varphi}n/\log\varphi\right)}{n\log\varphi}\right)\right)^2 = \left(\frac{\psi_0\left(\frac{\log n}{\log\varphi}\right)}{\log\varphi}\right)^2 \tag{17}$$
$$-\frac{2\psi_0\left(\frac{\log n}{\log\varphi}\right)}{\log\varphi}W\left(e^{\psi_0\left(\frac{\log n}{\log\varphi}\right)/\log\varphi}n/\log\varphi\right) + W\left(e^{\psi_0\left(\frac{\log n}{\log\varphi}\right)/\log\varphi}n/\log\varphi\right)^2.$$

For the middle term we use the first three terms of the asymptotic of W in (15) and, for the square, we will use the square of all of (15), which is, as $x\to\infty$,

$$W(x)^2 = (\log x)^2 + (\log\log x)^2 - 2\log x\log\log x + 2\log\log x + O\left(\frac{(\log\log x)^2}{\log x}\right).$$

To this end, we to determine strong estimates for the asymptotics of $\log x$, $\log\log x$ and $\log x\log\log x$ with $x = e^{\psi_0\left(\frac{\log n}{\log\varphi}\right)/\log\varphi}n/\log\varphi$. Here, we have

$$\log\left(e^{\psi_0\left(\frac{\log n}{\log\varphi}\right)/\log\varphi}n/\log\varphi\right) = \log n + \frac{\psi_0\left(\frac{\log n}{\log\varphi}\right)}{\log\varphi} - \log\log\varphi \tag{18}$$
$$= \log n\left(1 + \frac{\psi_0\left(\frac{\log n}{\log\varphi}\right)/\log\varphi - \log\log\varphi}{\log n}\right),$$

and so, using that as $y \to 0$, $\log(1+y) = y - y^2/2 + O(y^3)$, we have

$$\log\log\left(e^{\psi_0\left(\frac{\log n}{\log\varphi}\right)/\log\varphi} n/\log\varphi\right) = \log\log n + \frac{\psi_0\left(\frac{\log n}{\log\varphi}\right)/\log\varphi - \log\log\varphi}{\log n} + O\left(\frac{1}{(\log n)^2}\right). \tag{19}$$

So, using the above asymptotics,

$$W\left(e^{\psi_0\left(\frac{\log n}{\log\varphi}\right)/\log\varphi} n/\log\varphi\right) = \log n + \log\log n + \frac{\psi_0\left(\frac{\log n}{\log\varphi}\right)}{\log\varphi} - \log\log\varphi + O\left(\frac{\log\log n}{\log n}\right), \tag{20}$$

and

$$\begin{aligned} W\left(e^{\psi_0\left(\frac{\log n}{\log\varphi}\right)/\log\varphi} n/\log\varphi\right)^2 &= (\log n)^2 - 2\log n\log\log n \\ -2\log n\left(\frac{\psi_0\left(\frac{\log n}{\log\varphi}\right)}{\log\varphi} - \log\log\varphi\right) &+ (\log\log n)^2 + 2\log\log n\left(1 + \log\log\varphi - \frac{\psi_0\left(\frac{\log n}{\log\varphi}\right)}{\log\varphi}\right) \\ + \left(\frac{\psi_0\left(\frac{\log n}{\log\varphi}\right)}{\log\varphi} - \log\log\varphi\right)^2 &- 2\left(\frac{\psi_0\left(\frac{\log n}{\log\varphi}\right)}{\log\varphi} - \log\log\varphi\right) + O\left(\frac{(\log\log n)^2}{\log n}\right). \end{aligned} \tag{21}$$

Combining (16), (17), (20) and (21) gives the result. □

Proposition 7. *Let $f(s) := g(s) + h(s)$, where $g(s)$ and $h(s)$ are as in Propositions 4 and 5, respectively. Then, as $s \to 0^+$, the function $f(s)$ satisfies $f(s) = f(\varphi s) + O(s^2)$. That is, for sufficiently small $\alpha > 0$, or equivalently, for sufficiently large n,*

$$f(\alpha) = \psi_1\left(\frac{\log n}{\log\varphi}\right) + O\left(\frac{\log\log n}{\log n}\right),$$

for some 1-periodic function $\psi_1(x)$, and any $\varepsilon > 0$.

Proof. We proceed as in the proof of Lemma 3, and, as in that proof, separating out the $k = 0$ term of the sum for $f(\alpha)$ and denoting it $[k=0]h(s)$, as $s = \alpha \to 0^+$,

$$\begin{aligned} f(\alpha) = [k=0]h(\alpha) + O(\alpha^2) &= \frac{1}{\log\varphi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(\frac{\alpha}{\sqrt{5}}\right)^{-\frac{2\pi i n}{\log\varphi}}\Gamma\left(\tfrac{2\pi i n}{\log\varphi}\right)\zeta\left(1+\tfrac{2\pi i n}{\log\varphi}\right) + O(\alpha^2) \\ &= \frac{2}{\log\varphi}\sum_{n=1}^{\infty}\Re\left(\left(\frac{\alpha}{\sqrt{5}}\right)^{-\frac{2\pi i n}{\log\varphi}}\Gamma\left(\tfrac{2\pi i n}{\log\varphi}\right)\zeta\left(1+\tfrac{2\pi i n}{\log\varphi}\right)\right) + O(\alpha^2) \\ &= \frac{2}{\log\varphi}\sum_{n=1}^{\infty}\Re\left(\left(\cos\left(\tfrac{2\pi n}{\log\varphi}\log\left(\tfrac{\alpha}{\sqrt{5}}\right)\right) - i\sin\left(\tfrac{2\pi n}{\log\varphi}\log\left(\tfrac{\alpha}{\sqrt{5}}\right)\right)\right)\right. \\ &\qquad \left.\times\Gamma\left(\tfrac{2\pi i n}{\log\varphi}\right)\zeta\left(1+\tfrac{2\pi i n}{\log\varphi}\right)\right) + O(\alpha^2). \end{aligned} \tag{22}$$

Since both the sine and cosine functions are 2π-periodic, (22) gives that $f(s) = f(\varphi s) + O(s^2)$ as $s \to 0^+$. Applying Corollary 1, we obtain

$$\frac{2\pi j}{\log\varphi}\log\left(\tfrac{\alpha}{\sqrt{5}}\right) = -2\pi j\left(\frac{\log n}{\log\varphi}\right) + O\left(\frac{\log\log n}{\log n}\right).$$

Finally, we note that $O(\alpha^2) = O((\log n)^2/n^2) = O(\log\log n/\log n)$ and set $\psi_1(s) = [k=0]h(s)$ to finish the proof. □

We now have all of the elements to continue with our proof Theorem 1 in the case of non-distinct partitions of n.

Proof of Theorem 1. We evaluate the pieces of the asymptotic in Theorem 4. Towards this end, Proposition 6 combined with (20) gives,

$$\begin{aligned} n\alpha &= \frac{1}{\log\varphi} W\left(e^{\psi_0\left(\frac{\log n}{\log\varphi}\right)/\log\varphi} n/\log\varphi\right)\left(1+O\left(\frac{\log n}{n^2}\right)\right) \\ &= \frac{\log n}{\log\varphi} + \frac{\log\log n}{\log\varphi} + \frac{\psi_0\left(\frac{\log n}{\log\varphi}\right)}{(\log\varphi)^2} - \frac{\log\log\varphi}{\log\varphi} + O\left(\frac{\log\log n}{\log n}\right), \end{aligned}$$

so that

$$e^{n\alpha} = e^{\frac{1}{(\log\varphi)^2}\psi_0\left(\frac{\log n}{\log\varphi}\right) - \frac{\log\log\varphi}{\log\varphi}} n^{1/\log\varphi}(\log n)^{1/\log\varphi}\left(1+O\left(\frac{\log\log n}{\log n}\right)\right) \tag{23}$$

Corollary 2 gives

$$\left(\sqrt{\frac{1}{-\left.\frac{dn}{ds}\right|_{s=\alpha}}} + O\left(\frac{1}{n^{3/2}}\right)\right) = \frac{1}{n}\left(\frac{\log n}{\log\varphi}\right)^{1/2}\left(1+O\left(\frac{\log\log n}{\log n}\right)\right). \tag{24}$$

It remains to determine the small α, and hence large n, asymptotics of $F_2(e^{-\alpha})$. To this end, we combine Theorem 3 with Corollaries 1 and 3, and Proposition 7 to obtain, as $n\to\infty$,

$$\begin{aligned} \log F_2(e^{-\alpha}) &= \frac{(\log\alpha)^2}{2\log\varphi} - (\log\alpha)\left(\frac{c_3}{\log\varphi} - 1\right) + c_2 + 2\gamma + f(\alpha) + O(\alpha^2) \\ &= \frac{\log n}{2\log\varphi}\left(\log n - 2\log\log n - \frac{4\psi_0\left(\frac{\log n}{\log\varphi}\right)}{\log\varphi} + 2\log\log\varphi + 2c_3 - 2\log\varphi\right) \\ &\quad + \frac{\log\log n}{2\log\varphi}\left(\log\log n + 2 - \frac{4\psi_0\left(\frac{\log n}{\log\varphi}\right)}{\log\varphi} + 2\log\log\varphi - 2c_3 + 2\log\varphi\right) \\ &\quad + \frac{(\log\log\varphi)^2}{2\log\varphi} - \frac{2\psi_0\left(\frac{\log n}{\log\varphi}\right)}{2(\log\varphi)^2} + \frac{\log\log\varphi}{\log\varphi} + \log\log\varphi\left(\frac{c_3}{\log\varphi} - 1\right) \\ &\quad + c_2 + 2\gamma + \psi_1\left(\frac{\log n}{\log\varphi}\right) + O\left(\frac{(\log\log n)^2}{\log n}\right). \end{aligned}$$

Thus, we have

$$F_2(e^{-\alpha}) = \psi_2(n) n^{a(n)}(\log n)^{b(n)}\left(1+O\left(\frac{(\log\log n)^2}{\log n}\right)\right),$$

where $\psi_2(n)$ is the strictly positive bounded (above and below) function

$$\begin{aligned} \psi_2(n) &:= \exp\left(\frac{(\log\log\varphi)^2}{2\log\varphi} - \frac{2\psi_0\left(\frac{\log n}{\log\varphi}\right)}{2(\log\varphi)^2} + \frac{\log\log\varphi}{\log\varphi}\right. \\ &\qquad \left. + \log\log\varphi\left(\frac{c_3}{\log\varphi} - 1\right) + c_2 + 2\gamma + \psi_1\left(\frac{\log n}{\log\varphi}\right)\right), \\ a(n) &:= \frac{1}{2\log\varphi}\left(\log n - 2\log\log n - \frac{4\psi_0\left(\frac{\log n}{\log\varphi}\right)}{\log\varphi} + 2\log\log\varphi + 2c_3 - 2\log\varphi\right) \\ b(n) &:= \frac{1}{2\log\varphi}\left(\log\log n + 2 - \frac{4\psi_0\left(\frac{\log n}{\log\varphi}\right)}{\log\varphi} + 2\log\log\varphi - 2c_3 + 2\log\varphi\right). \end{aligned}$$

Combing the asymptotic for $F_2(e^{-\alpha})$ with equations (23) and (24) gives the desired result. □

4. Partitions over general linear recurrences with a dominant root

The results of the previous sections on Fibonacci partitions can be generalised to positive recurrence sequences P_n with an irreducible characteristic polynomial having a positive dominant real root and such that $P_1 = 1$. We will also assume that the values P_n are distinct. Here, we require $P_1 = 1$ so that there always exists a non-distinct partition of n over P_n. The restriction on the values of P_n being distinct is not a very strict one—since the recursion has a dominant real root there are at most finitely many repeated values, so an analysis analogous to moving between $F(x)$ and $F_2(x)$ is possible, and uncomplicated. For such P_n, we wish to asymptotically understand the number of solutions to

$$n = a_1P_1 + a_2P_2 + \cdots + a_kP_k + \cdots . \tag{25}$$

As before, we let $p_P(n)$ denote the number of non-distinct partitions of n over the sequence P_n, that is, solutions to (1) in nonnegative integers a_k.

The case of partitions over P_n is carried out exactly as in the case with the Fibonacci numbers F_n, but with the Fibonacci zeta function replaced by the zeta function $\zeta_P(z)$ which is the meromorphic continuation of the Dirichlet series $\sum_{k\geqslant 1} P_k^{-z}$ ($\Re(z) > 0$). Here, we consider the generating function,

$$F_P(x) := \sum_{n\geqslant 0} p_P(n)x^n = \prod_{k\geqslant 1} \left(1 - x^{P_k}\right)^{-1}.$$

To complete our analysis, we use the following result of Serrano Holdago and Navas Vicente [15], which is a generalisation of Navas [10].

Proposition 8 (Serrano Holdago and Navas Vicente, 2023). *Let $P(x)$ be the minimal polynomial with $\deg P(x) = r$ of the linear recurrence P_n (as described above) and let $\beta > 1$ be the dominant root of $P(x)$. Then the Dirichlet series $\zeta_P(z) := \sum_{k\geqslant 1} P_k^{-z}$ ($\Re(z) > 0$) can be analytically continued to a meromorphic function, also denoted $\zeta_P(z)$, all of whose singularities are simple poles at the points*

$$s(n, \boldsymbol{k}) := \frac{\log|\beta^{-k_1}\beta_2^{k_1-k_2}\cdots\beta_r^{k_{r-1}}|}{\log\beta} + i\cdot\frac{\arg(\beta_1^{-k_1}\beta_2^{k_1-k_2}\cdots\beta_r^{k_{r-1}}) + 2\pi n}{\log\beta},$$

where $\boldsymbol{k} = (k_1, \ldots, k_{r-1})$, $\beta_2, \ldots, \beta_r$ are the algebraic conjugates of β, $n \in \mathbb{Z}$ and the parameters $k_1, \ldots, k_{r-1}$ are integers satisfying $0 \leqslant k_{r-1} \leqslant k_{r-2} \leqslant \cdots \leqslant k_1$.

We adopt the terminology of Proposition 8 for the rest of this section along with the definitions of the real numbers $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_r$, which satisfy

$$P_n = \lambda\beta^n + \lambda_2\beta_2^n + \cdots + \lambda_r\beta_r^n.$$

Note that since P_n is strictly increasing, we necessarily have that $\lambda > 0$.

Continuing the analogy with the Fibonacci partitions, we need asymptotic results, as $s \to 0^+$, of the functions

$$\log F_P(e^{-s}) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} s^{-z}\Gamma(z)\zeta(1+z)\zeta_P(z)dz.$$

As before, we have a few different types of poles to consider:

- a triple pole at $z = 0$ in the case of $F_P(e^{-s})$,
- simple poles at countable (and separated) non-integer real values $z \leqslant -\frac{\log(\beta/|\beta_2|)}{\log\beta}$.
- double poles at $z \in -\mathbb{N}$, and
- simple poles off the real line at $z = s(n, \boldsymbol{k})$.

Now, it is clear from the previous analysis on Fibonacci partitions that the main contribution will come from the pole at $z = 0$, which comes only from $\boldsymbol{k} = \boldsymbol{0} = (0, \dots, 0)$. The negative real poles will give a cumulative contribution of $O\big(s^{\min\{\log(\beta/|\beta_2|)/\log\beta, 1-\varepsilon\}}\big)$ for any fixed $\varepsilon > 0$, since $s \log s = O(s^{1-\varepsilon})$ for any fixed $\varepsilon > 0$ as $s \to 0^+$. As well, the simple poles off the real line contribute a function $f_2(s)$ toward $\log F_P(e^{-s})$, that satisfies $f_2(s) = f_2(\beta s) + O\big(s^{\min\{\log(\beta/|\beta_2|)/\log\beta, 1-\varepsilon\}}\big)$. It remains to obtain the contributions from the pole at $z = 0$. For these, we require the following result.

Lemma 4. *Near $z = 0$, we have*

$$\zeta_P(z) = \frac{1}{\log\beta} \cdot \frac{1}{z} - \frac{\log\lambda}{\log\beta} - \frac{1}{2} + \left(\frac{(\log\lambda)^2}{2\log\beta} + \frac{\log\lambda}{2} + \frac{\log\beta}{12} + C_1 \right) z + O(z^2),$$

where $C_1 := \sum_{k\geqslant 1} \sum_{\boldsymbol{k}\neq\boldsymbol{0}} \frac{1}{k_1} \binom{k_1}{k_2} \cdots \binom{k_{r-2}}{k_{r-1}} \frac{(\lambda_r \beta_r^k)^{k_{r-1}}}{(\lambda\beta^k)^{z+k_1}} \left(\prod_{j=2}^{r-1} (\lambda_j \beta_j^k)^{k_{j+1}-k_j} \right)$.

Proof. Following Serrano Holdago and Navas Vicente [15], we write

$$\begin{aligned} \zeta_P(z) &= \sum_{k\geqslant 1} \sum_{\boldsymbol{k}} \binom{-z}{k_1} \binom{k_1}{k_2} \cdots \binom{k_{r-2}}{k_{r-1}} \frac{(\lambda_r \beta_r^k)^{k_{r-1}}}{(\lambda\beta^k)^{z+k_1}} \left(\prod_{j=2}^{r-1} (\lambda_j \beta_j^k)^{k_{j+1}-k_j} \right) \\ &= \sum_{k\geqslant 1} \frac{1}{(\lambda\beta^k)^z} + \sum_{k\geqslant 1} \sum_{\boldsymbol{k}\neq\boldsymbol{0}} \binom{-z}{k_1} \binom{k_1}{k_2} \cdots \binom{k_{r-2}}{k_{r-1}} \frac{(\lambda_r \beta_r^k)^{k_{r-1}}}{(\lambda\beta^k)^{z+k_1}} \left(\prod_{j=2}^{r-1} (\lambda_j \beta_j^k)^{k_{j+1}-k_j} \right) \\ &= \frac{\lambda^{-z}}{\beta^z - 1} + \sum_{k\geqslant 1} \sum_{\boldsymbol{k}\neq\boldsymbol{0}} \frac{\Gamma(z+k_1)}{\Gamma(z)\, k_1!} \binom{k_1}{k_2} \cdots \binom{k_{r-2}}{k_{r-1}} \frac{(\lambda_r \beta_r^k)^{k_{r-1}}}{(\lambda\beta^k)^{z+k_1}} \left(\prod_{j=2}^{r-1} (\lambda_j \beta_j^k)^{k_{j+1}-k_j} \right). \end{aligned}$$

where $\sum_{\boldsymbol{k}} := \sum_{k_1\geqslant 0} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{r-1}=0}^{k_{r-2}}$. Again, using that $\frac{1}{\Gamma(z)} = z + O(z^2)$ and $\Gamma(z + k_1) = (k_1 - 1)! + O(z)$ near $z = 0$, along with the asymptotic expansions

$$\lambda^{-z} = 1 - z\log\lambda + \frac{1}{2}(\log\lambda)^2 z^2 + O(z^3),$$

and

$$\frac{1}{\beta^z - 1} = \frac{1}{\log\beta} \cdot \frac{1}{z} - \frac{1}{2} + \frac{\log\beta}{12} z + O(z^2),$$

we have that

$$\zeta_P(z) = \frac{1}{\log\beta} \cdot \frac{1}{z} - \frac{\log\lambda}{\log\beta} - \frac{1}{2} + \left(\frac{(\log\lambda)^2}{2\log\beta} + \frac{\log\lambda}{2} + \frac{\log\beta}{12} + C_1 \right) z + O(z^2). \quad \square$$

In the following result, we use Lemma 4 to determine the asymptotic behaviour of the function $\log F_P(e^{-s})$ as $s \to 0^+$. All of the expansions of the functions involved, s^{-z}, $\Gamma(z)$ and $\zeta(1 + z)$, have been noted somewhere in the previous sections of this work—we use them below without further reference.

Proposition 9. *For any fixed $\varepsilon \in (0, 1)$, as $s \to 0^+$, the function $F_P(z)$ defined above satisfies,*

$$\log F_P(e^{-s}) = \frac{(\log s)^2}{2\log\beta} + \left(\frac{\log\lambda}{\log\beta} + \frac{1}{2} - \frac{2\gamma}{\log\beta} \right) \log s + C_2 + f_2(s) + O\big(s^{\min\{\log(\beta/|\beta_2|)/\log\beta, 1-\varepsilon\}}\big),$$

where

$$C_2 := \frac{1}{\log\beta} \left(\frac{1}{2} \left(\gamma^2 + \frac{\pi^2}{6} \right) + \gamma^2 - \gamma_1 \right) - \left(\frac{\log\lambda}{\log\beta} + \frac{1}{2} \right) 2\gamma + \left(\frac{(\log\lambda)^2}{2\log\beta} + \frac{\log\lambda}{2} + \frac{\log\beta}{12} + C_1 \right),$$

and $f_2(s) = f_2(\beta s) + O\big(s^{\min\{\log(\beta/|\beta_2|)/\log\beta, 1-\varepsilon\}}\big)$.

Proof. Note that near $z = 0$, we have

$$s^{-z}\Gamma(z)\zeta(1+z) = \frac{1}{z^2} + (2\gamma - \log s)\frac{1}{z} + \left(\frac{1}{2}\left(\gamma^2 + \frac{\pi^2}{6}\right) + \gamma^2 - \gamma_1 - 2\gamma\log s + \frac{(\log s)^2}{2}\right) + O(z),$$

so that, using Lemma 4, we have that

$$\begin{aligned}\operatorname*{Res}_{z=0}\left\{s^{-z}\Gamma(z)\zeta(1+z)\zeta_P(z)\right\} &= \frac{1}{\log\beta}\left(\frac{1}{2}\left(\gamma^2 + \frac{\pi^2}{6}\right) + \gamma^2 - \gamma_1 - 2\gamma\log s + \frac{(\log s)^2}{2}\right) \\ &\quad - \left(\frac{\log\lambda}{\log\beta} + \frac{1}{2}\right)(2\gamma - \log s) \\ &\quad + \left(\frac{(\log\lambda)^2}{2\log\beta} + \frac{\log\lambda}{2} + \frac{\log\beta}{12} + C_1\right) \\ &= \frac{(\log s)^2}{2\log\beta} + \left(\frac{\log\lambda}{\log\beta} + \frac{1}{2} - \frac{2\gamma}{\log\beta}\right)\log s + C_2.\end{aligned}$$

The contributions coming from the rest of the poles are as discussed before the statement of Lemma 4. □

On inspection, one notices that the dominant asymptotic terms of $\log F_P(e^{-s})$ are precisely the dominant asymptotic terms of $\log F_2(e^{-s})$, the function related to the Fibonacci partitions, after substituting β for φ. Of course, this is not so surprising, as the Fibonacci numbers F_k are just a special case of the more general sequence P_k. The property of note, here, is regarding the associated saddle point. Since the leading order behaviour is the same, the saddle points satisfy the same leading order asymptotics. In particular, Proposition 6 and its corollaries hold for the saddle point α_{F_P} related to non-distinct partitions of n over P_k, and so also then, do (23) and (24). We use these results, as well as this notation, below.

Proof of Theorem 2. By Proposition 9, Proposition 6 and its corollaries, the small-α_{F_P} asymptotics, or, equivalently, the large-n asymptotics satisfy,

$$\begin{aligned}\log F_P(e^{-\alpha_{F_P}}) &= \frac{(\log\alpha_{F_p})^2}{2\log\beta} - (\log\alpha_{F_p})\left(\frac{2\gamma}{\log\beta} - \frac{\log\lambda}{\log\beta} - \frac{1}{2}\right) \\ &\qquad + C_2 + f_2(\alpha_{F_p}) + O\Big(\alpha_{F_p}^{\min\{\log(\beta/|\beta_2|)/\log\beta, 1-\varepsilon\}}\Big) \\ &= \frac{\log n}{2\log\beta}\left(\log n - 2\log\log n - \frac{4\psi_3\Big(\frac{\log n}{\log\varphi}\Big)}{\log\beta} + 2\log\log\beta + 4\gamma - 2\log\lambda - \log\beta\right) \\ &\quad + \frac{\log\log n}{2\log\beta}\left(\log\log n + 2 - \frac{4\psi_3\Big(\frac{\log n}{\log\beta}\Big)}{\log\beta} + 2\log\log\beta - 4\gamma + 2\log\lambda + \log\beta\right) \\ &\qquad + \frac{(\log\log\beta)^2}{2\log\beta} - \frac{\psi_3\Big(\frac{\log n}{\log\beta}\Big)}{(\log\beta)^2} + (1 + 2\gamma - \log\lambda)\frac{\log\log\beta}{\log\beta} - \frac{1}{2}\log\log\beta \\ &\qquad + C_2 + \psi_4\Big(\frac{\log n}{\log\beta}\Big) + O\left(\frac{(\log\log n)^2}{\log n}\right),\end{aligned}$$

where $\psi_3(x)$ and $\psi_4(x)$ are explicitly computable 1-periodic functions that are analogous to $\psi_0(x)$ and $\psi_1(x)$, respectively. Thus, we have

$$F_P(e^{-\alpha_{F_P}}) = \psi_5\Big(\frac{\log n}{\log\beta}\Big)\, n^{c(n)}(\log n)^{d(n)}\left(1 + O\left(\frac{(\log\log n)^2}{\log n}\right)\right),$$

where $\psi_5(n)$ is the 1-periodic positive function

$$\psi_5(x) := \exp\left(\frac{(\log\log\beta)^2}{2\log\beta} - \frac{\psi_3(x)}{(\log\beta)^2} + (1+2\gamma-\log\lambda)\frac{\log\log\beta}{\log\beta} - \frac{1}{2}\log\log\beta + C_2 + \psi_4(x)\right),$$

$$c(n) := \frac{1}{2\log\beta}\left(\log n - 2\log\log n - \frac{4\psi_3\left(\frac{\log n}{\log\varphi}\right)}{\log\beta} + 2\log\log\beta + 4\gamma - 2\log\lambda - \log\beta\right),$$

$$d(n) := \frac{1}{2\log\beta}\left(\log\log n + 2 - \frac{4\psi_3\left(\frac{\log n}{\log\beta}\right)}{\log\beta} + 2\log\log\beta - 4\gamma + 2\log\lambda + \log\beta\right).$$

Combing the asymptotic for $F_P(e^{-\alpha_{F_P}})$ with equations (23) and (24) gives the desired result. □

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California State University, 400 West First Street, Chico, California 95929, USA
Email address: `mjcoons@csuchico.edu`

Aarhus University, Ny Munkegade 118, 8000 Aarhus C, Denmark
Email address: `sik@math.au.dk, mll@math.au.dk`

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