# Geometric compactifications of Calabi–Yau moduli and Weil–Petersson metric asymptotics




**Andres David Gomez Villegas**

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Supervised by Cristiano Spotti

Department of Mathematics
Aarhus University
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# Abstract

A long-standing goal in complex geometry is to understand the global geometry of moduli spaces of polarized Calabi–Yau manifolds. In this Thesis, we make progress towards this goal by studying the relationship between the asymptotic Weil–Petersson geometry of these moduli spaces and the degeneration of the canonical Ricci-flat Kähler metrics on the manifolds they parametrize, with particular emphasis on the moduli space $A_g$ of $g$-dimensional principally polarized abelian varieties (PPAV) and the moduli space $\mathscr{F}_{2d}$ of primitively polarized K3 surfaces of degree $2d$.

In both cases, the central differential-geometric phenomenon that must be accounted for is collapsing. For PPAV, Odaka classified the limits of the canonical flat Kähler metrics at the diameter and volume scales: after diameter normalization, the collapsing limits are real flat tori, whereas at fixed volume, the pointed limits are products of lower-dimensional PPAV with Euclidean spaces. Along semi-abelian degenerations, the diameter-scale limit realizes the essential skeleton as a flat torus, while the compact factor retained at the volume scale is the abelian part of the degeneration. These descriptions give rise to two complementary Gromov–Hausdorff compactifications of $A_g$. Building on this framework, we show that, along algebraic degenerations, the pointed Gromov–Hausdorff limits of $A_g$ endowed with its Weil–Petersson metric are products determined by the boundary strata of these compactifications, with a scale parameter retained in the tropical factor. The limiting Weil–Petersson metric splits accordingly, with the metric on each moduli factor induced by variation of the limiting geometric structures parametrized by that factor. This gives a complete modular interpretation of the asymptotic Weil–Petersson geometry of $A_g$ in terms of the multiscale collapsing geometry of the canonical flat Kähler metrics along degenerating sequences of abelian varieties.

For polarized K3 surfaces, we follow a similar approach and determine the pointed Gromov–Hausdorff limits of $\mathscr{F}_{2d}$ endowed with its Weil–Petersson metric along Type II and Type III algebraic degenerations. We describe the limiting spaces and their product metrics in terms of the boundary strata of two natural compactifications of $\mathscr{F}_{2d}$, namely the Baily–Borel and Satake-adjoint compactifications. Combining this description with the results of Odaka–Oshima and Ouyang–Tian, which show that the Satake-adjoint strata parametrize the diameter-scale limits of the Ricci-flat Kähler metrics, we obtain a partial modular interpretation of the Weil–Petersson asymptotics. We discuss the remaining steps towards completing this interpretation, namely relating the Baily–Borel factors to the volume-scale limits and, in Type III, interpreting the Weil–Petersson-type metric on the Satake-adjoint boundary stratum as induced by the infinitesimal variation of the limiting real Monge–Ampère metric under deformations of the integral affine structure on the tropical K3 surface. These results motivate a conjectural extension of this picture to general polarized Calabi–Yau manifolds.

# Resumé

Et mangeårigt mål inden for kompleks geometri er at forstå den globale geometri af modulirum for polariserede Calabi–Yau-mangfoldigheder. I denne afhandling gør vi fremskridt mod dette mål ved at undersøge sammenhængen mellem disse modulirums asymptotiske Weil–Petersson-geometri og degenerationen af de kanoniske Ricci-flade Kähler-metrikker på de mangfoldigheder, som modulirummene parametriserer, med særlig vægt på modulirummet $A_g$ for $g$-dimensionale principalt polariserede abelske varieteter (PPAV) og modulirummet $\mathcal{F}_{2d}$ for primitivt polariserede K3-flader af grad $2d$.

I begge tilfælde er såkaldt *kollaps* det centrale differentialgeometriske fænomen, der skal gøres rede for. Odaka klassificerede grænserne for de kanoniske flade Kähler-metrikker på diameter- og volumenskalaen for PPAV: Efter diameternormalisering er grænserne ved kollaps reelle flade torusser, mens de *punkterede* Gromov–Hausdorff-grænser ved fast volumen er produkter af PPAV af lavere dimension og euklidiske rum. Langs semiabelske degenerationer realiserer grænsen på diameterskalaen degenerationens *essentielle skelet* som en flad torus, mens den kompakte faktor, der bevares på volumenskalaen, er degenerationens abelske del. Disse beskrivelser giver anledning til to komplementære Gromov–Hausdorff-kompaktificeringer af $A_g$. På dette grundlag viser vi, at de *punkterede* Gromov–Hausdorff-grænser af $A_g$, udstyret med sin Weil–Petersson-metrik, langs algebraiske degenerationer er produkter bestemt af randstrataene i disse kompaktificeringer, med en skalaparameter bevaret i den tropiske faktor. Weil–Petersson-metrikken i grænsen har en tilsvarende produktstruktur, og metrikken på hver modulifaktor induceres af variationen af de geometriske strukturer i grænsen, som faktoren parametriserer. Dette giver en fuldstændig moduliteoretisk fortolkning af den asymptotiske Weil–Petersson-geometri på $A_g$ ud fra geometrien af de kanoniske flade Kähler-metrikkers kollaps på flere skalaer langs degenererende følger af abelske varieteter.

For polariserede K3-flader følger vi en tilsvarende fremgangsmåde og bestemmer de *punkterede* Gromov–Hausdorff-grænser af $\mathcal{F}_{2d}$, udstyret med sin Weil–Petersson-metrik, langs algebraiske degenerationer af type II og type III. Vi beskriver grænserummene og deres produktmetrikker ved hjælp af randstrataene i to naturlige kompaktificeringer af $\mathcal{F}_{2d}$, nemlig Baily–Borel- og Satake-adjoint-kompaktificeringerne. Ved at kombinere denne beskrivelse med resultaterne af Odaka–Oshima og Ouyang–Tian, som viser, at Satake-adjoint-strataene parametriserer grænserne på diameterskalaen for de Ricci-flade Kähler-metrikker, opnår vi en delvis moduliteoretisk fortolkning af Weil–Petersson-asymptotikken. Vi diskuterer de resterende skridt mod at fuldende denne fortolkning, nemlig at relatere Baily–Borel-faktorerne til grænserne på volumenskalaen og, i type III-tilfældet, at fortolke metrikken af Weil–Petersson-type på randstratummet i Satake-adjoint-kompaktificeringen som induceret af den infinitesimale variation af den reelle Monge–Ampère-metrik i grænsen under deformationer af den heltallige affine struktur på den tropiske K3-flade. Disse resultater motiverer en formodning om, at dette billede kan udvides til generelle polariserede Calabi–Yau-mangfoldigheder.

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# Introduction

One of the most compelling features of Kähler geometry is the rich interplay between its differential-geometric and algebro-geometric aspects. In the context of moduli theory, recent developments in our understanding of how Ricci-flat Kähler metrics degenerate along polarized algebraic families suggest that the geometry of their limits may provide a natural framework for studying the global geometry of moduli spaces of polarized Calabi–Yau manifolds and constructing their geometric compactifications. This Thesis explores this perspective with particular emphasis on studying the modular interpretation of the Weil–Petersson metric asymptotics on the moduli spaces of principally polarized abelian varieties and polarized K3 surfaces.

At the foundation of this approach lies Yau's solution to the Calabi conjecture [73]. Given a polarized family of Calabi–Yau manifolds $\pi\colon (\mathcal{X}^*, \mathcal{L}^*) \to \Delta^*$, it endows each fiber $X_t$ with a unique Ricci-flat Kähler metric $\omega_{CY,t}$ satisfying $[\omega_{CY,t}] = 2\pi c_1(\mathcal{L}^*|_{X_t})$. The algebraic problem of compactifying the moduli space thus leads naturally to the analytic problem of understanding the limiting behavior of these canonical metrics as $t \to 0$.

A central, and unavoidable, difficulty of this perspective comes from the Riemannian phenomenon of collapsing: relative to the scale at which the metrics are observed, certain directions can shrink away, giving rise to Gromov–Hausdorff limits of smaller real dimension than the original manifolds. Moreover, the geometry that survives in the limit depends on the choice of scale. Consequently, a moduli-theoretic framework that seeks to capture this multiscale geometry must therefore accommodate limiting objects of different dimensions and geometric nature, including spaces whose natural description lies outside of the category of algebraic varieties.

To go forward with this perspective, our investigation will rely on the fact that the moduli spaces themselves can be endowed with a natural modular Kähler metric, the Weil–Petersson metric. This metric is defined through $L^2$-size of the infinitesimal variation of the Ricci-flat metric in the direction of complex-structure deformation. Consequently it provides a direct link between the geometry of the parametrized manifolds and that of their moduli spaces, suggesting that the degeneration of the former should be reflected in the asymptotic geometry of the latter. Moreover, it was shown in [72, 71, 68] that on a one-parameter degeneration of polarized Calabi–Yau's the origin lies at finite Weil–Petersson distance if and only if the central fiber of the degeneration is a Calabi–Yau variety with at worst canonical singularities, which implies that the polarized Ricci-flat Kähler metrics have uniformly bounded diameter and are uniformly volume non-collapsed. In contrast to this, the degenerations we are interested are always at infinite Weil–Petersson distance and the Ricci-flat Kähler metric will collapse. This leads to the following question.

**Question.** How does the asymptotic Weil–Petersson geometry of the moduli space encode the limiting behavior of the Ricci-flat Kähler metrics along polarized degenerations of Calabi–Yau manifolds?

For the moduli spaces of principally polarized abelian varieties and polarized K3 surfaces, namely $A_g$ and $\mathcal{F}_{2d}$ respectively, this question admits a particularly concrete formulation. Their period descriptions identify them with arithmetic quotients of Hermitian symmetric domains and the Weil–Petersson metrics agree, up to normalization, with the invariant metrics. Furthermore, a lot of progress has been made in the characterization of the limiting behavior of the Ricci-flat Kähler metrics on abelian varieties and K3 surfaces. Thus, making it possible to analyze the geometry of the moduli spaces at infinity explicitly and to compare the resulting asymptotic behavior with the degeneration of the corresponding Ricci-flat Kähler metrics. Here we would like to note that an essential point of reference in our investigation is the pioneering work of Odaka on abelian varieties [51] and of Odaka–Oshima on K3 surfaces [53], which develops a geometric compactification framework relating boundary data to the collapsing limits of Ricci-flat Kähler metrics after diameter normalization. As we will see below, and in the development of this Thesis, their framework provides the natural geometric setting in which our asymptotic analysis of the Weil–Petersson metric acquires a modular interpretation.

## Main results

Now we give a concrete summary of the main results through which this perspective is developed in the Thesis.

### Weil–Petersson geometry of $A_g$

This results are based on joint work with Yanbo Fang in [16]. There, we work on the moduli space $A_g$ of complex $g$-dimensional principally polarized abelian varieties and building upon the classification of the Gromov–Hausdorff limits of abelian varieties by Odaka [51] we establish the following modular interpretation of the asymptotic Weil–Petersson geometry:

**Theorem.** *Let $(A_g, g_{WP})$ be the moduli space of complex $g$-dimensional principally polarized abelian varieties endowed with its canonical Kähler structure given by the Weil–Petersson metric. Then for every non-degenerate direction at infinity in $A_g$, there exists an integer $g'$ with $0 \leq g' < g$ such that the Weil–Petersson metric converges in the pointed Gromov–Hausdorff sense, to a Weil–Petersson type metric on*

$$(A_{g'} \times \mathscr{T}^{>0}_{g-g'},\, g'_{WP} + g''_{trWP}),$$

*where*

- *the moduli space $A_{g'}$ is naturally identified with a boundary stratum of the constant-volume Gromov–Hausdorff compactification $\bar{A}_g^{\mathrm{GH_{vol}}}$ of $A_g$, and $g'_{WP}$ denotes the usual Weil–Petersson metric on $A_{g'}$;*
- *$\mathscr{T}^{>0}_{g-g'}$ denotes the moduli space of real $(g-g')$-dimensional flat tori of arbitrary positive diameter, which at any fixed diameter $\mathscr{T}^{c}_{g-g'}$ is naturally realized as a boundary stratum of the constant-diameter Gromov–Hausdorff compactification $\bar{A}_g^{\mathrm{GH_{diam}}}$ of $A_g$. Furthermore, $g''_{trWP}$ is a "tropical Weil–Petersson type" metric on $\mathscr{T}^{>0}_{g-g'}$, defined via the variation of the limiting flat metrics.*

We refer the reader to Section 2.3.2 of Chapter 2, for a detailed formulation of the statement where a precise definition of what we mean by a *non-degenerate direction of infinity in $A_g$* is given. For the time being a geometric way of understanding this condition is to think of this non-degenerate directions as those coming from polarized degenerating families of principally polarized abelian varieties.

### Weil–Petersson geometry of $\mathcal{F}_{2d}$

This results are based on the ongoing joint work with Yanbo Fang and Cristiano Spotti [17]. Now we work on the moduli space $\mathcal{F}_{2d}$ of complex $K3$ surfaces with at worst ADE singularities and primitive polarization of degree $2d$. We endow $\mathcal{F}_{2d}$ with its Weil–Petersson metric $g_{WP}$. Furthermore, we assume that we have fixed a polarized algebraic degeneration $\pi : (\mathcal{X}^*, \mathcal{L}^*) \to \Delta^*$ of $K3$ surfaces and that such a degeneration admits a Type II or Type III Kulikov model. Consequently, if we denote by $N$ the logarithm of the monodromy, we have that

- Type II: $N \neq 0$ and $N^2 = 0$ and $\Pi = \operatorname{Im} N$ is a rational isotropic plane;
- Type III: $N^2 \neq 0$ and $N^3 = 0$, and $\ell = \operatorname{Im} N^2$ is a rational isotropic line.

The main result of our investigation is the following characterization of the Weil–Petersson metric asymptotic.

**Theorem.** *Let $(x_j)$ be a sequence in $\mathcal{F}_{2d}$ coming from the fixed polarized degeneration $\pi$ along a sequence $q_j \to 0$. Then*

$$\big(\mathcal{F}_{2d}, g_{\mathrm{WP}}, x_j\big) \xrightarrow{\mathrm{pGH}} \begin{cases} \big(\mathcal{F}_{2d}^{BB}(\Pi) \times \mathcal{C}(\mathcal{F}_{2d}^{\mathrm{Sat}}(\Pi)), g_{WP,\Pi}, \big([\tau_\infty], 0\big)\big), & \text{in Type II,} \\ \big(\mathcal{F}_{2d}^{BB}(\ell) \times \mathcal{C}(\mathcal{F}_{2d}^{\mathrm{Sat}}(\ell)), g_{WP,\ell}, \big([\widehat{Y}_\infty], 0\big)\big), & \text{in Type III.} \end{cases}$$

*The factors have the following interpretation.*

(i) *The* tropical factors *are the Riemannian cylinders $\mathcal{C}(\mathcal{F}_{2d}^{\mathrm{Sat}}(I))$, $I \in \{\ell, \Pi\}$, over the corresponding Satake-adjoint boundary strata.*

(ii) *The* holomorphic factors*, are naturally identified with the corresponding Baily–Borel boundary strata $\mathcal{F}_{2d}^{BB}(I)$, $I \in \{\ell, \Pi\}$.*

The geometric interpretation of the Satake-adjoint boundary strata is supplied by the work of Odaka–Oshima [53] and Ouyang–Tian [55]. We recall the part of this result relevant to our discussion.

**Theorem** (Odaka–Oshima; Ouyang–Tian)**.** *Let* $\mathrm{CMet}_1$ *denote the space of isometry classes of compact metric spaces of diameter-one endowed with the Gromov–Hausdorff topology. Let*

$$\Phi : \mathcal{F}_{2d} \longrightarrow \mathrm{CMet}_1,$$

*be the continuous map that for $(X, L) \in \mathcal{F}_{2d}$ maps $\Phi_{\mathrm{alg}}(X, L) = (X, \hat{g}_{X,L})$, where $\hat{g}_{X,L}$ is the diameter-normalized Ricci-flat Kähler metric in the polarization class $2\pi c_1(L)$. Then $\Phi$ extends continuously to the Satake-adjoint compactification $\overline{\mathcal{F}}_{2d}^{\mathrm{Sat}}$. Its image is the diameter-scale Gromov–Hausdorff compactification of $\mathcal{F}_{2d}$, and the boundary strata have the following geometric interpretation:*

(i) *The stratum $\mathcal{F}_{2d}^{\mathrm{Sat}}(\Pi)$ consists of a single point, whose geometric realization is the unit interval.*

(ii) *The stratum $\mathcal{F}_{2d}^{\mathrm{Sat}}(\ell)$ parametrizes diameter-one metrized tropical $K3$ surfaces.*

*In particular, convergence to a point of either stratum in the Satake-adjoint compactification implies Gromov–Hausdorff convergence of the corresponding diameter-normalized Ricci-flat $K3$ surfaces to the metric space represented by that point.*

Combining this description of the degenerating Ricci-flat metrics with our theorem on the asymptotic Weil–Petersson geometry gives the following compatibility between the diameter-scale limits of the surfaces and the tropical factors in the pointed limits of their moduli space.

**Corollary.** *Let $(x_j)$ be the sequence arising from the fixed degeneration $\pi$ and let $(X_j, \hat{g}_j)$ be the corresponding, diameter-normalized, Ricci-flat Kähler surfaces. Then, the tropical factor in the pointed Weil–Petersson limit is the Riemannian cylinder over the Satake-adjoint stratum parametrizing the associated diameter-scale limits, and the projection of its distinguished basepoint to this stratum represents the Gromov–Hausdorff limit of $(X_j, \hat{g}_j)$.*

Finally, to complete the modular interpretation of the Weil–Petersson metric asymptotics on $\mathcal{F}_{2d}$, two further ingredients are needed; we outline them below, together with the progress made toward establishing them in ongoing joint work with Yanbo Fang and Cristiano Spotti. The arguments developed so far provide a concrete basis for the following statements, although some steps in their complete justification remain to be settled.

- The first step concerns the interpretation of the holomorphic factors $\mathcal{F}_{2d}^{BB}(I)$, with $I \in \{\ell, \Pi\}$, in terms of the volume-scale limits of the Ricci-flat metrics. Our analysis of principally polarized abelian varieties suggests that these strata should parametrize the compact factors retained at this scale. As we explain in Chapter 3, $\mathcal{F}_{2d}^{BB}(\Pi)$ is a modular curve, and the point $[\tau_\infty]$ determines the common elliptic double curve $E$ of the Type II Kulikov model, whereas $\mathcal{F}_{2d}^{BB}(\ell)$ is a point.

  To establish this interpretation, the principal remaining analytic step is to prove that, for suitably *generic* sequences of basepoints $p_j \in X_j$, the Ricci-flat metrics at fixed total volume satisfy

$$(X_j, g_j, p_j) \xrightarrow[j\to\infty]{\text{pGH}} \begin{cases} \big(\mathbb{R} \times E, ds^2 + \lambda\, g_E, (0, q)\big), & \text{in Type II,} \\ \big(\mathbb{R}^2, g_{\text{flat}}, 0\big), & \text{in Type III.} \end{cases}$$

  Here the basepoints approach $q \in E$ in Type II and a triple point of the Kulikov central fiber in Type III, while $g_E$ is the flat metric determined by the limiting polarization.

- The second step is to complete the interpretation of the limiting metric $g_{WP,I}$ as a metric induced by variation of the geometric structures parametrized by the corresponding factors. Our asymptotic theorem already supplies locally symmetric product metrics, with the decomposition

$$g_{\text{WP},I} = g_{\text{WP},I}^{hol} + g_{\text{WP},I}^{trop}.$$

  Moreover, on the modular curve $\mathcal{F}_{2d}^{\text{BB}}(\Pi)$, the metric $g_{\text{WP},\Pi}^{hol}$ already agrees with the usual Weil–Petersson metric induced by variation of the elliptic curve, up to the constant fixed by our normalization. Once the convergence above is established, this also identifies it as the metric induced by variation of the compact factor of the volume-scale limit. In Type III the holomorphic stratum is a point, so its metric is trivial. The remaining task is to show that $g_{\text{WP},\ell}^{trop}$ is induced by the infinitesimal variation of the limiting metric on the tropical $K3$ surface as we deform its integral affine structure, with diameter fixed to one. This is precisely the heuristic approach outlined in Remark 3.4.18.

Together with the established diameter-scale interpretation, these remaining identifications would show that the Weil–Petersson asymptotic along the degenerations under consideration is jointly encoded, in the sense of the boundary parametrizations above, by the volume- and diameter-scale Gromov–Hausdorff compactifications. This would extend to $\mathcal{F}_{2d}$ the picture established for $A_g$, showing in both settings that the tropical Weil–Petersson metric is induced by the variation of the limiting metrics as their underlying tropical structures deform.

We refer to Conjecture 3.2.7 in Chapter 3 and the discussion following it for a more detailed formulation of the expected volume-scale convergence, including the genericity of the basepoints, and the evidence supporting it. Section 3.4.3 of that Chapter gives a detailed account of our asymptotic limits and their relation to the geometric compactifications, together with a discussion of the remaining steps in their modular interpretation.

Motivated by our analysis of principally polarized abelian varieties and polarized $K3$ surfaces, it is natural to ask whether similar phenomena occur for general polarized Calabi–Yau manifolds. The conjectural picture formulated at the end of Chapter 1 proposes the existence of complementary geometric compactifications of their moduli space, parametrizing, possibly up to birational class, the compact Calabi–Yau factors of generic volume-scale limits and the Monge–Ampère metric and affine structures arising at the diameter scale. We expect the corresponding boundary strata, equipped with metrics induced by variation of these limiting structures, to determine the pointed Gromov–Hausdorff asymptotics of the Weil–Petersson metric along algebraic directions at infinity, as a product of holomorphic and tropical factors together with a real scale direction. These expectations are grounded in the geometric picture emerging from the recent developments surveyed in Section 1.4. Moreover, the recent breakthrough of Bakker–Filipazzi–Mauri–Tsimerman [4], constructing a canonical Hodge-theoretic Baily–Borel compactification for moduli spaces of polarized Calabi–Yau varieties, provides a concrete candidate for the compactification associated with the volume scale, whose interpretation through generic pointed limits remains conjectural.

## Outline of the Thesis

The Thesis is organized as follows. In Chapter 1 we recall some fundamental notions in the theory of moduli spaces of Calabi–Yau manifolds. We begin by revisiting the analytic construction of the moduli space of polarized Calabi–Yau manifolds, before giving a brief account of Gromov–Hausdorff theory and explaining how it provides a suitable framework for studying geometric compactifications through limits of degenerating sequences. We then recall the definition and basic properties of the Weil–Petersson metric, emphasizing its role as a bridge between the geometry of the moduli space and the behavior of the Ricci-flat metrics on the corresponding Calabi–Yau manifolds. Next, we survey recent developments relating the degeneration of Calabi–Yau metrics to algebraic geometry and describe the emerging geometric picture in relation to our moduli problem. Finally, we formulate the aim of the Thesis and explain how our investigation seeks to advance a moduli-theoretic description of the multi-scale geometry of degenerating Ricci-flat Kähler metrics by studying the modular information encoded in the Weil–Petersson geometry at infinity.

Chapter 2 is a reproduction of the preprint *Modular interpretation of the Weil–Petersson metric asymptotics for abelian varieties*, coauthored with Yanbo Fang and available at `https://arxiv.org/abs/2602.23140`. There we study the Weil–Petersson geometry of the moduli space $A_g$ of principally polarized abelian varieties. We first recall the basic

moduli theory and fix notation and conventions for the Siegel upper half-space and the relevant reduction theory. We then describe its rational boundary components and the algebraic and differential-geometric data associated with them, before revisiting the work of Odaka and Odaka–Oshima on the Gromov–Hausdorff compactifications of $A_g$ [51, 53]. The main part of the chapter begins with the derivation of the explicit formula for the Weil–Petersson metric on $A_g$, which provides the starting point for our asymptotic analysis. After developing the necessary geometric estimates, we prove the main theorem and give its modular interpretation in terms of the limiting flat metrics at the volume and diameter scales. We conclude by discussing the relation with algebraic compactifications and possible extensions to other Calabi–Yau moduli spaces.

Chapter 3 develops the corresponding investigation for the moduli space $\mathcal{F}_{2d}$ of polarized $K3$ surfaces. We begin by recalling its period-theoretic description, followed by the Kulikov classification of degenerations and an overview of the known results and conjectures concerning the limiting Ricci-flat metrics. We then study the geometry of the type IV domain $\mathcal{D}_{2d}$, developing the horospherical coordinates and the reduction theory needed to analyze the geometry near its rational boundary components. The main part of the Chapter begins by recalling the locally symmetric description of the Weil–Petersson metric, after which we establish the quantitative Gromov–Hausdorff estimates underlying our asymptotic analysis. We apply these estimates to polarized algebraic degenerations of Types II and III, using their monodromy to identify the pointed Weil–Petersson limits. Finally, we relate the resulting factors to the Baily–Borel and Satake-adjoint compactifications and discuss their geometric interpretation in terms of limits at the volume and diameter scales, explaining the remaining steps needed to complete the modular interpretation. The original results presented in this chapter form part of ongoing joint work with Yanbo Fang and Cristiano Spotti [17].

**Note to the reader.** Chapter 2 reproduces the preprint [16], with minor formatting and layout adjustments and with its abstract and acknowledgments omitted. Its references have been incorporated into the bibliography at the end of the thesis. Apart from these changes, its content and organization have been preserved. Consequently, some definitions, background material, and explanations overlap with those presented elsewhere in the thesis. We ask for the reader's understanding regarding this repetition.

# Chapter 1

# Geometry of the moduli space of polarized Calabi–Yau manifolds

In this chapter we recall some fundamental notions in the theory of moduli spaces of Calabi–Yau manifolds.

## 1.1 Main definitions and properties

The purpose of this section is to recall the aspects of the deformation theory and moduli theory of polarized Calabi–Yau manifolds that will be needed in the sequel; we make no attempt to give an exhaustive account of this extensive subject. The local deformation theory of compact complex manifolds was developed in the foundational works of Kodaira–Spencer, Kodaira–Nirenberg–Spencer and Kuranishi [33, 32, 34, 40, 41]. We have also benefited from the particularly clear exposition of this theory in [22, 31, 14].

By a polarized Calabi–Yau manifold we denote a pair $(X, L)$, where $X$ is an $n$-dimensional compact Kähler manifold, satisfying $K_X \cong \mathcal{O}_X$, and $L \to X$ is an ample line bundle. We denote by $\lambda := c_1(L) \in H^2(X, \mathbb{Z})$ the polarization class. A fundamental property of such a pair is given by Yau's solution [73] of the Calabi conjecture. Thus for each pair $(X, L)$ there exists a unique Ricci-flat Kähler metric $\omega_{CY} \in 2\pi\lambda$. The existence and uniqueness of this canonical Kähler metric underlie much of the geometry of Calabi–Yau moduli spaces.

*Remark* 1.1.1. Given a compact Kähler manifold $(X, J)$, the Calabi–Yau condition often depends on the convention being used. Thus far, we have required the minimum $K_{(X,J)} \simeq \mathcal{O}_{(X,J)}$. However, it is customary to assume additional cohomological assumptions. For our purposes, we will further assume that $H^0(X, T^{1,0}_{X,J}) = 0$. See Remark 1.1.2 below for additional comments.

Now, given a pair $(X, L)$ the moduli space of polarized Calabi–Yau structures is defined as follows. First we set

$$\mathcal{J}^{\lambda}_{\text{CY}}(X) := \left\{ J \in \text{End}(TX) \;\middle|\; \begin{array}{l} J \quad \text{integrable complex structure,} \\ (X, J) \text{ is Calabi–Yau, and } \lambda \in \mathcal{K}_J \end{array} \right\},$$

where $\mathcal{K}_J$ denotes the Kähler cone associated with the complex structure $J$. Notice that the condition $\lambda \in \mathcal{K}_J$ means that $\lambda$, viewed as a real cohomology class, is represented by a $J$-Kähler form. Since $\lambda$ is integral, the Lefschetz $(1,1)$ theorem and the Kodaira embedding theorem imply that it is the first Chern class of an ample holomorphic line bundle on $(X, J)$. Notice, however, that the definition fixes the polarization class rather than a particular holomorphic line bundle.

The set $\mathcal{I}^{\lambda}_{\mathrm{CY}}(X)$ is not yet the set of isomorphism classes of $\lambda$-polarized Calabi–Yau complex structures on $X$, because two complex structures related by a diffeomorphism of $X$ determine isomorphic polarized complex manifolds. Consider the group

$$\mathrm{Diff}(X,\lambda) := \{f \in \mathrm{Diff}(X) \mid f^*\lambda = \lambda\}\,.$$

This group acts on $\mathcal{I}^{\lambda}_{\mathrm{CY}}(X)$ by pullback, namely $f^*J = df^{-1} \circ J \circ df$. Thus the map $f\colon (X, f^*J) \longrightarrow (X, J)$ is a biholomorphism, and in particular, the defining properties of $\mathcal{I}^{\lambda}_{\mathrm{CY}}(X)$ are preserved under this action.

Thus the moduli space of $\lambda$-polarized Calabi–Yau structures on $X$ is defined as

$$\mathcal{M}^{\lambda}_{\mathrm{CY}}(X) := \mathcal{I}^{\lambda}_{\mathrm{CY}}(X)/\,\mathrm{Diff}(X,\lambda).$$

We will endow this quotient with the structure of a complex-analytic space by constructing suitable local slices for the action of $\mathrm{Diff}(X,\lambda)$, using the corresponding polarized deformation spaces. Before we proceed with this analysis let us record the structure of the stabilizers of this action. For $J \in \mathcal{I}^{\lambda}_{\mathrm{CY}}(X)$ the stabilizer of $J$ under the action of $\mathrm{Diff}(X,\lambda)$ is the group of biholomorphisms preserving the polarization class

$$\mathrm{Aut}(X,J,\lambda) := \{f \in \mathrm{Aut}(X,J) \mid f^*\lambda = \lambda\}\,.$$

We notice that our assumption on $H^0(X, T^{1,0}_{X,J}) = 0$ implies that $\mathrm{Aut}(X,J,\lambda)$ is finite. Indeed, let $\omega_{\mathrm{CY},J}$ denote the unique Ricci-flat Kähler metric in the fixed Kähler class. If $f \in \mathrm{Aut}(X,J,\lambda)$, then $f^*\omega_{\mathrm{CY},J}$ is again a Ricci-flat Kähler metric compatible with $J$. Moreover, it belongs to the same Kähler class, since $f^*\lambda = \lambda$. The uniqueness part of Yau's theorem therefore gives $f^*\omega_{\mathrm{CY},J} = \omega_{\mathrm{CY},J}$, and hence

$$\mathrm{Aut}(X,J,\lambda) \subseteq \mathrm{Isom}(X,\omega_{\mathrm{CY},J}).$$

Here the isometry group is understood with respect to the Riemannian metric associated with $\omega_{\mathrm{CY},J}$. Since $X$ is compact, $\mathrm{Isom}(X,\omega_{\mathrm{CY},J})$ is a compact Lie group. Within this isometry group, $\mathrm{Aut}(X,J,\lambda)$ is precisely the subgroup consisting of isometries preserving the complex structure $J$. This is a closed condition, so $\mathrm{Aut}(X,J,\lambda)$ is a closed subgroup. Now the Lie algebra of $\mathrm{Aut}(X,J)$ is naturally identified with $H^0(X, T^{1,0}_{X,J})$: differentiating a one-parameter family of biholomorphisms produces a holomorphic vector field, and every holomorphic vector field on a compact complex manifold is complete and therefore integrates to a one-parameter group of biholomorphisms. Consequently, the Lie algebra of $\mathrm{Aut}(X,J,\lambda)$ is a real Lie subalgebra of $H^0(X, T^{1,0}_{X,J})$, and hence vanishes by assumption. It follows that $\mathrm{Aut}(X,J,\lambda)$ is a closed, zero-dimensional subgroup of a compact Lie group, and is therefore finite.

Thus the action of $\mathrm{Diff}(X,\lambda)$ on $\mathcal{I}^{\lambda}_{\mathrm{CY}}(X)$ has finite stabilizers. This finiteness is the source of the finite-quotient phenomena that occur in the local moduli theory. It is important, however, that finite stabilizers alone do not imply that the raw quotient $\mathcal{M}^{\lambda}_{\mathrm{CY}}(X)$ is an orbifold. Once one has constructed a local slice and shown that the relevant local polarized deformation space is smooth, a neighborhood of the point represented by $J$ is locally described as the quotient of that deformation space by the finite group $\mathrm{Aut}(X,J,\lambda)$. It is in this precise sense that one expects orbifold, or more generally finite-quotient, singularities rather than an everywhere smooth moduli space.

*Remark* 1.1.2. The assumption $H^0(X, T^{1,0}_{X,J}) = 0$ excludes an important class of examples that will play a central role in our investigation, namely abelian varieties. We can give a unified framework by requiring our moduli space to parametrize actual polarized pairs

$(X, L)$ and not just a fixed integral Kähler class $\lambda$. However, as we will see below in the definition of the moduli space of abelian varieties it is customary to parametrize abelian varieties with their origin and a polarization homomorphism, i.e. a polarization class. With this convention translations are not automorphisms in this moduli problem because morphisms of abelian varieties must preserve the origin.

We now proceed with the local properties of $\mathcal{M}^{\lambda}_{\mathrm{CY}}(X)$. Fix $J \in \mathcal{J}^{\lambda}_{\mathrm{CY}}(X)$ and let $\omega_{\mathrm{CY},J}$ be the Ricci-flat Kähler metric in the class $2\pi\lambda$. We use this metric to define the formal adjoint $\bar\partial^*$, the harmonic projection H, and the Green operator G for the $\bar\partial$-Laplacian on forms with values in $T^{1,0}_{X,J}$. After completing the relevant spaces in $C^{k,\alpha}$ norms, with $k \geq 3$ and $0 < \alpha < 1$, a sufficiently small element $\phi \in A^{0,1}(X, T^{1,0}_{X,J})$ determines an almost-complex structure $J_\phi$ by declaring its $(0,1)$-bundle to be the graph of $\phi$

$$T^{0,1}_{X,J_\phi} := \left\{ \bar v + \phi(\bar v) \;\middle|\; \bar v \in T^{0,1}_{X,J} \right\} \subset T_{\mathbb{C}}X.$$

Conversely, the $(0,1)$-bundle of every almost-complex structure sufficiently close to $J$ is transverse to $T^{1,0}_{X,J}$ and is therefore the graph of a unique small $\phi$. Thus the assignment $\phi \mapsto J_\phi$ gives a preliminary local chart around $J$ in the space of almost-complex structures. Gauge fixing and the integrability equation will cut out the relevant local deformation space inside this chart. With our sign convention, $J_\phi$ is integrable precisely when $\phi$ satisfies the Maurer–Cartan equation

$$\bar\partial\phi + \frac{1}{2}[\phi, \phi] = 0.$$

Next, we remove the pullback action of the group of diffeomorphisms. First, notice that if $V$ is a real vector field and $f_s$ is its flow, the infinitesimal variation induced by $f_s^* J$ is represented by $-\bar\partial V^{1,0}$. Thus the infinitesimal orbit of $\mathrm{Diff}_0(X)$ is $\operatorname{im}\bar\partial$. The Hodge decomposition

$$A^{0,1}(X, T^{1,0}X) = \mathcal{H}^{0,1}(X, T^{1,0}X) \oplus \operatorname{im}\bar\partial \oplus \operatorname{im}\bar\partial^*$$

shows that the condition $\bar\partial^*\phi = 0$ is transverse to these orbit directions. Moreover, the assumption $H^0(X, T^{1,0}X) = 0$ implies that $\bar\partial^*\bar\partial$ has trivial kernel on vector fields. Thus the implicit function theorem applied to the diffeomorphism action shows that every sufficiently small almost-complex structure is equivalent, by a unique sufficiently small diffeomorphism, to one satisfying $\bar\partial^*\phi = 0$. This is the gauge slice; the possible remaining identifications come from the finite group $\mathrm{Aut}(X, J, \lambda)$.

The gauge condition alone still defines an infinite-dimensional space. To isolate a finite-dimensional locus containing all integrable points, let

$$\mathcal{P}(t, \phi) := \phi - t + \frac{1}{2}\bar\partial^*\mathrm{G}[\phi, \phi],$$

where $t \in \mathcal{H}^{0,1}(X, T^{1,0}X)$ and $\phi$ is assumed to lie in the chosen Hölder completion. Since $D_\phi\mathcal{P}_{(0,0)}$ is the identity, the implicit function theorem gives an open ball $B_\varepsilon$ and a unique analytic map $t \mapsto \phi(t)$ such that $\mathcal{P}(t, \phi(t)) = 0$ and $\phi(0) = 0$. Equivalently,

$$\phi(t) = t - \frac{1}{2}\bar\partial^*\mathrm{G}[\phi(t), \phi(t)].$$

In particular, $\mathrm{H}\,\phi(t) = t$ and $\bar\partial^*\phi(t) = 0$. Hence the image of $t \mapsto \phi(t)$ is a finite-dimensional analytic submanifold of the gauge slice. Every sufficiently small gauge-fixed solution of the Maurer–Cartan equation belongs to this submanifold, by the Hodge decomposition.

It remains to impose integrability. Let $\kappa(t) := \frac{1}{2}H[\phi(t),\phi(t)]$ denote the Kuranishi obstruction map. A standard argument, using the Jacobi identity and elliptic estimates, shows that, after decreasing $\varepsilon$ if necessary, $\phi(t)$ satisfies the Maurer–Cartan equation if and only if $\kappa(t)=0$. Consequently, the local deformation space is the analytic germ

$$\mathrm{Def}(X,J) = \kappa^{-1}(0) \subset B_\varepsilon \subset \mathscr{H}^{0,1}(X,T^{1,0}X).$$

This may equivalently be regarded as an analytic germ in $H^1(X,T_X)$. Notice that finite-dimensionality follows already from the implicit-function-theorem locus, whereas integrability selects the possibly singular analytic subspace $\kappa^{-1}(0)$ inside it. Nevertheless, for the special case of Calabi–Yau manifolds the following result asserts that the local deformation space is actually smooth

**Theorem 1.1.3** (Bogomolov–Tian–Todorov)**.** *Let $(X,J)$ be a compact Kähler manifold with trivial canonical bundle. Then its deformations are unobstructed. Equivalently, the Kuranishi obstruction map vanishes identically near the origin, and* $\mathrm{Def}(X,J)$ *is smooth of complex dimension $h^1(X,T^{1,0}X)$.*

The unobstructedness result was first obtained by Bogomolov in a more restrictive setting and was subsequently proved, in the generality stated above, independently by Tian and Todorov; see [6, 69, 70].

We must finally impose the polarization condition. The local deformation theory of polarized compact Kähler manifolds was developed in the work of Schumacher and Fujiki [61, 62, 20]. In the Calabi–Yau setting, these ideas are used by Tian in [69, §6], and our presentation essentially follows his treatment. Let

$$\pi\colon \mathcal{X} \longrightarrow \mathrm{Def}(X,J)$$

be the Kuranishi family of $(X,J)$, with central fiber $X_0=(X,J)$ and fibers $X_t=(X,J_t)$. We first observe that every sufficiently small deformation $X_t$ is again a compact Kähler manifold. This follows from the Kodaira–Spencer stability theorem, according to which the Kähler property is open under small deformations of compact complex manifolds; see [34]. Furthermore, the nearby fibers also have trivial canonical bundle. Finally, we must require $\lambda \in \mathscr{K}_{J_t}$. This condition imposes two separate requirements. First, $\lambda$ must remain of Hodge type $(1,1)$ for the complex structure $J_t$. Second, it must admit a positive $J_t$-$(1,1)$ representative.

To describe the first condition, we begin by extending $\lambda$ to the nearby fibers. Let $B \subset \mathrm{Def}(X,J)$ be a sufficiently small contractible neighborhood of the origin and let $\mathbb{H}^2_{\mathbb{Z}}$ be the local system whose fiber at $t\in B$ is $H^2(X_t,\mathbb{Z})$. Consider the associated holomorphic vector bundle $\mathscr{H}^2 := \mathbb{H}^2_{\mathbb{C}} \otimes \mathcal{O}_B$ equipped with its Gauss–Manin connection, whose flat sections are precisely the locally constant sections of $\mathbb{H}^2_{\mathbb{C}}$. Since $B$ is contractible, the class $\lambda \in H^2(X,\mathbb{Z})$ extends uniquely to a flat integral section $\widetilde{\lambda} \in H^0(B,\mathbb{H}^2_{\mathbb{Z}})$, such that $\widetilde{\lambda}(0)=\lambda$. The Hodge decompositions of the fibers define a holomorphic subbundle $F^1\mathscr{H}^2$ whose fiber at $t$ is $H^{2,0}(X_t)\oplus H^{1,1}(X_t)$. Let

$$q\colon \mathscr{H}^2 \to \mathscr{H}^2/F^1\mathscr{H}^2$$

denote the quotient map and, regarding $\widetilde{\lambda}$ as a holomorphic section of $\mathscr{H}^2$, define

$$s_\lambda := q(\widetilde{\lambda}).$$

The fiber of the quotient bundle at $t$ is naturally identified with $H^{0,2}(X_t)$, and under this identification $s_\lambda(t)$ is the $(0,2)$-component of the transported class $\widetilde{\lambda}(t)$. Since $\widetilde{\lambda}(t)$ is

real, this component vanishes if and only if its $(2,0)$-component vanishes. Consequently, $\widetilde{\lambda}(t) \in H^{1,1}(X_t, \mathbb{R})$ if and only if $s_\lambda(t) = 0$. It follows that the locus on which the fixed topological class $\lambda$ remains of type $(1,1)$ is the analytic subspace $s_\lambda^{-1}(0) \subset B$.

It remains to verify that positivity imposes no additional local equations. Choose a smoothly varying family of Kähler metrics on the fibers $X_t$ whose value at $t = 0$ is the metric determined by $\omega_{\mathrm{CY},J}$; such a family exists by [34]. For $t \in s_\lambda^{-1}(0)$, let $\alpha_t$ denote the harmonic representative of the class $2\pi\lambda$ with respect to the chosen metric on $X_t$. Because $2\pi\lambda$ is of type $(1,1)$ on $X_t$ it follows that $\alpha_t$ is a real closed $(1,1)$-form. On the other hand, harmonic representatives depend smoothly on the parameter, and hence $\alpha_t \to \alpha_0 = \omega_{\mathrm{CY},J}$ as $t \to 0$. Lastly, given that positivity is an open condition on real $(1,1)$-forms, $\alpha_t$ remains positive for all sufficiently small $t \in s_\lambda^{-1}(0)$. Thus, after shrinking $\mathrm{Def}(X,J)$ if necessary, we get that $\lambda \in H^{1,1}(X_t, \mathbb{R})$ if and only if $\lambda \in \mathcal{K}_{J_t}$.

We may therefore define the local $\lambda$-polarized deformation space by

$$\mathrm{Def}(X, J, \lambda) := \{t \in \mathrm{Def}(X,J) \mid \lambda \in \mathcal{K}_{J_t}\} = s_\lambda^{-1}(0),$$

where the equality is understood as an equality of analytic germs at the origin. Hence, locally around the original polarized manifold, the only equations defining the polarized deformation space are the equations requiring that $\lambda$ remain of Hodge type $(1,1)$. Notice that this condition concerns the cohomology class $\lambda$, not the particular representative $\omega_{\mathrm{CY},J}$. The form $\omega_{\mathrm{CY},J}$ itself need not remain of type $(1,1)$ for $J_t$; it is the existence of some positive $J_t$-$(1,1)$ representative of the same class that is required.

Finally we show that the analytic germ $\mathrm{Def}(X, J, \lambda)$ is smooth. Since $s_\lambda$ is a holomorphic section of the vector bundle $\mathcal{H}^2/F^1\mathcal{H}^2$, it is enough to show that its differential at the origin is surjective. By the Bogomolov–Tian–Todorov theorem, $\mathrm{Def}(X,J)$ is locally an open neighborhood of the origin in $\mathcal{H}^{0,1}(X, T^{1,0}X)$, and hence we may identify $T_0 \mathrm{Def}(X,J) \simeq H^1(X, T_X)$. Let $v \in H^1(X, T_X)$ and let $\theta \in \mathcal{H}^{0,1}(X, T^{1,0}X)$ be its harmonic representative. Writing the corresponding deformation as $\phi(\tau v) = \tau\theta + O(\tau^2)$, our graph convention gives

$$\bar{Z}_\tau = \bar{Z} + \tau\theta(\bar{Z}) + O(\tau^2)$$

for a $(0,1)$-vector with respect to $J_{\tau v}$. Since $\omega_{\mathrm{CY},J}$ is of type $(1,1)$ with respect to $J$, we obtain

$$\left.\frac{d}{d\tau}\right|_{\tau=0} \omega_{\mathrm{CY},J}(\bar{Z}_\tau, \bar{W}_\tau) = (\theta \lrcorner \omega_{\mathrm{CY},J})(\bar{Z}, \bar{W}).$$

Thus $\theta \lrcorner \omega_{\mathrm{CY},J}$ represents the infinitesimal $(0,2)$-component of the polarization class. Furthermore, it is $\bar{\partial}$-closed and therefore

$$(ds_\lambda)_0 \colon H^1(X, T_X) \longrightarrow H^2(X, \mathcal{O}_X), \quad v \longmapsto [v \lrcorner \lambda] := \frac{1}{2\pi}[\theta \lrcorner \omega_{\mathrm{CY},J}].$$

It remains to show that this map is surjective. By Serre duality, its dual is the cup-product map

$$H^{n-2}(X, K_X) \longrightarrow H^{n-1}(X, \Omega_X^1 \otimes K_X), \qquad \eta \longmapsto \lambda \smile \eta.$$

Since $K_X \simeq \mathcal{O}_X$, this map is identified with the Lefschetz operator $L_\lambda$, hence it is injective. Thus the dual of $(ds_\lambda)_0$ is injective, so $(ds_\lambda)_0$ is surjective. Altogether, the holomorphic implicit function theorem now shows that $\mathrm{Def}(X, J, \lambda)$ is smooth near the origin, with

$$T_0 \mathrm{Def}(X, J, \lambda) = \ker\left( H^1(X, T_X) \xrightarrow{v \mapsto [v \lrcorner \lambda]} H^2(X, \mathcal{O}_X) \right)$$

and

$$\dim_{\mathbb{C}} \mathrm{Def}(X, J, \lambda) = h^1(X, T_X) - h^2(X, \mathcal{O}_X).$$

## 1.2 Metric compactifications

It is a fact that the moduli space of Calabi–Yau manifolds and in particular the polarized one is in general non-compact. This leads to the natural problem of constructing *geometric compactifications* of $\mathcal{M}_{\mathrm{CY}}^{\lambda}(X)$. This being a fundamental question in Kähler geometry.

Yau's theorem provides a differential-geometric perspective on this question. To every polarized Calabi–Yau manifold $(X, J, \lambda)$ it associates the unique Ricci-flat Kähler form $\omega_{\mathrm{CY},J} \in 2\pi\lambda$ and hence a compact metric space $\big(X, d_{\mathrm{CY},J}\big)$, where $d_{\mathrm{CY},J}$ is the distance induced by the corresponding Riemannian metric.

The crucial observation is that the assignment $\Phi_\lambda([J]) := [(X, d_{\mathrm{CY},J})]$ defines a canonical metric map from $\mathcal{M}_{\mathrm{CY}}^{\lambda}(X)$ to the space Met of isometry classes of compact metric spaces, endowed with the Gromov–Hausdorff distance. Hence, the Gromov–Hausdorff topology provides a suitable framework in which to formulate and study possible limits of degenerating sequences of Calabi–Yau manifolds. In this way, one may investigate the global geometry of $\mathcal{M}_{\mathrm{CY}}^{\lambda}(X)$ through the behavior of the corresponding canonical Ricci-flat metrics.

Before we proceed with a more detailed exposition of this idea, let us first recall some basic notions in Gromov–Hausdorff theory. Let $(X, d_X)$ and $(Y, d_Y)$ be compact metric spaces. Their Gromov–Hausdorff distance is defined by

$$d_{\mathrm{GH}}(X, Y) := \inf \left\{ d_H^Z(X, Y) \,\middle|\, X, Y \hookrightarrow Z \text{ isometrically} \right\}$$

where the infimum is taken over all metric spaces $(Z, d_Z)$ and $d_H^Z$ denotes the Hausdorff distance between compact subsets of $Z$. The resulting function is a genuine metric on Met and in particular $d_{GH}(X, Y) = 0$ if and only if $X$ and $Y$ are isometric.

The foundational result which will allow us to talk about convergence to singular spaces is the following theorem of Gromov.

**Theorem 1.2.1** (Gromov's precompactness)**.** *Let $\mathcal{S}$ be a family of compact metric spaces and suppose that $\mathcal{S}$ is uniformly totally bounded, in the following sense:*

1. *there exists $D > 0$ such that $\mathrm{diam}(Y) \leq D$ for every $Y \in \mathcal{S}$;*
2. *for every $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ such that every $Y \in \mathcal{S}$ admits an $\varepsilon$-dense subset containing at most $N(\varepsilon)$ points.*

*Then $\mathcal{S}$ is precompact with respect to the Gromov–Hausdorff distance. That is, every sequence in $\mathcal{S}$ admits a Gromov–Hausdorff convergent subsequence whose limit is a compact metric space.*

For Riemannian manifolds, the following criterion provides a convenient way to verify that a family is uniformly totally bounded.

**Proposition 1.2.2.** *Let $\mathcal{S}$ be a family of compact connected $m$-dimensional Riemannian manifolds satisfying*

$$\mathrm{Ric}_g \geq (m-1)kg \qquad \text{and} \qquad \mathrm{diam}(Y, g) \leq D,$$

*where $k \in \mathbb{R}$, and $D > 0$. Then, for every $\varepsilon > 0$, there exists an integer $N = N(m, k, D, \varepsilon)$ such that every $(Y, g) \in \mathcal{S}$ admits an $\varepsilon$-dense subset containing at most $N$ points. Consequently, $\mathcal{S}$ is uniformly totally bounded and hence Gromov–Hausdorff precompact.*

The proof is based on an illuminating application of the Bishop–Gromov volume comparison theorem, and we therefore briefly present the argument; see [13, §3]. Set $k_0 := \min\{k, 0\}$. The assumed curvature bound then implies $\mathrm{Ric}_g \geq (m-1)k_0 g$. Fix $\varepsilon > 0$ and suppose that $\varepsilon < D$, since otherwise any single point forms an $\varepsilon$-dense subset of $Y$. Choose a maximal $\varepsilon$-separated set $\{p_1, \ldots, p_N\} \subset Y$. By maximality, this set is $\varepsilon$-dense, while the balls $B(p_i, \varepsilon/2)$ are pairwise disjoint. Let $v_{k_0,m}(r)$ denote the volume of a ball of radius $r$ in the simply connected $m$-dimensional space form of constant sectional curvature $k_0$. Since $\mathrm{diam}(Y, g) \leq D$, we have $B(p_i, 2D) = Y$ for every $i$. The Bishop–Gromov volume comparison theorem, applied with radii $\varepsilon/2$ and $2D$, gives

$$\frac{\mathrm{Vol}(B_g(p_i, \varepsilon/2))}{\mathrm{Vol}(Y)} \geq \frac{v_{k_0,m}(\varepsilon/2)}{v_{k_0,m}(2D)}.$$

Summing over the pairwise disjoint balls, we obtain

$$\mathrm{Vol}(Y) \geq \sum_{i=1}^{N} \mathrm{Vol}(B(p_i, \varepsilon/2)) \geq N \frac{v_{k_0,m}(\varepsilon/2)}{v_{k_0,m}(2D)} \mathrm{Vol}(Y),$$

and hence

$$N \leq \frac{v_{k_0,m}(2D)}{v_{k_0,m}(\varepsilon/2)}.$$

This bound depends only on $m, k, D, \varepsilon$, and is therefore uniform over the family. Thus the family is uniformly totally bounded and consequently Gromov–Hausdorff precompact.

Returning to our original problem, it is important to notice that given a family $[J_i] \in \mathcal{M}^{\lambda}_{\mathrm{CY}}(X)$ we can not apply the above criterion directly because the diameters of the Ricci-flat metrics may still become unbounded along a degenerating sequence. To remedy this, one works with the following normalization of the Ricci-flat Kähler metrics

$$\widehat{g}_i := \mathrm{diam}(X_i, g_i)^{-2} g_i.$$

Then Gromov's theorem implies that the diameter-rescaled family is Gromov–Hausdorff precompact. Its closure in the diameter-normalized metric image, therefore provides a natural metric enlargement of the moduli space.

*Remark* 1.2.3. The diameter normalization selects the macroscopic, or largest, scale of the degeneration. Indeed, setting $D_i := \mathrm{diam}(X_i, g_i)$ and passing to a subsequence, suppose that

$$(X_i, \widehat{g}_i) \xrightarrow{\mathrm{GH}} (Y, d_Y).$$

The limit $(Y, d_Y)$ records the geometry visible at length scales comparable to $D_i$. Geometric features occurring at smaller scales $s_i \ll D_i$ may collapse and therefore become invisible in this limit. Further information about the degeneration can nevertheless be extracted by choosing points $p_i \in X_i$ and scales $\rho_i \to 0$, and considering the pointed rescalings

$$(X_i, \rho_i^{-2}\widehat{g}_i, p_i) = (X_i, (\rho_i D_i)^{-2} g_i, p_i).$$

Whenever such a sequence admits a pointed Gromov–Hausdorff limit, that limit describes the geometry at the original length scale $s_i = \rho_i D_i$. If further geometric concentration occurs, the procedure may be repeated at successively smaller scales. The resulting collection of limits can be organized naturally in a hierarchy, whose root is the diameter-scale limit and whose lower levels record the geometry appearing at smaller scales. Under suitable additional hypotheses, such a hierarchical structure is commonly referred to as a *bubble tree*. Thus the diameter-normalized limit captures the macroscopic geometry of the degeneration, while pointed rescaled limits provide a finer multiscale description.

## 1.3 Weil–Petersson geometry

We work on the smooth analytic germ $\mathrm{Def}(X,J,\lambda)$ and denote by $\omega_{\mathrm{CY},J}$ the unique Ricci-flat Kähler metric in the class $2\pi\lambda$. Given $v \in T_0\,\mathrm{Def}(X,J,\lambda)$ let $\rho(v) \in H^1\big(X,T^{1,0}X\big)$ be the Kodaira–Spencer class and let $\mu_v \in \mathscr{H}^{0,1}\big(X,T^{1,0}X\big)$ be its unique $\omega_{\mathrm{CY},J}$-harmonic representative. The Weil–Petersson metric on $\mathrm{Def}(X,J,\lambda)$ is defined by

$$g_{WP}(v,w) := \frac{1}{\mathrm{Vol}(X,\omega_{\mathrm{CY},J})}\int_X \langle \mu_v,\mu_w\rangle_{\omega_{\mathrm{CY},J}}\, dV_{\omega_{\mathrm{CY},J}}.$$

Let us make explicit the geometric meaning of this definition. Write the Ricci-flat Kähler form at the marked point as

$$\omega_{\mathrm{CY},J} = \sqrt{-1}\, g_{\alpha\bar\beta}\, dz^\alpha \wedge d\bar z^\beta,$$

and write the harmonic representative of $v \in T_0\,\mathrm{Def}(X,J,\lambda)$ locally as

$$\mu_v = (\mu_v)^\alpha_{\bar\beta}\, d\bar z^\beta \otimes \frac{\partial}{\partial z^\alpha}.$$

If $\mu_w$ is the harmonic representative corresponding to another tangent vector $w$, then their pointwise Hermitian inner product is given by

$$\langle \mu_v,\mu_w\rangle_{\omega_{\mathrm{CY},J}} = g^{\alpha\bar\beta} g_{\gamma\bar\delta} (\mu_v)^\gamma_{\bar\beta}\overline{(\mu_w)^\delta_{\bar\alpha}}.$$

It is useful to interpret this contraction in terms of an ordinary covariant tensor. We therefore define $(S_v)_{\bar\alpha\bar\beta} := g_{\gamma\bar\alpha}(\mu_v)^\gamma{}_{\bar\beta}$ and notice that

$$\big(\mu_v \lrcorner \omega_{\mathrm{CY},J}\big)_{\bar\alpha\bar\beta} = \sqrt{-1}\,\big((S_v)_{\bar\beta\bar\alpha} - (S_v)_{\bar\alpha\bar\beta}\big)\,.$$

We claim that

$$\mu_v \lrcorner \omega_{\mathrm{CY},J} = 0, \tag{1.1}$$

from which, it will then follow that

$$(S_v)_{\bar\alpha\bar\beta} = (S_v)_{\bar\beta\bar\alpha}. \tag{1.2}$$

This is the content of Tian's Lemma 7.1 [69]. For completeness, we spell out the Hodge-theoretic argument implicit in its proof. Since $v$ is tangent to the polarized deformation space, its Kodaira–Spencer class satisfies the infinitesimal polarization condition $[\mu_v \lrcorner \omega_{\mathrm{CY},J}]_{\bar\partial} = 0$ in $H^2(X,\mathcal{O}_X) \simeq H^{0,2}_{\bar\partial}(X)$. Consequently, there exists $\eta \in A^{0,1}(X)$ such that $\mu_v \lrcorner \omega_{\mathrm{CY},J} = \bar\partial\eta$ and in particular, $\mu_v \lrcorner \omega_{\mathrm{CY},J}$ is $\bar\partial$-closed. The key idea is to notice that it is also $\bar\partial$-co-closed and consequently $\bar\partial$-harmonic.

Let $\Omega$ be a holomorphic volume form on $X$. By the Bochner principle, $\Omega$ is parallel with respect to the Ricci-flat Kähler metric, so $\nabla\Omega = 0$. Then, $\mu_v \lrcorner \Omega$ is $\bar\partial$-harmonic and hence, using $\Delta_{\bar\partial} = \Delta_\partial$ and

$$\langle \Delta_\partial(\mu_v \lrcorner \Omega), \mu_v \lrcorner \Omega\rangle_{L^2} = \|\partial(\mu_v \lrcorner \Omega)\|^2_{L^2} + \|\partial^*(\mu_v \lrcorner \Omega)\|^2_{L^2} = 0,$$

we get that $\mu_v \lrcorner \Omega$ is $\partial$-closed. In local holomorphic coordinates, the later condition is $\nabla_i(\mu_v)^i{}_{\bar j} = 0$. Together with $\bar\partial^*\mu_v = 0$, this gives

$$\big(\bar\partial^*(\mu_v \lrcorner \omega_{\mathrm{CY},J})\big)_{\bar k} = \sqrt{-1}\, g_{i\bar k}\big(\bar\partial^*\mu_v\big)^i + \sqrt{-1}\,\nabla_i(\mu_v)^i{}_{\bar k} = 0.$$

Thus, $\mu_v \lrcorner \omega_{\mathrm{CY},J}$ is $\bar\partial$-harmonic. Finally, since $\mu_v \lrcorner \omega_{\mathrm{CY},J} = \bar\partial\eta$, integration by parts gives

$$\begin{aligned}\|\mu_v \lrcorner \omega_{\mathrm{CY},J}\|_{L^2}^2 &= \big\langle \bar\partial\eta, \mu_v \lrcorner \omega_{\mathrm{CY},J}\big\rangle_{L^2} \\ &= \big\langle \eta, \bar\partial^*(\mu_v \lrcorner \omega_{\mathrm{CY},J})\big\rangle_{L^2} \\ &= 0.\end{aligned}$$

Consequently, (1.1) holds and (1.2) then follows immediately.

Thus $S_v$ is a symmetric $(0,2)$-tensor rather than a differential $(0,2)$-form. Moreover, lowering the vector index is an isometry, so

$$\langle \mu_v, \mu_w\rangle_{\omega_{\mathrm{CY},J}} = g^{k\bar i} g^{\ell\bar j} (S_v)_{\bar i\bar j} \overline{(S_w)_{\bar k\bar\ell}}.$$

We now relate $S_v$ to the infinitesimal variation of the Ricci-flat metric. Let $\gamma\colon \Delta \longrightarrow \mathrm{Def}(X, J, \lambda)$ be a holomorphic curve with $\gamma(0) = 0$ and $\dot\gamma(0) = v$. Using the differentiable trivialization of the Kuranishi family and the harmonic gauge, the corresponding deformation of complex structures is represented by $\phi(t) = t\mu_v + O(t^2)$. Moreover, according to the graph convention used above, a $(0,1)$-vector for $J_t$ has the form

$$\bar Z_t = \bar Z + t\mu_v(\bar Z) + O(t^2).$$

Let $g_t$ denote the Riemannian metric associated with the Ricci-flat Kähler form $\omega_{\mathrm{CY},J_t}$, pulled back to the fixed differentiable manifold $X$, and set

$$h_v := \left.\frac{\partial}{\partial t}\right|_{t=0} g_t.$$

Since $g_t$ is Hermitian with respect to $J_t$, one has $g_t(\bar Z_t, \bar W_t) = 0$. Differentiating this at $t = 0$ gives

$$h_v(\bar Z, \bar W) + g_{\mathrm{CY},J}\big(\mu_v(\bar Z), \bar W\big) + g_{\mathrm{CY},J}\big(\bar Z, \mu_v(\bar W)\big) = 0.$$

The symmetry of $S_v$ therefore implies

$$(h_v)^{0,2}(\bar Z, \bar W) = -2S_v(\bar Z, \bar W),$$

or equivalently,

$$(h_v)^{0,2} = -2S_v.$$

Thus the symmetric tensor obtained by lowering the vector index of $\mu_v$ is, up to the factor $-2$ determined by our graph convention, precisely the $(0,2)$-part of the infinitesimal variation of the Ricci-flat metric.

It follows that the Weil–Petersson metric can equivalently be written as

$$g_{\mathrm{WP}}(v, \overline{w}) = \frac{1}{\mathrm{Vol}(X, \omega_{\mathrm{CY},J})} \int_X \langle S_v, S_w\rangle_{\omega_{\mathrm{CY},J}}\, dV_{\omega_{\mathrm{CY},J}}.$$

In particular,

$$\|v\|_{\mathrm{WP}}^2 = \frac{1}{4\mathrm{Vol}(X, \omega_{\mathrm{CY},J})} \left\|(h_v)^{0,2}\right\|_{L^2}^2.$$

In this precise sense, the Weil–Petersson metric measures the $L^2$-size of the infinitesimal variation of the Ricci-flat metric in the complex-structure direction represented by $v$. More precisely, it measures its anti-Hermitian component; the remaining components of the metric derivative depend on the choice of differentiable trivialization and gauge.

We next follow Tian's argument and express the Weil–Petersson metric in terms of the variation of a holomorphic volume form (see [69, §7]). Retain the nowhere-vanishing holomorphic $n$-form $\Omega$ used above. Since $\Omega$ is parallel with respect to $\omega_{\mathrm{CY},J}$, its pointwise norm is constant on $X$. Moreover, contraction with $\Omega$ satisfies the pointwise identity

$$\langle \mu_v \lrcorner \Omega, \mu_w \lrcorner \Omega \rangle_{\omega_{\mathrm{CY},J}} = |\Omega|^2_{\omega_{\mathrm{CY},J}} \langle \mu_v, \mu_w \rangle_{\omega_{\mathrm{CY},J}} .$$

It follows from the definition of the Weil–Petersson metric that

$$g_{\mathrm{WP}}(v, \overline{w}) = \frac{\displaystyle\int_X \langle \mu_v \lrcorner \Omega, \mu_w \lrcorner \Omega \rangle_{\omega_{\mathrm{CY},J}} \, dV_{\omega_{\mathrm{CY},J}}}{\displaystyle\int_X |\Omega|^2_{\omega_{\mathrm{CY},J}} \, dV_{\omega_{\mathrm{CY},J}}}. \tag{1.3}$$

We now use the Weil identity for primitive form to rewrite the preceding expression. The crucial observation is that the identity $\mu_v \lrcorner \omega_{\mathrm{CY},J} = 0$ implies that the $(n-1,1)$-form $\mu_v \lrcorner \Omega$ is primitive. To see this, let $L$ denote the Lefschetz operator $L\alpha = \omega_{\mathrm{CY},J} \wedge \alpha$. Since $\mu_v \lrcorner \Omega$ has total degree $n$, it is primitive if and only if $L(\mu_v \lrcorner \Omega) = 0$. Using the Leibniz rule for contraction by the vector-valued one-form $\mu_v$, we therefore obtain

$$\mu_v \lrcorner \big(\omega_{\mathrm{CY},J} \wedge \Omega\big) = \big(\mu_v \lrcorner \omega_{\mathrm{CY},J}\big) \wedge \Omega + \omega_{\mathrm{CY},J} \wedge \big(\mu_v \lrcorner \Omega\big).$$

Thus, given that $\omega_{\mathrm{CY},J} \wedge \Omega = 0$ and $\mu_v \lrcorner \omega_{\mathrm{CY},J} = 0$ we get that $L\big(\mu_v \lrcorner \Omega\big) = 0$, and the claim follows.

The Weil identity states that if $\alpha$ is a primitive form of type $(p,q)$ and total degree $k = p + q$, then

$$*\alpha = (-1)^{\frac{k(k+1)}{2}} (\sqrt{-1})^{p-q} \frac{L^{n-k}}{(n-k)!} \alpha.$$

Since $\omega_{\mathrm{CY},J}$ is real, the form $\overline{\mu_w \lrcorner \Omega}$ is primitive of type $(1, n-1)$. Taking $k = n$ in the Weil identity gives

$$*(\overline{\mu_w \lrcorner \Omega}) = (-1)^{\frac{n(n+1)}{2}} (\sqrt{-1})^{2-n} (\overline{\mu_w \lrcorner \Omega}) = -(\sqrt{-1})^{n^2} (\overline{\mu_w \lrcorner \Omega}).$$

Similarly, the $(0,n)$-form $\overline{\Omega}$ is primitive and satisfies $*\overline{\Omega} = (\sqrt{-1})^{n^2} \overline{\Omega}$. It follows that

$$\int_X \langle \mu_v \lrcorner \Omega, \mu_w \lrcorner \Omega \rangle_{\omega_{\mathrm{CY},J}} \, dV_{\omega_{\mathrm{CY},J}} = -(\sqrt{-1})^{n^2} \int_X (\mu_v \lrcorner \Omega) \wedge (\overline{\mu_w \lrcorner \Omega}),$$

whereas

$$\int_X |\Omega|^2_{\omega_{\mathrm{CY},J}} \, dV_{\omega_{\mathrm{CY},J}} = (\sqrt{-1})^{n^2} \int_X \Omega \wedge \overline{\Omega}.$$

Substituting these identities into the expression in Equation (1.3) for the Weil–Petersson metric, we obtain

$$g_{\mathrm{WP}}(v, \overline{w}) = -\frac{\displaystyle\int_X \big(\mu_v \lrcorner \Omega\big) \wedge \overline{\big(\mu_w \lrcorner \Omega\big)}}{\displaystyle\int_X \Omega \wedge \overline{\Omega}}. \tag{1.4}$$

Finally, we relate this expression to the curvature of the Hodge line bundle. Let $\mathcal{H}^n$ denote the degree-$n$ cohomology bundle over $\mathrm{Def}(X, J, \lambda)$. Its top Hodge subbundle $F^n\mathcal{H}^n$ has fiber $\big(F^n\mathcal{H}^n\big)_t = H^0(X_t, K_{X_t})$. Since $h^{n,0}(X_t) = 1$, the bundle $F^n\mathcal{H}^n$ is a holomorphic line bundle. After shrinking the deformation space if necessary, choose a nowhere-vanishing holomorphic family $\Omega_t \in H^0(X_t, K_{X_t})$. Its Hodge norm is defined by

$$\|\Omega_t\|^2_{\mathrm{H}} := (\sqrt{-1})^{n^2} \int_{X_t} \Omega_t \wedge \overline{\Omega_t}.$$

**Proposition 1.3.1** (Tian)**.** *The Hermitian metric $g_{\mathrm{WP}}$ is Kähler. Its associated Kähler form is locally given by*

$$\omega_{\mathrm{WP}} = -\sqrt{-1}\,\partial\bar\partial \log\left((\sqrt{-1})^{n^2}\int_{X_t}\Omega_t\wedge\overline{\Omega_t}\right). \tag{1.5}$$

*Equivalently,*

$$-\log\|\Omega_t\|^2_{\mathrm{H}} \tag{1.6}$$

*is a local Kähler potential for the Weil–Petersson metric.*

*Proof.* It is enough to verify the identity at the marked point in an arbitrary tangent direction. We assume that $\Omega_t$ is such that $\Omega_0 = \Omega$. Let $\gamma\colon \Delta \to \mathrm{Def}(X, J, \lambda)$ be a holomorphic curve satisfying $\gamma(0) = 0$ and $\dot\gamma(0) = v$, and write $\Omega_s := \Omega_{\gamma(s)}$. Set

$$\dot\Omega_0 := \frac{\partial}{\partial s}\bigg|_{s=0}\Omega_s.$$

Using a differentiable trivialization of the family, we regard $\Omega_s$ as a smoothly varying family of closed complex-valued $n$-forms on $X$.

We first determine the type decomposition of $\dot\Omega_0$ with respect to the central complex structure $J$. With our graph convention, a $(0,1)$-vector for the complex structure $J_{\gamma(s)}$ has the form

$$\bar Z_s = \bar Z + s\mu_v(\bar Z) + O(s^2).$$

Since $\Omega_s$ is of type $(n,0)$ with respect to $J_{\gamma(s)}$, one has $\iota_{\bar Z_s}\Omega_s = 0$. Differentiating at $s = 0$ gives

$$\iota_{\bar Z}\dot\Omega_0 + \iota_{\mu_v(\bar Z)}\Omega = 0.$$

To determine what this implies for the type components of $\dot\Omega_0$, write

$$\dot\Omega_0 = \sum_{q=0}^{n}\alpha_q, \quad \alpha_q := \big(\dot\Omega_0\big)^{n-q,q}.$$

Since contraction by a vector of type $(0,1)$ annihilates $\alpha_0$, the preceding identity becomes

$$\sum_{q=1}^{n}\iota_{\bar Z}\alpha_q = -\iota_{\mu_v(\bar Z)}\Omega.$$

For each $q \geq 1$, the form $\iota_{\bar Z}\alpha_q$ has type $(n-q, q-1)$, whereas $\iota_{\mu_v(\bar Z)}\Omega \in A^{n-1,0}(X)$, because $\mu_v(\bar Z)$ is of type $(1,0)$. By uniqueness of the type decomposition, it follows that

$$\iota_{\bar Z}\alpha_q = 0 \quad \text{for every } q \geq 2 \text{ and every } \bar Z \in T^{0,1}X.$$

An $(n-q,q)$-form with $q > 0$ is determined by its contractions with vectors of type $(0,1)$. Consequently, $\alpha_q = 0$ for every $q \geq 2$. Thus the only possible type components of $\dot\Omega_0$ are of types $(n,0)$ and $(n-1,1)$. For $q = 1$, the same contraction identity gives

$$\iota_{\bar Z}\alpha_1 = -\iota_{\mu_v(\bar Z)}\Omega.$$

With our convention for contraction by the vector-valued $(0,1)$-form $\mu_v$, one has

$$\iota_{\bar Z}\big(\mu_v \lrcorner\, \Omega\big) = \iota_{\mu_v(\bar Z)}\Omega.$$

It follows that

$$\big(\dot\Omega_0\big)^{n-1,1} = -\mu_v \lrcorner\, \Omega.$$

Therefore, the form $\dot\Omega_0 + \mu_v \lrcorner\, \Omega$ is of pure type $(n,0)$.

We next show that $\dot\Omega_0 + \mu_v \lrcorner\, \Omega$ is holomorphic. Since each $\Omega_s$ is closed and exterior differentiation commutes with differentiation in the parameter,

$$d\dot\Omega_0 = \frac{\partial}{\partial s}\Big|_{s=0} d\Omega_s = 0.$$

Moreover, as shown above, $\mu_v \lrcorner \Omega$ is $\bar\partial$-harmonic, hence $\bar\partial\big(\mu_v \lrcorner \Omega\big) = 0$. Since $X$ is Kähler, the identity $\Delta_\partial = \Delta_{\bar\partial}$ implies that $\mu_v \lrcorner\, \Omega$ is also $\partial$-harmonic and, in particular, $\partial\big(\mu_v \lrcorner\, \Omega\big) = 0$. Thus $d\big(\mu_v \lrcorner\, \Omega\big) = 0$, and consequently $d(\dot\Omega_0 + \mu_v \lrcorner\, \Omega) = 0$. Since $\dot\Omega_0 + \mu_v \lrcorner\, \Omega$ has type $(n,0)$, its $\partial$-derivative vanishes for dimensional reasons. Hence $\bar\partial(\dot\Omega_0 + \mu_v \lrcorner\, \Omega) = 0$, so $\dot\Omega_0 + \mu_v \lrcorner\, \Omega$ is a holomorphic $n$-form.

Because $\Omega$ is nowhere vanishing, there exists a holomorphic function $f_v$ on $X$ such that

$$\dot\Omega_0 + \mu_v \lrcorner\, \Omega = f_v \Omega.$$

The manifold $X$ is compact and connected, so every holomorphic function on $X$ is constant. Therefore, there exists $c_v \in \mathbb{C}$ such that

$$\dot\Omega_0 = c_v \Omega - \mu_v \lrcorner\, \Omega.$$

This is Tian's variation formula (cf Lemma 7.2 in [69]), with the sign determined by our graph convention.

We now differentiate the logarithm of the Hodge norm. Set

$$Q(s) := (\sqrt{-1})^{n^2} \int_X \Omega_s \wedge \overline{\Omega_s}.$$

Since the cohomology classes $[\Omega_s]$ depend holomorphically on $s$ and the intersection pairing is flat, differentiation gives

$$\begin{aligned} &- \frac{\partial^2}{\partial s\, \partial \bar s}\Big|_{s=0} \log Q(s) \\ &\quad = -\frac{\displaystyle\int_X \dot\Omega_0 \wedge \overline{\dot\Omega_0}}{\displaystyle\int_X \Omega \wedge \overline{\Omega}} + \frac{\displaystyle\Big(\int_X \dot\Omega_0 \wedge \overline{\Omega}\Big)\Big(\int_X \Omega \wedge \overline{\dot\Omega_0}\Big)}{\displaystyle\Big(\int_X \Omega \wedge \overline{\Omega}\Big)^2}. \end{aligned}$$

Since $\mu_v \lrcorner\, \Omega$ has type $(n-1,1)$, the mixed terms vanish for reasons of type:

$$(\mu_v \lrcorner\, \Omega) \wedge \overline{\Omega} = 0, \qquad \Omega \wedge (\overline{\mu_v \lrcorner\, \Omega}) = 0.$$

It follows that

$$\int_X \dot\Omega_0 \wedge \overline{\Omega} = c_v \int_X \Omega \wedge \overline{\Omega},$$

and

$$\int_X \Omega \wedge \overline{\dot\Omega_0} = \overline{c_v} \int_X \Omega \wedge \overline{\Omega}.$$

Moreover,

$$\int_X \dot\Omega_0 \wedge \overline{\dot\Omega_0} = |c_v|^2 \int_X \Omega \wedge \overline{\Omega} + \int_X (\mu_v \lrcorner\, \Omega) \wedge (\overline{\mu_v \lrcorner\, \Omega}).$$

Substituting these identities into the preceding expression, the terms containing $|c_v|^2$ cancel, and we obtain

$$-\frac{\partial^2}{\partial s\,\partial\bar{s}}\bigg|_{s=0}\log\left((\sqrt{-1})^{n^2}\int_X \Omega_s\wedge\overline{\Omega_s}\right)=-\frac{\displaystyle\int_X \big(\mu_v\lrcorner\,\Omega\big)\wedge\overline{\big(\mu_v\lrcorner\,\Omega\big)}}{\displaystyle\int_X \Omega\wedge\overline{\Omega}}.$$

By Equation (1.4), the right-hand side is precisely $g_{\mathrm{WP}}(v,\overline{v})$. Since $v$ was arbitrary, polarization gives the corresponding identity for $g_{\mathrm{WP}}(v,\overline{w})$. Therefore,

$$\omega_{\mathrm{WP}}=-\sqrt{-1}\,\partial\bar{\partial}\log\left((\sqrt{-1})^{n^2}\int_{X_t}\Omega_t\wedge\overline{\Omega_t}\right).$$

The right-hand side is a closed real $(1,1)$-form whose associated Hermitian metric is $g_{\mathrm{WP}}$. Hence $g_{\mathrm{WP}}$ is Kähler, and $-\log\|\Omega_t\|^2_{\mathrm{Hdg}}$ is a local Kähler potential. □

Some remarks are in order. If $\Omega_t$ is replaced by another holomorphic frame $\Omega'_t=f(t)\Omega_t$, then its Hodge norm is multiplied by $|f(t)|^2$. The corresponding local potentials therefore differ by $-\log|f(t)|^2$, whose complex Hessian vanishes. Hence the Kähler form in the proposition is independent of the chosen holomorphic family of volume forms.

We conclude this section by recalling a geometric characterization of finite Weil–Petersson distance, which relates the metric geometry of the moduli space to the behavior of the Ricci-flat metrics on the corresponding Calabi–Yau manifolds. The following theorem combines results of Wang [72], Tosatti [71], and Takayama [68]; the metric equivalence is recorded in [68, Corollary 1.7].

**Theorem 1.3.2** (Wang–Tosatti–Takayama)**.** *Let $\pi^*\colon(\mathcal{X}^*,\mathcal{L}^*)\to\Delta^*$ be a smooth projective family of connected $n$-dimensional Calabi–Yau manifolds, with $\mathcal{L}^*$ relatively ample, admitting a flat projective model over $\Delta$. Let $g_t$ be the Riemannian metric associated with the Ricci-flat Kähler form $\omega_{\mathrm{CY},t}$ in the class $2\pi c_1(\mathcal{L}^*|_{X_t})$. The following conditions are equivalent:*

1. *The origin lies at finite distance for the Weil–Petersson metric induced on $\Delta^*$.*
2. *The diameters are uniformly bounded as $t\to 0$: there exists $C>0$ such that $\operatorname{diam}(X_t,g_t)\le C$ for all sufficiently small $t\neq 0$.*
3. *The metrics are uniformly volume noncollapsed: there exists $c>0$ such that, for all sufficiently small $t\neq 0$, every $x\in X_t$, and every $0<r\le\operatorname{diam}(X_t,g_t)$,*

$$\operatorname{Vol}_{g_t}\big(B_{g_t}(x,r)\big)\ge c\,r^{2n}.$$

Here finite Weil–Petersson distance means that there exists a curve in $\Delta^*$ approaching the origin with finite length for the possibly degenerate metric induced by the family. The noncollapsing condition concerns the volumes of metric balls and allows singularities to form; it does not assert a uniform curvature bound or a positive uniform lower bound on the injectivity radius.

The equivalence of the two geometric conditions follows directly from Bishop–Gromov comparison and the constancy of the total volume. The polarization fixes

$$\operatorname{Vol}_{g_t}(X_t)=\frac{1}{n!}\int_{X_t}\omega_{\mathrm{CY},t}^n,$$

while $\mathrm{Ric}(g_t) = 0$ implies

$$\frac{\mathrm{Vol}_{g_t}\big(B_{g_t}(x,r)\big)}{r^{2n}} \geq \frac{\mathrm{Vol}_{g_t}(X_t)}{\mathrm{diam}(X_t,g_t)^{2n}}$$

for $0 < r \leq \mathrm{diam}(X_t, g_t)$. A uniform diameter bound therefore gives the stated noncollapsing estimate. Conversely, applying that estimate with $r = \mathrm{diam}(X_t, g_t)$ and bounding the volume of the ball by the total volume gives a uniform diameter bound.

This characterization also relates to the diameter normalization considered above. Setting $\widehat{g}_t := \mathrm{diam}(X_t, g_t)^{-2} g_t$, we have

$$\mathrm{Vol}_{\widehat{g}_t}(X_t) = \mathrm{diam}(X_t, g_t)^{-2n}\,\mathrm{Vol}_{g_t}(X_t).$$

Thus finite Weil–Petersson distance is equivalent to a positive uniform lower bound on the total volume of the diameter-normalized metrics. Moreover, Takayama's argument shows that noncollapsing along a single sequence $t_i \to 0$ already implies finite Weil–Petersson distance [68, §3.1]. Consequently, if the origin lies at infinite Weil–Petersson distance, then $\mathrm{diam}(X_t, g_t) \to \infty$ and $\mathrm{Vol}_{\widehat{g}_t}(X_t) \to 0$ as $t \to 0$. In the next section we discuss the algebro-geometric interpretation of this distinction.

*Remark* 1.3.3. A fundamental ingredient in the implication from noncollapsing to finite Weil–Petersson distance in Theorem 1.3.2 is the theory of Donaldson–Sun [15]; see [68, §3.1]. In the polarized Ricci-flat setting considered here, their theory shows that compact Gromov–Hausdorff limits of uniformly noncollapsed sequences are homeomorphic to normal projective Calabi–Yau varieties with at worst canonical singularities, and their metric structures are induced by singular Ricci-flat Kähler–Einstein metrics. Thus the limiting metric spaces retain the structure of algebraic varieties!

From a modular perspective, this suggests a partial compactification of $\mathcal{M}^{\lambda}_{\mathrm{CY}}(X)$ obtained by adjoining suitable polarized singular Calabi–Yau varieties equipped with their canonical metrics, while retaining the complex structures and polarization data. A beautiful realization of this idea is due to Yuguang Zhang [74, Theorem 1.1], who constructs such a partial compactification for moduli of polarized simply connected Calabi–Yau manifolds with fixed Hilbert polynomial, building on Donaldson–Sun's theory and the convergence results of Rong–Zhang [58]. One can also consider the metric completion of $(\mathcal{M}^{\lambda}_{\mathrm{CY}}(X), g_{\mathrm{WP}})$. In Zhang's setting, every added boundary point can be approached along a curve of finite Weil–Petersson length, and one-parameter degenerations at finite Weil–Petersson distance extend uniquely to his enlarged moduli space [74, Theorem 1.3]. These results provide a modular interpretation of the boundary reached by such degenerations.

Finally, it is important to emphasize that our investigation will focus on the Weil–Petersson geometry of $\mathcal{M}^{\lambda}_{\mathrm{CY}}(X)$ at infinity, and more precisely on its behavior along polarized degenerations whose origin lies at infinite Weil–Petersson distance. By the preceding theorem, the associated Ricci-flat Kähler metrics necessarily collapse after diameter normalization. As we shall see in the next section, this highly transcendental phenomenon necessarily leads to a change of perspective: the limiting metric spaces are no longer naturally described as algebraic varieties, and understanding their geometry requires us to look beyond the algebraic category. It will turn out that tropical and non-Archimedean analytic geometry provide a natural framework for this transition.

## 1.4 Degeneration of Calabi–Yau metrics and algebraic geometry

Let $\pi\colon (\mathcal{X}^*, \mathcal{L}^*) \longrightarrow \Delta^*$ be a polarized degeneration of $n$-dimensional Calabi–Yau manifolds. Thus, $\pi$ is a smooth projective morphism with connected fibers, and $\mathcal{L}^*$ is relatively

ample. By Yau's theorem, each fiber $X_t$ carries a unique Ricci-flat Kähler metric $\omega_{\mathrm{CY},t}$ satisfying $[\omega_{\mathrm{CY},t}] = 2\pi c_1(\mathscr{L}^*|_{X_t})$. It is then natural to ask the following question.

**Question 1.4.1.** How is the limiting behavior of $(X_t, \omega_{\mathrm{CY},t})$ as $t \to 0$ related to the algebro-geometric data encoded by the degeneration $\pi$?

This question has been at the center of several recent developments in Kähler geometry. In this section, we describe the emerging geometric picture arising from these developments, focusing on those aspects of the theory that will be relevant for our purposes. Our account is necessarily selective rather than comprehensive, and we refer the reader to the surveys [43, 65], whose clear expositions have substantially shaped our presentation.

First, we need to introduce the notion of essential skeleton $Sk(\mathcal{X}^*)$ whose dimension gives qualitative information about the severity of the degeneration. In what follows we assume that the family admits a projective model $\pi\colon \mathcal{X} \to \Delta$, where $\mathcal{X}$ is normal, $\pi$ is flat and projective, and $\mathcal{X}|_{\Delta^*}$ is identified with $\mathcal{X}^*$. Moreover, we may assume that $\mathcal{X}$ is smooth and that

$$\mathcal{X}_0 = \sum_{i \in I} E_i$$

is a reduced simple normal-crossings divisor. In this case $\pi$ is called a *semistable snc model*.

**Definition 1.4.2.** The dual intersection complex $\Delta(\mathcal{X}_0)$ is the regular cell complex whose vertices correspond to the irreducible components $E_i$ of $\mathcal{X}_0$. More generally, every connected component of a nonempty intersection $E_{i_0} \cap \cdots \cap E_{i_p}$ determines a $p$-dimensional cell, with the incidence relations induced by inclusions among strata.

*Remark* 1.4.3. It is important to notice that the dual complex of an arbitrary snc model is not an invariant of the punctured degeneration. For example, consider the trivial family $Y \times \Delta \longrightarrow \Delta$. Its central fiber is irreducible, so its dual complex consists of a single point. If one blows up a smooth proper subvariety $Z \subsetneq Y \times \{0\}$, then the new central fiber has two components, namely the strict transform of $Y \times \{0\}$ and the exceptional divisor. These components intersect, so the new dual complex contains an interval. Nevertheless, the family over $\Delta^*$, its monodromy, and its Ricci-flat metrics are unchanged. To remedy this one must either work with a suitable minimal dlt model or retain only the faces on which a relative holomorphic volume form has minimal weight. We follow the latter.

Choose a nowhere-vanishing relative holomorphic volume form

$$\Omega \in H^0\big(\mathcal{X}^*, K_{\mathcal{X}^*/\Delta^*}\big).$$

After multiplication by a meromorphic function of the base coordinate, $\Omega$ extends as a rational section of $K_{\mathcal{X}/\Delta}$. Since it has no zeros or poles over $\Delta^*$, its divisor is vertical and can be written as

$$\mathrm{div}_{\mathcal{X}}(\Omega) = \sum_{i \in I} a_i E_i.$$

Multiplying $\Omega$ by a power of $t$ shifts all the coefficients $a_i$ by the same integer. We therefore normalize $\Omega$ so that $\min_{i \in I} a_i = 0$.

**Definition 1.4.4.** The minimal-weight dual subcomplex

$$\Delta_{\min}(\mathcal{X}_0, \Omega) \subseteq \Delta(\mathcal{X}_0)$$

is the union of those cells corresponding to strata $E_{i_0} \cap \cdots \cap E_{i_p} \neq \varnothing$ for which $a_{i_0} = \cdots = a_{i_p} = 0$. Equivalently, it is the subcomplex spanned by the components on which the weight of $\Omega$ is minimal.

This subcomplex does not depend on the choice of $\Omega$. Indeed, if $\Omega'$ is another relative holomorphic volume form, then $\Omega' = f(t)\Omega$ for some nowhere-vanishing holomorphic function on $\Delta^*$. Writing $f(t) = t^k u(t)$, where $u(0) \neq 0$, shows that every $a_i$ is shifted by the same integer $k$. Hence the set of components attaining the minimum remains unchanged.

We now explain the intrinsic object underlying the preceding construction, recalling the necessary facts from non-Archimedean geometry without developing their proofs. Let $X_K^{\mathrm{an}}$ denote the Berkovich analytification of the generic fiber of the degeneration over $K = \mathbb{C}((t))$, equipped with its $t$-adic absolute value. Locally, this analytification consists of multiplicative seminorms on the coordinate rings of affine open subsets extending the absolute value on $K$, with the topology of pointwise convergence; see [11, Appendix A.2]. The volume form $\Omega$ determines an intrinsic weight function on $X_K^{\mathrm{an}}$, for its construction, see [48, §4].

**Definition 1.4.5.** The *essential skeleton* of the Calabi–Yau degeneration $\mathcal{X}^*$, denoted by $\mathrm{Sk}(\mathcal{X}^*)$, is the subset of $X_K^{\mathrm{an}}$ on which the weight function associated with $\Omega$ attains its minimum.

This definition is independent of the choice of $\Omega$. Any two nowhere-vanishing volume forms on the generic fiber differ by an element of $K^\times$, which changes their weight functions by an additive constant. Thus $\mathrm{Sk}(\mathcal{X}^*)$ is intrinsic to the punctured degeneration. Moreover, by [48, Theorem 4.7.5], the canonical embedding of $\Delta(\mathcal{X}_0)$ into $X_K^{\mathrm{an}}$ restricts to a homeomorphism

$$\Delta_{\min}(\mathcal{X}_0, \Omega) \xrightarrow{\sim} \mathrm{Sk}(\mathcal{X}^*).$$

Consequently, the minimum-weight subcomplex of any semistable snc model provides a finite polyhedral presentation of the essential skeleton, compatible with its canonical piecewise affine structure. The model determines this presentation, while the underlying subset of $X_K^{\mathrm{an}}$ is independent of that choice. We therefore define the degeneration index by

$$m(\mathcal{X}^*) := \dim_{\mathbb{R}} \mathrm{Sk}(\mathcal{X}^*) = \dim \Delta_{\min}(\mathcal{X}_0, \Omega).$$

Having introduced the essential skeleton and its degeneration index, we can now state the first conjectural picture, which relates these algebro-geometric invariants to the limiting behavior of the Ricci-flat metrics after diameter normalization. The following is due to Sun–Zhang [67, Conjecture 7.3].

**Conjecture 1.4.6.** *Let $\pi\colon (\mathcal{X}^*, \mathcal{L}^*) \to \Delta^*$ be a polarized degeneration of $n$-dimensional Calabi–Yau manifolds admitting a projective model as above, and assume that $m(\mathcal{X}^*) > 0$. For each $t \in \Delta^*$, let $\omega_{\mathrm{CY},t}$ be the Ricci-flat Kähler metric on $X_t$ in the class $2\pi c_1(\mathcal{L}^*|_{X_t})$, and set*

$$\widehat{\omega}_{\mathrm{CY},t} := \big(\mathrm{diam}_{\omega_{\mathrm{CY},t}}(X_t)\big)^{-2}\omega_{\mathrm{CY},t}.$$

*Then, as $t \to 0$, the spaces $(X_t, \widehat{\omega}_{\mathrm{CY},t})$ converge in the Gromov–Hausdorff sense to a compact metric space $(B, d_B)$ homeomorphic to $\mathrm{Sk}(\mathcal{X}^*)$. In particular, the topological dimension of $B$ is $m(\mathcal{X}^*)$.*

Several results and constructions support this geometric picture. These include the foundational work of Gross–Wilson [25] and Gross–Tosatti–Zhang [23, 24] on collapsing Ricci-flat metrics, particularly in the K3 setting, the study of K3 moduli compactifications by Odaka–Oshima [53], and Odaka's description of the Gromov–Hausdorff limits of principally polarized abelian varieties [51]. In higher dimensions, relevant developments include the gluing construction of Sun–Zhang [67], Li's work on the metric SYZ conjecture near large complex structure limits [42, 44], and his potential-theoretic analysis of

intermediate degenerations [45]. The general diameter estimates of Li–Tosatti [46] provide a further analytic ingredient. We refer to these works and to [43] for their precise statements, hypotheses, and relations to the conjecture.

*Remark* 1.4.7. The assumption $m(\mathcal{X}^*) > 0$ places us in the collapsing regime after diameter normalization. Indeed, the diameter estimates of Li–Tosatti [46, Theorem 1.1] imply that $\operatorname{diam}_{\omega_{\mathrm{CY},t}}(X_t) \to \infty$, whereas the polarization fixes $\operatorname{Vol}_{\omega_{\mathrm{CY},t}}(X_t)$. Consequently,

$$\operatorname{Vol}_{\widehat{\omega}_{\mathrm{CY},t}}(X_t) = \frac{\operatorname{Vol}_{\omega_{\mathrm{CY},t}}(X_t)}{\big(\operatorname{diam}_{\omega_{\mathrm{CY},t}}(X_t)\big)^{2n}} \longrightarrow 0.$$

Gromov's compactness theorem already guarantees the existence of subsequential limits; the conjecture asserts convergence without passing to a subsequence and identifies the topology of the limit with the essential skeleton.

*Remark* 1.4.8. The conjecture thus predicts that the Calabi–Yau manifolds collapse, after diameter normalization, from real dimension $2n$ to a space of topological dimension $m(\mathcal{X}^*) \leq n$, whose topology is encoded by the essential skeleton. Such a limit need not carry a natural structure of a polarized complex algebraic variety, even when its underlying topological space happens to admit an algebraic realization. This makes precise the change of perspective announced in the preceding section: a moduli-theoretic framework that records these limits at the diameter scale must accommodate real metric spaces modeled topologically on essential skeleta, bringing tropical and non-Archimedean geometry into the description of its boundary.

Conjecture 1.4.6 concerns the geometry visible after diameter normalization. As discussed in Section 1.2, a finer understanding of a metric degeneration requires considering its geometry at different scales and basepoints, as organized by the bubble-tree picture. Among these scales, the volume scale is naturally distinguished by the polarization, which fixes the total volume of the Ricci-flat metrics even when their diameters diverge.

The particular relevance of this scale to our investigation will emerge from the results of the later chapters. For abelian varieties and polarized K3 surfaces, we shall see that the pointed Gromov–Hausdorff asymptotics of the Weil–Petersson metric on their moduli spaces admit a modular interpretation in terms of the boundary geometry of the compactifications arising from volume-scale limits of the underlying Calabi–Yau metrics. Our emphasis on this scale is therefore motivated a posteriori by these results, which also suggest seeking an analogous relationship for more general Calabi–Yau manifolds. With this motivation in mind, we recall the following conjectural picture proposed by Spotti; see [64, Conjecture picture 2].

**Conjecture 1.4.9.** *Let $\pi\colon (\mathcal{X}^*, \mathcal{L}^*) \to \Delta^*$ be a polarized degeneration of $n$-dimensional Calabi–Yau manifolds admitting a projective model as above, and assume that $m(\mathcal{X}^*) > 0$. Normalize the Ricci-flat metrics to have volume one by setting*

$$\bar{\omega}_{\mathrm{CY},t} := \big(\operatorname{Vol}_{\omega_{\mathrm{CY},t}}(X_t)\big)^{-1/n} \omega_{\mathrm{CY},t}.$$

*Then the generic pointed Gromov–Hausdorff limits of $(X_t, \bar{\omega}_{\mathrm{CY},t}, p_t)$ as $t \to 0$ have the product form*

$$(Y, \omega_{\mathrm{CY},Y}) \times (\mathbb{R}^{m(\mathcal{X}^*)}, g_{\mathrm{flat}}),$$

*where $Y$ is a compact polarized Calabi–Yau variety of complex dimension $n - m(\mathcal{X}^*)$ with at worst canonical singularities, appearing as a minimal log canonical center of a minimal dlt model. The product is equipped with the metric induced by the possibly singular Ricci-flat metric on $Y$ and the Euclidean metric.*

*Remark* 1.4.10. As in [64], this statement should be understood as a conjectural picture: the precise meaning of generic basepoints, the interpretation of the limiting polarization, and the ambiguities associated with the choice of a minimal dlt model require further clarification.

Evidence for this picture comes from Odaka's analysis of volume-normalized limits of principally polarized abelian varieties [51, §2.5.3], the gluing construction of Sun–Zhang [67], and the metric ansatz underlying Li's study of intermediate degenerations [45]. For K3 surfaces, the collapsing analysis of Sun–Zhang [66] is also relevant; we refer to [64, §4.1.3] for its interpretation at the volume scale. The convergence of volume forms studied by Boucksom–Jonsson [11] provides a related perspective on the generic regions appearing in the statement.

The appearance of a lower-dimensional Calabi–Yau factor at the volume scale is suggested by the geometry underlying the gluing construction and the ansatz in [67, 45]. In these examples, a central role is played by a compact intersection stratum $Y$ of complex dimension $n - m(\mathcal{X}^*)$, which is Calabi–Yau by adjunction. Its normal directions supply the remaining ingredients of the local model: the logarithmic radial variables describe a region of the corresponding face of the dual complex, while the angular variables give real torus bundles over $Y$. More concretely, near a point where $m(\mathcal{X}^*) + 1$ components meet transversely, the semistable local equation $t = z_0 \cdots z_{m(\mathcal{X}^*)}$ leaves $m(\mathcal{X}^*)$ independent radial variables and the same number of angular variables. In the generic regions described by these ansätze, the torus directions shrink at the volume scale, the compact Calabi–Yau factor remains at finite size, and the radial directions extend over increasingly large distances. This provides the geometric motivation for a pointed limit of the form $Y \times \mathbb{R}^{m(\mathcal{X}^*)}$.

On the other hand, the terminology of log canonical centers gives an algebro-geometric description of the strata involved in this picture. For the snc pair $(\mathcal{X}, \mathcal{X}_0)$, its log canonical centers are precisely the irreducible components of the nonempty intersections of components of $\mathcal{X}_0$, and a center is minimal if it contains no smaller such stratum. On a minimal dlt model, these minimal centers provide the corresponding notion of deepest stratum, allowing singularities; see [50]. Their identification with the compact factors surviving at the volume scale is part of the conjectural picture.

## 1.5 Weil–Petersson geometry at infinity: a modular perspective

The preceding discussion brings together two geometric aspects of Calabi–Yau degenerations: the collapsing behavior of the Ricci-flat Kähler metrics on the fibers and the Weil–Petersson geometry at infinity of their moduli space. The aim of this thesis is to investigate the relation between them, taking steps towards a moduli-theoretic description of the multiscale geometry of degenerating Ricci-flat Kähler metrics. A natural motivation comes from the definition of the Weil–Petersson metric itself: since it measures infinitesimal variations of Calabi–Yau structures through their canonical Ricci-flat metrics, one may ask whether its asymptotic geometry encodes the variation of the limiting metric structures as well. This leads to the guiding question of our investigation.

**Question 1.5.1.** To what extent is the Weil–Petersson geometry at infinity of the moduli space determined by the multiscale behavior of the Ricci-flat Kähler metrics along polarized degenerations of Calabi–Yau manifolds?

To formulate this question more precisely, let $\pi\colon (\mathcal{X}^*, \mathcal{L}^*) \to \Delta^*$ be a polarized algebraic degeneration of $n$-dimensional Calabi–Yau manifolds, with associated Ricci-flat

Kähler metrics $\omega_{\mathrm{CY},t}$ in the classes $2\pi c_1(\mathcal{L}^*|_{X_t})$, and let $\gamma\colon \Delta^* \to \mathcal{M}^{\lambda}_{\mathrm{CY}}(X)$ be its classifying map. We restrict our attention to degenerations for which $\gamma(t)$ escapes to infinite Weil–Petersson distance as $t \to 0$, and write $d := m(\mathcal{X}^*) > 0$. We refer to the germ of $\gamma$ at the origin, considered up to finite base change, as an *algebraic direction at infinity*. Given $t_j \to 0$ and $x_j := \gamma(t_j)$, the asymptotic geometry we seek to understand is that of the pointed spaces

$$(\mathcal{M}^{\lambda}_{\mathrm{CY}}(X), g_{\mathrm{WP}}, x_j)$$

in the pointed Gromov–Hausdorff sense.

From a modular perspective, the question becomes whether these pointed limits can themselves be interpreted as spaces parametrizing the geometric structures arising from collapse, equipped with natural metrics induced by their variation. The Gromov–Hausdorff compactifications discussed above provide a framework for seeking such an interpretation: their boundary strata record limiting objects at specified scales, and one may ask which of these strata, together with their geometric structures, are needed to describe the Weil–Petersson asymptotics. Since different scales retain different information, the desired description may involve complementary boundary data from more than one compactification.

In this thesis we investigate this problem for principally polarized abelian varieties and polarized K3 surfaces. These cases combine a locally symmetric description of their moduli spaces with a detailed understanding of many aspects of the collapse of their Ricci-flat metrics, allowing us to compare the two geometries explicitly. The results and analysis developed in the corresponding chapters suggest the conjectural picture below, which we present here to place these investigations within a common framework. We refer to the introduction for a summary of the results and to the later chapters for the mechanisms that motivate this formulation; on a first reading, the guiding question above provides the necessary orientation, and the reader may return to the detailed picture after studying these examples.

The geometric ingredients have already appeared in Conjectures 1.4.6 and 1.4.9. At the volume scale, one expects compact Calabi–Yau factors of lower dimension, whose variation should contribute an ordinary Weil–Petersson metric. At the diameter scale, one expects real limiting spaces related to the essential skeleton, together with additional affine and metric structures whose variation should contribute a real or *tropical* analogue of the Weil–Petersson metric. The proposed description combines these contributions.

**Conjectural picture 1.5.2.** *Let $[\gamma]$ be an algebraic direction at Weil–Petersson infinity as above, and set $\Sigma_\gamma := \mathrm{Sk}(\mathcal{X}^*)$. There should exist an algebraic boundary stratum $\mathscr{M}^{\mathrm{lc}}_\gamma$ and a tropical moduli space $\mathscr{T}^{aff}_\gamma$, parametrizing limiting structures of the same type as those associated with $[\gamma]$, with the following properties.*

1. *The space $\mathscr{T}^{aff}_\gamma$ parametrizes triples $(X_\gamma, g_\gamma, \mathcal{A}_\gamma)$, where $X_\gamma$ is homeomorphic to $\Sigma_\gamma$, its regular part carries an integral affine structure $\mathcal{A}_\gamma$, and $g_\gamma$ is a compatible Monge–Ampère metric whose metric completion has diameter one. The diameter-normalized Gromov–Hausdorff limit of the degeneration, together with its limiting affine structure, determines a point of this space.*

2. *The space $\mathscr{M}^{\mathrm{lc}}_\gamma$ parametrizes compact polarized Calabi–Yau varieties $Y_\gamma$ of complex dimension $n-d$, with at worst canonical singularities, equipped with their Ricci-flat metrics $h_\gamma$. These are the compact factors appearing in generic pointed limits at the volume scale,*

$$(X_{t_j}, \bar{\omega}_{\mathrm{CY},t_j}, p_j) \xrightarrow{\mathrm{pGH}} (Y_\gamma \times \mathbb{R}^d, h_\gamma + g_{\mathrm{flat}}, p_\infty),$$

*and are identified with minimal log canonical centers of minimal dlt models of the corresponding degenerations.*

3. *For $t_j \to 0$ and $x_j := \gamma(t_j)$, one has*

$$\big(\mathcal{M}^{\lambda}_{\mathrm{CY}}(X), g_{\mathrm{WP}}, x_j\big) \xrightarrow{\mathrm{pGH}} \big(\mathscr{M}^{\mathrm{lc}}_{\gamma} \times \mathcal{C}(\mathscr{T}^{aff}_{\gamma}), d_{\gamma,\infty}, z_{\infty}\big),$$

*where $\mathcal{C}(\mathscr{T}^{aff}_{\gamma}) = \mathbb{R} \times \mathscr{T}^{aff}_{\gamma}$ is a Riemannian cylinder and $d_{\gamma,\infty}$ is induced by the product metric*

$$g_{\gamma,\infty} = g^{\mathrm{lc}}_{\mathrm{WP}} + g_{\mathrm{trWP}}.$$

*Here $g^{\mathrm{lc}}_{\mathrm{WP}}$ is the natural Weil–Petersson metric on $\mathscr{M}^{\mathrm{lc}}_{\gamma}$, $s$ is a logarithmic scale coordinate, and $g_{\mathrm{trWP}}$ is a canonical real metric induced by variation of the compatible metric and affine structures parametrized by $\mathscr{T}^{aff}_{\gamma}$, with their diameter kept equal to one. We call $g_{\mathrm{trWP}}$ the* tropical Weil–Petersson metric*.*

The spaces appearing in this picture parametrize variations of the limiting structures, while a fixed degeneration determines the limiting objects and the basepoint in the proposed asymptotic space. Their precise construction, including the geometric data that must be retained at the boundary and the definition of the tropical Weil–Petersson metric, forms part of the proposed framework. In this sense, the conjectural picture expresses the expectation that the boundary geometry of suitable Gromov–Hausdorff compactifications, considered at complementary scales, provides a modular description of the Weil–Petersson geometry at infinity.

Evidence for this conjectural picture comes from the developments discussed after Conjectures 1.4.6 and 1.4.9, which support the expected descriptions of the Ricci-flat limits at the diameter and volume scales. Further support comes from the recent breakthrough of Bakker–Filipazzi–Mauri–Tsimerman [4, Corollary 1.8 and §7.3], who construct a canonical Hodge-theoretic Baily–Borel compactification for moduli spaces of smooth polarized Calabi–Yau varieties, on which a suitable power of the Hodge line bundle extends amply. The appearance of minimal log canonical centers in their analysis of the boundary is particularly suggestive, since these same varieties are expected to provide the compact factors in generic volume-scale limits. Their construction therefore provides a concrete candidate for the compactification associated with the volume scale, although its interpretation through generic pointed limits of Ricci-flat metrics remains conjectural. In Chapter 3 we give a detailed description of this picture for polarized K3 surfaces, relating the Weil–Petersson asymptotics to the Baily–Borel and Satake-adjoint compactifications and explaining the remaining steps needed to complete their modular interpretation.

# Chapter 2

# Modular interpretation of the Weil–Petersson metric asymptotics for abelian varieties[1]

## 2.1 Introduction

Motivated by the problem of studying the possible *geometric* compactifications of the moduli space of polarized Calabi–Yau manifolds, we explore the relations between the collapsing of the Ricci flat Kähler metrics along degenerating families of polarized Calabi-Yau manifolds and the asymptotic behavior of the Weil–Petersson metric at infinity on their moduli space. In this paper, we first make progress on this question by investigating the concrete case of principally polarized abelian varieties.

In order to justify our approach, let us elaborate on what we mean by a *geometric compactification*. By Yau's celebrated solution of the Calabi conjecture, for any compact Kähler manifold $(X, \omega)$ with $c_1(X) = 0$ there exists a *unique* Ricci-flat Kähler metric $\omega_{CY}$ in the Kähler class $[\omega]$. Therefore, given a polarized degenerating family of Calabi-Yau manifolds $(\mathcal{X}^*, \mathcal{L}^*) \to \Delta_t^*$, as $t \to 0$ we approach the boundary of moduli with a fixed Kähler class and the complex structure degenerates. If we denote by $\omega_{CY,t}$ the unique Ricci-flat Kähler metric on $X_t$ in the class $2\pi c_1(\mathcal{L}|_{X_t})$, we say that a compactification is *geometric* if the boundary of the compactified moduli space parametrizes the metric limit of $\omega_{CY,t}$ as $t \to 0$. In the concrete case of principally polarized abelian varieties, where $\omega_{CY}$ is a flat Kähler metric on a complex torus, these limits were classified by Odaka in [51]. There he explicitly determines the Gromov–Hausdorff limits of principally polarized abelian varieties and provides a moduli theoretic framework to understand such limits.

Following this perspective, the Weil–Petersson metric is a natural candidate for studying the geometry at infinity on the moduli space of polarized Calabi-Yau manifold because, by definition, the Weil–Petersson metric is induced by the variation of the Ricci-flat Kähler metric on the Kähler class fixed by the polarization as the complex structure varies (see Section 2.3.1 for more details); along this lines, it has been proved by Wang [72] and Tosatti [71] that on a one-parameter degeneration of polarized Calabi-Yau's the origin lies at finite Weil–Petersson distance if and only if the central fiber of the degeneration is a Calabi-Yau variety with at worst canonical singularities, which implies that the polarized

[1] This chapter reproduces the preprint *Modular interpretation of the Weil–Petersson metric asymptotics for abelian varieties*, coauthored with Yanbo Fang and available at https://arxiv.org/abs/2602.23140, with minor formatting and layout adjustments and with its abstract and acknowledgments omitted. Its references have been incorporated into the bibliography at the end of the thesis.

Ricci-flat Kähler metrics have uniformly bounded diameter and are uniformly volume non-collapsed. In contrast to this set-up, the degenerations we are interested are always at infinite Weil–Petersson distance and the Ricci-flat Kähler metric will collapse, phenomena that is already present for abelian varieties as explicitly classified by Odaka.

The main theorem of this paper is the following result that morally says that the asymptotic behavior of the Weil–Petersson metric on the moduli of principally polarized abelian varieties is encoded in the multi-scale collapsing geometry of the (canonical) flat Kähler metric along degenerating sequences of abelian varieties.

**Theorem 2.1.1.** *Let $(A_g, g_{WP})$ be the moduli space of complex $g$-dimensional principally polarized abelian varieties endowed with its canonical Kähler structure given by the Weil–Petersson metric. Then for every non-degenerate direction at infinity in $A_g$, there exists an integer $g'$ with $0 \leq g' < g$ such that the Weil–Petersson metric converges in the pointed Gromov–Hausdorff sense, to a Weil–Petersson type metric on*

$$(A_{g'} \times \mathscr{T}^{>0}_{g-g'},\, g'_{WP} + g''_{trWP}), \tag{2.1}$$

*where*

- *the moduli space $A_{g'}$ is naturally identified with a boundary stratum of the constant-volume Gromov–Hausdorff compactification $\bar{A}_g^{\mathrm{GH}_{\mathrm{vol}}}$ of $A_g$, and $g'_{WP}$ denotes the usual Weil–Petersson metric on $A_{g'}$;*
- *$\mathscr{T}^{>0}_{g-g'}$ denotes the moduli space of real $(g-g')$-dimensional flat tori of arbitrary positive diameter, which at any fixed diameter $\mathscr{T}^{c}_{g-g'}$ is naturally realized as a boundary stratum of the constant-diameter Gromov–Hausdorff compactification $\bar{A}_g^{\mathrm{GH}_{\mathrm{diam}}}$ of $A_g$. Furthermore, $g''_{trWP}$ is a "tropical Weil–Petersson type" metric on $\mathscr{T}^{>0}_{g-g'}$, defined via the variation of the limiting flat metrics.*

A few remarks are in order.

*Remark* 2.1.2. Moduli spaces like $A_g$, that can be realized as locally symmetric spaces, naturally admit several distinct compactification frameworks. Consequently, identifying which of these compactifications carries a modular geometric interpretation is one of the guiding questions of our investigation. Theorem 2.1.1 suggest that the Weil–Petersson metric can indeed shed light on this problem: as the abelian varieties degenerate, their canonical flat metrics collapse, and this collapse is reflected in the asymptotic geometry of the Weil–Petersson metric.

*Remark* 2.1.3. The theorem's formulation requires a precise notion of a *non-degenerate direction at infinity in $A_g$*, which we give in Definition 2.3.6. Roughly speaking this says that, given a sequence $\tau_n \in A_g$ after choosing lifts $\tau_n = X_n + iY_n \in \mathfrak{S}_g$ contained in a Siegel set, one may assume that $Y_n$ has exactly $g - g'$ eigenvalues tending to infinity, while the remaining $g'$ eigenvalues stay uniformly bounded above and below. Non-degeneracy then amounts to ruling out the possibility that a sub-collection of the diverging eigenvalues separates into different scales, which would force an additional splitting and hence a lower-rank limit. (Cf. Theorem 3.1 in [51] for an algebraic characterization of this condition).

*Remark* 2.1.4. One immediate consequence of Theorem 2.1.1 is that along any such non-degenerate direction, $A_g$ collapses in the pointed Gromov–Hausdorff sense to the lower–dimensional space $A_{g'} \times \mathscr{T}^{>0}_{g-g'}$. From a Riemannian viewpoint, this behavior is expected given that it is classical that $A_g$ is a complete non-compact Riemannian orbifold with finite volume.

The organization of the paper is as follows. In §2, we introduce the necessary preliminary results and fix the notation and conventions used throughout. We first recall, in §2.1, the basic theory of moduli of complex abelian varieties and fix conventions for the Siegel upper half space together with the relevant reduction theory. Next, in §2.2, in the setting of the Siegel upper half-space, we recall the Lie theoretic notion of *boundary component* for a Hermitian symmetric domain and the algebraic and differential-geometric data naturally associated with it. Next, in §2.3 we revisit the work of [51, 53] on the Gromov–Hausdorff compactifications of $A_g$. The core of the paper is §3. In §3.1 we recall the definition of the Weil–Petersson metric and the standard derivation of its explicit formula on $A_g$, which serves as the starting point for our analysis in the remainder of the paper. Finally In §3.1 we prove the main theorem after developing the necessary technical preliminaries.

## 2.2 Preliminaries

### 2.2.1 Moduli space of principally polarized abelian varieties.

In this section we recall some classical notions about complex abelian varieties and set up our notation on the Siegel upper half space and the Siegel reduction theory, we follow closely [12]. For a detailed treatment and all of the proofs, we refer to [49, 30].

Recall that $A_g$, the moduli space of principally polarized abelian varieties of complex dimension $g$, is given the structure of complex analytic space by means of the identification

$$A_g \simeq \Gamma\backslash\mathfrak{S}_g,$$

where $\mathfrak{S}_g$ is the *Siegel upper half-space*, namely $\mathfrak{S}_g = \{\tau \in \mathrm{Sym}(g,\mathbb{C}) |\ \mathrm{Im}(\tau) > 0\}$ and $\Gamma = \mathrm{Sp}(2g,\mathbb{Z})/\{\pm 1\}$ is an arithmetic subgroup of the group $\mathrm{Sp}(2,\mathbb{R})/\{\pm 1\}$ of holomorphic automorphism of $\mathfrak{S}_g$, whose action is given by

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \tau \to M\cdot\tau = (A\tau + B)(C\tau + D)^{-1}, \quad \tau \in \mathfrak{S}_g, M \in \mathcal{G}.$$

Notice that, as an analytic space, $A_g$ has the structure of a complex orbifold, or V-manifold, due to the presence of abelian varieties with extra automorphism.

**Definition 2.2.1.** Let $n \geq 3$ and let $\Gamma(n)$ denote the principal congruence subgroup of level $n$,

$$\Gamma(n) := \{\, M \in \mathrm{Sp}(2g,\mathbb{Z}) \mid M \equiv I_{2g} \pmod{n} \,\}. \tag{2.2}$$

We define the moduli space of complex $g$-dimensional principally polarized abelian varieties with level-$n$ structure as the quotient

$$A_g(n) := \Gamma(n)\backslash\mathfrak{S}_g. \tag{2.3}$$

For $n \geq 3$, the group $\Gamma(n)$ is torsion-free; hence the action on $\mathfrak{S}_g$ is free and properly discontinuous, and $A_g(n)$ is a smooth complex manifold. Equivalently, the finite cover $A_g(n) \to A_g$ provides uniformizing charts near the orbifold points of $A_g$.

We now recall some basic notions from Siegel reduction theory that will be used throughout. For a point $\tau = X + iY \in \mathfrak{S}_g$ we denote the *Jacobi decomposition* of $Y$ as $Y = LDL^t$ where

$$L = \begin{pmatrix} 1 & & & & \\ l_{12} & 1 & & 0 & \\ l_{13} & l_{23} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ l_{1g} & l_{2g} & \cdots & l_{g-1,g} & 1 \end{pmatrix}, \quad D = \mathrm{diag}(d_1, \cdots, d_g).$$

With respect to this decomposition we have that a Siegel sets $\mathfrak{F}_g(u)$ is be given by

$$\mathfrak{F}_g(u) = \{X + i\,Y \in \mathfrak{S}_g \mid |x_{ij}| < u, |l_{ij}| < u,\ 1 < ud_1,\ d_i < ud_{i+1} \text{ for all } i, j\}. \tag{2.4}$$

Additionally, it is straightforward to check that a subset of the inequalities in (2.4) define the Siegel sets on $\mathrm{Sym}^+(g,\mathbb{R})$ for the action of the discrete subgroup $\mathrm{GL}(g,\mathbb{Z})$. More specifically we have that with respect to the Jacobi decomposition $Y = LDL^t$ for a point $Y \in \mathrm{Sym}^+(g,\mathbb{R})$ a Siegel set $\mathfrak{F}_g^+(u)$ is given by

$$\mathfrak{F}_g^+(u) = \{Y \in \mathrm{Sym}^+(g,\mathbb{R}) \mid |l_{ij}| < u,\ 1 < ud_1,\ d_i < ud_{i+1} \text{ for all } i, j\}. \tag{2.5}$$

Expression that will be needed later in our analysis.

Finally let us recall that for a suitable choice of $u \gg 1$ the Siegel set $\mathfrak{F}_g(u)$ is a *coarse fundamental domain* for the action of $\Gamma$ in the sense that they satisfy the following properties:

i. $\bigcup_{u>0} \mathfrak{F}_g(u) = \mathfrak{S}_g$.

ii. $\Gamma \cdot \mathfrak{F}_g(u) = \mathfrak{S}_g$ if $u$ is large enough.

iii. For all $u > 0$ the set $\{\gamma \in \Gamma \mid \gamma\,\mathfrak{F}_g(u) \cap \mathfrak{F}_g(u) \neq \varnothing\}$ is finite.

Analogously, $\mathfrak{F}_g^+(u)$ will be a coarse fundamental domain for the action of $\mathrm{GL}(g,\mathbb{Z})$.

### 2.2.2 Boundary components of $\mathfrak{S}_g$.

In this subsection we introduce the Lie-theoretic concepts underlying the notion of boundary component for a Hermitian symmetric domain. Our main goal is to describe the boundary components of the Siegel upper half space and the algebraic data naturally attached to them. As this material is technical and notation-heavy, we do not provide a comprehensive exposition but introduce only the necessary notions and results that we will need in later sections. For a detailed treatment and all of the proofs, we refer to [49].

Recall that as a Hermitian symmetric space, the Siegel upper half-space is given by $\mathfrak{S}_g = Sp(2g,\mathbb{R})/\mathrm{U}(g)$. Here, the unitary group $\mathrm{U}(g)$, which is the stabilizer of the base-point $iI_g \in \mathfrak{S}_g$, is embedded in $\mathrm{Sp}(2g,\mathbb{R})$ by $A + iB \to \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$. The Harish-Chandra embedding

$$\Phi : \mathfrak{S}_g \to \mathfrak{D}_g, \quad \Phi(\tau) = (\tau - iI_g)(\tau + iI_g)^{-1},$$

gives a realization of $\mathfrak{S}_g$ as a bounded domain $\mathfrak{D}_g$ in $\mathrm{Sym}_g(\mathbb{C}) \simeq \mathbb{C}^{g(g+1)/2}$, where

$$\mathfrak{D}_g = \{Z \in \mathrm{Sym}_g(\mathbb{C}) \,|\, I_g - Z\bar{Z} > 0\}.$$

The boundary components of $\mathfrak{S}_g$ are defined as an stratification of the topological closure $\overline{\mathfrak{D}}_g = \{Z \in \mathrm{Sym}_g(\mathbb{C}) \,|\, I_g - Z\bar{Z} \geq 0\}$. A distinguished collection of proper boundary components is obtained by fixing $0 \leq g' < g$ and requiring

$$\left\{ Z \in \overline{\mathfrak{D}}_g \,\middle|\, Z = \begin{pmatrix} Z' & 0 \\ 0 & I_{g-g'} \end{pmatrix},\ \text{and } Z' \in \mathfrak{D}_{g'} \right\}.$$

Furthermore, each boundary component is a $\mathrm{Sp}(2g,\mathbb{R})$-translate of one of the *distinguished* boundary components. It is clear that each *distinguished* boundary component can be identified with $\mathfrak{D}_{g'}$ for $0 \leq g' < g$, and by abuse of notation we identify them with their corresponding unbounded realization $\mathfrak{S}_{g'}$.

Associated with each boundary component $\mathfrak{S}_{g'}$ we have its stabilizer subgroup $P_{g'}$ in $\mathrm{Sp}(2g,\mathbb{R})$. Each $P_{g'}$ is a maximal real parabolic subgroup and can be realized as the stabilizer of the isotropic-co-isotropic flag $I \subset I^{\perp}$, where $I = \langle e_{g'+1}, \ldots, e_g \rangle$. Let us fix $g = g' + g''$ throughout, a direct computation will show that $P_{g'}$ consist of matrices

$$P_{g'} = \left\{ \begin{pmatrix} A' & 0 & B' & * \\ * & u & * & * \\ C' & 0 & D' & * \\ 0 & 0 & 0 & (u^t)^{-1} \end{pmatrix} \in \mathrm{Sp}(2g,\mathbb{R}) \;\middle|\; \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in \mathrm{Sp}(2g',\mathbb{R}),\, u \in \mathrm{GL}(g'',\mathbb{R}) \right\}.$$

Moreover, relative to the filtration $I \subset I^{\perp}$ we have the Levi decomposition $P_{g'} = N_{g'} \rtimes L_{g'}$, where $L_{g'}$ is the Levi factor given by the direct product $L = \mathrm{Sp}(I^{\perp}/I)\,\mathrm{GL}(I) = \mathrm{Sp}(2g',\mathbb{R})\,\mathrm{GL}(g'',\mathbb{R})$, and $N_{g'}$ is the unipotent radical of $P_{g'}$ which is given in terms of the matrices

$$N_{g'} = \left\{ \begin{pmatrix} I_{g'} & 0 & 0 & n \\ m^t & I_{g''} & n^t & b \\ 0 & 0 & I_{g'} & -m \\ 0 & 0 & 0 & I_{g''} \end{pmatrix} \;\middle|\; n^t m + b = m^t n + b^t \right\}.$$

Finally, following standard terminology we will refer to $L_{g'h} = \mathrm{Sp}(2g',\mathbb{R})$ as the Hermitian factor and to $L_{g'l} = \mathrm{GL}(g'',\mathbb{R})$ as the linear factor.

From a more geometric perspective it is worth noting that, as differentiable manifolds, the decomposition $P_{g'} \cong N_{g'}\, L_{g'h}\, L_{g'l}$ is a diffeomorphism. Furthermore, this decomposition gives the existence of a diffeomorphism

$$\mathfrak{S}_g \cong P_{g'}/(\mathrm{U}(g) \cap P_{g'}) \cong N_{g'} \times \mathfrak{S}_{g'} \times \mathrm{Sym}^+(g'',\mathbb{R}), \tag{2.6}$$

which we call the *horospherical* decomposition of $\mathfrak{S}_g$ relative to $P_{g'}$. To see this we first need to recall that $P_{g'}$ acts transitively on $\mathfrak{S}_{g'}$ and that the Levi factor is stable under the Cartan involution corresponding to $\mathrm{U}(g)$. Then we observe that

$$\mathrm{U}(g) \cap P_{g'} = \left\{ \begin{pmatrix} A' & 0 & B' & 0 \\ 0 & u & 0 & 0 \\ -B' & 0 & A' & 0 \\ 0 & 0 & 0 & u \end{pmatrix} \in \mathrm{Sp}(2g,\mathbb{R}) \;\middle|\; A' + iB' \in \mathrm{U}(g'),\, u \in \mathrm{O}(g'',\mathbb{R}) \right\},$$

which identifies the boundary component $\mathfrak{S}_{g'}$ with the symmetric space of $L_{g'h} = \mathrm{Sp}(2g',\mathbb{R})$ and the cone $\mathrm{Sym}^+(g'',\mathbb{R})$ with the symmetric space of $L_{g'l} = \mathrm{GL}(g'',\mathbb{R})$. Finally, with respect to this decomposition we define the map

$$\pi_{g'} : \mathfrak{S}_g \to \mathfrak{S}_{g'} \times \mathrm{Sym}^+(g'',\mathbb{R}),\ \tau = \begin{pmatrix} \tau' & \tau''' \\ {}^t\tau''' & \tau'' \end{pmatrix} \longmapsto (\tau', Y'' - {}^tY'''(Y')^{-1}Y'''). \tag{2.7}$$

The geometric significance of the decomposition of Equation (2.6) is that it defines a coordinate system of the quotient $\Gamma\backslash\mathfrak{S}_g$ adapted to the boundary component $\mathfrak{S}_{g'}$. To see this we have the following result that essentially tell us that for a *suitable neighborhood* of $\Gamma\backslash\mathfrak{S}_g$ the action of $\Gamma$ is equivalent to the action of $P_{g'} \cap \Gamma$.

**Proposition 2.2.2.** *Fix $0 \leq g' < g$, and let $\mathtt{P}_{g'} := \Gamma \cap \mathtt{P}_{g'}$. Then, there exist a $\mathtt{P}_{g'}$-invariant neighborhood $\mathcal{U}_{g'}$ of $\mathfrak{S}_{g'}$ in $\mathfrak{S}_g$ such that the inclusion $\mathcal{U}_{g'} \hookrightarrow \mathfrak{S}_g$ induces a holomorphic embedding*

$$\mathtt{P}_{g'}\backslash\mathcal{U}_{g'} \hookrightarrow \Gamma\backslash\mathfrak{S}_g.$$

*Proof.* The existence of such a $\mathtt{P}_{g'}$-invariant neighborhood is a consequence of the Piatetskii-Shapiro cylindrical topology (see [2], Chap. III, §1.). □

In practice, we can described such invariant neighborhood in terms of the Siegel sets as follows. Fix a $u > 0$ such that $(\Gamma \cap L_{g'h}) \cdot \mathfrak{F}_{g'}(u) = \mathfrak{S}_{g'}$, for all $1 \leq g' \leq g$ and for $r > 0$ sufficiently large, set

$$\mathcal{U}^r_{g'} := \left\{ \tau \in \mathfrak{F}_g(u) \,|\, d_{g'+1} > r \right\}. \tag{2.8}$$

This will define a family of $\mathtt{P}_{g'}$-invariant neighborhoods of $\mathfrak{F}_{g'}(u) \subset \mathfrak{S}_{g'}$. Moreover, we notice that if we have a sequence $\{\tau_n\}_{n\in\mathbb{N}}$ in $\mathfrak{F}_g(u)$ we conclude that $\tau_n \to \tau'$, for some $\tau' \in \mathfrak{F}_{g'}(u)$ if and only if $\pi_{g'}(\tau_n) \to \tau'$ and $d_{g'+1}(\tau_n) \to \infty$.

The relevance of this result to our analysis is that to study the asymptotic behavior of the Weil–Petersson metric near the boundary components of $\Gamma\backslash\mathfrak{S}_g$ we can instead work on the neighborhoods $\mathcal{U}_{g'}$ modulo the action of the simpler subgroup $\mathtt{P}_{g'}$. Before we proceed, let us introduce the following notation that we will use throughout the paper.

**Notation:** Fix $0 \leq g' < g$, and let $P_{g'}$ be the parabolic subgroup associated with the $\mathfrak{S}_{g'}$ boundary component of $\mathfrak{S}_g$. Then relative to the arithmetic subgroup $\Gamma$ we denote by

$$\mathtt{P}_{g'} = \Gamma \cap P_{g'}, \quad \mathtt{N}_{g'} := \Gamma \cap N_g, \quad \mathtt{L}_{g'} = \Gamma \cap L_{g'}.$$

Moreover, for any congruence subgroup $\Gamma(n)$ we have the analogous notation, i.e., $\mathtt{P}_{g'}(n)$, $\mathtt{N}_{g'}(n)$, and $\mathtt{L}_{g'}(n)$.

*Remark* 2.2.3. Notice, that given a general Levi decomposition $P = NL$, $\mathtt{L}$ is in principle only a finite index subgroup of $\mathrm{Im}(\mathtt{P} \to L)$. However, given that we have fixed the data of the isotropic-co-siotropic flag the Levi decomposition is such that the integral structure of $\mathtt{L}$ agrees with the one of $\mathtt{P}$

To understand the quotient $\mathtt{P}_{g'}\backslash\mathfrak{S}_g$, notice that $N_{g'}$ is a normal subgroup of $P_{g'}$. Thus, we have the exact sequence

$$1 \to N_{g'} \to P_{g'} \to L_{g'} \to 1. \tag{2.9}$$

This suggest that we can compute the quotient $A_g \simeq \Gamma\backslash\mathfrak{S}_g$ in two steps. First we take the quotient by $\mathtt{N}_{g'}$, which only acts on the $N_{g'}$ factor. By general theory of arithmetic groups [9] it follows that $\mathtt{N}_{g'}$ is a uniform lattice in $N_{g'}$ and consequently the quotient $N_{g'}/\mathtt{N}_{g'}$ is a compact nilmanifold. Thus, using the decomposition in Equation (2.6) and the map defined in Equation (2.7) we get the map

$$\mathtt{N}_{g'}\backslash\mathfrak{S}_g \longrightarrow \mathfrak{S}_{g'} \times \mathrm{Sym}^+(g'', \mathbb{R}),$$

that realizes $\mathtt{N}_{g'}\backslash\mathfrak{S}_g$ as a trivial fiber bundle over the base $\mathfrak{S}_{g'} \times \mathrm{Sym}^+(g - g', \mathbb{R})$ and with fibers the compact nilmanifold $\mathtt{N}_{g'}\backslash N_{g'}$. Next, we take the quotient by $\mathtt{L}_{g'}$. This has two effects: it replaces the base of the fiber bundle by

$$\mathtt{L}_{g'}\backslash\left(\mathfrak{S}_{g'} \times \mathrm{Sym}^+(g'', \mathbb{R})\right) \cong \left(Sp(2g', \mathbb{Z})\backslash\mathfrak{S}_{g'}\right) \times \left(\mathrm{GL}(g'', \mathbb{Z})\backslash\,\mathrm{Sym}^+(g'', \mathbb{R})\right)$$

and on the fibers it acts with the stabilizer subgroup of the base-point. Thus, given that in general $\Gamma$ is not torsion-free we might get a flat orbifold structure in the fibers. Now, since in Section 2.3 we will work with the quotient

$$\Gamma\backslash\mathcal{U}_{g'} \simeq \mathtt{P}_{g'}\backslash\mathcal{U}_{g'} \subset \mathtt{P}_{g'}\backslash\mathfrak{S}_g,$$

viewed as a Riemannian orbifold endowed with the Weil–Petersson metric, local arguments must a priori be carried out in orbifold charts. To avoid repeatedly invoking this language,

we will carry out most of our analysis on the Siegel sets i.e., we will consider instead the map

$$\pi_{g'} : \mathfrak{F}_g(u) \longrightarrow \mathfrak{F}_{g'}(u) \times \mathfrak{F}^+_{g''}(u) \tag{2.10}$$

with $u \gg 1$ suitably chosen.

### 2.2.3 Geometric compactifications of $A_g$.

There are several important compactification frameworks for modular varieties such as $\Gamma\backslash\mathfrak{S}_g$, which are arithmetic quotients of Hermitian symmetric domains. In this section we review two compactifications: the Satake–Baily–Borel (SBB) compactification and the tropical compactification, in the sense of Odaka. We place special emphasis on their differential-geometric interpretation, as developed in [51].

The Satake-Baily-Borel compactification $\bar{A}_g^{\text{さ}BB}$ [2] is defined by the following construction. First, let us define the partial compactification

$$\mathfrak{S}^*_g := \mathfrak{S}_g \sqcup \mathfrak{S}_{g-1} \sqcup \cdots \sqcup \mathfrak{S}_0,$$

obtained by adjoining the distinguished rational boundary components. This set $\mathfrak{S}^*_g$ is endowed with the Satake topology (see [2] Chapter III §6.1 for details). The Satake-Baily-Borel compactification is then defined as the quotient $\bar{A}_g^{\text{さ}BB} := \mathfrak{S}^*_g/\sim$, where $\sim$ is the equivalence relation extending the action of $\Gamma$ on $\mathfrak{S}_g$.

More recently, in [51], with the aim of understanding the relationship between the differential-geometric and the algebro-geometric compactifications of moduli, Odaka explicitly classifies the Gromov–Hausdorff limits of the flat metric along degenerating sequences of complex principally polarized abelian varieties and studies the geometry of the resulting Gromov–Hausdorff compactifications of $A_g$. In particular, the following connection between the Satake-Baily-Borel compactification and the Gromov–Hausdorff compactification at the volume scale is established.

**Theorem 2.2.4** ([51] Proposition 2.13)**.** *Fix $0 \leq g' < g$ and consider a sequence of pointed $g$-dimensional complex principally polarized abelian varieties $(X_n, p_n)$ which is converging to a point of $A_{g'} \subset \partial \bar{A}_g^{\text{さ}BB}$. Then, the pointed Gromov–Hausdorff limit of $(X_n, p_n)$ with fixed volume 1 is isometric to*

$$(X_\infty \times \mathbb{R}^{g''}, (p_\infty, 0)),$$

*where $X_\infty$ is the real $2g'$-dimensional flat torus underlying a complex $g'$-dimensional principally polarized abelian variety and the $\mathbb{R}^{g''}$ is given the flat metric.*

Therefore, as an immediate consequence of this result one obtains that the Satake-Baily-Borel compactification $\bar{A}_g^{\text{さ}BB}$ parametrizes the set of pointed Gromov– Haussdorff limits of complex $g$-dimensional principally polarized abelian varieties with fixed volume, or in other words that this compactification can be differential–geometrically recovered as parametrizing the volume scale limits of the *canonical* flat Kähler metric underlying a principally polarized abelian varieties.

On the other hand, the Gromov–Hausdorff limits of principally polarized abelian varieties at the diameter scale are classified as follows.

**Theorem 2.2.5** ([51] Theorem 2.3)**.** *Fix $0 \leq g' < g$, and let $X_n$ be any sequence of complex $g$-dimensional principally polarized abelian varieties converging to a point of $A_{g'} \subset$*

[2] Following [51] and [49] we use the *Hiragana* type character "さ" which is pronouced as "SA", the first syllable of Satake. we refer to the footnote in [51] for a brief explanatory note on the choice of notation.

$\partial \bar{A}_g^{\triangleleft BB}$. *After rescaling so that* $\operatorname{diam}(X_n) = 1$, *the sequence* $X_n$ *converges, in the Gromov–Hausdorff sense, to a* $(g-r)$*-dimensional real flat torus of diameter* 1, *for some* $r$ *satisfying* $g' \leq r < g$. *Conversely, every* $(g-r)$*-dimensional real flat torus of diameter* 1 *with* $g' \leq r < g$ *occurs as the Gromov–Hausdorff limit of such a rescaled sequence.*

Odaka goes on to construct $\bar{A_g}^T$, which he calls the *tropical geometric compactification* of the moduli space of principally polarized abelian varieties, as the Gromov–Hausdorff compactification of $A_g$ at the diameter scale. As a topological space $\bar{A_g}^T$ is obtained by attaching the moduli space of real flat tori with diameter 1 of dimension $1 \leq r \leq g$,

$$\bar{A_g}^T := A_g \bigsqcup_r \mathscr{T}_r, \qquad \text{where } \mathscr{T}_r := \mathrm{GL}(r,\mathbb{Z})\backslash \mathrm{Sym}^+(r,\mathbb{R})/\mathbb{R}_{>0}, \tag{2.11}$$

and putting the topology whose basis induces the complex analytic topology in $A_g$ and the Gromov–Hausdorff topology on the boundary.

One should emphasize that this compactification is no longer an algebraic variety. From our perspective, however, this is a feature rather than a drawback: it is the geometric constraint imposed by requiring the boundary to parametrize limits of the canonical flat Kähler metric on a principally polarized abelian variety, which collapse to real manifolds at diameter scale. Furthermore, the resulting limits have a very interesting connection with the algebro-geometric degenerations of principally polarized abelian varieties, as explained by the following result of Odaka.

**Theorem 2.2.6** ([51] Theorem 3.1)**.** *Let* $\pi^* : (\mathcal{X}^*, \mathcal{L}^*) \to \Delta_t^*$ *be a flat projective family of* $g$*-dimensional principally polarized abelian varieties which extends to some quasi-projective family over* $\Delta_t$. *Furthermore, we assume that we have semi-abelian reduction over* $0 \in \Delta_t$ *and that the Raynaud extension is the trivial extension. Let* $t_n \in \Delta_t^*$ *be an arbitrary sequence converging to* 0. *Then, the fiber* $\mathcal{X}_{t_n}$ *with the rescaled flat Kähler metric of diameter 1 collapses to a* $r$*-dimensional real torus where* $r$ *is the torus rank of* $\mathcal{X}_0$. *In Particular the collapsing limit is homeomorphic to the dual complex of the degeneration.*

Notice that the result above says that, if we fix $0 \leq g' < g$ and $X_n \in A_g$ is sequence converging to a point $A_{g'} \subset \partial \bar{A}_g^{\triangleleft BB}$ along an holomorphic family, the dimension $r = \dim(X_\infty)$ of the real flat torus appearing as the diameter-rescaled limit is completely determined by the corank $r = g - g'$ of the boundary component $A_{g-r}$. This contrasts with Theorem 2.2.5, which allows the dimension to satisfy $g' \leq r < g$.

As a consequence of this result, we obtain a correspondence between the boundary strata of the Satake–Baily–Borel compactification and the tropical compactification; more precisely, we relate the differential-geometric data encoded by the volume-scale and diameter-scale Gromov–Hausdorff compactifications of $A_g$ (see the diagram below).

*Strata of Gromov–Hausdorff compactification at volume scale*

$$\begin{array}{ccccccc} A_{g-1} & \sqcup & A_{g-2} & \sqcup & \cdots & \sqcup & * \\ \updownarrow & & \updownarrow & & & & \updownarrow \\ * & \sqcup & \mathscr{T}_2 & \sqcup & \cdots & \sqcup & \mathscr{T}_g \end{array}$$

*Strata of Gromov–Hausdorff compactification at diameter scale*

This, as Odaka remarks, is a manifestation of the tropicalization phenomenon where the degeneration order is reversed when passing from the algebro-geometric setting to the tropical one.

*Remark* 2.2.7. Notice that *a posteriori* one can argue that the duality between the strata of the diameter-scale and volume-scale Gromov–Hausdorff compactifications (equivalently, between the Satake–Baily–Borel and the tropical compactifications) agrees with the Lie-theoretic data of the horospherical decomposition

$$\mathfrak{S}_g \cong N_{g'} \times \mathfrak{S}_{g'} \times \mathrm{Sym}^+(g-g', \mathbb{R}),$$

relative to the parabolic subgroup $P_{g'}$ associated with the boundary component $A_{g'}$ to which the sequence $X_n$ converges. At this point we do not claim a deeper structural statement, as the duality was derived solely from the differential-geometric behavior of the canonical flat metrics under degeneration.

*Remark* 2.2.8. Odaka further shows that the boundary of $\bar{A}_g^T$ can be described as the dual complex of *any* toroidal compactification of $A_g$, by identifying $\bar{A}_g^T$ with what he calls the generalized Morgan–Shalen compactification of $A_g$, denoted $\bar{A}_g^{\mathrm{MSBJ}}$ (see the appendix of [51] for details). Morally, this provides an invariant framework for compactifying locally symmetric spaces by attaching the data of the essential skeleton/dual complex associated with a toroidal compactification. Finally, as shown in [53], the same compactification $\bar{A}_g^T$ also recovers one of the compactifications proposed by Satake, namely the Satake compactification for the adjoint representation, denoted $\bar{A}_g^{\mathrm{Sat},\tau_{\mathrm{ad}}}$ (see Section 2.2 of [53]).

## 2.3 Asymptotic behavior of the Weil–Petersson metric

### 2.3.1 The Weil–Petersson metric on $A_g$.

We recall the definition of the Weil – Petersson metric and briefly review the standard derivation of its explicit expression on $A_g$, which will be used throughout the sequel.

We work on the deformation space of a polarized Calabi–Yau manifold. Let $\pi : (\mathcal{X}, \mathcal{L}) \to S$ be a proper holomorphic polarized family of Calabi–Yau manifolds of dimension $n$. For each $s \in S$, let $\omega_s$ denote the unique Ricci-flat Kähler metric on $X_s$ in the class $2\pi c_1(\mathcal{L}|_{X_s})$. Given $v \in T_sS$, let $\rho_s(v) \in H^1(X_s, T_{X_s}) \cong H^{0,1}_{\bar\partial}(X_s, T^{1,0}_{X_s})$ be the Kodaira–Spencer class, and denote by $\mu_v$ its $\omega_s$-harmonic representative. The Weil–Petersson metric is the $L^2$ pairing

$$g_{WP}(v_s, \overline{w}_s) = \int_{X_s} \langle \mu_v, \mu_w \rangle_{\omega_s} \, dV_{\omega_s}, \qquad v, w \in T_sS.$$

From this, it follows that the Weil–Petersson metric is induced by the variation of the Ricci-flat Kähler metric on the corresponding fibers: the norm on the tangent space is given by the norm of the harmonic representatives of the Kodaira-Spencer classes with respect to the Ricci-flat Kähler metric on the corresponding fibers according to the Calabi-Yau theorem. In [69, 70] it is shown that $g_{WP}$ is indeed a Kähler metric and that it has negative sectional curvature.

A useful alternative description is in terms of a Kähler potential. As shown in [69], one has

$$\omega_{WP} = -\, i\, \partial\bar\partial \log \int_{X_s} i^{n^2}\, \Omega_s \wedge \overline{\Omega_s},$$

Here, $\Omega_s$ is a holomorphic section of the relative canonical bundle $K_{\mathcal{X}/S}$, and $\omega_{WP}$ is the Ricci-form associated with $g_{WP}$.

*Remark* 2.3.1. This expression identifies $\omega_{\mathrm{WP}}$ with the curvature form of the $L^2$ metric on the Hodge line bundle $\lambda = \pi_* \omega_{\mathcal{X}/S}$.

We now turn to the case of abelian varieties, where the preceding expression becomes explicit.

**Proposition 2.3.2** (cf. Schumacher [63])**.** *On $A_g$ the Weil–Petersson metric is given in terms of the left-invariant metric on $\mathfrak{S}_g$, as follows*

$$\omega_{WP} = \frac{i}{4}\mathrm{tr}\big(Y^{-1}d\tau Y^{-1}d\bar{\tau}\big). \tag{2.12}$$

*In particular, for abelian varieties the Weil–Petersson metric is determined entirely by the locally Hermitian symmetric space data of $A_g$.*

*Proof.* We work on the universal covering space $\mathfrak{S}_g$. There, we have that locally an holomorphic family is given by $\mathcal{X} \to \mathfrak{S}_g$, where $\mathcal{X} := (\mathfrak{S}_g \times \mathbb{C}^g)/\mathbb{Z}^{2g}$ and $\mathbb{Z}^{2g}$ acts on $\mathfrak{S} \times \mathbb{C}^g$ via

$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} : (\tau, v) \mapsto (\tau, v + n_1 + \tau \cdot n_2).$$

Equivalently, the torus $X_\tau \cong \mathbb{R}^{2g}/\mathbb{Z}^{2g}$, with differentiable coordinates $(x^i, y^i)$, corresponding to the modular point $\tau \in \mathfrak{S}_g$ has holomorphic coordinates given by the map $(x^i, y^i) \mapsto z^i = x^i + \tau^i_j\, y^j$.

Thus, an holomorphic section $\Omega_\tau$ of the relative canonical bundle is given by means of the standard translation-invariant holomorphic $g$-form

$$\Omega_\tau = dz^1 \wedge \cdots \wedge dz^g.$$

Then, a determinant computation yields

$$i^{g^2}\, \Omega_\tau \wedge \overline{\Omega}_\tau = 2^g\, \det(\mathrm{Im}(\tau))\, dx^1 \wedge \cdots \wedge dx^g \wedge dy^1 \wedge \cdots \wedge dy^g,$$

and consequently,

$$\int_{X_\tau} i^{g^2}\, \Omega_\tau \wedge \overline{\Omega}_\tau = 2^g\, \det(\mathrm{Im}(\tau)).$$

Finally, a straightforward computation using Jacobi's formula gives

$$\omega_{WP} = \frac{i}{4}\mathrm{tr}\big(Y^{-1}d\tau Y^{-1}d\bar{\tau}\big).$$

□

We conclude with two examples that will be revisited in the next section and will serve to motivate the general framework and results.

**Example 2.3.3.** We now take up the simplest instance of our construction, namely the moduli space of elliptic curves. Let

$$A_1 \;=\; \Gamma\backslash\mathcal{H}, \qquad \mathcal{H} = \{\tau = x + iy \in \mathbb{C} \mid y > 0\}, \qquad \Gamma = \mathrm{SL}(2, \mathbb{Z}).$$

Denote by $*_\infty \in A_1$ the cusp at infinity. It is well known that a fundamental system of neighborhoods of $*_\infty$ is given, for $d \gg 1$, by

$$\mathcal{U}_d \;=\; \{\tau \in \mathcal{H} \mid \mathrm{Im}(\tau) > d\}.$$

Let $P_\infty \subset \mathrm{SL}(2, \mathbb{R})$ be the parabolic subgroup stabilizing $*_\infty$, namely

$$P_\infty \;=\; \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a \in \mathbb{R}^\times,\ b \in \mathbb{R} \right\},$$

and set $\mathtt{P}_\infty := \Gamma \cap P_\infty$. Then $\mathtt{P}_\infty$ is generated by the translation

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad T \cdot \tau = \tau + 1,$$

and, in particular, $\mathtt{P}_\infty \simeq \mathbb{Z}$. Moreover, for $d \gg 1$ the $\Gamma$-action on $\mathcal{U}_d$ is effectively reduced to the cusp stabilizer: if $\gamma \in \Gamma$ satisfies $\gamma(\mathcal{U}_d) \cap \mathcal{U}_d \neq \varnothing$, then $\gamma \in \mathtt{P}_\infty$. Equivalently, the quotient of a sufficiently deep cusp neighborhood is modeled by the parabolic quotient

$$\Gamma\backslash\mathcal{U}_d \;\cong\; \mathtt{P}_\infty\backslash\mathcal{U}_d,$$

so $\Gamma\backslash\mathcal{U}_d$ is a standard cusp neighborhood in $A_1$.

Now, for $g = 1$, Equation (2.12) recovers the Weil–Petersson metric on $A_1$ in terms of the standard hyperbolic (Poincaré) metric:

$$g_{WP} \;=\; \frac{1}{2}\frac{|d\tau|^2}{\mathrm{Im}(\tau)^2} \;=\; \frac{1}{2}\frac{dx^2 + dy^2}{y^2}, \tag{2.13}$$

with associated Kähler form

$$\omega_{WP} \;=\; \frac{i}{4}\,\frac{d\tau \wedge d\bar{\tau}}{\mathrm{Im}(\tau)^2}.$$

In particular, at a point $\tau = x + iy$ we have

$$\|\partial_x\|_{WP} = \frac{1}{2}\frac{1}{y}, \qquad \|\partial_y\|_{WP} = \frac{1}{y}.$$

Thus, as $y = \mathrm{Im}(\tau) \to \infty$ (approaching the cusp), motion in the $x$–direction becomes arbitrarily short: a fixed displacement $\Delta x$ has WP-length $\sim |\Delta x|/y \to 0$. Furthermore, since the $\mathtt{P}_\infty$-action identifies $x \sim x + n$ for $n \in \mathbb{Z}$, the $x$–direction compactifies to an $\mathbb{S}^1$; combined with the previous estimate, this shows that the $\mathbb{S}^1$ collapses in the limit. A more explicit geometric description of this behavior is obtained via the following coordinate change on $\mathtt{P}_\infty\backslash\mathcal{U}_d$:

$$r = \log\, y, \qquad \theta = x \;(\mathrm{mod}\,\mathbb{Z}).$$

Then (2.13) becomes the standard cusp form

$$g_{WP} \;=\; \frac{1}{2}\Big(dr^2 + e^{-2r}d\theta^2\Big). \tag{2.14}$$

This makes the collapse explicit: along the end $r \to \infty$ the circle direction $\theta$ is multiplied by the shrinking factor $e^{-r}$, while the base direction $r$ has bounded size. In particular, the diameter of the fiber circles satisfies

$$\mathrm{diam}_{WP}(\mathbb{S}^1_r) = \frac{1}{\sqrt{2}}\,e^{-r} \;\xrightarrow[r\to\infty]{}\; 0.$$

Thus, the asymptotic Weil–Petersson geometry of $A_1$ exhibits a collapsing circle fibration over the ray $r > \log d$.

To give a coordinate-free description of this phenomenon, let us recall from Section 2.2.2 that, relative to the parabolic subgroup $P_\infty$ of the cusp, $\mathtt{P}_\infty\backslash\mathcal{U}_d \subset A_1$ admits the structure of a fibration given by the map

$$\pi : \mathtt{P}_\infty\backslash\mathcal{H} \to \mathrm{Sym}^+(1,\mathbb{R}) \cong \mathbb{R}^+, \tag{2.15}$$

whose fiber is a compact nilmanifold, here given by $\mathtt{N}_\infty\backslash N_\infty \cong \mathbb{S}^1$. Therefore, the analysis above is simply the explicit coordinate description of the collapsing behavior of this fibration onto its base $\mathbb{R}^+$.

**Example 2.3.4.** We now turn to investigate the asymptotic Weil–Petersson geometry of the moduli space of principally polarized abelian surfaces. The main point is that this case is rich enough to exhibit the basic structural features of the asymptotics that we aim at capturing in the general setting.

We work on $A_2 = \Gamma\backslash\mathfrak{S}_2$ and assume that $\mathfrak{F}_2(u) \subset \mathfrak{S}_2$ is a Siegel set adapted to the distinguished boundary components, namely $\mathscr{H}$, and $*_\infty$. Given the Jacobi's decomposition $\tau = X + iY = X + i(L\,D\,L^t)$

$$L = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \tag{2.16}$$

and the bounds imposed by the Siegel set $\mathfrak{F}_2(u)$ we have that a convenient set of local coordinates is

$$(\theta_1, \dots, \theta_4) = (x_1, x_2, x_3, l_{21}), \qquad (r_1, r_2) = (\log\, d_1, \log\, d_2). \tag{2.17}$$

With respect to this coordinates we have that the Weil–Petersson metric is given by

$$\begin{aligned} g_{WP} = \frac{1}{2}\Bigg[ & dr_1^2 + dr_2^2 + 2e^{r_1-r_2}\,d\theta_4^2 + \frac{\left(1+e^{r_1-r_2}\theta_4^2\right)^2}{e^{2r_1}}\,d\theta_1^2 + 2\,\frac{\left(1+2e^{r_1-r_2}\theta_4^2\right)}{e^{r_1}e^{r_2}}\,d\theta_2^2 \\ & + \frac{1}{e^{2r_2}}\,d\theta_3^2 - 4\theta_4\,\frac{\left(1+e^{r_1-r_2}\theta_4^2\right)}{e^{r_1}e^{r_2}}\,d\theta_1\,d\theta_2 + 2\,\frac{\theta_4^2}{e^{2r_2}}\,d\theta_1\,d\theta_3 - 4\,\frac{\theta_4}{e^{2r_2}}\,d\theta_2\,d\theta_3\Bigg]. \end{aligned}$$

We restrict our analysis to the regime corresponding to the boundary component $\mathscr{H}$. In terms of our choice of Siegel set data, this means that a sequence $\tau_n \in \mathfrak{F}_2(u) \subset \mathfrak{S}_2$ converges to a point $\tau_\infty \in \mathfrak{F}_1(u) \subset \mathscr{H}$ if and only if $(\tau_n)_1 \to \tau_\infty$ and $d_2(\tau_n) \to \infty$. In particular, if we work on the closed metric balls $\overline{B}_{WP}(\tau_n, R)$ with $R \gg 1$, then for $n$ sufficiently large we have $d_1 \ll d_2$ uniformly on $\overline{B}_{WP}(\tau_n, R)$.

Under the above setup, the Weil–Petersson metric admits the following asymptotic expansion

$$g_{WP} = \frac{1}{2}\Big(dr_1^2 + e^{-2r_1}d\theta_1^2\Big) + \frac{1}{2}\,dr_2^2 + O\big(e^{-r_2(\tau_n)}\big). \tag{2.18}$$

Heuristically, (2.18) should be interpreted as saying that, in the direction of the boundary component $\mathscr{H}$, the Weil–Petersson geometry decouples into a lower-dimensional hyperbolic factor (in the $(r_1, \theta_1)$-variables) together with an additional rank-one symmetric space direction (the $r_2$-variable), up to an error that is uniformly small on bounded $g_{WP}$-balls centered at $\tau_n$.

To make this precise, recall that relative to the parabolic subgroup $P_{\mathscr{H}}$ there is a natural fibration

$$\pi : \mathtt{P}_{\mathscr{H}}\backslash\mathfrak{S}_2 \longrightarrow \mathtt{L}_{\mathscr{H}}\backslash(\mathscr{H} \times \mathbb{R}^+), \quad \tau \longmapsto \big(\tau_1,\ y_3 - y_2 y_1^{-1}\big). \tag{2.19}$$

Endow the total space with the Weil–Petersson metric $g_{WP}$, and the base with the product metric

$$\frac{1}{2}\Big(g_{\mathscr{H}} + g_{\mathbb{R}^+}\Big),$$

where $\frac{1}{2}g_{\mathscr{H}}$ is the Weil–Petersson metric on the lower-dimensional quotient $\mathtt{L}_{\mathscr{H},h}\backslash\mathscr{H}$ and $\frac{1}{2}g_{\mathbb{R}^+}$ is the canonical (suitably normalized) symmetric-space metric on $\mathbb{R}^+$. A direct computation then shows that (2.18) can be rephrased as

$$g_{WP} = \frac{1}{2}\,\pi^*\Big(g_{\mathscr{H}} + g_{\mathbb{R}^+}\Big) + O\Big(e^{-r_2(\tau_n)}\Big) \qquad \text{on } \ \overline{B}_{WP}(\tau_n, R). \tag{2.20}$$

Here the error term is understood in the sense that, once $R > 0$ is fixed, the coefficients of the difference tensor

$$g_{WP} - \frac{1}{2}\,\pi^*\Big(g_{\mathcal{H}} + g_{\mathbb{R}^+}\Big)$$

are bounded by $C\,e^{-r_2(\tau_n)}$ throughout the closed ball $\overline{B}_{WP}(\tau_n, R)$, for some constant $C$ that depends on the Siegel set data, $R$, and $\tau_\infty$. Therefore, (2.20) should be viewed as a *uniform, local* approximation statement: it compares the two metrics on a bounded $g_{WP}$-neighborhood of the moving basepoint $\tau_n$.

One could now strengthen (2.20) by estimating the $C^0$-norm of the difference tensor on $\overline{B}_{WP}(\tau_n, R)$ with respect to either of the two metrics, thereby obtaining a genuine uniform $C^0$-closeness result. However, attention must be paid to the fact that for a fixed $R$, and $n$ the two tensors can not be made arbitrarily close. Instead we need to work for a fixed $R$ and let $\tau_n$ vary as $n \to \infty$. In particular, the space on which the comparison is made changes every time.

We do not pursue this analysis further. Instead, in the next section we will give a characterization of the asymptotics of $g_{WP}$ in terms of pointed Gromow-Hausdorff limits, which we find to be the most natural framework for describing the large-scale geometry in the non-compact setting relevant to our applications.

Finally, in light of the essential role played by the moduli space of real flat tori and its canonical invariant metric in our analysis of the asymptotic behaviour of the Weil–Petersson metric, we introduce the following

**Definition 2.3.5.** We define the tropical Weil–Petersson metric $g_{trWP}$ on the moduli space of real flat tori of dimension $r$ to be

$$g_{trWP} = \frac{1}{2} tr(P^{-1}\,dP\,P^{-1}\,dP). \tag{2.21}$$

Besides $g_{trWP}$ having the expected normalization appearing in the Weil–Petersson metric asymtotic in Examples 2.3.3, and 2.3.4 one can give a clear modular interpretation to it as follows. Let $T^r = \mathbb{R}^r/\mathbb{Z}^r$ be the standard smooth torus, and let $P \in \mathrm{Sym}^+(r, \mathbb{R})$. Then $P$ defines a translation–invariant (hence flat) Riemannian metric $g_P$ on $T^r$ by

$$g_P = \sum_{i,j=1}^{r} p_{ij}\,dx_i\,dx_j, \qquad P = (p_{ij}).$$

Therefore, interpreting a tangent vector $V_P \in T_P\,\mathrm{Sym}^+(r, \mathbb{R})$ as the germ of a one parameter family of flat tori and recalling that we have $T_P\,\mathrm{Sym}^+(r, \mathbb{R}) \cong \mathrm{Sym}(r, \mathbb{R})$ given by the map

$$\Psi_P : T_P\,\mathrm{Sym}^+(r, \mathbb{R}) \to \mathrm{Sym}(r, \mathbb{R}), V \to P^{-1/2}\,V\,P^{-1/2}$$

Then, we can interpret $g_{trWP}$ as the natural metric measuring the variation of the flat Riemannian metric on the standard $r$-torus.

### 2.3.2 Pointed Gromov–Hasudorff limits of $(A_g,\, g_{WP}, \tau_n)$.

In this section we give a proof of Theorem 2.1.1. First, we make some notions precise:

**Definition 2.3.6.** Set $0 \le g' < g$, and let $\tau_n$ be a sequence in $A_g$, such that $\tau_n \to \tau'$ for some $\tau' \in A_{g'}$. Let $(A_g, g_{WP}, \tau_n)$ be a sequence of pointed spaces. Then, by the asymptotics of the Weil–Petersson metric in the direction given by the sequence $\tau_n$, we mean the pointed Gromov–Hausdorff limit

$$(A_g, g_{WP}, \tau_n) \xrightarrow{pGH} (Z, d_Z, z).$$

Furthermore, we say that such a sequence defines a non-degenerate direction at infinity if and only if with respect to the Jacobi's decomposition $\tau_n = X_n + i\, L_n\, D_n L_n^t$ the following condition holds

$$\lim_{n\to\infty} \frac{d_i(\tau_n)}{d_j(\tau_n)} = k_{ij} \neq 0, \quad \text{for all } g' < i < j \leq g. \tag{2.22}$$

*Remark* 2.3.7. The reason it is natural to describe the asymptotic behavior of the Weil–Petersson metric in terms of pointed Gromov–Hausdorff limits is already suggested by our analysis in Examples 2.3.3, and 2.3.4. Since $A_g$ is non-compact, there is in general no background-independent notion of global convergence for the tensor $g_{WP}$ on all of $A_g$. Instead, one compares the geometry on larger and larger pointed balls: for a degenerating sequence $\tau_n$, one studies the metric spaces $\big(\overline{B}_{A_g}(\tau_n, R), g_{WP}\big)$ for fixed $R > 0$ and lets $n \to \infty$. Pointed Gromov–Hausdorff convergence is precisely designed to capture this type of local-to-global asymptotic behavior.

Before we proceed with our general argument we motivate the discussion by giving a detailed computation for the moduli of elliptic curves.

**Example 2.3.8** (Gromov–Hausdorff collapse of the moduli of elliptic curves)**.**

With the notation and set up of Example 2.3.3 we have the following

**Proposition 2.3.9.** *Let $\tau_n \in \mathtt{P}_\infty\backslash\mathcal{U}_d \subset A_1$ be a sequence such that $\mathrm{Im}(\tau_n) \to \infty$. Then, the fibration*

$$\pi : \mathtt{P}_\infty\backslash\mathcal{H} \to \mathbb{R}^+, \tag{2.23}$$

*collapses to its base, as $\mathrm{Im}(\tau_n) \to \infty$, in the pointed Gromov-Hausdroff sense. In particular we have that*

$$(A_1, g_{WP}, \tau_n) \xrightarrow{pGH} (\mathbb{R}^+_t, dt^2/2t^2, 1), \tag{2.24}$$

*in the pointed Gromov–Hausdorff sense.*

*Proof.* Although the statement is classical and surely familiar to experts, we provide a proof for completeness and for the convenience of readers who are not accustomed to Gromov–Hausdorff convergence.

As shown in Example 2.3.3, it is more convenient to work with the coordinates $r = \log y$ and $\theta = x \,(\mathrm{mod}\, \mathbb{Z})$ on $\mathtt{P}_\infty\backslash\mathcal{H}$, and $s = \log t$ on $\mathbb{R}^+$. In these coordinates, the Weil–Petersson metric and the base metric take the form

$$g_{WP} = \frac{1}{2}\Big(dr^2 + e^{-2r} d\theta^2\Big), \quad g_{\mathbb{R}} = \frac{1}{2} ds^2.$$

To show that the fibration collapses to its base, it suffices to prove that for any $R > 0$ and any $\epsilon > 0$ there exists an $N$ such that, for every $n > N$, the map

$$\pi : \overline{B}_{\mathtt{P}_\infty\backslash\mathcal{H}}(\tau_n, R) \to \overline{B}_{\mathbb{R}}(s_n, R)$$

is an $\epsilon$-pointed Gromov–Hausdorff approximation. To see this, note first that $\pi$ is a Riemannian submersion; in particular, it is 1–Lipschitz. Moreover, the fiber diameter over the closed ball $\overline{B}_{\mathbb{R}}(s_n, R)$ admits the uniform bound

$$\mathrm{diam}_{WP}\big(\pi^{-1}(s)\big) \leq C_R\, e^{-s_n}, \quad \text{for any } s \in \overline{B}_{\mathbb{R}}(s_n, R).$$

It follows from this observations that for any $\tau_p, \tau_q \in \overline{B}_{\mathtt{P}_\infty\backslash\mathcal{H}}(\tau_n, R)$ we have

$$\|d_{WP}(\tau_p, \tau_q) - d_{\mathbb{R}}(s_p, s_q)\| \leq C_R\, e^{-s_n}.$$

Therefore, there exists a constant $C'_R > 0$ such that

$$d_{GH}\Big(\overline{B}_{\mathrm{P}_\infty\setminus\mathcal{H}}(\tau_n, R), \overline{B}_{\mathbb{R}}(s_n, R)\Big) \leq C'_R\, e^{-s_n}.$$

Consequently, choosing $N > 0$ so that $C'_R\, e^{-s_N} \leq \epsilon$ yields the desired pointed Gromov–Hausdorff approximation.

Finally, to obtain the pointed Gromov–Hausdorff convergence in (2.24), choose $s_0 = s_n$. Then the translation $s \mapsto s' = s - s_0$ is an isometry of $(\mathbb{R}, g_{\mathbb{R}})$ sending the basepoint $s_n$ to 0. In the $t$-coordinate, this corresponds to the isometry $t \mapsto t' = e^{-s_n} t$, which sends $t_n = e^{s_n}$ to 1. Thus, composing $\pi$ with this translation yields the desired pointed Gromov–Hausdorff approximation. □

We now proceed with the proof of our main theorem. First let us summarize the set up and notation for our analysis. Throughout, we assume that we are working relative to the (maximal) chain of rational boundary components

$$\mathfrak{S}_{g-1} \succ \cdots \succ \mathfrak{S}_1 \succ \mathfrak{S}_0,$$

given by the complete isotropic flag

$$\langle e_g \rangle \subset \langle e_{g-1}, e_g \rangle \subset \cdots \subset \langle e_1, \cdots, e_g \rangle,$$

For each $0 \leq g' < g$, let $P_{g'}$ denote the corresponding parabolic subgroup associated with $\mathfrak{S}_{g'}$. We fix the Levi decomposition of $P_{g'} \cong N_{g'} L_{g'}$ associated with the isotropic-coisotropic flag $I_{g'} \subset I_{g'}^{\perp}$ such that

$$\mathfrak{S}_g \cong N_{g'} \mathfrak{S}_{g'} \operatorname{Sym}^+(g'', \mathbb{R}), \qquad g'' = g - g',$$

and set

$$\mathtt{P}_{g'} := \Gamma \cap P_{g'}, \quad \mathtt{N}_{g'} := \Gamma \cap N_{g'}, \quad \mathtt{L}_{g'} := \Gamma \cap L_{g'}.$$

We fix a suitable $u > 0$ and work with the Siegel sets $\mathfrak{F}_{g'}(u)$, for $0 \leq g' \leq g$. Furthermore, we assume they are adapted to the boundary components so that for any sequence $\tau_n \in \mathfrak{F}_g(u) \subset \mathfrak{S}_g$ we have

$$\tau_n \to \tau'_\infty \in \mathfrak{F}_{g'}, \quad \text{if and only if} \quad d_{g'+1}(\tau_n) \to \infty, \ \text{ and } \ \tau'_n \to \tau'_\infty. \tag{2.25}$$

Here, for any $\tau = X + iY \in \mathfrak{S}_g$ we use the block decomposition

$$\tau = \begin{pmatrix} \tau' & \tau''' \\ {}^t\tau''' & \tau'' \end{pmatrix}, \tag{2.26}$$

relative to the boundary component $\mathfrak{S}_{g'}$, that we extend in the obvious way to its real and imaginary part.

With this notation in place, we turn to a preliminary analysis that will allow for a cleaner presentation of the main ideas in the proof. The key observation, already visible in Examples 2.3.3 and 2.3.4, is that the geometry of $A_g$ near a given boundary component is governed by the fibration induced by the horospherical decomposition relative to the associated parabolic subgroup. Crucially, the base of this fibration is the product $A_{g'} \times \mathscr{T}_{g''}^{>0}$—precisely the space appearing in the conclusion of Theorem 2.1.1. Thus, showing that this fibration collapses to its base in the pointed Gromov–Hausdorff sense will establish the theorem.

Concretely, for each $0 \leq g' < g$, we restrict our analysis to the region

$$\mathcal{U}_{g',d} := \{\tau \in \mathfrak{F}_g(u) \mid d_{g'}(\tau) > d\},$$

with $d \gg 1$ sufficiently large. Then, by Proposition 2.2.2, for $d$ large enough the quotient $\mathtt{P}_{g'}\backslash\overline{\mathcal{U}}_{g',d}$ embeds into $\Gamma\backslash\mathfrak{S}_g$, so we may work on $\mathtt{P}_{g'}\backslash\mathfrak{S}_g$. Finally, using the horospherical decomposition of $\mathfrak{S}_g$ with respect to $P_{g'}$, we obtain

$$\pi_{g'} : \mathtt{P}_{g'}\backslash\mathfrak{S}_g \to \mathtt{L}_{g'}\backslash\big(\mathfrak{S}_{g'} \times \mathrm{Sym}^+(g'', \mathbb{R})\big), \tag{2.27}$$

where $\pi_{g'}(\tau) = (\tau', t)$ with $t = Y'' - {}^tY'''(Y')^{-1}Y'''$.

We can express this more suggestively by recalling that

$$\mathtt{L}_{g'} = \mathtt{L}_{g'h} \times \mathtt{L}_{g'l} = \mathrm{Sp}(2g', \mathbb{Z}) \times \mathrm{GL}(g'', \mathbb{Z}).$$

Thus, Equation (2.27) becomes

$$\pi_{g'} : \mathtt{P}_{g'}\backslash\mathfrak{S}_g \to A_{g'} \times \mathscr{T}^{>0}_{g''}. \tag{2.28}$$

This identification shows that the base of the fibration is precisely the product of the corresponding boundary strata in the volume-scale and diameter-scale Gromov–Hausdorff compactifications of $A_g$ (cf. Remark 2.2.7 and the diagram on page 34).

From our preliminary analysis, we can restate the main theorem as follows.

**Theorem 2.3.10.** *Let $\tau_n$ be a sequence in $\mathfrak{F}_g(u) \subset \mathfrak{S}_g$ such that $\tau_n \to \tau'_\infty \in \mathfrak{F}_{g'}(u)$ for some $0 \leq g' < g$. This naturally defines a sequence in $\mathtt{P}_{g'}\backslash\mathfrak{S}_g$, which we keep denoting by $\tau_n$. Then the fibration*

$$\pi_{g'} : \mathtt{P}_{g'}\backslash\mathfrak{S}_g \longrightarrow A_{g'} \times \mathscr{T}^{>0}_{g''},$$

*collapses to its base in the pointed Gromov–Hausdorff sense. More precisely, for each fixed $R > 0$,*

$$d_{GH}\Big(B(\tau_n, R),\, B(\pi_{g'}(\tau_n), R)\Big) \leq \frac{C_R}{\sqrt{d_{g'+1}(\tau_n)}} \xrightarrow[n\to\infty]{} 0, \tag{2.29}$$

*for some constant $C_R$ depending only on the Siegel set data, on $R$, and on the limit point $\tau'_\infty$. Furthermore, for $n$ sufficiently large, we may (and will) regard $\tau_n$ as a sequence in*

$$\mathtt{P}_{g'}\backslash\overline{\mathcal{U}}_{g',d} \simeq \Gamma\backslash\overline{\mathcal{U}}_{g',d} \subset A_g.$$

*Assuming moreover that $\tau_n$ satisfies*

$$\lim_{n\to\infty} \frac{d_i(\tau_n)}{d_j(\tau_n)} = k_{i,j} \neq 0 \quad \textit{for all } g'+1 \leq i < j \leq g, \tag{2.30}$$

*we obtain*

$$(A_g,\, g_{WP},\, \tau_n) \xrightarrow{pGH} (A_{g'} \times \mathscr{T}^{>0}_{g''},\, g_{WP} + g_{trWP},\, (\tau'_\infty, t_\infty)). \tag{2.31}$$

Let us first give a brief overview of the idea of the proof. Our goal is to show three properties of this fibration:

(i) The projection map $\pi_{g'}$ is length-nonincreasing;

(ii) Given a metric ball of arbitrary radius on the base, we have a uniform diameter bound on the the fibers depending only on the center and radius of the ball;

(iii) in the direction of the boundary, the fiber diameters vanish.

These three properties immediately imply that the fibration collapses to its base in the pointed Gromov–Hausdorff sense.

**Lemma 2.3.11.** *Fix $0 \leq g' < g$, and equip the total space and base of the fibration*

$$\pi_{g'} : \mathtt{P}_{g'} \backslash \mathfrak{S}_g \to A_{g'} \times \mathscr{T}_{g''}^{\geqslant 0},$$

*with the Riemannian structures given, respectively, by $g_{WP}$ and by the product metric $g'_{WP} + g''_{trWP}$. Then, the projection $\pi_{g'}$ is a Riemannian submersion and in particular it is 1–Lipschitz.*

*Proof.* Since being a Riemannian submersion is local property we may work on the universal covers [3]

$$\pi_{g'} : \mathfrak{S}_g \longrightarrow \mathfrak{S}_{g'} \times \mathrm{Sym}^+(g'', \mathbb{R}).$$

Furthermore, since $P_{g'}$ acts transitively and by isometries and $\pi_{g'}$ is $P_{g'}$-equivariant, it suffices to verify the claim at the base point $iI_g \in \mathfrak{S}_g$. Consider the differential

$$(\pi_{g'*})_{iI_g} : T_{iI_g}\mathfrak{S}_g \longrightarrow T_{(iI_{g'}, I_{g''})}\big(\mathfrak{S}_{g'} \times \mathrm{Sym}^+(g'', \mathbb{R})\big).$$

Using the identification $T_{iI_g}\mathfrak{S}_g \simeq \mathrm{Sym}(g, \mathbb{C})$, write a tangent vector as

$$V = V_X + iV_Y, \qquad V_X, V_Y \in \mathrm{Sym}(g, \mathbb{R}),$$

and decompose into blocks according to $g = g' + g''$:

$$V_X = \begin{pmatrix} V'_X & V'''_X \\ {}^tV'''_X & V''_X \end{pmatrix}, \qquad V_Y = \begin{pmatrix} V'_Y & V'''_Y \\ {}^tV'''_Y & V''_Y \end{pmatrix}.$$

A direct computation yields

$$(\pi_{g'*})_{iI_g}(V) = \big(V'_X + iV'_Y,\, V''_Y\big),$$

so $(\pi_{g'*})_{iI_g}$ is surjective and its kernel is

$$\ker(\pi_{g'*})_{iI_g} = \{\, V \in T_{iI_g}\mathfrak{S}_g :\ V'_X = 0,\ V'_Y = 0,\ V''_Y = 0 \,\}.$$

At $iI_g$ the Weil–Petersson metric gives

$$\|V\|^2_{iI_g} = \frac{1}{2}\big(tr(V_X^2) + tr(V_Y^2)\big),$$

and the block trace identities give

$$tr(V_X^2) = tr\big((V'_X)^2\big) + 2tr\big(V'''_X\, {}^tV'''_X\big) + tr\big((V''_X)^2\big),$$

and similarly for $V_Y$. It follows that the orthogonal complement of the kernel is precisely

$$\ker(\pi_{g'*})^{\perp}_{iI_g} = \left\{ V \in T_{iI_g}\mathfrak{S}_g :\ V = \begin{pmatrix} V'_X + iV'_Y & 0 \\ 0 & iV''_Y \end{pmatrix} \right\}.$$

For such $V$ one has

$$\|V\|^2_{iI_g} = \frac{1}{2}\big(tr\big((V'_X)^2\big) + tr\big((V'_Y)^2\big) + tr\big((V''_Y)^2\big)\big),$$

which is exactly the product metric on $T_{(iI_{g'}, I_{g''})}\big(\mathfrak{S}_{g'} \times \mathrm{Sym}^+(g'', \mathbb{R})\big)$. Therefore $(\pi_{g'*})_{iI_g}$ restricts to an isometry

$$\ker\big((\pi_{g'*})_{iI_g}\big)^{\perp} \xrightarrow{\ \simeq\ } T_{(iI_{g'}, I_{g''})}\big(\mathfrak{S}_{g'} \times \mathrm{Sym}^+(g'', \mathbb{R})\big),$$

and hence $\pi_{g'}$ is a Riemannian submersion. □

[3] Recall that, by a harmless abuse of notation, we denote by $\pi_{g'}$ both the projection on the universal covers and the induced projection on the quotients.

As a first step in obtaining a uniform bound on the fiber diameter, we derive in the following two lemmas an explicit bound on the norm of vertical tangent vectors in terms of the base point $(\tau', t)$.

**Lemma 2.3.12.** *Let $\mathfrak{F}_g(u) \subset \mathfrak{S}_g, \mathfrak{F}_{g'}(u) \subset \mathfrak{S}_{g'}$, and $\mathfrak{F}^+_{g''}(u) \subset \mathrm{Sym}^+(g'', \mathbb{R})$ be fixed Siegel sets and let $(\tau', t) \in \mathfrak{F}_g(u) \times \mathfrak{F}_{g''}(u)$ be a fixed point in the base of fibration suitably restricted to*

$$\pi_g : \mathfrak{F}_g(u) \longrightarrow \mathfrak{F}_{g'}(u) \times \mathfrak{F}^+_{g''}(u).$$

*let $V \in T_\tau \mathfrak{F}_g(u)$, be a vertical tangent vector at $\tau \in \pi^{-1}_{g'}(\tau', t) \cap \mathfrak{F}_g(u)$. Then, its norm satisfies the identity*

$$\|V\|^2_\tau = tr\big(t^{-1}\,{}^t V'''_X (Y')^{-1} V'''_X\big) + tr\big(t^{-1}\,{}^t V'''_Y (Y')^{-1} V'''_Y\big) + \frac{1}{2} tr\big(t^{-1}\widetilde{V}''_X\, t^{-1}\widetilde{V}''_X\big), \tag{2.32}$$

*where $\widetilde{V}''_X$ is defined below in Equation (2.37).*

*Proof.* On the fiber over $(\tau', t)$ we use the coordinates

$$(X''' + iY''',\, X''), \qquad \text{so that} \qquad \tau'' = X'' + i\big(t + {}^t Y'''(Y')^{-1} Y'''\big).$$

If $V \in T_\tau \mathfrak{F}_g(u)$ is vertical then

$$V' = 0, \quad \text{and} \quad V''_Y = {}^t V'''_Y (Y')^{-1} Y''' + {}^t Y'''(Y')^{-1} V'''_Y. \tag{2.33}$$

In order to simplify the norm formula

$$\|V\|^2_\tau = \frac{1}{2}\big(tr(Y^{-1} V_X Y^{-1} V_X) + tr(Y^{-1} V_Y Y^{-1} V_Y)\big), \tag{2.34}$$

we introduce the lower-triangular matrix

$$P := \begin{pmatrix} I_{g'} & 0 \\ -{}^t Y'''(Y')^{-1} & I_{g''} \end{pmatrix}.$$

A direct computation yields the identities

$$P\, Y\, P^t = \mathrm{diag}(Y', t), \qquad Y^{-1} = P^t\, \mathrm{diag}((Y')^{-1}, t^{-1})\, P. \tag{2.35}$$

Moreover, the same change of variables gives

$$P\, V_X\, P^t = \begin{pmatrix} 0 & V'''_X \\ {}^t V'''_X & \widetilde{V}''_X \end{pmatrix}, \qquad P\, V_Y\, P^t = \begin{pmatrix} 0 & V'''_Y \\ {}^t V'''_Y & 0 \end{pmatrix}, \tag{2.36}$$

where

$$\widetilde{V}''_X := V''_X - {}^t V'''_X (Y')^{-1} Y''' - {}^t Y'''(Y')^{-1} V'''_X \in \mathrm{Sym}(g'', \mathbb{R}). \tag{2.37}$$

Substituting (2.35) and (2.36) into (2.34) yields (2.32). □

**Lemma 2.3.13.** *Under the hypothesis of the previous lemma, there exists a constant $C > 0$, depending only on the Siegel set data and $\tau'$, such that for every vertical tangent vector $V \in T_\tau \mathfrak{F}_g(u)$,*

$$\|V\|^2_\tau \le \frac{C}{\lambda_{\min}(t)} \Big( \|V'''_X\|^2_{\mathcal{F}} + \|V'''_Y\|^2_{\mathcal{F}} \Big) + \frac{C}{\lambda_{\min}(t)^2} \|V''_X\|^2_{\mathcal{F}}; \tag{2.38}$$

*here $\pi(\tau) = (\tau', t)$, $\lambda_{min}(t)$ denotes the smallest eigenvalue of $t$, and by $\|\cdot\|_{\mathcal{F}}$ we denote the Frobenius norm.*

*Proof.* We have that $\|t^{-1}\|_{\mathrm{op}} = \lambda_{\min}(t)^{-1}$, where $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm. Using the identity

$$tr(t^{-1}\,{}^t B\, A^{-1}\, B) \leq \|t^{-1}\|_{\mathrm{op}}\, \|A^{-1}\|_{\mathrm{op}}\, \|B\|_{\mathcal{F}}^2,$$

the first two terms in (2.32) satisfy

$$tr\big(t^{-1}\,{}^t V_X'''(Y')^{-1}V_X'''\big) \leq \frac{\|(Y')^{-1}\|_{\mathrm{op}}}{\lambda_{\min}(t)}\, \|V_X'''\|_{\mathcal{F}}^2, \tag{2.39}$$

and similarly with $V_Y'''$. For the last term we bound

$$tr(t^{-1}\widetilde{V}_X''\, t^{-1}\widetilde{V}_X'') \leq \|t^{-1}\|_{\mathrm{op}}^2\, \|\widetilde{V}_X''\|_{\mathcal{F}}^2 = \frac{1}{\lambda_{\min}(t)^2}\, \|\widetilde{V}_X''\|_{\mathcal{F}}^2.$$

Finally, from the definition of $\widetilde{V}_X''$ we have

$$\|\widetilde{V}_X''\|_{\mathcal{F}} \leq \|V_X''\|_{\mathcal{F}} + 2\, \|Y'''\|_{\mathrm{op}}\, \|(Y')^{-1}\|_{\mathrm{op}}\, \|V_X'''\|_{\mathcal{F}}. \tag{2.40}$$

Since the base point $(\tau', t)$ is fixed and $\|L\|_{op}$ is uniformly bounded on a fixed Siegel set we get that $\|Y'''\|_{\mathrm{op}}$ is uniformly bounded on the fiber. Hence, there are constants $C_1$, and $C_2$ such that $\|(Y')^{-1}\|_{\mathrm{op}} = C_1$ and $\|Y'''\|_{\mathrm{op}} \leq C_2$, so (2.40) gives

$$\|\widetilde{V}_X''\|_{\mathcal{F}} \leq \|V_X''\|_{\mathcal{F}} + 2\, C_1 C_2\, \|V_X'''\|_{\mathcal{F}}. \tag{2.41}$$

Combining (2.39)–(2.41) and absorbing constants yields the desired inequality. □

With these results at hand we can give the following point-wise estimate of the fiber diameter

**Proposition 2.3.14.** *Let $\tau_n$ be a sequence in $\mathfrak{F}_g(u) \subset \mathfrak{S}_g$ such that $\tau_n \to \tau'_\infty \in \mathfrak{F}_{g'}(u) \subset \mathfrak{S}_{g'}$. This naturally defines a sequence in $\mathtt{P}_{g'}\backslash\mathfrak{S}_g$, that we keep denoting by $\tau_n$. Let*

$$\pi_{g'}(\tau_n) = \big(\tau'_n\,, t_n\big) \in A_{g'} \times \mathscr{T}_{g''}^{>0}.$$

*Then there exists a constant $C > 0$ (depending only on the Siegel set and on $\tau'_\infty$) such that*

$$\operatorname{diam}\big(\pi_{g'}^{-1}\big(\tau'_n\,, t_n\big)\big) \leq \frac{C}{\sqrt{\lambda_{\min}(t_n)}}. \tag{2.42}$$

*Furthermore, there exists $C' > 0$ such that*

$$\operatorname{diam}\big(\pi_{g'}^{-1}\big(\tau'_n\,, t_n\big)\big) \leq \frac{C'}{\sqrt{d_{g'+1}(\tau_n)}}. \tag{2.43}$$

*In particular, as $\tau_n \to \tau'_\infty$ the diameter of the compact fiber vanishes.*

*Proof.* Without loss of generality we work with the Siegel set description of the quotients, i.e.,

$$\pi_g : \mathfrak{F}_g(u) \longrightarrow \mathfrak{F}_{g'}(u) \times \mathfrak{F}_{g''}^{+}(u).$$

Fix $(\tau', t) \in \mathfrak{F}_{g'}(u) \times \mathfrak{F}_{g''}^{+}(u)$ and let

$$(X''' + iY'''\,,\, X''), \qquad \text{so that} \qquad \tau'' = X'' + i\big(t + {}^t Y'''(Y')^{-1}Y'''\big).$$

be the usual coordinates on the fiber over $(\tau', t)$.

It is clear from the definiton of Siegel set that $\pi_g^{-1}(\tau', t) \cap \mathfrak{F}_g(u)$ gives a lattice fundamental domain of uniformly bounded size for the compact nilmanifold, possibly with finite orbifold structure, appearing as the fiber of the fibration[4]. Thus, there exists a constant $R > 0$ (depending only on the chosen Siegel set) such that for any two points $\tau_p, \tau_q \in \pi_{g'}^{-1}(\tau', t) \cap \mathfrak{F}_g(u)$ the fiber-coordinate differences are bounded by $R$ in Frobenius norm, namely

$$\|\Delta X'''\|_{\mathcal{F}} \leq R, \qquad \|\Delta Y'''\|_{\mathcal{F}} \leq R, \qquad \|\Delta X''\|_{\mathcal{F}} \leq R. \tag{2.44}$$

Given such $\tau_p, \tau_q$, connect them by a piecewise smooth path $\gamma(s)$ contained in the fiber, chosen so that $(\tau', t)$ remains constant and $X'''$ varies linearly on the first segment and, respectively, $Y'''$ and $X''$ on the second and third. Then $\dot\gamma(s)$ is vertical, and along each linear segment

$$\|V_X'''\|_{\mathcal{F}} = \|\Delta X'''\|_{\mathcal{F}}, \qquad \|V_Y'''\|_{\mathcal{F}} = \|\Delta Y'''\|_{\mathcal{F}}, \qquad \|V_X''\|_{\mathcal{F}} = \|\Delta X''\|_{\mathcal{F}},$$

and consequently

$$\|V_X'''\|_{\mathcal{F}},\ \|V_Y'''\|_{\mathcal{F}},\ \|V_X''\|_{\mathcal{F}}\ \leq R.$$

Applying Lemma 2.3.13 to $V = \dot\gamma(s)$ yields

$$\|\dot\gamma(s)\|^2 \leq \frac{C_0}{\lambda_{\min}(t)}\Big(\|V_X'''\|_{\mathcal{F}}^2 + \|V_Y'''\|_{\mathcal{F}}^2\Big) + \frac{C_0}{\lambda_{\min}(t)^2}\,\|V_X''\|_{\mathcal{F}}^2 \leq \frac{C_1}{\lambda_{\min}(t)},$$

for a constant $C_1$ depending only on the Siegel set and $\tau'$. In particular, $\|\dot\gamma(s)\| \leq C_2/\sqrt{\lambda_{\min}(t)}$. Integrating along $\gamma$ gives

$$d(\tau_p, \tau_q) \leq \operatorname{length}(\gamma) \leq \frac{C_{\tau'}}{\sqrt{\lambda_{\min}(t)}},$$

uniformely over all such $\tau_p, \tau_q \in \pi_{g'}^{-1}(\tau', t) \cap \mathfrak{F}_g(u)$, with $C_{\tau'}$ depending only on the Siegel set and $\tau'$. Taking the supremum over $\tau_p, \tau_q$ yields

$$\operatorname{diam}\big(\pi_{g'}^{-1}(\tau', t)\big) \leq \frac{C_{\tau'}}{\sqrt{\lambda_{\min}(t)}}.$$

Then, applying this to the sequence $(\tau_n', t_n)$ and noticing that $\tau_n' \to \tau_\infty$ we get (2.42).

Finally, (2.43) follows from the fact that, given a point $(\tau', t)$ and any $\tau \in \pi_{g'}^{-1}(\tau', t)$, the Siegel set inequalities $d_i \leq u\, d_{i+1}$ imply the existence of $c > 0$ such that

$$\lambda_{\min}(t) \geq c\, d_{g'+1}(\tau). \tag{2.45}$$

To see this, write the Jacobi's decomposition $Y = LDL^t$ in block form as

$$\begin{pmatrix} L' & 0 \\ L''' & L'' \end{pmatrix} \begin{pmatrix} D' & 0 \\ 0 & D'' \end{pmatrix} \begin{pmatrix} {}^tL' & {}^tL''' \\ 0 & {}^tL'' \end{pmatrix}. \tag{2.46}$$

This gives $t = L''\, D''\, {}^tL''$ with $D'' = \operatorname{diag}(d_{g'+1}, \cdots, d_g)$. Let $x \in \mathbb{R}^{g''}$. Then

$${}^t x\, t\, x = {}^t\big({}^tL'' x\big)\, D''\, \big({}^tL'' x\big) \geq d_{min}\, \|{}^tL'' x\|^2.$$

[4]Recall that given a point $(\tau', t)$ in the base of the fibration $\pi_g : \mathtt{P}_{g'}\backslash\mathfrak{S}_g \to A_{g'} \times \mathscr{T}_{g''}^{>0}$ the fiber over it is homeomorpich to the (orbifold) compact nimanifold given by $\mathtt{L}_{g',(\tau',t)}\backslash(\mathtt{N}_{g'}\backslash N_{g'})$, where $\mathtt{L}_{g',(\tau',t)}$ is the isotropy group of $(\tau', t)$

Using $\|{}^tL''x\| \geq \frac{\|x\|}{\|(L'')^{-1}\|_{\mathrm{op}}}$ and taking the infimum over $\|x\| = 1$ yields

$$\lambda_{min}(t) \geq \frac{d_{min}}{\|(L'')^{-1}\|_{\mathrm{op}}^2},$$

where $d_{min} = \min\{d_i \,|\, g'+1 \leq i \leq g\}$. Thus (2.45) follows from $\|(L'')\|_{\mathrm{op}}$ being uniformly bounded on the Siegel set (and consequently $\|(L'')^{-1}\|_{\mathrm{op}}$), together with $d_{min} \geq d_{g'+1}/\max\{1, u^{g''-1}\}$. □

Finally we proceed with the proof of the main theorem

*Proof of Theorem 2.3.10.* First, observe that for every $n$ and every $R > 0$,

$$\pi_{g'}\big(B(\tau_n, R)\big) = B\big(\pi_{g'}(\tau_n), R\big).$$

Indeed, by Lemma 2.3.11 the map $\pi_{g'}$ is 1-Lipschitz, hence for any $\tau \in B(\tau_n, R)$ we have

$$d\big(\pi_{g'}(\tau), \pi_{g'}(\tau_n)\big) \leq d(\tau, \tau_n) \leq R,$$

which gives the inclusion $\pi_{g'}(B(\tau_n, R)) \subset B(\pi_{g'}(\tau_n), R)$. For the reverse inclusion, fix $(\tau', t) \in B(\pi_{g'}(\tau_n), R)$ and let $\beta$ be a minimizing geodesic in the base joining $\pi_{g'}(\tau_n)$ to $(\tau', t)$. Since $\pi_{g'}$ is a Riemannian submersion, $\beta$ admits a horizontal lift $\tilde{\beta}$ starting at $\tau_n$, and the lift preserves length. In particular,

$$\mathrm{leng}(\tilde{\beta}) = \mathrm{leng}(\beta) = d\big(\pi_{g'}(\tau_n), (\tau', t)\big) \leq R,$$

so $\tilde{\beta}(1) \in B(\tau_n, R)$ and $\pi_{g'}(\tilde{\beta}(1)) = (\tau', t)$. This proves $B(\pi_{g'}(\tau_n), R) \subset \pi_{g'}(B(\tau_n, R))$, hence the desired equality.

Next, we quantify the distortion of distances in terms of the fiber diameter. Define

$$\delta(\tau_n; R) := \sup\Big\{ \mathrm{diam}\big(\pi_{g'}^{-1}(\tau', t)\big) \;\Big|\; d\big(\pi_{g'}(\tau_n), (\tau', t)\big) \leq R \Big\}.$$

Then for all $\tau_p, \tau_q \in B(\tau_n, R)$, the usual argument using the 1-Lipschitz property of $\pi_{g'}$ and a path connecting $\tau_p$ and $\tau_q$ consisting of an horizontal lift of a path in the base, concatenated with a vertical path inside the fiber over $\pi_{g'}(\tau_q)$, gives

$$0 \leq d(\tau_p, \tau_q) - d\big(\pi_{g'}(\tau_p), \pi_{g'}(\tau_q)\big) \leq \delta(\tau_n; R).$$

Consequently,

$$d_{GH}\Big(B(\tau_n, R),\, B(\pi_{g'}(\tau_n), R)\Big) \leq \frac{1}{2}\,\delta(\tau_n; R).$$

It remains to estimate $\delta(\tau_n; R)$, i.e. to control fiber diameters uniformly over the base ball $B(\pi_{g'}(\tau_n), R)$. Fix $(\tau', t) \in B(\pi_{g'}(\tau_n), R)$. We claim that

$$e^{-\sqrt{2}R}\,\lambda_{\min}(t_n) \leq \lambda_{\min}(t) \leq e^{\sqrt{2}R}\,\lambda_{\min}(t_n). \tag{2.47}$$

In particular, there exists a constant $C$ (depending only on the chosen Siegel set data and on $\tau'_\infty$) such that

$$\delta(\tau_n; R) \leq \frac{C\, e^{R/\sqrt{2}}}{\sqrt{\lambda_{\min}(t_n)}}. \tag{2.48}$$

Granting (2.47), the estimate (2.48) follows immediately from Proposition 2.3.14, Equation (2.43), together with the definition of $\delta(\tau_n; R)$. Moreover, combining this with (2.45) and the fact that $\tau_n \to \tau'_\infty$ forces $d_{g'+1}(\tau_n) \to \infty$, one obtains (2.29).

We now prove (2.47). Recall the standard distance formula on $\mathrm{Sym}^+(g'',\mathbb{R})$:

$$d(t_n,t)^2=\frac{1}{2}\sum_i \log^2(\mu_i),$$

where $\mu_1\leq\cdots\leq\mu_{g''}$ are the eigenvalues of

$$A:=t_n^{-1/2}\,t\,t_n^{-1/2}.$$

Since $(\tau',t)\in B(\pi_{g'}(\tau_n),R)$, we have $d(t_n,t)\leq R$, and hence

$$\frac{1}{\sqrt{2}}\,|\log\mu_i|\leq R\qquad\text{for all } i,$$

so $e^{-\sqrt{2}R}\leq\mu_i\leq e^{\sqrt{2}R}$. In particular $\mu_{\min}(A)\geq e^{-\sqrt{2}R}$ and $\mu_{\max}(A)\leq e^{\sqrt{2}R}$. Using the inequalities

$$\lambda_{\min}(t)\ \geq\ \lambda_{\min}(t_n)\,\mu_{\min}(A),\qquad\text{and}\qquad \lambda_{\min}(t)\ \leq\ \lambda_{\min}(t_n)\,\mu_{\max}(A),$$

we obtain exactly (2.47).

Finally, observe that the condition in Equation (2.30) can be precisely rephrased as saying that there exist a point $\bar{t}_\infty\in \mathtt{L}_{g'l}\backslash\,\mathrm{Sym}^+(g'',\mathbb{R})/\mathbb{R}_{>0}$ such that the corresponding sequence $\bar{t}_n\to\bar{t}_\infty$. To see this, just notice that

$$\begin{aligned}\bar{t}_\infty=\lim_{n\to\infty}\frac{1}{d_g(\tau_n)}L''_nD''^t_nL''_n&=\lim_{n\to\infty}\mathrm{diag}\left(\frac{d_{g'+1}(\tau_n)}{d_g(\tau_n)},\cdots,\frac{d_{g-1}(\tau_n)}{d_g(\tau_n)},1\right)\\&=\mathrm{diag}\big(k_{g'+1,g},\cdots,k_{g-1,g},1\big).\end{aligned}\tag{2.49}$$

Thus, the pointed Gromov–Hausdorff convergence in Equation (2.31) follows by remembering that for any $R>0$ and for $n$ sufficiently large $B(\tau_n,R)\subset A_g$ can be assumed to be in the region where $\Gamma\sim\mathtt{P}_{g'}$ and consequently we can use Equation (2.29) together. This finishes the proof □

### 2.3.3 Final remarks

The above results establish an explicit connection between the Weil–Petersson metric asymptotics and the geometric compactifications of $A_g$ in terms of Gromov–Hausdorff limits of the (canonical) flat Kähler metric along degenerating sequences of abelian varieties. Furthermore, they show that at infinity the moduli space collapses to a lower-dimensional space and in particular such space does not longer have the structure of a complex algebraic variety. This result, although enigmatic, is consistent with the expectation that any *geometric compactification* of the moduli space of polarized Calabi–Yau manifolds must account for the fact that, due to collapsing phenomena, the limiting objects are in general not algebraic varieties.

However, let us emphasize that the limiting Weil–Petersson geometry of $A_g$ is not at all detached from an algebro-geometric interpretation. First, the holomorphic factor in the limit, namely $(A_{g'},g'_{WP})$, can be naturally identified with a boundary stratum of the Satake–Baily–Borel compactification $\bar{A}_g^{\mathrm{a}BB}$ of $A_g$. Which by construction is the minimal projective compactification of $A_g$. On the other hand, the non-holomorphic factor $(\mathscr{T}^{>0}_{g''},g''_{trWP})$, that we will refer to as the tropical factor following Odaka, can be naturally identified with a stratum of the dual intersection complex of the boundary divisor of any toroidal compactification of $A_g$ (This in the sense explained by [51]). Moreover, from

Theorem 2.2.6 (Theorem 3.1 in [51]) one can see that the holomorphic factor $A_{g'}$ and the tropical factor $\mathscr{T}_{g''}^{>0}$ can be thought as parameterizing respectively the abelian and the compact torus part in a semi-abelian degeneration.

The proof of Theorem 2.1.1 relies on the fact that $A_g$ admits an explicit realization as a locally symmetric space (although the final theorem gives a modular interpretation which is independent from that). In particular, our approach is not specific to abelian varieties and it applies to moduli spaces whose global structure is governed by an arithmetic quotient of a symmetric space, such as the moduli spaces of polarized K3 surfaces or general polarized hyperähler manifolds. Let us briefly outline here this perspective and describe on-going work toward these extensions.

It is well known that the moduli space $\mathcal{F}_{2d}$ of (polarized) K3 surfaces of degree $2d$ can be realized as a locally symmetric space

$$\mathcal{F}_{2d} = \Gamma_{2d}\backslash\mathcal{D}_{2d}, \tag{2.50}$$

where $\mathcal{D}_{2d}$ is a Hermitian symmetric domain of type IV and $\Gamma_{2d}$ is an arithmetic group acting on $\mathcal{F}_{2d}$. It is not difficult to see that the constructions introduced in Section 2.2.2 generalized accordingly to $\mathcal{D}_{2d}$. In particular, we have that there are only two types of boundary components $E \subset \overline{\mathcal{D}}_{2d}$ and associated to each (rational) maximal parabolic subgroup $P_E$ we have the corresponding horospherical decomposition of $\mathcal{D}_{2d}$. In on-going work we show that our result in Theorem 2.3.10 extend to the case of $\mathcal{F}_{2d}$ and, similarly, the picture connects naturally with the works of Odaka and Oshima concerning diameter scale limits, as well as the differential geometric meaning for the Satake-Baily-Borel in terms of volume limits as suggested in [64] Conjectural picture 2.

Finally, let us conclude with two more speculative comments. Firstly, note that along the collapsing process of the Weil–Petersson metric, important metric information is lost. For instance, the limiting asympotitic alone is not sufficient to reconstruct the full Weil–Petersson metric at infinity. For example, the baby case of moduli of flat metrics with cone angle on the projective line (see, e.g., [64] Section 2) suggests that information of *higher scales bubbling* may be important as well: in such case the Weil–Petersson cusps degenerate to a real line, but the full metric can be recover as a Calabi's ansatz from a collapsed complex link, see [7] Proposition 5, which is naturally identified with moduli of non-collapsing minimal cylindrical bubbles. Secondly, note that a very recent breakthrough [4] has constructed Satake-Baily-Borel type compactification for general polarized Calabi-Yau varieties. In particular, it is natural to expect that similar holomorphic plus tropical splitting phenomena as we have described can occur in much greater generality for the Weil–Petersson asymptotic: the holomorphic part should relate to the Weil–Petersson metric generated by moduli of log canonical centers in the above generalized Satake-Baily-Borel type compactification, while a more mysterious tropical part should relate to aspects of collapsing at fixed diameter scale/dual complexes of degenerations (see, for instance, the results and conjectural pictures in [45, 52, 64, 67]).

# Chapter 3

# Asymptotic Weil–Petersson geometry of the moduli space of polarized K3 surfaces

In this chapter, we continue our investigation by studying the asymptotic Weil–Petersson geometry of the moduli space of polarized $K3$ surfaces.

The chapter is organized as follows. We begin in § 3.1 by recalling the period-theoretic description of the moduli space, followed in § 3.2 by the Kulikov classification of degenerations and an overview of the known results and conjectures concerning the limiting Ricci-flat metrics. In § 3.3, we study the geometry of the type IV domain $\mathcal{D}_{2d}$. The main part of the chapter begins in § 3.4.1 by recalling the locally symmetric description of the Weil–Petersson metric, after which we establish in § 3.4.2 the quantitative Gromov–Hausdorff estimates underlying our asymptotic analysis. Finally, in § 3.4.3, we apply these estimates to polarized algebraic degenerations of Types II and III, and identify their pointed Weil–Petersson limits. We then relate the resulting factors to the Baily–Borel and Satake-adjoint compactifications and discuss their geometric interpretation in terms of limits at the volume and diameter scales, explaining the remaining steps needed to complete the modular interpretation.

## 3.1 Moduli space of polarized K3 surfaces

In this section we review the classical construction of the moduli space of polarized $K3$ surfaces and set up our notation and conventions. We work on $\mathcal{F}_{2d}$, the moduli space of complex $K3$ surfaces with at worst ADE singularities and primitive polarization of degree $2d$. The global Torelli theorem for algebraic $K3$ surfaces [57], together with the surjectivity of the period map [39], asserts that $\mathcal{F}_{2d}$ is completely characterized, as a complex analytic space, in terms of the corresponding period domain. We begin by revisiting this construction.

Let $X$ be a smooth complex $K3$ surface. It is well known that $H^2(X,\mathbb{Z})$ carries a natural lattice structure given by the cup product pairing

$$(\alpha,\beta)=\int_X \alpha\wedge\beta.$$

Moreover, this lattice is even, unimodular, and has signature $(3, 19)$. Therefore, by the classification of even unimodular indefinite lattices, it is isomorphic to the lattice[1]

$$\Lambda_{K3} := H^{\oplus 3} \oplus E_8(-1)^{\oplus 2}.$$

Now, consider a pair $(X, L)$ where $L$ is an ample line bundle such that $c_1(L) \in H^2(X, \mathbb{Z})$ is primitive and satisfies $c_1(L)^2 = 2d$. Thus, a marking of a primitively polarized $K3$ surface $(X, L)$ is an isometry

$$\varphi : H^2(X, \mathbb{Z}) \longrightarrow \Lambda_{K3}$$

such that $\varphi(c_1(L)) = \ell$, for a primitive vector $\ell \in \Lambda_{K3}$ with $(\ell, \ell) = 2d$.

Let $\Omega_X$ be a nowhere vanishing holomorphic two-form on $X$. Since $X$ is a $K3$ surface, one has $H^{2,0}(X) = \mathbb{C}\,\Omega_X$, and therefore $H^{2,0}(X)$ is a complex line inside $H^2(X, \mathbb{C})$. Thus, for a marked $K3$ surface the period map is naturally defined as the line

$$[\varphi(\Omega_X)] \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C}).$$

The Hodge–Riemann bilinear relations impose two conditions on this line

$$(\Omega_X, \Omega_X) = 0, \quad \text{and,} \quad (\Omega_X, \overline{\Omega}_X) > 0,$$

whereas the polarization condition and the marking impose $(\ell, \varphi(\Omega_X)) = 0$.

Thus we conclude that the period line of a primitively polarized $K3$ surface with a marking defines a point in the projective space $\mathbb{P}(\ell^\perp \otimes \mathbb{C})$; here by $\ell^\perp$ we denote the primitive lattice $\Lambda_{2d} := \ell^\perp \subset \Lambda_{K3}$. Since $\ell$ has positive square, $\Lambda_{2d}$ has signature $(2, 19)$. The period domain for primitively polarized $K3$ surfaces of degree $2d$ is therefore

$$\mathcal{D}_{2d} = \{[\omega] \in \mathbb{P}(\Lambda_{2d} \otimes \mathbb{C}) : (\omega, \omega) = 0,\ (\omega, \overline{\omega}) > 0\}\,.$$

This is a Hermitian symmetric domain of type IV and has complex dimension 19. It has two connected components, and one fixes one of them once and for all.

*Remark* 3.1.1. The lattice $\Lambda_{2d}$ admits the explicit description

$$\Lambda_{2d} \cong H^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2d \rangle.$$

Indeed, after choosing the marking appropriately, we may assume that the primitive polarization vector is $h = e + df$ in one of the hyperbolic summands $H = \mathbb{Z}e \oplus \mathbb{Z}f$, where $(e, e) = (f, f) = 0$ and $(e, f) = 1$. The orthogonal complement of $h$ in this summand is generated by $e - df$, whose square is $-2d$, and the asserted decomposition follows.

Finally, the arithmetic group

$$\Gamma_{2d} = \{g \in O^+(\Lambda_{K3}) : g(\ell) = \ell\}$$

acts properly discontinuously on $\mathcal{D}_{2d}$. By the global Torelli theorem and the surjectivity of the period map, the arithmetic quotient

$$\mathcal{F}_{2d} \simeq \Gamma_{2d} \backslash \mathcal{D}_{2d}$$

is the coarse moduli space of primitively polarized $K3$ surfaces of degree $2d$ with at worst ADE singularities. For a singular surface, its period is computed on the minimal resolution, where the pullback of the polarization is primitive, big and nef. The curves on which this pullback has degree zero are contracted to the ADE singularities. In particular, the locus parametrizing smooth primitively polarized $K3$ surfaces is obtained by removing from $\mathcal{D}_{2d}$ the hyperplanes orthogonal to classes $\delta \in \Lambda_{2d}$ with $(\delta, \delta) = -2$ before taking the quotient.

[1] Here $H$ denotes the rank two hyperbolic lattice with the symmetric bilinear form given by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and the lattice $E_8(-1)$ denotes the negative definite, even, unimodular lattice of rank 8 associated with the $E_8$ root system.

## 3.2 Degenerations of $K3$ surfaces

Having recalled the period-theoretic construction of the moduli space $\mathcal{F}_{2d}$, we now turn to the study of its *geometry at infinity*. From a moduli-theoretic perspective, approaching infinity in $\mathcal{F}_{2d}$ corresponds to considering degenerating families of polarized $K3$ surfaces; the associated differential-geometric problem is to understand the asymptotic behavior of their Ricci-flat Kähler metrics. We therefore begin by recalling the basics of the Kulikov–Persson–Pinkham theory, which provides the natural framework for our analysis.

Let $\Delta \subset \mathbb{C}$ be a disk centered at the origin and let $\Delta^* := \Delta \setminus \{0\}$. Consider a one-parameter family of primitively polarized $K3$ surfaces

$$\pi^* \colon (\mathcal{X}^*, \mathcal{L}^*) \longrightarrow \Delta^*.$$

The existence theorem of Kulikov and Persson–Pinkham [38, 56] provides a particularly well-behaved model of the underlying family near the origin; for a clear exposition of this, see [18].

**Theorem 3.2.1.** *Assume that the underlying family $\mathcal{X}^*$ admits a proper flat extension $\mathcal{Y} \to \Delta$ whose central fiber has Kähler irreducible components. Then, there exist a base change $\Delta' \to \Delta$ and a sequence of bimeromorphic modifications $\mathcal{Y}' \to \mathcal{X}$ of the pullback, such that*

$$\pi \colon \mathcal{X} \longrightarrow \Delta'$$

*is a proper flat family satisfying the following properties:*

1. *The restriction of $\pi$ over $\Delta'^*$ is an extension of the pulled-back family*

$$\mathcal{X}^* \times_{\Delta^*} \Delta'^* \to \Delta'^*.$$

2. *The total space $\mathcal{X}$ is smooth.*
3. *The central fiber $\mathcal{X}_0$ is a reduced simple normal crossings divisor.*
4. *The relative canonical bundle is trivial.*

*A model satisfying these properties is called a* Kulikov model *of the degeneration.*

One of the most important features of Kulikov models is that their central fibers admit a complete classification. More precisely, let $\mathcal{X}_0 = \bigcup_i V_i$ be the decomposition of the central fiber into its irreducible components, and for $i \neq j$ set $D_{ij} := V_i \cap V_j$. Then the following holds.

**Theorem 3.2.2.** *Let $\pi \colon \mathcal{X} \to \Delta$ be a Kulikov model of a degeneration of $K3$ surfaces. Then exactly one of the following three possibilities occurs:*

(i) (Type I). *The central fiber $\mathcal{X}_0$ is a smooth $K3$ surface.*

(ii) (Type II). *The central fiber has double curves but no triple points. More precisely, it is a chain $\mathcal{X}_0 = V_0 \cup V_1 \cup \cdots \cup V_r$, where $V_0$ and $V_r$ are rational surfaces and the intermediate components are elliptic ruled surfaces. The only nonempty intersections are $D_{i,i+1} = V_i \cap V_{i+1}$, and all these double curves are isomorphic to a fixed smooth elliptic curve $E$. The dual complex $\Delta(\mathcal{X}_0)$ is a closed interval subdivided into subintervals.*

(iii) (Type III). *The central fiber has triple points. Every component $V_i$ is a rational surface, and its double locus $D_i := \sum_{j\neq i} D_{ij}$ is an anticanonical cycle of rational curves. Every double curve contains exactly two triple points, and the dual complex $\Delta(\mathcal{X}_0)$ is a triangulation of $S^2$.*

We now turn from the algebro-geometric description of a degeneration to its differential-geometric counterpart. By Yau's solution of the Calabi conjecture, $\pi^*\colon (\mathcal{X}^*, \mathcal{L}^*) \to \Delta^*$ determines a one-parameter family $(X_t, \omega_t)$ of Ricci-flat Kähler $K3$ surfaces with $[\omega_t] = 2\pi c_1\big(\mathcal{L}^*|_{X_t}\big)$. The natural differential-geometric question is to understand the asymptotic behaviour of $\omega_t$ as $t \to 0$. We now recall the main results in this direction.

**Theorem 3.2.3** (Sun–Zhang)**.** *Let $X$ be the oriented smooth four-manifold underlying a $K3$ surface, and let $(g_i)$ be a sequence of hyperkähler metrics on $X$ such that*

$$\operatorname{diam}(X, g_i) = 1 \quad \textit{and} \quad \operatorname{Vol}(X, g_i) \longrightarrow 0. \tag{3.1}$$

*After passing to a subsequence, $(X, g_i)$ converges in the Gromov–Hausdorff topology to a compact metric space $(Y, d_Y)$. Moreover, precisely one of the following possibilities occurs:*

(i) $\dim Y = 3$*, and $(Y, d_Y)$ is a flat orbifold of the form $T^3/\{\pm 1\}$;*

(ii) $\dim Y = 2$*, and $Y$ is homeomorphic to $S^2$, with its regular part carrying a singular real Monge-Ampere metric with local integral monodromy;*

(iii) $\dim Y = 1$*, and $(Y, d_Y)$ is isometric to the unit interval $([0,1], d_{\mathrm{eucl}})$.*

This landmark structural result is due to [66, Theorem 1.1]. However, it is important to note that this is, a priori, a purely Riemannian classification. In other words, it describes all possible collapsed limits of arbitrary sequences of hyperkähler metrics, but does not by itself determine which of the three alternatives occurs when the metrics arise from a prescribed polarized degeneration. This leads naturally to the following question.

**Question 3.2.4.** Let $\pi^*\colon (\mathcal{X}^*, \mathcal{L}^*) \to \Delta^*$ be a one-parameter degeneration of primitively polarized $K3$ surfaces, and let

$$\widehat{g}_t := \operatorname{diam}(X_t, g_t)^{-2} g_t$$

be the diameter-normalized Ricci-flat metric associated with the polarization. How does the Kulikov type of the degeneration determine the Gromov–Hausdorff limit of $(X_t, \widehat{g}_t)$ as $t \to 0$? In particular, how is this metric limit related to the algebro-geometric data encoded by a Kulikov model?

We now restrict our attention to Type II and Type III degenerations, for which the diameter-normalized Ricci-flat metrics collapse. The answer to Question 3.2.4 reveals a striking correspondence between the differential- and the algebro-geometric aspect of the theory:

**Theorem 3.2.5.** *The topology and dimension of the Gromov-Hausdorff limits of $(X, \widehat{g}_t)$ are governed by the essential skeleton of the degeneration*[2]*, which may be identified with the dual complex of the central fiber of any Kulikov model. More precisely:*

[2]The essential skeleton is a canonical piecewise-linear space associated with the degeneration and is invariant under birational modifications that are isomorphisms over $\Delta^*$ [37, 48, 50]. For any Kulikov model $\mathcal{X} \to \Delta$, the essential skeleton can be identified, up to piecewise-linear homeomorphism, with the dual complex of the central fiber $\mathcal{X}_0$ [50]. For further details and a clear exposition, we refer to [11, §5.5 and §7.1].

(i) *If the degeneration is of Type II, then*

$$(X_t, \widehat{g}_t) \xrightarrow[t\to 0]{\mathrm{GH}} ([0,1], d_{\mathrm{eucl}})\,.$$

(ii) *If the degeneration is of Type III, then*

$$(X_t, \widehat{g}_t) \xrightarrow[t\to 0]{\mathrm{GH}} (B, d_B),$$

*where $(B, d_B)$ is a diameter-one metrized tropical $K3$ surface. In particular, the Gromov–Hausdorff limit is a singular real Monge–Ampère metric space homeomorphic to $S^2$.*

Here, a metrized tropical $K3$ surface is a compact metric space $(B, d_B)$ whose underlying topological space is homeomorphic to $S^2$. More precisely, there is a finite subset $\mathrm{Sing}(B) \subset B$ such that $B^{\mathrm{reg}} := B \setminus \mathrm{Sing}(B)$ carries two Legendre-dual integral affine structures and a Riemannian metric $g_B$ which is Monge–Ampère with respect to both of them. Thus, in local affine coordinates, one may write

$$g_B = \sum_{i,j} \frac{\partial^2 \varphi}{\partial x_i \partial x_j}\, dx_i\, dx_j, \qquad \det\left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right) = \mathrm{const},$$

for a strictly convex potential $\varphi$, and $(B, d_B)$ is the metric completion of $(B^{\mathrm{reg}}, g_B)$, normalized to have diameter one (cf. [53, §4.3]).

The picture summarized in the preceding theorem emerged from several complementary developments, differing both in scope and in the notion of convergence involved. Its foundations were laid, at roughly the same time, by Gross–Wilson and Kontsevich–Soibelman. Gross–Wilson initiated the metric study of large-complex-structure limits of $K3$ surfaces by analyzing collapsing Kähler classes on elliptically fibred $K3$ surfaces [25]. In parallel, Kontsevich–Soibelman placed such degenerations within the broader framework of mirror symmetry and the SYZ conjecture, and subsequently developed the corresponding non-Archimedean viewpoint [36, 37].

Gross–Tosatti–Zhang later extended the metric analysis to arbitrary projective elliptic $K3$ fibrations, without restrictions on the types of singular fibres [23, 24]. Through hyperkähler rotation, the results of Gross–Wilson and Gross–Tosatti–Zhang provide models for Type III complex-structure degenerations, although they do not begin with an arbitrary polarized Kulikov degeneration. The non-Archimedean perspective was further developed in a complementary measure-theoretic direction by Boucksom–Jonsson, who proved that normalized Calabi–Yau volume forms converge weakly in the hybrid topology to a Lebesgue-type measure supported on the essential skeleton [11].

On the Type II side, Hein–Sun–Viaclovsky–Zhang constructed families of hyperkähler metrics collapsing to an interval and described the associated collapsing fibrations [28]. Their construction, however, does not begin with a prescribed polarized degeneration. Sun–Zhang later treated genuine polarized complex-structure degenerations within a special class of Calabi–Yau hypersurface families whose central fiber is a transverse union of two smooth Fano hypersurfaces; in complex dimension two, these yield particular Type II degenerations [67].

At the moduli-theoretic level, Odaka–Oshima studied the diameter-normalized Gromov–Hausdorff limits of polarized $K3$ surfaces through a natural compactification of $\mathscr{F}_{2d}$ admitting both Lie-theoretic and algebro-geometric descriptions; see Section 3.4.3 for details. They established the Type III statement for arbitrary polarized Type III degenerations,

while the corresponding Type II interval limit remained conditional on their continuity conjecture [53]. From a complementary Riemannian perspective, Sun–Zhang established the trichotomy for collapsed limits of arbitrary sequences of unit-diameter hyperkähler metrics on the $K3$ manifold [66]. Ouyang subsequently proved the global integrality of the limiting two-dimensional singular special Kähler structures, thereby completing the description of the two-dimensional stratum [54]. Finally, Ouyang–Tian proved the full continuity conjecture of Odaka–Oshima. Its restriction to the polarized moduli space yields the general Type II interval limit and completes the moduli-theoretic description of the Gromov–Hausdorff compactification [55].

Going forward with our analysis, we would like to point out that, although the diameter-normalized Gromov–Hausdorff limits in Theorem 3.2.5 capture the global geometry of the degeneration, they generally lose the geometric information occurring at scales that become negligible relative to $\operatorname{diam}(X_t, g_t)$. After rescaling about suitable points, these regions may give rise to nontrivial complete noncompact Ricci-flat limits, and such *bubbling* may occur at several successive scales. A finer differential-geometric description of the degeneration therefore requires understanding its full multiscale, or *bubble-tree*, structure, rather than only its diameter-normalized limit. This leads to the following refined question.

**Question 3.2.6.** Can the full bubble-tree decomposition of $(X_t, \widehat{g}_t)$ as $t \to 0$ be described in terms of the algebro-geometric data encoded by a Kulikov model? More precisely, can one determine the nontrivial pointed rescaling limits

$$\big(X_{t_i}, \lambda_i^{-2}\widehat{g}_{t_i}, p_i\big), \quad t_i \to 0, \quad p_i \in X_{t_i},$$

arising at the successive concentration scales, together with the hierarchy of scales and the way in which the resulting limits are attached, from the geometry of the central fiber?

To the author's knowledge, Question 3.2.6 remains open in the stated generality. Nevertheless, the works [67] provide a natural Riemannian framework for the problem, as well as remarkably complete descriptions in several special settings. For our purposes, however, we will not attempt to describe our expectations for the full hierarchy of scales occurring in the bubble tree. Instead, we restrict our attention to the volume normalization scale. This restriction is motivated by our later application in Section 3.4.3 where the corresponding Gromov–Hausdorff compactifications will be shown to encode the asymptotic Weil–Petersson geometry of $\mathscr{F}_{2d}$. Accordingly, we formulate the expected geometric picture as follows (see Conjectural Picture 2 in [64, § 4.1.3]).

**Conjecture 3.2.7.** *Let $s := \dim \mathrm{Sk}(\mathcal{X})$ and let $Z \subset \mathcal{X}_0$ be a minimal log canonical center of the simple normal-crossings pair $(\mathcal{X}, \mathcal{X}_0)$. For a suitably* generic *sequence of basepoints $p_i \in X_{t_i}$ specializing to a point $q \in Z$, we expect the volume-normalized pointed convergence*

$$\big(X_{t_i}, \lambda_i^{-2}\widehat{g}_{t_i}, p_i\big) \xrightarrow[i\to\infty]{\mathrm{pGH}} \big(\mathbb{R}^s \times Z, g_{\mathbb{R}^s} + g_Z, (0, q)\big),$$

*where $g_Z$ denotes a Ricci-flat metric arising on the compact factor at the chosen scale; its Kähler class is part of the limiting geometry. In particular,*

*(i) For a Type II degeneration, $s = 1$ and the minimal log canonical centers are the elliptic double curves. Thus the expected limit is*

$$\big(\mathbb{R} \times E, ds^2 + \lambda g_E, (0, q)\big),$$

*where $g_E$ is the unit-area flat metric compatible with the complex structure of $E$, and $\lambda > 0$ is the area of the elliptic curve at the chosen scale.*

*(ii) For a Type III degeneration, $s = 2$ and the minimal log canonical centers are the triple points. In this case the compact factor is zero-dimensional, and the expected limit reduces to* $\big(\mathbb{R}^2, g_{\mathrm{flat}}, 0\big)$.

Some remarks are in order. First, we use the term *generic* informally to mean that the basepoints approach the chosen deepest stratum in a direction corresponding to the relative interior of the associated maximal cell of the dual complex. Second, the conjecture isolates the volume-normalization scale rather than attempting to describe the full hierarchy of collapsing scales. In special Type II settings, however, this hierarchy is understood in considerably greater detail. Hein–Sun–Viaclovsky–Zhang constructed families of hyperkähler metrics collapsing to an interval and described the associated collapsing fibrations, whose regular fibers are diffeomorphic to either 3-tori or compact Heisenberg nilmanifolds, as well as the Tian–Yau and Taub–NUT bubbles arising at smaller scales [28]. Complementary structural results for general one-dimensional collapses were obtained by Honda–Sun–Zhang [29]: away from finitely many points of the limiting interval, the collapse occurs with uniformly bounded curvature, and Fukaya's theory yields local infranil fibrations over the regular subintervals. Sun–Zhang subsequently showed that the fibers over the regular set may be taken to be nilmanifolds; in the one-dimensional case, these are 3-tori or compact Heisenberg nilmanifolds [66]. Finally, at the volume-normalization scale considered in the conjecture, their analysis of the special polarized hypersurface degenerations studied in [67] establishes, in complex dimension two, the product limit $(\mathbb{R} \times E, ds^2 + g_E)$ predicted in part (i). In the Type III setting, the conclusion of part (ii) can also be extracted, for basepoints lying over the regular locus, from the local adiabatic estimates of Gross–Tosatti–Zhang in the uniform form developed by Odaka–Oshima [23, 24, 53]. At the volume-normalization scale the special Kähler base expands to its Euclidean tangent plane while the elliptic fibers obtained after hyperkähler rotation continue to collapse, yielding $(\mathbb{R}^2, g_{\mathrm{flat}})$; identifying this analytic regular-locus condition with specialization toward a prescribed maximal cell of a Kulikov model, however, requires an additional comparison.

## 3.3 Geometry of $\mathcal{D}_{2d}$ and Siegel reduction

Before continuing with our analysis, we need to take a Lie theoretic interlude and recollect some fundamental facts from the theory of Hermitian symmetric spaces. We specialize to the type IV domain $\mathcal{D}_{2d}$ the general theory developed in [2, Chapter III, §§3–6] and [10, Chapter III, §§1–2 and 6]. For type IV domains, closely related descriptions appear in [35, §§2.6–2.15] and [47, §1A–1B]; see also the foundational treatment of the boundary of moduli spaces of polarized K3 surfaces in [59]. Our organization is guided by the analogous treatment of Siegel modular varieties in [16, §2.2]; reproduced in Chapter 2 . Throughout we will denote $V_{\mathbb{Q}} := \Lambda_{2d} \otimes \mathbb{Q}$ and similarly for $\mathbb{R}$, and $\mathbb{C}$. Additionally we let $\mathbb{G}$ be the algebraic group defined over $\mathbb{Q}$ such that $\mathbb{G}(\mathbb{R}) = O(2, 19)$.

Recall that $\mathcal{D}_{2d}$ has the structure of a Hermitian symmetric domain given by

$$\mathrm{SO}(2, 19)^{\circ}/(\mathrm{SO}(2) \times \mathrm{SO}(19)).$$

Here the superscript $\circ$ denotes the connected component of the identity. One advantage of viewing $\mathcal{D}_{2d}$ as a Hermitian symmetric domain is that the Borel and Harish–Chandra embedding provides a natural framework for describing its geometry at infinity. Let

$$\check{\mathcal{D}}_{2d} = \{[\omega] \in \mathbb{P}(V_{\mathbb{C}}) : (\omega, \omega) = 0\}$$

denote its compact dual. The embedding $\mathcal{D}_{2d} \hookrightarrow \check{\mathcal{D}}_{2d}$ realizes $\mathcal{D}_{2d}$ as an open subset of $\check{\mathcal{D}}_{2d}$, and, in a suitable affine chart of the compact dual, this embedding yields the realization of $\mathcal{D}_{2d}$ as a bounded domain. The closure of this bounded realization decomposes as the disjoint union

$$\overline{\mathcal{D}}_{2d} = \mathcal{D}_{2d} \sqcup \bigsqcup_{\alpha} \mathcal{D}_{\alpha},$$

where each proper boundary component $\mathcal{D}_{\alpha}$ is itself a Hermitian symmetric domain of lower dimension, with points included as zero-dimensional components. Furthermore, the normalizer of every proper boundary component is a maximal parabolic subgroup.

Now, taking into account the $\mathbb{Q}$-structure on $\mathbb{G}$ allows one to distinguish the rational boundary components: a boundary component is called *rational* if its normalizer in $\mathbb{G}$ is defined over $\mathbb{Q}$. For the type IV domain $\mathcal{D}_{2d}$, these rational boundary components admit a particularly concrete description. They are indexed by the nonzero rational isotropic subspaces $I \subset V_{\mathbb{Q}}$. Since $V_{\mathbb{Q}}$ has signature $(2, 19)$, such a subspace has dimension one or two. We write $\ell$ for an isotropic line and $\Pi$ for an isotropic plane. If $\mathcal{D}_I$ denotes the boundary component indexed by $I$, then its normalizer is precisely the stabilizer of $I$. Accordingly, we set

$$P_I := \mathrm{Stab}_{\mathbb{G}}(I).$$

Thus, isotropic lines and isotropic planes determine the two types of rational maximal parabolic subgroups, denoted by $P_\ell$ and $P_\Pi$, respectively.

Associated with each parabolic $P_I$, and after fixing a basepoint for the action of $\mathrm{SO}(2, 19)^\circ$ on $\mathcal{D}_{2d}$ we have a unique Levi decomposition

$$P_I = N_I \rtimes L_I,$$

where $N_I$ is the unipotent radical and $L_I$ is a Levi factor. The significance of such a decomposition is that it encodes the geometry at infinity of a given boundary component. Let us explain this in detail. Each $P_I$ acts transitively and by diffeomorphism on $\mathcal{D}_{2d}$. Additionally, recalling that each Levi can further be decomposed as

$$L_I = \left[L_{I,h} \cdot L_{I,l} \cdot M_I\right] \qquad (\text{morally} \quad L_I = L_{I,h} \times L_{I,l})$$

into its Hermitian and linear factor we get the following decomposition at the level of the symmetric space

$$\mathcal{D}_{2d} \cong N_I \times X_{I,h} \times X_{I,l}, \tag{3.2}$$

that we denote as the *horospherical decomposition* associated with the rational boundary component $I$. Let us write $B_I = X_{I,h} \times X_{I,l}$ and define the natural projection map

$$\pi_I : \mathcal{D}_{2d} \to B_I. \tag{3.3}$$

Much of our subsequent analysis will be devoted to understanding the differential-geometric properties of $\pi_I$ and of the fibration induced by it after passing to the appropriate arithmetic quotients. In particular, in Section 3.3.2, we will see that this quotient fibration provides a natural model for the geometry of $\mathcal{F}_{2d}$ at infinity in the direction of the rational boundary component indexed by $I$.

*Remark* 3.3.1. Notice that the factor $X_{I,h}$ is precisely the Hermitian symmetric domain $\mathcal{D}_I$ appearing as boundary component in the Borel and Harish-chandra embedding. In particular, we will see in the next section that isotropic lines give zero-dimensional boundary components, while isotropic planes give one-dimensional boundary components.

### 3.3.1 Horospherical coordinates

As the horospherical decomposition (3.2) will play an important role in Section 3.4, we now describe it more concretely. Suppose $I \subset V_{\mathbb{Q}}$ is a rational isotropic subspace and let $I^{\perp} = \{v \in V_{\mathbb{Q}} \mid (v,u) = 0,\ \forall u \in I\}$. Because $I$ is isotropic we have $I \subset I^{\perp}$. Then we have the following filtration

$$0 \subset I \subset I^{\perp} \subset V_{\mathbb{Q}}, \tag{3.4}$$

and it follows that $P_I$ is exactly the stabilizer of this filtration. Now consider the graded vector space $Gr(V_{\mathbb{Q}}) = I \oplus (I^{\perp}/I) \oplus (V_{\mathbb{Q}}/I^{\perp})$. Then the following holds

**Proposition 3.3.2.** *Let $I \subset V_{\mathbb{Q}}$ be a rational isotropic subspace, let $P_I$ be its stabilizer, and let $N_I$ be the unipotent radical of $P_I$. Setting $W := I^{\perp}/I$, the natural action of $P_I$ on the graded pieces of the filtration $0 \subset I \subset I^{\perp} \subset V_{\mathbb{Q}}$ defines a short exact sequence*

$$1 \longrightarrow N_I \longrightarrow P_I \xrightarrow{\rho_I} \mathrm{GL}(I) \times \mathrm{SO}(W) \longrightarrow 1.$$

*Proof.* Let $g \in P_I$. Since $g$ preserves both $I$ and $I^{\perp}$, it induces the linear maps $g_I := g|_I$, and $g_W : W \to W, [v] \mapsto [gv]$. Additionally, the quotient $W = I^{\perp}/I$ carries the nondegenerate bilinear form defined by $([v],[w])_W := (v,w)$, where $v, w \in I^{\perp}$. Hence, since $g$ preserves the bilinear form, the induced map $g_W$ belongs to $O(W)$. Moreover, the bilinear form induces a canonical identification $V_{\mathbb{Q}}/I^{\perp} \simeq \mathrm{Hom}_{\mathbb{Q}}(I, \mathbb{Q})$, under which the action of $g$ on $V_{\mathbb{Q}}/I^{\perp}$ is the contragredient of its action on $I$. Consequently,

$$\det(g) = \det(g_I)\det(g_W)\det(g_I)^{-1} = \det(g_W).$$

Since $g \in \mathrm{SO}(V_{\mathbb{Q}})$, it follows that $\det(g_W) = 1$, and hence $g_W \in \mathrm{SO}(W)$. We therefore obtain a homomorphism

$$\rho_I : P_I \longrightarrow \mathrm{GL}(I) \times \mathrm{SO}(W), \quad g \longmapsto (g_I, g_W).$$

We next prove that $\rho_I$ is surjective. We may choose a rational isotropic subspace $J \subset V_{\mathbb{Q}}$ such that $I \cap J = \{0\}$ and the bilinear form induces a perfect pairing $I \times J \to \mathbb{Q}$. Setting $W_J := (I \oplus J)^{\perp}$, we obtain a decomposition

$$V_{\mathbb{Q}} = I \oplus W_J \oplus J, \tag{3.5}$$

where the natural map $W_J \to I^{\perp}/I$ is an isometry. Relative to dual bases of $I$ and $J$, define $s_{I,J} : \mathrm{GL}(I) \times \mathrm{SO}(W_J) \to P_I$ by

$$s_{I,J}(a,b) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (a^{-1})^t \end{pmatrix}.$$

This transformation preserves the bilinear form, stabilizes $I$, and has determinant 1. Thus $s_{I,J}(a,b) \in P_I$. Under the identification $W_J \simeq W$, one has

$$\rho_I \circ s_{I,J} = \mathrm{id}_{\mathrm{GL}(I)\times\mathrm{SO}(W)},$$

so $\rho_I$ is surjective.

It remains to determine its kernel. If $g \in \ker(\rho_I)$, then $g$ acts trivially on $I$ and on $I^{\perp}/I$. Its induced action on $V_{\mathbb{Q}}/I^{\perp} \simeq \mathrm{Hom}_{\mathbb{Q}}(I, \mathbb{Q})$ is therefore also trivial. Hence, relative to the decomposition $V_{\mathbb{Q}} = I \oplus W_J \oplus J$, it has the form

$$g = \begin{pmatrix} \mathrm{id}_I & * & * \\ 0 & \mathrm{id}_{W_J} & * \\ 0 & 0 & \mathrm{id}_J \end{pmatrix}.$$

In particular, $g$ is unipotent. Thus $\ker(\rho_I)$ is a normal unipotent subgroup of $P_I$, and therefore $\ker(\rho_I) \subseteq N_I$. Conversely, since $\mathrm{GL}(I) \times \mathrm{SO}(W)$ is reductive, the image $\rho_I(N_I)$ is a connected normal unipotent subgroup of a reductive group and must therefore be trivial. Hence $N_I \subseteq \ker(\rho_I)$. □

The proposition canonically identifies the Levi quotient as

$$P_I/N_I \cong \mathrm{GL}(I) \times \mathrm{SO}(I^\perp/I).$$

The choice of the complementary isotropic subspace $J$ made in the proof determines a splitting of this exact sequence. Its image $L_{I,J} := s_{I,J}\big(\mathrm{GL}(I) \times \mathrm{SO}(W_J)\big)$ is a Levi factor of $P_I$, and therefore

$$P_I = N_I \rtimes L_{I,J}, \qquad L_{I,J} \cong \mathrm{GL}(I) \times \mathrm{SO}(W_J).$$

Thus, although the Levi quotient is canonically determined by $I$, its realization as a subgroup of $P_I$ depends on the choice of $J$.

We now relate this splitting to the geometry of $\mathcal{D}_{2d}$. Throughout we work with the real points of $\mathbb{G}$. First, we note that the horospherical decomposition associated with $P_I$ is defined relative to a choice of basepoint $o \in \mathcal{D}_{2d}$. Let $K_o$ be the stabilizer of the basepoint, which is a maximal compact subgroup of $\mathrm{SO}(2,19)^\circ$. Since $P_I$ acts transitively on $\mathcal{D}_{2d}$, this choice gives the homogeneous-space realization $\mathcal{D}_{2d} \cong P_I/(P_I \cap K_o)$. Thus, the geometric description of the action of $P_I$ depends on how its Levi decomposition interacts with $K_o$.

The maximal compact subgroup $K_o$, in turn, determines a Cartan involution $\Theta_o$ of $\mathrm{SO}(2,19)^\circ$. Among all Levi factors of $P_I$, this involution singles out the distinguished Levi $L_{I,o} := P_I \cap \Theta_o(P_I)$, which is the unique $\Theta_o$-stable Levi of $P_I$. The Levi factor $L_{I,J}$ constructed above is instead determined by the choice of $J$ and need not coincide with $L_{I,o}$ for an arbitrary basepoint $o$. However, the subspace $J$ also determines an opposite parabolic subgroup $P_J$, and $L_{I,J} = P_I \cap P_J$. It is therefore enough to construct a basepoint $o_J \in \mathcal{D}_{2d}$ whose associated Cartan involution satisfies $\Theta_J(P_I) = P_J$. We now carry out this construction.

Let $r := \dim I$ and choose bases $(u_1, \dots, u_r)$ of $I_\mathbb{R}$ and $(v_1, \dots, v_r)$ of $J_\mathbb{R}$ such that $(u_i, v_j) = \delta_{ij}$. Set $e_i := u_i + v_i$ and $f_i := u_i - v_i$. Since $I_\mathbb{R}$ and $J_\mathbb{R}$ are totally isotropic, we have $(e_i, e_j) = 2\delta_{ij}$, $(f_i, f_j) = -2\delta_{ij}$, and $(e_i, f_j) = 0$. Consequently, the vectors $e_1, \dots, e_r$ span a positive-definite subspace, whereas $f_1, \dots, f_r$ span a negative-definite subspace.

Since $I_\mathbb{R} \oplus J_\mathbb{R}$ has signature $(r, r)$, the space $W_{J,\mathbb{R}}$ has signature $(2-r, 19-r)$. Choose a positive-definite subspace $U_W \subset W_{J,\mathbb{R}}$ of dimension $2 - r$. Thus, $U_W = \{0\}$ if $r = 2$, whereas for $r = 1$ we choose a positive-definite line in $W_{J,\mathbb{R}}$, which is possible because $W_{J,\mathbb{R}}$ has signature $(1, 18)$. Define

$$\Pi_J := \mathrm{Span}_\mathbb{R}\{e_1, \dots, e_r\} \oplus U_W.$$

Then $\Pi_J$ is a positive-definite real 2-plane in $V_\mathbb{R}$. After choosing an orientation of $\Pi_J$, it determines a point $o_J := [\Pi_J] \in \mathcal{D}_{2d}$.

Let $s_J : V_\mathbb{R} \to V_\mathbb{R}$ be the involution acting as $+1$ on $\Pi_J$ and as $-1$ on $\Pi_J^\perp$. Since the decomposition $V_\mathbb{R} = \Pi_J \oplus \Pi_J^\perp$ is orthogonal, $s_J$ preserves the bilinear form. Consequently, conjugation by $s_J$ defines an involutive automorphism

$$\Theta_J : G \longrightarrow G, \qquad \Theta_J(g) := s_J g s_J.$$

At the level of the Lie algebra $\mathfrak{g} = \mathfrak{so}(2,19)$, the induced involution is $\theta_J(X) := s_J X s_J$. Its fixed-point subalgebra consists of the infinitesimal orthogonal transformations preserving both $\Pi_J$ and $\Pi_J^\perp$. Therefore,

$$\mathfrak{g}^{\theta_J} = \mathfrak{so}(\Pi_J) \oplus \mathfrak{so}(\Pi_J^\perp) \cong \mathfrak{so}(2) \oplus \mathfrak{so}(19).$$

This is a maximal compact subalgebra of $\mathfrak{so}(2,19)$. Consequently, $\theta_J$ is a Cartan involution and $\Theta_J$ is the corresponding group-level Cartan involution associated with the basepoint $o_J$. It remains to verify that this Cartan involution is compatible with the choice of $J$. Since $s_J(e_i) = e_i$ and $s_J(f_i) = -f_i$, we have $s_J(u_i) = v_i$ and $s_J(v_i) = u_i$. Hence $s_J$ exchanges $I_{\mathbb{R}}$ and $J_{\mathbb{R}}$, and therefore $\Theta_J(P_I) = P_J$. It follows that

$$P_I \cap \Theta_J(P_I) = P_I \cap P_J = L_{I,J}.$$

Thus, $L_{I,J}$ is precisely the distinguished, and hence unique, $\Theta_J$-stable Levi factor of $P_I$.

We now make the preceding abstract description explicit for $P_\Pi$ and $P_\ell$.

*3.3.1.1 Horospherical decomposition associated with $P_\Pi$*

Let $\Pi \subset V_{\mathbb{Q}}$ be a rational isotropic plane. Choose generators $u_1, u_2 \in \Pi$ and isotropic vectors $v_1, v_2 \in V_{\mathbb{Q}}$ such that $(u_i, v_j) = \delta_{ij}$. Set $\Pi' = \mathbb{Q}\, v_2 \oplus \mathbb{Q} v_1$ and $W_{\mathbb{Q}} = (\Pi \oplus \Pi')^{\perp}$. We use the ordered decomposition

$$V_{\mathbb{Q}} = \mathbb{Q}\, u_1 \oplus \mathbb{Q}\, u_2 \oplus W_{\mathbb{Q}} \oplus \mathbb{Q}\, v_2 \oplus \mathbb{Q}\, v_1,$$

and similarly over $\mathbb{R}$. Since $V_{\mathbb{R}}$ has signature $(2,19)$, the subspace $W_{\mathbb{R}}$ is negative definite of dimension 17.

*Remark* 3.3.3. Notice that the ordered basis chosen above differs from the convention used in the previous section. We follow this new convention in order to make the basis compatible with the complete rational isotropic flag

$$0 \subset \ell = \mathbb{Q}u_1 \subset \Pi = \mathbb{Q}u_1 \oplus \mathbb{Q}u_2 \subset V_{\mathbb{Q}}$$

considered in 3.3.3.

Let $Q_W$ denote the Gram matrix of the restriction of the bilinear form to $W_{\mathbb{R}}$, and set

$$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

With respect to this decomposition an element of $P_\Pi$ is of the form

$$g(A,B,X,Z) = \begin{pmatrix} A & -ARX^tQ_W & A\left(Z - \frac{1}{2}RX^tQ_WX\right) \\ 0 & B & BX \\ 0 & 0 & R(A^{-1})^tR \end{pmatrix},$$

where

$$A \in GL^+(\Pi_{\mathbb{R}}), \qquad B \in SO(W_{\mathbb{R}}), \qquad X \in \mathrm{Hom}(\Pi'_{\mathbb{R}}, W_{\mathbb{R}}),$$

and $Z \in \mathrm{Hom}(\Pi'_{\mathbb{R}}, \Pi_{\mathbb{R}})$ satisfies $Z^tR + RZ = 0$. Equivalently, relative to the ordered basis above, the matrix of $Z$ is of the form $Z_\lambda = diag(\lambda, -\lambda)$ with $\lambda \in \mathbb{R}$. In particular we have that the unipotent radical $N_\Pi$ is a two-step nilpotent group whose elements are given by $n(X,Z) = g(1,1,X,Z)$. Moreover, its center $U_\Pi$ is the one-dimensional subgroup given by $n(0,Z_\lambda)$. On the other hand, the Levi factor will be given by

$$L^\circ_\Pi \simeq GL^+(\Pi_{\mathbb{R}}) \times SO(W_{\mathbb{R}}).$$

Using $GL^+(\Pi_{\mathbb{R}}) \simeq \mathbb{R}_{>0} \times SL(\Pi_{\mathbb{R}})$, we obtain

$$L_{\Pi,h} \simeq SL(\Pi_{\mathbb{R}}), \qquad L_{\Pi,l} \simeq \mathbb{R}_{>0}, \qquad M_\Pi \simeq SO(W_{\mathbb{R}}).$$

From this we get that, at the level of the symmetric space, the horospherical decomposition associated with the real parabolic $P_{\Pi}$ is

$$\mathcal{D}_{2d} \simeq N_{\Pi} \times \mathcal{H} \times \mathbb{R}_{>0}. \tag{3.6}$$

Finally we give a coordinate description of the decomposition above. We work in the affine chart $(\omega, v_2) = 1$. Thus every point $[\omega] \in \mathcal{D}_{2d}$ has a unique representative of the form

$$\omega(\tau, \zeta_W, \sigma) = \tau u_1 + u_2 + \zeta_W + \sigma v_1 - \left(\tau\sigma + \frac{1}{2}(\zeta_W, \zeta_W)\right) v_2, \tag{3.7}$$

where $\tau, \sigma \in \mathbb{C}$, $\zeta_W \in W_{\mathbb{C}}$, and the condition $(\omega, \omega) = 0$ determines the coefficient of $v_2$. On the other hand, if we denote by $\mathcal{Q}$ the positive definite quadratic form $\mathcal{Q}(w) = -(w, w)$ in $W_{\mathbb{R}}$, then the condition $(\omega, \overline{\omega}) > 0$ is equivalent to

$$\operatorname{Im}\tau \, \operatorname{Im}\sigma > \frac{1}{2}\mathcal{Q}(\operatorname{Im}\zeta_W). \tag{3.8}$$

Choosing the connected component for which $\operatorname{Im}\tau > 0$, we obtain

$$\mathcal{D}_{2d} = \left\{(\tau, \zeta_W, \sigma) : \operatorname{Im}\tau > 0, \quad \operatorname{Im}\sigma - \frac{\mathcal{Q}(\operatorname{Im}\zeta_W)}{2\operatorname{Im}\tau} > 0\right\}. \tag{3.9}$$

*Remark* 3.3.4. The description above gives the realization of $\mathcal{D}_{2d}$ as a Siegel domain of the third kind, namely

$$\mathcal{D}_{2d} \cong \left\{(\tau, \zeta_W, \sigma) \in \mathcal{H} \times W_{\mathbb{C}} \times \mathbb{C} : \operatorname{Im}\sigma > \frac{\mathcal{Q}(\operatorname{Im}\,\zeta_W)}{2\operatorname{Im}\tau}\right\}.$$

Accordingly we will refer to the coordinates $(\tau, \zeta_W, \sigma)$ as Siegel domain coordinates.

We now pass from the Siegel-domain coordinates to the horospherical coordinates associated with $P_{\Pi}$. To distinguish the two coordinate systems, we write $\tau_s$ for the Siegel-domain coordinate and $\tau_H$ for the corresponding horospherical coordinate. The idea is that, relative to the basepoint $o \in \mathcal{D}_{2d}$ determined by the opposite isotropic plane $\Pi'$, every point can be written uniquely as the action of an element of $N_{\Pi}$ on a point with $\zeta_W = 0$ and purely imaginary $\sigma$.

Recall from Section 3.3.1 that the positive plane defining the basepoint $o$ is given by

$$\Pi_o = \operatorname{Span}_{\mathbb{R}}\{e_1, e_2\} = \operatorname{Span}_{\mathbb{R}}\{u_1 + v_1, u_2 + v_2\}.$$

We orient this plane by the complex vector $\omega_o = e_2 + ie_1 = \omega(i, 0, i)$, so that

$$o = [\omega(i, 0, i)].$$

First we compute the action of the Levi factor. Its elements have the form

$$\ell(A, h) = \begin{pmatrix} A & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & R(A^{-1})^t R \end{pmatrix}, \quad A \in \mathrm{GL}^+(2, \mathbb{R}), \quad h \in \mathrm{SO}(W_{\mathbb{R}}, Q_W).$$

writing

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \Delta = \det A > 0,$$

and applying $\ell(A,h)$ to an arbitrary Siegel representative gives

$$\ell(A,h)\cdot[\omega(\tau,\zeta_W,\sigma)] = \left[\omega\left(\frac{a\tau+b}{c\tau+d}, \frac{h\zeta_W}{c\tau+d}, \frac{\sigma}{\Delta} - \frac{c\mathcal{Q}(\zeta_W)}{2\Delta(c\tau+d)}\right)\right]. \tag{3.10}$$

In particular

$$\ell(A,h)\cdot o = \left[\omega\left(\frac{ia+b}{ic+d}, 0, \frac{i}{\det A}\right)\right].$$

Notice that the factor $h$ does not occur in this expression and therefore the $\mathrm{SO}(W_{\mathbb{R}}, Q_W)$ factor fixes $o$.

For $\tau_H = x+iy \in \mathcal{H}$ and $t>0$, define

$$A_{\tau_H,t} := \frac{1}{\sqrt{ty}}\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}, \qquad \ell_{\tau_H,t} := \ell(A_{\tau_H,t}, I_W).$$

Then a direct calculation gives the vector identity

$$\ell_{\tau_H,t}\,\omega_o = \frac{1}{\sqrt{ty}}\omega(\tau_H, 0, it).$$

Thus it follows that $\ell_{\tau_H,t}\cdot o = [\omega(\tau_H,0,it)]$ and the Levi orbit is precisely

$$\mathcal{S}_\Pi := \{[\omega(\tau_H,0,it)] : \tau_H \in \mathcal{H},\ t>0\}.$$

We next compute the action of the nilpotent radical. Let

$$X = \begin{pmatrix} x_2 & x_1\end{pmatrix} \in \mathrm{Hom}(\Pi'_{\mathbb{R}}, W_{\mathbb{R}}), \quad x_2 = X(v_2), \quad x_1 = X(v_1).$$

The corresponding element of $N_\Pi$ is

$$n(X, Z_\lambda) = \begin{pmatrix} I_2 & -RX^tQ_W & Z_\lambda - \frac{1}{2}RX^tQ_WX \\ 0 & I_W & X \\ 0 & 0 & I_2\end{pmatrix}.$$

Since

$$Z_\lambda - \frac{1}{2}RX^tQ_WX = \begin{pmatrix} \lambda + \frac{1}{2}\mathcal{Q}(x_1,x_2) & \frac{1}{2}\mathcal{Q}(x_1) \\ \frac{1}{2}\mathcal{Q}(x_2) & -\lambda + \frac{1}{2}\mathcal{Q}(x_1,x_2)\end{pmatrix},$$

a direct multiplication gives, for $\tau_H \in \mathcal{H}$ and $t>0$,

$$n(X,Z_\lambda)\,\omega(\tau_H,0,it) = \begin{pmatrix} \alpha \\ \beta \\ it(x_1 - \tau_H x_2) \\ -it\tau_H \\ it\end{pmatrix},$$

where

$$\alpha = \tau_H + it\left(-\lambda\tau_H - \frac{\tau_H}{2}\mathcal{Q}(x_1,x_2) + \frac{1}{2}\mathcal{Q}(x_1)\right),$$
$$\beta = 1 + it\left(-\lambda - \frac{1}{2}\mathcal{Q}(x_2)\tau_H + \frac{1}{2}\mathcal{Q}(x_1,x_2)\right).$$

The scalar $\beta$ is the coefficient of $u_2$. It is nonzero, otherwise the positive plane represented by the image would be contained in $v_2^\perp$, whereas $v_2^\perp/\mathbb{R}\,v_2$ has positive index one and

therefore cannot contain a positive-definite two-plane. We must consequently divide the vector above by $\beta$ in order to return to the affine normalization in which the coefficient of $u_2$ is equal to one. Set $\eta := x_1 - \tau_H x_2$. If $(\tau_s, \zeta_W, \sigma)$ are the Siegel coordinates of the resulting point, then

$$\sigma = \frac{it}{\beta}, \qquad \zeta_W = \sigma\eta.$$

Moreover,

$$\alpha - \tau_H\beta = \frac{it}{2}\mathcal{Q}(\eta),$$

and hence

$$\tau_s = \tau_H + \frac{\sigma}{2}\mathcal{Q}(\eta).$$

Equivalently, the action of $N_\Pi$ on the Levi section is described by

$$\begin{aligned} \zeta_W &= \sigma(x_1 - \tau_H x_2), \\ \tau_s &= \tau_H + \frac{\sigma}{2}\mathcal{Q}(x_1 - \tau_H x_2), \\ \frac{1}{\sigma} &= -\lambda - \frac{i}{t} - \frac{1}{2}\mathcal{Q}(x_2)\tau_H + \frac{1}{2}\mathcal{Q}(x_1, x_2). \end{aligned} \tag{3.11}$$

Since $P_\Pi = N_\Pi \rtimes L_\Pi$ and $L_\Pi$ is stable under the Cartan involution determined by $o$, multiplication induces a diffeomorphism

$$N_\Pi \times L_\Pi/(L_\Pi \cap K_o) \longrightarrow \mathcal{D}_{2d}, \quad (n, \ell(L_\Pi \cap K_o)) \longmapsto n\ell \cdot o.$$

Using the section above, this becomes

$$N_\Pi \times \mathcal{H} \times \mathbb{R}_{>0} \longrightarrow \mathcal{D}_{2d}, \qquad (n, \tau_H, t) \longmapsto n \cdot [\omega(\tau_H, 0, it)].$$

Consequently, every point of $\mathcal{D}_{2d}$ admits a unique factorization

$$[\omega(\tau_s, \zeta_W, \sigma)] = n(X, Z_\lambda) \cdot [\omega(\tau_H, 0, it)]. \tag{3.12}$$

For completeness, we now give the inverse transformation explicitly. Put

$$\rho := \operatorname{Im}(\tau_s)\operatorname{Im}(\sigma) - \frac{1}{2}\mathcal{Q}(\operatorname{Im}\zeta_W) > 0$$

and define

$$\tau_H := \tau_s - \frac{\mathcal{Q}(\zeta_W)}{2\sigma}, \qquad \eta := \frac{\zeta_W}{\sigma}.$$

A direct computation gives

$$\operatorname{Im}(\sigma)\ \operatorname{Im}(\tau_H) = \rho + \frac{|\sigma|^2}{2}\mathcal{Q}(\operatorname{Im}\eta).$$

In particular, $\operatorname{Im}(\tau_H) > 0$, so $\tau_H \in \mathcal{H}$. The real vectors $x_1, x_2$, the positive real number $t$, and the central coordinate $\lambda$ are then recovered by

$$\begin{aligned} x_2 &= -\frac{\operatorname{Im}(\eta)}{\operatorname{Im}(\tau_H)}, \\ x_1 &= \operatorname{Re}(\eta) + \operatorname{Re}(\tau_H)x_2, \\ t &= \frac{\operatorname{Im}(\tau_H)|\sigma|^2}{\rho}, \\ \lambda &= -\operatorname{Re}\left(\frac{1}{\sigma}\right) - \frac{\operatorname{Re}(\tau_H)}{2}\mathcal{Q}(x_2) + \frac{1}{2}\mathcal{Q}(x_1, x_2). \end{aligned} \tag{3.13}$$

Indeed, the first two identities are equivalent to $\eta = x_1 - \tau_H x_2$, while the last two follow by taking the imaginary and real parts, respectively, of the third identity in (3.11). The positivity of $t$ follows from $\rho > 0$ and $\mathrm{Im}(\tau_H) > 0$. These formulas also prove directly the existence and uniqueness of the factorization (3.12).

*3.3.1.2 Horospherical decomposition associated with $P_\ell$*

Now consider the rational boundary component associated with an isotropic line $\ell \subset V_{\mathbb{Q}}$. Let $\ell = \mathbb{Q}u_1 \subset V_{\mathbb{Q}}$ be a rational isotropic line and choose an isotropic vector $v_1 \in V_{\mathbb{Q}}$ such that $(u_1, v_1) = 1$. Set $\ell' = \mathbb{Q}v_1$, and $W_\ell = (\ell \oplus \ell')^\perp$. Thus, $V_{\mathbb{Q}} = \ell \oplus W_\ell \oplus \ell'$, and similarly over $\mathbb{R}$.

Relative to this decomposition, the real parabolic subgroup $P_\ell = \mathrm{Stab}(\ell_{\mathbb{R}})$ is given by matrices of the form

$$g(a, A, x) = \begin{pmatrix} a & -ax^t Q_{W_\ell} & -\dfrac{a}{2}x^t Q_{W_\ell} x \\ 0 & A & Ax \\ 0 & 0 & a^{-1} \end{pmatrix},$$

where $a \in \mathbb{R}_{>0}$,$A \in SO(W_{\ell,\mathbb{R}})^\circ$, $x \in W_{\ell,\mathbb{R}}$, and $Q_{W_\ell}$ denotes the matrix of the restriction of the bilinear form to $W_{\ell,\mathbb{R}}$. We write

$$n(x) = \begin{pmatrix} 1 & -x^t Q_{W_\ell} & -\dfrac{1}{2}x^t Q_{W_\ell} x \\ 0 & I & x \\ 0 & 0 & 1 \end{pmatrix}, \quad \ell(a, A) = \begin{pmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a^{-1} \end{pmatrix}.$$

Then $g(a, A, x) = \ell(a, A)n(x)$. In particular, the unipotent radical is abelian and satisfies $N_\ell \simeq (W_{\ell,\mathbb{R}}, +)$, while the connected component of the Levi factor is

$$L_\ell \simeq \mathbb{R}_{>0} \times SO(W_{\ell,\mathbb{R}})^\circ.$$

We next specify a basepoint. In terms of the rational decomposition

$$V_{\mathbb{Q}} = \mathbb{Q}u_1 \oplus \mathbb{Q}u_2 \oplus W_{\Pi,\mathbb{Q}} \oplus \mathbb{Q}v_2 \oplus \mathbb{Q}v_1,$$

we have $W_{\ell,\mathbb{R}} = \mathbb{R}u_2 \oplus W_{\Pi,\mathbb{R}} \oplus \mathbb{R}v_2$. Thus, in the notation of the general construction, we may take $U_W = \mathbb{R}e_2 \subset W_{\ell,\mathbb{R}}$. The corresponding positive definite plane is $\Pi_{\ell'} = \mathbb{R}e_1 \oplus \mathbb{R}e_2$. We choose its orientation so that the corresponding point $o \in \mathcal{D}_{2d}$ is represented by $\omega_o = e_2 + ie_1$. Projectively, multiplying this representative by $-i$ gives $o = [e_1 - ie_2]$.

We first recall the tube-domain realization. We work in the affine chart $(\omega, u_1) = 1$. A point of this chart can be written as

$$\omega = \alpha u_1 + \zeta_{W_\ell} + v_1, \qquad \zeta_{W_\ell} \in W_{\ell,\mathbb{C}}, \quad \alpha \in \mathbb{C}.$$

The isotropy condition $(\omega, \omega) = 0$ gives $\alpha = -\frac{1}{2}(\zeta_{W_\ell}, \zeta_{W_\ell})$. Let

$$\omega(\zeta_{W_\ell}) := -\frac{1}{2}(\zeta_{W_\ell}, \zeta_{W_\ell})u_1 + \zeta_{W_\ell} + v_1. \tag{3.14}$$

Hence, we obtain $\big(\omega(\zeta_{W_\ell}), \overline{\omega(\zeta_{W_\ell})}\big) = 2(\mathrm{Im}(\zeta_{W_\ell}), \mathrm{Im}(\zeta_{W_\ell}))$. Therefore, the positivity condition is equivalent to $(\mathrm{Im}(\zeta_{W_\ell}), \mathrm{Im}(\zeta_{W_\ell})) > 0$.

Let $\mathcal{T}_{W_\ell}$ denote the connected component of

$$\{Y \in W_{\ell,\mathbb{R}} : (Y, Y) > 0\}$$

containing $e_2$. In accordance with the basepoint, we write

$$\zeta_{W_\ell} = X - iY, \qquad X \in W_{\ell,\mathbb{R}}, \quad Y \in \mathcal{T}_{W_\ell}.$$

This gives the tube-domain realization

$$\mathcal{D}_{2d} \simeq W_{\ell,\mathbb{R}} \times \mathcal{T}_{W_\ell}.$$

We now proceed with the computation for the horospherical coordinates. First, notice that our choice of basepoint satisfies $o = [\omega(-ie_2)]$. Therefore, the action of the unipotent radical is as follows.

$$n(x) \cdot [\omega(\zeta_{W_\ell})] = [\omega(\zeta_{W_\ell} + x)] \tag{3.15}$$

Thus, in the real coordinates $\zeta_{W_\ell} = X - iY$, the action is

$$n(x) : (X, Y) \longmapsto (X + x, Y),$$

and in particular, $N_\ell$ acts simply transitively on the first factor $W_{\ell,\mathbb{R}}$.

*Remark* 3.3.5. The normalization $(\omega, u_1) = 1$ is chosen precisely so that the upper-triangular unipotent radical $N_\ell$ displayed above acts by translations on the corresponding affine coordinate. With the alternative normalization $(\omega, v_1) = 1$, an element $n(x) \in N_\ell$ acts instead by a fractional linear transformation.

We next consider the action of the Levi factor. We have

$$\begin{aligned} \ell(a, A)\, \omega(\zeta_{W_\ell}) &= -\frac{a}{2}(\zeta_{W_\ell}, \zeta_{W_\ell})u_1 + A\zeta_{W_\ell} + a^{-1}v_1 \\ &= a^{-1}\omega(aA\zeta_{W_\ell}), \end{aligned}$$

where we have used the fact that $A$ preserves the bilinear form on $W_{\ell,\mathbb{R}}$. Therefore, projectively,

$$\ell(a, A) \cdot [\omega(\zeta_{W_\ell})] = [\omega(aA\,\zeta_{W_\ell})]. \tag{3.16}$$

Equivalently,

$$l(a, A) : (X, Y) \longmapsto (aAX, aAY).$$

Applying (3.16) to the basepoint gives

$$\ell(a, A) \cdot o = [\omega(-iaAe_2)].$$

The group $SO(W_{\ell,\mathbb{R}})^\circ$ acts transitively on the unit hyperboloid contained in $\mathcal{T}_{W_\ell}$, while the factor $\mathbb{R}_{>0}$ acts by positive homotheties. It follows that

$$L_\ell \cdot o = \big\{ [\omega(-iY)] \,\big|\, Y \in \mathcal{T}_{W_\ell} \big\}.$$

Moreover, the stabilizer of $o$ in $L_\ell$ is

$$L_\ell \cap K_o = \big\{ \ell(1, A) \,\big|\, Ae_2 = e_2 \big\} \simeq SO(18).$$

Consequently, $L_\ell/(L_\ell \cap K_o) \simeq \mathcal{T}_{W_\ell}$. More explicitly, if $Y = aAe_2$, then $(Y, Y) = 2a^2$, and hence the radial coordinate of the Levi factor is

$$a = \sqrt{\frac{(Y, Y)}{2}}. \tag{3.17}$$

Combining the actions of $N_\ell$ and $L_\ell$, every point has the unique factorization

$$[\omega(\zeta_{W_\ell})] = n(x)\, \ell(a, A) \cdot [\omega(-ie_2)], \quad \zeta_{W_\ell} = x - iaAe_2. \tag{3.18}$$

### 3.3.2 Quotient by the arithmetic subgroup

So far we have worked at the level of the symmetric space without taking into account the quotient by the arithmetic subgroup $\Gamma_{2d}$. To proceed with our analysis first we need to recall a fundamental fact from the geometry of Hermitian locally symmetric domains. First, we fix some notation. Given a rational isotropic subspace $I$ we have an exact sequence of Lie Groups

$$1 \longrightarrow N_I \longrightarrow P_I \xrightarrow{\lambda_I} L_I \longrightarrow 1. \tag{3.19}$$

Let us denote by

$$\Gamma_I := \Gamma_{2d} \cap P_I, \quad \Gamma_{I,N} := \Gamma_I \cap N_I, \quad \Gamma_{I,L} := \lambda_I(\Gamma_I) \subset L_I.$$

It is clear that the following exact sequence holds

$$1 \longrightarrow \Gamma_{I,N} \longrightarrow \Gamma_I \xrightarrow{\lambda_I} \Gamma_{I,L} \longrightarrow 1. \tag{3.20}$$

The key fact we will need is

**Theorem 3.3.6.** *Let $I$ be a rational isotropic subspace. There exists a $\Gamma_I$-invariant neighborhood $\mathcal{U}_I \subset \mathcal{D}_{2d}$ of the boundary component associated with $I$ and an embedding*

$$\Gamma_I \backslash \mathcal{U}_I \hookrightarrow \Gamma_{2d} \backslash \mathcal{D}_{2d}.$$

*Proof.* This follows from the precise-invariance property of sufficiently small cusp neighborhoods in the Satake topology; see [2, Chapter III, Theorem 6.1(iv)]. The description of these neighborhoods in Siegel domain coordinates is given in [2, Chapter III, Theorem 6.3]. □

The relevance of Theorem 3.3.6 is that it provides a local model for the arithmetic quotient near the boundary component determined by $I$. More precisely, the image of $\mathcal{U}_I$ in $\mathcal{F}_{2d}$ is naturally identified with $\Gamma_I \backslash \mathcal{U}_I$: any two points of $\mathcal{U}_I$ that are identified by $\Gamma_{2d}$ are already identified by $\Gamma_I$. Hence, after restricting to $\mathcal{U}_I$, we may replace the quotient by $\Gamma_{2d}$ with the quotient by $\Gamma_I$ and use the preceding short exact sequence to analyze it in two stages, first by taking the quotient by $\Gamma_{I,N}$ and then by $\Gamma_{I,L}$

$$\Gamma_I \backslash \mathcal{D}_{2d} \simeq \Gamma_{I,L} \backslash (\Gamma_{I,N} \backslash \mathcal{D}_{2d}). \tag{3.21}$$

Let us describe the geometry of this two-step quotient in detail. Recall that, relative to the horospherical decomposition associated with $I$, we have the trivial fiber bundle $\pi_I : \mathcal{D}_{2d} \to B_I = X_{I,h} \times X_{I,l}$ with fiber $N_I$. Furthermore, given that the $N_I$-orbits are precisely the fibers of $\pi_I$ it is sufficient to consider the quotient $\mathcal{N}_I := \Gamma_{I,N} \backslash N_I$. Since $N_I$ is the unipotent radical of a rational parabolic subgroup and $\Gamma_{2d}$ is arithmetic, the subgroup $\Gamma_{I,N}$ is a lattice in $N_I$. In particular, $\mathcal{N}_I$ is compact. Moreover, $N_I$ is a simply connected nilpotent Lie group, and a nontrivial unipotent element has infinite order. Hence $\Gamma_{I,N}$ is torsion-free and $\mathcal{N}_I$ is a smooth compact nilmanifold. Finally, to understand the quotient by $\Gamma_{I,L}$, let $b \in B_I$ and denote its image in the quotient $\Gamma_{I,L} \backslash B_I$ by $[b]$. The $P_I$-equivariance of $\pi_I$ implies that it descends to a map

$$\overline{\pi}_I : \Gamma_I \backslash \mathcal{D}_{2d} \longrightarrow \Gamma_{I,L} \backslash B_I. \tag{3.22}$$

Set $H_b := \mathrm{Stab}_{\Gamma_{I,L}}(b)$ and notice that since $\Gamma_{I,L}$ is discrete and acts properly discontinuously on $B_I$, the group $H_b$ is finite. Thus one has that the fiber over $[b] \in \Gamma_{I,L} \backslash B_I$

is naturally isomorphic to $H_b\backslash\mathcal{N}_I$, where $H_b$ acts on $\mathcal{N}_I$ by the affine nilmanifold automorphisms induced by the action of $\Gamma_I$. Consequently, the fibers are finite quotients of compact nilmanifolds and may have finite quotient singularities.

Let us now describe the base in the two cases under consideration. For $I = \Pi$, recall that $B_\Pi = \mathcal{H} \times \mathbb{R}_{>0}$. An element of $\Gamma_{\Pi,L}$ preserves the rank-two lattice $\Pi \cap \Lambda_{2d}$, and hence its $\mathrm{GL}(\Pi)$-component has determinant one. It therefore acts trivially on the $\mathbb{R}_{>0}$-factor, and we obtain

$$\Gamma_{\Pi,L}\backslash B_\Pi \simeq \left(\Gamma_{\Pi,L}\backslash\mathcal{H}\right) \times \mathbb{R}.$$

Here the action on $\mathcal{H}$ is understood through the effective image of $\Gamma_{\Pi,L}$ in $\mathrm{PSL}(2,\mathbb{R})$. After choosing a basis of $\Pi \cap \Lambda_{2d}$, this image is a finite-index subgroup of $\mathrm{PSL}(2,\mathbb{Z})$. Thus $\Gamma_{\Pi,L}\backslash\mathcal{H}$ is a modular curve and gives a finite orbifold covering of $A_1 = \mathrm{PSL}(2,\mathbb{Z})\backslash\mathcal{H}$.

For $I = \ell$, we have $B_\ell = \mathcal{T}_{W_\ell}$. Since an element of $\Gamma_{\ell,L}$ preserves the rank-one lattice $\ell \cap \Lambda_{2d}$, its positive scalar component is equal to one. The action therefore preserves the radial coordinate of $\mathcal{T}_{W_\ell}$. Writing $Y = e^\rho \widehat{Y}$, with $(\widehat{Y},\widehat{Y}) = 2$, identifies $\mathcal{T}_{W_\ell}$ with $\mathbb{R} \times \mathbb{H}^{18}$, and hence

$$\Gamma_{\ell,L}\backslash B_\ell \simeq \left(\Gamma_{\ell,L}\backslash\mathbb{H}^{18}\right) \times \mathbb{R}.$$

### 3.3.3 Siegel sets and coordinates at infinity

In this section we recall the notion of a Siegel set associated with a *minimal* rational parabolic subgroup. Siegel sets will play an essential role in our analysis, since reduction theory shows that they provide coarse fundamental domains for the action of arithmetic subgroups; see Theorem 3.3.7 below.

Let

$$0 \subset \ell \subset \Pi \subset V_{\mathbb{Q}} \tag{3.23}$$

be a complete rational isotropic flag. Its stabilizer $P_0 := \mathrm{Stab}_{\mathbb{G}}(\ell \subset \Pi)$ is a minimal rational parabolic subgroup of $\mathbb{G}$. We first describe abstractly the Siegel sets associated to $P_0(\mathbb{R})$; in what follows we will use $P_0$ for both the rational and real minimal parabolic. Fix a base point $o \in \mathcal{D}_{2d}$ and let $K_o$ be the maximal compact subgroup associated with $o$. We then have the Langlands decomposition $P_0 = N_0 A M_0$, where $L_{0,o} = AM_0$ is the unique $\theta_o$-stable Levi factor of $P_0$ and the group $A$ is the connected split component of $L_{0,o}$. Notice that in the present case, $\dim A = 2$.

The next step in the construction is to define a truncated positive chamber in $A$. To this end, we first recall the system of simple restricted roots determined by $P_0$. Let $\mathfrak{a} = \mathrm{Lie}(A)$ and consider the restricted-root decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Sigma(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_\alpha.$$

The minimal parabolic $P_0$ determines a positive system by declaring

$$\Sigma^+(P_0,A) := \{\alpha \in \Sigma(\mathfrak{g},\mathfrak{a}) \mid \mathfrak{g}_\alpha \subset \mathfrak{n}_0\},$$

where $\mathfrak{n}_0 = \mathrm{Lie}(N_0)$. Let $\Delta(P_0,A) \subset \Sigma^+(P_0,A)$ be the corresponding set of simple roots. For $c > 0$, the truncated positive chamber is defined abstractly by

$$A_c^+ := \{a \in A \mid \alpha(a) \geq c \text{ for every } \alpha \in \Delta(P_0,A)\}.$$

Thus $A_c^+$ is obtained by imposing a uniform lower bound on each simple-root character.

Finally, we require the $N_0M_0$-coordinates to vary in a fixed compact set, while the $A$-coordinate ranges over the truncated positive chamber $A_c^+$. More precisely, given a

compact subset $\Omega \subset N_0 M_0$, the corresponding Siegel set in $SO^\circ(2,19)$ is $\widetilde{\mathfrak{S}}_{\Omega,c} = \Omega A_c^+ K_o$, and its image in the symmetric domain is

$$\mathfrak{S}_{\Omega,c} = \Omega A_c^+ o \subset \mathcal{D}_{2d}.$$

The importance of Siegel sets lies in the following fundamental results of reduction theory that roughly speaking say that for any arithmetic subgroup $\Gamma \subset \mathbb{G}(\mathbb{Q})$, finitely many rational translates of a suitable Siegel set cover $\mathcal{D}_{2d}$ modulo the action of $\Gamma$, while only finitely many $\Gamma$-translates of such sets can meet a fixed one

**Theorem 3.3.7.** *Let $\Gamma \subset \mathbb{G}(\mathbb{Q}) \cap SO^\circ(2,19)$ be an arithmetic subgroup, and fix the minimal rational parabolic subgroup $P_0$ and the base point $o$ as above. Then the following statements hold.*

*(i) There exist a compact subset $\Omega \subset N_0 M_0$, a constant $c > 0$, and a finite subset $F \subset \mathbb{G}(\mathbb{Q}) \cap SO^\circ(2,19)$ such that*

$$\mathcal{D}_{2d} = \Gamma \cdot F \cdot \mathfrak{S}_{\Omega,c};$$

*(ii) for every compact subset $\Omega \subset N_0 M_0$, every $c > 0$, and every pair $g_1, g_2 \in \mathbb{G}(\mathbb{Q}) \cap SO^\circ(2,19)$, the set*

$$\{\gamma \in \Gamma \mid g_1 \mathfrak{S}_{\Omega,c} \cap \gamma g_2 \mathfrak{S}_{\Omega,c} \neq \varnothing\}$$

*is finite.*

For proofs of the covering and finiteness assertions, we refer the reader to [10, §III.2] and [2, Chapter II, §4.1, p. 69].

*Remark* 3.3.8. In particular, choosing $\Omega$, $c$, and $F$ as in (i), the set $\mathfrak{F} := F \cdot \mathfrak{S}_{\Omega,c}$ is a coarse fundamental domain for the action of $\Gamma$: it satisfies $\Gamma\mathfrak{F} = \mathcal{D}_{2d}$, and $\{\gamma \in \Gamma \mid \mathfrak{F} \cap \gamma\mathfrak{F} \neq \varnothing\}$ is finite.

We now specialize this abstract construction to the coordinates determined by a rational basis adapted to the complete flag. As usual, we choose the rational decomposition $V_{\mathbb{Q}} = \mathbb{Q}u_1 \oplus \mathbb{Q}u_2 \oplus W_{0,\mathbb{Q}} \oplus \mathbb{Q}v_2 \oplus \mathbb{Q}v_1$ such that $\ell = \mathbb{Q}u_1$, $\Pi = \mathbb{Q}u_1 \oplus \mathbb{Q}u_2$, and $(u_i, v_j) = \delta_{ij}$, $(u_i, u_j) = (v_i, v_j) = 0$. Additionally, we take the base point $o$ to be represented by the oriented positive plane $\Pi_o = \mathrm{Span}_{\mathbb{R}}\{u_1 + v_1, u_2 + v_2\}$.

With this choice, the split component $A$ is

$$A = \left\{ a(r,s) = \mathrm{diag}\left(r, s, \mathrm{id}_{W_{0,\mathbb{R}}}, s^{-1}, r^{-1}\right) \;\middle|\; r, s > 0 \right\}.$$

Define characters $\varepsilon_1, \varepsilon_2 \in X^*(A)$ by $\varepsilon_1\big(a(r,s)\big) = r$, $\varepsilon_2\big(a(r,s)\big) = s$. Since $P_0$ is upper triangular with respect to the ordered decomposition above, the positive roots are

$$\Sigma^+(P_0, A) = \{\varepsilon_1 - \varepsilon_2,\ \varepsilon_2,\ \varepsilon_1,\ \varepsilon_1 + \varepsilon_2\},$$

and the corresponding simple roots are $\alpha_1 = \varepsilon_1 - \varepsilon_2$, and $\alpha_2 = \varepsilon_2$. Their values on $A$ are

$$\alpha_1\big(a(r,s)\big) = \frac{r}{s}, \qquad \alpha_2\big(a(r,s)\big) = s.$$

Therefore the abstract definition of the truncated chamber becomes

$$A_c^+ = \left\{ a(r,s) \in A \;\middle|\; \frac{r}{s} \geq c,\ s \geq c \right\}.$$

Consequently, the Siegel sets associated with the chosen complete flag take the form

$$\mathfrak{S}_{\Omega,c} = \left\{ \omega\, a(r,s) \cdot o \;\middle|\; \omega \in \Omega, \quad \frac{r}{s} \geq c, \quad s \geq c \right\}.$$

*Remark* 3.3.9. The compactness condition on $\Omega$ is essential. For each $I \in \{\ell, \Pi\}$, the subgroup $\Gamma_{I,N} = \Gamma_I \cap N_I$ is a uniform lattice in $N_I$, meaning that it is discrete and that $\Gamma_{I,N}\backslash N_I$ is compact. Hence one may choose a compact subset $\Omega_I \subset N_I$ such that $N_I = \Gamma_{I,N}\Omega_I$. After enlarging $\Omega$, if necessary, one may assume that its $N_I$-component contains such a compact fundamental set. Thus every point of the compact nilmanifold $\Gamma_{I,N}\backslash N_I$ admits a representative in the fixed compact set $\Omega_I$.

In the final part of this section, we relate the parameters defining the Siegel sets to the horospherical coordinates introduced in Section 3.3.1. This comparison will clarify the geometric meaning of the two simple-root parameters $r/s$ and $s$, which may be viewed as describing two independent *directions at infinity* in moduli.

**Proposition 3.3.10.** *The following relations between the two simple-root parameters and the horospherical coordinates hold:*

(i) *Relative to the $P_\Pi$-horospherical coordinates of a point $[\omega] \in \mathcal{D}_{2d}$ (see Equation 3.12) we get*

$$r = \sqrt{\frac{\operatorname{Im}\tau_H}{t}}, \qquad s = \frac{1}{\sqrt{t \operatorname{Im}\tau_H}}. \tag{3.24}$$

(ii) *Relative to the $P_\ell$-horospherical coordinates of a point $[\omega] \in \mathcal{D}_{2d}$ (see Equation 3.17) one has that the radial and normalized hyperboloid coordinates are*

$$r = \sqrt{\frac{(Y,Y)}{2}}, \qquad \widehat{Y} := \frac{Y}{r} = su_2 + s^{-1}v_2. \tag{3.25}$$

*Moreover, fix a compact subset $\Omega \subset N_0M_0$ and a constant $c > 0$, and let $x_i = \omega_i a(r_i, s_i)\cdot o \in \mathfrak{S}_{\Omega,c}$, with $\omega_i \in \Omega$, be a sequence leaving every compact subset of $\mathcal{D}_{2d}$. After passing to a subsequence, the asymptotic behavior of $(x_i)$ is classified by the two simple-root parameters as follows.*

(iii) *The sequence goes in the direction at infinity associated with $P_\Pi$ if and only if*

$$s_i \longrightarrow \infty, \qquad \frac{r_i}{s_i} \quad \textit{remains uniformly bounded.}$$

(iv) *The sequence goes in the direction at infinity associated with $P_\ell$ if and only if*

$$\frac{r_i}{s_i} \longrightarrow \infty.$$

*Proof.* The assertions follow directly by comparing the formulas for the $P_\Pi$- and $P_\ell$-horospherical decompositions obtained in Section 3.3.1 with the action of $a(r,s)$ on the basepoint; the asymptotic statements then follow by letting the two simple-root parameters diverge, while compactness of $\Omega$ ensures that the bounded $N_0M_0$-factor does not change the resulting direction at infinity. □

## 3.4 Weil-Petersson geometry of $\mathcal{F}_{2d}$

In this section we characterize the asymptotic behavior of the Weil–Petersson metric at infinity in $\mathcal{F}_{2d}$ and make the necessary step towards a modular geometric interpretation of these asymptotics.

### 3.4.1 Preliminaries

We retain the definition and normalization of the Weil–Petersson metric introduced in the previous chapters, and recall only the period-theoretic formula that will be used below. Let $U \subset \mathcal{D}_{2d}$ be a sufficiently small local uniformizing neighborhood in the smooth polarized locus, and let $f : \mathcal{X} \to U$ be the corresponding marked family, with $X_s := f^{-1}(s)$. Consider the Hodge line bundle $\mathcal{H}^{2,0} := f_* K_{\mathcal{X}/U}$. Possibly after shrinking $U$, we choose a nowhere-vanishing holomorphic frame $\Omega$ of $\mathcal{H}^{2,0}$, and denote by $\Omega_s \in H^0(X_s, K_{X_s})$ its restriction to the fiber $X_s$. Its Hodge norm is

$$\|\Omega\|^2_{\mathrm{H}}(s) := \int_{X_s} \Omega_s \wedge \overline{\Omega_s}.$$

Tian's formula [69] identifies the Weil–Petersson form with the curvature of this Hodge metric

$$\omega_{\mathrm{WP}} = -\, i\, \partial\overline{\partial} \log \|\Omega\|^2_{\mathrm{H}}. \tag{3.26}$$

The fundamental property that we will exploit is that the Weil–Petersson metric agrees, up to an overall positive constant, with the invariant symmetric-space metric on $\mathcal{D}_{2d}$. To see this, observe first that, under the marking $\varphi_s$, the Hodge norm is given by the lattice pairing

$$\|\Omega\|^2_{\mathrm{H}}(s) = \big(\varphi_s[\Omega_s], \overline{\varphi_s[\Omega_s]}\big).$$

Since the action of $\mathrm{SO}(2,19)^\circ$ on $\mathcal{D}_{2d}$ is holomorphic and preserves the lattice pairing, it follows that (3.26) is $\mathrm{SO}(2,19)^\circ$-invariant. Moreover, $\mathcal{D}_{2d}$ is an irreducible Hermitian symmetric domain. Therefore, its invariant Kähler metric is unique up to multiplication by a positive constant. Consequently, there exists $c > 0$ such that $\omega_{\mathrm{WP}} = c\,\omega_{\mathcal{D}}$, where $\omega_{\mathcal{D}}$ denotes the symmetric-space Kähler form. We normalize the symmetric-space metric accordingly, so that its Kähler form agrees with $\omega_{\mathrm{WP}}$.

Consequently, $(\mathcal{D}_{2d}, g_{\mathrm{WP}})$ is a complete and simply connected Riemannian symmetric space of noncompact type. Since $\mathcal{D}_{2d}$ is irreducible, its invariant metric, and hence every positive multiple of it, is Einstein with negative Einstein constant (see [5, Theorem 7.73, Corollary 7.74, and Theorem 7.78]). Therefore, $g_{\mathrm{WP}}$ is a complete Kähler–Einstein metric satisfying $\mathrm{Ric}(g_{\mathrm{WP}}) = \lambda g_{\mathrm{WP}}$ for some $\lambda < 0$. Finally the metric $\omega_{\mathrm{WP}}$ descends to a complete locally symmetric Kähler orbifold metric on $\mathcal{F}_{2d}$.

We now make precise what we mean by its asymptotic behavior at infinity. Following the approach introduced in [16, Definition 3.6], reproduced in Chapter 2, we formulate this notion in terms of pointed Gromov–Hausdorff convergence.

**Definition 3.4.1.** Let $(x_j)$ be a sequence in $\mathcal{F}_{2d}$ escaping every compact subset. A pointed proper metric space $(Z, d_Z, z_\infty)$ is called a *Weil–Petersson asymptotic limit* of $\mathcal{F}_{2d}$ along $(x_j)$ if

$$\big(\mathcal{F}_{2d}, d_{\mathrm{WP}}, x_j\big) \xrightarrow{\mathrm{pGH}} (Z, d_Z, z_\infty),$$

where $d_{\mathrm{WP}}$ is the distance induced by the Weil–Petersson metric.

*Remark* 3.4.2. The ingredients underlying the analysis of such limits have appeared in several complementary forms in the literature. The horospherical description of rational cusps and the reduction-theoretic control provided by Siegel sets are classical; see [2, Chapter III, §§4–6], while the rootwise dependence of the invariant metric in rational horospherical coordinates is made explicit in [10, Lemma III.20.7]. Hattori determined unrescaled pointed limits along rational geodesic rays for arithmetic quotients of $SO(n)\backslash SL(n,\mathbb{R})$ [26, §4, in particular Lemma 4.2 and Proposition C], and subsequently described the rescaled

tangent cone at infinity for higher-rank arithmetic locally symmetric spaces [27]. These results provide important models for the analysis below, but they do not directly yield the statement required here for the type-IV quotient $\mathscr{F}_{2d}$, namely a quantitative comparison, uniform on balls of fixed radius, for arbitrary sequences approaching a prescribed rational boundary component through a Siegel set. We establish this comparison directly by adapting to the type-IV setting the approach developed for Siegel modular varieties in [16, §3.2]; reproduced in Chapter 2.

### 3.4.2 Gromov–Hausdorff analysis

Before beginning the analytic estimates, we recall the geometric setup established in Section 3.3. We retain the complete rational isotropic flag $0 \subset \ell \subset \Pi \subset V_{\mathbb{Q}}$ and the chosen basepoint $o \in \mathscr{D}_{2d}$, and fix $I \in \{\ell, \Pi\}$. We work with representatives in a suitable Siegel set $\mathfrak{S}_{\Omega,c} = \Omega A_c^+ o$, whose compact coordinates range over the fixed set $\Omega$, while the simple-root parameters describe the approach to infinity, as explained in Section 3.3.3.

For the rational boundary component determined by $I$, Theorem 3.3.6 provides a $\Gamma_I$-invariant neighborhood $\mathcal{U}_I \subset \mathscr{D}_{2d}$ whose quotient $\Gamma_I\backslash\mathcal{U}_I$ embeds in $\mathscr{F}_{2d}$. Consequently, the image of $\mathcal{U}_I$ in the moduli space is naturally identified with $\Gamma_I\backslash\mathcal{U}_I$, so that in the direction at infinity associated with $P_I$ we may replace the action of $\Gamma$ by that one of $\Gamma_I$ and use the horospherical coordinates associated with $P_I$.

Recall that the horospherical projection $\pi_I : \mathscr{D}_{2d} \to B_I$ has the $N_I$-orbits as its fibers, where $B_I \simeq L_I/(L_I \cap K_o)$ is the corresponding Levi base. The action of $P_I$ on $B_I$ factors through the Levi projection $\lambda_I : P_I \to L_I$, defined by $\lambda_I(nl) = l$, and the equivariance identity $\pi_I(p \cdot x) = \lambda_I(p) \cdot \pi_I(x)$ implies that $\pi_I$ descends to the quotient fibration

$$\overline{\pi}_I : \Gamma_I\backslash\mathscr{D}_{2d} \longrightarrow \Gamma_{I,L}\backslash B_I.$$

As recalled in Section 3.3.2, its fibers are finite quotients of the compact nilmanifold $\mathcal{N}_I = \Gamma_{I,N}\backslash N_I$. The restriction of this fibration to $\Gamma_I\backslash\mathcal{U}_I$ therefore gives the local model for our analysis near the chosen boundary component.

We equip $\mathscr{D}_{2d}$ with the Weil–Petersson metric $g_{\mathrm{WP}}$, which agrees with the invariant symmetric-space metric under the normalization fixed above, and equip $B_I$ with the $L_I$-invariant metric obtained by restricting the same normalized Killing form to the symmetric part of the Lie algebra of $L_I$. Both arithmetic quotients carry the corresponding quotient metrics. With these choices, we first establish the Riemannian-submersion property of $\pi_I$ and then estimate the diameters of the fibers of $\overline{\pi}_I$; together, these results will allow us to compare the pointed geometry of the total space with that of the Levi base on balls of fixed radius.

**Proposition 3.4.3.** *For each $I \in \{\ell, \Pi\}$, the map $\pi_I : \mathscr{D}_{2d} \to B_I$ is a $P_I$-equivariant Riemannian submersion. In particular, $\pi_I$ is distance non-increasing, that is*

$$d_{B_I}\big(\pi_I(x), \pi_I(x')\big) \leq d_{\mathrm{WP}}(x, x').$$

*Moreover, the horizontal distribution is the orthogonal complement of the tangent spaces to the $N_I$-orbits.*

*Proof.* Since $P_I$ acts transitively and by isometries on $\mathscr{D}_{2d}$, while $L_I$ acts by isometries on $B_I$, the $P_I$-equivariance shows that the differential of $\pi_I$ at any point is obtained from its differential at the basepoint $o$ by composing with isometries on the source and target. Consequently, it is enough to prove that $\pi_I$ is a Riemannian submersion at $o$.

Let $\theta$ be the Cartan involution associated with the basepoint $o$. At the level of Lie algebras, write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the corresponding Cartan decomposition, where

$$\mathfrak{k} = \{X \in \mathfrak{g} : \theta X = X\}, \qquad \mathfrak{p} = \{X \in \mathfrak{g} : \theta X = -X\}.$$

We now compute the differential of $\pi_I$ at the basepoint. Recall we have the natural identification $T_o\mathcal{D}_{2d} \simeq \mathfrak{p}$ and for $X \in \mathfrak{g}$ the tangent vector at $o$ corresponds to

$$\mathrm{pr}_{\mathfrak{p}}(X) = \frac{1}{2}(X - \theta X).$$

Consequently, we have

$$T_o(N_I \cdot o) = \left\{ \frac{1}{2}(Z - \theta Z) : Z \in \mathfrak{n}_I \right\}.$$

Moreover, since $L_I$ is $\theta$-stable, the Cartan decomposition restricts to $\mathfrak{l}_I$ and hence $T_{b_o}B_I \simeq \mathfrak{l}_I \cap \mathfrak{p}$, where $b_o := \pi_I(o)$.

To describe $(d\pi_I)_o$, let $Z + A \in \mathfrak{n}_I \rtimes \mathfrak{l}_I$ and write $A = A_{\mathfrak{k}} + H$, where $A_{\mathfrak{k}} \in \mathfrak{l}_I \cap \mathfrak{k}$, and $H \in \mathfrak{l}_I \cap \mathfrak{p}$. Since $A_{\mathfrak{k}}$ belongs to the isotropy algebra at $o$, the tangent vector determined by $Z + A$ corresponds, under the identification $T_o\mathcal{D}_{2d} \simeq \mathfrak{p}$, to

$$\mathrm{pr}_{\mathfrak{p}}(Z + H) = \frac{1}{2}(Z - \theta Z) + H.$$

On the other hand, the differential of the Levi projection sends $Z + A$ to $A$, whose class in $\mathfrak{l}_I/(\mathfrak{l}_I \cap \mathfrak{k})$ is represented by $H$. Therefore,

$$(d\pi_I)_o \left( \frac{1}{2}(Z - \theta Z) + H \right) = H. \tag{3.27}$$

Since the action of $P_I$ on $\mathcal{D}_{2d}$ is transitive, every tangent vector at $o$ arises in this way. It follows that

$$\ker(d\pi_I)_o = \left\{ \frac{1}{2}(Z - \theta Z) : Z \in \mathfrak{n}_I \right\} = T_o(N_I \cdot o).$$

It remains to prove that the orthogonal complement of this vertical space is precisely $T_{b_o}B_I$. First, notice that since $P_I$ and $\theta(P_I)$ are both maximal parabolic subgroups with common Levi factor $L_I$, one has the vector-space decomposition

$$\mathfrak{g} = \mathfrak{n}_I \oplus \mathfrak{l}_I \oplus \theta\mathfrak{n}_I.$$

Thus if $X \in \mathfrak{p}$, we write $X = Z + A + \theta W$, with $Z, W \in \mathfrak{n}_I$, and $A \in \mathfrak{l}_I$. The condition $\theta X = -X$, together with the uniqueness of this decomposition, gives $W = -Z$, and $\theta A = -A$. Thus $A \in \mathfrak{l}_I \cap \mathfrak{p}$, and it follows that

$$\mathfrak{p} = \{Z - \theta Z : Z \in \mathfrak{n}_I\} \oplus (\mathfrak{l}_I \cap \mathfrak{p}).$$

We now show that this is an orthogonal direct sum. Let $\kappa_{\mathfrak{g}}$ denote the Killing form of $\mathfrak{g}$. We want to show that $\kappa_{\mathfrak{g}}(Z - \theta Z, H) = 0$ for every $Z \in \mathfrak{n}_I$ and $H \in \mathfrak{l}_I$. To see this, recall that there exists an element $E_I \in \mathfrak{z}(\mathfrak{l}_I) \cap \mathfrak{p}$ such that $\mathfrak{l}_I$ is the zero eigenspace of $\mathrm{ad}(E_I)$ and $\mathfrak{n}_I$ is the sum of its positive eigenspaces[3]. Suppose first that $Z \in \mathfrak{n}_I$ is eigenvector of

[3]Such an element is called a grading element. In the coordinates introduced above, set

$$H(x, y) := \left. \frac{d}{dt} \right|_{t=0} a(e^{tx}, e^{ty}) \in \mathfrak{a}.$$

Since $\alpha_1(H(x, y)) = x - y$ and $\alpha_2(H(x, y)) = y$, one may take $E_\ell = H(1, 0)$ and $E_\Pi = H(1, 1)$. Indeed, $E_\ell$ vanishes on the Levi root $\alpha_2$ and satisfies $\alpha_1(E_\ell) = 1$, whereas $E_\Pi$ vanishes on the Levi root $\alpha_1$ and satisfies $\alpha_2(E_\Pi) = 1$. Consequently, $\mathrm{ad}(E_I)$ vanishes on $\mathfrak{l}_I$ and has positive eigenvalues on the root spaces forming $\mathfrak{n}_I$.

$\mathrm{ad}(E_I)$ and $H \in \mathfrak{l}_I$. Then we get $[E_I, Z] = rZ$, for some $r > 0$ and $[E_I, H] = 0$. Hence invariance of the Killing form gives $\kappa_{\mathfrak{g}}(Z, H) = 0$. The same argument applies to $\theta Z$, which belongs to a negative eigenspace of $\mathrm{ad}(E_I)$. By linearity, it follows that $\kappa_{\mathfrak{g}}(Z - \theta Z, H) = 0$.

Now, the invariant metric on $\mathcal{D}_{2d}$ is determined at $o$ by

$$\langle X, Y \rangle_o = -c\,\kappa_{\mathfrak{g}}(X, \theta Y), \quad c > 0.$$

Then, for $X, Y \in \mathfrak{p}$, this becomes $\langle X, Y \rangle_o = c\,\kappa_{\mathfrak{g}}(X, Y)$ and consequently

$$\{Z - \theta Z : Z \in \mathfrak{n}_I\} \perp \mathfrak{l}_I \cap \mathfrak{p}.$$

Hence we have proved that $\ker(d\pi_I)_o^{\perp} = \mathfrak{l}_I \cap \mathfrak{p}$. Taking $Z = 0$ in (3.27) gives $(d\pi_I)_o(H) = H$ for every $H \in \mathfrak{l}_I \cap \mathfrak{p}$. Thus

$$(d\pi_I)_o\big|_{\ker(d\pi_I)_o^{\perp}} : \ker(d\pi_I)_o^{\perp} \longrightarrow T_{b_o}B_I$$

is the identity under the natural identifications. Because the metric on $B_I$ is defined by restricting the same normalized Killing form, this map is an isometry. Hence $\pi_I$ is a Riemannian submersion at the basepoint. Therefore $\pi_I$ is a Riemannian submersion everywhere. Finally, since the fibers are the $N_I$-orbits, their tangent spaces are precisely the vertical spaces $\mathcal{V}_x = T_x(N_I \cdot x)$. Thus the horizontal distribution is $\big(T_x(N_I \cdot x)\big)^{\perp}$. □

*Remark* 3.4.4. The preceding statement is not specific to the type-IV domain $\mathcal{D}_{2d}$. More generally, let

$$D = G/K$$

be an irreducible Hermitian symmetric space of noncompact type, let $P = N \rtimes L \subset G$ be a parabolic subgroup, and choose a $\theta$-stable Levi factor $L$. Then the natural horospherical projection

$$D \simeq N \times L/(L \cap K) \longrightarrow L/(L \cap K)$$

is a Riemannian submersion when both spaces are equipped with the invariant metrics induced by the Killing form of $\mathfrak{g}$. The proof is identical, mutatis mutandis, to the one given below, and we leave the details to the interested reader.

With the preceding result at hand, the strategy for determining the asymptotic behavior of the Weil–Petersson metric, in the sense of pointed Gromov–Hausdorff convergence, along the direction at infinity associated with $P_I$ becomes clear. The horospherical projection separates the geometry into horizontal directions, which reproduce isometrically the geometry of the base $B_I$, and vertical direction tangent to the $N_I$-orbits. Thus, after passing to the corresponding arithmetic quotients, it remains to prove that the diameters of the nilpotent fibers tend uniformly to zero on metric balls of every fixed radius. Once this is established, the asymptotic geometry is captured by the projected regions in $\Gamma_{I,L}\backslash B_I$. The following proposition records the general Gromov–Hausdorff analysis underlying this strategy.

**Proposition 3.4.5.** *Let*

$$f_j : (M_j, g_j) \longrightarrow (B_j, h_j)$$

*be a sequence of surjective Riemannian submersions between compact connected Riemannian manifolds. Set*

$$\varepsilon_j := \sup_{b \in B_j} \mathrm{diam}_{g_j}\big(f_j^{-1}(b)\big).$$

*Then*

$$d_{\mathrm{GH}}(M_j, B_j) \leq \frac{1}{2}\varepsilon_j.$$

*In particular, if $\varepsilon_j \to 0$, then the maps $f_j$ are Gromov–Hausdorff approximations and the spaces $M_j$ and $B_j$ have the same Gromov–Hausdorff limit.*

Before we proceed with our analysis, it is useful to isolate, in the simplest possible setting, the basic mechanism responsible for the shrinking of the fiber diameters. We therefore focus on points in the $A_c^+$-orbit of the basepoint and show how the $A$-action rescales the tangent directions arising from the restricted root spaces.

Let $X_\alpha \in \mathfrak{g}_\alpha$ and denote by $X_\alpha^\#$ the associated fundamental vector field, then the invariance of the metric gives

$$\left\| X_\alpha^\#(a \cdot o) \right\|_{\mathrm{WP}} = \left\| \left(\mathrm{Ad}(a^{-1})X_\alpha\right)^\#(o) \right\|_{\mathrm{WP}}.$$

In addition, since $X_\alpha$ belongs to the restricted root space $\mathfrak{g}_\alpha$, one has $\mathrm{Ad}(a^{-1})X_\alpha = \alpha(a)^{-1}X_\alpha$, and therefore

$$\left\| X_\alpha^\#(a \cdot o) \right\|_{\mathrm{WP}} = \alpha(a)^{-1} \left\| X_\alpha^\#(o) \right\|_{\mathrm{WP}}. \tag{3.28}$$

Thus the metric length of the direction determined by $\mathfrak{g}_\alpha$ decays by the factor $\alpha(a)^{-1}$ whenever $\alpha(a) \to +\infty$. Since the vertical tangent spaces are generated by the fundamental vector fields associated with the root spaces occurring in $\mathfrak{n}_I$, this calculation exhibits the basic mechanism underlying the collapse of the compact nilpotent quotient fibers.

Combining this calculation with Proposition 3.4.3, which identifies the vertical tangent spaces with the tangent spaces to the $N_I$-orbits, we see that the desired fiber-diameter estimates are governed by the root spaces occurring in $\mathfrak{n}_I$ and by the corresponding root characters. More explicitly, recall that the chosen simple positive restricted root system is

$$\Delta(P_0, A) = \{\alpha_1, \alpha_2\}, \qquad \alpha_1 = \varepsilon_1 - \varepsilon_2, \qquad \alpha_2 = \varepsilon_2,$$

and the corresponding set of positive restricted roots is

$$\Sigma^+(P_0, A) = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}.$$

For $I \in \{\ell, \Pi\}$, the maximal parabolic $P_I$ is obtained by omitting one of the two simple roots: the omitted root is $\alpha_\ell = \alpha_1$ for $P_\ell$ and $\alpha_\Pi = \alpha_2$ for $P_\Pi$. The Lie algebra of its nilpotent radical is

$$\mathfrak{n}_I = \bigoplus_{\alpha \in \Sigma_I^+} \mathfrak{g}_\alpha, \qquad \Sigma_I^+ := \left\{ \alpha \in \Sigma^+(P_0, A) \;\middle|\; n_{\alpha_I}(\alpha) > 0 \right\},$$

where $n_{\alpha_I}(\alpha)$ denotes the coefficient of $\alpha_I$ in the expansion of $\alpha$ in terms of the simple roots. Explicitly,

$$\mathfrak{n}_\ell = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2},$$

whereas

$$\mathfrak{n}_\Pi = \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2}.$$

For a general point represented in the Siegel set, the remaining coordinates introduce additional adjoint actions that may mix or distort these root directions. The following lemma shows that this additional distortion is uniformly controlled, so that the factors $\alpha(a)^{-1}$ continue to govern the size of the vertical directions.

**Lemma 3.4.6.** *Let $I \in \{\ell, \Pi\}$ and fix any norm $\|\cdot\|_0$ on $\mathfrak{n}_I$. Then there exists a constant $C_0 > 0$, depending only on $I$, $\|\cdot\|_0$, the Siegel set data $\mathfrak{S}_{\Omega,c}$ and the normalization of $g_{\mathrm{WP}}$, such that the following holds.*

*For every $a \in A_c^+$, every $b \in \pi_I\big(\Omega a \cdot o\big) \subset B_I$, and every $X \in \mathfrak{n}_I$, writing*

$$X = \sum_{\alpha \in \Sigma_I^+} X_\alpha, \quad X_\alpha \in \mathfrak{g}_\alpha,$$

*one has*

$$\big\|X^{\#}(\iota_I(b))\big\|_{\mathrm{WP}} \leq C_0 \sum_{\alpha \in \Sigma_I^+} \alpha(a)^{-1} \|X_\alpha\|_0. \tag{3.29}$$

*Here $\iota_I : B_I \to \mathcal{D}_{2d}$ denotes the Levi section determined by the chosen $\theta$-stable Levi factor $L_I$. In particular, the constant $C_0$ is uniform in $a$, $b$, and $X$.*

*Proof.* Let us first notice that the vertical directions that we wish to estimate are generated by $N_I$, whereas the compact factor $\Omega$ defining the Siegel set is expressed in terms of the minimal parabolic and is contained in $N_0 M_0$. We must therefore first relate the $N_0$-coordinates supplied by the Siegel set to the $N_I L_I$-decomposition associated with $P_I$. The decomposition

$$N_0 = N_I \rtimes U_I, \qquad U_I = N_0 \cap L_I,$$

provides this relation: the $N_I$-component acts along the fibers of $\pi_I$, while the remaining $U_I M_0$-component determines the corresponding point of the Levi section.

We begin by making the relation between $N_0$ and $N_I$ precise. Let $\mathfrak{n}_0$ be the Lie algebra of $N_0$, namely the unipotent radical of the minimal parabolic $P_0$ associated with the complete flag $0 \subset \ell \subset \Pi$. Set $\mathfrak{u}_I := \mathfrak{n}_0 \cap \mathfrak{l}_I$. The restricted-root decomposition gives $\mathfrak{n}_0 = \mathfrak{n}_I \rtimes \mathfrak{u}_I$ and since the exponential map is a diffeomorphism for the connected unipotent groups under consideration, this Lie-algebra decomposition integrates to $N_0 = N_I \rtimes U_I$.

Thus every $n_0 \in N_0$ has a unique decomposition $n_0 = n_I u$, with $n_I \in N_I$, and $u \in U_I$. In what follows we write $\eta = n_0 m$, with $n_0 \in N_0$, $m \in M_0$, and decompose $n_0 = n_I u$. Therefore, $n_I u m a$ is the $N_I L_I$-decomposition of $\eta a$. Set $q_\eta := \lambda_I(\eta) \in L_I$, and notice that $\iota_I(b) = q_\eta a \cdot o$.

We now proceed with the estimate of the norm of the fundamental vector field. First, we have

$$\big\|X^{\#}(q_\eta a \cdot o)\big\|_{\mathrm{WP}} = \Big\|\big(\mathrm{Ad}\big((q_\eta a)^{-1}\big)X\big)^{\#}(o)\Big\|_{\mathrm{WP}}.$$

Setting $q_{\eta,a} := a^{-1} q_\eta a$ we obtain

$$\mathrm{Ad}((q_\eta a)^{-1})X = \mathrm{Ad}(q_{\eta,a}^{-1})\,\mathrm{Ad}(a^{-1})X.$$

Here notice that factor $\mathrm{Ad}(a^{-1})$ accounts for the rootwise rescaling that we wish to retain. It therefore remains to control uniformly the additional distortion introduced by $\mathrm{Ad}(q_{\eta,a}^{-1})$. Since $L_I$ normalizes $N_I$ and the adjoint representation is continuous, it is enough to show that the elements $q_{\eta,a}$ range in a fixed relatively compact subset of $L_I$.

First, notice that since $\Omega$ is compact, the corresponding $u$- and $m$-components range in fixed compact subsets of $U_I$ and $M_0$, respectively. Moreover, $M_0$ centralizes $A$, and hence $a^{-1} q_\eta a = (a^{-1} u a) m$. The $m$-component is therefore already uniformly controlled. It remains to show that the elements $a^{-1} u a$ range in a relatively compact subset of $U_I$. This does not follow merely from the compactness of the $u$-components, since the conjugating elements $a \in A_c^+$ need not remain in a compact subset of $A$. To control this conjugation, we use the restricted-root decomposition of $\mathfrak{u}_I$.

Since

$$\mathfrak{u}_I = \bigoplus_{\substack{\gamma\in\Sigma^+\\ n_{\alpha_I}(\gamma)=0}} \mathfrak{g}_\gamma,$$

we may write

$$\log u = \sum_{\substack{\gamma\in\Sigma^+\\ n_{\alpha_I}(\gamma)=0}} Y_\gamma, \qquad Y_\gamma \in \mathfrak{g}_\gamma.$$

Because the $u$-components range in a compact subset of $U_I$, the elements $\log u$ range in a compact subset of $\mathfrak{u}_I$. Consequently, their root components $Y_\gamma$ are uniformly bounded. Using the action of $A$ on the restricted-root spaces, we obtain

$$\begin{aligned}\log(a^{-1}ua) &= \mathrm{Ad}(a^{-1})(\log u)\\ &= \sum_{\substack{\gamma\in\Sigma^+\\ n_{\alpha_I}(\gamma)=0}} \gamma(a)^{-1}Y_\gamma. \end{aligned} \tag{3.30}$$

Since the root components $Y_\gamma$ are uniformly bounded and only finitely many roots occur, it remains to obtain a uniform upper bound for the factors $\gamma(a)^{-1}$. To this end, let $\gamma\in\Sigma^+$ be one of the roots occurring in $\mathfrak{u}_I$ and write its simple-root expansion as

$$\gamma = \sum_{\delta\in\Delta} n_\delta(\gamma)\delta.$$

Since $a\in A_c^+$, one has $\delta(a)\geq c$ for every $\delta\in\Delta$, and therefore

$$\gamma(a) = \prod_{\delta\in\Delta}\delta(a)^{n_\delta(\gamma)} \geq c^{\mathrm{ht}(\gamma)}, \quad \mathrm{ht}(\gamma) := \sum_{\delta\in\Delta} n_\delta(\gamma).$$

Thus every term $\gamma(a)^{-1}Y_\gamma$ in (3.30) is uniformly bounded and since only finitely many roots occur, it follows that the elements $\log(a^{-1}ua)$ range in a bounded subset of $\mathfrak{u}_I$. Hence the set

$$\left\{a^{-1}q_\eta a \;\middle|\; \eta\in\Omega,\ a\in A_c^+\right\}$$

has compact closure in $L_I$ which we denote by $C_{\Omega,c}$.

The compactness just established gives the uniform bound anticipated above: there exists a constant $C_0>0$ such that

$$\left\|\left(\mathrm{Ad}(\widetilde{q}^{-1})Z\right)^{\#}(o)\right\|_{\mathrm{WP}} \leq C_0\|Z\|_0$$

for every $\widetilde{q}\in C_{\Omega,c}$ and every $Z\in\mathfrak{n}_I$. Applying this estimate with $\widetilde{q}=q_{\eta,a}$ and $Z=\mathrm{Ad}(a^{-1})X$ and writing $X=\sum_{\alpha\in\Sigma_I^+}X_\alpha$, the action of $A$ gives $\mathrm{Ad}(a^{-1})X=\sum_{\alpha\in\Sigma_I^+}\alpha(a)^{-1}X_\alpha$. Therefore,

$$\begin{aligned}\left\|X^{\#}(\iota_I(b))\right\|_{\mathrm{WP}} &= \left\|\left(\mathrm{Ad}(q_{\eta,a}^{-1})\,\mathrm{Ad}(a^{-1})X\right)^{\#}(o)\right\|_{\mathrm{WP}}\\ &\leq C_0\left\|\mathrm{Ad}(a^{-1})X\right\|_0\\ &\leq C_0\sum_{\alpha\in\Sigma_I^+}\alpha(a)^{-1}\|X_\alpha\|_0.\end{aligned}$$

This proves (3.29). □

The preceding lemma shows that, up to a uniform multiplicative constant, the dependence of the lengths of vertical tangent vectors on the point in the Siegel set is entirely captured by the root characters evaluated at its $A$-coordinate. We now use this infinitesimal estimate to obtain a bound for the diameters of the fibers.

**Proposition 3.4.7.** *For each $I \in \{\ell, \Pi\}$, there exists a constant $C_I > 0$ such that, for every point $[b] \in \Gamma_{I,L}\backslash B_I$ represented by $b \in B_I$ arising from the Siegel set $\mathfrak{S}_{\Omega,c}$, with $A$-coordinate $a_b \in A_c^+$, one has*

$$\operatorname{diam}\big(\overline{\pi}_I^{-1}([b])\big) \leq C_I \sum_{\alpha \in \Sigma_I^+} \alpha(a_b)^{-1}. \tag{3.31}$$

*In particular, there exists a constant $C_I' > 0$ such that*

$$\operatorname{diam}\big(\overline{\pi}_I^{-1}([b])\big) \leq \frac{C_I'}{\alpha_I(a_b)}. \tag{3.32}$$

*Proof.* We first estimate the diameter of the intermediate nilmanifold fiber $\mathcal{N}_{I,b} = \Gamma_{I,N}\backslash N_I$, equipped with the metric induced from the $N_I$-orbit through a chosen lift $b \in B_I$. By our previous discussion in Sections 3.3.2 and 3.3.3 (see for example Remark 3.3.9) we may choose a compact subset $\Omega_I \subset N_I$ such that $N_I = \Gamma_{I,N}\Omega_I$. Thus every point of $\mathcal{N}_{I,b}$ has at least one representative in $\Omega_I$.

Following the notation in the previous lemma let $b^\circ := \iota_I(b) \in \mathcal{D}_{2d}$ and let $x, x' \in \mathcal{N}_{I,b}$ be arbitrary. We choose representatives $n, n' \in \Omega_I$, so that

$$x = [n \cdot b^\circ]_{\Gamma_{I,N}}, \quad x' = [n' \cdot b^\circ]_{\Gamma_{I,N}},$$

and denote by $g := n'n^{-1}$ the left-displacement that sends $n \cdot b^\circ$ to $n' \cdot b^\circ$ under the left $N_I$-action. Set $X := \log(n^{-1}n')$ and notice that by the equivariance of the exponential map under conjugation, we obtain

$$g = n \exp(X) n^{-1} = \exp\big(\operatorname{Ad}(n)X\big).$$

Thus $X$ is obtained by first translating $n$ to the identity and then identifying the resulting relative displacement $n^{-1}n'$ with the unique element of $\mathfrak{n}_I$ whose exponential equals it. Since the exponential path $t \mapsto \exp(tX)$ joins the identity to $n^{-1}n'$, left-translating it by $n$ and applying it to $b^\circ$ gives a curve joining $n \cdot b^\circ$ to $n' \cdot b^\circ$, namely

$$\gamma(t) := n \exp(tX) \cdot b^\circ = \exp\big(t \operatorname{Ad}(n)X\big) \cdot (n \cdot b^\circ), \quad t \in [0,1].$$

Hence its projection to $\mathcal{N}_{I,b}$ joins $x$ to $x'$. Moreover, it follows that $\big\|\dot\gamma(t)\big\|_{\mathrm{WP}} = \big\|X^\#(b^\circ)\big\|_{\mathrm{WP}}$ for every $t \in [0,1]$ and consequently,

$$\operatorname{Length}(\gamma) = \big\|X^\#(\iota_I(b))\big\|_{\mathrm{WP}}.$$

Applying (3.29) to $X$ gives

$$\begin{aligned} d_{\mathcal{N}_{I,b}}(x, x') &\leq \big\|X^\#(b)\big\|_{\mathrm{WP}} \\ &\leq C_0 \sum_{\alpha \in \Sigma_I^+} \alpha(a)^{-1} \|X_\alpha\|_0. \end{aligned}$$

To turn this estimate for a fixed pair $x, x'$ into a diameter bound, it remains to control the root components $X_\alpha$ uniformly as the representatives $n, n'$ vary in $\Omega_I$. For this purpose, define

$$\mathcal{C}_I := \big\{\log(n^{-1}n') \;\big|\; n, n' \in \Omega_I\big\} \subset \mathfrak{n}_I,$$

and let $Y = \sum_{\alpha\in\Sigma_I^+} Y_\alpha \in \mathcal{C}_I$. Thus, given that $\mathcal{C}_I$ is compact and only finitely many roots occur, it follows that there exists a constant $C_{\Omega,I} > 0$ such that $\|Y_\alpha\|_0 \le C_{\Omega,I}$ for every $Y \in \mathcal{C}_I$ and every root $\alpha$ occurring in $\mathfrak{n}_I$.

Since $X = \log(n^{-1}n')$ belongs to $\mathcal{C}_I$, it follows that

$$d_{\mathcal{N}_{I,b}}(x,x') \le C_0 C_{\Omega,I} \sum_{\alpha\in\Sigma_I^+} \alpha(a_b)^{-1}.$$

The right-hand side is independent of $x$ and $x'$. Taking the supremum over all pairs of points in $\mathcal{N}_{I,b}$ therefore yields

$$\operatorname{diam}\bigl(\mathcal{N}_{I,b}\bigr) \le C_I \sum_{\alpha\in\Sigma_I^+} \alpha(a_b)^{-1}, \tag{3.33}$$

where $C_I := C_0 C_{\Omega,I}$.

It remains to pass from the intermediate nilmanifold fiber $\mathcal{N}_{I,b}$ to its quotient $H_b\backslash\mathcal{N}_{I,b}$, where $H_b = \operatorname{Stab}_{\Gamma_{I,L}}(b)$ is the finite stabilizer of $b$. Endowing the quotient with the metric quotient we get that

$$\operatorname{diam}\bigl(H_b\backslash\mathcal{N}_{I,b}\bigr) \le \operatorname{diam}\bigl(\mathcal{N}_{I,b}\bigr).$$

Combining this with (3.33) proves the first assertion.

Finally, let $\alpha = m_1\alpha_1 + m_2\alpha_2$ be a positive root occurring in $\mathfrak{n}_I$. Its coefficient along $\alpha_I$ is a positive integer. If $c \ge 1$, then $\alpha_1(a) \ge 1$, $\alpha_2(a) \ge 1$, and consequently

$$\alpha(a) = \alpha_1(a)^{m_1}\alpha_2(a)^{m_2} \ge \alpha_I(a).$$

Therefore $\alpha(a)^{-1} \le \alpha_I(a)^{-1}$. Since only finitely many positive roots occur in $\mathfrak{n}_I$, their number can be absorbed into the constant, giving

$$\operatorname{diam}\bigl(\pi_I^{-1}([b])\bigr) \le \frac{C_I'}{\alpha_I(a)}.$$

If the truncation constant is only assumed to satisfy $c > 0$, the same conclusion holds with an additional multiplicative constant depending on $c$. Indeed, for the $B_2$-root system one has

$$\alpha(a) \ge \min\{1, c^2\}\,\alpha_I(a)$$

for every positive root occurring in $\mathfrak{n}_I$. □

Combining the Riemannian-submersion property of $\overline{\pi}_I$ with the preceding fiber-diameter estimate, we now show that, along sequences for which $\alpha_I(a_j) \to \infty$, the restrictions of $\overline{\pi}_I$ to metric balls of any fixed radius are Gromov–Hausdorff approximations whose errors tend to zero.

**Theorem 3.4.8.** *Fix $I \in \{\ell, \Pi\}$ and consider the fibration*

$$\overline{\pi}_I : \Gamma_I\backslash\mathcal{D}_{2d} \longrightarrow \Gamma_{I,L}\backslash B_I.$$

*Let $x_j \in \Gamma_I\backslash\mathcal{D}_{2d}$ be a sequence. For each $j$, choose a representative $b_j \in B_I$ of $\overline{\pi}_I(x_j) = [b_j]$ arising from the chosen Siegel set, with A-coordinate $a_j \in A_c^+$. Assume moreover that $\alpha_I(a_j) \to \infty$. Then, for every $R > 0$, there exists a constant $C_{I,R} > 0$, independent of $j$, such that*

$$d_{\mathrm{GH}}\left(\overline{B}_R(x_j), \overline{B}_R([b_j])\right) \le \frac{C_{I,R}}{\alpha_I(a_j)}.$$

*In particular, the restrictions of $\overline{\pi}_I$ to balls of every fixed radius are Gromov–Hausdorff approximations whose errors tend to zero.*

*Proof.* Fix $R > 0$ and write $b_j = l_j \cdot o$. Since $B_I = L_I/(L_I \cap K_o)$ is a complete homogeneous Riemannian manifold, there exists a compact subset $Q_R \subset L_I$, depending only on $R$, such that every $b \in \overline{B}_R(b_j)$ can be written as $b = l_j c \cdot o$ for some $c \in Q_R$. Note that, after translating by $l_j^{-1}$, this simply amounts to choosing representatives in $L_I$ of the compact ball $\overline{B}_R(o) \subset B_I$. Because $L_I$ normalizes $N_I$, its adjoint action preserves $\mathfrak{n}_I$. Compactness of $Q_R$ therefore gives a constant $D_R \geq 1$, independent of $j$, such that

$$\left\| \big(\mathrm{Ad}(c^{-1})Z\big)^{\#}(o) \right\|_{\mathrm{WP}} \leq D_R \left\| Z^{\#}(o) \right\|_{\mathrm{WP}}$$

for every $c \in Q_R$ and every $Z \in \mathfrak{n}_I$. Thus, if $b = l_j c \cdot o \in \overline{B}_R(b_j)$ and $X \in \mathfrak{n}_I$, the invariance of the metric gives

$$\begin{aligned} \left\| X^{\#}\big(\iota_I(b)\big) \right\|_{\mathrm{WP}} &= \left\| \big(\mathrm{Ad}(c^{-1})\,\mathrm{Ad}(l_j^{-1})X\big)^{\#}(o) \right\|_{\mathrm{WP}} \\ &\leq D_R \left\| \big(\mathrm{Ad}(l_j^{-1})X\big)^{\#}(o) \right\|_{\mathrm{WP}} \\ &= D_R \left\| X^{\#}\big(\iota_I(b_j)\big) \right\|_{\mathrm{WP}}. \end{aligned}$$

The metrics on $\mathcal{N}_{I,b}$ and $\mathcal{N}_{I,b_j}$ are induced by the corresponding left-invariant metrics on $N_I$, and hence the preceding comparison of their norms at the identity gives

$$\mathrm{diam}\big(\mathcal{N}_{I,b}\big) \leq D_R\,\mathrm{diam}\big(\mathcal{N}_{I,b_j}\big)$$

for every $b \in \overline{B}_R(b_j)$. Furthermore, applying the simple-root comparison from the proof of Proposition 3.4.7 to the nilmanifold estimate (3.33) gives, after increasing $C_I'$ if necessary,

$$\mathrm{diam}\big(\mathcal{N}_{I,b_j}\big) \leq \frac{C_I'}{\alpha_I(a_j)}.$$

Consequently,

$$\sup_{b \in \overline{B}_R(b_j)} \mathrm{diam}\big(\mathcal{N}_{I,b}\big) \leq \frac{D_R C_I'}{\alpha_I(a_j)}.$$

We now pass to the arithmetic quotient. If $[b] \in \overline{B}_R([b_j])$, then, since $\Gamma_{I,L}$ acts properly discontinuously and by isometries on the proper metric space $B_I$, the class $[b]$ admits a representative $b \in B_I$ satisfying $d_{B_I}(b, b_j) \leq R$. Moreover, as recalled in the proof of Proposition 3.4.7, the fiber over $[b]$ is the finite quotient

$$\overline{\pi}_I^{-1}([b]) \simeq H_b \backslash \mathcal{N}_{I,b}.$$

Since taking a finite metric quotient cannot increase diameter, it follows that

$$\sup_{[b] \in \overline{B}_R([b_j])} \mathrm{diam}\big(\overline{\pi}_I^{-1}([b])\big) \leq \frac{D_R C_I'}{\alpha_I(a_j)}.$$

It remains to translate this uniform fiber estimate into the claimed Gromov–Hausdorff estimate. By Proposition 3.4.3, the map $\pi_I$ is a Riemannian submersion, and the induced map $\overline{\pi}_I$ on the arithmetic quotients is therefore a submetry. In particular,

$$\overline{\pi}_I\big(\overline{B}_R(x_j)\big) = \overline{B}_R([b_j]).$$

Let $x, x' \in \overline{B}_R(x_j)$. Since $\overline{\pi}_I$ is distance non-increasing, we have

$$d\big(\overline{\pi}_I(x), \overline{\pi}_I(x')\big) \leq d(x, x').$$

Conversely, horizontally lift a minimizing geodesic joining $\overline{\pi}_I(x)$ to $\overline{\pi}_I(x')$, starting at $x$. Its endpoint belongs to the fiber over $\overline{\pi}_I(x')$, and hence

$$\begin{aligned} d(x,x') &\leq d\big(\overline{\pi}_I(x),\overline{\pi}_I(x')\big) + \operatorname{diam}\big(\overline{\pi}_I^{-1}(\overline{\pi}_I(x'))\big) \\ &\leq d\big(\overline{\pi}_I(x),\overline{\pi}_I(x')\big) + \frac{D_R C_I'}{\alpha_I(a_j)}. \end{aligned}$$

Thus the graph of the restriction of $\overline{\pi}_I$ to $\overline{B}_R(x_j)$ is a correspondence with $\overline{B}_R([b_j])$ whose distortion is at most $D_R C_I'/\alpha_I(a_j)$. Therefore,

$$d_{\mathrm{GH}}\left(\overline{B}_R(x_j),\overline{B}_R([b_j])\right) \leq \frac{D_R C_I'}{2\alpha_I(a_j)}.$$

Taking $C_{I,R} := D_R C_I'/2$ proves the assertion. Since $\alpha_I(a_j) \to \infty$, the errors of these Gromov–Hausdorff approximations tend to zero for every fixed $R > 0$. □

The preceding theorem reduces determining the Weil–Petersson asymptotic in the sense of Definition 3.4.1 to the convergence of the projected pointed sequence $(\Gamma_{I,L}\backslash B_I, [b_j])$. Pointed precompactness guarantees the existence of convergent subsequences, but does not by itself identify their limits or ensure convergence of the original sequence. In the next section, for sequences arising from geometric degenerations, we identify these limits and establish the modular interpretation of the resulting Weil–Petersson asymptotics.

### 3.4.3 Relation with geometric compactifications of $\mathcal{F}_{2d}$

Having completed our Lie-theoretic intermezzo, we now resume the modular movement by returning to the one-parameter degenerations considered in Section 3.2. Let

$$f : (\mathcal{X}^*, \mathcal{L}^*) \longrightarrow \Delta^*$$

be a one-parameter family of primitively polarized $K3$ surfaces which, after a finite base change, admits a Kulikov model over $\Delta$. Choose a sequence $q_j \in \Delta^*$ with $q_j \to 0$, and let $x_j \in \mathcal{F}_{2d}$ be the moduli point represented by the polarized fiber $(X_j, L_j)$. After choosing a marking over the universal cover of $\Delta^*$ and applying elements of $\Gamma_{2d}$ if necessary, we choose representatives of the points $x_j$ in a Siegel set adapted to the rational boundary component approached by the period map $\Delta^* \to \mathcal{F}_{2d}$ . We continue to denote these representatives by $x_j$, and we denote their projections to the corresponding Levi base by $[b_j] \in \Gamma_{I,L}\backslash B_I$.

The relation between the Kulikov type and the rational boundary component is a standard consequence of the Hodge theory of degenerations; for a clear exposition in the setting of polarized $K3$ surfaces, see [59, §2.2, especially pp. 15–19]. We recall only the part of this theory that will be used below. After the preceding base change, the local monodromy $T$ is unipotent, and we set $N = \log T$. On the universal cover, with $q = e^{2\pi i z}$ and $\operatorname{Im} z \to \infty$, Schmid's nilpotent orbit theorem [60, Theorem 4.9] determines a limiting Hodge line $[\omega_\infty] \in \check{\mathcal{D}}_{2d}$. After choosing a nonzero representative $\omega_\infty \in V_{\mathbb{C}}$ of this line, the lifted period line is asymptotic to $[\exp(zN)\,\omega_\infty]$ (for a detailed account see [21]). Accordingly, determining the asymptotic behavior of $(x_j)$ reduces to expressing the nilpotent-orbit data in the horospherical coordinates associated with $P_I$ introduced in Sections 3.3.1.1 and 3.3.1.2.

The cases relevant to sequences escaping to infinity are Types II and III. In Type II one has $N \neq 0$ and $N^2 = 0$, and $\Pi = \operatorname{Im} N$ is a rational isotropic plane, whereas in Type III one has $N^2 \neq 0$ and $N^3 = 0$, and $\ell = \operatorname{Im} N^2$ is a rational isotropic line. In both

cases, expressing the nilpotent-orbit asymptotics of the period map in the horospherical coordinates associated with $P_I$ yields the quantitative estimates needed to prove that the projected sequence of pointed spaces $(\Gamma_{I,L}\backslash B_I, [b_j])$ converges in the pointed Gromov–Hausdorff sense, where $I = \Pi$ in Type II and $I = \ell$ in Type III.

The general mechanism relating nilpotent-orbit asymptotics to the Siegel-reduction analysis of $\mathcal{F}_{2d}$ is already present in the work of Odaka–Oshima [53], whose analysis has substantially influenced the discussion below. The conceptual content of the statements that follow should therefore be regarded as implicit in their work, and we make no claim of priority for it. Since our argument requires this mechanism to be expressed in the precise horospherical coordinates fixed in Section 3.3, together with quantitative estimates for the corresponding simple-root parameters, we give the relevant calculations in sufficient detail.

**Proposition 3.4.9.** *Let $(x_j)$ be the sequence obtained from the fibers of the fixed degeneration above.*

(i) *If the degeneration admits a Type III Kulikov model, then $I = \ell = \mathbb{Q}u_1$. After choosing the signs of $u_1$ and $v_1$ so that $-N(v_1) \in \mathcal{T}_{W_\ell}$, the classes of the normalized hyperboloid coordinates $\widehat{Y}_j$, defined in (3.25), converge in $\Gamma_{\ell,L}\backslash\mathbb{H}^{18}$ to the class $[\widehat{Y}_\infty]$ (see definition below in the proof). Consequently,*

$$\big(\Gamma_{\ell,L}\backslash B_\ell, [b_j]\big) \xrightarrow{\text{pGH}} \Big(\big(\Gamma_{\ell,L}\backslash\mathbb{H}^{18}\big) \times \mathbb{R}, \big([\widehat{Y}_\infty], 0\big)\Big).$$

(ii) *If the degeneration admits a Type II Kulikov model, then $I = \Pi$, and the horospherical coordinates $[\tau_{H,j}]$ converge in $\Gamma_{\Pi,L}\backslash\mathcal{H}$ to a point $[\tau_\infty]$. Under the natural finite map $\Gamma_{\Pi,L}\backslash\mathcal{H} \longrightarrow A_1$, the point $[\tau_\infty]$ maps to the isomorphism class of the elliptic curve $E$ occurring as the common double curve of the Kulikov model. Consequently,*

$$\big(\Gamma_{\Pi,L}\backslash B_\Pi, [b_j]\big) \xrightarrow{\text{pGH}} \big(\big(\Gamma_{\Pi,L}\backslash\mathcal{H}\big) \times \mathbb{R}, \big([\tau_\infty], 0\big)\big).$$

*In particular, both projected pointed sequences converge without passing to a subsequence.*

The proof rests on three auxiliary facts. We establish these facts before proving the proposition. The first translates the Type III monodromy normal form into the line-horospherical coordinates and determines the asymptotic direction of the corresponding nilpotent orbit.

**Lemma 3.4.10.** *Suppose that $N$ is the monodromy logarithm of a Type III degeneration, that $\operatorname{Im} N^2 = \mathbb{Q}u_1$, and that $v_1$ is the dual isotropic vector satisfying $(u_1, v_1) = 1$. Then $N(v_1) \in W_\ell$ and under the Lie-algebra identification $\mathfrak{n}_\ell \simeq W_{\ell,\mathbb{R}}$, induced by $x \mapsto n(x)$, the monodromy logarithm $N$ corresponds to the vector $N(v_1)$. More precisely,*

$$N = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} n\big(\epsilon N(v_1)\big). \tag{3.34}$$

*Consequently, after complexification, $\exp(zN) = n\big(zN(v_1)\big)$ and hence*

$$\exp(zN) \cdot [\omega(\zeta_{W_\ell})] = \big[\omega\big(\zeta_{W_\ell} + zN(v_1)\big)\big].$$

*In particular, if $\Phi(z)$ denotes the lifted period map and its Siegel domain coordinate is written as $\zeta_{W_\ell}(z) = X(z) - iY(z)$, then, as $\operatorname{Im} z \to \infty$,*

$$Y(z) = -(\operatorname{Im} z)N(v_1) + O(1).$$

*Consequently,* $-N(v_1) \in \mathcal{T}_{W_\ell}$, *and*

$$\widehat{Y}(z) \longrightarrow -\frac{N(v_1)}{\sqrt{(N(v_1), N(v_1))/2}}. \tag{3.35}$$

*Proof.* The Type III monodromy normal form of [19, Lemma 1.1], applied to the dual isotropic vectors $u_1$ and $v_1$, gives

$$N(\xi) = (\xi, u_1)N(v_1) - \big(\xi, N(v_1)\big)u_1, \quad \xi \in V_{\mathbb{Q}}.$$

Furthermore, in the notation of the decomposition used in Section 3.3, one has $N(v_1) \in W_\ell$, while [19, Proposition 1.10] gives

$$\big(N(v_1), N(v_1)\big) > 0.$$

It remains to interpret the displayed normal form in terms of the horospherical coordinates. For $x \in W_{\ell,\mathbb{R}}$, recall that

$$n(\epsilon x) = \begin{pmatrix} 1 & -\epsilon x^t Q_{W_\ell} & -\dfrac{\epsilon^2}{2} x^t Q_{W_\ell} x \\ 0 & I & \epsilon x \\ 0 & 0 & 1 \end{pmatrix}.$$

Write $\xi = au_1 + w + bv_1$, with $w \in W_{\ell,\mathbb{R}}$. A direct matrix multiplication gives

$$n(\epsilon x)\xi = \left(a - \epsilon(x, w) - \frac{\epsilon^2 b}{2}(x, x)\right) u_1 + \big(w + \epsilon b x\big) + bv_1.$$

Differentiating at $\epsilon = 0$ and using $(\xi, u_1) = b$ and $(\xi, x) = (w, x)$, we obtain

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} n(\epsilon x)(\xi) = (\xi, u_1)x - (\xi, x)u_1.$$

Taking $x = N(v_1)$ and comparing with the Friedman–Scattone normal form gives (3.34).

Thus, under the Lie-algebra identification $\mathfrak{n}_\ell \simeq W_{\ell,\mathbb{R}}$, the monodromy logarithm $N$ corresponds precisely to the vector $N(v_1)$. Since $\epsilon \mapsto n\big(\epsilon N(v_1)\big)$ is a one-parameter subgroup, it is the exponential of its infinitesimal generator. Therefore, after complexification, this gives $\exp(zN) = n\big(zN(v_1)\big)$. We now translate this identity into the Siegel domain coordinates. Using the matrix defining $n(x)$ and Equation (3.14) we find $n(x)\omega(\zeta_{W_\ell}) = \omega(\zeta_{W_\ell} + x)$. Consequently,

$$\exp(zN) \cdot [\omega(\zeta_{W_\ell})] = \left[\omega\big(\zeta_{W_\ell} + zN(v_1)\big)\right].$$

Let $\Phi\colon \mathcal{H} \to \mathcal{D}_{2d}$ denote the lifted period map. With the monodromy convention fixed above, it satisfies $\Phi(z+1) = T\Phi(z) = \exp(N)\Phi(z)$. Therefore the untwisted period map defined by

$$\Psi(q) := \exp(-zN)\Phi(z), \quad q = e^{2\pi i z},$$

is well defined on $\Delta^*$. Schmid's nilpotent orbit theorem implies that $\Psi$ extends holomorphically across $q = 0$ as a map to the compact dual $\check{\mathcal{D}}_{2d}$, with $\Psi(0) = [\omega_\infty]$. Equivalently, the lifted period map admits the exact factorization

$$\Phi(z) = \exp(zN)\Psi(q).$$

Additionally, since the normal form also gives $N^2(\xi) = -\big(N(v_1), N(v_1)\big)(\xi, u_1)u_1$ and $N^2\omega_\infty \neq 0$ for a Type III degeneration, we have $(\omega_\infty, u_1) \neq 0$. Consequently, after normalizing the local representative of $\Psi(q)$ by the condition $(\omega, u_1) = 1$, the map $\Psi(q)$ remains in the affine chart used in Section 3.3 for $q$ sufficiently close to zero. We may therefore write $\Psi(q) = [\omega(\zeta_0(q))]$, where $\zeta_0(q)$ is holomorphic, and hence bounded, as $q \to 0$. In particular, the translation formula now gives

$$\zeta_{W_\ell}(z) = \zeta_0(q) + zN(v_1).$$

Writing $\zeta_{W_\ell}(z) = X(z) - iY(z)$ and using the fact that $N(v_1)$ is real, we obtain

$$Y(z) = -(\operatorname{Im} z)N(v_1) + O(1).$$

Since $(N(v_1), N(v_1)) > 0$, the corresponding radial coordinate satisfies

$$\sqrt{\frac{(Y(z), Y(z))}{2}} = (\operatorname{Im} z)\sqrt{\frac{(N(v_1), N(v_1))}{2}} + O(1).$$

It follows that

$$\widehat{Y}(z) \longrightarrow -\frac{N(v_1)}{\sqrt{(N(v_1), N(v_1))/2}}.$$

This proves the assertion. □

A similar analysis also holds for the Type II degenerations.

**Lemma 3.4.11.** *Suppose that $N$ is the monodromy logarithm of a Type II degeneration, so that $N^2 = 0$ and $\Pi = \operatorname{Im} N$. Let $[\omega_\infty]$ be the limiting Hodge line determined by the nilpotent orbit. Then $N\omega_\infty$ is a nonzero vector in $\Pi_{\mathbb{C}}$ and, relative to the basis $u_1, u_2$ of $\Pi$ used in the Siegel-domain realization, there is a unique $\tau_\infty \in \mathcal{H}$ such that*

$$[N\omega_\infty] = [\tau_\infty u_1 + u_2].$$

*Let $q_j \to 0$, and choose the representatives of the corresponding period points in the adapted Siegel set as above. Let $(\tau_{s,j}, \zeta_{W,j}, \sigma_j)$ denote the Siegel-domain coordinates of the chosen representatives, and write their $P_\Pi$-horospherical factorizations as*

$$[\omega(\tau_{s,j}, \zeta_{W,j}, \sigma_j)] = n(X_j, Z_{\lambda_j}) \cdot [\omega(\tau_{H,j}, 0, it_j)].$$

*where $n(X_j, Z_{\lambda_j}) \in N_\Pi$ is the nilpotent component, whereas $(\tau_{H,j}, t_j)$ are the coordinates of the corresponding projection $b_j \in B_\Pi$. Then*

$$\tau_{s,j} = \tau_\infty + O\left(\frac{1}{-\log|q_j|}\right), \quad \zeta_{W,j} = O\left(\frac{1}{-\log|q_j|}\right),$$

*and there exists a constant $C > 1$ such that, for all sufficiently large $j$,*

$$\frac{1}{C(-\log|q_j|)} \leq |\sigma_j| \leq \frac{C}{-\log|q_j|}.$$

*Consequently, $\tau_{H,j} \to \tau_\infty$ and*

$$\frac{1}{C(-\log|q_j|)} \leq t_j \leq \frac{C}{-\log|q_j|}, \qquad t_j \longrightarrow 0.$$

*In particular, $[\tau_{H,j}] \to [\tau_\infty]$ in $\Gamma_{\Pi,L}\backslash\mathcal{H}$. Moreover, the corresponding simple-root parameters satisfy*

$$s_j \longrightarrow \infty, \qquad \frac{r_j}{s_j} = \operatorname{Im}\tau_{H,j} \longrightarrow \operatorname{Im}\tau_\infty.$$

*Thus the period points approach infinity in the plane-cusp regime of Proposition 3.3.10.*

*Proof.* The standard description of the limiting mixed Hodge structure of a Type II degeneration shows that $N\omega_\infty \neq 0$ and that $\mathbb{C}N\omega_\infty \subset \Pi_{\mathbb{C}}$ is the $(1,0)$-line of the polarized weight-one Hodge structure carried by $\Pi$; see [59, §2.2]. In particular, relative to the oriented basis $u_1, u_2$ used in the Siegel-domain realization, this line belongs to the affine chart in which the coefficient of $u_2$ is nonzero, and its coordinate in this chart lies in $\mathscr{H}$. We now identify this coordinate explicitly from the nilpotent-orbit expansion. We now record the form of $N$ relative to the decomposition fixed in Section 3.3. Since the monodromy preserves the intersection form, its logarithm is skew-symmetric:

$$(N\xi, \eta) + (\xi, N\eta) = 0.$$

It follows that $\ker N = (\operatorname{Im} N)^\perp = \Pi^\perp$. In particular, $N$ vanishes on $\Pi \oplus W_{\mathbb{R}}$ and is determined by its values on $v_1, v_2$. The relations $(u_i, v_j) = \delta_{ij}$ and the skew-symmetry of $N$ imply that there exists a nonzero real number $m$ such that

$$N(v_1) = mu_2, \quad N(v_2) = -mu_1. \tag{3.36}$$

Let $\Phi\colon \mathscr{H} \to \mathscr{D}_{2d}$ be the lifted period map. As in the proof of Lemma 3.4.10, the untwisted period map $\Psi(e^{2\pi i z}) = \exp(-zN)\Phi(z)$ extends holomorphically, with $\Psi(0) = [\omega_\infty]$ and $\Phi(z) = \exp(zN)\Psi(q)$. Because the coefficient of $u_2$ in $N\omega_\infty$ is nonzero, the coefficient of $v_1$ in $\omega_\infty$ is nonzero. After shrinking the disk and choosing a holomorphic lift of $\Psi$, we may therefore normalize this coefficient to be one and write

$$\Psi(q) = [\omega_0(q)], \qquad \omega_0(q) = a_1(q)u_1 + a_2(q)u_2 + \zeta_0(q) + v_1 + \delta(q)v_2,$$

where all the coefficients are holomorphic near $q = 0$. Applying Equation (3.36) gives $N\omega_0(q) = mu_2 - m\delta(q)u_1$ and consequently it follows that

$$[N\omega_\infty] = [-\delta(0)u_1 + u_2].$$

Since the preceding Hodge-theoretic description shows that this is a positive weight-one Hodge line, one has $-\delta(0) \in \mathscr{H}$. By the uniqueness of its expression in the affine chart $[\tau u_1 + u_2]$, we conclude that $\tau_\infty = -\delta(0)$.

Next we compute the leading behavior of $\Phi(z)$. First, note that $N^2 = 0$ and consequently $\exp(zN) = 1 + zN$. Therefore

$$\exp(zN)\omega_0(q) = \big(a_1(q) - mz\delta(q)\big)u_1 + \big(a_2(q) + mz\big)u_2 + \zeta_0(q) + v_1 + \delta(q)v_2.$$

Dividing by the coefficient $a_2(q) + mz$ of $u_2$ places this vector in the affine chart used to define the Siegel-domain coordinates. We obtain the exact identities

$$\tau_s(z) = \frac{a_1(q) - mz\delta(q)}{a_2(q) + mz}, \qquad \zeta_W(z) = \frac{\zeta_0(q)}{a_2(q) + mz}, \qquad \sigma(z) = \frac{1}{a_2(q) + mz}.$$

For each $q_j$, choose a lift $z_j$ whose real part remains bounded. Then

$$\operatorname{Im} z_j = \frac{-\log|q_j|}{2\pi} \longrightarrow \infty.$$

Since $a_2(q)$ is bounded and $m \neq 0$, there exists a constant $C > 1$ such that, for all sufficiently large $j$,

$$\frac{1}{C}\operatorname{Im} z_j \leq |a_2(q_j) + mz_j| \leq C \operatorname{Im} z_j.$$

After increasing $C$ to absorb the factor $2\pi$, the formula for $\sigma$ gives

$$\frac{1}{C(-\log|q_j|)} \le |\sigma_j| \le \frac{C}{-\log|q_j|}.$$

Furthermore,

$$\tau_{s,j} - \tau_\infty = \frac{a_1(q_j) + \delta(0)a_2(q_j) + mz_j\big(\delta(0) - \delta(q_j)\big)}{a_2(q_j) + mz_j}.$$

The numerator remains bounded: the first two terms are bounded by holomorphicity, while $\delta(q) - \delta(0) = O(|q|)$ and hence $z_j\big(\delta(q_j) - \delta(0)\big) \to 0$. The denominator has order $\operatorname{Im} z_j$, and therefore

$$\tau_{s,j} = \tau_\infty + O\left(\frac{1}{-\log|q_j|}\right).$$

The corresponding formula for $\zeta_{W,j}$ immediately gives

$$\zeta_{W,j} = O\left(\frac{1}{-\log|q_j|}\right).$$

We now pass to the full $P_\Pi$-horospherical factorization. Since $\mathcal{Q}$ is quadratic, the exact inverse-coordinate formula gives

$$\begin{aligned}\tau_{H,j} &= \tau_{s,j} - \frac{\mathcal{Q}(\zeta_{W,j})}{2\sigma_j} \\ &= \tau_{s,j} - \frac{\mathcal{Q}(\zeta_0(q_j))}{2\big(a_2(q_j) + mz_j\big)}.\end{aligned}$$

The second term tends to zero, and hence $\tau_{H,j} \longrightarrow \tau_\infty$. These formulas also show that the nilpotent components $n(X_j, Z_{\lambda_j})$ remain in a fixed bounded subset of $N_\Pi$, since this will not play a substantial role in our analysis we leave the details to the reader.

It remains to estimate $t_j$. Taking imaginary parts in the identity

$$\frac{1}{\sigma_j} = -\lambda_j - \frac{i}{t_j} - \frac{1}{2}\mathcal{Q}(x_{2,j})\tau_{H,j} + \frac{1}{2}\mathcal{Q}(x_{1,j}, x_{2,j})$$

and using $1/\sigma_j = a_2(q_j) + mz_j$, we obtain

$$\frac{1}{t_j} = -m\operatorname{Im} z_j - \operatorname{Im} a_2(q_j) - \frac{1}{2}\mathcal{Q}(x_{2,j})\operatorname{Im}\tau_{H,j}.$$

The last two terms remain bounded, and hence

$$\frac{1}{t_j} = -m\operatorname{Im} z_j + O(1).$$

Since $t_j > 0$ and $\operatorname{Im} z_j \to \infty$, our conventions force $m < 0$. It follows that, after increasing $C$ once more,

$$\frac{1}{C(-\log|q_j|)} \le t_j \le \frac{C}{-\log|q_j|}.$$

In particular, $t_j \to 0$.

Finally, the convergence $\tau_{H,j} \to \tau_\infty$ implies $[\tau_{H,j}] \to [\tau_\infty]$ in $\Gamma_{\Pi,L}\backslash\mathcal{H}$ and Proposition 3.3.10 gives

$$r_j = \sqrt{\frac{\operatorname{Im}\tau_{H,j}}{t_j}}, \qquad s_j = \frac{1}{\sqrt{t_j \operatorname{Im}\tau_{H,j}}}.$$

Since $t_j \to 0$ and $\operatorname{Im}\tau_{H,j} \to \operatorname{Im}\tau_\infty > 0$, it follows that

$$s_j \longrightarrow \infty, \qquad \frac{r_j}{s_j} = \operatorname{Im}\tau_{H,j} \longrightarrow \operatorname{Im}\tau_\infty.$$

This is precisely the plane-cusp regime of Proposition 3.3.10. □

The final result we need records the product structure of the invariant metrics on the Levi bases.

**Lemma 3.4.12.** *Equip $B_I$ with the $L_I$-invariant metric induced by the restriction of the normalized Killing form.*

1. *Let $I = \ell$ and write $Y = e^\rho \widehat{Y}$. Then we have a Riemannian product decomposition of $B_\ell \simeq \mathbb{H}^{18} \times \mathbb{R}$. Moreover, $\Gamma_{\ell,L}$ acts trivially on the second factor, and therefore*

$$\Gamma_{\ell,L}\backslash B_\ell \simeq \left(\Gamma_{\ell,L}\backslash \mathbb{H}^{18}\right) \times \mathbb{R}$$

   *is a Riemannian product.*

2. *Let $I = \Pi$. Similarly, writing the radial coordinate of $B_\Pi = \mathcal{H} \times \mathbb{R}_{>0}$ logarithmically gives a Riemannian product decomposition*

$$\Gamma_{\Pi,L}\backslash B_\Pi \simeq \left(\Gamma_{\Pi,L}\backslash \mathcal{H}\right) \times \mathbb{R}.$$

*In particular, radial translations are isometries in both cases.*

*Proof.* We treat the two boundary components separately.

*The $I = \ell$ case.* Recall that $L_\ell^\circ \simeq \mathbb{R}_{>0} \times SO(W_{\ell,\mathbb{R}})^\circ$ and that $B_\ell = \mathbb{R}_{>0} \times SO(W_{\ell,\mathbb{R}})^\circ / SO(18)$. Under this identification, the factor $\mathbb{R}_{>0}$ acts by radial dilations of $\mathcal{T}_{W_\ell}$, whereas $SO(W_{\ell,\mathbb{R}})^\circ$ acts on the normalized hyperboloid. Thus, writing $Y = e^\rho \widehat{Y}$, with $(\widehat{Y}, \widehat{Y}) = 2$, the action of $\ell(a, A) \in L_\ell$ is

$$(\widehat{Y}, \rho) \longmapsto (A\widehat{Y}, \rho + \log a).$$

Moreover, the two factors are orthogonal for the metric under consideration. To see this note that the Lie algebra of the $\mathbb{R}_{>0}$-factor is the center of $\mathfrak{l}_\ell$, while $\mathfrak{so}(W_{\ell,\mathbb{R}})$ is its derived algebra, and invariance of the Killing form gives

$$\kappa_\mathfrak{g}(Z, [X_1, X_2]) = \kappa_\mathfrak{g}([Z, X_1], X_2) = 0$$

whenever $Z$ belongs to the central summand. Consequently, the restriction of the normalized Killing form has no mixed term between the hyperbolic and radial directions. Its restriction to the first factor is the invariant hyperbolic metric, while its restriction to the second factor is a constant multiple of $d\rho^2$. Hence $B_\ell \simeq \mathbb{H}^{18} \times \mathbb{R}$ is a Riemannian product. Finally, if $\ell(a, A) \in \Gamma_{\ell,L}$, then its action on the rank-one lattice $\ell \cap \Lambda_{2d}$ is multiplication by $a$. Since this action preserves the lattice and $a > 0$, necessarily $a = 1$. Therefore $\Gamma_{\ell,L}$ fixes $\rho$ and acts only on $\widehat{Y}$, proving that

$$\Gamma_{\ell,L}\backslash B_\ell \simeq \left(\Gamma_{\ell,L}\backslash \mathbb{H}^{18}\right) \times \mathbb{R}$$

is a Riemannian product.

*The $I = \Pi$ case.* Here $L_\Pi^\circ \simeq \mathbb{R}_{>0} \times SL(\Pi_\mathbb{R}) \times SO(W_\mathbb{R})$ and $B_\Pi \simeq \mathcal{H} \times \mathbb{R}_{>0}$. As in the line case, the central $\mathbb{R}_{>0}$-summand is orthogonal to $\mathfrak{sl}(\Pi_\mathbb{R})$ for the restriction of the Killing form. Since $t$ is a multiplicative coordinate on this central factor, the corresponding

metric is a constant multiple of $d(\log t)^2$, and therefore writing the radial coordinate logarithmically gives a Riemannian product $B_\Pi \simeq \mathcal{H} \times \mathbb{R}$. Finally, if $\ell(A,h) \in \Gamma_{\Pi,L}$, then $A$ preserves the rank-two lattice $\Pi \cap \Lambda_{2d}$. Since $A \in GL^+(\Pi_\mathbb{R})$, this implies $\det A = 1$. The Levi action sends the radial coordinate to $t/\det A$, so $\Gamma_{\Pi,L}$ fixes $t$ and acts only on the $\mathcal{H}$-factor. It follows that

$$\Gamma_{\Pi,L}\backslash B_\Pi \simeq \left(\Gamma_{\Pi,L}\backslash\mathcal{H}\right) \times \mathbb{R}$$

as a Riemannian product.

In summary, in the first case a radial translation is $\rho \mapsto \rho + c$, or equivalently $Y \mapsto e^c Y$, while in the second it is $\log t \mapsto \log t + c$, or equivalently $t \mapsto e^c t$. These are translations of the Euclidean factor and are therefore isometries. □

Having established the necessary result we proceed.

*Proof of Proposition 3.4.9.* We consider the two Kulikov types separately.

*The Type III case.* Set

$$\widehat{Y}_\infty := -\frac{N(v_1)}{\sqrt{(N(v_1),N(v_1))/2}}.$$

By Lemma 3.4.10, one has $\widehat{Y}_\infty \in \mathbb{H}^{18}$ and $\widehat{Y}_j \to \widehat{Y}_\infty$ and consequently, the same holds in $\Gamma_{\ell,L}\backslash\mathbb{H}^{18}$. Now, write $Y_j = e^{\rho_j}\widehat{Y}_j$. By Lemma 3.4.12, there is a Riemannian product decomposition

$$\Gamma_{\ell,L}\backslash B_\ell \simeq \left(\Gamma_{\ell,L}\backslash\mathbb{H}^{18}\right) \times \mathbb{R},$$

and $\Gamma_{\ell,L}$ acts trivially on the second factor. The radial translation $\rho \mapsto \rho - \rho_j$ therefore descends to an isometry of $\Gamma_{\ell,L}\backslash B_\ell$ and sends the basepoint $[b_j]$ to $\left([\widehat{Y}_j],0\right)$. Hence

$$\left(\Gamma_{\ell,L}\backslash B_\ell, [b_j]\right) \cong \left(\left(\Gamma_{\ell,L}\backslash\mathbb{H}^{18}\right) \times \mathbb{R}, \left([\widehat{Y}_j],0\right)\right).$$

Since $[\widehat{Y}_j] \to [\widehat{Y}_\infty]$, it follows that

$$\left(\Gamma_{\ell,L}\backslash B_\ell, [b_j]\right) \xrightarrow{\text{pGH}} \left(\left(\Gamma_{\ell,L}\backslash\mathbb{H}^{18}\right) \times \mathbb{R}, \left([\widehat{Y}_\infty],0\right)\right).$$

*The Type II case.* By Lemma 3.4.11, the $P_\Pi$-horospherical coordinates of the corresponding Levi projections satisfy $\tau_{H,j} \to \tau_\infty$ and $t_j \to 0$. In particular, $[\tau_{H,j}] \to [\tau_\infty]$ in $\Gamma_{\Pi,L}\backslash\mathcal{H}$. It remains to identify the modular meaning of the limiting point. By the standard Hodge-theoretic description of Type II Kulikov degenerations, the image of $[\tau_\infty]$ under the natural finite map

$$\Gamma_{\Pi,L}\backslash\mathcal{H} \longrightarrow A_1$$

is precisely the isomorphism class of the common elliptic double curve $E$ of the Kulikov model; see [59, §2.2, especially pp. 15–19]. The point $[\tau_\infty]$ in the finer quotient may retain additional marking or level data. Finally, Lemma 3.4.12 gives a Riemannian product decomposition

$$\Gamma_{\Pi,L}\backslash B_\Pi \simeq \left(\Gamma_{\Pi,L}\backslash\mathcal{H}\right) \times \mathbb{R},$$

where the second factor is the logarithmic radial coordinate and $\Gamma_{\Pi,L}$ acts trivially on it. The radial translation

$$(\tau,t) \longmapsto \left(\tau, \frac{t}{t_j}\right)$$

therefore descends to an isometry and sends the basepoint represented by $(\tau_{H,j}, t_j)$ to the point represented by $(\tau_{H,j}, 1)$, whose logarithmic radial coordinate is zero. Thus

$$\big(\Gamma_{\Pi,L}\backslash B_\Pi, [b_j]\big) \cong \big(\big(\Gamma_{\Pi,L}\backslash\mathcal{H}\big) \times \mathbb{R}, \big([\tau_{H,j}], 0\big)\big)\,.$$

Since $[\tau_{H,j}] \to [\tau_\infty]$, we conclude that

$$\big(\Gamma_{\Pi,L}\backslash B_\Pi, [b_j]\big) \xrightarrow{\text{pGH}} \big(\big(\Gamma_{\Pi,L}\backslash\mathcal{H}\big) \times \mathbb{R}, \big([\tau_\infty], 0\big)\big)\,.$$

In both cases the relevant component of the Levi base converges along the original sequence, rather than merely along a subsequence, and hence both projected pointed sequences converge without passing to a subsequence. □

Combining Proposition 3.4.9 with Theorem 3.4.8 gives the desired Weil–Petersson asymptotics.

**Corollary 3.4.13.** *Let $(x_j)$ be obtained by sampling a fixed polarized algebraic degeneration along a sequence $q_j \to 0$. If the degeneration is of Type II or Type III, then the pointed sequence $(\mathcal{F}_{2d}, d_{\text{WP}}, x_j)$ converges without passing to a subsequence, and its Weil–Petersson asymptotic limit is the corresponding pointed Levi space appearing in Proposition 3.4.9.*

*Proof.* In the Type III case, the preceding proof shows that the sequence approaches the cusp associated with $P_\ell$ and that the normalized hyperbolic component converges to $[\widehat{Y}_\infty]$; in particular, the omitted simple-root parameter satisfies $\alpha_\ell(a_j) \to \infty$. In the Type II case one has $\alpha_\Pi(a_j) = s_j \to \infty$. Theorem 3.4.8 therefore identifies the pointed Weil–Petersson geometry on every ball of fixed radius with the corresponding pointed Levi geometry, with an error tending to zero, while Proposition 3.4.9 gives the limit of the latter. □

*Remark* 3.4.14. It is important to notice that the proposition gives a necessary asymptotic description of every sequence coming from a fixed algebraic degeneration. We do not claim the converse: the convergence conditions displayed above do not by themselves imply that an arbitrary sequence in $\mathcal{F}_{2d}$ is contained in, or arises from, a one-parameter algebraic degeneration.

We can now conclude our investigation by giving a differential-geometric modular interpretation of the Weil–Petersson asymptotics established in Corollary 3.4.13. Roughly speaking our result says that the asymptotic Weil–Petersson geometry of $\mathcal{F}_{2d}$ is encoded by the diameter- and volume-scale Gromov-Hausdorff compactifications of $\mathcal{F}_{2d}$. This establishes a direct connection between the asymptotic of the Weil-Petersson metric and the multiscale limiting behavior of the Ricci-flat Kähler metric on polarized families of K3 surfaces, in the sense developed in Section 3.2. However, let us point out that our interpretation is conditional on Conjecture 3.2.7. Establishing this result is part of ongoing joint work with Yanbo Fang and Cristiano Spotti [17] (see Remark 3.4.17).

**Theorem 3.4.15.** *Assume Conjecture 3.2.7 holds, and let $(x_j)$ be a sequence in $\mathcal{F}_{2d}$ coming from a fixed polarized degeneration of K3 surfaces along a sequence $q_j \to 0$. Denote the corresponding Ricci-flat Kähler surfaces by $(X_j, g_j)$ and their diameter-normalized metrics by $\hat{g}_j$, as in Section 3.2. If the degeneration admits a Type II or Type III Kulikov model, then*

$$\big(\mathcal{F}_{2d}, g_{\text{WP}}, x_j\big) \xrightarrow{\text{pGH}} \begin{cases} \big(\mathcal{F}_{2d}^{\text{BB}}(\Pi) \times \mathcal{C}(\mathcal{F}_{2d}^{\text{Sat}}(\Pi)), g_{\text{WP},\Pi}, \big([\tau_\infty], 0\big)\big), & \text{in Type II,} \\ \big(\mathcal{F}_{2d}^{\text{BB}}(\ell) \times \mathcal{C}(\mathcal{F}_{2d}^{\text{Sat}}(\ell)), g_{\text{WP},\ell}, \big([\widehat{Y}_\infty], 0\big)\big), & \text{in Type III,} \end{cases}$$

*where the products carry the invariant metrics induced by the normalization of the Weil–Petersson metric fixed above. Their factors have the following modular interpretation.*

(i) *The* holomorphic factors, *namely* $\Gamma_{\Pi,L}\backslash\mathcal{H}$ *in Type II and a point* $*_\ell$ *in Type III, are naturally identified with the corresponding Baily–Borel boundary strata* $\mathcal{F}_{2d}^{\mathrm{BB}}(I)$, $I \in \{\ell, \Pi\}$. *These strata parametrize the compact factors of the generic volume-normalized Gromov–Hausdorff limits of* $(X_j, g_j)$. *In Type II these limits are* $\mathbb{R} \times E$, *and* $[\tau_\infty]$ *determines the common elliptic double curve* $E$; *in Type III the limit is* $\mathbb{R}^2$ *and the compact factor is a point.*

(ii) *The* tropical factors *are the Riemannian cylinders* $\mathcal{C}(\mathcal{F}_{2d}^{\mathrm{Sat}}(I))$, $I \in \{\ell, \Pi\}$, *over the corresponding Satake-adjoint boundary strata. In Type II this stratum is a point, so the tropical factor is* $\mathbb{R}$; *in Type III it is* $\Gamma_{\ell,L}\backslash\mathbb{H}^{18}$, *so the tropical factor is* $\left(\Gamma_{\ell,L}\backslash\mathbb{H}^{18}\right) \times \mathbb{R}$. *Each* $\mathcal{F}_{2d}^{\mathrm{Sat}}(I)$ *parametrizes diameter-normalized Gromov–Hausdorff limits of* $(X_j, \hat{g}_j)$, *namely the unit interval in Type II and the corresponding metrized tropical* $K3$ *surfaces in Type III.*

(iii) *Each asymptotic product carries a natural modular metric*

$$g_{\mathrm{WP},I} = g_{\mathrm{WP},I}^{hol} + g_{\mathrm{WP},I}^{trop}$$

*that is induced by the variation of the limiting metric structure they parametrize (see below for details in Remark 3.4.17 and Remark 3.4.18).*

*Thus, in the sense of these boundary parametrizations, the Weil–Petersson asymptotic is jointly encoded by the volume- and diameter-scale Gromov–Hausdorff compactifications.*

Before giving a conditional proof of this statement, we make several remarks to clarify this interpretation.

*Remark* 3.4.16. Here we briefly explain the connection between the diameter-scale Gromov-Hausdorff compactification and the Satake-adjoint compactification of $\mathcal{F}_{2d}$. First let us recall the moduli-theoretic framework introduced by Odaka–Oshima [53] for the limiting behavior of Ricci-flat Kähler metrics on polarized K3 surfaces, together with its subsequent completion by Ouyang–Tian [55]. Let $\overline{\mathcal{F}}_{2d}^{\mathrm{Sat}}$ denote the Satake compactification of $\mathcal{F}_{2d}$ associated with the adjoint representation. Recall that it admits the stratification

$$\overline{\mathcal{F}}_{2d}^{\mathrm{Sat}} = \mathcal{F}_{2d} \sqcup \coprod_{[\ell]} \mathcal{F}_{2d}^{\mathrm{Sat}}(\ell) \sqcup \coprod_{[\Pi]} \mathcal{F}_{2d}^{\mathrm{Sat}}(\Pi),$$

where $[\ell]$ and $[\Pi]$ range over the relevant $\Gamma_{2d}$-equivalence classes of rational isotropic lines and planes in $\Lambda_{2d} \otimes \mathbb{Q}$, respectively. Here $\mathcal{F}_{2d}^{\mathrm{Sat}}(\ell) = \Gamma_{\ell,L}\backslash\mathbb{H}^{18}$ whereas each $\mathcal{F}_{2d}^{\mathrm{Sat}}(\Pi)$ is a point $*_\Pi$. Let $\mathrm{CMet}_1$ denote the space of isometry classes of compact metric spaces of diameter one, endowed with the Gromov–Hausdorff topology. Following Odaka–Oshima, the *geometric realization map*

$$\Phi_{\mathrm{alg}} \colon \overline{\mathcal{F}}_{2d}^{\mathrm{Sat}} \longrightarrow \mathrm{CMet}_1,$$

is defined as follows [53, §§4.3–4.4]:

(i) For $(X, L) \in \mathcal{F}_{2d}$, set $\Phi_{\mathrm{alg}}(X, L) = (X, \hat{g}_{X,L})$, where $\hat{g}_{X,L}$ is obtained by diameter-normalizing the Ricci-flat Kähler metric in the polarization class $2\pi c_1(L)$, as in Section 3.2.

(ii) For $[\widehat{Y}_\infty] \in \mathcal{F}_{2d}^{\mathrm{Sat}}(\ell)$, set $\Phi_{\mathrm{alg}}([\widehat{Y}_\infty]) = (B_{[\widehat{Y}_\infty]}, d_{[\widehat{Y}_\infty]})$, the associated diameter-one metrized tropical $K3$ surface.

(iii) Each point of $\mathcal{F}_{2d}^{\mathrm{Sat}}(\Pi)$ is sent to $([0,1], d_{\mathrm{eucl}})$.

The main result of this moduli-theoretic framework is that $\Phi_{\mathrm{alg}}$ is continuous. Odaka–Oshima prove continuity of $\Phi_{\mathrm{alg}}$ away from the zero-dimensional strata $*_\Pi$ [53, Theorem 4.20], developing uniform adiabatic estimates for collapsing Ricci-flat metrics on polarized $K3$ surfaces. Their construction gives the identification of the diameter-normalized GH closure with the image of $\Phi_{\mathrm{alg}}$ conditional on their global continuity conjecture [53, Conjecture 4.3]. Ouyang–Tian subsequently establish this continuity in full generality by a different approach, thereby confirming the compactification picture [55, Theorem 1.2, Corollaries 1.3–1.4 and §3.6].

An important feature of $\overline{\mathcal{F}}_{2d}^{\mathrm{Sat}}$ is that it no longer has the structure of an algebraic variety! However we believe this is a feature rather than a drawback, given that from a modular point of view they parametrize limits that are no longer algebraic varieties but resemble more simplicial objects. Furthermore, Odaka–Oshima also show [53, Theorem 2.1] that $\overline{\mathcal{F}}_{2d}^{\mathrm{Sat}}$ has a deep algebro-geometric interpretation: It can be identified with the Morgan–Shalen–Boucksom–Jonsson compactification $\overline{\mathcal{F}}_{2d}^{\mathrm{MSBJ}}$, constructed by Odaka in [51], whose boundary can be described as the dual complex of *any* toroidal compactification of $\mathcal{F}_{2d}$.

*Remark* 3.4.17. We now explain the expected connection between the volume-scale Gromov–Hausdorff compactification and the Baily–Borel compactification of $\mathcal{F}_{2d}$. The latter was constructed by Baily and Borel [3] by adjoining the arithmetic quotients of the rational boundary components of $\mathcal{D}_{2d}$, endowed with the Satake topology. In our setting, it admits the stratification

$$\overline{\mathcal{F}}_{2d}^{\mathrm{BB}} = \mathcal{F}_{2d} \sqcup \coprod_{[\ell]} \mathcal{F}_{2d}^{\mathrm{BB}}(\ell) \sqcup \coprod_{[\Pi]} \mathcal{F}_{2d}^{\mathrm{BB}}(\Pi),$$

where the indices run over the same $\Gamma_{2d}$-equivalence classes of rational isotropic lines and planes as above, but now $\mathcal{F}_{2d}^{\mathrm{BB}}(\ell) = \{*_\ell\}$ and $\mathcal{F}_{2d}^{\mathrm{BB}}(\Pi) = \Gamma_{\Pi,L}\backslash\mathcal{H}$. The closure of each one-dimensional stratum is obtained by adjoining the zero-dimensional cusps corresponding to incident isotropic lines $\ell \subset \Pi$. The Baily–Borel compactification is a normal projective algebraic variety, realized as the Proj of a graded ring of automorphic forms. Moreover it is *minimal* in the sense given by Borel's extension theorem [8]: at neat arithmetic level, every smooth projective compactification with simple normal-crossing boundary admits a unique morphism to the corresponding Baily–Borel compactification extending the identity on the open locally symmetric variety. Comprehensive accounts of this construction and its properties can be found in [3, 2, 10].

The identification of the holomorphic factors of our Weil–Petersson asymptotics with these Baily–Borel boundary strata is unconditional: both the pointed convergence and this identification follow from the preceding results independently of Conjecture 3.2.7. Moreover, in Type II, the metric on $\Gamma_{\Pi,L}\backslash\mathcal{H}$ is its natural Weil–Petersson metric, up to the constant determined by our ambient normalization, and is induced by variation of the elliptic curve $E$; in Type III, the holomorphic stratum is a point.

Assuming Conjecture 3.2.7, these same strata acquire a differential-geometric interpretation: they parametrize the compact factors of the generic volume-normalized Gromov–Hausdorff limits of $(X_j, g_j)$, namely the elliptic curve $E$ in the Type II limit $\mathbb{R} \times E$ and the point in the Type III limit $\mathbb{R}^2$. This suggests that $\overline{\mathcal{F}}_{2d}^{\mathrm{BB}}$ realizes the appropriate volume-scale Gromov–Hausdorff compactification of $\mathcal{F}_{2d}$. Spotti has already outlined this expected identification in his survey [64, §4.1.3, Observation 2], where he proposes an argument combining Sun–Zhang's analysis of collapsing $K3$ metrics with period calculations and Friedman–Scattone's description of the Baily–Borel boundary in terms of Kulikov degenerations.

A precise global formulation and proof of this identification are the subject of ongoing joint work with Yanbo Fang and Cristiano Spotti [17]. The formulation must retain the cusp labels and finite covering data of the arithmetic boundary; we refer to [64, §4.1.3, Remark 4] and to this work in preparation for further discussion of these subtleties.

The complementary roles of the two compactifications are summarized in Table 3.1. The last column records the underlying Weil–Petersson asymptotic space, with the basepoints and metric normalizations specified in the theorem.

| **Type** | **Baily–Borel**<br>Holomorphic stratum | **Adjoint Satake**<br>Tropical stratum | **WP asymptotic** |
|---|---|---|---|
| II | $\Gamma_{\Pi,L}\backslash\mathcal{H}$ | $*_{\Pi}$ | $\left(\Gamma_{\Pi,L}\backslash\mathcal{H}\right)\times\mathbb{R}$ |
| III | $*_{\ell}$ | $\Gamma_{\ell,L}\backslash\mathbb{H}^{18}$ | $\left(\Gamma_{\ell,L}\backslash\mathbb{H}^{18}\right)\times\mathbb{R}$ |

**Table 3.1:** Boundary factors of the Weil–Petersson asymptotics.

*Remark* 3.4.18. Under the geometric realization map, each point $[\widehat{Y}_\infty]\in\mathcal{F}_{2d}^{\mathrm{Sat}}(\ell)$ determines a diameter-one metrized tropical $K3$ surface, namely an integral affine sphere with singularities equipped with a compatible Monge–Ampere metric, as recalled in Theorem 3.2.5. We expect the invariant metric $g_{\mathrm{WP},I}^{trop}$ to admit a modular interpretation in terms of these limiting structures, with their diameter kept fixed. The guiding idea is that the tropical volume, before diameter normalization, is governed by the same quadratic form that controls the Weil–Petersson geometry. Indeed, Friedman–Scattone identify the quadratic monodromy invariant with the number of triple points of a Kulikov central fiber [19, Definition 1.4 and Proposition 7.1], which in turn determines the affine volume of its dual complex [1, Proposition 2.22]. Together with the tropical period interpretation of Odaka–Oshima [53, Remarks 4.2 and 4.26], this suggests a connection between variation of tropical volume and the invariant hyperbolic geometry of $\mathcal{F}_{2d}^{\mathrm{Sat}}(\ell)$. This discussion is intended as a heuristic explanation. The comparison with the McLean geometry, the treatment of diameter normalization, and the precise construction will be developed in joint work with Yanbo Fang and Cristiano Spotti [17].

*Conditional Proof of Theorem 3.4.15.* The pointed convergence and the invariant product metrics follow from Corollary 3.4.13 and Lemma 3.4.12. The holomorphic factors are the Baily–Borel strata associated with $\Pi$ and $\ell$, respectively, and Proposition 3.4.9 identifies $[\tau_\infty]$ with the period of $E$. Conjecture 3.2.7 therefore gives their asserted volume-scale interpretation. On the tropical side, the description of the adjoint Satake boundary and its compatibility with monodromy identify the corresponding strata and the point $[\widehat{Y}_\infty]$ [53, §§4.2 and 4.5]; continuity of the geometric realization map then gives the diameter-scale interpretation [55, Corollary 1.3]. □

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