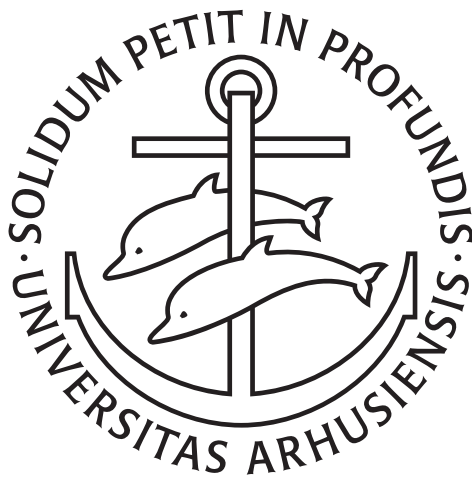


Completing derived representation theory of algebras

Metrics on triangulated categories and applications



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Ph.D. Dissertation

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27 February, 2026

Abstract

The construction of a metric completion of a triangulated category is a novel method to build new triangulated categories. Its key idea is to transfer the originally topological notion of convergence to algebra. Employing the structure of a metric on a triangulated category, we are allowed to measure lengths of morphisms and determine which sequences of objects and morphisms are convergent. The metric completion then emerges out of the limit points of convergent sequences.

In this thesis, we study metrics and metric completions of triangulated categories arising in derived representation theory with a particular focus on finite dimensional algebras. The thesis has three main chapters.

In [Paper A](#), we define a preimage metric on a triangulated category under a triangulated functor, show when it induces an equivalence between the completions of two triangulated categories, and demonstrate applications for triangulated categories with Serre functors. Furthermore, we classify all possible metric completions of bounded derived categories $\mathbf{D}^b(\text{mod-}A)$ of hereditary finite dimensional algebras of finite representation type as precisely the thick subcategories of $\mathbf{D}^b(\text{mod-}A)$.

[Paper B](#) provides a classification result on metric completions of bounded derived categories $\mathbf{D}^b(\text{mod-}A)$ of well-behaved hereditary rings such as hereditary commutative noetherian rings and hereditary finite dimensional algebras of tame representation type. There are precisely two types of triangulated categories arising as completions of such derived categories - thick subcategories of $\mathbf{D}^b(\text{mod-}A)$ and bounded derived categories $\mathbf{D}^b(\text{mod-}B)$ of a universal localisation B of A . The lattice theory of metrics on a triangulated category, also developed in [Paper B](#), turns out to be a particularly contributive tool to study metrics on derived categories of hereditary rings.

The final chapter, [Project C](#), reports on an ongoing project generalising the methods and results of [Paper B](#) to a broader setting. Firstly, we show that the completion of the category of compact elements of a compactly generated triangulated category with respect to a countably generated constant additive good metric is equivalent to Verdier's quotient of the category with respect to a corresponding thick subcategory. We then use this fact to prove an analogous result as in [Paper B](#) for bounded derived categories $\mathbf{D}^b(\text{coh-}\mathbb{X})$ of coherent sheaves on a weighted projective line \mathbb{X} of domestic or tubular type - the completions here are again either thick subcategories, or localisations.

Resumé

Konstruktionen af en metrisk fuldstændiggørelse af en trianguleret kategori er en ny metode til at bygge nye triangulerede kategorier. Dens hovedidé er at overføre det oprindeligt topologiske koncept om konvergens til algebra. Ved at anvende strukturen af en metrik på en trianguleret kategori, får vi lov til at måle længderne af morfier og bestemme, hvilke følger af objekter og morfier der er konvergente. Den metriske fuldstændiggørelse opstår derefter ud af grænsepunkterne for konvergente følger.

I denne afhandling studerer vi metrikker og metriske fuldstændiggørelser af triangulerede kategorier, der opstår i deriveret repræsentationsteori med et særligt fokus på endeligt-dimensionelle algebraer. Afhandlingen har tre hovedkapitler.

I [Paper A](#) definerer vi en urbillede-metrik på en trianguleret kategori under en trianguleret funktor, viser hvornår den inducerer en ækvivalens mellem fuldstændiggørelserne af to triangulerede kategorier og demonstrerer anvendelser på triangulerede kategorier med Serre-funktorer. Derudover klassificerer vi alle mulige metriske fuldstændiggørelser af begrænsede deriverede kategorier $\mathbf{D}^b(\text{mod-}A)$ af arvelige endeligt-dimensionelle algebraer af endelig repræsentationstype som præcist de tykke delkategorier af $\mathbf{D}^b(\text{mod-}A)$.

[Paper B](#) giver et klassifikationsresultat af metriske fuldstændiggørelser af begrænsede deriverede kategorier $\mathbf{D}^b(\text{mod-}A)$ af tilstrækkeligt pæne arvelige ringe såsom arvelige kommutative noetherske ringe og arvelige endeligt-dimensionelle algebraer af tam repræsentationstype. Der er præcist to typer triangulerede kategorier, der opstår som fuldstændiggørelser af sådanne deriverede kategorier - tykke delkategorier af $\mathbf{D}^b(\text{mod-}A)$ og begrænsede deriverede kategorier $\mathbf{D}^b(\text{mod-}B)$ af en universel lokalisering B af A . Gitterteorien for metrikker på en trianguleret kategori, også udviklet i [Paper B](#), viser sig at være et særligt nyttigt værktøj til at studere metrikker på deriverede kategorier af arvelige ringe.

Det sidste kapitel, [Project C](#), rapporterer om et igangværende projekt, der generaliserer metoderne og resultaterne fra [Paper B](#) til en bredere kontekst. For det første viser vi, at fuldstændiggørelsen af kategorien af kompakte objekter i en kompakt genereret trianguleret kategori med hensyn til en tælleligt genereret konstant additiv god metrik er ækvivalent med Verdiers kvotient af kategorien med hensyn til en tilsvarende tyk delkategori. Vi bruger derefter denne kendsgerning til at bevise et analogt resultat som i [Paper B](#) for begrænsede deriverede kategorier $\mathbf{D}^b(\text{coh-}\mathbb{X})$ af kohærente knipper på en vægtet projektiv linje \mathbb{X} af domesticeret eller tubulær type - fuldstændiggørelserne her er igen enten tykke delkategorier eller lokaliseringer.

Contents

Abstract		i
Resumé		iii
Preface		vii
Introduction		1
1	Notation and conventions	1
2	Fundamental notions	2
2.1	Triangulated categories	2
2.2	Derived categories	9
2.3	Hom-finite categories	14
2.4	Finite dimensional algebras	17
2.5	Metrics and metric completions	18
3	Mathematical context	21
4	Overview of the main results	23
	References	27
Paper A	Metric completions of triangulated categories from finite dimensional algebras	35
1	Introduction	35
2	Preliminaries	38
2.1	Notation and conventions	38
2.2	Metrics on triangulated categories	40
2.3	Completions of triangulated categories	42
2.4	Good extensions	45
3	Completions of certain derived categories	46
4	Functors between completions	51
4.1	Induced functors on completions	51
4.2	Preimages and images of metrics	55
4.3	Categories with Serre functors	60
5	Good metrics and improvements	63
5.1	Examples	66
	References	69

Paper B	Metric completions of triangulated categories from hereditary rings	75
1	Introduction	75
2	Preliminaries	78
2.1	Notation and conventions	78
2.2	Metrics and completions	78
2.3	t -structures and approximation sequences	82
2.4	Derived categories	83
2.5	Commutative algebra	84
2.6	Finite dimensional algebras	85
3	Constant metrics, smashing subcategories, and ring epimorphisms .	87
4	Computational tools for hereditary and noetherian rings	90
5	Lattice theory of metrics	96
5.1	Decomposition of metrics	99
6	Completions for commutative noetherian rings	101
6.1	General completion lemmas	101
6.2	The computation	102
7	Completions for tame algebras	104
	References	109
Project C	Metric completions from weighted projective lines	113
5	Completions of compacts	113
6	Completions from Grothendieck categories	115
7	Weighted projective lines	116
8	Derived categories of weighted projective lines	119
9	Completions	122
10	Examples	126
	References	131

Preface

This thesis is submitted to conclude my Ph.D. studies at Aarhus University. It consists of four parts - the [Introduction](#) and three main chapters:

- [Paper A](#) is a preprint titled “*Metric completions of triangulated categories from finite dimensional algebras*” available online at <https://arxiv.org/abs/2409.01828v3>. It has been accepted for publication in Applied Categorical Structures.
- [Paper B](#) is a preprint titled “*Metric completions of triangulated categories from hereditary rings*” available online at <https://arxiv.org/abs/2508.20283v2>. It is submitted for publication to a journal and currently undergo review process.
- [Project C](#) is a report “*Metric completions from weighted projective lines*” on current results of an ongoing research project.

[Paper A](#) and [Paper B](#) are single-author, stand-alone research papers. As chapters of this thesis, they are self-contained and have their own introductions and preliminaries sections. For the sake of consistency, the [Introduction](#) and [Project C](#) share common numbering of sections. All four chapters have their own bibliography sections.

In accordance with GSNS rules, parts of this thesis were also used in the progress report for the qualifying examination. Concretely, the progress report served as a draft version of [Paper A](#). None of the parts of the thesis have been used to obtain any previous academic degree.

Acknowledgements

I would like to express my sincere gratitude to my supervisor, Sira Gratz, for her guidance and unwavering support during the whole three years of my Ph.D. studies in Aarhus.

I wish to thank my officemates Esben Bonnerup, Mary Elworth, and David Nkansah for creating a fun workplace and many interesting mathematical discussions. I also thank the algebraists Esther Banaian, Raphael Bennett-Tennenhaus, Erlend Børve, Charley Cummings, Karin Jacobsen, Peter Jørgensen, Carlo Klapproth, Anders Kortegård, Davide Morigi, Mads Sandøy, Amit Shah, Greg Stevenson, and many other friends and members of the Department of Mathematics for making my studies at Aarhus enjoyable. In addition, I appreciate the help of Esben Bonnerup, Peter Jørgensen, and Mads Sandøy with translating the thesis abstract into Danish.

I am grateful to Henning Krause and the BIREP group for their hospitality during my three-month research stay at the Bielefeld University. Thanks are also due to Jan Šťovíček

and Gustavo Jasso for helpful suggestions which eventually led to the results of [Paper B](#) and [Project C](#).

Finally, I would like to thank my family for their constant encouragement and moral support.

This work was supported by a research grant (VIL42076) from VILLUM FONDEN.

Introduction

This introductory chapter provides an overview of the mathematical content of the thesis. Section 2 is an exposition of the key notions from homological algebra and representation theory studied in this thesis, including the pivotal topic - metric completions of triangulated categories. In Section 3, we illustrate the relevance of the theory of metric completions in contemporary research. The author's contribution to the theory of metric completions presented in this thesis is summarised in Section 4.

1 Notation and conventions

In this short section, we declare what mathematical and notational conventions we use in the rest of the [Introduction](#) and [Project C](#). As [Paper A](#) and [Paper B](#) are stand-alone articles, their own conventions take precedence.

Unless explicitly stated otherwise, by *a category*, we mean a category \mathcal{A} which is locally small; that is $\text{Hom}_{\mathcal{A}}(X, Y)$ forms a set. We shall write $A \in \mathcal{A}$ to express that A is an object of \mathcal{A} , and $\mathcal{B} \subseteq \mathcal{A}$ expresses that \mathcal{B} is a subcategory of \mathcal{A} . All subcategories are considered to be full.

An adjoint pair of functors is depicted as $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$, meaning that F is a left adjoint to G .

Any functor $F : \mathcal{A} \rightarrow \mathcal{B}$ between additive categories is automatically assumed to be additive.

Let \mathcal{A} be an additive category and $A \in \mathcal{A}$. We denote the two orthogonal categories to A as $A^{\perp} := \{B \in \mathcal{A} : \text{Hom}_{\mathcal{A}}(A, B) \simeq 0\}$ and ${}^{\perp}A := \{B \in \mathcal{A} : \text{Hom}_{\mathcal{A}}(B, A) \simeq 0\}$. An analogous notation \mathcal{B}^{\perp} and ${}^{\perp}\mathcal{B}$ is employed when $\mathcal{B} \subseteq \mathcal{A}$. The subcategory $\text{add}(A) \subseteq \mathcal{A}$ consists, by definition, of finite coproducts of direct summand of $A \in \mathcal{A}$.

For an additive category, the category $\text{Mod-}\mathcal{A}$ of modules over \mathcal{A} consists of contravariant functors $\mathcal{A}^{\text{op}} \rightarrow \text{Ab}$ from \mathcal{A} to Ab . The category \mathcal{A} embeds into $\text{Mod-}\mathcal{A}$ via Yoneda's embedding $\mathfrak{y} : \mathcal{A} \rightarrow \text{Mod-}\mathcal{A}$, $A \mapsto \text{Hom}_{\mathcal{A}}(-, A)$.

Even though, in general, the module category $\text{Mod-}\mathcal{A}$ does not have to be locally small, the relevant subcategories of module categories studied in this thesis (such as $\mathfrak{L}(\mathcal{A})$ or $\mathfrak{S}(\mathcal{A})$ from [Definition 2.63](#)) consist of countably presented functors only and thus are locally small. For this reason, we will not be further addressing the set-theoretical issues related to $\text{Mod-}\mathcal{A}$.

If R is a ring, then $\text{Mod-}R$ is the category of right R -modules, and $\text{mod-}R$ is the category of finitely presented right modules. By default, modules over a ring will be considered to be right modules, unless stated otherwise.

2 Fundamental notions

This section contains mathematical preliminaries for the whole thesis. In Subsection 2.1, we review the basic theory of triangulated categories. The most important example of a triangulated category for the purposes of this thesis are derived categories, discussed in Subsection 2.2. The specifics of working with finite dimensional algebras and K -linear categories over a field K are commented in Subsection 2.3 and Subsection 2.4. We finish by describing what metrics on triangulated categories are and how to use them to construct a metric completion of the triangulated category (Subsection 2.5).

2.1 Triangulated categories

Triangulated categories were independently introduced by Verdier [70] and Puppe [62]. Here, we mainly follow the exposition by Neeman from [47, Chapter 1].

Definition 2.1 ([47, Definition 1.1.1]). Let \mathcal{A} be an additive category with an autoequivalence $\Sigma : \mathcal{A} \rightarrow \mathcal{A}$. We call a diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

in \mathcal{A} a *candidate triangle* if $g \circ f \simeq 0$, $h \circ g \simeq 0$, and $\Sigma f \circ h \simeq 0$.

Definition 2.2 ([47, Definition 1.1.2],[71, Definition 10.2.1]). Let \mathcal{T} be an additive category. The category \mathcal{T} together with an autoequivalence $\Sigma : \mathcal{T} \rightarrow \mathcal{T}$ and a class Δ of candidate triangles is called *triangulated* if the following axioms hold:

TR0: The class Δ is closed under isomorphisms of candidate triangles, and the candidate triangle

$$X \xrightarrow{\text{id}_X} X \rightarrow 0 \rightarrow \Sigma X$$

belongs to Δ for every $X \in \mathcal{T}$.

TR1: Any morphism $f : X \rightarrow Y$ in \mathcal{T} can be completed to a candidate triangle

$$X \xrightarrow{f} Y \rightarrow Z \rightarrow \Sigma X$$

belonging to Δ .

TR2: Any candidate triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

belongs to Δ if and only if the candidate triangle

$$Y \xrightarrow{-g} Z \xrightarrow{-h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

does so.

TR3: Any commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow u & & & & \downarrow v & & \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \end{array}$$

in \mathcal{T} whose rows are candidate triangles belonging to Δ can be completed to a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow u & & \downarrow v & & \downarrow w & & \downarrow \Sigma f \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X'. \end{array}$$

TR4: Any three candidate triangles from Δ of the form

$$X \xrightarrow{f} Y \rightarrow Z \rightarrow \Sigma X, \quad Y \xrightarrow{g} Y' \rightarrow Y'' \rightarrow \Sigma Y, \quad \text{and} \quad X \xrightarrow{gf} Y' \rightarrow Z' \rightarrow \Sigma X$$

can be completed to a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \text{id}_X \downarrow & & \downarrow g & & \downarrow & & \downarrow \text{id}_{\Sigma X} \\ X & \xrightarrow{gf} & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Y'' & \xrightarrow{\text{id}_{Y''}} & Y'' & \longrightarrow & 0 \\ \downarrow & & \downarrow v & & \downarrow & & \downarrow \\ \Sigma X & \xrightarrow{\Sigma f} & \Sigma Y & \longrightarrow & \Sigma Z & \longrightarrow & \Sigma^2 X \end{array}$$

in \mathcal{T} whose rows and columns are candidate triangles belonging to Δ .

In this case, the functor Σ is called *the shift* on \mathcal{T} , and the candidate triangles from Δ are called *distinguished triangles* (or *triangles* for short).

We shall denote the shift functor on every triangulated category as Σ without further notice.

The object Z determined by the distinguished triangle

$$X \xrightarrow{f} Y \rightarrow Z \rightarrow \Sigma X$$

given by the axiom TR2 applied to a morphism $f : X \rightarrow Y$ in \mathcal{T} is unique up to a non-unique isomorphism and is often called *the cone* of f and denoted $\text{cone } f$. The construction of a cone is generally not functorial. Also, the object Y is called *an extension* of Z by X .

The axiom TR4 is known as *the octahedral axiom*. It is an open problem to determine whether it follows from the preceding axioms TR0-TR3. In [43, Theorem 1.8] and [47, Proposition 1.4.6], Neeman proves that the octahedral axiom is equivalent to the following reformulation TR4'.

Theorem 2.3. Let \mathcal{T} be an additive category with an autoequivalence $\Sigma : \mathcal{T} \rightarrow \mathcal{T}$ and a class of distinguished triangles satisfying the axioms TR0-TR3. Then the axiom TR4 holds if and only if the axiom TR4' below holds.

TR4': Any commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow u & & \downarrow v & & & & \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \end{array}$$

in \mathcal{T} whose rows are distinguished triangles can be completed to a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow u & & \downarrow v & & \downarrow w & & \downarrow \Sigma u \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X'. \end{array}$$

such that the candidate triangle (so-called “*mapping cone*”)

$$Y \oplus X' \xrightarrow{\begin{bmatrix} -g & 0 \\ v & f' \end{bmatrix}} Z \oplus Y' \xrightarrow{\begin{bmatrix} -h & 0 \\ w & g' \end{bmatrix}} \Sigma X \oplus Z' \xrightarrow{\begin{bmatrix} -\Sigma f & 0 \\ \Sigma u & h' \end{bmatrix}} \Sigma Y \oplus \Sigma X'$$

is distinguished.

Definition 2.4 ([47, Definition 1.1.7]). Let \mathcal{T} be a triangulated category, \mathcal{A} an abelian category, and $H : \mathcal{T} \rightarrow \mathcal{A}$ a functor. We say that H is *homological* if for every distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

in \mathcal{T} there is a long exact sequence

$$\cdots \rightarrow H(\Sigma^{-1}Z) \xrightarrow{H(\Sigma^{-1}h)} H(X) \xrightarrow{H(f)} H(Y) \xrightarrow{H(g)} H(Z) \xrightarrow{H(h)} H(\Sigma X) \rightarrow \cdots$$

in \mathcal{A} .

Similarly, a contravariant functor $H : \mathcal{T}^{\text{op}} \rightarrow \mathcal{A}$ from \mathcal{T} to \mathcal{A} is called *cohomological* if it satisfies the dual condition.

Example 2.5 ([47, Lemma 1.1.10, Remark 1.1.11]). Let \mathcal{T} be a triangulated category. For every object $X \in \mathcal{T}$, the covariant representable functor $\text{Hom}_{\mathcal{T}}(X, -) : \mathcal{T} \rightarrow \text{Ab}$ is homological, and the contravariant representable functor $\text{Hom}_{\mathcal{T}}(-, X) : \mathcal{T}^{\text{op}} \rightarrow \text{Ab}$ from \mathcal{T} to Ab is cohomological.

Proposition 2.6 ([47, Corollary 1.2.6, Corollary 1.2.7]). Let \mathcal{T} be a triangulated category. A distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

is *split*, meaning that it is isomorphic to the distinguished triangle

$$X \xrightarrow{\begin{bmatrix} \text{id}_X \\ 0 \end{bmatrix}} X \oplus Z \xrightarrow{[0 \text{ id}_Z]} Z \xrightarrow{0} \Sigma X,$$

if and only if $h \simeq 0$.

In particular, the morphism f is an isomorphism if and only if $Z \simeq 0$.

Definition 2.7. Let \mathcal{X}, \mathcal{Z} be subcategories of a triangulated category \mathcal{T} . We denote $\mathcal{X} * \mathcal{Z}$ the subcategory of \mathcal{T} consisting of extensions of the objects of \mathcal{Z} by the objects of \mathcal{X} . In other words,

$$\mathcal{X} * \mathcal{Z} := \{Y \in \mathcal{T} : \exists X \in \mathcal{X}, \exists Z \in \mathcal{Z}, \exists \text{ a distinguished triangle } X \rightarrow Y \rightarrow Z \rightarrow \Sigma X\}.$$

In particular, the subcategory \mathcal{X} is called *extension-closed* if $\mathcal{X} = \mathcal{X} * \mathcal{X}$.

2.1.1 Triangulated functors and subcategories

Definition 2.8 ([47, Definition 2.1.1]). Let $F : \mathcal{T} \rightarrow \mathcal{S}$ be a functor between triangulated categories and $\varphi : F \circ \Sigma \Rightarrow \Sigma \circ F$ a natural transformation. We say that the pair (F, φ) is a *triangulated functor* if for every distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

in \mathcal{T} the candidate triangle

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{\varphi_X \circ F(h)} \Sigma F(X)$$

in \mathcal{S} is distinguished.

Most of the time, we will omit the natural transformation while referring to a triangulated functor because it will be implicit from the context. However, formally speaking, the natural transformation is an inseparable part of the data defining a triangulated functor. Indeed, for example, the shift $\Sigma : \mathcal{T} \rightarrow \mathcal{T}$ is not a triangulated functor while accompanied by the identity natural transformation $\Sigma \Rightarrow \Sigma$, but it is triangulated with respect to the “minus identity” natural transformation with components $-\text{id}_X : \Sigma X \rightarrow \Sigma X$ for each $X \in \mathcal{T}$.

Lemma 2.9 ([47, Lemma 5.3.6]). Any adjoint to a triangulated functor is again triangulated.

Definition 2.10. Let \mathcal{T} be a triangulated category. A replete subcategory $\mathcal{S} \subseteq \mathcal{T}$ is called a *triangulated subcategory* if $\Sigma \mathcal{S} = \mathcal{S}$ and for every distinguished triangle

$$X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$$

in \mathcal{T} with $X, Y \in \mathcal{S}$ it holds that $Z \in \mathcal{S}$.

Furthermore, we say that the triangulated subcategory \mathcal{S} is *thick* if it is closed under direct summands.

It follows from the definition, that if \mathcal{S} is a triangulated subcategory of \mathcal{T} , then the inclusion functor $\nu : \mathcal{S} \hookrightarrow \mathcal{T}$ together with the identity natural transformation $\nu \circ \Sigma \Rightarrow \Sigma \circ \nu$ is a triangulated functor.

The smallest triangulated subcategory of \mathcal{T} containing a given class of objects $\mathcal{C} \subseteq \mathcal{T}$ is said to be *generated* by \mathcal{C} and denoted $\langle \mathcal{C} \rangle$. Analogously, the thick subcategory generated by \mathcal{C} is denoted $\text{thick}(\mathcal{C})$.

If a triangulated subcategory \mathcal{S} of \mathcal{T} is generated by finitely many objects, then it is called *finitely generated*; and if it is generated by finitely many or countably many objects, then it is called *countably generated*.

We recall a fundamental construction of *Verdier's quotient* (or *Verdier's localisation*) of a triangulated category by a thick subcategory.

Theorem 2.11 ([47, Theorem 2.1.8]). Let \mathcal{S} be a thick subcategory of \mathcal{T} . Then there exists a triangulated category \mathcal{T}/\mathcal{S} and a triangulated functor $F : \mathcal{T} \rightarrow \mathcal{T}/\mathcal{S}$ such that $\text{Ker } F = \mathcal{S}$ which is universal with respect to this property, i.e. any other triangulated functor $F' : \mathcal{T} \rightarrow \mathcal{T}'$ with $\mathcal{S} \subseteq \text{Ker } F'$ factors uniquely through F .

Remark 2.12. Under the above notation, if \mathcal{S} is essentially small, then $\text{Hom}_{\mathcal{T}/\mathcal{S}}(X, Y)$ is a set for each $X, Y \in \mathcal{T}$ by [47, Proposition 2.2.1]. However, if \mathcal{S} has proper class of objects, it can, in principle, happen that $\text{Hom}_{\mathcal{T}/\mathcal{S}}(X, Y)$ does not form a set for some $X, Y \in \mathcal{T}$, so \mathcal{T}/\mathcal{S} is not a locally small category any more. Luckily, all Verdier's quotients considered in this thesis yield locally small categories with Hom-sets, so we shall not further address this set-theoretical issue.

2.1.2 Structures on triangulated categories

Throughout Subsection 2.1.2, we work with a fixed triangulated category \mathcal{T} .

Definition 2.13 ([30, Definition 2.2]). Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{T}$. We say that $(\mathcal{X}, \mathcal{Y})$ is a *torsion pair* if the following conditions hold:

- $\mathcal{X} = \text{add}(\mathcal{X})$ and $\mathcal{Y} = \text{add}(\mathcal{Y})$,
- $\mathcal{Y} \subseteq \mathcal{X}^\perp$,
- and $\mathcal{T} = \mathcal{X} * \mathcal{Y}$.

When $(\mathcal{X}, \mathcal{Y})$ is a torsion pair, it automatically holds that $\mathcal{X} = {}^\perp \mathcal{Y}$ and $\mathcal{Y} = \mathcal{X}^\perp$.

Definition 2.14 ([6, Définition 1.3.1],[7]). We say that a torsion pair $(\mathcal{T}^{\leq 0}, \mathcal{T}^{> 0})$ on \mathcal{T} is a *t-structure* if $\Sigma \mathcal{T}^{\leq 0} \subseteq \mathcal{T}^{\leq 0}$.

In that case, we call $\mathcal{T}^{\leq 0}$ an *aisle*, $\mathcal{T}^{> 0}$ a *coaisle*, and for each $n \in \mathbb{Z}$ we denote $\mathcal{T}^{\leq n} := \Sigma^{-n} \mathcal{T}^{\leq 0}$ and $\mathcal{T}^{> n} := \Sigma^{-n} \mathcal{T}^{> 0}$.

Moreover, the *t-structure* $(\mathcal{T}^{\leq 0}, \mathcal{T}^{> 0})$ is said to be:

- *bounded above* if $\mathcal{T} = \bigcup_{n \in \mathbb{Z}} \mathcal{T}^{\leq n}$,
- *bounded below* if $\mathcal{T} = \bigcup_{n \in \mathbb{Z}} \mathcal{T}^{> n}$,
- and *bounded* if it is bounded above and bounded below.

We recall that a subcategory \mathcal{B} of an additive category \mathcal{A} is called *reflective* if the inclusion functor $\mathcal{B} \rightarrow \mathcal{A}$ has a left adjoint. Dually, the subcategory \mathcal{B} is *coreflective* if the inclusion $\mathcal{B} \rightarrow \mathcal{A}$ has a right adjoint. If \mathcal{B} is both reflective and coreflective, then it is *bireflective*. (Some authors call bireflective subcategories *admissible*.)

An aisle of a t -structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{> 0})$ is a coreflective subcategory of \mathcal{T} with the right adjoint $\mathrm{tr}^{\leq 0} : \mathcal{T} \rightarrow \mathcal{T}^{\leq 0}$ to the inclusion called *the truncation*. Dually, the coaisle $\mathcal{T}^{> 0}$ is reflective with the truncation functor $\mathrm{tr}^{> 0} : \mathcal{T} \rightarrow \mathcal{T}^{> 0}$ as a left adjoint to the respective inclusion. Applying the two truncations to an object $X \in \mathcal{T}$ yields a distinguished triangle

$$\mathrm{tr}^{\leq 0} X \rightarrow X \rightarrow \mathrm{tr}^{> 0} X \rightarrow \Sigma \mathrm{tr}^{\leq 0} X$$

in \mathcal{T} called *the approximation triangle*.

A pair $(\mathcal{U}, \mathcal{V})$ of thick subcategories of \mathcal{T} is clearly a torsion pair if and only if it is a t -structure. We can even find out whether thick subcategory sits in a torsion pair solely based on the existence of adjoints to the inclusion functor.

Proposition 2.15. Let \mathcal{S} be a thick subcategory of \mathcal{T} . Then:

- $({}^\perp \mathcal{S}, \mathcal{S})$ is a torsion pair if and only if \mathcal{S} is reflective.
- $(\mathcal{S}, \mathcal{S}^\perp)$ is a torsion pair if and only if \mathcal{S} is coreflective.
- $({}^\perp \mathcal{S}, \mathcal{S})$ and $(\mathcal{S}, \mathcal{S}^\perp)$ are simultaneously torsion pairs if and only if \mathcal{S} is bireflective.

Proof. By [48, Proposition 1.4] and [40, Theorem 2.9], the thick subcategory \mathcal{S} is coreflective if and only if it is covering. Then [30, Proposition 2.3] implies that $(\mathcal{S}, \mathcal{S}^\perp)$ is a torsion pair. Conversely, if $(\mathcal{S}, \mathcal{S}^\perp)$ is a torsion pair, then the right adjoint to the inclusion $\mathcal{S} \hookrightarrow \mathcal{T}$ is given by the truncation.

The proof of the remaining statements is analogous. \square

Triples $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ of thick subcategories such that $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{V}, \mathcal{W})$ are torsion pairs are known as *TTF-triples* (TTF= *torsion torsion-free*) and correspond to so-called recollements of triangulated categories.

Definition 2.16 ([6, Section 1.4],[7]). By *recollement* we mean a diagram

$$\begin{array}{ccccc} & & i^* & & j_! \\ & \swarrow & \leftarrow & \swarrow & \leftarrow \\ \mathcal{V} & \xrightarrow{i_*} & \mathcal{T} & \xrightarrow{j^*} & \mathcal{U} \\ & \searrow & \leftarrow & \searrow & \leftarrow \\ & & i^! & & j_* \end{array}$$

of triangulated categories where

- $i^* : \mathcal{T} \rightleftarrows \mathcal{V} : i_*$, $i_* : \mathcal{V} \rightleftarrows \mathcal{T} : i^!$, $j_! : \mathcal{U} \rightleftarrows \mathcal{T} : j^*$, and $j^* : \mathcal{T} \rightleftarrows \mathcal{U} : j_*$ are adjoint pairs of triangulated functors,
- i_* , $j_!$, and j_* are fully faithful,
- $j^* \circ i_* \simeq 0$,
- and $(j_! \mathcal{U}, i_* \mathcal{V})$ and $(i_* \mathcal{V}, j_* \mathcal{U})$ are torsion pairs.

By [6, Sous-section 1.4.4] (see also [59, Subsection 2.1]), a TTF triple $(\mathcal{U}, \mathcal{V}, \mathcal{W})$ in \mathcal{T} corresponds to a recollement of the same form as in Definition 2.16 with $j_! : \mathcal{U} \hookrightarrow \mathcal{T}$ and $i_* : \mathcal{V} \hookrightarrow \mathcal{T}$ being the respective inclusions; and this correspondence is bijective. Note that i^* is isomorphic to Verdier's quotient functor $\mathcal{T} \rightarrow \mathcal{T}/\mathcal{U}$ and j^* is isomorphic to Verdier's quotient functor $\mathcal{T} \rightarrow \mathcal{T}/\mathcal{V}$. This forces $\mathcal{V} \simeq \mathcal{T}/\mathcal{U}$ and $\mathcal{U} \simeq \mathcal{T}/\mathcal{V} \simeq \mathcal{W}$.

2.1.3 Triangulated categories with coproducts

Throughout Subsection 2.1.3, we work with a fixed triangulated category \mathcal{T} which admits all coproducts.

Using countable coproducts, we can construct homotopy colimits of \mathbb{N} -directed diagrams, which turn out to be extremely useful for our further study of metric completions (Subsection 2.5.1). The definition can be found in [15, Definition 2.1]; here we use the formulation from Paper A - Subsection 2.1.3.

Definition 2.17. Let $\mathbf{E} = E_1 \xrightarrow{f_{1,2}} E_2 \xrightarrow{f_{2,3}} \cdots$ be an \mathbb{N} -directed diagram in \mathcal{T} . We define its *homotopy colimit*, denoted $\mathop{\mathrm{Hocolim}}\limits_{\rightarrow} \mathbf{E}$, as the cone in the distinguished triangle

$$\coprod_{n \in \mathbb{N}} E_n \xrightarrow{\varphi} \coprod_{n \in \mathbb{N}} E_n \longrightarrow \mathop{\mathrm{Hocolim}}\limits_{\rightarrow} \mathbf{E} \longrightarrow \Sigma \coprod_{n \in \mathbb{N}} E_n$$

where the first map is given by the infinite matrix

$$\varphi := \begin{bmatrix} \mathrm{id}_{E_1} & 0 & 0 & \cdots \\ -f_{1,2} & \mathrm{id}_{E_2} & 0 & \cdots \\ 0 & -f_{2,3} & \mathrm{id}_{E_3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Remark 2.18 ([47, Proposition 1.6.8]). As \mathcal{T} has countable coproducts, we can use homotopy colimits to prove that it is idempotent complete. If $e : X \rightarrow X$ is an idempotent map in \mathcal{T} , its image is isomorphic to the homotopy colimit of the \mathbb{N} -directed diagram $X \xrightarrow{e} X \xrightarrow{e} \cdots$ and therefore exists in \mathcal{T} .

In particular, the homotopy colimit of a diagram of $X \xrightarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} \cdots$ consisting of identities on X is again X , and the homotopy colimit of zero maps $X \xrightarrow{0} X \xrightarrow{0} \cdots$ is the zero object.

Definition 2.19 ([46, Definition 1.6, Definition 1.7]). An object $T \in \mathcal{T}$ is called *compact* if the functor $\mathrm{Hom}_{\mathcal{T}}(T, -)$ commutes with coproducts. The subcategory of all compact objects of \mathcal{T} is denoted \mathcal{T}^c .

The category \mathcal{T} is *compactly generated* if there exists a set $\mathcal{C} \subseteq \mathcal{T}^c$ such that $\mathcal{C}^\perp = 0$ in \mathcal{T} .

Remark 2.20. The category \mathcal{T}^c is a thick subcategory of \mathcal{T} .

As taking homomorphisms from compacts sends coproducts to coproducts, it also sends homotopy colimits to directed colimits, in the sense of the following lemma.

Lemma 2.21 ([46, Lemma 2.8]). Let $\mathbf{E} = E_1 \xrightarrow{f_{1,2}} E_2 \xrightarrow{f_{2,3}} \cdots$ be an \mathbb{N} -directed diagram in \mathcal{T} . Pick $T \in \mathcal{T}^c$. Then $\mathrm{Hom}_{\mathcal{T}}(T, \mathop{\mathrm{Hocolim}}\limits_{\rightarrow} \mathbf{E}) \simeq \varinjlim \mathrm{Hom}_{\mathcal{T}}(T, \mathbf{E})$.

The theory of localisation functors with adjoints has been extensively studied by Bousfield [12], [13] and Ravenel [65]. See also [27], [33], or [45]. We present the terminology adjusted to the setting of triangulated categories.

Definition 2.22 ([1, Definition 2.1]). A triangulated subcategory \mathcal{S} of \mathcal{T} is called *localising* if it is closed under taking coproducts. Given a class \mathcal{C} of objects of \mathcal{T} , *the localising subcategory generated by \mathcal{C}* , denoted $\text{Loc}(\mathcal{C})$, is the smallest localising subcategory of \mathcal{T} containing \mathcal{C} .

A localising subcategory \mathcal{S} of \mathcal{T} is called *smashing* if it is coreflective and \mathcal{S}^\perp is localising.

Proposition 2.23 ([58, Proposition 4.4.14], [1, Proposition 2.9]). Assume \mathcal{T} is compactly generated. Let $\mathcal{S} \subseteq \mathcal{T}$ be a smashing subcategory. Then $(\mathcal{S}, \mathcal{S}^\perp, (\mathcal{S}^\perp)^\perp)$ is a TTF-triple. In other words, there is a recollement diagram

$$\begin{array}{ccccc} & & i^* & & \\ & \curvearrowright & & \curvearrowleft & \\ \mathcal{S}^\perp & \xrightarrow{i_*} & \mathcal{T} & \xleftarrow{j^*} & \mathcal{S} \\ & \curvearrowleft & & \curvearrowright & \\ & & i^! & & \\ & & j_! & & \end{array}$$

The following fact has been proved in [45, Proposition]. See also [1, Corollary 2.8], [18, Proposition 3.8], or [52, Theorem 2.3].

Proposition 2.24. Any localising subcategory of \mathcal{T} generated by a set of compact objects is smashing.

Whether every smashing subcategory is necessarily generated by a set of compact objects used to be an open problem known as *the telescope conjecture*, but nowadays the conjecture is disproved; see [31] and [39, Example 7.8] for counterexamples. Nevertheless, there also exist positive cases [44] and [39, Theorem A], which we briefly discuss in Section 3 of Paper B, and the telescope conjecture remains open for many specific triangulated categories.

2.2 Derived categories

In this subsection, we give a brief overview of two prototypical examples of triangulated categories, the homotopy category and the derived category. We follow the exposition [38, Chapter 4].

We fix an abelian category \mathcal{A} and consider its category of cochain complexes $\text{Ch}(\mathcal{A})$. Its objects are \mathbb{Z} -directed diagrams

$$X^\bullet = (X^n, d^n)_{n \in \mathbb{Z}} = \cdots \rightarrow X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \rightarrow \cdots$$

satisfying $d^{n+1} \circ d^n = 0$ for each $n \in \mathbb{Z}$, and its morphisms are the usual morphisms of diagrams. The category $\text{Ch}(\mathcal{A})$ is again abelian and contains \mathcal{A} as a full subcategory of complexes concentrated at degree 0. A cochain complex $X^\bullet = (X^n, d^n)_{n \in \mathbb{Z}}$ is *bounded* if $X^n \simeq 0$ for all but finitely many $n \in \mathbb{Z}$.

Given a cochain complex $X^\bullet \in \text{Ch}(\mathcal{A})$ and $n \in \mathbb{Z}$, we define its *n-th cohomology* $H^n(X^\bullet)$ as the object $\text{Ker } d^n / \text{Im } d^{n-1} \in \mathcal{A}$. This yields a functor $H^n : \text{Ch}(\mathcal{A}) \rightarrow \mathcal{A}$.

A complex $X^\bullet \in \text{Ch}(\mathcal{A})$ is *acyclic* if $H^n(X^\bullet) \simeq 0$ for all $n \in \mathbb{Z}$; and a morphism $f^\bullet : X^\bullet \rightarrow Y^\bullet$ is a *quasi-isomorphism* if $H^n(f^\bullet)$ is an isomorphism for all $n \in \mathbb{Z}$.

We also define an autoequivalence $\Sigma : \text{Ch}(\mathcal{A}) \rightarrow \text{Ch}(\mathcal{A})$ by

$$X^\bullet = (X^n, d^n)_{n \in \mathbb{Z}} \quad \mapsto \quad \Sigma X^\bullet = (X^{n+1}, -d^{n+1})_{n \in \mathbb{Z}}.$$

Definition 2.25. A morphism $f^\bullet : X^\bullet \rightarrow Y^\bullet$ in $\text{Ch}(\mathcal{A})$ is *null-homotopic* if there exists a family $(h^n : X^n \rightarrow Y^{n-1})_{n \in \mathbb{Z}}$ in \mathcal{A} , depicted as

$$\begin{array}{ccccccccc} \dots & \longrightarrow & X^{-1} & \xrightarrow{d_X^{-1}} & X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & X^2 & \longrightarrow & \dots \\ & & \downarrow f^{-1} & \swarrow h^0 & \downarrow f^0 & \swarrow h^1 & \downarrow f^1 & \swarrow h^2 & \downarrow f^2 & & \\ \dots & \longrightarrow & Y^{-1} & \xrightarrow{d_Y^{-1}} & Y^0 & \xrightarrow{d_Y^0} & Y^1 & \xrightarrow{d_Y^1} & Y^2 & \longrightarrow & \dots \end{array},$$

such that $f^n = d_Y^{n-1} \circ h_n + h^{n+1} \circ d_X^n$ for all $n \in \mathbb{Z}$.

The collection of all null-homotopic maps forms an ideal in the category $\text{Ch}(\mathcal{A})$.

Definition 2.26. The *homotopy category* $\mathbf{K}(\mathcal{A})$ of \mathcal{A} is the quotient category of $\text{Ch}(\mathcal{A})$ by the ideal \mathfrak{J} of the null-homotopic maps, i.e. $\text{Hom}_{\mathbf{K}(\mathcal{A})}(X, Y) := \text{Hom}_{\text{Ch}(\mathcal{A})}(X, Y) / \mathfrak{J}(X, Y)$ for all $X, Y \in \text{Ch}(\mathcal{A})$.

Theorem 2.27 ([38, Lemma 4.1.1, Proposition 3.3.2]). The category $\mathbf{K}(\mathcal{A})$ is triangulated with the shift functor induced by the shift Σ on $\text{Ch}(\mathcal{A})$.

The cone of a morphism $f^\bullet : X^\bullet \rightarrow Y^\bullet$ in the triangulated category $\mathbf{K}(\mathcal{A})$ has an explicit description as *the mapping cone*

$$\text{cone } f^\bullet = \dots \rightarrow X^0 \oplus Y^{-1} \xrightarrow{\begin{bmatrix} -d_X^0 & 0 \\ f^0 & d_Y^{-1} \end{bmatrix}} X^1 \oplus Y^0 \xrightarrow{\begin{bmatrix} -d_X^1 & 0 \\ f^1 & d_Y^0 \end{bmatrix}} X^2 \oplus Y^1 \rightarrow \dots,$$

which fits into a distinguished triangle

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{\nu^\bullet} \text{cone } f^\bullet \xrightarrow{\pi^\bullet} \Sigma X^\bullet$$

with $\nu^n : Y^n \rightarrow X^{n+1} \oplus Y^n$ and $\pi^n : X^{n+1} \oplus Y^n \rightarrow X^{n+1}$ being the canonical inclusion and the canonical projection, respectively, for each $n \in \mathbb{Z}$.

Definition 2.28. The *derived category* $\mathbf{D}(\mathcal{A})$ of \mathcal{A} is Verdier's quotient of $\mathbf{K}(\mathcal{A})$ by the thick subcategory generated by acyclic complexes.

The derived category $\mathbf{D}(\mathcal{A})$ possess *the standard t-structure* $(\mathbf{D}(R)^{\leq 0}, \mathbf{D}(R)^{> 0})$ with

$$\begin{aligned} \mathbf{D}(R)^{\leq 0} &:= \{X \in \mathbf{D}(R) : \forall n \in \mathbb{N}, H^n(X) = 0\} \\ \text{and } \mathbf{D}(R)^{> 0} &:= \{X \in \mathbf{D}(R) : \forall n \in \mathbb{N}_0, H^{-n}(X) = 0\}. \end{aligned}$$

The composition $\mathcal{A} \hookrightarrow \text{Ch}(\mathcal{A}) \rightarrow \mathbf{K}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{A})$ yields a fully faithful canonical functor $\mathcal{A} \hookrightarrow \mathbf{D}(\mathcal{A})$ [38, Lemma 4.1.11], and all quasi-isomorphism in $\text{Ch}(\mathcal{A})$ are sent to isomorphisms in $\mathbf{D}(\mathcal{A})$. If

$$0 \rightarrow X \xrightarrow{f} Y \rightarrow Z \rightarrow 0$$

is a short exact sequence in \mathcal{A} , then Z is quasi-isomorphic to cone f as constructed above. Hence, the short exact sequence becomes a distinguished triangle

$$X \xrightarrow{f} Y \rightarrow Z \rightarrow \Sigma X$$

in $\mathbf{D}(\mathcal{A})$ [38, Lemma 4.1.12].

The derived category $\mathbf{D}(\mathcal{A})$ has a thick subcategory $\mathbf{D}^b(\mathcal{A})$, called *the bounded derived category* of \mathcal{A} , consisting of complexes of *bounded cohomology*, i.e. complexes X^\bullet such that the complex $\coprod_{n \in \mathbb{Z}} \Sigma^{-n} H^n(X^\bullet) \in \text{Ch}(\mathcal{A})$ is bounded. In the whole text, we will always implicitly consider \mathcal{A} as a subcategory of $\mathbf{D}^b(\mathcal{A}) \subseteq \mathbf{D}(\mathcal{A})$.

2.2.1 Grothendieck categories

For the purposes of the following definition, we recall that a *noetherian object* of a category satisfies, by definition, the ascending chain condition on subobjects.

Definition 2.29. A *Grothendieck category* \mathcal{G} is an abelian category with a generator and exact directed colimits. If every object of \mathcal{G} is a directed colimit of its noetherian subobjects, then \mathcal{G} is considered to be *locally noetherian*.

Remark 2.30. As mentioned in Remark 2.12, Verdier's quotient does not have to be a locally small category, and this, in particular, applies to derived categories. Nevertheless, the derived category $\mathbf{D}(\mathcal{G})$ of a Grothendieck category is always locally small [38, Proposition 4.3.8]. All derived categories considered in this thesis will be full subcategories of a derived category of a Grothendieck category and thus avoid the set-theoretical issue.

The global dimension of an abelian category \mathcal{A} is the smallest $n \in \mathbb{N}_0$ such that $\text{Ext}_{\mathcal{A}}^{n+1}(-, -) \simeq 0$ in \mathcal{A} (in the sense of Yoneda Ext [71, Vista 3.4.6]), or ∞ if such a number does not exist. Abelian categories of global dimension 0 or 1 are called *hereditary*.

Proposition 2.31 ([38, Proposition 4.2.11]). Let \mathcal{G} be a Grothendieck category. Then for all $X, Y \in \mathcal{G}$ and $n \in \mathbb{N}_0$ there is an isomorphism $\text{Ext}_{\mathcal{G}}^n(X, Y) \simeq \text{Hom}_{\mathbf{D}(\mathcal{G})}(X, \Sigma^n Y)$ which is natural in both variables and compatible with the Yoneda Ext composition.

By [38, Example 4.3.12], the derived category $\mathbf{D}(\mathcal{G})$ of a Grothendieck category has products and coproducts.

Consider an \mathbb{N} -directed diagram $\mathbf{E} = E_1 \xrightarrow{f_{1,2}} E_2 \xrightarrow{f_{2,3}} \dots$ in a Grothendieck category \mathcal{G} (or potentially in $\text{Ch}(\mathcal{G})$). Its directed colimit in \mathcal{G} (or $\text{Ch}(\mathcal{G})$) has a canonical presentation (cf. [24, Lemma 2.12]) as a short exact sequence

$$0 \rightarrow \coprod_{n \in \mathbb{N}} E_n \rightarrow \coprod_{n \in \mathbb{N}} E_n \rightarrow \varinjlim \mathbf{E} \rightarrow 0,$$

which gets sent by the canonical functor $\mathcal{G} \hookrightarrow \mathbf{D}(\mathcal{G})$ to the distinguished triangle

$$\coprod_{n \in \mathbb{N}} E_n \xrightarrow{\varphi} \coprod_{n \in \mathbb{N}} E_n \longrightarrow \mathbf{Hocolim} \mathbf{E} \longrightarrow \Sigma \coprod_{n \in \mathbb{N}} E_n$$

defining the homotopy colimit in $\mathbf{D}(\mathcal{G})$. Of course, while $\mathbf{Hocolim} \mathbf{E}$ is isomorphic to $\mathbf{lim} \mathbf{E}$ as an object of $\mathbf{D}(\mathcal{G})$, it is no longer a colimit of the diagram. This allows us to compute homotopy colimits of \mathbb{N} -directed diagrams in $\mathbf{D}(\mathcal{G})$ as directed colimits of cochain complexes.

Proposition 2.32 ([34],[72, Proposition 2.5]). Let \mathcal{G} be a locally noetherian Grothendieck category of finite global dimension. Then $\mathbf{D}(\mathcal{G})$ is compactly generated, and the compact objects of $\mathbf{D}(\mathcal{G})$ are precisely objects isomorphic to a bounded complex of noetherian objects from \mathcal{G} .

In case of a locally noetherian Grothendieck category \mathcal{G} of finite global dimension, the standard t -structure $(\mathbf{D}(\mathcal{G})^{\leq 0}, \mathbf{D}(\mathcal{G})^{> 0})$ on $\mathbf{D}(\mathcal{G})$ then restricts to a t -structure

$$(\mathbf{D}(\mathcal{G})^{\leq 0} \cap \mathbf{D}(\mathcal{G})^c, \mathbf{D}(\mathcal{G})^{> 0} \cap \mathbf{D}(\mathcal{G})^c)$$

on $\mathbf{D}(\mathcal{G})^c$.

Besides the category of quasi-coherent sheaves $\mathbf{Qcoh}\text{-}\mathbb{X}$ over a weighted projective line \mathbb{X} central to [Project C](#), the Grothendieck categories mainly considered in this thesis are module categories $\mathbf{Mod}\text{-}R$ of a ring R . A derived category of a ring is always compactly generated. We shall use $\mathbf{D}(R)$ as an abbreviation for $\mathbf{D}(\mathbf{Mod}\text{-}R)$.

Proposition 2.33 ([15, Proposition 6.4],[38, Lemma 9.2.1]). Let R be a ring. Then $\mathbf{D}(R)$ is compactly generated and $\mathbf{D}(R)^c = \mathbf{thick}(R)$. The compact objects of $\mathbf{D}(R)$ are precisely *the perfect complexes*, i.e. complexes quasi-isomorphic to a bounded complex of finitely generated projective R -modules.

Corollary 2.34. Let R be a right noetherian ring of finite global dimension. Then $\mathbf{Mod}\text{-}R$ is locally noetherian, and $\mathbf{D}(R)^c \simeq \mathbf{D}^b(\mathbf{mod}\text{-}R)$.

Proof. By [24, Lemma 2.5], every R -module is a directed colimit of finitely presented modules, and those are precisely the noetherian objects of $\mathbf{Mod}\text{-}R$. This makes $\mathbf{Mod}\text{-}R$ into a locally noetherian Grothendieck category. The rest follows from [Proposition 2.32](#). \square

2.2.2 Hereditary categories

The structure of derived categories of hereditary abelian categories can be fully understood using the original structure of \mathcal{A} .

Proposition 2.35 ([38, Proposition 4.4.15]). Let \mathcal{A} be an abelian category. Then \mathcal{A} is hereditary if and only if every complex $X^\bullet \in \mathbf{Ch}(\mathcal{A})$ is quasi-isomorphic to its cohomology, or, more precisely, if

$$\coprod_{n \in \mathbb{Z}} \Sigma^{-n} H^n(X^\bullet) \simeq X^\bullet \simeq \coprod_{n \in \mathbb{Z}} \Sigma^{-n} H^n(X^\bullet)$$

in $\mathbf{D}(\mathcal{A})$.

Definition 2.36 ([28, Section 1],[14, Lemma 4.2]). A subcategory \mathcal{W} of an abelian category \mathcal{A} is called *wide* if it is closed under extensions, kernels, and cokernels.

The smallest wide subcategory of \mathcal{A} containing a class $\mathcal{B} \subseteq \mathcal{A}$ is denoted $\text{wide}(\mathcal{B})$.

For hereditary categories, wide subcategories shed light on thick subcategories of their bounded derived categories.

Theorem 2.37 ([14, Theorem 5.1]). Let \mathcal{A} be a hereditary abelian category. Then the assignments

$$\mathcal{S} \mapsto H^0(\mathcal{S}) \quad \text{and} \quad \text{thick}(\mathcal{W}) \leftrightarrow \mathcal{W}$$

form mutually inverse bijections

$$\{\text{thick subcategories of } \mathbf{D}^b(\mathcal{A})\} \leftrightarrow \{\text{wide subcategories of } \mathcal{A}\}.$$

Due to this correspondence, some authors call wide subcategories “thick”; however, in this thesis, we keep the term “thick” for triangulated subcategories only.

2.2.3 Derived functors

We give a very brief commentary on derived functors and refer to [38, Section 4.3] for details.

Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an exact functor between abelian categories. Then F can be extended to $F : \text{Ch}(\mathcal{A}) \rightarrow \text{Ch}(\mathcal{B})$. Assume that Verdier’s quotient functor $\mathbf{K}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{A})$ has a right adjoint \mathbf{i} . For each $X \in \mathbf{D}(\mathcal{A})$ we consider the morphism $X \rightarrow \mathbf{i}X$ given by the unit of the adjunction. Then the assignment $\mathbf{R}F : \mathbf{D}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{B})$, $X \mapsto F(\mathbf{i}X)$, is a well-defined triangulated functor called *the right derived functor* of F ([38, Proposition 4.3.11]).

Conversely, assume $\mathbf{K}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{A})$ has a left adjoint \mathbf{p} , and consider the morphism $\mathbf{p}X \rightarrow X$ given by the counit of the adjunction for every $X \in \mathbf{D}(\mathcal{A})$. Then the assignment $\mathbf{L}F : \mathbf{D}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{B})$, $X \mapsto F(\mathbf{p}X)$, is a well-defined triangulated functor called *the left derived functor* of F .

The only instances of derived functors used in this thesis will be the derived functors of Hom and the tensor product in the module category.

Example 2.38 ([38, Example 4.3.16]). Let R and S be rings and M be an R - S -bimodule. Then the right derived functor

$$\mathbf{R} \text{Hom}_S(M, -) : \mathbf{D}(S) \rightarrow \mathbf{D}(R) \quad \text{of} \quad \text{Hom}_S(M, -) : \text{Mod-}S \rightarrow \text{Mod-}R$$

is the right adjoint to the left derived functor

$$- \otimes_R^{\mathbf{L}} M : \mathbf{D}(R) \rightarrow \mathbf{D}(S) \quad \text{of} \quad - \otimes_R M : \text{Mod-}R \rightarrow \text{Mod-}S.$$

Furthermore, for every $L \in \text{Mod-}R$, $N \in \text{Mod-}S$, and $n \in \mathbb{N}_0$ we have

$$\text{Ext}_S^n(M, N) \simeq H^n(\mathbf{R} \text{Hom}_S(M, N)) \quad \text{and} \quad \text{Tor}_n^R(L, M) \simeq H^n\left(L \otimes_R^{\mathbf{L}} M\right).$$

2.3 Hom-finite categories

In this subsection, we fix a commutative ring \mathbb{k} and a field K . We denote the standard K -duality $\mathrm{Hom}_K(-, K) : \mathrm{mod}\text{-}K \rightarrow K\text{-mod}$ with the star symbol $(-)^*$.

An additive category \mathcal{A} is \mathbb{k} -linear if $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is a \mathbb{k} -module for all $X, Y \in \mathcal{A}$ and the composition of morphisms in \mathcal{A} is \mathbb{k} -bilinear. An additive functor $\mathcal{A} \rightarrow \mathcal{B}$ is \mathbb{k} -linear if the corresponding map $\mathrm{Hom}_{\mathcal{A}}(X, Y) \rightarrow \mathrm{Hom}_{\mathcal{B}}(FX, FY)$ is \mathbb{k} -linear. We will always implicitly assume that functors between \mathbb{k} -linear categories are \mathbb{k} -linear.

A \mathbb{k} -linear category \mathcal{A} is said to be Hom-finite if $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is a finite length \mathbb{k} -module for all $X, Y \in \mathcal{A}$; and more generally, the category \mathcal{A} is Ext-finite if $\mathrm{Ext}_{\mathcal{A}}^n(X, Y)$ is a finite length \mathbb{k} -module for all $X, Y \in \mathcal{A}$ and $n_0 \in \mathbb{N}$. If a \mathbb{k} -linear abelian category \mathcal{A} is Ext-finite, then $\mathbf{D}^b(\mathcal{A})$ is Hom-finite.

A \mathbb{k} -algebra is a ring R together with a ring homomorphism from \mathbb{k} to the center of R giving R a structure of a \mathbb{k} -module. A finitely generated \mathbb{k} -algebra is a \mathbb{k} -algebra which is finitely generated as a \mathbb{k} -module. The categories $\mathrm{Mod}\text{-}R$ and $\mathbf{D}(R)$ are \mathbb{k} -linear whenever R is a \mathbb{k} -algebra.

2.3.1 Tilting

Definition 2.39 ([19, Section 3.1],[26]). Let \mathcal{A} be an abelian category. We say that $T \in \mathcal{A}$ is a tilting object if:

- $\mathrm{Ext}_{\mathcal{A}}^n(T, -) \simeq 0$ for all $n \geq 2$,
- $\mathrm{Ext}_{\mathcal{A}}^1(T, T) \simeq 0$,
- and for every $0 \neq X \in \mathcal{A}$ we have $\mathrm{Hom}_{\mathcal{A}}(T, X) \neq 0$ or $\mathrm{Ext}_{\mathcal{A}}^1(T, X) \neq 0$.

Theorem 2.40 ([25, Theorem III.2.10],[19, Theorem 3.2.5]). Let \mathcal{A} be a K -linear, Ext-finite abelian category. If $T \in \mathcal{A}$ is a tilting object, then the functor

$$\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(T, -) : \mathbf{D}^b(\mathcal{A}) \rightarrow \mathbf{D}^b(\mathrm{mod}\text{-}\mathrm{End}_{\mathcal{A}}(T))$$

is a triangulated equivalence.

2.3.2 Krull-Schmidt categories

Definition 2.41. We say that an additive category \mathcal{A} is a Krull-Schmidt category if every non-zero object of \mathcal{A} can be decomposed as a finite coproduct of indecomposable objects with local endomorphism rings.

Remark 2.42. By [68, Theorem 6.1], a Hom-finite \mathbb{k} -linear category is Krull-Schmidt if and only if every indecomposable object has a local endomorphism ring. So in particular, the category $\mathrm{mod}\text{-}R$ is Krull-Schmidt if R is a finite dimensional K -algebra.

Theorem 2.43 ([36, Corollary 4.3]). Let \mathcal{A} be a Krull-Schmidt category. Then any non-zero object of \mathcal{A} has a unique decomposition into a finite coproduct of indecomposable objects up to isomorphisms and permutations of the summands.

2.3.3 Exceptional sequences

Definition 2.44 ([60, Definition 3.12]). A K -linear triangulated category \mathcal{T} is *proper* if $\coprod_{n \in \mathbb{Z}} \text{Hom}_{\mathcal{T}}(X, \Sigma^n Y)$ is a finite dimensional vector space for all $X, Y \in \mathcal{T}$.

Definition 2.45 ([10, Section 2]). Let \mathcal{T} be a K -linear triangulated category. An object $E \in \mathcal{T}$ is *exceptional* if $\text{End}_{\mathcal{T}}(E) \simeq K$ and $\text{Hom}_{\mathcal{T}}(E, \Sigma^n E) \simeq 0$ for all $n \in \mathbb{Z} \setminus \{0\}$.

A tuple (E_1, \dots, E_n) of exceptional objects $E_1, \dots, E_n \in \mathcal{T}$ for some $n \in \mathbb{N}_0$ is said to be an *exceptional sequence* if $\text{Hom}_{\mathcal{T}}(X_j, \Sigma^m X_i) \simeq 0$ for all $m \in \mathbb{Z}$ and $1 \leq i < j \leq n$.

Definition 2.46. Let \mathcal{A} be an essentially small abelian category. We define *the Grothendieck group* $K_0(\mathcal{A})$ as the quotient of the free abelian group on isomorphism classes $[A]$ of objects $A \in \mathcal{A}$ by the relations $[Y] = [X] + [Z]$ for each short exact sequence

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

in \mathcal{A} .

Similarly, let \mathcal{T} be an essentially small triangulated category. We define *the Grothendieck group* $K_0(\mathcal{T})$ as the quotient of the free abelian group on isomorphism classes $[T]$ of objects $T \in \mathcal{T}$ by the relations $[Y] = [X] + [Z]$ for each distinguished triangle

$$X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$$

in \mathcal{T} .

If $(\mathcal{U}, \mathcal{V})$ is a torsion pair consisting of triangulated subcategories of an essentially small triangulated category \mathcal{T} , then the adjoint functors to the inclusions $\mathcal{U} \hookrightarrow \mathcal{T}$ and $\mathcal{V} \hookrightarrow \mathcal{T}$ induce a splitting $K_0(\mathcal{T}) \simeq K_0(\mathcal{U}) \oplus K_0(\mathcal{V})$.

Lemma 2.47 ([38, Lemma 4.1.18]). Let \mathcal{A} be an essentially small abelian category. Then the map

$$K_0(\mathbf{D}^b(\mathcal{A})) \rightarrow K_0(\mathcal{A}), \quad [X^\bullet] \mapsto \sum_{n \in \mathbb{Z}} (-1)^n [H^n(X^\bullet)]$$

is a group isomorphism.

Theorem 2.48 ([10, Theorem 3.2.a]). Let \mathcal{T} be a proper K -linear triangulated category over a field K . Let (E_1, \dots, E_n) be an exceptional sequence in \mathcal{T} for some $n \in \mathbb{N}_0$. Then $\langle E_1, \dots, E_n \rangle \subseteq \mathcal{T}$ is bireflective.

Corollary 2.49. Let \mathcal{T} be a proper K -linear triangulated category. Let (E_1, \dots, E_n) be an exceptional sequence in \mathcal{T} for some $n \in \mathbb{N}_0$. Denote $\mathcal{S} := \langle E_1, \dots, E_n \rangle \subseteq \mathcal{T}$. Then $K_0(\mathcal{S}) \simeq \mathbb{Z}^n$ is a direct summand of $K_0(\mathcal{T})$.

Proof. By Theorem 2.48, there is a torsion pair $(\langle E_n \rangle, \langle E_1, \dots, E_{n-1} \rangle)$ on \mathcal{S} . As

$$K_0(\langle E_n \rangle) \simeq K_0(\mathbf{D}^b(\text{mod-}K)) \simeq K_0(\text{mod-}K) \simeq \mathbb{Z}$$

is then a direct summand of $K_0(\mathcal{S})$, we get $K_0(\mathcal{S}) \simeq \mathbb{Z}^n$ by induction. But then again, Theorem 2.48 guarantees $(\mathcal{S}, \mathcal{S}^\perp)$ to be a torsion pair on \mathcal{T} . Thus $K_0(\mathcal{S})$ embeds into $K_0(\mathcal{T})$ as a direct summand. \square

Corollary 2.50. Let \mathcal{T} be a proper K -linear triangulated category. Let (E_1, \dots, E_m) and (F_1, \dots, F_n) be two exceptional sequences in \mathcal{T} for some $m, n \in \mathbb{N}_0$.

Denote $\mathcal{U} := \langle E_1, \dots, E_m \rangle \subseteq \mathcal{T}$ and $\mathcal{V} := \langle F_1, \dots, F_n \rangle \subseteq \mathcal{T}$. If $\mathcal{U} \subseteq \mathcal{V}$, then $m \leq n$.

Proof. The pair of thick subcategories $(\mathcal{U}, \mathcal{U}^\perp \cap \mathcal{V})$ of \mathcal{V} is a torsion pair on \mathcal{V} . It follows that $K_0(\mathcal{U}) \simeq \mathbb{Z}^m$ is a direct summand of $K_0(\mathcal{V}) \simeq \mathbb{Z}^n$. Hence $m \leq n$. \square

Proposition 2.51. Let \mathcal{T} be a proper K -linear triangulated category generated by an exceptional sequence. Then the following statements are equivalent for a thick subcategory $\mathcal{S} \subseteq \mathcal{T}$:

- \mathcal{S} is reflective.
- \mathcal{S} is coreflective.
- \mathcal{S} is bireflective.

Proof. This proposition has been implicitly proved in [11, Section 2] for $K = \mathbb{C}$ but the same argument works for any field. By [11, Corollary on pg. 530], the category \mathcal{T} is left and right saturated - see [11, Definition 2.5] for the definition. If \mathcal{S} is reflective, then it is right saturated by [11, Proposition 2.8], so [11, Proposition 2.6] implies that \mathcal{S} is coreflective. The proof of the remaining implication is dual. \square

Remark 2.52. The assertion of Proposition 2.51 holds even in some other, more general settings, for example if \mathcal{S} is idempotent complete and possesses a strong generator; see e.g. [60, Proposition 3.17].

2.3.4 Serre functor

Definition 2.53 ([11], [38, Section 6.3]). Let \mathcal{A} be a Hom-finite, K -linear category. A K -linear autoequivalence $\mathbb{S} : \mathcal{A} \rightarrow \mathcal{A}$ is called a *Serre functor* if there exists an isomorphism $\text{Hom}_{\mathcal{A}}(X, Y) \simeq \text{Hom}_{\mathcal{A}}(Y, \mathbb{S}(X))^*$ for all $X, Y \in \mathcal{A}$ natural in both variables.

Definition 2.54 ([19, Section 1.8]). Let \mathcal{A} be a Hom-finite, K -linear abelian category. A K -linear autoequivalence $\tau : \mathcal{A} \rightarrow \mathcal{A}$ is called an *Auslander-Reiten translation* if there exists an isomorphism $\text{Ext}_{\mathcal{A}}^1(X, Y)^* \simeq \text{Hom}_{\mathcal{A}}(Y, \tau X)$ for all $X, Y \in \mathcal{A}$ natural in both variables.

If \mathcal{A} has an Auslander-Reiten translation, it is said to *satisfy Serre duality*. Any such category \mathcal{A} is necessarily Ext-finite and hereditary because $\text{Ext}_{\mathcal{A}}^1(X, -) \simeq \text{Hom}_{\mathcal{A}}(-, \tau X)^*$ is right exact. It follows that $\Sigma\tau$ is a Serre functor on $\mathbf{D}^b(\mathcal{A})$.

Remark 2.55 ([38, Remark 6.3.2.2]). If a K -linear triangulated category possesses a Serre functor \mathbb{S} , then \mathbb{S} is automatically a triangulated functor.

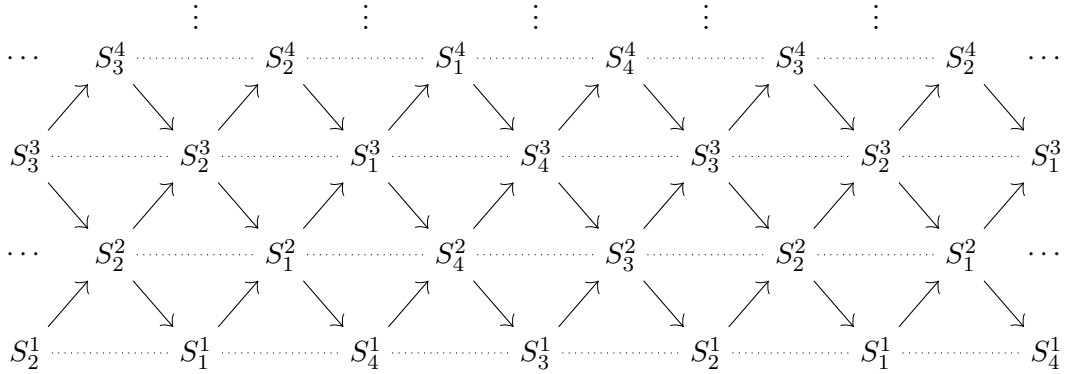
2.3.5 Tubes

We recall the definition and properties of stable tubes over a fixed field K .

Definition 2.56. Let \mathcal{H} be a Hom-finite, K -linear, length, abelian category satisfying Serre duality. Assume that the Auslander-Reiten translate $\tau : \mathcal{H} \rightarrow \mathcal{H}$ has finite order, and let $n \in \mathbb{N}$ be minimal such that $\tau^n \simeq \text{Id}_{\mathcal{H}}$. We say that \mathcal{H} is a *tube* of rank n if τ acts transitively on simple objects, i.e. for all simples $U, V \in \mathcal{H}$ there exists $m \in \mathbb{N}$ satisfying $\tau^m U = V$. Tube of rank 1 is *homogenous*.

By [19, Proposition 1.8.2], any Hom-finite, K -linear, length, abelian category \mathcal{A} satisfying Serre duality is uniserial and decomposes as a coproduct $\mathcal{A} = \coprod_{i \in I} \mathcal{A}_i$ of wide subcategories \mathcal{A}_i generated by τ -orbits of simple objects. Each such component fixed by some power of τ is a tube.

Example 2.57. Since every tube \mathcal{H} of rank $n \in \mathbb{N}$ is uniserial, every indecomposable object of \mathcal{H} is uniquely determined by its socle. Let S_1, \dots, S_n be an ordering of the simples of \mathcal{H} such that $\tau S_n \simeq S_1$ and $\tau S_i \simeq S_{i+1}$ for each $1 \leq i < n$. For each $1 \leq i \leq n$ and $j \in \mathbb{N}$ we denote S_i^j the unique indecomposable object of \mathcal{H} with socle S_i and length j . Then we can depict the category of indecomposable objects of \mathcal{H} using the Auslander-Reiten quiver (see [67, pg. 61], [3, pg. 287]) of \mathcal{H} as follows ($n = 4$ in the example):



where the dotted lines indicate the τ -orbits.

Proposition 2.58 ([23, Proposition 2.4.2]). A tube of rank $n \in \mathbb{N}$ has $\binom{2n}{n}$ wide subcategories.

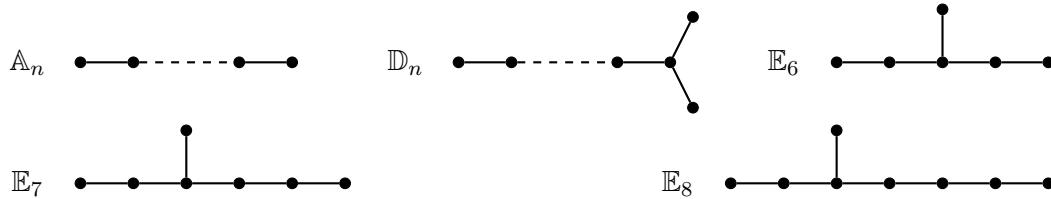
2.4 Finite dimensional algebras

Finite dimensional algebra A is by definition a finitely generated algebra over a field K . The abelian category $\text{Mod-}A$ is K -linear, while $\text{mod-}A$ is K -linear, Krull-Schmidt, and Ext-finite. Consequently, the bounded derived category $\mathbf{D}^b(\text{mod-}A)$ is K -linear, Krull-Schmidt, and Hom-finite. Moreover, the category $\mathbf{D}^b(\text{mod-}A)$ has a Serre functor if and only if A has finite global dimension (see e.g. [38, Theorem 6.4.13]).

If K is algebraically closed, then any finite dimensional K -algebra is Morita equivalent to the quotient ring KQ/I of a path algebra KQ of a finite quiver Q modulo an admissible

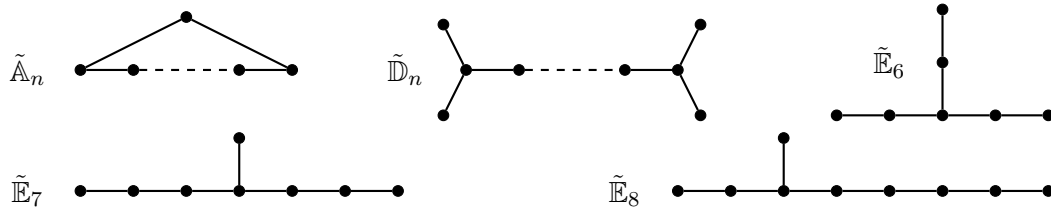
ideal $I \leq KQ$ (see [2, Theorem II.3.7]). And any hereditary finite dimensional algebra is Morita equivalent to KQ for a finite acyclic quiver Q (see [2, Theorem VII.1.7.b]).

Depending on the underlying diagram of the acyclic quiver Q , we can identify the representation type of the path algebra KQ . The algebra is *representation-finite*, meaning that there are only finitely many isomorphism classes of indecomposable finite dimensional modules, if and only if the underlying diagram is (simply laced) *Dynkin* (see e.g. [35, Corollary 5.3.3]); that is one of the diagrams:



where $n \in \mathbb{N}$ stands for the number of vertices.

For the underlying diagram *Euclidean* (also known as *extended Dynkin*)



the algebra KQ is of the *tame* type, and while $\text{mod-}KQ$ has infinitely many pairwise non-isomorphic indecomposable objects, they can still all be systematically classified (see e.g. [35, Theorem 5.3.1]). Here, for $n \in \mathbb{N}$ the corresponding diagram has $n + 1$. We elaborate more on the structure of indecomposables for a tame hereditary algebra in Section 2.6 of Paper B.

For other diagrams, the algebra is *wild* in the sense of Drozd's trichotomy (see e.g. [20, Corollary C]), and the indecomposables of $\text{mod-}A$ defy any reasonable classification.

2.5 Metrics and metric completions

Fix \mathcal{S} a triangulated category. In this final introductory subsection, we go through the definition of a metric on \mathcal{S} , the metric completion of \mathcal{S} and their most important properties. The concepts have been developed by Neeman in [49], while we mostly follow the notation from [21]. We also refer to expositions in the preliminary sections of Paper A and Paper B.

Definition 2.59. Consider a family $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$ of subcategories of \mathcal{S} . We say that \mathcal{M} is a *metric* on \mathcal{S} if for each $n \in \mathbb{N}$ it holds that:

- (i) $B_{n+1} \subseteq B_n$.
- (ii) and $B_n * B_n = B_n$.

Moreover, a metric \mathcal{M} is considered to be *good* if

- (iii) $\Sigma^{-1}B_{n+1} \cup B_{n+1} \cup \Sigma B_{n+1} \subseteq B_n$ for all $n \in \mathbb{N}$.

And a metric \mathcal{M} is considered to be *additive* if

- (iv) B_n is closed under direct summands in \mathcal{S} for all $n \in \mathbb{N}$.

Metrics on triangulated categories have been originally defined by Neeman in [49, Definition 1.2]. In practice, one works most often with good metrics introduced later in [50, Definition 10] since the additional condition iii) makes the subsequent construction of a completion less technical. The additive metrics appear in [21] without being named explicitly; the terminology of additive metrics comes from Paper B (Definition 2.8).

A metric $\{B_n\}_{n \in \mathbb{N}}$ on \mathcal{S} imposes on every morphism $f : X \rightarrow Y$ in \mathcal{S} a length in the spirit of Lawvere's norms on categories ([41]); the morphism f has length $\leq \frac{1}{n}$ if and only if $\text{cone } f \in B_n$ for some $n \in \mathbb{N}$.

Example 2.60. For any triangulated subcategory $\mathcal{C} \subseteq \mathcal{S}$, setting $B_n := \mathcal{C}$ for every $n \in \mathbb{N}$ defines a good metric $\{B_n\}_{n \in \mathbb{N}} = \{\mathcal{C}\}_{n \in \mathbb{N}}$ on \mathcal{S} . We call such a metric a *constant metric* (Subsection 2.2.2 of Paper B) and we shall not be making a formal distinction between a constant good metric and the corresponding triangulated category. A length of a morphism with respect to the constant metric \mathcal{C} is zero if and only if its cone lies in \mathcal{C} , otherwise the length can be interpreted as infinite.

The subcategory \mathcal{C} is thick if and only if the constant metric \mathcal{C} is additive.

A Cauchy sequence is a diagram of morphisms which are eventually becoming shorter. For the sake of this introduction, we shall discuss Cauchy sequences and completions with respect to good metrics only. For the general treatment see Paper A.

Definition 2.61 ([55, Definition 1.6]). Let $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$ be a good metric on \mathcal{S} . Consider an \mathbb{N} -directed diagram $\mathbf{E} = (E_n, e_n)_{n \in \mathbb{N}} = E_1 \xrightarrow{e_1} E_2 \xrightarrow{e_2} E_3 \rightarrow \cdots$ in \mathcal{S} . We denote the composition $e_{m,n} := e_n \circ e_{n-1} \cdots \circ e_{m+1} \circ e_m$ for any $m \leq n$. We say that \mathbf{E} is a *Cauchy sequence* with respect to \mathcal{M} if for each $M \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such for all $n \geq m \geq N$ we have $\text{cone } e_{m,n} \in B_M$.

Definition 2.62 ([21, Definition 3.4]). Let $\mathbf{E} = (E_n, e_n)$ be a Cauchy sequence in \mathcal{S} with respect to some good metric. We define *the module colimit* of \mathbf{E} as the functor $\underline{\text{Mocolim}} \mathbf{E} := \varinjlim \mathbf{E} \in \text{Mod-}\mathcal{T}$.

Definition 2.63. Let $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$ be a good metric on \mathcal{S} . We define the following subcategories of $\text{Mod-}\mathcal{S}$:

- $\mathfrak{L}_{\mathcal{M}}(\mathcal{S}) := \left\{ F \in \text{Mod-}\mathcal{S} : F \simeq \underline{\text{Mocolim}} \mathbf{E} \text{ for some Cauchy sequence } \mathbf{E} \text{ in } \mathcal{S} \right\}$
called *the pre-completion* of \mathcal{S} with respect to \mathcal{M} .
- $\mathfrak{C}_{\mathcal{M}}(\mathcal{S}) := \{ F \in \text{Mod-}\mathcal{S} : \exists n \in \mathbb{N}, F(B_n) \simeq 0 \}$ the category of *compactly supported* elements of $\text{Mod-}\mathcal{S}$ with respect to \mathcal{M} .
- $\mathfrak{S}_{\mathcal{M}}(\mathcal{S}) := \mathfrak{L}_{\mathcal{M}}(\mathcal{S}) \cap \mathfrak{C}_{\mathcal{M}}(\mathcal{S})$ called *the completion* of \mathcal{S} with respect to \mathcal{M} .

We will omit the index if the good metric is implicit from the context.

Theorem 2.64 ([55, Theorem 2.14]). For any good metric \mathcal{M} on \mathcal{S} , the completion $\mathfrak{S}_{\mathcal{M}}(\mathcal{S})$ is a triangulated category.

The triangulated structure on $\mathfrak{S}_{\mathcal{M}}(\mathcal{S})$ is induced by the original triangulated structure on \mathcal{S} . The shift of a functor $F \in \mathfrak{S}_{\mathcal{M}}(\mathcal{S})$ is given by $\Sigma F := F \circ \Sigma^{-1}$. And a candidate triangle in $\mathfrak{S}_{\mathcal{M}}(\mathcal{S})$ is proclaimed to be distinguished if it is a module colimit of a Cauchy sequence of distinguished triangles from \mathcal{S} .

Example 2.65. If we interpret the zero subcategory of \mathcal{S} as a constant metric on \mathcal{S} , we get $\mathfrak{S}_0(\mathcal{S}) \simeq \mathcal{S}$. This is because every Cauchy sequence $\mathbf{E} = (E_n, e_n)_{n \in \mathbb{N}}$ with respect to 0 satisfies that e_n is an isomorphism for all but finitely many $n \in \mathbb{N}$. Thus $\mathfrak{L}_0(\mathcal{S}) \simeq \mathfrak{J}(\mathcal{S})$. Every functor from $\text{Mod-}\mathcal{S}$ is compactly supported at 0 by the definition of an additive functor. We conclude $\mathfrak{S}_0(\mathcal{S}) = \mathfrak{L}_0(\mathcal{S}) \cap \mathfrak{C}_0(\mathcal{S}) \simeq \mathfrak{J}(\mathcal{S})$, so $\mathfrak{S}_0(\mathcal{S}) \simeq \mathcal{S}$ follows by Yoneda's lemma.

Example 2.66 ([55, Example 1.13]). On the other hand, if we take \mathcal{S} as a constant metric on itself, we get $\mathfrak{S}_{\mathcal{S}}(\mathcal{S}) \simeq 0$. The only functor $F \in \text{Mod-}\mathcal{S}$ vanishing on the whole of \mathcal{S} is the zero functor $F \simeq 0$. Hence, there are no non-zero compactly supported elements.

We also provide an example of a (generally) non-constant metric where the subsequent completion is fully understood.

Theorem 2.67 ([21, Theorem 1.1]). Let $(\mathcal{S}^{\leq 0}, \mathcal{S}^{> 0})$ be a t -structure on \mathcal{S} . Then the family $\{\mathcal{S}^{\leq -n}\}_{n \in \mathbb{N}}$ is a good metric on \mathcal{S} and $\mathfrak{S}(\mathcal{S}) \simeq \bigcup_{n \in \mathbb{N}} \mathcal{S}^{> -n}$.

Corollary 2.68. Let \mathcal{C} be a coreflective thick subcategory of \mathcal{S} . Then it holds that $\mathfrak{S}_{\mathcal{C}}(\mathcal{S}) \simeq \mathcal{C}^{\perp} \subseteq \mathcal{S}$.

Proof. As \mathcal{C} is coreflective, it is the aisle of the t -structure $(\mathcal{C}, \mathcal{C}^{\perp})$ on \mathcal{S} . But then $\mathfrak{S}_{\mathcal{C}}(\mathcal{S}) \simeq \mathcal{C}^{\perp}$ by Theorem 2.67 because, in this case, the coaisle \mathcal{C}^{\perp} is stable under Σ . \square

2.5.1 Good extensions

Assume that there exists a fully faithful triangulated functor $\nu : \mathcal{S} \hookrightarrow \mathcal{T}$. Then the functors from $\text{Mod-}\mathcal{T}$ act on \mathcal{S} by restriction, giving us a functor $\text{Mod-}\mathcal{T} \rightarrow \text{Mod-}\mathcal{S}$, $F \mapsto F \circ \nu$. If we limit our choice of functors from $\text{Mod-}\mathcal{T}$ to representable functors, we obtain *the restricted Yoneda functor* $\exists : \mathcal{T} \rightarrow \text{Mod-}\mathcal{S}$, $T \mapsto \text{Hom}_{\mathcal{T}}(\nu(-), T)$ (see [55, Notation 3.1] and Definition 2.12 of Paper B).

The notation of the restricted Yoneda functor will be used in the following definition.

Definition 2.69 ([49, Definition 3.5]). Let \mathcal{T} be a triangulated category with countable coproducts. Assume that \mathcal{S} is equipped with a good metric \mathcal{M} . A fully faithful triangulated functor $\nu : \mathcal{S} \rightarrow \mathcal{T}$ is called *a good extension* with respect to \mathcal{M} if $\text{Mocolim}_{\rightarrow} \mathbf{E} \simeq \exists(\text{Hocolim}_{\rightarrow} \nu(\mathbf{E}))$ holds for every Cauchy sequence $\mathbf{E} = (E_n, e_n)$ in \mathcal{S} .

Definition 2.70 ([55, Definition 3.11], Definition 2.14 of Paper B). Let \mathcal{M} be a good metric on \mathcal{S} and $\mathcal{S} \hookrightarrow \mathcal{T}$ a good extension with respect to \mathcal{M} . We define the following subcategories of \mathcal{T} :

- $\mathfrak{L}'_{\mathcal{M}}(\mathcal{S}) := \left\{ T \in \mathcal{T} : T \simeq \text{Hocolim}_{\rightarrow} \nu(\mathbf{E}) \text{ for some Cauchy sequence } \mathbf{E} \text{ in } \mathcal{S} \right\}$,
- $\mathfrak{C}'_{\mathcal{M}}(\mathcal{S}) := \exists^{-1}(\mathfrak{C}_{\mathcal{M}}(\mathcal{S}))$,

- $\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) := \mathfrak{L}'_{\mathcal{M}}(\mathcal{S}) \cap \mathfrak{E}'_{\mathcal{M}}(\mathcal{S})$.

The point of the whole construction of $\mathfrak{S}'_{\mathcal{M}}(\mathcal{S})$ is that it allows us to exhibit the metric completion $\mathfrak{S}_{\mathcal{M}}(\mathcal{S})$ as a triangulated subcategory of the good extension. Again, we will omit the lower index indicating the metric if it is not necessary.

Theorem 2.71 ([55, Theorem 3.23]). Assume \mathcal{S} is equipped with a good metric and let $\mathcal{S} \hookrightarrow \mathcal{T}$ be a good extension. Then $\mathfrak{S}'(\mathcal{S})$ is a triangulated subcategory of \mathcal{T} and $\exists \downarrow_{\mathfrak{S}'(\mathcal{S})}: \mathfrak{S}'(\mathcal{S}) \xrightarrow{\simeq} \mathfrak{S}(\mathcal{S})$ is a well-defined triangulated equivalence.

The following fact is a direct consequence of Lemma 2.21. As the compact elements respect homotopy colimits, they possess an inherent good extension working for any choice of a metric.

Theorem 2.72 ([55, Example 3.9]). Let \mathcal{T} be a triangulated category with countable coproducts. Then $\mathcal{T}^c \rightarrow \mathcal{T}$ is a good extension with respect to all good metrics on \mathcal{T}^c .

3 Mathematical context

The central topic of this thesis, metric completions of triangulated categories (Subsection 2.5), is a unique method to construct new triangulated categories. Using the idea behind convergence of sequences of elements of a metric space, one can construct a metric completion of a triangulated category by adding formal colimits of Cauchy sequences of objects and morphisms to the triangulated category in such a way that the resulting category is again triangulated.

Despite being discovered relatively recently, metric completions have found many applications in homological algebra, representation theory, and algebraic geometry during the last years. In this section, we give a brief summary of the history and the most state-of-art results regarding the topic.

Metrics and metric completions of triangulated categories have been initially introduced in 2018 by Neeman in [49] with the purpose to provide an alternative proof to following Rickard's derived Morita theory result:

Theorem 3.1 ([66, Theorem 6.4, Corollary 8.3]). Let R and S be coherent rings. Then $\mathbf{D}(R)^c \simeq \mathbf{D}^c(S)$ if and only if $\mathbf{D}^b(\text{mod-}R) \simeq \mathbf{D}^b(\text{mod-}S)$.

Neeman's argument from [49] is to consider a specific metric \mathcal{M} on $\mathbf{D}(R)^c$ such that $\mathfrak{S}_{\mathcal{M}}(\mathcal{S}) \simeq \mathbf{D}^b(\text{mod-}R)$. Similarly, one can find a metric \mathcal{N} on $\mathbf{D}^b(\text{mod-}R)^{\text{op}}$ satisfying $\mathfrak{S}_{\mathcal{N}}\left(\mathbf{D}^b(\text{mod-}R)^{\text{op}}\right)^{\text{op}} \simeq \mathbf{D}(R)^c$. Since both of the metrics in question are intrinsic to the given categories and do not depend on the ring itself, we can see that $\mathbf{D}(R)^c$ uniquely determines $\mathbf{D}^b(\text{mod-}R)$ and vice versa, thus proving Rickard's theorem.

The same metrics-based strategy works even for a larger class of triangulated categories such as noetherian, weakly approximable categories (see [17],[49],[53]) and thus offers a generalisation of derived Morita theory to e.g. a derived category of a separated noetherian scheme. This has also led to a subsequent discovery of a notion of an excellent metric [56]; that is a metric \mathcal{M} on a triangulated category \mathcal{S} satisfying the necessary properties to induce another metric on $\mathfrak{S}_{\mathcal{M}}(\mathcal{S})^{\text{op}}$ such that the process of taking the completion twice yields again the desired derived Morita equivalence in the spirit of Theorem 3.1.

What makes the technique of metric completions such a groundbreaking result, however, are not necessarily only its applications to derived Morita theory, but also the fact that it allows us to construct a new triangulated category from the data of an old one. All we need to do is to simply pick a metric on the triangulated category and the corresponding completion is guaranteed to be a triangulated category. This is something almost unheard of.

Besides the standard constructions of a triangulated subcategory and Verdier's quotient, there are not that many methods to reproduce a triangulated category. For example, the construction of a relative stable category ([5],[8]) delivers a new triangulated category. Under certain technical conditions, one can also equip an orbit category of triangulated category with with a triangulated structure [32]. Some of the rare such methods rely on rather strong additional assumptions such as the existence of a separable ring object and a tensor-triangulated structure in [4], or a suitable enhancement.

On the other hand, the condition for a family of subcategories of a triangulated category to be a good metric (Definition 2.59) is relatively easy to satisfy. For example, any triangulated subcategory automatically determines the corresponding constant metric, and any t -structure provides us with three metrics - the aisle metric, the coaisle metric, and the t -structure metric (see [21] and Definition 5.7 of Paper B). One is then, in principle, capable of constructing many different completions of a given triangulated category with respect to different choices of a metric.

The method of metric completions has already spawned an array of applications in various areas of research, e.g. [9],[16],[21],[22],[51],[54],[57],[63],[64],[69]. We point out a few examples.

The topological nature of convergence of Cauchy sequences was exploited by Sun and Zhang in [69] to study functoriality of the completion construction. They define a notion of a compression functor - it is a triangulated functor with continuous behaviour with respect to good metrics on the source and the target categories.

Theorem 3.2 ([69, Theorem 4.3.3], Theorem 1.2 of Paper A). Let \mathcal{S} and \mathcal{T} be triangulated categories equipped with metrics and $F : \mathcal{T} \rightleftarrows \mathcal{S} : G$ adjoint triangulated functors. If F and G are compressions, then the precomposition $- \circ F : \text{Mod-}\mathcal{S} \rightarrow \text{Mod-}\mathcal{T}$ induces a triangulated functor $- \circ F : \mathfrak{S}(\mathcal{S}) \rightarrow \mathfrak{S}(\mathcal{T})$.

Theorem 3.2 is then used in [69] to construct half-recollements of completions. We provide more details on the theory of compression functors in Section 4 of Paper A.

The question of lifting t -structures along completions have been addressed in [9]. In [9, Theorem 1.6], the authors identify a condition for a t -structure on a triangulated category to be extendable with respect to a good metric - an extendable t -structure then naturally induces a t -structure on the completion. Moreover, the results of [9] establish a necessary condition for the existence of a bounded t -structure on a triangulated category under a finiteness assumption on the relative finitistic dimension of the category.

Theorem 3.3 ([9, Theorem 1.5.a]). Let \mathcal{S} be an essentially small triangulated category. Assume \mathcal{S}^{op} has finite relative finitistic dimension with respect to an object of \mathcal{S} . Then there exists a specific good additive metric on \mathcal{S} such that $\mathfrak{S}(\mathcal{S}) = \mathcal{S}$ whenever \mathcal{S} has a bounded t -structure.

Consequently, if $\mathfrak{S}(\mathcal{S}) \neq \mathcal{S}$ under the conditions above, then \mathcal{S} cannot have a bounded t -structure at all. We comment a little bit more on the definitions of the metric in question and the relative finitistic dimension in Section 3 of Paper A.

Theorem 2.67 by Cummings and Gratz about completions with respect to good additive metrics determined by aisles of t -structures is a part of a much larger study of metric completions via t -structure approximations in [21]. For the Igusa-Todorov discrete cluster categories of type \mathbb{A} (see [29]), [21, Theorem 1.2] provides an explicit combinatorial description of all metric completions of the category taken with respect to good additive metrics determined by coaisles of t -structures. On top of that, [21, Corollary 6.24] relates metric completions of discrete cluster categories to the relevant Paquette-Yildirim combinatorial completions (see [61]).

We finish this section by mentioning that there also exist other approaches to studying triangulated categories by investigating converging sequences of objects, dissimilar to Neeman's Cauchy sequences with respect to a good metric. Among those are the sequential completion of a triangulated category [37] or the reconstruction of the dualisable objects in certain tensor-triangulated categories [42].

4 Overview of the main results

The overarching goal of this thesis is to deepen our understanding of metric completions as a method of creating new triangulated categories and to study metrics on triangulated categories as independent structures.

A prominent feature of the construction of a metric completion is how general it is. A triangulated category typically possess many different metrics, each of them spawning a corresponding completion. Nevertheless, the list of examples of metric and completions in the existing literature is rather limited. Most of the contemporary results employ only those metrics on a triangulated category \mathcal{S} determined by either aisles, or coaisles of t -structures on \mathcal{S} or t -structures on a good extension \mathcal{T} of \mathcal{S} . And besides the Cummings-Gratz combinatorial description of completions of discrete cluster categories [21], no other systematic classification of metric completions of a specific triangulated category has been done.

Since any triangulated subcategory and any t -structure on a triangulated category automatically provides us with good metrics on the triangulated category (see Definition 5.7 in Paper B), trying to classify all metrics would certainly also require to understand triangulated subcategories and t -structures. Our efforts to study metrics and metrics completions are, therefore, targeted primarily towards bounded derived categories $\mathbf{D}^b(\text{mod-}A)$ of a finite dimensional algebra A . The category of finite dimensional modules $\text{mod-}A$ and the bounded derived category $\mathbf{D}^b(\text{mod-}A)$ are Krull-Schmidt, Hom-finite, and overall relatively well-understood, with strong structure-theoretical tools at our disposition.

The first question addressed in this project was of an almost philosophical nature. What actually is a metric on a triangulated category? As a matter of fact, there are two different notions (Definition 2.59) - a metric, introduced in [49], and a good metric appearing in Neeman's survey [50]. Even the definition of a completion with respect to a metric and with respect to a good metric are slightly different; the former being more

complicated. In Section 5 of Paper A, we provide a comparison between the two notions, and prove the following:

Theorem 4.1 (Paper A - Corollary 5.9). For any (even non-good) metric \mathcal{M} on a triangulated category \mathcal{S} , there exists a canonically determined good metric \mathcal{N} , called *the improvement* of \mathcal{M} , such that $\mathfrak{S}_{\mathcal{M}}(\mathcal{S}) \subseteq \mathfrak{S}_{\mathcal{N}}(\mathcal{S})$.

So all we can achieve by using more general, non-good metrics, is to obtain triangulated subcategories of completions with respect to good metrics. It is then perhaps not surprising that in practice almost all metrics on triangulated categories are assumed to be good, and even the very recent update [55] of [49] replaces the definition of a metric by a good metric.

We mentioned that the most known examples of good metrics are coming from t -structures. In the effort to find other instances of good metrics, we consider “moving” metric between different triangulated categories using triangulated functors. We define a notion of preimage metric (Definition 4.11 from Paper A) taken along a triangulated functor. Using the theory of compression functors from [69], we then proceed to show when a preimage metric yields the same completion as the original metric.

Theorem 4.2 (Paper A - Theorem 1.3). Consider a diagram of triangulated functors

$$\begin{array}{ccc} & F & \\ \curvearrowleft & & \curvearrowright \\ \mathcal{S} & \xrightarrow{G} & \mathcal{T} \\ \curvearrowright & & \curvearrowleft \\ & J & \end{array}$$

consisting of two adjoint pairs $F : \mathcal{T} \rightleftarrows \mathcal{S} : G$ and $G : \mathcal{S} \rightleftarrows \mathcal{T} : J$. Let $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$ be a metric on \mathcal{T} and denote $\mathcal{N} := \{G^{-1}(B_n)\}_{n \in \mathbb{N}}$ the preimage metric of \mathcal{M} under G .

If F and J are fully faithful, then the induced functors $- \circ G : \mathfrak{S}_{\mathcal{M}}(\mathcal{T}) \rightarrow \mathfrak{S}_{\mathcal{N}}(\mathcal{S})$ and $- \circ F : \mathfrak{S}_{\mathcal{N}}(\mathcal{S}) \rightarrow \mathfrak{S}_{\mathcal{M}}(\mathcal{T})$ are well-defined, mutually inverse triangulated equivalences.

We then show applications of Theorem 4.2 for a bounded derived category $\mathbf{D}^b(\text{mod-}A)$ of a finite dimensional algebra A of finite global dimension in Section 4.3 of Paper A.

Our search for classification results for all completions on specific triangulated categories started with the bounded derived category $\mathbf{D}^b(\text{mod-}A)$ of a hereditary, finite dimensional algebra A of finite representation type. The representation theory of A is crystal clear with only finitely many indecomposable objects in $\mathbf{D}^b(\text{mod-}A)$ up to shifts. A completion of a triangulated category consists of directed colimits of Cauchy sequences, and this allows, in principle, for new objects which are not present in the original triangulated category to appear in the completion. However, it turns out that for these representation-finite algebras, no new objects are allowed in the completion no matter which metric we choose.

Theorem 4.3 (Paper A - Theorem 1.1, Corollary 3.9). Let A be a hereditary, finite dimensional algebra of finite representation type. Let $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$ be a good metric on the category $\mathbf{D}^b(\text{mod-}A)$. Denote $\mathcal{B} := \bigcap_{n=1}^{\infty} B_n$. Then

$$\mathfrak{S}_{\mathcal{M}}\left(\mathbf{D}^b(\text{mod-}A)\right) \simeq \mathcal{B}^{\perp} \subseteq \mathbf{D}^b(\text{mod-}A).$$

More specifically, the completions of $\mathbf{D}^b(\text{mod-}A)$ are precisely the thick subcategories of $\mathbf{D}^b(\text{mod-}A)$.

In the situation above, a thick subcategory $\mathcal{C} \subseteq \mathbf{D}^b(\text{mod-}A)$ is equivalent to the completion taken with respect to the constant metric ${}^\perp\mathcal{C}$. Of course, there are many non-constant good metrics on $\mathbf{D}^b(\text{mod-}A)$, but Theorem 4.3 tells us that for such a good metric \mathcal{M} there exists a thick subcategory $\mathcal{D} \subseteq \mathbf{D}^b(\text{mod-}A)$ such that \mathcal{M} and the constant metric \mathcal{D} determine the same completion. There exists an innate notion of an equivalence of metrics (see Definition 2.4 in Paper A), and any two equivalent metrics yield the same completion. However, this notion is not refined enough; we can have non-equivalent metrics giving the same completion regardless. The metrics \mathcal{M} and \mathcal{D} can be, in fact, taken as examples.

In order to obtain a better insight to when two good metrics determine the same completion, we introduce the lattice of good metrics on a triangulated category in Section 5 of Paper B. We prove that the equivalence classes of good metrics indeed form a lattice and show some of its basic properties. For a bounded derived categories of hereditary rings, we observe that two metrics determine the same completion if the two metrics differ only in cohomological degrees $n \in \mathbb{Z}$ with $n \ll 0$, or $n \gg 0$; the lattice theory of metrics allows for a more precise formulation of this property as $\mathcal{M} = \mathcal{M}_\infty \vee \mathcal{N}$ (see Definition 5.9 from Paper B).

Proposition 4.4 (Paper B - Proposition 5.10). Let R be a hereditary, finitely generated algebra over a commutative noetherian ring. Equip $\mathcal{S} := \mathbf{D}^b(\text{mod-}R)$ with additive good metrics \mathcal{M} and \mathcal{N} . If $\mathcal{M} = \mathcal{M}_\infty \vee \mathcal{N}$, then $\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) = \mathfrak{S}'_{\mathcal{N}}(\mathcal{S})$.

Motivated by the classification result in Theorem 4.3, we ask what happens if we relax the representation type of the algebra A in question from finite to tame. This time, we are finally able to produce completions of $\mathbf{D}^b(\text{mod-}A)$ which are not just thick subcategories but also contain some new objects. With the help of the aforementioned lattice theory of metrics, we prove:

Theorem 4.5 (Paper B - Theorem 1.1). Let R be a connected hereditary ring and equip $\mathbf{D}^b(\text{mod-}R)$ with an additive good metric $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$. Denote $\mathcal{B} := \bigcap_{n \in \mathbb{N}} B_n$ and let $R \rightarrow R_{\mathcal{B}}$ be the universal localisation of R uniquely determined by \mathcal{B} via [39, Theorem 8.1].

- 1) Suppose that R is a commutative noetherian ring. Then:
 - i) If \mathcal{B} is countably generated as a thick subcategory and $\mathcal{M} = \mathcal{M}_\infty \vee \mathcal{B}$, then $\mathfrak{S}_{\mathcal{M}}(\mathbf{D}^b(\text{mod-}R)) \simeq \mathfrak{S}_{\mathcal{B}}(\mathbf{D}^b(\text{mod-}R)) \simeq \mathbf{D}^b(\text{mod-}R_{\mathcal{B}})$.
 - ii) Otherwise, we have $\mathfrak{S}_{\mathcal{M}}(\mathcal{S}) \simeq \mathcal{B}^\perp \cap \mathfrak{T}$ where $\mathfrak{T} \subseteq \mathbf{D}^b(\text{mod-}R)$ is the thick subcategory generated by all finitely generated torsion modules.
- 2) Suppose that R is a finite dimensional algebra of tame representation type over an algebraically closed field. Then:
 - i) If \mathcal{B} is countably generated as a thick subcategory and $\mathcal{M} = \mathcal{M}_\infty \vee \mathcal{B}$, then $\mathfrak{S}_{\mathcal{M}}(\mathbf{D}^b(\text{mod-}R)) \simeq \mathfrak{S}_{\mathcal{B}}(\mathbf{D}^b(\text{mod-}R)) \simeq \mathbf{D}^b(\text{mod-}R_{\mathcal{B}})$.
 - ii) Otherwise, we have $\mathfrak{S}_{\mathcal{M}}(\mathcal{S}) \simeq \mathcal{B}^\perp \cap \mathfrak{R}$ where $\mathfrak{R} \subseteq \mathbf{D}^b(\text{mod-}R)$ is the thick subcategory generated by all regular modules.

In the last part of this thesis, [Project C](#), we gather our current results from an ongoing research project whose aim is to generalise [Theorem 4.5](#) and other tools from [Paper B](#). As a first step, we observe that the appearance of Verdier's localisations as metric completions is a rather common phenomenon for constant metrics.

Theorem 4.6 (Corollary [5.4](#)). Let \mathcal{T} be a compactly generated \mathbb{k} -linear triangulated category over a commutative ring \mathbb{k} . If $\mathcal{C} \subseteq \mathcal{T}^c$ is a countably generated thick subcategory and \mathcal{T}^c is Hom-finite, then $\mathfrak{S}_{\mathcal{C}}(\mathcal{T}^c) \simeq \mathcal{T}^c/\mathcal{C}$.

Finally, we are able to prove an analogous statement to [Theorem 4.5](#) for the bounded derived category $\mathbf{D}^b(\text{coh-}\mathbb{X})$ of coherent sheaves on a weighted projective line \mathbb{X} of domestic or tubular type. Indecomposable objects of $\text{coh-}\mathbb{X}$ can be assigned a numerical invariant called a *slope* attaining values in $\mathbb{Q} \cup \{\infty\}$. The abelian category $\text{coh-}\mathbb{X}$ has a tilting object, so $\mathbf{D}^b(\text{coh-}\mathbb{X})$ is triangle equivalent to a bounded derived category of finite dimensional algebra, while not necessarily a hereditary one. In fact, for \mathbb{X} domestic $\mathbf{D}^b(\text{coh-}\mathbb{X}) \simeq \mathbf{D}^b(\text{mod-}A)$ where A is a tame hereditary finite dimensional algebra, so [Theorem 4.5](#) applies directly. But for \mathbb{X} tubular, the following main result of [Project C](#) is new.

Theorem 4.7 ([Theorem 9.8](#)). Let \mathbb{X} be a weighted projective line of a domestic or tubular type over an algebraically closed field. Equip $\mathbf{D}^b(\text{coh-}\mathbb{X})$ with an additive good metric $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$ and denote $\mathcal{B} := \bigcap_{n \in \mathbb{N}} B_n$. Then:

- (i) If $\mathcal{M} = \mathcal{M}_{\infty} \vee \mathcal{B}$ and \mathcal{B} is countably generated, then

$$\mathfrak{S}_{\mathcal{M}}(\mathbf{D}^b(\text{coh-}\mathbb{X})) \simeq \mathfrak{S}_{\mathcal{B}}(\mathbf{D}^b(\text{coh-}\mathbb{X})) \simeq \mathbf{D}^b(\text{coh-}\mathbb{X})/\mathcal{B}.$$

In this case, it holds that $\mathfrak{S}_{\mathcal{M}}(\mathbf{D}^b(\text{coh-}\mathbb{X})) \simeq \mathcal{B}^{\perp} \cap \mathbf{D}^b(\text{coh-}\mathbb{X})$ if and only if \mathcal{B} is generated by an exceptional sequence.

- (ii) Otherwise, there exists $q \in \mathbb{Q} \cup \{\infty\}$ depending on \mathcal{M} such that

$$\mathfrak{S}_{\mathcal{M}}(\mathbf{D}^b(\text{coh-}\mathbb{X})) \simeq \mathcal{B}^{\perp} \cap \mathcal{H}^{(q)}$$

where $\mathcal{H}^{(q)} \subseteq \mathbf{D}^b(\text{coh-}\mathbb{X})$ is a thick subcategory generated by all indecomposable objects from $\text{coh-}\mathbb{X}$ of slope q .

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The following chapter “*Paper A*” is a copy of the third version of the author’s preprint “*Metric completions of triangulated categories from finite dimensional algebras*”, which was uploaded on 21 January 2026 to the online repository arXiv under the code 2409.01828v3. The preprint is available at <https://arxiv.org/abs/2409.01828v3>. The text of the chapter is identical to the text of the preprint, except for formatting, typesetting, page layout, page numbers, and page headers. The table of contents, the ORCID identifier, and the footnotes indicating the date of submission, the Mathematics Subject Classification, the keywords, the supporting grant, and the contact details of the author have been removed from the version presented in this thesis.

Paper A

Metric completions of triangulated categories from finite dimensional algebras

Cyril Matoušek

Abstract

In this paper, we study metric completions of triangulated categories in a representation-theoretic context. We provide a concrete description of completions of bounded derived categories of hereditary finite dimensional algebras of finite representation type. In order to investigate completions of bounded derived categories of algebras of finite global dimension, we define image and preimage metrics under a triangulated functor and use them to induce a triangulated equivalence between two completions. Furthermore, for a given metric on a triangulated category we construct a new, closely related good metric called the improvement and compare the respective completions.

1 Introduction

Since the introduction of triangulated categories by Verdier [35] and Puppe [31], there have not been many known ways to construct a new triangulated category from an already existing one without the presence of an enhancement or a separable monoidal structure [5]. Together with other rare, advanced methods such as orbit categories [19] and relative stable categories [8], [6], Neeman's recent groundbreaking result from [27] goes against the prevailing belief that triangulated categories do not reproduce.

In [27], Neeman constructs completions of triangulated categories using metric techniques. His method resembles the construction of a completion of a metric space, while the concept of defining a metric (or a norm) on a category evolves from the influential paper [22] by Lawvere from 1973. Neeman also employs his previous ideas about approximability of triangulated categories from [29]. By defining a metric on a triangulated category, we assign rational-valued lengths to morphisms in the category, and then formally add colimits of Cauchy sequences (i.e. \mathbb{N} -shaped diagrams where morphisms eventually get shorter and shorter) to our category. If we then restrict ourselves to so-called compactly supported elements, i.e. those colimits of Cauchy sequences which behave nicely with respect to very short morphisms, we obtain a triangulated category called the completion.

Neeman's original motivation in [27] was to generalise Rickard's results on derived Morita equivalences from [32]. He shows that for a noetherian and weakly approximable triangulated category \mathcal{T} there exist intrinsically determined metrics on the triangulated categories of compact elements \mathcal{T}^c of \mathcal{T} and the bounded category \mathcal{T}_c^b (in the sense of [29,

Definition 0.16]). Using these canonical metrics, one can show that \mathcal{T}^c and \mathcal{T}_c^b are completions of each other (up to switching to opposite categories). For a coherent ring R , setting $\mathcal{T} := \mathbf{D}(\text{Mod-}R)$ recovers Rickard's original result [32, Proposition 8.2]. Another particular instance of such a phenomenon is the case of a derived category $\mathbf{D}(X)$ of a separated, noetherian scheme X and the subcategories $\mathbf{D}^{\text{perf}}(X)$ (perfect complexes) and $\mathbf{D}_{\text{coh}}^b(X)$ (complexes with bounded, coherent cohomology). For more examples and applications, we refer to [27], [30] and Section 5 of [14]. The last two aforementioned papers elaborate further on the method of metric completions in the context of approximable triangulated categories.

Neeman summarizes his method of metric completions in his survey paper [28] where he also introduces a more refined version of a metric, a so-called good metric. While the definition of a good metric is more restrictive, it makes statements of theorems and their proofs slicker and less technical. We shall provide more details on good metrics in Section 5.

Even though Neeman's result on completing triangulated categories with respect to a metric is relatively new, metric completions have already found their use in several recent papers. Metric completions have been used by Sun and Zhang in [33] to study recollements of triangulated categories, and their main theorem [33, Theorem 4.4] states that under certain conditions the construction of a completion transforms a recollement of triangulated categories into a right recollement of the corresponding completions. In [11], Biswas, Chen, Rahul, Parker, and Zheng use completions (and invariance under the completion construction) with respect to certain natural good metrics on a triangulated category as a criterion for existence (and mutual equivalence) of bounded t -structures on said triangulated category ([11, Theorem 1.5]). Cummings and Gratz calculate completions of discrete cluster categories of type \mathbb{A} with respect to good metrics in [15], and give their combinatorial descriptions.

In this paper, we study metrics and metric completions in a representation-theoretic setting. While the results from [27] and [11] focus mainly on metrics coming from t -structures with motivation in algebraic geometry, our aim is to systematically calculate completions with respect to any arbitrary metric on a bounded derived category $\mathbf{D}^b(\text{mod-}A)$ of a sufficiently nice finite dimensional algebra A .

We provide an explicit formula for calculating completions of $\mathbf{D}^b(\text{mod-}A)$ for A a hereditary algebra of finite representation type (e.g. a path algebra of a simply laced Dynkin quiver) with respect to any metric. A metric on a triangulated category formally consists of a decreasing chain of subcategories (see Definition 2.1) that should be thought of as shrinking open balls around zero, so in our case we can express the completion in terms of these open balls.

Theorem 1.1 (Theorem 3.7). Let A be a hereditary finite dimensional algebra (over a field) of finite representation type. Let $\{B_n\}_{n \in \mathbb{N}}$ be a good metric on the category $\mathbf{D}^b(\text{mod-}A)$. Denote $\mathcal{B} := \bigcap_{n=1}^{\infty} B_n$.

The completion of $\mathbf{D}^b(\text{mod-}A)$ with respect to the good metric $\{B_n\}_{n \in \mathbb{N}}$ is equal to $\mathcal{B}^\perp \subseteq \mathbf{D}^b(\text{mod-}A)$ where $\mathcal{B}^\perp := \bigcap_{B \in \mathcal{B}} \text{Ker Hom}_{\mathbf{D}^b(\text{mod-}A)}(B, -)$.

The condition on $\{B_n\}_{n \in \mathbb{N}}$ of being a good metric is, in fact, not necessary. Theorem 3.7 is stated more generally and encompasses also the case where the metric is not good.

A noteworthy corollary of our theorem is that the thick subcategories of $\mathbf{D}^b(\text{mod-}A)$ are precisely the completions of $\mathbf{D}^b(\text{mod-}A)$ with respect to some metric (see Corollary 3.9).

Our second main result is concerned with functors between completions of triangulated categories of the form $\mathbf{D}^b(\text{Mod-}A)$ for a finite dimensional algebra of finite global dimension, or even more generally between completions of K -linear triangulated categories (over a field K) with Serre functors. Our starting point is Theorem 4.3 from Sun and Zhang's paper [33], that triangulated functors between triangulated categories under certain continuity conditions induce triangulated functors on their completions, whose generalisation for general (non-good) metrics is presented in this paper.

Theorem 1.2 ([33, Theorem 4.3, pt. 3], Theorem 4.9). Let $F : \mathcal{T} \rightleftarrows \mathcal{S} : G$ be adjoint triangulated functors between triangulated categories equipped with metrics. If F and G are compressions, then the precomposition $- \circ F : \text{Mod-}\mathcal{S} \rightarrow \text{Mod-}\mathcal{T}$ induces a triangulated functor $- \circ F : \mathfrak{S}(\mathcal{S}) \rightarrow \mathfrak{S}(\mathcal{T})$ between the completion $\mathfrak{S}(\mathcal{S})$ of \mathcal{S} and the completion $\mathfrak{S}(\mathcal{T})$ of \mathcal{T} .

In other words, we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{S}(\mathcal{S}) & \xrightarrow{-\circ F|_{\mathfrak{S}(\mathcal{S})}} & \mathfrak{S}(\mathcal{T}) \\ \downarrow & & \downarrow \\ \text{Mod-}\mathcal{S} & \xrightarrow{-\circ F} & \text{Mod-}\mathcal{T} \end{array}$$

with the upper horizontal functor $- \circ F|_{\mathfrak{S}(\mathcal{S})}$ being triangulated and the vertical functors being inclusions.

Utilising Theorem 1.2, we can express a completion of triangulated category as a completion of another using a related metric. For that purpose we define and study the image and the preimage of a metric under a triangulated functor. It turns out that the preimage of a metric is always a metric (see Lemma 4.12) and the image of a metric is a metric if it is taken under a full triangulated functor (see Lemma 4.14). In the following theorem we use a preimage metric in such a way, that all relevant functors are compressions and consequently Theorem 1.2 applies.

Theorem 1.3 (Theorem 4.23). Consider a diagram of triangulated functors

$$\begin{array}{ccc} & F & \\ \mathcal{S} & \xleftrightarrow{G} & \mathcal{T} \\ & J & \end{array}$$

consisting of two adjoint pairs $F : \mathcal{T} \rightleftarrows \mathcal{S} : G$ and $G : \mathcal{S} \rightleftarrows \mathcal{T} : J$. Let $\{B_n\}_{n \in \mathbb{N}}$ be a metric on \mathcal{T} . Denote the completion of \mathcal{T} with respect to $\{B_n\}_{n \in \mathbb{N}}$ as $\mathfrak{S}(\mathcal{T})$ and the completion of \mathcal{S} with respect to the preimage metric $\{G^{-1}(B_n)\}_{n \in \mathbb{N}}$ as $\mathfrak{S}(\mathcal{S})$.

If F and J are fully faithful, then the induced functor $- \circ G : \mathfrak{S}(\mathcal{T}) \rightarrow \mathfrak{S}(\mathcal{S})$ given by Theorem 1.2 is a well-defined triangulated equivalence with a triangulated inverse $- \circ F$.

Many of recent articles ([28], [33], [11], [15]) about metric completions of triangulated categories work with good metrics only. We finish this paper by a short comparison of generic metrics and good metrics in Section 5. We show that every metric has a naturally

given improvement which is always a good metric (see Definition 5.2 and Proposition 5.7). A completion of a triangulated category with respect to a given metric then embeds into the completion with respect to the original metric's improvement as a triangulated subcategory (see Corollary 5.9).

Acknowledgements

I wish to express my gratitude to my supervisor, Sira Gratz, for introducing me to the topic of metrics on triangulated categories and for her valuable comments regarding the paper. I thank Yaohua Zhang for making me aware of the content of Remark 4.24. I would also like to thank everyone from the “Categories, Clusters and Completions” and “Homological Algebra” groups at Aarhus University. This work was supported by a research grant (VIL42076) from VILLUM FONDEN.

2 Preliminaries

In this section, we recall definitions, theorems and notation related to completions of triangulated categories via metric techniques introduced by Neeman in [27], and his follow up survey paper [28].

2.1 Notation and conventions

2.1.1 General category theory

If we have an adjoint pair F, G of functors between two categories \mathcal{A}, \mathcal{B} , i.e. $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{A}$, we write it as $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$. By this notation, we mean that F is the left adjoint of G , and G is the right adjoint of F .

For \mathcal{C} a category, we will simply denote $C \in \mathcal{C}$ (by an abuse of notation) as a shortcut for $C \in \text{Ob}(\mathcal{C})$. All subcategories are considered full. For \mathcal{A}, \mathcal{B} categories, we use the notation $\mathcal{A} \subseteq \mathcal{B}$ for denoting that \mathcal{A} is a subcategory of \mathcal{B} .

Given a natural transformation $\varphi : G \Rightarrow H$ between two functors $G, H : \mathcal{B} \rightarrow \mathcal{C}$ with components $\varphi_B : G(B) \rightarrow H(B)$ for $B \in \mathcal{B}$ and a functor $F : \mathcal{A} \rightarrow \mathcal{B}$, we shall denote $\varphi_F : G \circ F \Rightarrow H \circ F$ the natural transformation with components $\varphi_{F(A)} : GF(A) \rightarrow HF(A)$ for each $A \in \mathcal{A}$.

Ab denotes the category of abelian groups.

2.1.2 Additive categories

Let \mathcal{A} be an additive category. We denote the category of additive contravariant functors from \mathcal{A} to Ab as $\text{Mod-}\mathcal{A}$, and $Y : \mathcal{A} \rightarrow \text{Mod-}\mathcal{A}$ the Yoneda embedding $A \mapsto \text{Hom}_{\mathcal{A}}(-, A)$.

For a class of objects $\mathcal{C} \subseteq \text{Ob}(\mathcal{A})$ we denote the following subcategories of \mathcal{A} :

- $\text{Summ}(\mathcal{C})$ the subcategory of \mathcal{A} consisting of objects isomorphic to direct summands of elements from \mathcal{C} ,
- $\text{Coproduct}(\mathcal{C})$ the subcategory of \mathcal{A} consisting of objects isomorphic to coproducts of elements from \mathcal{C} ,

- $\text{coprod}(\mathcal{C})$ the subcategory of \mathcal{A} consisting of objects isomorphic to finite coproducts of elements from \mathcal{C} ,
- $\text{Add}(\mathcal{C})$ the subcategory $\text{Summ}(\text{Coproduct}(\mathcal{C}))$,
- $\text{add}(\mathcal{C})$ the subcategory $\text{Summ}(\text{coprod}(\mathcal{C}))$,
- \mathcal{C}^\perp the subcategory $\{A \in \mathcal{A} : \forall C \in \mathcal{C}, \text{Hom}_{\mathcal{A}}(C, A) = 0\}$,
- ${}^\perp\mathcal{C}$ the subcategory $\{A \in \mathcal{A} : \forall C \in \mathcal{C}, \text{Hom}_{\mathcal{A}}(A, C) = 0\}$.

If $\mathcal{C} = \{C\}$ for a single object $C \in \mathcal{C}$, we often omit the brackets and write $\text{Summ}(C)$ etc. instead.

2.1.3 Triangulated categories

The notational conventions about triangulated categories here are primarily based on [25] and [11, Subsection 2.1].

By default, we will denote the shift functor on a triangulated category \mathcal{T} as Σ . The triangulated subcategory of \mathcal{T} consisting of its compact elements is denoted as \mathcal{T}^c . And for subcategories $\mathcal{A}, \mathcal{B} \subseteq \mathcal{T}$ we denote $\mathcal{A} * \mathcal{B}$ the subcategory of \mathcal{T} with objects

$$\{X \in \mathcal{T} : \exists A \in \mathcal{A}, \exists B \in \mathcal{B}, \exists \text{ a triangle } A \rightarrow X \rightarrow B \rightarrow \Sigma A \text{ in } \mathcal{T}\}.$$

A subcategory $\mathcal{S} \subseteq \mathcal{T}$ is *extension-closed* if $\mathcal{S} = \mathcal{S} * \mathcal{S}$.

For every object X of a triangulated category \mathcal{T} and $N, M \in \mathbb{Z}$ we define a set $X[N, M] := \{\Sigma^{-i}X : i \in \mathbb{Z}, N \leq i \leq M\}$. We also define:

- $X[N, \infty] := \{\Sigma^{-i}X : i \in \mathbb{Z}, N \leq i\}$,
- $X[-\infty, M] := \{\Sigma^{-i}X : i \in \mathbb{Z}, i \leq M\}$,
- $X[-\infty, \infty] := \{\Sigma^{-i}X : i \in \mathbb{Z}\}$.

The smallest subcategory of \mathcal{T} closed under extensions and direct summands containing $X[N, M]$ will be denoted $\langle X \rangle^{[N, M]}$.

More generally, for a class $\mathcal{C} \subseteq \text{Ob}(\mathcal{T})$ we define $\mathcal{C}[N, M] := \bigcup_{C \in \mathcal{C}} C[N, M]$. We treat the cases $\mathcal{C}[N, \infty]$, $\mathcal{C}[-\infty, M]$, $\mathcal{C}[-\infty, \infty]$, and $\langle \mathcal{C} \rangle^{[N, M]}$ analogously. In particular, the notation $\langle \mathcal{C} \rangle := \langle \mathcal{C} \rangle^{[-\infty, \infty]}$ stands for *the thick subcategory* of \mathcal{T} generated by \mathcal{C} .

Assuming that the category \mathcal{T} has countable coproducts, we define a *homotopy colimit* $\text{Hocolim} E_*$ of a directed diagram $E_* := E_1 \xrightarrow{f_{1,2}} E_2 \xrightarrow{f_{2,3}} \dots$ in \mathcal{T} by the distinguished triangle

$$\coprod_{n \in \mathbb{N}} E_n \xrightarrow{\varphi} \coprod_{n \in \mathbb{N}} E_n \longrightarrow \text{Hocolim} E_* \longrightarrow \Sigma \coprod_{n \in \mathbb{N}} E_n$$

where the first map is given by the infinite matrix

$$\varphi := \begin{bmatrix} \text{id}_{E_1} & 0 & 0 & \cdots \\ -f_{1,2} & \text{id}_{E_2} & 0 & \cdots \\ 0 & -f_{2,3} & \text{id}_{E_3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

2.1.4 Homotopy categories and derived categories

Let \mathcal{A} be an exact category. We use the following notation for relevant related categories:

- $\mathbf{K}(\mathcal{A})$ the homotopy category of \mathcal{A} ,
- $\mathbf{D}(\mathcal{A})$ the derived category of \mathcal{A} ,
- $\mathbf{D}^b(\mathcal{A})$ the bounded derived category of \mathcal{A} .

2.1.5 Rings and modules

By ring, we always mean associative, unitary ring whose operation of multiplication generally does not have to be commutative. By module, we mean a right module over a given ring unless stated otherwise. A K -algebra over a field K is a ring containing K as a subring of its center.

For a ring R , we use the following notation for relevant related categories:

- $\text{Mod-}R$ the category of R -modules,
- $\text{mod-}R$ the category of finitely generated R -modules,
- $\text{ind-}R$ the category of finitely generated indecomposable R -modules,
- $\mathbf{D}(R)$ abbreviation for $\mathbf{D}(\text{Mod-}R)$,
- $\mathbf{D}^b(R)$ abbreviation for $\mathbf{D}^b(\text{Mod-}R)$.

Furthermore, for R, S rings and ${}_R M_S$ an R - S -bimodule we denote

$$- \otimes_R^{\mathbf{L}} M : \mathbf{D}(R) \rightarrow \mathbf{D}(S) \text{ the left derived functor of } - \otimes_R M : \text{Mod-}R \rightarrow \text{Mod-}S$$

and

$$\mathbf{R} \text{Hom}_S(M, -) : \mathbf{D}(S) \rightarrow \mathbf{D}(R)$$

the right derived functor of $\text{Hom}_S(M, -) : \text{Mod-}S \rightarrow \text{Mod-}R$.

2.2 Metrics on triangulated categories

As explained in [28, Heuristic 9], we are aiming to assign lengths (with values in rational numbers) to morphisms in a triangulated theory \mathcal{T} in such a way that the length is invariant under homotopy cartesian squares, i.e. if

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow g' \\ C & \xrightarrow{f'} & D \end{array}$$

is a homotopy cartesian square, then the morphisms f and f' have equal length. We recall that the commutative square as above is called *homotopy cartesian* if there exists a distinguished triangle

$$A \xrightarrow{\begin{bmatrix} f \\ -g \end{bmatrix}} B \oplus C \xrightarrow{[g' \ f']} D \longrightarrow \Sigma A.$$

Consequently, we may measure a length of a morphism by looking at its cone, because for every distinguished triangle $A \xrightarrow{f} B \rightarrow C \rightarrow \Sigma A$ there exists a homotopy cartesian square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & C. \end{array}$$

In other words, the length of all morphisms is determined by the lengths of the zero morphisms $0 \rightarrow C$ for all $C \in \mathcal{T}$. This motivates the following definition of a metric on a triangulated category as a decreasing sequence of neighbourhoods of zero.

This definition of a metric comes from [27, Definition 1.2], while the definition of a good metric is from [28, Definition 10].

Definition 2.1. Let \mathcal{T} be a triangulated category. Let $\{B_n\}_{n \in \mathbb{N}}$ be a non-increasing chain $B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots$ of subcategories of \mathcal{T} . We say that $\{B_n\}_{n \in \mathbb{N}}$ is a *metric* if $B_n * B_n = B_n$ and $0 \in B_n$ for all $n \in \mathbb{N}$.

Furthermore, we say that a metric $\{B_n\}_{n \in \mathbb{N}}$ is a *good metric* if it satisfies the *rapid decrease condition*, i.e. $\Sigma^{-1}B_{n+1} \cup B_{n+1} \cup \Sigma B_{n+1} \subseteq B_n$ for all $n \in \mathbb{N}$.

Definition 2.2 ([27, Definition 1.6]). Let \mathcal{T} be a triangulated category with a metric $\{B_n\}_{n \in \mathbb{N}}$. Let $E_* := E_1 \xrightarrow{f_{1,2}} E_2 \xrightarrow{f_{2,3}} \dots$ be a directed diagram in \mathcal{T} . For $m' > m \in \mathbb{N}$, we denote the composition $f_{m,m'} := f_{m',m'-1} \circ \dots \circ f_{m+1,m+2} \circ f_{m,m+1}$ and $f_{m,m} := \text{id}_{E_m}$.

We say that E_* is a *Cauchy sequence* (with respect to the metric $\{B_n\}_{n \in \mathbb{N}}$) if for all $i \in \mathbb{N}$ there is an index $M \in \mathbb{N}$ such that for every $m' \geq m \geq M$ the triangle $E_m \xrightarrow{f_{m,m'}} E_{m'} \rightarrow D_{m,m'} \rightarrow \Sigma E_m$ satisfies $D_{m,m'} \in B_i$.

We say that E_* is a *good Cauchy sequence* (with respect to the metric $\{B_n\}_{n \in \mathbb{N}}$) if for all $i \in \mathbb{N}$ and $j \in \mathbb{Z}$ there is an index $M \in \mathbb{N}$ such that for every $m' \geq m \geq M$ the triangle $E_m \xrightarrow{f_{m,m'}} E_{m'} \rightarrow D_{m,m'} \rightarrow \Sigma E_m$ satisfies $\Sigma^j D_{m,m'} \in B_i$.

Remark 2.3. The existence of a metric $\{B_n\}_{n \in \mathbb{N}}$ on \mathcal{T} does not, strictly speaking, assign to a morphism $f : a \rightarrow b$ a precise number to be the length of f . Instead, we are merely interpreting $\text{cone}(f) \in B_n$ for some fixed $n \in \mathbb{N}$ as the length of f being less or equal to $\frac{1}{n}$. Since the cone of every isomorphism, the zero object, is contained in the intersection $\bigcap_{n \in \mathbb{N}} B_n$, all isomorphisms are of length $\leq \frac{1}{n}$ for all $n \in \mathbb{N}$. In this sense, we can say that isomorphisms are of zero length. And Cauchy sequences are sequences of composable morphisms whose lengths are eventually getting smaller and smaller (see [28, Definition 2]).

The original definition of a good metric $\{B_n\}_{n \in \mathbb{N}}$ in [28, Definition 10] contains an additional assumption that $B_1 = \mathcal{T}$. This was done to ensure that every morphism has a length, concretely length ≤ 1 . Since we are interested mainly in Cauchy sequences $E_* := E_1 \xrightarrow{f_{1,2}} E_2 \xrightarrow{f_{2,3}} \dots$ anyway, where we impose the length of $E_m \xrightarrow{f_{m,m'}} E_{m'}$ to be small for indices $m' \geq m \geq M$ for some fixed $M \in \mathbb{N}$ onwards, the requirement of every morphism having lengths ≤ 1 is not necessary. We can simply skip any finite initial segment of a Cauchy sequence.

Therefore, in this text, we will stick to the variant of the definition of a good metric from [11, Definition 2.8] where we allow morphisms without any length by omitting the condition

$B_1 = \mathcal{T}$. This will simplify the notation and formulations of certain statements and their proofs.

Similarly to the case of metrics on metric spaces, we can define an equivalence relation on metrics on triangulated categories.

Definition 2.4 ([27, Definition 1.2]). Let \mathcal{T} be a triangulated category and two metrics $\{B_n\}_{n \in \mathbb{N}}$ and $\{C_n\}_{n \in \mathbb{N}}$. We say that $\{C_n\}_{n \in \mathbb{N}}$ is *finer* than (or a *refinement* of) $\{B_n\}_{n \in \mathbb{N}}$ if for every $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that $C_m \subseteq B_n$.

In this case, we also say that $\{B_n\}_{n \in \mathbb{N}}$ is *coarser* than $\{C_n\}_{n \in \mathbb{N}}$, and we denote this by $\{C_n\}_{n \in \mathbb{N}} \leq \{B_n\}_{n \in \mathbb{N}}$.

We say that $\{B_n\}_{n \in \mathbb{N}}$ and $\{C_n\}_{n \in \mathbb{N}}$ are *equivalent* if

$$\{C_n\}_{n \in \mathbb{N}} \leq \{B_n\}_{n \in \mathbb{N}} \quad \text{and} \quad \{B_n\}_{n \in \mathbb{N}} \leq \{C_n\}_{n \in \mathbb{N}}.$$

Remark 2.5 ([27, Remark 1.7]). Equivalent metrics yield the same (good) Cauchy sequences.

Remark 2.6. Equivalence of metrics does not preserve goodness. Let $\mathbf{D}(K)$ be the category of graded vector spaces over a field K . Consider a good metric $\{C_n\}_{n \in \mathbb{N}}$ given by the cohomology functor $H : \mathbf{D}(K) \rightarrow \text{Mod-}K$ via the formula

$$C_n = \{T \in \mathbf{D}(K) : \forall i > -n, H^i(T) = 0\}.$$

This is a particular case of a metric of “type i)” from [28, Example 12].

The metric $\{C_n\}_{n \in \mathbb{N}}$ is indeed good because for each $n \in \mathbb{N}$ and $T \in C_{n+1}$, the property $H^i(T) = 0$ for $i > -n - 1$ implies $H^j(\Sigma^{-1}T) = 0$ whenever $j > -n$, thus justifying $\Sigma C_{n+1} \cup C_{n+1} \cup \Sigma^{-1}C_{n+1} \subseteq C_n$.

Define a metric $\{B_n\}_{n \in \mathbb{N}}$ by the formula $B_n := C_{\lceil \frac{n}{2} \rceil}$ for all $n \in \mathbb{N}$. This metric is not good since $\Sigma^n K \in B_{2n} = C_n$ for all $n \in \mathbb{N}$ while $\Sigma^{n-1} K \notin B_{2n-1} = C_n$.

We observe that $\{B_n\}_{n \in \mathbb{N}}$ and $\{C_n\}_{n \in \mathbb{N}}$ are equivalent, even though $\{C_n\}_{n \in \mathbb{N}}$ is good, while $\{B_n\}_{n \in \mathbb{N}}$ is not.

2.3 Completions of triangulated categories

The construction of a completion of a triangulated category \mathcal{T} resembles the one of a completion of a metric space. We are formally adjoining “limit points” of Cauchy sequences - in our case directed colimits of images of Cauchy sequences under the Yoneda embedding to obtain the pre-completion $\mathfrak{L}(\mathcal{T})$ (see Definition 2.7).

However, unlike the completion of a metric space, the pre-completion $\mathfrak{L}(\mathcal{T})$ is not yet what we wanted because it is not known, whether it is, in general, a triangulated category. To get hold of the true completion $\mathfrak{S}(\mathcal{T}) = \mathfrak{C}(\mathcal{T}) \cap \mathfrak{L}(\mathcal{T})$, we must restrict ourselves to those elements of $\mathfrak{L}(\mathcal{T})$ which are compactly supported. This ensures that the completion $\mathfrak{S}(\mathcal{T})$ is triangulated with a shift functor compatible with the original shift on \mathcal{T} .

The reasoning behind the introduction of compactly supported elements is explained in Remark 2.13 and Remark 2.14.

A more precise definition follows.

Definition 2.7 ([27, Definition 1.10]). Let \mathcal{T} be a triangulated category with a metric $\{B_n\}_{n \in \mathbb{N}}$. We define the following subcategories of $\text{Mod-}\mathcal{T}$ (all of them with respect to $\{B_n\}_{n \in \mathbb{N}}$):

- (i) *the compactly supported elements* as $\mathfrak{C}(\mathcal{T}) := \bigcap_{j \in \mathbb{Z}} \bigcup_{i \in \mathbb{N}} Y(\Sigma^j B_i)^\perp$,
- (ii) *the weakly compactly supported elements* as $\mathfrak{WC}(\mathcal{T}) := \bigcup_{i \in \mathbb{N}} Y(B_i)^\perp$,
- (iii) *the pre-completion* of \mathcal{T} as

$$\mathfrak{L}(\mathcal{T}) := \left\{ F \in \text{Mod-}\mathcal{T} : \exists \text{ a good Cauchy sequence } E_* \text{ in } \mathcal{T}, F \simeq \varinjlim Y(E_*) \right\},$$

- (iv) and *the completion* of \mathcal{T} as $\mathfrak{S}(\mathcal{T}) := \mathfrak{C}(\mathcal{T}) \cap \mathfrak{L}(\mathcal{T})$.

We also define an autoequivalence $\Sigma : \text{Mod-}\mathcal{T} \rightarrow \text{Mod-}\mathcal{T}$ by the formula

$$[\Sigma F](T) := F(\Sigma^{-1}T)$$

for all $F \in \text{Mod-}\mathcal{T}$ and $T \in \mathcal{T}$.

A directed diagram

$$\begin{array}{ccccccc} a_1 & \xrightarrow{f_1} & b_1 & \xrightarrow{g_1} & c_1 & \xrightarrow{h_1} & \Sigma a_1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ a_2 & \xrightarrow{f_2} & b_2 & \xrightarrow{g_2} & c_2 & \xrightarrow{h_2} & \Sigma a_2 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

of distinguished triangles in \mathcal{T} is said to be a *good Cauchy sequence of distinguished triangles* provided that each of the sequences a_* , b_* , c_* , and Σa_* is good Cauchy.

A sequence $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ in $\mathfrak{S}(\mathcal{T})$ is then called a *distinguished triangle* if it is isomorphic to the directed colimit of a good Cauchy sequence of distinguished triangles $a_* \xrightarrow{f_*} b_* \xrightarrow{g_*} c_* \xrightarrow{h_*} \Sigma a_*$ in \mathcal{T} embedded into $\text{Mod-}\mathcal{T}$ via Yoneda's embedding.

In case that the category \mathcal{T} is not skeletally small, natural transformations between two functors in $\text{Mod-}\mathcal{T}$ does not have to form a set. However, both of the categories $\mathfrak{L}(\mathcal{T})$ and $\mathfrak{S}(\mathcal{T})$, which we are mainly interested in, consist of countably presented functors only. This makes them genuine categories with small Hom-sets regardless of any set-theoretical issues with \mathcal{T} .

Since we will be often working with directed colimits $\varinjlim Y(E_*)$ of (good) Cauchy sequences E_* from \mathcal{T} calculated inside the module category $\text{Mod-}\mathcal{T}$, we introduce a special notation and terminology for such a colimit.

Definition 2.8 ([15, Definition, 3.4]). Let \mathcal{T} be a triangulated category with a metric. Let E_* be a Cauchy sequence in \mathcal{T} . We call $\varinjlim Y(E_*) \in \mathfrak{L}(\mathcal{T})$ *the module colimit* of E_* and we denote it as $\text{Mocolim} E_*$.

Remark 2.9 ([27, Remark 1.11]). Equivalent metrics yield the same (weakly) compactly supported elements and the same pre-completion. Consequently, they yield the same completion.

Remark 2.10. It holds that $\Sigma(\mathfrak{S}(\mathcal{T})) = \mathfrak{S}(\mathcal{T})$. In other words, Σ restricts to an autoequivalence of $\mathfrak{S}(\mathcal{T})$. This is explained in more detail in [27, Chapter 2].

We finally get to Neeman's result that the completion (as above) of triangulated category is triangulated itself.

Theorem 2.11 ([27, Theorem 2.11]). Let \mathcal{T} be a triangulated category with shift Σ and a metric $\{B_n\}_{n \in \mathbb{N}}$. Then the category $\mathfrak{S}(\mathcal{T})$ together with its autoequivalence Σ and its distinguished triangles from Definition 2.7 is triangulated.

In order to illustrate the importance of compactly supported elements, we state one of the auxiliary lemmas needed in the proof of Theorem 2.11. Not only is it an essential tool to show that the octahedral axiom holds in $\mathfrak{S}(\mathcal{T})$ but it is an interesting statement by itself. It will, in fact, be used again in Section 3.

Lemma 2.12 ([27, Lemma 2.8]). Let \mathcal{T} be a triangulated category with a metric. Let $F \in \mathfrak{C}(\mathcal{T})$ be a cohomological functor on \mathcal{T} and let $E = \text{Mocolim} E_* \in \mathfrak{L}(\mathcal{T})$ for some good Cauchy sequence E_* . Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$ every map $Y(E_n) \rightarrow F$ in $\text{Mod-}\mathcal{T}$ factors uniquely as $Y(E_n) \xrightarrow{\varphi} E \rightarrow F$ where φ is the canonical map into the colimit.

Lemma 2.12 concerns the unique factorisation of a certain morphism to a compactly supported cohomological functor F through a morphism from an element E of the pre-completion. As the completion $\mathfrak{S}(\mathcal{T}) = \mathfrak{C}(\mathcal{T}) \cap \mathfrak{L}(\mathcal{T})$ by definition consists of the elements of the pre-completion $\mathfrak{L}(\mathcal{T})$ which are compactly supported, it applies to any pair of elements E and F from the completion.

The following two remarks are based on [28, Definition 4] and the proof of aforementioned Lemma 2.12 in [27, Lemma 2.8].

Remark 2.13. Let \mathcal{T} be a triangulated category with a metric $\{B_n\}_{n \in \mathbb{N}}$. Let $F \in \mathfrak{C}(\mathcal{T})$. If F is a cohomological functor on \mathcal{T} , then it sends sufficiently small morphisms to isomorphisms.

By the definition of compactly supported elements, we can find $n \in \mathbb{N}$ such that $F \in Y(\Sigma^{-1}B_n)^\perp \cap Y(B_n)^\perp$. Let $f : C \rightarrow D$ in \mathcal{T} be of length at most $\frac{1}{n}$, i.e. there exists a triangle

$$C \xrightarrow{f} D \rightarrow E \rightarrow \Sigma C$$

with $E \in B_n$. Applying the cohomological functor F to this triangle yields a long exact sequence

$$\dots \rightarrow F(E) \rightarrow F(D) \xrightarrow{f} F(C) \rightarrow F(\Sigma^{-1}E) \rightarrow \dots$$

of abelian groups. By Yoneda's lemma it holds that

$$F(E) \simeq \text{Hom}(Y(E), F) = 0 \quad \text{and} \quad F(\Sigma^{-1}E) \simeq \text{Hom}(Y(\Sigma^{-1}E), F) = 0,$$

so $F(f)$ must be an isomorphism thanks to the exactness of our sequence.

This fact is a key step in the proof of Lemma 2.12, and consequently of the octahedral axiom for $\mathfrak{S}(\mathcal{T})$ as well.

Remark 2.14. Note that a functor $F \in \mathfrak{L}(\mathcal{T})$ is automatically cohomological as a directed colimit of representable functors, which are known to be cohomological. Therefore, any functor in the completion $\mathfrak{S}(\mathcal{T}) = \mathfrak{C}(\mathcal{T}) \cap \mathfrak{L}(\mathcal{T})$ is both compactly supported and cohomological, so it satisfies the property of “sending small morphisms to isomorphisms” from Remark 2.13.

2.4 Good extensions

A good extension is an indispensable tool to explicitly calculate completions of triangulated categories without referring to the functor category. If a triangulated category embeds into another triangulated category with coproducts in such a way that the embedding is a good extension, the completion is then triangle equivalent to a suitable subcategory of the extension.

The following definition is comprised of Notation 3.1, Definition 3.5, and Definition 3.8 from [27]. For the definition of a homotopy colimit, we refer to Subsection 2.1.3.

Definition 2.15. Let \mathcal{S}, \mathcal{T} be triangulated categories and $F : \mathcal{S} \rightarrow \mathcal{T}$ a fully faithful, triangulated functor. We define *the restricted Yoneda functor* $\mathcal{Y} : \mathcal{T} \rightarrow \text{Mod-}\mathcal{S}$ by setting $T \mapsto \text{Hom}(F(-), T)$.

Suppose that \mathcal{S} has a metric $\{B_n\}_{n \in \mathbb{N}}$. We say that F is a *good extension* (with respect to $\{B_n\}_{n \in \mathbb{N}}$) if \mathcal{T} has countable coproducts and for every good Cauchy sequence E_* the canonical map $\text{Mocolim} E_* \rightarrow \mathcal{Y}(\text{Hocolim} F(E_*))$ is an isomorphism.

In this setting, we define the following subcategories of \mathcal{T} :

$$\mathfrak{C}'(\mathcal{S}) := \mathcal{Y}^{-1}(\mathfrak{C}(\mathcal{S})),$$

$$\mathfrak{WC}'(\mathcal{S}) := \mathcal{Y}^{-1}(\mathfrak{WC}(\mathcal{S})),$$

$$\mathfrak{L}'(\mathcal{S}) := \left\{ T \in \mathcal{T} : \exists \text{ a good Cauchy sequence } E_* \text{ in } \mathcal{S}, T \simeq \text{Hocolim} F(E_*) \right\},$$

$$\mathfrak{S}'(\mathcal{S}) := \mathfrak{C}'(\mathcal{S}) \cap \mathfrak{L}'(\mathcal{S}).$$

Remark 2.16 ([27, Observation 3.2]). We can describe the categories $\mathfrak{C}'(\mathcal{S})$ and $\mathfrak{WC}'(\mathcal{S})$ explicitly as $\mathfrak{C}'(\mathcal{S}) = \bigcap_{j \in \mathbb{Z}} \bigcup_{i \in \mathbb{N}} F(\Sigma^j B_i)^\perp$ and $\mathfrak{WC}'(\mathcal{S}) = \bigcup_{i \in \mathbb{N}} F(B_i)^\perp$.

We can calculate completions inside good extensions.

Theorem 2.17 ([27, Theorem 3.15]). Let \mathcal{S} be a triangulated category with a metric, and $\mathcal{S} \hookrightarrow \mathcal{T}$ a good extension. Then $\mathcal{Y} \upharpoonright_{\mathfrak{S}'(\mathcal{S})} : \mathfrak{S}'(\mathcal{S}) \rightarrow \mathfrak{S}(\mathcal{S})$ is a well-defined triangulated equivalence.

We mention a technical lemma from the proof of the theorem.

Lemma 2.18 ([27, Lemma 3.13]). Let \mathcal{S} be a triangulated category with a metric, and $\mathcal{S} \hookrightarrow \mathcal{T}$ a good extension. Let $E \in \mathfrak{L}'(\mathcal{S})$ and $X \in \mathfrak{C}'(\mathcal{S})$. Then the map

$$\mathcal{Y} : \text{Hom}_{\mathcal{T}}(E, X) \rightarrow \text{Hom}_{\text{Mod-}\mathcal{S}}(\mathcal{Y}(E), \mathcal{Y}(X))$$

is an isomorphism.

The isomorphism from Lemma 2.18 enables us to translate the unique factorisation property of Lemma 2.12 into the setting of the good extension. This will be useful again in Section 3.

Lemma 2.19. Let \mathcal{S} be a triangulated category with a metric, and $F : \mathcal{S} \rightarrow \mathcal{T}$ a good extension. Let $X \in \mathfrak{C}'(\mathcal{S})$ and let $E = \text{Hocolim} F(E_*) \in \mathfrak{L}'(\mathcal{T})$ for some good Cauchy sequence E_* . Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$ every map $F(E_n) \rightarrow X$ in \mathcal{T} factors uniquely as $F(E_n) \rightarrow E \rightarrow X$.

Proof. Since $\mathcal{Y}(X)$ is a cohomological functor, Lemma 2.12 gives us an index $N \in \mathbb{N}$ and for each $n \geq N$ a unique factorisation of $\mathcal{Y} \circ F(E_n) \simeq Y(E_n) \rightarrow \mathcal{Y}(X)$ as

$$\mathcal{Y} \circ F(E_n) \simeq Y(E_n) \rightarrow \text{Mocolim} E_* \simeq \mathcal{Y}(E) \rightarrow \mathcal{Y}(X).$$

We then lift the factorisation to \mathcal{T} using Lemma 2.18. □

We finish the section by mentioning the prototypical example of a good extension, i.e. the embedding of compact elements $\mathcal{T}^c \hookrightarrow \mathcal{T}$.

Theorem 2.20 ([27, Example 3.6]). Let \mathcal{T} be a triangulated category with arbitrary coproducts. Then the inclusion $\mathcal{T}^c \hookrightarrow \mathcal{T}$ is a good extension with respect to all metrics on \mathcal{T}^c .

Proof. Let $E_* := E_1 \rightarrow E_2 \rightarrow \dots$ be a directed diagram in \mathcal{T}^c . Then we can use [24, Lemma 2.8] to show that the canonical map $\varinjlim Y(E_*) \rightarrow \mathcal{Y}(\text{Hocolim} E_*)$ is an isomorphism regardless of whether the sequence E_* is good Cauchy or not. Hence the inclusion is a good extension for every possible metric on \mathcal{T}^c . □

3 Completions of certain derived categories

This section is dedicated to calculating all completions of the bounded derived category $\mathbf{D}^b(\text{mod-}A)$ of finitely generated modules over a hereditary, finite dimensional algebra of finite representation type. Such a category is equal to the category of $\mathbf{D}^c(A)$ of the compact elements (equivalently perfect complexes) of the derived category $\mathbf{D}(A)$. See for example [13, Proposition 6.4]. We are using the fact that the subcategory of perfect complexes is equal to $\mathbf{D}^b(\text{mod-}A)$ for A of finite global dimension.

Throughout the section, we will be always implicitly using Theorem 2.17 as our computation tool because embedding $\mathbf{D}^b(\text{mod-}A) \hookrightarrow \mathbf{D}(A)$ is a good extension for every metric (see Theorem 2.20). In other words, all the completions mentioned below will be of the form $\mathfrak{S}'(\mathcal{T})$ in the sense of the notation from Definition 2.15.

Firstly, we show that in the case of a hereditary category \mathcal{A} the completion of $\mathbf{D}(\mathcal{A})^c$ is always bounded in cohomology. Recall that any object of $\mathbf{D}(\mathcal{A})$ for \mathcal{A} hereditary is quasi-isomorphic to its cohomology (see for example [20, Proposition 4.4.15]).

Proposition 3.1. Let \mathcal{A} be a hereditary Abelian category with coproducts. Consider the subcategory of perfect complexes $\mathbf{D}(\mathcal{A})^c$ of the derived category $\mathbf{D}(\mathcal{A})$ of \mathcal{A} . Let \mathcal{S} be a triangulated subcategory of $\mathbf{D}(\mathcal{A})^c$. Then for every metric on \mathcal{S} , we have $\mathfrak{S}'(\mathcal{S}) \subseteq \mathbf{D}^b(\mathcal{A})$.

Proof. Let $E \in \mathfrak{S}'(\mathcal{S})$ where $E = \mathop{\mathrm{Hocolim}}\limits_{\rightarrow} E_*$ for some Cauchy sequence E_* .

By Lemma 2.19, for every $X \in \mathfrak{C}'(\mathcal{S})$ there exists $N \in \mathbb{N}$ such that every map $E_N \rightarrow X$ in $\mathbf{D}(\mathcal{A})$ factors uniquely as $E_N \rightarrow E \rightarrow X$. We find such an $N \in \mathbb{N}$ for the choice $X := E \in \mathfrak{C}'(\mathcal{S})$.

Suppose that E_N has cohomology concentrated in degrees $-n, \dots, n$ for some $n \in \mathbb{N}$. Denote $Q := \coprod_{i=-n}^{n+1} \Sigma^{-i} H^i(E) \in \mathbf{D}^b(\mathcal{A})$. Since \mathcal{A} is hereditary, Q is a direct summand of E , so we can consider the idempotent map $e : E \rightarrow E$ having Q as its image.

Because there are no morphisms $E_N \rightarrow \Sigma^i Z$ for every $Z \in \mathcal{A}$ and every

$$i \in \mathbb{Z}, \quad i \notin \{-n, \dots, n+1\},$$

the canonical map $E_N \rightarrow E$ factors through Q . But then the compositions $E_N \rightarrow E \xrightarrow{\mathrm{id}_E} E$ and $E_N \rightarrow E \xrightarrow{e} E$ are equal. By the uniqueness of such a factorisation, we get $e = \mathrm{id}_E$ and $E \simeq Q \in \mathbf{D}^b(\mathcal{A})$. \square

For the rest of the section, we narrow our attention to finite dimensional algebras.

Lemma 3.2. Let A be a hereditary finite dimensional K -algebra and \mathcal{S} a triangulated subcategory of $\mathbf{D}^b(\mathrm{mod}\text{-}A)$. Let $i \in \mathbb{Z}$. Let $E = \mathop{\mathrm{Hocolim}}\limits_{\rightarrow} E_* \in \mathfrak{S}'(\mathcal{S})$ for some good Cauchy sequence E_* with respect to a fixed metric on \mathcal{S} . If $H^i(E)$ contains $\bigoplus_{j \in J} M$ as a direct summand for some $0 \neq M \in \mathrm{mod}\text{-}A$ and some indexing set J , then J is finite.

Proof. Since $\Sigma^{-i} M$ is a direct summand of E , it has to be compactly supported (see Definition 2.7). By Lemma 2.19, there is $N \in \mathbb{N}$ such that the map

$$\mathrm{Hom}_{\mathbf{D}(A)}(\varphi, \Sigma^{-i} M) : \mathrm{Hom}_{\mathbf{D}(A)}(E, \Sigma^{-i} M) \rightarrow \mathrm{Hom}_{\mathbf{D}(A)}(E_N, \Sigma^{-i} M)$$

is a K -linear isomorphism, where $\varphi : E_N \rightarrow E$ is the canonical map.

Because $E_N, \Sigma^{-i} M \in \mathbf{D}^b(\mathrm{mod}\text{-}A)$, the dimension of $\mathrm{Hom}_{\mathbf{D}(A)}(E_N, \Sigma^{-i} M)$ is finite. As $\bigoplus_{j \in J} \Sigma^{-i} M$ is a direct summand of E , we know

$$\begin{aligned} 0 \neq \prod_{j \in J} \mathrm{Hom}_{\mathbf{D}(A)}(\Sigma^{-i} M, \Sigma^{-i} M) &\simeq \mathrm{Hom}_{\mathbf{D}(A)}\left(\bigoplus_{j \in J} \Sigma^{-i} M, \Sigma^{-i} M\right) \\ &\subseteq \mathrm{Hom}_{\mathbf{D}(A)}(E, \Sigma^{-i} M) \simeq \mathrm{Hom}_{\mathbf{D}(A)}(E_N, \Sigma^{-i} M). \end{aligned}$$

If J was infinite, the dimension of $\mathrm{Hom}_{\mathbf{D}(A)}(E_N, \Sigma^{-i} M)$ would be infinite, but that is impossible. Hence, the set J is finite. \square

We recall the following result of Auslander about modules over artinian rings of finite representation type. It applies, in particular, to path algebras over a simply laced Dynkin quiver.

Theorem 3.3 ([3, Corollary 4.8]). Let R be an artinian ring of finite representation type. Then every (possibly infinitely generated) R -module decomposes as a direct sum of finitely generated indecomposables, i.e. $\mathrm{Mod}\text{-}R = \mathrm{Coproduct}(\mathrm{ind}\text{-}R)$.

Lemma 3.4. Let A be a hereditary finite dimensional K -algebra of finite representation type. Then for every metric on $\mathbf{D}^b(\mathrm{mod}\text{-}A)$ we have

$$\mathfrak{S}'(\mathbf{D}^b(\mathrm{mod}\text{-}A)) \subseteq \mathbf{D}^b(\mathrm{mod}\text{-}A).$$

Proof. Since A is a hereditary ring, Proposition 3.1 gives us the inclusion of categories $\mathfrak{S}'(\mathbf{D}^b(\text{mod-}A)) \subseteq \mathbf{D}^b(\text{Mod-}A)$. Let $X \in \mathfrak{S}'(\mathbf{D}^b(\text{mod-}A))$ and let $i \in \mathbb{Z}$. By Theorem 3.3, we have $H^i(X) \simeq \bigoplus_{M \in \text{ind-}A} M^{(J_M)}$ for some suitable indexing sets J_M . By Lemma 3.2, the set J_M is finite for all $M \in \text{ind-}A$, and $\text{ind-}A$ is also finite. Hence $H^i(X) \in \text{mod-}A$ and $X \in \mathbf{D}^b(\text{mod-}A)$. \square

Lemma 3.5. Let \mathcal{T} be a triangulated category with a good metric $\{B_n\}_{n \in \mathbb{N}}$. Then $\mathcal{B} := \bigcap_{n=1}^{\infty} B_n$ is a triangulated subcategory of \mathcal{T} .

Proof. Let $T \in \mathcal{B}$ and let $n \in \mathbb{N}$. Then $T \in B_{n+1}$. Because the metric is good, we have the inclusion $\Sigma^{-1}B_{n+1} \cup B_{n+1} \cup \Sigma B_{n+1} \subseteq B_n$. Then $\Sigma^{\pm 1}T \in B_n$. Since n was arbitrary, we get that \mathcal{B} is closed under shifts. The subcategory \mathcal{B} then has to be a triangulated subcategory as a subcategory closed under shifts and extensions. \square

Recall that a pair $(\mathcal{U}, \mathcal{V})$ of subcategories of a triangulated category \mathcal{T} is called a *torsion pair* if $\mathcal{V} \subseteq \mathcal{U}^\perp$ and $\mathcal{U} * \mathcal{V} = \mathcal{T}$ (see [18, Definition 2.2]). In that case, it automatically holds that $\mathcal{V} = \mathcal{U}^\perp$ and $\mathcal{U} = {}^\perp \mathcal{V}$. We will make use of the following well-known fact that, in our setting of derived categories of representation-finite algebras, every thick subcategory is a part of a torsion pair.

Lemma 3.6. Let A be a hereditary finite dimensional K -algebra of finite representation type. Let \mathcal{S} be a thick subcategory of $\mathbf{D}^b(\text{mod-}A)$. Then there exists a triangulated subcategory \mathcal{U} of $\mathbf{D}^b(\text{mod-}A)$, such that $(\mathcal{U}, \mathcal{S})$ is a torsion pair.

Proof. As A is hereditary and of finite representation type, the thick subcategory \mathcal{S} is a preenveloping class (even functorially finite) in \mathcal{T} . By [26, Proposition 1.4] and [21, Theorem 2.9], the subcategory \mathcal{S} is even enveloping.

Then by [18, Proposition 2.4], the category \mathcal{S} is a right class of a torsion pair $(\mathcal{U}, \mathcal{S})$ in $\mathbf{D}^b(\text{mod-}A)$. \square

Finally, we get to the explicit computation of all completions of $\mathbf{D}^b(\text{mod-}A)$ for A a hereditary finite dimensional K -algebra of finite representation type. If the field K is algebraically closed, every such a (connected) algebra A is Morita equivalent to KQ for some simply laced Dynkin quiver Q (see e.g. [2, pg. 243]).

Theorem 3.7. Let A be a hereditary finite dimensional K -algebra of finite representation type. Let $\{B_n\}_{n \in \mathbb{N}}$ be a metric on the category $\mathbf{D}^b(\text{mod-}A)$. Denote $\mathcal{B} := \bigcap_{n=1}^{\infty} B_n$. Then

$$\mathfrak{S}'(\mathbf{D}^b(\text{mod-}A)) = \mathbf{D}^b(\text{mod-}A) \cap \mathcal{B}[-\infty, \infty]^\perp \subseteq \mathbf{D}(A).$$

Moreover, if $\{B_n\}_{n \in \mathbb{N}}$ is good, then $\mathfrak{S}'(\mathbf{D}^b(\text{mod-}A)) = \mathbf{D}^b(\text{mod-}A) \cap \mathcal{B}^\perp$.

Proof. “ \subseteq ” The inclusion $\mathfrak{S}'(\mathbf{D}^b(\text{mod-}A)) \subseteq \mathbf{D}^b(\text{mod-}A)$ follows from Lemma 3.4. If $X \in \mathfrak{S}'(\mathbf{D}^b(\text{mod-}A))$, then X is compactly supported, so consequently for every $j \in \mathbb{Z}$ there exists $i \in \mathbb{N}$ with $X \in \Sigma^j B_i^\perp$. Since $\Sigma^j \mathcal{B} \subseteq \Sigma^j B_i$, we get that $X \in \mathcal{B}[-\infty, \infty]^\perp$.

“ \supseteq ” Let $X \in \mathbf{D}^b(\text{mod-}A) \cap \mathcal{B}[-\infty, \infty]^\perp$. Then X is the homotopy colimit of the constant Cauchy sequence $X \xrightarrow{\text{id}_X} X \xrightarrow{\text{id}_X} \dots$ of elements of $\mathbf{D}^b(\text{mod-}A)$, so

$$X \in \mathfrak{L}'(\mathbf{D}^b(\text{mod-}A)).$$

Suppose that the cohomology of X is concentrated in degrees $-k+1, \dots, k$ for some $k \in \mathbb{N}$. Fix $j \in \mathbb{N}_0$. For any $M \in \text{ind-}A \setminus \text{Summ}(\mathcal{B}[-\infty, \infty])$ and any $i \in \mathbb{Z}$, there exists $n \in \mathbb{N}$ satisfying $M \notin \Sigma^i \text{Summ}(B_n)$. Since j and k are fixed, and there are (up to isomorphism) only finitely many finitely generated indecomposable A -modules, we can actually find $n \in \mathbb{N}$ big enough that

$$M \notin \text{Summ}(B_n)[-k-j, k+j]$$

holds for every $M \in \text{ind-}A \setminus \text{Summ}(\mathcal{B}[-\infty, \infty])$.

Let $Z \in B_n$, and denote $P := \coprod_{i=-k-j}^{k+j} \Sigma^{-i} H^i(Z)$. By Theorem 3.3, all cohomologies of P decompose as direct sums of indecomposable finitely generated A -modules. This yields $P \in \text{Add}(\mathcal{B}[-\infty, \infty])$ due to the choice of n . But then

$$\text{Hom}_{\mathbf{D}(A)}(\Sigma^j Z, X) = \text{Hom}_{\mathbf{D}(A)}(\Sigma^j P, X) = 0$$

because $X \in \mathcal{B}[-\infty, \infty]^\perp = \text{Add}(\mathcal{B}[-\infty, \infty])^\perp$. Using $X \in \text{Add}(\mathcal{B}[-\infty, \infty])^\perp$ again, we also get

$$\text{Hom}_{\mathbf{D}(A)}(\Sigma^{-j} Z, X) = \text{Hom}_{\mathbf{D}(A)}(\Sigma^{-j} P, X) = 0.$$

Hence, we obtain $X \in \Sigma^{\pm j} B_n^\perp$.

Since j was picked arbitrarily, we conclude

$$X \in \mathfrak{L}'(\mathbf{D}^b(\text{mod-}A)) \cap \mathfrak{C}'(\mathbf{D}^b(\text{mod-}A)) = \mathfrak{S}'(\mathbf{D}^b(\text{mod-}A)).$$

For the proof of the final statement, we further assume that the metric $\{B_n\}_{n \in \mathbb{N}}$ is good. Then \mathcal{B} is a triangulated subcategory of $\mathbf{D}^b(\text{mod-}A)$ by Lemma 3.5, so $\mathcal{B} = \mathcal{B}[-\infty, \infty]$. The claim now directly follows. \square

The main message of Theorem 3.7 is that for any (no matter how chaotic) metric $\{B_n\}_{n \in \mathbb{N}}$ on the category $\mathbf{D}^b(\text{mod-}A)$, the completion $\mathfrak{S}'(\mathbf{D}^b(\text{mod-}A))$ depends only on the intersection $\mathcal{B} := \bigcap_{n=1}^\infty B_n$ of all balls. To the author's knowledge, other instances of classification results for completions in the literature involve much more well-behaved metrics; e.g. metrics determined by aisles ([15, Theorem 1.1]), metrics determined by coaisles on discrete cluster categories ([15, Theorem 1.2]), or metrics $\left\{ \langle G \rangle^{[-\infty, -n]} \right\}_{n \in \mathbb{N}}$ determined by shifts of a classical generator G of a triangulated category \mathcal{S} under a finiteness condition on the relative finitistic dimension $\text{fin. dim}(\mathcal{S}^{\text{op}}, G) < \infty$ ([11, Theorem 3.11]).

Here, the relative finitistic dimension of a triangulated category \mathcal{S} at an object G , defined in [11, Definition 1.3] as

$$\text{fin. dim}(\mathcal{S}, G) := \inf \left\{ n \in \mathbb{N} : G[-\infty, -1]^\perp \subseteq \Sigma^n \langle G \rangle^{[0, \infty]} \right\},$$

is a generalisation of notion (due to Bass [7]) of the little finitistic dimension

$$\text{fin. dim}(R) := \sup \{ \text{proj. dim } M : M \in \text{mod-}R, \text{proj. dim } M < \infty \}$$

of a ring R in a sense that $\text{fin. dim}(\mathbf{D}(R)^c, R) = \text{fin. dim}(R)$ (see [11, Lemma 4.1.5]).

In this context, while the algebra A from Theorem 3.7 itself serves as a classical (even strong) generator of the category $\mathbf{D}^b(\text{mod-}A)$ and satisfies

$$\text{fin. dim}(\mathbf{D}^b(\text{mod-}A)^{\text{op}}, A) < \infty \quad \text{and} \quad \mathfrak{S}'(\mathbf{D}^b(\text{mod-}A)) = \mathbf{D}^b(\text{mod-}A)$$

with respect to the metric $\{\langle A \rangle^{[-\infty, -n]}\}_{n \in \mathbb{N}}$ as in [11, Theorem 3.11], our theorem also provides a description of completions with respect to metrics not related to the strong generator - beyond other things, metrics determined by shifts of any (non-generating) object M with $\text{fin. dim}(\mathbf{D}^b(\text{mod-}A)^{\text{op}}, M) = \infty$.

There exist, however, cases where the methods from Theorem 3.7, [11] and [15] all apply at the same time. We offer an insight in the following remark.

Remark 3.8. Let A be a hereditary finite dimensional K -algebra of finite representation type, and let $(\mathcal{U}, \mathcal{V})$ be a t -structure on $\mathbf{D}^b(\text{mod-}A)$. The t -structure determines an aisle metric $\{\Sigma^n \mathcal{U}\}_{n \in \mathbb{N}}$ (see [15, Definition 4.2]) on $\mathbf{D}^b(\text{mod-}A)$, and the metric is good. Theorem 3.7 then yields

$$\mathfrak{S}'(\mathbf{D}^b(\text{mod-}A)) = \left(\bigcap_{n=1}^{\infty} \Sigma^n \mathcal{U} \right)^{\perp} = \bigcup_{n=1}^{\infty} \Sigma^{-n} \mathcal{V} \subseteq \mathbf{D}^b(\text{mod-}A)$$

giving us a specific instance of a general result [15, Theorem 1.1] about completions with respect to aisle metrics being unions of shifts of coaisles.

In particular, if the t -structure $(\mathcal{U}, \mathcal{V})$ is bounded below, we have $\bigcap_{n=1}^{\infty} \Sigma^n \mathcal{U} = 0$ and $\mathfrak{S}'(\mathbf{D}^b(\text{mod-}A)) = \bigcup_{n=1}^{\infty} \Sigma^{-n} \mathcal{V} = \mathbf{D}^b(\text{mod-}A)$ as noticed in [11, Corollary 3.6] and implicitly proven in [10, Theorem 5.1] in the first version of said paper [11].

Corollary 3.9. Let A be a hereditary finite dimensional K -algebra of finite representation type. Then completions of $\mathbf{D}^b(\text{mod-}A)$ are precisely the thick subcategories of $\mathbf{D}^b(\text{mod-}A)$.

Proof. Let $\mathfrak{S}'(\mathbf{D}^b(\text{mod-}A))$ be a completion with respect to a metric $\{B_n\}_{n \in \mathbb{N}}$. If we denote $\mathcal{B} := \bigcap_{n=1}^{\infty} B_n$, by Theorem 3.7 we get

$$\mathfrak{S}'(\mathbf{D}^b(\text{mod-}A)) = \mathbf{D}^b(\text{mod-}A) \cap \mathcal{B}[-\infty, \infty]^{\perp}.$$

In other words, the completion $\mathfrak{S}'(\mathbf{D}^b(\text{mod-}A))$ is Hom-orthogonal to the suspension closed subcategory $\mathcal{B}[-\infty, \infty]$ of $\mathbf{D}^b(\text{mod-}A)$. Then $\mathfrak{S}'(\mathbf{D}^b(\text{mod-}A))$ must necessarily be a triangulated subcategory of $\mathbf{D}^b(\text{mod-}A)$, which is closed under direct summands. Hence, it is a thick subcategory.

Conversely, let \mathcal{S} be a thick subcategory of $\mathbf{D}^b(\text{mod-}A)$. Lemma 3.6 gives us a triangulated subcategory $\mathcal{U} \subseteq \mathbf{D}^b(\text{mod-}A)$ such that $(\mathcal{U}, \mathcal{S})$ is a torsion pair, implying $\mathcal{S} = \mathcal{U}^{\perp}$. The constant sequence of subcategories $\{\mathcal{U}\}_{n \in \mathbb{N}}$ is a good metric on $\mathbf{D}^b(\text{mod-}A)$. By Theorem 3.7, the completion with respect to this metric is \mathcal{S} . \square

Example 3.10. Let K be a field. There are only two completions of $\mathbf{D}^b(\text{mod-}K)$, namely $\mathbf{D}^b(\text{mod-}K)$ and 0.

There are only five completions of $\mathbf{D}^b(\text{mod-}KA_2)$, namely all the thick subcategories $\mathbf{D}^b(\text{mod-}KA_2)$, $\langle S(1) \rangle$, $\langle S(2) \rangle$, $\langle P(2) \rangle$, and 0 (see e.g. [16, Example 5.1.3]).

We finish the section by a non-example.

Example 3.11. Theorem 3.7 does not hold for a general hereditary ring. Let us set $\mathcal{T} := \mathbf{D}^b(\text{mod-}\mathbb{Z})$ and let \mathcal{S} be the triangulated subcategory of \mathcal{T} generated by finitely

generated torsion abelian groups. We consider the constant metric $\{\mathcal{S}\}_{n \in \mathbb{N}}$ on \mathcal{T} . We claim that $\mathfrak{S}'(\mathcal{T}) = \langle \mathbb{Q} \rangle \subseteq \mathbf{D}(\text{Mod-}\mathbb{Z})$. Thanks to Proposition 3.1, we know so far that $\mathfrak{S}'(\mathcal{T}) \subseteq \mathbf{D}^b(\text{Mod-}\mathbb{Z})$.

Let $E = \text{Hocolim} E_* \in \mathfrak{S}'(\mathcal{T})$ for some good Cauchy sequence E_* in \mathcal{T} . Let $i \in \mathbb{Z}$ and $M := H^i(E) \in \text{Mod-}\mathbb{Z}$. Then $\Sigma^{-i}M$ is compactly supported as a direct summand of $E \in \mathfrak{C}'(\mathcal{T})$. Hence $\text{Hom}_{\mathbf{D}(\text{Mod-}\mathbb{Z})}(\mathcal{S}, \Sigma^{-i}M) = 0$. We infer that

$$\text{Hom}_{\mathbb{Z}} \left(\bigoplus_{n=2}^{\infty} \mathbb{Z}/n\mathbb{Z}, M \right) \simeq 0 \simeq \text{Ext}_{\mathbb{Z}}^1 \left(\bigoplus_{n=2}^{\infty} \mathbb{Z}/n\mathbb{Z}, M \right),$$

so M is torsion-free and injective (by Baer's criterion [4, Theorem 1], see also [1, Lemma 18.3] or [34, Proposition 1.2]). Therefore $M \simeq \mathbb{Q}^{(J)}$ for some indexing set J .

To show that J is finite, we use an argument analogous to the proof of Lemma 3.2. Since $\Sigma^{-i}\mathbb{Q} \in \mathbf{D}(\text{Mod-}\mathbb{Z})$ is compactly supported, Lemma 2.19 gives us $N \in \mathbb{N}$ such that the map

$$\text{Hom}_{\mathbf{D}(\text{Mod-}\mathbb{Z})}(\varphi, \Sigma^{-i}\mathbb{Q}) : \text{Hom}_{\mathbf{D}(\text{Mod-}\mathbb{Z})}(E, \Sigma^{-i}\mathbb{Q}) \rightarrow \text{Hom}_{\mathbf{D}(\text{Mod-}\mathbb{Z})}(E_N, \Sigma^{-i}\mathbb{Q})$$

is an abelian group isomorphism (so a bijection), where $\varphi : E_N \rightarrow E$ is the canonical map. But $\text{Hom}_{\mathbf{D}(\text{Mod-}\mathbb{Z})}(E_N, \Sigma^{-i}\mathbb{Q})$ is at most countable as a set. If J was infinite, we would get that

$$\text{Hom}_{\mathbf{D}(\text{Mod-}\mathbb{Z})}(E, \Sigma^{-i}\mathbb{Q}) \simeq \text{Hom}_{\mathbf{D}(\text{Mod-}\mathbb{Z})}(\Sigma^{-i}\mathbb{Q}^{(J)}, \Sigma^{-i}\mathbb{Q}) \simeq \mathbb{Q}^J$$

is uncountable, which is impossible. Hence J is finite.

So far we have shown that $\mathfrak{S}'(\mathcal{T}) \subseteq \langle \mathbb{Q} \rangle$. To prove the equality, it is enough to observe that $\mathbb{Q} \in \mathfrak{L}'(\mathcal{T})$. We look at the following diagram in $\text{mod-}\mathbb{Z}$

$$C_* := \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \xrightarrow{3\cdot} \mathbb{Z} \xrightarrow{4\cdot} \mathbb{Z} \hookrightarrow \dots$$

with $\varinjlim C_* = \mathbb{Q}$ in $\text{Mod-}\mathbb{Z}$. We have for all $i \in \mathbb{N}$ that $\text{Coker}(i \cdot -)$ is a torsion group, so if we view \mathbb{Z} as a complex concentrated in degree 0 and interpret C_* as a diagram in $\mathbf{D}^b(\text{mod-}\mathbb{Z})$, then C_* is Cauchy with $\text{Hocolim} C_* = \mathbb{Q}$ in $\mathbf{D}(\text{Mod-}\mathbb{Z})$. We conclude $\mathfrak{S}'(\mathcal{T}) = \langle \mathbb{Q} \rangle$.

4 Functors between completions

This section is divided into three parts. In Subsection 4.1, we elaborate on the theory of induced functors on completions from [33]. Subsection 4.2 contains the main new result of this section (Theorem 4.23) - we introduce the preimage and image metrics on triangulated categories and use them to induce equivalences between their completions. We give an application of the result for categories with Serre functors in Subsection 4.3.

4.1 Induced functors on completions

In this subsection, we recall one of the main results from [33] by Sun and Zhang, namely Theorem 1.1 (later stated as Theorem 4.3). They prove that an adjoint pair of triangulated

functors between triangulated categories equipped with metrics induces a triangulated functor between the respective completions, assuming that the adjoint functors are, in some sense, “continuous” with respect to the metrics. The proper term used by Sun and Zhang is that of a compression functor.

Since the original theorem deals with good metrics only, we reprove it here in the full generality of (potentially) non-good metrics. The proofs in [33] require only minimal adjustments to include the stronger hypothesis, but we repeat them for the convenience of the reader.

Definition 4.1 ([33, Definition 4.1]). Let \mathcal{T}, \mathcal{S} be triangulated categories with metrics $\{B_n\}_{n \in \mathbb{N}}$ and $\{C_n\}_{n \in \mathbb{N}}$, respectively. We say that a triangulated functor $F : \mathcal{T} \rightarrow \mathcal{S}$ is a *compression* if for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $F(B_m) \subseteq C_n$.

Remark 4.2. Let $\{B_n\}_{n \in \mathbb{N}}$ and $\{C_n\}_{n \in \mathbb{N}}$ be metrics on a triangulated category \mathcal{T} . Notice that $\{B_n\}_{n \in \mathbb{N}} \leq \{C_n\}_{n \in \mathbb{N}}$ if and only if the identity functor $\text{id} : \mathcal{T} \rightarrow \mathcal{T}$ from \mathcal{T} equipped with the metric $\{B_n\}_{n \in \mathbb{N}}$ to \mathcal{T} equipped with the metric $\{C_n\}_{n \in \mathbb{N}}$ is a compression.

Lemma 4.3 ([33, Lemma 4.2]). Let \mathcal{T}, \mathcal{S} be triangulated categories with metrics $\{B_n\}_{n \in \mathbb{N}}$ and $\{C_n\}_{n \in \mathbb{N}}$, respectively, and let $F : \mathcal{T} \rightarrow \mathcal{S}$ be a compression. Then F preserves (good) Cauchy sequences.

Proof. We prove the statement only for good Cauchy sequences, because the other case is analogous.

Let $E_* := E_1 \rightarrow E_2 \rightarrow \dots$ be a good Cauchy sequence in \mathcal{T} . We check that its image $F(E_*) := F(E_1) \rightarrow F(E_2) \rightarrow \dots$ is a good Cauchy sequence in \mathcal{S} . Let $i \in \mathbb{N}$ and $j \in \mathbb{Z}$. We find $k \in \mathbb{N}$ such that $F(B_k) \subseteq C_i$. We find $M \in \mathbb{N}$ such that for every $m' \geq m \geq M$ the triangle $E_m \rightarrow E_{m'} \rightarrow D_{m,m'} \rightarrow \Sigma E_m$ satisfies $\Sigma^j D_{m,m'} \in B_k$. We have that

$$F(E_m) \rightarrow F(E_{m'}) \rightarrow F(D_{m,m'}) \rightarrow \Sigma F(E_m)$$

is also a triangle because F is a triangulated functor. Since $\Sigma^j D_{m,m'} \in B_k$, it holds that $F(\Sigma^j D_{m,m'}) \simeq \Sigma^j F(D_{m,m'}) \in C_i$. Thus, the sequence $F(E_*)$ is good Cauchy. \square

Lemma 4.4 ([33, Theorem 4.3, pt. 2]). Let \mathcal{T}, \mathcal{S} be triangulated categories with metrics $\{B_n\}_{n \in \mathbb{N}}$ and $\{C_n\}_{n \in \mathbb{N}}$, respectively, and let $F : \mathcal{T} \rightarrow \mathcal{S}$ be a compression.

Then $- \circ F : \text{Mod-}\mathcal{S} \rightarrow \text{Mod-}\mathcal{T}$ induces functors

$$- \circ F : \mathfrak{C}(\mathcal{S}) \rightarrow \mathfrak{C}(\mathcal{T}) \quad \text{and} \quad - \circ F : \mathfrak{WC}(\mathcal{S}) \rightarrow \mathfrak{WC}(\mathcal{T}).$$

Proof. We prove the statement only for compactly supported elements, because the other case is analogous.

Let $G \in \mathfrak{C}(\mathcal{S}) = \bigcap_{j \in \mathbb{Z}} \bigcup_{i \in \mathbb{N}} Y(\Sigma^j C_i)^\perp$. Let $j \in \mathbb{Z}$. We find $i, k \in \mathbb{N}$ such that $G \in Y(\Sigma^j C_i)^\perp$ and $F(B_k) \subseteq C_i$. Let $X \in B_k$. By Yoneda’s lemma, we have

$$\begin{aligned} \text{Hom}\left(Y(\Sigma^j X), G \circ F\right) &\simeq G\left(F(\Sigma^j X)\right) \simeq \text{Hom}\left(Y\left(F(\Sigma^j X)\right), G\right) \\ &\simeq \text{Hom}\left(Y(\Sigma^j F(X)), G\right) = 0 \end{aligned}$$

since $F(X) \in C_i$ and $G \in Y(\Sigma^j C_i)^\perp$.

Because j was arbitrary, we obtain $G \circ F \in \mathfrak{C}(\mathcal{T}) = \bigcap_{j \in \mathbb{Z}} \bigcup_{i \in \mathbb{N}} Y(\Sigma^j B_i)^\perp$. \square

Lemma 4.5 ([33, Theorem 4.3, pt. 1]). Let $F : \mathcal{T} \rightleftarrows \mathcal{S} : G$ be adjoint triangulated functors between triangulated categories with metrics $\{B_n\}_{n \in \mathbb{N}}$ and $\{C_n\}_{n \in \mathbb{N}}$, respectively. If G is a compression, then $- \circ F : \text{Mod-}\mathcal{S} \rightarrow \text{Mod-}\mathcal{T}$ induces a functor $- \circ F : \mathfrak{L}(\mathcal{S}) \rightarrow \mathfrak{L}(\mathcal{T})$.

In particular, if $E = \text{Mocolim}_{\rightarrow} E_* \in \mathfrak{L}(\mathcal{S})$ for some good Cauchy sequence E_* in \mathcal{S} , then $E \circ F \simeq \text{Mocolim}_{\rightarrow} G(E_*)$.

Proof. Let $E \simeq \text{Mocolim}_{\rightarrow} E_* \in \mathfrak{L}(\mathcal{S})$ be as in the statement of the lemma. Then by adjunction, we obtain

$$\begin{aligned} E \circ F &= \left(\text{Mocolim}_{\rightarrow} E_* \right) \circ F \simeq \varinjlim \text{Hom}_{\mathcal{S}}(F(-), E_*) \simeq \varinjlim \text{Hom}_{\mathcal{T}}(-, G(E_*)) \\ &= \text{Mocolim}_{\rightarrow} G(E_*), \end{aligned}$$

where $\text{Mocolim}_{\rightarrow} G(E_*) \in \mathfrak{L}(\mathcal{T})$ because the compression functor G preserves good Cauchy sequences by Lemma 4.3. \square

We will heavily rely on the following well-known category-theoretical fact and its consequences.

Proposition 4.6 ([23, Theorem IV.3.1]). Let $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ be adjoint functors. Let $\eta : 1_{\mathcal{A}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow 1_{\mathcal{B}}$ be the unit and the counit of the adjunction, respectively. Then F is fully faithful if and only if η is a natural isomorphism. In this case, G is essentially surjective.

Dually, G is fully faithful if and only if ε is a natural isomorphism. And in this case, F is essentially surjective.

For the next corollary, we recall that a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is *conservative* provided that whenever $F(f)$ is an isomorphism in \mathcal{B} for a morphism f in \mathcal{A} , then f has to be an isomorphism as well.

Corollary 4.7. Let $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ be adjoint functors. If one of the functors F , G is conservative and the other one is fully faithful, then F and G are mutually inverse equivalences of categories.

Proof. By symmetry, it is sufficient to assume that F is conservative and G is fully faithful. Let $\eta : 1_{\mathcal{A}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow 1_{\mathcal{B}}$ be the unit and the counit of the adjunction, respectively.

Let $A \in \mathcal{A}$. By the adjunction, the identity map $1_{F(A)}$ decomposes as

$$F(A) \xrightarrow{F(\eta_A)} FGF(A) \xrightarrow{\varepsilon_{F(A)}} F(A).$$

By Proposition 4.6, the map $\varepsilon_{F(A)}$ is an isomorphism, so $F(\eta_A)$ is an isomorphism as well. Since F is conservative, we have that $\eta_A : A \rightarrow GF(A)$ is an isomorphism. As A was arbitrary, we obtain that G is essentially surjective.

We conclude that G is an equivalence of categories. By the uniqueness of adjoints, F is its inverse. \square

Corollary 4.8 ([33, Corollary 3.2]). Let $F : \mathcal{A} \rightleftarrows \mathcal{B} : G$ be adjoint additive functors between additive categories. Then $- \circ F : \text{Mod-}\mathcal{B} \rightleftarrows \text{Mod-}\mathcal{A} : - \circ G$ is an adjoint pair.

Furthermore, if G is fully faithful, then $- \circ F$ is fully faithful. Conversely, if F is fully faithful, then $- \circ G$ is fully faithful.

Finally, we can present the theorem by Sun and Zhang about induced functors. As mentioned above, the statement presented here is more general than in the original article because it also concerns metrics which are not good.

Theorem 4.9 ([33, Theorem 4.3, pt. 3]). Let $F : \mathcal{T} \rightleftarrows \mathcal{S} : G$ be adjoint triangulated functors between triangulated categories with metrics $\{B_n\}_{n \in \mathbb{N}}$ and $\{C_n\}_{n \in \mathbb{N}}$, respectively. If F and G are compressions, then the functor $- \circ F : \text{Mod-}\mathcal{S} \rightarrow \text{Mod-}\mathcal{T}$ induces a triangulated functor $- \circ F : \mathfrak{S}(\mathcal{S}) \rightarrow \mathfrak{S}(\mathcal{T})$.

Proof. By Lemma 4.4 and Lemma 4.5, we get that the functor $- \circ F : \mathfrak{S}(\mathcal{S}) \rightarrow \mathfrak{S}(\mathcal{T})$ is well-defined. We only need to check that it is triangulated.

By Theorem 2.11, we know that $\mathfrak{S}(\mathcal{S}), \mathfrak{S}(\mathcal{T})$ are triangulated categories with shifts given by $\Sigma E := E \circ \Sigma^{-1}$ for either $E \in \mathfrak{S}(\mathcal{S})$, or $E \in \mathfrak{S}(\mathcal{T})$, respectively. (For clarity, in this proof we denote shifts in completions by bold Σ instead of Σ .)

We define a natural isomorphism $\psi : (- \circ F) \circ \Sigma \Rightarrow \Sigma \circ (- \circ F)$ with components ψ_E given for each $E \in \mathfrak{S}(\mathcal{S})$ by the equation

$$\begin{aligned} [(- \circ F) \circ \Sigma](E) &= (- \circ F)(E \circ \Sigma^{-1}) = E \circ \Sigma^{-1} \circ F \simeq E \circ F \circ \Sigma^{-1} \\ &= \Sigma(E \circ F) = [\Sigma \circ (- \circ F)](E) \end{aligned}$$

where the existence of the natural isomorphism in the middle follows from the fact that F is a triangulated functor.

We fix a triangle in $\mathfrak{S}(\mathcal{S})$, which is necessarily isomorphic to

$$\text{Mocolim}_{\rightarrow} A_* \xrightarrow{a} \text{Mocolim}_{\rightarrow} B_* \xrightarrow{b} \text{Mocolim}_{\rightarrow} C_* \xrightarrow{c} \Sigma \left(\text{Mocolim}_{\rightarrow} A_* \right)$$

for a good Cauchy sequence of triangles $A_* \rightarrow B_* \rightarrow C_* \rightarrow \Sigma A_*$ in \mathcal{S} .

Under the notation for natural transformations from Subsection 2.1.1 and after setting $\delta := \psi_{\text{Mocolim}_{\rightarrow} A_* \circ c_F}$, the candidate triangle

$$\begin{aligned} \left(\text{Mocolim}_{\rightarrow} A_* \right) \circ F \xrightarrow{a_F} \left(\text{Mocolim}_{\rightarrow} B_* \right) \circ F \xrightarrow{b_F} \\ \xrightarrow{b_F} \left(\text{Mocolim}_{\rightarrow} C_* \right) \circ F \xrightarrow{\delta} \Sigma \left[\left(\text{Mocolim}_{\rightarrow} A_* \right) \circ F \right] \end{aligned}$$

is isomorphic (as a candidate triangle) to the distinguished triangle

$$\text{Mocolim}_{\rightarrow} G(A_*) \rightarrow \text{Mocolim}_{\rightarrow} G(B_*) \rightarrow \text{Mocolim}_{\rightarrow} G(C_*) \rightarrow \Sigma \left(\text{Mocolim}_{\rightarrow} G(A_*) \right),$$

by Lemma 4.5 and the fact that G is a compression. Hence, it is a distinguished triangle itself. Consequently, the functor $- \circ F$ is triangulated. \square

Corollary 4.10 ([33, Corollary on pg. 102]). Let $F : \mathcal{T} \rightleftarrows \mathcal{S} : G$ be adjoint triangulated functors between triangulated categories with metrics $\{B_n\}_{n \in \mathbb{N}}$ and $\{C_n\}_{n \in \mathbb{N}}$, respectively. If F and G are compressions and G has a right adjoint which is a compression, then the triangulated functors $- \circ F : \mathfrak{S}(\mathcal{S}) \rightleftarrows \mathfrak{S}(\mathcal{T}) : - \circ G$ form an adjoint pair.

Proof. The right adjoint to the triangulated functor G is automatically triangulated by [25, Lemma 5.3.6]. The rest then straightforwardly follows from Theorem 4.9 and Corollary 4.8. \square

4.2 Preimages and images of metrics

This subsection begins with a brief overview. Precise definitions and statements follow.

Assume we have a triangulated functor $F : \mathcal{T} \rightarrow \mathcal{S}$. Given a metric on \mathcal{S} , we construct a so-called preimage metric on \mathcal{T} rendering the functor F a compression with respect to these two metrics. Conversely, we want to define an image of a metric on \mathcal{T} under \mathcal{S} - however, this construction is only possible while F is full.

We then use the image and preimage metrics together with the theory of induced functors between completions from [33] recalled in Subsection 4.1 to prove that under certain condition on an existence of sufficiently nice adjoints we can calculate a completion of a triangulated category as a completion of a different triangulated category with respect to a preimage metric (see Theorem 4.23). The conditions for the proposition to work are satisfied in the presence of a Serre functor (see Subsection 4.3).

Definition 4.11. Let \mathcal{T}, \mathcal{S} be triangulated categories and $F : \mathcal{T} \rightarrow \mathcal{S}$ be a functor. Let $\{B_n\}_{n \in \mathbb{N}}$ be a metric on \mathcal{S} . We define *the preimage of $\{B_n\}_{n \in \mathbb{N}}$ under F* as the sequence $\{F^{-1}(B_n)\}_{n \in \mathbb{N}}$ of subcategories of \mathcal{T} .

Lemma 4.12. Let \mathcal{T}, \mathcal{S} be triangulated categories and $F : \mathcal{T} \rightarrow \mathcal{S}$ be a triangulated functor with its natural isomorphism $\varphi : F\Sigma \Rightarrow \Sigma F$. If $\{B_n\}_{n \in \mathbb{N}}$ is a metric on \mathcal{S} , then $\{F^{-1}(B_n)\}_{n \in \mathbb{N}}$ is a metric on \mathcal{T} . Furthermore, if $\{B_n\}_{n \in \mathbb{N}}$ is good, so is $\{F^{-1}(B_n)\}_{n \in \mathbb{N}}$.

Proof. Since $0 \in B_n$ for all $n \in \mathbb{N}$ and F is additive, we have $0 \in F^{-1}(B_n)$. Also $F^{-1}(B_1) \supseteq F^{-1}(B_2) \supseteq F^{-1}(B_3) \supseteq \dots$ is a non-increasing chain of subcategories.

We need to check that $F^{-1}(B_n) * F^{-1}(B_n) = F^{-1}(B_n)$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. We fix a triangle $U \xrightarrow{f} V \xrightarrow{g} W \xrightarrow{h} \Sigma U$ in \mathcal{T} with $U, W \in F^{-1}(B_n)$. We know that

$$F(U) \xrightarrow{F(f)} F(V) \xrightarrow{F(g)} F(W) \xrightarrow{\varphi_{U \circ F(h)}} \Sigma F(U)$$

is a triangle because F is triangulated. As $F(U), F(W) \in B_n$, so is $F(V)$ since B_n is extension-closed. Hence $V \in F^{-1}(B_n)$, and $F^{-1}(B_n)$ is also extension-closed. This finishes the proof of the fact that the preimage of a metric is a metric.

Now for the proof of the second statement, assume that the metric $\{B_n\}_{n \in \mathbb{N}}$ is good. In order to verify that the metric $\{F^{-1}(B_n)\}_{n \in \mathbb{N}}$ is also good, we need to check

$$\Sigma^{-1}F^{-1}(B_{n+1}) \cup F^{-1}(B_{n+1}) \cup \Sigma F^{-1}(B_{n+1}) \subseteq F^{-1}(B_n)$$

for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. We already know $F^{-1}(B_{n+1}) \subseteq F^{-1}(B_n)$. Let us have $X \in F^{-1}(B_{n+1})$. Because $F(X) \in B_{n+1}$, we get $\Sigma F(X) \in B_n$. However, there is an isomorphism $\varphi_X : F\Sigma(X) \rightarrow \Sigma F(X)$, and since B_n is extension-closed (thus closed under isomorphisms), it holds that $\Sigma X \in F^{-1}(B_n)$. The proof that $\Sigma^{-1}X \in F^{-1}(B_n)$ is analogous. \square

Definition 4.13. Let \mathcal{T}, \mathcal{S} be triangulated categories and $F : \mathcal{T} \rightarrow \mathcal{S}$ be a functor. Let $\{B_n\}_{n \in \mathbb{N}}$ be a metric on \mathcal{T} . We define *the image of $\{B_n\}_{n \in \mathbb{N}}$ under F* as the sequence $\{F(B_n)\}_{n \in \mathbb{N}}$, where $F(B_n)$ denotes the subcategory

$$F(B_n) = \{Z \in \mathcal{S} : Z \simeq F(X), X \in B_n\} \quad \text{for any } n \in \mathbb{N}.$$

Lemma 4.14. Let \mathcal{T}, \mathcal{S} be triangulated categories and $F : \mathcal{T} \rightarrow \mathcal{S}$ be a full triangulated functor with its natural isomorphism $\varphi : F\Sigma \Rightarrow \Sigma F$. If $\{B_n\}_{n \in \mathbb{N}}$ is a metric on \mathcal{S} , then $\{F(B_n)\}_{n \in \mathbb{N}}$ is a metric on \mathcal{T} .

Furthermore, if $\{B_n\}_{n \in \mathbb{N}}$ is a good metric on \mathcal{S} , then $\{F(B_n)\}_{n \in \mathbb{N}}$ is a good metric on \mathcal{T} .

Proof. Since $0 \in B_n$ for all $n \in \mathbb{N}$ and F is additive, we have $0 \in F(B_n)$. Also $\mathcal{S} \supseteq F(B_2) \supseteq F(B_3) \supseteq \dots$ is a non-increasing chain of subcategories.

We need to check that $F(B_n) * F(B_n) = F(B_n)$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. We fix a triangle in \mathcal{S} of the form

$$F(X) \rightarrow V \rightarrow F(Z) \xrightarrow{w} \Sigma F(X)$$

for some $X, Z \in B_n$. Using the fullness of F , we find a map

$$h : Z \rightarrow \Sigma X \text{ satisfying } F(h) = \varphi_X^{-1} \circ w : F(Z) \rightarrow F(\Sigma X).$$

We complete h to a triangle $X \xrightarrow{f} U \xrightarrow{g} Z \xrightarrow{h} \Sigma X$. Since $B_n * B_n = B_n$, we have $U \in B_n$. We know that

$$F(X) \xrightarrow{F(f)} F(U) \xrightarrow{F(g)} F(Z) \xrightarrow{\varphi_X \circ F(h)} \Sigma F(X)$$

is a triangle because F is triangulated. And since $\varphi_X \circ F(h) = w$, this triangle is isomorphic to the triangle

$$F(X) \rightarrow V \rightarrow F(Z) \xrightarrow{w} \Sigma F(X).$$

In particular, it holds that $V \simeq F(U) \in F(B_n)$, so $F(B_n)$ is extension closed.

Now furthermore assume that the metric $\{B_n\}_{n \in \mathbb{N}}$ is good. In order to verify that the metric $\{F(B_n)\}_{n \in \mathbb{N}}$ is also good, we need to check

$$\Sigma^{-1}F(B_{n+1}) \cup F(B_{n+1}) \cup \Sigma F(B_{n+1}) \subseteq F(B_n)$$

for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. We already know that it holds that $F(B_{n+1}) \subseteq F(B_n)$. Let $F(X) \in F(B_{n+1})$ for some $X \in B_{n+1}$. We get $\Sigma X \in B_n$. However, there is an isomorphism $\varphi_X : F\Sigma(X) \rightarrow \Sigma F(X)$. Thus $\Sigma F(X) \simeq F\Sigma(X) \in F(B_n)$.

The proof that $\Sigma^{-1}X \in F^{-1}(B_n)$ is analogous. \square

Example 4.15. The statement of Lemma 4.14 does hold in general without the assumption on the functor $F : \mathcal{T} \rightarrow \mathcal{S}$ being full.

Let K_1 and K_2 be two copies of the same field K and consider the coproduct category $\mathcal{T} := \mathbf{D}^b(\text{mod-}K_1) \amalg \mathbf{D}^b(\text{mod-}K_2)$. We set $\mathcal{S} := \mathbf{D}^b(\text{mod-}KA_2)$ to be the bounded derived category of the path algebra of the Dynkin quiver A_2 . The notation $S(1)$ and $S(2)$ will stand for the two non-isomorphic simple modules in the module category $\text{mod-}KA_2$.

Then the triangulated functor $F : \mathcal{T} \rightarrow \mathcal{S}$ given by $K_1 \mapsto S(1)$ and $K_2 \mapsto S(2)$ is not full, and $F(\mathcal{T}) \subsetneq \mathcal{S}$, but $F(\mathcal{T}) * F(\mathcal{T}) = \mathcal{S}$. If we interpret the category \mathcal{T} as a constant metric $\{\mathcal{T}\}_{n \in \mathbb{N}}$ on itself, then the image $\{F(\mathcal{T})\}_{n \in \mathbb{N}}$ of $\{\mathcal{T}\}_{n \in \mathbb{N}}$ is not a metric on \mathcal{S} at all because $F(\mathcal{T})$ is not extension-closed.

We prove a relation between a preimage metric under a functor and an image metric under a fully faithful adjoint.

Lemma 4.16. Let $F : \mathcal{T} \rightleftarrows \mathcal{S} : G$ be adjoint triangulated functors between triangulated categories. Let $\{B_n\}_{n \in \mathbb{N}}$ be a metric on \mathcal{T} .

If F is fully faithful, then $\{F(B_n)\}_{n \in \mathbb{N}} \leq \{G^{-1}(B_n)\}_{n \in \mathbb{N}}$.

Proof. We know that $\{F(B_n)\}_{n \in \mathbb{N}}$ and $\{G^{-1}(B_n)\}_{n \in \mathbb{N}}$ are metrics by Lemma 4.14 and Lemma 4.12, respectively. For all $n \in \mathbb{N}$ we have $GF(B_n) = B_n$ because of the natural isomorphism $1_{\mathcal{T}} \simeq GF$ from Proposition 4.6 and the fact that both subcategories are closed under isomorphisms. Hence $F(B_n) \subseteq G^{-1}(B_n)$, so $\{F(B_n)\}_{n \in \mathbb{N}} \leq \{G^{-1}(B_n)\}_{n \in \mathbb{N}}$. \square

Lemma 4.17. Let $F : \mathcal{T} \rightleftarrows \mathcal{S} : G$ be adjoint triangulated functors between triangulated categories. Let $\{B_n\}_{n \in \mathbb{N}}$ be a metric on \mathcal{S} .

If G is fully faithful, then $\{G(B_n)\}_{n \in \mathbb{N}} \leq \{F^{-1}(B_n)\}_{n \in \mathbb{N}}$.

Proof. Dual to the proof of Lemma 4.16. \square

The following two lemmas help us identify which Cauchy sequences have a trivial module colimit. To prove Lemma 4.18, we use an argument from [15, Remark 3.14], where it is shown that $\text{Mocolim} E_* \simeq 0$ if and only if $E_* := E_1 \xrightarrow{f_{1,2}} E_2 \xrightarrow{f_{2,3}} \dots$ is a so-called *null sequence*, i.e. a sequence where for every $m \in \mathbb{N}$ we can find $n \geq m$ satisfying $f_{m,n} = 0$.

Lemma 4.18. Let \mathcal{T} be a triangulated category with a metric $\{B_n\}_{n \in \mathbb{N}}$. Let

$$E_* := E_1 \xrightarrow{f_{1,2}} E_2 \xrightarrow{f_{2,3}} \dots$$

be a good Cauchy sequence. If $\text{Mocolim} E_* \simeq 0$ in $\mathfrak{L}(\mathcal{T})$, then for every $i \in \mathbb{N}$ and $j \in \mathbb{Z}$ there exists $M \in \mathbb{N}$ such that for all $m \geq M$ we have $\Sigma^j E_m \in \text{Summ}(B_i)$.

Proof. Fix $i \in \mathbb{N}$ and $j \in \mathbb{Z}$. We find $M \in \mathbb{N}$ such that for every $m' \geq m \geq M$ the triangle $E_m \xrightarrow{f_{m,m'}} E_{m'} \rightarrow D_{m,m'} \rightarrow \Sigma E_m$ satisfies $\Sigma^{j-1} D_{m,m'} \in B_i$. Let us have $m \geq M$. Since $\text{Mocolim} E_* \simeq 0$ in $\mathfrak{L}(\mathcal{T})$, we have, in particular, that $\varinjlim_{n \in \mathbb{N}} \text{Hom}_{\mathcal{T}}(E_m, E_n) \simeq 0$ in Ab . Then [17, Lemma 2.2] implies that there exists some $n \geq m$ satisfying $f_{m,n} = 0$.

Consider the triangle $E_m \xrightarrow{f_{m,n}} E_n \rightarrow D_{m,n} \rightarrow \Sigma E_m$. Using $n \geq m \geq M$, we obtain $\Sigma^{j-1} D_{m,n} \in B_i$. Since $f_{m,n} = 0$, this triangle splits, so $D_{m,n} \simeq E_n \oplus \Sigma E_m$. But then

$$\Sigma^{j-1} D_{m,n} \simeq \Sigma^{j-1} E_n \oplus \Sigma^j E_m \in B_i.$$

Hence, we get $\Sigma^j E_m \in \text{Summ}(B_i)$. \square

Lemma 4.19. Let \mathcal{T} be a triangulated category with a metric $\{B_n\}_{n \in \mathbb{N}}$. Let E_* be a good Cauchy sequence satisfying $\text{Mocolim} E_* \in \mathfrak{WC}(\mathcal{T})$. Then the following statements are equivalent:

- (i) $\text{Mocolim} E_* \simeq 0$.
- (ii) For every $i \in \mathbb{N}$ and $j \in \mathbb{Z}$ there exists $M \in \mathbb{N}$ such that for all $m \geq M$ we have that $\Sigma^j E_m \in \text{Summ}(B_i)$.
- (iii) For every $i \in \mathbb{N}$ there exists $M \in \mathbb{N}$ such that for all $m \geq M$ we have that $E_m \in \text{Summ}(B_i)$.

Proof. “i) \Rightarrow ii)” By Lemma 4.18.

“ii) \Rightarrow iii)” Condition iii) is a special case of ii) when $j = 1$.

“iii) \Rightarrow i)” Since $\text{Mocolim} E_* \in \mathfrak{WC}(\mathcal{T})$, we know that there exists an index $i \in \mathbb{N}$ such that $\text{Mocolim} E_* \in Y(B_i)^\perp$.

Using the condition on E_* from the statement of the lemma, we may, without loss of generality, assume that for every $n \in \mathbb{N}$ it holds that $E_n \in \text{Summ}(B_i)$ because otherwise we can find $M \in \mathbb{N}$ such that for every $m \geq M$ it holds that $E_m \in \text{Summ}(B_i)$ and then cut off objects E_k for small indices $0 \leq k < M$.

Let $n \in \mathbb{N}$ and let $F_n \in \mathcal{T}$ be such that $E_n \oplus F_n \in B_i$. Then after applying Yoneda’s embedding we get $Y(E_n \oplus F_n) = Y(E_n) \oplus Y(F_n)$, so

$$\begin{aligned} 0 &\simeq \text{Hom} \left(Y(E_n \oplus F_n), \text{Mocolim} E_* \right) \\ &\simeq \text{Hom} \left(Y(E_n), \text{Mocolim} E_* \right) \oplus \text{Hom} \left(Y(F_n), \text{Mocolim} E_* \right). \end{aligned}$$

In particular, we have $0 \simeq \text{Hom} \left(Y(E_n), \text{Mocolim} E_* \right)$.

Finally, we obtain

$$\begin{aligned} \text{End} \left(\text{Mocolim} E_* \right) &= \text{Hom} \left(\varinjlim_{n \in \mathbb{N}} Y(E_n), \text{Mocolim} E_* \right) \\ &\simeq \varprojlim_{n \in \mathbb{N}} \text{Hom} \left(Y(E_n), \text{Mocolim} E_* \right) \simeq \varprojlim_{n \in \mathbb{N}} 0 \simeq 0, \end{aligned}$$

so the identity on $\text{Mocolim} E_*$ is equal to the zero morphism.

Hence, we get that $\text{Mocolim} E_* \simeq 0$. \square

We recall a well-know characterisation of conservative triangulated functors (see for example [9, Example 4.11]).

Lemma 4.20. Let \mathcal{T}, \mathcal{S} be triangulated categories and let $F : \mathcal{T} \rightarrow \mathcal{S}$ be a triangulated functor. Then F is conservative if and only if $\text{Ker } F = 0$.

We prove two more technical lemmas and then we proceed with the main result of this section.

Lemma 4.21. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an essentially surjective additive functor between additive categories. Let $\mathcal{C} \subseteq \text{Ob}(\mathcal{B})$ be a class of objects. Then

$$F^{-1}(\text{Summ}(\mathcal{C})) = \text{Summ}(F^{-1}(\mathcal{C})).$$

Proof. Without loss of generality assume that \mathcal{C} is closed under isomorphisms.

“ \subseteq ” Let $X \in F^{-1}(\text{Summ}(\mathcal{C}))$. Then $F(X) \in \text{Summ}(\mathcal{C})$, so there is some $Y \in \mathcal{B}$ such that $F(X) \oplus Y \in \mathcal{C}$. Since F is essentially surjective, we have $Y \simeq F(Z)$ for some $Z \in \mathcal{A}$. Then $F(X \oplus Z) \simeq F(X) \oplus Y \in \mathcal{C}$, so $X \oplus Z \in F^{-1}(\mathcal{C})$. Hence $X \in \text{Summ}(F^{-1}(\mathcal{C}))$.

“ \supseteq ” Let $X \in \text{Summ}(F^{-1}(\mathcal{C}))$. Then there is $Y \in \mathcal{A}$ such that $X \oplus Y \in F^{-1}(\mathcal{C})$. It follows that $F(X) \oplus F(Y) \simeq F(X \oplus Y) \in \mathcal{C}$, so $F(X) \in \text{Summ}(\mathcal{C})$. In other words, we have $X \in F^{-1}(\text{Summ}(\mathcal{C}))$. \square

Lemma 4.22. Let $F : \mathcal{T} \rightleftarrows \mathcal{S} : G$ be adjoint triangulated functors between triangulated categories. Let $\{B_n\}_{n \in \mathbb{N}}$ be a metric on \mathcal{T} . Denote the completion of \mathcal{T} with respect to $\{B_n\}_{n \in \mathbb{N}}$ as $\mathfrak{S}(\mathcal{T})$ and the completion of \mathcal{S} with respect to the preimage metric $\{G^{-1}(B_n)\}_{n \in \mathbb{N}}$ as $\mathfrak{S}(\mathcal{S})$.

If F is fully faithful, then the induced functor $- \circ F : \mathfrak{S}(\mathcal{S}) \rightarrow \mathfrak{S}(\mathcal{T})$ is a well-defined conservative triangulated functor.

Proof. We know that $\{G^{-1}(B_n)\}_{n \in \mathbb{N}}$ and $\{F(B_n)\}_{n \in \mathbb{N}}$ are metrics by Lemma 4.12 and Lemma 4.14. The functor G is trivially a compression with respect to the metrics

$$\{G^{-1}(B_n)\}_{n \in \mathbb{N}} \quad \text{and} \quad \{B_n\}_{n \in \mathbb{N}}.$$

As F is fully faithful, we have $\{F(B_n)\}_{n \in \mathbb{N}} \leq \{G^{-1}(B_n)\}_{n \in \mathbb{N}}$ by Lemma 4.16, so F is a compression with respect to $\{B_n\}_{n \in \mathbb{N}}$ and $\{G^{-1}(B_n)\}_{n \in \mathbb{N}}$. Therefore, we may use Theorem 4.9 to establish that the functor $- \circ F : \mathfrak{S}(\mathcal{S}) \rightarrow \mathfrak{S}(\mathcal{T})$ is well-defined and triangulated.

It remains to be checked that $- \circ F$ is conservative. Let $E \in \mathfrak{S}(\mathcal{S})$ satisfy $E \circ F = 0$. We know that $E \simeq \text{Mocolim} E_*$ for some good Cauchy sequence E_* in \mathcal{S} . By Lemma 4.5, we have $E \circ F \simeq \text{Mocolim} G(E_*)$ where $G(E_*)$ is a good Cauchy sequence by Lemma 4.3. By Lemma 4.18, the good Cauchy sequence $G(E_*)$ satisfies that for every $i \in \mathbb{N}$ there exists $M \in \mathbb{N}$ such that for all $m \geq M$ we have $G(E_m) \in \text{Summ}(B_i)$.

Since G is essentially surjective by Proposition 4.6, Lemma 4.21 yields

$$E_m \in G^{-1}(\text{Summ}(B_i)) = \text{Summ}(G^{-1}(B_i)).$$

Because $i \in \mathbb{N}$ was arbitrary, Lemma 4.19 gives us $E \simeq 0$. We obtain $\text{Ker}(- \circ F) = 0$. By Lemma 4.20, the functor $- \circ F$ is conservative. \square

Theorem 4.23. Consider a diagram of triangulated functors

$$\begin{array}{ccc} & F & \\ \curvearrowleft & & \curvearrowright \\ \mathcal{S} & \xrightarrow{G} & \mathcal{T} \\ \curvearrowright & & \curvearrowleft \\ & J & \end{array}$$

consisting of two adjoint pairs $F : \mathcal{T} \rightleftarrows \mathcal{S} : G$ and $G : \mathcal{S} \rightleftarrows \mathcal{T} : J$. Let $\{B_n\}_{n \in \mathbb{N}}$ be a metric on \mathcal{T} . Denote the completion of \mathcal{T} with respect to $\{B_n\}_{n \in \mathbb{N}}$ as $\mathfrak{S}(\mathcal{T})$ and the completion of \mathcal{S} with respect to the preimage metric $\{G^{-1}(B_n)\}_{n \in \mathbb{N}}$ as $\mathfrak{S}(\mathcal{S})$. If F and J are fully faithful, then the induced functor $- \circ G : \mathfrak{S}(\mathcal{T}) \rightarrow \mathfrak{S}(\mathcal{S})$ is a well-defined triangulated equivalence with a triangulated inverse $- \circ F$.

Proof. We know that $\{G^{-1}(B_n)\}_{n \in \mathbb{N}}$ is a metric by Lemma 4.12, while $\{F(B_n)\}_{n \in \mathbb{N}}$ and $\{J(B_n)\}_{n \in \mathbb{N}}$ are metrics by Lemma 4.14. We need to verify that all three functors in the diagram are compressions. The functor G is clearly a compression with respect to $\{G^{-1}(B_n)\}_{n \in \mathbb{N}}$ and $\{B_n\}_{n \in \mathbb{N}}$. The functor F is a compression with respect to $\{B_n\}_{n \in \mathbb{N}}$ and $\{G^{-1}(B_n)\}_{n \in \mathbb{N}}$ because Lemma 4.16 guarantees that $\{F(B_n)\}_{n \in \mathbb{N}} \leq \{G^{-1}(B_n)\}_{n \in \mathbb{N}}$. Symmetrically, the functor J is a compression with respect to $\{B_n\}_{n \in \mathbb{N}}$ and $\{G^{-1}(B_n)\}_{n \in \mathbb{N}}$ since $\{J(B_n)\}_{n \in \mathbb{N}} \leq \{G^{-1}(B_n)\}_{n \in \mathbb{N}}$ by Lemma 4.17.

By Theorem 4.9, we obtain two induced triangulated functors between the completions. Namely $- \circ F : \mathfrak{S}(\mathcal{S}) \rightarrow \mathfrak{S}(\mathcal{T})$ and $- \circ G : \mathfrak{S}(\mathcal{T}) \rightarrow \mathfrak{S}(\mathcal{S})$, where $- \circ F$ is a left adjoint to $- \circ G$ by Corollary 4.8. The same corollary also gives us that $- \circ G$ is fully faithful.

By Lemma 4.22, the functor $- \circ F$ is conservative. We may conclude that $- \circ F$ and $- \circ G$ are mutually inverse equivalences using Corollary 4.7. \square

Remark 4.24. Alternatively, a proof of Theorem 4.23 can be obtained using completions of recollements. For a good metric $\{B_n\}_{n \in \mathbb{N}}$ on \mathcal{T} , the following argument is due to Zhang.

Under the assumptions of Theorem 4.23, there exists a recollement of triangulated categories

$$\begin{array}{ccccc} & & & F & \\ & \curvearrowright & & \curvearrowleft & \\ \text{Ker } G & \xrightarrow{\quad} & \mathcal{S} & \xrightarrow{\quad G \quad} & \mathcal{T} \\ & \curvearrowleft & & \curvearrowright & \\ & & & J & \end{array}$$

The functors F , G and J are compressions by the same reasoning as in the proof of Theorem 4.23. After equipping the category $\text{Ker } G$ with the constant good metric $\{\text{Ker } G\}_{n \in \mathbb{N}}$, all the functors of the recollement become compressions. Then [33, Theorem 1.2] applies, and we obtain a right recollement

$$\begin{array}{ccccc} \mathfrak{S}(\text{Ker } G) & \xrightarrow{\quad} & \mathfrak{S}(\mathcal{S}) & \xrightarrow{\quad - \circ F \quad} & \mathfrak{S}(\mathcal{S}) \\ & \curvearrowleft & & \curvearrowright & \\ & & & - \circ G & \end{array}$$

However, the completion $\mathfrak{S}(\text{Ker } G)$ is zero by [27, Example 1.12]. Thus $- \circ F$ and $- \circ G$ are mutually inverse triangulated equivalences.

Corollary 4.25. Let \mathcal{A} be an abelian category with enough injectives, enough projectives, exact products, and exact coproducts (e.g. $\text{Mod-}R$ for a ring R). Let $L : \mathbf{K}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{A})$ be the localisation functor from the homotopy category of \mathcal{A} to the derived category of \mathcal{A} . Then for every metric $\{B_n\}_{n \in \mathbb{N}}$ on $\mathbf{D}(\mathcal{A})$, there is a triangulated equivalence of categories $- \circ L : \mathfrak{S}(\mathbf{D}(\mathcal{A})) \rightarrow \mathfrak{S}(\mathbf{K}(\mathcal{A}))$ where the completion $\mathfrak{S}(\mathbf{K}(\mathcal{A}))$ is calculated with respect to the metric $\{L^{-1}(B_n)\}_{n \in \mathbb{N}}$.

Proof. Under the conditions on \mathcal{A} , every object in $\mathbf{K}(\mathcal{A})$ has both a \mathbf{K} -injective and a \mathbf{K} -projective resolution by [20, Proposition 4.3.4]. Using [20, Proposition 4.3.1], this is equivalent to L having both fully faithful right adjoint and a fully faithful left adjoint. The statement now directly follows from Theorem 4.23. \square

4.3 Categories with Serre functors

In the end of this section, we we illustrate Theorem 4.23 in the context of finite dimensional algebras of finite global dimension. Let A be a finite dimensional algebra over a field K . It is well-known (see e.g. [20, Theorem 6.4.13]) that A has finite global dimension if and only the category $\mathbf{D}^b(\text{mod-}A)$ possesses a Serre functor $\mathbb{S} : \mathbf{D}^b(\text{mod-}A) \rightarrow \mathbf{D}^b(\text{mod-}A)$.

A Serre functor (originally defined in [12, pg. 519]) is a K -linear triangulated autoequivalence on a K -linear triangulated category \mathcal{T} such that there is an isomorphism $\text{Hom}_{\mathcal{T}}(B, C) \simeq D \text{Hom}_{\mathcal{T}}(C, \mathbb{S}(B))$ for all $B, C \in \mathcal{T}$ natural in both variables, where $D = \text{Hom}_K(-, K) : K\text{-mod} \rightarrow \text{mod-}K$ is the standard duality.

The following argument is essentially contained in the proof of [12, Proposition 3.6].

Lemma 4.26. Let K be a field. Let \mathcal{T} and \mathcal{S} be K -linear triangulated categories with Serre functors $\mathbb{S}_{\mathcal{T}}$ and $\mathbb{S}_{\mathcal{S}}$, respectively. Let $F : \mathcal{T} \rightleftarrows \mathcal{S} : G$ be adjoint K -linear, triangulated functors. Then F has a left adjoint and G has a right adjoint.

Proof. Let $B \in \mathcal{S}$ and $C \in \mathcal{T}$. Then

$$\begin{aligned} \mathrm{Hom}_{\mathcal{T}}(G(B), C) &\simeq D \mathrm{Hom}_{\mathcal{T}}(C, \mathbb{S}_{\mathcal{T}} G(B)) \simeq D \mathrm{Hom}_{\mathcal{T}}(\mathbb{S}_{\mathcal{T}}^{-1}(C), G(B)) \\ &\simeq D \mathrm{Hom}_{\mathcal{S}}(F \mathbb{S}_{\mathcal{T}}^{-1}(C), B) \simeq D \mathrm{Hom}_{\mathcal{S}}(\mathbb{S}_{\mathcal{S}} F \mathbb{S}_{\mathcal{T}}^{-1}(C), \mathbb{S}_{\mathcal{S}}(B)) \\ &\simeq D \mathrm{Hom}_{\mathcal{S}}(\mathbb{S}_{\mathcal{S}} F \mathbb{S}_{\mathcal{T}}^{-1}(C), \mathbb{S}_{\mathcal{S}}(B)) \simeq \mathrm{Hom}_{\mathcal{S}}(B, \mathbb{S}_{\mathcal{S}} F \mathbb{S}_{\mathcal{T}}^{-1}(C)). \end{aligned}$$

Since all the isomorphisms above are natural, we get that $\mathbb{S}_{\mathcal{S}} F \mathbb{S}_{\mathcal{T}}^{-1}$ is a right adjoint to G .

The proof that F has a left adjoint is dual. \square

Remark 4.27. Under the notation of Lemma 4.26 and its proof, the right adjoint of G is equal to $\mathbb{S}_{\mathcal{S}} F \mathbb{S}_{\mathcal{T}}^{-1}$. Since both of the Serre functors $\mathbb{S}_{\mathcal{S}}$, $\mathbb{S}_{\mathcal{T}}$ are autoequivalences, we get that the left adjoint F to G is fully faithful if and only if the right adjoint $\mathbb{S}_{\mathcal{S}} F \mathbb{S}_{\mathcal{T}}^{-1}$ to G is such. The same condition also holds for the adjoints of F .

Consequently, if we have a pair of adjoint functors between K -linear, triangulated categories with Serre functors, Lemma 4.26 guarantees that both of the functors from the adjoint pair have also an adjoint on the other side. This is helpful while checking whether the conditions of Theorem 4.23 hold or not.

Since $\mathbf{D}^b(\mathrm{mod}\text{-}A) \hookrightarrow \mathbf{D}(A)$ is a good extension whenever the algebra A has finite global dimension (see Section 3), we are able to take advantage of the following translation of Theorem 4.9 into the setting of good extensions.

Proposition 4.28. Let $\overline{F} : \overline{\mathcal{T}} \rightleftarrows \overline{\mathcal{S}} : \overline{G}$ be adjoint triangulated functors between triangulated categories with countable coproducts. Suppose that there are good extension $\nu_{\mathcal{T}} : \mathcal{T} \hookrightarrow \overline{\mathcal{T}}$ and $\nu_{\mathcal{S}} : \mathcal{S} \hookrightarrow \overline{\mathcal{S}}$ with respect to metrics $\{B_n\}_{n \in \mathbb{N}}$ on \mathcal{T} and $\{C_n\}_{n \in \mathbb{N}}$ on \mathcal{S} , respectively. Assume that \overline{F} and \overline{G} restrict via $\nu_{\mathcal{T}}$ and $\nu_{\mathcal{S}}$ to an adjoint pair $F : \mathcal{T} \rightleftarrows \mathcal{S} : G$, i.e. we have the following diagram

$$\begin{array}{ccc} \overline{\mathcal{T}} & \begin{array}{c} \xleftarrow{\overline{G}} \\ \xrightarrow{\overline{F}} \end{array} & \overline{\mathcal{S}} \\ \nu_{\mathcal{T}} \uparrow & & \uparrow \nu_{\mathcal{S}} \\ \mathcal{T} & \begin{array}{c} \xleftarrow{G} \\ \xrightarrow{F} \end{array} & \mathcal{S} \end{array}$$

where the square with F and \overline{F} commutes, and the square with G and \overline{G} commutes (both up to natural isomorphisms). If F and G are compressions and \overline{G} preserves coproducts,

then we have a commutative (up to natural isomorphisms) diagram of triangulated functors

$$\begin{array}{ccc}
\mathfrak{G}'(\mathcal{T}) & \xleftarrow{\overline{G}|_{\mathfrak{G}'(\mathcal{S})}} & \mathfrak{G}'(\mathcal{S}) \\
\mathcal{Y}|_{\mathfrak{G}'(\mathcal{T})} \downarrow \simeq & & \downarrow \mathcal{Y}|_{\mathfrak{G}'(\mathcal{S})} \\
\mathfrak{G}(\mathcal{T}) & \xleftarrow{-\circ F} & \mathfrak{G}(\mathcal{S}),
\end{array}$$

where the equivalences $\mathcal{Y}|_{\mathfrak{G}'(\mathcal{T})}$ and $\mathcal{Y}|_{\mathfrak{G}'(\mathcal{S})}$ are given by Theorem 2.17.

Proof. The existence of the induced functor $-\circ F$ on the completions is established by Theorem 4.9. Theorem 2.17 guarantees that the vertical maps $\mathcal{Y}|_{\mathfrak{G}'(\mathcal{T})}$ and $\mathcal{Y}|_{\mathfrak{G}'(\mathcal{S})}$ exist and are triangulated equivalences.

Let $X \in \mathfrak{G}'(\mathcal{S})$. Then $X \simeq \underline{\text{Hocolim}} \nu_{\mathcal{S}}(E_*)$ for some good Cauchy sequence in \mathcal{S} . Let

$$\coprod_{i \in \mathbb{N}} \nu_{\mathcal{S}}(E_i) \rightarrow \coprod_{i \in \mathbb{N}} \nu_{\mathcal{S}}(E_i) \rightarrow X \rightarrow \coprod_{i \in \mathbb{N}} \Sigma \nu_{\mathcal{S}}(E_i)$$

be the triangle defining $\underline{\text{Hocolim}} \nu_{\mathcal{S}}(E_*)$ in $\overline{\mathcal{S}}$. Since \overline{G} is triangulated and preserves coproducts, the triangle

$$\coprod_{i \in \mathbb{N}} \overline{G} \nu_{\mathcal{S}}(E_i) \rightarrow \coprod_{i \in \mathbb{N}} \overline{G} \nu_{\mathcal{S}}(E_i) \rightarrow \overline{G}(X) \rightarrow \coprod_{i \in \mathbb{N}} \Sigma \overline{G} \nu_{\mathcal{S}}(E_i)$$

shows that $\overline{G}(X) = \underline{\text{Hocolim}} \overline{G} \nu_{\mathcal{S}}(E_*)$. We infer that $\overline{G}(X) \simeq \underline{\text{Hocolim}} \nu_{\mathcal{T}} G(E_*)$ because of the commutativity rule $\overline{G} \nu_{\mathcal{S}} \simeq \nu_{\mathcal{T}} G$. Since $G(E_*)$ is a Cauchy sequence by Lemma 4.3, we get $\overline{G}(X) \in \mathfrak{L}'(\mathcal{T})$.

Using the good extension $\nu_{\mathcal{T}}$ we get $\mathcal{Y}(\overline{G}(X)) \simeq \underline{\text{Mocolim}} G(E_*)$. Also it holds that $\mathcal{Y}(X) \simeq \underline{\text{Mocolim}} E_*$ since $\nu_{\mathcal{S}}$ is a good extension. By Lemma 4.5,

$$\mathcal{Y}(X) \circ F \simeq \underline{\text{Mocolim}} G(E_*) \in \mathfrak{G}(\mathcal{T}) \subseteq \mathfrak{C}(\mathcal{T}).$$

This proves that $\overline{G}(X) \in \mathcal{Y}^{-1}(\mathfrak{C}(\mathcal{T})) = \mathfrak{C}'(\mathcal{T})$, so $\overline{G}(X) \in \mathfrak{G}'(\mathcal{T}) = \mathfrak{L}'(\mathcal{T}) \cap \mathfrak{C}'(\mathcal{T})$.

We conclude that $\overline{G}|_{\mathfrak{G}'(\mathcal{S})}: \mathfrak{G}'(\mathcal{S}) \rightarrow \mathfrak{G}'(\mathcal{T})$ is well defined. The commutativity of the diagram follows from the already proven chain of isomorphisms

$$\mathcal{Y}(\overline{G}(X)) \simeq \underline{\text{Mocolim}} G(E_*) \simeq \mathcal{Y}(X) \circ F.$$

□

With that in mind, we can proceed to our example.

Example 4.29. Let $\mathcal{T} := \mathbf{D}^b(\text{mod-}K)$ be the bounded derived category of finite dimensional K -vector spaces with the standard shift Σ , i.e. the category of finite dimensional graded vector spaces. Consider the metric $\{B_n\}_{n \in \mathbb{N}}$ given by the cohomology functor $H: \mathbf{D}^b(\text{mod-}K) \rightarrow \text{mod-}K$ via the formula

$$B_n = \left\{ T \in \mathbf{D}^b(\text{mod-}K) : \forall i > -n, H^i(T) = 0 \right\} \text{ for } n \in \mathbb{N}.$$

The metric $\{B_n\}_{n \in \mathbb{N}}$ is an example of a metric of “type i)” from [28, Example 12]. The completion with respect to $\{B_n\}_{n \in \mathbb{N}}$ is triangle equivalent to \mathcal{T} perceived as a subcategory $\mathfrak{S}'(\mathcal{T}) = \mathbf{D}^b(\text{mod-}K) = \mathcal{T}$ of a good extension $\mathbf{D}(K)$ (using Theorem 2.17 and Theorem 2.20).

Let us fix the orientation $2 \rightarrow 1$ for the Dynkin quiver A_2 and look at the derived category $\mathcal{S} := \mathbf{D}^b(\text{mod-}KA_2)$ with the standard shift Σ . The module category $\text{mod-}KA_2$ contains three indecomposable modules, namely

$$S(1) = P(1) = 0 \rightarrow K, \quad P(2) = I(1) = K \xrightarrow{1} K, \quad \text{and} \quad I(2) = S(2) = K \rightarrow 0.$$

We have an adjoint pair of functors

$$- \otimes_K^{\mathbf{L}} P(2) : \mathbf{D}^b(\text{mod-}K) \rightleftarrows \mathbf{D}^b(\text{mod-}KA_2) : \mathbf{R} \text{Hom}_{KA_2}(P(2), -).$$

This gives us a preimage metric

$$\left\{ \mathbf{R} \text{Hom}_{KA_2}(P(2), -)^{-1}(B_n) \right\}_{n \in \mathbb{N}}$$

on the category $\mathbf{D}^b(\text{mod-}KA_2)$, where we can calculate

$$\mathbf{R} \text{Hom}_{KA_2}(P(2), -)^{-1}(B_n) = \left\{ X \in \mathcal{S} : \forall i > -n, H^i(X) \in \text{coprod}(S(1)) \right\}$$

for $n \in \mathbb{N}$. The completion $\mathfrak{S}'_G(\mathcal{S}) \subseteq \mathbf{D}(KA_2)$ with respect to this metric computed inside the good extension satisfies $\mathfrak{S}'_G(\mathcal{S}) = \langle S(2) \rangle$ (the explicit calculation can be done using Theorem 3.7).

Both the categories \mathcal{T} and \mathcal{S} have Serre functors. The functor $- \otimes_K^{\mathbf{L}} P(2)$ is fully faithful and its right adjoint $\mathbf{R} \text{Hom}_{KA_2}(P(2), -)$ has a fully faithful right adjoint by Lemma 4.26 and Remark 4.27. Hence Theorem 4.23 applies. After translating the statement of the theorem for good extensions (as in Proposition 4.28) we get that the restriction $\mathbf{R} \text{Hom}_{KA_2}(P(2), -) : \mathfrak{S}'_G(\mathcal{S}) = \langle S(2) \rangle \rightarrow \mathfrak{S}'(\mathcal{T}) = \mathcal{T}$ is a triangulated equivalence of categories.

5 Good metrics and improvements

The purpose of this section is to compare the properties of a good metric and a generic (*a priori* not good) metric. The notion of good metric is, on one hand, stronger, since it requires the additional rapid decrease condition. On the other hand, it makes the calculation of the completion easier as described below.

We also make an argument that a general (non-good) metric can be improved to a good metric (called improvement). Then the completion with respect to the original metric is a triangulated subcategory of the completion of the improvement metric.

Lemma 5.1. Let \mathcal{T} be a triangulated category with a good metric $\{B_n\}_{n \in \mathbb{N}}$.

Let $E_* := E_1 \rightarrow E_2 \rightarrow \dots$ be a directed diagram in \mathcal{T} . Then E_* is a good Cauchy sequence if and only if it is a Cauchy sequence.

Proof. “ \Rightarrow ” Every good Cauchy sequence is a Cauchy sequence by the definition.

“ \Leftarrow ” Fix $i \in \mathbb{N}$ and $j \in \mathbb{Z}$. We find $M \in \mathbb{N}$ such that for any choice of numbers $m' \geq m \geq M$ the triangle $E_m \rightarrow E_{m'} \rightarrow D_{m,m'} \rightarrow \Sigma E_m$ satisfies $D_{m,m'} \in B_{i+|j|}$. Then the rapid decrease gives us $\Sigma^{\pm 1} D_{m,m'} \in B_{i+|j|-1}$, $\Sigma^{\pm 2} D_{m,m'} \in B_{i+|j|-2}$ etc.

In particular, $\Sigma^{\pm j} D_{m,m'} \in B_i$. Hence, the sequence E_* is good Cauchy. \square

Definition 5.2. Let \mathcal{T} be a triangulated category with a metric $\{B_n\}_{n \in \mathbb{N}}$. We define the improvement of $\{B_n\}_{n \in \mathbb{N}}$ as the sequence of subcategories $\{C_n\}_{n \in \mathbb{N}}$ of \mathcal{T} given inductively by the rule $C_1 := B_1$ and

$$C_n := B_n \cap \Sigma C_{n-1} \cap \Sigma^{-1} C_{n-1} \text{ for } n > 1.$$

Remark 5.3. If $\{B_n\}_{n \in \mathbb{N}}$ is a good metric, it is equal to its own improvement. The rapid decrease property of good metrics gives us $B_n \subseteq \Sigma B_{n-1}$, $B_n \subseteq \Sigma^{-1} B_{n-1}$, and consequently $B_n = B_n \cap \Sigma B_{n-1} \cap \Sigma^{-1} B_{n-1}$ for $n > 1$.

Lemma 5.4. Let \mathcal{T} be a triangulated category with a metric $\{B_n\}_{n \in \mathbb{N}}$ together with its improvement $\{C_n\}_{n \in \mathbb{N}}$. Then $C_n = \bigcap_{i=-n+1}^{n-1} \Sigma^i B_{n-|i|}$ for all $n \in \mathbb{N}$.

Proof. Denote $G_n := \bigcap_{i=-n+1}^{n-1} \Sigma^i B_{n-|i|}$ for all $n \in \mathbb{N}$. We prove $C_n = G_n$ by induction on $n \in \mathbb{N}$. The case $n = 1$ is trivial.

Suppose that we already know $C_n = G_n$ for some $n \in \mathbb{N}$. Let $X \in G_{n+1}$. From the induction premise and $X \in G_{n+1} = \bigcap_{i=-n}^n \Sigma^i B_{n+1-|i|}$, we infer $X \in B_{n+1}$,

$$\Sigma X \in \bigcap_{i=-n+1}^0 \Sigma^i B_{n-|i|} \cap \bigcap_{i=1}^{n-1} \Sigma^i B_{n+2-|i|} \subseteq \bigcap_{i=-n+1}^{n-1} \Sigma^i B_{n-|i|} = G_n = C_n,$$

and similarly $\Sigma^{-1} X \in C_n$, so $X \in C_{n+1}$ and $G_{n+1} \subseteq C_{n+1}$.

Conversely, let $X \in C_{n+1} \subseteq B_{n+1}$. Then $\Sigma X \in C_n = G_n = \bigcap_{i=-n+1}^{n-1} \Sigma^i B_{n-|i|}$, so we have $X \in \bigcap_{i=-n}^0 \Sigma^i B_{n+1-|i|}$.

An analogous argument with $\Sigma^{-1} X$ yields $X \in \bigcap_{i=0}^n \Sigma^i B_{n+1-|i|}$. Hence, we conclude that $X \in G_{n+1} = \bigcap_{i=-n}^n \Sigma^i B_{n+1-|i|}$ and $G_{n+1} = C_{n+1}$. \square

Proposition 5.5. Let \mathcal{T} be a triangulated category with a metric $\{B_n\}_{n \in \mathbb{N}}$. Let $\{C_n\}_{n \in \mathbb{N}}$ be the improvement of $\{B_n\}_{n \in \mathbb{N}}$. Then $\{C_n\}_{n \in \mathbb{N}}$ is a good metric.

Proof. We only need to check the closure under extensions for the other conditions of being a good metric (Definition 2.1) are obvious from the definition of the improvement. Let $n \in \mathbb{N}$. By Lemma 5.4, we have $C_n = \bigcap_{i=-n+1}^{n-1} \Sigma^i B_{n-|i|}$. This implies that C_n is extension-closed as an intersection of extension-closed subcategories of \mathcal{T} . \square

Remark 5.6. The improvement $\{C_n\}_{n \in \mathbb{N}}$ of $\{B_n\}_{n \in \mathbb{N}}$ in the sense of Definition 5.2 is a refinement of $\{B_n\}_{n \in \mathbb{N}}$ since $C_n \subseteq B_n$ for all $n \in \mathbb{N}$.

Proposition 5.7. Let \mathcal{T} be a triangulated category with a metric $\{B_n\}_{n \in \mathbb{N}}$ together with its improvement $\{C_n\}_{n \in \mathbb{N}}$. Then $\{C_n\}_{n \in \mathbb{N}}$ is the coarsest good metric (up to equivalence of metrics) which is finer than $\{B_n\}_{n \in \mathbb{N}}$.

Proof. Let $\{A_n\}_{n \in \mathbb{N}}$ be a good metric finer than $\{B_n\}_{n \in \mathbb{N}}$. We claim that it also holds that $\{A_n\}_{n \in \mathbb{N}} \leq \{C_n\}_{n \in \mathbb{N}}$. We show that for all $n \in \mathbb{N}$ there exists $m_n \in \mathbb{N}$ with $A_{m_n} \subseteq C_n$ by induction on n . For the base case, we can simply find $m_1 \in \mathbb{N}$ such that $A_{m_1} \subseteq B_1 = C_1$.

Now, suppose that we already have $m_n \in \mathbb{N}$ with $A_{m_n} \subseteq C_n$. We find $m \in \mathbb{N}$, $m \geq m_n$, such that $A_m \subseteq B_{n+1}$. By the rapid decrease property of $\{A_n\}_{n \in \mathbb{N}}$ and the induction premise, we have

$$\Sigma^{-1}A_{m+1} \cup A_{m+1} \cup \Sigma A_{m+1} \subseteq A_m \subseteq A_{m_n} \subseteq C_n.$$

Hence, it holds that

$$A_{m+1} \subseteq \left\{ T \in B_{n+1} : \Sigma^{\pm 1}T \in C_n \right\} = C_{n+1},$$

so we may finish the proof by setting $m_{n+1} := m + 1$. \square

Proposition 5.8. Let \mathcal{T} be a triangulated category with a metric $\{B_n\}_{n \in \mathbb{N}}$ together with its improvement $\{C_n\}_{n \in \mathbb{N}}$. Let $E_* := E_1 \rightarrow E_2 \rightarrow \cdots$ be a directed diagram in \mathcal{T} . Then the following statements are equivalent:

- (i) E_* is a good Cauchy sequence with respect to $\{B_n\}_{n \in \mathbb{N}}$.
- (ii) E_* is a good Cauchy sequence with respect to $\{C_n\}_{n \in \mathbb{N}}$.
- (iii) E_* is a Cauchy sequence with respect to $\{C_n\}_{n \in \mathbb{N}}$.

Proof. “i) \Rightarrow iii)” Fix $n \in \mathbb{N}$, and find $M \in \mathbb{N}$ such that for every $m' \geq m \geq M$ the triangle $E_m \rightarrow E_{m'} \rightarrow D_{m,m'}$ satisfies

$$\Sigma^i D_{m,m'} \in B_n \quad \text{for all } i \in \{-n+1, -n+2, \dots, n-2, n-1\}.$$

Then

$$D_{m,m'} \in \bigcap_{i=-n+1}^{n-1} \Sigma^i B_n \subseteq \bigcap_{i=-n+1}^{n-1} \Sigma^i B_{n-|i|} = C_n$$

by Lemma 5.4. Hence E_* is Cauchy with respect to the improvement.

“ii) \Leftrightarrow iii)” By Lemma 5.1.

“ii) \Rightarrow i)” Is obvious because $C_n \subseteq B_n$ for all $n \in \mathbb{N}$. \square

Corollary 5.9. Let \mathcal{T} be a triangulated category with a metric $\{B_n\}_{n \in \mathbb{N}}$ together with its improvement $\{C_n\}_{n \in \mathbb{N}}$. Denote by $\mathfrak{L}_B(\mathcal{T})$, $\mathfrak{C}_B(\mathcal{T})$ and $\mathfrak{S}_B(\mathcal{T})$ the pre-completion, the compactly supported elements and the completion with respect to the metric $\{B_n\}_{n \in \mathbb{N}}$, respectively. We also introduce the analogous notation $\mathfrak{L}_C(\mathcal{T})$, $\mathfrak{C}_C(\mathcal{T})$ and $\mathfrak{S}_C(\mathcal{T})$ for $\{C_n\}_{n \in \mathbb{N}}$. Then $\mathfrak{S}_B(\mathcal{T})$ is a triangulated subcategory of $\mathfrak{S}_C(\mathcal{T})$.

Proof. Proposition 5.8 yields $\mathfrak{L}_B(\mathcal{T}) = \mathfrak{L}_C(\mathcal{T})$. Since $\{C_n\}_{n \in \mathbb{N}} \leq \{B_n\}_{n \in \mathbb{N}}$, we have $\mathfrak{C}_B(\mathcal{T}) \subseteq \mathfrak{C}_C(\mathcal{T})$, which implies $\mathfrak{S}_B(\mathcal{T}) \subseteq \mathfrak{S}_C(\mathcal{T})$. \square

Lemma 5.10. Let \mathcal{T} be a triangulated category with shift Σ and a good metric $\{B_n\}_{n \in \mathbb{N}}$. Then $\mathfrak{C}(\mathcal{T}) = \mathfrak{WC}(\mathcal{T})$.

Proof. The inclusion $\mathfrak{C}(\mathcal{T}) \subseteq \mathfrak{WC}(\mathcal{T})$ is trivial. For the other inclusion, pick $j \in \mathbb{Z}$ and $F \in \mathfrak{WC}(\mathcal{T})$. So $F \in Y(B_i)^\perp$ for some $i \in \mathbb{N}$. By the goodness of $\{B_n\}_{n \in \mathbb{N}}$, we have $\Sigma^j B_{i+|j|} \subseteq B_i$. Hence $F \in Y(\Sigma^j B_{i+|j|})^\perp$. Since j was arbitrary, the proof follows. \square

Remark 5.11. After combining Lemmas 5.1 and 5.10, we see that if we have a good metric, it does not matter whether we calculate the completion using good or ordinary Cauchy sequences, weakly or ordinarily compactly supported elements - the result is always the same.

5.1 Examples

We have shown that the completion with respect to the improvement contains the completion with respect to the original metric (Corollary 5.9), and that completions with respect to good metrics can be computed using only (*a priori* non-good) Cauchy sequences (Remark 5.11).

This subsection consists of examples and counterexamples attesting that we cannot, in full generality, say more. The above conditions are indeed necessary.

The completions in all the examples below are calculated inside a good extension (as in Theorem 2.20) using Theorem 3.7.

Example 5.12. Let \mathcal{T} be a triangulated category with a metric $\{B_n\}_{n \in \mathbb{N}}$ together with its improvement $\{C_n\}_{n \in \mathbb{N}}$. Thanks to Proposition 5.8 we know that both metrics yield the same good Cauchy sequences, so the pre-completion $\mathfrak{L}(\mathcal{T})$ is the same regardless of which of these metric we use. However, this is not, in general, the case for the compactly supported elements.

Let $\mathcal{T} := \mathbf{D}^b(\text{mod-}K)$ be the bounded derived category of the category of finite dimensional K -vector spaces, i.e. the category of finite dimensional graded K -vector spaces. Consider a metric $\{B_n\}_{n \in \mathbb{N}}$ given by the cohomology functor $H : \mathbf{D}^b(\text{mod-}K) \rightarrow \text{mod-}K$ via the formula

$$B_n = \{T \in \mathbf{D}^b(\text{mod-}K) : \forall i > -n, i \neq 0, H^i(T) = 0\} \text{ for } n \in \mathbb{N}.$$

The improvement $\{C_n\}_{n \in \mathbb{N}}$ of $\{B_n\}_{n \in \mathbb{N}}$ can be calculated using Lemma 5.4, and we obtain

$$C_1 = B_1 \text{ and } C_n = \{T \in \mathbf{D}^b(\text{mod-}K) : \forall i > -n, H^i(T) = 0\} \text{ for } n > 2.$$

From the perspective of graded vector spaces, the only difference between the metric $\{B_n\}_{n \in \mathbb{N}}$ and its improvement $\{C_n\}_{n \in \mathbb{N}}$ is that $\{B_n\}_{n \in \mathbb{N}}$ allows non-zero vector spaces in degree 0.

Note that $\{C_n\}_{n \in \mathbb{N}}$ is equivalent to the metric of “type i)” from [28, Example 12]. The completion of \mathcal{T} with respect to $\{C_n\}_{n \in \mathbb{N}}$ is equal to $\mathcal{T} = 0^\perp \cap \mathcal{T}$ by Theorem 3.7 since $\bigcap_{n=1}^\infty C_n = 0$. However, the completion of \mathcal{T} with respect to $\{B_n\}_{n \in \mathbb{N}}$ is $0 = K[-\infty, \infty]^\perp$ by Theorem 3.7 because $K \in \bigcap_{n=1}^\infty B_n$. Indeed, no non-zero object of $T \in \mathcal{T}$ is compactly supported with respect to $\{B_n\}_{n \in \mathbb{N}}$ as one can always find $j \in \mathbb{Z}$ and a non-zero morphism $\Sigma^j K \rightarrow T$.

Example 5.13. If we try to compute a “completion” using non-good Cauchy sequences with respect to a non-good metric, we may not get a triangulated category.

Look at the bounded derived category $\mathcal{T} := \mathbf{D}^b(\text{mod-}\mathbb{Z})$ of the category of finitely generated abelian groups. Let $\mathcal{C} \subseteq \text{mod-}\mathbb{Z}$ be the subcategory consisting of finitely generated torsion groups. Consider the constant (non-good) metric $\{B_n\}_{n \in \mathbb{N}} = \{\mathcal{C}\}_{n \in \mathbb{N}}$, i.e.

$$B_n = \{T \in \mathbf{D}^b(\text{mod-}\mathbb{Z}) : \forall i \in \mathbb{Z}, i \neq 0, H^i(T) = 0 \wedge H^0(T) \in \mathcal{C}\} \text{ for all } n \in \mathbb{N}.$$

We make use of the good extension $\mathbf{D}^b(\text{mod-}\mathbb{Z}) \hookrightarrow \mathbf{D}(\mathbb{Z})$ provided by Theorem 2.20. Similarly as in Example 3.11, we look at the following diagram in $\text{mod-}\mathbb{Z}$

$$C_* := \mathbb{Z} \xrightarrow{2 \cdot -} \mathbb{Z} \xrightarrow{3 \cdot -} \mathbb{Z} \xrightarrow{4 \cdot -} \mathbb{Z} \hookrightarrow \dots$$

with $\varinjlim C_* = \mathbb{Q}$ in $\text{Mod-}\mathbb{Z}$. We have for all $i \in \mathbb{N}$ that $\text{Coker}(i \cdot -) \in \mathcal{C}$, so if we view \mathbb{Z} as a complex concentrated in degree 0 and interpret C_* as a diagram in $\mathbf{D}^b(\text{mod-}\mathbb{Z})$, then C_* is Cauchy (but not good Cauchy) with $\text{Hocolim} C_* = \mathbb{Q}$ in $\mathbf{D}(\mathbb{Z})$.

Trivially, we have $\mathbb{Q} \in (\Sigma^j \mathcal{C})^\perp$ for $j > 0$. Because \mathbb{Q} is torsion-free and injective, we also have $\mathbb{Q} \in (\Sigma^j \mathcal{C})^\perp$ for $j \leq 0$. All together, we get that $\mathbb{Q} \in \mathcal{L}'(\mathcal{T}) \cap \mathcal{C}'(\mathcal{T})$.

On the other hand, it holds that $\Sigma \mathbb{Q} \notin \mathcal{L}'(\mathcal{T})$ since for any Cauchy sequence E_* the (-1) -st cohomology

$$H^{-1}(\text{Hocolim} E_*) \simeq \varinjlim H^{-1}(E_*) \in \text{mod-}\mathbb{Z}$$

is a directed colimit of an eventually constant sequence $H^{-1}(E_*)$ of finitely generated abelian groups. We conclude that $\mathcal{L}'(\mathcal{T}) \cap \mathcal{C}'(\mathcal{T})$ is not triangulated.

We finish by observing that the property of $\mathcal{L}'(\mathcal{T}) \cap \mathcal{C}'(\mathcal{T})$ not being triangulated transfers to $\mathcal{L}(\mathcal{T}) \cap \mathcal{C}(\mathcal{T})$ as well. Indeed, we have $\text{Mocolim} C_* = \mathcal{Y}(\mathbb{Q})$ by the goodness of the extension $\mathbf{D}^b(\text{mod-}\mathbb{Z}) \hookrightarrow \mathbf{D}(\mathbb{Z})$. Since

$$\Sigma \mathcal{Y}(\mathbb{Q}) = \Sigma \text{Hom}(-, \mathbb{Q}) = \text{Hom}(\Sigma^{-1} -, \mathbb{Q}) \simeq \text{Hom}(-, \Sigma \mathbb{Q}) = \mathcal{Y}(\Sigma \mathbb{Q})$$

and $\mathcal{Y} \upharpoonright \mathcal{L}'(\mathcal{T})$ is full and conservative by [27, Remark 21] (see also Lemma 2.18), we obtain then any potential Cauchy sequence converging towards $\Sigma \mathcal{Y}(\mathbb{Q})$ in $\text{Mod-}\mathcal{T}$ would have homotopy colimit $\Sigma \mathbb{Q}$ in $\mathbf{D}(\mathbb{Z})$, which is impossible. Hence the category $\mathcal{L}(\mathcal{T}) \cap \mathcal{C}(\mathcal{T})$ is not triangulated.

Example 5.14. If we try to compute a “completion” using weakly compactly supported elements, we may not get a triangulated category.

We consider the bounded derived category $\mathcal{T} := \mathbf{D}^b(\text{mod-}K)$ of the category of graded finite dimensional vector spaces with the standard shift Σ . Consider a metric $\{B_n\}_{n \in \mathbb{N}}$ given by the cohomology functor $H : \mathbf{D}^b(\text{mod-}K) \rightarrow \text{mod-}K$ via the formula

$$B_n = \{T \in \mathbf{D}^b(\text{mod-}K) : \forall i \in \mathbb{Z}, i \neq 0, H^i(T) = 0\} \text{ for } n \in \mathbb{N}.$$

We make use of the good extension $\mathbf{D}^b(\text{mod-}K) \hookrightarrow \mathbf{D}(K)$ and we view ordinary vector spaces from $\text{Mod-}K$ as complexes concentrated in degree 0. Since it holds that $K \in B_n$ for all $n \in \mathbb{N}$, we have $K \notin \mathcal{W}\mathcal{C}'(\mathcal{T})$. However $\Sigma K \in \mathcal{L}'(\mathcal{T}) \cap \mathcal{W}\mathcal{C}'(\mathcal{T})$, so the subcategory is not closed under the shift, nor triangulated.

We may than infer that $\mathcal{L}(\mathcal{T}) \cap \mathcal{W}\mathcal{C}(\mathcal{T})$ is not triangulated using the same argument as in Example 5.13.

Conflict of Interests

The author has no relevant financial or non-financial interests to disclose.

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The following chapter “*Paper B*” is a copy of the second version of the author’s preprint “*Metric completions of triangulated categories from hereditary rings*”, which was uploaded on 27 January 2026 to the online repository arXiv under the code 2508.20283v2. The preprint is available at <https://arxiv.org/abs/2508.20283v2>. The text of the chapter is identical to the text of the preprint, except for formatting, typesetting, page layout, page numbers, and page headers. The table of contents and the footnotes indicating the date of submission, the Mathematics Subject Classification, the keywords, the supporting grant, and the contact details of the author have been removed from the version presented in this thesis.

Paper B

Metric completions of triangulated categories from hereditary rings

Cyril Matoušek

Abstract

The focus of this article is on metric completions of triangulated categories arising in the representation theory of hereditary finite dimensional algebras and commutative rings. We explicitly describe all completions of bounded derived categories with respect to additive good metrics for two classes of rings - hereditary commutative noetherian rings and hereditary algebras of tame representation type over an algebraically closed field. To that end, we develop and study the lattice theory of metrics on triangulated categories. Moreover, we establish a link between metric completions of bounded derived categories of a ring and the ring's universal localisations.

1 Introduction

Neeman's method of constructing a metric completion of a triangulated category, inspired by the Cauchy completion of a metric space, is one of the few ([6], [7], [10], [22]), rather scarce tools to build a new triangulated category from an existing old one. Developed in [33] as a particular case of Lawvere's norm on a category ([27]), the data of a metric on triangulated category postulate how far individual morphisms of the category are from being isomorphisms.

Under the interpretation of a morphism close to an isomorphism being short, we can define Cauchy sequences of objects and morphisms as \mathbb{N} -directed diagrams with morphisms eventually getting shorter and shorter. A completion of the triangulated category then consists of formal directed colimits of all Cauchy sequences, but restricted to only those objects which perceive short morphisms as actual isomorphisms - we call those special objects "compactly supported". The crucial point of [33] is that the metric completion satisfies all the axioms of triangulated categories ([40],[48]) with the triangulated structure induced by the original triangulated structure of the initial category.

Since its introduction in 2018, the technique of metric completions for triangulated categories has found broad range of applications in homological algebra, algebraic geometry, and related areas. In [33] (and the updated version [35]), Neeman generalises Rickard's derived Morita theory results [44] to noetherian, weakly approximable triangulated categories by expressing two triangulated categories as mutual metric completions of each other. He then encapsulates this phenomenon by defining excellent metrics on triangulated categories in [36] with the property that the process of taking the completion and then the opposite category is an involution. In [16], Cummings and Gratz calculate metric completions of Igusa-Todorov ([21]) discrete cluster categories as subcategories of their

Paquette-Yıldırım ([39]) combinatorial completion. Sun and Zhang study recollements of completions of triangulated categories in [47]. Biswas, Chen, Manali Rahul, Parker, and Zheng formulate a necessary condition for an existence of a bounded t -structure on a compactly generated triangulated using specific metric completions in [11]. Manali Rahul ([41],[42]) also uses metric techniques to obtain semi-orthogonal decompositions of triangulated categories and Brown representability flavoured results in algebraic geometry.

Despite the outstanding theoretical strength of the theorem establishing a triangulated structure on a metric completion, the completions are rather difficult to compute in practice for a general triangulated category. For example, the calculations of completions in [16] rely on the combinatorial nature of triangulated categories specific to the setting of discrete cluster categories. Many other results involving metrics on triangulated categories express the respective completion as a triangulated subcategory of a bigger, ambient category called good extension (see 2.2.4), whose existence, however, cannot be guaranteed for an arbitrary triangulated category. What is more, while the definition of a metric is relatively generous and easy to satisfy, the vast majority of completion-related results utilise only very specific choices of metrics determined by either aisles, or coaisles of t -structures.

This lack of diverse examples of metrics and the corresponding completions serves as the main motivation behind this article. We present a systematic study of metrics and metric completions for the bounded derived category $\mathbf{D}^b(\text{mod-}R)$ for a large class of hereditary rings R . We do not limit ourselves to metrics arising from t -structures only, but we explicitly describe the completions with respect to all possible additive good metrics on $\mathbf{D}^b(\text{mod-}R)$.

Technically, the data of a metric on a triangulated category \mathcal{S} consist of a family $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$ of subcategories of \mathcal{S} corresponding to cones of short morphisms. We use this specific data to describe the corresponding completions. The key observation is that for an additive good metric $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$ on $\mathbf{D}^b(\text{mod-}R)$ the intersection $\mathcal{B} := \bigcap_{n \in \mathbb{N}} B_n$ is a thick subcategory of $\mathbf{D}^b(\text{mod-}R)$. Such a thick subcategory corresponds to a universal localisation $R \rightarrow R_{\mathcal{B}}$ uniquely determined by the property $\mathcal{B} \otimes_R^{\mathbf{L}} R_{\mathcal{B}} = 0$ in $\mathbf{D}(R)$ (see [25]), and there is an embedding of derived categories $\mathbf{D}(R_{\mathcal{B}}) \hookrightarrow \mathbf{D}(R)$. With the existence of such an embedding kept in mind, we prove that the completion $\mathfrak{S}_{\mathcal{M}}(\mathbf{D}^b(\text{mod-}R))$ of $\mathbf{D}^b(\text{mod-}R)$ with respect to \mathcal{M} is equivalent either to $\mathbf{D}^b(\text{mod-}R_{\mathcal{B}})$, or a thick subcategory of $\mathcal{B}^{\perp} = \text{Ker Hom}_{\mathbf{D}^b(\text{mod-}R)}(\mathcal{B}, -) \subseteq \mathbf{D}^b(\text{mod-}R)$ depending on whether the lattice-theoretic condition $\mathcal{M} = \mathcal{M}_{\infty} \vee \mathcal{B}$ that the metric \mathcal{M} “converges locally uniformly” towards \mathcal{B} in terms of cohomology (see Section 5) is satisfied.

Theorem 1.1. Let R be a connected hereditary ring and $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$ an additive good metric on $\mathbf{D}^b(\text{mod-}R)$.

- 1) (Theorem 6.4) Suppose that R is a commutative noetherian ring. Then:
 - i) If \mathcal{B} is countably generated as a thick subcategory and $\mathcal{M} = \mathcal{M}_{\infty} \vee \mathcal{B}$, then $\mathfrak{S}_{\mathcal{M}}(\mathbf{D}^b(\text{mod-}R)) \simeq \mathbf{D}^b(\text{mod-}R_{\mathcal{B}})$.
 - ii) Otherwise, we have $\mathfrak{S}_{\mathcal{M}}(\mathcal{S}) \simeq \mathcal{B}^{\perp} \cap \mathfrak{T}$ where $\mathfrak{T} \subseteq \mathbf{D}^b(\text{mod-}R)$ is the thick subcategory generated by all finitely generated torsion modules.

- 2) (Theorem 7.2) Suppose that R is a finite dimensional algebra of tame representation type over an algebraically closed field. Then:
- i) If \mathcal{B} is countably generated as a thick subcategory and $\mathcal{M} = \mathcal{M}_\infty \vee \mathcal{B}$, then $\mathfrak{S}_{\mathcal{M}}(\mathbf{D}^b(\text{mod-}R)) \simeq \mathbf{D}^b(\text{mod-}R_{\mathcal{B}})$.
 - ii) Otherwise, we have $\mathfrak{S}_{\mathcal{M}}(\mathcal{S}) \simeq \mathcal{B}^\perp \cap \mathfrak{R}$ where $\mathfrak{R} \subseteq \mathbf{D}^b(\text{mod-}R)$ is the thick subcategory generated by all regular modules.

This result complements author's earlier work [29, Section 3] on completions for representation-finite algebras. For every finite dimensional algebra A of finite representation finite, all the completions of $\mathbf{D}^b(\text{mod-}A)$ are just the thick subcategories of $\mathbf{D}^b(\text{mod-}A)$. Our current result tell us that once we abandon the realm of representation-finite algebras and consider a tame algebra R instead, we can actually encounter new elements in the completion $\mathfrak{S}'_{\mathcal{M}}(\mathbf{D}^b(\text{mod-}R))$, such as the universal localisation $R_{\mathcal{B}}$, which do not lie in the initial category $\mathbf{D}^b(\text{mod-}R)$.

While the fact that a metric completion of a triangulated category \mathcal{S} can, in fact, shrink to a subcategory of \mathcal{S} can seem counter-intuitive at the first glance, the reason behind such a phenomenon resides in the two-step process of creating the completion. As mentioned before, unlike the usual metric space completion, or e.g. the sequential completion of triangulated category ([24]), we are not only adding directed colimits of Cauchy sequences to our category during the construction, but we are also removing objects which are not compactly supported with respect to the given metric. We discuss how compactly supported elements look like for bounded derived categories of rings in Section 3, where we present their description using universal localisations for hereditary rings together with partial results for general commutative noetherian rings.

Section 4 consists of several auxiliary results about completions for bounded derived categories for finitely generated algebras over commutative noetherian rings, which are both useful for proving Theorem 1.1 and of an independent interest. For instance, for a hereditary ring, Proposition 4.7 gives us that any completion with respect to an additive good metric is an idempotent complete triangulated category, while Proposition 4.12 fully describes completions with respect to constant metrics determined by countably generated thick subcategories.

The statement of Theorem 1.1 involves a lattice-theory condition on a metric on a triangulated category. The result that metrics form a lattice is new, and is proven in Section 5, together with further applications and examples linked to the lattices of metrics. In Proposition 5.10, we give a precise formulation of the condition of a cohomological locally uniform convergence of a metric towards a specific thick subcategory, and show its consequences for the resulting metric completions.

Finally, Section 6 and Section 7 are dedicated to proving the main result Theorem 1.1 in our two primary settings - hereditary commutative noetherian rings and hereditary tame algebras. We analyse additive good metrics on bounded derived categories using the ring's prime spectrum in the commutative case and the structure of regular modules in the tame algebra case.

Acknowledgements

I wish to express my gratitude to Sira Gratz for her supervision and for her valuable comments regarding this paper. A special thanks goes to Jan Šťovíček for introducing me to the topic of universal localisations. I would also like to thank Isaac Bird, Charley Cummings, David Nkansah, Greg Stevenson, and Alexandra Zvonareva for useful discussions. This work was supported by a research grant (VIL42076) from VILLUM FONDEN.

2 Preliminaries

2.1 Notation and conventions

If A is an object of a category \mathcal{A} , we will only write $A \in \mathcal{A}$ instead of $A \in \text{Ob}(\mathcal{A})$. Similarly, if \mathcal{A} and \mathcal{B} are categories, we will employ the shortcut $\mathcal{A} \subseteq \mathcal{B}$ to express that \mathcal{A} is a subcategory of \mathcal{B} .

For a class of objects \mathcal{C} of an additive category \mathcal{A} , the full subcategory of \mathcal{A} of objects isomorphic to finite coproducts of objects from \mathcal{C} will be denoted as $\text{coprod}(\mathcal{C})$.

We will always implicitly denote the shift functor of any triangulated category as Σ . Let \mathcal{S} be a triangulated category. By \mathcal{S}^c we mean the triangulated subcategory consisting of compact elements of \mathcal{S} . Given a class of objects $\mathcal{C} \subseteq \mathcal{S}$, the notation $\langle \mathcal{C} \rangle$ stands for the triangulated subcategory of \mathcal{S} generated by \mathcal{C} , while $\text{thick}(\mathcal{C})$ stands for the thick subcategory of \mathcal{S} generated by \mathcal{C} .

All rings are assumed to be unital and associative. A \mathbb{k} -algebra over a commutative ring \mathbb{k} is a ring R with a \mathbb{k} -module structure determined by a ring homomorphism $\mathbb{k} \rightarrow R$ with the image lying in the centre of R .

2.2 Metrics and completions

Here we recall the most important facts and definitions about completions of triangulated categories via metric techniques introduced by Neeman in [35] and [34]. We also make use of additional terminology and notational conventions from [16]. In this subsection, we fix a triangulated category \mathcal{S} .

2.2.1 Sequences

Definition 2.1 ([16, Definition 3.2, Subsection 3.2]). An \mathbb{N} -directed diagram of the form $E_1 \xrightarrow{e_1} E_2 \xrightarrow{e_2} \cdots$ in \mathcal{S} is called a *sequence* and denoted $\mathbf{E} = (E_n, e_n)_{n \in \mathbb{N}}$. For $n \leq m \in \mathbb{N}$, the map $e_{n,m}$ denotes the composition $e_m \circ \cdots \circ e_{n+1} \circ e_n$.

Let $\mathbf{I} = (i_m)_{m \in \mathbb{N}} = i_1 < i_2 < \cdots$ be a strictly increasing sequence of natural numbers. Then we define a *subsequence* $\mathbf{E}_{\mathbf{I}} = (E_{i_n}, e_{i_n, i_{n+1}})_{n \in \mathbb{N}}$ of \mathbf{E} .

Assume that for every $n \in \mathbb{N}$ we have direct summand F_n of E_n together with the canonical inclusion $\nu_n : F_n \rightarrow E_n$ and the canonical projection $\pi_n : E_n \rightarrow F_n$. Then the sequence $\mathbf{F} = (F_n, \pi_{n+1} \circ e_n \circ \nu_n)$ is called a *component* of \mathbf{E} .

Notation. We denote the category of additive contravariant functors $\mathcal{S}^{\text{op}} \rightarrow \text{Ab}$ as $\text{Mod-}\mathcal{T}$. Furthermore, the notation for Yoneda's embedding $X \mapsto \text{Hom}_{\mathcal{S}}(-, X)$ is $\mathfrak{y} : \mathcal{S} \rightarrow \text{Mod-}\mathcal{S}$. Here Ab stands for the category of abelian groups.

Following [16, Definition 3.4], to a sequence $\mathbf{E} = (E_n, e_n)_{n \in \mathbb{N}}$ in \mathcal{S} we associate a functor $\underline{\text{Mocolim}} \mathbf{E} := \varinjlim \mathfrak{A}(\mathbf{E}) \in \text{Mod-}\mathcal{T}$ called *the module colimit* of \mathbf{E} , which is the directed colimit of the image of \mathbf{E} in $\text{Mod-}\mathcal{S}$ under Yoneda's embedding.

Definition 2.2 ([16, Definition 3.11, Definition 3.16]). Let \mathcal{W} be a replete subcategory of \mathcal{S} . Let $\mathbf{E} = (E_n, e_n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{T} . We say that \mathbf{E} *stabilises at \mathcal{W} from $N \in \mathbb{N}$ onwards* if for all $N \leq n \leq m$ we have $\text{cone } e_{n,m} \in \mathcal{W}$. We then simply say that \mathbf{E} *stabilises at \mathcal{W}* if it stabilises at \mathcal{W} from some $N \in \mathbb{N}$ onwards.

A functor $F \in \text{Mod-}\mathcal{S}$ is *compactly supported* at \mathcal{W} if $F(\mathcal{W}) = 0$. In this spirit, the sequence \mathbf{E} is *compactly supported* at \mathcal{W} if the functor $\underline{\text{Mocolim}} \mathbf{E}$ is such.

We say that \mathbf{E} is *\mathcal{W} -trivial* if for every $n \in \mathbb{N}$ the entry E_n does not have any non-zero direct summands which lie in \mathcal{W} .

In particular, an object $X \in \mathcal{S}$ is *compactly supported* with respect to a replete subcategory $\mathcal{W} \subseteq \mathcal{S}$ if $\mathfrak{A}(X)$ is compactly supported at \mathcal{W} . This is by Yoneda's lemma equivalent to $X \in \mathcal{W}^\perp$. Here we are employing the following notation of ‘‘Hom-perpendicularity’’ for a general additive category \mathcal{A} and a subcategory $\mathcal{B} \subseteq \mathcal{A}$:

- \mathcal{B}^\perp denotes the full subcategory $\{A \in \mathcal{A} : \forall B \in \mathcal{B}, \text{Hom}_{\mathcal{A}}(B, A) = 0\}$,
- ${}^\perp\mathcal{B}$ denotes the full subcategory $\{A \in \mathcal{A} : \forall B \in \mathcal{B}, \text{Hom}_{\mathcal{A}}(A, B) = 0\}$.

2.2.2 Metrics on triangulated categories

Definition 2.3 ([35, Definition 1.2]). Let $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$ be a decreasing chain $B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots$ of full subcategories of \mathcal{S} . We say that \mathcal{M} is a *good metric* on \mathcal{S} if for every $n \in \mathbb{N}$:

- $0 \in B_n$,
- B_n is extension-closed, i.e. $Y \in B_n$ whenever $X, Z \in B_n$ and there exists a triangle $X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$ in \mathcal{S} , and
- $\Sigma^{-1}B_{n+1} \cup B_{n+1} \cup \Sigma B_{n+1} \subseteq B_n$.

The categories B_n of a metric $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$ should be thought as open balls (or open neighbourhoods) shrinking around the zero object. As the cone of every isomorphism is zero, metrics measure how close a morphism is to an isomorphism in the following sense - a morphism $f : X \rightarrow Y$ in \mathcal{S} is ‘‘of distance $\leq \frac{1}{n}$ from an isomorphism’’ if $\text{cone } f \in B_n$. With metrics, we can define Cauchy sequences as those sequences whose morphisms are progressively getting shorter.

Definition 2.4 ([35, Definition 1.6],[16, Remark 3.12]). Equip \mathcal{S} with a good metric $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$. Let $\mathbf{E} = (E_n, e_n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{S} . We say that \mathbf{E} *stabilises at $n \in \mathbb{N}$ with respect to \mathcal{M}* if it stabilises at B_n .

We call \mathbf{E} a *Cauchy sequence with respect to \mathcal{M}* if it stabilises at every $n \in \mathbb{N}$ with respect to \mathcal{M} .

We say that \mathbf{E} is *n -trivial with respect to \mathcal{M}* for $n \in \mathbb{N}$ if it is B_n -trivial.

Definition 2.5 ([35, Definition 1.11],[16, Remark 3.12]). Equip \mathcal{S} with a good metric $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$. Let $F \in \text{Mod-}\mathcal{S}$. We say that F is *compactly supported at* $n \in \mathbb{N}$ with respect to \mathcal{M} if it is compactly supported at B_n . And we call F *compactly supported with respect to* \mathcal{M} if it is compactly supported at some $n \in \mathbb{N}$ with respect to \mathcal{M} .

We will omit the name of the metric with respect to which we are considering the stabilisation, the triviality, and the compactly supported elements if such a metric is implicit. And again, any consideration for a sequence $\mathbf{E} = (E_n, e_n)_{n \in \mathbb{N}}$ to be compactly supported refers to the functor $\text{Mocolim } \mathbf{E}$, and for an object $X \in \mathcal{S}$ it refers to the functor $\mathfrak{z}(X)$.

Remark 2.6. For a sequence $\mathbf{E} = (E_n, e_n)_{n \in \mathbb{N}}$ to be Cauchy with respect to a good metric $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$, it means that for every $M \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for all $N \leq n \leq m$, the morphism $e_{n,m}$ is “of distance $\leq M$ from an isomorphism”, i.e. $\text{cone } e_{n,m} \in B_M$.

By [35, Lemma 1.9], it is enough to test whether a sequence is Cauchy on the morphisms $e_n : E_n \rightarrow E_{n+1}$ only. The sequence \mathbf{E} is Cauchy if for every $M \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for all $N \leq n$ we have $\text{cone } e_n \in B_M$.

Definition 2.7 ([35, Definition 1.2]). Let $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$ and $\mathcal{N} = \{C_n\}_{n \in \mathbb{N}}$ be two metrics on \mathcal{S} . We say that \mathcal{N} is *finer* than \mathcal{M} if for every $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that $C_m \subseteq B_n$, and we denote this by $\mathcal{N} \leq \mathcal{M}$.

We proclaim \mathcal{M} and \mathcal{N} to be *equivalent* if $\mathcal{M} \leq \mathcal{N}$ and $\mathcal{N} \leq \mathcal{M}$.

Good metrics with balls closed under direct summands have already been studied in [16]. We propose the following terminology for them.

Definition 2.8. Let $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$ be a good metric on \mathcal{S} . We say that \mathcal{M} is *additive* if the ball B_n is closed under direct summands for all $n \in \mathbb{N}$.

By an abuse of notation, we will not distinguish between a triangulated subcategory $\mathcal{C} \subseteq \mathcal{S}$ and the corresponding *constant metric* $\{\mathcal{C}\}_{n \in \mathbb{N}}$. Under this identification, additive constant metrics correspond precisely to thick subcategories of \mathcal{S} .

2.2.3 Completions of triangulated categories

Definition 2.9 ([35, Definition 1.11]). Equip \mathcal{S} with a good metric \mathcal{M} . We define the following full subcategories of $\text{Mod-}\mathcal{T}$ with respect to \mathcal{M} :

- the category $\mathfrak{C}_{\mathcal{M}}(\mathcal{S})$ of all compactly supported functors from $\text{Mod-}\mathcal{S}$,
- the *pre-completion* of \mathcal{S} as

$$\mathfrak{L}_{\mathcal{M}}(\mathcal{S}) := \left\{ F \in \text{Mod-}\mathcal{S} : \exists \text{ a Cauchy sequence } \mathbf{E} \text{ in } \mathcal{S}, F \simeq \text{Mocolim } \mathbf{E} \right\},$$

- and the *completion* of \mathcal{S} as $\mathfrak{S}_{\mathcal{M}}(\mathcal{S}) := \mathfrak{C}_{\mathcal{M}}(\mathcal{S}) \cap \mathfrak{L}_{\mathcal{M}}(\mathcal{S})$.

If there is no space for confusion, we will omit the subscript \mathcal{M} , and write only $\mathfrak{C}(\mathcal{S})$, $\mathfrak{L}(\mathcal{S})$, and $\mathfrak{S}(\mathcal{S})$. And since the (pre)completion and the compactly supported elements are the same for two equivalent metrics, we will also write $\mathfrak{C}_{\mathfrak{M}}(\mathcal{S})$, $\mathfrak{L}_{\mathfrak{M}}(\mathcal{S})$, and $\mathfrak{S}_{\mathfrak{M}}(\mathcal{S})$.

for an equivalence class \mathfrak{M} of good metrics, while meaning the respective categories for any/all metrics from \mathfrak{M} . Also, for a thick subcategory $\mathcal{C} \subseteq \mathcal{S}$, the categories $\mathfrak{C}_{\mathcal{C}}(\mathcal{S})$, $\mathfrak{L}_{\mathcal{C}}(\mathcal{S})$, and $\mathfrak{S}_{\mathcal{C}}(\mathcal{S})$ are meant with respect to \mathcal{C} interpreted as a constant metric.

The most important fact about metric completions of triangulated categories is that they are triangulated categories themselves.

Theorem 2.10 ([35, Theorem 2.14]). If \mathcal{S} is equipped with a good metric, then $\mathfrak{S}(\mathcal{S})$ is a triangulated category.

The triangulated structure on $\mathfrak{S}(\mathcal{S})$ is naturally induced by the triangulated structure on \mathcal{S} - the distinguished triangles of $\mathfrak{S}(\mathcal{S})$ are (compactly supported) module colimits of Cauchy sequences of distinguished triangles from \mathcal{S} , and the shift functor on $\mathfrak{S}(\mathcal{S})$ is the restriction of $- \circ \Sigma^{-1} : \text{Mod-}\mathcal{S} \rightarrow \text{Mod-}\mathcal{S}$ for the shift Σ on \mathcal{S} .

The key idea behind the completions is that compactly supported homological functors from $\text{Mod-}\mathcal{S}$ are precisely those which send “morphisms close to isomorphisms” in \mathcal{S} with respect to the given metric to isomorphisms in Ab . Restricting the pre-completion to compactly supported elements then ensures that every morphism in the completion has its well-defined cone.

This relation between Cauchy sequences and compactly supported elements is encapsulated in the following lemma:

Lemma 2.11 ([35, Lemma 2.10]). Assume \mathcal{S} is equipped with a good metric. Let $F \in \text{Mod-}\mathcal{S}$ be a cohomological functor compactly supported at $s \in \mathbb{N}$.

Let $E = \text{Mocolim}_{\rightarrow} \mathbf{E} \in \mathfrak{L}(\mathcal{S})$ for some Cauchy sequence $\mathbf{E} = (E_n, e_n)_{n \in \mathbb{N}}$. If \mathbf{E} stabilises at $s + 1$ from some $N \in \mathbb{N}$ onwards, then

$$\text{Hom}_{\text{Mod-}\mathcal{S}}(\varphi_n, F) : \text{Hom}_{\text{Mod-}\mathcal{S}}(E, F) \rightarrow \text{Hom}_{\text{Mod-}\mathcal{S}}(\mathfrak{z}(E_n), F)$$

is an isomorphism for all $n \geq N$ where $\varphi_n : \mathfrak{z}(E_n) \rightarrow E$ is the colimit injection.

2.2.4 Good extensions

Calculating completions of the triangulated category \mathcal{S} directly from the definition as subcategory of $\text{Mod-}\mathcal{S}$ can be rather hard in practice. However, sometimes there exists a computational tool, the so-called good extension.

Definition 2.12 ([35, Notation 3.1]). Let $\nu : \mathcal{S} \hookrightarrow \mathcal{T}$ be a fully faithful triangulated functor. By *the restricted Yoneda functor* we mean $\exists : \mathcal{T} \rightarrow \text{Mod-}\mathcal{S}$, $X \mapsto \text{Hom}_{\mathcal{T}}(\nu(-), X)$.

Definition 2.13 ([33, Definition 3.5]). Assume \mathcal{S} is equipped with a good metric \mathcal{M} . Let $\nu : \mathcal{S} \hookrightarrow \mathcal{T}$ be a fully faithful triangulated functor, and denote \exists the restricted Yoneda functor. Then ν is a *good extension with respect to \mathcal{M}* if:

- \mathcal{T} has countable coproducts, and
- the canonical map $\text{Mocolim}_{\rightarrow} \mathbf{E} \rightarrow \exists(\text{Hocolim}_{\rightarrow} \nu(\mathbf{E}))$ in $\text{Mod-}\mathcal{S}$ is an isomorphism for every Cauchy sequence \mathbf{E} in \mathcal{S} .

Definition 2.14 ([35, Definition 3.11]). Let $\nu : \mathcal{S} \hookrightarrow \mathcal{T}$ be a good extension with respect to a good metric \mathcal{M} on \mathcal{S} . We define the following full subcategories of \mathcal{T} :

- $\mathfrak{C}'_{\mathcal{M}}(\mathcal{S}) := \exists^{-1}(\mathfrak{C}_{\mathcal{M}}(\mathcal{S}))$ where \exists is the restricted Yoneda functor,
- $\mathfrak{L}'_{\mathcal{M}}(\mathcal{S}) := \left\{ X \in \mathcal{T} : \exists \text{ a Cauchy sequence } \mathbf{E} \text{ in } \mathcal{S}, X \simeq \text{Hocolim } \nu(\mathbf{E}) \right\},$
- $\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) := \mathfrak{C}'_{\mathcal{M}}(\mathcal{S}) \cap \mathfrak{L}'_{\mathcal{M}}(\mathcal{S}).$

Note that by [35, Observation 3.2], for a metric $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$ we can alternatively describe $\mathfrak{C}'_{\mathcal{M}}(\mathcal{S})$ as $\bigcup_{n \in \mathbb{N}} \nu(B_n)^\perp \subseteq \mathcal{T}$.

As in the case of $\mathfrak{C}(\mathcal{S})$, $\mathfrak{L}(\mathcal{S})$, and $\mathfrak{S}(\mathcal{S})$, we will not mention the metric while writing $\mathfrak{C}'(\mathcal{S})$, $\mathfrak{L}'(\mathcal{S})$, and $\mathfrak{S}'(\mathcal{S})$ unless necessary.

If a good extension exists, we can calculate the completion as a triangulated subcategory of the good extension.

Theorem 2.15 ([35, Theorem 3.23]). Let $\mathcal{S} \hookrightarrow \mathcal{T}$ be a good extension with respect to a good metric on \mathcal{S} . Then the restricted Yoneda functor \exists determines a well-defined triangulated equivalence $\exists \upharpoonright_{\mathfrak{S}'(\mathcal{S})} : \mathfrak{S}'(\mathcal{S}) \rightarrow \mathfrak{S}(\mathcal{S})$.

The prototypical example of a good extension is the inclusion of compact elements.

Theorem 2.16 ([35, Example 3.9],[32, Lemma 2.8]). Let \mathcal{T} be a triangulated category with coproducts. Then the inclusion $\mathcal{T}^c \hookrightarrow \mathcal{T}$ serves as a good extension with respect to any possible choice of a good metric on \mathcal{T}^c .

2.3 t -structures and approximation sequences

The notion of a t -structure on a triangulated category was introduced in [8, Définition 1.3.1], see [9] for a republished version. In this subsection, we recall its definition and the construction of approximation sequences for a Cauchy sequence.

Definition 2.17. Let $(\mathcal{S}^{\leq 0}, \mathcal{S}^{> 0})$ be a pair of full subcategories of a triangulated category \mathcal{S} . Assume that $\mathcal{S}^{\leq 0}$ and $\mathcal{S}^{> 0}$ are closed under direct summands. We fix a notation $\mathcal{S}^{\leq n} := \Sigma^{-n} \mathcal{S}^{\leq 0}$ and $\mathcal{S}^{> n} := \Sigma^{-n} \mathcal{S}^{> 0}$ for all $n \in \mathbb{Z}$.

We call $(\mathcal{S}^{\leq 0}, \mathcal{S}^{> 0})$ a t -structure on \mathcal{S} if:

- $\mathcal{S}^{> 0} \subseteq (\mathcal{S}^{\leq 0})^\perp,$
- for every $Y \in \mathcal{S}$ there exist $X \in \mathcal{S}^{\leq 0}, Z \in \mathcal{S}^{> 0},$ and a distinguished triangle

$$X \rightarrow Y \rightarrow Z \rightarrow \Sigma X,$$

- and $\mathcal{S}^{\leq -1} \subseteq \mathcal{S}^{< 0}.$

In this case, the subcategory $\mathcal{S}^{\leq 0}$ is called *an aisle*, and $\mathcal{S}^{> 0}$ is called *a coaisle*.

For a t -structure $(\mathcal{S}^{\leq 0}, \mathcal{S}^{> 0})$ on \mathcal{S} and $Y \in \mathcal{S}$, the distinguished triangle

$$X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$$

with $X \in \mathcal{S}^{\leq 0}$ and $Z \in \mathcal{S}^{> 0}$ is unique up to an isomorphism and is usually called *the approximation triangle*. This name is justified by the fact that $X \rightarrow Y$ is an $\mathcal{S}^{\leq 0}$ -cover of Y (or a minimal right $\mathcal{S}^{\leq 0}$ -approximation). The $\mathcal{S}^{\leq 0}$ -cover given by an approximation triangle is functorial, we denote this *truncation functor* as $\mathrm{tr}^{\leq 0}$. Dually $Y \rightarrow Z$ is an $\mathcal{S}^{> 0}$ -envelope of Y (a minimal left $\mathcal{S}^{> 0}$ -approximation) with a corresponding functor $\mathrm{tr}^{> 0}$. The same approximation theory works for the shifts of the truncation functors $\mathrm{tr}^{\leq n}$ and $\mathrm{tr}^{> n}$.

For the theory of metric completions, it is important to consider approximation sequences of (primarily Cauchy) sequences.

Definition 2.18 ([16, Definition 4.4]). Let $(\mathcal{S}^{\leq 0}, \mathcal{S}^{> 0})$ be a t -structure on a triangulated category \mathcal{S} , and let $\mathbf{E} = (E_n, e_n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{S} . Fix $m \in \mathbb{Z}$.

We call the sequence $\mathrm{tr}^{\leq m} \mathbf{E} = (\mathrm{tr}^{\leq m} E_n, \mathrm{tr}^{\leq m} e_n)_{n \in \mathbb{N}}$ an $\mathcal{S}^{\leq m}$ -approximation of \mathbf{E} and the sequence $\mathrm{tr}^{> m} \mathbf{E} = (\mathrm{tr}^{> m} E_n, \mathrm{tr}^{> m} e_n)_{n \in \mathbb{N}}$ an $\mathcal{S}^{> m}$ -approximation of \mathbf{E} .

Even if \mathbf{E} is a Cauchy sequence with respect to some good metric, the approximation sequences do not have to be Cauchy in general. We can see some positive cases, however, in [16, Section 4].

2.4 Derived categories

Let R be a ring. The category of right R -modules is denoted $\mathrm{Mod}\text{-}R$, the notation $\mathrm{mod}\text{-}R$ is reserved for the category of finitely presented right R -modules. The category of finitely presented indecomposable modules is denoted $\mathrm{ind}\text{-}R$.

We will mainly work with two triangulated categories associated to R , namely $\mathbf{D}(R) = \mathbf{D}(\mathrm{Mod}\text{-}R)$, the derived category of R , and $\mathbf{D}^b(\mathrm{mod}\text{-}R)$, the bounded derived category of finitely presented R -modules. The category of compact elements $\mathbf{D}(R)^c \subseteq \mathbf{D}(R)$ is equal to the category of perfect complexes $\mathbf{D}^{\mathrm{perf}}(R) \subseteq \mathbf{D}(R)$ over R , i.e. complexes quasi-isomorphic to bounded complexes of finitely generated projective R -modules. The category $\mathbf{D}^{\mathrm{perf}}(R)$ is thus also equal to the essential image of the embedding $\mathbf{K}^b(\mathrm{proj}\text{-}R) \hookrightarrow \mathbf{D}(R)$ of the bounded homotopy category of finitely generated projective R -modules.

By Theorem 2.16, the inclusion $\mathbf{D}^{\mathrm{perf}}(R) \hookrightarrow \mathbf{D}(R)$ is a good extension with respect to all possible good metrics on $\mathbf{D}^{\mathrm{perf}}(R)$. This good extension is a strong computational tool for completions of $\mathbf{D}^{\mathrm{perf}}(R)$, giving us $\mathfrak{S}'(\mathbf{D}^{\mathrm{perf}}(R)) \subseteq \mathbf{D}(R)$. We can even say more for R hereditary.

Proposition 2.19 ([29, Proposition 3.1]). Assume R is right hereditary. Then for every good metric on $\mathbf{D}^{\mathrm{perf}}(R)$, the completion $\mathfrak{S}'(\mathbf{D}^{\mathrm{perf}}(R)) \subseteq \mathbf{D}(R)$ is a triangulated subcategory of the bounded derived category of (all) R -modules $\mathbf{D}^b(\mathrm{Mod}\text{-}R)$.

The standard t -structure $(\mathbf{D}(R)^{\leq 0}, \mathbf{D}(R)^{> 0})$ on $\mathbf{D}(R)$ is determined by the cohomology functor $H^0 \simeq \mathrm{Hom}_{\mathbf{D}(R)}(R, -) : \mathbf{D}(R) \rightarrow \mathrm{Ab}$ via

$$\mathbf{D}(R)^{\leq 0} := \left\{ X \in \mathbf{D}(R) : \forall i \in \mathbb{Z}, i > 0 \Rightarrow H^i(X) = 0 \right\}.$$

If R is noetherian and of finite global dimension, then $\mathbf{D}^b(\text{mod-}R) \simeq \mathbf{D}^{\text{perf}}(R)$, the standard t -structure on $\mathbf{D}(R)$ restricts to a t -structure on $\mathbf{D}^b(\text{mod-}R)$, and the inclusion $\mathbf{D}^b(\text{mod-}R) \hookrightarrow \mathbf{D}(R)$ becomes a good extension with respect to all possible good metrics on $\mathbf{D}^b(\text{mod-}R)$.

Assume for the rest of the subsection that R is noetherian and hereditary. The noetherian condition ensures that $\text{mod-}R$ is an abelian category, and we may consider its wide subcategories. A *wide* subcategory of an abelian category can be characterised as a subcategory closed under taking extensions, kernels, and cokernels (see [20, Section 1], [13, Lemma 4.2]). In particular, a wide subcategory is closed under images and direct summands. By $\text{wide}(\mathcal{C})$, we will denote the smallest wide subcategory of $\text{mod-}R$ containing $\mathcal{C} \subseteq \text{mod-}R$.

By Brüning's [13, Theorem 5.1], the assignment $\mathcal{C} \mapsto H^0(\mathcal{C})$ establishes a bijection

$$\{\text{thick subcategories of } \mathbf{D}^b(\text{mod-}R)\} \leftrightarrow \{\text{wide subcategories of } \text{mod-}R\}.$$

As R is hereditary, every object $X \in \mathbf{D}(R)$ is quasi-isomorphic to its cohomology

$$X \simeq \coprod_{i \in \mathbb{Z}} \Sigma^{-i} H^i(X).$$

Thus the preimage of a wide subcategory $\mathcal{W} \subseteq \text{mod-}R$ under the aforementioned bijection is $\text{thick}(\mathcal{W}) \subseteq \mathbf{D}^b(\text{mod-}R)$.

2.5 Commutative algebra

Let R be a commutative ring. The *Zariski topology* on the prime spectrum $\text{Spec}(R)$ is the topology where closed sets are of the form $V(I) = \{\mathfrak{p} \in \text{Spec}(R) : I \leq \mathfrak{p}\}$ for ideals $I \leq R$. A subset of $\text{Spec}(R)$ is *specialisation closed* if it is a union of closed sets. The topological space $\text{Spec}(R)$ is connected if and only if R is connected as a ring.

For every prime ideal $\mathfrak{p} \in \text{Spec}(R)$, we denote $k(\mathfrak{p}) := R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ its *residue field*. A finitely generated R -module M is *supported* at $\mathfrak{p} \in \text{Spec}(R)$ if its localisation $M_{\mathfrak{p}}$ at \mathfrak{p} is non-zero. For an arbitrary complex $X \in \mathbf{D}(R)$, we employ the generalisation of a *support* due to Foxby [18] $\text{Supp}(X) := \{\mathfrak{p} \in \text{Spec}(R) : X \otimes_{\mathbb{L}}^R k(\mathfrak{p}) \neq 0\}$. For a class $\mathcal{C} \subseteq \mathbf{D}(R)$, we set $\text{Supp}(\mathcal{C}) := \bigcup_{C \in \mathcal{C}} \text{Supp}(C)$. A subset $\Phi \subset \text{Spec}(R)$ is called *coherent* [23] if for every $X \in \mathbf{D}(R)$ it holds that $\text{Supp}(X) \subseteq \Phi$ if and only if $\text{Supp}(\coprod_{i \in \mathbb{Z}} H^i(X)) \subseteq \Phi$.

Lemma 2.20. Let P be a finitely generated projective R -module. Then the rank function $\text{Spec}(R) \rightarrow \mathbb{N}_0$, $\mathfrak{p} \mapsto \dim_{k(\mathfrak{p})} P \otimes_R k(\mathfrak{p})$ is locally constant.

In particular, if Φ is a connected component of $\text{Spec}(R)$ and $\text{Supp}(P) \cap \Phi \neq \emptyset$, then $\Phi \subseteq \text{Supp}(P)$.

Proof. This is the implication “a) \Rightarrow c)” in [12, Chapitre II, §5, №2, Théorème 1]. \square

Our main focus in this paper is on hereditary noetherian rings. If the commutative ring R is such, it decomposes as a finite direct product $R \simeq \prod_{n=1}^N D_n$ of Dedekind domains D_1, \dots, D_N for some $N \in \mathbb{N}$; see for example [2, Corollary 5.5]. Lemma 2.20 then tell us that $\text{Spec}(D_1), \dots, \text{Spec}(D_N)$ are precisely the connected components of $\text{Spec}(R)$. Therefore, we can often reduce our reasoning to the case of Dedekind domains.

There are several equivalent definitions of a *Dedekind domain*; one such characterisation is that it is a hereditary integral domain. In particular, every principal ideal domain is a Dedekind domain. Some authors do not allow a field to be considered a Dedekind domain; however, we do not impose such a restriction.

For the rest of this subsection, we fix a Dedekind domain D . The structure of finitely generated D -modules is well-understood.

Theorem 2.21. Every finitely generated D -module is a direct sum of a projective module P and a torsion module T of the form

$$T \simeq \bigoplus_{i=1}^n R/\mathfrak{p}_i^{l_i} \quad \text{for some } n \in \mathbb{N}_0, \mathfrak{p}_1, \dots, \mathfrak{p}_n \in \text{Spec}(D), \text{ and } l_1, \dots, l_n \in \mathbb{N}.$$

This decomposition is unique up to isomorphisms and reordering of the summands.

Thus the support of a finitely generated module M over D is either the whole $\text{Spec}(D)$ if M contains a non-zero projective direct summand, or finite.

Let P and Q be indecomposable finitely generated projective D -modules. Then P is generated by either one element (and is isomorphic to D) or two elements, and it holds that $P \oplus Q \simeq D \oplus (P \otimes_D Q)$. This makes the tensor product \otimes_D into a commutative group operation on the *Picard group* $\text{Pic}(D)$ of D , which consists of equivalence classes of indecomposable finitely generated projective D -modules up to isomorphisms and free direct summands. The structure of indecomposable finitely generated projectives can be rather complicated; in fact, any abelian group is the Picard group of some Dedekind domain [14].

2.6 Finite dimensional algebras

Let K be an algebraically closed field. By Drozd's trichotomy, every finite dimensional K -algebra is either of representation-finite type, tame, or wild. Every connected hereditary finite dimensional K -algebra is Morita equivalent to a path algebra KQ for some finite acyclic connected quiver Q , and we can determine its representation type based on the underlying diagram of the quiver Q - the algebra is representation-finite if and only if the diagram is simply laced Dynkin (type \mathbb{A}, \mathbb{D} , and \mathbb{E}); and it is tame if and only if the diagram is Euclidean (or extended Dynkin - type $\tilde{\mathbb{A}}, \tilde{\mathbb{D}}$, or $\tilde{\mathbb{E}}$).

In this subsection, we fix an Euclidean quiver Q with $N \in \mathbb{N}$ vertices, and we shall consider the tame algebra KQ . Both $\text{mod-}KQ$ and $\mathcal{S} := \mathbf{D}^b(\text{mod-}KQ)$ are K -linear Krull-Schmidt categories.

Consider the Auslander-Reiten translation $\tau : \mathcal{S} \rightarrow \mathcal{S}$ (see e.g. [43]). The category $\text{ind-}KQ$ (up to isomorphisms) consists of three components:

- the preprojectives

$$\mathcal{P} := \{M \in \text{ind-}KQ : \exists P \in \text{ind-}KQ, P \text{ projective}, \exists n \in \mathbb{N}_0, M \simeq \tau^{-n}P\},$$

- the preinjectives

$$\begin{aligned} \mathcal{Q} &:= \{M \in \text{ind-}KQ : \exists I \in \text{ind-}KQ, I \text{ injective}, \exists n \in \mathbb{N}_0, M \simeq \tau^n I\} \\ &= \{M \in \text{ind-}KQ : \exists P \in \text{ind-}KQ, P \text{ projective}, \exists n \in \mathbb{N}, M \simeq \tau^n \Sigma P\}, \text{ and} \end{aligned}$$

- the regular modules \mathcal{R} .

The morphisms in $\text{ind-}KQ$ go only in one direction, meaning that $\mathcal{P} \subseteq \mathcal{R}^\perp$ and $\mathcal{P}, \mathcal{R} \subseteq \mathcal{Q}^\perp$ in $\text{mod-}KQ$.

The regular component $\text{coprod}(\mathcal{R}) = \text{wide}(\mathcal{R}) \simeq \coprod_{i \in \mathbb{P}_K^1} \mathfrak{t}_i$ decomposes as a coproduct of so-called tubes indexed by the projective line \mathbb{P}_K^1 . All the tubes are mutually *orthogonal*, i.e. $\mathfrak{t}_i, \Sigma \mathfrak{t}_i \subseteq \mathfrak{t}_j^\perp$ in \mathcal{S} for all $i \neq j \in \mathbb{P}_K^1$. Each tube \mathfrak{t} is an orbit under the autoequivalence τ , and the minimal $n \in \mathbb{N}$ such that τ^n acts as the identity on \mathfrak{t} is called *the rank of \mathfrak{t}* . Tubes of rank 1 are *homogenous*. All tubes have finite rank, and at most 3 tubes can be non-homogenous. By [17, Proposition 2.4.2], every tube of rank $n \in \mathbb{N}$ has $\binom{2n}{n}$ wide subcategories.

Definition 2.22. Let $M \in \text{mod-}KQ$. Then M is called *rigid* if $\text{Ext}_{KQ}^1(M, M) = 0$. A rigid module M is called *exceptional* if $\text{End}_{KQ}(M) \simeq K$.

Let $n \in \mathbb{N}$. An n -tuple (X_1, \dots, X_n) of finitely generated KQ -modules is called *an exceptional sequence of length n* if it consists of exceptional objects, and for all $1 \leq i < j \leq n$ we have $\text{Hom}_{KQ}(X_j, X_i) \simeq 0 \simeq \text{Ext}_{KQ}^1(X_j, X_i)$. If n equals the number of vertices of Q , then the exceptional sequence is called *complete*.

Any complete exceptional sequence (X_1, \dots, X_N) generates the module category $\text{mod-}KQ$ in the sense that $\text{wide}(X_1, \dots, X_N) = \text{mod-}KQ$. Furthermore, the category $\text{wide}(X_1, \dots, X_n)$ for an exceptional sequence of length $n \in \mathbb{N}$ is equivalent to $\text{mod-}KQ'$ for some finite acyclic quiver Q' with n vertices, and the embedding $\text{mod-}KQ' \hookrightarrow \text{mod-}KQ$ is exact and induces isomorphisms on Hom and Ext . Additionally, the category $\text{wide}(X_1, \dots, X_n)^\perp \subseteq \text{mod-}KQ$ possesses an exceptional sequence of length $N - n$, and $\text{wide}(X_1, \dots, X_n) = {}^\perp(\text{wide}(X_1, \dots, X_n)^\perp)$ (see [15] for details).

A module $S \in \mathcal{R}$ without proper non-zero regular submodules is *simple regular*. Every tube of rank $n \in \mathbb{N}$ contains precisely n different simple regulars up to isomorphism.

Lemma 2.23. Let $S \in \text{ind-}KQ$ be simple regular from a homogenous tube. Then

$$\Sigma \mathcal{P} \cap S^\perp = \emptyset = \mathcal{Q} \cap S^\perp$$

in \mathcal{S} .

Proof. The notation $(-)^* = \text{Hom}_K(-, K)$ shall denote the standard K -dual. Since S simple regular lies in a homogenous tube, then its dimension vector is sincere (see e.g. [46, Corollary XI.3.9]), meaning $\text{Hom}(P, S) \neq 0$ for every projective $P \in \text{ind-}KQ$. Any preinjective is of the form $\tau^n \Sigma P$ for some $n \in \mathbb{N}$ and P indecomposable projective. Using that $\tau \Sigma : \mathcal{S} \rightarrow \mathcal{S}$ is a Serre functor on \mathcal{S} (see [43, Theorem II.1.3]), we get

$$\begin{aligned} 0 \neq \text{Hom}_{\mathcal{S}}(P, S) &\simeq \text{Hom}_{\mathcal{S}}(P, \tau^{-n-1} S) \simeq \text{Hom}_{\mathcal{S}}(\tau^{-n-1} S, \tau \Sigma P)^* \\ &\simeq \text{Hom}_{\mathcal{S}}(S, \tau^n \Sigma P)^*. \end{aligned}$$

The proof that $\text{Hom}_{\mathcal{S}}(S, \Sigma P') \neq 0$ for all $P' \in \mathcal{P}$ is dual. □

We intend to use Brüning's correspondence to understand thick subcategories of \mathcal{S} . We will denote $\mathfrak{R} := \text{thick}(\mathcal{R}) \simeq \coprod_{i \in \mathbb{P}_K^1} \coprod_{j \in \mathbb{Z}} \Sigma^j \mathfrak{t}_i \subseteq \mathcal{S}$ the thick subcategory generated by all the regular tubes.

Theorem 2.24 ([17, Theorem 3.2.15],[26, Proposition 6.14]). Let \mathcal{C} be a thick subcategory of \mathcal{S} . Then at least one of the following conditions holds:

- (i) $\mathcal{C} = \text{thick}(X_1, \dots, X_n)$ for an exceptional sequence (X_1, \dots, X_n) in $\text{mod-}KQ$,
- (ii) $\mathcal{C} \subseteq \mathfrak{R}$.

Lemma 2.25. There exists a number $M \in \mathbb{N}$ (depending on Q) such that for any strictly decreasing chain $\mathcal{C}_1 \supsetneq \mathcal{C}_2 \supsetneq \dots \supsetneq \mathcal{C}_M$ of thick subcategories of \mathcal{S} we can find indices $1 \leq i < j \leq M$ and a homogenous regular tube $\mathfrak{t} \subseteq \text{mod-}KQ$ with $\mathcal{C}_i, \mathcal{C}_j \subseteq \mathfrak{R}$, $\mathfrak{t} \subseteq \mathcal{C}_i$ and $\mathfrak{t} \cap \mathcal{C}_j = 0$.

Proof. Let $\mathfrak{U} \subseteq \mathfrak{R}$ be the thick subcategory generated by all the non-homogenous tubes, and $\mathfrak{V} \subseteq \mathfrak{R}$ be the thick subcategory generated by all the homogenous tubes. Every thick subcategory of \mathfrak{R} is of the form $\mathcal{U} \oplus \mathcal{V}$ for some thick subcategories $\mathcal{U} \subseteq \mathfrak{U}$ and $\mathcal{V} \subseteq \mathfrak{V}$. Since there are only finitely many non-homogenous tubes, and each tube has only finite number of wide subcategories, there are only finitely many thick subcategories of \mathfrak{U} ; let us denote this number by M' .

Set $M := N + M' + 2$. Let $\mathcal{C}_1 \supsetneq \mathcal{C}_2 \supsetneq \dots \supsetneq \mathcal{C}_M$ be a strictly decreasing chain of thick subcategories of \mathcal{S} . If \mathcal{C}_i and \mathcal{C}_j for some $1 \leq i < j \leq M$ are generated by exceptional sequences, then the exceptional sequence generating \mathcal{C}_j must be strictly shorter than the exceptional sequence generating \mathcal{C}_i because the equality would yield $\mathcal{C}_i = \mathcal{C}_j$. Therefore, at most $N + 1 < M$ thick subcategories in the chain can be generated by an exceptional sequence. Theorem 2.24 then tells us that the remaining thick subcategories must be then generated by regular elements.

We restrict ourselves to a sub-chain $\mathcal{D}_1 \supsetneq \mathcal{D}_2 \supsetneq \dots \supsetneq \mathcal{D}_{M'+1}$ of thick subcategories of \mathcal{S} with $\mathcal{D}_i \subseteq \mathfrak{R}$ for all $1 \leq i \leq M' + 1$. By the pigeonhole principle, there exist $1 \leq i < j \leq M' + 1$ and thick subcategories $\mathcal{U} \subseteq \mathfrak{U}$ and $\mathcal{V}, \mathcal{W} \subseteq \mathfrak{V}$ such that $\mathcal{D}_i = \mathcal{U} \oplus \mathcal{V}$ and $\mathcal{D}_j = \mathcal{U} \oplus \mathcal{W}$. This forces $\mathcal{W} \subsetneq \mathcal{V}$, and these two subcategories must differ by at least one homogenous tube. \square

3 Constant metrics, smashing subcategories, and ring epimorphisms

We fix a ring A across this whole section, and denote $\mathcal{S} := \mathbf{D}^{\text{perf}}(A)$. We shall make use of the fact that the inclusion $\mathcal{S} \hookrightarrow \mathbf{D}(A)$ is a good extension, so all the categories of pre-completions, completions, and compactly supported elements will be calculated inside $\mathbf{D}(A)$.

In this short section, we intend to relate the completion $\mathfrak{S}'(\mathcal{S}) \subseteq \mathbf{D}(A)$ with respect to a constant metric to a subcategory $\mathbf{D}(B) \subseteq \mathbf{D}(A)$ determined by a homological ring epimorphism $A \rightarrow B$ (see Proposition 3.3 and Corollary 3.5).

Recall that if \mathcal{T} is a category with coproducts, then a triangulated subcategory of \mathcal{T} is *localising* if it is closed under (infinite) coproducts. For $\mathcal{C} \subseteq \mathcal{T}$ a class, the smallest localising subcategory of \mathcal{T} containing \mathcal{C} is called *a localising subcategory generated by \mathcal{C}* and denoted $\text{Loc}(\mathcal{C})$. A localising subcategory $\mathcal{L} \subseteq \mathcal{T}$ is called *smashing* if the inclusion $\mathcal{L} \hookrightarrow \mathcal{T}$ admits a right adjoint and \mathcal{L}^\perp is closed under coproducts. Smashing subcategories give rise to recollements of triangulated categories (defined in [8, Subsection 1.4]).

Lemma 3.1. Let \mathcal{T} be a compactly generated triangulated category. Consider the good extension $\mathcal{T}^c \hookrightarrow \mathcal{T}$. Let $\mathcal{C} \subseteq \mathcal{T}^c$ be a triangulated subcategory. Then there exists a recollement diagram

$$\begin{array}{ccccc} & & i^* & & j^! \\ & \swarrow & \leftarrow & \searrow & \swarrow \\ \mathfrak{C}'_{\mathcal{C}}(\mathcal{T}^c) & \xrightarrow{i_*} & \mathcal{T} & \xrightarrow{j^*} & \text{Loc}(\mathcal{C}) \\ & \nwarrow & \leftarrow & \swarrow & \nwarrow \\ & & i^! & & j_* \end{array}$$

with $i_* : \mathfrak{C}'_{\mathcal{C}}(\mathcal{T}^c) \hookrightarrow \mathcal{T}$ and $j_! : \text{Loc}(\mathcal{C}) \hookrightarrow \mathcal{T}$ being the respective inclusions.

Proof. $\text{Loc}(\mathcal{C})$ is a smashing subcategory of \mathcal{T} by [4, Corollary 2.8]. The existence of the recollement diagram above then follows from [4, Proposition 2.9] (see also [38, Chapter 4]) where we identify $\text{Loc}(\mathcal{C})^\perp = \mathcal{C}^\perp = \mathfrak{C}'_{\mathcal{C}}(\mathcal{T}^c)$ using [35, Lemma 3.8]. \square

Although every localising subcategory of $\mathbf{D}(A)$ (more generally, of a compactly generated triangulated category) generated by perfect complexes (compact objects) is smashing, the converse is the assertion of the so-called *telescope conjecture*. While there exist counterexamples to the conjecture (e.g. [25, Example 7.8]) in its full generality, it has been proven for some classes of rings, such as hereditary rings by Krause and Šťovíček [25, Theorem A] and commutative noetherian rings by Neeman [31].

Theorem 3.2 ([31, Theorem 3.3, Corollary 3.4]). Let A be a commutative noetherian ring. Then there are bijections between:

- thick subcategories of $\mathbf{D}^{\text{perf}}(A)$,
- smashing subcategories of $\mathbf{D}(A)$, and
- specialisation closed subsets of $\text{Spec}(A)$.

More specifically, to a thick subcategory \mathcal{C} of $\mathbf{D}^{\text{perf}}(A)$ we assign the localising subcategory $\text{Loc}(\mathcal{C})$ of $\mathbf{D}(A)$ and the specialisation closed subset $\text{Supp}(\mathcal{C})$, respectively.

Sometimes, the validity of the telescope conjecture also provides us for each smashing subcategory of $\mathbf{D}(A)$ with a corresponding *homological epimorphism*; that is a ring epimorphism $A \rightarrow B$ such that $B \otimes_A B \simeq B$ and $\text{Tor}_n^A(B, B) = 0$ for all $n \in \mathbb{N}$, or equivalently (via [19, Theorem 4.4]), such that the induced functor $\mathbf{D}(B) \rightarrow \mathbf{D}(A)$ is fully faithful.

Proposition 3.3. Let $\mathcal{C} \subseteq \mathcal{S}$ be a thick subcategory. Assume further that at least one of these conditions holds:

- (i) A is hereditary, or
- (ii) A is commutative noetherian, and $\text{Supp}(\mathcal{C})$ has a coherent complement in $\text{Spec}(A)$.

Then there exists a homological epimorphism $A \rightarrow B$ satisfying $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) \subseteq \mathbf{D}^{\text{perf}}(B)$ where we identify $\mathbf{D}(B)$ with $\text{Im}(i_*)$ from the recollement diagram

$$\begin{array}{ccccc} & & -\otimes_A^{\mathbf{L}} B & & j^! \\ & \swarrow & \leftarrow & \searrow & \swarrow \\ \mathbf{D}(B) & \xrightarrow{i_*} & \mathbf{D}(A) & \xrightarrow{j^*} & \text{Loc}(\mathcal{C}) \\ & \nwarrow & \leftarrow & \swarrow & \nwarrow \\ & & i^! & & j_* \end{array}$$

with $j_! : \text{Loc}(\mathcal{C}) = \text{Ker}(- \otimes_A^{\mathbf{L}} B) \hookrightarrow \mathbf{D}(A)$ and $i^! = \mathbf{R} \text{Hom}_A(B, -)$.

Furthermore, the equality $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) = \mathbf{D}^{\text{perf}}(B)$ holds if and only if $B \in \mathfrak{S}'_{\mathcal{C}}(\mathcal{S})$ and $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S})$ is idempotent complete.

Proof. We start by obtaining a recollement diagram from Lemma 3.1 applied to the good extension $\mathcal{S} \hookrightarrow \mathbf{D}(A)$ and the subcategory \mathcal{C} .

If “i)” holds, then the desired homological epimorphism $A \rightarrow B$ such that

$$\text{Loc}(\mathcal{C}) = \text{Ker}(- \otimes_A^{\mathbf{L}} B) \quad \text{and} \quad \mathbf{D}(B) \simeq \text{Loc}(\mathcal{C})^{\perp}$$

exists by [25, Section 3, Theorem 8.1].

Now, assume “ii)” holds. By the bijection from Theorem 3.2, the category $\text{Loc}(\mathcal{C})$ consists of objects of $\mathbf{D}(A)$ supported at $\text{Supp}(\mathcal{C})$. As $\text{Supp}(\mathcal{C})$ has a coherent complement, [5, Theorem 4.9, Corollary 4.10] yields the existence of a homological (even flat) epimorphism $A \rightarrow B$ again satisfying $\text{Loc}(\mathcal{C}) = \text{Ker}(- \otimes_A^{\mathbf{L}} B)$ and $\mathbf{D}(B) \simeq \text{Loc}(\mathcal{C})^{\perp}$.

Regardless of whether “i)” or “ii)” happens, it follows for the completion of \mathcal{S} that $\mathfrak{C}'_{\mathcal{C}}(\mathcal{S}) = \text{Loc}(\mathcal{C})^{\perp} = \mathbf{D}(B)$, so $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) \subseteq \mathbf{D}(B)$.

Let $E = \text{Hocolim} \mathbf{E} \in \mathfrak{S}'_{\mathcal{C}}(\mathcal{S})$ for a Cauchy sequence $\mathbf{E} = (E_n, e_n)_{n \in \mathbb{N}}$. We may w.l.o.g. assume $\text{cone}(e_{n,m}) \in \mathcal{C}$ for all $n \leq m \in \mathbb{N}$. Since $\mathcal{C} \subseteq \text{Ker}(- \otimes_A^{\mathbf{L}} B)$, we get that

$$E \simeq E \otimes_A^{\mathbf{L}} B \simeq \left(\text{Hocolim} \mathbf{E} \right) \otimes_A^{\mathbf{L}} B \simeq \text{Hocolim} \left(\mathbf{E} \otimes_A^{\mathbf{L}} B \right)$$

is a homotopy colimit of a constant sequence $\mathbf{E} \otimes_A^{\mathbf{L}} B$ in $\mathbf{D}(B)$ consisting of isomorphisms. Suppose $E_1 \simeq P^{\bullet}$ for a bounded complex P^{\bullet} of finitely generated projective A -modules. Then $E \simeq E_1 \otimes_A^{\mathbf{L}} B \simeq P^{\bullet} \otimes_A B$ is isomorphic to a bounded complex of finitely generated projective B -modules, so $E \in \mathbf{D}^{\text{perf}}(B)$. As E was arbitrary, we get $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) \subseteq \mathbf{D}^{\text{perf}}(B)$.

The final part of the statement follows directly from $\mathbf{D}^{\text{perf}}(B) = \text{thick}(B)$. \square

The ring B above is obtained as the endomorphism ring $\text{End}_A(i_* i^* A)$ using the recollement notation of Lemma 3.1 and is uniquely determined by the thick subcategory $\mathcal{C} \subseteq \mathcal{S}$.

A universal localisation is a generalisation of a localisation of a commutative ring at a multiplicative set.

Definition 3.4 ([45, Theorem 4.1]). Let $f : A \rightarrow B$ be a ring homomorphism and Φ a set of R -homomorphisms between finitely generated projective R -modules. The ring homomorphism f is called Φ -inverting if for all $\alpha : P \rightarrow Q$ in Φ , the B -module homomorphism $\alpha \otimes_A \text{id}_B : P \otimes_A B \rightarrow Q \otimes_A B$ is an isomorphism.

The ring homomorphism f (together with the ring B) is called a *universal localisation of A at Φ* if f is Φ -inverting, and for every Φ -inverting ring homomorphism $f' : A \rightarrow B'$ there exists a unique ring homomorphism $g : B \rightarrow B'$ such that $f' = gf$.

Under favourable conditions, universal localisations and homological epimorphisms coincide, thus allowing us to express metric completions with respect to constant metrics as subcategories of derived categories of universal localisations.

Corollary 3.5. Assume A is a hereditary ring or a commutative noetherian ring of Krull dimension at most 1. Let $\mathcal{C} \subseteq \mathcal{S}$ be a thick subcategory. Then there exists a universal localisation $A \rightarrow B$ such that $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) \subseteq \mathbf{D}^{\text{perf}}(B)$ in the sense of Proposition 3.3.

Proof. Under the conditions on A , every homological epimorphism $A \rightarrow B$ is a universal localisation. This is due to [25, Theorem 6.1] in the hereditary case and due to [5, Theorem 5.7] in the case of a commutative noetherian ring of Krull dimension at most 1.

The only thing remaining to be checked is that conditions of Proposition 3.3 are, in fact, satisfied. For A hereditary, there is nothing to check. If A is commutative noetherian of Krull dimension at most 1, then all the subsets of $\text{Spec}(A)$ are coherent by [23, Corollary 4.3], including $\text{Spec}(A) \setminus \text{Supp}(\mathcal{C})$. With that we may conclude the proof. \square

Note that if R is hereditary, then $R_{\mathcal{C}}$ is also hereditary by [37, Lemma 1.4], so $\mathbf{D}^{\text{perf}}(R_{\mathcal{C}}) \simeq \mathbf{D}^b(\text{mod-}R_{\mathcal{C}})$.

Notation. Given a hereditary ring R and a thick subcategory $\mathcal{C} \subseteq \mathbf{D}^{\text{perf}}(R)$, we shall denote the universal localisation determined by \mathcal{C} in the sense of Corollary 3.5 as $R_{\mathcal{C}}$. We will then always implicitly treat $\mathbf{D}(R_{\mathcal{C}})$ as a thick subcategory of $\mathbf{D}(R)$ and the completion $\mathfrak{S}'_{\mathcal{C}}(\mathbf{D}^{\text{perf}}(R)) \subseteq \mathbf{D}(R)$ as a triangulated subcategory of

$$\mathbf{D}^b(\text{mod-}R_{\mathcal{C}}) \simeq \mathbf{D}^{\text{perf}}(R_{\mathcal{C}}) \subseteq \mathbf{D}(R)$$

using Proposition 3.3.

4 Computational tools for hereditary and noetherian rings

This section contains multiple new results about completions of bounded derived categories of certain, mostly noetherian hereditary, rings. Those facts will be useful later for the classification of completions over commutative hereditary noetherian rings (Section 6) and hereditary tame algebras (Section 7). Among other things, for a suitable hereditary ring R , we show how to construct Cauchy sequences with uniformly bounded cohomology (Corollary 4.4); that $\mathfrak{S}'(\mathbf{D}^b(\text{mod-}R))$ is idempotent complete (Proposition 4.7); and explicitly describe completions of $\mathfrak{S}'(\mathbf{D}^b(\text{mod-}R))$ with respect to constant additive metrics (Proposition 4.12). We also recall the construction of \mathcal{B} -trivial sequences by Cummings and Gratz in Proposition 4.2.

Through this whole section we work with a ring R which is a finitely generated \mathbb{k} -algebra over a commutative noetherian ring \mathbb{k} . Typical choices of \mathbb{k} will be $\mathbb{k} = K$ for K a field, thus making R into a finite dimensional K -algebra, or $\mathbb{k} = R$ when R itself is commutative noetherian.

We fix a notation $\mathcal{S} := \mathbf{D}^{\text{perf}}(R)$. Note that $\mathcal{S} \simeq \mathbf{D}^b(\text{mod-}R)$ whenever R is of finite global dimension. We also denote $\mathcal{T} := \mathbf{D}(R)$ and note that the inclusion $\mathcal{S} \hookrightarrow \mathcal{T}$ is a good extension, so all (pre)completions will be calculated inside \mathcal{T} .

Lemma 4.1. Let $0 \in \mathcal{C}$ be a replete subcategory of $\mathcal{K} := \mathbf{K}^b(\text{proj-}R)$. Then for any two objects $X, Y \in \mathcal{K}$ it holds that $\text{Hom}_{\mathcal{K}}(X, Y)$ is a noetherian \mathbb{k} -module.

In particular, every object $U \in \mathcal{K}$ contains a maximal direct summand $V \in \mathcal{C}$.

Proof. Let $X, Y \in \mathcal{K}$. Then the \mathbb{k} -module $\text{Hom}_{\mathcal{K}}(X, Y)$ is a quotient of the \mathbb{k} -module $\text{Hom}_{\text{Ch}^b(\text{proj-}R)}(X, Y)$ by null-homotopic maps, while $\text{Hom}_{\text{Ch}^b(\text{proj-}R)}(X, Y)$ is a \mathbb{k} -submodule of $\text{End}_R(F)$ for some finitely generated free R -module F . As R is a noetherian \mathbb{k} -module, $\text{Hom}_{\mathcal{K}}(X, Y)$ is then a noetherian \mathbb{k} -module as well.

Now let $U \in \mathcal{K}$. There definitely exists some direct summand of U belonging to \mathcal{C} , namely the zero object. If there did not exist a maximal direct summand from \mathcal{C} of U , we would be able to use induction to construct an infinite, strictly increasing chain of \mathbb{k} -submodules of $\text{End}_{\mathcal{K}}(U) = \text{Hom}_{\mathcal{K}}(U, U)$, which is impossible in a noetherian module. \square

We state a variant of [16, Lemma 3.17] for perfect complexes over \mathbb{k} -algebras.

Proposition 4.2. Let \mathcal{B} and \mathcal{C} be replete subcategories of \mathcal{S} such that \mathcal{C} is closed under extensions and direct summands. Let \mathbf{E} be a sequence in \mathcal{S} that is compactly supported at \mathcal{B} . Then there exists a subsequence $\mathbf{E}_{\mathbf{I}}$ of \mathbf{E} and a \mathcal{B} -trivial component $\tilde{\mathbf{E}}_{\mathbf{I}}$ of $\mathbf{E}_{\mathbf{I}}$ such that

$$\text{Mocolim}_{\rightarrow} \mathbf{E} \simeq \text{Mocolim}_{\rightarrow} \tilde{\mathbf{E}}_{\mathbf{I}},$$

and $\tilde{\mathbf{E}}_{\mathbf{I}}$ is compactly supported at \mathcal{B} . Moreover, if \mathbf{E} stabilises at \mathcal{C} , then so does $\tilde{\mathbf{E}}_{\mathbf{I}}$.

Proof. The exact same proposition was proved by Cummings and Gratz in [16, Lemma 3.17] for \mathcal{S} being a Krull-Schmidt triangulated category. The Krull-Schmidt property of \mathcal{S} is used in the beginning of the proof in order to pick a maximal direct summand of E belonging to \mathcal{B} for any object $E \in \mathcal{S}$. However, by Lemma 4.1 we may perform the same selection of a maximal direct summand in the category $\mathcal{S} = \mathbf{D}^{\text{perf}}(R) \simeq \mathbf{K}^b(\text{proj-}R)$. Therefore, the original proof of [16, Lemma 3.17] also applies *verbatim* as a proof of this proposition. \square

Corollary 4.3. Equip \mathcal{S} with an additive good metric. Let \mathcal{B} be a replete subcategory of \mathcal{S} . If \mathbf{E} is a Cauchy sequence compactly supported at \mathcal{B} , then there exists a \mathcal{B} -trivial component \mathbf{F} of a subsequence of \mathbf{E} such that \mathbf{F} is Cauchy and compactly supported at \mathcal{B} , and $\text{Mocolim}_{\rightarrow} \mathbf{E} \simeq \text{Mocolim}_{\rightarrow} \mathbf{F}$.

For hereditary rings, the above construction can be also used to change a generic Cauchy sequence into a Cauchy sequence with all entries uniformly bounded in cohomology.

Corollary 4.4. Assume R is a hereditary ring, and equip \mathcal{S} with an additive good metric. Let \mathbf{E} be a Cauchy sequence. If $E := \text{Hocolim}_{\rightarrow} \mathbf{E} \in \mathcal{T}$ has cohomology concentrated in degrees N, \dots, M for some $N \leq M \in \mathbb{Z}$, then there exists a Cauchy sequence $\mathbf{F} = (F_n, f_n)_{n \in \mathbb{N}}$ such that $\text{Hocolim}_{\rightarrow} \mathbf{F} \simeq E$ and for all $n \in \mathbb{N}$ the object F_n has cohomology concentrated in degrees $N, \dots, M + 1$.

Proof. Let $(\mathcal{S}^{\leq 0}, \mathcal{S}^{> 0})$ be the standard t -structure on $\mathcal{S} \simeq \mathbf{D}^b(\text{mod-}R)$. The fact that $E \in \mathcal{T}$ has cohomology concentrated in degrees N, \dots, M implies that \mathbf{E} is compactly supported with respect to the full subcategory $\mathcal{B} := \mathcal{S}^{\leq N-1} \cup \mathcal{S}^{> M+1}$.

Since the conditions of Corollary 4.3 are satisfied, there exists an \mathcal{B} -trivial Cauchy sequence $\mathbf{F} = (F_n, f_n)_{n \in \mathbb{N}}$ in \mathcal{S} with $\text{Hocolim}_{\rightarrow} \mathbf{F} \simeq E$ holding in \mathcal{T} . As \mathbf{F} is \mathcal{B} -trivial, for all $n \in \mathbb{N}$ the object F_n does not contain a direct summand from \mathcal{B} . This is equivalent to F_n having cohomology concentrated in degrees $N, \dots, M + 1$ because F_n is a coproduct of its cohomologies due to R being hereditary. \square

In a series of two lemmas and a proposition, we now aim to prove that $\mathfrak{S}'(\mathcal{S})$ is idempotent complete for any choice of an additive good metric on \mathcal{S} whenever R is hereditary.

The proof of the following lemma uses the idea behind Brüning's correspondence between wide categories of a hereditary abelian category and thick subcategories of its bounded derived category, see [13, Lemma 5.2].

Lemma 4.5. Assume R is a hereditary ring. Equip \mathcal{S} with an additive good metric $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$. Let $\mathcal{C} \subseteq \text{mod-}R$ and $n \in \mathbb{N}$. If $\mathcal{C} \subseteq B_{n+2}$ as objects of \mathcal{S} , then $\text{wide}(\mathcal{C}) \subseteq B_n$.

Proof. We show that the closure of \mathcal{C} under extensions, kernels and cokernels (in $\text{Mod-}R$) lies in B_n . Let $M, N \in \mathcal{C}$. Any extension of M by N already lies in $B_{n+2} \subseteq B_n$ by the definition of a good metric. Let $f : M \rightarrow N$ be a homomorphism of R -modules. Consider the distinguished triangle

$$M \xrightarrow{f} N \rightarrow \text{cone } f \rightarrow \Sigma M$$

in \mathcal{S} . Since R is hereditary, we get $\text{cone } f \simeq \Sigma \text{Ker } f \oplus \text{Coker } f$. As $M \in B_{n+2}$, we have $\Sigma M \in B_{n+1}$. And since B_{n+1} is closed under extensions and direct summands, and $N \in B_{n+1}$, we obtain $\Sigma \text{Ker } f, \text{Coker } f \in B_{n+1}$, allowing us to conclude the proof with $\text{Ker } f, \text{Coker } f \in B_n$. \square

Lemma 4.6. Assume R is a hereditary ring. Equip \mathcal{S} with an additive good metric $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$. Let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{> 0})$ be the standard t -structure on \mathcal{T} . Let $M \in \text{Mod-}R$ and $T \in \mathcal{T}^{> 1}$. If $M \oplus T \in \mathfrak{S}'(\mathcal{S})$, then $T, M \in \mathfrak{S}'(\mathcal{S})$.

Proof. Assume $M \oplus T \in \mathfrak{S}'(\mathcal{S})$. By Corollary 4.4, there exists a good Cauchy sequence $\mathbf{E} = (E_n, e_n)_{n \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$ the object E_n has cohomology concentrated in non-negative degrees and $\text{Hocolim } \mathbf{E} \simeq M \oplus T$. Suppose that $M \oplus T$ is compactly supported at $s \in \mathbb{N}$. Then by Corollary 4.3 we can even w.l.o.g. assume that \mathbf{E} is s -trivial. Since

$$\varinjlim H^1(\mathbf{E}) \simeq H^1(\text{Hocolim } \mathbf{E}) \simeq H^1(M \oplus T) \simeq 0,$$

and all R -modules in the directed diagram $H^1(\mathbf{E})$ are finitely generated, we obtain that for all $n \in \mathbb{N}$ there exists $m > n$ such that $H^1(e_{n,m}) = 0$. This is why, we can (after restricting to a subsequence) w.l.o.g. assume that $H^1(e_n) = 0$ for all $n \in \mathbb{N}$.

As R is of finite global dimension, it holds that $(\mathcal{T}^{\leq 0} \cap \mathcal{S}, \mathcal{T}^{> 0} \cap \mathcal{S})$ is a t -structure on \mathcal{S} . We consider two approximation sequences of \mathbf{E} ; the $\mathcal{T}^{\leq 0}$ -approximation $\mathbf{X} = (X_n, f_n)_{n \in \mathbb{N}}$ and the $\mathcal{T}^{> 0}$ -approximation $\mathbf{Y} = (Y_n, g_n)_{n \in \mathbb{N}}$. For all $n \in \mathbb{N}$ we have $E_n \simeq X_n \oplus Y_n$ (because R is hereditary), and the object X_n has cohomology concentrated in degree 0.

Fix an index $t \geq s$. Let $N \in \mathbb{N}$ be such that for all $n \geq N$ it holds that

$$C_n := \text{cone } e_n \in B_{t+3}.$$

Pick $n \geq N$. The objects $\Sigma^{-1}H^1(C_n)$ and $H^0(C_{n+1})$ are direct summands of C_n and C_{n+1} , respectively, so they both lie in B_{t+3} . Hence both the modules $H^1(C_n)$ and $H^0(C_{n+1})$ lie in B_{t+2} . Consider the distinguished triangles

$$E_n \xrightarrow{e_n} E_{n+1} \rightarrow C_n \rightarrow \Sigma E_n \quad \text{and} \quad E_{n+1} \xrightarrow{e_{n+1}} E_{n+2} \rightarrow C_{n+1} \rightarrow \Sigma E_{n+1}.$$

The long exact sequences in cohomology

$$H^1(E_n) \xrightarrow{0} H^1(E_{n+1}) \rightarrow H^1(C_n) \quad \text{and} \quad H^0(C_{n+1}) \rightarrow H^1(E_{n+1}) \xrightarrow{0} H^1(E_{n+2})$$

express $H^1(E_{n+1})$ as an image of an R -homomorphism $H^0(C_{n+1}) \rightarrow H^1(C_n)$ of objects from B_{t+2} . Lemma 4.5 gives us $H^1(E_{n+1}) \in B_t \subseteq B_s$. As \mathbf{E} is assumed to be s -trivial, this forces $H^1(E_{n+1}) \simeq 0$.

By the triangulated 3×3 -lemma (see [30, Lemma 2.6]), there exists a diagram

$$\begin{array}{ccccccc} X_{n+1} & \longrightarrow & E_{n+1} & \longrightarrow & Y_{n+1} & \xrightarrow{0} & \Sigma X_{n+1} \\ \downarrow f_{n+1} & & \downarrow e_{n+1} & & \downarrow g_{n+1} & & \downarrow \\ X_{n+2} & \longrightarrow & E_{n+2} & \longrightarrow & Y_{n+2} & \xrightarrow{0} & \Sigma X_{n+2} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C_{n+1}^X & \longrightarrow & C_{n+1} & \longrightarrow & C_{n+1}^Y & \longrightarrow & \Sigma C_{n+1}^X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma X_{n+1} & \longrightarrow & \Sigma E_{n+1} & \longrightarrow & \Sigma Y_{n+1} & \xrightarrow{0} & \Sigma^2 X_{n+1} \end{array}$$

where all rows and columns are distinguished triangles and all squares are commutative with the exception of the right bottom square which is anticommutative. Taking cohomologies and using $H^0(\mathbf{X}) = \mathbf{X}$ and $H^0(\mathbf{Y}) \simeq 0 \simeq H^{-1}(\mathbf{Y})$ gives us a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^{-1}(C_{n+1}^X) & \longrightarrow & H^{-1}(C_{n+1}) & \longrightarrow & 0 \longrightarrow H^0(C_{n+1}^X) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X_{n+1} & \longrightarrow & H^0(E_{n+1}) & \longrightarrow & 0 \longrightarrow 0 \\ & & \downarrow f_{n+1} & & \downarrow H^0(e_{n+1}) & & \downarrow \\ 0 & \longrightarrow & X_{n+2} & \longrightarrow & H^0(E_{n+2}) & \longrightarrow & 0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^0(C_{n+1}^X) & \longrightarrow & H^0(C_{n+1}) & \longrightarrow & H^0(C_{n+1}^Y) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & H^1(E_{n+1}) & \longrightarrow & H^1(Y_{n+1}) \longrightarrow 0 \end{array}$$

of R -modules with exact rows and columns. Using exactness and $H^1(E_{n+1}) \simeq 0$, we obtain $H^{-1}(C_{n+1}^X) \simeq H^{-1}(C_{n+1}) \in B_{t+1}$ and $H^0(C_{n+1}^X) \simeq H^0(C_{n+1}) \in B_t$.

In other words, we have $C_{n+1}^X \simeq \Sigma H^{-1}(C_{n+1}^X) \oplus H^0(C_{n+1}^X) \in B_t$. Since t and n were arbitrary, we may conclude that \mathbf{X} is a Cauchy sequence. Because

$$\text{Hocolim} \mathbf{X} \simeq \varinjlim H^0(\mathbf{E}) \simeq H^0(\text{Hocolim} \mathbf{E}) \simeq H^0(M \oplus T) \simeq M,$$

we get $M \in \mathcal{L}'(\mathcal{S})$. But M is also compactly supported as a direct summand of an element of the completion. Hence $M \in \mathcal{C}'(\mathcal{S})$.

Finally, consider the split distinguished triangle

$$M \rightarrow M \oplus T \rightarrow T \rightarrow \Sigma M$$

in \mathcal{T} . Since $M, M \oplus T \in \mathfrak{G}'(\mathcal{S})$ and $\mathfrak{G}'(\mathcal{S})$ is a triangulated subcategory of \mathcal{T} , we obtain $T \in \mathfrak{G}'(\mathcal{S})$. \square

Proposition 4.7. Assume R is a hereditary ring, and equip \mathcal{S} with an additive good metric. Then $\mathfrak{G}'(\mathcal{S})$ is idempotent complete.

Proof. We start by observing that if two objects X, Y of the good extension \mathcal{T} satisfy $X \oplus Y \in \mathfrak{G}'(\mathcal{S})$, then from the existence of the triangles

$$X \oplus Y \xrightarrow{\begin{bmatrix} \text{id}_X & 0 \\ 0 & 0 \end{bmatrix}} X \oplus Y \rightarrow Y \oplus \Sigma Y \rightarrow \Sigma X \oplus \Sigma Y$$

and

$$\Sigma Y \oplus \Sigma^2 Y \xrightarrow{\begin{bmatrix} 0 & 0 \\ \text{id}_{\Sigma Y} & 0 \end{bmatrix}} Y \oplus \Sigma Y \rightarrow Y \oplus \Sigma^3 Y \rightarrow \Sigma^2 Y \oplus \Sigma^3 Y$$

in $\mathfrak{G}'(\mathcal{S})$ we can infer $Y \oplus \Sigma Y^3 \in \mathfrak{G}'(\mathcal{S})$.

Let $C \in \mathfrak{G}'(\mathcal{S})$. Then C is bounded in cohomology by Proposition 2.19. Since R is hereditary, this means $C \simeq \coprod_{i=-l}^l \Sigma^{-i} H^i(C)$ for some $l \in \mathbb{N}$. By the above observation, $H^{-l}(C) \oplus \Sigma^{-3} H^{-l}(C) \in \mathfrak{G}'(\mathcal{S})$. We get $H^{-l}(C) \in \mathfrak{G}'(\mathcal{S})$ by Lemma 4.6. An induction argument on $-l \leq i \leq l$ then proves that $\mathfrak{G}'(\mathcal{S})$ is closed under taking cohomologies.

All we are left with to prove is that for modules $M, N \in \text{Mod-}R$ with $M \in \mathfrak{G}'(\mathcal{S})$ and N a direct summand of M , we have $N \in \mathfrak{G}'(\mathcal{S})$. However, this follows from the same trick as before; the initial observation gives us $N \oplus \Sigma^{-3} N \in \mathfrak{G}'(\mathcal{S})$, and Lemma 4.6 yields $N \in \mathfrak{G}'(\mathcal{S})$. We conclude that the completion $\mathfrak{G}'(\mathcal{S})$ is idempotent complete. \square

Definition 4.8. Let \mathcal{C} be a triangulated category. We say that \mathcal{C} is *countably generated* if there exists an at most countable subset $\mathcal{D} \subseteq \mathcal{T}$ such that $\langle \mathcal{D} \rangle = \mathcal{T}$.

We shall now show that (for R hereditary) elements of a completion of \mathcal{S} also lie in a completion with respect to a specific countably generated constant metric. This is useful for reducing arguments about generic metrics to constant metrics.

Lemma 4.9. Assume R is a hereditary ring. Equip \mathcal{S} with an additive good metric $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$. Let $E \in \mathfrak{G}'_{\mathcal{M}}(\mathcal{S})$. Then for every $t \in \mathbb{N}$ there exist $s \geq t$ and a countably generated thick subcategory \mathcal{C} of \mathcal{S} such that $\mathcal{C} \subseteq \text{thick}(H^0(B_s))$ and $E \in \mathfrak{G}'_{\mathcal{C}}(\mathcal{S})$.

Proof. By Proposition 2.19, there exists $l \in \mathbb{N}$ such that E has cohomology concentrated in degrees $-l, \dots, l-1$. By Corollary 4.4, we can find a Cauchy (with respect to \mathcal{M}) sequence $\mathbf{E} = (E_n, e_n)_{n \in \mathbb{N}}$ with $E \simeq \text{Hocolim} \mathbf{E}$ such that for all $n \in \mathbb{N}$ the entry E_n has cohomology concentrated in degrees $-l, \dots, l$. Denote $C_n := \text{cone } e_n$ for all $n \in \mathbb{N}$.

Fix $t \in \mathbb{N}$. Let $s \geq t$ be such that \mathbf{E} is compactly supported at s . Let $N \in \mathbb{N}$ be such that for all $n \geq N$ we have $C_n \in B_{s+2l+2}$. Define $\mathcal{C} := \text{thick}(C_n : N \leq n)$. Then \mathbf{E} is Cauchy with respect to \mathcal{C} and $E \in \mathfrak{L}'_{\mathcal{C}}(\mathcal{S})$.

Since $C_n = \coprod_{i=-l-1}^l \Sigma^{-i} H^i(C_n)$ for all $n \geq N$, the category \mathcal{C} is also generated (as a thick subcategory) by

$$\left\{ H^i(C_n) : -l-1 \leq i \leq l, N \leq n \right\} \subseteq B_{s+l+1} \subseteq B_s,$$

so $\mathcal{C} \subseteq \text{thick}(H^0(B_s))$.

It remains to be shown that E is compactly supported at \mathcal{C} . Pick indices $n \geq N$, $-l-1 \leq i \leq l$ and $j \in \mathbb{Z}$. If $-l-1 \leq j \leq l$, then $\Sigma^j H^i(C_n) \in \Sigma^j B_{s+l+1} \subseteq B_s \subseteq {}^\perp E$, so $\text{Hom}_{\mathcal{T}}(\Sigma^j H^i(C_n), E) = 0$. If $j < -l-1$ or $j > l$, then it once again holds that $\text{Hom}_{\mathcal{T}}(\Sigma^j H^i(C_n), E) = 0$ because E has cohomology concentrated in degrees $-l, \dots, l-1$ and R is hereditary.

Therefore, all the shifts of the set $\{H^i(C_n) : -l-1 \leq i \leq l, N \leq n\}$ of generators of \mathcal{C} lie in ${}^\perp E$. Because ${}^\perp E$ is closed under extensions, we conclude that \mathbf{E} is compactly supported at \mathcal{C} and $E \in \mathfrak{S}'_{\mathcal{C}}(\mathcal{S})$. \square

In the next lemma, we employ Quillen's small object argument to build special Cauchy sequences with respect to constant metrics starting at a prescribed object by repeatedly taking suitable approximations.

Lemma 4.10. Let \mathcal{C} be a countably generated triangulated subcategory of \mathcal{S} . Then for every object $X \in \mathcal{S}$ there exists a Cauchy sequence $\mathbf{E} = (E_n, e_n)_{n \in \mathbb{N}}$ with respect to \mathcal{C} such that $X \simeq E_1$, $\text{Hocolim} \mathbf{E} \in \mathfrak{S}'_{\mathcal{C}}(\mathcal{S})$, and $\text{cone } e_n \in \mathcal{C}$ for all $n \in \mathbb{N}$.

Proof. Let \mathcal{C}' be an at most countable generating subset of \mathcal{C} . Using a bijection $\mathbb{N} \times \mathbb{N} \simeq \mathbb{N}$, we can find an enumeration $(C_n)_{n \in \mathbb{N}}$ of \mathcal{C}' such that for all $C \in \mathcal{C}'$ and $l \in \mathbb{Z}$ there exists an infinite subset $I \subseteq \mathbb{N}$ satisfying $\Sigma^l C \simeq C_i$ for each $i \in I$.

Let $X \in \mathcal{C}$. We construct the sequence $\mathbf{E} = (E_n, e_n)_{n \in \mathbb{N}}$ by induction on n . We start by defining $E_1 := X$. Assume that E_n is already constructed for some $n \in \mathbb{N}$. Since $\mathcal{S} \simeq \mathbf{K}^b(\text{proj-}R)$, the \mathbb{k} -module $\text{Hom}_{\mathcal{S}}(C_n, E_n)$ is generated by a finite subset J_n due to Lemma 4.1. We then define e_n and E_{n+1} by the distinguished triangle

$$\coprod_{j \in J_n} C_n \xrightarrow{\varphi_n} E_n \xrightarrow{e_n} E_{n+1} \rightarrow \coprod_{j \in J_n} \Sigma C_n$$

where $\varphi_n = [j]_{j \in J_n}$ is the natural coproduct map. This construction makes φ_n into a $\text{coprod}(C_n)$ -precover. With this definition $\text{cone } e_n \in \mathcal{C}$ for all $n \in \mathbb{N}$, so \mathbf{E} is indeed Cauchy.

Denote $E := \text{Hocolim} \mathbf{E}$. We need to prove that E is compactly supported with respect to \mathcal{C} . Fix $C \in \mathcal{C}'$ and $l \in \mathbb{Z}$. There are infinitely many indices $m \in \mathbb{N}$ satisfying $\Sigma^l C \simeq C_m$, and for any such an m applying $\text{Hom}_{\mathcal{S}}(C_m, -)$ to the distinguished triangle defining E_{m+1} provides us with an exact sequence

$$\text{Hom}_{\mathcal{S}} \left(C_m, \coprod_{j \in J_m} C_m \right) \xrightarrow{\varphi_m \circ -} \text{Hom}_{\mathcal{S}}(C_m, E_m) \xrightarrow{e_m \circ -} \text{Hom}_{\mathcal{S}}(C_m, E_{m+1})$$

where $\varphi_m \circ -$ is clearly an epimorphism, so $e_m \circ - = \text{Hom}_{\mathcal{S}}(C_m, e_m) = 0$ by the exactness. We infer $0 \simeq \varinjlim \text{Hom}_{\mathcal{T}}(\Sigma^l C, \mathbf{E}) \simeq \text{Hom}_{\mathcal{T}}(\Sigma^l C, E)$.

Consequently, as all the shifts of \mathcal{C}' lie in ${}^\perp E$, which is extension-closed, so does $\mathcal{C} = \langle \mathcal{C}' \rangle$. Hence \mathbf{E} is compactly supported at \mathcal{C} and $E \in \mathfrak{S}'_{\mathcal{C}}(\mathcal{S})$. \square

Remark 4.11. Quillen’s small object argument, where we construct precovers by taking coproducts over “all possible morphisms”, has many applications in algebra and topology. In the context of triangulated categories, we can mention e.g. the construction of compactly generated t -structures by Alonso, Jeremías, and Souto in [1, Theorem A.1].

For hereditary \mathbb{k} -algebras, we can use Corollary 3.5 to express completions with respect to certain constant metrics as precisely bounded derived categories of corresponding universal localisations.

Proposition 4.12. Assume R is a hereditary ring. Let \mathcal{C} be a countably generated thick subcategory of \mathcal{S} . Then $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) \simeq \mathbf{D}^b(\text{mod-}R_{\mathcal{C}})$.

Proof. We employ Lemma 4.10 to the object $X := R$, and we obtain a Cauchy sequence $\mathbf{E} = (E_n, e_n)_{n \in \mathbb{N}}$ in \mathcal{S} with respect to \mathcal{C} such that

$$E := \text{Hocolim} \mathbf{E} \in \mathfrak{S}'_{\mathcal{C}}(\mathcal{S}), \quad R \simeq E_1, \quad \text{and} \quad \text{cone } e_n \in \mathcal{C} \text{ for all } n \in \mathbb{N}.$$

Since $E \in \mathbf{D}(R_{\mathcal{C}}) = \mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) \subseteq \mathcal{T}$ under the identification from Corollary 3.5, it holds that $E \simeq E \otimes_R^{\mathbf{L}} R_{\mathcal{C}}$. After applying $-\otimes_R^{\mathbf{L}} R_{\mathcal{C}}$ to the sequence \mathbf{E} , we get a sequence $\mathbf{E} \otimes_R^{\mathbf{L}} R_{\mathcal{C}}$ in the category $\mathbf{D}(R_{\mathcal{C}})$. Because $\text{cone } e_n \in \mathcal{C} \subseteq \text{Ker}(-\otimes_R^{\mathbf{L}} R_{\mathcal{C}})$ for all $n \in \mathbb{N}$, the sequence $\mathbf{E} \otimes_R^{\mathbf{L}} R_{\mathcal{C}}$ consists of isomorphisms only, so

$$R_{\mathcal{C}} \simeq R \otimes_R^{\mathbf{L}} R_{\mathcal{C}} \simeq E_1 \otimes_R^{\mathbf{L}} R_{\mathcal{C}} \simeq \text{Hocolim} \mathbf{E} \otimes_R^{\mathbf{L}} R_{\mathcal{C}} \simeq E \otimes_R^{\mathbf{L}} R_{\mathcal{C}} \simeq E.$$

As $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S})$ is idempotent complete by 4.7 and $R_{\mathcal{C}} \in \mathfrak{S}'_{\mathcal{C}}(\mathcal{S})$, we get the desired equality $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) = \mathbf{D}^b(\text{mod-}R_{\mathcal{C}})$ by Proposition 3.3. \square

5 Lattice theory of metrics

This section is dedicated to developing the lattice theory of good metrics on a triangulated category. We also present its application to the representation theory of algebras of finite representation type in Proposition 5.5 and hereditary rings in Proposition 5.10.

Through this whole section \mathcal{T} will denote an essentially small triangulated category. Denote also \sim the equivalence of good metrics and $[\mathcal{M}]$ the block of equivalence of a good metric \mathcal{M} on \mathcal{T} under \sim .

Notation. Let be a triangulated category \mathcal{T} and $\mathcal{A} \subseteq \mathcal{T}$ a class. We define *extension closure* of \mathcal{A} as the smallest full subcategory of \mathcal{T} containing \mathcal{A} and 0 which is closed under extensions. We denote it as $\overline{\mathcal{A}}$.

Proposition 5.1. Let \mathcal{M} be the class of all good metrics on \mathcal{T} . Then \mathcal{M}/\sim forms a lattice. For good metrics $\mathcal{A} = \{A_n\}_{n \in \mathbb{N}}$ and $\mathcal{C} = \{C_n\}_{n \in \mathbb{N}}$ on \mathcal{T} the meet is given by $[\mathcal{A}] \wedge [\mathcal{C}] := [\{A_n \cap C_n\}_{n \in \mathbb{N}}]$ and the join is given by $[\mathcal{A}] \vee [\mathcal{C}] := \left[\left\{ \overline{A_n \cup C_n} \right\}_{n \in \mathbb{N}} \right]$.

Proof. Let $\mathcal{A} = \{A_n\}_{n \in \mathbb{N}}$, $\mathcal{C} = \{C_n\}_{n \in \mathbb{N}} \in \mathcal{M}$. It is obvious that $\{A_n \cap C_n\}_{n \in \mathbb{N}}$ is a good metric. We must check that $[\{A_n \cap C_n\}_{n \in \mathbb{N}}]$ does not depend on the choice of the representative elements for $[\mathcal{A}]$ and $[\mathcal{C}]$.

Suppose that $\mathcal{A} \sim \mathcal{B}$ and $\mathcal{C} \sim \mathcal{D}$ for some other good metrics $\mathcal{B} = \{B_n\}_{n \in \mathbb{N}}$ and $\mathcal{D} = \{D_n\}_{n \in \mathbb{N}}$ on \mathcal{T} . We want to prove $\{A_n \cap C_n\}_{n \in \mathbb{N}} \sim \{B_n \cap D_n\}_{n \in \mathbb{N}}$. Due to the symmetry of \cap and transitivity of \sim , we may w.l.o.g. assume $\mathcal{C} = \mathcal{D}$. Clearly, we have

$$\{A_n \cap B_n \cap C_n\}_{n \in \mathbb{N}} \leq \{A_n \cap C_n\}_{n \in \mathbb{N}} \quad \text{and} \quad \{A_n \cap B_n \cap C_n\}_{n \in \mathbb{N}} \leq \{B_n \cap C_n\}_{n \in \mathbb{N}}.$$

Since $\mathcal{A} \sim \mathcal{B}$, for every $n \in \mathbb{N}$ we can find $m \geq n$ such that $A_m \subseteq B_n$. Then

$$A_m \subseteq A_n \cap B_n \quad \text{and} \quad A_m \cap C_m \subseteq A_n \cap B_n \cap C_m \subseteq A_n \cap B_n \cap C_n.$$

Hence $\{A_n \cap B_n \cap C_n\}_{n \in \mathbb{N}} \sim \{A_n \cap C_n\}_{n \in \mathbb{N}}$. Similarly, with the role of \mathcal{A} and \mathcal{B} swapped, we obtain $\{A_n \cap B_n \cap C_n\}_{n \in \mathbb{N}} \sim \{B_n \cap C_n\}_{n \in \mathbb{N}}$.

Consequently, $\{A_n \cap C_n\}_{n \in \mathbb{N}} \sim \{B_n \cap C_n\}_{n \in \mathbb{N}}$, so $[\{A_n \cap C_n\}_{n \in \mathbb{N}}]$ is choice-free. Furthermore, it is clear that any metric finer than both \mathcal{A} and \mathcal{C} is finer than $\{A_n \cap C_n\}_{n \in \mathbb{N}}$ as well. Hence the meet $[\mathcal{A}] \wedge [\mathcal{C}]$ exists and is equal to the desired equivalence class $[\{A_n \cap C_n\}_{n \in \mathbb{N}}]$.

Now, we tackle the join operation. It is easy to see that $\{\overline{A_n \cup C_n}\}_{n \in \mathbb{N}}$ is a good metric.

We must now verify that $\left[\{\overline{A_n \cup C_n}\}_{n \in \mathbb{N}} \right]$ does not depend on the choice of the representative elements for \mathcal{A} and \mathcal{C} . Similarly as above, we may assume that $\mathcal{A} \sim \mathcal{B}$ for some $\mathcal{B} = \{B_n\}_{n \in \mathbb{N}} \in \mathcal{M}$. W.l.o.g. it is enough to show $\left[\{\overline{A_n \cup C_n}\}_{n \in \mathbb{N}} \right] = \left[\{\overline{B_n \cup C_n}\}_{n \in \mathbb{N}} \right]$. Fix $n \in \mathbb{N}$. We find $m \geq n$ such that $A_m \subseteq B_n$. Then

$$\overline{A_m \cup C_m} \subseteq \overline{A_m \cup C_n} \subseteq \overline{B_n \cup C_n}.$$

Hence it holds that $\{\overline{A_n \cup C_n}\}_{n \in \mathbb{N}} \leq \{\overline{B_n \cup C_n}\}_{n \in \mathbb{N}}$. A dual argument then yields $\{\overline{A_n \cup C_n}\}_{n \in \mathbb{N}} \sim \{\overline{B_n \cup C_n}\}_{n \in \mathbb{N}}$.

Because any metric coarser than both \mathcal{A} and \mathcal{C} is coarser than $\{\overline{A_n \cup C_n}\}_{n \in \mathbb{N}}$ as well, we see that the join

$$[\mathcal{A}] \vee [\mathcal{C}] := \left[\{\overline{A_n \cup C_n}\}_{n \in \mathbb{N}} \right]$$

exists and is well-defined. We conclude that \mathcal{M} / \sim has a structure of a lattice. \square

Definition 5.2. By $\mathbb{M}_{\mathcal{T}}$ we shall denote the lattice given by Proposition 5.1 applied to \mathcal{T} and call it a *lattice of good metrics* on \mathcal{T} .

By abuse of notation, we will (from now on) not distinguish between a good metric \mathcal{M} on \mathcal{T} and its equivalence class $[\mathcal{M}] \in \mathbb{M}_{\mathcal{T}}$ unless necessary. Also, for two good metrics \mathcal{M} and \mathcal{N} on \mathcal{T} we will write $\mathcal{M} \vee \mathcal{N}$ and $\mathcal{M} \wedge \mathcal{N}$ instead of $[\mathcal{M}] \vee [\mathcal{N}]$ and $[\mathcal{M}] \wedge [\mathcal{N}]$, respectively.

Lemma 5.3. Equip \mathcal{T} with two good metrics $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$ and $\mathcal{N} = \{C_n\}_{n \in \mathbb{N}}$. Then $\mathfrak{C}_{\mathcal{M} \vee \mathcal{N}}(\mathcal{T}) = \mathfrak{C}_{\mathcal{M}}(\mathcal{T}) \cap \mathfrak{C}_{\mathcal{N}}(\mathcal{T})$.

Proof. This is clear, because for all $F \in \text{Mod-}\mathcal{T}$ and a fixed $n \in \mathbb{N}$ it holds that $F(\overline{B_n \cup C_n}) = 0$ if and only if $F(B_n) = 0$ and $F(C_n) = 0$. \square

Remark 5.4. The lattice $\mathbb{M}_{\mathcal{T}}$ is bounded with the constant metric \mathcal{T} being the largest element and the constant metric 0 being the smallest.

However, an infinite intersection is not an infinite meet in general because it is not compatible with equivalences of metrics. More precisely, consider a metric $\{B_n\}_{n \in \mathbb{N}}$ such that $\bigcap_{n \in \mathbb{N}} B_n = 0$ and $[\{B_n\}_{n \in \mathbb{N}}] \neq [0]$ (e.g. any aisle metric $\{\mathcal{T}^{\leq n}\}_{n \in \mathbb{N}}$ for a non-degenerate t -structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{> 0})$, see [34, Example 12 i), [16, Definition 4.2]). Then all the metrics of the form $\{B_{kn}\}_{n \in \mathbb{N}}$ for some $k \in \mathbb{N}$ are equivalent.

Therefore, the meet satisfies $\bigwedge_{k=1}^{\infty} [\{B_{kn}\}_{n \in \mathbb{N}}] = [\{B_n\}_{n \in \mathbb{N}}]$, but this block of equivalence is not equal to $[\{\bigcap_{k=1}^{\infty} B_{kn}\}_{n \in \mathbb{N}}] = [\{0\}_{n \in \mathbb{N}}]$.

We show that for a representation-finite hereditary algebra, additive good metrics can be decomposed into “indecomposable metrics”. If K is an algebraically closed field, every connected representation-finite hereditary algebra is Morita equivalent to the path algebra KQ of a simply laced Dynkin quiver. For any finite dimensional indecomposable KQ -module M it then holds that $\langle M \rangle = \text{coprod} \{\Sigma^i M : i \in \mathbb{Z}\}$ in $\mathbf{D}^b(\text{mod-}KQ)$.

Proposition 5.5. Let K be a field and Q a simply laced Dynkin quiver.

Let $\mathcal{N} := \{B_n\}_{n \in \mathbb{N}}$ be an additive good metric on the category $\mathbf{D}^b(\text{mod-}KQ)$. Then

$$\mathcal{N} = \bigvee_{M \in \text{ind-}KQ} (\mathcal{N} \wedge \langle M \rangle)$$

in $\mathbb{M}_{\mathbf{D}^b(\text{mod-}KQ)}$.

Proof. It is enough to prove that two specific representatives from each of the equivalence classes of good metrics are equivalent. We shall do so for the metrics \mathcal{N} and $\left\{ \overline{\bigcup_{M \in \text{ind-}KQ} B_n \cap \langle M \rangle} \right\}_{n \in \mathbb{N}}$.

“ \leq ” Let $n \in \mathbb{N}$. For every indecomposable object from B_n , which is necessarily of the form $\Sigma^i M$ for some $M \in \text{ind-}KQ$ and $i \in \mathbb{Z}$, we have $\Sigma^i M \in B_n \cap \langle M \rangle$.

Since the metric \mathcal{N} is additive, all objects from B_n are extensions of indecomposables from B_n . Therefore, we have

$$B_n \subseteq \overline{\bigcup_{M \in \text{ind-}KQ} B_n \cap \langle M \rangle}.$$

This proves $\mathcal{N} \leq \bigvee_{M \in \text{ind-}KQ} (\mathcal{N} \wedge \langle M \rangle)$.

“ \geq ” Let $n \in \mathbb{N}$. For all $M \in \text{ind-}KQ$, we have $B_n \cap \langle M \rangle \subseteq B_n$. Hence

$$\overline{\bigcup_{M \in \text{ind-}KQ} B_n \cap \langle M \rangle} \subseteq B_n$$

for any $M \in \text{ind-}KQ$.

Consequently, we get $\bigvee_{M \in \text{ind-}KQ} (\mathcal{N} \wedge \langle M \rangle) \leq \mathcal{N}$ by the property of the join. The claim now follows. \square

Example 5.6. Let $\{K_m\}_{m \in \mathbb{N}}$ be a countable family of fields. We shall work with the coproduct triangulated category $\mathcal{S} := \coprod_{m=1}^{\infty} \mathbf{D}(K_m)$. The indecomposables of \mathcal{S} are precisely the fields of the form K_m for some $m \in \mathbb{N}$ concentrated in degree zero in the m -th component of the category, and all their shifts.

For every $m \in \mathbb{N}$ we define a metric $\{B_n^m\}_{n \in \mathbb{N}}$ by the formula

$$B_n^m = \text{coprod} \left\{ \Sigma^i K_m : i \geq n \right\} \quad \text{for all } n \in \mathbb{N}.$$

We show that the infinite join $\bigvee_{m=1}^{\infty} \{B_n^m\}_{n \in \mathbb{N}}$ does not exist using a diagonal argument.

Assume for a contradiction that there exists a metric $\{C_n\}_{n \in \mathbb{N}}$ on \mathcal{T} satisfying

$$\{C_n\}_{n \in \mathbb{N}} = \bigvee_{m=1}^{\infty} \{B_n^m\}_{n \in \mathbb{N}}.$$

Since $\{C_n\}_{n \in \mathbb{N}}$ is coarser than $\{B_n^m\}_{n \in \mathbb{N}}$ for any choice of $m \in \mathbb{N}$, there exists a strictly increasing function $f_m : \mathbb{N} \rightarrow \mathbb{N}$ with $B_{f_m(n)}^m \subseteq C_n$ for all $n \in \mathbb{N}$.

The metric $\left\{ \overline{\bigcup_{m \in \mathbb{N}} B_{f_m(n)}^m} \right\}_{n \in \mathbb{N}} \leq \{C_n\}_{n \in \mathbb{N}}$ is coarser than $\{B_n^m\}_{n \in \mathbb{N}}$ for all indices $m \in \mathbb{N}$, so by the definition of a join we actually get an equivalence

$$\left\{ \overline{\bigcup_{m \in \mathbb{N}} B_{f_m(n)}^m} \right\}_{n \in \mathbb{N}} \sim \{C_n\}_{n \in \mathbb{N}}.$$

Now consider the metric $\left\{ \overline{\bigcup_{m \in \mathbb{N}} B_{f_m(mn)}^m} \right\}_{n \in \mathbb{N}} \leq \{C_n\}_{n \in \mathbb{N}}$. It is also coarser than $\{B_n^m\}_{n \in \mathbb{N}}$ for all indices $m \in \mathbb{N}$, so by the definition of a join, we also get an equivalence $\left\{ \overline{\bigcup_{m \in \mathbb{N}} B_{f_m(n)}^m} \right\}_{n \in \mathbb{N}} \sim \left\{ \overline{\bigcup_{m \in \mathbb{N}} B_{f_m(mn)}^m} \right\}_{n \in \mathbb{N}} \sim \{C_n\}_{n \in \mathbb{N}}$.

This means, in particular, that there exists $n \in \mathbb{N}$ satisfying

$$\overline{\bigcup_{m \in \mathbb{N}} B_{f_m(n)}^m} \subseteq \overline{\bigcup_{m \in \mathbb{N}} B_{f_m(m)}^m}$$

but it is impossible because $B_{f_m(n)}^m \not\subseteq B_{f_m(m)}^m$ for all $m > n$. We have reached a contradiction.

Not only we have proven that $\bigvee_{m=1}^{\infty} \{B_n^m\}_{n \in \mathbb{N}}$ does not exist, and the lattice $\mathbb{M}_{\mathcal{S}}$ is not complete, but also that a decomposition statement such as Proposition 5.5 cannot hold in general for triangulated categories with infinitely many indecomposables. No metric on \mathcal{S} coarser than $\{B_n^m\}_{n \in \mathbb{N}}$ for all $m \in \mathbb{N}$ can be written as an infinite join of $\{B_n^m\}_{n \in \mathbb{N}}$'s.

5.1 Decomposition of metrics

Definition 5.7. Let \mathcal{S} be a thick subcategory of \mathcal{T} . Let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{> 0})$ be a t -structure on \mathcal{T} . We define the following good metrics (up to equivalence) on \mathcal{S} :

- (i) *the aisle metric* as the good metric $\mathcal{S}_- := \{\mathcal{T}^{\leq n} \cap \mathcal{S}\}_{n \in \mathbb{N}}$,
- (ii) *the coaisle metric* as the good metric $\mathcal{S}_+ := \{\mathcal{T}^{> n} \cap \mathcal{S}\}_{n \in \mathbb{N}}$,
- (iii) *the t -structure metric* as the good metric $\mathcal{S}_{\infty} := \mathcal{S}_- \vee \mathcal{S}_+$ in $\mathbb{M}_{\mathcal{S}}$.

If the subcategory \mathcal{S} is not specified, we implicitly take $\mathcal{S} = \mathcal{T}$.

Remark 5.8. Similar metrics, namely metrics given by specific shifts of kernels of a fixed homological functor $\mathcal{T} \rightarrow \mathcal{A}$ into an abelian category \mathcal{A} , have been studied by Neeman in [34, Example 12]. The notions of aisle and coaisle metrics were formally introduced by Cummings and Gratz in [16, Definition 4.2]. The notion of a t -structure metric is new, while it may in some cases coincide with the metric from [35, Example 3.4] called Krause’s metric due its connection to Krause’s construction of the sequential completion of a triangulated category from [24].

Definition 5.9. Let \mathcal{S} be a thick subcategory of \mathcal{T} . Let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{> 0})$ be a t -structure on \mathcal{T} . Let \mathcal{M} be a good metric on \mathcal{S} . We define the following good metrics (up to equivalence) on \mathcal{S} :

- (i) *the aisle submetric* of \mathcal{M} as the good metric $\mathcal{M}_- := \mathcal{M} \wedge \mathcal{S}_-$ in $\mathbb{M}_{\mathcal{S}}$,
- (ii) *the coaisle submetric* of \mathcal{M} as the good metric $\mathcal{M}_+ := \mathcal{M} \wedge \mathcal{S}_+$ in $\mathbb{M}_{\mathcal{S}}$,
- (iii) *the t -structure submetric* of \mathcal{M} as the good metric $\mathcal{M}_{\infty} := \mathcal{M} \wedge \mathcal{S}_{\infty}$ in $\mathbb{M}_{\mathcal{S}}$,

For bounded derived categories of hereditary algebras, the t -structure submetric of a good metric does not contribute much to the completion in the following sense:

Proposition 5.10. Let R be a hereditary, finitely generated algebra over a commutative noetherian ring. Equip $\mathcal{S} := \mathbf{D}^b(\text{mod-}R)$ with the standard t -structure $(\mathcal{S}^{\leq 0}, \mathcal{S}^{> 0})$ and additive good metrics $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$ and $\mathcal{N} = \{C_n\}_{n \in \mathbb{N}}$. If $\mathcal{M} = \mathcal{M}_{\infty} \vee \mathcal{N}$, then $\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) = \mathfrak{S}'_{\mathcal{N}}(\mathcal{S})$.

Proof. The (pre)completions and compactly supported elements are calculated inside the good extension $\mathcal{S} \hookrightarrow \mathbf{D}(R)$ using Theorem 2.15.

“ \supseteq ” Let $E \in \mathfrak{S}'_{\mathcal{N}}(\mathcal{S})$. Then $E \in \mathfrak{L}'_{\mathcal{N}}(\mathcal{S}) \subseteq \mathfrak{L}'_{\mathcal{M}}(\mathcal{S})$. By Proposition 2.19, we know $E \in \mathbf{D}^b(\text{Mod-}R)$, so $E \in \mathfrak{C}'_{\mathcal{M}_{\infty}}(\mathcal{S})$. Since also $E \in \mathfrak{C}'_{\mathcal{N}}(\mathcal{S})$, by Lemma 5.3 we have $E \in \mathfrak{C}'_{\mathcal{M}_{\infty} \vee \mathcal{N}}(\mathcal{S}) = \mathfrak{C}'_{\mathcal{M}}(\mathcal{S})$. Thus $E \in \mathfrak{S}'_{\mathcal{M}}(\mathcal{S})$.

“ \subseteq ” Let $E \in \mathfrak{S}'_{\mathcal{M}}(\mathcal{S})$. Then $E \in \mathfrak{C}'_{\mathcal{M}}(\mathcal{S}) \subseteq \mathfrak{C}'_{\mathcal{N}}(\mathcal{S})$ by Lemma 5.3. By Proposition 2.19, there exists $l \in \mathbb{N}$ such that E has cohomology concentrated in degrees $-l + 1, \dots, l - 1$. By Corollary 4.4, we can find a Cauchy sequence (with respect to \mathcal{M}) $\mathbf{E} = (E_n, e_n)_{n \in \mathbb{N}}$ with $E \simeq \text{Hocolim} \mathbf{E}$ such that for all $n \in \mathbb{N}$ the entry E_n has cohomology concentrated in degrees $-l + 1, \dots, l$.

Let $m \in \mathbb{N}$. We find $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$\text{cone } e_n \in \overline{\mathcal{S}^{\leq -l} \cup \mathcal{S}^{> l} \cup C_m}.$$

After applying the double truncation functor $\text{tr}^{\leq l} \circ \text{tr}^{> -l}$ we obtain that

$$\text{cone } e_n = \text{tr}^{\leq l} \circ \text{tr}^{> -l}(\text{cone } e_n) \in \text{tr}^{\leq l} \circ \text{tr}^{> -l} \left(\overline{\mathcal{S}^{\leq -l} \cup \mathcal{S}^{> l} \cup C_m} \right) \subseteq C_m.$$

This implies that \mathbf{E} stabilises at C_m . As m was arbitrary, \mathbf{E} is Cauchy with respect to \mathcal{N} . Hence $E \in \mathfrak{L}'_{\mathcal{N}}(\mathcal{S})$ and consequently $E \in \mathfrak{S}'_{\mathcal{N}}(\mathcal{S})$. \square

The condition $\mathcal{M} = \mathcal{M}_\infty \vee \mathcal{N}$ from Proposition 5.10 can be interpreted as some sort of local uniform convergence of \mathcal{M} towards \mathcal{N} . Up to the exception of finitely many indices $n \in \mathbb{N}$, objects with bounded cohomology cannot distinguish between the metrics $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$ and $\mathcal{N} = \{C_n\}_{n \in \mathbb{N}}$.

The last lemma of this section describes a scenario where the conditions of Proposition 5.10 are not satisfied.

Lemma 5.11. Let R be a hereditary, finitely generated algebra over a commutative noetherian ring. Equip $\mathcal{S} := \mathbf{D}^b(\text{mod-}R)$ with the standard t -structure $(\mathcal{S}^{\leq 0}, \mathcal{S}^{> 0})$ and an additive good metric $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$. If $\mathcal{M} \not\sim \mathcal{M}_\infty \vee \mathcal{B}$, where $\mathcal{B} := \bigcap_{n \in \mathbb{N}} B_n$, then for every $E \in \mathfrak{G}'_{\mathcal{M}}(\mathcal{S})$ there exist an additive good metric $\mathcal{N} = \{C_n\}_{n \in \mathbb{N}} \sim \mathcal{M}$ on \mathcal{S} and a family of finitely generated R -modules $(X_n)_{n \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$:

- $\text{Hom}_{\mathbf{D}(R)}(\Sigma^j X_n, E) = 0$ for all $j \in \mathbb{Z}$,
- $X_n \in C_n$,
- $X_n \notin \text{thick}(H^0(C_{n+1}))$.

Proof. The intersection $\mathcal{B} = \bigcap_{n \in \mathbb{N}} B_n$ is, by [29, Lemma 3.5], a thick subcategory of \mathcal{S} , so the expression $\mathcal{M}_\infty \vee \mathcal{B}$ makes sense. Since $\mathcal{M} \not\sim \mathcal{M}_\infty \vee \mathcal{B}$, there exists $l \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ we have $B_n \not\subseteq \overline{\mathcal{B} \cup \mathcal{S}^{\leq -l} \cup \mathcal{S}^{> l}}$. The completion $\mathfrak{G}'_{\mathcal{M}}(\mathcal{S})$ is calculated inside the good extension $\mathcal{S} \hookrightarrow \mathbf{D}(R)$ using Theorem 2.15.

Pick $E \in \mathfrak{G}'_{\mathcal{M}}(\mathcal{S})$. Using Proposition 2.19, we find $k \in \mathbb{N}$ such that E has cohomology concentrated in degrees $-k, \dots, k-1$. Suppose E is compactly supported at some $s_1 \in \mathbb{N}$. Because $B_{s_1+l+k} \not\subseteq \overline{\mathcal{B} \cup \mathcal{S}^{\leq -l} \cup \mathcal{S}^{> l}}$, we can find $X_1 \in \text{mod-}R$ and $-l \leq i \leq l$ such that $\Sigma^i X_1 \in B_{s_1+l+k} \setminus \overline{\mathcal{B} \cup \mathcal{S}^{\leq -l} \cup \mathcal{S}^{> l}}$.

Then $X_1 \in \Sigma^{-i} B_{s_1+l+k} \subseteq B_{s_1+k}$, and $\Sigma^j X_1 \in B_{s_1}$ for all $-k \leq j \leq k$. Due to $E \in B_{s_1}^\perp$ and the bounds on the cohomology of E , we get $\text{Hom}_{\mathbf{D}(R)}(\Sigma^j X_1, E) = 0$ for all $j \in \mathbb{Z}$. Because $X_1 \notin \mathcal{B}$, there exists $s_2 > s_1+2$ such that $X_1 \notin B_{s_2-2}$. Then $X_1 \notin \text{thick}(H^0(B_{s_2}))$ since otherwise we would have $X_1 \in \text{wide}(H^0(B_{s_2})) \subseteq B_{s_2-2}$ due to Lemma 4.5.

We continue the construction of s_n and X_n for general $n \in \mathbb{N}$ inductively by the same process as above. We finish the proof by defining $\mathcal{N} := \{B_{s_n}\}_{n \in \mathbb{N}}$ and noticing $\mathcal{N} \sim \mathcal{M}$. \square

6 Completions for commutative noetherian rings

This section contains one of the main results of this paper (Theorem 6.4); that is an explicit description of all completions of the bounded derived category of a hereditary commutative noetherian ring with respect to additive good metrics.

6.1 General completion lemmas

Before we start the actual computation of the completions in our special case, we need the following two facts about general completions of triangulated categories.

Lemma 6.1. Let \mathcal{S} be a triangulated category equipped with a good metric and $X \in \mathcal{S}$. If $\mathfrak{y}(X) \in \mathfrak{C}(\mathcal{S})$, then $\mathfrak{y}(X) \in \mathfrak{G}(\mathcal{S})$.

Proof. There is an equality $\mathfrak{z}(X) = \underline{\text{Mocolim}} \mathbf{E}$ for the constant Cauchy sequence $\mathbf{E} = (X, \text{id}_X)_{n \in \mathbb{N}}$ in \mathcal{S} , so $\mathfrak{z}(X) \in \mathfrak{L}(\mathcal{S})$. Since we already know that $\mathfrak{z}(X)$ is compactly supported, we get $\mathfrak{z}(X) \in \mathfrak{S}(\mathcal{S})$. \square

The above argument directly translates into the setting of good extensions, giving us that $\mathfrak{C}'(\mathcal{S}) \cap \mathcal{S} \subseteq \mathfrak{S}'(\mathcal{S})$.

Lemma 6.2. Equip a triangulated category \mathcal{S} with a good metric $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$. Let $\mathbf{E} = (E_n, e_n)_{n \in \mathbb{N}}$ be a Cauchy sequence compactly supported at $s \in \mathbb{N}$, and $E := \underline{\text{Mocolim}} \mathbf{E}$. Let $\mathbf{F} = (F_n, f_n)_{n \in \mathbb{N}}$ be any (non-necessarily Cauchy) sequence satisfying $F_n \in B_s^\perp$ for all $n \in \mathbb{N}$. If $E \simeq \underline{\text{Mocolim}} \mathbf{F}$, then E is a direct summand in $\text{Mod-}\mathcal{S}$ of an object from $\mathfrak{z}(\mathcal{S})$.

Proof. By Lemma 2.11, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and all

$$X \in \{\mathfrak{z}(F_m) : m \in \mathbb{N}\} \cup \{E\}$$

every map $\mathfrak{z}(E_n) \rightarrow X$ in $\text{Mod-}\mathcal{T}$ factors uniquely through the colimit injection $\varphi_n : \mathfrak{z}(E_n) \rightarrow E$.

By Yoneda's lemma and $E \simeq \underline{\text{Mocolim}} \mathbf{F}$, we get $\varphi_N : \mathfrak{z}(E_N) \rightarrow E$ factors as

$$\mathfrak{z}(E_N) \xrightarrow{g} \mathfrak{z}(F_M) \xrightarrow{\psi_M} E$$

for some $M \in \mathbb{N}$ where ψ_M is the colimit injection. By the choice of N , the map g factors as $\mathfrak{z}(E_N) \xrightarrow{\varphi_N} E \xrightarrow{h} \mathfrak{z}(F_M)$. But then the equality $\text{id}_E \varphi_N = \psi_M g = \psi_M h \varphi_N$ gives us two factorisations of the same map $\mathfrak{z}(E_N) \rightarrow E$ through φ_N . Since this factorisation is unique, we obtain $\text{id}_E = \psi_M h$, thus exhibiting E as a direct summand of $\mathfrak{z}(F_M)$. \square

Once again, the above lemma translates into the setting of good extensions. In particular, if \mathcal{T} is a triangulated category with coproducts and $\mathcal{T}^c \hookrightarrow \mathcal{T}$ the corresponding good extension, then for any sequence \mathbf{E} satisfying the conditions of Lemma 6.2, it necessarily holds that $\underline{\text{Hocolim}} \mathbf{E} \in \mathcal{T}^c \subseteq \mathcal{T}$ because \mathcal{T}^c is idempotent complete.

6.2 The computation

As every hereditary commutative noetherian ring is a direct product of Dedekind domains, we can reduce all the computations for completions to Dedekind domains only - in a general case we can proceed componentwise. Therefore, through this whole section we shall work with a Dedekind domain D and $\mathcal{S} := \mathbf{D}^b(\text{mod-}D) \simeq \mathbf{D}^{\text{perf}}(D)$ its bounded derived category. We denote $\mathcal{T} := \mathbf{D}(D)$ and note that the inclusion $\mathcal{S} \hookrightarrow \mathcal{T}$ is a good extension, so all (pre)completions will be calculated inside \mathcal{T} .

We shall denote $\mathfrak{T} \subseteq \mathcal{S}$ the thick subcategory generated by all finitely generated torsion D -modules.

Lemma 6.3. Let $\mathcal{D} \subsetneq \mathcal{C}$ be thick subcategories of \mathcal{S} . Then:

- (i) It holds that $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) \cap \mathfrak{S}'_{\mathcal{D}}(\mathcal{S}) \subseteq \mathcal{C}^\perp \cap \mathfrak{T}$.
- (ii) If \mathcal{C} is not countably generated, then $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) = \mathcal{C}^\perp \cap \mathfrak{T}$.

Proof. “Part i)” Let $E \in \mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) \cap \mathfrak{S}'_{\mathcal{D}}(\mathcal{S})$. We find a sequence $\mathbf{E} = (E_n, e_n)_{n \in \mathbb{N}}$ with $E \simeq \underline{\text{Hocolim}} \mathbf{E}$ which is Cauchy with respect to \mathcal{D} , and therefore with respect to \mathcal{C} as well. W.l.o.g. suppose cone $e_n \in \mathcal{D}$ for all $n \in \mathbb{N}$. By Corollary 4.3, we can also w.l.o.g. assume that \mathbf{E} is \mathcal{C} -trivial. Theorem 3.2 yields $\text{Supp}(\mathcal{D}) \subsetneq \text{Supp}(\mathcal{C})$. We pick a prime ideal $\mathfrak{p} \in \text{Supp}(\mathcal{C}) \setminus \text{Supp}(\mathcal{D})$. Then $\mathcal{D} \otimes_D^{\mathbf{L}} k(\mathfrak{p}) = 0$.

Assume for contradiction that there exists $m \in \mathbb{N}$ and $i \in \mathbb{Z}$ such that $H^i(E_m)$ has a projective direct summand. Since D is connected, Lemma 2.20 gives us $\text{Supp}(E_m) = \text{Spec}(D)$. As $-\otimes_D^{\mathbf{L}} k(\mathfrak{p})$ annihilates cone e_n for all $n \in \mathbb{N}$, the sequence $\mathbf{E} \otimes_D^{\mathbf{L}} k(\mathfrak{p})$ in $\mathbf{D}(D)$ is constant, so

$$E \otimes_D^{\mathbf{L}} k(\mathfrak{p}) \simeq \underline{\text{Hocolim}} \mathbf{E} \otimes_D^{\mathbf{L}} k(\mathfrak{p}) \simeq E_m \otimes_D^{\mathbf{L}} k(\mathfrak{p}) \neq 0.$$

But that is impossible because $\text{Supp}(\mathfrak{C}'_{\mathcal{C}}(\mathcal{S})) = \text{Supp}(D_{\mathcal{C}}) = \text{Spec}(D) \setminus \text{Supp}(\mathcal{C})$ by [31, Lemma 3.5].

We now know that no entry of \mathbf{E} contains a shift of a projective module as a direct summand. Because \mathbf{E} is also \mathcal{C} -trivial, we get by Theorem 2.21 that every entry of \mathbf{E} is supported at $\text{Supp}(D_{\mathcal{C}})$. In other words, every entry of \mathbf{E} is compactly supported at \mathcal{C} . Taking $\mathbf{F} = \mathbf{E}$ in Lemma 6.2 then gives us that $E \in \mathcal{S}$. We already know that $\text{Supp}(E) \neq \text{Spec}(D)$, so E cannot contain a shift of a projective module as a direct summand. Thus we may conclude $E \in \mathcal{C}^{\perp} \cap \mathfrak{T}$.

“Part ii)” Assume furthermore that \mathcal{C} is not countably generated. Let us have $E \in \mathfrak{S}'_{\mathcal{C}}(\mathcal{S})$. By Lemma 4.9, there exists a countably generated thick subcategory $\mathcal{D} \subseteq \mathcal{C}$ such that $E \in \mathfrak{S}'_{\mathcal{D}}(\mathcal{S})$. Since \mathcal{C} is not countably generated, we have $\mathcal{D} \subsetneq \mathcal{C}$. Hence part i) applies, so $E \in \mathcal{C}^{\perp} \cap \mathfrak{T}$. This proves $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) \subseteq \mathcal{C}^{\perp} \cap \mathfrak{T}$. However, the other inclusion $\mathcal{C}^{\perp} \cap \mathfrak{T} \subseteq \mathfrak{S}'_{\mathcal{C}}(\mathcal{S})$ holds by Lemma 6.1. \square

We now formulate and prove the main result. Recall that for a good metric $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$, the intersection $\mathcal{B} := \bigcap_{n \in \mathbb{N}} B_n$ is a triangulated subcategory by [29, Lemma 3.5]. Furthermore, if \mathcal{M} is additive, then \mathcal{B} is thick.

Theorem 6.4. Equip \mathcal{S} with the standard t -structure $(\mathcal{S}^{\leq 0}, \mathcal{S}^{> 0})$ and an additive good metric $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$. Consider the thick subcategory $\mathcal{B} := \bigcap_{n \in \mathbb{N}} B_n$ of \mathcal{S} . Then:

- (i) If $\mathcal{M} = \mathcal{M}_{\infty} \vee \mathcal{B}$, and if \mathcal{B} is countably generated, then

$$\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) = \mathfrak{S}'_{\mathcal{B}}(\mathcal{S}) = \mathbf{D}^b(\text{mod-}D_{\mathcal{B}}) \subseteq \mathbf{D}(D).$$

- (ii) Otherwise, we have $\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) = \mathcal{B}^{\perp} \cap \mathfrak{T} \subseteq \mathcal{S}$.

Proof. If the condition of “case i)” happens, the result follows from Proposition 5.10 and Proposition 4.12. For the rest of the proof, we then assume that the condition of “case i)” is not satisfied. There are two possible explanations.

The first possibility is that $\mathcal{M} = \mathcal{M}_{\infty} \vee \mathcal{B}$ and \mathcal{B} is not countably generated. Then we obtain

$$\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) = \mathfrak{S}'_{\mathcal{B}}(\mathcal{S}) = \mathcal{B}^{\perp} \cap \mathfrak{T}$$

by Proposition 5.10 and Lemma 6.3.

All we are left with is to prove $\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) = \mathcal{B}^{\perp} \cap \mathfrak{T}$ when $\mathcal{M} \not\supseteq \mathcal{M}_{\infty} \vee \mathcal{B}$.

“ \supseteq ” Let $T \in \mathcal{B}^\perp \cap \mathfrak{T}$. We can w.l.o.g. assume that $0 \neq T$ is an indecomposable torsion module supported at a single prime ideal $\mathfrak{p} \in \text{Spec}(D)$ concentrated in degree zero, otherwise we just build T as an extension of suitable shifts of indecomposable summands of its cohomologies.

By Lemma 6.1, it is enough to prove that $T \in \mathfrak{C}'_{\mathcal{M}}(\mathcal{S})$. Using $T \notin \mathcal{B}$, we find $n \in \mathbb{N}$ such that $T, \Sigma^{-1}T \notin B_n$. If there existed $X \in H^0(B_{n+2})$ such that $\text{Hom}_{\mathbf{D}(D)}(X, T) \neq 0$, then $\mathfrak{p} \in \text{Supp}(X)$, so $T \in \text{wide}(X) \subseteq B_n$ by Lemma 4.5, which is impossible. Analogously, it is possible to show that there does not exist $X \in H^1(B_{n+2})$ such that $\text{Hom}_{\mathbf{D}(D)}(\Sigma^{-1}X, T) \neq 0$.

This means $H^0(B_{n+1}), \Sigma^{-1}H^1(B_{n+1}) \subseteq {}^\perp T$. Since there are no non-zero morphisms from $\mathcal{S}^{\leq -1}$ and $\mathcal{S}^{> 1}$ to T , we see that T is compactly supported at $n+2$. This finishes the proof of this inclusion.

“ \subseteq ” Let $E \in \mathfrak{S}'_{\mathcal{M}}(\mathcal{S})$. By Lemma 4.9, there exist $s \in \mathbb{N}$ and a thick subcategory $\mathcal{E} \subseteq \text{thick}(H^0(B_s))$ such that $E \in B_s^\perp$ and $E \in \mathfrak{S}'_{\mathcal{E}}(\mathcal{S})$.

Since $\mathcal{M} \succeq \mathcal{M}_\infty \vee \mathcal{B}$, we can by Lemma 5.11 w.l.o.g. (up to replacing \mathcal{M} by an equivalent metric) assume, that there exists a D -module $X \in B_s$ such that

$$X \notin \text{thick}(H^0(B_{s+1})) \quad \text{and} \quad \text{Hom}_{\mathbf{D}(R)}(\Sigma^j X, E) = 0 \quad \text{for all } j \in \mathbb{Z}.$$

We use Lemma 4.9 again to find a thick subcategory $\mathcal{D} \subseteq \text{thick}(H^0(B_{s+1}))$ with $E \in \mathfrak{S}'_{\mathcal{D}}(\mathcal{S})$. Then for $\mathcal{C} := \text{thick}(\mathcal{D} \cup \mathcal{E} \cup \{X\})$ we have $\mathcal{D} \subsetneq \mathcal{C}$ and $E \in \mathfrak{S}'_{\mathcal{C}}(\mathcal{S})$. Lemma 6.3 then gives us $E \in \mathcal{C}^\perp \cap \mathfrak{T}$. We finish the proof by observing that $E \in \mathcal{B}^\perp$ because E is compactly supported. \square

Example 6.5. If D is a Dedekind domain of an uncountable spectrum, then $\mathfrak{S}'_{\mathfrak{T}}(\mathcal{S}) = 0$.

Example 6.6. Set $D = \mathbb{Z}$ and $\mathcal{M} = \langle \mathbb{Z}/2\mathbb{Z} \rangle$ as a constant metric on \mathcal{S} . Then

$$\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) = \mathbf{D}^b \left(\text{mod-}\mathbb{Z} \left[\frac{1}{2} \right] \right).$$

One of the possible Cauchy sequences expressing the localisation ring $\mathbb{Z}[\frac{1}{2}]$ as an element of $\mathfrak{L}'_{\mathcal{M}}(\mathcal{S})$ is $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 2} \dots$.

Similarly, for the constant metric \mathfrak{T} , we get $\mathfrak{S}'_{\mathfrak{T}}(\mathcal{S}) = \mathbf{D}^b(\text{mod-}\mathbb{Q})$. But for a non-constant metric $\mathcal{N} = \{ \langle \mathbb{Z}/p\mathbb{Z} : p \text{ prime}, p \geq n \rangle \}_{n \in \mathbb{N}}$ we get $\mathfrak{S}'_{\mathcal{N}}(\mathcal{S}) = \mathfrak{T}$.

Example 6.7. Let D be a Dedekind domain with $\text{Pic}(D)$ non-torsion, i.e. there exists a projective maximal ideal $I \leq R$ such that I^n is not a principal ideal for all $n \in \mathbb{N}$. There is an ideal $J \leq R$ such that $IJ = (a)$ is principal (generated by $a \in D$). Set $\mathcal{M} = \langle D/I \rangle$. Then $\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) = \mathbf{D}^b(\text{mod-}R_{\mathcal{M}})$ and the universal localisation $R_{\mathcal{M}}$ can be expressed as a homotopy colimit of the Cauchy sequence $D \xrightarrow{\cdot a} J \xrightarrow{\cdot a} J^2 \xrightarrow{\cdot a} \dots$. Unlike Example 6.6, here, the universal localisation $R_{\mathcal{M}}$ is not a classical localisation at a multiplicative subset of D due to [28, Chapitre IV, Proposition 4.6].

7 Completions for tame algebras

In this final section, we present the second main result of this paper (Theorem 7.2) about completions of the bounded derived category of a hereditary tame finite dimensional al-

gebra. The section follows the same structure as Section 6 but with the statements and proofs adjusted to the setting of finite dimensional algebras.

Through this whole section we work with an Euclidean quiver Q and its path algebra KQ over an algebraically closed field K . Any connected hereditary tame finite dimensional algebra over K is Morita equivalent to a path algebra over an Euclidean quiver, so we may reduce our argumentation from the tame algebra to the case of KQ . We once again denote $\mathcal{S} := \mathbf{D}^b(\text{mod-}KQ) \simeq \mathbf{D}^{\text{perf}}(KQ)$, $\mathcal{T} := \mathbf{D}(KQ)$ and note that the inclusion $\mathcal{S} \hookrightarrow \mathcal{T}$ is a good extension, so all (pre)completions will be calculated inside \mathcal{T} .

We recall that \mathcal{P} , \mathcal{R} , and \mathcal{Q} stand for the preprojective, regular, and preinjective components of $\text{ind-}KQ$, respectively, and that $\mathfrak{R} = \text{thick}(\mathcal{R}) \subseteq \mathcal{S}$.

Lemma 7.1. Let $\mathcal{D} \subsetneq \mathcal{C}$ be thick subcategories of \mathcal{S} generated by regular modules. Let $S \in \text{ind-}KQ$ be a simple regular module from a homogenous tube. Then:

- (i) If $S \in \mathcal{C} \setminus \mathcal{D}$, then it holds that $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) \cap \mathfrak{S}'_{\mathcal{D}}(\mathcal{S}) \subseteq \mathcal{C}^{\perp} \cap \mathfrak{R}$.
- (ii) If \mathcal{C} is not countably generated, then $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) = \mathcal{C}^{\perp} \cap \mathfrak{R}$.

Proof. “Part i)” Let $E \in \mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) \cap \mathfrak{S}'_{\mathcal{D}}(\mathcal{S})$. We find a sequence $\mathbf{E} = (E_n, e_n)_{n \in \mathbb{N}}$ with $E \simeq \text{Hocolim } \mathbf{E}$ which is Cauchy with respect to \mathcal{D} , and therefore with respect to \mathcal{C} as well. W.l.o.g. suppose $\text{cone } e_n \in \mathcal{D}$ for all $n \in \mathbb{N}$. By Corollary 4.3, we can also w.l.o.g. assume that \mathbf{E} is \mathcal{C} -trivial.

By the mutual orthogonality of tubes, we have $\mathcal{D} \subseteq S^{\perp}$. This means that $\text{Hom}_{\mathcal{S}}(S, -)$ annihilates $\text{cone } e_n$ for all $n \in \mathbb{N}$, so the sequence $\text{Hom}_{\mathcal{S}}(S, \mathbf{E})$ is constant. Hence

$$\text{Hom}_{\mathcal{S}}(S, E_n) \simeq \varinjlim \text{Hom}_{\mathcal{S}}(S, \mathbf{E}) \simeq \text{Hom}_{\mathcal{S}}(S, E) \simeq 0$$

for all $n \in \mathbb{N}$. As S is a simple regular from a homogenous tube, we have $\text{Hom}_{KQ}(S, Q) \neq 0$ and $\text{Ext}_{KQ}^1(S, P) \neq 0$ for all $Q \in \mathcal{Q}$ and $P \in \mathcal{P}$ by Lemma 2.23. This implies that $H^0(E)$ does not contain a preinjective direct summand, and $H^{-1}(E)$ does not contain a preprojective direct summand.

By repeating the same argument for arbitrary shifts of S , we obtain that for all $n \in \mathbb{N}$ the cohomology of E_n consists of regular modules only. We then even have $H^i(E_n) \in \mathcal{C}^{\perp}$ for all $n \in \mathbb{N}$ and $i \in \mathbb{Z}$ by the \mathcal{C} -triviality of \mathbf{E} and the orthogonality of tubes. Taking $\mathbf{F} = \mathbf{E}$ in Lemma 6.2 then gives us that $E \in \mathcal{S}$. In the same manner as before, we can consequently deduce that E cannot contain a shift of neither a preprojective, nor preinjective module as a direct summand. Thus we may conclude $E \in \mathcal{C}^{\perp} \cap \mathfrak{R}$.

“Part ii)” Assume furthermore that \mathcal{C} is not countably generated. Then there exists some uncountable indexing set $I \subseteq \mathbb{P}_K^1$ such that $\mathcal{C} = \coprod_{i \in I} \coprod_{j \in \mathbb{Z}} \Sigma^j \mathbf{t}'_i$, where $0 \neq \mathbf{t}'_i$ is a wide subcategory of the tube \mathbf{t}_i and all but finitely many tubes \mathbf{t}_i are homogenous. Let us have $E \in \mathfrak{S}'_{\mathcal{D}}(\mathcal{S})$. By Lemma 4.9, there exists a countably generated thick subcategory $\mathcal{D} \subseteq \mathcal{C}$ such that $E \in \mathfrak{S}'_{\mathcal{D}}(\mathcal{S})$. Here \mathcal{D} is necessarily contained in a thick subcategory generated by countably many tubes. Since I is uncountable, we can find $i \in I$ such that $\mathbf{t}_i = \mathbf{t}'_i$ is homogenous and $\mathbf{t}_i \cap \mathcal{D} = 0$. Hence part i) applies, so $E \in \mathcal{C}^{\perp} \cap \mathfrak{R}$. This proves $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) \subseteq \mathcal{C}^{\perp} \cap \mathfrak{R}$. However, the other inclusion $\mathcal{C}^{\perp} \cap \mathfrak{R} \subseteq \mathfrak{S}'_{\mathcal{C}}(\mathcal{S})$ holds by Lemma 6.1. \square

Theorem 7.2. Equip \mathcal{S} with the standard t -structure $(\mathcal{S}^{\leq 0}, \mathcal{S}^{> 0})$ and an additive good metric $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$. Consider the thick subcategory $\mathcal{B} := \bigcap_{n \in \mathbb{N}} B_n$ of \mathcal{S} . Then:

(i) If $\mathcal{M} = \mathcal{M}_\infty \vee \mathcal{B}$, and if \mathcal{B} is countably generated, then

$$\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) = \mathfrak{S}'_{\mathcal{B}}(\mathcal{S}) = \mathbf{D}^b(\text{mod-}KQ_{\mathcal{B}}) \subseteq \mathbf{D}(KQ).$$

(ii) Otherwise, we have $\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) = \mathcal{B}^\perp \cap \mathfrak{R} \subseteq \mathcal{S}$.

Proof. If the condition of “case i)” happens, the result follows from Proposition 5.10 and Proposition 4.12. For the rest of the proof, we then assume that the condition of “case i)” is not satisfied. There are two possible explanations.

The first possibility is that $\mathcal{M} = \mathcal{M}_\infty \vee \mathcal{B}$ and \mathcal{B} is not countably generated. As \mathcal{B} then cannot be generated by an exceptional sequence, it holds that $\mathcal{B} \subseteq \mathfrak{R}$ by Theorem 2.24. Then we obtain

$$\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) = \mathfrak{S}'_{\mathcal{B}}(\mathcal{S}) = \mathcal{B}^\perp \cap \mathcal{S}$$

by Proposition 5.10 and Lemma 7.1.

All we are left with is to prove $\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) = \mathcal{B}^\perp \cap \mathfrak{R}$ when $\mathcal{M} \not\geq \mathcal{M}_\infty \vee \mathcal{B}$.

“ \supseteq ” Let $T \in \mathcal{B}^\perp \cap \mathfrak{R}$. We can w.l.o.g. assume that $0 \neq T$ is an indecomposable regular module from a single tube concentrated in degree zero, otherwise we just build T as an extension of suitable shifts of indecomposable summands of its cohomologies.

By Lemma 6.1, it is enough to prove that $T \in \mathfrak{S}'_{\mathcal{M}}(\mathcal{S})$. For all $n \in \mathbb{N}$ we define

$$\mathfrak{H}_n := \{X \in \text{ind-}KQ : X \in B_{n+2}, \text{Hom}_{KQ}(X, T) \neq 0\}.$$

By Lemma 4.5, for every $n \in \mathbb{N}$ it holds that $\text{wide}(\mathfrak{H}_n) \subseteq B_n$. Then we have an infinite decreasing chain $\text{thick}(\mathfrak{H}_1) \supseteq \text{thick}(\mathfrak{H}_2) \supseteq \dots$ of thick subcategories of \mathcal{S} . If this chain did not stabilise, then a repeated application of Lemma 2.25 would give us two different homogenous tubes $\mathbf{u}, \mathbf{v} \not\subseteq {}^\perp T$, which is impossible for a regular module T . Hence the chain of thick subcategories above stabilises, meaning that $\mathfrak{H}_n = \mathfrak{H}_m$ for some $N \in \mathbb{N}$ and all $m, n \geq N$, and $\mathfrak{H}_N \subseteq \mathcal{B}$. If \mathfrak{H}_N was not zero, then it could not hold that $T \in \mathcal{B}^\perp$. Therefore, we obtain $H^0(B_N) \subseteq {}^\perp T$. Analogously, it is possible to show that $H^1(B_M) \subseteq {}^\perp T$ for some $M \in \mathbb{N}$.

Since there are no non-zero morphisms from $\mathcal{S}^{\leq -1}$ and $\mathcal{S}^{> 1}$ to T , we see that T is compactly supported at $\max\{N, M\}$. This finishes the proof of this inclusion.

“ \subseteq ” We find $M \in \mathbb{N}$ such that for any strictly decreasing chain of thick subcategories of \mathcal{S} some homogenous regular tube must appear at one stage and disappear at a later one as (more precisely) stated in Lemma 2.25.

Let $E \in \mathfrak{S}'_{\mathcal{M}}(\mathcal{S})$. Since $\mathcal{M} \not\geq \mathcal{M}_\infty \vee \mathcal{B}$, we can by Lemma 5.11 w.l.o.g. (up to replacing \mathcal{M} by an equivalent metric) assume, that there exist KQ -modules X_1, \dots, X_M such that for all $1 \leq i \leq M$ we have $X_i \in B_i$, $X \notin \text{thick}(H^0(B_{i+1}))$ and $\text{Hom}_{\mathbf{D}(R)}(\Sigma^j X_i, E) = 0$ for every $j \in \mathbb{Z}$.

By Lemma 4.9, there exist thick subcategories $\mathcal{E}_1, \dots, \mathcal{E}_M$ of \mathcal{S} satisfying for each $1 \leq i \leq M$ that $\mathcal{E}_i \subseteq \text{thick}(H^0(B_i))$ and $E \in \mathfrak{S}'_{\mathcal{E}_i}(\mathcal{S})$. Then for each $1 \leq i \leq M$ we define $\mathcal{C}_i := \text{thick}\left(\bigcup_{j=i}^M (\mathcal{E}_j \cup \{X_j\})\right)$, ensuring that $E \in \mathfrak{S}'_{\mathcal{C}_i}(\mathcal{S})$. Now we have a strictly decreasing chain $\mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \dots \supseteq \mathcal{C}_M$ of thick subcategories of \mathcal{S} , so Lemma 2.25 provides us with indices $1 \leq i < j \leq M$ and a homogenous regular tube $\mathbf{t} \subseteq \text{mod-}KQ$ such that $\mathcal{C}_i, \mathcal{C}_j \subseteq \mathfrak{R}$, $\mathbf{t} \subseteq \mathcal{C}_i$ and $\mathbf{t} \cap \mathcal{C}_j = 0$.

The conditions of Lemma 7.1, part i), are now satisfied. Thus $E \in \mathcal{C}_i^\perp \cap \mathfrak{R}$. We finish the proof by observing that $E \in \mathcal{B}^\perp$ because E is compactly supported. \square

We see that if we want the completion $\mathfrak{S}'_{\mathcal{M}}(\mathcal{S})$ to contain new elements which are not from the original category \mathcal{S} , then the good metric \mathcal{M} must be of the form $\mathcal{M} = \mathcal{M}_{\infty} \vee \mathcal{B}$ with \mathcal{B} countably generated. However, this condition is not sufficient.

Corollary 7.3. Equip \mathcal{S} with the standard t -structure $(\mathcal{S}^{\leq 0}, \mathcal{S}^{> 0})$ and an additive good metric $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$. Assume that the thick subcategory $\mathcal{B} := \bigcap_{n \in \mathbb{N}} B_n$ of \mathcal{S} is generated by an exceptional sequence in $\text{mod-}KQ$. If $\mathcal{M} = \mathcal{M}_{\infty} \vee \mathcal{B}$, then $\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) = \mathcal{B}^{\perp} \cap \mathcal{S}$.

Proof. Theorem 7.2 i) gives us $\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) = \mathfrak{S}'_{\mathcal{B}}(\mathcal{S}) = \mathbf{D}^b(\text{mod-}KQ_{\mathcal{B}})$. The category $H^0(\mathcal{B})$ is generated by an exceptional sequence as a wide subcategory of $\text{mod-}KQ$, so it is equivalent to a module category. Hence $H^0(\mathcal{B}) = \text{wide}(P)$ for a rigid object $P \in \text{mod-}KQ$. By [25, Proposition 4.2], this forces the universal localisation $KQ_{\mathcal{B}}$ to be finite dimensional over K . It follows that $\mathbf{D}^b(\text{mod-}KQ_{\mathcal{B}}) \subseteq \mathbf{D}^b(\text{mod-}KQ)$, and consequently $\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) = \mathcal{B}^{\perp} \cap \mathcal{S}$. \square

In light of the above corollary, we may conclude the discussion of completions of tame algebras by constating that new objects appear in the completion only for specific metrics determined by countable collections of regular modules. If \mathcal{B} is generated by preprojective or preinjective modules, it is generated by an exceptional sequence, and the completion $\mathfrak{S}'_{\mathcal{M}}(\mathcal{S})$ is then a thick subcategory of \mathcal{S} .

For a hereditary finite dimensional algebra A of finite representation type, it is known by [29, Corollary 3.9] that any completion of $\mathbf{D}^b(\text{mod-}A)$ is just a thick subcategory $\mathbf{D}^b(\text{mod-}A)$. One can obtain an alternative proof of this fact by mimicking the argument of Corollary 7.3 - all indecomposable modules of A are both preprojective and preinjective, so no new object can appear in the completion.

Example 7.4. Let Q be the Kronecker quiver with a fixed orientation $2 \rightrightarrows 1$. Every regular tube is homogenous, and the simple regulars are of the form

$$R_{(1:k)} = K \xrightarrow[k]{1} K \quad \text{and} \quad R_{(0:1)} = K \xrightarrow[1]{0} K \quad \text{for every } k \in K.$$

Fix $\lambda \in \mathbb{P}_K^1$, and denote $\mathcal{C} := \langle R_{\lambda} \rangle \subseteq \mathcal{S}$. The preprojective modules are precisely

$$P_n = K^n \xrightarrow{\begin{bmatrix} \text{id}_{n \times n} \\ 0 \end{bmatrix}} K^{n+1} \xrightarrow{\begin{bmatrix} 0 \\ \text{id}_{n \times n} \end{bmatrix}}$$

for $n \in \mathbb{N}_0$. We can arrange the whole preprojective component into a Cauchy sequence $\mathbf{E} = (P_{n-1}, e_n)_{n \in \mathbb{N}}$ with respect to \mathcal{C} where $\text{cone}(P_{n-1} \xrightarrow{e_n} P_n) \simeq R_{\lambda}$ for all $n \in \mathbb{N}$. It holds that $E := \text{Hocolim} \mathbf{E} \in \mathbf{D}(KQ)$ is compactly supported at \mathcal{C} . Under the identification $\mathfrak{C}'_{\mathcal{C}}(\mathcal{S}) \simeq \mathbf{D}(KQ_{\mathcal{C}})$ within $\mathbf{D}(KQ)$, the universal localisation $KQ_{\mathcal{C}}$ corresponds to $\text{End}_{KQ}(E \oplus E)$, which is Morita equivalent to $\text{End}_{KQ}(E)$. We then have $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) \simeq \mathbf{D}^b(\text{mod-}KQ_{\mathcal{C}}) \simeq \langle E, R_{\mu} \rangle_{\lambda \neq \mu \in \mathbb{P}_K^1}$.

Remark 7.5. For an arbitrary Euclidean quiver Q , we can build Cauchy sequences of preprojectives with respect to thick subcategories generated by simple regulars in a fashion similar to Example 7.4. For every $P \in \mathcal{P}$ and R a simple regular, there exists a $P' \in \mathcal{P}$

and a short exact sequence $0 \rightarrow P \rightarrow P' \rightarrow R \rightarrow 0$ yielding a corresponding distinguished triangle in \mathcal{S} . The construction of such Cauchy sequences has appeared under the name of *special chains* in [3, Lemma 7.3].

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Project C

Metric completions from weighted projective lines

This last chapter presents two results from our current research on metric completions. One is an elaboration on the argument of Proposition 4.12 from Paper B about completions with respect to constant metrics, which we extend to compactly generated triangulated categories in Section 5. The other main result is the generalisation of Theorem 1.1 of Paper B from a bounded derived category of a tame hereditary algebra to a bounded derived category of coherent sheaves over a weighted projective line of a domestic or tubular type. The journey towards the desired generalisation (Theorem 9.8) involves three intermediate steps - we build on the theory of Grothendieck categories (Section 6), quasi-coherent and coherent sheaves on weighted projective lines (Section 7), and the derived categories thereof (Section 8). We then discuss sample examples in Section 10.

5 Completions of compacts

In this section, we fix a commutative ring \mathbb{k} and a \mathbb{k} -linear triangulated categories \mathcal{S} and \mathcal{T} . We further assume that \mathcal{T} is compactly generated and that for all $X, Y \in \mathcal{S}$ the \mathbb{k} -module $\mathrm{Hom}_{\mathcal{S}}(X, Y)$ is finitely generated. This would be the case, for example, if \mathcal{S} were Hom-finite. We recall that $\mathcal{T}^c \hookrightarrow \mathcal{T}$ is a good extension with respect to all good metrics on \mathcal{T}^c by Theorem 2.72, so we will be calculating completions of \mathcal{T}^c inside \mathcal{T} .

The following theorem has been proved by Neeman [14, Theorem 2.1] based on [16], [18], and [19]. See also [10, Corollary 12.7] for a generalisation for cohomological quotient functors.

Theorem 5.1. Let \mathcal{T} be a compactly generated triangulated category and $\mathcal{C} \subseteq \mathcal{T}^c$ a thick subcategory. Then the localisation functor $\mathcal{T} \rightarrow \mathcal{T}/\mathrm{Loc}(\mathcal{C})$ restricts to a triangulated functor $\mathcal{T}^c \rightarrow (\mathcal{T}/\mathrm{Loc}(\mathcal{C}))^c$.

Furthermore, the induced functor $F : \mathcal{T}^c/\mathcal{C} \rightarrow (\mathcal{T}/\mathrm{Loc}(\mathcal{C}))^c$ is fully faithful and $(\mathcal{T}/\mathrm{Loc}(\mathcal{C}))^c$ is generated by the image of F as a thick subcategory.

Lemma 5.2. Let \mathcal{S} be a triangulated category and $\mathcal{C} \subseteq \mathcal{S}$ be a countably generated triangulated subcategory. Then for every object $X \in \mathcal{S}$ there exists a Cauchy sequence $\mathbf{E} = (E_n, e_n)_{n \in \mathbb{N}}$ with respect to \mathcal{C} such that $X \simeq E_1, \mathrm{Mocolim} \mathbf{E} \in \mathfrak{S}_{\mathcal{C}}(\mathcal{S})$, and $\mathrm{cone} e_n \in \mathcal{C}$ for all $n \in \mathbb{N}$.

Proof. This lemma has been proved in Paper B as Lemma 4.10 for $\mathcal{S} = \mathbf{D}(R)^c$ for R a finitely generated \mathbb{k} -algebra. The proof, however, applies to any \mathbb{k} -linear category with finitely generated Hom-spaces as imposed on \mathcal{S} and does not even require good extensions. \square

We state a reformulation of Lemma 3.1 from Paper B.

Proposition 5.3. Let $\mathcal{C} \subseteq \mathcal{T}^c$ be a thick subcategory. Then there exists a recollement diagram

$$\begin{array}{ccccc} & & F & & \\ & \curvearrowright & & \curvearrowleft & \\ \mathcal{T}/\mathrm{Loc}(\mathcal{C}) & \xrightarrow{\quad} & \mathcal{T} & \xrightarrow{\quad} & \mathrm{Loc}(\mathcal{C}) \\ & \curvearrowleft & & \curvearrowright & \\ & & \nu & & \end{array}$$

with $\nu : \mathrm{Loc}(\mathcal{C}) \hookrightarrow \mathcal{T}$ being the inclusion and $F : \mathcal{T} \rightarrow \mathcal{T}/\mathrm{Loc}(\mathcal{C})$ being the smashing localisation functor.

Furthermore, the functor F induces the following commutative diagram of triangulated functors

$$\begin{array}{ccccc} \mathfrak{C}'_{\mathcal{C}}(\mathcal{T}^c) & \xleftarrow{\simeq} & \mathcal{T}/\mathrm{Loc}(\mathcal{C}) & \xleftarrow{F} & \mathcal{T} \\ \uparrow & & \uparrow & & \uparrow \\ \mathfrak{S}'_{\mathcal{C}}(\mathcal{T}^c) & \xleftarrow{J} & \mathcal{T}^c/\mathcal{C} & \xleftarrow{\bar{F}} & \mathcal{T}^c \end{array}$$

with J and all the vertical arrows being fully faithful.

Proof. The existence of the recollement diagram is a consequence of Proposition 2.23 and Proposition 2.24. The identification $\mathcal{T}/\mathrm{Loc}(\mathcal{C}) \simeq \mathfrak{C}'_{\mathcal{C}}(\mathcal{T}^c)$ has been discussed in Lemma 3.1 of Paper B and [15, Lemma 3.8], and we may consider $\mathcal{T}/\mathrm{Loc}(\mathcal{C})$ as a thick subcategory of \mathcal{T} . The embedding $\mathfrak{S}'_{\mathcal{C}}(\mathcal{T}^c) \hookrightarrow \mathfrak{C}'_{\mathcal{C}}(\mathcal{T}^c)$ is by the definition of a completion. And the functor F restricts to \bar{F} by Theorem 5.1.

We need to prove that the embedding $\mathfrak{S}'_{\mathcal{C}}(\mathcal{T}^c) \hookrightarrow \mathcal{T}/\mathrm{Loc}(\mathcal{C})$ factors through the embedding $\mathcal{T}^c/\mathcal{C} \hookrightarrow \mathcal{T}/\mathrm{Loc}(\mathcal{C})$. Let $E = \mathrm{Hocolim} \mathbf{E} \in \mathfrak{S}'_{\mathcal{C}}(\mathcal{S})$ for a Cauchy sequence $\mathbf{E} = (E_n, e_n)_{n \in \mathbb{N}}$. We may w.l.o.g. assume that $\mathrm{cone} e_n \in \mathcal{C}$ holds for every $n \in \mathbb{N}$. The smashing localisation F preserves coproducts, hence also homotopy colimits, so $F(E) \simeq \mathrm{Hocolim} F(\mathbf{E})$. But as $\mathrm{cone} e_n \in \mathcal{C} \subseteq \mathrm{Ker} F$ for all $n \in \mathbb{N}$, the sequence $F(\mathbf{E})$ is constant in $\mathcal{T}/\mathrm{Loc}(\mathcal{C})$. We get $F(E) \simeq F(E_1) \simeq \bar{F}(E_1) \in \mathcal{T}^c/\mathcal{C}$. Since E is compactly supported, we obtain $E \simeq F(E) \in \mathcal{T}^c/\mathcal{C}$. This gives is the desired factorisation. \square

Corollary 5.4. Assume $\mathcal{S} = \mathcal{T}^c$. Let $\mathcal{C} \subseteq \mathcal{T}^c$ be a thick subcategory. If \mathcal{C} is countably generated, then $\mathfrak{S}'_{\mathcal{C}}(\mathcal{T}^c) \simeq \mathcal{T}^c/\mathcal{C}$.

Proof. Let $F : \mathcal{T} \rightarrow \mathcal{T}/\mathrm{Loc}(\mathcal{C})$ be the smashing localisation functor and $\bar{F} : \mathcal{T}^c \rightarrow \mathcal{T}^c/\mathcal{C}$ its restriction. By Proposition 5.3, the completion $\mathfrak{S}'_{\mathcal{C}}(\mathcal{T}^c)$ is a triangulated subcategory of $\mathcal{T}^c/\mathcal{C}$ and $F(E) \simeq E$ for each $E \in \mathfrak{S}'_{\mathcal{C}}(\mathcal{T}^c) \subseteq \mathcal{T}$.

Let us pick any element of $\mathcal{T}^c/\mathcal{C}$; it is necessarily of the form $\bar{F}(E_1)$ for some $E_1 \in \mathcal{C}$. By Lemma 5.2, the element E_1 is a part of a Cauchy sequence $\mathbf{E} = (E_n, e_n)_{n \in \mathbb{N}}$ with $E = \mathrm{Hocolim} \mathbf{E} \in \mathfrak{S}'_{\mathcal{C}}(\mathcal{S})$ and $\mathrm{cone} e_n \in \mathcal{C}$ for all $n \in \mathbb{N}$. This implies that $F(\mathrm{cone} e_n) = 0$ for each $n \in \mathbb{N}$, so the sequence $F(\mathbf{E})$ in $\mathcal{T}/\mathrm{Loc}(\mathcal{C})$ is constant. As F preserves coproducts and homotopy colimits, we get

$$\bar{F}(E_1) \simeq F(E_1) \simeq \mathrm{Hocolim} F(\mathbf{E}) \simeq F(E) \simeq E.$$

This proves the desired inclusion $\mathcal{T}^c/\mathcal{C} \subseteq \mathfrak{S}'_{\mathcal{C}}(\mathcal{S})$. \square

Corollary 5.5. Assume $\mathcal{S} = \mathcal{T}^c$. Let $\mathcal{C} \subseteq \mathcal{T}^c$ be a countably generated thick subcategory. Then $\mathfrak{S}'_{\mathcal{C}}(\mathcal{T}^c) = \mathcal{C}^{\perp} \cap \mathcal{T}^c$ if and only if \mathcal{C} is a coreflective category of \mathcal{T}^c .

Proof. “ \Leftarrow ” By Corollary 2.68.

“ \Rightarrow ” Let $F : \mathcal{T} \rightleftarrows \mathcal{T}/\text{Loc}(\mathcal{C}) : G$ be the adjoint pair taking part in the recollement from Proposition 5.3 where F is the smashing localisation functor and G is fully faithful. We may consider G to be an inclusion. The functor F restricts to $\bar{F} : \mathcal{T}^c \rightarrow \mathcal{T}^c/\mathcal{C}$. Denote $\bar{G} : \mathcal{C}^{\perp} \cap \mathcal{T}^c \hookrightarrow \mathcal{T}^c$ the restriction of the inclusion G .

By Corollary 5.4, we get $\mathcal{T}^c/\mathcal{C} \simeq \mathfrak{S}'_{\mathcal{C}}(\mathcal{T}^c) \simeq \mathcal{C}^{\perp} \cap \mathcal{T}^c$. This makes the functors

$$\bar{F} : \mathcal{T}^c \rightleftarrows \mathcal{C}^{\perp} \cap \mathcal{T}^c : \bar{G}$$

adjoint and $\mathcal{C}^{\perp} \cap \mathcal{T}^c$ a reflective subcategory of \mathcal{T}^c . Hence, the subcategory \mathcal{C} is coreflective. \square

6 Completions from Grothendieck categories

In this section, let \mathbb{k} be a commutative ring and \mathcal{G} a locally noetherian, \mathbb{k} -linear, hereditary Grothendieck category. Assume furthermore, that for any two noetherian objects $X, Y \in \mathcal{G}$ the \mathbb{k} -module $\text{Hom}_{\mathcal{G}}(X, Y)$ is finitely generated. We set $\mathcal{T} := \mathbf{D}(\mathcal{G})$ and $\mathcal{S} := \mathbf{D}(\mathcal{G})^c$. As in Section 5, the inclusion $\mathcal{S} \rightarrow \mathcal{T}$ is a good extension with respect to all possible metrics. Notice that by Proposition 2.32 and Proposition 2.35 it holds that \mathcal{S} consists of finite coproducts of shifts of noetherian objects of \mathcal{G} .

We shall state several results from Section 4 and Section 5 of Paper B, which directly generalise from the original setting of derived categories of hereditary, finitely generated algebras, to the current setting of derived categories of locally noetherian, hereditary Grothendieck categories. As the original proofs need only formal modifications to include the stronger hypothesis, we omit their proofs here.

Lemma 6.1 (Paper B - Corollary 4.3). Equip \mathcal{S} with an additive good metric. Let \mathcal{B} be a replete subcategory of \mathcal{S} . If \mathbf{E} is a Cauchy sequence compactly supported at \mathcal{B} , then there exists a \mathcal{B} -trivial component \mathbf{F} of a subsequence of \mathbf{E} such that \mathbf{F} is Cauchy and compactly supported at \mathcal{B} , and $\text{Mocolim} \mathbf{E} \simeq \text{Mocolim} \mathbf{F}$.

Lemma 6.2 (Paper B - Lemma 4.5). Let $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$ be an additive good metric on \mathcal{S} , and \mathcal{C} a class consisting of noetherian objects of \mathcal{G} . If $\mathcal{C} \subseteq B_{n+2}$ as objects of \mathcal{S} , then $\text{wide}(\mathcal{C}) \subseteq B_n$.

Proposition 6.3 (Paper B - Proposition 4.7). For all additive good metrics on \mathcal{S} , the completion $\mathfrak{S}'(\mathcal{S}) \subseteq \mathcal{T}$ is idempotent complete.

Corollary 6.4. Let $\mathcal{C} \subseteq \mathcal{S}$ be a thick subcategory. Then $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) \simeq (\mathcal{T}/\text{Loc}(\mathcal{C}))^c$.

Proof. Corollary 5.4 tells us that $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) \simeq \mathcal{S}/\mathcal{C}$. By Theorem 5.1, Verdier’s quotient \mathcal{S}/\mathcal{C} generates $(\mathcal{T}/\text{Loc}(\mathcal{C}))^c$ as a thick subcategory. Since the completion is closed under direct summands by Proposition 6.3, the claim follows. \square

Lemma 6.5 (Paper B - Lemma 4.9). Let $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$ be an additive good metric on \mathcal{S} and $E \in \mathfrak{S}'_{\mathcal{M}}(\mathcal{S})$. Then for every $n \in \mathbb{N}$ there exist a countably generated thick subcategory \mathcal{C} of \mathcal{S} such that $\mathcal{C} \subseteq \text{thick}(H^0(B_n))$ and $E \in \mathfrak{S}'_{\mathcal{C}}(\mathcal{S})$.

The following two results about lattice-theoretical properties of metrics are immediate generalisations of Proposition 5.10 and Lemma 5.11 in Paper B from the setting of $\mathbf{D}^{\text{perf}}(R)$ for a hereditary algebra R to $\mathbf{D}(\mathcal{G})^c$ for specific hereditary Grothendieck categories.

Proposition 6.6 (Paper B - Proposition 5.10). Equip \mathcal{S} with the standard t -structure $(\mathcal{S}^{\leq 0}, \mathcal{S}^{> 0})$ and additive good metrics $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$ and $\mathcal{N} = \{C_n\}_{n \in \mathbb{N}}$. If $\mathcal{M} = \mathcal{M}_{\infty} \vee \mathcal{N}$, then $\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) = \mathfrak{S}'_{\mathcal{N}}(\mathcal{S})$.

Lemma 6.7 (Paper B - Lemma 5.11). Equip \mathcal{S} with the standard t -structure $(\mathcal{S}^{\leq 0}, \mathcal{S}^{> 0})$ and an additive good metric $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$. If $\mathcal{M} \not\sim \mathcal{M}_{\infty} \vee \mathcal{B}$, where $\mathcal{B} := \bigcap_{n \in \mathbb{N}} B_n$, then for every $E \in \mathfrak{S}'_{\mathcal{M}}(\mathcal{S})$ there exist an additive good metric $\mathcal{N} = \{C_n\}_{n \in \mathbb{N}} \sim \mathcal{M}$ on \mathcal{S} and a family of noetherian objects $(X_n)_{n \in \mathbb{N}}$ of \mathcal{G} such that for all $n \in \mathbb{N}$:

- $\text{Hom}_{\mathbf{D}(R)}(\Sigma^j X_n, E) = 0$ for all $j \in \mathbb{Z}$,
- $X_n \in C_n$,
- $X_n \notin \text{thick}(H^0(C_{n+1}))$.

Lemma 6.8. Equip \mathcal{S} with the standard t -structure $(\mathcal{S}^{\leq 0}, \mathcal{S}^{> 0})$ and an additive good metric $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$. If $\mathcal{M} \not\sim \mathcal{M}_{\infty} \vee \mathcal{B}$, where $\mathcal{B} := \bigcap_{n \in \mathbb{N}} B_n$, then for every $E \in \mathfrak{S}'_{\mathcal{M}}(\mathcal{S})$ there exists an additive good metric $\mathcal{N} = \{C_n\}_{n \in \mathbb{N}}$ on \mathcal{S} such that for all $n \in \mathbb{N}$:

- $E \in \mathfrak{S}'_{\mathcal{N}}(\mathcal{S})$,
- $C_n \subseteq \text{thick}(H^0(B_n))$ is a thick subcategory,
- $C_{n+1} \neq C_n$.

Proof. By Lemma 6.7, we can assume that there exist a family $(X_n)_{n \in \mathbb{N}}$ of noetherian objects of \mathcal{G} such that $X_n \in B_n \setminus \text{thick}(H^0(B_{n+1}))$ and $\text{thick}(X_n) \subseteq {}^{\perp}E$. Using Lemma 6.5, we pick a thick subcategory $\mathcal{E}_n \subseteq \text{thick}(H^0(B_n))$ satisfying $E \in \mathfrak{S}'_{\mathcal{E}_n}(\mathcal{S})$ for each $n \in \mathbb{N}$.

Then for every $n \in \mathbb{N}$ we set $C_n := \text{thick}(\mathcal{E}_m, X_m : m \geq n)$. By the choice of the family $(X_n)_{n \in \mathbb{N}}$, the thick subcategories $C_1 \supseteq C_2 \supseteq \dots$ form a strictly decreasing chain and, therefore, an additive good metric $\mathcal{N} := \{C_n\}_{n \in \mathbb{N}}$. Because $E \in \mathfrak{S}'_{C_n}(\mathcal{S})$ for all $n \in \mathbb{N}$, it also holds that $E \in \mathfrak{S}'_{\mathcal{N}}(\mathcal{S})$. \square

7 Weighted projective lines

Weighted projective lines and their categories of coherent sheaves were introduced in [7]. Roughly speaking, a weighted projective line is obtained from the usual projective line by assigning weights to finitely many distinguished points. Weighted points then give rise to exceptional simple object in the category of coherent sheaves. While a weighted projective line was originally defined in [7] using graded sheaf theory, here we follow a more direct approach from [3] and [13] where coherent sheaves are constructed as a quotient of suitable

category of graded modules. For the construction of quasi-coherent sheaves on weighted projective lines, we refer to [1, pg. 75] and [2].

Let K be an algebraically closed field. Let $k \in \mathbb{N}$ and $\mathbf{p} = (p_1, \dots, p_k) \in \mathbb{N}^k$. Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k)$ be a k -tuple of pairwise distinct points on \mathbb{P}_K^1 . By a *weighted projective line* we understand the triple $\mathbb{X} = (\mathbb{P}_K^1, \boldsymbol{\lambda}, \mathbf{p})$.

Let $\mathbb{L}(\mathbf{p})$ be a rank one abelian group with generators $\vec{x}_1, \dots, \vec{x}_k$ subjected to relations $p_1 \vec{x}_1 = \dots = p_n \vec{x}_n$. We specify two distinguished elements of $\mathbb{L}(\mathbf{p})$, *the canonical element* $\vec{c} := p_1 \vec{x}_1$ and *the dualising element* $\vec{\omega} := (k-2)\vec{c} - \sum_{i=1}^k \vec{x}_i$, and a partial order on $\mathbb{L}(\mathbf{p})$ by $\vec{l} \geq 0 \Leftrightarrow \vec{l} \in \sum_{i=1}^k \mathbb{N}_0 \vec{x}_i$. The group $\mathbb{L}(\mathbf{p})$ is almost totally ordered in a sense that for all $\vec{l} \in \mathbb{L}(\mathbf{p})$ we have $\vec{l} \geq 0$ or $\vec{l} \leq \vec{c} + \vec{\omega}$.

We define a K -algebra

$$S(\mathbf{p}, \boldsymbol{\lambda}) := K[u, v, x_1, \dots, x_k] / (f_1, \dots, f_k) \quad \text{where} \quad f_i := x_i^{p_i} - (\lambda_i^1 u - \lambda_i^0 v)$$

for each $1 \leq i \leq k$ and $\lambda_i = [\lambda_i^0, \lambda_i^1] \in \boldsymbol{\lambda}$. The algebra $S(\mathbf{p}, \boldsymbol{\lambda})$ is $\mathbb{L}(\mathbf{p})$ -graded by imposing $\deg(u) = \deg(v) = \vec{c}$ and $\deg(x_i) = \vec{x}_i$ for each $1 \leq i \leq k$. Thus $S(\mathbf{p}, \boldsymbol{\lambda}) = \bigoplus_{\vec{l} \in \mathbb{L}(\mathbf{p})} S(\mathbf{p}, \boldsymbol{\lambda})_{\vec{l}}$ decomposes as a direct sum of its homogenous components $S(\mathbf{p}, \boldsymbol{\lambda})_{\vec{l}}$ of degree \vec{l} . It holds that $S(\mathbf{p}, \boldsymbol{\lambda})_{\vec{l}} = 0$ whenever $\vec{l} \not\geq 0$, and $S(\mathbf{p}, \boldsymbol{\lambda})_0 = K$. Consequently, the algebra $S(\mathbf{p}, \boldsymbol{\lambda})$ is graded local with the unique graded maximal ideal $\bigoplus_{0 \neq \vec{l} \in \mathbb{L}(\mathbf{p})} S(\mathbf{p}, \boldsymbol{\lambda})_{\vec{l}}$.

We consider the category $\text{Mod}^{\mathbb{L}(\mathbf{p})} - S(\mathbf{p}, \boldsymbol{\lambda})$ of $\mathbb{L}(\mathbf{p})$ -graded $S(\mathbf{p}, \boldsymbol{\lambda})$ -modules and its subcategories:

- $\text{mod}^{\mathbb{L}(\mathbf{p})} - S(\mathbf{p}, \boldsymbol{\lambda})$ finitely generated graded $S(\mathbf{p}, \boldsymbol{\lambda})$ -modules,
- $\text{mod}_0^{\mathbb{L}(\mathbf{p})} - S(\mathbf{p}, \boldsymbol{\lambda})$ graded $S(\mathbf{p}, \boldsymbol{\lambda})$ -modules of finite length,
- and $\text{Mod}_0^{\mathbb{L}(\mathbf{p})} - S(\mathbf{p}, \boldsymbol{\lambda}) := \varinjlim \text{mod}_0^{\mathbb{L}(\mathbf{p})} - S(\mathbf{p}, \boldsymbol{\lambda})$ the directed colimits of objects from $\text{mod}_0^{\mathbb{L}(\mathbf{p})} - S(\mathbf{p}, \boldsymbol{\lambda})$.

On a graded module $M = \bigoplus_{\vec{x} \in \mathbb{L}(\mathbf{p})} M_{\vec{x}}$, we can perform a graded shift by $\vec{y} \in \mathbb{L}(\mathbf{p})$ to obtain $M(\vec{y})$ with homogenous components $M(\vec{y})_{\vec{x}} := M_{\vec{x} + \vec{y}}$.

As $\text{Mod}_0^{\mathbb{L}(\mathbf{p})} - S(\mathbf{p}, \boldsymbol{\lambda})$ is a Serre subcategory of $\text{Mod}^{\mathbb{L}(\mathbf{p})} - S(\mathbf{p}, \boldsymbol{\lambda})$, we may apply Serre's construction and define the category of *quasi-coherent sheaves* over \mathbb{X} as the quotient

$$\text{Qcoh} - \mathbb{X} := \text{Mod}^{\mathbb{L}(\mathbf{p})} - S(\mathbf{p}, \boldsymbol{\lambda}) / \text{Mod}_0^{\mathbb{L}(\mathbf{p})} - S(\mathbf{p}, \boldsymbol{\lambda}).$$

This construction restricts to $\text{mod}^{\mathbb{L}(\mathbf{p})} - S(\mathbf{p}, \boldsymbol{\lambda})$ giving us the category of *coherent sheaves* over \mathbb{X} as $\text{coh} - \mathbb{X} := \text{mod}^{\mathbb{L}(\mathbf{p})} - S(\mathbf{p}, \boldsymbol{\lambda}) / \text{mod}_0^{\mathbb{L}(\mathbf{p})} - S(\mathbf{p}, \boldsymbol{\lambda})$.

The subcategory of finite length objects of $\text{coh} - \mathbb{X}$ (also called *torsion sheaves*) is denoted $\text{coh}_0 - \mathbb{X}$. On the other hand, the subcategory $\text{vect} - \mathbb{X}$ of *vector bundles* consists precisely of objects of $\text{coh} - \mathbb{X}$ without any non-zero finite length subobjects.

The graded shift action of $\mathbb{L}(\mathbf{p})$ on $\text{Mod}^{\mathbb{L}(\mathbf{p})} - S(\mathbf{p}, \boldsymbol{\lambda})$ induces a graded shift on $\text{Qcoh} - \mathbb{X}$. The image \mathcal{O} of $S(\mathbf{p}, \boldsymbol{\lambda})$ in $\text{coh} - \mathbb{X}$ under the quotient functor is called *the structure sheaf*; its shifts $\mathcal{O}(\vec{x})$ by some $\vec{x} \in \mathbb{L}(\mathbf{p})$ are *line bundles*.

Remark 7.1 ([3, Corollary 7.4.4]). The category $\text{coh} - \mathbb{X}$ does not depend on the choice of coordinates of the weighted points $\lambda_i = [\lambda_i^0, \lambda_i^1] \in \boldsymbol{\lambda}$, $1 \leq i \leq k$, during the construction of $S(\mathbf{p}, \boldsymbol{\lambda})$.

The basic properties of $\text{coh}_0\text{-}\mathbb{X}$ are summarised in the theorem below.

Theorem 7.2 ([8, Theorem 2.1],[13, Theorem 2.2]). The category $\text{coh-}\mathbb{X}$ is an abelian, noetherian, hereditary, K -linear, Hom-finite, Krull-Schmidt category. Moreover:

- (i) Every object $X \in \text{coh-}\mathbb{X}$ uniquely decomposes as a direct sum $X = Y \oplus Z$ with $Y \in \text{vect-}\mathbb{X}$ and $Z \in \text{coh}_0\text{-}\mathbb{X}$.
- (ii) For all $\vec{x}, \vec{y} \in \mathbb{L}(\mathbf{p})$ it holds that $\text{Hom}_{\text{coh-}\mathbb{X}}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{y})) \simeq S(\mathbf{p}, \boldsymbol{\lambda})_{\vec{y} - \vec{x}}$.
- (iii) Simple objects of $\text{coh}_0\text{-}\mathbb{X}$ are up to isomorphism parametrised by \mathbb{P}_K^1 . Each simple is of one of the two following forms:
 - (a) for $\lambda \in \mathbb{P}_K^1 \setminus \boldsymbol{\lambda}$, $[\lambda^0, \lambda^1] \in \mathbb{P}_K^1$, there is a unique *ordinary* simple S_λ determined by the short exact sequence

$$0 \rightarrow \mathcal{O} \xrightarrow{\lambda^1 u - \lambda^0 v} \mathcal{O}(\vec{c}) \rightarrow S_\lambda \rightarrow 0,$$

- (b) for $1 \leq i \leq k$ and $1 \leq j \leq p_i$ there is a unique *exceptional* simple $S_{\lambda_{i,j}}$ determined by the short exact sequence

$$0 \rightarrow \mathcal{O}((j-1)\vec{x}_i) \xrightarrow{x_i} \mathcal{O}(j\vec{x}) \rightarrow S_{\lambda_{i,j}} \rightarrow 0.$$

- (iv) The category $\text{coh-}\mathbb{X}$ satisfies Serre duality with the graded shift $X \mapsto X(\vec{\omega})$ being the Auslander-Reiten translation τ . In particular, we have $S(\vec{\omega}) \simeq S$ for every exceptional simple S , and $S_{\lambda_{i,j}}(\vec{\omega}) \simeq S_{\lambda_{i,(j \bmod p_i)+1}}$ for all $1 \leq i \leq k$ and $1 \leq j \leq p_i$.

We can see that the Auslander-Reiten translation $\tau : \text{coh-}\mathbb{X} \rightarrow \text{coh-}\mathbb{X}$ restricts to an Auslander-Reiten translation on torsion sheaves and that simple objects concentrated at a fixed point $\lambda \in \mathbb{P}_K^1$ form a τ -orbit. But $\text{coh}_0\text{-}\mathbb{X}$ is a length category, so [3, Proposition 1.8.2] implies $\text{coh}_0\text{-}\mathbb{X} = \coprod_{\lambda \in \mathbb{P}_K^1} \mathcal{H}_\lambda$ where the wide subcategories \mathcal{H}_λ are generated by τ -orbits of simple objects. If $\lambda = \lambda_i \in \boldsymbol{\lambda}$ for some $1 \leq i \leq k$, then \mathcal{H}_λ is a tube of rank p_i . Otherwise, the tube \mathcal{H}_λ is homogenous.

Lemma 7.3 ([13, Lemma 2.4]). Let $X \in \text{coh-}\mathbb{X}$. Then $X \simeq 0$ if and only if $[X] = 0$ in $K_0(\text{coh-}\mathbb{X})$.

Definition 7.4 ([13, Section 2.2]). We define the following linear forms on $K_0(\text{coh-}\mathbb{X})$:

- (i) *The rank* is the linear form $\text{rank} : K_0(\text{coh-}\mathbb{X}) \rightarrow \mathbb{Z}$ uniquely determined by imposing $\text{rank}([X]) = 1$ for every line bundle X .
- (ii) Let p be the least common multiple of p_1, \dots, p_k . *The degree* is the linear form $\text{deg} : K_0(\text{coh-}\mathbb{X}) \rightarrow \mathbb{Z}$ characterised by $\text{rank}([\mathcal{O}]) = 0$, $\text{rank}([S]) = p$ for every ordinary simple S , and $\text{rank}([S_{\lambda_{i,j}}]) = \frac{p}{p_i}$ for every exceptional simple $S_{\lambda_{i,j}}$ given by $1 \leq i \leq k$ and $1 \leq j \leq p_i$.

For $X \in \text{coh-}\mathbb{X}$ we denote $\text{rank}(X) := \text{rank}([X])$ and $\text{deg}(X) := \text{deg}([X])$. And for $0 \neq X \in \text{coh-}\mathbb{X}$ we define *the slope* of X as $\mu(X) := \frac{\text{deg}(X)}{\text{rank}(X)} \in \mathbb{Q} \cup \{\infty\}$ where the slope is infinite precisely when $X \in \text{coh}_0\text{-}\mathbb{X}$.

Definition 7.5 ([13, Section 3.1]). We say that $0 \neq X \in \text{coh-}\mathbb{X}$ is *semistable* if for every subobject $0 \neq Y$ of X we get $\mu(Y) \leq \mu(X)$.

Let $q \in \mathbb{Q} \cup \{\infty\}$. We define the category $\mathcal{H}^{(q)}$ as the replete subcategory of $\text{coh-}\mathbb{X}$ consisting of the zero object and all the semistable objects X with $\mu(X) = q$.

Proposition 7.6 ([13, Lemma 3.1]). Let $q \in \mathbb{Q} \cup \{\infty\}$. Then $\mathcal{H}^{(q)}$ is abelian, hereditary, length, K -linear category which is extension-closed as a subcategory of $\text{coh-}\mathbb{X}$.

Note that $\mathcal{H}^{(\infty)} = \text{coh}_0\text{-}\mathbb{X}$.

Definition 7.7. We define the *Euler characteristic* of \mathbb{X} as $\chi_{\mathbb{X}} := 2 - \sum_{i=1}^k \left(1 - \frac{1}{p_i}\right)$.

We say that \mathbb{X} is:

- *domestic* if $\chi_{\mathbb{X}} > 0$,
- *tubular* if $\chi_{\mathbb{X}} = 0$,
- or *wild* if $\chi_{\mathbb{X}} < 0$.

Remark 7.8. As discussed in [8, pg. 12], the weighted projective line \mathbb{X} is tubular if and only if the tuple \mathbf{p} is a permutation of $(2, 2, 2, 2)$, $(3, 3, 3)$, $(2, 4, 4)$, or $(2, 3, 6)$.

Remark 7.9. The Euler characteristic is tied to the degree of the dualising element $\vec{\omega}$ as

$$\chi_{\mathbb{X}} = -\frac{\deg(\mathcal{O}(\vec{\omega}))}{p}$$

holds for p being the least common multiple of p_1, \dots, p_k . By [13, Lemma 2.5], we get

$$\mu(\tau X) = \mu(X(\vec{\omega})) = \mu(X) + \deg(\vec{\omega}) = \mu(X) - p\chi_{\mathbb{X}}.$$

In other words, the Euler characteristic tells us how τ changes the slope of an object.

Proposition 7.10 ([7, Proposition 5.5]). If $\chi_{\mathbb{X}} \geq 0$, then every indecomposable object in $\text{coh-}\mathbb{X}$ is semistable.

Moreover, if \mathbb{X} is domestic, then every indecomposable vector bundle is exceptional.

8 Derived categories of weighted projective lines

In this section, we fix a weighted projective line $\mathbb{X} = (\mathbb{P}_K^1, \boldsymbol{\lambda}, \mathbf{p})$ with $k \in \mathbb{N}$ weighted points over an algebraically closed field K .

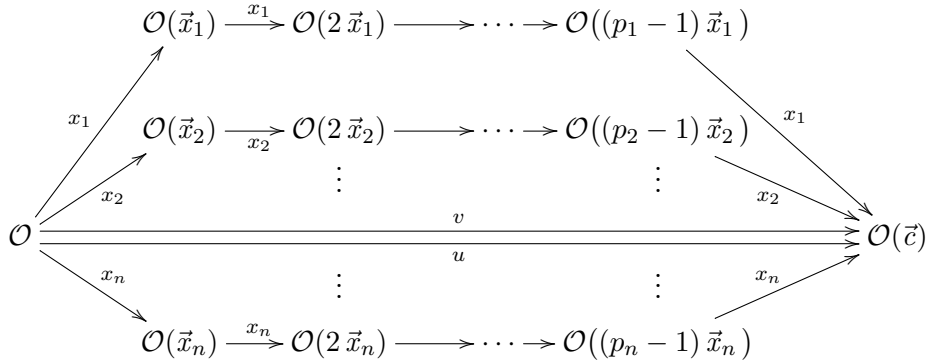
The quasi-coherent sheaves $\text{Qcoh-}\mathbb{X}$ form a hereditary, locally noetherian Grothendieck category ([11, pg. 1481]) with $\text{coh-}\mathbb{X}$ being its noetherian objects. The derived category $\mathbf{D}(\text{Qcoh-}\mathbb{X})$ is then compactly generated with compact elements being isomorphic to bounded complexes of objects from $\text{coh-}\mathbb{X}$ (see Proposition 2.32). Hence, we may identify $\mathbf{D}^b(\text{coh-}\mathbb{X}) \simeq \mathbf{D}(\text{Qcoh-}\mathbb{X})^c$ as a triangulated subcategory of $\mathbf{D}(\text{Qcoh-}\mathbb{X})$. Every object of $\mathbf{D}(\text{Qcoh-}\mathbb{X})$ decomposes as a coproduct of its cohomologies.

Lemma 8.1 (See eg. [6, pg. 38]). The category $\mathbf{D}^b(\text{coh-}\mathbb{X})$ is generated by an exceptional sequence. Line bundles of the form $\mathcal{O}(\vec{l})$ for $0 \leq \vec{l} \leq \vec{c}$ can be arranged into a particular example of an exceptional sequence

$$\left(\mathcal{O}, \mathcal{O}(\vec{x}_1), \dots, \mathcal{O}((p_1 - 1)\vec{x}_1), \dots, \mathcal{O}(\vec{x}_k), \dots, \mathcal{O}((p_k - 1)\vec{x}_k), \mathcal{O}(\vec{c}) \right).$$

of length $2 + \sum_{i=1}^k (p_i - 1)$ generating $\mathbf{D}^b(\text{coh-}\mathbb{X})$.

The direct sum $\bigoplus_{0 \leq \vec{l} \leq \vec{c}} \mathcal{O}(\vec{l})$ is known as *the canonical tilting bundle* (see [13, Theorem 2.9] and [3, Theorem 7.1.1]) and its endomorphism ring is isomorphic to *the canonical algebra* A (originally defined by Ringel in [17, Section 3.7]) of type $(\mathbf{p}, \boldsymbol{\lambda})$ which is the path algebra of the quiver



bound by the relations $x_i^{p_i} = (\lambda_i^1 u - \lambda_i^0 v)$ for each $1 \leq i \leq n$. By Theorem 2.40, the category $\mathbf{D}^b(\text{coh-}\mathbb{X})$ is then equivalent to $\mathbf{D}^b(\text{mod-}A)$.

Theorem 8.2. If \mathbb{X} is domestic, then there exists a tilting object $T \in \text{coh-}\mathbb{X}$ such that $A := \text{End}_{\text{coh-}\mathbb{X}}(T)$ is isomorphic to a path algebra of a Euclidean quiver over K . Hence A is a tame finite dimensional K -algebra and $\mathbf{D}^b(\text{coh-}\mathbb{X}) \simeq \mathbf{D}^b(\text{mod-}A)$.

Theorem 8.3 ([12, Proposition 10.18, Theorem 10.19]). Let $q, q' \in \mathbb{Q} \cup \{\infty\}$. If \mathbb{X} is tubular, then there exists a triangulated autoequivalence $\mathbf{D}^b(\text{coh-}\mathbb{X}) \xrightarrow{\simeq} \mathbf{D}^b(\text{coh-}\mathbb{X})$ restricting to an equivalence $\mathcal{H}^{(q)} \xrightarrow{\simeq} \mathcal{H}^{(q')}$.

In particular, every $\mathcal{H}^{(q)}$ decomposes as a coproduct of tubes, i.e. $\mathcal{H}^{(q)} \simeq \coprod_{\lambda \in \mathbb{P}_K^1} \mathcal{H}_\lambda^{(q)}$ where each $\mathcal{H}_\lambda^{(q)}$ is equivalent to the tube $\mathcal{H}_\lambda \subseteq \text{coh}_0\text{-}\mathbb{X}$.

We shall use the notation $\mathcal{H}^{(q)} := \text{thick}(\mathcal{H}^{(q)})$ for the corresponding thick categories generated by semistable objects of a fixed slope $q \in \mathbb{Q} \cup \{\infty\}$. If $q = \infty$ or \mathbb{X} is tubular, we also denote $\mathcal{H}_\lambda^{(q)} := \text{thick}(\mathcal{H}_\lambda^{(q)})$ for each $\lambda \in \mathbb{P}_K^1$. In that case, the thick categories generated by tubes of the same slope are mutually orthogonal, i.e.

$$\mathcal{H}_\lambda^{(q)} \subseteq \left(\mathcal{H}_{\lambda'}^{(q)} \right)^\perp \quad \text{and} \quad \mathcal{H}_{\lambda'}^{(q)} \subseteq \left(\mathcal{H}_\lambda^{(q)} \right)^\perp \quad \text{in } \mathbf{D}^b(\text{coh-}\mathbb{X}) \text{ for any } \lambda \neq \lambda' \in \mathbb{P}_K^1.$$

Proposition 8.4. A thick subcategory of $\mathbf{D}^b(\text{coh-}\mathbb{X})$ is bireflective if and only if it is generated by an exceptional collection.

Proof. “ \Rightarrow ” By [6, Corollary 8.4].

“ \Leftarrow ” By Theorem 2.48. \square

Corollary 8.5. Pick $n \in \mathbb{N}_0$. Let (E_1, \dots, E_n) and (F_1, \dots, F_n) be two exceptional sequences in $\mathbf{D}^b(\text{coh-}\mathbb{X})$, and denote $\mathcal{U} := \langle E_1, \dots, E_n \rangle$, $\mathcal{V} := \langle F_1, \dots, F_n \rangle \subseteq \mathbf{D}^b(\text{coh-}\mathbb{X})$. If $\mathcal{U} \subseteq \mathcal{V}$, then $\mathcal{U} = \mathcal{V}$.

Proof. As \mathcal{U} is bireflective, the pair $(\mathcal{U}, \mathcal{U}^\perp \cap \mathcal{V})$ is a t -structure on \mathcal{V} , and

$$K_0(\mathcal{V}) \simeq K_0(\mathcal{U}) \oplus K_0(\mathcal{U}^\perp \cap \mathcal{V}).$$

The isomorphisms $K_0(\mathcal{U}) \simeq \mathbb{Z}^n \simeq K_0(\mathcal{V})$ from Corollary 2.49 force $K_0(\mathcal{U}^\perp \cap \mathcal{V}) \simeq 0$. However, [6, Corollary 8.10] tells us that the only triangulated subcategory of $\mathbf{D}^b(\text{coh-}\mathbb{X})$ with zero Grothendieck group is the zero category. The equality $\mathcal{U} = \mathcal{V}$ directly follows. \square

Remark 8.6. Corollary 8.5 is specific to coherent sheaves over weighted projective lines because $\mathbf{D}^b(\text{coh-}\mathbb{X})$ does not contain *phantom* subcategories; those are non-zero bireflective triangulated subcategories with vanishing Grothendieck group. A counterexample to the statement of Corollary 8.5 for a K -linear triangulated possessing phantom subcategories can be found in [9, Theorem 1.1].

Lemma 8.7. Let \mathcal{S} be a thick subcategory of $\mathbf{D}^b(\text{coh-}\mathbb{X})$. If every indecomposable object of \mathcal{S} is exceptional, then \mathcal{S} is bireflective.

Proof. By Corollary 2.50 and Lemma 8.1, the length of any exceptional sequence in $\mathbf{D}^b(\text{coh-}\mathbb{X})$ is bounded by $2 + \sum_{i=1}^k (p_i - 1)$. Let (E_1, \dots, E_m) be an exceptional sequence in \mathcal{S} with $m \in \mathbb{N}$ maximal possible, and $\mathcal{E} := \langle E_1, \dots, E_m \rangle \subseteq \mathcal{S}$. Assume for contradiction, that $\mathcal{E} \subsetneq \mathcal{S}$. Choose $X \in \mathcal{S} \setminus \mathcal{E}$. Since $(\mathcal{E}, \mathcal{E}^\perp)$ is a t -structure on $\mathbf{D}^b(\text{coh-}\mathbb{X})$, there exists a distinguished triangle

$$E \rightarrow X \rightarrow Y \rightarrow \Sigma E$$

in \mathcal{S} with $Y \in \mathcal{E}^\perp$. Pick an indecomposable direct summand E_0 of Y . Then the exceptional sequence (E_0, E_1, \dots, E_m) in \mathcal{S} has $m + 1$ terms. A contradiction. \square

Proposition 8.8. If \mathbb{X} is domestic, then any thick subcategory of $\mathbf{D}^b(\text{coh-}\mathbb{X})$ containing a vector bundle is bireflective.

Proof. By [4, Proposition 7.1] and Proposition 8.4. \square

Lemma 8.9. Let $q, q' \in \mathbb{Q} \cup \{\infty\}$, $q \neq q'$, and let \mathcal{S} be a thick subcategory of $\mathbf{D}^b(\text{coh-}\mathbb{X})$ containing $X, Y \in \text{coh-}\mathbb{X}$ with $\mu(X) = q$ and $\mu(Y) = q'$. If $\chi_{\mathbb{X}} \geq 0$, then \mathcal{S} is bireflective.

Proof. Note that X or Y is a vector bundle contained in \mathcal{S} . Firstly, if \mathbb{X} is domestic, then the statement now directly follows from Proposition 8.8. Secondly, if every indecomposable object of \mathcal{S} is exceptional, then \mathcal{S} is bireflective by Lemma 8.7.

Finally, suppose that \mathbb{X} is tubular and \mathcal{S} contains a non-exceptional indecomposable object. W.l.o.g. the object X is non-exceptional and indecomposable. By Theorem 8.3, we may also w.l.o.g. assume $q = \infty$. Since $\mathcal{S} \cap \text{coh-}\mathbb{X}$ now contains a vector bundle and a non-exceptional indecomposable torsion sheaf X , it holds by [4, Proposition 6.2, Theorem 6.3]

that $\mathcal{S} = \langle E_1, \dots, E_n \rangle^\perp$ for some exceptional sequence (E_1, \dots, E_n) of objects from $\text{coh}_0\text{-}\mathbb{X}$ and index $n \in \mathbb{N}_0$.

Considering that $\langle E_1, \dots, E_n \rangle$ is bireflective, this already implies that \mathcal{S} is reflective. Using Proposition 2.51, we get that \mathcal{S} is bireflective. \square

We now state variants of Lemma 2.25 from Paper B for domestic and tubular weighted projective lines.

Lemma 8.10. Assume \mathbb{X} is tubular. Then there exists a number $M \in \mathbb{N}$ (depending on \mathbb{X}) such that for any strictly decreasing chain $\mathcal{S}_1 \supsetneq \mathcal{S}_2 \supsetneq \dots \supsetneq \mathcal{S}_M$ of thick subcategories of $\mathbf{D}^b(\text{coh-}\mathbb{X})$ we can find $1 \leq i < j \leq M$, $q \in \mathbb{Q} \cup \{\infty\}$, and $\lambda \in \mathbb{P}_K^1 \setminus \boldsymbol{\lambda}$ with $\mathcal{S}_i, \mathcal{S}_j \subseteq \mathcal{H}^{(q)}$ such that $\mathcal{H}_\lambda^{(q)} \subseteq \mathcal{S}_i$ and $\mathcal{H}_\lambda^{(q)} \cap \mathcal{S}_j = 0$.

Proof. If $\mathcal{S} \subsetneq \mathcal{S}'$ are bireflective subcategories of $\mathbf{D}^b(\text{coh-}\mathbb{X})$, they are generated by exceptional collections due to Proposition 8.4. Then the exceptional sequence generating \mathcal{S} must be shorter than the one generating \mathcal{S}' because of Corollary 2.50 and Corollary 8.5. Using induction, we can then infer that any strictly decreasing chain $\mathcal{S}_1 \supsetneq \mathcal{S}_2 \supsetneq \dots \supsetneq \mathcal{S}_N$, $N \in \mathbb{N}$, of thick subcategories of $\mathbf{D}^b(\text{coh-}\mathbb{X})$ can include at most $3 + \sum_{i=1}^k (p_i - 1)$ bireflective subcategories.

We are now left with proving the statement for $M := 1 + \prod_{i=1}^k \binom{2p_i}{p_i}$ and a strictly decreasing chain $\mathcal{S}_1 \supsetneq \mathcal{S}_2 \supsetneq \dots \supsetneq \mathcal{S}_M$ of non-bireflective thick subcategories of $\mathbf{D}^b(\text{coh-}\mathbb{X})$. Then by Lemma 8.9 we get $\mathcal{S}_1 \subseteq \mathcal{H}^{(q)}$ for some $q \in \mathbb{Q} \cup \{\infty\}$. As $\mathcal{H}^{(q)} \simeq \prod_{\lambda \in \mathbb{P}_K^1} \mathcal{H}_\lambda^{(q)}$ and thick $(\prod_{\lambda \in \boldsymbol{\lambda}} \mathcal{H}_\lambda^{(q)})$ has only $\prod_{i=1}^k \binom{2p_i}{p_i}$ thick subcategories (see Proposition 2.58), there must consequently exist indices $1 \leq i < j \leq M$ such that the subcategories \mathcal{S}_i and \mathcal{S}_j differ by \mathcal{S}_i containing a homogenous tube $\mathcal{H}_\lambda^{(q)}$ for some $\lambda \in \mathbb{P}_K^1 \setminus \boldsymbol{\lambda}$, while $\mathcal{H}_\lambda^{(q)} \cap \mathcal{S}_j = 0$. \square

Lemma 8.11. Assume \mathbb{X} is domestic. Then there exists a number $M \in \mathbb{N}$ (depending on \mathbb{X}) such that for any strictly decreasing chain $\mathcal{S}_1 \supsetneq \mathcal{S}_2 \supsetneq \dots \supsetneq \mathcal{S}_M$ of thick subcategories of $\mathbf{D}^b(\text{coh-}\mathbb{X})$ we can find $1 \leq i < j \leq M$ and $\lambda \in \mathbb{P}_K^1 \setminus \boldsymbol{\lambda}$ with $\mathcal{S}_i, \mathcal{S}_j \subseteq \mathcal{H}^{(\infty)}$ such that $\mathcal{H}_\lambda \subseteq \mathcal{S}_i$ and $\mathcal{H}_\lambda \cap \mathcal{S}_j = 0$.

Proof. Considering that any thick subcategory of $\mathbf{D}^b(\text{coh-}\mathbb{X})$ containing a vector bundle is bireflective by Proposition 8.8, the proof of the lemma is identical to the proof of Lemma 8.10 in the special case of $q = \infty$. \square

9 Completions

As before, in this section, we fix a weighted projective line $\mathbb{X} = (\mathbb{P}_K^1, \boldsymbol{\lambda}, \mathbf{p})$ with $k \in \mathbb{N}$ weighted points over an algebraically closed field K . We also set $\mathcal{S} := \mathbf{D}^b(\text{coh-}\mathbb{X})$ and $\mathcal{T} := \mathbf{D}(\text{Qcoh-}\mathbb{X})$. We know that $\mathcal{S} \hookrightarrow \mathcal{T}$ is a good extension with respect to all metrics due to Theorem 2.16 because \mathcal{T} has coproducts and $\mathcal{S} = \mathcal{T}^c$. Hence all completions $\mathfrak{S}'(\mathcal{S})$ will be calculated as subcategories of \mathcal{T} .

This section is dedicated to the classification of completions of \mathcal{S} (Theorem 9.8) when \mathbb{X} is domestic or tubular. We mimic the overall strategy of Section 7 of Paper B.

Lemma 9.1. Let $\lambda \in \mathbb{P}_K^1 \setminus \boldsymbol{\lambda}$. Let $0 \neq X \in \mathcal{H}_\lambda$ and $V \in \text{vect-}\mathbb{X}$. Then $\text{Hom}_{\mathcal{S}}(X, \Sigma V) \neq 0$.

Proof. By [7, Corollary 2.7], there exists a short exact sequence

$$0 \rightarrow V \rightarrow L \rightarrow T \rightarrow 0$$

in $\text{coh-}\mathbb{X}$ with L a direct sum of line bundles and $0 \neq T \in \mathcal{H}_\lambda$. This sequence does not split, so $\text{Hom}_{\mathcal{S}}(T, \Sigma V) \neq 0$. The tube \mathcal{H}_λ is homogenous, so any of its elements is filtered by the unique simple of the tube. This implies $\text{Hom}_{\mathcal{S}}(X, \Sigma V) \neq 0$. \square

We will need to employ the theory of preimage metrics developed in [Paper A](#).

Lemma 9.2. Let $F : \mathcal{U} \rightarrow \mathcal{V}$ be a triangulated equivalence between triangulated categories. Let $\mathcal{M} := \{B_n\}_{n \in \mathbb{N}}$ be a good metric on \mathcal{U} . For the image metric $\mathcal{N} := \{F(B_n)\}_{n \in \mathbb{N}}$ on \mathcal{V} we get $\mathfrak{S}_{\mathcal{M}}(\mathcal{U}) \simeq \mathfrak{S}_{\mathcal{N}}(\mathcal{V})$.

Proof. The family \mathcal{N} of subcategories of \mathcal{V} is a well-defined good metric by Lemma 4.14 from [Paper A](#) because F is full. Then \mathcal{M} is a preimage metric of \mathcal{N} under F . Moreover, the inverse F^{-1} is a fully faithful left and right adjoint to F . Consequently, all the assumptions of Theorem 4.23 from [Paper A](#) are satisfied, so the completions $\mathfrak{S}_{\mathcal{M}}(\mathcal{U})$ and $\mathfrak{S}_{\mathcal{N}}(\mathcal{V})$ are equivalent. \square

This is based on Lemma 7.1 from [Paper B](#). The formulations are adjusted to the setting of weighted projective lines.

Proposition 9.3. Let $\mathcal{D} \subsetneq \mathcal{C}$ be thick subcategories of \mathcal{S} generated by torsion sheaves. Let $\lambda \in \mathbb{P}_K^1 \setminus \boldsymbol{\lambda}$ and pick indecomposable $T \in \mathcal{H}_\lambda$. Then:

- (i) If $T \in \mathcal{C} \setminus \mathcal{D}$, then it holds that $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) \cap \mathfrak{S}'_{\mathcal{D}}(\mathcal{S}) \subseteq \mathcal{C}^\perp \cap \mathcal{H}^{(\infty)}$.
- (ii) If \mathcal{C} is not countably generated, then $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) = \mathcal{C}^\perp \cap \mathcal{H}^{(\infty)}$.

Proof. “Part i)” Pick $E \in \mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) \cap \mathfrak{S}'_{\mathcal{D}}(\mathcal{S})$ and choose a Cauchy sequence $\mathbf{E} = (E_n, e_n)_{n \in \mathbb{N}}$ with respect to \mathcal{D} satisfying $E \simeq \text{Hocolim } \mathbf{E}$. The sequence \mathbf{E} is also Cauchy with respect to \mathcal{C} . We may assume $\text{cone } e_n \in \mathcal{D}$ for all $n \in \mathbb{N}$; otherwise we just remove finitely many entries from \mathbf{E} . Up to replacing \mathcal{E} by a subsequence of a component via Lemma 6.1, the sequence \mathbf{E} is \mathcal{C} -trivial as well.

Since $T \notin \mathcal{D}$, we infer $\mathcal{D} \subseteq T^\perp$ using the tube orthogonality. Fix $j \in \mathbb{Z}$. The functor $\text{Hom}_{\mathcal{S}}(\Sigma^j T, -)$ vanishes on $\text{cone } e_n$ for all $n \in \mathbb{N}$. Hence, the sequence $\text{Hom}_{\mathcal{S}}(\Sigma^j T, \mathbf{E})$ is constant. For each $n \in \mathbb{N}$ we get

$$\text{Hom}_{\mathcal{S}}(\Sigma^j T, E_n) \simeq \varinjlim \text{Hom}_{\mathcal{S}}(\Sigma^j T, \mathbf{E}) \simeq \text{Hom}_{\mathcal{S}}(\Sigma^j T, E) \simeq 0.$$

Using $\text{Hom}_{\mathcal{S}}(\Sigma^{-1} T, V) \neq 0$ for all $V \in \text{vect-}\mathbb{X}$ from Lemma 9.1, we get that for all $n \in \mathbb{N}$ the object E_n does contain any shift of vector bundle as a direct summand. In other words, we get $E_n \in \mathcal{H}^{(\infty)}$ for all $n \in \mathbb{N}$. Because \mathbf{E} is also \mathcal{C} -trivial, the mutual orthogonality of tubes implies $E_n \in \mathcal{C}^\perp$ for all $n \in \mathbb{N}$.

An application of Lemma 6.2 from [Paper B](#) for the choice of sequences $\mathbf{F} = \mathbf{E}$ proves $E \in \mathcal{S}$. We again use $\text{Hom}_{\mathcal{S}}(\Sigma^{-1} T, V) \neq 0$ for all $V \in \text{vect-}\mathbb{X}$ to deduce $E \in \mathcal{H}^{(\infty)}$. Thus we may conclude $E \in \mathcal{C}^\perp \cap \mathcal{H}^{(\infty)}$.

“Part ii)” The category \mathcal{C} is now not countably generated. Therefore, we can find an uncountable subset $I \subseteq \mathbb{P}_K^1$ and for each $\lambda \in I$ a thick subcategory

$$\mathbf{t}_\lambda \subseteq \mathcal{H}_\lambda^{(\infty)} \quad \text{such that} \quad \mathcal{C} \simeq \coprod_{\lambda \in I} \mathbf{t}_\lambda.$$

Let $E \in \mathfrak{S}'_{\mathcal{C}}(\mathcal{S})$. By Lemma 6.5, we get $E \in \mathfrak{S}'_{\mathcal{D}}(\mathcal{S})$ for some countably generated thick subcategory $\mathcal{D} \subseteq \mathcal{C}$. We have $\mathcal{D} \subseteq \coprod_{\lambda \in J} \mathbf{t}_\lambda$ for some $J \subseteq I$ countable. Because I is uncountable, there exists $\lambda \in I \setminus (\lambda \cup J)$ satisfying $\mathbf{t}_\lambda = \mathcal{H}_\lambda^{(\infty)}$ and $\mathcal{H}_\lambda^{(\infty)} \cap \mathcal{D} = 0$.

Consequently, part i) applies, so $E \in \mathcal{C}^\perp \cap \mathcal{H}^{(\infty)}$. As E was arbitrary, the inclusion $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) \subseteq \mathcal{C}^\perp \cap \mathcal{H}^{(\infty)}$ follows. Finally, Lemma 6.1 from Paper B then ensures the validity of the other inclusion $\mathcal{C}^\perp \cap \mathcal{H}^{(\infty)} \subseteq \mathfrak{S}'_{\mathcal{C}}(\mathcal{S})$. \square

Corollary 9.4. Assume \mathbb{X} is tubular. Let $q \in \mathbb{Q} \cup \{\infty\}$. Let $\mathcal{D} \subsetneq \mathcal{C}$ be thick subcategories of $\mathcal{H}^{(q)} \subseteq \mathcal{S}$. Pick an indecomposable object $T \in \mathcal{H}_\lambda^{(q)}$ for some $\lambda \in \mathbb{P}_K^1 \setminus \lambda$. Then:

- (i) If $T \in \mathcal{C} \setminus \mathcal{D}$, then it holds that $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) \cap \mathfrak{S}'_{\mathcal{D}}(\mathcal{S}) \subseteq \mathcal{C}^\perp \cap \mathcal{H}^{(q)}$.
- (ii) If \mathcal{C} is not countably generated, then $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) = \mathcal{C}^\perp \cap \mathcal{H}^{(q)}$.

Proof. By Theorem 8.3, there exists a triangulated autoequivalence $F : \mathcal{S} \rightarrow \mathcal{S}$ such that $F(\mathcal{H}^{(q)}) = \text{coh}_0\text{-}\mathbb{X}$. By Lemma 9.2, the autoequivalence F induces a triangulated equivalence $\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) \simeq \mathfrak{S}'_{F(\mathcal{M})}(\mathcal{S})$ for any good metric \mathcal{M} on \mathcal{S} . Since $F(\mathcal{C}) \subseteq \mathcal{H}^{(\infty)}$, the statement now directly follows from Proposition 9.3. \square

For the purpose of describing completions of \mathcal{S} , we introduce a technical notion of a slope of metric. We stress that not every metric has to have a well-defined slope.

Definition 9.5. Equip \mathcal{S} with an additive good metric $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$. If there exists a unique $q \in \mathbb{Q} \cup \{\infty\}$ such that for every $n \in \mathbb{N}$ it holds that $\mathcal{H}^{(q)} \cap B_n \neq 0$, then we say that \mathcal{M} has *slope* q and denote $\mu(\mathcal{M}) := q$.

Lemma 9.6. Assume $\chi_{\mathbb{X}} \geq 0$. Let \mathcal{C} be a thick subcategory of \mathcal{S} . If \mathcal{C} is not countably generated, then it has a well-defined slope as constant metric on \mathcal{S} , and

$$\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) = \mathcal{C}^\perp \cap \mathcal{H}^{(\mu(\mathcal{C}))}.$$

Proof. Because \mathcal{C} is not countably generated, it is not generated by an exceptional collection, so it is not bireflective by Proposition 8.4. Then by Lemma 8.9, there exists $q \in \mathbb{Q} \cup \{\infty\}$ such that $\mathcal{C} \subseteq \mathcal{H}^{(q)}$. If \mathbb{X} is domestic, then necessarily $q = \infty$ due to Proposition 8.8. This forces $q = \mu(\mathcal{C})$. By Proposition 9.3 and Corollary 9.4, we get $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) = \mathcal{C}^\perp \cap \mathcal{H}^{(\mu(\mathcal{C}))}$. \square

Lemma 9.7. Assume $\chi_{\mathbb{X}} \geq 0$. Let \mathcal{C} be a thick subcategory of \mathcal{S} . Equip \mathcal{S} with the standard t -structure $(\mathcal{S}^{\leq 0}, \mathcal{S}^{> 0})$ and an additive good metric $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$. Consider the thick subcategory $\mathcal{B} := \bigcap_{n \in \mathbb{N}} B_n$ of \mathcal{S} . If $\mathcal{M} \not\sim \mathcal{M}_\infty \vee \mathcal{B}$, then \mathcal{M} has a well-defined slope.

Proof. By Lemma 6.7 applied to the choice of $E = 0 \in \mathfrak{S}'_{\mathcal{M}}(\mathcal{S})$, we get that (up to replacing \mathcal{M} by an equivalent metric) there exists a family $(X_n)_{n \in \mathbb{N}}$ of objects of $\text{coh-}\mathbb{X}$ such that for all $n \in \mathbb{N}$ it holds that $X_n \in B_n$ but simultaneously $X_n \notin \text{thick}(H^0(B_{n+1}))$. This implies that $\text{thick}(H^0(B_1)) \supsetneq \text{thick}(H^0(B_2)) \supsetneq \dots$ is an infinite, strictly decreasing chain of thick subcategories of \mathcal{S} . By Lemma 8.10 and Lemma 8.11, there exist $n \in \mathbb{N}$ and $q \in \mathbb{Q} \cup \{\infty\}$ satisfying

$$\text{thick}(H^0(B_n)) \subseteq \mathcal{H}^{(q)}.$$

This q is then necessarily unique and equal to the slope of \mathcal{M} . \square

We may now finally state the final result of the chapter.

Theorem 9.8. Assume $\chi_{\mathbb{X}} \geq 0$. Equip \mathcal{S} with the standard t -structure $(\mathcal{S}^{\leq 0}, \mathcal{S}^{> 0})$ and an additive good metric $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$. Set $\mathcal{B} := \bigcap_{n \in \mathbb{N}} B_n$; it is a thick subcategory of \mathcal{S} . Then:

- (i) If $\mathcal{M} = \mathcal{M}_{\infty} \vee \mathcal{B}$, and if \mathcal{B} is countably generated, then

$$\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) \simeq \mathfrak{S}'_{\mathcal{B}}(\mathcal{S}) \simeq \mathcal{S}/\mathcal{B} \hookrightarrow \mathcal{T}.$$

In this case, it holds that $\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) \simeq \mathcal{B}^{\perp} \cap \mathcal{S} \hookrightarrow \mathcal{S}$ if and only if \mathcal{B} is generated by an exceptional sequence.

- (ii) Otherwise, we have

$$\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) = \mathcal{B}^{\perp} \cap \mathcal{H}^{(\mu(\mathcal{M}))}.$$

Proof. The proof will be done by considering the following cases:

- (a) Condition i) holds, i.e. $\mathcal{M} = \mathcal{M}_{\infty} \vee \mathcal{B}$ and \mathcal{B} is countably generated.
 (b) Condition ii) holds, $\mathcal{M} = \mathcal{M}_{\infty} \vee \mathcal{B}$, and \mathcal{B} is not countably generated.
 (c) Condition ii) holds, $\mathcal{M} \succeq \mathcal{M}_{\infty} \vee \mathcal{B}$.

“**Case a)**” By Proposition 5.4 and Proposition 6.6, we get $\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) \simeq \mathfrak{S}'_{\mathcal{B}}(\mathcal{S}) \simeq \mathcal{S}/\mathcal{B}$. Corollary 5.5 tells us that the equivalence $\mathfrak{S}'_{\mathcal{B}}(\mathcal{S}) \simeq \mathcal{B}^{\perp} \cap \mathcal{S}$ holds if and only if \mathcal{B} is coreflective. As \mathcal{S} is generated by an exceptional sequence, Proposition 2.51 and Proposition 8.4 imply that \mathcal{B} is coreflective if and only if \mathcal{B} is bireflective, or equivalently if and only if \mathcal{B} is generated by an exceptional sequence.

“**Case b)**” Assume that $\mathcal{M} = \mathcal{M}_{\infty} \vee \mathcal{B}$ and \mathcal{B} is not countably generated. Then for all but finitely many $n \in \mathbb{N}$ it holds that $H^0(B_n) = H^0(\mathcal{B})$. This implies $\mu(\mathcal{M}) = \mu(\mathcal{B})$. By Proposition 6.6 and Lemma 9.6, we get

$$\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) \simeq \mathfrak{S}'_{\mathcal{B}}(\mathcal{S}) \simeq \mathcal{B}^{\perp} \cap \mathcal{H}^{(\mu(\mathcal{M}))}.$$

“**Case c)**” Assume $\mathcal{M} \succeq \mathcal{M}_{\infty} \vee \mathcal{B}$. By Lemma 9.7, the metric \mathcal{M} has a well-defined slope $q \in \mathbb{Q} \cup \{\infty\}$. We prove $\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) = \mathcal{B}^{\perp} \cap \mathcal{H}^{(q)}$.

“ \supseteq ” It is enough to prove that any indecomposable coherent sheaf $T \in \mathcal{B}^{\perp} \cap \mathcal{H}^{(q)}$ is an element of the completion $\mathfrak{S}'_{\mathcal{M}}(\mathcal{S})$ because any other object of the right-hand side category is a direct sum of shifts of indecomposable coherent sheaves from $\mathcal{B}^{\perp} \cap \mathcal{H}^{(q)}$.

Denote $\mathcal{C}_n := \text{thick}(X \in \text{coh-}\mathbb{X} : X \in B_n, \text{Hom}_{\mathcal{S}}(X, T \oplus \Sigma T) \neq 0)$ for each $n \in \mathbb{N}$. Consider the decreasing chain $\mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \dots$ of thick subcategories of \mathcal{S} . If the chain did not stabilise, then by Lemma 8.10 or Lemma 8.11, there would exist $n \in \mathbb{N}$ such that

$$\text{thick}(\mathcal{H}_{\lambda}^{(q')}, \mathcal{H}_{\lambda'}^{(q')}) \subseteq \mathcal{C}_n \subseteq \mathcal{H}^{(q')} \quad \text{for some } \lambda, \lambda' \in \mathbb{P}_K^1, \lambda \neq \lambda', q' \in \mathbb{Q} \cup \{\infty\},$$

forcing $q = q'$ (by the uniqueness of the slope of a metric) and $T = 0$; a contradiction with the choice of T .

Hence, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have $\mathcal{C}_m = \mathcal{C}_n$. Assume for contradiction that there exists an indecomposable object $Y \in B_{N+1}$ with $\text{Hom}_{\mathcal{S}}(Y, T) \neq 0$. Then Y is concentrated in cohomological degrees either 0, or 1. We get $\Sigma Y, Y \in B_N$, so $Y \in \mathcal{C}_N = \mathcal{C}_n$ for all $n \geq N$. By Lemma 6.2, this means that $Y \in \mathcal{B}$, contradicting $T \in \mathcal{B}^\perp$.

We conclude that T is compactly supported at $N+1$. Using Lemma 6.1 from Paper B, we obtain $T \in \mathfrak{S}'_{\mathcal{M}}(\mathcal{S})$.

“ \subseteq ” Let $E \in \mathfrak{S}'_{\mathcal{M}}(\mathcal{S})$. Since $\mathcal{M} \succeq \mathcal{M}_\infty \vee \mathcal{B}$, Lemma 6.8 provides us with an infinite, strictly decreasing chain $\mathcal{C}_1 \supsetneq \mathcal{C}_2 \supsetneq \dots$ of thick subcategories of \mathcal{S} such that $E \in \mathfrak{S}'_{\mathcal{C}_n}(\mathcal{S})$ and $\mathcal{C}_n \subseteq \text{thick}(H^0(B_n))$ for each $n \in \mathbb{N}$. Using Lemma 8.10 or Lemma 8.11 we find $n > m \geq N$, $q' \in \mathbb{Q} \cup \{\infty\}$, and $\lambda \in \mathbb{P}_K^1 \setminus \boldsymbol{\lambda}$ such that $\mathcal{H}_{\lambda}^{(q')} \subseteq \mathcal{C}_m \subseteq \mathcal{H}^{(q')}$ and $\mathcal{H}_{\lambda}^{(q')} \cap \mathcal{C}_n = 0$. By the definition of a slope of a metric, we obtain $q = q'$. Thus Proposition 9.3 and Corollary 9.4 apply, so $E \in \mathcal{H}^{(q)}$. As E is compactly supported with respect to \mathcal{M} , inevitably $E \in \mathcal{B}^\perp$. We may now conclude the proof. \square

10 Examples

We now illustrate Theorem 9.8 with examples. We adhere to the long-standing assumptions in this chapter - \mathbb{X} is a weighted projective line $\mathbb{X} = (\mathbb{P}_K^1, \boldsymbol{\lambda}, \mathbf{p})$ with $k \in \mathbb{N}$ weighted points over an algebraically closed field K , $\mathcal{S} := \mathbf{D}^b(\text{coh-}\mathbb{X})$, and $\mathcal{T} := \mathbf{D}(\text{Qcoh-}\mathbb{X})$. The inclusion $\mathcal{S} \hookrightarrow \mathcal{T}$ is a good extension and every completion $\mathfrak{S}'(\mathcal{S})$ is considered to be a subcategory of \mathcal{T} .

Example 10.1. Let $\lambda = [\lambda^0, \lambda^1] \in \mathbb{P}_K^1 \setminus \boldsymbol{\lambda}$. Consider the short exact sequence

$$0 \rightarrow \mathcal{O} \xrightarrow{\lambda^1 u - \lambda^0 v} \mathcal{O}(\vec{c}) \rightarrow S_\lambda \rightarrow 0,$$

Define $\mathcal{C} := \mathcal{H}_\lambda$. We know by Theorem 9.8, that $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) \simeq \mathcal{S}/\mathcal{C}$. We shall explicitly express the completion as a bounded derived category of a graded localisation ring.

Denote $y := \lambda^1 u - \lambda^0 v \in S(\mathbf{p}, \boldsymbol{\lambda})$, which is a homogenous element of degree $\vec{c} \in \mathbb{L}(\mathbf{p})$, and $Y := \{y^n : n \in \mathbb{N}_0\}$ the homogenous multiplicative set generated by y . The localisation $S(\mathbf{p}, \boldsymbol{\lambda}) \rightarrow S(\mathbf{p}, \boldsymbol{\lambda})_{Y^{-1}}$ of graded rings and the corresponding graded localisation functor

$$F : \text{mod}^{\mathbb{L}(\mathbf{p})} - S(\mathbf{p}, \boldsymbol{\lambda}) \rightarrow \text{mod}^{\mathbb{L}(\mathbf{p})} - S(\mathbf{p}, \boldsymbol{\lambda})_{Y^{-1}}$$

induce an isomorphism $\text{mod}^{\mathbb{L}(\mathbf{p})} - S(\mathbf{p}, \boldsymbol{\lambda}) / \text{Ker } F \rightarrow \text{mod}^{\mathbb{L}(\mathbf{p})} - S(\mathbf{p}, \boldsymbol{\lambda})_{Y^{-1}}$ where the quotient is taken in the sense of Serre (see e.g. [3, Section 1.2]).

Since $y \in S(\mathbf{p}, \boldsymbol{\lambda})$ is not invertible and $S(\mathbf{p}, \boldsymbol{\lambda})$ is graded local, we get

$$\text{mod}_0^{\mathbb{L}(\mathbf{p})} - S(\mathbf{p}, \boldsymbol{\lambda}) \subseteq \text{Ker } F.$$

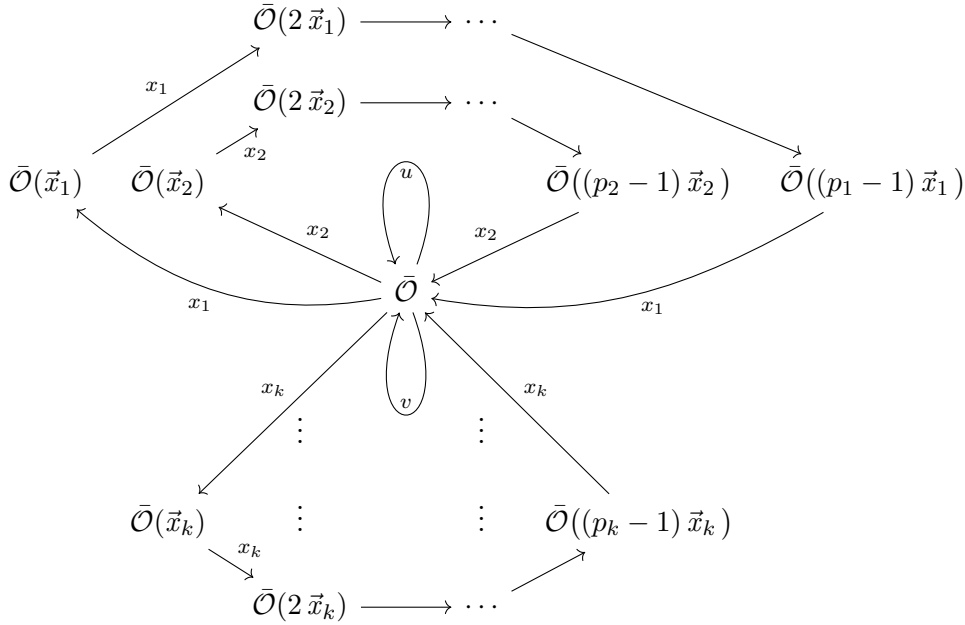
In fact, the image of $\text{Ker } F$ in $\text{coh-}\mathbb{X}$ is precisely \mathcal{H}_λ . By the universal property of Serre's quotients and the definition of $\text{coh-}\mathbb{X}$, the functor F factors through $\text{coh-}\mathbb{X}$, thus giving us an isomorphism

$$\bar{F} : \text{coh-}\mathbb{X} / \mathcal{H}_\lambda \rightarrow \text{mod}^{\mathbb{L}(\mathbf{p})} - S(\mathbf{p}, \boldsymbol{\lambda})_{Y^{-1}}.$$

As \bar{F} is exact, it extends to the derived level

$$\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) \simeq \mathbf{D}^b(\text{coh-}\mathbb{X}) / \mathcal{C} \simeq \mathbf{D}^b\left(\text{mod}^{\mathbb{L}(\mathbf{p})} - S(\mathbf{p}, \boldsymbol{\lambda})_{Y^{-1}}\right).$$

Denote $\bar{X} \in \mathfrak{S}'_{\mathcal{C}}(\mathcal{S})$ the image of $X \in \text{coh-}\mathbb{X}$ under Verdier's quotient functor $\mathcal{S} \rightarrow \mathcal{S}/\mathcal{C}$. Note that $\bar{X} \simeq \bar{X}(l\vec{c})$ for each $X \in \text{coh-}\mathbb{X}$ and $l \in \mathbb{Z}$. An instance of a Cauchy sequence in \mathcal{S} with respect to the constant metric \mathcal{C} converging to an object $\bar{\mathcal{O}}$ of the completion is $\mathcal{O} \xrightarrow{y} \mathcal{O}(\vec{c}) \xrightarrow{y} \mathcal{O}(2\vec{c}) \xrightarrow{y} \dots$. We can visualise the image $\bigoplus_{0 \leq l \leq \vec{c}} \bar{\mathcal{O}}(l) \in \mathfrak{S}'_{\mathcal{C}}(\mathcal{S})$ of the canonical tilting bundle as:



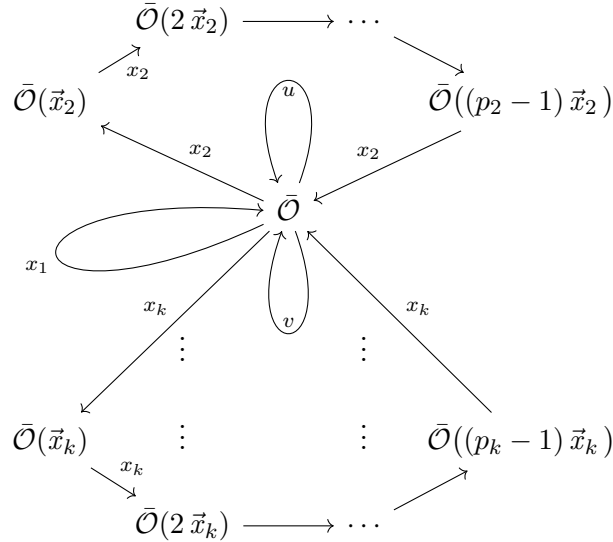
where the multiplication by $y = \lambda^1 u - \lambda^0 v$ is invertible on the top of the usual canonical relations $x_i^{p_i} = (\lambda_i^1 u - \lambda_i^0 v)$ for each $1 \leq i \leq k$.

Example 10.2. Consider $\lambda_1 = [\lambda_1^0, \lambda_1^1] \in \boldsymbol{\lambda}$ and set $\mathcal{C} := \mathcal{H}_{\lambda_1}$. To describe the completion $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) \simeq \mathcal{S}/\mathcal{C}$, we proceed in the same way as in Example 10.1, but this time the tube \mathcal{H}_{λ_1} is not homogenous, so the thick subcategory \mathcal{C} is generated by p_1 simple objects given by short exact sequences

$$\begin{aligned} 0 &\rightarrow \mathcal{O} \xrightarrow{x_1} \mathcal{O}(\vec{x}_1) \rightarrow S_{\lambda_1,1} \rightarrow 0 \\ 0 &\rightarrow \mathcal{O}(\vec{x}_1) \xrightarrow{x_1} \mathcal{O}(2\vec{x}_1) \rightarrow S_{\lambda_1,2} \rightarrow 0 \\ &\vdots \\ 0 &\rightarrow \mathcal{O}((p_1-1)\vec{x}_1) \xrightarrow{x_1} \mathcal{O}(\vec{c}) \rightarrow S_{\lambda_1,p_1} \rightarrow 0. \end{aligned}$$

We need to consider the homogenous multiplicative set $Y := \{x_1^n : n \in \mathbb{N}_0\}$ generated by x_1 . We can again show $\mathfrak{S}'_{\mathcal{C}}(\mathcal{S}) \simeq \mathbf{D}^b(\text{mod}^{\mathbb{L}(\mathbf{p})} - S(\mathbf{p}, \boldsymbol{\lambda})_{Y^{-1}})$.

Denote again $\bar{X} \in \mathfrak{S}'_{\mathcal{C}}(\mathcal{S})$ the image of a $X \in \text{coh-}\mathbb{X}$ under Verdier's quotient functor $\mathcal{S} \rightarrow \mathcal{S}/\mathcal{C}$, so $\bar{X} \simeq \bar{X}(l\vec{x}_1)$ for every $X \in \text{coh-}\mathbb{X}$ and $l \in \mathbb{Z}$. The image $\bigoplus_{0 \leq \vec{l} \leq \vec{c}} \bar{\mathcal{O}}(\vec{l}) \in \mathfrak{S}'_{\mathcal{C}}(\mathcal{S})$ of the canonical tilting bundle is



where the multiplication by x_1 (and also by $x_1^{p_1} = \lambda_1^1 u - \lambda_1^0 v$) is invertible on the top of the usual canonical relations $x_i^{p_i} = (\lambda_i^1 u - \lambda_i^0 v)$ for each $2 \leq i \leq k$.

If we calculate the completion of \mathcal{S} with respect to a thick subcategory generated only by a proper subset of exceptional simples $S_{\lambda_{i,1}}, S_{\lambda_{i,2}}, \dots, S_{\lambda_{i,p_i}}$ for some $1 \leq i \leq k$, then we, in general, no longer obtain a derived category of a graded localisation.

In Example 10.3 and Example 10.4, we discuss the effects of the cardinality of the field K on the metric completions of \mathcal{S} .

Example 10.3. Assume K is uncountable. Then any thick subcategory $\mathcal{C} \subseteq \mathcal{S}$ can be expressed as $\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) \simeq \mathcal{C}$ for some good additive metric \mathcal{M} on \mathcal{S} .

If \mathcal{C} is bireflective, so it is a coaisle of a t -structure $({}^\perp \mathcal{C}, \mathcal{C})$ on \mathcal{S} . Hence $\mathfrak{S}'_{{}^\perp \mathcal{C}}(\mathcal{S}) \simeq \mathcal{C}$ by Corollary 2.68.

Suppose that \mathcal{C} is not bireflective. Then $\mathcal{C} \subseteq \mathcal{H}^{(q)}$ for some $q \in \mathbb{Q} \cup \{\infty\}$ by Proposition 8.8 and Lemma 8.9. If \mathbb{X} is domestic, then necessarily $q = \infty$.

Firstly, assume that \mathcal{C} is generated by finitely many elements. Then $\mathcal{D} := \mathcal{H}^{(q)} \cap {}^\perp \mathcal{C}$ is generated by uncountably many elements, since \mathbb{P}_K^1 is uncountable and $\text{coh}_0\text{-}\mathbb{X}$ has to contain uncountably many tubes. Theorem 9.8 gives us $\mathfrak{S}'_{\mathcal{D}}(\mathcal{S}) \simeq \mathcal{D}^\perp \cap \mathcal{H}^{(q)} \simeq \mathcal{C}$, where the second equivalence is implied by [5, Theorem 2.3.25].

Secondly, assume that \mathcal{C} is not generated by finitely many elements. We find a countable subset $I = \{i_n\}_{n \in \mathbb{N}} \subseteq \mathbb{P}_K^1 \setminus \boldsymbol{\lambda}$ satisfying $\mathcal{H}_{\lambda_{i_n}}^{(q)} \subseteq \mathcal{C}$ for each $n \in \mathbb{N}$. Then

$$\mathcal{M} = \{B_n\}_{n \in \mathbb{N}} = \left\{ \left({}^\perp \mathcal{C} \cap \mathcal{H}^{(q)} \right) \oplus \prod_{m \geq n} \mathcal{H}_{\lambda_{i_m}}^{(q)} \right\}_{n \in \mathbb{N}}$$

is an additive good metric on \mathcal{S} with $\mathcal{B} := \bigcap_{n \in \mathbb{N}} B_n = {}^\perp \mathcal{C} \cap \mathcal{H}^{(q)}$. As $\mathcal{M} \neq \mathcal{M}_\infty \vee \mathcal{B}$, Theorem 9.8 yields $\mathfrak{S}'_{\mathcal{M}}(\mathcal{S}) \simeq \mathcal{B}^\perp \cap \mathcal{H}^{(q)} \simeq \mathcal{C}$.

Example 10.4. The statement of Example 10.3 does not hold when K is countable.

Let \mathcal{C} be, for example, the thick subcategory generated by all non-homogenous tubes of torsion sheaves.

Considering that in our situation any thick subcategory of \mathcal{S} is countably generated, if there existed a thick subcategory \mathcal{D} such that $\mathfrak{S}'_{\mathcal{D}}(\mathcal{S}) \simeq \mathcal{C}$, then \mathcal{D} would be bireflective in \mathcal{S} by Theorem 9.8. But then $\mathcal{C} \simeq \mathcal{D}^\perp \cap \mathcal{S}$ would also be bireflective in \mathcal{S} by Proposition 2.51, which is nevertheless not true.

Our only other chance to exhibit \mathcal{C} as a completion is to find a good additive metric $\mathcal{M} = \{B_n\}_{n \in \mathbb{N}}$ satisfying

$$\mathcal{B} := \bigcap_{n \in \mathbb{N}} B_n = {}^\perp \mathcal{C} \cap \mathcal{H}^{(\infty)} = \prod_{\lambda \in \mathbb{P}_K^1 \setminus \lambda} \mathcal{H}_\lambda$$

and $\mathcal{M} \neq \mathcal{M}_\infty \vee \mathcal{B}$.

Assume for contradiction that such a metric \mathcal{M} exists. Because \mathcal{B} contains all homogenous tubes and $\mathcal{B} \subseteq \text{thick}(H^0(B_n))$ for all $n \in \mathbb{N}$, Lemma 8.10 and Lemma 8.11 force the chain $\text{thick}(H^0(B_1)) \supseteq \text{thick}(H^0(B_2)) \supseteq \dots$ of thick subcategories to stabilise at some $N \in \mathbb{N}$. By Lemma 6.2, we then get $\text{thick}(H^0(B_N)) = \mathcal{B}$. But this is impossible, since the condition $\mathcal{M} \neq \mathcal{M}_\infty \vee \mathcal{B}$ ensures that $H^0(B_N) \subseteq \text{thick}(H^0(B_N))$ contains an indecomposable object from $\mathcal{S} \setminus \mathcal{B}$.

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