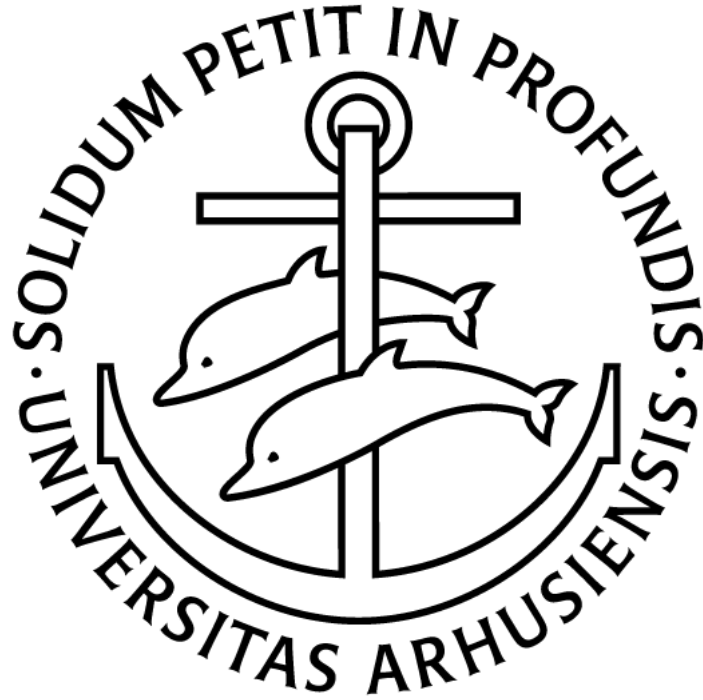


Enumerative Geometry on the Nested Hilbert Scheme of Points via Moduli of Weak Algebraic Structures



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PhD Thesis

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Preface

This thesis consists mainly of two papers written during the course of my three-year doctoral studies. They are presented in their original wording without substantial changes, except for formatting and layout adjustments. Citations for the individual papers are not provided at the end of each chapter. Instead, all sources are collected in a single bibliography at the end of the thesis.

- Chapter 1 is a copy of the paper titled "Virtual K-theoretic invariants of the nested Hilbert scheme on $\mathbb{C}^2$ " written solely by me. It is available as an arXiv preprint (arXiv:2606.30172).
- Chapter 2 is a copy of the paper "Virtual invariants from the non-associative Hilbert scheme" written by me and with my PhD-supervisor, Gergely Bérczi, as coauthor. It is available as an arXiv preprint (arXiv:2512.11538v3).

These papers can be read separately as finished works, but together they represent a way of working with virtual enumerative geometry on the Hilbert scheme via moduli spaces parameterizing weaker algebraic structures. In chapter 1 this is done by reproducing existing invariants of the Hilbert scheme of points via the non-commutative nested Hilbert scheme to derive new calculations of these invariants. In chapter 2 this is instead done by defining new virtual invariants on the Hilbert scheme of points using the non-associative Hilbert scheme.

Declarations

This thesis was written solely by me except when stated explicitly otherwise (i.e. chapter 2 which is a copy of a paper with my PhD-supervisor, Gergely Bérczi, as coauthor).

Acknowledgments

I would like to thank my PhD-supervisor, Gergely Bérczi, for three interesting years of exploring many different aspects of the Hilbert scheme of points and for sharing his great geometric and computational intuition with me.

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Abstract

This thesis explores the enumerative geometry of the nested Hilbert scheme of points on affine n -space by embedding it into smooth moduli spaces that parameterize related objects satisfying weaker algebraic conditions. Using these embeddings, we construct perfect obstruction theories on the nested Hilbert scheme of points and compute the associated virtual invariants. The thesis consists of two chapters, each of which can be read independently.

In chapter 1, we construct a nested version of the non-commutative Hilbert scheme introduced by Nori. We use this construction to define a perfect obstruction theory on the nested Hilbert scheme of points on the affine plane and prove that this is equivalent to the perfect obstruction theory constructed by Gholampour-Sheshmani-Yau. We then consider the morphism of nested Hilbert schemes of points that forgets the largest subscheme. Using our setup, we show that the pushforward of the virtual structure sheaf under this map is a twist of the virtual structure sheaf on the target. Using localization, we prove that the twist is by a constant class which we identify explicitly. As a corollary, we are able to prove a closed formula for the multivariate generating function of the torus-equivariant virtual Euler characteristic.

In chapter 2, we construct a smooth moduli space called the non-associative Hilbert scheme which extends the construction of Kazarian from a punctual locus to the full nested Hilbert scheme of points. We exhibit the nested Hilbert scheme of points on affine n -space as the associativity locus inside this smooth moduli space and, from this description, endow it with a new perfect obstruction theory. Using localization, we obtain an explicit formula for the virtual fundamental class in terms of torus-fixed points. We consider a punctual version of the non-associative Hilbert scheme whose intersection with the associativity locus is a distinguished locus of the punctual nested Hilbert scheme of points. We refer to this locus as the nilpotently filtered Hilbert scheme of points. Using a partial resolution, we derive an efficient iterated residue formula for computing virtual integrals of tautological classes on the nilpotently filtered Hilbert scheme of points. Finally, we show that certain formal virtual integrals on the nested Hilbert scheme of points can be expressed in terms of the nilpotently filtered Hilbert scheme of points, so that these integrals also enjoy the iterated residue formula.

Resumé

Denne afhandling udforsker den enumerative geometri af det nastede Hilbert skema af punkter på det affine n -rum, ved at indlejre det ind i glatte modulirum, der parametriserer relaterede objekter, der opfylder svagere algebraiske betingelser. Ved at bruge disse indlejringer konstruerer vi perfekte obstruktionsteorier på det nastede Hilbert skema af punkter og udregner de tilhørende virtuelle invarianter. Afhandlingen består af to kapitler, som kan læses uafhængigt af hinanden.

I kapitel 1 konstruerer vi en indlejret version af det ikke-kommutative Hilbert skema introduceret af Nori. Vi bruger dette til at definere en perfekt obstruktionsteori på det nastede Hilbert skema af punkter på det affine plan og beviser, at dette er ækvivalent til den perfekte obstruktionsteori konstrueret af Gholampour-Sheshmani-Yau. Vi betragter derefter morfien mellem nastede Hilbert skemaer af punkter, der glemmer det største underskema. Ved at bruge vores opsætning beviser vi, at det direkte billede af det virtuelle strukturknippe under denne afbildning er et tensorprodukt af det virtuelle strukturknippe på codomænet og en anden klasse. Ved at bruge lokalisering beviser vi, at denne klasse er konstant, og identificerer den. Som et korollar kan vi bevise en lukket formel for den multivariate genererende funktion af den torus-ækvivariante virtuelle Euler-karakteristik.

I kapitel 2 konstruerer vi et glat modulirum, som vi kalder det ikke-associative Hilbert skema, der generaliserer Kazarians konstruktion fra en punktuelt locus til det fulde nastede Hilbert skema af punkter. Vi beskriver det nastede Hilbert skema af punkter på det affine n -rum som associativitets-locusen i det glatte modulirum, og fra denne beskrivelse giver vi den en ny perfekt obstruktionsteori. Ved at bruge lokalisering får vi en eksplicit formel for den virtuelle fundamentalklasse beskrevet ved torusfikserede punkter. Vi betragter en punktuelt version af det ikke-associative Hilbert skema, hvis snit med associativitets-locusen er en specifik locus i det punktuelle nastede Hilbert skema af punkter. Vi kalder denne for det nilpotent-filtrerede Hilbert skema af punkter. Ved at bruge en partiel resolution udleder vi en effektiv itereret residue formel til at udregne virtuelle integraler af tautologiske klasser på det nilpotent-filtrerede Hilbert skema af punkter. Til sidst viser vi, at visse formelle virtuelle integraler på det nastede Hilbert skema af punkter kan blive udtrykt i form af det nilpotent-filtrerede Hilbert skema af punkter, så disse integraler også kan regnes via den itererede residue formel.

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Introduction

The Hilbert scheme of points and moduli of weaker algebraic structures

This dissertation is about enumerative geometry on the nested Hilbert scheme of points on $\mathbb{A}^n$. Our focus is on constructing perfect obstruction theories and calculating the associated virtual invariants. We do so by viewing the nested Hilbert scheme of points as the moduli space parameterizing finite dimensional quotient algebras of the polynomial ring and embed it into a smooth moduli space parameterizing similar but weaker algebraic structures, i.e. the non-commutative and the non-associative Hilbert scheme of points.

The Hilbert scheme of d points of a scheme X is a moduli space parameterizing zero-dimensional length d subschemes of X , that is

$$\mathrm{Hilb}^d(X) = \{Z \subset X \mid \dim Z = 0, \text{length } Z = d\}.$$

Similarly, for a collection of integers $0 \leq n_1 \leq n_2 \leq \dots \leq n_r$ one can consider *the nested Hilbert scheme of $\mathbf{n}$ points* of X which is the moduli space of nestings of zero-dimensional subschemes of lengths prescribed by $\mathbf{n}$, that is

$$\mathrm{NHilb}^{\mathbf{n}}(X) = \{Z_r \subset \dots \subset Z_1 \subset X \mid \dim Z_i = 0, \text{length } Z_i = n_i\}.$$

When X is smooth of dimension 1 or 2 the Hilbert scheme of points $\mathrm{Hilb}^d(X)$ is smooth of the expected dimension for all d [Fog68]. When X is of dimension ≥ 3 the Hilbert scheme of points is no longer smooth, nor irreducible [Iar72]. For the nested Hilbert scheme of points, the situation is slightly worse. The nested Hilbert scheme of points on a smooth curve is smooth of the expected dimension [Che98], but when X is of dimension ≥ 2 the nested Hilbert scheme of points is no longer smooth, nor irreducible [RT21].

To do enumerative geometry related to the (nested) Hilbert scheme of points, we need to be able to understand its intersection theory. For the Hilbert scheme of points on a surface this is a well studied question where the theory has been shown to carry many rich structures [Nak99, Göt02, OP10]. However, the above makes this study difficult in higher dimensions, as we lack a well behaved fundamental class.

One remedy to this is to study the intersection theory of the *geometric subsets* of [Ren12]. These are constructible subsets built from set theoretic operations of the subsets of $\mathrm{Hilb}^d(X)$ of specified isomorphism classes of the subschemes. The theory of tautological integrals on these subschemes is developed in [Bá, Bb].

Another way to remedy this, is to replace the fundamental class by a *virtual fundamental class* as defined in [BF97]. Similarly, working K -theoretically one can introduce a *virtual structure sheaf* [BF97, Remark 5.4]. Both of these enjoy many of the functorial properties

we are used to in intersection theory (see for example [Man11, Qu18]), but their construction requires the extra input of a *perfect obstruction theory* and the class depends heavily on the chosen obstruction theory. For the Hilbert scheme of points, several natural choices of such obstruction theories exist depending on X . The nested Hilbert scheme of $n_1 \leq \dots \leq n_r$ points on a smooth projective surface admits a perfect obstruction theory of virtual dimension $n_1 + n_r$ [GSY20]. The Hilbert scheme of points on a projective Calabi-Yau threefold being a special case of the moduli of stable sheaves admits a *symmetric perfect obstruction theory*, in particular of virtual dimension 0 [BF08, Tho00b]. The associated invariants are a special case of the Donaldson-Thomas invariants. The Hilbert scheme of a projective Calabi-Yau fourfold admits a natural obstruction theory which is not perfect. Using a square root Euler class it is however still possible to define a virtual fundamental class and structure sheaf [OT23].

The *non-commutative Hilbert scheme* $\mathrm{ncHilb}^d(\mathbb{A}^n)$ introduced in [Nor78] is a smooth moduli space containing the Hilbert scheme of points $\mathrm{Hilb}^d(\mathbb{A}^n)$ as a closed subscheme corresponding to the *commutativity locus*. It has played a special role in calculating virtual invariants of many of the obstruction theories found above. The power of the non-commutative Hilbert scheme is that it typically allows a "standard model" interpretation of the abstractly defined virtual theory of the Hilbert scheme of points.

The notions of a perfect obstruction theory, a symmetric perfect obstruction theory and the setup of Oh-Thomas [OT23] on a scheme X all admit "standard models" that is, prototypical examples that produce them, which are rarely available globally. For a perfect obstruction theory the standard model is when $X = Z(s)$ is exhibited as the zero locus of a section $s \in H^0(Y, E)$ of a vector bundle E on a smooth scheme Y . For a symmetric perfect obstruction theory it is when $X = Z(df)$ is exhibited as a critical locus of a potential $f : Y \rightarrow \mathbb{A}^1$ of a smooth ambient space Y . For the Oh-Thomas theory it is when $X = Z(s)$ is exhibited as the zero locus of an isotropic section $s \in H^0(Y, E)$ of an orthogonal bundle (E, q) on a smooth ambient space Y .

If we let $X = \mathrm{Hilb}^d(\mathbb{A}^3)$ be the Hilbert scheme of points on $\mathbb{A}^3$ and $Y = \mathrm{ncHilb}^d(\mathbb{A}^3)$ its non-commutative variant then [BBS13] exhibits X as a critical locus of Y , giving the standard model of the symmetric-perfect obstruction theory. It, along with its generalization to Quot schemes of points, is used to calculate various K -theoretic virtual invariants of the Hilbert scheme of points and the Quot scheme of points. For examples of this, see [Oko17, BR21, FMR21].

Similarly, if we let $X = \mathrm{Quot}_r^d(\mathbb{A}^4)$ be the Quot scheme of points on $\mathbb{A}^4$ and $Y = \mathrm{ncQuot}_r^d(\mathbb{A}^4)$ its non-commutative variant (which in particular includes the (non-commutative) Hilbert scheme for $r = 1$) then Kool-Rennemo [KR] exhibits X as a zero locus of an isotropic section of an orthogonal bundle on Y , giving us the standard model of Oh-Thomas. Using this, they are able to calculate K -theoretic invariants which they identify with the instanton partition function of Nekrasov-Piazzalunga [NP19].

In chapter 1 we continue this trend. We construct a nested version of the non-commutative Hilbert scheme and use it to construct a perfect obstruction theory on the nested Hilbert scheme of $\mathbb{A}^2$ equivalent to that of [GSY20]. We then use this model to calculate K -theoretic virtual invariants, proving in particular a closed formula for the multivariate generating series of the torus-equivariant virtual Euler characteristics.

One could ask whether one could simply define virtual invariants of the Hilbert scheme of

points for $\mathbb{A}^n$ for any n using the non-commutative Hilbert scheme as above. The main problem is that the virtual dimension quickly becomes negative. Indeed the scheme $\text{ncHilb}^d(\mathbb{A}^n)$ has dimension

$$(n-1) \cdot d^2 + d$$

while the naive bundle used to cut out the commutativity locus is of rank

$$\frac{n(n-1)}{2}d^2$$

giving us a virtual dimension (that is the degree of the Chow group in which the virtual fundamental class lives) of

$$d - d^2 \frac{(n-2)(n-1)}{2}$$

which is negative for $n \geq 3$. For $n = 3$ one is able to reduce the number of equations giving a zero-dimensional virtual class, but it is not clear if this is possible or how to do this for $n > 3$. In chapter 2 we try to remedy this by using another smooth ambient space to define invariants. We extend the non-associative Hilbert scheme of Kazarian [Kaz17] and construct a smooth moduli space $\text{naNHilb}^d(\mathbb{A}^n)$ containing $\text{NHilb}^d(\mathbb{A}^n)$ as a closed subscheme and show that the latter is cut out by a bundle whose rank depends only on $\underline{d}$. In particular, we are able to construct a positive dimensional virtual fundamental class on $\text{NHilb}^d(\mathbb{A}^n)$ as long as n is large enough compared to $\underline{d}$. We calculate the virtual fundamental class using localization and prove an efficient iterated residue formula for calculating certain virtual tautological integrals.

Summary of results

Virtual K-theoretic invariants of the nested Hilbert scheme on $\mathbb{C}^2$

In this paper we construct a nested version of the non-commutative Hilbert scheme. From it, we can reproduce the perfect obstruction theory of [GSY20]. The main application is a formula for a certain pushforward of the virtual structure sheaf, which allows us to prove a closed formula for the multivariate generating series of the torus-equivariant virtual Euler characteristic.

Let us first recall the construction of the non-commutative Hilbert scheme due to Nori [Nor78]. Briefly, the Hilbert scheme of d points on $\mathbb{A}^n$ can be thought of as the moduli space of ideals of the n -variable polynomial ring with a quotient algebra of finite dimension d

$$\text{Hilb}^d(\mathbb{A}^n) = \{I \subset \mathbb{C}[x_1, \dots, x_n] \text{ ideal} \mid \dim_{\mathbb{C}} \mathbb{C}[x_1, \dots, x_n]/I = d\}.$$

The idea of the non-commutative Hilbert scheme is simply to consider the moduli space of left ideals of the non-commutative polynomial ring satisfying the same dimension requirement

$$\text{ncHilb}^d(\mathbb{A}^n) = \{I \subset \mathbb{C}\langle x_1, \dots, x_n \rangle \text{ left ideal} \mid \dim_{\mathbb{C}} \mathbb{C}\langle x_1, \dots, x_n \rangle/I = d\}.$$

Equivalently, it is the moduli space parameterizing d -dimensional vector spaces V (corresponding to the quotient $\mathbb{C}\langle x_1, \dots, x_n \rangle/I$) equipped with operators $A_1, \dots, A_n \in \text{End}_{\mathbb{C}}(V)$ (corresponding to left multiplication with $x_1, \dots, x_n$) and a vector $v \in V$ (corresponding to the unit 1) satisfying the cyclicity condition

$$\mathbb{C}\langle A_1, \dots, A_n \rangle v = V.$$

This allows a quotient description

$$(\text{End}_{\mathbb{C}}(V)^{\oplus n} \oplus V)^{\text{st}} / \text{GL}_d$$

where the superscript st denotes the tuples satisfying the cyclicity condition, see for example [Rei05]. The moduli is smooth and contains the Hilbert scheme of points as the locus where I contains the commutator ideal or equivalently where the operators $A_1, \dots, A_n$ commute.

In our construction of the nested non-commutative Hilbert scheme we do essentially the same. We think of the nested Hilbert scheme of points on $\mathbb{A}^n$ as parameterizing chains of ideals

$$I_{r+1} \subset I_r \subset \dots \subset I_1 \subset I_0 = \mathbb{C}[x_1, \dots, x_n]$$

where $\dim_{\mathbb{C}} I_i/I_{i+1} = d_i$ where

$$d_i = \text{length } Z_{i+1} - \text{length } Z_i$$

where Z_i is the subscheme corresponding to I_i . We can then consider the moduli of chains of left ideals of $\mathbb{C}\langle x_1, \dots, x_n \rangle$ satisfying the same dimension requirement. Similarly to the case of the ordinary non-commutative Hilbert scheme, we can think of it as parameterizing a flag of vector spaces

$$V_{\bullet} = (V_0 \supset \dots \supset V_{r+1} = 0)$$

with $\dim_{\mathbb{C}} V_i/V_{i+1} = d_i$ (corresponding to $V_i = I_i/I_{i+1}$) equipped with flag-preserving operators $A_1, \dots, A_n$ (corresponding to left multiplication by $x_1, \dots, x_n$) and a vector $v \in V_0$ (corresponding to the unit 1) satisfying the cyclicity condition

$$\mathbb{C}\langle A_1, \dots, A_n \rangle v = V_0.$$

This allows us to construct the non-commutative nested Hilbert scheme as

$$\text{ncNHilb}^d(\mathbb{A}^n) = (\text{End}_{\mathbb{C}}^{\text{fil}}(V_{\bullet}) \oplus V_0)^{\text{st}}/P_d$$

where the superscript fil denotes the flag-preserving operators, the superscript st denotes the tuples satisfying the stability condition and $P_d \subset \text{Aut}_{\mathbb{C}}(V_{\bullet})$ is the parabolic group of flag-preserving automorphisms. As we are taking the quotient of a non-reductive group, it is not a priori clear that this produces a well defined scheme. However, one can easily prove that the action is free so that the quotient stack construction provides us with a smooth algebraic space. One can then by hand construct a Zariski cover using the functor of points, proving that it is in fact a scheme.

The non-commutative nested Hilbert scheme $\text{ncNHilb}^d(\mathbb{A}^n)$ comes equipped with a universal flag of bundles $\mathcal{V}_{\bullet}$ with universal operators $\Phi_1, \dots, \Phi_n \in \text{End}^{\text{fil}}(\mathcal{V}_{\bullet})$ and a universal section $\mathfrak{v} \in H^0(\text{ncNHilb}^d(\mathbb{A}^n), \mathcal{V}_0)$. These are by definition constructed such that for a point x corresponding to a tuple $(V_{\bullet}, A_1, \dots, A_n, v)$ we have $(\mathcal{V}_{\bullet})_x = V_{\bullet}$, $(\Phi_i)_x = A_i$ and $\mathfrak{v}_x = v$. In the case where $n = 2$ we define

$$\mathcal{E}_{\text{naive}} = \mathcal{E}nd^{\text{fil}}(\mathcal{V}_{\bullet})$$

the bundle of flag-preserving endomorphisms of sheaves and let

$$s_{\text{com}} = [\Phi_1, \Phi_2] \in H^0(\text{ncNHilb}^d(\mathbb{A}^n), \mathcal{E}_{\text{naive}})$$

be the section corresponding to the commutator of the universal operators. It then follows that the nested Hilbert scheme of points is the zero locus of this section thus equipping it with a perfect obstruction theory. We further prove that if we consider the quotient $\mathcal{E}$ of $\mathcal{E}_{\text{naive}}$ given by

$$\mathcal{E} = \mathcal{H}om^{\text{fil}}(\mathcal{V}_{\bullet}, \mathcal{V}_{\bullet}/(\mathfrak{v}))$$

then we still have $\text{NHilb}^d(\mathbb{A}^2) = Z(\bar{s}_{\text{com}})$ where $\bar{s}_{\text{com}}$ is the image of s_{com} in $\mathcal{E}$. All these constructions admit natural T -equivariant extensions for $T = \mathbb{G}_m^2$ the two dimensional torus. We then compare the K -theory classes with that of [GSY20] leading to the following result.

Theorem 0.0.1. *Let $\underline{d} = (d_0, \dots, d_r) \in \mathbb{Z}^{r+1}$ and define*

$$\mathcal{E}_{\underline{d}} = \begin{cases} \mathcal{H}om^{\text{fil}}(\mathcal{V}_{\bullet}, \mathcal{V}_{\bullet}/(\mathfrak{v})) & \text{if } d_0 \neq 0 \\ \mathcal{E}nd^{\text{fil}}(\mathcal{V}_{\bullet}) & \text{if } d_0 = 0 \end{cases}$$

as a bundle on $\text{ncNHilb}^{\underline{d}}(\mathbb{A}^2)$. Let $s_{\text{com}} \in H^0(\text{ncNHilb}^{\underline{d}}(\mathbb{A}^2), \mathcal{E}_{\underline{d}})$ be the commutator of the universal operators and let

$$\mathbb{E} \rightarrow \mathbb{L}_{\text{NHilb}^{\underline{d}}(\mathbb{A}^2)}$$

be the induced T -equivariant perfect obstruction theory on $\text{NHilb}^{\underline{d}}(\mathbb{A}^2)$ given by exhibiting it as $Z(s_{\text{com}})$. Let

$$\mathbb{E}_{\text{GSY}} \rightarrow \mathbb{L}_{\text{NHilb}^{\underline{d}}(\mathbb{A}^2)}$$

be the restriction of the T -equivariant perfect obstruction theory of $\text{NHilb}^{\underline{d}}(\mathbb{P}^2)$ defined in [GSY20] to the Zariski open $\text{NHilb}^{\underline{d}}(\mathbb{A}^2) \subset \text{NHilb}^{\underline{d}}(\mathbb{P}^2)$. Then in the K -theory $K_T^0(\text{NHilb}^{\underline{d}}(\mathbb{A}^2))$ we have

$$[\mathbb{E}] = [\mathbb{E}_{\text{GSY}}].$$

In particular, by [Sie04, Theorem 4.6] and [Tho22, Corollary 4.5] the corresponding virtual fundamental classes and virtual structure sheaves agree.

During the proof of the theorem above, we also explicitly calculate the virtual tangent bundle. Let $\mathcal{Q}_i$ be the tautological bundle on $\text{NHilb}^{\underline{d}}(\mathbb{A}^2)$ corresponding to the i 'th subscheme that is,

$$\mathcal{Q}_i = p_* q^* \mathcal{O}_{\mathcal{Z}_i}$$

where $\mathcal{Z}_i \subset \text{NHilb}^{\underline{d}}(\mathbb{A}^2) \times \mathbb{A}^2$ is the universal family of the i 'th subscheme and where

$$\mathbb{A}^2 \xleftarrow{q} \text{NHilb}^{\underline{d}}(\mathbb{A}^2) \times \mathbb{A}^2 \xrightarrow{p} \text{NHilb}^{\underline{d}}(\mathbb{A}^2)$$

are the projections. Then

$$T_{\text{NHilb}^{\underline{d}}(\mathbb{A}^2)}^{\text{vir}} = \mathcal{Q}_{r+1} + \frac{\mathcal{Q}_1^{\vee}}{t_1 t_2} - \frac{(1-t_1)(1-t_2)}{t_1 t_2} \sum_{i=0}^r \mathcal{Q}_{j+1}^{\vee} (\mathcal{Q}_{j+1} - \mathcal{Q}_j).$$

Next, we consider the map

$$\begin{aligned} p : \text{NHilb}^{(d_0, \dots, d_r)}(\mathbb{A}^2) &\longrightarrow \text{NHilb}^{(d_0, \dots, d_{r-1})}(\mathbb{A}^2) \\ [I_{r+1} \subset \dots \subset I_1 \subset \mathbb{C}[x, y]] &\longmapsto [I_r \subset \dots \subset I_1 \subset \mathbb{C}[x, y]] \end{aligned}$$

which forgets the largest subscheme. This extends to a map

$$\begin{aligned} p : \text{ncNHilb}^{(d_0, \dots, d_r)}(\mathbb{A}^2) &\longrightarrow \text{ncNHilb}^{(d_0, \dots, d_{r-1})}(\mathbb{A}^2) \\ (V_{\bullet}, A_1, A_2, v) &\longmapsto (V_{\bullet}/V_r, \overline{A}_1, \overline{A}_2, \overline{v}) \end{aligned}$$

where $\overline{A}_1, \overline{A}_2, \overline{v}$ denotes the induced operators and vector on the quotient $V_{\bullet}/V_r$. We further have a surjection of bundles

$$\mathcal{E}_{(d_0, \dots, d_r)} \twoheadrightarrow p^* \mathcal{E}_{(d_0, \dots, d_{r-1})}$$

given by either

$$\mathcal{E}nd^{\text{fil}}(\mathcal{V}_{\bullet}) \twoheadrightarrow \mathcal{E}nd^{\text{fil}}(\mathcal{V}_{\bullet}/\mathcal{V}_r)$$

or

$$\mathcal{H}om^{\text{fil}}(\mathcal{V}_{\bullet}, \mathcal{V}_{\bullet}/\mathfrak{v}) \twoheadrightarrow \mathcal{H}om^{\text{fil}}(\mathcal{V}_{\bullet}/\mathcal{V}_r, \mathcal{V}_{\bullet}/(\mathcal{V}_r + \mathfrak{v}))$$

depending on d_0 . Using the explicit description of the virtual structure sheaf in the standard model, we use this map to derive a factorization

$$\mathcal{O}_{(d_0, \dots, d_r)}^{\text{vir}} = E \cdot p^* \mathcal{O}_{(d_0, \dots, d_{r-1})}^{\text{vir}}$$

for a certain excess class $E \in K_T^0(\text{NHilb}^{(d_0, \dots, d_r)}(\mathbb{A}^2))$. We then calculate $p_* E$ via localization and show that it is a constant class via combinatorial calculations. Using the projection formula, we arrive at the main result.

Theorem 0.0.2. *Let*

$$p : \text{NHilb}^{(d_0, \dots, d_r)}(\mathbb{A}^2) \rightarrow \text{NHilb}^{(d_0, \dots, d_{r-1})}(\mathbb{A}^2)$$

be the map of nested Hilbert schemes as above and let

$$q : \text{NHilb}^{(d_0, \dots, d_{r-1})}(\mathbb{A}^2) \rightarrow \text{pt}$$

be the structure map. We then have

$$p_* \mathcal{O}_{(d_0, \dots, d_r)}^{\text{vir}} = q^*(\alpha_{d_r}) \cdot \mathcal{O}_{(d_0, \dots, d_{r-1})}^{\text{vir}}$$

where

$$\alpha_{d_r} = \chi_T(\text{Hilb}^{d_r}(\mathbb{A}^2), \wedge_{-1}^\bullet(\omega_{\mathbb{A}^2}^{[d_r]})) \in K_T^0(\text{pt})$$

for $\omega_{\mathbb{A}^2} = t_1 t_2 \mathcal{O}_{\mathbb{A}^2}$ the dualizing sheaf and $\omega_{\mathbb{A}^2}^{[d_r]}$ the corresponding tautological bundle.

At last, using that $\mathcal{O}_{(d_0)}^{\text{vir}} = \mathcal{O}_{\text{Hilb}^{d_0}(\mathbb{A}^2)}$ and known formulas for α_{d_r} (see for example [WZ14, Theorem 3.2]) we derive a closed formula for the generating series.

Corollary 0.0.1. *We have*

$$\sum_{d_0, \dots, d_r} q_0^{d_0} \cdots q_r^{d_r} \chi_T(\text{NHilb}^{(d_0, \dots, d_r)}(\mathbb{A}^2), \mathcal{O}_{(d_0, \dots, d_r)}^{\text{vir}}) = \text{PE} \left(\frac{q_1 + (q_2 + \cdots + q_r)(1 - t_1 t_2)}{(1 - t_1)(1 - t_2)} \right)$$

where PE denotes the plethystic exponential i.e.,

$$\text{PE} \left(\frac{q_1 + (q_2 + \cdots + q_r)(1 - t_1 t_2)}{(1 - t_1)(1 - t_2)} \right) = \exp \left(\sum_{n \geq 1} \frac{q_1^n + (q_2^n + \cdots + q_r^n)(1 - t_1^n t_2^n)}{n(1 - t_1^n)(1 - t_2^n)} \right).$$

Virtual invariants from the non-associative Hilbert scheme

In this paper we construct a smooth moduli space called the non-associative nested Hilbert scheme. We embed the nested Hilbert scheme of points as an associativity locus and use this description to define a perfect obstruction theory of the Hilbert scheme of points providing new virtual invariants. We calculate the virtual fundamental class by localization and prove an efficient iterated residue formula for calculating virtual tautological integrals.

We again view the nested Hilbert scheme of points on $\mathbb{A}^n$ as parameterizing nestings of ideals

$$I_{r+1} \subset \cdots \subset I_1 \subset I_0 = \mathbb{C}[x_1, \dots, x_n]$$

with $\dim_{\mathbb{C}} I_i/I_{i+1} = d_i$ for a dimension vector $(d_0, \dots, d_r)$. The idea behind the non-associative Hilbert scheme is simply to consider the moduli space which parameterizes chains of ideals inside

$$\mathbb{C}\{x_1, \dots, x_n\}_c$$

the free commutative non-associative algebra on n generators. The data of the quotient can then equivalently be encoded by a flag of vector spaces $V_\bullet$ (corresponding to $V_i = I_i/I_{i+1}$) equipped with a commutative but not necessarily associative multiplication rule

$$\psi_2 : \text{Sym}^2 V_\bullet \rightarrow V_\bullet$$

which treats V_i as an ideal i.e. $\psi_2(V_0 \otimes V_i) \subset V_i$ and is equipped with a unit $v \in V_0$. The quotient map (and hence the ideals) can then be reconstructed by picking n -generators

$$\psi_1 : (\mathbb{C}^n)^\vee = \text{span}_{\mathbb{C}}(x_1, \dots, x_n) \rightarrow V_0.$$

Them being generators translates to the stability condition that the smallest subspace closed under ψ_2 and containing the image of the generators $\text{im } \psi_1$ and the unit vector v is V_0 itself. This allows us to construct the non-associative Hilbert scheme as a quotient. Concretely, we let

$$W \subset \text{Hom}_{\mathbb{C}}((\mathbb{C}^n)^\vee, V_0) \oplus \text{Hom}_{\mathbb{C}}^{\text{fil}}(\text{Sym}^2 V_\bullet, V_\bullet) \oplus V_0$$

be the affine subspace of tuples (ψ_1, ψ_2, v) where $\psi_2(v \otimes (-)) = (-)$ i.e., where v is a unit for the multiplication rule ψ_2 . We then define *the non-associative nested Hilbert scheme* by

$$\text{naNHilb}^{\underline{d}}(\mathbb{A}^n) = (W)^{\text{st}}/P_{\underline{d}}.$$

where the superscript st denotes the tuples satisfying the stability condition above and where $P_{\underline{d}} \subset \text{Aut}_{\underline{d}}(V_0)$ is the parabolic group of flag-preserving automorphisms. Again, to take the quotient by a non-reductive group, we note that the action is free so that the quotient stack is a smooth algebraic space. We can then, as in the case of the non-commutative nested Hilbert scheme, explicitly construct a Zariski cover, proving that the quotient is in fact a smooth scheme. The nested Hilbert scheme embeds as the locus where the ideals contain the associator ideal or equivalently the locus where ψ_2 is associative. In the case where $d_0 = 1$ we consider the locus where ψ_1 factors through V_1 and the multiplication satisfies $\psi_2(V_1 \otimes V_i) \subset V_{i+1}$ which we denote by $\text{naNHilb}_0^{\underline{d}}(\mathbb{A}^n)$. On the associativity locus this corresponds to requiring

$$I_1 = \mathfrak{m}_0 = (x_1, \dots, x_n)$$

and that

$$\mathfrak{m}_0 \cdot I_i \subset I_{i+1}.$$

We call this locus *the nilpotently filtered nested Hilbert scheme of points* and denote it by $\text{NHilb}_{\text{nil-fil}}^{\underline{d}}(\mathbb{A}^n)$. We see that the above condition in particular implies that

$$\mathfrak{m}_0^i \subset I_i \subset \mathfrak{m}_0$$

so that the locus is in the punctual Hilbert scheme of points $\text{NHilb}_0^{\underline{d}}(\mathbb{A}^n)$, i.e. the locus where the subschemes are all supported on $\mathbf{0} \in \mathbb{A}^n$. Similarly to the work of [Kaz17] we prove that this non-associative locus admits the structure of a tower of Grassmannian bundles. In fact, Kazarian's model sits as the smaller locus where the stronger condition

$$\psi_2(V_i \otimes V_j) \subset V_{i+j}$$

is required for all i, j . In the case where $\underline{d} = (1, \dots, 1)$ we can construct an extension of the Hilbert-Chow morphism

$$\text{naHC} : \text{naNHilb}^{\underline{d}}(\mathbb{A}^n) \rightarrow (\mathbb{A}^n)^{r+1}$$

and prove that the nilpotently filtered nested Hilbert scheme of points is equal to the punctual nested Hilbert scheme of points

$$\text{NHilb}_{\text{nil-fil}}^{\underline{d}}(\mathbb{A}^n) = \text{NHilb}_0^{\underline{d}}(\mathbb{A}^n)$$

although this is not the case for general $\underline{d}$.

We then proceed to describe the nested Hilbert scheme of points as the zero locus of a section of a bundle, which equips it with a perfect obstruction theory. As in the case of the non-commutative Hilbert scheme, we have a universal flag of bundles with universal maps $(\mathcal{V}_\bullet, \Psi_1, \Psi_2, \mathfrak{v})$ such that the fiber over a point x corresponding to $(V_\bullet, \psi_1, \psi_2, v)$ is given by

$$((\mathcal{V}_\bullet)_x, (\Psi_1)_x, (\Psi_2)_x, \mathfrak{v}_x) = (V_\bullet, \psi_1, \psi_2, v).$$

We can then consider the associator

$$\Psi_2(\Psi_2(-, -), -) - \Psi_2(-, \Psi_2(-, -))$$

which a priori is a section of $\mathcal{H}om(\mathcal{V}_0^{\otimes 3}, \mathcal{V}_0)$. We can however ignore the unit in the source, as associativity is automatic for the unit. By Ψ_2 being filtration preserving we can further restrict to the morphisms satisfying

$$\mathcal{V}_i \otimes \mathcal{V}_j \otimes \mathcal{V}_k \mapsto \mathcal{V}_{\max\{i, j, k\}}$$

and by commutativity of Ψ_2 we can further restrict to the morphisms that are alternating in the two outer coordinates. If we let

$$\mathcal{E}_{\text{ass}} \subset \mathcal{H}om((\mathcal{V}_0/\mathfrak{v})^{\otimes 3}, \mathcal{V}_0)$$

be the subbundle of maps satisfying the filtration requirement and being alternating in the outer coordinates we then get $\text{NHilb}^{\underline{d}}(\mathbb{A}^n) = Z(s_{\text{ass}})$ for

$$s_{\text{ass}} \in H^0(\text{naNHilb}^{\underline{d}}(\mathbb{A}^n), \mathcal{E}_{\text{ass}})$$

the associator section. This equips the nested Hilbert scheme of points with a perfect obstruction theory. Similarly, we have $\text{NHilb}_{\text{nil-fil}}^{\underline{d}}(\mathbb{A}^n) = Z(s_{\text{ass}}|_{\text{naNHilb}_0^{\underline{d}}(\mathbb{A}^n)})$ which equips the nilpotently filtered nested Hilbert scheme of points with a perfect obstruction theory. All this admits natural T -equivariant structures for $T = \mathbb{G}_m^n$ the n -dimensional torus.

We then calculate the equivariant virtual tangent space $T_{\lambda_\bullet}^{\text{vir}}$ at fixed points corresponding to nested Young diagrams $\lambda_\bullet$. If we let $d = d_0 + \dots + d_r$ and define

$$(w(0), w(1), \dots, w(d-1)) = (\underbrace{0, 0, \dots, 0}_{d_0 \text{ times}}, \underbrace{1, 1, \dots, 1}_{d_1 \text{ times}}, \dots, \underbrace{r, r, \dots, r}_{d_r \text{ times}})$$

then for a nested Young diagram

$$\lambda_\bullet = (\lambda_1 \subset \dots \subset \lambda_{r+1})$$

we can choose vectors

$$\mathbf{u}_0, \dots, \mathbf{u}_{d-1} \in \mathbb{Z}_{\geq 0}^n$$

such that

$$\lambda_i = \{\mathbf{u}_k \mid w(k) \leq i-1\}.$$

For any vector $\mathbf{v} \in \mathbb{Z}_{\geq 0}^n$ let

$$t^{\mathbf{v}} = t_1^{v_1} \dots t_n^{v_n} \in \mathbb{Z}[t_1^\pm, \dots, t_n^\pm] \cong K_T^0(\text{pt})$$

be the corresponding K -class. The virtual tangent space is then given by

$$T_{\text{NHilb}^{\underline{d}}(\mathbb{A}^n), \lambda_\bullet}^{\text{vir}} = T_{\text{naNHilb}^{\underline{d}}(\mathbb{A}^n), \lambda_\bullet} - \mathcal{E}_{\text{ass}, \lambda_\bullet}.$$

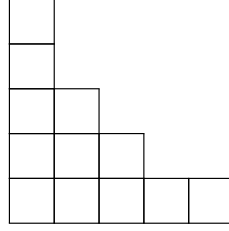
where $T_{\text{naNHilb}^d(\mathbb{A}^n), \lambda_\bullet}$ is given by

$$(t_1 + t_2 + \cdots + t_n) \sum_{k=0}^{d-1} t^{-\mathbf{u}_k} + \sum_{1 \leq i \leq j \leq d-1} \sum_{\substack{0 \leq k \leq d-1 \\ w(j) \leq w(k)}} t^{\mathbf{u}_i + \mathbf{u}_j - \mathbf{u}_k} - \sum_{j=1}^{d-1} \sum_{\substack{0 \leq k \leq d-1 \\ w(j) \leq w(k)}} t^{\mathbf{u}_j - \mathbf{u}_k}$$

and $\mathcal{E}_{\text{ass}, \lambda_\bullet}$ is given by

$$\sum_{\substack{1 \leq i, j, k \leq d-1 \\ i < k}} \sum_{\substack{0 \leq m \leq d-1 \\ w(k), w(j) \leq w(m)}} t^{\mathbf{u}_i + \mathbf{u}_j + \mathbf{u}_k - \mathbf{u}_m}.$$

We then prove that the fixed part $T_{\text{NHilb}^d(\mathbb{A}^n), \lambda_\bullet}^{\text{vir}, \text{fix}}$ has rank 0 if and only if $\lambda_\bullet \subset \mathbb{Z}_{\geq 0}^n$ is contained in the union of planes of the form



where we here use the French notation for Young diagrams. We call nested Young diagrams satisfying this *admissible*.

This gives us an explicit localization formula for the virtual fundamental class. Explicitly, if we let $s_i = c_1(t_i) \in A_T^*(\text{pt})$ and for $\mathbf{v} \in \mathbb{Z}^n$ let

$$\mathbf{v}(\underline{s}) = v_1 s_1 + \cdots + v_n s_n$$

then the virtual fundamental class $[\text{NHilb}^d(\mathbb{A}^n)]^{\text{vir}}$ can be expressed as

$$\sum_{\lambda_\bullet} [\lambda_\bullet] \cap \frac{\prod_{\substack{1 \leq i, j, k \leq d-1 \\ i < k}} \prod_{\substack{0 \leq m \leq d-1 \\ w(k), w(j) \leq w(m)}} (\mathbf{u}_i + \mathbf{u}_j + \mathbf{u}_k - \mathbf{u}_m)(\underline{s}) \prod_{j=1}^{d-1} \prod_{\substack{0 \leq k \leq d-1 \\ w(j) \leq w(k)}} \mathbf{u}_j - \mathbf{u}_k(\underline{s})}{\prod_{i=1}^n \prod_{k=0}^{d-1} (e_i - \mathbf{u}_k)(\underline{s}) \prod_{1 \leq i \leq j \leq d-1} \prod_{\substack{0 \leq k \leq d-1 \\ w(j) \leq w(k)}} (\mathbf{u}_i + \mathbf{u}_j - \mathbf{u}_k)(\underline{s})}.$$

where the sum is over all admissible nested partitions $\lambda_\bullet$ and where we for each one choose vectors $\mathbf{u}_0, \dots, \mathbf{u}_{d-1}$ as above. Here, we use the notation

$$\prod_{i \in I}^{\sim} a_i := \prod_{\substack{i \in I \\ a_i \neq 0}} a_i.$$

Next, we focus on the nilpotently filtered locus. We construct a partial resolution $\widetilde{\text{naNHilb}}_0^d(\mathbb{A}^n)$ equipped with maps

$$\begin{array}{ccc} \widetilde{\text{naNHilb}}_0^d(\mathbb{A}^n) & \xrightarrow{\mu} & \text{Flag}_{\hat{d}}(\mathbb{A}^n) \\ \downarrow \tau & & \\ \text{naNHilb}_0^d(\mathbb{A}^n) & & \end{array}$$

where $\underline{d} = (1, d_1, \dots, d_r)$, $\hat{d} = (d_1, \dots, d_r)$ and where we assume $n \geq d_1 + \cdots + d_r$. We prove that μ is smooth, and that τ is proper and birational. If we equip

$$\widetilde{\text{NHilb}}_{\text{nil-fil}}^d(\mathbb{A}^n) = \tau^{-1}(\text{NHilb}_{\text{nil-fil}}^d(\mathbb{A}^n)) = Z(\tau^* s_{\text{ass}})$$

with the corresponding perfect obstruction theory we then get

$$\tau_*[\widetilde{\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^d(\mathbb{A}^n)}]^{\mathrm{vir}} = [\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^d(\mathbb{A}^n)]^{\mathrm{vir}}.$$

We can further equip μ with a relative perfect obstruction theory compatible with the above perfect obstruction theory on the source and the smooth obstruction theory on the target. This allows us to rewrite the virtual class as a weighted sum of virtual classes on the fibers of the torus fixed points on the flag variety using localization. Inspecting the formula, we can then use this to rewrite tautological virtual integrals over $\widetilde{\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^d(\mathbb{A}^n)}$ (and equivalently $\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^d(\mathbb{A}^n)$ by the projection formula) as an iterated residue depending only on the virtual integral on a single fiber. We perform virtual localization on the fiber and prove that the residue vanishes at all points except the Porteous point corresponding to the nested partition $\lambda_\bullet$ where

$$\lambda_i = \{0, e_1, e_2, \dots, e_{d_0+\dots+d_{i-1}}\}$$

where e_k denotes the standard unit vectors. By embedding

$$\mathrm{naNHilb}^d(\mathbb{A}^n) \hookrightarrow \mathrm{naNHilb}^d(\mathbb{A}^N)$$

we show that this formula also works when $n \leq d_1 + \dots + d_r$. This gives the main result, which is essentially just a very efficient way of calculating our virtual tautological integrals on the nilpotently filtered locus.

Theorem 0.0.3. *Let $\underline{d} = (1, d_1, \dots, d_r) \in \mathbb{Z}_{\geq 0}^{r+1}$, $d = 1 + d_1 + \dots + d_r$, $n \in \mathbb{Z}_{\geq 0}$, and $T = (\mathbb{C}^*)^n$. Let $A_T^*(\mathrm{pt}) = \mathbb{Z}[s_1, \dots, s_n]$ where $s_1, \dots, s_n$ are the Chern roots of $\mathbb{C}^n$ given the standard torus action. For any $q \geq 0$ let W be a q -dimensional T -representation with Chern roots $\theta_1, \dots, \theta_q$ and consider the T -equivariant rank q bundle*

$$E = W \otimes \mathbb{A}^n$$

on $\mathbb{A}^n$. Let

$$-\eta_0, -\eta_1, \dots, -\eta_{d-1}$$

be the Chern roots of the tautological bundle $\mathcal{O}_{\mathbb{A}^n}^{[\underline{d}]}$ so that

$$E^{[\underline{d}]} \cong W \otimes \mathcal{O}_{\mathbb{A}^n}^{[\underline{d}]}$$

has Chern roots $\theta_i - \eta_j$. Let

$$P = P(\theta_1, \dots, \theta_q, \eta_1, \dots, \eta_{d-1})$$

be a polynomial in the Chern classes of $E^{[\underline{d}]}$. Then

$$\int_{[\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^d(\mathbb{A}^n)]^{\mathrm{vir}}} P$$

can be calculated via the iterated residue formula

$$\mathrm{Res}_{\underline{z}=\infty} \frac{P(\theta_1, \dots, \theta_q, \underline{z}) \prod_{1 \leq m \neq l \leq d-1} (z_m - z_l) \prod_{\substack{1 \leq i, j, k \leq d-1 \\ i < k}} \prod_{\substack{0 \leq m \leq d-1 \\ w(k), w(j) \leq w(m)}} (z_i + z_j + z_k - z_m) d\underline{z}}{\prod_{\substack{1 \leq l < m \leq d-1 \\ w(l) < w(m)}} (z_m - z_l) \prod_{i=1}^n \prod_{l=1}^{d-1} (s_i - z_l) \prod_{1 \leq i \leq j \leq d-1} \prod_{\substack{1 \leq k \leq d-1 \\ w(j) < w(k)}} (z_i + z_j - z_k)}$$

Here $\underline{z} = (z_1, \dots, z_{d-1})$ and the iterated residue is defined as the coefficient of $(z_1 \cdots z_{d-1})$ in the expansion of the rational expression in the domain $z_1 \ll \dots \ll z_{d-1}$.

By looking at the inclusion $\mathrm{naNHilb}_0^d(\mathbb{A}^n) \hookrightarrow \mathrm{naNHilb}^d(\mathbb{A}^n)$ we are able to express certain integrals on the nested Hilbert scheme of points (defined in the formal sense via localization) as integrals on the nilpotently filtered locus, so that these also enjoy the above residue formula. For $\underline{d} = (1, \dots, 1)$ the nilpotently filtered nested Hilbert scheme of points is equal to the punctual Hilbert scheme which contains all fixed points, making this possible for any integral. For general $\underline{d}$ we need to restrict to integrals of Chow cohomology classes supported on the nilpotently filtered locus. The result is the following.

Proposition 0.0.1. *Let $\alpha \in A_{\mathrm{NHilb}_{\mathrm{nil-fil}}^d(\mathbb{A}^n), T}^*(\mathrm{NHilb}^d(\mathbb{A}^n))$ be a local T -equivariant Chow cohomology class supported on the nilpotently filtered locus. Then*

$$\int_{[\mathrm{NHilb}^d(\mathbb{A}^n)]^{\mathrm{vir}}} \alpha = \int_{[\mathrm{NHilb}_{\mathrm{nil-fil}}^d(\mathbb{A}^n)]^{\mathrm{vir}}} \frac{\alpha|_{\mathrm{NHilb}_{\mathrm{nil-fil}}^d(\mathbb{A}^n)}}{e^T(\mathcal{E}_{\mathrm{punct}}|_{\mathrm{NHilb}_{\mathrm{nil-fil}}^d(\mathbb{A}^n)})}.$$

If $\underline{d} = (1, 1, \dots, 1)$ then

$$j_* \left(\frac{[\mathrm{NHilb}_0^{r+1}(\mathbb{A}^n)]^{\mathrm{vir}}}{e^T(\mathcal{E}_{\mathrm{punct}}|_{\mathrm{NHilb}_0^{r+1}(\mathbb{A}^n)})} \right) = [\mathrm{NHilb}^{r+1}(\mathbb{A}^n)]^{\mathrm{vir}}$$

and in particular for any $\alpha \in A_T^*(\mathrm{NHilb}^{r+1}(\mathbb{A}^n))$ we get

$$\int_{[\mathrm{NHilb}^{r+1}(\mathbb{A}^n)]^{\mathrm{vir}}} \alpha = \int_{[\mathrm{NHilb}_0^{r+1}(\mathbb{A}^n)]^{\mathrm{vir}}} \frac{\alpha|_{\mathrm{NHilb}_0^{r+1}(\mathbb{A}^n)}}{e^T(\mathcal{E}_{\mathrm{punct}}|_{\mathrm{NHilb}_0^{r+1}(\mathbb{A}^n)})}.$$

Future work

Other virtual classes from the nested non-commutative Hilbert and Quot scheme

It would be interesting to see if the construction of the non-commutative nested Hilbert scheme could be used to define other perfect obstruction theories of interest.

The nested Quot scheme

Recall that the rank r Quot scheme of d points $\mathrm{Quot}_r^d(X)$ is a generalization of the Hilbert scheme of points, parameterizing zero-dimensional length d quotients of the sheaf $\mathcal{O}_X^r$. The critical locus description of the Hilbert scheme of d points on $\mathbb{A}^3$ in the non-commutative Hilbert scheme generalizes almost directly to the Quot scheme of points [BR21]. The construction of a smooth non-commutative nested Quot scheme can be done exactly in the same way as for the non-commutative Hilbert scheme, by just keeping track of multiple cyclic vectors. It would be interesting to see if one could use this to provide a perfect obstruction theory on the nested Quot scheme. This would require a careful analysis of how to reduce the rank of the naïve commutativity bundle as lemma 1.5.1 is a priori special to the case of one cyclic vector.

The only perfect obstruction theories of the nested Quot scheme (sometimes called the hyperquot scheme) that the author is aware of is for the nested Quot scheme on a smooth curve [MR24]. For general nested Quot schemes of quotients of a perfect sheaf on a projective variety, an obstruction theory (not necessarily perfect) is given in [MPR26] via the construction of a derived enhancement of the nested Quot scheme. They prove that for the Quot scheme of quotients of a locally free sheaf on a smooth curve, the K -theory of the obstruction theory agrees with that of the perfect obstruction theory of [MR24].

Note that exhibiting a scheme as the zero locus of a section of a bundle on a smooth scheme canonically equips it with a quasi-smooth derived enhancement via the Koszul complex viewed as a commutative differential graded algebra. It would therefore be very interesting to see if one could use the non-commutative nested Quot scheme to reconstruct the perfect obstruction theory of [MR24] in a way so that the associated quasi-smooth derived enhancement agrees with the derived enhancement of [MPR26].

The nested Hilbert scheme of points on a threefold

It would similarly be interesting to see whether one could reduce the rank of the naïve bundle of the commutator section on the non-commutative nested Hilbert scheme of $\mathbb{A}^3$ to produce a perfect obstruction theory of dimension zero. Recall that in the non-nested case we have a critical locus description given by

$$\mathrm{Hilb}^d(\mathbb{A}^3) = Z(df) \subset \mathrm{ncHilb}^d(\mathbb{A}^3)$$

where $f([A_1, A_2, A_3, v]) = \mathrm{Tr}(A_1[A_2, A_3])$. This equips $\mathrm{Hilb}^d(\mathbb{A}^3)$ with a perfect symmetric obstruction theory which in particular has virtual dimension zero. This means that we should be able to reduce the number of equations locally from $3d^2$ to

$$2d^2 + d = \dim \mathrm{ncNHilb}^d(\mathbb{A}^3).$$

This can be done globally and rather explicitly. Indeed, consider the commutator section

$$([\Phi_2, \Phi_3], [\Phi_3, \Phi_1], [\Phi_1, \Phi_2]) \in \mathcal{E}nd(\mathcal{V})^{\oplus 3}$$

and note that it lies in the kernel of the map of bundles

$$\begin{aligned} \mathcal{E}nd(\mathcal{V})^{\oplus 3} &\longrightarrow \mathcal{H}om(\mathcal{V}, \overline{\mathcal{V}}) \\ (\phi_1, \phi_2, \phi_3) &\longmapsto q([\Phi_1, \phi_1] + [\Phi_2, \phi_2] + [\Phi_3, \phi_3]) \end{aligned}$$

by the Jacobi identity. Here $q : \mathcal{V} \rightarrow \overline{\mathcal{V}}$ is the quotient of the universal cyclic vector. We use cyclicity to show that the above map is in fact an epimorphism, thus proving that the commutator section factors through a subbundle of rank

$$3d^2 - d(d-1) = 2d^2 + d$$

as desired. Taking the trace-dual this is equivalent to showing that the map

$$\begin{aligned} \mathcal{H}om(\overline{\mathcal{V}}, \mathcal{V}) &\longrightarrow \mathcal{E}nd(\mathcal{V})^{\oplus 3} \\ \phi &\longmapsto ([\Phi_1, \phi q], [\Phi_2, \phi q], [\Phi_3, \phi q]) \end{aligned}$$

is a monomorphism. But if ϕ is in the kernel of this map, then ϕq commutes with Φ_1, Φ_2, Φ_3 and disappears on the cyclic vector. We must then have $\phi q = 0$ and so $\phi = 0$ since q is an epimorphism. Both descriptions fail when we add a nesting. For example in the case $\underline{d} = (1, \dots, 1)$ we have that $A_1[A_2, A_3]$ is strictly lower triangular so that for the similarly defined potential we get

$$f : (A_1, A_2, A_3, v) \mapsto \mathrm{Tr} A_1[A_2, A_3]$$

it holds that $f \equiv 0$. Similarly, the above proof cannot be used to show that

$$\mathcal{E}nd^{\mathrm{fil}}(\mathcal{V}_\bullet)^{\oplus 3} \rightarrow \mathcal{H}om^{\mathrm{fil}}(\mathcal{V}_\bullet, \overline{\mathcal{V}}_\bullet)$$

is an epimorphism, as the filtration preserving maps are not self-dual under the trace pairing. In fact, it is clear that this map cannot be an epimorphism. For example, in the case $\underline{d} = (1, \dots, 1)$ we have that $[\Phi_i, \phi]$ is strictly lower triangular and so the map at least fails to be surjective in the locus where Φ_1, Φ_2, Φ_3 are linearly dependent.

It would therefore require new ideas adapted to the nested case to see whether a reduction of the rank is possible.

Relating the non-associative invariants to other known invariants

As explained in the introduction to chapter 2 we would like to relate the virtual invariants constructed from the non-associative Hilbert scheme to other known virtual invariants. Our plan is to use the rational map

$$\begin{aligned} \mathrm{naNHilb}^d(\mathbb{A}^n) &\dashrightarrow \mathrm{ncNHilb}^d(\mathbb{A}^n) \\ (V_\bullet, \psi_1, \psi_2, v) &\longmapsto (V_\bullet, \psi_2(\psi_1(x_1), -), \dots, \psi_2(\psi_1(x_n), -), v) \end{aligned}$$

along with the technical lemma 1.3.1 to provide a relative perfect obstruction theory on the Hilbert scheme of points. In the case of $n = 2$ we have a map between the associativity bundle $\mathcal{E}_{\mathrm{ass}}$ and the commutativity bundle $\mathcal{E}_{\mathrm{com}}$ as needed in lemma 1.3.1. It is explicitly given by

$$\begin{aligned} \xi : \mathcal{E}_{\mathrm{ass}} = \mathcal{H}om^{\mathrm{fil}, \mathrm{out-alt}}(\overline{\mathcal{V}}_\bullet \otimes \overline{\mathcal{V}}_\bullet \otimes \overline{\mathcal{V}}_\bullet, \mathcal{V}_\bullet) &\longrightarrow \mathcal{H}om^{\mathrm{fil}}(\mathcal{V}_\bullet, \overline{\mathcal{V}}_\bullet) = f^* \mathcal{E}_{\mathrm{com}} \\ \kappa &\longmapsto (w \mapsto \kappa(y_1, w, y_2)) \end{aligned}$$

where $y_i := \psi_1(x_i)$. We see that $\xi(s_{\mathrm{ass}}) = f^* s_{\mathrm{com}}$. This follows simply because for

$$s_{\mathrm{ass}}(a_1, a_2, a_3) = \Psi_2(a_1, \Psi_2(a_2, a_3)) - \Psi_2(\Psi_2(a_1, a_2), a_3)$$

we get

$$\begin{aligned} \xi(s_{\mathrm{ass}})(a) &= \Psi_2(y_1, \Psi_2(a, y_2)) - \Psi_2(\Psi_2(y_1, a), y_2) \\ &= \Psi_2(y_1, \Psi_2(y_2, a)) - \Psi_2(y_2, \Psi_2(y_1, a)) \\ &= f^*(\Phi_1 \Phi_2(a) - \Phi_2 \Phi_1(a)) \\ &= f^*(s_{\mathrm{com}})(a) \end{aligned}$$

where we have used that $f^* \Phi_i(-) = \Psi_2(y_i, -)$ by definition. It however fails the transversality condition of lemma 1.3.1. Indeed, the map ξ vanishes on the locus where the generators y_1, y_2 are linearly dependent, simply because the domain of ξ are maps which are alternating in the outer coordinates. The differential of s_{com} can be identified with

$$\begin{aligned} \mathcal{E}nd^{\mathrm{fil}}(\mathcal{V}_\bullet)^{\oplus 2} \oplus \mathcal{V}_0 &\longrightarrow \mathcal{H}om^{\mathrm{fil}}(\mathcal{V}_\bullet, \overline{\mathcal{V}}_\bullet) \\ (\phi_1, \phi_2, w) &\longmapsto [\phi_1, \Phi_2] + [\Phi_1, \phi_2] \end{aligned}$$

and on the locus where y_1, y_2 are linearly dependent we have that $f^* \Phi_1, f^* \Phi_2$ are linearly dependent. It follows that on this locus the image of $f^* ds_{\mathrm{com}}$ is equivalent to the image of

$$\begin{aligned} \mathcal{E}nd^{\mathrm{fil}}(\mathcal{V}_\bullet) &\longrightarrow \mathcal{H}om^{\mathrm{fil}}(\mathcal{V}_\bullet, \overline{\mathcal{V}}_\bullet) \\ \phi &\longmapsto [\Phi_1, \phi] \end{aligned}$$

but this is not surjective in general. For example when $\underline{d} = (1, \dots, 1)$ then $[\Phi_1, \phi]$ will be strictly lower triangular and therefore cannot hit all lower triangular maps. We conclude that surjectivity of

$$(\xi, -f^* ds_{\mathrm{com}})$$

fails so that the transversality requirement of lemma 1.3.1 fails. We hope to be able to modify this map to something that satisfies this requirement.

Chapter 1

Virtual K-theoretic invariants of the nested Hilbert scheme on $\mathbb{C}^2$

Abstract

We construct a nested version of the non-commutative Hilbert scheme and embed the nested Hilbert scheme of points on $\mathbb{C}^n$ as the commutativity locus. In the $\mathbb{C}^2$ -case, we exhibit this locus as the zero locus of two different sections of bundles and use this description to equip the nested Hilbert scheme of points with a perfect obstruction theory equivalent to that of Gholampour, Sheshmani and Yau. We study the torus equivariant pushforward of the virtual structure sheaf under the map of nested Hilbert schemes forgetting the largest subscheme of the nesting. Using a map of the bundles on the non-commutative Hilbert scheme, we prove that this pushforward is a twist of the virtual structure sheaf on the lower level. Using localization, we show that the twist is by a constant class with values corresponding to the equivariant Euler characteristic of a tautological class of the Hilbert scheme of points. From this, we derive a closed formula for the multivariate generating series of the equivariant virtual Euler characteristic of the nested Hilbert scheme of points.

1.1 Introduction

In the theory of virtual classes on the Hilbert scheme of points, the non-commutative Hilbert scheme introduced by Nori [Nor78] has played a central role. The description of $\text{Hilb}^d(\mathbb{A}^3)$ as a critical locus of a potential $\text{ncHilb}^d(\mathbb{A}^3)$ was used by Behrend, Bryan and Szendrői [BBS13] in the study of motivic Donaldson-Thomas invariants. Okounkov [Oko17] used the same description to prove the Nekrasov formula for the generating series of the twisted virtual Euler characteristic on $\text{Hilb}^d(\mathbb{A}^3)$.

Theorem 1.1.1. ([Oko17]).

$$\sum_{d \geq 0} \chi_T(\text{Hilb}^d(\mathbb{A}^3), \hat{\mathcal{O}}^{\text{vir}})(-q)^d = \text{PE} \left(\frac{[t_1 t_2][t_1 t_3][t_2 t_3]}{[\mathfrak{t}^{\frac{1}{2}} q][\mathfrak{t}^{\frac{1}{2}} q^{-1}][t_1][t_2][t_3]} \right)$$

where $[x] = x^{1/2} - x^{-1/2}$, $\mathfrak{t} = t_1 t_2 t_3$ and

$$\text{PE}(f(q; t_1, t_2, t_3)) = \exp \left(\sum_{n \geq 1} \frac{f(q^n; t_1^n, t_2^n, t_3^n)}{n} \right).$$

Though this critical locus description is special to dimension 3, it is not special to the rank 1 case; Beentjes–Ricolfi [BR21] proves that the Quot scheme $\text{Quot}_r^d(\mathbb{A}^3)$ is the critical

locus of a potential on a non-commutative Quot scheme $\mathrm{ncQuot}_r^d(\mathbb{A}^3)$. Using this description, Fasola, Monavari and Ricolfi [FMR21] and Arbesfeld and Kononov [AK] proved the formula for the generating series of the twisted virtual Euler characteristic on $\mathrm{Quot}_r^d(\mathbb{A}^3)$ conjectured by Awata and Kanno [AK09].

Theorem 1.1.2. ([FMR21], [AK]).

$$\sum_{d \geq 0} \chi_T(\mathrm{Quot}_r^d(\mathbb{A}^3), \hat{\mathcal{O}}^{\mathrm{vir}})((-1)^r q)^d = \mathrm{PE} \left(\frac{[t^r][t_1 t_2][t_1 t_3][t_2 t_3]}{[t][t^{\frac{1}{2}} q][t^{\frac{1}{2}} q^{-1}][t_1][t_2][t_3]} \right).$$

In the $\mathbb{A}^4$ -case Kool and Rennemo realize $\mathrm{Quot}_r^d(\mathbb{A}^4)$ as the zero locus of an isotropic section of an orthogonal bundle on $\mathrm{ncQuot}_r^d(\mathbb{A}^4)$ setting it in the "standard model" version of the invariants defined by Oh and Thomas [OT23]. They then consider the virtual Euler characteristic of the twisted structure sheaf tensored with a specific bundle. They identify the generating function with the instanton partition function $\mathbf{Z}_r^{\mathrm{NP}}$ of Nekrasov and Piazzalunga [NP19] and prove their conjectured formula.

Theorem 1.1.3. ([KR].)

$$\mathbf{Z}_r^{\mathrm{NP}} = \mathrm{PE} \left(\frac{[t_1 t_2][t_1 t_3][t_2 t_3][y]}{[y^{\frac{1}{2}} q][y^{\frac{1}{2}} q^{-1}][t_1][t_2][t_3][t_4]} \right)$$

where $y = y_1 \cdots y_r$ and $t_1 t_2 t_3 t_4 = 1$.

In this paper we continue the scheme of calculating K -theoretic virtual invariants using a non-commutative version of the Hilbert scheme, applying it to the nested Hilbert scheme of points on $\mathbb{A}^2$. For a smooth projective surface S , integers $n_0 \geq n_2 \geq \cdots \geq n_N \geq 0$ and cohomology classes $\beta_0, \dots, \beta_{N-1} \in H^2(S, \mathbb{Z})$ Gholampour, Sheshmani and Yau [GSY20] define perfect obstruction theory on the nested Hilbert scheme $S_\beta^{[\mathbf{n}]}$ parameterizing

$$(Z_0, \dots, Z_N), (C_0, \dots, C_{r-1})$$

where Z_i is a zero-dimensional subscheme of length n_i and C_i is a divisor with $[C_i] = \beta_i$ such that

$$I_{Z_i}(-C_i) \subset I_{Z_{i+1}}$$

for all $i = 0, \dots, N-1$. This includes the case of the nested Hilbert scheme of points for $\beta_0 = \beta_2 = \cdots = \beta_{N-1} = 0$. Gholampour and Thomas [GT20a] exhibit the nested Hilbert scheme of points on S as a degeneracy locus of $S^{[n_0]} \times \cdots S^{[n_N]}$. From this, they equip it with a perfect obstruction theory equivalent to that of [GSY20]. They further use a Thom-Porteous formula to express the pushforward of the virtual fundamental class to $S^{[n_0]} \times \cdots S^{[n_N]}$ in terms of Carlsson-Okounkov operators in the sense of [CO12]. For integers $r \geq 1$ and $n_0 \geq n_2 \geq \cdots \geq n_N \geq n_{N+1} = 0$ Bonelli, Fasola and Tanzini [BFT24] define the moduli space $\mathcal{N}(r, \underline{n})$ of stable representations of the nested instantons quiver of numerical type $(r, \mathbf{n})$. They prove that $\mathcal{N}(1, \mathbf{n}) \cong (\mathbb{A}^2)^{[\hat{\mathbf{n}}]}$ where $\hat{n}_i = n_0 - n_{N+1-i}$. They further equip $\mathcal{N}(r, \underline{n})$ with a perfect obstruction theory and prove for $r = 1$ that this is equivalent to that of [GSY20].

In this paper, we consider the nested Hilbert scheme of points $\mathrm{NHilb}^{(d_0, \dots, d_N)}(\mathbb{A}^2) = S^{[\hat{\mathbf{n}}]} \cong \mathcal{N}(1, \mathbf{n})$ where $d_i = n_i - n_{i+1}$. We exhibit it as the zero locus of a section of a bundle on a non-commutative version $\mathrm{ncNHilb}^{(d_0, \dots, d_N)}(\mathbb{A}^2)$ and prove that the induced perfect obstruction theory agrees with that of [GSY20], [GT20a] and [BFT24]. We will use this to compute pushforwards of virtual structure sheaves allowing us to prove the following formula for the generating series.

Theorem 1.1.4.

$$\sum_{d_0, \dots, d_N \geq 0} q_0^{d_0} \cdots q_N^{d_N} \chi_T(\mathrm{NHilb}^{(d_0, \dots, d_N)}(\mathbb{A}^2), \mathcal{O}^{\mathrm{vir}}) = \mathrm{PE} \left(\frac{q_0 + (q_1 + \cdots + q_N)(1 - \mathfrak{t})}{(1 - t_1)(1 - t_2)} \right)$$

where $\mathfrak{t} = t_1 t_2$.

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1.2 The results

For any $r \geq 0$ and any $\underline{d} = (d_0, \dots, d_r) \in \mathbb{Z}_{\geq 0}^r$ we construct the nested non-commutative Hilbert scheme

$$\mathrm{ncNHilb}^{\underline{d}}(\mathbb{A}^2) = \{(N_{\bullet}, A_1, \dots, A_n, v) \mid A_i \in \mathrm{End}_{\mathbb{C}}^{\mathrm{fil}}(N_{\bullet}), v \in N_0, \mathbb{C}\langle A_1, \dots, A_n \rangle v = N_0\} / \sim$$

which parameterizes flag-preserving operators $A_i : N_{\bullet} \rightarrow N_{\bullet}$ with a cyclic vector $v \in N_{\bullet}$ for a flag of vector spaces

$$N_{\bullet} = N_0 \supset \cdots \supset N_{r+1} = 0$$

with $\dim_{\mathbb{C}} N_i / N_{i+1} = d_i$. We construct an embedding from the nested Hilbert scheme of points on $\mathbb{C}^n$

$$\begin{aligned} \mathrm{NHilb}^{\underline{d}}(\mathbb{A}^n) &= \{Z_1 \subset \cdots \subset Z_{r+1} \subset \mathbb{C}^n \mid \dim Z_i = 0, \text{length}(Z_i) = d_0 + \cdots + d_{i-1}\} \\ &= \{I_{r+1} \subset \cdots \subset I_1 \subset I_0 = \mathbb{C}[x_1, \dots, x_n] \mid \dim_{\mathbb{C}} I_i / I_{i+1} = d_i\} \end{aligned}$$

which maps a nesting of ideals to the flag of vector spaces $N_{\bullet}$ with $N_i = I_i / I_{i+1}$ with operators given by left multiplication with x_j and the cyclic vector given by the image of the unit $\bar{1} \in I_0 / I_{r+1}$. We identify the image of this embedding with the locus of $\mathrm{ncNHilb}^{\underline{d}}(\mathbb{A}^2)$ where the operators $A_1, \dots, A_n$ commute. In the case where $n = 2$ this locus is given by the vanishing of

$$[A_1, A_2] \in \mathrm{End}_{\mathbb{C}}^{\mathrm{fil}}(N_{\bullet})$$

and we prove using cyclicity that the vanishing of

$$\overline{[A_1, A_2]} \in \mathrm{Hom}_{\mathbb{C}}^{\mathrm{fil}}(N_{\bullet}, N_{\bullet} / \langle v \rangle)$$

is sufficient. This allows us to exhibit the nested Hilbert scheme of points on $\mathbb{C}^2$ as the zero locus of a section of two different tautological bundles, thus equipping it with perfect obstruction theories. For $d_0 \neq 0$ we prove that the second bundle respectively first bundle gives a perfect obstruction theory with the K -theory of the virtual tangent bundle equal to the virtual tangent bundles of

$$\mathrm{NHilb}^{(d_0, \dots, d_r)}(\mathbb{A}^2), \text{ respectively } \mathrm{NHilb}^{(0, d_0, \dots, d_r)}(\mathbb{A}^2)$$

equipped with the perfect obstruction theories of Gholampour, Sheshmani and Yau [GSY20]. In particular, the virtual invariants agree with that of [GSY20]. We show that the morphism

$$p : \mathrm{NHilb}^{(d_0, \dots, d_r)}(\mathbb{A}^2) \rightarrow \mathrm{NHilb}^{(d_0, \dots, d_{r-1})}(\mathbb{A}^2)$$

extends to the non-commutative nested Hilbert scheme. Letting $\mathcal{E}_{\underline{d}}$ be one of the bundles above we construct a surjection

$$\mathcal{E}_{(d_0, \dots, d_r)} \twoheadrightarrow p^* \mathcal{E}_{(d_0, \dots, d_{r-1})}$$

which in turn gives us a factorization

$$\mathcal{O}_{(d_0, \dots, d_r)}^{\text{vir}} = p_*(\mathcal{F}) \cdot \mathcal{O}_{(d_0, \dots, d_{r-1})}^{\text{vir}}$$

in K -theory for a certain explicit tautological class $\mathcal{F}$. Using localization, we then calculate $p_* \mathcal{F}$ and show that it is a constant class depending only on d_r . Explicitly, we have the following main result.

Theorem 1.2.1. *For $r \in \mathbb{Z}_{\geq 0}$ let $\underline{d} = (d_0, \dots, d_r) \in \mathbb{Z}_{\geq 0}^{r+1}$ be a dimension vector and let $\hat{\underline{d}} = (d_0, \dots, d_{r-1}) \in \mathbb{Z}_{\geq 0}^r$. Let*

$$p_{\underline{d}, \hat{\underline{d}}} : \text{NHilb}^{\underline{d}}(\mathbb{A}^2) \rightarrow \text{NHilb}^{\hat{\underline{d}}}(\mathbb{A}^2)$$

and

$$q_{\hat{\underline{d}}} : \text{NHilb}^{\hat{\underline{d}}}(\mathbb{A}^2) \rightarrow \text{pt}$$

be the projections. Let $\mathcal{O}_{\underline{d}}^{\text{vir}}, \mathcal{O}_{\hat{\underline{d}}}^{\text{vir}}$ be the virtual structure sheaves from the perfect obstruction theory of definition 1.5.1 or equivalently [GSY20]. We have an identity

$$(p_{\underline{d}, \hat{\underline{d}}})_* \mathcal{O}_{\underline{d}}^{\text{vir}} = (q_{\hat{\underline{d}}}^* \alpha_{d_r}) \cdot \mathcal{O}_{\hat{\underline{d}}}^{\text{vir}}$$

where

$$\alpha_{d_r} = \chi_T(\text{NHilb}^{d_r}(\mathbb{A}^2), \wedge_{-1}^{\bullet}(\omega_{\mathbb{A}^2}^{[d_r]}))$$

for $\omega_{\mathbb{A}^2} = \Omega_{\mathbb{C}^2}^2$ the dualizing sheaf on $\mathbb{C}^2$ and $\omega_{\mathbb{A}^2}^{[d_r]}$ the corresponding tautological bundle.

Note that in the non-nested case we have $\mathcal{O}_{(d)}^{\text{vir}} = \mathcal{O}_{\text{Hilb}^d(\mathbb{A}^n)}$, i.e. the virtual structure sheaf is the structure sheaf. Using this and the known closed formulas for the T -equivariant Euler characteristic of $\mathcal{O}_{\text{Hilb}^d(\mathbb{A}^n)}$ and $\wedge_{-1}^{\bullet}(\omega_{\mathbb{A}^2}^{[d]})$ the above then immediately implies a closed formula for the multivariate generating series of the equivariant virtual holomorphic Euler characteristic of the nested Hilbert scheme of points on $\mathbb{C}^2$.

Corollary 1.2.1. *For any $r \geq 0$ let*

$$\mathbf{Z}_{\text{GSY}}(q_0, \dots, q_r; t_1, t_2) := \sum_{d_0, \dots, d_r} q_0^{d_0} \cdots q_r^{d_r} \chi_T(\text{NHilb}^{(d_0, \dots, d_r)}(\mathbb{A}^2), \mathcal{O}_{(d_0, \dots, d_r)}^{\text{vir}})$$

be the generating function for the T -equivariant virtual holomorphic Euler characteristic of the nested Hilbert scheme with perfect obstruction theory equivalent to that of Gholampour, Sheshmani and Yau [GSY20]. Then

$$Z_{\text{GSY}}(q_0, \dots, q_r; t_1, t_2) = \text{PE} \left(\frac{q_0 + (q_1 + \cdots + q_r)(1 - t_1 t_2)}{(1 - t_1)(1 - t_2)} \right).$$

Proof. Set

$$\beta_{d_0} = \chi_T(\text{Hilb}^{d_0}(\mathbb{A}^2), \mathcal{O}^{\text{vir}}) = \chi_T(\text{Hilb}^{d_0}(\mathbb{A}^2), \mathcal{O}).$$

The above then implies that

$$\chi_T(\text{NHilb}^{(d_0, \dots, d_r)}(\mathbb{A}^2), \mathcal{O}_{(d_0, \dots, d_r)}^{\text{vir}}) = \beta_{d_0} \alpha_{d_1} \alpha_{d_2} \cdots \alpha_{d_r}.$$

We note that the classic known formulas show that

$$\alpha_{d_i} = [x^{d_i}] \text{PE} \left(\frac{x(1-t_1t_2)}{(1-t_1)(1-t_2)} \right)$$

while

$$\beta_{d_1} = [x^{d_1}] \text{PE} \left(\frac{x}{(1-t_1)(1-t_2)} \right).$$

To see this we note that $\omega_{\mathbb{A}^2} = t_1t_2\mathcal{O}_{\mathbb{A}^2}$. The results then follow from [WZ14, Theorem 3.2] with respectively $u = 1, v = 0$ and $A = (1, 1)$ and $u = v = 0$ and $A = (0, 0)$. We now conclude

$$\begin{aligned} Z_{\text{GSY}}(q_0, \dots, q_r; t_1, t_2) &= \left(\sum_{d_0 \geq 0} q_0^{d_0} \beta_{d_0} \right) \left(\sum_{d_1 \geq 0} q_1^{d_1} \alpha_{d_1} \right) \cdots \left(\sum_{d_r \geq 0} q_r^{d_r} \alpha_{d_r} \right) \\ &= \text{PE} \left(\frac{q_0}{(1-t_1)(1-t_2)} \right) \prod_{i=1}^r \text{PE} \left(\frac{q_i(1-t_1t_2)}{(1-t_1)(1-t_2)} \right) \\ &= \text{PE} \left(\frac{q_0 + (q_1 + \cdots + q_r)(1-t_1t_2)}{(1-t_1)(1-t_2)} \right). \end{aligned}$$

□

1.3 Preliminaries

1.3.1 Notation and conventions

For an algebraic group G and a G -scheme X we will denote by $K_0^G(X)$ the K -theory of G -equivariant coherent sheaves and by $K_G^0(X)$ the K -theory of G -equivariant locally free sheaves.

We will use the convention that $T = \mathbb{G}_m^n$ acts on $\mathbb{A}^n$ by

$$(t_1, \dots, t_n) \cdot (p_1, \dots, p_n) = (t_1^{-1}p_1, \dots, t_n^{-1}p_n)$$

so that $\mathcal{O}_{\mathbb{A}^n} = \mathbb{C}[x_1, \dots, x_n]$ is the T -representation where $t.x_i = t_i x_i$.

We will keep track of the number of points of the nested Hilbert scheme of points by keeping track of increments. The subschemes will be indexed as ascending chains starting at 1, while the increments will be indexed starting at 0. In other words, the nested Hilbert scheme $\text{NHilb}^{(d_0, \dots, d_r)}(\mathbb{A}^n)$ of $(d_0, \dots, d_r)$ points on $\mathbb{A}^n$ will parameterize finite subschemes

$$Z_{r+1} \subset \cdots \subset Z_1 \subset \mathbb{A}^n$$

such that $\text{length } Z_1 = d_0$ and

$$\text{length } Z_{i+1} - \text{length } Z_i = d_i.$$

If we let I_i be the ideal of Z_i with $I_0 = \mathcal{O}_{\mathbb{A}^n}$ we are equivalently parameterizing

$$I_{r+1} \subset \cdots \subset I_1 \subset I_0$$

such that $\dim_{\mathbb{C}} I_i / I_{i+1} = d_i$.

1.3.2 Perfect obstruction theories and zero loci

Recall that for a scheme X a perfect obstruction theory as defined in [BF97] is a map

$$\phi : \mathbb{E} \rightarrow \mathbb{L}_X$$

in the derived category $D^{[-1,0]}(X)$ such that $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective. Here, $\mathbb{L}_X = \tau_{-1 \geq} L_X^\bullet$ is the truncation of the cotangent complex of X as defined in [Ill71]. If

$$\mathbb{E} = [E^{-1} \rightarrow E^0]$$

then the perfect obstruction theory gives rise to a cone

$$\mathfrak{C} \hookrightarrow E_1 = (E^{-1})^\vee$$

which in turn gives rise to the *virtual fundamental class*

$$[X]^{\text{vir}} = i^*[\mathfrak{C}] \in A_{\text{vd}}(X)$$

where $i : X \rightarrow E_1$ is the zero section and i^* its Gysin map. Here,

$$\text{vd} = \text{rk } \mathbb{E} = \text{rk } E^0 - \text{rk } E^{-1}$$

is the virtual dimension and $A_*(X)$ is the Chow group of X . Similarly, we can define a *virtual structure sheaf*

$$[Li^* \mathcal{O}_{\mathfrak{C}}] \in K_0(X)$$

for $K_0(X)$ the K -theory of coherent sheaves on X . Suppose X is proper. For any Chow cohomology class $\alpha \in A^*(X)$ and any class $V \in K^0(X)$ in the K -theory of locally free sheaves we may define virtual invariants by

$$\int_{[X]^{\text{vir}}} \alpha \in A_*(\text{pt}) \cong \mathbb{Z}$$

and

$$\chi^{\text{vir}}(X, V) := \chi(X, V \otimes_X \mathcal{O}_X^{\text{vir}}) \in K_0(\text{pt}) \cong \mathbb{Z}.$$

By [FG10] these are related by the virtual Hirzebruch-Riemann-Roch formula

$$\chi^{\text{vir}}(X, V) = \int_{[X]^{\text{vir}}} \text{ch}(V) \cdot \text{Td}(T_X^{\text{vir}})$$

where

$$T_X^{\text{vir}} = [\mathbb{E}^\vee] = [E_0] - [E_1] \in K^0(X)$$

is the virtual tangent bundle. Both the virtual fundamental class (see [Sie04, Theorem 4.6]) and the virtual structure sheaf (see [Tho22, Corollary 4.5]) only depend on the K -theory class $[\mathbb{E}]$.

Example 1.3.1. We are primarily interested in the following simpler situation. Suppose that there exists a smooth scheme Y with a locally free sheaf E equipped with a section $s \in H^0(Y, E)$ such that

$$X = Z(S) \subset Y.$$

Let $I \subset \mathcal{O}_Y$ be the ideal sheaf of X in Y . Then the truncated cotangent bundle of X can be expressed as

$$\mathbb{L}_X = \left[I/I^2 \xrightarrow{d} \Omega_Y|_X \right].$$

The section defines a map

$$s^\vee : E^\vee \rightarrow \mathcal{O}_Y$$

with image I so that

$$\begin{array}{ccc} \mathbb{E} & = & [E^\vee|_X \xrightarrow{dos^\vee} \Omega_Y|_X] \\ & \downarrow s^\vee & \parallel \\ \mathbb{L}_X & = & [I/I^2 \xrightarrow{d} \Omega_Y|_X] \end{array}$$

defines a perfect obstruction theory. In this case the virtual invariants are well known. The virtual fundamental class is simply Fultons refined Euler class [Ful98, Section 14.1], that is

$$[X]^{\text{vir}} = 0^!([Y])$$

where $0 : Y \rightarrow E$ is the zero section and $0^!$ is the refined Gysin map. For the virtual structure sheaf we see that the K -theory class of the Koszul complex $\text{Kos}^\bullet(E^\vee, s^\vee)$ associated to $(E^\vee, s^\vee)$

$$\wedge_{-1}^\bullet[E^\vee] = \sum_{i \geq 0} (-1)^i [\wedge^i E^\vee] = \sum_{i \geq 0} (-1)^i [\mathcal{H}^i(\text{Kos}(E^\vee, s^\vee))] \in K_0(Y)$$

is supported on X and defines the virtual structure sheaf $\mathcal{O}_X^{\text{vir}} \in K_0(X)$ (this is explained in [Oko17, Section 3.2]).

1.3.3 Torus actions and localization

We fix a split torus $T = \mathbb{G}_m^n$. We let $t_1, \dots, t_n \in K_T^0(\text{pt})$ be the classes corresponding to the one-dimensional representations of the characters given by projection to the i 'th coordinate. Let $s_i = c_1(t_i)$. Then

$$K_0^T(\text{pt}) \cong K_T^0(\text{pt}) \cong \mathbb{Z}[t_1^\pm, \dots, t_n^\pm]$$

and

$$A_*^T(\text{pt}) \cong A_T^*(\text{pt}) \cong \mathbb{Z}[s_1, \dots, s_n].$$

Let X be a scheme with a T -action. A T -equivariant perfect obstruction theory is a map

$$\mathbb{E} \rightarrow \mathbb{L}_X$$

in $D^{[-1,0]}(\text{Coh}^T(X))$ such that the underlying morphism in $D^{[-1,0]}(X)$ is a perfect obstruction theory [GP99]. Given such, the virtual fundamental class and structure sheaf admits equivariant refinements, i.e.

$$[X]^{\text{vir}} \in A_{\text{vd}}^T(X)$$

and

$$\mathcal{O}_X^{\text{vir}} \in K_0^T(X).$$

In particular, if X is proper and $\alpha \in A_T^i(X)$ and $V \in K_T^0(X)$ we get equivariant virtual invariants

$$\int_{[X]^{\text{vir}}} \alpha \in A_*^T(\text{pt}) \cong \mathbb{Z}[s_1, \dots, s_n]$$

and

$$\chi_T^{\text{vir}}(X, V) = \chi_T(X, V \otimes_X \mathcal{O}_X^{\text{vir}}) \in K_0^T(\text{pt}) \cong \mathbb{Z}[t_1, \dots, t_n]$$

which are again related by the virtual Hirzebruch-Riemann-Roch formula

$$\text{ch}_T(\chi_T^{\text{vir}}(X, V)) = \int_{[X]^{\text{vir}}} \text{ch}_T(V) \cdot \text{Td}_T(T_X^{\text{vir}}).$$

Example 1.3.2. Suppose $X = Z(s) \subset Y$ for Y a smooth scheme with a T -action and $s \in H_T^0(Y, E)$ a section of a global T -equivariant section of an equivariant sheaf E . Then example 1.3.1 naturally produces a T -equivariant perfect obstruction theory. We again have

$$[X]^{\text{vir}} = 0^! [Y]$$

for $0^!$ the T -equivariant refined Gysin map and

$$\mathcal{O}^{\text{vir}} = [\wedge_{-1}^\bullet E^\vee] = [\text{Kos}^\bullet(E^\vee, s^\vee)]$$

where the Koszul complex is now of T -equivariant sheaves.

The main way to calculate these classes is through virtual localization. The general formula of Thomason [Tho92] states that for $\iota : X^T \hookrightarrow X$ the inclusion of the T -fixed locus the map

$$\iota_* : K_0^T(X^T) \rightarrow K_0^T(X)$$

becomes an isomorphism after applying

$$- \otimes_{K_T^0(\text{pt})} K_T^0(\text{pt})_{\text{loc}}$$

where

$$K_T^0(\text{pt})_{\text{loc}} = K_T^0(\text{pt}) \left[\frac{1}{1 - t^a} \mid a \in \mathbb{Z}^n \right]$$

i.e. after we localize in the coefficients $1 - t^a = 1 - t_1^{a_1} \cdots t_n^{a_n}$.

Definition 1.3.1. When X^T is finite and reduced it follows that

$$K_0^T(X)_{\text{loc}} = \bigoplus_{x \in X^T} K_0^T(\text{pt})_{\text{loc}}.$$

For $V \in K_0^T(X)$ we will denote by

$$\text{cont}_x(V) \in K_0^T(\text{pt})_{\text{loc}}$$

the corresponding unique class corresponding to $x \in X^T$ such that

$$\iota_* \sum_{x \in X^T} \text{cont}_x(V) = V.$$

We will refer to it as the contribution of V at $x \in X$.

In the case of the virtual fundamental class, we have a formula for the unique class. When we restrict the perfect obstruction theory to the fixed locus, it splits into a fixed and a moving part

$$\iota^* \mathbb{E} = \mathbb{E}^{\text{fix}} \oplus \mathbb{E}^{\text{mov}}$$

corresponding to the summands with trivial respectively non-trivial weights in the eigensheaf decomposition. The induced map

$$\iota^* \phi^{\text{fix}} : \mathbb{E}^{\text{fix}} \rightarrow (i_* \mathbb{L}_X)^{\text{fix}} \rightarrow \mathbb{L}_{X^T}$$

is a perfect obstruction thus inducing a virtual structure sheaf $\mathcal{O}_{X^T}^{\text{vir}}$. If we define the virtual normal bundle as $N^{\text{vir}} = (\iota^* T_X^{\text{vir}})^{\text{mov}}$ where $T_X^{\text{vir}} = \mathbb{E}^\vee$ is the virtual tangent bundle, then [Qu18, Theorem 3.3] states that

$$\mathcal{O}_X^{\text{vir}} = \iota_* \left(\frac{\mathcal{O}_{X^T}^{\text{vir}}}{\wedge_{-1}^\bullet [N^{\text{vir}, \vee}]} \right).$$

In particular, this allows us to define the virtual Euler characteristic for non-proper X .

Definition 1.3.2. Let X be a scheme with a T -action and a T -equivariant perfect obstruction theory. Suppose X^T is proper. For $V \in K_T^0(X)$ we define

$$\chi_T^{\text{vir}}(X, V) := \chi_T^{\text{vir}} \left(X^T, \frac{V|_{X^T}}{\wedge_{-1}^{\bullet}[N^{\text{vir}, V}]} \right) \in K_0^T(\text{pt})_{\text{loc}}.$$

By the above, it agrees with the usual virtual Euler characteristic whenever X is proper.

Example 1.3.3. Suppose X^T is finite and reduced. Note that for a sum of one-dimensional representations $V = \sum \alpha_i - \sum \beta_j \in K_T^0(\text{pt})$ we have that

$$\wedge_{-1}^{\bullet} V = \frac{\prod (1 - \alpha_i)}{\prod (1 - \beta_j)}.$$

We further note that $\mathbb{L}_{\text{pt}} = 0$ so that an equivariant perfect obstruction theory is simply given by a T -representation E with $\mathbb{E} = E[-1]$. It follows that

$$\mathcal{O}_{\text{pt}}^{\text{vir}} = \wedge_{-1}^{\bullet}[E].$$

If $\mathbb{E}$ has only trivial weights (as will be the case for $\mathbb{E}_x^{\text{fix}}$) we get

$$\mathcal{O}_{\text{pt}}^{\text{vir}} = \begin{cases} \mathcal{O}_{\text{pt}} & \text{if } \mathbb{E} = 0 \\ 0 & \text{else} \end{cases}$$

since this is true for $\wedge_{-1}^{\bullet}[E]$. We conclude that the localization formula in this case is given by

$$\mathcal{O}_X^{\text{vir}} = \iota_* \sum_{\substack{x \in X^T \\ \mathbb{E}_x^{\text{fix}} = 0}} \frac{[\mathcal{O}_x]}{\wedge_{-1}^{\bullet}(T_{X,x}^{\text{vir}, V})}$$

and in particular writing $T_{X,x}^{\text{vir}} = \sum \alpha_{x,i} - \sum \beta_{x,j}$ we get

$$\chi_T^{\text{vir}}(X, V) = \sum_{\substack{x \in X^T \\ \mathbb{E}_x^{\text{fix}} = 0}} \frac{V_x \prod (1 - \bar{\beta}_{x,j})}{\prod (1 - \bar{\alpha}_{x,i})}$$

where $\overline{(-)} : K_T^0(\text{pt}) \rightarrow K_T^0(\text{pt})$ is given by linearly extending $t_i^{\pm} \mapsto t_i^{\mp}$.

1.3.4 Relative perfect obstruction theories

Let $f : X \rightarrow Y$ be a map of schemes. A relative perfect obstruction theory is a map

$$\phi_f : \mathbb{E}_f \rightarrow \mathbb{L}_f$$

in $D^{[-1,0]}(X)$ satisfying that $h^0(\phi)$ is an isomorphism and that $h^{-1}(\phi)$ is surjective. A compatible triple is a commuting diagram

$$\begin{array}{ccccc} f^* \mathbb{E}_Y & \longrightarrow & \mathbb{E}_X & \longrightarrow & \mathbb{E}_f \\ \downarrow f^* \phi_Y & & \downarrow \phi_X & & \downarrow \phi_f \\ f^* \mathbb{L}_Y & \longrightarrow & \mathbb{L}_X & \longrightarrow & \mathbb{L}_f \end{array}$$

where the rows are distinguished triangles with the lower row the fundamental triangle and such that ϕ_X, ϕ_Y and ϕ_f are perfect obstruction theories. Given a compatible triple [Man11] shows that we get a virtual pullback

$$f^! : A_*(Y) \rightarrow A_*(X)$$

such that $f^! [Y]^{\text{vir}} = [X]^{\text{vir}}$. Similarly, [Qu18] constructs a virtual pullback

$$f^! : K_0(Y) \rightarrow K_0(X)$$

such that $f^! \mathcal{O}_Y^{\text{vir}} \rightarrow \mathcal{O}_X^{\text{vir}}$. Both of these pullbacks are bivariant classes in the sense of [Ful98]. We briefly show a way to construct a compatible relative perfect obstruction theory when we have two different zero loci descriptions.

Lemma 1.3.1. *For $i = 1, 2$ let Y_i be a smooth scheme with a locally free sheaf $E_i \in \text{Coh}(Y_i)$ and a section $s_i \in H^0(Y_i, E_i)$. Let $Z_i = Z(s_i)$ equipped with the perfect obstruction theory $\phi_i : \mathbb{E}_i \rightarrow \mathbb{L}_{Z_i}$ coming from being exhibited as a zero locus of a section of a locally free sheaf. Suppose we have a morphism of schemes*

$$f : Y_1 \rightarrow Y_2$$

and a morphism of sheaves

$$\psi : E_1 \rightarrow f^* E_2$$

such that $\psi(s_1) = f^* s_2$ satisfying the transversality condition that the map of sheaves on Z_1

$$f^*(E_2^\vee|_{Z_2}) \xrightarrow{\begin{pmatrix} -ds_2^\vee \\ \psi^\vee \end{pmatrix}} f^*(\Omega_{Y_2|Z_2}) \oplus E_1^\vee|_{Z_1}$$

is injective. Then from f, ψ we can define a map fitting into a commutative diagram

$$\begin{array}{ccc} f^* \mathbb{E}_2 & \dashrightarrow & \mathbb{E}_1 \\ \downarrow f^* \phi_2 & & \downarrow \phi_1 \\ f^* \mathbb{L}_{Z_2} & \longrightarrow & \mathbb{L}_{Z_1} \end{array}$$

such that the induced map of cones

$$\mathbb{E}_{1,2} := \text{cone}(f^* \mathbb{E}_2 \rightarrow \mathbb{E}_1) \rightarrow \text{cone}(f^* \mathbb{L}_{Z_2} \rightarrow \mathbb{L}_{Z_1}) = \mathbb{L}_{Z_1/Z_2}$$

is a relative perfect obstruction theory.

Proof. The condition $\psi s_1 = s_2 f$ implies in particular that f restricts to a map of schemes

$$f : Z_1 \rightarrow Z_2.$$

The condition further implies that

$$\begin{array}{ccc} f^*(E_2^\vee|_{Z_2}) & \xrightarrow{\psi^\vee} & E_1^\vee|_{Z_1} \\ \downarrow df^* s_2^\vee & & \downarrow ds_1^\vee \\ f^*(\Omega_{Y_2|Z_2}) & \xrightarrow{df} & \Omega_{Y_1|Z_1} \end{array}$$

commutes giving us the desired map $f^* \mathbb{E}_2 \rightarrow \mathbb{E}_1$ fitting into the commutative diagram. Since $f^* \mathbb{E}_2, \mathbb{E}_1$ are perfect of amplitude $[-1, 0]$ we have that the cone $\mathbb{E}_{1,2}$ is perfect of amplitude $[-2, 0]$. It is explicitly given by the complex

$$f^*(E_2^\vee|_{Z_2}) \xrightarrow{\begin{pmatrix} -ds_2^\vee \\ \psi^\vee \end{pmatrix}} f^*(\Omega_{Y_2|Z_2}) \oplus E_1^\vee|_{Z_1} \xrightarrow{(df, ds_1^\vee)} \Omega_{Y_1|Z_1}$$

so that the transversality condition exactly implies that $h^{-2}(\mathbb{E}_{1,2}) = 0$. Finally, taking the map of long exact sequences induced by the map of triangles we get the commuting diagram

$$\begin{array}{ccccccccccc}
h^{-1}(\mathbb{E}_1) & \longrightarrow & h^{-1}(\mathbb{E}_{1,2}) & \longrightarrow & h^0(f^*\mathbb{E}_2) & \longrightarrow & h^0(\mathbb{E}_1) & \longrightarrow & h^0(\mathbb{E}_{1,2}) & \longrightarrow & 0 \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \wr & & \downarrow \wr & & \downarrow & & \parallel \quad \parallel \\
h^{-1}(\mathbb{L}_{Z_1}) & \longrightarrow & h^{-1}(\mathbb{L}_{Z_1/Z_2}) & \longrightarrow & h^0(f^*\mathbb{L}_{Z_2}) & \longrightarrow & h^0(\mathbb{L}_{Z_1}) & \longrightarrow & h^0(\mathbb{L}_{Z_1/Z_2}) & \longrightarrow & 0 \longrightarrow 0
\end{array}$$

and from the 4-lemma we conclude that $h^{-1}(\mathbb{E}_{1,2}) \rightarrow h^{-1}(\mathbb{L}_{Z_1/Z_2})$ is an epimorphism and from the 5-lemma we conclude that $h^0(\mathbb{E}_{1,2}) \rightarrow h^0(\mathbb{L}_{Z_1/Z_2})$ is an isomorphism. We conclude that $\mathbb{E}_{1,2} \rightarrow \mathbb{L}_{Z_1/Z_2}$ is a perfect obstruction theory as desired. $\square$

Remark 1.3.1. Note that the transversality requirement is automatic if ψ is surjective. In this case we get a short exact sequence

$$0 \rightarrow K \rightarrow E_1 \rightarrow f^*E_2 \rightarrow 0$$

for $K = \ker \psi$. This yields the identity

$$\wedge_{-1}^\bullet[E_1^\vee] = (\wedge_{-1}^\bullet[K^\vee]) \cdot (f^* \wedge_{-1}^\bullet[E_2^\vee])$$

in $K_0(Y)$ and so in particular by the definition of the virtual structure sheaf we get that

$$\mathcal{O}_{Z_1}^{\text{vir}} = (\wedge_{-1}^\bullet[K^\vee|_{Z_1}]) \cdot \mathcal{O}_{Z_2}^{\text{vir}}$$

in $K_0(Z_1)$.

1.3.5 Fixed points of the nested Hilbert scheme of points

Recall that for the induced $T = \mathbb{G}_m^n$ action on $\text{NHilb}^d(\mathbb{A}^n)$ the fixed locus is finite and reduced and corresponds to nestings of monomial ideals. These ideals are in particular in bijection with higher dimensional Young diagrams. We briefly recall the definition.

Definition 1.3.3. An n -dimensional Young diagram of size d is a subset $\lambda \subset \mathbb{Z}_{\geq 0}^n$ which satisfies that for all $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}_{\geq 0}^n$ and $\mathbf{b} = (b_1, \dots, b_n) \in \lambda$ then $a_i \leq b_i$ for $i = 1, \dots, n$ implies $\mathbf{a} \in \lambda$. A nested n -dimensional Young diagram $\lambda_\bullet$ of size $\underline{d} = (d_0, \dots, d_r)$ is a chain

$$\emptyset = \lambda_0 \subset \lambda_1 \subset \dots \subset \lambda_{r+1}$$

such that λ_i is an n -dimensional Young diagram for $i = 1, \dots, r+1$ and such that

$$|\lambda_{i+1}| - |\lambda_i| = d_i.$$

We recall that for a nested n -dimensional Young diagram of size $\underline{d}$ we get a nesting of monomial ideals

$$I_{r+1} \subset \dots \subset I_1$$

by setting

$$I_i = I_{\lambda_i} = \langle \mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \notin \lambda_i \rangle \subset \mathbb{C}[x_1, \dots, x_n].$$

This bijectively describes all T -fixed points of $\text{NHilb}^d(\mathbb{A}^n)$ when $\lambda_\bullet$ ranges over all n -dimensional nested Young diagrams of size $\underline{d}$. We recall that with our convention of torus action we have

$$[\mathcal{O}_{Z_i}] = [\mathbb{C}[x_1, \dots, x_n]/I_{\lambda_i}] = \sum_{\mathbf{a} \in \lambda_i} \mathbf{t}^{\mathbf{a}} \in K_0^T(\text{pt}) \cong \mathbb{Z}[t_1^\pm, \dots, t_n^\pm].$$

1.4 The non-commutative nested Hilbert scheme

In this section, we will show the existence of smooth moduli spaces which we call the nested non-commutative Hilbert scheme. The non-commutative Hilbert scheme was first introduced by Nori [Nor78] and has been used successfully to derive various results on the Hilbert scheme of points. A nested version for the non-commutative Hilbert scheme with a full dimension vector (i.e. where the difference in lengths of the subschemes of the nesting are constant 1) on $\mathbb{C}^2$ was considered in [OR18] using a different construction to avoid non-reductive quotients.

This section is devoted to defining the nested non-commutative Hilbert scheme on $\mathbb{C}^n$ for any dimension vector. We will introduce it as the quotient stack of a parabolic subgroup acting freely on a smooth variety. This immediately implies that the corresponding stack is smooth and an algebraic space. We will then identify its functor of points and provide explicit Zariski coverings, taking our inspiration from the proof of the representability of the Hilbert scheme of points found in [GLS07]. We can then construct an explicit embedding from the nested Hilbert scheme of points using the functor of points.

1.4.1 Constructions

For this section, we fix $r \geq 0$, a dimension vector $\underline{d} = (d_0, \dots, d_r) \in \mathbb{Z}_{\geq 0}^{r+1}$ and a reference flag of $\mathbb{C}$ -vectorspaces

$$N_{\bullet} = (N_0 \supset N_1 \supset \dots \supset N_r = 0)$$

such that $\dim_{\mathbb{C}} N_i/N_{i+1} = d_i$. We will denote by $\mathrm{Hom}_{\mathbb{C}}^{\mathrm{fil}}$ and $\mathrm{End}_{\mathbb{C}}^{\mathrm{fil}}$ the flag preserving $\mathbb{C}$ -linear homomorphisms and endomorphisms. We let $P_{\underline{d}} \subset \mathrm{GL}(N_0)$ be the parabolic subgroup of flag-preserving automorphisms.

Definition 1.4.1. We define

$$\widetilde{\mathrm{ncNHilb}^{\underline{d}}(\mathbb{A}^n)} \subset \mathrm{End}_{\mathbb{C}}^{\mathrm{fil}}(N_{\bullet})^{\oplus n} \oplus N_0$$

given by the Zariski open locus of tuples $(A_1, \dots, A_n, v)$ where

$$\mathbb{C}\langle A_1, \dots, A_n \rangle v = N_0.$$

Equivalently, it consists of the tuples $(A_1, \dots, A_n, v)$ where the smallest subspace of N_0 closed under $A_1, \dots, A_n$ and containing v is N_0 itself.

Definition 1.4.2. We let $P_{\underline{d}}$ act on $\mathrm{End}_{\mathbb{C}}^{\mathrm{fil}}(N_{\bullet})^{\oplus n} \oplus N_0$ by

$$g.(A_1, \dots, A_n, v) = (A_1^g, \dots, A_n^g, gv)$$

where $A_i^g = gA_i g^{-1}$.

Lemma 1.4.1. *The action of $P_{\underline{d}}$ restricts to a free action on $\widetilde{\mathrm{ncNHilb}^{\underline{d}}(\mathbb{A}^n)}$.*

Proof. Let $(A_1, \dots, A_n, v) \in \widetilde{\mathrm{ncNHilb}^{\underline{d}}(\mathbb{A}^n)}$ and let $g \in P_{\underline{d}}$. If $M \subset N_0$ is invariant under $A_1^g, \dots, A_n^g$ and contains gv then $g^{-1}M$ is invariant under $A_1, \dots, A_n$ and $v \in g^{-1}M$ hence by stability $g^{-1}M = N_0$ and so $M = N_0$ proving that $(A_1^g, \dots, A_n^g, gv)$ is stable. This shows that $\widetilde{\mathrm{ncNHilb}^{\underline{d}}(\mathbb{A}^n)}$ is closed under the $P_{\underline{d}}$ action. If $(A_1^g, \dots, A_n^g, gv) = (A_1, \dots, A_n, v)$ then $\ker(g - \mathrm{id}_{N_0})$ contains v and is invariant under $A_1, \dots, A_n$ hence $g = \mathrm{id}$ thus proving that the action is free. $\square$

Definition 1.4.3. We define

$$\text{ncNHilb}^{\underline{d}}(\mathbb{A}^n) = \left[\widetilde{\text{ncNHilb}^{\underline{d}}(\mathbb{A}^n)} / P_{\underline{d}} \right]$$

which we call the *the non-commutative nested Hilbert scheme*. Here $[X/G]$ denotes the resulting quotient stack. Note that since we are using a smooth algebraic group acting freely on a smooth space the resulting quotient is a smooth algebraic space. When $r = 1$ so that $\underline{d} = d_0$ we will drop the "N" prefix in the notation and simply write $\text{ncHilb}^{d_0}(\mathbb{A}^n)$ which we will simply refer to as *the non-commutative Hilbert scheme*. In this case it agrees with the original construction, see for example [Rei05]

Next we identify the functor of points by making the obvious generalization of the stable data above from vector spaces to vector bundles.

Definition 1.4.4. Let $S \in \text{Sch}_{/\mathbb{C}}$. A *non-commutative stable tuple* of type $\underline{d}, n$ on S consists of the data $(\mathcal{F}_{\bullet}, \phi_1, \dots, \phi_n, s)$ where

1. $\mathcal{F}_{\bullet}$ is a flag of locally free sheaves

$$\mathcal{F}_{\bullet} = (\mathcal{F}_0 \supset \dots \supset \mathcal{F}_m = 0)$$

such that $\mathcal{F}_i / \mathcal{F}_{i+1}$ is locally free of rank d_i .

2. $\phi_1, \dots, \phi_n$ are operators

$$\phi_1, \dots, \phi_n : \mathcal{F}_0 \rightarrow \mathcal{F}_0$$

such that $\phi_i(\mathcal{F}_j) \subset \mathcal{F}_j$.

3. $s \in H^0(S, \mathcal{F}_0)$ is a global section such that the induced map

$$\begin{aligned} \mathcal{O}_S \langle x_1, \dots, x_n \rangle &\longrightarrow \mathcal{F}_0 \\ f(x_1, \dots, x_n) &\longmapsto f(\phi_1, \dots, \phi_n)s \end{aligned}$$

is an epimorphism.

A *framed non-commutative stable tuple* of type $(\underline{d}, n)$ is the data of $(\mathcal{F}_{\bullet}, \phi_1, \dots, \phi_n, s, \beta)$ where $(\mathcal{F}_{\bullet}, \phi_1, \dots, \phi_n, s)$ is a non-commutative stable tuple and

$$\beta : \mathcal{O}_S \otimes_{\mathbb{C}} N_{\bullet} \rightarrow \mathcal{F}_{\bullet}$$

is an isomorphism of flags. Two non-commutative stable tuples $(\mathcal{F}_{\bullet}, \phi_1, \dots, \phi_n, s)$ and $(\mathcal{F}'_{\bullet}, \phi'_1, \dots, \phi'_n, s')$ (respectively framed non-commutative stable tuples $(\mathcal{F}_{\bullet}, \phi_1, \dots, \phi_n, s, \beta)$ and $(\mathcal{F}'_{\bullet}, \phi'_1, \dots, \phi'_n, s', \beta')$) are said to be equivalent if there exists an isomorphism of flags $\rho : \mathcal{F}_{\bullet} \rightarrow \mathcal{F}'_{\bullet}$ such that $\phi'_i = \rho \phi_i \rho^{-1}$ and $s' = \rho s$ (respectively also $\beta' = \rho \beta$).

Definition 1.4.5. We define the functor

$$\text{ncHilb}^{\underline{d}, n}, : \text{Sch}_{/\mathbb{C}}^{\text{op}} \rightarrow \text{Set}$$

as follows. For $S \in \text{Sch}_{/\mathbb{C}}$ we let

$$\text{ncHilb}^{\underline{d}, n}(S) = \{(\mathcal{F}_{\bullet}, \phi_1, \dots, \phi_n, s) \text{ non-commutative stable tuple}\} / \sim$$

and for $f : S' \rightarrow S$ we simply define

$$f^*(\mathcal{F}_{\bullet}, \phi_1, \dots, \phi_n, s) = (f^* \mathcal{F}_{\bullet}, f^* \phi_1, \dots, f^* \phi_n, f^* s).$$

We similarly define a framed version

$$\widetilde{nc\mathcal{Hilb}^{d,n}} : \text{Sch}_{/\mathbb{C}}^{\text{op}} \rightarrow \text{Set}$$

by

$$\widetilde{nc\mathcal{Hilb}^{d,n}}(S) = \{(\mathcal{F}_\bullet, \phi_1, \dots, \phi_n, s, \beta) \text{ framed non-commutative stable tuple}\} / \sim$$

and

$$f^*(\mathcal{F}_\bullet, \phi_1, \dots, \phi_n, s, \beta) = (f^*\mathcal{F}_\bullet, f^*\phi_1, \dots, f^*\phi_n, f^*s, f^*\beta).$$

Proposition 1.4.1. *The map*

$$\widetilde{nc\mathcal{Hilb}^{d,n}} \rightarrow nc\mathcal{Hilb}^{d,n}$$

given by forgetting the framing is isomorphic to the functor of points of the quotient map

$$\widetilde{nc\mathcal{NHilb}^d(\mathbb{A}^n)} \rightarrow nc\mathcal{NHilb}^d(\mathbb{A}^n).$$

Proof. Set $X = \text{End}_{\mathbb{C}}^{\text{fil}}(N_\bullet)^{\oplus n} \oplus N_0$ viewing the vector space as a scheme (i.e. by taking $\text{Spec Sym}^\bullet(-)^\vee$). Let $V_\bullet = N_\bullet \otimes_{\mathbb{C}} \mathcal{O}_X$ which is a flag of locally free sheaves on X . We construct universal maps on $V_\bullet$ as follows. We set $d = |d| = d_0 + \dots + d_r$ and choose a basis $v_1, \dots, v_d$ of N_0 such that

$$\text{span}(v_{d_0+\dots+d_{i-1}+1}, \dots, v_d) = N_i.$$

For $1 \leq i \leq d$ define we $w(i) = l$ if

$$d_0 + \dots + d_{l-1} < i \leq d_0 + \dots + d_l$$

i.e. if and only if $v_i \in N_l \setminus N_{l+1}$. Writing everything in the coordinates of the basis we get

$$X \cong \text{Spec } \mathbb{C}[a_{i,j}^k, \alpha_m]$$

where the indices ranges over $1 \leq k \leq n$, $1 \leq i, j \leq d$ with $w(j) \leq w(i)$ and $1 \leq m \leq d$. Explicitly, for a point $(A_1, \dots, A_n, v) \in X$ we have

$$A_k(v_j) = \sum_i a_{i,j}^k v_i, \quad v = \sum_m \alpha_m v_m.$$

We define

$$\tilde{\Phi}_k : V_0 \rightarrow V_0, \quad \tilde{\mathfrak{v}} : \mathcal{O}_X \rightarrow V_0$$

by

$$\tilde{\Phi}_k(v_j \otimes 1) = \sum_i v_i \otimes a_{i,j}^k, \quad \tilde{\mathfrak{v}}(1) = \sum_m v_m \otimes \alpha_m.$$

Note also that the identity map $\beta_0 := \text{id}_{V_0}$ is a framing of $V_\bullet$. By definition $(V_\bullet, \tilde{\Phi}_1, \dots, \tilde{\Phi}_n, \tilde{\mathfrak{v}}, \beta_0)$ is a framed non-commutative tuple when restricted to $\widetilde{nc\mathcal{NHilb}^d(\mathbb{A}^2)}$, which then in turn classifies a map

$$\widetilde{nc\mathcal{NHilb}^d(\mathbb{A}^2)} \rightarrow \widetilde{nc\mathcal{Hilb}^{d,n}}.$$

Conversely, suppose we have a map $S \rightarrow \widetilde{\text{ncHilb}^d}^n$ classified by a framed non-commutative tuple $(\mathcal{F}_\bullet, \varphi_1, \dots, \varphi_n, s, \beta)$. Then we can view $(\beta^{-1}\varphi_1\beta, \dots, \beta^{-1}\varphi_n\beta, \beta^{-1}s)$ as a map

$$\mathcal{O}_S \rightarrow \left(\text{End}^{\text{fil}}(N_\bullet)^{\oplus n} \oplus N_0 \right) \otimes_{\mathbb{C}} \mathcal{O}_S$$

which in turn classifies a map $f : S \rightarrow X$ in such a way that

$$f^*(V_\bullet, \widetilde{\Phi}_1, \dots, \widetilde{\Phi}_n, \widetilde{\mathbf{v}}, \beta_0) \sim (\mathcal{F}_\bullet, \varphi_1, \dots, \varphi_n, s, \beta)$$

via $\beta : f^*V_\bullet = N_\bullet \otimes_{\mathbb{C}} \mathcal{O}_S \rightarrow \mathcal{F}_\bullet$. The various conditions on being a non-commutative tuple exactly translates to f factoring through $\text{ncNHilb}^d(\mathbb{A}^n)$, giving us the desired inverse.

Next we note that $V_\bullet = N_\bullet \otimes_{\mathbb{C}} \mathcal{O}$ on $\widetilde{\text{ncNHilb}^d}(\mathbb{A}^2)$ has a natural P_d -equivariant structure by letting $P_d = \text{Aut}_{\mathbb{C}}^{\text{fil}}(N_\bullet)$ act by multiplication on the fibers. This structure naturally makes $\widetilde{\Phi}_1, \dots, \widetilde{\Phi}_n$ and $\widetilde{\mathbf{v}}$ equivariant. By properties of the quotient stack, pullback along the quotient induces an equivalence of categories

$$\text{QCoh}(\text{ncNHilb}^d(\mathbb{A}^2)) \cong \text{QCoh}^{P_d}(\widetilde{\text{ncNHilb}^d}(\mathbb{A}^2))$$

where the left hand side is equivariant quasi-coherent sheaves. From this we get an induced non-commutative tuple $(\mathcal{V}_\bullet, \Phi_1, \dots, \Phi_n, \mathbf{v})$ on $\text{ncNHilb}^d(\mathbb{A}^2)$ such that $\pi^*(\mathcal{V}_\bullet, \Phi_1, \dots, \Phi_n, \mathbf{v}) = (V_\bullet, \widetilde{\Phi}_1, \dots, \widetilde{\Phi}_n, \widetilde{\mathbf{v}})$, which in turn gives us a map

$$\text{ncNHilb}^d(\mathbb{A}^n) \rightarrow \widetilde{\text{ncHilb}^d}^n.$$

Conversely, given a map $S \rightarrow \widetilde{\text{ncHilb}^d}^n$ classified by a non-associative tuple $(\mathcal{F}_\bullet, \varphi_1, \dots, \varphi_n, s)$ we can pick an open cover $\{U_i\}_i$ of S and trivializations $\beta_i : \mathcal{O}_{U_i} \otimes_{\mathbb{C}} N_\bullet \rightarrow \mathcal{F}_\bullet|_{U_i}$. It follows that $(\mathcal{F}_\bullet|_{U_i}, \varphi_1, \dots, \varphi_n, s, \beta_i)$ is a framed non-associative tuple. Using the above, these classify maps

$$f_i : U_i \rightarrow \widetilde{\text{ncNHilb}^d}(\mathbb{A}^2).$$

On overlaps $U_{i,j} = U_i \cap U_j$ the transition function of β_i, β_j determines a map

$$\gamma_{j,i} : U_{j,i} \rightarrow P_d$$

such that $\gamma_{j,i} \cdot \beta_i = \beta_j$. It follows that

$$(\pi f_i)|_{U_{i,j}} = (\pi f_j)|_{U_{i,j}}$$

so that $\{\pi f_i\}_i$ glues together to a map

$$S \rightarrow \text{ncNHilb}^d(\mathbb{A}^2)$$

giving us the desired inverse. $\square$

Remark 1.4.1. We note that the proposition shows the existence of a universal non-commutative tuple on $\text{ncNHilb}^d(\mathbb{A}^n)$. We will denote the universal tuple by $(\mathcal{V}_\bullet, \Phi_1, \dots, \Phi_n, \mathbf{v})$

It now only remains to show that our moduli space is representable by a scheme. Since it is an algebraic space it satisfies fppf descent and hence also Zariski descent. It therefore only remains to construct a Zariski open cover of schemes. We note that the universal tuple in turn induces a universal epimorphism

$$q_{nc} : \mathbb{C}\langle x_1, \dots, x_n \rangle \otimes_{\mathbb{C}} \mathcal{O} \cong \mathcal{O}\langle x_1, \dots, x_n \rangle \twoheadrightarrow \mathcal{V}_0$$

on $\text{ncNHilb}^d(\mathbb{A}^n)$. Inspired by [GLS07] we consider loci of sections of these epimorphisms.

Definition 1.4.6. For fixed $\gamma : N_0 \rightarrow \mathbb{C}\langle x_1, \dots, x_n \rangle$ we let

$$\text{ncNHilb}_{\gamma}^d(\mathbb{A}^n) \subset \text{ncNHilb}^d(\mathbb{A}^n)$$

be the loci where the composition

$$N_0 \otimes_{\mathbb{C}} \mathcal{O} \xrightarrow{\gamma \otimes \text{id}} \mathcal{O}\langle x_1, \dots, x_n \rangle \xrightarrow{q_{nc}} \mathcal{V}_0$$

is an isomorphism (or equivalently surjective). Note that this is a Zariski open as it is defined as the surjectivity locus of a map of sheaves.

Proposition 1.4.2. *Let $\gamma : N_0 \rightarrow \mathbb{C}\langle x_1, \dots, x_n \rangle$. Then $\text{ncNHilb}_{\gamma}^d(\mathbb{A}^n)$ is representable by a scheme.*

Proof. Consider the flag variety $\text{Flag}_{\underline{d}}(N_0)$ which is the moduli of flags on N_0 of dimension prescribed by $\underline{d}$. Let $d = d_0 + \dots + d_r$ and consider $\widetilde{\text{ncHilb}^d(\mathbb{A}^n)}$ i.e. the non-nested version. We have a map

$$\text{ncNHilb}_{\gamma}^d(\mathbb{A}^n) \rightarrow \text{Flag}_{\underline{d}}(N_0) \times \widetilde{\text{ncHilb}^d(\mathbb{A}^n)}$$

which on the functor of points for $S \in \text{Sch}/_{\mathbb{C}}$ is given by

$$(\mathcal{F}_{\bullet}, \phi_1, \dots, \phi_n, s) \mapsto ((q_{nc}(\gamma \otimes 1))^{-1} \mathcal{F}_{\bullet}, (\mathcal{F}_0, \phi_1, \dots, \phi_n, s, q_{nc}(\gamma \otimes \text{id})))$$

i.e. in the second coordinate we forget the flag and use $q_{nc}(\gamma \otimes 1)$ as framing while we in the first coordinate use the flag structure induced on $N_0 \otimes_{\mathbb{C}} \mathcal{O}_S$ by $\mathcal{F}_{\bullet}$ via $q_{nc}(\gamma \otimes 1)$. The target of this map is a scheme, so we are done if we can show that the map is a closed immersion. It is clearly a monomorphism on the functor of points. We can identify the image of the map as the locus of

$$(\mathcal{G}_{\bullet}, (\mathcal{F}_0, \phi_1, \dots, \phi_n, s, \beta))$$

where $\phi_1, \dots, \phi_n$ are flag-preserving wrt. the flag structure $\beta(\mathcal{G}_{\bullet})$ on $\mathcal{F}_0$ and where $\beta = q_{nc}(\gamma_1 \otimes \text{id})$. These conditions are Zariski closed, thus finishing the proof. $\square$

Proposition 1.4.3. *The subschemes $\{\text{ncNHilb}_{\gamma}^d(\mathbb{A}^n)\}_{\gamma \in \text{Hom}_{\mathbb{C}}(N_0, \mathbb{C}\langle x_1, \dots, x_n \rangle)}$ provide a Zariski cover of $\text{ncNHilb}^d(\mathbb{A}^n)$. In particular, $\text{ncNHilb}^d(\mathbb{A}^n)$ is a scheme.*

Proof. This follows simply because the maps q_{nc} admit sections γ Zariski locally. Indeed, for each closed point $p \in \text{ncNHilb}^d(\mathbb{A}^n)$ the map is given by

$$q_{nc}|_p : \mathbb{C}\langle x_1, \dots, x_n \rangle \twoheadrightarrow \mathcal{V}_{0,p_1}$$

for which such a section clearly exists. $\square$

1.4.2 Embedding of the nested Hilbert scheme

Consider the nested Hilbert scheme of points $\text{NHilb}^d(\mathbb{A}^n)$. The universal subschemes

$$\mathcal{Z}_1 \subset \mathcal{Z}_2 \subset \dots \subset \mathcal{Z}_{r+1} \subset \mathbb{A}_{\mathbb{C}}^n \times \text{NHilb}^d(\mathbb{A}^n) \cong \mathbb{A}_{\text{NHilb}^d(\mathbb{A}^n)}^n$$

give rise to surjections of locally free $\mathcal{O} = \mathcal{O}_{\text{NHilb}^d(\mathbb{A}^n)}$ -algebras

$$\mathcal{O}[x_1, \dots, x_n] \twoheadrightarrow \text{pr}_* \mathcal{O}_{\mathcal{Z}_r} \twoheadrightarrow \dots \twoheadrightarrow \text{pr}_* \mathcal{O}_{\mathcal{Z}_1}.$$

We consider the flag $\mathcal{F}_{\bullet}$ given by setting $\mathcal{F}_0 = \text{pr}_* \mathcal{O}_{\mathcal{Z}_{r+1}}$ and

$$\mathcal{F}_i = \ker(\text{pr}_* \mathcal{O}_{\mathcal{Z}_r} \twoheadrightarrow \text{pr}_* \mathcal{O}_{\mathcal{Z}_i}).$$

We consider the map

$$\mathcal{O}\langle x_1, \dots, x_n \rangle \twoheadrightarrow \mathcal{O}[x_1, \dots, x_n] \twoheadrightarrow \mathcal{F}_{\bullet}$$

which induces a non-commutative tuple which in turn classifies a map to the non-commutative nested Hilbert scheme.

Proposition 1.4.4. *The map*

$$\mathrm{NHilb}^d(\mathbb{A}^n) \rightarrow \mathrm{ncNHilb}^d(\mathbb{A}^n).$$

classified by the tuple above is a closed immersion. Its image agrees with the locus where $\Phi_1, \dots, \Phi_n$ commute which we will call the commutativity locus.

Proof. We construct an inverse from the commutativity locus via the functor of points. The commutativity locus simply classifies tuples where the associated quotient map q_{nc} factors through $\mathcal{O}_S[x_1, \dots, x_n]$. In this case an S -valued point is therefore completely classified up to isomorphism by a flag $\mathcal{F}_\bullet$ of locally free sheaves with an epimorphism

$$q : \mathcal{O}_S[x_1, \dots, x_n] \twoheadrightarrow \mathcal{F}_0$$

such that $q^{-1}(\mathcal{F}_i)$ is closed under left multiplication by $x_1, \dots, x_n$, i.e. is a left and therefore also a two-sided ideal. This in turn induces a series of epimorphisms

$$\mathcal{O}_S[x_1, \dots, x_n] \twoheadrightarrow \mathcal{F}_0 \twoheadrightarrow \mathcal{F}_0/\mathcal{F}_{r-1} \twoheadrightarrow \dots \twoheadrightarrow \mathcal{F}_0/\mathcal{F}_1$$

which allows us to identify $\mathcal{F}_0/\mathcal{F}_i$ with a locally free $\mathcal{O}_S$ -algebra $\mathcal{O}_{Z_i}$ giving us the desired S -valued point

$$Z_1 \subset \dots \subset Z_r \subset \mathbb{A}_{\mathbb{C}}^n \times S$$

on $\mathrm{NHilb}^d(\mathbb{A}^n)$. One can easily check that these constructions are the inverse of each other. Clearly the commutativity locus is Zariski closed as it can be given by the zero locus of the commutators

$$\bigoplus_{1 \leq i < j \leq n} [\Phi_i, \Phi_j] \in H^0 \left(\bigoplus_{i,j} \mathcal{E}nd^{\mathrm{fil}}(\mathcal{V}_\bullet) \right).$$

□

Remark 1.4.2. Recall that for a quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}(\mathbb{A}^n)$ we can construct tautological sheaves

$$\mathcal{F}^{[d_0+\dots+d_r]} \twoheadrightarrow \mathcal{F}^{[d_0+\dots+d_{r-1}]} \twoheadrightarrow \dots \twoheadrightarrow \mathcal{F}^{[d_0]}$$

on $\mathrm{NHilb}^d(\mathbb{A}^n)$ by setting

$$\mathcal{F}^{[d_0+\dots+d_{i-1}]} = p_{i*} q_i^*(\mathcal{F})$$

where

$$\mathbb{A}_{\mathbb{C}^n}^n \xleftarrow{q_i} \mathcal{Z}_i \xrightarrow{p_i} \mathrm{NHilb}^d(\mathbb{A}^n)$$

are the projection maps of the universal family. We note that by construction we have

$$\mathcal{V}_0|_{\mathrm{NHilb}^d} = \mathcal{O}_{\mathbb{A}^n}^{[d_0+\dots+d_r]}$$

and for each $i \geq 1$ we have a short exact sequence

$$0 \rightarrow \mathcal{V}_i|_{\mathrm{NHilb}^d(\mathbb{A}^n)} \rightarrow \mathcal{O}_{\mathbb{A}^n}^{[d_0+\dots+d_r]} \rightarrow \mathcal{O}_{\mathbb{A}^n}^{[d_0+\dots+d_{i-1}]} \rightarrow 0$$

where $\mathcal{V}_\bullet$ is the universal flag on $\mathrm{ncNHilb}^d(\mathbb{A}^n)$.

1.4.3 The torus action

We let $T = \mathbb{G}_m^n$ which we let act on $\mathbb{A}^n$ by

$$(t_1, \dots, t_i) \cdot (p_1, \dots, p_i) = (t_1^{-1} p_1, \dots, t_n^{-1} p_n).$$

This equips $\mathbb{C}[x_1, \dots, x_n]$ with a natural T space structure where $t.x_i = t_i x_i$ and $t.1 = 1$.

With this in mind, we let T act on $\text{ncNHilb}^d(\mathbb{A}^n)$ by

$$t.(A_1, \dots, A_n, v) = (t_1^{-1} A_1, \dots, t_n^{-1} A_n, v).$$

This commutes with the $P_{\underline{d}}$ -action and therefore descends to an action on the quotient. Similarly, equipping the $P_{\underline{d}}$ -equivariant sheaf $V_{\bullet}$ with the trivial T -action gives us an action on the corresponding universal flag $\mathcal{V}_{\bullet}$. In this way, the universal operators are equivariant maps

$$\Phi_i : t_i \mathcal{V}_{\bullet} \rightarrow \mathcal{V}_{\bullet}$$

where the twist t_i is by tensoring with the one-dimensional representation of character t_i . We can therefore view it as a section of

$$t_i^{-1} \mathcal{E}nd^{\text{fil}}(\mathcal{V}_{\bullet})$$

The universal cyclic vector gives an equivariant map

$$\mathfrak{v} : \mathcal{O} \rightarrow \mathcal{V}_{\bullet}$$

where $\mathcal{O}$ is given the trivial T -equivariant structure. We equip $\text{NHilb}^d(\mathbb{A}^n)$ with the T -action induced from the action on $\mathbb{A}^n$. With these actions the embedding

$$\text{NHilb}^d(\mathbb{A}^n) \hookrightarrow \text{ncNHilb}^d(\mathbb{A}^n)$$

and the maps between equivariant sheaves

$$\mathcal{V}_i|_{\text{NHilb}^d(\mathbb{A}^n)} \hookrightarrow \mathcal{O}_{\mathbb{A}^n}^{[d_0 + \dots + d_r]}$$

are also equivariant.

1.5 The commutativity bundle and the perfect obstruction theory

1.5.1 Reduction of commutativity equations

Naively the set of commutativity equations can be structured into a section of the equivariant bundle

$$\bigoplus_{1 \leq i < j \leq n} t_i^{-1} t_j^{-1} \mathcal{E}nd^{\text{fil}}(\mathcal{V}_{\bullet}).$$

Letting

$$h(\underline{d}) = \sum_{0 \leq i \leq j \leq r} d_i d_j = \text{rk } \mathcal{E}nd^{\text{fil}}(\mathcal{V}_{\bullet})$$

and

$$d = \sum_{i=0}^r d_i$$

we see that

$$\dim \text{ncNHilb}^d(\mathbb{A}^n) = n \cdot h(\underline{d}) + d - h(\underline{d})$$

giving us a virtual dimension of

$$d - \frac{(n-1)(n-2)}{2} h(\underline{d}).$$

For $n = 2$ this gives us a virtual dimension of d . For $n \geq 3$ the virtual dimension becomes negative since

$$\frac{d(d+1)}{d} \leq h(\underline{d}) \leq d^2.$$

In this section we show that the rank of this bundle can be reduced in the $n = 2$ case giving us a virtual dimension of $d + d_0$. We note that in the non-nested case $r = 0$ the Hilbert scheme $\text{Hilb}^d(\mathbb{A}^2)$ is smooth of dimension $2d$. In particular, the above virtual class of virtual dimension d is not the fundamental one. This means that we are able to locally reduce the number of commutativity equations by d . We will show that the cyclicity condition allows us to do this globally. We do so by showing that if the commutator is in the span of the cyclic vector then it is forced to vanish.

Lemma 1.5.1. *Let V be a vector space with operators $A_1, A_2 \in \text{End}(V)$ and $v \in V$ cyclic. If $\text{im}([A_1, A_2]) \subset \text{span}\{v\}$ then $[A_1, A_2] = 0$.*

Proof. Set $B = [A_1, A_2]$. We can find a functional $\phi : V \rightarrow \mathbb{C}$ such that $Bw = \phi(w)v$. Let

$$V_m = \text{span}\{A_{i_1} \cdots A_{i_k} v \mid k \leq m, i_j \in \{1, 2\}\}.$$

We have $V = \bigcup_{m \geq 0} V_m$ and show by induction that $\phi|_{V_m} \equiv 0$. We note that for any operator $L : V \rightarrow V$ it holds that

$$\text{tr}(BL) = \phi(Lv)$$

since we have $BL(w) = \phi(Lw)v$. In particular,

$$\phi(v) = \text{Tr}[A_1, A_2] = 0$$

proving that $\phi|_{V_0} \equiv 0$. Suppose now that $\phi|_{V_m} \equiv 0$ for $m \geq 0$. Let $L = A_{i_1} \cdots A_{i_{m+1}}$ so that $Lv \in V_{m+1}$. We first prove that $Lv = A_2^k A_1^l v$ for some l, k with $l + k = m + 1$. Indeed, if L contains an inversion we can write it in the form $L = FA_1 A_2 G$ where F, G are non-commutative words in A_1, A_2 of length $\leq m$. Then

$$FA_1 A_2 Gv = F(A_2 A_1 + B)Gv = FA_2 A_1 Gv + FBGv = FA_2 A_1 Gv - \phi(Gv)Fv$$

but $\phi(Gv) = 0$ by the induction hypothesis, so we have decreased the number of inversions by 1. Inductively continuing this procedure allows us to write $Lv = A_2^k A_1^l v$. If $k = 0$ then

$$\phi(Lv) = \text{tr}([A_1, A_2]A_1^l) = \text{tr}(A_1 A_2 A_1^l) - \text{tr}(A_2 A_1 A_1^l)$$

which disappears by invariance of the trace under cyclic shifts. For $k > 0$ we first see that

$$\text{tr}([L, A_1]A_2) = \text{tr}(LA_1 A_2) - \text{tr}(A_1 LA_2) = \text{tr}(BL) = \phi(Lv).$$

We apply the Leibniz rule $[XY, Z] = X[Y, Z] + [X, Z]Y$ to see

$$\begin{aligned} [L, A_1] &= [A_2^k A_1^l, A_1] \\ &= A_2^k [A_1^l, A_1] + [A_2^k, A_1] A_1^l \\ &= [A_2^k, A_1] A_1^l \\ &= \sum_{i=1}^k A_2^{i-1} [A_2, A_1] A_2^{k-i} A_1^l \\ &= - \sum_{i=1}^k A_2^{i-1} B A_2^{k-i} A_1^l. \end{aligned}$$

It follows that

$$\begin{aligned}
\phi(Lv) &= \text{tr}([L, A_1]A_2) \\
&= - \sum_{i=1}^k \text{tr}(A_2^{i-1} B A_2^{k-i} A_1^l A_2) \\
&= - \sum_{i=1}^k \text{tr}(B A_2^{k-i} A_1^l A_2^i) \\
&= - \sum_{i=1}^k \phi(A_2^{k-i} A_1^l A_2^i v)
\end{aligned}$$

but by inverting inversions we have $A_2^{k-i} A_1^l A_2^i v = A_2^k A_1^l v = Lv$ proving

$$\phi(Lv) = -k\phi(Lv)$$

so that $\phi(Lv) = 0$ as desired. $\square$

This means that the commutativity locus of $\text{ncNHilb}^d(\mathbb{A}^2)$ is cut out by the commutator section on the quotient sheaf

$$\mathcal{H}om^{\text{fl}}(\mathcal{V}_\bullet, \bar{\mathcal{V}}_\bullet)$$

where $\bar{\mathcal{V}}_0 = \text{coker}(\mathfrak{v})$ for $\mathfrak{v} : \mathcal{O} \rightarrow \mathcal{V}_0$ the universal cyclic vector. We quickly prove that this is in fact locally free; the main point is simply proving that $\bar{\mathcal{V}}_\bullet$ is a flag of bundles.

Lemma 1.5.2. *Consider the universal cyclic vector $\mathfrak{v} : \mathcal{O} \rightarrow \mathcal{V}_0$. The cokernel $\bar{\mathcal{V}}_0$ is locally free and the subsheaves*

$$\bar{\mathcal{V}}_i := (\mathcal{V}_i + \text{im } \mathfrak{v}) / \text{im } \mathfrak{v} \subset \bar{\mathcal{V}}_0$$

are inclusions of locally free sheaves with locally free quotients.

Proof. Let $\mathcal{L} := \text{im } \mathfrak{v} \subset \mathcal{V}_0$. By the cyclic condition the map $\mathfrak{v}$ has local constant rank 1 so that $\mathcal{L}$ is a subbundle and so

$$\bar{\mathcal{V}}_0 = \mathcal{V}_0 / \mathcal{L}$$

is locally free. If $\mathcal{V}_i = \mathcal{V}_0$ then $\bar{\mathcal{V}}_i = \bar{\mathcal{V}}_0$ is locally free. If $\mathcal{V}_i \subsetneq \mathcal{V}_0$ then $\mathcal{L} \cap \mathcal{V}_i = 0$ since if $\mathfrak{v}$ factors through $\mathcal{V}_i$ then $\mathcal{V}_i$ would be closed under the universal operators and contain the universal cyclic vector, hence by cyclicity $\mathcal{V}_i = \mathcal{V}_0$. It follows that

$$\mathcal{V}_i + \mathcal{L} \cong \mathcal{V}_i \oplus \mathcal{L}$$

is locally free and

$$\bar{\mathcal{V}}_i \cong \mathcal{V}_i$$

is also locally free. $\square$

1.5.2 The perfect obstruction theory

The above allows us to construct a virtual class on the nested Hilbert scheme of points on $\mathbb{C}^2$ with dimension matching the fundamental class in the smooth non-nested case. We note that equivariantly the commutativity section lives in

$$\mathcal{H}om^{\text{fl}}(\mathcal{V}_\bullet, t_1^{-1} t_2^{-1} \bar{\mathcal{V}}_\bullet) = t_1^{-1} t_2^{-1} \mathcal{H}om^{\text{fl}}(\mathcal{V}_\bullet, \bar{\mathcal{V}}_\bullet)$$

where $t_1^{-1} t_2^{-1}$ means twisting by the corresponding character. Note that if $d_0 \neq 0$ the rank of this bundle is given by

$$h(\underline{d}) - d_0$$

so that

$$\dim \text{ncNHilb}^d(\mathbb{A}^n) - \text{rk} = h(\underline{d}) + d - (h(\underline{d}) - d_0) = d + d_0.$$

Definition 1.5.1. Let $\underline{d}$ be a dimension vector. Define

$$\mathcal{E} = t_1^{-1} t_2^{-1} \mathcal{H}om^{\text{fil}}(\mathcal{V}_\bullet, \overline{\mathcal{V}}_\bullet)$$

and

$$\mathcal{E}_{\text{naive}} = t_1^{-1} t_2^{-1} \mathcal{E}nd^{\text{fil}}(\mathcal{V}_\bullet)$$

as equivariant locally free sheaves on $\text{ncNHilb}^{\underline{d}}(\mathbb{A}^2)$ and let $s_{\text{com}} = [\Phi_1, \Phi_2]$ be the commutator section. Since the nested Hilbert scheme is its zero locus $Z(s_{\text{com}}) = \text{NHilb}^{\underline{d}}(\mathbb{A}^2)$ (viewed either as a section of $\mathcal{E}$ and $\mathcal{E}'$) this defines two perfect obstruction theories of $\text{NHilb}^{\underline{d}}(\mathbb{A}^2)$ of virtual dimensions d and $d + d_i$ respectively, where i is chosen to be the smallest such that $d_i \neq 0$. Concretely, if we let $I \subset \mathcal{O}_{\text{ncNHilb}^{\underline{d}}(\mathbb{A}^2)}$ they are given by restricting the map of complexes

$$\begin{array}{ccc} [\mathcal{E}_*^\vee & \xrightarrow{ds_{\text{com}}^\vee} & \Omega_{\text{ncNHilb}^{\underline{d}}(\mathbb{A}^2)}] \\ \downarrow s_{\text{com}}^\vee & & \parallel \\ [I/I^2 & \xrightarrow{d} & \Omega_{\text{ncNHilb}^{\underline{d}}(\mathbb{A}^2)}] \end{array}$$

from $\text{ncNHilb}^{\underline{d}}(\mathbb{A}^2)$ to $\text{NHilb}^{\underline{d}}(\mathbb{A}^2)$ where $* \in \{\emptyset, \text{naive}\}$. We let

$$\mathbb{E} \rightarrow \mathbb{L}_{\text{NHilb}^{\underline{d}}(\mathbb{A}^2)}$$

be the perfect obstruction theory defined by the bundle

$$\mathcal{E}_{\underline{d}} = \begin{cases} \mathcal{E} & \text{if } d_0 \neq 0 \\ \mathcal{E}_{\text{naive}} & \text{if } d_0 = 0 \end{cases}$$

so that the virtual dimension is $d + d_0$.

As a first step, we identify ds_{com} with an explicit map of tautological bundles.

Lemma 1.5.3. *We have a short exact sequence*

$$0 \rightarrow \mathcal{E}nd^{\text{fil}}(\mathcal{V}_\bullet) \rightarrow t_1^{-1} \mathcal{E}nd^{\text{fil}}(\mathcal{V}_\bullet) \oplus t_2^{-1} \mathcal{E}nd^{\text{fil}}(\mathcal{V}_\bullet) \oplus \mathcal{V}_0 \rightarrow T_{\text{ncNHilb}^{\underline{d}}(\mathbb{A}^2)} \rightarrow 0$$

of T -equivariant sheaves on $\text{ncNHilb}^{\underline{d}}(\mathbb{A}^2)$. Using this, the map

$$ds_{\text{com}} : T_{\text{ncNHilb}^{\underline{d}}(\mathbb{A}^2)}|_{\text{NHilb}^{\underline{d}}(\mathbb{A}^2)} \rightarrow \mathcal{E}_{\text{naive}}|_{\text{NHilb}^{\underline{d}}(\mathbb{A}^2)}$$

identifies with

$$\begin{aligned} (t_1^{-1} \mathcal{E}nd^{\text{fil}}(\mathcal{V}_\bullet) \oplus t_2^{-1} \mathcal{E}nd^{\text{fil}}(\mathcal{V}_\bullet) \oplus \mathcal{V}_0) / \mathcal{E}nd^{\text{fil}}(\mathcal{V}_\bullet) &\longrightarrow t_1^{-1} t_2^{-1} \mathcal{E}nd^{\text{fil}}(\mathcal{V}_\bullet) \\ (\phi_1, \phi_2, v) &\longmapsto [\phi_1, \Phi_2] + [\Phi_1, \phi_2] \end{aligned}$$

after restricting the latter to $\text{NHilb}^{\underline{d}}(\mathbb{A}^2)$. The similar map with target $\mathcal{E}|_{\text{NHilb}^{\underline{d}}(\mathbb{A}^2)}$ is obtained by simply composing the above with the quotient $\mathcal{E}_{\text{naive}} \rightarrow \mathcal{E}$.

Proof. We consider the Atiyah sequence

$$0 \rightarrow \mathfrak{p} \otimes_{\mathbb{C}} \mathcal{O} \rightarrow T_{\text{ncNHilb}^{\underline{d}}(\mathbb{A}^2)} \xrightarrow{\sim} \pi^* T_{\text{ncNHilb}^{\underline{d}}(\mathbb{A}^2)} \rightarrow 0$$

for the principal $P_{\underline{d}}$ -bundle $\pi : \text{ncNHilb}^{\underline{d}}(\mathbb{A}^2) \rightarrow \text{ncNHilb}^{\underline{d}}(\mathbb{A}^2)$ where $\mathfrak{p}$ is the Lie algebra of $P_{\underline{d}}$. Setting $V_\bullet = N_\bullet \otimes_{\mathbb{C}} \mathcal{O}_{\text{ncNHilb}^{\underline{d}}(\mathbb{A}^2)}$ for $N_\bullet$ the reference flag we see that

$$T_{\text{ncNHilb}^{\underline{d}}(\mathbb{A}^2)} \xrightarrow{\sim} t_1^{-1} \mathcal{E}nd^{\text{fil}}(V_\bullet) \oplus t_2^{-1} \mathcal{E}nd^{\text{fil}}(V_\bullet) \oplus V_0$$

while

$$\mathfrak{p} \otimes_{\mathbb{C}} \mathcal{O} \cong \mathcal{E}nd^{\text{fil}}(V_{\bullet}).$$

Note that $V_{\bullet}$ with its natural $P_{\underline{d}}$ -equivariant structure descends to the universal flag $\mathcal{V}_{\bullet}$ under the equivalence

$$\widetilde{\text{QCoh}(\text{ncNHilb}^d(\mathbb{A}^2))}^{P_{\underline{d}}} \cong \text{QCoh}(\text{ncNHilb}^d(\mathbb{A}^2)).$$

Using faithful flatness of π , we therefore get that the Atiyah sequence descends to the desired short exact sequence

$$0 \rightarrow \mathcal{E}nd^{\text{fil}}(\mathcal{V}_{\bullet}) \rightarrow t_1^{-1} \mathcal{E}nd^{\text{fil}}(\mathcal{V}_{\bullet}) \oplus t_2^{-1} \mathcal{E}nd^{\text{fil}}(\mathcal{V}_{\bullet}) \oplus \mathcal{V}_0 \rightarrow T_{\text{ncNHilb}^d(\mathbb{A}^2)} \rightarrow 0$$

where the first map is given by $\phi \mapsto ([\phi, \Phi_1], [\phi, \Phi_2], \phi \mathfrak{v})$. Similarly, on the commuting locus $Z(s_{\text{com}}) = \text{NHilb}^d(\mathbb{A}^2)$ the map

$$ds_{\text{com}} : T_{\text{ncNHilb}^d(\mathbb{A}^2)}|_{\text{NHilb}^d(\mathbb{A}^2)} \rightarrow \mathcal{E}_{\text{naive}}|_{\text{NHilb}^d(\mathbb{A}^2)}$$

of equivariant sheaves is induced by the map

$$t_1^{-1} \mathcal{E}nd^{\text{fil}}(V_{\bullet}) \oplus t_2^{-1} \mathcal{E}nd^{\text{fil}}(V_{\bullet}) \rightarrow t_1^{-1} t_2^{-1} \mathcal{E}nd^{\text{fil}}(V_{\bullet})$$

given by $(\phi_1, \phi_2, v) \mapsto [\phi_1, \tilde{\Phi}_2] + [\tilde{\Phi}_1, \phi_2]$ which descends to the map of the lemma. $\square$

As a first step, we consider the non-nested case $\underline{d} = (d)$. Here, we simply have the Hilbert scheme of points $\text{Hilb}^d(\mathbb{A}^2)$ and we will prove that $\mathbb{E} \rightarrow \mathbb{L}_{\text{Hilb}^d(\mathbb{A}^2)}$ is indeed just the smooth perfect obstruction theory. We similarly have that $\text{NHilb}^{(0,d)}(\mathbb{A}^n) = \text{Hilb}^d(\mathbb{A}^n)$ and we prove that $\mathbb{E} \rightarrow \mathbb{L}_{\text{NHilb}^{(0,d)}(\mathbb{A}^2)}$ has obstruction bundle $h^{-1}(\mathbb{E})^{\vee}$ given by the dual of the tautological bundle of the canonical line bundle of $\mathbb{C}^2$.

Lemma 1.5.4. *Let $d \in \mathbb{Z}$. The map $\mathbb{E} \rightarrow \mathbb{L}_{\text{Hilb}^d(\mathbb{A}^2)}$ is an equivalence, i.e. the perfect obstruction theory is the smooth one.*

Proof. By smoothness of $\text{Hilb}^d(\mathbb{A}^2)$ we have $\mathbb{L}_{\text{Hilb}^d(\mathbb{A}^2)} = \Omega_{\text{Hilb}^d(\mathbb{A}^2)}[0]$. Since we also have that the virtual dimension agrees with the actual dimension it will suffice to prove that

$$\mathcal{E}^{\vee} \rightarrow \Omega_{\text{ncHilb}^d(\mathbb{A}^2)}$$

is injective. Using the trace pairing and lemma 1.5.3 we can explicitly identify this map with

$$\mathcal{H}om(\bar{\mathcal{V}}, \mathcal{V}) \rightarrow \ker(\mathcal{E}nd(\mathcal{V})^{\oplus 2} \oplus \mathcal{V} \rightarrow \mathcal{E}nd(\mathcal{V}))$$

which is given by $\phi \mapsto ([\Phi_2, \phi q], [\phi q, \Phi_1], 0)$ for $q : \mathcal{V} \rightarrow \bar{\mathcal{V}}$ the quotient map. If ϕ is in the kernel of this map, then ϕq is an endomorphism of $\mathcal{V}$ which vanishes on the cyclic vector and commutes with Φ_1, Φ_2 hence it must be 0. Since q is an epimorphism we conclude that $\phi = 0$ as desired. $\square$

Lemma 1.5.5. *Let $d \in \mathbb{Z}$. For $\mathbb{E} \rightarrow \mathbb{L}_{\text{NHilb}^{(0,d)}(\mathbb{A}^2)}$ we have $h^{-1}(\mathbb{E})^{\vee} \cong (\omega_{\mathbb{A}^2}^{[d]})^{\vee}$ i.e. the obstruction bundle is the dual of the tautological bundle of the dualizing bundle.*

Proof. It will suffice to identify

$$t_1^{-1} t_2^{-1} \mathcal{V}|_{\text{Hilb}^d(\mathbb{A}^2)} \cong \ker(\mathcal{E}_{\text{naive}}^{\vee} \rightarrow \Omega_{\text{ncHilb}^d(\mathbb{A}^2)})|_{\text{Hilb}^d(\mathbb{A}^2)}$$

since

$$t_1^{-1} t_2^{-1} \mathcal{V}|_{\text{Hilb}^d(\mathbb{A}^2)} = t_1^{-1} t_2^{-1} \mathcal{O}_{\mathbb{A}^2}^{[d]} = (t_1^{-1} t_2^{-1} \mathcal{O}_{\mathbb{A}^2})^{[d]} = \omega_{\mathbb{A}^2}^{[d]}$$

by construction. Repeating the arguments in the proof of lemma 1.5.4 we can directly identify the above kernel with the operators of

$$t_1^{-1}t_2^{-1}\mathcal{E}nd(\mathcal{V}) \cong \mathcal{E}_{\text{naive}}^{\vee}$$

which commutes with the universal operators Φ_1, Φ_2 . Clearly such an operator is uniquely determined by its value on the cyclic vector by the cyclicity condition. Conversely, on the commutativity locus we have an algebra structure on $\mathcal{E}$ viewing it as a quotient of $\mathcal{O}[x, y]$ by a left (hence two-sided) ideal. It follows that for any $w \in \mathcal{V}$ we have an operator $L_w \in \mathcal{E}nd(\mathcal{V})$ given by left multiplication with w . This commutes with Φ_1, Φ_2 and evaluates to w on the cyclic vector. We conclude that the evaluation map

$$\text{ev}_{\mathfrak{v}} : t_1^{-1}t_2^{-1}\mathcal{E}nd(\mathcal{V})|_{\text{Hilb}^d(\mathbb{A}^2)} \rightarrow t_1^{-1}t_2^{-1}\mathcal{V}|_{\text{Hilb}^d(\mathbb{A}^2)}$$

restricts to an isomorphism on the above kernel as desired. $\square$

Next, we will prove that our virtual class is equal to that of Gholampour, Sheshmani and Yau [GSY20].

Proposition 1.5.1. *Let $\underline{d} = (d_0, \dots, d_r) \in \mathbb{Z}_{\geq 0}^{r+1}$. Let $\mathbb{A}^2 \subset \mathbb{P}^2$ as the complement of the hyperplane at ∞ . Let $\widetilde{\mathbb{E}}_{\text{GSY}} \rightarrow \mathbb{L}_{\text{NHilb}^{\underline{d}}(\mathbb{P}^2)}$ be the perfect obstruction theory defined in [GSY20] for the Hilbert scheme of a smooth projective surface. Let $\mathbb{E}_{\text{GSY}} \rightarrow \mathbb{L}_{\text{NHilb}^{\underline{d}}(\mathbb{P}^2)}$ be the restriction to the Zariski open $\text{NHilb}^{\underline{d}}(\mathbb{A}^2) \hookrightarrow \text{NHilb}^{\underline{d}}(\mathbb{P}^2)$ induced by $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$. Then in T -equivariant K -theory we have the identity*

$$[\mathbb{E}_{\text{GSY}}] = [\mathbb{E}]$$

where $\mathbb{E}$ is the perfect obstruction theory constructed above definition 1.5.1. In particular, both theories induce the same virtual invariants, i.e. the same virtual fundamental class and virtual structure sheaf agree.

Proof. We dualize and show that virtual tangent bundles agree. To avoid cumbersome notation we will for $\mathcal{F}$ a coherent/locally free sheaf simply denote its K -theory class by $\mathcal{F}$ instead of $[\mathcal{F}]$. By definition

$$\mathbb{E}^{\vee} = \begin{cases} T_{\text{ncNHilb}^{\underline{d}}(\mathbb{A}^2)}|_{\text{NHilb}^{\underline{d}}(\mathbb{A}^2)} - \mathcal{E}|_{\text{NHilb}^{\underline{d}}(\mathbb{A}^2)} & \text{if } d_0 \neq 0 \\ T_{\text{ncNHilb}^{\underline{d}}(\mathbb{A}^2)}|_{\text{NHilb}^{\underline{d}}(\mathbb{A}^2)} - \mathcal{E}_{\text{naive}}|_{\text{NHilb}^{\underline{d}}(\mathbb{A}^2)} & \text{if } d_0 = 0 \end{cases}$$

By lemma 1.5.3 we can rewrite $T_{\text{ncNHilb}^{\underline{d}}(\mathbb{A}^2)}|_{\text{NHilb}^{\underline{d}}(\mathbb{A}^2)}$ as

$$\mathcal{V}_0 + (t_1^{-1} + t_2^{-1} - 1)\mathcal{E}nd^{\text{fil}}(\mathcal{V}_{\bullet}) = \mathcal{V}_0 + (t_1^{-1} + t_2^{-1} - 1) \sum_{0 \leq i \leq j \leq r-1} (\text{gr}^i \mathcal{V}_{\bullet})^{\vee} \cdot \text{gr}^j \mathcal{V}_{\bullet}.$$

Letting $\mathcal{Q}_i = \mathcal{O}_{\mathbb{A}^2}^{[d_0, \dots, d_{i-1}]}$ be the tautological sheaf for the i 'th subscheme of the nesting, we have for each $0 < i \leq r+1$ a short exact sequence

$$0 \rightarrow \mathcal{V}_i \rightarrow \mathcal{Q}_{r+1} \rightarrow \mathcal{Q}_i \rightarrow 0$$

and if we set $\mathcal{Q}_0 = 0$ by convention then the above extends to $i = 0$. In particular, we get

$$\text{gr}^i(\mathcal{V}_{\bullet}) = \mathcal{V}_i - \mathcal{V}_{i+1} = \mathcal{Q}_{i+1} - \mathcal{Q}_i.$$

We can then rewrite

$$\begin{aligned} \sum_{0 \leq i \leq j \leq r} (\mathrm{gr}^i \mathcal{V}_\bullet)^\vee \cdot \mathrm{gr}^j \mathcal{V}_\bullet &= \sum_{0 \leq i \leq j \leq r} (\mathcal{Q}_{i+1}^\vee - \mathcal{Q}_i^\vee)(\mathcal{Q}_{j+1} - \mathcal{Q}_j) \\ &= \sum_{j=0}^r (\mathcal{Q}_{j+1} - \mathcal{Q}_j) \sum_{i=0}^j (\mathcal{Q}_{i+1}^\vee - \mathcal{Q}_i^\vee) \\ &= \sum_{j=0}^r \mathcal{Q}_{j+1}^\vee (\mathcal{Q}_{j+1} - \mathcal{Q}_j) \end{aligned}$$

where the third equality uses that the sum is alternating. It follows that

$$T_{\mathrm{ncHilb}^d(\mathbb{A}^2)}|_{\mathrm{NHilb}^d(\mathbb{A}^2)} = \mathcal{Q}_{r+1} + (t_1^{-1} + t_1^{-1} - 1) \sum_{j=0}^r \mathcal{Q}_{j+1}^\vee (\mathcal{Q}_{j+1} - \mathcal{Q}_j).$$

If $d_0 \neq 0$ we have

$$\mathcal{E}|_{\mathrm{NHilb}^d(\mathbb{A}^2)} = t_1^{-1} t_2^{-1} \mathcal{H}om^{\mathrm{fil}}(\mathcal{V}_\bullet, \bar{\mathcal{V}}_\bullet) = t_1^{-1} t_2^{-1} \sum_{0 \leq i \leq j \leq r} (\mathrm{gr}^i \mathcal{V}_\bullet)^\vee (\mathrm{gr}^j \bar{\mathcal{V}}_\bullet)$$

where

$$\bar{\mathcal{V}}_i = \begin{cases} \mathcal{V}_0 - 1 & i = 0 \\ \mathcal{V}_i & i > 0 \end{cases}$$

hence

$$\mathcal{E}|_{\mathrm{NHilb}^d(\mathbb{A}^2)} = t_1^{-1} t_2^{-1} \left(-\mathrm{gr}^0 \mathcal{V}_\bullet + \sum_{0 \leq i \leq j \leq r} (\mathrm{gr}^i \mathcal{V}_\bullet)^\vee (\mathrm{gr}^j \mathcal{V}_\bullet) \right).$$

For $d_0 = 0$ we have

$$\mathcal{E}_{\mathrm{naive}}|_{\mathrm{NHilb}^d(\mathbb{A}^2)} = t_1^{-1} t_2^{-1} \mathcal{H}om^{\mathrm{fil}}(\mathcal{V}_\bullet, \mathcal{V}_\bullet) = t_1^{-1} t_2^{-1} \sum_{0 \leq i \leq j \leq r} (\mathrm{gr}^i \mathcal{V}_\bullet)^\vee (\mathrm{gr}^j \mathcal{V}_\bullet)$$

and so using that $\mathrm{gr}^0(\mathcal{V}_\bullet) = 0$ we get the same identity

$$\mathcal{E}_{\mathrm{naive}}|_{\mathrm{NHilb}^d(\mathbb{A}^2)} = t_1^{-1} t_2^{-1} \left(-\mathrm{gr}^0 \mathcal{V}_\bullet + \sum_{0 \leq i \leq j \leq r} (\mathrm{gr}^i \mathcal{V}_\bullet)^\vee (\mathrm{gr}^j \mathcal{V}_\bullet) \right).$$

Using the above we can then identify the bundle term in either case with

$$-t_1^{-1} t_2^{-1} \mathcal{Q}_1^\vee + t_1^{-1} t_2^{-1} \sum_{j=0}^r \mathcal{Q}_{j+1}^\vee (\mathcal{Q}_{j+1} - \mathcal{Q}_j).$$

Using the identity

$$t_1^{-1} + t_2^{-1} - 1 - t_1^{-1} t_2^{-1} = -\frac{(1-t_1)(1-t_2)}{t_1 t_2}$$

and the above we now arrive at the formula

$$\mathbb{E}^\vee = \mathcal{Q}_{r+1} + \frac{\mathcal{Q}_1^\vee}{t_1 t_2} - \frac{(1-t_1)(1-t_2)}{t_1 t_2} \sum_{j=0}^r \mathcal{Q}_{j+1}^\vee (\mathcal{Q}_{j+1} - \mathcal{Q}_j).$$

On the other side, we use the virtual tangent space found in [GT20a, Theorem 2]. Let $n_i = d_0 + \dots + d_{i-1}$ for $i = 1, \dots, r+1$. Then

$$\widetilde{\mathbb{E}}_{\mathrm{GSY}}^\vee \cong \left[T_{\mathrm{Hilb}^{n_{r+1}}(\mathbb{P}^2)} \oplus \dots \oplus T_{\mathrm{Hilb}^{n_1}(\mathbb{P}^2)} \rightarrow \mathcal{E}xt_{\widetilde{\mathcal{P}}}^1(\widetilde{\mathcal{I}}_{r+1}, \widetilde{\mathcal{I}}_r)_0 \oplus \dots \oplus \mathcal{E}xt_{\widetilde{\mathcal{P}}}^1(\widetilde{\mathcal{I}}_2, \widetilde{\mathcal{I}}_1)_0 \right]$$

where

$$\tilde{p}: \mathrm{NHilb}^d(\mathbb{P}^2) \times \mathbb{P}^2 \rightarrow \mathrm{NHilb}^d(\mathbb{P}^2)$$

is the projection, $\tilde{\mathcal{I}}_i$ are the ideal sheaves of the universal subschemes $\tilde{\mathcal{Z}}_i \subset \mathrm{NHilb}^d(\mathbb{P}^2) \times \mathbb{P}^2$ and $\mathcal{E}xt_{\tilde{p}}^i(-, -)$ is the i 'th cohomology sheaf of

$$R\mathcal{H}om_{\tilde{p}}(-, -) := \tilde{p}_* R\mathcal{H}om(-, -)$$

with the subscript 0 denoting the trace free part. Let $j: \mathbb{A}^2 \hookrightarrow \mathbb{P}^2$ and $\iota: \mathrm{NHilb}^d(\mathbb{A}^2) \hookrightarrow \mathrm{NHilb}^d(\mathbb{P}^2)$ be the open immersions. Clearly $\iota^* T_{\mathrm{Hilb}^{n_i}(\mathbb{P}^2)} \cong T_{\mathrm{Hilb}^{n_i}(\mathbb{A}^2)}$ and since our perfect obstruction theory for $r = 1$ is the smooth one the above calculations show that

$$\iota^* T_{\mathrm{Hilb}^{n_i}(\mathbb{P}^2)} = \mathcal{Q}_i + \frac{\mathcal{Q}_i^\vee}{t_1 t_2} - \frac{(1 - t_1)(1 - t_2)}{t_1 t_2} \mathcal{Q}_i^\vee \mathcal{Q}_i.$$

To make the notation less cumbersome for the calculation of the ext-term, we will set $S = \mathbb{A}^2$, $\tilde{S} = \mathbb{P}^2$, $X = \mathrm{NHilb}^d(\mathbb{A}^2)$ and $\tilde{X} = \mathrm{NHilb}^d(\mathbb{P}^2)$. We will let $\mathcal{Z}_k \subset X \times S$ be the universal families with ideals $\mathcal{I}_k$. Note that by definition $(\iota \times \mathrm{id})^* \tilde{\mathcal{Z}}_k \subset X \times \tilde{S}$ is supported on $X \times S$ and agrees with $\mathcal{Z}_k$. We finally let ω_S and $\omega_{\tilde{S}}$ be the dualizing bundles. We consider the diagram

$$\begin{array}{ccccc} X \times S & \xrightarrow{\mathrm{id} \times j} & X \times \tilde{S} & \xrightarrow{\iota \times \mathrm{id}} & \tilde{X} \times \tilde{S} \\ & \searrow p & \downarrow p' & & \downarrow \tilde{p} \\ & & X & \xrightarrow{\iota} & \tilde{X} \end{array}$$

where the square is cartesian. Using that

$$\tilde{\mathcal{I}}_k = \mathcal{O}_{\tilde{X} \times \tilde{S}} - \mathcal{O}_{\tilde{\mathcal{Z}}_k}$$

in K -theory we get that $R\mathcal{H}om_{\tilde{p}}(\tilde{\mathcal{I}}_{i+1}, \tilde{\mathcal{I}}_i)$ equals

$$R\mathcal{H}om_{\tilde{p}}(\mathcal{O}_{\tilde{X} \times \tilde{S}}, \mathcal{O}_{\tilde{X} \times \tilde{S}}) - R\mathcal{H}om_{\tilde{p}}(\mathcal{O}_{\tilde{X} \times \tilde{S}}, \mathcal{O}_{\tilde{\mathcal{Z}}_i}) - R\mathcal{H}om_{\tilde{p}}(\mathcal{O}_{\tilde{\mathcal{Z}}_{i+1}}, \mathcal{O}_{\tilde{X} \times \tilde{S}}) + R\mathcal{H}om_{\tilde{p}}(\mathcal{O}_{\tilde{\mathcal{Z}}_{i+1}}, \mathcal{O}_{\tilde{\mathcal{Z}}_i})$$

in K -theory. We see that

$$R\mathcal{H}om_{\tilde{p}}(\mathcal{O}_{\tilde{X} \times \tilde{S}}, \mathcal{O}_{\tilde{X} \times \tilde{S}}) = R\tilde{p}_* \mathcal{O}_{\tilde{X} \times \tilde{S}} = \mathcal{O}_{\tilde{X}}$$

by the usual calculation of the cohomology of $\mathcal{O}_{\mathbb{P}^2}$. Similarly, we have

$$R\mathcal{H}om_{\tilde{p}}(\mathcal{O}_{\tilde{X} \times \tilde{S}}, \mathcal{O}_{\tilde{\mathcal{Z}}_i}) = R\tilde{p}_* \mathcal{O}_{\tilde{\mathcal{Z}}_i}.$$

and by Grothendieck duality we have quasi-isomorphisms

$$R\mathcal{H}om_{\tilde{p}}(\mathcal{O}_{\tilde{\mathcal{Z}}_{i+1}}, \mathcal{O}_{\tilde{X} \times \tilde{S}})[2] \cong R\mathcal{H}om_{\tilde{p}}(\omega_{\tilde{S}}^{-1}, \mathcal{O}_{\tilde{\mathcal{Z}}_{i+1}})^\vee \cong (R\tilde{p}_*(\omega_{\tilde{S}} \otimes \mathcal{O}_{\tilde{\mathcal{Z}}_{i+1}}))^\vee$$

so that in K -theory we get

$$R\mathcal{H}om_{\tilde{p}}(\mathcal{O}_{\tilde{\mathcal{Z}}_{i+1}}, \mathcal{O}_{\tilde{X} \times \tilde{S}}) = (R\tilde{p}_*(\omega_{\tilde{S}} \otimes \mathcal{O}_{\tilde{\mathcal{Z}}_{i+1}}))^\vee$$

as the shift $[2]$ contributes with the sign $(-1)^2$. Let

$$R\mathcal{H}om_{\tilde{p}}(\tilde{\mathcal{I}}_{i+1}, \tilde{\mathcal{I}}_i) = \mathcal{E}xt_{\tilde{p}}^0(\tilde{\mathcal{I}}_{i+1}, \tilde{\mathcal{I}}_i) - \mathcal{E}xt_{\tilde{p}}^1(\tilde{\mathcal{I}}_{i+1}, \tilde{\mathcal{I}}_i) + \mathcal{E}xt_{\tilde{p}}^2(\tilde{\mathcal{I}}_{i+1}, \tilde{\mathcal{I}}_i).$$

We first note that

$$\mathcal{H}om_{\tilde{p}}(\tilde{\mathcal{I}}_{i+1}, \tilde{\mathcal{I}}_i) \hookrightarrow \mathcal{H}om_{\tilde{p}}(\tilde{\mathcal{I}}_{i+1}, \mathcal{O}_{\tilde{X} \times \tilde{S}}) \cong R\tilde{p}_* \mathcal{O}_{\tilde{X} \times \tilde{S}}$$

and the inclusion $\tilde{\mathcal{I}}_{i+1} \subset \tilde{\mathcal{I}}_i$ gives us an inverse providing us with an identification

$$\mathcal{E}xt_{\tilde{p}}^0(\tilde{\mathcal{I}}_{i+1}, \tilde{\mathcal{I}}_i) = R\tilde{p}_* \mathcal{O}_{\tilde{X} \times \tilde{S}} = \mathcal{O}_{\tilde{X}}.$$

For a closed point corresponding to $I_{i+1} \subset I_i$ we have

$$\mathrm{Hom}_{\tilde{S}}(I_i, I_{i+1} \otimes_{\tilde{S}} \omega_{\tilde{S}}) \hookrightarrow \mathrm{Hom}_{\tilde{S}}(I_i, \omega_{\tilde{S}}) = H^0(\tilde{S}, \omega_{\tilde{S}}) = 0$$

where the last calculation follows since $\omega_{\tilde{S}} = \mathcal{O}(-3)$. We conclude by Serre duality that $\mathrm{Ext}_{\tilde{S}}^2(I_{i+1}, I_i) = 0$ and hence by Nakayama that

$$\mathcal{E}xt_{\tilde{p}}^2(\tilde{\mathcal{I}}_{i+1}, \tilde{\mathcal{I}}_i) = 0.$$

Note also that

$$\mathcal{E}xt_{\tilde{p}}^1(\tilde{\mathcal{I}}_{i+1}, \tilde{\mathcal{I}}_i)_0 = \mathcal{E}xt_{\tilde{p}}^1(\tilde{\mathcal{I}}_{i+1}, \tilde{\mathcal{I}}_i)$$

since $R^1\tilde{p}_* \mathcal{O}_{\tilde{X} \times \tilde{S}} = 0$. We conclude that $\iota^* \mathcal{E}xt_{\tilde{p}}^1(\tilde{\mathcal{I}}_{i+1}, \tilde{\mathcal{I}}_i)_0$ equals

$$\iota^* R\tilde{p}_* \mathcal{O}_{\tilde{Z}_i} + \iota^* (R\tilde{p}_* (\omega_{\tilde{S}} \otimes \mathcal{O}_{\tilde{Z}_{i+1}}))^{\vee} - \iota^* R\mathcal{H}om_{\tilde{p}}(\mathcal{O}_{\tilde{Z}_{i+1}}, \mathcal{O}_{\tilde{Z}_i}).$$

By cohomology and base change we have $\iota^* R\tilde{p}_* = Rp'_*(\iota \times \mathrm{id})^*$ and using that $(\iota \times \mathrm{id})^* \mathcal{O}_{\tilde{Z}_k} = (\mathrm{id} \times j)_* \mathcal{O}_{Z_k}$ we see that the above equals

$$Rp_* \mathcal{O}_{Z_i} + (Rp_*(\omega_S \otimes \mathcal{O}_{Z_{i+1}}))^{\vee} - R\mathcal{H}om_p(\mathcal{O}_{Z_{i+1}}, \mathcal{O}_{Z_i}).$$

Note that p is affine with $p_* \mathcal{O}_{Z_k} = \mathcal{Q}_k$ by definition. It follows that

$$Rp_* \mathcal{O}_{Z_i} = \mathcal{Q}_i$$

and since $\omega_S = p^*(t_1 t_2 \mathcal{O}_X)$ we also get

$$(Rp_*(\omega_S \otimes \mathcal{O}_{Z_{i+1}}))^{\vee} = (t_1 t_2 \mathcal{Q}_{i+1})^{\vee} = \frac{\mathcal{Q}_{i+1}^{\vee}}{t_1 t_2}.$$

Since $\mathcal{O}_{Z_{i+1}}$ is a perfect complex and $X \times S$ has the resolution property we can find a finite resolution of T -equivariant bundles, which is necessarily of the form

$$E_m \otimes_X \mathcal{O}_{X \times S} \rightarrow E_{m-1} \otimes_X \mathcal{O}_{X \times S} \rightarrow \cdots \rightarrow E_1 \otimes_X \mathcal{O}_{X \times S} \rightarrow \mathcal{O}_{Z_{i+1}}$$

for E_i locally free T -equivariant sheaves on X by K -theoretic homotopy invariance since $S = \mathbb{A}^2$. Then in K -theory

$$R\mathcal{H}om_p(\mathcal{O}_{Z_{i+1}}, \mathcal{O}_{Z_i}) = \sum_{j=1}^m (-1)^j Rp_* R\mathcal{H}om_{X \times S}(Lp^* E_j, \mathcal{O}_{Z_i}) = \sum_{j=1}^m (-1)^j R\mathcal{H}om_X(E_j, Rp_* \mathcal{O}_{Z_i})$$

so that

$$R\mathcal{H}om_p(\mathcal{O}_{Z_{i+1}}, \mathcal{O}_{Z_i}) = \sum_{j=1}^m (-1)^j E_j^{\vee} \mathcal{Q}_i.$$

But we see that

$$\mathcal{Q}_{i+1} = p_* \mathcal{O}_{Z_{i+1}} = \sum_{j=1}^m (-1)^j p_*(E_j \otimes_X \mathcal{O}_{X \times S})$$

and as an $\mathcal{O}_X$ algebra we have $\mathcal{O}_{X \times S} = \mathcal{O}_X[x_1, x_2]$ which has T -character

$$\sum_{(a,b) \in \mathbb{Z}^2} t_1^a t_2^b \mathcal{O}_X = \frac{1}{(1-t_1)(1-t_2)} \mathcal{O}_X$$

so that

$$\mathcal{Q}_{i+1} = \sum (-1)^j \frac{1}{(1-t_1)(1-t_2)} E_j$$

hence

$$\sum (-1)^j E_j^\vee = (1-t_1^{-1})(1-t_2^{-1}) \mathcal{Q}_{i+1}^\vee = \frac{(1-t_1)(1-t_2)}{t_1 t_2} \mathcal{Q}_{i+1}^\vee.$$

Combining this we see that

$$\iota^* \mathcal{E} x t_{\tilde{p}}^1(\tilde{\mathcal{I}}_i, \tilde{\mathcal{I}}_{i+1})_0 = \mathcal{Q}_i + \frac{\mathcal{Q}_{i+1}^\vee}{t_1 t_2} - \frac{(1-t_1)(1-t_2)}{t_1 t_2} \mathcal{Q}_{i+1}^\vee \mathcal{Q}_i$$

and so $\mathbb{E}_{\text{GSY}}^\vee$ equals

$$\begin{aligned} & \sum_{i=1}^{r+1} \left(\mathcal{Q}_i + \frac{\mathcal{Q}_i^\vee}{t_1 t_2} - \frac{(1-t_1)(1-t_2)}{t_1 t_2} \mathcal{Q}_i^\vee \mathcal{Q}_i \right) - \sum_{i=1}^r \left(\mathcal{Q}_i + \frac{\mathcal{Q}_{i+1}^\vee}{t_1 t_2} - \frac{(1-t_1)(1-t_2)}{t_1 t_2} \mathcal{Q}_{i+1}^\vee \mathcal{Q}_i \right) \\ &= \mathcal{Q}_{r+1} + \frac{\mathcal{Q}_1^\vee}{t_1 t_2} - \frac{(1-t_1)(1-t_2)}{t_1 t_2} \sum_{j=0}^r \mathcal{Q}_{j+1}^\vee (\mathcal{Q}_{j+1} - \mathcal{Q}_j) \end{aligned}$$

as desired. $\square$

We end this section by proving that two maps between nested Hilbert schemes forgetting the largest or the smallest subscheme may be equipped with relative perfect obstruction theories compatible with the obstruction theories above. Recall that for the dimension vector $(d_0, \dots, d_r)$ we keep track of a nesting of $r+1$ ideals

$$I_{r+1} \subset \dots \subset I_1$$

in $I_0 = \mathbb{C}[x, y]$ via the flag of vectorspaces $V_\bullet$ with

$$V_i = I_i / I_{r+1}.$$

The map forgetting the first ideal simply forgets the V_1 part of the flag and changes the dimension vector to

$$(d_0 + d_1, d_2, \dots, d_r).$$

The map forgetting the last, i.e. $(r+1)$ 'th ideal takes $V_\bullet$ to the quotient flag $W_\bullet$ where

$$W_i = V_i / V_r \cong I_i / I_r$$

and changes the dimension vector to

$$(d_0, \dots, d_{r-1}).$$

We start by showing the case of forgetting the smallest subscheme, i.e. the first ideal.

Lemma 1.5.6. *Let $\underline{d} = (d_0, \dots, d_r) \in \mathbb{Z}_{\geq 0}^{r+1}$ be a dimension vector. Set*

$$\underline{d}^{(1)} = (d_0 + d_1, d_2, \dots, d_r) \in \mathbb{Z}_{\geq 0}^r$$

and let

$$f = f_{\underline{d}, \underline{d}^{(1)}} : \text{ncNHilb}^{\underline{d}}(\mathbb{A}^2) \rightarrow \text{ncNHilb}^{\underline{d}^{(1)}}(\mathbb{A}^2)$$

be the map which is given by the non-commutative tuple $(\mathcal{V}_\bullet^{\hat{1}}, \Phi_1, \Phi_2, \mathfrak{v})$ where

$$\mathcal{V}_\bullet^{\hat{1}} = (\mathcal{V}_0 \supset \mathcal{V}_2 \supset \mathcal{V}_3 \cdots \supset \mathcal{V}_{r+1} = 0)$$

i.e., where we simply forget the first part of the filtration. There exists a map of equivariant bundles

$$\psi : \mathcal{E}_{\underline{d}} \rightarrow f^* \mathcal{E}_{\underline{d}^{(1)}}$$

such that the pair (f, ψ) satisfies the conditions of lemma 1.3.1. In particular, the corresponding forgetful map

$$\mathrm{NHilb}^{\underline{d}}(\mathbb{A}^2) \rightarrow \mathrm{NHilb}^{\underline{d}^{(1)}}(\mathbb{A}^2)$$

can be equipped with a relative perfect obstruction theory compatible with the obstruction theories defined in definition 1.5.1.

Proof. Suppose first that $d_0 \neq 0$. Define ψ as the inclusion

$$t_1^{-1} t_2^{-1} \mathcal{H}om^{\mathrm{fil}}(\mathcal{V}_{\bullet}, \overline{\mathcal{V}}_{\bullet}) \hookrightarrow t_1^{-1} t_2^{-1} \mathcal{H}om^{\mathrm{fil}}(\mathcal{V}_{\bullet}^{\hat{1}}, \overline{\mathcal{V}}_{\bullet}^{\hat{1}}).$$

Clearly ψ maps the commutator of the universal maps to itself. It remains to show the transversality condition. Using faithful flatness of the group quotient and injectivity of the map

$$\pi^* \Omega_{\mathrm{ncNHilb}^{\underline{d}}(\mathbb{A}^2)} \rightarrow \Omega_{\widetilde{\mathrm{ncNHilb}^{\underline{d}}(\mathbb{A}^2)}}$$

we can reduce the calculation to $\widetilde{\mathrm{ncNHilb}^{\underline{d}}(\mathbb{A}^2)}$. If we set $V_{\bullet} = N_{\bullet} \otimes \mathcal{O}_{\widetilde{\mathrm{ncNHilb}^{\underline{d}}(\mathbb{A}^2)}}$ then

$$\pi^* \mathcal{E}_{\underline{d}} = \mathcal{H}om^{\mathrm{fil}}(V_{\bullet}, \overline{V}_{\bullet}),$$

$$\pi^* f^* \mathcal{E}_{\underline{d}^{(1)}} = \mathcal{H}om^{\mathrm{fil}}(V_{\bullet}^{\hat{1}}, \overline{V}_{\bullet}^{\hat{1}})$$

and

$$f^* T_{\mathrm{ncNHilb}^{\underline{d}^{(1)}}(\mathbb{A}^2)} \widetilde{\phantom{f^* T_{\mathrm{ncNHilb}^{\underline{d}^{(1)}}(\mathbb{A}^2)}}} = \mathcal{E}nd^{\mathrm{fil}}(V_{\bullet}^{\hat{1}})^{\oplus 2} \oplus V_0.$$

By dualizing we should show that the map

$$(\mathcal{E}nd^{\mathrm{fil}}(V_{\bullet}^{\hat{1}})^{\oplus 2} \oplus V_0) \oplus \mathcal{H}om^{\mathrm{fil}}(V_{\bullet}, \overline{V}_{\bullet}) \xrightarrow{(-\mathrm{ds}_{\mathrm{com}}, \phi)} \mathcal{H}om^{\mathrm{fil}}(V_{\bullet}^{\hat{1}}, \overline{V}_{\bullet}^{\hat{1}})$$

is surjective on the commutativity locus. Equivalently, by taking the quotient by the second summand of the source which is an inclusion we should show that the map

$$\mathcal{E}nd^{\mathrm{fil}}(V_{\bullet}^{\hat{1}})^{\oplus 2} \oplus V_0 \xrightarrow{-\mathrm{ds}_{\mathrm{com}}} \mathcal{H}om^{\mathrm{fil}}(V_{\bullet}^{\hat{1}}, \overline{V}_{\bullet}^{\hat{1}}) / \mathcal{H}om^{\mathrm{fil}}(V_{\bullet}, \overline{V}_{\bullet}) = \mathcal{H}om(V_1/V_2, \overline{V}_0/\overline{V}_1)$$

is a surjection. We can remove the V_0 summand from the source, since $\mathrm{ds}_{\mathrm{com}}$ is zero on it. Taking the dual and using the trace pairing we should show injectivity of

$$\mathcal{H}om(\overline{V}_0/\overline{V}_1, V_1/V_2) \rightarrow (\mathcal{E}nd(V_{\bullet}^{\hat{1}}) / \mathcal{H}om^{\mathrm{fil}}(V_{\bullet}^{\hat{1}}, V_{\bullet+1}^{\hat{1}}))^{\oplus 2}.$$

If $\varphi : \overline{V}_0/\overline{V}_1 \rightarrow V_1/V_2$ is in the kernel of this map then for $k = 1, 2$ the map

$$[\tilde{\Phi}_k, \varphi]$$

must take $V_0^{\hat{1}} = V_0$ to $V_1^{\hat{1}} = V_2$ and therefore vanish as a map $V_0/V_1 \rightarrow V_1/V_2$. But this means that φ commutes with $\tilde{\Phi}_k$ and vanishes on the cyclic vector so that it must be 0. This finishes the proof in the $d_0 \neq 0$ case. In the case where $d_0 = 0$ we have $\mathcal{V}_0 = \mathcal{V}_1$ so the flag-preserving maps of $\mathcal{V}_{\bullet}$ and $\mathcal{V}_{\bullet}^{\hat{1}}$ are the same. When $d_1 = 0$ we can simply take the identity map

$$\psi : t_1^{-1} t_2^{-1} \mathcal{H}om^{\mathrm{fil}}(\mathcal{V}_{\bullet}, \mathcal{V}_{\bullet}) = t_1^{-1} t_2^{-1} \mathcal{H}om^{\mathrm{fil}}(\mathcal{V}_{\bullet}^{\hat{1}}, \mathcal{V}_{\bullet}^{\hat{1}})$$

while in the case where $d_1 \neq 0$ we can take the quotient map

$$\psi : t_1^{-1} t_2^{-1} \mathcal{H}om^{\mathrm{fil}}(\mathcal{V}_{\bullet}, \mathcal{V}_{\bullet}) = t_1^{-1} t_2^{-1} \mathcal{H}om^{\mathrm{fil}}(\mathcal{V}_{\bullet}^{\hat{1}}, \mathcal{V}_{\bullet}^{\hat{1}}) \twoheadrightarrow t_1^{-1} t_2^{-1} \mathcal{H}om^{\mathrm{fil}}(\mathcal{V}_{\bullet}^{\hat{1}}, \overline{\mathcal{V}}_{\bullet}^{\hat{1}}).$$

Both maps are surjective, so that the transversality requirement is automatic. $\square$

Next, we focus on the map forgetting the largest subscheme. Here, we are able to show a bit more and provide a factorization of the virtual structure sheaf.

Lemma 1.5.7. *Let $\underline{d} = (d_0, \dots, d_r) \in \mathbb{Z}_{\geq 0}^{r+1}$ be a dimension vector. Set*

$$\hat{\underline{d}} = (d_0, \dots, d_{r-1}) \in \mathbb{Z}_{\geq 0}^r$$

and let

$$p = p_{\underline{d}, \hat{\underline{d}}} : \text{ncNHilb}^{\underline{d}}(\mathbb{A}^2) \rightarrow \text{ncNHilb}^{\hat{\underline{d}}}(\mathbb{A}^2)$$

be the map given by the non-commutative tuple $(\mathcal{V}_\bullet / \mathcal{V}_r, \Phi_1, \Phi_2, \mathfrak{v})$. There exists a map of equivariant bundles

$$\psi : \mathcal{E}_{\underline{d}} \rightarrow p^* \mathcal{E}_{\hat{\underline{d}}}$$

such that the pair (p, ψ) satisfies the conditions of lemma 1.3.1. In particular, the forgetful map

$$p : \text{NHilb}^{\underline{d}}(\mathbb{A}^2) \rightarrow \text{NHilb}^{\hat{\underline{d}}}(\mathbb{A}^2)$$

can be equipped with a relative perfect obstruction theory compatible with the obstruction theories defined in definition 1.5.1. Furthermore, there exists a class $\mathcal{E}_{\underline{d}/\hat{\underline{d}}} \in K_T^0(\text{NHilb}^{\underline{d}}(\mathbb{A}^2))$ such that

$$\mathcal{O}_{\underline{d}}^{\text{vir}} = \mathcal{E}_{\underline{d}/\hat{\underline{d}}} \cdot p^* \mathcal{O}_{\hat{\underline{d}}}^{\text{vir}}.$$

In particular, by the projection formula we get that

$$p_* \mathcal{O}_{\underline{d}}^{\text{vir}} = p_*(\mathcal{E}_{\underline{d}/\hat{\underline{d}}}) \cdot \mathcal{O}_{\hat{\underline{d}}}^{\text{vir}}.$$

Proof. We simply define ψ as the quotient map

$$\psi : \mathcal{E}_{\underline{d}} = t_1^{-1} t_2^{-1} \mathcal{H}om^{\text{fil}}(\mathcal{V}_\bullet, \bar{\mathcal{V}}_\bullet) \rightarrow t_1^{-1} t_2^{-1} \mathcal{H}om^{\text{fil}}(\mathcal{V}_\bullet / \mathcal{V}_r, \bar{\mathcal{V}}_\bullet / \bar{\mathcal{V}}_r) = p^* \mathcal{E}_{\hat{\underline{d}}}$$

which clearly takes the commutator of the universal operators to itself. Surjectivity of ψ automatically implies the transversality requirement, so that the pair (p, ψ) satisfies the conditions of lemma 1.3.1. If we let $\mathcal{F} = \ker(\psi)$ we have

$$\mathcal{E}_{\underline{d}} = \mathcal{F} + p^* \mathcal{E}_{\hat{\underline{d}}} \in K_T^0(\text{ncNHilb}^{\underline{d}}(\mathbb{A}^2))$$

and so

$$\wedge_{-1}^\bullet(\mathcal{E}_{\underline{d}}^\vee) = \wedge_{-1}^\bullet(\mathcal{F}^\vee) \cdot p^*(\wedge_{-1}^\bullet(\mathcal{E}_{\hat{\underline{d}}}^\vee))$$

hence $\mathcal{E}_{\underline{d}/\hat{\underline{d}}} := \wedge_{-1}^\bullet(\mathcal{F}^\vee)|_{\text{NHilb}^{\underline{d}}(\mathbb{A}^2)}$ has the desired property. $\square$

Remark 1.5.1. In the case where we forget the smallest subscheme we get a similar but slightly different factorization of the virtual structure sheaf. Let f, ψ be as in lemma 1.5.6. When $d_0 = d_1 = 0$ we see that ψ is an isomorphism, so that we have

$$\mathcal{O}_{\underline{d}}^{\text{vir}} = f^* \mathcal{O}_{\underline{d}^{(1)}}^{\text{vir}}.$$

When $d_0 = 0$ and $d_1 \neq 0$ we see that ψ is a surjection so by setting $\mathcal{E}_{\underline{d}/\underline{d}^{(1)}} = \wedge_{-1}^\bullet(\ker \psi)^\vee$ we get

$$\mathcal{O}_{\underline{d}}^{\text{vir}} = \mathcal{E}_{\underline{d}/\underline{d}^{(1)}} \cdot f^* \mathcal{O}_{\underline{d}^{(1)}}^{\text{vir}}.$$

In the case where $d_0 \neq 0$ we see that the map ψ is injective so by setting $\mathcal{K}_{\underline{d}/\underline{d}^{(1)}} = \wedge_{-1}^\bullet(\text{coker } \psi)^\vee$ we get

$$\mathcal{K}_{\underline{d}/\underline{d}^{(1)}} \cdot \mathcal{O}_{\underline{d}}^{\text{vir}} = f^* \mathcal{O}_{\underline{d}^{(1)}}^{\text{vir}}.$$

1.6 Localization calculations

1.6.1 Localization of the factorization

In this section we directly compute $p_*(\mathcal{E}_{\underline{d}/\hat{d}})$ via localization. Note that if we can prove that

$$p_*(\mathcal{E}_{\underline{d}/\hat{d}}) = q^* \chi(\mathrm{Hilb}^{d_r}(\mathbb{A}^2), \wedge_{-1}^\bullet(\omega_{\mathbb{A}^2}^{[d_r]}))$$

then this will immediately imply theorem 1.2.1. Recall that we proved in proposition 1.5.1 that

$$T_{\underline{d}}^{\mathrm{vir}} = \mathbb{E}_{\underline{d}}^\vee = \mathcal{Q}_{r+1} + \frac{\mathcal{Q}_1^\vee}{t_1 t_2} - \frac{(1-t_1)(1-t_2)}{t_1 t_2} \sum_{j=0}^r \mathcal{Q}_{j+1}^\vee (\mathcal{Q}_{j+1} - \mathcal{Q}_j)$$

where $\mathcal{Q}_i$ is the tautological bundle of the i 'th subscheme. In particular, if we for a nested Young diagram $\lambda_\bullet$ set

$$Q_{\lambda_i} = \mathcal{O}_{\mathbb{C}[x,y]/I_{\lambda_i}} \in K_T^0(\mathrm{pt})$$

we get that

$$T_{\underline{d}, \lambda_\bullet}^{\mathrm{vir}, \vee} = Q_{\lambda_{r+1}}^\vee + t_1 t_2 Q_{\lambda_1} - (1-t_1)(1-t_2) \sum_{j=0}^r Q_{\lambda_{j+1}} (Q_{\lambda_{j+1}}^\vee - Q_{\lambda_j}^\vee).$$

Lemma 1.6.1. *Let $\underline{d} = (d_0, \dots, d_r) \in \mathbb{Z}_{\geq 0}^r$ be a dimension vector and let $\mathcal{E}_{\underline{d}/\hat{d}} \in K_T^0(\mathrm{NHilb}^{\underline{d}}(\mathbb{A}^2))$ be the class of lemma 1.5.7. Let $\lambda_\bullet$ be a nested partition of type $\underline{d}$ and let*

$$\mathrm{cont}_{\lambda_\bullet}(\mathcal{E}_{\underline{d}/\hat{d}}) \in K_T^0(\mathrm{pt})$$

be the contribution of the Thomason localization of $\mathcal{E}_{\underline{d}/\hat{d}}$ at the fixed point corresponding to $\lambda_\bullet$. Then

$$\mathrm{cont}_{\lambda_\bullet}(\mathcal{E}_{\underline{d}/\hat{d}}) = \frac{1}{\wedge_{-1}^\bullet(\mathcal{C}_{\lambda_r, \lambda_{r+1}})}$$

where

$$\mathcal{C}_{\lambda_r, \lambda_{r+1}} = (Q_{\lambda_{r+1}}^\vee - Q_{\lambda_r}^\vee)(1 - (1-t_1)(1-t_2)Q_{\lambda_{r+1}}).$$

In particular, the contribution depends only on $\lambda_r \subset \lambda_{r+1}$.

Proof. By definition we see that

$$\mathrm{cont}_{\lambda_\bullet}(\mathcal{E}_{\underline{d}/\hat{d}}) = \frac{1}{\wedge_{-1}^\bullet(T_{\underline{d}, \lambda_\bullet}^{\mathrm{vir}, \vee} - p^* T_{\hat{d}, \lambda_\bullet}^{\mathrm{vir}, \vee})}.$$

We then directly see that

$$\begin{aligned} T_{\underline{d}, \lambda_\bullet}^{\mathrm{vir}, \vee} &= Q_{\lambda_{r+1}}^\vee + t_1 t_2 Q_{\lambda_1} - (1-t_1)(1-t_2) \sum_{j=0}^r Q_{\lambda_{j+1}} (Q_{\lambda_{j+1}}^\vee - Q_{\lambda_j}^\vee) \\ &= \sum_{i=0}^r (Q_{\lambda_{j+1}}^\vee - Q_{\lambda_j}^\vee) + t_1 t_2 Q_{\lambda_1} - (1-t_1)(1-t_2) \sum_{j=0}^r Q_{\lambda_{j+1}} (Q_{\lambda_{j+1}}^\vee - Q_{\lambda_j}^\vee) \\ &= Q_{\lambda_1}^\vee + t_1 t_2 Q_{\lambda_1} - (1-t_1)(1-t_2) Q_{\lambda_1}^\vee Q_{\lambda_1} + \sum_{i=1}^r (Q_{\lambda_{j+1}}^\vee - Q_{\lambda_{j-1}}^\vee)(1 - (1-t_1)(1-t_2)Q_{\lambda_{j+1}}). \end{aligned}$$

In particular, we see that

$$T_{\underline{d}, \lambda_\bullet}^{\mathrm{vir}, \vee} - p^* T_{\hat{d}, \lambda_\bullet}^{\mathrm{vir}, \vee} = (Q_{\lambda_{r+1}}^\vee - Q_{\lambda_r}^\vee)(1 - (1-t_1)(1-t_2)Q_{\lambda_{r+1}})$$

as desired. $\square$

With this in mind, we make the following definition.

Definition 1.6.1. For any nesting of Young diagrams $\lambda \subset \mu$, we define the expression

$$P_{\mu/\lambda} = \frac{1}{\wedge_{-1}^{\bullet}(\mathcal{C}_{\lambda,\mu})}$$

and for fixed λ we let

$$F_{\lambda}(u) = \sum_{\lambda \subset \mu} u^{|\mu| - |\lambda|} P_{\mu/\lambda} \in \mathbb{Q}(t_1, t_2)[[u]].$$

Remark 1.6.1. We note that for a nested partition $\lambda_{\bullet} = \lambda_1 \subset \cdots \subset \lambda_r$ of type $\hat{d}$ and

$$p : \text{NHilb}^{\hat{d}}(\mathbb{A}^2) \rightarrow \text{NHilb}^{\hat{d}}(\mathbb{A}^2)$$

we get

$$\text{cont}_{\lambda_{\bullet}}(p_*(\mathcal{E}_{\hat{d}/\hat{d}})) = \sum_{\substack{\lambda_r \subset \mu \\ |\mu| - |\lambda_r| = d_r}} P_{\mu/\lambda_r} = [u^{d_r}] F_{\lambda_r}(u).$$

We also note that for a Young diagram μ with $|\mu| = d$ we have

$$P_{\mu/\emptyset} = \text{cont}_{\mu}(\wedge_{-1}^{\bullet} \omega_{\mathbb{A}^2}^{[d]})$$

so that

$$F_{\emptyset}(u) = \sum_{d \geq 0} u^d \chi_T(\text{NHilb}^d(\mathbb{A}^2), \wedge_{-1}^{\bullet}(\omega_{\mathbb{A}^2}^{[d]})) = \text{PE} \left(\frac{u(1 - t_1 t_2)}{(1 - t_1)(1 - t_2)} \right).$$

We conclude that theorem 1.2.1 follows directly if we can prove that

$$F_{\lambda}(u) = F_{\emptyset}(u).$$

1.6.2 Independence of the generating series

We will now prove that $F_{\lambda}(u)$ is independent of λ finishing the proof theorem 1.2.1. We start by introducing some notation to ease the later calculations.

Setup 1.6.1. For any subset $A \subset \mathbb{Z}^2$ we write $x, y, z, v, w, \dots \in A$ when we really mean

$$x, y, z, v, w, \dots = t_1^{a_1} t_2^{a_2} \in \mathbb{Z}[t_1^{\pm}, t_2^{\pm}]$$

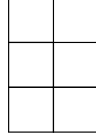
for some $(a_1, a_2) \in A$. In this notation it follows that we have

$$\mathcal{C}_{\lambda,\mu} = \sum_{z \in \mu \setminus \lambda} z^{-1} \left(1 - (1 - t_1)(1 - t_2) \sum_{v \in \mu} v \right)$$

for a nesting $\lambda \subset \mu$ of Young diagrams. For a Young diagram λ and boxes $x \in \lambda$ and $y \in \mathbb{Z}^2 \setminus \lambda$ we will say that x is a removable box if $\lambda \setminus \{x\}$ is still a Young diagram and we will say that y is an addable box if $\lambda \cup \{y\}$ is still a partition. We denote by $R(\lambda)$, $A(\lambda)$ the set of removable respectively addable boxes of λ . Note that

$$|R(\lambda)| + 1 = |A(\lambda)|$$

as can easily be proven by induction on the number of boxes of $|\lambda|$. We will typically abuse notation by identifying the elements of these sets by their algebraic elements above. For example, for the Young diagram λ of the form



we would write

$$R(\lambda) = \{t_1 t_2, t_2^2\}, \quad A(\lambda) = \{t_1^2, t_1 t_2^2, t_2^3\}.$$

We will further define $\mathfrak{t} = t_1 t_2$ and

$$C = \frac{(1 - \mathfrak{t})}{(1 - t_1)(1 - t_2)} = \chi_T(\mathbb{A}^2, \wedge_{-1}^\bullet(\omega_{\mathbb{A}^2}))$$

and

$$\zeta(x) = \frac{(1 - x)(1 - \mathfrak{t}x)}{(1 - t_1 x)(1 - t_2 x)}$$

since these expressions will appear a lot in our calculations below.

Note that for a nesting of Young diagrams $\lambda \subset \mu$ we directly see that

$$P_{\mu/\lambda} = \frac{1}{\wedge_{-1}^\bullet(\mathcal{C}_{\lambda, \mu})} = \prod_{z \in \mu \setminus \lambda} \frac{1}{(1 - z^{-1})} \prod_{v \in \mu} \frac{(1 - \frac{v}{z})(1 - \mathfrak{t} \frac{v}{z})}{(1 - t_1 \frac{v}{z})(1 - t_2 \frac{v}{z})}$$

that is

$$\prod_{z \in \mu \setminus \lambda} \frac{1}{(1 - z^{-1})} \prod_{v \in \mu} \zeta\left(\frac{v}{z}\right).$$

There is a lot of cancellation happening in this expression as the following lemma shows.

Lemma 1.6.2. *For any Young diagram μ we have*

$$\frac{1}{(1 - z^{-1})} \prod_{v \in \mu} \frac{(1 - \frac{v}{z})(1 - t_1 t_2 \frac{v}{z})}{(1 - t_1 \frac{v}{z})(1 - t_2 \frac{v}{z})} = \frac{\prod_{x \in R(\mu)} (1 - t_1 t_2 \frac{x}{z})}{\prod_{y \in A(\mu)} (1 - \frac{y}{z})}$$

where the identity is of meromorphic functions in the variable z .

Proof. We proceed by induction on $|\mu|$. For $\mu = \emptyset$ the left hand side is

$$\frac{1}{(1 - z^{-1})}.$$

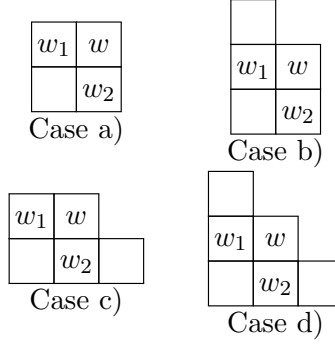
We have $R(\emptyset) = \emptyset$ while $A(\emptyset) = \{1\}$ so that the right hand side is

$$\frac{1}{(1 - \frac{1}{z})}$$

as desired. Suppose the statement is true for μ and let $w \in A(\mu)$ with $\mu' = \mu \cup \{w\}$. The induction hypothesis now immediately implies that

$$\frac{1}{(1 - z^{-1})} \prod_{v \in \mu'} \frac{(1 - \frac{v}{z})(1 - t_1 t_2 \frac{v}{z})}{(1 - t_1 \frac{v}{z})(1 - t_2 \frac{v}{z})} = \frac{(1 - \frac{w}{z})(1 - t_1 t_2 \frac{w}{z})}{(1 - t_1 \frac{w}{z})(1 - t_2 \frac{w}{z})} \frac{\prod_{x \in R(\mu)} (1 - t_1 t_2 \frac{x}{z})}{\prod_{y \in A(\mu)} (1 - \frac{y}{z})}.$$

We consider 4 different cases for the placement of the box w



where we denote the neighbors of w by w_1 and w_2 . Note that $t_1 w_1 = w$ and $t_2 w_2 = w$. Note that case b), d) respectively case c), d) also includes the case where w is on the y -respectively x -axis.

a) Here we have

$$R(\mu') = (R(\mu) \setminus \{w_1, w_2\}) \cup \{w\}, \quad A(\mu') = A(\mu) \setminus \{w\}$$

so that

$$\begin{aligned} \frac{\prod_{x \in R(\mu)} (1 - t_1 t_2 \frac{x}{z})}{\prod_{y \in A(\mu)} (1 - \frac{y}{z})} &= \frac{(1 - t_1 t_2 \frac{w_1}{z})(1 - t_1 t_2 \frac{w_2}{z})(1 - t_1 t_2 \frac{w}{z})^{-1} \prod_{x \in R(\mu')} (1 - t_1 t_2 \frac{x}{z})}{(1 - \frac{w}{z}) \prod_{y \in A(\mu')} (1 - \frac{y}{z})} \\ &= \frac{(1 - t_2 \frac{w}{z})(1 - t_1 \frac{w}{z}) \prod_{x \in R(\mu')} (1 - t_1 t_2 \frac{x}{z})}{(1 - \frac{w}{z})(1 - t_1 t_2 \frac{w}{z}) \prod_{y \in A(\mu')} (1 - \frac{y}{z})} \end{aligned}$$

which is what we want.

b) Here we have

$$R(\mu') = (R(\mu) \setminus \{w_2\}) \cup \{w\}, \quad A(\mu') = (A(\mu) \setminus \{w\}) \cup \{t_2 w\}$$

so that

$$\begin{aligned} \frac{\prod_{x \in R(\mu)} (1 - t_1 t_2 \frac{x}{z})}{\prod_{y \in A(\mu)} (1 - \frac{y}{z})} &= \frac{(1 - t_1 t_2 \frac{w_2}{z})(1 - t_1 t_2 \frac{w}{z})^{-1} \prod_{x \in R(\mu')} (1 - t_1 t_2 \frac{x}{z})}{(1 - \frac{w}{z})(1 - t_2 \frac{w}{z})^{-1} \prod_{y \in A(\mu')} (1 - \frac{y}{z})} \\ &= \frac{(1 - t_2 \frac{w}{z})(1 - t_1 \frac{w}{z}) \prod_{x \in R(\mu')} (1 - t_1 t_2 \frac{x}{z})}{(1 - \frac{w}{z})(1 - t_1 t_2 \frac{w}{z}) \prod_{y \in A(\mu')} (1 - \frac{y}{z})} \end{aligned}$$

which is what we want.

c) Here we have

$$R(\mu') = (R(\mu) \setminus \{w_1\}) \cup \{w\}, \quad A(\mu') = (A(\mu) \setminus \{w\}) \cup \{t_1 w\}$$

so that

$$\begin{aligned} \frac{\prod_{x \in R(\mu)} (1 - t_1 t_2 \frac{x}{z})}{\prod_{y \in A(\mu)} (1 - \frac{y}{z})} &= \frac{(1 - t_1 t_2 \frac{w_1}{z})(1 - t_1 t_2 \frac{w}{z})^{-1} \prod_{x \in R(\mu')} (1 - t_1 t_2 \frac{x}{z})}{(1 - \frac{w}{z})(1 - t_1 \frac{w}{z})^{-1} \prod_{y \in A(\mu')} (1 - \frac{y}{z})} \\ &= \frac{(1 - t_2 \frac{w}{z})(1 - t_1 \frac{w}{z}) \prod_{x \in R(\mu')} (1 - t_1 t_2 \frac{x}{z})}{(1 - \frac{w}{z})(1 - t_1 t_2 \frac{w}{z}) \prod_{y \in A(\mu')} (1 - \frac{y}{z})} \end{aligned}$$

which is what we want.

d) Here we have

$$R(\mu') = R(\mu) \cup \{w\}, \quad A(\mu') = (A(\mu) \setminus \{w\}) \cup \{t_1 w, t_2 w\}$$

so that

$$\begin{aligned} \frac{\prod_{x \in R(\mu)} (1 - t_1 t_2 \frac{x}{z})}{\prod_{y \in A(\mu)} (1 - \frac{y}{z})} &= \frac{(1 - t_1 t_2 \frac{w}{z})^{-1} \prod_{x \in R(\mu')} (1 - t_1 t_2 \frac{x}{z})}{(1 - \frac{w}{z})(1 - t_1 \frac{w}{z})^{-1} (1 - t_2 \frac{w}{z})^{-1} \prod_{y \in A(\mu')} (1 - \frac{y}{z})} \\ &= \frac{(1 - t_2 \frac{w}{z})(1 - t_1 \frac{w}{z}) \prod_{x \in R(\mu')} (1 - t_1 t_2 \frac{x}{z})}{(1 - \frac{w}{z})(1 - t_1 t_2 \frac{w}{z}) \prod_{y \in A(\mu')} (1 - \frac{y}{z})} \end{aligned}$$

which is what we want. □

Definition 1.6.2. With this in mind we define

$$\omega_\mu(x) = \frac{1}{1 - x^{-1}} \prod_{z \in \mu} \zeta\left(\frac{z}{x}\right) = \frac{\prod_{r \in R(\mu)} (1 - \mathfrak{t} \frac{r}{x})}{\prod_{a \in A(\mu)} (1 - \frac{a}{x})}$$

as a meromorphic function. For a nesting of partitions $\lambda \subset \mu$ we therefore have

$$P_{\mu/\lambda} = \prod_{z \in \mu \setminus \lambda} \omega_\mu(z).$$

Recall 1.6.1. We note that ω satisfies the branching rule, where for $w \in A(\mu)$ we have

$$\begin{aligned} \omega_{\mu \cup \{w\}}(x) &= \zeta\left(\frac{w}{x}\right) \omega_\mu(x) \\ &= \frac{(1 - \frac{w}{x})(1 - \mathfrak{t} \frac{w}{x})}{(1 - t_1 \frac{w}{x})(1 - t_2 \frac{w}{x})} \frac{\prod_{r \in R(\mu)} (1 - \mathfrak{t} \frac{r}{x})}{\prod_{a \in A(\mu)} (1 - \frac{a}{x})} \\ &= \frac{(1 - \mathfrak{t} \frac{w}{x})}{(1 - t_1 \frac{w}{x})(1 - t_2 \frac{w}{x})} \frac{\prod_{r \in R(\mu)} (1 - \mathfrak{t} \frac{r}{x})}{\prod_{a \in A(\mu) \setminus \{w\}} (1 - \frac{a}{x})} \end{aligned}$$

and so in particular

$$\omega_{\mu \cup \{w\}}(w) = C \frac{\prod_{r \in R(\mu)} (1 - \mathfrak{t} \frac{r}{w})}{\prod_{a \in A(\mu) \setminus \{w\}} (1 - \frac{a}{w})}.$$

We note also the following symmetry of ζ

$$\zeta\left(\frac{1}{\mathfrak{t}x}\right) = \frac{(1 - \frac{1}{\mathfrak{t}x})(1 - \mathfrak{t} \frac{1}{\mathfrak{t}x})}{(1 - t_1 \frac{1}{\mathfrak{t}x})(1 - t_2 \frac{1}{\mathfrak{t}x})} = \frac{\mathfrak{t}^{-1} x^{-2} (\mathfrak{t}x - 1)(x - 1)}{t_2^{-1} t_1^{-1} x^{-2} (t_2 x - 1)(t_1 x - 1)} = \zeta(x).$$

We will frequently use these rules in the computations below.

Recall that our goal is to show that $F_\lambda(u)$ is independent of λ , so that

$$F_\lambda(u) = F_\varnothing(u) = \text{PE} \left(\frac{u(1 - t_1 t_2)}{(1 - t_1)(1 - t_2)} \right).$$

Suppose therefore $w \in A(\lambda)$ is an addable box and let $\lambda' = \lambda \cup \{w\}$ be the diagram obtained by adding w to λ . We want to prove that $F_\lambda(u) = F_{\lambda'}(u)$. We can split the sum as

$$F_\lambda(u) = \sum_{\lambda' \subset \mu} u^{|\mu| - |\lambda|} P_{\mu/\lambda} + \sum_{\substack{\lambda \subset \nu \\ w \notin \nu}} u^{|\nu| - |\lambda|} P_{\nu/\lambda}.$$

Sending $\nu \mapsto \nu \cup \{w\}$ provides a bijection

$$\{\lambda \subset \nu \mid w \notin \nu\} \cong \{\lambda \subset \mu \mid w \in R(\omega)\}.$$

We note further that

$$P_{\mu/\lambda} = \prod_{z \in \mu \setminus \lambda} \omega_\mu(z) = \omega_\mu(w) \prod_{z \in \mu \setminus \lambda'} \omega_\mu(z) = \omega_\mu(w) P_{\mu/\lambda'}$$

and for $\mu = \nu \cup \{w\}$ we have

$$P_{\mu/\lambda'} = \prod_{z \in (\nu \cup \{w\}) \setminus (\lambda \cup \{w\})} \omega_{\nu \cup \{w\}}(z) = \prod_{z \in \nu \setminus \lambda} \zeta(w/z) \omega_\nu(z) = P_{\nu/\lambda} \prod_{z \in \nu \setminus \lambda} \zeta(w/z)$$

so setting

$$\kappa_{\mu/\lambda'}(x) = \prod_{z \in \mu \setminus \lambda'} \zeta(x/z)^{-1} = \prod_{z \in \mu \setminus \lambda'} \frac{(1 - t_1 x/z)(1 - t_2 x/z)}{(1 - x/z)(1 - tx/z)}$$

we obtain

$$P_{\nu/\lambda} = \kappa_{\mu/\lambda'}(w) P_{\mu/\lambda'}.$$

Using this, it follows that we may rewrite

$$F_\lambda(u) = \sum_{\lambda' \subset \mu} (u \cdot \omega_\mu(w)) u^{|\mu| - |\lambda'|} P_{\mu/\lambda'} + \sum_{\substack{\lambda' \subset \mu \\ w \in R(\mu)}} \kappa_{\mu/\lambda'}(w) u^{|\mu| - |\lambda'|} P_{\mu/\lambda'}.$$

We note that if $w \notin R(\mu)$ for $\lambda' \subset \mu$ we have $\kappa_{\mu/\lambda'}(w) = 0$. Indeed, since $w \notin R(\mu)$ we must have $t_1 w \in \mu$ or/and $t_2 w \in \mu$ so that the numerator of $\kappa_{\mu/\lambda'}(x)$ has a zero of degree 1 or 2 at $x = w$. Conversely, since $w \in \lambda'$ we see that the denominator has a zero - which is necessarily of degree 1 - at $x = w$ if and only if $t_1 t_2 w \in \mu$, but then $t_1 w, t_2 w \in \mu$ so that the numerator has a zero of degree 2. It follows that

$$F_\lambda(u) = \sum_{\lambda' \subset \mu} (u \cdot \omega_\mu(w) + \kappa_{\mu/\lambda'}(w)) u^{|\mu| - |\lambda'|} P_{\mu/\lambda'}$$

and in particular, we see that

$$F_\lambda(u) - F_{\lambda'}(u) = \sum_{\lambda' \subset \mu} (u \cdot \omega_\mu(w) + \kappa_{\mu/\lambda'}(w) - 1) u^{|\mu| - |\lambda'|} P_{\mu/\lambda'}.$$

This motivates the following definition.

Definition 1.6.3. Let λ be a Young diagram. We set

$$D_\lambda(x) = \sum_{\lambda \subset \mu} (u \cdot \omega_\mu(x) + \kappa_{\mu/\lambda}(x) - 1) u^{|\mu| - |\lambda|} P_{\mu/\lambda}$$

and

$$D_{\lambda,n}(x) = \sum_{\substack{\lambda \subset \mu \\ |\mu \setminus \lambda| = n}} P_{\mu/\lambda} (\kappa_{\mu/\lambda}(x) - 1) + \sum_{\substack{\lambda \subset \nu \\ |\nu \setminus \lambda| = n-1}} P_{\nu/\lambda} \omega_\nu(x)$$

the coefficient of u^n in $D_\lambda(u)$. Similarly we let

$$c_n(\lambda) = \sum_{\substack{\lambda \subset \mu \\ |\mu| - |\lambda| = n}} P_{\mu/\lambda}$$

the coefficient of u^n in $F_\lambda(u)$.

The above discussion now shows the following lemma.

Lemma 1.6.3. *Let λ be a partition and let $w \in R(\lambda)$ be a removable box. Then $c_n(\lambda) = c_n(\lambda \setminus \{w\})$ if and only if $D_{\lambda,n}(w) = 0$. In particular, if $D_{\lambda,n}(w) = 0$ for all λ and all $w \in R(\lambda)$ then $c_n = c_n(\lambda)$ is independent of λ .*

With this in mind we begin the study of $D_{\lambda,n}(x)$. We start by identifying its poles.

Lemma 1.6.4. *$D_{\lambda,n}(x)$ is a meromorphic function. It has poles exactly at $x = a/t$ for $a \in A(\lambda)$ and these poles are simple with residue*

$$\text{Res}_{x=a/t} D_{\lambda,n}(x) = \frac{a \cdot \omega_{\lambda \cup \{a\}}(a)}{tC} \cdot c_{n-1}(\lambda \cup \{a\}).$$

Proof. Note first that

$$\zeta(x/z) = \zeta(z/tx)$$

and so

$$\omega_\lambda(tx) \cdot \kappa_{\mu/\lambda}(x)^{-1} = \frac{1}{1 - (tx)^{-1}} \prod_{z \in \lambda} \zeta(z/tx) \cdot \prod_{z \in \mu \setminus \lambda} \zeta(z/tx) = \omega_\mu(tx)$$

so that

$$\kappa_{\mu/\lambda}(x) = \frac{\omega_\lambda(tx)}{\omega_\mu(tx)} = \frac{\prod_{r \in R(\lambda)} (1 - r/x) \prod_{a \in A(\mu)} (1 - a/tx)}{\prod_{a \in A(\lambda)} (1 - a/tx) \prod_{r \in R(\mu)} (1 - r/x)}$$

i.e. the possible poles are for $x = b \in R(\mu) \setminus R(\lambda)$ or $x = a/t$ for $a \in A(\lambda)$. Note that if $a \in A(\lambda)$ then $a/t_1, a/t_2 \in \lambda$ hence $a/t \in \lambda \setminus R(\lambda)$. In particular $a/t \notin R(\mu)$ so all these possible poles are distinct and at most of order 1. On the other hand we have

$$\omega_\nu(x) = \frac{\prod_{r \in R(\nu)} (1 - tr/x)}{\prod_{a \in A(\nu)} (1 - a/x)}$$

which has simple poles at $x = b \in A(\nu)$. Note that if we choose any box b outside λ which is addable for a ν or removable for a μ we have a bijection

$$\{\nu \supset \lambda \mid |\nu \setminus \lambda| = n - 1, b \in A(\nu)\} \longleftrightarrow \{\mu \supset \lambda \mid |\mu \setminus \lambda| = n, b \in R(\mu) \setminus R(\lambda)\}$$

simply given by either removing or adding the box b . For a fixed b we will show that the residue cancels for such a pair (μ, ν) with $\mu = \nu \cup \{b\}$. We note that

$$\text{Res}_{x=b} \omega_\nu(x) dx = b \frac{\prod_{r \in R(\nu)} (1 - tr/b)}{\prod_{a \in A(\nu) \setminus \{b\}} (1 - a/b)} = \frac{b}{C} \omega_\mu(b)$$

where $\mu = \nu \cup \{b\}$ and where we have used the branching rule. On the other hand

$$\text{Res}_{x=b} \zeta(x/b)^{-1} dx = -\frac{b}{C}$$

so that

$$\text{Res}_{x=b} \kappa_{\mu/\lambda}(x) = -\frac{b}{C} \prod_{z \in \nu \setminus \lambda} \zeta(b/z)^{-1}.$$

Hence the two summands of $\text{Res}_{x=b} D_{\lambda,n}(x)$ corresponding to μ and ν with $\mu = \nu \cup \{b\}$ are given by

$$P_{\mu/\lambda} \left(-\frac{b}{C} \prod_{z \in \nu \setminus \lambda} \zeta(b/z)^{-1} \right) + P_{\nu/\lambda} \frac{b}{C} \omega_\mu(b)$$

but

$$P_{\mu/\lambda} = \prod_{z \in \nu \cup \{b\}/\lambda} \omega_{\nu \cup \{b\}}(z) = \omega_{\mu}(b) \prod_{z \in \nu} \zeta(b/z) \omega_{\nu}(z) = \omega_{\mu} P_{\nu/\lambda}(b) \prod_{z \in \nu} \zeta(b/z)$$

which shows that $\text{Res}_{x=b} D_{\lambda,n}(x) = 0$. For the remaining possible poles we see that for $a \in A(\lambda)$ we get a pole for each μ such that $a \notin A(\mu)$ i.e. $a \in \mu$. We see that

$$\text{Res}_{x=a/\mathfrak{t}} \zeta(x/a)^{-1} dx = \frac{a}{\mathfrak{t}C}$$

and so for μ with $\lambda \cup \{a\} \subset \mu$ we get

$$\text{Res}_{x=a/\mathfrak{t}} \kappa_{\mu/\lambda}(x) dx = \frac{a}{\mathfrak{t}C} \prod_{z \in \mu \setminus (\lambda \cup \{a\})} \zeta(a/\mathfrak{t}z)^{-1} = \frac{a}{\mathfrak{t}C} \prod_{z \in \mu \setminus (\lambda \cup \{a\})} \zeta(z/a)^{-1}.$$

We note that

$$P_{\mu/\lambda} = \prod_{z \in \mu \setminus \lambda} \omega_{\mu}(z) = \omega_{\mu}(a) P_{\mu/(\lambda \cup \{a\})}$$

and that

$$w_{\mu}(a) = \prod_{z \in \mu} \zeta(z/a)$$

so that

$$\omega_{\mu}(a) \prod_{z \in \mu \setminus (\lambda \cup \{a\})} \zeta(z/a)^{-1} = \omega_{\lambda \cup a}(a)$$

hence we get

$$\text{Res}_{x=a/\mathfrak{t}} D_{\lambda,n}(x) = \frac{a \omega_{\lambda \cup \{a\}}(a)}{\mathfrak{t}C} \sum_{\substack{\lambda \cup \{a\} \subset \mu \\ |\mu| - |\lambda \cup \{a\}| = n-1}} P_{\mu/(\lambda \cup \{a\})} = \frac{a \omega_{\lambda \cup \{a\}}(a)}{\mathfrak{t}C} \cdot c_{n-1}(\lambda \cup \{a\})$$

□

Next, we identify the limit.

Lemma 1.6.5. *Let λ be a diagram. We have*

$$\lim_{x \rightarrow \infty} D_{\lambda,n}(x) = c_{n-1}(\lambda).$$

Proof. We see that

$$\kappa_{\mu/\lambda}(x) = \prod_{z \in \mu \setminus \lambda} \zeta(x/z)^{-1} = \prod_{z \in \mu \setminus \lambda} \frac{(1 - t_1 x/z)(1 - t_2 x/z)}{(1 - x/z)(1 - \mathfrak{t}x/z)}$$

which tends to

$$\prod_{z \in \mu \setminus \lambda} \frac{(t_1/z)(t_2/z)}{(1/z)(\mathfrak{t}/z)} = 1$$

when $x \rightarrow \infty$. Meanwhile

$$\omega_{\nu}(x) = \frac{\prod_{r \in R(\mu)} (1 - \mathfrak{t} \frac{r}{x})}{\prod_{a \in A(\mu)} (1 - \frac{a}{x})} \rightarrow 1$$

as $x \rightarrow \infty$. It follows that

$$D_{\lambda,n}(x) \rightarrow \sum_{\substack{\lambda \subset \nu \\ |\nu \setminus \lambda| = n-1}} P_{\nu/\lambda} = c_{n-1}(\lambda).$$

□

Theorem 1.6.1. *The expression $c_n(\lambda)$ is independent of λ for all $n \geq 0$.*

Proof. We proceed by induction on n with the induction start $n = 0$ being trivial as $c_0(\lambda) = 1$. Assuming that $c_m = c_m(\lambda)$ is independent of λ for $m < n$ we will show that

$$D_{\lambda,n}(x) = xc_{n-1} \frac{\prod_{r \in R(\lambda)}(x-r)}{\prod_{a \in A(\lambda)}(x-a/\mathfrak{t})}$$

which in particular implies that $D_{\lambda,n}(w) = 0$ for all $w \in R(\lambda)$ which then implies that $c_n(\lambda)$ is independent of λ . By Liouville's theorem it will suffice to show that the right hand side has the same residues and limit at $x \rightarrow \infty$. Clearly the right hand side has simple poles at $x = b/\mathfrak{t}$ for $b \in A(\lambda)$ and the residue is given

$$c_{n-1} \frac{b}{\mathfrak{t}} \frac{\prod_{r \in R(\lambda)}(b/\mathfrak{t} - r)}{\prod_{a \in A(\lambda) \setminus \{b\}}(b/\mathfrak{t} - a/\mathfrak{t})} = c_{n-1} \frac{b}{\mathfrak{t}} \frac{\prod_{r \in R(\lambda)}(1 - \mathfrak{t}r/b)}{\prod_{a \in A(\lambda) \setminus \{b\}}(1 - a/b)}.$$

Conversely we see using the branching rule that

$$\omega_{\lambda \cup \{b\}}(b) = C \frac{\prod_{r \in R(\lambda)}(1 - \mathfrak{t}r/b)}{\prod_{a \in A(\lambda) \setminus \{b\}}(1 - a/b)}$$

so that the residue of $D_{\lambda,n}(x)$ at $x = b/\mathfrak{t}$ is

$$\frac{b \cdot \omega_{\lambda \cup \{b\}}(b)}{\mathfrak{t}C} \cdot c_{n-1}(\lambda \cup \{b\}) = \frac{b}{\mathfrak{t}} \frac{\prod_{r \in R(\lambda)}(1 - \mathfrak{t}r/b)}{\prod_{a \in A(\lambda) \setminus \{b\}}(1 - a/b)} c_{n-1}$$

which agrees with the right hand side. Lastly, using $|A| = |R| + 1$ we can rewrite the right hand side to

$$c_{n-1} \frac{\prod_{r \in R(\lambda)}(1 - r/x)}{\prod_{a \in A(\lambda)}(1 - a/\mathfrak{t}x)}$$

so that when $x \rightarrow \infty$ we get

$$c_{n-1}$$

which agrees with

$$\lim_{x \rightarrow \infty} D_{\lambda,n}(x) = c_{n-1}(\lambda).$$

□

Proof of theorem 1.2.1. Let $p = p_{\underline{d}, \hat{\underline{d}}}$. We have by lemma 1.5.7 that

$$p_*(\mathcal{O}_{\underline{d}}^{\text{vir}}) = p_*(\mathcal{E}_{\underline{d}/\hat{\underline{d}}}) \cdot \mathcal{O}_{\hat{\underline{d}}}^{\text{vir}}.$$

For each nested Young diagram of type $\underline{d}$

$$\lambda_{\bullet} = \lambda_{r+1} \supset \lambda_{r-1} \supset \cdots \supset \lambda_1$$

corresponding to a fixed point of $\text{NHilb}^{\underline{d}}(\mathbb{A}^2)$ we then see by lemma 1.6.1 that

$$\text{cont}_{\lambda_{\bullet}}(\mathcal{E}_{\underline{d}/\hat{\underline{d}}}) = P_{\lambda_{r+1}/\lambda_r}$$

so that for a nested Young diagram $\lambda_{\bullet}$ of type $\hat{\underline{d}}$ we have

$$\text{cont}_{\lambda_{\bullet}} p_*(\mathcal{E}_{\underline{d}/\hat{\underline{d}}}) = \sum_{\substack{\lambda_r \subset \mu \\ |\mu| = |\lambda_r|}} P_{\mu/\lambda_r} = c_{d_r}(\lambda_r).$$

By theorem 1.6.1 this is independent of λ_r , and so in particular $p_*(\mathcal{E}_{\underline{d}/\hat{d}})$ is constant. One then explicitly sees that

$$c_{d_r}(\lambda_r) = c_{d_r}(\emptyset) = \sum_{\mu \vdash d_r} P_{\mu/\emptyset}$$

agrees with the localization formula for $\chi_T(\text{Hilb}^{d_r}(\mathbb{A}^2), \wedge_{-1}^\bullet(\omega_{\mathbb{A}^2}^{[d_r]}))$. Indeed, note first that

$$\omega_{\mathbb{A}^2}^{[d_r]} = t_1 t_2 \mathcal{O}_{\mathbb{A}^2}^{[d_r]}.$$

We have by definition that

$$P_{\mu/\emptyset} = \frac{1}{\wedge_{-1}^\bullet(\mathcal{C}_{\emptyset, \mu})}$$

where

$$\mathcal{C}_{\emptyset, \mu} = Q_\mu^\vee - (1 - t_1)(1 - t_2)Q_\mu^\vee Q_\mu.$$

Using that

$$T_{\text{Hilb}^{d_r}(\mathbb{A}^2), \mu}^\vee = Q_\mu^\vee + t_1 t_2 Q_\mu - (1 - t_1)(1 - t_2)Q_\mu^\vee Q_\mu$$

we get

$$\mathcal{C}_{\emptyset, \mu} = T_{\text{Hilb}^{d_r}(\mathbb{A}^2), \mu}^\vee - t_1 t_2 Q_{d_r} = T_{\text{Hilb}^{d_r}(\mathbb{A}^2), \mu}^\vee - t_1 t_2 \mathcal{O}_{\mathbb{A}^2, \mu}^{[d_r]}$$

hence

$$P_{\mu/\emptyset} = \frac{\wedge_{-1}^\bullet(t_1 t_2 \mathcal{O}_{\mathbb{A}^2, \mu}^{[d_r]})}{\wedge_{-1}^\bullet(T_{\text{Hilb}^{d_r}(\mathbb{A}^2), \mu}^\vee)} = \text{cont}_\mu(\wedge_{-1}^\bullet t_1 t_2 \mathcal{O}_{\mathbb{A}^2}^{[d_r]}).$$

We therefore get

$$\sum_{\mu \vdash d_r} P_{\mu/\emptyset} = \sum_{\mu \vdash d_r} \text{cont}_\mu(t_1 t_2 \mathcal{O}_{\mathbb{A}^2}^{[d_r]}) = \chi_T(\text{Hilb}^{d_r}(\mathbb{A}^2), t_1 t_2 \mathcal{O}_{\mathbb{A}^2}^{[d_r]})$$

as desired. □

Chapter 2

Virtual invariants from the non-associative Hilbert scheme

Abstract

We introduce a non-associative model for the Hilbert scheme of points in arbitrary dimension. We define a smooth ambient space, which we call the non-associative Hilbert scheme, containing the classical nested Hilbert scheme $\mathrm{NHilb}^{\underline{d}}(\mathbb{A}^n)$ as the associativity, cut out by an explicit section of an associativity bundle. This construction yields canonical perfect obstruction theories and virtual fundamental classes on $\mathrm{NHilb}^{\underline{d}}(\mathbb{A}^n)$ for all $(n, \underline{d})$. Using virtual localization, we obtain closed formulas for these virtual classes as sums over admissible nested partitions. Over the punctual locus, we rewrite these as a single multivariable iterated residue formula governing all virtual integrals. Our construction works for all n , produces positive-dimensional virtual classes when n is large compared to the number of points, and we expect that they extend the non-commutative matrix model and virtual class construction on Calabi-Yau threefolds.

Introduction

Hilbert schemes of points are fundamental objects in algebraic and enumerative geometry. For a smooth surface S , the Hilbert scheme $\mathrm{Hilb}^d(S)$ of length- d zero-dimensional subschemes is a smooth, irreducible variety of dimension $2d$ [Fog68]. The topology and intersection theory of $\mathrm{Hilb}^d(S)$ are very well understood, and they carry rich structures such as Nakajima's Heisenberg algebra action [Nak99], quantum cohomology structures in the work of Okounkov-Pandharipande [OP10], and relations to curve counting and modular forms via K3 surfaces.

In higher dimension, the geometry quickly becomes more subtle. For a threefold X , the Hilbert scheme $\mathrm{Hilb}^d(X)$ is typically singular for $d > 1$ but admits a symmetric perfect obstruction theory in the Calabi-Yau case. Behrend-Fantechi's theory of intrinsic normal cones and perfect obstruction theories produces a virtual fundamental class $[\mathrm{Hilb}^d(X)]^{\mathrm{vir}}$ [BF97], and Behrend's constructible function describes Donaldson-Thomas invariants as weighted Euler characteristics [Beh09]. Applied to Calabi-Yau threefolds, this yields degree-zero Donaldson-Thomas (DT) invariants as weighted Euler characteristics on $\mathrm{Hilb}^d(X)$.

In the special important case when $X = \mathbb{A}^3$, $\mathrm{Hilb}^d(\mathbb{A}^3)$ parametrizes ideals of colength d in $\mathbb{C}[x_1, x_2, x_3]$ and is highly singular. Nevertheless, it can be modeled as a critical locus of a function on a smooth space of non-commuting matrices: one considers triples of matrices $(A_1, A_2, A_3) \in \mathrm{End}(V)^3$ together with a vector $v \in V$ generating V under the A_i , and imposes the commuting relations $[A_i, A_j] = 0$ via a potential [BF08, BBS13]. This non-commuting matrix model leads to explicit formulae for DT invariants of $\mathbb{A}^3$, recovering

the MacMahon product formula for the generating series of zero-dimensional DT invariants [MNOP06a, MNOP06b]. From our point of view, this is the first instance of a general philosophy: one embeds a singular Hilbert scheme into a smoother, more flexible moduli space (here of unconstrained or non-commuting matrices) and then imposes the constraints (here of commutativity) as equations cutting out the original moduli space.

Moduli of sheaves.

The special role of threefolds and fourfolds in enumerative geometry is by now well established. On a Calabi-Yau threefold X , moduli spaces of stable sheaves or stable pairs carry symmetric perfect obstruction theories and hence zero-dimensional virtual classes. Integrals over these classes define Donaldson-Thomas invariants, which may be viewed as algebraic counterparts of BPS state counts in string theory and are related to Gromov-Witten invariants via the DT/PT correspondence [Tho00a, PT09].

For Calabi-Yau fourfolds, the picture is more delicate: the derived moduli stack of (semi)stable sheaves is expected to be (-2) -shifted symplectic in the sense of Pantev-Toën-Vaquié-Vezzosi. Using this structure, Borisov-Joyce showed that such a derived moduli space carries a real virtual fundamental class of half the expected complex dimension, constructed via derived differential geometry and Kuranishi atlases [BJ17]. Their theory applies in particular to moduli spaces of stable sheaves on Calabi-Yau fourfolds (including Hilbert schemes of points), but the resulting cycle is a priori only defined in real homology and depends on subtle orientation data.

Building on this, Oh-Thomas constructed an algebraic virtual cycle for moduli spaces of sheaves on Calabi-Yau fourfolds, using a localized square-root Euler class for orthogonal bundles and a theory of complex Kuranishi structures. They proved a localization formula and showed that their algebraic cycle agrees with the Borisov-Joyce cycle after inverting 2 [OT23, OT24]. This provides a robust framework for defining DT-type invariants on Calabi-Yau fourfolds and for comparing different constructions.

In parallel, Feyzbakhsh-Thomas have developed a powerful wall-crossing and stability-conditions approach to Donaldson-Thomas invariants on Calabi-Yau threefolds. They express general rank- r DT invariants in terms of simpler rank 0 or rank 1 theories via Bridgeland stability and wall crossing [FT23, FT24].

Within this sheaf-theoretic framework, Hilbert schemes of points on Calabi-Yau fourfolds play the role of moduli of zero-dimensional sheaves. Let X be a smooth projective Calabi-Yau fourfold and $\text{Hilb}^d(X)$ its Hilbert scheme of length- d subschemes. Using the Borisov-Joyce theory together with orientability results of Cao-Gross-Joyce for moduli of sheaves on Calabi-Yau fourfolds [CGJ20], one obtains a virtual class on $\text{Hilb}^d(X)$. Cao-Kool then defined zero-dimensional DT_4 invariants by integrating the Euler classes of tautological bundles against this virtual class, and proved deformation invariance and a number of structural properties [CK18]. These invariants can be viewed as fourfold analogues of degree-zero DT invariants on Calabi-Yau threefolds.

More recently, Bojko has used the Gross-Joyce-Tanaka wall-crossing formalism to study topological information encoded by the virtual fundamental classes of $\text{Hilb}^d(X)$ as d varies. Assuming the conjectural wall-crossing framework, Bojko proves the Cao-Kool conjectures for zero-dimensional sheaf counting and shows that the collection of virtual classes on $\text{Hilb}^d(X)$ is equivalent to the data of all descendent integrals. As a consequence, many generating series of invariants are expressed in terms of explicit universal power series and are related to Nekrasov’s “genus” for $\text{Hilb}^d(\mathbb{C}^4)$ and to Segre-Verlinde correspondences involving Quot schemes on elliptic surfaces [Boj21].

On the local and toric side, Nekrasov [Nek20] introduced an equivariant K-theoretic “Magnificent Four” partition function, which can be interpreted as a new genus of Hilbert schemes of points on $\mathbb{C}^4$. Cao-Kool-Monavari [CKM22] extended this to curves and stable

pairs, formulating a K-theoretic DT/PT correspondence for toric Calabi–Yau fourfolds and verifying it via a vertex formalism.

In a different but related direction, Oberdieck et al [COT24, COT22] have developed curve-counting theories on holomorphic symplectic fourfolds, including holomorphic symplectic analogues of Gopakumar–Vafa invariants. These are often expressed in terms of invariants of Hilbert schemes of points on K3 surfaces and their higher-dimensional analogues and connect the 4-dimensional Hilbert scheme geometry to modularity and Jacobi forms.

The non-associative nested Hilbert scheme.

For a smooth variety X of dimension n and a dimension vector $\underline{d} = (d_0, \dots, d_r)$, the nested Hilbert scheme

$$\mathrm{NHilb}^{\underline{d}}(X) = \{\xi_1 \subset \dots \subset \xi_{r+1} \subset X \mid \dim \xi_i = 0, \text{length}(\xi_i) = d_0 + \dots + d_{i-1}\}$$

parametrizes chains of zero-dimensional subschemes of prescribed lengths. Even on surfaces, these nested schemes are typically singular with poorly understood component structure. Nonetheless, they have proven extremely useful in sheaf counting theories. Gholampour–Sheshmani–Yau [GSY20] constructed virtual classes on nested Hilbert schemes of points and curves on surfaces and showed that they recover reduced stable pair and Seiberg–Witten theory. Gholampour–Thomas [GT20a, GT20b] then expressed nested Hilbert schemes as virtual resolutions of degeneracy loci of maps of bundles on smooth ambient varieties, and used Thom–Porteous-type formulas to compute the resulting virtual cycles and their integrals. This provides a powerful technique for expressing Vafa–Witten and local DT/PT invariants in terms of tautological classes on nested Hilbert schemes of surfaces.

In this work we use a different smooth ambient space for Hilbert scheme of points. We will reinterpret the work of Bérczi–Szenes [BS12, B17] and Kazarian [Kaz17] where they showed that a certain filtered part of the punctual Hilbert scheme embeds into a smooth non-associative Hilbert scheme and used this to compute Thom polynomials of Morin singularity classes.

In this paper, we extend the non-associative Hilbert scheme model to the whole nested Hilbert schemes. Fix a dimension vector $\underline{d} = (d_0, \dots, d_r) \in \mathbb{Z}_{\geq 0}^{r+1}$ and $n \geq 0$. Consider the nested Hilbert scheme of $d = d_0 + \dots + d_r$ points in $\mathbb{A}^n$ given as the moduli space

$$\begin{aligned} \mathrm{NHilb}^{\underline{d}}(\mathbb{A}^n) &= \{\xi_1 \subset \dots \subset \xi_{r+1} \subset \mathbb{A}^n \mid \dim \xi_i = 0, \text{Length} \xi_i = d_0 + \dots + d_{i-1}\} \\ &= \{I_{r+1} \subset \dots \subset I_1 \subset I_0 = \mathbb{C}[x_1, \dots, x_n] \mid \dim_{\mathbb{C}} I_i/I_{i+1} = d_i\}. \end{aligned}$$

We define the non-associative nested Hilbert scheme of $d = d_0 + \dots + d_r$ points on $\mathbb{A}^n$ as the moduli space

$$\mathrm{naNHilb}^{\underline{d}}(\mathbb{A}^n) = \{I_{r+1} \subset \dots \subset I_1 \subset I_0 = \mathbb{C}\{x_1, \dots, x_n\}_c \mid \dim_{\mathbb{C}} I_i/I_{i+1} = d_i\}$$

where $\mathbb{C}\{x_1, \dots, x_n\}_c$ is the free unital, commutative, non-associative algebra on n -generators and the I_i 's are ideals. We prove that this is a smooth variety and that $\mathrm{NHilb}^{\underline{d}}(\mathbb{A}^n)$ embeds in $\mathrm{naNHilb}^{\underline{d}}(\mathbb{A}^n)$ as the locus where I_1 is contained in the associativity ideal or equivalently where the quotient algebras are associative.

The key point is that associativity can be imposed by a section of a vector bundle over the smooth ambient space $\mathrm{naNHilb}^{\underline{d}}(\mathbb{A}^n)$. That is, the associator conditions are encoded in a vector bundle E_{ass} on $\mathrm{naNHilb}^{\underline{d}}(\mathbb{A}^n)$ and a global section

$$s_{\text{ass}} \in \Gamma(\mathrm{naNHilb}^{\underline{d}}(\mathbb{A}^n), E_{\text{ass}})$$

whose zero locus is precisely $\mathrm{NHilb}^{\underline{d}}(\mathbb{A}^n)$. Thus $\mathrm{NHilb}^{\underline{d}}(\mathbb{A}^n)$ has a canonical perfect obstruction theory as the zero locus of a section on a smooth variety, and hence a virtual fundamental class

$$[\mathrm{NHilb}^{\underline{d}}(\mathbb{A}^n)]^{\mathrm{vir}} \in A_*(\mathrm{NHilb}^{\underline{d}}(\mathbb{A}^n)).$$

This provides an elementary and concrete construction of a virtual class for nested Hilbert schemes of points in all dimensions.

Our approach should be contrasted with the non-commuting matrix models used in threefold DT theory. There, one replaces the commutative coordinate ring by a free associative (non-commutative) algebra of matrices, and the Hilbert scheme is realized as a critical locus of a function cutting out the commuting relations. In our non-associative model, we instead keep commutativity but drop associativity, realizing the Hilbert scheme as the zero locus of associativity equations inside a smooth commutative but non-associative ambient space. This shift turns out to be well adapted to the combinatorics of monomial ideals and yields ambient varieties that are themselves built as towers of Grassmannian fibrations, in the spirit of Kazarian's work. Crucially, our construction produces virtual classes of positive dimension whenever the ambient dimension n is sufficiently large compared to the number of points (or total length) $d = d_0 + \cdots + d_r$, and our iterated residue formula for virtual integration then provides a uniform way to define and compute these virtual classes for all pairs $(n, \underline{d})$.

Virtual localization and iterated residue formulas.

The affine torus $T = (\mathbb{C}^*)^n$ acts on $\mathbb{A}^n$ by scaling each coordinate, and this action lifts to both $\mathrm{naNHilb}^{\underline{d}}(\mathbb{A}^n)$ and $\mathrm{NHilb}^{\underline{d}}(\mathbb{A}^n)$. Our construction of the obstruction theory is T -equivariant, so we obtain an equivariant virtual class

$$[\mathrm{NHilb}^{\underline{d}}(\mathbb{A}^n)]^{\mathrm{vir}} \in A_*^T(\mathrm{NHilb}^{\underline{d}}(\mathbb{A}^n)).$$

Then we apply the Graber–Pandharipande virtual localization formula to compute the restriction of this class in the localized equivariant Chow group $A_*^T(\mathrm{NHilb}^{\underline{d}}(\mathbb{A}^n))_{\mathrm{loc}}$ as a sum over T -fixed points. These fixed points correspond to chains of monomial ideals, i.e., to nested partitions. We obtain an explicit iterated residue formula expressing $[\mathrm{NHilb}^{\underline{d}}(\mathbb{A}^n)]^{\mathrm{vir}}$ as a weighted sum over admissible $(n-1)$ -dimensional nested partitions of length $\underline{d}$, with weights given by rational functions in the torus weights (Theorem 2.1.1).

We also single out the nilpotently filtered locus $\mathrm{NHilb}_{\mathrm{nil-fil}}^{\underline{d}}(\mathbb{A}^n)$, consisting of nested ideals

$$I_{r+1} \subset \cdots \subset I_1 \subset I_0 = \mathbb{C}[x_1, \dots, x_n]$$

such that I_1 is the maximal ideal at the origin and $I_1 \cdot I_i \subset I_{i+1}$ for all i . This locus lies inside the punctual nested Hilbert scheme and has a natural virtual class induced by our embedding into the non-associative model. When $\underline{d} = (1, \dots, 1)$ is a full flag, we show that $\mathrm{NHilb}_{\mathrm{nil-fil}}^{\underline{d}}(\mathbb{A}^n)$ coincides with the usual punctual nested Hilbert scheme, and more generally we study its geometry for arbitrary $\underline{d}$. When $r = 1$, $d_1 = d$, we identify

$$\mathrm{NHilb}_{\mathrm{nil-fil}}^{\underline{d}}(\mathbb{A}^n) = \{I \subset \mathbb{C}[x_1, \dots, x_n] \mid \dim_{\mathbb{C}} \mathbb{C}[x]/I = d+1, \mathfrak{m}^2 \subset I \subset \mathfrak{m}\},$$

which is empty whenever $d > n$; this illustrates that the nilpotently filtered locus is much smaller than the full punctual Hilbert scheme.

For virtual integrals supported on the nilpotently filtered locus $\mathrm{NHilb}_{\mathrm{nil-fil}}^{\underline{d}}(\mathbb{A}^n)$, we refine the localization calculation to an iterated residue formula. When $d_1 + \cdots + d_r \leq n$ we construct a partial resolution of $\mathrm{NHilb}_{\mathrm{nil-fil}}^{\underline{d}}(\mathbb{A}^n)$ fibering over a flag variety $\mathrm{Flag}_{\hat{\underline{d}}}(\mathbb{C}^n)$ (where $\hat{\underline{d}} = (d_1, \dots, d_r)$) and reduce integrals to a fiber over a reference flag. Following [BS12],

Atiyah-Bott localization on this resolved space can be transformed into a multi-variable rational form whose virtual integral is given by an iterated residue at infinity. A key feature of our analysis is that all but one of the candidate residues vanish; the sole contribution comes from the “Porteous” nested partition consisting only of unit vectors in the natural order. The resulting formula expresses virtual integrals on $\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^d(\mathbb{A}^n)$ as a single explicit iterated residue (Theorem 2.1.2). This generalizes earlier iterated residue techniques of [BS12] in singularity theory and Thom polynomial computations to the context of nested Hilbert schemes of points and virtual integration.

Finally, using an embedding

$$\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^d(\mathbb{A}^n) \hookrightarrow \mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^d(\mathbb{A}^N)$$

for $N \gg n$ and comparing the corresponding virtual structures, we show that the same residue formula continues to hold even when $d_1 + \cdots + d_r > n$.

Tautological versus virtual intersection theory.

The curvilinear Hilbert scheme, denoted by $\mathrm{CHilb}_0^d(\mathbb{A}^n)$, parametrizes infinitesimally collinear subschemes of a smooth variety and forms a small, distinguished component of the punctual Hilbert scheme. In [B17] it is shown that this curvilinear component forms a natural projective compactification of the jet moduli space $J_k(1, n)/\mathrm{Diff}_k$. This perspective led to a systematic theory of tautological intersection numbers on curvilinear and full Hilbert schemes of points. In a series of papers [B17, B1, B2, B3], tautological integrals were expressed in terms of explicit iterated residue formulas for all tautological integral on the main component of the Hilbert schemes of points in arbitrary dimension. These formulas recover and generalise classical surface results of Göttsche and Nakajima, and they apply to a wide range of enumerative problems, including counting curves and hypersurfaces with prescribed singularities and higher-dimensional analogues of Lehn’s conjecture and Segre–Verlinde type formulas. A striking structural property of Hilbert scheme of points proved in [BS] is that all monomial ideals (i.e. torus fixed points) lie on the curvilinear component, strongly suggesting that the curvilinear locus controls the topology of the entire Hilbert scheme.

The present work develops a parallel virtual intersection theory on the full Hilbert schemes of points, not just on the main (smoothable) component. Crucially, our construction yields virtual classes of positive dimension whenever the ambient dimension n is sufficiently large compared to the total length of the subscheme, and the virtual localization formulas can again be written as iterated residues, now incorporating the contributions of the obstruction theory. In this sense, the virtual intersection theory developed here is formally very close to the tautological intersection theory of [B17, B1, B2, B3]: in both settings, universal intersection numbers on Hilbert schemes of points are governed by explicit multi-variable residue formulas over the same combinatorial data of monomial ideals and partitions. In forthcoming work [BM25], this framework is used to reconstruct Donaldson–Thomas invariants of Calabi–Yau threefolds from the non-associative Hilbert scheme model, revealing a direct bridge between the tautological and virtual theories.

Outline of the paper.

In Section 2.3 we define the non-associative nested Hilbert scheme $\mathrm{naNHilb}^d(\mathbb{A}^n)$ and prove its smoothness and basic geometric properties, including the embedding of $\mathrm{NHilb}^d(\mathbb{A}^n)$ as the associativity locus. In 2.4 we construct the punctual locus, prove that it can be described as a tower of Grassmannian fibrations and construct a non-associative Hilbert–Chow morphism. In Section 2.5 we construct the induced perfect obstruction theory and virtual class on $\mathrm{NHilb}^d(\mathbb{A}^n)$ and derive an explicit T -equivariant localization formula, expressed as a sum over admissible nested partitions. Section 2.6 is devoted to the nilpotently filtered

locus: we define $\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^d(\mathbb{A}^n)$, analyze its geometry, construct a partial resolution over a flag variety and prove the iterated residue formula for virtual integrals.

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2.1 The results

Fix a dimension vector $\underline{d} = (d_0, \dots, d_r) \in \mathbb{Z}_{\geq 0}^{r+1}$ and $n \in \mathbb{Z}_{\geq 0}$. The main object of interest in this paper is the nested Hilbert scheme of $d = d_0 + \dots + d_r$ points in $\mathbb{A}^n$ given as the moduli space

$$\begin{aligned} \mathrm{NHilb}^{\underline{d}}(\mathbb{A}^n) &= \{\xi_1 \subset \dots \subset \xi_{r+1} \subset \mathbb{A}^n \mid \dim \xi_i = 0, \text{Length } \xi_i = d_0 + \dots + d_{i-1}\} \\ &= \{I_{r+1} \subset \dots \subset I_1 \subset I_0 = \mathbb{C}[x_1, \dots, x_n] \mid \dim_{\mathbb{C}} I_i/I_{i+1} = d_i\}. \end{aligned}$$

We study it by defining a new smooth variety for which this (highly singular) scheme embeds. We use this to define virtual fundamental lass on the nested Hilbert scheme and develop localization and iterated residue formulas for efficiently computing virtual invariants.

The non-associative Hilbert scheme.

Extending the work of [Kaz17] we construct $\mathrm{naNHilb}^{\underline{d}}(\mathbb{A}^n)$ the non-associative nested Hilbert scheme of $d = d_0 + \dots + d_r$ points on $\mathbb{A}^n$. It is the moduli space

$$\mathrm{naNHilb}^{\underline{d}}(\mathbb{A}^n) = \{I_{r+1} \subset \dots \subset I_1 \subset I_0 = \mathbb{C}\{x_1, \dots, x_n\}_c \mid \dim_{\mathbb{C}} I_i/I_{i+1} = d_i\}$$

where $\mathbb{C}\{x_1, \dots, x_n\}_c$ is the free unital, commutative, non-associative algebra on n -generators and the I_i 's are ideals. We prove that this is a smooth variety and that $\mathrm{NHilb}^{\underline{d}}(\mathbb{A}^n)$ embeds in $\mathrm{naNHilb}^{\underline{d}}(\mathbb{A}^n)$ as the locus where I_1 is contained in the associator ideal or equivalently where the quotient algebras are associative.

In the case where $d_0 = 1$ we construct a punctual locus $\mathrm{naNHilb}_0^{\underline{d}}(\mathbb{A}^n) \subset \mathrm{naNHilb}^{\underline{d}}(\mathbb{A}^n)$ which intersects the Hilbert scheme in the nilpotently filtered locus

$$\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^{\underline{d}}(\mathbb{A}^n) = \left\{ [I_{r+1} \subset \dots \subset I_1 \subset I_0] \in \mathrm{NHilb}^{\underline{d}}(\mathbb{A}^n) \mid I_1 = \mathfrak{m}, I_1 \cdot I_i \subset I_{i+1} \right\}$$

where $\mathfrak{m} = (x_1, \dots, x_n)$. Using the methods of [Kaz17] we prove $\mathrm{naNHilb}_0^{\underline{d}}(\mathbb{A}^n)$ admits the structure of a tower of Grassmannian fibrations. In the special case where $d_i = 1$ for all $0 \leq i \leq r$ we are further able to construct an extension

$$\mathrm{naNHilb}^{\underline{d}}(\mathbb{A}^n) \rightarrow (\mathbb{A}^n)^{r+1}$$

of the (ordered) Hilbert-Chow morphism on $\mathrm{NHilb}^{\underline{d}}(\mathbb{A}^n)$. We use this to prove that $\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^{\underline{d}}(\mathbb{A}^n)$ identifies with the punctual Hilbert scheme of points

$$\mathrm{NHilb}_0^{\underline{d}}(\mathbb{A}^n) = \{\xi_1 \subset \dots \subset \xi_{r+1} \subset \mathbb{A}^n \mid \dim \xi_i = 0, \text{Length } \xi_i = i, \text{Supp } \xi_i = \{0\}\}.$$

We always have $\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^{\underline{d}}(\mathbb{A}^n) \subset \mathrm{NHilb}_0^{\underline{d}}(\mathbb{A}^n)$ but the equality only holds for the full dimension vector where $d_i = 1$ for all i . On the other end of the spectrum where $r = 1$, $d_1 = d$ we can identify

$$\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^{\underline{d}}(\mathbb{A}^n) = \{I \subset \mathbb{C}[x_1, \dots, x_n] \mid \dim_{\mathbb{C}} \mathbb{C}[x_1, \dots, x_n]/I = d+1, \mathfrak{m}^2 \subset I \subset \mathfrak{m}\}.$$

This is much smaller than the punctual locus and becomes empty whenever $d > n$.

Virtual fundamental class and localization.

One can easily prove that the associativity locus is the zero locus of a section of a vector bundle on $\text{naNHilb}^d(\mathbb{A}^n)$. This equips $\text{NHilb}^d(\mathbb{A}^n)$ with a perfect obstruction theory and therefore also a virtual fundamental class. Setting $T = (\mathbb{C}^*)^n$ this procedure can be made T -equivariant and we are able to calculate the virtual class via a localization formula. We obtain the following result, where we use the notation

$$\widetilde{\prod} \gamma := \prod_{\gamma \neq 0} \gamma$$

for non-zero products.

Theorem 2.1.1. *Let $\underline{d} = (d_0, \dots, d_r) \in \mathbb{Z}_{\geq 0}^{r+1}$, $d = d_0 + \dots + d_r$, $n \in \mathbb{Z}_{\geq 0}$ and $T = (\mathbb{C}^*)^n$. Let $A_T^*(\text{pt}) = \mathbb{Z}[s_1, \dots, s_n]$ where $s_1, \dots, s_n$ are the Chern roots of $\mathbb{C}^n$ given the standard torus action. In the Chow group $A_*^T(\text{NHilb}^d(\mathbb{A}^n))_{\text{loc}}$ localized in linear functions of $s_1, \dots, s_n$ the virtual fundamental class $[\text{NHilb}^d(\mathbb{A}^n)]^{\text{vir}}$ equals*

$$\sum_{\lambda_{\bullet}} [\lambda_{\bullet}] \cap \frac{\prod_{1 \leq i, j, k \leq d-1} \prod_{i < k} \prod_{0 \leq m \leq d-1} \frac{(\mathbf{u}_i + \mathbf{u}_j + \mathbf{u}_k - \mathbf{u}_m)(\underline{s})}{w(k), w(j) \leq w(m)} \prod_{j=1}^{d-1} \prod_{0 \leq k \leq d-1} \frac{\mathbf{u}_j - \mathbf{u}_k(\underline{s})}{w(j) \leq w(k)}}{\prod_{i=1}^n \prod_{k=0}^{d-1} (e_i - \mathbf{u}_k)(\underline{s}) \prod_{1 \leq i \leq j \leq d-1} \prod_{0 \leq k \leq d-1} \frac{(\mathbf{u}_i + \mathbf{u}_j - \mathbf{u}_k)(\underline{s})}{w(j) \leq w(k)}}.$$

Here the sum is over all admissible $(n-1)$ -dimensional nested partitions of length $\underline{d}$

$$\lambda_{\bullet} = (\lambda_1 \subset \dots \subset \lambda_{r+1} \subset \mathbb{Z}_{\geq 0}^n)$$

viewed as a fixed point $\lambda_{\bullet} \in \text{NHilb}^d(\mathbb{A}^n)^T$ and where we have a chosen enumeration

$$\lambda_{k+1} = \{\mathbf{u}_0, \dots, \mathbf{u}_{d_0 + \dots + d_{k-1} - 1}\} \subset \mathbb{Z}_{\geq 0}^n$$

for each $\lambda_{\bullet}$. For $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{Z}^n$ we define

$$\mathbf{u}(\underline{s}) = u_1 s_1 + \dots + u_n s_n.$$

Being admissible means that λ_{r+1} contains no vectors with 3 or more non-zero coordinates and that the projection to a two-dimensional plane sits inside the partition given by figure 1 below.

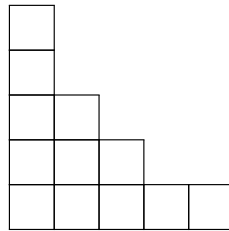


Figure 2.1

The iterated residue formula.

Similarly, when $d_0 = 1$ the nilpotently filtered Hilbert scheme $\text{NHilb}_{\text{nil-fil}}^d(\mathbb{A}^n)$ admits a virtual fundamental class. When the flag is full, i.e., $d_i = 1$ for all i we are able to describe virtual integrals on $\text{NHilb}^d(\mathbb{A}^n)$ in terms of a virtual integral on $\text{NHilb}_{\text{nil-fil}}^d(\mathbb{A}^n)$. When the flag is not full we can still perform this procedure, provided the integrand is supported

on the nilpotently filtered locus. We spent the last section of the paper developing an iterated residue for virtual integrals over the nilpotently filtered locus. If we let $\underline{d} = (1, \hat{d}) = (1, d_1, \dots, d_r)$ and assume

$$d = 1 + d_1 + \dots + d_r \leq n + 1$$

we can construct a partial resolution of $\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^{\underline{d}}(\mathbb{A}^n)$ that fibers over the flag variety $\mathrm{Flag}_{\hat{d}}(\mathbb{C}^n)$. We can use this to derive a residue formula depending on

$$\int_{[H_{\mathrm{id}}]^{\mathrm{vir}}} \alpha$$

where H_{id} is the fiber in the partial resolution over the reference flag

$$f_{\bullet} = (\mathrm{span}(e_1, \dots, e_{d_0}) \subset \dots \subset \mathrm{span}(e_1, \dots, e_{\hat{d}})) \in \mathrm{Flag}_{\hat{d}}(\mathbb{C}^n).$$

Calculating this integral via localization, we get a sum of iterated residues and we can prove that all residues vanish except for the one at the Porteous point i.e., the nested partition containing only unit vectors nested in the natural way. By considering the map

$$\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^{\underline{d}}(\mathbb{A}^n) \hookrightarrow \mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^{\underline{d}}(\mathbb{A}^N)$$

for $n < N$ we can prove that this formula remains valid even when $d > n + 1$. We obtain the following result.

Theorem 2.1.2. *Let $\underline{d} = (1, d_1, \dots, d_r) \in \mathbb{Z}_{\geq 0}^{r+1}$, $d = 1 + d_1 + \dots + d_r$, $n \in \mathbb{Z}_{\geq 0}$, and $T = (\mathbb{C}^*)^n$. Let $A_T^*(\mathrm{pt}) = \mathbb{Z}[s_1, \dots, s_n]$ where $s_1, \dots, s_n$ are the Chern roots of $\mathbb{C}^n$ given the standard torus action. Pick a q -dimensional T -representation W with Chern roots $\theta_1, \dots, \theta_q$ and consider the T -equivariant rank q bundle*

$$E = W \otimes \mathbb{A}^n$$

on $\mathbb{A}^n$. Let

$$-\eta_0, -\eta_1, \dots, -\eta_{d-1}$$

be the Chern roots of the tautological bundle $\mathcal{O}_{\mathbb{A}^n}^{[\underline{d}]}$ so that

$$E^{[\underline{d}]} \cong W \otimes \mathcal{O}_{\mathbb{A}^n}^{[\underline{d}]}$$

has Chern roots $\theta_i - \eta_j$. Let

$$P = P(\theta_1, \dots, \theta_q, \eta_1, \dots, \eta_{d-1})$$

be a polynomial in the Chern classes of $E^{[\underline{d}]}$. Then

$$\int_{[\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^{\underline{d}}(\mathbb{A}^n)]^{\mathrm{vir}}} P$$

can be calculated via the iterated residue formula

$$\mathrm{Res}_{\underline{z}=\infty} \frac{P(\theta_1, \dots, \theta_q, \underline{z}) \prod_{1 \leq m \neq l \leq d-1} (z_m - z_l) \prod_{\substack{1 \leq i, j, k \leq d-1 \\ i < k}} \prod_{\substack{0 \leq m \leq d-1 \\ w(k), w(j) \leq w(m)}} (z_i + z_j + z_k - z_m) d\underline{z}}{\prod_{\substack{1 \leq l < m \leq d-1 \\ w(l) < w(m)}} (z_m - z_l) \prod_{i=1}^n \prod_{l=1}^{d-1} (s_i - z_l) \prod_{1 \leq i \leq j \leq d-1} \prod_{\substack{1 \leq k \leq d-1 \\ w(j) < w(k)}} (z_i + z_j - z_k)}$$

Here $\underline{z} = (z_1, \dots, z_{d-1})$ and the iterated residue is defined as the coefficient of $(z_1 \cdots z_{d-1})$ in the expansion of the rational expression in the domain $z_1 \ll \dots \ll z_{d-1}$.

In the follow-up paper [BM25], we will further develop the theory by comparing the virtual invariants obtained in this paper with other known virtual invariants using a rational map from the non-associative to the non-commutative nested Hilbert scheme. We note that several virtual invariants of the Hilbert scheme of points can be constructed explicitly using the non-commutative Hilbert scheme. The degree 0 rank 1 DT-invariants on $\text{Hilb}^d(\mathbb{A}^3)$ can be described using the critical locus description in the non-commutative Hilbert scheme, see [BBS13, Oko17]. The Hilbert scheme of points on $\mathbb{A}^4$ can be exhibited as a zero locus of an isotropic section of an orthogonal bundle on the non-commutative Hilbert scheme thus providing a virtual class in the sense of [OT23], see [KR]. In [Min] a nested version of the non-commutative Hilbert scheme is constructed and it is shown that $\text{NHilb}^d(\mathbb{A}^2)$ can be exhibited as a zero locus of a section of a vector bundle in a way that recovers the virtual class of [GSY20].

In more detail, the existence and smoothness of the nested non-commutative Hilbert scheme

$$\text{ncNHilb}^d(\mathbb{A}^n),$$

is proven in [Min]. It is the moduli space parameterizing $(N_\bullet, A_1, \dots, A_n, v)$, where $N_\bullet$ is a flag of vector spaces of dimension $\underline{d}$, each A_i is a flag-preserving operator, and $v \in N_0$ is a cyclic vector, meaning that every element of $N_\bullet$ is a linear combination of $A_{i_1} A_{i_2} \cdots A_{i_m} v$. This construction specializes to the ordinary non-commutative Hilbert scheme in the case where $\underline{d} = (d)$. We consider the rational map

$$\begin{aligned} \text{naNHilb}^d(\mathbb{A}^n) &\dashrightarrow \text{ncNHilb}^d(\mathbb{A}^n) \\ [N_\bullet, \psi_1, \psi_2, v] &\longmapsto [N_\bullet, \psi_2(\psi_1(x_1), -), \dots, \psi_2(\psi_1(x_n), -), v] \end{aligned}$$

where $(\mathbb{C}^n)^\vee = \text{span}(x_1, \dots, x_n)$. In other words, it is the same flag $N_\bullet$ with the same unit v , and the operators are defined by left multiplication with $\psi_1(x_i)$. The domain of definition is the Zariski open locus where $N_\bullet$ is spanned by v and by elements obtained by iterative left multiplication by $\psi_1(x_1), \dots, \psi_1(x_n)$. The nested Hilbert scheme of points lies in the domain of this map and the map restricts to the identity map of the nested Hilbert scheme of points viewed either as the associativity or the commutativity locus.

We hope to use this map to construct relative perfect obstruction theories on the identity map of the (nested) Hilbert scheme of points compatible with our obstruction theory in the domain and well-known obstruction theories that can be constructed from the non-commutative (nested) Hilbert scheme in the codomain.

Remark 2.1.1. One might be tempted to think that the map is well defined everywhere, since $y_1 = \psi_1(x_1), \dots, y_n = \psi_1(x_n)$ generate $N_\bullet$ as a non-associative algebra. This is, however, not the case. The essential issue is that elements of the form

$$(y_i y_j)(y_k y_m)$$

cannot *a priori* be written as elements obtained by iterative left multiplication by the y_ℓ .

Similarly, if we simply denote the operators by

$$A_i := \psi_2(\psi_1(x_i), -)$$

we can define a potential

$$\begin{aligned} \text{naHilb}^d(\mathbb{A}^3) &\xrightarrow{g} \mathbb{A}^1 \\ [N_\bullet, \psi_1, \psi_2, v] &\longmapsto \text{Tr } A_1[A_2, A_3] \end{aligned}$$

which is well defined on the whole non-associative Hilbert scheme. One easily sees that

$$\begin{array}{ccccc}
 & & \mathbb{A}^1 & & \\
 & \nearrow g & & \nwarrow f & \\
 \text{naHilb}^d(\mathbb{A}^3) & \dashrightarrow & & \dashrightarrow & \text{ncHilb}^d(\mathbb{A}^3) \\
 & \nwarrow & & \nearrow & \\
 & & \text{Hilb}^d(\mathbb{A}^3) & &
 \end{array}$$

commutes and that the nested Hilbert scheme lies in the domain of definition of the rational map.

Remark 2.1.2. Note that the critical locus $\text{Crit}(g)$ is exactly the locus where the operators $\psi_2(\psi_1(x_i), -)$ commutes. This commutativity yields associativity conditions. Indeed, let A be a commutative non-associative algebra and for $a \in A$ denote by L_a the operator given by left multiplication of a . If $L_a L_b = L_b L_a$ then for all $c \in A$ we have

$$(ac)b = b(ac) = L_b L_a(c) = L_a L_b(c) = a(bc) = a(cb)$$

i.e. the associator $[a, c, b]$ vanishes for all $c \in A$.

Remark 2.1.3. One might be tempted to think that $\text{NHilb}^d(\mathbb{A}^3) = \text{Crit}(g)$. We have $\text{NHilb}^d(\mathbb{A}^3) \subset \text{Crit}(g)$ but the inclusion is in general strict, even if we restrict to the domain of definition of the rational map. Let us consider the following counter example. Let $N = \text{span}(v_0, v_1, v_2)$. Let v_0 be the unit and define a multiplication rule by

$$v_1 v_1 = v_2, v_1 v_2 = v_2 v_1 = v_2 \text{ and } v_2 v_2 = 0.$$

Set $\psi_1(x_1) = \psi_1(x_2) = \psi_1(x_3) = v_1$. This is in the domain of definition of the rational map since everything is spanned by the unit v_0 and left multiplications of v_1 with itself. Clearly the operators $\psi_2(\psi_1(x_i), -)$ and $\psi_2(\psi_1(x_j), -)$ commutes since $\psi_1(x_i) = \psi_1(x_j)$. However, the algebra is not associative as

$$(v_1 v_1) v_2 = v_2 v_2 = 0 \neq v_2 = v_1 v_2 = v_1(v_1 v_2).$$

The rational map defined in this section provides us a direct approach to express DT invariants as tautological virtual integrals on the positive dimensional virtual classes in higher dimensions.

2.2 Preliminaries

2.2.1 Non-associative algebras

We provide the definition of a non-associative $\mathcal{O}_S$ -algebra and describe how to construct quotients of the free non-associative algebra.

Definition 2.2.1. Let S be a scheme. A (quasi-coherent) non-associative $\mathcal{O}_S$ -algebra is by definition a pair $(\mathcal{F}, \psi)$ of a quasi-coherent sheaf $\mathcal{F} \in \text{QCoh}(S)$ equipped with a multiplication rule

$$\psi : \mathcal{F} \otimes_S \mathcal{F} \rightarrow \mathcal{F}.$$

A (quasi-coherent) unital non-associative $\mathcal{O}_S$ algebra is a triple $(\mathcal{F}, \psi, s)$ where $(\mathcal{F}, \psi)$ is a non-associative $\mathcal{O}_S$ algebra and $s : \mathcal{O}_S \rightarrow \mathcal{F}$ is a global section such that

$$\mathcal{F} \simeq \mathcal{O}_S \otimes_S \mathcal{F} \xrightarrow{s \oplus \text{id}} \mathcal{F} \otimes_S \mathcal{F} \xrightarrow{\psi} \mathcal{F}$$

and

$$\mathcal{F} \simeq \mathcal{F} \otimes_S \mathcal{O}_S \xrightarrow{\text{id} \oplus s} \mathcal{F} \otimes_S \mathcal{F} \xrightarrow{\psi} \mathcal{F}$$

agrees with $\text{id}_{\mathcal{F}}$. A non-associative $\mathcal{O}_S$ -algebra $(\mathcal{F}, \psi)$ is said to be commutative if ψ factors through

$$\text{Sym}_S^2 \mathcal{F} = \mathcal{F} \otimes_S \mathcal{F} / (a \otimes b - b \otimes a).$$

A morphism of non-associative $\mathcal{O}_S$ -algebras $f : (\mathcal{F}, \psi) \rightarrow (\mathcal{G}, \varphi)$ is a morphism of $\mathcal{O}_S$ -modules $f : \mathcal{F} \rightarrow \mathcal{G}$ such that

$$\begin{array}{ccc} \mathcal{F} \otimes_S \mathcal{F} & \xrightarrow{\psi} & \mathcal{F} \\ \downarrow f \otimes f & & \downarrow f \\ \mathcal{G} \otimes_S \mathcal{G} & \xrightarrow{\varphi} & \mathcal{G} \end{array}$$

commutes. A morphism of unital non-associative $\mathcal{O}_S$ -algebras $f : (\mathcal{F}, \psi, s) \rightarrow (\mathcal{G}, \varphi, t)$ is a morphism $f : \mathcal{F} \rightarrow \mathcal{G}$ of non-associative $\mathcal{O}_S$ -algebras such that $t = fs$.

Lemma 2.2.1. *Let $S \in \text{Sch}_{\mathbb{C}}$, let $(\mathcal{F}, \psi)$ be a unital non-associative algebra and let $\mathcal{G} \in \text{QCoh}(S)$. Let $\pi : \mathcal{F} \rightarrow \mathcal{G}$ be an epimorphism of $\mathcal{O}_S$ -modules with kernel $\mathcal{I}$. Then there exists a necessarily unique unital non-associative algebra structure φ on $\mathcal{G}$ making π a morphism of unital non-associative algebras if and only if $\mathcal{I}$ is a two-sided ideal. If ψ is commutative resp. associative, then the same will hold true for φ .*

Proof. Suppose first that φ exists. Then

$$\pi(\psi(\mathcal{I} \otimes_S \mathcal{F})) = \varphi(\pi(\mathcal{I}) \otimes_S \pi(\mathcal{F})) = 0$$

and

$$\pi(\psi(\mathcal{F} \otimes_S \mathcal{I})) = \varphi(\pi(\mathcal{F}) \otimes_S \pi(\mathcal{I})) = 0$$

proving that $\psi(\mathcal{I} \otimes_S \mathcal{F}), \psi(\mathcal{F} \otimes_S \mathcal{I}) \subset \mathcal{I}$. Conversely, if $\mathcal{I}$ is a two-sided ideal. We have two exact sequences given by

$$\mathcal{I} \otimes_S \mathcal{F} \rightarrow \mathcal{F} \otimes_S \mathcal{F} \rightarrow \mathcal{G} \otimes_S \mathcal{F} \rightarrow 0$$

and

$$\mathcal{G} \otimes_S \mathcal{I} \rightarrow \mathcal{G} \otimes_S \mathcal{F} \rightarrow \mathcal{G} \otimes_S \mathcal{G} \rightarrow 0.$$

Since $\pi(\psi(\mathcal{I} \otimes_S \mathcal{F})) = 0$ it follows that $\pi\psi$ uniquely induce a map $\mathcal{F} \otimes_S \mathcal{G} \rightarrow \mathcal{G}$. Since $\pi(\psi(\mathcal{F} \otimes_S \mathcal{I})) = 0$ it follows that this induced map takes $\mathcal{G} \otimes_S \mathcal{I}$ to 0 and hence induces a unique map

$$\varphi : \mathcal{G} \otimes_S \mathcal{G} \rightarrow \mathcal{G}.$$

Let $\tau_{\mathcal{G}}$ be the swap map of $\mathcal{G} \otimes_S \mathcal{G}$. Commutativity of φ can be checked by the vanishing of the map $\varphi - \varphi\tau_{\mathcal{G}}$, which by surjectivity of π can be checked after precomposing with $\pi \otimes \pi$. This composition yields the map $\pi(\psi - \psi\tau_{\mathcal{F}})$ which vanishes if ψ is commutative. Similarly, associativity can be checked by the vanishing of the map $\varphi(\text{id} \otimes \varphi) - \varphi(\varphi \otimes \text{id})$, which again by surjectivity of π can be checked after precomposing with $\pi \otimes \pi \otimes \pi$. This composition gives the map $\pi(\psi(\text{id} \otimes \psi) - \psi(\psi \otimes \text{id}))$ which vanishes if ψ is associative. $\square$

Remark 2.2.1. We note that if $(\mathcal{F}, \psi)$ is also equipped with a unital structure s then we similarly get a unique unital structure t on $\mathcal{G}$ by setting $t = \pi s$.

Lemma 2.2.2. *Let $S \in \text{Sch}_{\mathbb{C}}$ and let $\text{Qlg}(S)$ be the category of unital non-associative $\mathcal{O}_S$ algebras. The forgetful functor*

$$\text{Qlg}(S) \rightarrow \text{QCoh}(S)$$

admits a left adjoint $F_{\text{Qlg}}(S)$.

Proof. One could argue briefly using the adjoint functor theorem. Instead, we give concrete construction. We define for each $k \geq 1$ the set $B(k)$ corresponding to the set of ways to put bracket around a length k expression. One can define this inductively setting $B(1) = \{*\}$ and

$$B(k) = \coprod_{i+j=k} B(i) \times B(j).$$

For a $\mathcal{F} \in \text{QCoh}(S)$ we then set

$$F_{\text{Qlg}}(\mathcal{F}) := \mathcal{O}_S \oplus \bigoplus_{k=1}^{\infty} \bigoplus_{x \in B(k)} \mathcal{F}^{\otimes_S k}.$$

The first summand is the unit, and the multiplication rule

$$F_{\text{QLG}}(\mathcal{F}) \otimes_S F_{\text{QLG}}(\mathcal{G}) \rightarrow F_{\text{QLG}}(\mathcal{F})$$

is given by first distributing the tensor product over the direct sums and then mapping $\mathcal{F}^{\otimes_S i} \otimes_S \mathcal{F}^{\otimes_S j}$ corresponding to $x \in B(i)$ and $y \in B(j)$ to the summand corresponding to $(x, y) \in B(i + j)$. Given the datum of a map $f : \mathcal{F} \rightarrow \mathcal{G}$ in $\text{QCoh}(S)$ and a unital non-associative structure $s \in H^0(S, \mathcal{G})$, $\psi : \mathcal{G} \otimes_S \mathcal{G} \rightarrow \mathcal{G}$ one can then easily via induction construct a necessarily unique map

$$F_{\text{Qlg}}(\mathcal{F}) \rightarrow \mathcal{G}$$

of unital non-associative algebras such that the map agrees with f when restricted to $\bigoplus_{x \in B(1)} \mathcal{F} \cong \mathcal{F}$. This proves that the construction is left adjoint to the forgetful functor. $\square$

Lemma 2.2.3. *Let $S \in \text{Sch}_{\mathbb{C}}$ and let $\text{Qlg}(S)$, $\text{QClg}(S)$, $\text{QAlg}(S)$ and $\text{QCalg}(S)$ be the categories of respectively unital non-associative, unital commutative non-associative, unital associative and unital commutative associative $\mathcal{O}_S$ -algebras. The inclusion functors*

$$\begin{array}{ccc} & \text{QClg}(S) & \\ \nearrow & & \searrow \\ \text{QCalg}(S) & & \text{Qlg}(S) \\ \searrow & & \nearrow \\ & \text{QAlg}(S) & \end{array}$$

admits left adjoints. In particular, using lemma 2.2.2 we see that the forgetful functor from any of these categories to $\text{QCoh}(S)$ admits left adjoints.

Proof. For $(\mathcal{F}, \psi, s) \in \text{Qlg}(S)$ we define the commutator ideal as the smallest two-sided ideal containing

$$\psi(x, y) - \psi(y, x)$$

for all $x, y \in \mathcal{F}$. Taking the quotient of $\mathcal{F}$ by this ideal yields the left adjoints of the upper right and lower left arrows of the diamond. Similarly, we can define the associator ideal generated by

$$\psi(x, \psi(y, z)) - \psi(\psi(x, y), z)$$

for $x, y, z \in \mathcal{F}$. Taking the quotient of $\mathcal{F}$ by this ideal yields the left adjoints of the lower right and upper left arrows of the diamond. $\square$

Remark 2.2.2. We will denote the left adjoints of the forgetful functors $\mathcal{C} \rightarrow \text{QCoh}(S)$ by $F_{\mathcal{C}}$. Let $n \in \mathbb{N}$. We will throughout the paper use the following standard notation

$$\begin{aligned}\mathcal{O}_S\{x_1, \dots, x_n\} &:= F_{\text{Qlg}}((\mathcal{O}_S^n)^\vee), \\ \mathcal{O}_S\{x_1, \dots, x_n\}_c &:= F_{\text{QClg}}((\mathcal{O}_S^n)^\vee), \\ \mathcal{O}_S\langle x_1, \dots, x_n \rangle &:= F_{\text{QAlg}}((\mathcal{O}_S^n)^\vee) \text{ and} \\ \mathcal{O}_S[x_1, \dots, x_n] &:= F_{\text{QCAlg}}((\mathcal{O}_S^n)^\vee)\end{aligned}$$

for the various free algebras on n generators. Notice that we take the dual to keep the convention of a covariant equivalence between vector bundles and locally free sheaves of finite rank.

Lemma 2.2.4. *Let $S \in \text{Sch}_{\mathbb{C}}$ and let $\mathcal{F}, \mathcal{G}$ be quasi-coherent sheaves. Let ψ, s be the multiplication rule and unit of $F_{\text{Qlg}}(\mathcal{F})$. The following data is equivalent.*

1. *A map $f : \mathcal{F} \rightarrow \mathcal{G}$ and a unital non-associative algebra structure (φ, t) on $\mathcal{G}$ such that the induced map $\pi : F_{\text{Qlg}}(\mathcal{F}) \rightarrow \mathcal{G}$ is an epimorphism.*
2. *An epimorphism $\pi : F_{\text{Qlg}}(\mathcal{F}) \rightarrow \mathcal{G}$ of quasi-coherent sheaves such that the kernel is a two-sided ideal.*

Proof. By adjunction and lemma 2.2.1 both types of data exactly describes a surjection π of unital non-associative algebras. $\square$

Lemma 2.2.5. *Let $S \in \text{Sch}_{\mathbb{C}}$ and let $\mathcal{F}, \mathcal{G}$ be quasi-coherent sheaves. Let (φ, t) be a unital non-associative algebra structure on $\mathcal{G}$ and let $\pi : F_{\text{Qlg}}(\mathcal{F}) \rightarrow \mathcal{G}$ be an epimorphism of unital non-associative algebras. Then $\mathcal{G}$ is associative (resp. commutative) if and only if π factors through $F_{\text{Qlg}}(\mathcal{F}) \rightarrow F_{\text{QAlg}}(\mathcal{F})$ (resp. $F_{\text{Qlg}}(\mathcal{F}) \rightarrow F_{\text{QClg}}(\mathcal{F})$).*

Proof. If $\mathcal{G}$ is associative (resp. commutative) then we get such a factorization since F_{QAlg} (resp. F_{QClg}) is the composite of F_{Qlg} with the left adjoint of the inclusion $\text{QAlg}(S) \rightarrow \text{Qlg}(S)$ (resp. $\text{QClg}(S) \rightarrow \text{Qlg}(S)$). Conversely, if such a factorization exists associativity or commutativity of $\mathcal{G}$ will then directly follow from lemma 2.2.1 $\square$

2.2.2 Obstruction theories and virtual classes and pullbacks

Let $X/\mathbb{C}$ be a scheme and let

$$\mathbb{L}_X = \tau_{\geq -1} L_X^\bullet \in D^{[-1, 0]}(X)$$

be the truncation of the full cotangent complex $L_X^\bullet$ as defined in [Ill71]. A perfect obstruction theory of $X \rightarrow Y$ as defined in [BF97] is a map

$$\phi : \mathbb{E} \rightarrow \mathbb{L}_X$$

in $D^{[-1, 0]}(X)$, where $\mathbb{E}$ is a perfect complex of perfect amplitude $[-1, 0]$ and where $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective. If we have

$$\mathbb{E} = [E^{-1} \rightarrow E^0]$$

then a perfect obstruction theory gives rise to a cone

$$\mathfrak{C} \hookrightarrow E_1 = (E^{-1})^\vee$$

which in term gives rise to a virtual fundamental class

$$[X]^{\text{vir}} = 0^*[\mathfrak{C}] \in A_{\text{vd}}(X)$$

where $0 : X \rightarrow E_1$ is the zero section and 0^* its Gysin. Here,

$$\mathrm{vd} = \chi(\mathbb{E}) = \mathrm{rk} E^0 - \mathrm{rk} E^{-1}.$$

Similarly, for $f : X \rightarrow Y$ a map of schemes a relative perfect obstruction theory is a map

$$\phi : \mathbb{E} \rightarrow \mathbb{L}_{X/Y} = \tau_{\geq -1} L_{X/Y}^\bullet$$

in $D^{[-1,0]}(X)$ where $\mathbb{E}$ and ϕ satisfies the same definitions as above. It is shown in [Man11] that ϕ defines a virtual pullback

$$f^* = f_{\mathbb{E}}^* : A_*(Y) \rightarrow A_{*+\chi(\mathbb{E})}(X).$$

If $Y = S$ is a smooth (or pure dimensional) one may define a virtual class by

$$[X]^{\mathrm{vir}} = f^*[S].$$

This recovers the usual definition for $S = \mathrm{Spec} \mathbb{C}$. It is further shown that if we have a compatible triple i.e. a distinguished triangle

$$f^* \mathbb{E}_{Y/S} \rightarrow \mathbb{E}_{X/S} \rightarrow \mathbb{E}_{X/Y} \rightarrow f^* \mathbb{E}_{Y/S}[1]$$

of perfect obstruction theories with a map to the canonical distinguished triangle

$$f^* \mathbb{L}_{Y/S} \rightarrow \mathbb{L}_{X/S} \rightarrow \mathbb{L}_{X/Y} \rightarrow f^* \mathbb{L}_{Y/S}[1]$$

then the corresponding virtual pullbacks are functorial and so in particular

$$f^*[Y]^{\mathrm{vir}} = [X]^{\mathrm{vir}}.$$

In this paper, we are interested in the following simple case.

Example 2.2.1. Suppose $\pi : Y \rightarrow S$ is a smooth morphism over a smooth base scheme S . Then

$$\mathbb{L}_{Y/S} \cong \Omega_{Y/S}$$

and $\phi = \mathrm{id}_{\mathbb{L}_{X/Y}}$ defines a perfect obstruction theory with virtual pullback being the usual flat pullback π^* . Note that

$$[Y]^{\mathrm{vir}} = \pi^*[S] = [Y].$$

If $i : X \hookrightarrow Y$ is a closed immersion with ideal sheaf I then

$$\mathbb{L}_{X/Y} = [I/I^2 \rightarrow 0].$$

If we can find a vector bundle

$$E = \mathrm{Spec}_Y \mathrm{Sym}^\bullet \mathcal{E}^\vee$$

with a section $s \in H^0(Y, \mathcal{E})$ such that $X = Z(s)$ then the map

$$\mathcal{E}^\vee|_X \xrightarrow{s^\vee} I/I^2$$

defines a relative perfect obstruction theory. Its virtual pullback is given by $0^!$ the refined Gysin map of the zero section $0 : Y \rightarrow E$ along s . Similarly, we see that

$$\mathbb{L}_{X/S} = [I/I^2 \rightarrow \Omega_{Y/S}]$$

and that the diagram

$$\begin{array}{ccc} \mathcal{E}^\vee|_X & \xrightarrow{dos^\vee} & \Omega_{Y/S}|_X \\ \downarrow s^\vee & & \parallel \\ I/I^2 & \xrightarrow{d} & \Omega_{Y/S}|_X \end{array}$$

defines a perfect obstruction theory on X/S . We can turn the above theories into a compatible triple and so the virtual pullback of X/S is simply $0^! \pi^*$. In particular,

$$[X]^{\mathrm{vir}} = 0^![Y].$$

2.2.3 Torus equivariant obstruction theories

Let $T = (\mathbb{C}^*)^m$ be an algebraic torus and let $\hat{T} = \text{Hom}(T, \mathbb{C}^*) \cong \mathbb{Z}^m$ its characters. Letting $T = \{(t_1, \dots, t_m)\}$ we get a ring isomorphism

$$K_0^T(\text{pt}) \xrightarrow{\text{Tr}} \mathbb{Z}[t^\mu \mid \mu \in \hat{T}] \cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$$

sending a T -representation V to the character $t \mapsto \text{Tr}_V(t)$. We will therefore sometimes abuse notation and denote an element of $K_0^T(\text{pt})$ simply by its trace. Similarly, for the equivariant Chow cohomology group we obtain a graded ring isomorphism

$$\text{Sym}_{\mathbb{Z}}^\bullet[\hat{T}] \longrightarrow \mathbb{Z}[s_1, \dots, s_m] \cong A_T^*(\text{pt})$$

which maps a character to the first Chern class of the induced 1-dimensional representation. Here, s_i is the first Chern class of the 1-dimensional representation of weight t_i . If $X \rightarrow Y$ is a T -equivariant map of T -schemes equipped with a T -equivariant perfect obstruction theory $\mathbb{E} \rightarrow \mathbb{L}_{X/Y}$, then the virtual pullback extends to a map of T -equivariant Chow groups. In particular, for $Y = \text{Spec } \mathbb{C}$ we obtain a T -equivariant virtual class

$$[X]^{\text{vir}} \in A_{\chi(\mathbb{E})}^T(X).$$

The main way of calculating this class is through the virtual localization formula of [GP99]. Recall that a T -equivariant sheaf on a scheme with trivial T -action splits into eigensheaves and therefore also into a moving and fixed part.

Proposition 2.2.1. *[GP99] Let $X/\mathbb{C}$ be a T -equivariant scheme with a T -equivariant perfect obstruction theory $\mathbb{E}$. Suppose X admits a T -equivariant embedding into a smooth T -scheme Y . Let $Y^T = \bigcup_i Y_i$ be the irreducible components of the fixed locus and set $X_i = Y_i \cap X$. Then the composition*

$$\mathbb{E}|_{X_i}^{\text{fix}} \rightarrow \mathbb{L}_X|_{X_i}^{\text{fix}} \rightarrow \mathbb{L}_{X_i}$$

is a perfect obstruction theory equipping X_i with a virtual class. If we set

$$N_{X_i/X}^{\text{vir}} = (\mathbb{E}|_{X_i}^{\text{mov}})^\vee = [(E_{\text{mov}}^0)^\vee \rightarrow (E_{\text{mov}}^{-1})^\vee]$$

then the identity

$$[X]^{\text{vir}} = \iota_* \sum_i \frac{[X_i]^{\text{vir}}}{e^T(N_{X_i/X}^{\text{vir}})}$$

holds true in the localized Chow ring $A_*^T(X)_{\text{loc}}$ where we have inverted linear functions of $s_1, \dots, s_m$. Here

$$e^T(N_{X_i/X}^{\text{vir}}) = \frac{e^T((E_{\text{mov}}^0)^\vee)}{e^T((E_{\text{mov}}^{-1})^\vee)}$$

is a fraction of T -equivariant Euler classes.

Remark 2.2.3. In the case where X is non-proper with a proper fixed locus X^T we will for $\alpha \in A_T^*(X)$ define virtual integrals via the localization formula above

$$\int_{[X]^{\text{vir}}} \alpha := \sum_i \int_{[X_i]^{\text{vir}}} \frac{\alpha|_{X_i}}{e^T(N_{X_i/X}^{\text{vir}})}.$$

We will spend the rest of this section providing consequences and examples of virtual localization that we will need later. First we show that the localization formula is particularly easy to use if X^T is zero-dimensional, reduced, and proper. Essentially $[x]^{\text{vir}}$ for $x \in X^T$ is either $[x]$ or 0 depending only on whether or not $\text{rk } E_{\text{fix}}^0 = \text{rk } E_{\text{fix}}^{-1}$

Lemma 2.2.6. *Let $T = (\mathbb{C}^*)^n$ and let $\varphi : \mathbb{E} = [E^{-1} \rightarrow E^0] \rightarrow \mathbb{L}_{\text{pt}}$ be a T -equivariant perfect obstruction on $\text{pt} = \text{Spec } \mathbb{C}$. If $\mathbb{E}^{fix} = \mathbb{E}$ and*

$$\text{vd} = \text{rk } E^0 - \text{rk } E^{-1} \neq 0$$

then $[\text{pt}]^{\text{vir}} = 0$. If $\text{vd} = 0$ then $[\text{pt}]^{\text{vir}} = [\text{pt}]$.

Proof. Clearly $\mathbb{L}_{\text{pt}} = 0$ and so any T -equivariant perfect obstruction theory is of the form $\mathbb{E} = [V \rightarrow 0]$ for a V a finite dimensional T -representation. If $\mathbb{E} = \mathbb{E}^{fix}$ then V is a trivial representation. The intrinsic normal cone of pt is pt and so the virtual fundamental class is

$$0^*[\text{pt}]$$

where $0 : \text{pt} \rightarrow V$ is the zero section and 0^* its Gysin. Since

$$\text{vd} = -\dim V$$

the result now clearly follows. $\square$

Example 2.2.2. Suppose we have a smooth T -scheme Y with a T -equivariant vector bundle $E = \text{Spec}_Y \text{Sym}^\bullet \mathcal{E}^\vee$ and section $s \in H^0(Y, \mathcal{E})$. Then

$$X = Z(s)$$

can be equipped with a T -equivariant perfect obstruction theory as in example 2.2.1. In the case where X^T is zero dimensional, reduced, and proper we then get

$$[X]^{\text{vir}} = \sum_x [x] \cap \frac{e^T(\mathcal{E}_x^{\text{mov}})}{e^T(T_{Y,x}^{\text{mov}})}$$

where the sum is over all fixed points $x \in X$ such that $\mathcal{E}_x^{\text{fix}} = T_{Y,x}^{\text{fix}}$.

Next, we show a variation of the localization formula, allowing us to work more flexibly with different subschemes of the fixed-point locus.

Proposition 2.2.2. *Let X and Y be T -schemes with T -equivariant perfect obstruction theories $\mathbb{E}_X$ and $\mathbb{E}_Y$. Assume we have a T -equivariant map*

$$g : X \rightarrow Y$$

equipped with a relative perfect obstruction theory $\mathbb{E}_g$ compatible with $\mathbb{E}_X$ and $\mathbb{E}_Y$. Let $Y^T = \bigcup_i Y_i$ as in proposition 2.2.1. Let $j : g^{-1}(Y^T) \hookrightarrow X$ be the inclusion. Then

$$[X]^{\text{vir}} = j_* \sum_i \frac{g^! [Y_i]^{\text{vir}}}{e^T(g^* N_{Y_i/Y}^{\text{vir}})}$$

where $g^! : A_*^T(Y^T) \rightarrow A_*^T(X^T)$ is the (refined) virtual pullback associated to $\mathbb{E}_g$.

Proof. Compatibility ensures that

$$[X]^{\text{vir}} = g^! [Y].$$

Virtual localization on Y shows

$$[Y] = \iota_* \sum_i \frac{[Y_i]^{\text{vir}}}{e^T(N_{Y_i/Y}^{\text{vir}})}$$

for $\iota : Y^T \hookrightarrow Y$. Since virtual pullback is compatible with pushforward, we see that

$$g^! \iota_* = j_* g^!$$

thus, finishing the proof. $\square$

Example 2.2.3. Suppose in the above that Y is smooth and equipped with the perfect obstruction theory $\mathbb{E}_Y = \mathbb{L}_Y = \Omega_Y$. Then the formula above is simply

$$[X]^{\text{vir}} = j_* \sum_i \frac{g^! [Y_i]}{e^T(g^* N_{Y_i/Y})}.$$

We end this section by showing how virtual integrals can in some cases be calculated by another virtual integral on a smaller locus.

Proposition 2.2.3. *Let X be a T -scheme. Let E be a T -equivariant vector bundle over X , $s \in H^0(X, E)^T$ and set*

$$j : Z = Z(s) \hookrightarrow X.$$

Suppose we have perfect obstruction theories $\mathbb{E}_Z$ and $\mathbb{E}_X$ compatible with the relative perfect obstruction theory $\mathbb{E}_{Z/X}$ induced by E . Let $j' : Z^T \hookrightarrow X^T$ be the induced inclusion and suppose it is regular of codimension 0 (i.e., also open). Then for

$$\alpha \in A_{Z,T}^*(X)$$

a local Chow cohomology class supported on Z we have

$$\int_{[X]^{\text{vir}}} \alpha = \int_{[Z]^{\text{vir}}} \frac{j^* \alpha}{e^T(j^* E)}$$

If it further holds that $Z^T = X^T$ then

$$[X]^{\text{vir}} = \frac{j_* [Z]^{\text{vir}}}{e^T(E)}$$

in the localized Chow group of X . In particular, the above formula holds for all $\alpha \in A_T^*(X)$.

Proof. Form the pullback diagram

$$\begin{array}{ccc} Z^T & \xhookrightarrow{j'} & X^T \\ \downarrow \iota' & & \downarrow \iota \\ Z & \xhookrightarrow{j} & X \end{array}$$

and let $\gamma' \in A_*^T(Z^T)$ and $\gamma \in A_*^T(X^T)$ be the unique classes such that $\iota'_* \gamma' = [Z]^{\text{vir}}$ and $\iota_* \gamma = [X]^{\text{vir}}$. If we let

$$j^! : A_*^T(X^T) \rightarrow A_*^T(Z^T)$$

be the refined virtual pullback of j , we see from compatibility with pushforward that

$$j^! \gamma = \gamma'.$$

Note further that the virtual pullback is the refined Gysin of $0 : X \rightarrow E$ and that j' is regular with excess bundle $E|_{Z^T}$. It follows by the excess intersection formula that

$$j^! (\gamma) = e^T(E|_{Z^T}) \cap j'^* \gamma.$$

Let now $p : X^T \rightarrow \text{pt.}$ By definition

$$\int_{[X]^{\text{vir}}} \alpha = p_*(\iota^* \alpha \cap \gamma).$$

On the other hand

$$\int_{[Z]^{\text{vir}}} \frac{j^* \alpha}{e^T(j^* E)} = p_* j'_* \left(\frac{j'^* \iota^* \alpha}{e^T(E|_{Z^T})} \cap \gamma' \right)$$

and using that $\gamma' = j^! \gamma = e^T(E|_{Z^T}) \cap j'^* \gamma$ this equals

$$p_* j'_* (j'^* \iota^* \alpha \cap j'^* \gamma) = p_* j'_* j'^* (\iota_* \alpha \cap \gamma).$$

By definition of local Chow cohomology we can find a bivariant class $c \in A^*(Z \hookrightarrow X)$ such that $\alpha = j_* c$ and in particular $\iota^* \alpha = \iota^* j_* c = j'_* \iota^* c$. The integral now equals

$$p_* j'_* j'^* (j'_* \iota^* c \cap \gamma) = p_* j'_* j'^* j'_* (\iota^* c \cdot \gamma)$$

and by the self-intersection formula we have $j'^* j'_* = \text{id}$ so that we get

$$p_* j'_* (\iota^* c \cdot \gamma) = p_* (j'_* \iota^* c \cap \gamma) = p_* (\iota^* \alpha \cap \gamma)$$

as desired. If we assume $j' = \text{id}$ we can find a unique σ such that $j'_* \sigma = \gamma$. But then

$$e^T(E|_{Z^T}) \cap \sigma = j^! j'_* \sigma = j^! \gamma = \gamma'$$

showing that

$$\sigma = \frac{\gamma'}{e^T(E|_{Z^T})}$$

as desired. $\square$

Example 2.2.4. We are primarily interested in the following example. Let Y be a smooth T -scheme with T -equivariant vector bundles E, F and $s \in H^0(Y, E)^T$ and $u \in H^0(Y, F)^T$. Suppose that $W = Z(s)$ is smooth. Set $X = Z(u)$ and equip it with a perfect obstruction theory coming from F and Y . Then $Z = W \cap X = Z(s|_X)$ is cut out of X by $s|_X$ and so we have a relative perfect obstruction theory on $j : Z \hookrightarrow X$. Equipping Z with a perfect obstruction theory coming from F and W we obtain a compatible triple. If we further assume that X^T is zero dimensional, reduced, and proper and that Z^T is non-empty then $j' : Z^T \hookrightarrow X^T$ is regular of codimension 0.

2.3 Construction of the non-associative Hilbert scheme

In this section we construct a moduli functor and prove that it is representable by a smooth scheme $\text{naHilb}^{\underline{d}}(\mathbb{A}^n)$ which we call the non-associative nested Hilbert scheme of $\underline{d}$ points. The proof will be in two steps. First we exhibit the functor as the quotient stack of a smooth scheme by a linear algebraic group acting freely. This proves that the functor is representable by a smooth algebraic space. Next we explicitly construct a cover of Zariski open subfunctors which we prove are representable. This proves that the algebraic space must in fact be a scheme.

2.3.1 The definition

Fix a dimension vector $\underline{d} = (d_0, \dots, d_r) \in \mathbb{N}^{r+1}$, $r \geq 0$. We recall that the nested Hilbert scheme $\text{NHilb}^{\underline{d}}(\mathbb{A}^n)$ parameterize nesting of closed subschemes

$$\xi_1 \hookrightarrow \xi_2 \hookrightarrow \dots \hookrightarrow \xi_{r+1} \hookrightarrow \mathbb{A}^n$$

such that Z_k is finite of length $d_0 + d_1 + \dots + d_{k-1}$. Equivalently it parameterize flags of ideals

$$I_{r+1} \subset I_r \subset \dots \subset I_1 \subset \mathbb{C}[x_1, \dots, x_n]$$

such that $\mathbb{C}[x_1, \dots, x_n]/I_k$ has an underlying vector space of dimension $d_0 + \dots + d_{k-1}$. Equivalent to that, we see that it parameterize a surjective map of $\mathbb{C}$ -algebras

$$\mathbb{C}[x_1, \dots, x_n] \twoheadrightarrow \mathbb{C}[x_1, \dots, x_n]/I_{r+1} =: N$$

and a flag of ideals on the codomain

$$N = N_0 \supset N_1 \supset \cdots \supset N_{r+1} = 0$$

satisfying that N_k/N_{k+1} is a d_k dimensional vector space. The idea behind the non-associative Hilbert scheme is simply to take this last description and relax the condition of associativity. That is, if we denote by $\mathbb{C}\{x_1, \dots, x_n\}_c$ the free unital and commutative (but not associative) $\mathbb{C}$ -algebra on n generators, the non-associative Hilbert scheme parameterizes surjections of unital, commutative (but not necessarily associative) algebras

$$\mathbb{C}\{x_1, \dots, x_n\}_c \twoheadrightarrow N$$

and a flag on ideals

$$N = N_0 \supset N_1 \supset \cdots \supset N_{r+1} = 0$$

such that N_k/N_{k+1} is d_k dimensional as a $\mathbb{C}$ -vector space. To construct this space, we simply specify its functor of points. We recall that for the Hilbert scheme going from a $\mathbb{C}$ -valued point to an S -valued we simply replace $\mathbb{C}$ with $\mathcal{O}_S$ and vector space of a given dimension with vector bundle of a given rank in the above definitions. Doing the same for the non-associative Hilbert scheme, we arrive at the following definition

Definition 2.3.1. Let $\underline{d} = (d_0, \dots, d_r) \in \mathbb{N}^{r+1}$ be a dimension vector. We define a functor

$$naHilb^{n, \underline{d}} : \text{Sch}_{/\mathbb{C}}^{\text{op}} \rightarrow \text{Set}$$

as follows. For $S \in \text{Sch}_{/\mathbb{C}}$ we set

$$naHilb^{n, \underline{d}}(S) = \{\pi : \mathcal{O}_S\{x_1, \dots, x_n\}_c \twoheadrightarrow (\mathcal{F}_\bullet, \psi, s)\} / \text{iso}.$$

Here

$$\mathcal{F}_\bullet = (\mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots \supset \mathcal{F}_{r+1} = 0)$$

is a flag of locally free sheaves on S such that $\mathcal{F}_k/\mathcal{F}_{k+1}$ is locally free of rank d_k , $\mathcal{F}_0$ is equipped with a unital commutative non-associative $\mathcal{O}_S$ -algebra structure (ψ, s) which satisfies

$$\psi(\mathcal{F}_0 \otimes_S \mathcal{F}_k) \subset \mathcal{F}_k.$$

The surjection π is a morphism of unital non-associative $\mathcal{O}_S$ -algebras. Two such

$$(\mathcal{F}_\bullet, \psi, s, \pi) \quad \text{and} \quad (\mathcal{G}_\bullet, \psi', s', \pi')$$

are considered isomorphic if there exist an isomorphism of unital non-associative $\mathcal{O}_S$ -algebras $\varphi : \mathcal{F}_0 \xrightarrow{\sim} \mathcal{G}_0$ preserving the flags such that $\varphi\pi = \pi'$.

Remark 2.3.1. We note that by lemma 2.2.4 a point $[\mathcal{F}_\bullet, \psi, s, \pi]$ is uniquely determined by either $[\mathcal{F}_\bullet, \pi]$ or $[\mathcal{F}_\bullet, \psi, s, f]$ where f is the restriction of π to $(\mathcal{O}_S^n)^\vee$. Depending on the situation, we will use either of the three to describe a point on the non-associative Hilbert scheme. When using the last formulation, we will usually use the notation $[\mathcal{F}_\bullet, \psi_2, s, \psi_1]$, i.e. where $\psi_2 = \psi$ and $\psi_1 = f$ which agrees with the notation of [Kaz17].

2.3.2 Smoothness

We now exhibit the above moduli functor as the functor of points of a quotient stack. The idea is to fix a flag of $\mathbb{C}$ -vector spaces $N_\bullet$. We then construct a space parameterizing the different ways we can equip $N_\bullet$ with a unital commutative non-associative algebra structure and a surjection

$$\mathbb{C}\{x_1, \dots, x_n\}_c \twoheadrightarrow N_\bullet$$

of unital non-associative algebras. Then we quotient out the choice of flag, by taking the group quotient of the group of flag preserving automorphisms of $N_\bullet$. This data is completely determined by the non-associative algebra structure (ψ_2, v) on $N_\bullet$ along with a map of $\mathbb{C}$ -vector spaces

$$\psi_1 : (\mathbb{C}^n)^\vee \rightarrow N_\bullet.$$

The data we want to parameterize is therefore (ψ_1, ψ_2, v) , but we need to find a condition so that the associated map of algebras is surjective. This leads to the following stability condition, which we define globally

Definition 2.3.2. Let $S \in \text{Sch}/\mathbb{C}$ and let $(\mathcal{F}, \psi_2, s)$ be a unital non-associative $\mathcal{O}_S$ algebra. Let

$$\psi_1 : (\mathcal{O}_S)^\vee \rightarrow \mathcal{F}$$

be a map of quasi-coherent sheaves. We say that $(\mathcal{F}, \psi_1, \psi_2, s)$ is stable if the only non-associative $\mathcal{O}_S$ -subalgebra of $\mathcal{F}$ for which ψ_1 factors through is $\mathcal{F}$ itself.

Lemma 2.3.1. Let $S \in \text{Sch}/\mathbb{C}$ and let $(\mathcal{F}, \psi_2, s)$ be a unital non-associative $\mathcal{O}_S$ algebra and let

$$\psi_1 : (\mathcal{O}_S)^\vee \rightarrow \mathcal{F}$$

be a map of quasi-coherent sheaves. Then $(\mathcal{F}, \psi_1, \psi_2, s)$ is stable if and only if the associated map

$$\mathcal{O}_S\{x_1, \dots, x_n\} \rightarrow \mathcal{F}$$

is surjective.

Proof. This is clear as the image of the map

$$\mathcal{O}_S\{x_1, \dots, x_n\} \rightarrow \mathcal{F}$$

can be characterized as the smallest non-associative $\mathcal{O}_S$ -subalgebra containing ψ_1 . $\square$

We make the following definition.

Definition 2.3.3. Let $\underline{d} = (d_0, \dots, d_r) \in \mathbb{N}^{r+1}$ be a dimension vector. Fix a flag

$$N_0 \supset N_1 \supset \dots \supset N_{r+1} = 0$$

of $\mathbb{C}$ -vector spaces with $\dim_{\mathbb{C}} N_k/N_{k+1} = d_k$. We define

$$M_{n, \underline{d}} \subset \text{Hom}_{\mathbb{C}}((\mathbb{C}^n)^\vee, N_0) \oplus \text{Hom}_{\mathbb{C}}(\text{Sym}^2 N_0, N_0) \oplus N_0$$

to be the smooth locally closed subscheme consisting of tuples (ψ_1, ψ_2, v) satisfying the following properties.

1. For all $0 \leq k \leq r+1$ it holds that $\psi_2(N_0 \otimes_{\mathbb{C}} N_k) \subset N_k$.
2. For all $x \in N_0$ we have $\psi_2(v, x) = x$.
3. The tuple (N_0, ψ_1, ψ_2, v) is stable.

We note that condition (1) restricts us to a subspace, condition (2) restricts us to an affine subspace and conditions (3) restrict us to a Zariski open set so that $M_{n, \underline{d}}$ is a smooth locally closed subscheme. Next, we introduce the group action.

Definition 2.3.4. Let $\underline{d} \in \mathbb{N}^{r+1}$ and let $N_\bullet$ be as in definition 2.3.3. We let

$$P_{\underline{d}} \subset \text{Aut}_{\mathbb{C}}(N_0)$$

be the parabolic group of linear automorphisms of N_0 preserving the flag. We let $P_{\underline{d}}$ act on

$$\text{Hom}_{\mathbb{C}}((\mathbb{C}^n)^\vee, N_0) \oplus \text{Hom}(\text{Sym}^2 N_0, N_0) \oplus N_0$$

by

$$g.(\psi_1, \psi_2, v) = (g\psi_1, \psi_2^g, gv)$$

where

$$\psi_2^g(x, y) = g\psi_2(g^{-1}x, g^{-1}y).$$

Lemma 2.3.2. *The $P_{\underline{d}}$ -action restricts to a free action on $M_{n, \underline{d}}$*

Proof. Let $(\psi_1, \psi_2, v) \in M_{n, \underline{d}}$ and let $g \in P_{\underline{d}}$. We check that $(g\psi_1, \psi_2^g, gv)$ still satisfies the three conditions.

1. Since g is flag preserving it is clear that

$$\psi_2^g(N_0 \otimes_{\mathbb{C}} N_k) \subset N_k$$

for all $0 \leq k \leq r+1$.

2. For $x \in N_0$ we see that

$$\begin{aligned} \psi_2^g(gv, x) &= g\psi_2(g^{-1}gv, g^{-1}x) \\ &= g\psi_2(v, g^{-1}x) \\ &= gg^{-1}x \\ &= x. \end{aligned}$$

3. Suppose $M \subset N_0$ is a subspace containing v and $\text{im } g\psi_1$ and which is closed under ψ_2^g . Then $g^{-1}M$ contains v and $\text{im } \psi_1$. For $g^{-1}x, g^{-1}y \in g^{-1}M$ we see that

$$\begin{aligned} \psi_2(g^{-1}x, g^{-1}y) &= g^{-1}g\psi_2(g^{-1}x, g^{-1}y) \\ &= g^{-1}\psi_2^g(x, y) \end{aligned}$$

which is in $g^{-1}M$ since $\psi_2^g(x, y) \in M$. We conclude by stability of (ψ_1, ψ_2, v) that $g^{-1}M = N_0$ and thus $M = N_0$.

This proves that the $P_{\underline{d}}$ -action restricts to an action on $M_{n, \underline{d}}$. Next, we assume that $(\psi_1, \psi_2, v) \in M_{n, \underline{d}}$ and $g \in P_{\underline{d}}$ is chosen so that $(\psi_1, \psi_2, v) = (g\psi_1, \psi_2^g, gv)$. Let $M \subset N_0$ be the subspace of $x \in N_0$ where $gx = x$. By definition $v \in M$ and $\text{im } \psi_1 \in M$. If $x, y \in M$ then

$$\begin{aligned} g\psi_2(x, y) &= g\psi_2(g^{-1}x, g^{-1}y) \\ &= \psi_2^g(x, y) \\ &= \psi_2(x, y) \end{aligned}$$

so that M is closed under ψ_2 . We conclude by stability that $M = N_0$ so that $g = \text{id}_{N_0}$. This shows that $P_{\underline{d}}$ acts freely on $M_{n, \underline{d}}$. $\square$

Definition 2.3.5. Let $\underline{d} \in \mathbb{N}^{r+1}$ and $n \in \mathbb{N}$. We define the non-associative nested Hilbert scheme of $\underline{d}$ points in $\mathbb{A}^n$ to be the quotient stack

$$\text{naNHilb}^{\underline{d}}(\mathbb{A}^n) := [M_{n,\underline{d}}/P_{\underline{d}}].$$

In the case where $r = 0$ so $d \in \mathbb{N}$ we use the notation

$$\text{naHilb}^d(\mathbb{A}^n) := [M_{n,d}/\text{GL}_d]$$

which we will refer to as the non-associative Hilbert scheme of d points in $\mathbb{A}^n$.

Corollary 2.3.1. *The stack $\text{naNHilb}^{\underline{d}}(\mathbb{A}^n)$ is a smooth algebraic space.*

Proof. This follows since it is the quotient of a smooth algebraic group acting freely on a smooth scheme. $\square$

We are now ready to prove that $\text{naNHilb}^{\underline{d}}(\mathbb{A}^n)$ represents the moduli functor $\text{naHilb}^{n,\underline{d}}$. We first introduce the functor which $M_{n,\underline{d}}$ represents.

Definition 2.3.6. Let $\underline{d} \in \mathbb{N}^{r+1}$ and $n \in \mathbb{N}$ and fix a reference flag of $\mathbb{C}$ -vector spaces

$$N_{\bullet} = (N_0 \supset \cdots \supset N_{r+1})$$

as in definition 2.3.3. We define

$$\overline{\text{naHilb}}^{n,\underline{d}} : \text{Sch}_{\mathbb{C}}^{\text{op}} \rightarrow \text{Set}$$

as follows. For $S \in \text{Sch}_{\mathbb{C}}$ we set

$$\overline{\text{naHilb}}^{n,\underline{d}}(S) := \{(\mathcal{F}_{\bullet}, \psi, s, \pi, \beta)\} / \text{iso}$$

where $(\mathcal{F}_{\bullet}, \psi, s, \pi) \in \text{naHilb}^{n,\underline{d}}(S)$ and

$$\beta : N_{\bullet} \otimes_{\mathbb{C}} \mathcal{O}_S \rightarrow \mathcal{F}_{\bullet}$$

is an isomorphism of flags. An isomorphism of such tuples $(\mathcal{F}_{\bullet}, \psi, s, \pi, \beta)$ and $(\mathcal{G}_{\bullet}, \psi', s', \pi', \beta')$ is an isomorphism of flags $\varphi : \mathcal{F}_{\bullet} \rightarrow \mathcal{G}_{\bullet}$ of non-associative unital algebras which preserves the flag structures and satisfies $\varphi\pi = \pi'$ and $\varphi\beta = \beta'$. There is a natural transformation

$$\begin{aligned} \overline{\text{naHilb}}^{n,\underline{d}}(S) &\longrightarrow \text{naHilb}^{n,\underline{d}}(S) \\ [\mathcal{F}_{\bullet}, \psi, \pi, \beta] &\longmapsto [\mathcal{F}_{\bullet}, \psi, \pi] \end{aligned}$$

which forgets β .

Proposition 2.3.1. *The forgetful natural transformation*

$$\overline{\text{naHilb}}^{n,\underline{d}} \rightarrow \text{naHilb}^{n,\underline{d}}$$

identifies with the functor of points of the quotient map

$$M_{n,\underline{d}} \rightarrow [M_{n,\underline{d}}/P_{\underline{d}}] = \text{naNHilb}^{\underline{d}}(\mathbb{A}^n).$$

Proof. For $d = d_0 + \cdots + d_r$, we let $v_0, \dots, v_d$ be a basis of N such that

$$N_i = \text{span}(v_{d_0+\cdots+d_{i-1}+1}, v_{d_0+\cdots+d_{i-1}+2}, \dots, v_d)$$

for all $i = 0, \dots, r$. Let X be the affine scheme corresponding to the vector space

$$\text{Hom}_{\mathbb{C}}((\mathbb{C}^n)^{\vee}, N_0) \oplus \text{Hom}_{\mathbb{C}}(\text{Sym}^2 N_0, N_0) \oplus N_0.$$

Using the basis, we identify X with

$$\operatorname{Spec} \mathbb{C} \left[(x_l)_t, a_{i,j}^k, w_m \right]_{l,t,i,j,k,m}.$$

Here, for $(\psi_1, \psi_2, w) \in X$ we have $\psi_1(e_l^\vee) = x_l$, w_i the coordinates of w and

$$\psi_2(v_i, v_j) = \sum_{k=1}^d a_{i,j}^k v_k.$$

Consider the flag of locally free sheaves $V_\bullet$ on X given by

$$V_\bullet := N_\bullet \otimes_{\mathbb{C}} \mathcal{O}_X$$

with $\beta_0 = \operatorname{id}_{V_\bullet}$. We have a distinguished section $\bar{s} \in H^0(X, V_0)$ given by

$$\sum_{m=1}^d v_m \otimes w_m$$

and we can similarly construct

$$\begin{aligned} \bar{\Psi}_1 : (\mathcal{O}_X^n)^\vee &\longrightarrow V_0 \\ (e_l)^\vee &\longmapsto \sum_{t=1}^d v_t \otimes (x_l)_t \end{aligned}$$

and

$$\begin{aligned} \bar{\Psi}_2 : \operatorname{Sym}^2 V_0 &\longrightarrow V_0 \\ (v_i \otimes 1) \cdot (v_j \otimes 1) &\longmapsto \sum_{k=1}^d v_k \otimes a_{i,j}^k \end{aligned}$$

which are both maps of sheaves. We restrict the sheaves and maps to the locally closed subscheme $M_{n,\underline{d}}$. By condition 1., 2. in the definition of $M_{n,\underline{d}}$, it is clear that $(V_\bullet, \bar{\Psi}_1, \bar{s})$ is a unital commutative non-associative $\mathcal{O}_{M_{n,\underline{d}}}$ -algebra which satisfies

$$\bar{\Psi}_2(V_0 \otimes_{M_{n,\underline{d}}} V_k) \subset V_k$$

By the universal property of the free unital commutative non-associative algebra we obtain from $\bar{\Psi}_1$ a map

$$\bar{\omega} : \mathcal{O}_{M_{n,\underline{d}}} \{x_1, \dots, x_n\}_c \rightarrow V_0$$

of unital non-associative algebras which is surjective by condition 4. in the definition of $M_{n,\underline{d}}$. It follows that the tuple $(V_\bullet, \bar{\Psi}_2, \bar{s}, \bar{\omega}, \beta_0)$ defines a $M_{n,\underline{d}}$ -valued point of $\overline{naHilb}^{n,\underline{d}}$ giving us a natural transformation

$$M_{n,\underline{d}} \rightarrow \overline{naHilb}^{n,\underline{d}}.$$

We prove it is an isomorphism. Let $S \in \operatorname{Sch}_{\mathbb{C}}$ and let $[\mathcal{F}_\bullet, \psi, s, \pi, \beta] \in \overline{naHilb}^{n,\underline{d}}(S)$. From this data we obtain a section

$$\beta^{-1}s : \mathcal{O}_S \rightarrow N_\bullet \otimes_{\mathbb{C}} \mathcal{O}_S.$$

Let ψ_1 be the restriction of π to $(\mathcal{O}_S^n)^\vee$. By adjunction of $\beta^{-1}\psi(\beta \otimes \beta)$ and $\beta^{-1}\psi_1$ we similarly obtain sections

$$\mathcal{O}_S \rightarrow \operatorname{Hom}_{\mathbb{C}}(\operatorname{Sym}_{\mathbb{C}}^2 N_\bullet, N_\bullet) \otimes_{\mathbb{C}} \mathcal{O}_S$$

and

$$\mathcal{O}_S \rightarrow \operatorname{Hom}_{\mathbb{C}}((\mathbb{C}^n)^\vee, N_\bullet) \otimes_{\mathbb{C}} \mathcal{O}_S.$$

Combined these three sections provides a map $f : S \rightarrow X$ and from the different assumptions on $[\mathcal{F}_\bullet, \psi, s, \pi, \beta]$ we see that f factors through $M_{n,\underline{d}}$. We note that by construction

$$f^*(V_\bullet, \bar{\Psi}_2, \bar{s}, \bar{\varpi}, \beta_0) = (\mathcal{F}_\bullet, \psi, s, \pi, \beta)$$

and one can directly check that this procedure provides an inverse to

$$M_{n,\underline{d}} \rightarrow \overline{naHilb}^{n,\underline{d}}.$$

We now equip $V_\bullet = N_\bullet \otimes_{\mathbb{C}} \mathcal{O}_{M_{n,\underline{d}}}$ with $P_{\underline{d}}$ equivariant structure by letting $P_{\underline{d}}$ act naturally on $N_\bullet$ (being a subgroup of its linear automorphisms). One can easily check that $\bar{\Psi}_2, \bar{s}$ and $\bar{\varpi}$ are equivariant wrt. to this structure. Letting

$$q : M_{n,\underline{d}} \rightarrow [M_{n,\underline{d}}/P_{\underline{d}}]$$

be the quotient map, it follows that we have an induced flag of locally free sheaves

$$\mathcal{V}_\bullet \in \mathrm{QCoh}^{P_{n,\underline{d}}}(M_{n,\underline{d}}) = \mathrm{QCoh}([M_{n,\underline{d}}/P_{\underline{d}}])$$

equipped with a section $\mathfrak{s} \in H^0([M_{n,\underline{d}}/P_{\underline{d}}], \mathcal{V}_0)$ and maps

$$\Psi_2 : \mathrm{Sym}^2 \mathcal{V}_0 \rightarrow \mathcal{V}_0,$$

$$\varpi : \mathcal{O}_{[M_{n,\underline{d}}/P_{\underline{d}}]} \{x_1, \dots, x_n\}_c \rightarrow \mathcal{V}_0$$

such that

$$q^*(\mathcal{V}_\bullet, \Psi_2, \mathfrak{s}, \varpi) = (V_\bullet, \bar{\Psi}_2, \bar{s}, \bar{\varpi}).$$

This defines a $[M_{n,\underline{d}}/P_{\underline{d}}]$ valued point of $naHilb^{n,\underline{d}}$ giving us a natural transformation

$$[M_{n,\underline{d}}/P_{\underline{d}}] \rightarrow naHilb^{n,\underline{d}}.$$

We explicitly define the inverse. Let $[\mathcal{F}_\bullet, \psi, s, \pi] \in naHilb^{n,\underline{d}}(S)$ and pick a cover of $(S_i)_{i \in I}$ of S trivializing $\mathcal{F}_\bullet$ by explicit choices

$$\beta_i : N_\bullet \otimes_{\mathbb{C}} \mathcal{O}_{S_i} \rightarrow \mathcal{F}_\bullet|_{S_i}$$

of isomorphisms of flags. For each i the data $(\mathcal{F}_\bullet|_{S_i}, \psi_i, s_i, \pi_i, \beta_i)$ determines a map

$$f_i : S_i \rightarrow M_{n,\underline{d}}.$$

On overlaps $S_{j,i}$ the change from β_i to β_j determines a map

$$g_{j,i} : S_{j,i} \rightarrow P_{\underline{d}}$$

satisfying $g_{j,i} \cdot f_i = f_j$. This means that the maps qf_i and qf_j agrees on $S_{j,i}$ so by gluing we get a map

$$f : S \rightarrow [M_{n,\underline{d}}/P_{\underline{d}}].$$

By construction this satisfies that

$$f^*(\mathcal{V}_\bullet, \Psi_2, \mathfrak{s}, \varpi) \simeq (\mathcal{F}_\bullet, \psi, s, \pi)$$

and one can now easily check that this is inverse to the above construction. $\square$

Remark 2.3.2. By the moduli description we have that $naHilb^d(\mathbb{A}^n)$ comes equipped with a universal non-associative algebra, a universal quotient map and a universal flag of ideals. This corresponds exactly to $(\mathcal{V}_\bullet, \Psi_2, \mathfrak{s}, \varpi)$ as in the proof of the proposition above. We will denote the restriction of ϖ to $(\mathcal{O}^n)^\vee$ by Ψ_1 .

2.3.3 Representability

We prove that the smooth algebraic space $\mathrm{naHilb}^{\underline{d}}(\mathbb{A}^n)$ is representable by a scheme. It will suffice to find a Zariski cover of schemes. We consider the following opens.

Definition 2.3.7. Fix $\underline{d} \in \mathbb{N}^{r+1}$, $n \in \mathbb{N}$ and a flag $N_\bullet$ as in definition 2.3.3. For any morphism

$$\gamma : N_0 \rightarrow \mathbb{C}\{x_1, \dots, x_n\}_c$$

we define $\mathrm{naHilb}_\gamma^{n,\underline{d}} \subset \mathrm{naHilb}^{n,\underline{d}}$ to be the subfunctor which has S -valued points corresponding to $[\mathcal{F}_\bullet, \psi, s, \pi]$ such that

$$\pi(1_{\mathcal{O}_S} \otimes \gamma) : \mathcal{O}_S \otimes_{\mathbb{C}} N_0 \rightarrow \mathcal{F}_0$$

is an isomorphism of $\mathcal{O}_S$ -modules.

Lemma 2.3.3. *The subfunctor $\mathrm{naHilb}_\gamma^{n,\underline{d}} \subset \mathrm{naHilb}^{n,\underline{d}}$ is Zariski open. When γ varies over all $\mathrm{Hom}_{\mathbb{C}}(N_0, \mathbb{C}\{x_1, \dots, x_n\}_c)$ this provides a cover of $\mathrm{naHilb}^{n,\underline{d}}$.*

Proof. For an S -valued point classified by $[\mathcal{F}_\bullet, \psi, s, \pi]$ the pullback along the inclusion of the subfunctor corresponds to the locus of S where

$$\pi(1_{\mathcal{O}_S} \otimes \gamma)$$

is surjective. This is clearly a Zariski open locus. Furthermore, the map π admits a section γ Zariski locally on S , proving that the subfunctors provides a cover for varying γ . $\square$

Lemma 2.3.4. *The functor $\mathrm{naHilb}_\gamma^{n,\underline{d}}$ is representable by a scheme.*

Proof. We let $\mathrm{Flag}_{\underline{d}}(N_0)$ be the flag variety of type $\underline{d}$ on N_0 . It is a scheme having S -valued points corresponding to flags of locally free sheaves

$$0 = \mathcal{G}_{r+1} \subset \dots \subset \mathcal{G}_0 = N_0 \otimes_{\mathbb{C}} \mathcal{O}_S$$

such that $\mathcal{G}_k/\mathcal{G}_{k+1}$ is locally free of rank d_k . Let $d = d_0 + \dots + d_r \in \mathbb{N}$. We consider the map

$$\mathrm{naHilb}_\gamma^{n,\underline{d}} \rightarrow \mathrm{Flag}_{\underline{d}}(N_0) \times \overline{\mathrm{naHilb}}^{n,d} \cong \mathrm{Flag}_{\underline{d}}(N_0) \times M_{n,d}$$

which on S -valued points is given by

$$[\mathcal{F}_\bullet, \psi, s, \pi] \mapsto ([\mathcal{F}_\bullet], [\mathcal{F}_0, \psi, s, \pi, \pi(1_{\mathcal{O}_S} \otimes \gamma)]).$$

We claim that it is a closed immersion thus finishing the proof. Indeed, let $([\mathcal{G}_\bullet], [\mathcal{F}_0, \psi, s, \pi, \beta])$ be an S -valued point of $\mathrm{Flag}_{\underline{d}}(N_0) \times M_{n,d}$. The pullback along the above map identifies with the locus of S where $\pi(1_{\mathcal{O}_S} \otimes \gamma) = \beta$ and where the map induced by ψ and β on $\mathcal{G}_0 = N_0 \otimes_{\mathbb{C}} \mathcal{O}_S$ satisfies

$$\psi(\mathcal{G}_0 \otimes_S \mathcal{G}_k) \subset \mathcal{G}_k.$$

Both of these conditions are Zariski closed, as they can be described as the vanishing of sections of locally free sheaves. $\square$

Corollary 2.3.2. *The algebraic space $\mathrm{naHilb}^{\underline{d}}(\mathbb{A}^n)$ is representable by a scheme.*

Proof. This follows from the above since it admits a Zariski cover of open subschemes. $\square$

2.3.4 Embedding of the Hilbert scheme

In this section we construct a map from the nested Hilbert scheme of points to the non-associative nested Hilbert scheme. We prove that the map is a closed immersion and identifies the Hilbert scheme of points with the zero locus of a section of a vector bundle on the non-associative Hilbert scheme. This equips the nested Hilbert scheme of points with a perfect obstruction theory.

Definition 2.3.8. We define

$$\iota : \mathrm{NHilb}^d(\mathbb{A}^n) \rightarrow \mathrm{naNHilb}^d(\mathbb{A}^n)$$

via the functor of points. An S -valued point

$$[\mathcal{I}_{r+1} \subset \cdots \mathcal{I}_1 \subset \mathcal{O}_S[x_1, \dots, x_n]]$$

is mapped to $[\mathcal{F}_\bullet, \psi, s, \pi]$ where $(\mathcal{F}_0, \psi, s)$ is the commutative unital $\mathcal{O}_S$ -algebra given by $\mathcal{O}_S[x_1, \dots, x_n]/\mathcal{I}_{r+1}$, the flag of ideals is defined by $\mathcal{F}_k = \mathcal{I}_k/\mathcal{I}_{r+1}$ and the quotient map is given by

$$\pi : \mathcal{O}_S\{x_1, \dots, x_n\}_c \twoheadrightarrow \mathcal{O}_S[x_1, \dots, x_n] \twoheadrightarrow \mathcal{F}_0.$$

Definition 2.3.9. The associativity locus of $\mathrm{naNHilb}^d(\mathbb{A}^n)$ is the subfunctor which has S -valued points given by $[\mathcal{F}_\bullet, \psi, s, \pi]$ where ψ is associative. By lemma 2.2.5 it is the same as the locus where π factors through $\mathcal{O}_S[x_1, \dots, x_n]$. It follows that the locus is Zariski closed.

Lemma 2.3.5. *The map ι is a closed immersion identifying $\mathrm{NHilb}^d(\mathbb{A}^n)$ with the associativity locus.*

Proof. It is clear by definition that ι maps the nested Hilbert scheme to this locus. Conversely, given an S -valued point $[\mathcal{F}_\bullet, \psi, s, \pi]$ of the associativity locus we let

$$\tilde{\pi} : \mathcal{O}_S[x_1, \dots, x_n] \twoheadrightarrow \mathcal{F}_0$$

be the induced map and define an S -valued point on the nested Hilbert scheme by setting

$$\mathcal{I}_k := \tilde{\pi}^{-1}(\mathcal{F}_k).$$

It is easy to see that this operation is inverse to that of ι . □

2.3.5 The induced perfect obstruction theory

We wish to prove that this locus can also be described as the zero locus of a section of a vector bundle. We note that $\mathrm{naNHilb}^d$ comes equipped with a universal flag $\mathcal{V}_\bullet$ of locally free sheaves and a universal multiplication rule Ψ_2 . We note that the associativity conditions can now directly be checked by the vanishing of the associator section

$$\Psi_2(\mathrm{id} \otimes \Psi_2) - \Psi_2(\Psi_2 \otimes \mathrm{id})$$

of $\mathrm{Hom}(\mathcal{V}_0 \otimes \mathcal{V}_0 \otimes \mathcal{V}_0, \mathcal{V}_0)$. We notice that we can reduce the rank of the sheaf a bit. Firstly, we note that the symmetry of Ψ_2 implies that

$$\Psi_2(\Psi_2(x, y), x) - \Psi_2(x, \Psi_2(y, x)) = \Psi_2(x, \Psi_2(y, x)) - \Psi_2(x, \Psi_2(y, x)) = 0$$

so that the above section is alternating in the outer coordinates. Secondly, we see that

$$\Psi_2(\mathcal{V}_i \otimes \mathcal{V}_j) \subset \mathcal{V}_{\max\{i, j\}}$$

so that the above trilinear map takes $\mathcal{V}_i \otimes \mathcal{V}_j \otimes \mathcal{V}_k$ to $\mathcal{V}_{\max\{i,j,k\}}$. At last, if we let $e = \mathfrak{s}(1)$ be the unit we see that

$$\begin{aligned}\Psi_2(\Psi_2(e, y), z) - \Psi_2(e, \Psi_2(y, z)) &= \Psi_2(y, z) - \Psi_2(y, z) = 0 \\ \Psi_2(\Psi_2(x, e), z) - \Psi_2(x, \Psi_2(e, z)) &= \Psi_2(x, z) - \Psi_2(x, z) = 0 \\ \Psi_2(\Psi_2(x, y), e) - \Psi_2(x, \Psi_2(y, e)) &= \Psi_2(x, y) - \Psi_2(x, y) = 0\end{aligned}$$

so that the above trilinear map takes

$$(\mathfrak{s} \otimes \text{id} \otimes \text{id}) \oplus (\text{id} \otimes \mathfrak{s} \otimes \text{id}) \oplus (\text{id} \otimes \text{id} \otimes \mathfrak{s})$$

to zero. With this in mind, we make the following definition.

Definition 2.3.10. Let $(\mathcal{V}_\bullet, \Psi_2, \mathfrak{s})$ be the universal unital non-associative algebra of $\text{naNHilb}^d(\mathbb{A}^n)$. We define the associativity sheaf

$$\mathcal{E}_{\text{ass}} \subset \mathcal{H}om((\mathcal{V}_0/\mathfrak{s}) \otimes (\mathcal{V}_0/\mathfrak{s}) \otimes (\mathcal{V}_0/\mathfrak{s}), \mathcal{V}_0)$$

as the sub-sheaf of trilinear maps which satisfies the following properties.

1. They are alternating in the outer coordinates.
2. They take $\mathcal{V}_i \otimes \mathcal{V}_j \otimes \mathcal{V}_k$ to $\mathcal{V}_{\max\{i,j,k\}}$.

It comes equipped with a canonical associator section s_{ass} given by

$$\Psi_2(\Psi_2 \otimes \text{id}) - \Psi_2(\text{id} \otimes \Psi_2).$$

Remark 2.3.3. Note that in the special case where $d_0 = 1$, the inclusion $\mathcal{V}_1 \subset \mathcal{V}_0$ induces an isomorphism $\mathcal{V}_1 \cong \mathcal{V}_0/\mathfrak{s}$.

Since we can scheme theoretically identify $\text{NHilb}^d(\mathbb{A}^n)$ with the associativity locus which can again be identified with $Z(s_{\text{ass}})$ we obtain the following result.

Corollary 2.3.3. *The closed embedding $\iota : \text{NHilb}^d(\mathbb{A}^n) \rightarrow \text{naNHilb}^d(\mathbb{A}^n)$ identifies with the closed immersion $Z(s_{\text{ass}})$. In particular, it comes equipped with a perfect obstruction theory given by*

$$\begin{array}{ccc} \mathbb{E}_{\text{ass}} & := & \left[\begin{array}{ccc} (\mathcal{E}^{\text{ass}})^\vee|_{\text{NHilb}^d(\mathbb{A}^n)} & \xrightarrow{ds_{\text{ass}}} & \Omega_{\text{naNHilb}^d(\mathbb{A}^n)}|_{\text{NHilb}^d(\mathbb{A}^n)} \\ \downarrow & \downarrow s_{\text{ass}} & \parallel \\ \mathbb{L}_{\text{NHilb}^d(\mathbb{A}^n)} & \cong & \left[\mathcal{C}_{\text{NHilb}^d(\mathbb{A}^n)/\text{naNHilb}^d(\mathbb{A}^n)} \xrightarrow{d} \Omega_{\text{naNHilb}^d(\mathbb{A}^n)}|_{\text{NHilb}^d(\mathbb{A}^n)} \right] \end{array} \right] \end{array}$$

where $\mathcal{C}_{\text{NHilb}^d(\mathbb{A}^n)/\text{naNHilb}^d(\mathbb{A}^n)}$ is the conormal sheaf of the closed immersion ι . This equips it with a virtual class

$$\left[\text{NHilb}^d(\mathbb{A}^n) \right]^{\text{vir}} \in A_*(\text{NHilb}^d(\mathbb{A}^n))$$

which can also be described as the refined Gysin map of the zero section of $\mathcal{E}$ applied to the fundamental class of $\text{naNHilb}^d(\mathbb{A}^n)$.

2.4 The punctual locus in the full flag case

In this section we will restrict ourselves to the full flag case, i.e., a dimension vector $\underline{d} = (1, 1, \dots, 1) \in \mathbb{N}^{r+1}$. We will denote by

$$\text{naNHilb}^{r+1}(\mathbb{A}^n) := \text{naNHilb}^{(1,1,\dots,1)}(\mathbb{A}^n)$$

and

$$\text{NHilb}^{r+1}(\mathbb{A}^n) := \text{NHilb}^{(1,1,\dots,1)}(\mathbb{A}^n)$$

to ease the notation. We will prove two nice properties enjoyed by the non-associative Hilbert scheme with a full flag. Firstly, in this context we are able to extend the nested Hilbert-Chow morphism to a non-associative nested Hilbert-Chow morphism. Secondly, we can construct a punctual locus of the non-associative Hilbert scheme which we prove admits the structure of a tower of projective fibrations.

2.4.1 The non-associative Hilbert-Chow morphism

Recall that for the nested Hilbert scheme of points with full dimension vector, we have an ordered Hilbert-Chow morphism

$$\begin{aligned} \text{NHilb}^{(1,\dots,1)}(\mathbb{A}^n) &\xrightarrow{HC} (\mathbb{A}^n)^{r+1} \\ (I_{r+1} \subset I_r \subset \dots \subset I_0 = \mathbb{C}[x_1, \dots, x_n]) &\longmapsto (\text{Supp}(I_0/I_1), \text{Supp}(I_1/I_2), \dots, \text{Supp}(I_r/I_{r+1})). \end{aligned}$$

Since

$$I_i/I_{i+1} = (I_i/I_{r+1})/(I_{i+1}/I_{r+1}) = N_i/N_{i+1}$$

this suggest what to do in the non-associative case. We note that if $(N_\bullet, \psi_1, s, \psi_2)$ is a $\mathbb{C}$ -valued point of the non-associative Hilbert scheme, we cannot quite talk about

$$\text{Supp}(N_i/N_{i+1})$$

as we only have a $\mathbb{C}$ -module structure and not a $\mathbb{C}[x_1, \dots, x_n]$ -module structure. However, promoting it to a $\mathbb{C}[x_1, \dots, x_n]$ -module is the same as a choice of n commuting endomorphisms of N_i/N_{i+1} . By definition ψ_2 induces a map

$$\psi_2 : N_0 \otimes_{\mathbb{C}} (N_i/N_{i+1}) \rightarrow N_i/N_{i+1}.$$

We can then simply take the n -elements

$$\psi_1 = \bigoplus_{i=1}^n \iota_i : (\mathbb{C}^n)^\vee \rightarrow M_i$$

and define the endomorphisms by $\psi_2(\iota_i, -)$. Without associativity these might not commute. However, this is true for dimension reasons once we restrict to the one-dimensional space N_i/N_{i+1} . Having this picture in mind, we make the following construction.

Proposition 2.4.1. *There exist a map*

$$\text{naHilb}^r(\mathbb{A}^n) \xrightarrow{\text{naHC}} (\mathbb{A}^n)^{r+1}$$

which we will call the non-associative Hilbert-Chow morphism. The composite

$$\text{NHilb}^{r+1}(\mathbb{A}^n) \xrightarrow{\iota} \text{naNHilb}^{r+1}(\mathbb{A}^n) \xrightarrow{\text{naHC}} (\mathbb{A}^n)^{r+1}$$

agrees with the ordinary nested Hilbert-Chow morphism HC .

Proof. We construct the morphism directly. Let $(\mathcal{V}_\bullet, \Psi_2, \mathfrak{s}, \Psi_1)$ be the universal flag and maps of $\text{naNHilb}^{r+1}(\mathbb{A}^n)$. Let $\mathcal{L}_i = \mathcal{V}_i/\mathcal{V}_{i+1}$ which is a line bundle. Then the maps Ψ_2 and Ψ_1 induces a map

$$\Psi_2(\Psi_1 \otimes \text{id}) : (\mathcal{O}^n)^\vee \otimes \mathcal{L}_i \rightarrow \mathcal{L}_i$$

and so, we obtain a map

$$(\mathcal{O}^n)^\vee \rightarrow \mathcal{L}_i \otimes \mathcal{L}_i^{-1} \simeq \mathcal{O}$$

which induces a map $\text{naNHilb}^{r+1}(\mathbb{A}^n) \rightarrow \mathbb{A}^n$. Doing this for $i = 0, \dots, r$ we obtain the desired map

$$\text{naNHilb}^{r+1}(\mathbb{A}^n) \rightarrow (\mathbb{A}^n)^{r+1}.$$

Going through the definitions of ι and HC on the functor of points one easily sees that $\text{naHC} \circ \iota = HC$. $\square$

2.4.2 The punctual locus

We construct a closed locus $\text{naNHilb}_0^{r+1}(\mathbb{A}^n) \hookrightarrow \text{naNHilb}^{r+1}(\mathbb{A}^n)$ in such a way that it intersects with the associativity locus exactly in the punctual nested Hilbert scheme. The first idea would be to simply take the preimage of the origin under the Hilbert-Chow morphism. However, it turns out that choosing a slightly smaller locus yields better formal properties. To motivate this locus recall that the first point of the ordered Hilbert Chow morphism is given by $\text{Supp}(I_0/I_1)$. For this to be zero we need multiplication with x_i to vanish on I_0/I_1 which is the same as requiring $I_1 = (x_1, \dots, x_n)$. Now for $\text{Supp}(I_k/I_{k+1})$ to be zero we need $x_i \cdot I_k \subset I_{k+1}$ for $i = 1, \dots, n$, but since $I_1 = (x_1, \dots, x_n)$ this is the same as requiring

$$(I_1/I_{r+1}) \cdot (I_k/I_{r+1}) \subset (I_{k+1}/I_{r+1}).$$

Translating these conditions to the non-associative Hilbert scheme, we arrive at the following definition

Definition 2.4.1. Consider $\text{naNHilb}^{r+1}(\mathbb{A}^n)$ with the universal flag and maps $(\mathcal{V}_\bullet, \Psi_2, s, \Psi_1)$. The punctual nested non-associative Hilbert scheme is the closed subscheme

$$\text{naNHilb}_0^{r+1}(\mathbb{A}^n) \subset \text{naNHilb}^{r+1}(\mathbb{A}^n)$$

where Ψ_1 factors through $\mathcal{V}_1$ and where Ψ_2 maps $\mathcal{V}_1 \otimes \mathcal{V}_k$ to $\mathcal{V}_{k+1}$ for all $k = 1, \dots, r$. If we let $\mathcal{Q}$ be the quotient of $\mathcal{H}om(\text{Sym}^2 \mathcal{V}_1, \mathcal{V}_1)$ by the subsheaf of maps taking $\mathcal{V}_1 \otimes \mathcal{V}_k$ to $\mathcal{V}_{k+1}$ then we can similarly describe the punctual locus as the zero locus of the section of

$$\mathcal{H}om((\mathcal{O}^n)^\vee, \mathcal{V}_0/\mathcal{V}_1) \oplus \mathcal{Q}$$

given by $\Psi_1 \oplus \Psi_2$.

We prove that this actually is an extension of the punctual locus

$$\text{NHilb}_0^{r+1}(\mathbb{A}^n) = HC^{-1}(\mathbf{0}) \subset \text{NHilb}^{r+1}(\mathbb{A}^n)$$

of the nested Hilbert scheme of points.

Lemma 2.4.1. *It holds that*

$$\text{naHC}(\text{naNHilb}_0^{r+1}(\mathbb{A}^n)) \subset \mathbf{0}.$$

Proof. Let again $\mathcal{L}_i = \mathcal{V}_i/\mathcal{V}_{i+1}$. Since Ψ_1 factors through $\mathcal{V}_1$ on the punctual locus it follows that the map

$$\Psi_2(\Psi_1 \otimes \text{id}) : (\mathcal{O}^n)^\vee \otimes \mathcal{L}_0 \rightarrow \mathcal{L}_0$$

vanishes. Similarly, we can factor

$$\Psi_2(\Psi_1 \otimes \text{id}) : (\mathcal{O}^n)^\vee \otimes \mathcal{L}_k \rightarrow \mathcal{V}_1 \otimes \mathcal{L}_k \xrightarrow{\Psi_2} \mathcal{L}_k$$

and use that $\Psi_2(\mathcal{V}_1 \otimes \mathcal{V}_k) \subset \mathcal{V}_{k+1}$ to see that this map also vanishes. $\square$

Lemma 2.4.2. *There exists a closed immersion $\iota_0 : \text{NHilb}_0^{r+1}(\mathbb{A}^n) \rightarrow \text{naNHilb}_0^{r+1}(\mathbb{A}^n)$ such that the diagram*

$$\begin{array}{ccc} \text{NHilb}_0^{r+1}(\mathbb{A}^n) & \xrightarrow{\iota_0} & \text{naNHilb}_0^{r+1}(\mathbb{A}^n) \\ \downarrow & & \downarrow \\ \text{NHilb}^{r+1}(\mathbb{A}^n) & \xrightarrow{\iota} & \text{naNHilb}^{r+1}(\mathbb{A}^n) \end{array}$$

is a pullback.

Proof. We prove the statement on the functor of points. Given $S \in \text{Sch}_{\mathbb{C}}$ and

$$[\mathcal{I}_{r+1} \subset \cdots \subset \mathcal{I}_1 \subset \mathcal{I}_0 = \mathcal{O}_S[x_1, \dots, x_n]]$$

of $\text{NHilb}^{r+1}(\mathbb{A}^n)$ we should prove that it factors through $\text{NHilb}_0^{r+1}(\mathbb{A}^n)$ if and only if it factors through $\text{naNHilb}_0^{r+1}(\mathbb{A}^n)$ after composing it with ι . The "if" part follows from the above, since $\text{naHC}(\text{naNHilb}_0^{r+1}(\mathbb{A}^n)) \subset \mathbf{0}$ and $\text{naHC} \circ \iota = \text{HC}$. Conversely, if the point is punctual, then $x_i \mathcal{I}_k \subset \mathcal{I}_{k+1}$ for all $i = 1, \dots, n$ and all $k = 0, \dots, r$. For $k = 0$ this means that $x_i \in \mathcal{I}_1$ for all i hence $\mathcal{I}_1 = (x_1, \dots, x_n)$. Since $x_i \cdot \mathcal{I}_k \subset \mathcal{I}_{k+1}$ we conclude that

$$(\mathcal{I}_1/\mathcal{I}_{r+1}) \cdot (\mathcal{I}_k/\mathcal{I}_{r+1}) \subset (\mathcal{I}_{k+1}/\mathcal{I}_{r+1})$$

thus, proving that its image under ι is in $\text{naNHilb}_0^{r+1}(\mathbb{A}^n)$. $\square$

2.4.3 Projective tower structure

We prove that the punctual non-associative Hilbert scheme can be exhibited as an iterated tower of projective fibrations. The proof is essentially the same as in [Kaz17, Proposition 6.3] with a slightly more general space. Before proceeding, we show how the nested non-associative Hilbert scheme generally maps to a nested non-associative Hilbert scheme with a truncated dimension vector.

Definition 2.4.2. Let $\underline{d} = (d_0, \dots, d_r)$ and consider $\text{naNHilb}^{\underline{d}}(\mathbb{A}^n)$ with universal flag and maps $(\mathcal{V}_\bullet, \Psi_2, \mathfrak{s}, \varpi)$. Define the flag $\mathcal{F}_\bullet$ with dimension vector $(d_0, \dots, d_{r-1})$ by setting

$$\mathcal{F}_k := \mathcal{V}_k/\mathcal{V}_r.$$

We get an induced unital non-associative algebra structure from $\Psi_2, \mathfrak{s}$ exhibiting $\mathcal{F}_\bullet$ as a quotient algebra of $\mathcal{V}_\bullet$, so that we also get an induced surjection via ϖ . This data determines a map

$$\text{naNHilb}^{(d_0, \dots, d_r)}(\mathbb{A}^n) \rightarrow \text{naNHilb}^{(d_0, \dots, d_{r-1})}(\mathbb{A}^n).$$

Remark 2.4.1. Since a quotient of an associative algebra is again associative this map preserves the associativity locus. One can easily see that on the nested Hilbert scheme of points this is simply the map

$$[I_{r+1} \subset I_r \subset \cdots \subset I_1 \subset \mathbb{C}[x_1, \dots, x_n]] \mapsto [I_r \subset \cdots \subset I_1 \subset \mathbb{C}[x_1, \dots, x_n]]$$

which forgets the largest subscheme of the nesting.

Lemma 2.4.3. *The natural map*

$$\mathrm{naNHilb}^{r+1}(\mathbb{A}^n) \rightarrow \mathrm{naNHilb}^r(\mathbb{A}^n)$$

restricts to the punctual locus i.e., to a map

$$\mathrm{naNHilb}_0^{r+1}(\mathbb{A}^n) \rightarrow \mathrm{naNHilb}_0^r(\mathbb{A}^n).$$

Proof. If Ψ_1 factors through $\mathcal{V}_1 \subset \mathcal{V}_0$ then the induced map factors through $\mathcal{V}_1/\mathcal{V}_r \subset \mathcal{V}_0/\mathcal{V}_r$. Similarly, if Ψ_2 maps $\mathcal{V}_1 \otimes \mathcal{V}_k$ to $\mathcal{V}_{k+1}$ then the induced map maps $(\mathcal{V}_1/\mathcal{V}_r) \otimes (\mathcal{V}_k/\mathcal{V}_r)$ to $(\mathcal{V}_{k+1}/\mathcal{V}_r)$. $\square$

Before proving the projective tower statement, we show that the unit does not contain any additional information in the punctual case.

Definition 2.4.3. Let $S \in \mathrm{Sch}_S$ and let $\mathcal{O}_S\{x_1, \dots, x_n\}_c^+ \subset \mathcal{O}_S\{x_1, \dots, x_n\}_c$ be the ideal spanned by $x_1, \dots, x_n$. We view it as the free non-unital non-associative algebra of $(\mathcal{O}_S^n)^\vee$.

Lemma 2.4.4. *Let $S \in \mathrm{Sch}_S$. The map*

$$\begin{aligned} (\mathrm{naNHilb}_0^{r+1}(\mathbb{A}^n))(S) &\longrightarrow \{(\mathcal{G}_\bullet, \psi_2, \psi_1)\} / \text{iso} \\ [\mathcal{F}_\bullet, \psi_2, s, \psi_1] &\longmapsto [(\mathcal{F}_1)_\bullet, \psi_2, \psi_1] \end{aligned}$$

which forgets $\mathcal{F}_0$ and the unit s is a bijection. Here the tuples in the right-hand side set consists of a flag of sheaves

$$\mathcal{G}_1 \supset \mathcal{G}_2 \supset \dots \mathcal{G}_{r+1} = 0$$

such that $\mathcal{G}_k/\mathcal{G}_{k+1}$ is an invertible sheaf for all $k = 1, \dots, r$, a non-unital commutative non-associative algebra structure $\psi_2 : \mathrm{Sym}^2 \mathcal{G}_1 \rightarrow \mathcal{G}_1$ such that $\psi_2(\mathcal{G}_1 \otimes_S \mathcal{G}_k) \subset \mathcal{G}_{k+1}$ and a map $\psi_1 : (\mathcal{O}_S^n)^\vee \rightarrow \mathcal{G}$ such that the induced map $\mathcal{O}_S\{x_1, \dots, x_n\}_c^+ \rightarrow \mathcal{G}$ is an epimorphism.

Proof. We first prove that the map is well defined. Indeed, since we are in the punctual locus we have $\psi_2(\mathcal{F}_1 \otimes_S \mathcal{F}_k) \subset \mathcal{F}_{k+1}$ and ψ_1 factors through $\mathcal{F}_1$. We have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_S\{x_1, \dots, x_n\}_c^+ & \longrightarrow & \mathcal{O}_S\{x_1, \dots, x_n\}_c & \longrightarrow & \mathcal{O}_S \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow s \\ 0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{F}_0 & \longrightarrow & \mathcal{F}_0/\mathcal{F}_1 \longrightarrow 0 \end{array}$$

with exact rows. The right vertical map is an isomorphism, and the middle vertical map is an epimorphism, so the 5-lemma implies that the left vertical map is also an epimorphism. This shows that the map is well defined. Conversely given a point $[\mathcal{G}_\bullet, \psi_2, \psi_1]$ we define $\mathcal{F}_\bullet$ by setting

$$\mathcal{F}_0 = \mathcal{O}_S \oplus \mathcal{G}_1$$

and $\mathcal{F}_k = \mathcal{G}_k$ for $k = 1, \dots, r$. We note that

$$\mathrm{Sym}^2 \mathcal{F}_0 \cong \mathcal{O}_S \oplus \mathcal{G}_1 \oplus \mathrm{Sym}^2 \mathcal{G}_1 \cong \mathcal{F}_0 \oplus \mathrm{Sym}^2 \mathcal{G}_1$$

and extend ψ_2 to a map

$$\mathrm{Sym}^2 \mathcal{F}_0 \rightarrow \mathcal{F}_0$$

by letting it act by the identity on the first summand and by ψ_2 on the second. We let s be the inclusion

$$\mathcal{O}_S \hookrightarrow \mathcal{O}_S \oplus \mathcal{G}_1 = \mathcal{F}_0$$

and let ψ_1 be the composite

$$(\mathcal{O}_S^n)^\vee \xrightarrow{\psi_1} \mathcal{G}_1 \hookrightarrow \mathcal{O}_S \oplus \mathcal{G}_1 = \mathcal{F}_0.$$

We have exactly the same commutative diagram as above. Since the left vertical map is an epimorphism and the right vertical map is an isomorphism it follows that the middle vertical map is an epimorphism. This proves that the map is well defined. One easily checks that these maps are inverse to each other. $\square$

Lemma 2.4.5. *Let $\underline{d} = (1, d_1, \dots, d_r) \in \mathbb{N}^r$, let $S \in \text{Sch}_{\mathbb{C}}$ and $(\mathcal{F}_\bullet, \psi_2, s, \psi_1)$ defined as follows.*

1. $\mathcal{F}_\bullet$ is a flag of sheaves

$$\mathcal{F}_0 \supset \mathcal{F}_1 \supset \dots \mathcal{F}_{r+1} = 0$$

with $\mathcal{F}_0/\mathcal{F}_1$ invertible and $\mathcal{F}_k/\mathcal{F}_{k+1}$ locally free of rank d_k .

2. (ψ_2, s) is a unital non-associative algebra structure on $\mathcal{F}_0$ such that $\psi_2(\mathcal{F}_1 \otimes_S \mathcal{F}_k) \subset \mathcal{F}_{k+1}$ and $s \notin \mathcal{F}_1$.

3. $\psi_1 : (\mathcal{O}_S^n)^\vee \rightarrow \mathcal{F}_1$ is a map of sheaves.

Then the induced map

$$\mathcal{O}_S\{x_1, \dots, x_n\}_c \rightarrow \mathcal{F}_0$$

is surjective if and only if the map

$$\psi_1 \oplus \psi_2 : (\mathcal{O}_S^n)^\vee \oplus \text{Sym}^2 \mathcal{F}_1 \rightarrow \mathcal{F}_1$$

is surjective.

Proof. We first note that since $s : \mathcal{O}_S \rightarrow \mathcal{F}_0/\mathcal{F}_1$ is an isomorphism (since $\mathcal{F}_0/\mathcal{F}_1$ is an invertible sheaf) it follows that the map

$$\mathcal{O}_S\{x_1, \dots, x_n\}_c \rightarrow \mathcal{F}_0$$

is surjective if and only if

$$\mathcal{O}_S\{x_1, \dots, x_n\}_c^+ \rightarrow \mathcal{F}_1$$

is surjective. Clearly this implies that $\psi_1 \oplus \psi_2$ is surjective since any element of $\mathcal{F}_1$ is a linear combination of products of elements in the image of ψ_1 i.e., elements either in the image ψ_1 or in the image of ψ_2 . To prove the converse, we proceed by induction on r . For $r = 1$ we see that the multiplication map takes $\mathcal{F}_1 \otimes_S \mathcal{F}_1$ to $\mathcal{F}_2 = 0$ so that $\psi_2 = 0$. Therefore, if $\psi_1 \oplus \psi_2$ is surjective then ψ_1 is surjective so that

$$\mathcal{O}_S\{x_1, \dots, x_n\}_c^+ \rightarrow \mathcal{F}_1$$

is also surjective. Let now $\mathcal{F}_\bullet, \psi_1, \psi_2$ be given with $\psi_1 \oplus \psi_2$ surjective. By the induction hypothesis the composite

$$\mathcal{O}_S\{x_1, \dots, x_n\}_c^+ \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_1/\mathcal{F}_r$$

is surjective so that the map

$$\mathcal{O}_S\{x_1, \dots, x_n\}_c^+ \oplus \mathcal{F}_r \rightarrow \mathcal{F}_1$$

given by the quotient and the inclusion is surjective. In particular, the map

$$(\mathcal{O}_S^n)^\vee \oplus \text{Sym}^2(\mathcal{O}_S\{x_1, \dots, x_n\}_c^+ \oplus \mathcal{F}_r) \rightarrow (\mathcal{O}_S^n)^\vee \oplus \text{Sym}^2 \mathcal{F}_1 \rightarrow \mathcal{F}_1$$

is surjective. We have

$$\mathrm{Sym}^2(\mathcal{O}_S\{x_1, \dots, x_n\}_c^+ \oplus \mathcal{F}_r) \cong \mathrm{Sym}^2(\mathcal{O}_S\{x_1, \dots, x_n\}_c^+) \oplus (\mathcal{O}_S\{x_1, \dots, x_n\}_c^+ \otimes \mathcal{F}_r) \oplus \mathrm{Sym}^2 \mathcal{F}_r$$

and since $\psi_2(\mathcal{F}_1 \otimes_S \mathcal{F}_r) \subset \mathcal{F}_{r+1} = 0$ it follows that the above map vanishes on the last two summands. We conclude that the map

$$(\mathcal{O}_S^n)^\vee \oplus \mathrm{Sym}^2(\mathcal{O}_S\{x_1, \dots, x_n\}_c^+) \rightarrow \mathcal{F}_1$$

is surjective. The image of $\mathcal{O}_S\{x_1, \dots, x_n\}_c^+$ is closed under ψ_2 so that it contains the image of the above map. We conclude that it is the whole of $\mathcal{F}_1$. $\square$

Corollary 2.4.1. *An S -valued point of $\mathrm{naHilb}_0^{r+1}(\mathbb{A}^n)$ is the same as a tuple $(\mathcal{F}_\bullet, \psi_2, \psi_1)$ where*

1. $\mathcal{F}_\bullet$ is a flag in $\mathrm{QCoh}(S)$

$$\mathcal{F}_1 \supset \mathcal{F}_2 \supset \dots \supset \mathcal{F}_{r+1} = 0$$

such that $\mathcal{F}_k/\mathcal{F}_{k+1}$ is invertible.

2. ψ_2 is a non-associative algebra structure on $\mathcal{F}_1$ such that $\psi_2(\mathcal{F}_1 \otimes_S \mathcal{F}_k) \subset \mathcal{F}_{k+1}$.

3. $\psi_1 : (\mathcal{O}_S^n)^\vee \rightarrow \mathcal{F}_1$ is a map of sheaves such that $\psi_1 \oplus \psi_2$ is surjective.

It follows that $\mathrm{naHilb}_0^{r+1}(\mathbb{A}^n)$ comes equipped with universal such structure, which we will denote by $(\mathcal{N}_\bullet, \Psi_2, \Psi_1)$.

Proposition 2.4.2. *Consider $\mathrm{naHilb}^r(\mathbb{A}^n)$ with universal structure $(\mathcal{N}_\bullet, \Psi_2, \Psi_1)$. Then the natural map*

$$\mathrm{naHilb}_0^{r+1}(\mathbb{A}^n) \rightarrow \mathrm{naHilb}_0^r(\mathbb{A}^n)$$

identifies with the structure map

$$\mathbb{P}_{\mathrm{naHilb}^r(\mathbb{A}^n)}((\ker \Psi_1 \oplus \Psi_2)^\vee) \rightarrow \mathrm{naHilb}^r(\mathbb{A}^n).$$

Proof. We should prove that given $S \in \mathrm{Sch}_{\mathbb{C}}$ and a map $S \rightarrow \mathrm{naHilb}^r(\mathbb{A}^n)$ classified by $[\mathcal{F}_\bullet, \psi_2, \psi_1]$ the set of lifts

$$\begin{array}{ccc} & \mathrm{naHilb}_0^{r+1}(\mathbb{A}^n) & \\ & \nearrow \gamma & \downarrow \\ S & \longrightarrow & \mathrm{naHilb}_0^r(\mathbb{A}^n) \end{array}$$

is in bijection with the set of $(\mathcal{L}, q)$ where $\mathcal{L}$ is an invertible sheaf on S and

$$q : \ker \psi_1 \oplus \psi_2 \rightarrow \mathcal{L}$$

is a surjective map. We recall that a lift is classified by $[\tilde{\mathcal{F}}_\bullet, \tilde{\psi}_2, \tilde{\psi}_1]$ such that

$$\tilde{\mathcal{F}}_k/\tilde{\mathcal{F}}_r \simeq \mathcal{F}_k$$

with the maps induced by $\tilde{\psi}_2, \tilde{\psi}_1$ agreeing with ψ_1, ψ_2 . If we have a lift then we consider the commutative diagram

$$\begin{array}{ccccccc} \tilde{\mathcal{F}}_1 \otimes_S \tilde{\mathcal{F}}_r & \longrightarrow & (\mathcal{O}_S^n)^\vee \oplus \mathrm{Sym}^2 \tilde{\mathcal{F}}_1 & \longrightarrow & (\mathcal{O}_S^n)^\vee \oplus \mathrm{Sym}^2 \mathcal{F}_1 & \longrightarrow & 0 \\ \downarrow \tilde{\psi}_2 & & \downarrow \tilde{\psi}_1 \oplus \tilde{\psi}_2 & & \downarrow \psi_1 \oplus \psi_2 & & \\ 0 & \longrightarrow & \tilde{\mathcal{F}}_r & \longrightarrow & \tilde{\mathcal{F}}_1 & \longrightarrow & \mathcal{F}_1 \longrightarrow 0 \end{array}$$

with exact rows. The middle and right vertical maps are surjective by assumption and since $\tilde{\psi}_2(\tilde{\mathcal{F}}_1 \otimes_S \tilde{\mathcal{F}}_r) \subset \tilde{\mathcal{F}}_{r+1} = 0$ we conclude that the left vertical map is zero. If we set $\mathcal{L} := \tilde{\mathcal{F}}_r$ the snake lemma provides the desired map

$$q : \ker(\psi_1 \oplus \psi_2) \rightarrow \tilde{\mathcal{F}}_r.$$

We note that we in fact have a map

$$(\mathcal{O}_S^n)^\vee \oplus \text{Sym}^2 \mathcal{F}_1 \rightarrow \tilde{\mathcal{F}}_1$$

which factors as q when restricted to $\ker(\psi_1 \oplus \psi_2)$. Conversely, if we are given $(\mathcal{L}, q)$ we let $\tilde{\mathcal{F}}_1$ be the pushout of q along the inclusion of the kernel into $(\mathcal{O}_S^n)^\vee \oplus \text{Sym}^2 \mathcal{F}_1$. The left and outer square of the diagram

$$\begin{array}{ccccc} \ker(\psi_1 \oplus \psi_2) & \xrightarrow{q} & \mathcal{L} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ (\mathcal{O}_S^n)^\vee \oplus \text{Sym}^2 \mathcal{F}_1 & \longrightarrow & \tilde{\mathcal{F}}_1 & \longrightarrow & \mathcal{F}_1 \end{array}$$

are pushouts hence so is the right square by the pasting lemma. This proves that $\tilde{\mathcal{F}}_1$ is an extension of $\mathcal{F}_1$ by $\mathcal{L}$. We define $\tilde{\mathcal{F}}_k \subset \tilde{\mathcal{F}}_1$ to be the preimage of $\mathcal{F}_k \subset \mathcal{F}_1$ for $k = 1, \dots, r-1$ and $\tilde{\mathcal{F}}_r = \mathcal{L}$. We define $\tilde{\psi}_1 \oplus \tilde{\psi}_2$ as the composite

$$(\mathcal{O}_S^n)^\vee \oplus \text{Sym}^2 \tilde{\mathcal{F}}_1 \rightarrow (\mathcal{O}_S^n)^\vee \oplus \text{Sym}^2 \mathcal{F}_1 \rightarrow \tilde{\mathcal{F}}_1$$

where the second map is from the pushout and the first map is the identity on the first factor and the quotient on the second. Since $\tilde{\mathcal{F}}_1 \otimes_S \tilde{\mathcal{F}}_r$ is mapped to 0 by the quotient map it is mapped to 0 = $\tilde{\mathcal{F}}_{r+1}$ by ψ_2 . Since the composite

$$\text{Sym}^2 \mathcal{F}_1 \rightarrow \tilde{\mathcal{F}}_1 \rightarrow \mathcal{F}_1$$

agrees with ψ_2 we also see from the definition of $\tilde{\mathcal{F}}_k$ that $\tilde{\psi}_2(\tilde{\mathcal{F}}_1 \otimes_S \tilde{\mathcal{F}}_k) \subset \tilde{\mathcal{F}}_{k+1}$ thus proving that we have a well-defined S -valued point on $\text{naNHilb}_0^{r+1}(\mathbb{A}^n)$. We prove these constructions are inverse to each other. Indeed, in the above we see that $\tilde{\mathcal{F}}_r = \mathcal{L}$ and that q agrees with the restriction of $(\mathcal{O}_S^n)^\vee \oplus \text{Sym}^2 \mathcal{F}_1 \rightarrow \tilde{\mathcal{F}}_1$ to $\ker(\psi_1 \oplus \psi_2)$ so that we regain $(\mathcal{L}, q)$ when applying both constructions. Conversely, if we start with $[\tilde{\mathcal{F}}_\bullet, \tilde{\psi}_1, \tilde{\psi}_2]$ we have a commutative diagram

$$\begin{array}{ccc} \ker(\psi_1 \oplus \psi_2) & \longrightarrow & \tilde{\mathcal{F}}_r \\ \downarrow & & \downarrow \\ (\mathcal{O}_S^n)^\vee \oplus \text{Sym}^2 \mathcal{F}_1 & \longrightarrow & \tilde{\mathcal{F}}_1 \end{array}$$

which we need to prove is a pushout. Indeed, let

$$\mathcal{K} = \ker \left((\mathcal{O}_S^n)^\vee \oplus \text{Sym}^2 \mathcal{F}_1 \rightarrow \tilde{\mathcal{F}}_1 \right)$$

and

$$\mathcal{K}' = \ker \left(\ker(\psi_1 \oplus \psi_2) \rightarrow \tilde{\mathcal{F}}_r \right).$$

We have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker(\psi_1 \oplus \psi_2) & \longrightarrow & (\mathcal{O}_S^n)^\vee \oplus \text{Sym}^2 \mathcal{F}_1 & \longrightarrow & \mathcal{F}_1 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \tilde{\mathcal{F}}_r & \longrightarrow & \tilde{\mathcal{F}}_1 & \longrightarrow & \mathcal{F}_1 \longrightarrow 0 \end{array}$$

with exact rows so that $\mathcal{K}' \rightarrow \mathcal{K}$ is an isomorphism by the snake lemma. Given any $\mathcal{G}$ and a commuting diagram

$$\begin{array}{ccc} \ker(\psi_1 \oplus \psi_2) & \longrightarrow & \tilde{\mathcal{F}}_r \\ \downarrow & & \downarrow \\ (\mathcal{O}_S^n)^\vee \oplus \mathrm{Sym}^2 \mathcal{F}_1 & \longrightarrow & \mathcal{G} \end{array}$$

it then follows that $\mathcal{K}' \rightarrow \mathcal{K} \rightarrow \mathcal{G}$ is zero and hence $\mathcal{K} \rightarrow \mathcal{G}$ is zero proving that the lower horizontal map factors through a unique map $\tilde{\mathcal{F}}_1 \rightarrow \mathcal{G}$. This proves that the diagram is a pushout thus finishing the proof. $\square$

Example 2.4.1. Let us calculate this tower explicitly for small r . We first note that for $r = 0$ we have $\mathrm{naHilb}_0^1(\mathbb{A}^n) \cong \mathrm{Spec} \mathbb{C}$ with universal bundle $\mathcal{N}_1 = 0$ and $\Psi_1 \oplus \Psi_2 : (\mathbb{C}^n)^\vee \oplus 0 \rightarrow 0$. This has kernel $(\mathbb{C}^n)^\vee$ so that

$$\mathrm{naHilb}_0^2(\mathbb{A}^n) \cong \mathbb{P}(\mathbb{C}^n) \cong \mathbb{P}^{n-1}.$$

The above proof gives the recipe for building the universal bundle. We simply define

$$0 = \mathcal{N}_2 \subset \mathcal{N}_1 = \mathcal{O}(1)$$

with multiplication map

$$\Psi_2 : \mathrm{Sym}^2 \mathcal{N}_1 \rightarrow 0 = \mathcal{N}_2$$

and

$$\Psi_1 : (\mathcal{O}_{\mathbb{P}^{n-1}}^n)^\vee \rightarrow \mathcal{O}(1)$$

dual to the universal sub-linebundle

$$\mathcal{O}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^{n-1}}^n.$$

Continuing this procedure we get

$$\mathrm{naHilb}_0^3(\mathbb{A}^n) \cong \mathbb{P}_{\mathbb{P}^{n-1}}(\ker(\Psi_1 \oplus \Psi_2)^\vee) \cong \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}}^n / \mathcal{O}_{\mathbb{P}^{n-1}}(-1) \oplus \mathrm{Sym}^2 \mathcal{O}_{\mathbb{P}^{n-1}}(-1)).$$

The universal bundle is then constructed as follows. Denote by $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ and $\mathcal{N}_1$ the pullback of the bundles to $\mathrm{naHilb}_0^3(\mathbb{A}^n)$ and let $\mathcal{O}(-1)$ be the universal line bundle on $\mathrm{naHilb}_0^3(\mathbb{A}^n)$. We define the new universal bundle $\tilde{\mathcal{N}}_1$ as the pushout

$$\begin{array}{ccc} (\mathcal{O}^n / \mathcal{O}_{\mathbb{P}^{n-1}}(-1))^\vee \oplus \mathrm{Sym}^2(\mathcal{N}_1) & \longrightarrow & \mathcal{O}(1) \\ \downarrow & & \downarrow \\ (\mathcal{O}^n)^\vee \oplus \mathrm{Sym}^2 \mathcal{N}_1 & \longrightarrow & \tilde{\mathcal{N}}_1 \end{array}$$

where the top horizontal map is the dual of the inclusion of the universal line bundle. We set $\tilde{\mathcal{N}}_2 = \mathcal{O}(1)$, so that $\tilde{\mathcal{N}}_1 / \tilde{\mathcal{N}}_2 \simeq \mathcal{N}_1$. The universal maps are given by

$$(\mathcal{O}^n)^\vee \oplus \mathrm{Sym}^2 \tilde{\mathcal{N}}_1 \rightarrow (\mathcal{O}^n)^\vee \oplus \mathrm{Sym}^2 \mathcal{N}_1 \rightarrow \tilde{\mathcal{N}}_1$$

where the first map is the identity and quotient and the second map is the structural map from the pushout.

Corollary 2.4.2. *The scheme $\mathrm{naHilb}_0^{r+1}(\mathbb{A}^n)$ is smooth and projective.*

Proof. This follows directly from the projective tower description since

$$\mathrm{naHilb}_0^1(\mathbb{A}^n) \cong \mathrm{Spec} \mathbb{C}.$$

$\square$

2.4.4 Other dimension vectors and relations to the work of Kazarian

We prove a slight generalization of the above where we allow different dimension vectors. We construct the moduli functor of non-unital non-associative algebras with a flag structure satisfying a nilpotency condition. We prove that it embeds into the non-associative Hilbert scheme by freely adjoining a unit. We prove that its intersection with the associativity locus is contained in the punctual Hilbert scheme but will in general not be the whole punctual Hilbert scheme. A proof completely similar to the section above shows that the locus admits a description as a tower of Grassmannian fibrations. At last, we show that it contains a closed locus of algebras satisfying a stronger nilpotency condition which agrees with the non-associative Hilbert schemes appearing in the work of Kazarian [Kaz17].

Definition 2.4.4. Let $\underline{d} = (1, d_1, \dots, d_r) \in \mathbb{N}^{r+1}$ be a dimension vector beginning with 1. We define a moduli functor $\text{naNHilb}_0^{\underline{d}}(\mathbb{A}^n)$ by

$$S \in \text{Sch}_{\mathbb{C}} \mapsto \{(\mathcal{F}_{\bullet}, \psi_1, \psi_2)\} / \text{iso}.$$

The tuples of the right-hand set consists of the following data.

1. A flag of locally free sheaves

$$\mathcal{F}_{\bullet} = (\mathcal{F}_1 \supset \dots \supset \mathcal{F}_{r+1} = 0)$$

with $\mathcal{F}_k / \mathcal{F}_{k+1}$ locally free of rank d_k

2. A non-unital non-associative algebra structure ψ_2 on $\mathcal{F}_1$ such that $\psi_2(\mathcal{F}_1 \otimes_S \mathcal{F}_k) \subset \mathcal{F}_{k+1}$
3. A map of sheaves $\psi_1 : (\mathcal{O}_S^n)^{\vee} \rightarrow \mathcal{F}_1$ such that $\psi_1 \oplus \psi_2$ is surjective.

Lemma 2.4.6. Let $\underline{d} = (1, d_1, \dots, d_n) \in \mathbb{N}^{r+1}$ and let $(\mathcal{V}_{\bullet}, \Psi_2, s, \Psi_1)$ be the universal structure on $\text{naNHilb}^{\underline{d}}(\mathbb{A}^n)$. There exists a closed immersion

$$\text{naNHilb}_0^{\underline{d}}(\mathbb{A}^n) \rightarrow \text{naNHilb}^{\underline{d}}(\mathbb{A}^n)$$

identifying the domain with the locus where Ψ_1 factors through $\mathcal{V}_1$ and where $\Psi_2(\mathcal{V}_1 \otimes \mathcal{V}_k) \subset \mathcal{V}_{k+1}$.

Proof. On the functor of points, we simply map $(\mathcal{F}_{\bullet}, \psi_2, \psi_1)$ to $(\mathcal{G}_{\bullet}, \psi_2, s, \psi_1)$ where $\mathcal{G}_0 = \mathcal{O}_S \oplus \mathcal{F}_0$ and $\mathcal{G}_k = \mathcal{F}_k$ for $k = 1, \dots, r$ with induced algebra structure from adjoining a unit. From lemma 2.4.5 we see that this is indeed a point on $\text{naNHilb}^{\underline{d}}(\mathbb{A}^n)$. Conversely, if we have an S -valued point on the described locus we can simply forget the unit and the 0'th piece of the filtration to obtain an S -valued point on $\text{naNHilb}_0^{\underline{d}}(\mathbb{A}^n)$. One checks that these constructions are inverse to each other. $\square$

Remark 2.4.2. From the moduli description we see that the scheme $\text{naNHilb}_0^{\underline{d}}$ comes equipped with a universal non-unital non-associative algebra. We denote this universal data by $(\mathcal{N}_{\bullet}, \psi_1, \psi_2)$.

Lemma 2.4.7. Let $\underline{d} = (1, d_1, \dots, d_r) \in \mathbb{N}^{r+1}$. The natural map

$$\text{naNHilb}^{(1, d_1, \dots, d_r)}(\mathbb{A}^n) \rightarrow \text{naNHilb}^{(1, d_1, \dots, d_{r-1})}(\mathbb{A}^n)$$

restricts to a map

$$\text{naNHilb}_0^{(1, d_1, \dots, d_r)}(\mathbb{A}^n) \rightarrow \text{naNHilb}_0^{(1, d_1, \dots, d_{r-1})}(\mathbb{A}^n).$$

Proof. If Ψ_1 factors through $\mathcal{V}_1$ then the induced map factors through $\mathcal{V}_1 / \mathcal{V}_r$. If $\Psi_2(\mathcal{V}_1 \otimes \mathcal{V}_k) \subset \mathcal{V}_{k+1}$ then the induced map takes $(\mathcal{V}_1 / \mathcal{V}_r) \otimes (\mathcal{V}_k / \mathcal{V}_r)$ to $\mathcal{V}_{k+1} / \mathcal{V}_r$. $\square$

Proposition 2.4.3. *Let $(\mathcal{N}_\bullet, \Psi_2, \Psi_1)$ be the universal structure on $\text{naNHilb}_0^{(1,d_1,\dots,d_{r-1})}(\mathbb{A}^n)$. The natural map*

$$\text{naNHilb}_0^{(1,d_1,\dots,d_r)}(\mathbb{A}^n) \rightarrow \text{naNHilb}_0^{(1,d_1,\dots,d_{r-1})}(\mathbb{A}^n)$$

identifies with the structure map

$$\text{Gr}_{\text{naNHilb}_0^{(1,d_1,\dots,d_{r-1})}(\mathbb{A}^n)}(d_r, \ker(\Psi_1 \oplus \Psi_2)^\vee) \rightarrow \text{naNHilb}_0^{(1,d_1,\dots,d_{r-1})}(\mathbb{A}^n).$$

Proof. We should prove that given $S \in \text{Sch}_\mathbb{C}$ and a map $S \rightarrow \text{naNHilb}_0^{(1,d_1,\dots,d_{r-1})}(\mathbb{A}^n)$ classified by $[\mathcal{F}_\bullet, \psi_2, \psi_1]$ the set of lifts

$$\begin{array}{ccc} & \text{naNHilb}_0^{(1,d_1,\dots,d_r)}(\mathbb{A}^n) & \\ & \downarrow & \\ S & \xrightarrow{\quad} & \text{naNHilb}_0^{(1,d_1,\dots,d_{r-1})}(\mathbb{A}^n) \end{array}$$

is in bijection with the set of $(\mathcal{G}, q)$ where $\mathcal{G}$ is a locally free sheaf of rank d_r on S and

$$q : \ker \psi_1 \oplus \psi_2 \rightarrow \mathcal{G}$$

is a surjective map. The proof is exactly the same as that of proposition 2.4.2. $\square$

Corollary 2.4.3. *Let $\underline{d} = (1, d_1, \dots, d_r)$. The scheme $\text{naNHilb}_0^{\underline{d}}(\mathbb{A}^n)$ is smooth and projective.*

Proof. This follows inductively from the above since $\text{naNHilb}_0^1(\mathbb{A}^n) \cong \text{Spec } \mathbb{C}$. $\square$

Lemma 2.4.8. *Let $\underline{d} = (1, d_1, \dots, d_r)$. Let $S \in \text{Sch}_\mathbb{C}$ and let $S \rightarrow \text{NHilb}^{\underline{d}}(\mathbb{A}^n)$ be a map of schemes. If*

$$S \rightarrow \text{NHilb}^{\underline{d}}(\mathbb{A}^n) \xrightarrow{\iota} \text{naNHilb}^{\underline{d}}(\mathbb{A}^n)$$

factors through $\text{naNHilb}_0^{\underline{d}}(\mathbb{A}^n)$, then the first map factors through $\text{NHilb}_0^{\underline{d}}(\mathbb{A}^n)$.

Proof. Let the map be classified by

$$\mathcal{I}_{r+1} \subset \mathcal{I}_r \subset \dots \subset \mathcal{I}_1 \subset \mathcal{I}_0 = \mathcal{O}_S[x_1, \dots, x_n].$$

We set

$$\mathcal{F}_k = \mathcal{I}_k / \mathcal{I}_{r+1}$$

equipped with ψ_2 corresponding to the multiplication map of the algebra, s its unit and ψ_1 the restriction of the quotient map to the linear functions. Since s does not factor through $\mathcal{F}_1$ while ψ_1 does it follows that we have a map

$$\mathcal{O}_S[x_1, \dots, x_n] / (x_1, \dots, x_n) \rightarrow \mathcal{F}_0 / \mathcal{F}_1 \cong \mathcal{O}_S[x_1, \dots, x_n] / \mathcal{I}_1$$

which is an isomorphism by rank reasons. This means that

$$\mathcal{I}_1 \cong \mathfrak{m} := (x_1, \dots, x_n) \subset \mathcal{O}_S[x_1, \dots, x_n].$$

Since

$$\psi_2(\mathcal{F}_1, \mathcal{F}_k) \subset \mathcal{F}_{k+1}$$

this means

$$(\mathcal{I}_1 / \mathcal{I}_{r+1}) \cdot (\mathcal{I}_k / \mathcal{I}_{r+1}) \subset (\mathcal{I}_{k+1} / \mathcal{I}_{r+1})$$

i.e.

$$\mathcal{I}_1 \cdot \mathcal{I}_k \subset \mathcal{I}_{k+1}.$$

We conclude by induction that

$$\mathfrak{m}^k = \mathcal{I}_1^k \subset \mathcal{I}_k \subset \mathcal{I}_1 = \mathfrak{m}$$

which shows that $\mathcal{I}_k \subset \mathcal{O}_S[x_1, \dots, x_n]$ is punctual. $\square$

Corollary 2.4.4. *Let $\underline{d} = (1, d_1, \dots, d_r)$. We have an inclusion of schemes*

$$\text{naNHilb}_0^{\underline{d}}(\mathbb{A}^n) \cap \text{NHilb}^{\underline{d}}(\mathbb{A}^n) \subset \text{NHilb}_0^{\underline{d}}(\mathbb{A}^n).$$

Warning 2.4.1. In general (unless $\underline{d} = (1, 1, \dots, 1)$) we do not have an equality in the inclusion above. For a $\mathbb{C}$ -point $I_{r+1} \subset \dots \subset I_1 \subset \mathbb{C}[x_1, \dots, x_n]$ to factor through $\text{naNHilb}_0^{\underline{d}}(\mathbb{A}^n)$ means exactly that $\mathfrak{m} \subset I_1$ (and thus $\mathfrak{m} = I_1$ since they have the same colength) and that $I_1 \cdot I_k \subset I_{k+1}$ for all k . An easy example of a punctual point not satisfying this is

$$(x^3) \subset (x) \subset \mathbb{C}[x]$$

in $\text{NHilb}_0^{(1,2)}(\mathbb{A}^1)$, as $(x) \cdot (x) = (x^2) \not\subset (x^3)$.

Remark 2.4.3. (Relation to [Kaz17]) One can consider a version of definition 2.4.4 where we require $\psi_2(\mathcal{F}_i \otimes_S \mathcal{F}_j) \subset \mathcal{F}_{i+j}$ for all $1 \leq i, j \leq r$. We denote this locus by $\text{naFHilb}_0^{\underline{d}}(\mathbb{A}^n)$ and again, note that we get a closed immersion

$$\text{naFHilb}_0^{\underline{d}}(\mathbb{A}^n) \subset \text{naNHilb}^{\underline{d}}(\mathbb{A}^n)$$

by adjoining a unit and that the projection map on the latter restricts to a map

$$\text{naFHilb}_0^{(1, d_1, \dots, d_r)}(\mathbb{A}^n) \rightarrow \text{naFHilb}_0^{(1, d_1, \dots, d_{r-1})}(\mathbb{A}^n).$$

Let $(\mathcal{N}_\bullet, \Psi_2, \Psi_1)$ be the universal structure for $\text{naFHilb}^{(1, \dots, d_{r-1})}$. Set $S_r \subset \text{Sym}^2 \mathcal{N}_1$ spanned by $\mathcal{N}_i \otimes \mathcal{N}_j$ where $i + j > r$. We get an induced maps

$$\overline{\Psi}_2 : \text{Sym}^2 \mathcal{N}_1 / S_r \rightarrow \mathcal{N}_1$$

and one can again using the same arguments as the section above show that

$$\text{naFHilb}_0^{(1, d_1, \dots, d_r)}(\mathbb{A}^n) \cong \text{Gr}_{\text{naFHilb}_0^{(1, d_1, \dots, d_{r-1})}(\mathbb{A}^n)}(d_r, \ker(\Psi_1 \oplus \overline{\Psi}_2)^\vee)$$

and that the above projection is simply the structure map. In fact, this is exactly the proof of theorem 6.3 of [Kaz17] and $\text{naFHilb}^{\underline{d}}(\mathbb{A}^n)$ agrees with the notion of the non-associative Hilbert scheme appearing in [Kaz17]. Its intersection with the associative locus yields what we refer to as *the filtered Hilbert scheme*, which we informally describe as the following closed locus of the punctual nested Hilbert scheme

$$\text{FHilb}_0^{\underline{d}}(\mathbb{A}^n) = \{I_{r+1} \subset I_r \subset \dots \subset I_1 = \mathfrak{m} \mid I_i \cdot I_j \subset I_{i+j}, 1 \leq i, j \leq r\}$$

where $\mathfrak{m} = (x_1, \dots, x_n) \subset \mathbb{C}[x_1, \dots, x_n]$.

2.5 Virtual localization

In this section we extend the standard torus action of $T = \mathbb{G}_m^n$ on $\text{NHilb}^{\underline{d}}(\mathbb{A}^n)$ to the non-associative Hilbert scheme. We prove that $\mathcal{E}_{\text{ass}}$ can be equipped with T -equivariant structure making the section s_{ass} T -equivariant. It follows that our perfect obstruction theory admits a T -equivariant enhancement. We then apply virtual localization and compute the local contributions for each fixed point.

2.5.1 The torus action

Setup 2.5.1. In this section we set $T = (\mathbb{C}^*)^n$ an n -dimensional torus. We follow the notation of section 2.2.3 so that

$$K_T^0(\text{pt}) \cong \mathbb{Z}[t^\mu \mid \mu \in \hat{T}] \cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$$

and

$$A_T^*(T) \cong \text{Sym}_{\mathbb{Z}}^\bullet(\hat{T}) \cong \mathbb{Z}[s_1, \dots, s_n].$$

Explicitly, t_i is the class of a one-dimensional T -representation of weight e_i while s_i is its first Chern class.

Definition 2.5.1. We define a torus action on $M_{n,\underline{d}}$ by

$$\mathbf{t} \cdot (\psi_1, \psi_2, v) = (\psi_1^{\mathbf{t}}, \psi_2, v)$$

where $\psi_1^{\mathbf{t}}$ denotes the map given by

$$(\mathbb{C}^n)^\vee \xrightarrow{\mathbf{t}^{-1}} (\mathbb{C}^n)^\vee \xrightarrow{\psi_1} N_0$$

where T acts on $(\mathbb{C}^n)^\vee$ in the standard way. Clearly this commutes with the $P_{\underline{d}}$ -action and thus induces a T -action on the quotient

$$[M_{n,\underline{d}}/P_{\underline{d}}] = \text{naNHilb}^{\underline{d}}(\mathbb{A}^n).$$

Lemma 2.5.1. *The universal flag bundle $\mathcal{V}_\bullet$ on $\text{naNHilb}^{\underline{d}}(\mathbb{A}^n)$ carries natural T -equivariant structure such that the universal maps $\Psi_1, \Psi_2, \mathfrak{s}$ are equivariant.*

Proof. We simply define $V_\bullet = N_\bullet \otimes \mathcal{O}_{M_{n,\underline{d}}}$ on $M_{n,\underline{d}}$. It admits a $P_{\underline{d}}$ -equivariant structure identifying it with $\mathcal{V}_\bullet$ under

$$\text{QCoh}([M_{n,\underline{d}}/P_{\underline{d}}]) \simeq \text{QCoh}^{P_{\underline{d}}}(M_{n,\underline{d}}).$$

Equipping $V_\bullet$ with the fiberwise trivial T -equivariant structure induces the desired T -equivariant structure on $\mathcal{V}_\bullet$. $\square$

Corollary 2.5.1. *The perfect obstruction theory $\mathbb{E}_{\text{ass}} \rightarrow \mathbb{L}_{\text{NHilb}^{\underline{d}}(\mathbb{A}^n)}$ admits a T -equivariant refinement.*

Proof. This follows since the T -equivariant structure on $\mathcal{V}_\bullet$ naturally equips $\mathcal{E}_{\text{ass}}$ with T -equivariant structure. The section s_{ass} is then T -equivariant since it is built from Ψ_2 . $\square$

2.5.2 Weight calculations

We will calculate

$$T_{\text{naNHilb}^{\underline{d}}(\mathbb{A}^n), p}$$

and

$$\mathcal{E}_{\text{ass}, p}$$

as elements of $K_0^T(\text{pt})$ for $p \in \text{NHilb}^{\underline{d}}(\mathbb{A}^n)^T$ a fixed point. We recall that the fixed locus $\text{NHilb}^{\underline{d}}(\mathbb{A}^n)^T$ is 0-dimensional, reduced, and proper. Its points are in bijection with flags of monomial ideals, which is again in an order reversing bijection with flags of $(n+1)$ -dimensional partitions.

Setup 2.5.2. Let $\lambda_\bullet = (\lambda_1 \subset \lambda_2 \subset \cdots \subset \lambda_{r+1})$ be a flag of $(n+1)$ -dimensional partitions (i.e. $\lambda_i \subset (\mathbb{Z}_{\geq 0})^n$) with $|\lambda_i| - |\lambda_{i-1}| = d_{i-1}$. Let

$$I_{\lambda_{r+1}} \subset \cdots \subset I_{\lambda_1} \subset \mathbb{C}[x_1, \dots, x_n]$$

be the associated flag of monomial ideals. By abuse of notation, we will denote by

$$\lambda_\bullet \in \mathrm{NHilb}^d(\mathbb{A}^n)^T \hookrightarrow \mathrm{naNHilb}^d(\mathbb{A}^n)^T$$

the associated T -fixed point. We denote by $N_i^\lambda \in K_T^0(\mathrm{pt})$ the K -theory class of $I_{\lambda_i}/I_{\lambda_{r+1}}$ where we set $I_{\lambda_0} = \mathbb{C}[x_1, \dots, x_n]$. We note that

$$N_0^\lambda = \mathcal{O}_{\mathbb{A}^n, I_{\lambda_\bullet}}^{[d]}$$

where the right-hand side is the stalk of the tautological bundle of the structure sheaf.

The tangent space

Proposition 2.5.1. *Let $\underline{d} \in (\mathbb{Z}_{\geq 0})^{r+1}$, $n \in \mathbb{N}$ and let $\lambda_\bullet \in \mathrm{NHilb}^d(\mathbb{A}^n)^T$. We have an equality*

$$T_{\mathrm{naNHilb}^d(\mathbb{A}^n), \lambda_\bullet} = (t_1 + t_2 + \cdots + t_n + 1)N_0^\lambda + \mathrm{Hom}^{\mathrm{fil}}(\mathrm{Sym}^2 N_\bullet^\lambda, N_\bullet^\lambda) - 2\mathrm{End}^{\mathrm{fil}}(N_\bullet^\lambda)$$

in $K_T^0(\mathrm{pt})$. Here, the superscript fil denotes filtration preserving maps and the filtration structure on $\mathrm{Sym}^2 N_\bullet^\lambda$ is given by letting the i 'th part be the image of $N_0^\lambda \otimes N_i^\lambda$.

Proof. Since $M_{n, \underline{d}} \rightarrow \mathrm{naNHilb}^d(\mathbb{A}^n)$ is a T -equivariant principal $P_{\underline{d}}$ -bundle we have an equality

$$T_{\mathrm{naNHilb}^d(\mathbb{A}^n), \lambda_\bullet} = T_{M_{n, \underline{d}}, p} - \mathfrak{p}_{\underline{d}}$$

for a choice of lift $p \in M_{n, \underline{d}}$ of $\lambda_\bullet$. The T -structure on $T_{M_{n, \underline{d}}, p}$ is constructed as follows. We let

$$\rho : T \rightarrow G$$

uniquely defined by $\rho(t).p = t.p$. Twisting the T -action on $M_{n, \underline{d}}$ with $(\rho)^{-1}$ yields a T -action for which p is fixed and thus a T -space structure on the tangent space of p . The T -space structure on $\mathfrak{p}_{\underline{d}}$ is given by the adjoint action precomposed with $(\rho)^{-1}$. We choose the lift p as follows. We can choose a basis (v_u) of N_0 indexed by the $u \in \lambda_{r+1}$ such that $v_u \in N_i$ if and only if $u \notin \lambda_i$. We define

$$\psi_2(v_u \otimes v_{u'}) = v_{u+u'},$$

$$v = v_0$$

and

$$\psi_1(x_i) = \begin{cases} v_{e_i} & \text{if } e_i \in \lambda_{r+1} \\ 0 & \text{else} \end{cases}$$

This defines the lift p . Next, we calculate ρ . From the identity $\rho(t).p = t.p$ we immediately see that

$$\rho(t).v_0 = v_0$$

and for $e_i \in \lambda_{r+1}$ we have

$$\rho(t)v_{e_i} = \rho(t)\psi_1(x_i) = \psi_1^t(x_i) = \psi_1(t_i x_i) = t_i v_{e_i}.$$

Using that $\psi_2^{\rho(t)} = \psi_2$ we further see that

$$\begin{aligned} \rho(t)v_{u+u'} &= \rho(t)\psi_2(v_u \otimes v_{u'}) \\ &= \rho(t)\psi_2((\rho(t)^{-1}\rho(t)v_u) \otimes (\rho(t)^{-1}\rho(t)v_{u'})) \\ &= \psi_2^{\rho(t)}((\rho(t)v_u) \otimes (\rho(t)v_{u'})) \\ &= \psi_2((\rho(t)v_u) \otimes (\rho(t)v_{u'})). \end{aligned}$$

Combining these facts an induction argument shows that $\rho(t)v_u = t^u v_u$ i.e., the T -space structure on N_i induced from ρ^{-1} is isomorphic to N_i^λ . From this we immediately get that

$$\mathfrak{p}_d = \text{End}^{\text{fil}}(N_\bullet^\lambda) \in K_T^0(\text{pt})$$

when $\mathfrak{p}_d$ is equipped with the adjoint action composed with ρ^{-1} . The variety $M_{n,d}$ is a Zariski open of the affine sub-space

$$W = f^{-1}(\text{id}_{N_0}) \subset V := \text{Hom}((\mathbb{C}^n)^\vee, N_0) \oplus \text{Hom}^{\text{fil}}(\text{Sym}^2 N_0, N_0) \oplus N_0$$

where

$$\begin{aligned} f : V &\longrightarrow \text{End}^{\text{fil}}(N_\bullet) \\ (\psi_1, \psi_2, v) &\longmapsto (x \mapsto \psi_2(v, x)). \end{aligned}$$

This map is an equivariant submersion when we identify $N_\bullet$ with $N_\bullet^\lambda$. It follows that

$$T_{M_{n,d},p} = T_{W,p} = T_{V,p} - T_{\text{End}^{\text{fil}}(N_\bullet),\text{id}} = T_{V,p} - \text{End}^{\text{fil}}(N_\bullet^\lambda)$$

from which the statement now immediately follows since

$$T_{V,p} = (t_1 + t_2 + \cdots + t_n)N_0^\lambda + \text{Hom}^{\text{fil}}(\text{Sym}^2 N_\bullet^\lambda, N_\bullet^\lambda) + N_0^\lambda.$$

□

To further compute this, we introduce notation to keep track of the equivariant weights.

Setup 2.5.3. Let

$$\lambda_\bullet = (\lambda_1 \subset \lambda_2 \subset \cdots \subset \lambda_{r+1} \subset (\mathbb{Z}_{\geq 0})^n)$$

be an n -dimensional nested partition of length $\underline{d} = (d_0, \dots, d_r)$ and set $d = d_0 + \cdots + d_r$. In the following we choose an enumeration

$$\mathbf{u}_0, \mathbf{u}_2, \dots, \mathbf{u}_{d-1} \in (\mathbb{Z}_{\geq 0}^n)$$

of the partition λ_{r+1} in such a way that

$$\{\mathbf{u}_0, \dots, \mathbf{u}_k\}$$

is a partition for all $0 \leq k \leq d-1$ and that

$$\{\mathbf{u}_0, \dots, \mathbf{u}_{d_0+\dots+d_{i-1}-1}\} = \lambda_i.$$

For $0 \leq k \leq d-1$ we define $w(k) = i$ if

$$d_0 + \cdots + d_{i-2} - 1 < k \leq d_0 + \cdots + d_{i-1} - 1,$$

i.e.

$$(w(0), w(1), \dots, w(d-1)) = (\underbrace{0, 0, \dots, 0}_{d_0 \text{ times}}, \underbrace{1, 1, \dots, 1}_{d_1 \text{ times}}, \dots, \underbrace{r, r, \dots, r}_{d_r \text{ times}}).$$

It follows that $x^{\mathbf{u}_k} \in N_\bullet^\lambda$ is in N_i^λ if and only if $w(k) \geq i$. Note also that

$$N_0 = \text{span}(x^{\mathbf{u}_k} \mid 0 \leq k \leq d-1)$$

with $x^{\mathbf{u}_k}$ a T -eigenvector of weight $t^{-\mathbf{u}_k}$.

Lemma 2.5.2. *Let $\lambda_\bullet$ be an n -dimensional nested partition with enumeration $\mathbf{u}_0, \dots, \mathbf{u}_{d-1}$ as in setup 2.5.3. Then the trace of $T_{\text{naHilb}^d(\mathbb{A}^n), \lambda_\bullet}$ is given by*

$$(t_1 + t_2 + \dots + t_n) \sum_{k=0}^{d-1} t^{-\mathbf{u}_k} + \sum_{1 \leq i \leq j \leq d-1} \sum_{\substack{0 \leq k \leq d-1 \\ w(j) \leq w(k)}} t^{\mathbf{u}_i + \mathbf{u}_j - \mathbf{u}_k} - \sum_{j=1}^{d-1} \sum_{\substack{0 \leq k \leq d-1 \\ w(j) \leq w(k)}} t^{\mathbf{u}_j - \mathbf{u}_k}.$$

Proof. We have

$$T_{\text{naHilb}^d(\mathbb{A}^n), \lambda_\bullet} = (t_1 + t_2 + \dots + t_n + 1) N_0^\lambda + \text{Hom}^{\text{fil}}(\text{Sym}^2 N_\bullet^\lambda, N_\bullet^\lambda) - 2 \text{End}^{\text{fil}}(N_\bullet^\lambda)$$

so, using the T -eigenvector basis $x^{\mathbf{u}_1}, \dots, x^{\mathbf{u}_{d-1}}$ we see that the trace identifies with

$$(t_1 + t_2 + \dots + t_n + 1) \sum_{k=0}^{d-1} t^{-\mathbf{u}_k} + \sum_{0 \leq i \leq j \leq d-1} \sum_{\substack{0 \leq k \leq d-1 \\ w(j) \leq w(k)}} t^{\mathbf{u}_i + \mathbf{u}_j - \mathbf{u}_k} - 2 \sum_{\substack{0 \leq j, k \leq d-1 \\ w(j) \leq w(k)}} t^{\mathbf{u}_j - \mathbf{u}_k}.$$

We see some immediate cancellation. The " $i = 0$ " terms of the second sum cancels with one copy of the third sum. The last copy of the first sum corresponding to the coefficient 1 cancels with the " $j = 0$ " terms of the third sum. This leaves us with

$$(t_1 + t_2 + \dots + t_n) \sum_{k=0}^{d-1} t^{-\mathbf{u}_k} + \sum_{1 \leq i \leq j \leq d-1} \sum_{\substack{0 \leq k \leq d-1 \\ w(j) \leq w(k)}} t^{\mathbf{u}_i + \mathbf{u}_j - \mathbf{u}_k} - \sum_{j=1}^{d-1} \sum_{\substack{0 \leq k \leq d-1 \\ w(j) \leq w(k)}} t^{\mathbf{u}_j - \mathbf{u}_k}.$$

□

Remark 2.5.1. From here on the cancellation gets trickier. A term of the form $t^{\mathbf{u}_j - \mathbf{u}_k}$ cancels with either a term in the first or second sum depending on $\mathbf{u}_j \in (\mathbb{Z}_{\geq 0})^n$. If $\mathbf{u}_j = e_i$ is a unit vector, it will cancel with $t_i t^{-\mathbf{u}_k}$ from the first sum. If it is not a unit vector, we can rewrite $\mathbf{u}_j = \mathbf{u}_{i_1} + \mathbf{u}_{i_2}$ for $1 \leq i_1 \leq i_2 \leq j$. Since $i_2 \leq j$ we have $w(i_2) \leq w(j) \leq w(k)$ and so the term cancels with $t^{\mathbf{u}_{i_1} + \mathbf{u}_{i_2} - \mathbf{u}_k}$ from the second sum. Another way to keep track of only the positive weights is via the following multiset definition.

Lemma 2.5.3. *For a nested partition*

$$\lambda_\bullet = (\lambda_1 \subset \lambda_2 \subset \dots \subset \lambda_{r+1})$$

of length d we recursively define multisets

$$S_m^T(\lambda_\bullet)$$

for $-1 \leq m \leq r$ as follows. Choose an enumeration $\mathbf{u}_0, \dots, \mathbf{u}_{d-1}$ as in setup 2.5.3. We set

$$S_{-1}^T(\lambda_\bullet) = \{e_1, \dots, e_n\}$$

and recursively define

$$S_m^T(\lambda_\bullet) = (S_{m-1}^T(\lambda_\bullet) \cup \{\mathbf{u}_i + \mathbf{u}_j \mid 1 \leq i \leq j, w(j) = m\}) \setminus \{\mathbf{u}_j \mid 1 \leq j, w(j) = m\}.$$

Then the trace of $T_{\text{naHilb}^d(\mathbb{A}^n), \lambda_\bullet}$ is given by

$$\sum_{m=0}^r \sum_{\mathbf{v} \in \lambda_{m+1} \setminus \lambda_m} \sum_{\mathbf{u} \in S_m^T(\lambda_\bullet)} t^{\mathbf{u} - \mathbf{v}}.$$

Proof. One easily sees by induction that

$$S_k^T(\lambda_\bullet) = (\{e_1, \dots, e_n\} \cup \{\mathbf{u}_i + \mathbf{u}_j \mid 1 \leq i \leq j, w(j) \leq k\}) \setminus \{\mathbf{u}_j \mid w(j) \leq k\}$$

and since $\mathbf{u}_k \in \lambda_{m+1} \setminus \lambda_m$ if and only if $w(k) = m$ the result now easily follows from the above calculation. $\square$

Lastly, we compute the rank of the fixed part.

Corollary 2.5.2. *Let $\lambda_\bullet$ be an n -dimensional nested partition with enumeration $\mathbf{u}_0, \dots, \mathbf{u}_{d-1}$ as in setup 2.5.3. Define*

$$W_T(\mathbf{u}_k) = \begin{cases} 0 & \mathbf{u}_k = e_i \text{ for some } i = 1, \dots, n \\ \#\{(i, j) \mid 1 \leq i \leq j \leq d-1, \mathbf{u}_i + \mathbf{u}_j = \mathbf{u}_k\} - 1 & \text{else} \end{cases}$$

for $k = 1, \dots, d-1$ and e_i the i 'th unit vector. Then the trace of $T_{\text{naNHilb}^d(\mathbb{A}^n), \lambda_\bullet}^{\text{fix}}$ is

$$\sum_{k=1}^{d-1} W_T(\mathbf{u}_k).$$

Proof. The positive trivial weights of

$$(t_1 + t_2 + \dots + t_n) \sum_{k=0}^{d-1} t^{-\mathbf{u}_k} + \sum_{1 \leq i \leq j \leq d-1} \sum_{\substack{0 \leq k \leq d-1 \\ w(j) \leq w(k)}} t^{\mathbf{u}_i + \mathbf{u}_j - \mathbf{u}_k} - \sum_{j=1}^{d-1} \sum_{\substack{0 \leq k \leq d-1 \\ w(j) \leq w(k)}} t^{\mathbf{u}_j - \mathbf{u}_k}.$$

are exactly of the form $t_i t^{-\mathbf{u}_k}$ for $\mathbf{u}_k = e_i$ or $t^{\mathbf{u}_i + \mathbf{u}_j - \mathbf{u}_k}$ for $\mathbf{u}_i + \mathbf{u}_j = \mathbf{u}_k$, $i \leq j$ and $w(j) \leq w(k)$. We note that since $\mathbf{u}_j \leq \mathbf{u}_k$ we have $j \leq k$ so the condition $w(j) \leq w(k)$ is redundant. For each $1 \leq k$ we have exactly one negative trivial weight of the form $-t^{\mathbf{u}_k - \mathbf{u}_k}$ from the third sum. It follows that the trace of $T_{\text{naNHilb}^d(\mathbb{A}^n), \lambda_\bullet}^{\text{fix}}$ is given by

$$\sum_{k=1}^{d-1} Q_k(t)$$

where

$$Q_k(t) = t_i t^{-\mathbf{u}_k} - t^{\mathbf{u}_k - \mathbf{u}_k} = 0$$

if $\mathbf{u}_k = e_i$ for some $i = 1, \dots, n$ and

$$Q_k(t) = \sum_{\substack{1 \leq i \leq j \leq d-1 \\ \mathbf{u}_i + \mathbf{u}_j = \mathbf{u}_k}} t^{\mathbf{u}_i + \mathbf{u}_j - \mathbf{u}_k} - t^{\mathbf{u}_k - \mathbf{u}_k}$$

else. It is clear that $Q_k(t) = W_T(\mathbf{u}_k)$ from which the statement follows directly $\square$

Corollary 2.5.3. *The fixed part of the tangent space $T_{\text{naNHilb}^d(\mathbb{A}^n), \lambda_\bullet}^{\text{fix}}$ depends only on λ_{r+1} and not on the chosen nesting.*

Proof. The above formula only depends on a choice of enumeration $\mathbf{u}_0, \dots, \mathbf{u}_{d-1}$. However, the allowable choices of enumerations does depend on the nesting, so we should check that the formula is independent of the enumeration. Clearly the number of unit vectors is independent of the enumeration and for $u = \mathbf{u}_k \in \lambda_{r+1}$ not a unit vector, we see that

$$\#\{(i, j) \mid 1 \leq i \leq j \leq d-1, \mathbf{u}_i + \mathbf{u}_j = \mathbf{u}_k\} = \#\{\{v, w\} \mid v, w \in \lambda_{r+1}, v + w = u\}$$

which is clearly independent of the choice of enumeration. $\square$

The associativity bundle

We calculate the equivariant K-theory of $\mathcal{E}_{\text{ass}, \lambda_\bullet}$ and $\mathcal{E}_{\text{ass}, \lambda_\bullet}^{\text{fix}}$.

Lemma 2.5.4. *Let $\underline{d} \in (\mathbb{Z}_{\geq 0})^{r+1}$, $n \in \mathbb{N}$ and let $\lambda_\bullet \in \text{NHilb}^{\underline{d}}(\mathbb{A}^n)^T$. Let $\mathcal{V}_\bullet$ be the universal flag on $\text{naNHilb}^{\underline{d}}(\mathbb{A}^n)$. We have an equality*

$$\mathcal{V}_{\bullet, \lambda_\bullet} = N_\bullet^\lambda$$

in $K_T^0(\text{pt})$.

Proof. Let $U \subset M_{n, \underline{d}}$ be the preimage of $\lambda_\bullet$ under the quotient map. Using the lift $p \in U$ as in the proof of proposition 2.5.1 we get a T -equivariant isomorphism $U \cong P_{\underline{d}}$ where $P_{\underline{d}}$ is equipped with a T -structure coming from ρ . It follows that the induced T -representation on $\mathcal{V}_{\bullet, \lambda_\bullet} \cong N_\bullet^\lambda$ is the one coming from the $P_{\underline{d}}$ action precomposed with ρ^{-1} . This again naturally identifies with $N_\bullet^\lambda$, see the proof of proposition 2.5.1. $\square$

Corollary 2.5.4. *Let $\underline{d} \in (\mathbb{Z}_{\geq 0})^{r+1}$, $n \in \mathbb{N}$ and let $\lambda_\bullet \in \text{NHilb}^{\underline{d}}(\mathbb{A}^n)^T$. Let*

$$N_\bullet^{+, \lambda} = N_\bullet^\lambda / (1 \cdot \mathbb{C})$$

for $1 \in N_\bullet^\lambda$ the image of $1 \in \mathbb{C}[x_1, \dots, x_n]$. We have an identification

$$\mathcal{E}_{\text{ass}, \lambda_\bullet} = \text{Hom}^{\text{fil, alt}}(N_\bullet^{+, \lambda} \otimes N_\bullet^{+, \lambda} \otimes N_\bullet^{+, \lambda}, N_\bullet^\lambda).$$

where the right-hand side is trilinear maps which are alternating in the outer coordinates and takes $N_i \otimes N_j \otimes N_k$ to $N_{\max\{i, j, k\}}$. In particular, if we choose an enumeration $\mathbf{u}_0, \dots, \mathbf{u}_{d-1}$ as in setup 2.5.3, then the trace of $\mathcal{E}_{\text{ass}, \lambda_\bullet}$ is given by

$$\sum_{\substack{1 \leq i, j, k \leq d-1 \\ i < k}} \sum_{\substack{0 \leq m \leq d-1 \\ w(k), w(j) \leq w(m)}} t^{\mathbf{u}_i + \mathbf{u}_j + \mathbf{u}_k - \mathbf{u}_m}.$$

Again, it will be convenient for later notation to have a definition similar to lemma 2.5.3.

Lemma 2.5.5. *We define for $0 \leq m \leq r$ the set $S_m^{\text{ass}}(\lambda_\bullet)$ as follows. Picking an enumeration $\mathbf{u}_0, \dots, \mathbf{u}_{d-1}$ we define*

$$S_m^{\text{ass}}(\lambda_\bullet) = \{\mathbf{u}_i + \mathbf{u}_j + \mathbf{u}_k \mid 1 \leq i, j, k \leq d-1, i < k, w(k), w(j) \leq w(m)\}.$$

Then the trace of $\mathcal{E}_{\text{ass}, \lambda_\bullet}$ is given by

$$\sum_{m=0}^r \sum_{\mathbf{v} \in \lambda_{m+1} \setminus \lambda_m} \sum_{\mathbf{u} \in S_m^{\text{ass}}(\lambda_\bullet)} t^{\mathbf{u} - \mathbf{v}}.$$

Corollary 2.5.5. *Let $\lambda_\bullet$ be an n -dimensional nested partition with enumeration $\mathbf{u}_0, \dots, \mathbf{u}_{d-1}$ as in setup 2.5.3. Define*

$$W_B(\mathbf{u}_m) = \#\{(i, j, k) \in \{1, \dots, d-1\}^3 \mid i < k, \mathbf{u}_i + \mathbf{u}_j + \mathbf{u}_k = \mathbf{u}_m\}$$

for $1 \leq m \leq d-1$. Then the trace of $\mathcal{E}_{\text{ass}, \lambda_\bullet}^{\text{fix}}$ is

$$\sum_{m=1}^{d-1} W_B(\mathbf{u}_m)$$

Proof. This follows almost directly from the above formula. The only thing we need to notice is that if $\mathbf{u}_i + \mathbf{u}_j + \mathbf{u}_k = \mathbf{u}_m$ for $1 \leq i, j, k \leq d-1$ then $\mathbf{u}_j, \mathbf{u}_k \leq \mathbf{u}_m$ so $i, k \leq m$ hence it already holds that $w(j), w(k) \leq w(m)$. $\square$

Corollary 2.5.6. *The fixed part of the associativity bundle $\mathcal{E}_{\text{ass}, \lambda_\bullet}^{\text{fix}}$ at nested partition $\lambda_\bullet$ depends only on λ_{r+1} and not on the nesting.*

Proof. For any enumeration $\mathbf{u}_1, \dots, \mathbf{u}_{d-1}$ we see for $u = \mathbf{u}_m$ that

$$\begin{aligned} W_B(\mathbf{u}_m) &= \#\{(i, j, k) \in \{1, \dots, d-1\}^3 \mid i < k, \mathbf{u}_i + \mathbf{u}_j + \mathbf{u}_k = \mathbf{u}_m\} \\ &= \#\{(v, s, w) \in \lambda_{r+1} \times \lambda_{r+1} \times \lambda_{r+1} \mid v \neq w, v + s + w = u\}/2 \end{aligned}$$

and the last expression clearly only depends on λ_{r+1} . $\square$

2.5.3 The fixed virtual class

The T -equivariant perfect obstruction theory $\mathbb{E}$ induces a virtual class on the fixed locus $\text{NHilb}^d(\mathbb{A}^n)^T$. Since the latter is zero dimensional, reduced and proper we are in the situation of example 2.2.2. With that in mind we make the following definition.

Definition 2.5.2. Let $\lambda_\bullet$ be a flag of partitions corresponding to a fixed point $\lambda_\bullet \in \text{NHilb}^d(\mathbb{A}^n)$. We will say that $\lambda_\bullet$ is admissible if $[\lambda_\bullet]^{\text{vir}} \neq [0]$ i.e., if

$$\text{rk } \mathcal{E}_{\text{ass}, [\lambda_\bullet]}^{\text{fix}} = \text{rk } T_{\text{naNHilb}^d(\mathbb{A}^n), [\lambda_\bullet]}^{\text{fix}}.$$

We recall that if we choose an enumeration $\mathbf{u}_0, \dots, \mathbf{u}_{d-1}$ of $\lambda_\bullet$ as in setup 2.5.3 then we have

$$\text{rk } \mathcal{E}_{\text{ass}, \lambda_\bullet}^{\text{fix}} = \sum_{m=0}^{d-1} W_B(\mathbf{u}_m) \text{ and } \text{rk } T_{\text{naNHilb}^d(\mathbb{A}^n), \lambda_\bullet}^{\text{fix}} = \sum_{m=0}^{d-1} W_T(\mathbf{u}_m)$$

where

$$W_T(\mathbf{u}_m) = \begin{cases} 0 & \mathbf{u}_m = e_i \text{ for some } i = 1, \dots, n \\ \#\{(i, j) \mid 1 \leq i \leq j \leq d-1, \mathbf{u}_i + \mathbf{u}_j = \mathbf{u}_m\} - 1 & \text{else} \end{cases}$$

and

$$W_B(\mathbf{u}_m) = \#\{(i, j, k) \in \{1, \dots, d-1\}^3 \mid i < k, \mathbf{u}_i + \mathbf{u}_j + \mathbf{u}_k = \mathbf{u}_m\}.$$

We also note that admissibility of $\lambda_\bullet$ only depends on λ_{r+1} .

Lemma 2.5.6. *Let $\lambda_\bullet$ be an nested partition with chosen enumeration $\mathbf{u}_0, \dots, \mathbf{u}_{d-1}$ as in setup 2.5.3. Then*

$$W_T(\mathbf{u}_m) \leq W_B(\mathbf{u}_m)$$

for $m = 1, \dots, d-1$.

Proof. We can assume that $\mathbf{u}_m$ is not a unit vector, since in that case we simply get $0 \leq 0$. For each i let

$$\phi(i) = \min \{j \mid \mathbf{u}_j \text{ is a unit vector and } \mathbf{u}_j \leq \mathbf{u}_i\}.$$

Note that $\phi(m) < m$ since $\mathbf{u}_m$ is not a unit vector. Let κ be the map $\kappa(\mathbf{u}_i) = i$ be the map extracting the index. It will suffice to prove that

$$\begin{aligned} \{(i, j, k) \mid i < k, \mathbf{u}_i + \mathbf{u}_j + \mathbf{u}_k = \mathbf{u}_m\} &\longrightarrow \{(i, j) \mid \phi(m) < i \leq j, \mathbf{u}_i + \mathbf{u}_j = \mathbf{u}_m\} \\ (i, j, k) &\longmapsto (\min\{\kappa(\mathbf{u}_i + \mathbf{u}_j), k\}, \max\{\kappa(\mathbf{u}_i + \mathbf{u}_j), k\}) \end{aligned}$$

is a surjection. Indeed, the cardinality of the codomain corresponds to

$$\#\{(i, j) \mid i \leq j, \mathbf{u}_i + \mathbf{u}_j = \mathbf{u}_m\} - 1$$

since we have removed the single element of the set corresponding to the pair

$$(\kappa(\mathbf{u}_{\phi(m)}), \kappa(\mathbf{u}_m - \mathbf{u}_{\phi(m)})).$$

To show surjectivity we let $\mathbf{u}_i + \mathbf{u}_j = \mathbf{u}_m$ with $\phi(m) < i \leq j$. We must have either $\mathbf{u}_{\phi(m)} \leq \mathbf{u}_i$ or $\mathbf{u}_{\phi(m)} \leq \mathbf{u}_j$. In the first case we let $\mathbf{u}_k = \mathbf{u}_i - \mathbf{u}_{\phi(m)}$, then clearly (i, j) is the image of $(\phi(m), k, j)$. Similarly in the second case we can just take $\mathbf{u}_k = \mathbf{u}_j - \mathbf{u}_{\phi(m)}$ then (i, j) is in the image of $(\phi(m), j, i)$. $\square$

Proposition 2.5.2. *Let $\lambda_\bullet$ be a flag of partitions. Then $\lambda_\bullet$ is admissible if and only if the following conditions hold.*

1. For all $1 \leq k \leq n$ and $\alpha > 0$ we have

$$\alpha e_k \in \lambda_{r+1} \implies \alpha \leq 4.$$

2. For all $1 \leq k_1 < k_2 \leq n$ and all $\alpha_1, \alpha_2 > 0$ we have

$$\alpha_1 e_{k_1} + \alpha_2 e_{k_2} \in \lambda_{r+1} \implies \alpha_1 + \alpha_2 \leq 3.$$

3. For all $m \geq 3$ and $1 \leq k_1 < k_2 < \dots < k_m \leq m$ and all $\alpha_1, \dots, \alpha_m > 0$ we have

$$\sum_{i=1}^m \alpha_i e_{k_i} \notin \lambda_{r+1}.$$

Proof. Choose an enumeration $\mathbf{u}_0, \dots, \mathbf{u}_{d-1}$ as in setup 2.5.3. One can then explicitly check that for

$$\mathbf{u}_m \in \{\alpha e_k, \alpha_1 e_{k_1} + \alpha_2 e_{k_2} \mid 1 \leq k, k_1, k_2 \leq n, \alpha \leq 4, \alpha_1 + \alpha_2 \leq 3\}$$

we have $W_T(\mathbf{u}_m) = W_B(\mathbf{u}_m)$. This shows that these partitions are admissible. One can further check that

$$\begin{aligned} W_T(5e_k) &< W_B(5e_k), \\ W_T(\alpha_1 e_{k_1} + \alpha_2 e_{k_2}) &< W_B(\alpha_1 e_{k_1} + \alpha_2 e_{k_2}), \\ W_T(e_{k_1} + e_{k_2} + e_{k_3}) &< W_B(e_{k_1} + e_{k_2} + e_{k_3}) \end{aligned}$$

for $\alpha_1 + \alpha_2 = 4$. It follows that any partition not contained in

$$\{\alpha e_k, \alpha_1 e_{k_1} + \alpha_2 e_{k_2} \mid 1 \leq k, k_1, k_2 \leq n, \alpha \leq 4, \alpha_1 + \alpha_2 \leq 4\}$$

is non-admissible. $\square$

Remark 2.5.2. The theorem states that an admissible nested partition is admissible if and only if it does not contain any vector with 3 or more non-zero coordinates and for every choice of two coordinates the projected partition must be contained in the boxes corresponding to figure 2 below.

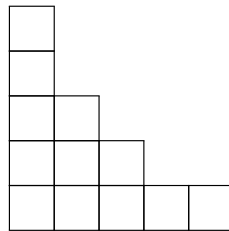


Figure 2.2

2.5.4 The localization formula

We now apply the virtual localization formula of [GP99]. As explained in example 2.2.2 the formula yields

$$[\mathrm{NHilb}^d(\mathbb{A}^n)]^{\mathrm{vir}} = \sum_{\lambda_\bullet} [\lambda_\bullet] \cap \frac{e^T(\mathcal{E}_{\mathrm{ass}, \lambda_\bullet})}{e^T(T_{\mathrm{naNHilb}^d(\mathbb{A}^n), \lambda_\bullet})}$$

where the sum is over the admissible partitions. The above calculations now immediately give us the formula below.

Remark 2.5.3. We use the following notation

$$\widetilde{\prod}_{\gamma \in \Gamma} \gamma = \prod_{0 \neq \gamma \in \Gamma} \gamma.$$

for non-zero products in the formulas to come. Note that if V is a T -representation with weights $\mathrm{Wt}(V) \subset \mathbb{Z}^n \cong \hat{T}$ then

$$e^T(V^{\mathrm{mov}}) = \widetilde{\prod}_{\mathbf{u} \in \mathrm{Wt}(V)} \mathbf{u}(\underline{s}) \in A_T^*(\mathrm{pt}) \cong \mathbb{Z}[s_1, \dots, s_n]$$

where we for $\mathbf{u} = (u_1, \dots, u_n)$ define

$$\mathbf{u}(\underline{s}) = u_1 s_1 + \dots + u_n s_n.$$

Theorem 2.5.1. Let $\underline{d} = (d_0, \dots, d_r) \in \mathbb{Z}_{\geq 0}^{r+1}$, $d = d_0 + \dots + d_r$ and $n \in \mathbb{Z}_{\geq 0}$. In the Chow group $A_*^T(\mathrm{NHilb}^d(\mathbb{A}^n))_{\mathrm{loc}}$ localized in linear functions of $s_1, \dots, s_n$ the virtual fundamental class $[\mathrm{NHilb}^d(\mathbb{A}^n)]^{\mathrm{vir}}$ equals

$$\sum_{\lambda_\bullet} [\lambda_\bullet] \cap \frac{\prod_{\substack{1 \leq i, j, k \leq d-1 \\ i < k}} \prod_{\substack{0 \leq m \leq d-1 \\ w(k), w(j) \leq w(m)}} (\mathbf{u}_i + \mathbf{u}_j + \mathbf{u}_k - \mathbf{u}_m)(\underline{s}) \prod_{j=1}^{d-1} \prod_{\substack{0 \leq k \leq d-1 \\ w(j) \leq w(k)}} \mathbf{u}_j - \mathbf{u}_k(\underline{s})}{\prod_{i=1}^n \prod_{k=0}^{d-1} (e_i - \mathbf{u}_k)(\underline{s}) \prod_{1 \leq i \leq j \leq d-1} \prod_{\substack{0 \leq k \leq d-1 \\ w(j) \leq w(k)}} (\mathbf{u}_i + \mathbf{u}_j - \mathbf{u}_k)(\underline{s})}.$$

Here the sum is over all admissible $(n-1)$ -dimensional nested partitions $\lambda_\bullet$ of length $\underline{d}$ and where we for each $\lambda_\bullet$ have a chosen enumeration $\mathbf{u}_0, \dots, \mathbf{u}_{d-1}$ as in setup 2.5.3. Alternatively, using the multisets of lemma 2.5.3 and lemma 2.5.5 we have

$$[\mathrm{NHilb}^d(\mathbb{A}^n)]^{\mathrm{vir}} = \sum_{\lambda_\bullet} [x_{\lambda_\bullet}] \cap \prod_{m=0}^r \prod_{\mathbf{v} \in \lambda_{m+1} \setminus \lambda_m} \frac{\prod_{\substack{\mathbf{u} \in S_m^{\mathrm{ass}}(\lambda_\bullet) \\ \mathbf{u} \neq \mathbf{v}}} (\mathbf{u} - \mathbf{v})(\underline{s})}{\prod_{\substack{\mathbf{u} \in S_m^T(\lambda_\bullet) \\ \mathbf{u} \neq \mathbf{v}}} (\mathbf{u} - \mathbf{v})(\underline{s})}$$

where the sum is again over admissible nested partitions $\lambda_\bullet$ of length $\underline{d}$.

Example 2.5.1. Let us consider the (non-nested) Hilbert scheme $\mathrm{Hilb}^{d+1}(\mathbb{A}^3)$ of d points on $\mathbb{A}^n$. Then the formula reads

$$\sum_{\lambda_\bullet} [x_{\lambda_\bullet}] \cap \frac{\prod_{\substack{1 \leq i, j, k \leq d-1 \\ i < k}} \prod_{m=0}^{d-1} (\mathbf{u}_i + \mathbf{u}_j + \mathbf{u}_k - \mathbf{u}_m)(\underline{s}) \prod_{j=1}^{d-1} \prod_{k=0}^{d-1} (\mathbf{u}_j - \mathbf{u}_k)(\underline{s})}{\prod_{i=1}^n \prod_{k=0}^{d-1} (e_i - \mathbf{u}_k)(\underline{s}) \prod_{1 \leq i \leq j \leq d-1} \prod_{k=0}^{d-1} (\mathbf{u}_i + \mathbf{u}_j - \mathbf{u}_k)(\underline{s})}$$

Let us calculate this explicitly for $d = 3$, i.e., the Hilbert scheme of three points. We have two types of partitions. They are either given by $\{0, e_i, 2e_i\}$ or $\{0, e_i, e_j\}$ for $i \neq j$ and are both admissible. For the first type of partition, we get a contribution of

$$\frac{4s_i 3s_i 2s_i 5s_i 4s_i 3s_i s_i (-s_i) 2s_i s_i}{2s_i s_i 3s_i 2s_i s_i 4s_i 3s_i 2s_i \prod_{k=1}^n s_k (s_k - 2s_i) \prod_{k \neq i} (s_k - s_i)}$$

which simplifies to

$$\frac{-10s_i^2}{\prod_{k=1}^n s_k(s_k - 2s_i) \prod_{k \neq i} (s_k - s_i)} = \frac{10}{\prod_{k \neq i} s_k(s_k - s_i)(s_k - 2s_i)}$$

For the other type of partition, we get

$$\frac{(2s_i + s_j)(s_i + s_j)2s_i(s_i + 2s_j)2s_j(s_i + s_j)s_i(s_i - s_j)s_j(s_j - s_i)}{2s_i s_i(2s_i - s_j)(s_i + s_j)s_j s_i 2s_j(2s_j - s_i)s_j \prod_{k=1}^n s_k \prod_{k \neq i} (s_k - s_i) \prod_{k \neq j} (s_k - s_j)}$$

which can be reduced to

$$\frac{(2s_i + s_j)(s_i + 2s_j)(s_i + s_j)}{s_i(2s_i - s_j)s_j(2s_j - s_i) \prod_{k=1}^n s_k \prod_{k \neq i, j} (s_k - s_i)(s_k - s_j)}.$$

We directly see that the virtual dimension is given by $3n - 3$. Let us specialize to $n = 3$ with a virtual dimension of 6. Consider the tautological bundle $\mathcal{O}_{\mathbb{A}^3}^{[3]}$ and let $c_2 = c_2^T((\mathcal{O}_{\mathbb{A}^3}^{[3]})^\vee)$ the second equivariant Chern class of the dual bundle. Let us integrate c_2^3 against the virtual class. For $\lambda = \{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_3\}$ the Chern roots of $(\mathcal{O}_{\mathbb{A}^3}^{[3]})^\vee$ are $\mathbf{u}_0(\underline{s}), \mathbf{u}_1(\underline{s}), \mathbf{u}_3(\underline{s})$. It follows that for $\lambda = \{0, e_i, 2e_i\}$ we have

$$c_2^3|_\lambda = (2s_i^2)^3 = 8s_i^6$$

while for $\lambda = \{0, e_i, e_j\}$ we have

$$c_2|_\lambda = (s_i s_j)^3 = s_i^3 s_j^3.$$

For $\{i, j, k\} = \{1, 2, 3\}$ the contribution at the partition $\{0, e_i, 2e_i\}$ is then

$$\frac{80s_i^6}{s_j(s_j - s_i)(s_j - 2s_i)s_k(s_k - s_i)(s_k - 2s_i)}$$

while the contribution at $\{0, e_i, e_j\}$ is

$$\frac{(2s_i + s_j)(s_i + 2s_j)(s_i + s_j)s_i^3 s_j^3}{s_i^2(2s_i - s_j)s_j^2(2s_j - s_i)s_k(s_k - s_i)(s_k - s_j)} = \frac{(2s_i + s_j)(s_i + 2s_j)(s_i + s_j)s_i s_j}{(2s_i - s_j)(2s_j - s_i)s_k(s_k - s_i)(s_k - s_j)}.$$

Summing over all 3 choices for each of the two types of partitions we get

$$\int_{[\text{Hilb}^3(\mathbb{A}^3)]^{\text{vir}}} c_2^T(\mathcal{O}_{\mathbb{A}^3}^{[3]})^3 = 20 \frac{(s_1 + s_2 + s_3)^3}{s_1 s_2 s_3} - 31 \frac{(s_1 s_2 + s_1 s_3 + s_2 s_3)(s_1 + s_2 + s_3)}{s_1 s_2 s_3} + 11.$$

In particular, if we restrict to the Calabi-Yau torus $T_{\text{CY}} = \{(t_1, t_2, t_3) \in T \mid t_1 t_2 t_3 = 1\}$ the virtual integral evaluates to 11.

2.6 Iterated residue formula

In this section we choose a dimension vector of the form

$$\underline{d} = (1, d_1, \dots, d_r) \in (\mathbb{Z}_{\geq 0})^{r+1}.$$

We let $\hat{d} = (d_1, \dots, d_r) \in (\mathbb{Z}_{\geq 0})^r$ so that $\underline{d} = (1, \hat{d})$. We show how tautological virtual integrals on $\text{NHilb}^{\underline{d}}(\mathbb{A}^n)$ can in some cases be expressed in terms of virtual integrals defined via $\text{naNHilb}_0^{\underline{d}}(\mathbb{A}^n)$. We then develop an iterated residue formula for these latter types of integrals. The key to proving this formula is a partial resolution

$$\widetilde{\text{naNHilb}_0^{\underline{d}}(\mathbb{A}^n)} \rightarrow \text{naNHilb}_0^{\underline{d}}(\mathbb{A}^n)$$

which fibers over the flag variety $\text{Flag}_{\hat{d}}(\mathbb{C}^n)$.

2.6.1 Reducing the integral to the nilpotently filtered locus

Let $(\mathcal{V}_\bullet, \psi_2, s, \psi_1)$ be the universal structure on $\text{naNHilb}^d(\mathbb{A}^n)$. Let $\mathcal{Q}$ be the quotient of $\mathcal{H}om^{\text{fil}}(\text{Sym}^2 \mathcal{V}_1, \mathcal{V}_1)$ by the subsheaf of maps which takes $\mathcal{V}_1 \otimes \mathcal{V}_k$ to $\mathcal{V}_{k+1}$. We let

$$\mathcal{E}_{\text{punct}} = \mathcal{H}om((\mathcal{O}^n)^\vee, \mathcal{V}_0/\mathcal{V}_1) \oplus \mathcal{Q}$$

and note that $\text{naNHilb}_0^d(\mathbb{A}^n) \subset \text{naNHilb}^d(\mathbb{A}^n)$ identifies with the zero locus of a (T -equivariant) section of $\mathcal{E}_{\text{punct}}$. Since $\text{naNHilb}_0^d(\mathbb{A}^n)$ is smooth the embedding is regular with normal bundle $\mathcal{E}_{\text{punct}}|_{\text{naNHilb}_0^d(\mathbb{A}^n)}$. This immediately allows us to compute the tangent space.

Lemma 2.6.1. *Let $\lambda_\bullet$ be a nested partition such that the corresponding fixed point is in $\text{naNHilb}_0^d(\mathbb{A}^n)$. Let $\mathbf{u}_0, \dots, \mathbf{u}_{d-1}$ be an enumeration of $\lambda_\bullet$ in the sense of setup 2.5.3. Then the trace of $T_{\text{naNHilb}_0^d(\mathbb{A}^n), \lambda_\bullet}$ is given by*

$$(t_1 + t_2 + \dots + t_n) \sum_{k=1}^{d-1} t^{-\mathbf{u}_k} + \sum_{1 \leq i \leq j \leq d-1} \sum_{\substack{1 \leq k \leq d-1 \\ w(j) < w(k)}} t^{\mathbf{u}_i + \mathbf{u}_j - \mathbf{u}_k} - \sum_{j=1}^{d-1} \sum_{\substack{1 \leq k \leq d-1 \\ w(j) \leq w(k)}} t^{\mathbf{u}_j - \mathbf{u}_k}.$$

Proof. It is clear that

$$T_{\text{naNHilb}_0^d(\mathbb{A}^n)} = T_{\text{naNHilb}^d(\mathbb{A}^n)} - \mathcal{E}_{\text{punct}}.$$

Since $d_0 = 1$ we see that

$$N_0^\lambda / N_1^\lambda = \text{span}_{\mathbb{C}}(x^{\mathbf{u}_0})$$

which is the trivial T -representation. We conclude

$$\mathcal{H}om((\mathcal{O}^n)^\vee, \mathcal{V}_0/\mathcal{V}_1)|_{\lambda_\bullet} = \sum_{i=1}^n t_i.$$

Furthermore, we get

$$\begin{aligned} \mathcal{Q}_{\lambda_\bullet} &= \sum_{1 \leq i \leq j \leq d-1} \sum_{\substack{1 \leq k \leq d-1 \\ w(j) \leq w(k)}} t^{\mathbf{u}_i + \mathbf{u}_j - \mathbf{u}_k} - \sum_{1 \leq i \leq j \leq d-1} \sum_{\substack{1 \leq k \leq d-1 \\ w(j)+1 \leq w(k)}} t^{\mathbf{u}_i + \mathbf{u}_j - \mathbf{u}_k} \\ &= \sum_{1 \leq i \leq j \leq d-1} \sum_{\substack{1 \leq k \leq d-1 \\ w(j) = w(k)}} t^{\mathbf{u}_i + \mathbf{u}_j - \mathbf{u}_k}. \end{aligned}$$

Recall that for $d_0 = 1$ we have that $T_{\text{naNHilb}^d(\mathbb{A}^n), \lambda_\bullet}$ equals

$$(t_1 + t_2 + \dots + t_n) \sum_{k=0}^{d-1} t^{-\mathbf{u}_k} + \sum_{1 \leq i \leq j \leq d-1} \sum_{\substack{1 \leq k \leq d-1 \\ w(j) \leq w(k)}} t^{\mathbf{u}_i + \mathbf{u}_j - \mathbf{u}_k} - \sum_{j=1}^{d-1} \sum_{\substack{1 \leq k \leq d-1 \\ w(j) \leq w(k)}} t^{\mathbf{u}_j - \mathbf{u}_k}.$$

From this, we see that $T_{\text{naNHilb}^d(\mathbb{A}^n), \lambda_\bullet} - \mathcal{E}_{\text{punct}, \lambda_\bullet}$ is equal to

$$(t_1 + t_2 + \dots + t_n) \sum_{k=1}^{d-1} t^{-\mathbf{u}_k} + \sum_{1 \leq i \leq j \leq d-1} \sum_{\substack{1 \leq k \leq d-1 \\ w(j) < w(k)}} t^{\mathbf{u}_i + \mathbf{u}_j - \mathbf{u}_k} - \sum_{j=1}^{d-1} \sum_{\substack{1 \leq k \leq d-1 \\ w(j) \leq w(k)}} t^{\mathbf{u}_j - \mathbf{u}_k}.$$

□

Let us denote by $\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^{\underline{d}}(\mathbb{A}^n) \subset \mathrm{naNHilb}_0^{\underline{d}}(\mathbb{A}^n)$ the intersection of the punctual locus and associativity locus. Note that if the dimension vector is full, then this agrees with the punctual Hilbert scheme of points, but in general it will be a closed subscheme of the latter. Note that $\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^{\underline{d}}(\mathbb{A}^n)$ is cut out of $\mathrm{naNHilb}_0^{\underline{d}}(\mathbb{A}^n)$ via a section of $\mathcal{E}_{\mathrm{ass}}|_{\mathrm{naNHilb}_0^{\underline{d}}(\mathbb{A}^n)}$. We can therefore equip it with a perfect obstruction theory. Note that we are in the situation of example 2.2.4 and so the inclusion

$$\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^{\underline{d}}(\mathbb{A}^n) \hookrightarrow \mathrm{NHilb}^{\underline{d}}(\mathbb{A}^n)$$

admits a relative perfect obstruction theory from $\mathcal{E}_{\mathrm{punct}}$ compatible with the two previously defined perfect obstruction theories. We naturally want to apply proposition 2.2.3, so we try to describe the fixed-point locus of $\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^{\underline{d}}(\mathbb{A}^n)$. We note that for $\underline{d} = (1, 1, \dots, 1)$ we get the punctual Hilbert scheme, which contains the full fixed-point locus. In general, we get the following.

Lemma 2.6.2. *Let*

$$I_{r+1} \subset \dots \subset I_1 \subset I_0 = \mathbb{C}[x_1, \dots, x_n]$$

be a flag of monomial ideals with corresponding flag of partitions

$$\lambda_1 \subset \lambda_2 \subset \dots \subset \lambda_{r+1} \subset (\mathbb{Z}_{\geq 0})^n.$$

Then the corresponding fixed point of $\mathrm{NHilb}^{\underline{d}}(\mathbb{A}^n)$ factors through $\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^{\underline{d}}(\mathbb{A}^n)$ if and only if for all $k \geq 1$, $\mathbf{u} \in \lambda_{k+1} \setminus \lambda_k$ and $i = 1, \dots, n$ we have

$$\mathbf{u} + e_i \notin \lambda_{k+1}.$$

Proof. Note first that $\dim_{\mathbb{C}} I_k/I_{k+1} = d_k$. In particular since $\dim_{\mathbb{C}} I_0/I_1 = 1$ we must have

$$I_1 = \mathfrak{m} = (x_1, \dots, x_n)$$

so that

$$\lambda_1 = \{\mathbf{0}\}.$$

We note that the corresponding fixed point is in $\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^{\underline{d}}(\mathbb{A}^n)$ if and only if

$$I_1 \cdot I_k \subset I_{k+1}$$

for all k . This means that for all $k \geq 0$ and all $\mathbf{u} \notin \lambda_k$ and $\mathbf{v} \notin \lambda_1$ we have

$$\mathbf{u} + \mathbf{v} \notin \lambda_{k+1}.$$

This statement is trivially true if $\mathbf{u} \notin \lambda_{k+1}$ so we can restrict to $\mathbf{u} \in \lambda_{k+1} \setminus \lambda_k$. Since $\mathbf{v} \notin \lambda_1$ if and only if $\mathbf{v} \neq 0$ we see that it is sufficient to consider $\mathbf{v} = e_i$ for $i = 1, \dots, n$. $\square$

We can now apply proposition 2.2.3 to get the following result.

Proposition 2.6.1. *Let $\alpha \in A_{\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^{\underline{d}}(\mathbb{A}^n), T}^*(\mathrm{NHilb}^{\underline{d}}(\mathbb{A}^n))$ be a local T -equivariant Chow cohomology class supported on the nilpotently filtered locus. Then*

$$\int_{[\mathrm{NHilb}^{\underline{d}}(\mathbb{A}^n)]^{\mathrm{vir}}} \alpha = \int_{[\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^{\underline{d}}(\mathbb{A}^n)]^{\mathrm{vir}}} \frac{\alpha|_{\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^{\underline{d}}(\mathbb{A}^n)}}{e^T(\mathcal{E}_{\mathrm{punct}}|_{\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^{\underline{d}}(\mathbb{A}^n)})}.$$

If $\underline{d} = (1, 1, \dots, 1)$ then

$$j_* \left(\frac{[\mathrm{NHilb}_0^{r+1}(\mathbb{A}^n)]^{\mathrm{vir}}}{e^T(\mathcal{E}_{\mathrm{punct}}|_{\mathrm{NHilb}_0^{r+1}(\mathbb{A}^n)})} \right) = [\mathrm{NHilb}^{r+1}(\mathbb{A}^n)]^{\mathrm{vir}}$$

and in particular for any $\alpha \in A_T^*(\mathrm{NHilb}^{r+1}(\mathbb{A}^n))$ we get

$$\int_{[\mathrm{NHilb}^{r+1}(\mathbb{A}^n)]^{\mathrm{vir}}} \alpha = \int_{[\mathrm{NHilb}_0^{r+1}(\mathbb{A}^n)]^{\mathrm{vir}}} \frac{\alpha|_{\mathrm{NHilb}_0^{r+1}(\mathbb{A}^n)}}{e^T(\mathcal{E}_{\mathrm{punct}}|_{\mathrm{NHilb}_0^{r+1}(\mathbb{A}^n)})}.$$

2.6.2 The partial resolution

We construct a partial resolution given as a birational map

$$\tau : \widetilde{\text{naNHilb}}_0^d(\mathbb{A}^n) \rightarrow \text{naNHilb}_0^d(\mathbb{A}^n)$$

such that the domain fibers over the flag variety

$$\mu : \widetilde{\text{naNHilb}}_0^d(\mathbb{A}^n) \rightarrow \text{Flag}_{\hat{d}}(\mathbb{C}^n).$$

Using Atiyah-Bott localization on the flag variety this allows us to rewrite virtual integrals $\text{NHilb}_{\text{nil-fil}}^d(\mathbb{A}^n)$ as an iterated residue formula depending on virtual integrals on the fibers of μ . We first define the variety.

Definition 2.6.1. Let $\underline{d} = (1, \hat{d}) = (1, d_1, \dots, d_r) \in \mathbb{Z}_{\geq 0}^{r+1}$ and $n \geq d_1 + \dots + d_r$. Let $\Psi_1 : (\mathcal{O}^n)^\vee \rightarrow \mathcal{V}_1$ be the universal map of $\text{naNHilb}_0^d(\mathbb{A}^n)$ and let

$$(\mathcal{O}^n)^\vee \twoheadrightarrow \mathcal{F}_1 \supset \mathcal{F}_2 \supset \dots \supset \mathcal{F}_{r+1} = 0$$

be the dualized universal flag on the flag variety $\text{Flag}_{\hat{d}}(\mathbb{C}^n)$. We let $\widetilde{\text{naNHilb}}_0^d(\mathbb{A}^n)$ be the closed locus of $\text{naNHilb}_0^d(\mathbb{A}^n) \times \text{Flag}_{\hat{d}}(\mathbb{C}^n)$ where Ψ_1 factors through a (necessarily unique) map

$$\mathcal{F}_1 \rightarrow \mathcal{V}_1$$

which respects the flag structures. We denote by

$$\begin{array}{ccc} \widetilde{\text{naNHilb}}_0^d(\mathbb{A}^n) & \xrightarrow{\mu} & \text{Flag}_{\hat{d}}(\mathbb{C}^n) \\ \downarrow \tau & & \\ \text{naNHilb}_0^d(\mathbb{A}^n) & & \end{array}$$

the projections to the non-associative Hilbert scheme and the flag variety.

Lemma 2.6.3. *The map $\mu : \widetilde{\text{naNHilb}}_0^d(\mathbb{A}^n) \rightarrow \text{Flag}_{\hat{d}}(\mathbb{C}^n)$ is smooth.*

Proof. For each $1 \leq k \leq r$ let

$$X_k \subset \text{naNHilb}_0^{(1, d_1, \dots, d_k)}(\mathbb{A}^n) \times \text{Flag}_{\hat{d}}(\mathbb{C}^n)$$

be the pullback of $\widetilde{\text{naNHilb}}_0^{(1, d_1, \dots, d_k)}(\mathbb{A}^n)$ under $1 \times q_k$ where $q_k : \text{Flag}_{\hat{d}}(\mathbb{C}^n) \rightarrow \text{Flag}_{(d_1, \dots, d_k)}(\mathbb{C}^n)$. Projection on the non-associative Hilbert scheme clearly induce maps

$$\widetilde{\text{naNHilb}}_0^d(\mathbb{A}^n) = X_r \rightarrow X_{r-1} \rightarrow \dots \rightarrow X_1.$$

On X_k we have a universal flag

$$(\mathcal{O}^n)^\vee \twoheadrightarrow \mathcal{F}_1 \supset \dots \supset \mathcal{F}_{r+1} = 0$$

from $\text{Flag}_{\hat{d}}(\mathbb{C}^n)$ and a universal flag

$$\mathcal{V}_1 \supset \dots \supset \mathcal{V}_{k+1} = 0$$

with universal maps Ψ_1, Ψ_2 such that

$$\Psi_1 : (\mathcal{O}^n)^\vee \rightarrow \mathcal{V}_1$$

factors through $(\mathcal{O}^n)^\vee \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_1/\mathcal{F}_{k+1}$. If we let $\tilde{\Psi}_1 : \mathcal{F}_1/\mathcal{F}_{k+1} \rightarrow \mathcal{V}_1$ be the induced map then a proof completely analogous to that of proposition 2.4.2 then shows that

$$X_{k+1} \rightarrow X_k$$

identifies with the structure map of the Grassmanian bundle

$$\mathrm{Gr}_{X_k}(d_r, (\ker \tilde{\Psi}_1 \oplus \Psi_2)^\vee).$$

In particular, $X_{k+1} \rightarrow X_k$ is smooth and it will suffice to show that $X_1 \rightarrow \mathrm{Flag}_{\hat{d}}(\mathbb{C}^n)$ is smooth. We note that

$$\mathrm{naNHilb}_0^{(1,d_1)}(\mathbb{A}^n) = \mathrm{Gr}_{\mathrm{pt}}(d_1, \mathbb{C}^n)$$

with universal bundle $\mathcal{V}_1$ coinciding with the dual universal bundle of the Grassmannian and universal maps given by $\Psi_2 = 0$ and $\Psi_1 : (\mathcal{O}^n)^\vee \rightarrow \mathcal{V}_1$ the universal map of the Grassmannian. We see that $X_1 \subset \mathrm{Gr}_{\mathrm{pt}}(d_1, \mathbb{C}^n) \times \mathrm{Flag}_{\hat{d}}(\mathbb{C}^n)$ is now simply the locus where Ψ_1 factors through $\mathcal{F}_1/\mathcal{F}_2$ so that the map

$$X_1 \rightarrow \mathrm{Flag}_{\hat{d}}(\mathbb{C}^n)$$

simply identifies with the structure map of

$$\mathrm{Gr}_{\mathrm{Flag}_{\hat{d}}}(d_1, \mathcal{F}_1/\mathcal{F}_2)$$

which is again smooth. □

Corollary 2.6.1. *The scheme $\widetilde{\mathrm{naNHilb}}_0^d(\mathbb{A}^n)$ is smooth and projective.*

Proof. Smoothness follows from the above. Projectiveness follows since it is a closed subscheme of a product of projective schemes. □

Lemma 2.6.4. *The map $\tau : \widetilde{\mathrm{naNHilb}}_0^d(\mathbb{A}^n) \rightarrow \mathrm{naNHilb}_0^d(\mathbb{A}^n)$ is projective birational.*

Proof. Projectiveness is immediate since it is a morphism of projective schemes. Let $U \subset \mathrm{naNHilb}_0^d(\mathbb{A}^n)$ be the Zariski open subset where Ψ_1 is surjective. Note that it is non-empty, containing for example the Porteous-type points, i.e., monomial ideals where the corresponding partition only contains unit vectors. We construct an inverse of τ on U . Explicitly, we define $U \rightarrow \mathrm{Flag}_{\hat{d}}(\mathbb{C}^n)$ via the flag dual to

$$\Psi_1 : (\mathcal{O}^n)^\vee \rightarrow \mathcal{V}_\bullet$$

and the universal property of the flag variety. □

Definition 2.6.2. We define

$$\widetilde{\mathrm{NHilb}}_{\mathrm{nil}\text{-}\mathrm{fil}}^d(\mathbb{A}^n) := \tau^{-1} \left(\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^d(\mathbb{A}^n) \right).$$

Note that it is equipped with a T -equivariant perfect obstruction theory being exhibited as the zero locus of a section of $\tau^* \mathcal{E}_{\mathrm{ass}}$.

Lemma 2.6.5. *We have*

$$\tau_* [\widetilde{\mathrm{NHilb}}_{\mathrm{nil}\text{-}\mathrm{fil}}^d(\mathbb{A}^n)]^{\mathrm{vir}} = [\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^d(\mathbb{A}^n)]^{\mathrm{vir}}.$$

In particular, for $\alpha \in A_T^*(\mathrm{NHilb}_0^d(\mathbb{A}^n))$ we have

$$\int_{[\mathrm{NHilb}_{\mathrm{nil}\text{-}\mathrm{fil}}^d(\mathbb{A}^n)]^{\mathrm{vir}}} \alpha = \int_{[\widetilde{\mathrm{NHilb}}_{\mathrm{nil}\text{-}\mathrm{fil}}^d(\mathbb{A}^n)]^{\mathrm{vir}}} \tau^* \alpha.$$

Proof. This follows by realizing the virtual fundamental classes as refined Euler classes in the sense of Fulton. Since τ is proper of degree 1 (being birational) the pushforward statement follows directly from [Ful98, Proposition 14.1, (d), (iii)]. The integration statement is now simply the projection formula. □

2.6.3 Localization over the flag variety

The key to proving that tautological virtual integration on $\widetilde{\mathrm{NHilb}}_0^d(\mathbb{A}^n)$ admits an iterated residue formula is to apply Atiyah-Bott localization on the flag variety. We will conduct this procedure in this section and provide fixed points and tangent space calculations which we will need later.

Lemma 2.6.6. *Let $\underline{d} = (1, \hat{d}) = (1, d_1, \dots, d_r) \in \mathbb{Z}_{\geq 0}^{r+1}$, $n \geq 0$ and $d = 1 + d_1 + \dots + d_r$. Let $e_1, \dots, e_n \in \mathbb{C}^n$ be the standard unit vectors. The fixed locus of $\mathrm{Flag}_{\hat{d}}(\mathbb{C}^n)$ is 0-dimensional, reduced, and proper. Letting the symmetric group $\sigma \in S_n$ act on the reference flag*

$$f_{\bullet} = \mathrm{span}(e_1, \dots, e_{d_1}) \subset \mathrm{span}(e_1, \dots, e_{d_1+d_2}) \subset \dots \subset \mathrm{span}(e_1, \dots, e_{d_1+\dots+d_r})$$

by $e_i \mapsto e_{\sigma(i)}$ we obtain a bijection of sets

$$\mathrm{Flag}_{\hat{d}}(\mathbb{C}^n)^T \cong S_n / (S_{d_1} \times \dots \times S_{d_r} \times S_{n-(d-1)}).$$

The T -equivariant Euler class of the tangent space at $\sigma(f_{\bullet})$ is given by

$$e^T(T_{\mathrm{Flag}_{\hat{d}}(\mathbb{C}^n), \sigma(f_{\bullet})}) = \prod_{j=1}^{d-1} \prod_{\substack{i=1 \\ w(i) > w(j)}}^n (s_{\sigma(i)} - s_{\sigma(j)})$$

in $A_*^T(\mathrm{pt}) \cong \mathbb{Z}[s_1, \dots, s_n]$. Here

$$(w(1), \dots, w(n)) = (\underbrace{1, \dots, 1}_{d_1 \text{ times}}, \underbrace{2, \dots, 2}_{d_2 \text{ times}}, \dots, \underbrace{r, \dots, r}_{d_r \text{ times}}, \underbrace{r+1, \dots, r+1}_{n-(d-1) \text{ times}}).$$

Proof. Using the reference flag, we get

$$\begin{aligned} \mathrm{GL}_n / P_{f_{\bullet}} &\xrightarrow{\sim} \mathrm{Flag}_{\hat{d}}(\mathbb{C}^n) \\ g &\longmapsto g(f_{\bullet}) \end{aligned}$$

where $P_{f_{\bullet}}$ is the parabolic group of linear automorphisms preserving $f_{\bullet}$, from which the above can be easily derived. $\square$

Definition 2.6.3. For $\sigma(f_{\bullet}) \in \mathrm{Flag}_{\hat{d}}(\mathbb{C}^n)$ we set

$$H_{\sigma} := \mu^{-1}(\sigma(f_{\bullet})) \cap \widetilde{\mathrm{NHilb}}_{\mathrm{nil-fil}}^d(\mathbb{A}^n)$$

and let $\iota_{\sigma} : H_{\sigma} \hookrightarrow \widetilde{\mathrm{NHilb}}_{\mathrm{nil-fil}}^d(\mathbb{A}^n)$ be the inclusion. Note that H_{σ} inherits a perfect obstruction theory being cut out of the smooth variety $\mu^{-1}(\sigma(f_{\bullet}))$ by a section of $\tau^* \mathcal{E}_{\mathrm{ass}}|_{\mu^{-1}(\sigma(f_{\bullet}))}$.

We can now rewrite the virtual class of the nilpotently filtered locus as a sum of virtual classes of $[H_{\sigma}]$.

Corollary 2.6.2. *For $\sigma \in S_n / (S_{d_1} \times \dots \times S_{n-(d-1)})$ set $T_{\sigma} = T_{\mathrm{Flag}_{\hat{d}}(\mathbb{C}^n), \sigma(f_{\bullet})}$ and $p_{\sigma} : H_{\sigma} \rightarrow \mathrm{pt}$. We have an identity*

$$[\widetilde{\mathrm{NHilb}}_{\mathrm{nil-fil}}^d(\mathbb{A}^n)]^{\mathrm{vir}} = \sum_{\sigma \in S_n / (S_{d_1} \times \dots \times S_{n-(d-1)})} \iota_{\sigma,*} \frac{[H_{\sigma}]^{\mathrm{vir}}}{e^T(p_{\sigma}^* T_{\sigma})}$$

in the localized Chow group $A_*^T(\widetilde{\mathrm{NHilb}}_{\mathrm{nil-fil}}^d(\mathbb{A}^n))_{\mathrm{loc}}$. In particular, for $\alpha \in A_T^*(\widetilde{\mathrm{NHilb}}_{\mathrm{nil-fil}}^d(\mathbb{A}^n))$ it holds that

$$\int_{[\widetilde{\mathrm{NHilb}}_{\mathrm{nil-fil}}^d(\mathbb{A}^n)]^{\mathrm{vir}}} \alpha = \sum_{\sigma \in S_n / (S_{d_1} \times \dots \times S_{n-(d-1)})} \frac{\int_{[H_{\sigma}]^{\mathrm{vir}}} \iota_{\sigma}^* \alpha}{\prod_{j=1}^{d-1} \prod_{\substack{1 \leq i \leq n \\ w(i) > w(j)}} (s_{\sigma(i)} - s_{\sigma(j)})}$$

in the localized Chow group $A_*^T(\mathrm{pt})_{\mathrm{loc}}$.

Proof. We apply proposition 2.2.2 to $g = \mu \circ \iota$ for

$$\iota : \widetilde{\mathrm{NHilb}}_{\mathrm{nil-fil}}^d(\mathbb{A}^n) \hookrightarrow \widetilde{\mathrm{naNHilb}}_0^d(\mathbb{A}^n).$$

Letting I be the ideal of the closed immersion ι we have

$$\mathbb{L}_g = [I/I^2 \rightarrow \iota^* \Omega_\mu]$$

and so can simply define the relative perfect obstruction theory by

$$\begin{array}{ccc} \mathbb{E}_g & := & [(\iota_* \tau^* \mathcal{E}^{\mathrm{ass}})^\vee \longrightarrow \iota^* \Omega_\mu] \\ \downarrow & & \downarrow s_{\mathrm{ass}} \quad \parallel \\ \mathbb{L}_g & \cong & [I/I^2 \longrightarrow \iota^* \Omega_\mu] \end{array}$$

which is clearly compatible with the perfect obstruction theory on $\widetilde{\mathrm{NHilb}}_{\mathrm{nil-fil}}^d(\mathbb{A}^n)$. We conclude that

$$[\widetilde{\mathrm{NHilb}}_{\mathrm{nil-fil}}^d(\mathbb{A}^n)]^{\mathrm{vir}} = \sum_{\sigma} \iota_{\sigma,*} g^! \frac{[\sigma(f_\bullet)]}{e^T(T_\sigma)}.$$

If we give μ the smooth relative perfect obstruction theory and ι the perfect obstruction theory from $(\tau^* \mathcal{E}_{\mathrm{ass}}, \tau^* s_{\mathrm{ass}})$ these are clearly compatible with the given perfect obstruction theory on g . Functoriality of the virtual pullback now shows that

$$g^! = \iota^! \mu^! = 0^! \mu^*$$

for $0^!$ the refined Gysin of the zero section of $\tau^* \mathcal{E}_{\mathrm{ass}}$ and μ^* the flat pullback. Since

$$0^! \mu^* [\sigma(f_\bullet)] = 0^! [\mu^{-1}(\sigma(f_\bullet))] = [H_\sigma]^{\mathrm{vir}}$$

we see that

$$g^! \frac{[\mu^{-1}(\sigma(f_\bullet))]}{e^T(T_\sigma)} = \frac{[H_\sigma]^{\mathrm{vir}}}{e^T(p_\sigma^* T_\sigma)}$$

from which the result follows. The integration formula is a direct consequence of the projection formula. $\square$

Next we study the structure of $[H_\sigma]^{\mathrm{vir}}$.

Lemma 2.6.7. *The map*

$$\mu^{-1}(\sigma(f_\bullet)) \hookrightarrow \widetilde{\mathrm{naNHilb}}_0^d(\mathbb{A}^n) \xrightarrow{\tau} \mathrm{naNHilb}_0^d(\mathbb{A}^n)$$

is a closed immersion. Explicitly, letting $(\mathbb{C}^n)^\vee \twoheadrightarrow \sigma(f_\bullet^\vee)$ be the dual flag associated with $\sigma(f_\bullet)$ it identifies with the locus where the universal map $\Psi_1 : (\mathcal{O}^n)^\vee \rightarrow \mathcal{V}_1$ factors through a (necessarily unique) map of flags

$$\mathcal{O} \otimes \sigma(f_\bullet^\vee) \rightarrow \mathcal{V}_\bullet.$$

In particular, if we let $\mathcal{E}_\sigma$ be the quotient of the sheaf

$$\mathcal{H}om((\mathcal{O}^n)^\vee, \mathcal{V}_1) / \mathcal{H}om^{\mathrm{fil}}(\mathcal{O} \otimes \sigma(f_\bullet^\vee), \mathcal{V}_\bullet)$$

then $\mu^{-1}(\sigma(f_\bullet))$ is the zero locus in $\mathrm{naNHilb}_0^d(\mathbb{A}^n)$ of the section of $\mathcal{E}_\sigma$ induced by Ψ_1 .

Proof. This is immediate from the definition of $\widetilde{\mathrm{naNHilb}}_0^d(\mathbb{A}^n)$. $\square$

Corollary 2.6.3. *Let $\lambda_\bullet = (\lambda_1 \subset \dots \subset \lambda_{r+1})$ be a nested partition satisfying the hypothesis of lemma 2.6.2. The corresponding fixed point of $\text{NHilb}_{\text{nil-fil}}^d(\mathbb{A}^n)$ is in H_σ if and only if*

$$e_{\sigma(j)} \notin \lambda_i$$

whenever $w(j) \geq i - 1$.

Proof. Recall that

$$\sigma(f_i) = \text{span}(e_{\sigma(1)}, \dots, e_{\sigma(d_1 + \dots + d_i)})$$

with $\sigma(f_0) = \{0\}$. The dual flag is

$$\sigma(f_i^\vee) = \ker(\sigma(f_r)^\vee \rightarrow \sigma(f_{i-1})^\vee)$$

which as a quotient of $(\mathbb{C}^n)^\vee = \text{span}(x_1, \dots, x_n)$ can be described as

$$\begin{aligned} I_i^\sigma / I_{r+1}^\sigma &:= \text{span}(x_k \mid k \neq \sigma(1), \dots, \sigma(d_1 + \dots + d_{i-1})) / \text{span}(x_k \mid k \neq \sigma(1), \dots, \sigma(d_1 + \dots + d_r)) \\ &= \text{span}(x_{\sigma(j)} \mid w(j) \geq i - 1) / \text{span}(x_{\sigma(j)} \mid w(j) \geq r) \end{aligned}$$

where $I_1^\sigma = (\mathbb{C}^n)^\vee$. By definition

$$\mathcal{V}_i|_\lambda = I_{\lambda_i} / I_{\lambda_{r+1}}$$

with $\psi_1 : (\mathbb{C}^n)^\vee \rightarrow I_{\lambda_1} / I_{\lambda_{r+1}}$ the restriction of the quotient map along

$$(\mathbb{C}^n)^\vee \subset (x_1, \dots, x_n) = I_{\lambda_1}.$$

It follows that ψ_1 factors through a map of flags $\sigma(f_\bullet^\vee) \rightarrow \mathcal{V}_\bullet|_\lambda$ if and only if

$$I_i \subset I_{\lambda_i}$$

for all i i.e., if and only if

$$e_{\sigma(j)} \notin \lambda_i$$

whenever $w(j) \geq i - 1$. □

Corollary 2.6.4. *Let $\lambda_\bullet$ be a nested partition such that the corresponding fixed point is in H_σ . Let $\mathbf{u}_0, \dots, \mathbf{u}_{d-1}$ be an enumeration of $\lambda_\bullet$ in the sense of setup 2.5.3. Then the trace of $T_{\mu^{-1}(\sigma(f_\bullet)), \lambda_\bullet}$ is given by*

$$\sum_{i=1}^{d-1} \sum_{\substack{1 \leq k \leq d-1 \\ w(i) \leq w(k)}} t^{e_{\sigma(i)} - \mathbf{u}_k} + \sum_{1 \leq i \leq j \leq d-1} \sum_{\substack{1 \leq k \leq d-1 \\ w(j) < w(k)}} t^{\mathbf{u}_i + \mathbf{u}_j - \mathbf{u}_k} - \sum_{j=1}^{d-1} \sum_{\substack{1 \leq k \leq d-1 \\ w(j) \leq w(k)}} t^{\mathbf{u}_j - \mathbf{u}_k}.$$

Proof. We note that

$$T_{\mu^{-1}(\sigma(f_\bullet))} = T_{\text{naNHilb}_0^d(\mathbb{A}^n)}|_{\sigma(f_\bullet)} - \mathcal{E}_\sigma|_{\sigma(f_\bullet)}$$

which follows directly from the above. We directly see that

$$\mathcal{E}_\sigma = (t_1 + \dots + t_n) \sum_{k=1}^{d-1} t^{-\mathbf{u}_k} - \sum_{i=1}^{d-1} \sum_{\substack{1 \leq k \leq d-1 \\ w(i) \leq w(k)}} t^{e_{\sigma(i)} - \mathbf{u}_k}$$

which combined with the previous calculation of

$$T_{\text{naNHilb}_0^d(\mathbb{A}^n), \lambda_\bullet}$$

yields the result. □

Again, the cancellation afterwards is trickier. We instead add a recursive definition of the multiset of weights similar to that of lemma 2.5.3.

Lemma 2.6.8. *Let $\lambda_\bullet$ be a nested partition such that the associated fixed point is in H_σ . For $0 \leq m \leq r$ we define multisets*

$$S_m^{T,\sigma}(\lambda_\bullet) \subset S_m^T(\lambda_\bullet)$$

recursively as follows. Pick an enumeration $\mathbf{u}_0, \dots, \mathbf{u}_{d-1}$ of $\lambda_\bullet$. Set

$$S_0^{T,\sigma}(\lambda_\bullet) = \emptyset$$

and define $S_m^{T,\sigma}(\lambda_\bullet)$ recursively as

$$\left(S_{m-1}^{T,\sigma}(\lambda_\bullet) \cup \{e_{\sigma(i)} \mid w(i) = m\} \cup \{\mathbf{u}_i + \mathbf{u}_j \mid 1 \leq i \leq j, w(j) = m-1\} \right) \setminus \{\mathbf{u}_j \mid 1 \leq j, w(j) = m\}.$$

Then the trace of $T_{\mu^{-1}(\sigma(f_\bullet)), \lambda_\bullet}$ is given by

$$\sum_{m=0}^r \sum_{\mathbf{v} \in \lambda_{m+1} \setminus \lambda_m} \sum_{\mathbf{u} \in S_m^{T,\sigma}(\lambda_\bullet)} t^{\mathbf{u}-\mathbf{v}}.$$

2.6.4 The residue formula

Setup 2.6.1. Let $\underline{d} = (1, \hat{d}) = (1, d_1, \dots, d_r) \in \mathbb{Z}_{\geq 0}^{r+1}$, $n \geq 0$ and $d = 1 + d_1 + \dots + d_r$. Let W be a q -dimensional T -representation, let $E = \tilde{W} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{A}^n}$ be the associated T -equivariant bundle on $\mathbb{A}^n$. Let $E^{[\underline{d}]}$ be the associated tautological bundle on $\text{NHilb}^d(\mathbb{A}^n)$ of rank $m = d \cdot q$. We note that

$$E^{[\underline{d}]} = W \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{A}^n}^{[\underline{d}]}.$$

We let $\theta_1, \dots, \theta_q$ be the Chern roots of W and $v_0, v_1, \dots, v_{d-1}$ be the Chern roots of $\mathcal{O}^{[\underline{d}]}$ ordered such that $\mathcal{O}^{[d_0+d_1+\dots+d_i]}$ has Chern roots $v_0, v_1, \dots, v_{d_1+\dots+d_i}$. Then $E^{[\underline{d}]}$ has Chern roots

$$\theta_i + v_j$$

for $1 \leq i \leq k$ and $0 \leq j \leq d-1$. Any polynomial in the Chern classes of $E^{[\underline{d}]}$ is therefore a bisymmetric polynomial in $\theta_1, \dots, \theta_q$ and $v_0, \dots, v_{d-1}$. When we develop the tautological integral formulas we will therefore just pick $\theta_1, \dots, \theta_q \in A_T^1(\text{pt})$ and consider such bisymmetric polynomials. As it is more convenient, we will also substitute v_i with $\eta_i := -v_i$. Note that we can then choose η_i such that

$$\eta_i|_{\lambda_\bullet} = \mathbf{u}_i(\underline{s})$$

for a fitting enumeration $\mathbf{u}_0, \dots, \mathbf{u}_{d-1}$ of a partition $\lambda_\bullet$. Note also that $\eta_0 = v_0 = 0$, as it corresponds to the first Chern class of the trivial summand coming from the unit of the algebra.

Proposition 2.6.2. *Let $\theta_1, \dots, \theta_q \in A_T^1(\text{pt})$ and let*

$$P = P(\theta_1, \dots, \theta_q, \eta_1, \dots, \eta_{d-1})$$

be a bisymmetric polynomial as in setup 2.6.1. Let

$$A_*^T(\text{pt}) \cong \mathbb{Z}[s_1, \dots, s_n].$$

Then there exists a polynomial Q in $k + (d-1)$ variables with coefficients in $\mathbb{Z}$ such that for all

$$\sigma \in S_n / (S_{d_1} \times \dots \times S_{d_r} \times S_{n-(d-1)})$$

we have

$$\int_{[H_\sigma]^{\text{vir}}} P = Q(\theta_1, \dots, \theta_q, s_{\sigma(1)}, \dots, s_{\sigma(d-1)}).$$

Proof. We first note that

$$\mathcal{O}_{\mathbb{A}^n}^{[d]} = \mathcal{V}_\bullet|_{\text{NHilb}^d(\mathbb{A}^n)}$$

for $\mathcal{V}_\bullet$ the universal flag on the non-associative nested Hilbert scheme. We apply virtual localization

$$\int_{[H_{\text{id}}]^{\text{vir}}} P = \sum_{\lambda_\bullet \in H_{\text{id}}^T} P_{\lambda_\bullet} \cdot \frac{e^T(\mathcal{E}_{\text{ass}, \lambda_\bullet})}{e^T(T_{\mu^{-1}(\sigma(f_\bullet)), \lambda_\bullet})}.$$

We can choose an enumeration $\mathbf{u}_0, \dots, \mathbf{u}_{d-1}$ of $\lambda_\bullet$ satisfying the properties of setup 2.5.3 and such that

$$\eta_i|_{\lambda_\bullet} = \mathbf{u}_i(\underline{s}).$$

Note that for $\lambda_\bullet \in H_{\text{id}}^T$ we have that $e_j \notin \lambda_\bullet$ for $j > d-1$. It follows that $P_{\lambda_\bullet}$ is a polynomial in $\theta_1, \dots, \theta_q, s_1, \dots, s_{d-1}$. From the explicit calculation of their K -theory class above, we can choose polynomials $Q_{1, \lambda_\bullet}, Q_{2, \lambda_\bullet}$ such that

$$e^T(\mathcal{E}_{\text{ass}, \lambda_\bullet}) = Q_{1, \lambda_\bullet}(\theta_1, \dots, \theta_q, s_1, \dots, s_{d-1})$$

and

$$e^T(T_{\mu^{-1}(\sigma(f_\bullet)), \lambda_\bullet}) = Q_{2, \lambda_\bullet}(\theta_1, \dots, \theta_q, s_1, \dots, s_{d-1}).$$

We now simply define

$$Q := \sum_{\lambda_\bullet \in H_{\text{id}}^T} P_{\lambda_\bullet} \frac{Q_{1, \lambda_\bullet}}{Q_{2, \lambda_\bullet}}.$$

For

$$\sigma \in S_n / (S_{d_1} \times \dots \times S_{d_r} \times S_{n-(d-1)})$$

and a nested partition $\lambda_\bullet$ we define $\sigma(\lambda_\bullet)$ by simply substituting $e_i \mapsto e_{\sigma(i)}$. The assignment $\lambda_\bullet \mapsto \sigma(\lambda_\bullet)$ defines a bijection

$$H_{\text{id}}^T \rightarrow H_\sigma^T$$

and one directly see from the concrete formulas that

$$e^T(\mathcal{E}_{\text{ass}, \sigma(\lambda_\bullet)}) = Q_{1, \lambda_\bullet}(\theta_1, \dots, \theta_k, s_{\sigma(1)}, \dots, s_{\sigma(d-1)})$$

and

$$e^T(T_{\mu^{-1}(\sigma(f_\bullet)), \sigma(\lambda_\bullet)}) = Q_{2, \lambda_\bullet}(\theta_1, \dots, \theta_q, s_{\sigma(1)}, \dots, s_{\sigma(d-1)}).$$

Applying virtual localization to $[H_\sigma]^{\text{vir}}$ and comparing gives the result. $\square$

We can now obtain a iterated residue formula for virtual tautological integrals on $\text{NHilb}_{\text{nil-fil}}^d(\mathbb{A}^n)$. The main input is the following iterated residue formula for integration on the flag variety $\text{Flag}_{\hat{d}}(\mathbb{C}^n)$ generalizing [BS12, Proposition 5.4].

Proposition 2.6.3. *Let $\hat{d} = (d_1, \dots, d_r) \in \mathbb{Z}_{\geq 0}^r$ and set $k = |\hat{d}| := d_1 + \dots + d_r \leq n$. Let Q be a polynomial on $\mathbb{C}^k$ and let $s_1, \dots, s_n$ be parameters.*

Partition $\{1, \dots, n\}$ into blocks

$$B_1 = \{1, \dots, d_1\}, B_2 = \{d_1+1, \dots, d_1+d_2\}, \dots, B_r = \{k-d_r+1, \dots, k\}, B_{r+1} = \{k+1, \dots, n\},$$

and define a weight function

$$w : \{1, \dots, n\} \rightarrow \{1, \dots, r+1\}, \quad w(i) = p \text{ if and only if } i \in B_p.$$

Then

$$\sum_{\sigma \in S_n / (S_{d_1} \times \cdots \times S_{d_r} \times S_{n-k})} \frac{Q(s_{\sigma(1)}, \dots, s_{\sigma(k)})}{\prod_{j=1}^k \prod_{\substack{1 \leq i \leq n \\ w(i) > w(j)}} (s_{\sigma(i)} - s_{\sigma(j)})} = \operatorname{Res}_{\mathbf{z}=\infty} \frac{\prod_{1 \leq m \neq \ell \leq k} (z_m - z_\ell) Q(\mathbf{z}) dz_1 \cdots dz_k}{\prod_{\substack{1 \leq \ell < m \leq k \\ w(\ell) < w(m)}} (z_m - z_\ell) \prod_{i=1}^n \prod_{\ell=1}^k (s_i - z_\ell)}. \quad (2.1)$$

Proof. Consider the meromorphic k -form on $(\mathbb{P}^1)^k$

$$\omega_k(\mathbf{z}) = \frac{\prod_{1 \leq m \neq \ell \leq k} (z_m - z_\ell) Q(\mathbf{z}) dz_1 \cdots dz_k}{\prod_{\substack{1 \leq \ell < m \leq k \\ w(\ell) < w(m)}} (z_m - z_\ell) \prod_{i=1}^n \prod_{\ell=1}^k (s_i - z_\ell)}, \quad \mathbf{z} = (z_1, \dots, z_k).$$

We will compute the iterated residue

$$\operatorname{Res}_{\mathbf{z}=\infty} \omega_k := \operatorname{Res}_{z_1=\infty} \cdots \operatorname{Res}_{z_k=\infty} \omega_k$$

by successively applying the Residue Theorem on $\mathbb{P}^1$ in the variables $z_k, z_{k-1}, \dots, z_1$, as in [BS12].

Step 1: Isolating the dependence on z_k . Write the full Vandermonde product as

$$\prod_{1 \leq m \neq \ell \leq k} (z_m - z_\ell) = \left(\prod_{1 \leq m \neq \ell \leq k-1} (z_m - z_\ell) \right) \left(\prod_{\ell < k} (z_k - z_\ell)(z_\ell - z_k) \right).$$

Among the cross-block factors in the denominator

$$\prod_{\substack{1 \leq \ell < m \leq k \\ w(\ell) < w(m)}} (z_m - z_\ell),$$

the only ones involving z_k are those with $m = k$ and $\ell < k$, namely

$$\prod_{\substack{\ell < k \\ w(\ell) < w(k)}} (z_k - z_\ell).$$

Thus, we can factor ω_k as

$$\omega_k = F_{k-1}(\mathbf{z}') \frac{\prod_{\ell < k} (z_k - z_\ell)(z_\ell - z_k)}{\prod_{\substack{\ell < k \\ w(\ell) < w(k)}} (z_k - z_\ell) \prod_{i=1}^n (s_i - z_k)} dz_k,$$

where $\mathbf{z}' = (z_1, \dots, z_{k-1})$ and

$$F_{k-1}(\mathbf{z}') = \frac{\prod_{1 \leq m \neq \ell \leq k-1} (z_m - z_\ell) Q(\mathbf{z}) dz_1 \cdots dz_{k-1}}{\prod_{\substack{1 \leq \ell < m \leq k-1 \\ w(\ell) < w(m)}} (z_m - z_\ell) \prod_{i=1}^n \prod_{\ell=1}^{k-1} (s_i - z_\ell)}.$$

For each $\ell < k$ we have:

- If $w(\ell) < w(k)$, then a denominator factor $(z_k - z_\ell)$ cancels one factor in $(z_k - z_\ell)(z_\ell - z_k)$, leaving $(z_\ell - z_k)$.
- If $w(\ell) = w(k)$, there is no denominator factor, and we simply keep $(z_k - z_\ell)(z_\ell - z_k) = -(z_k - z_\ell)^2$.

Hence, as a function of z_k ,

$$\omega_k = \Phi_k(\mathbf{z}') \frac{R_k(z_k; \mathbf{z}')}{\prod_{i=1}^n (s_i - z_k)} dz_k,$$

where $\Phi_k(\mathbf{z}')$ is independent of z_k and $R_k(z_k; \mathbf{z}')$ is a polynomial in z_k . In particular, the only poles in the z_k variable are at $z_k = s_1, \dots, s_n$.

Step 2: Residue in z_k . By the Residue Theorem on $\mathbb{P}^1$,

$$\text{Res}_{z_k=\infty} \omega_k = - \sum_{j=1}^n \text{Res}_{z_k=s_j} \omega_k.$$

For a simple pole in $s_j - z_k$, we have

$$\text{Res}_{z_k=s_j} \frac{R_k(z_k; \mathbf{z}')}{\prod_{i=1}^n (s_i - z_k)} dz_k = - \frac{R_k(s_j; \mathbf{z}')}{\prod_{i \neq j} (s_i - s_j)}.$$

Thus, the minus sign from the Residue Theorem cancels this minus sign, giving

$$\text{Res}_{z_k=\infty} \omega_k = \sum_{j=1}^n \Phi_k(\mathbf{z}') \frac{R_k(s_j; \mathbf{z}')}{\prod_{i \neq j} (s_i - s_j)}.$$

Now track the factors involving z_k more explicitly. As observed above, for each $\ell < k$ we obtain after substituting $z_k = s_j$ at least one factor $(z_\ell - s_j)$ in the numerator. On the other hand, in $F_{k-1}(\mathbf{z}')$ we still have the denominator

$$\prod_{i=1}^n (s_i - z_\ell), \quad \ell = 1, \dots, k-1.$$

Hence for each $\ell < k$ the factor $(z_\ell - s_j)$ cancels the factor $(s_j - z_\ell)$ in this product, leaving

$$\prod_{i \neq j} (s_i - z_\ell)$$

in the denominator and at most one $(z_\ell - s_j)$ factor remaining in the numerator. In particular, after the z_k -residue, the pole at $z_\ell = s_j$ disappears for all $\ell < k$.

Collecting all factors, we obtain (up to an overall sign depending only on k)

$$\text{Res}_{z_k=\infty} \omega_k = \sum_{j=1}^n \frac{\prod_{1 \leq m \neq \ell \leq k-1} (z_m - z_\ell) Q(z_1, \dots, z_{k-1}, s_j) \prod_{\ell=1}^{k-1} (z_\ell - s_j) dz_1 \cdots dz_{k-1}}{\prod_{\substack{1 \leq \ell < m \leq k-1 \\ w(\ell) < w(m)}} (z_m - z_\ell) \prod_{\ell=1}^{k-1} \prod_{i \neq j} (s_i - z_\ell) \prod_{i \neq j} (s_i - s_j)}. \quad (2.2)$$

This has the same structure as the original integrand, but with one fewer variable and with j excluded from the set of indices appearing in the denominators.

Step 3: Iteration. We now repeat the same procedure with z_{k-1} , then z_{k-2} , and so on.

Inductively, after having taken residues in $z_k, \dots, z_{m+1}$ we obtain a sum over ordered $(k-m)$ -tuples of distinct indices $(j_{m+1}, \dots, j_k)$, and for each such choice an integrand in the remaining variables $z_1, \dots, z_m$ of the same form:

$$\frac{\prod_{1 \leq a \neq b \leq m} (z_a - z_b) Q(\dots) dz_1 \cdots dz_m}{\prod_{\substack{1 \leq \ell < m \leq m \\ w(\ell) < w(m)}} (z_m - z_\ell) \prod_{\ell=1}^m \prod_{i \in I_m} (s_i - z_\ell)} \cdot (\text{constant depending on } s_{j_{m+1}}, \dots, s_{j_k}),$$

where $I_m \subset \{1, \dots, n\}$ is the set of indices not yet used. In particular, the only poles in z_m are at $z_m = s_i$ with $i \in I_m$.

The same argument as in Step 2 shows that:

- At the z_m step, we pick an index $j_m \in I_m$ and obtain a factor $1/\prod_{i \in I_m \setminus \{j_m\}} (s_i - s_{j_m})$.
- The pole at $z_\ell = s_{j_m}$ disappears for every $\ell < m$; hence j_m cannot reappear at later steps. Thus, the indices $j_1, \dots, j_k$ are all distinct.
- Tracking the cross-block factors, one checks that for the chosen index j_m with weight $w(j_m)$, the only factors that survive in the final denominator are those with $w(i) > w(j_m)$. The factors with $w(i) \leq w(j_m)$ are cancelled against factors arising from the Vandermonde numerator at earlier steps.

Proceeding inductively down to z_1 , we thus obtain a sum over ordered k -tuples of distinct indices $(j_1, \dots, j_k)$, or equivalently over cosets $\sigma \in S_n/(S_{d_1} \times \cdots \times S_{d_r} \times S_{n-k})$, of terms of the form

$$\frac{Q(s_{j_1}, \dots, s_{j_k})}{\prod_{m=1}^k \prod_{\substack{i \in \{1, \dots, n\} \\ w(i) > w(j_m)}} (s_i - s_{j_m})} = \frac{Q(s_{\sigma(1)}, \dots, s_{\sigma(k)})}{\prod_{j=1}^k \prod_{\substack{1 \leq i \leq n \\ w(i) > w(j)}} (s_{\sigma(i)} - s_{\sigma(j)})}.$$

Step 4: signs. At each step there is a sign coming from exchanging factors $(z_m - z_\ell)$ with $(z_\ell - z_m)$, and a global sign from expanding the Vandermonde product $\prod_{1 \leq m \neq \ell \leq k} (z_m - z_\ell)$. The total sign is $(-1)^{\sum_p \binom{d_p}{2}}$ both from these exchanges and from the Vandermonde, so these sign contributions cancel in the final expression.

Thus, the iterated residue $\text{Res}_{\mathbf{z}=\infty} \omega_k$ is exactly the left-hand side of (2.1), which proves the identity. $\square$

Corollary 2.6.5. *Let $\theta_1, \dots, \theta_q \in A_T^1(\text{pt})$ and let*

$$P = P(\theta_1, \dots, \theta_q, \eta_1, \dots, \eta_{d-1})$$

be a bisymmetric polynomial as in setup 2.6.1. Then

$$\int_{[\text{NHilb}_{\text{nil-fl}}^d(\mathbb{A}^n)]^{\text{vir}}} P = \text{Res}_{\mathbf{z}=\infty} \frac{\prod_{1 \leq m \neq l \leq d-1} (z_m - z_l) Q(\theta_1, \dots, \theta_q, z_1, \dots, z_{d-1}) d\mathbf{z}}{\prod_{\substack{1 \leq l < m \leq d-1 \\ w(l) < w(m)}} (z_m - z_l) \prod_{i=1}^n \prod_{l=1}^{d-1} (s_i - z_l)}$$

where $\mathbf{z} = (z_1, \dots, z_{d-1})$ and

$$Q(\theta_1, \dots, \theta_q, s_1, \dots, s_{d-1}) = \int_{[H_{\text{id}}]^{\text{vir}}} P.$$

Explicitly, one can calculate Q as

$$\sum_{\lambda_{\bullet}} P_{\lambda_{\bullet}} \frac{\prod_{\substack{1 \leq i, j, k \leq d-1 \\ i < k}} \prod_{\substack{0 \leq m \leq d-1 \\ w(k), w(j) \leq w(m)}} (\mathbf{u}_i + \mathbf{u}_j + \mathbf{u}_k - \mathbf{u}_m)(\underline{s}) \prod_{j=1}^{d-1} \prod_{\substack{1 \leq k \leq d-1 \\ w(j) \leq w(k)}} (\mathbf{u}_j - \mathbf{u}_k)(\underline{s})}{\prod_{i=1}^n \prod_{\substack{1 \leq k \leq d-1 \\ w(i) \leq w(k)}} (e_i - \mathbf{u}_k)(\underline{s}) \prod_{1 \leq i \leq j \leq d-1} \prod_{\substack{1 \leq k \leq d-1 \\ w(j) < w(k)}} (\mathbf{u}_i + \mathbf{u}_j - \mathbf{u}_k)(\underline{s})}$$

or using the notation of lemma 2.5.5 and lemma 2.6.8

$$\sum_{\lambda_{\bullet}} P_{\lambda_{\bullet}} \prod_{m=1}^r \prod_{\mathbf{v} \in \lambda_{m+1} \setminus \lambda_m} \frac{\prod_{\substack{\mathbf{u} \in S_m^{ass}(\lambda_{\bullet}) \\ \mathbf{u} \neq \mathbf{v}}} (\mathbf{u} - \mathbf{v})(\underline{s})}{\prod_{\substack{\mathbf{u} \in S_m^{T, id}(\lambda_{\bullet}) \\ \mathbf{u} \neq \mathbf{v}}} (\mathbf{u} - \mathbf{v})(\underline{s})}$$

Here the sums are taken over all admissible nested partitions $\lambda_{\bullet}$ satisfying $\mathbf{u} + e_i \notin \lambda_{k+1}$ for all $\mathbf{u} \in \lambda_{k+1} \setminus \lambda_k$ and all $k \geq 1$ and that $e_j \notin \lambda_i$ whenever $w(j) \geq i - 1$ and for $\mathbf{u}_0, \dots, \mathbf{u}_{d-1}$ an enumeration of $\lambda_{\bullet}$ as in setup 2.5.3 we define

$$P_{\lambda_{\bullet}} = P(\theta_1, \dots, \theta_q, \lambda_{\bullet}(\underline{z})) = P(\theta_1, \dots, \theta_q, \mathbf{u}_1(\underline{s}), \dots, \mathbf{u}_{d-1}(\underline{s})).$$

2.6.5 Residue vanishing

The above formula shows that the virtual integral can be exhibited as a sum of iterated residues, i.e., we have that

$$\sum_{\lambda_{\bullet}} \text{Res}_{\underline{z}=\infty} \frac{P(\theta_1, \dots, \theta_q, \lambda_{\bullet}(\underline{z})) \prod_{1 \leq m \neq l \leq d-1} (z_m - z_l) \prod_{m=1}^r \prod_{\mathbf{v} \in \lambda_{m+1} \setminus \lambda_m} \prod_{\substack{\mathbf{u} \in S_m^{ass}(\lambda_{\bullet}) \\ \mathbf{u} \neq \mathbf{v}}} (\mathbf{u} - \mathbf{v})(\underline{z}) d\underline{z}}{\prod_{\substack{1 \leq l < m \leq d-1 \\ w(l) < w(m)}} (z_m - z_l) \prod_{i=1}^n \prod_{l=1}^{d-1} (s_i - z_l) \prod_{m=1}^r \prod_{\mathbf{v} \in \lambda_{m+1} \setminus \lambda_m} \prod_{\substack{\mathbf{u} \in S_m^{T, id}(\lambda_{\bullet}) \\ \mathbf{u} \neq \mathbf{v}}} (\mathbf{u} - \mathbf{v})(\underline{z})}$$

calculates

$$\int_{[\text{NHilb}_{\text{nil-fil}}^d(\mathbb{A}^n)]^{\text{vir}}} P(\theta_1, \dots, \theta_q, \eta_1, \dots, \eta_{d-1}).$$

In this section we prove that most of these residues vanish. In fact, we prove that the only non-vanishing residue is the one related to the Porteous point

$$\lambda_k = \{0, e_1, e_2, \dots, e_{d_1 + \dots + d_{k-1}}\}.$$

To prove the vanishing of an iterated residue we use the following criterion from [BS12].

Proposition 2.6.4. [BS12, Proposition 6.3] *Let $p(\underline{z})$ and $q(\underline{z})$ be polynomials in the variables $z_1, \dots, z_r$ and assume that $q(\underline{z}) = \prod_{i=1}^N L_i(\underline{z})$ is a product of linear factors. Let $d\underline{z} = dz_1 \cdots dz_r$. The iterated residue*

$$\text{Res}_{\underline{z}=\infty} \frac{p(\underline{z}) d\underline{z}}{q(\underline{z})}$$

vanishes if there exist $1 \leq M \leq r$ satisfying either of the following conditions

- $\deg(p(\underline{z}); \{r, r-1, \dots, M\}) + r - M + 1 < \deg(q(\underline{z}); \{r, r-1, \dots, M\})$

or

- $\deg(q(\underline{z}); \{M\}) = \text{lead}(q(\underline{z}), M)$ and $\deg(p(\underline{z}); \{M\}) + 1 < \deg(q(\underline{z}); \{M\})$.

Recall that

$$\text{lead}(q(\underline{z}), M) := \# \{i; \max\{l; \text{coef}(L_i, z_l) \neq 0\} = M\}$$

i.e., the number of linear terms L_i where the highest indexed variable appearing is z_M .

Proposition 2.6.5. *Let $\theta_1, \dots, \theta_q \in A_T^1(\text{pt})$ and let*

$$P = P(\theta_1, \dots, \theta_q, \eta_1, \dots, \eta_{d-1})$$

be a bisymmetric polynomial as in setup 2.6.1. Let $\lambda_\bullet$ be a nested partition not of Porteuos type and assume that the corresponding fixed point is in H_{id} . Then

$$\text{Res}_{\underline{z}=\infty} \frac{P(\theta_1, \dots, \theta_q, \lambda_\bullet(\underline{z})) \prod_{1 \leq m \neq l \leq d-1} (z_m - z_l) \prod_{m=1}^r \prod_{\mathbf{v} \in \lambda_{m+1} \setminus \lambda_m} \prod_{\substack{\mathbf{u} \in S_m^{\text{ass}}(\lambda_\bullet) \\ \mathbf{u} \neq \mathbf{v}}} (\mathbf{u} - \mathbf{v})(\underline{z}) d\underline{z}}{\prod_{\substack{1 \leq l < m \leq d-1 \\ w(l) < w(m)}} (z_m - z_l) \prod_{i=1}^n \prod_{l=1}^{d-1} (s_i - z_l) \prod_{m=1}^r \prod_{\mathbf{v} \in \lambda_{m+1} \setminus \lambda_m} \prod_{\substack{\mathbf{u} \in S_m^{T, \text{id}}(\lambda_\bullet) \\ \mathbf{u} \neq \mathbf{v}}} (\mathbf{u} - \mathbf{v})(\underline{z})} = 0.$$

Proof. For the proof we let $p(\underline{z})$ and $q(\underline{z})$ be the numerator and the denominator of the expression in the iterated residue formula. Since $\lambda_\bullet$ is not Porteuos we can pick the largest $1 \leq M \leq d-1$ such that $e_M \notin \lambda_\bullet$. We first note that $e_M \notin \lambda_\bullet$ means that z_M does not appear in the term

$$P(\theta_1, \dots, \theta_q, \lambda_\bullet(\underline{z}))$$

nor in the term

$$\prod_{m=1}^r \prod_{\mathbf{v} \in \lambda_{m+1} \setminus \lambda_m} \prod_{\substack{\mathbf{u} \in S_m^{\text{ass}}(\lambda_\bullet) \\ \mathbf{u} \neq \mathbf{v}}} (\mathbf{u} - \mathbf{v})(\underline{z}).$$

We proceed by reverse induction on M . Suppose first that $M = d-1$. Since z_{d-1} is the variable with the largest index it follows by definition that

$$\deg(q(\underline{z}); \{d-1\}) = \text{lead}(q(\underline{z}), d-1).$$

We note that

$$e_{d-1} \notin S_m^{T, \text{id}}(\lambda_\bullet)$$

for $m < r$ and that e_{d-1} appears exactly once in $S_r^{T, \text{id}}(\lambda_\bullet)$. It follows that it appears in exactly $|\lambda_{r+1} \setminus \lambda_r| = d_r$ linear terms of

$$\prod_{m=1}^r \prod_{\mathbf{v} \in \lambda_{m+1} \setminus \lambda_m} \prod_{\substack{\mathbf{u} \in S_m^{T, \text{id}}(\lambda_\bullet) \\ \mathbf{u} \neq \mathbf{v}}} (\mathbf{u} - \mathbf{v})(\underline{z}).$$

Clearly it appears in n linear terms of

$$\prod_{i=1}^n \prod_{l=1}^{d-1} (s_i - z_l)$$

and $d_1 + \dots + d_{r-1}$ linear terms of

$$\prod_{\substack{1 \leq l < m \leq d-1 \\ w(l) < w(m)}} (z_m - z_l).$$

We conclude that

$$\deg(q(\underline{z}); \{d-1\}) = d-1 + n.$$

For $p(\underline{z})$ we see that z_{d-1} only appears in

$$\prod_{1 \leq m \neq l \leq d-1} (z_m - z_l)$$

where it appears in $2(d-1) - 2$ terms. From this we see that

$$\deg(p(\underline{z}); \{d-1\}) + 1 = 2(d-1) - 1 < d-1 + n = \deg(q(\underline{z}); \{d-1\})$$

since $n \geq d-1$ by assumption. We conclude the vanishing from proposition 2.6.4. For general M we can pick an enumeration $\mathbf{u}_0, \dots, \mathbf{u}_{d-1}$ of $\lambda_\bullet$ in such a way that

$$\mathbf{u}_M \neq e_M, \mathbf{u}_{M+1} = e_{M+1}, \dots, \mathbf{u}_{d-1} = e_{d-1}$$

We note that $e_M \in S_m^{T, \text{id}}(\lambda_\bullet)$ if and only if $m \geq w(M)$ and in this case it appears exactly once. It follows that z_M appears in $d_{w(M)} + d_{w(M)+1} + \dots + d_r$ linear terms of

$$\prod_{m=1}^r \prod_{\mathbf{v} \in \lambda_{m+1} \setminus \lambda_m} \prod_{\substack{\mathbf{u} \in S_m^{T, \text{id}}(\lambda_\bullet) \\ \mathbf{u} \neq \mathbf{v}}} (\mathbf{u} - \mathbf{v})(\underline{z}).$$

These terms are exactly of the form

$$z_M - \mathbf{u}_i(\underline{z})$$

for $d_1 + \dots + d_{w(M)-1} + 1 \leq i \leq d-1$ and we see that z_M is a leading term if and only if $i \leq M$. We note that the terms, where z_M is not leading, are of the form

$$z_M - z_i$$

for $i > M$. From

$$\prod_{i=1}^n \prod_{l=1}^{d-1} (s_i - z_l)$$

we have n linear terms containing z_M and z_M is the leading term in all of them. At last

$$\prod_{\substack{1 \leq l < m \leq d-1 \\ w(l) < w(m)}} (z_m - z_l)$$

has $d_1 + \dots + d_{w(M)-1} + d_{w(M)+1} + \dots + d_r$ linear terms containing z_M . The terms where it is not leading are of the form

$$z_m - z_M$$

for $m > M$. From this we see that

$$\deg(q(\underline{z}); \{M\}) = d-1 + n + d_{w(M)+1} + \dots + d_r \geq d-1 + n.$$

For $p(\underline{z})$ we again see that z_M only appears in

$$\prod_{1 \leq m \neq l \leq d-1} (z_m - z_l)$$

where it appears in $2(d-1) - 2$ terms so that

$$\deg(p(\underline{z}); \{M\}) + 1 = 2(d-1) - 1 < d-1 + n \leq \deg(q(\underline{z}); \{M\}).$$

We see that the terms $z_M - z_i$ and $z_m - z_M$ for $i, m > M$ where z_M is not leading each cancels with a unique term of

$$\prod_{1 \leq m \neq l \leq d-1} (z_m - z_l).$$

Let

$$\frac{p(\underline{z})}{q(\underline{z})} = \frac{p'(\underline{z})}{q'(\underline{z})}$$

be the result of canceling those term. We now see that

$$\deg(q'(\underline{z}); \{M\}) = \text{lead}(q'(\underline{z}), M)$$

and since q' and p' is obtained from q and p by removing the same number of linear terms with z_M it still holds that

$$\deg(p'(\underline{z}); \{M\}) + 1 < \deg(q'(\underline{z}); \{M\})$$

so that the residue vanishes by proposition 2.6.4. $\square$

Writing out the terms in the residue formula explicitly for the Porteous point, we arrive at the following formula.

Corollary 2.6.6. *Let $\theta_1, \dots, \theta_q \in A_T^1(\text{pt})$ and let*

$$P = P(\theta_1, \dots, \theta_q, \eta_1, \dots, \eta_{d-1})$$

be a bisymmetric polynomial as in setup 2.6.1. Then

$$\int_{[\text{NHilb}_{\text{nil-fil}}^d(\mathbb{A}^n)]^{\text{vir}}} P$$

can be calculated via the iterated residue formula

$$\text{Res}_{\underline{z}=\infty} \frac{P(\theta_1, \dots, \theta_q, \underline{z}) \prod_{1 \leq m \neq l \leq d-1} (z_m - z_l) \prod_{\substack{1 \leq i, j, k \leq d-1 \\ i < k}} \prod_{\substack{0 \leq m \leq d-1 \\ w(k), w(j) \leq w(m)}} (z_i + z_j + z_k - z_m) d\underline{z}}{\prod_{\substack{1 \leq l < m \leq d-1 \\ w(l) < w(m)}} (z_m - z_l) \prod_{i=1}^n \prod_{l=1}^{d-1} (s_i - z_l) \prod_{1 \leq i \leq j \leq d-1} \prod_{\substack{1 \leq k \leq d-1 \\ w(j) < w(k)}} (z_i + z_j - z_k)}.$$

2.6.6 Removing the restriction on the number of points

We recall that for all the machinery with the partial resolution to work we had to assume that $d_1 + \dots + d_r = d - 1 \leq n$. However, the iterated residue formula of corollary 2.6.6 makes sense for any $\underline{d}, n$. In this section we will prove that this is actually the case. The key is to increase n sufficiently and compare the two integrals

Setup 2.6.2. Let $n \leq N$ and set

$$\mathbb{A}^n \hookrightarrow \mathbb{A}^n \times \mathbb{A}^{N-n} \simeq \mathbb{A}^N.$$

In this section we set $T = (\mathbb{C}^*)^N$. We note that the above immersion is T -invariant and that there is a natural T -action on $\text{naNHilb}^{\underline{d}}(\mathbb{A}^n)$ by composing with the projection $(\mathbb{C}^*)^N \rightarrow (\mathbb{C}^*)^n$.

Lemma 2.6.9. *There exist a T -equivariant extension of the immersion*

$$\text{NHilb}^{\underline{d}}(\mathbb{A}^n) \hookrightarrow \text{NHilb}^{\underline{d}}(\mathbb{A}^N)$$

to

$$\text{naNHilb}^{\underline{d}}(\mathbb{A}^n) \rightarrow \text{naNHilb}^{\underline{d}}(\mathbb{A}^N).$$

The latter exhibits $\text{naNHilb}^{\underline{d}}(\mathbb{A}^n)$ as the zero locus of a section of the bundle

$$\mathcal{H}om(\mathcal{I}, \mathcal{V}_0)$$

where $\mathcal{V}_0$ is the universal flag bundle and $\mathcal{I}$ is the kernel of the map

$$(\mathcal{O}^N)^\vee \twoheadrightarrow (\mathcal{O}^n)^\vee$$

given by projection onto the first n summands. When $d_0 = 1$ the immersion restricts to an immersion

$$\mathrm{naNHilb}_0^d(\mathbb{A}^n) \hookrightarrow \mathrm{naNHilb}_0^d(\mathbb{A}^N)$$

which exhibits the domain as the zero locus of a section of

$$\mathcal{H}om(\mathcal{I}, \mathcal{V}_1).$$

Proof. We let $M_{n,\underline{d}}$ and $M_{N,\underline{d}}$ be as in definition 2.3.3 both defined using the same reference flag $N_\bullet$. We define

$$\begin{aligned} M_{n,\underline{d}} &\hookrightarrow M_{N,\underline{d}} \\ (\psi_1, \psi_2, v) &\longmapsto (\psi_1 \circ q, \psi_2, v) \end{aligned}$$

where $q : (\mathbb{C}^N)^\vee \twoheadrightarrow (\mathbb{C}^n)^\vee$ is projection onto the first n summands. One easily checks that this is T - and $P_{\underline{d}}$ -equivariant, thus given us the desired closed immersion. The description of a zero locus of a section of the above vector bundle also follows directly from the definition, as an S -valued point $(\mathcal{F}_\bullet, \psi_1, \psi_2, s)$ of $\mathrm{naNHilb}^d(\mathbb{A}^N)$ factors through $\mathrm{naNHilb}^d(\mathbb{A}^n)$ if and only if ψ_1 factors through $(\mathcal{O}^N)^\vee \twoheadrightarrow (\mathcal{O}^n)^\vee$. Note that on $\mathrm{naNHilb}_0^d(\mathbb{A}^N)$ we have that ψ_1 factors through $\mathcal{V}_1$, so we can restrict to this bundle. $\square$

Lemma 2.6.10. *Let $i : \mathrm{NHilb}^d(\mathbb{A}^n) \hookrightarrow \mathrm{NHilb}^d(\mathbb{A}^N)$ be the inclusion, let $\mathcal{I}$ be the kernel of*

$$(\mathcal{O}^N)^\vee \twoheadrightarrow (\mathcal{O}^n)^\vee$$

on $\mathrm{NHilb}^d(\mathbb{A}^N)$ and set

$$\mathcal{E}_n = \mathcal{H}om(\mathcal{I}, \mathcal{V}_0)|_{\mathrm{NHilb}^d(\mathbb{A}^n)}, \quad \mathcal{E}_{n,0} = \mathcal{H}om(\mathcal{I}, \mathcal{V}_1)|_{\mathrm{NHilb}_{\mathrm{nil-fil}}^d(\mathbb{A}^n)}.$$

We have

$$i_*[\mathrm{NHilb}^d(\mathbb{A}^n)]^{\mathrm{vir}} = e^T(\mathcal{E}_n) \cap [\mathrm{NHilb}^d(\mathbb{A}^N)]^{\mathrm{vir}}$$

and

$$i_*[\mathrm{NHilb}_{\mathrm{nil-fil}}^d(\mathbb{A}^n)]^{\mathrm{vir}} = e^T(\mathcal{E}_{n,0}) \cap [\mathrm{NHilb}_{\mathrm{nil-fil}}^d(\mathbb{A}^N)]^{\mathrm{vir}}.$$

In particular,

$$\int_{[\mathrm{NHilb}^d(\mathbb{A}^n)]^{\mathrm{vir}}} \alpha = \int_{[\mathrm{NHilb}^d(\mathbb{A}^N)]^{\mathrm{vir}}} e^T(\mathcal{E}_n) \cdot \alpha$$

and

$$\int_{[\mathrm{NHilb}_{\mathrm{nil-fil}}^d(\mathbb{A}^n)]^{\mathrm{vir}}} \alpha = \int_{[\mathrm{NHilb}_{\mathrm{nil-fil}}^d(\mathbb{A}^N)]^{\mathrm{vir}}} e^T(\mathcal{E}_{n,0}) \cdot \alpha.$$

Proof. Let j be the extension of i to the non-associative Hilbert scheme. Note that $j^*\mathcal{V}_\bullet$ is still the universal bundle on $\mathrm{naNHilb}^d(\mathbb{A}^n)$. It follows that the associativity bundle also pullback to the associativity bundle. We can therefore form the following pullback squares.

$$\begin{array}{ccccc} \mathrm{NHilb}^d(\mathbb{A}^n) & \longrightarrow & \mathrm{naNHilb}^d(\mathbb{A}^n) & \xrightarrow{j} & \mathrm{naNHilb}^d(\mathbb{A}^N) \\ \downarrow i & & \downarrow j & & \downarrow 0_n \\ \mathrm{NHilb}^d(\mathbb{A}^N) & \longrightarrow & \mathrm{naNHilb}^d(\mathbb{A}^N) & \xrightarrow{s_n} & \mathcal{H}om(\mathcal{I}, \mathcal{V}_\bullet) \\ \downarrow & & \downarrow s_{\mathrm{ass}} & & \\ \mathrm{naNHilb}^d(\mathbb{A}^N) & \xrightarrow{0_{\mathrm{ass}}} & \mathcal{E}_{\mathrm{ass}} & & \end{array}$$

and from this we get

$$\begin{aligned}
i_*[\mathrm{NHilb}^d(\mathbb{A}^n)]^{\mathrm{vir}} &= i_*0_{\mathrm{ass}}^![\mathrm{naNHilb}^d(\mathbb{A}^n)] \\
&= 0_{\mathrm{ass}}^!j_*[\mathrm{naNHilb}^d(\mathbb{A}^n)] \\
&= 0_{\mathrm{ass}}^!(e^T(\mathcal{H}om(\mathcal{I}, \mathcal{V}_\bullet)) \cap [\mathrm{naNHilb}^d(\mathbb{A}^N)]) \\
&= e^T(\mathcal{H}om(\mathcal{I}, \mathcal{V}_\bullet)|_{\mathrm{NHilb}^d(\mathbb{A}^N)}) \cap 0_{\mathrm{ass}}^![\mathrm{naNHilb}^d(\mathbb{A}^N)] \\
&= e^T(\mathcal{E}_n) \cap [\mathrm{NHilb}^d(\mathbb{A}^N)]^{\mathrm{vir}}
\end{aligned}$$

where we have used that j is regular so that

$$[\mathrm{naNHilb}^d(\mathbb{A}^n)] = 0_n^![\mathrm{naNHilb}^d(\mathbb{A}^N)]$$

is a refined Euler class of $\mathcal{H}om(\mathcal{I}, \mathcal{V}_0)$ i.e., $j_*[\mathrm{naNHilb}^d(\mathbb{A}^n)] = e^T(\mathcal{H}om(\mathcal{I}, \mathcal{V}_\bullet)) \cap [\mathrm{naNHilb}^d(\mathbb{A}^N)]$. The proof for the nilpotently filtered locus is exactly the same. $\square$

Theorem 2.6.1. *Let $\theta_1, \dots, \theta_q \in A_T^1(\mathrm{pt})$ and let*

$$P = P(\theta_1, \dots, \theta_q, \eta_1, \dots, \eta_{d-1})$$

be a bisymmetric polynomial as in setup 2.6.1. Then

$$\int_{[\mathrm{NHilb}_{\mathrm{nil-fil}}^d(\mathbb{A}^n)]^{\mathrm{vir}}} P$$

can be calculated via the iterated residue formula

$$\mathrm{Res}_{\tilde{z}=\infty} \frac{P(\theta_1, \dots, \theta_q, \tilde{z}) \prod_{1 \leq m \neq l \leq d-1} (z_m - z_l) \prod_{1 \leq i, j, k \leq d-1} \prod_{i < k} \prod_{0 \leq m \leq d-1} \prod_{w(k), w(j) \leq w(m)} (z_i + z_j + z_k - z_m) d\tilde{z}}{\prod_{1 \leq l < m \leq d-1} \prod_{w(l) < w(m)} (z_m - z_l) \prod_{i=1}^n \prod_{l=1}^{d-1} (s_i - z_l) \prod_{1 \leq i \leq j \leq d-1} \prod_{1 \leq k \leq d-1} \prod_{w(j) < w(k)} (z_i + z_j - z_k)}$$

Proof. If $n \geq d-1$ this is simply the content of corollary 2.6.6. If not, we can find $N \geq d-1$. Then

$$\int_{[\mathrm{NHilb}_{\mathrm{nil-fil}}^d(\mathbb{A}^n)]^{\mathrm{vir}}} P = \int_{[\mathrm{NHilb}_{\mathrm{nil-fil}}^d(\mathbb{A}^N)]^{\mathrm{vir}}} e^T(\mathcal{E}_{n,0}) \cdot P.$$

We note that

$$e^T(\mathcal{E}_{n,0}) = \prod_{i=n+1}^N \prod_{l=1}^{d-1} (s_i - \eta_l).$$

If we apply corollary 2.6.6 to calculate $\int_{[\mathrm{NHilb}_{\mathrm{nil-fil}}^d(\mathbb{A}^N)]^{\mathrm{vir}}} P'$ for $P' = e^T(\mathcal{E}_{n,0}) \cdot \alpha$ we get

$$\mathrm{Res}_{\tilde{z}=\infty} \frac{P'(\theta_1, \dots, \theta_q, \tilde{z}) \prod_{1 \leq m \neq l \leq d-1} (z_m - z_l) \prod_{1 \leq i, j, k \leq d-1} \prod_{i < k} \prod_{0 \leq m \leq d-1} \prod_{w(k), w(j) \leq w(m)} (z_i + z_j + z_k - z_m) d\tilde{z}}{\prod_{1 \leq l < m \leq d-1} \prod_{w(l) < w(m)} (z_m - z_l) \prod_{i=1}^N \prod_{l=1}^{d-1} (s_i - z_l) \prod_{1 \leq i \leq j \leq d-1} \prod_{1 \leq k \leq d-1} \prod_{w(j) < w(k)} (z_i + z_j - z_k)}$$

but

$$P'(\theta_1, \dots, \theta_q, \tilde{z}) = P(\theta_1, \dots, \theta_q, \tilde{z}) \cdot \prod_{i=n+1}^N \prod_{l=1}^{d-1} (s_i - z_l)$$

and we get the cancellation

$$\frac{\prod_{i=n+1}^N \prod_{l=1}^{d-1} (s_i - z_l)}{\prod_{i=1}^N \prod_{l=1}^{d-1} (s_i - z_l)} = \frac{1}{\prod_{i=1}^n \prod_{l=1}^{d-1} (s_i - z_l)}$$

giving us the desired formula. $\square$

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