U N I V E R S I T Y O F A A R H U S
D E P A R T M E N T O F M A T H E M A T I C S

ISSN: 1397-4076

NORMALITY OF CERTAIN NILPOTENT VARIETIES IN POSITIVE CHARACTERISTIC

By Jesper Funch Thomsen

Preprint Series No.: 1

January 1999

Ny Munkegade, Bldg. 530
8000 Aarhus C., Denmark

http://www.imf.au.dk
institut@imf.au.dk

NORMALITY OF CERTAIN NILPOTENT VARIETIES IN POSITIVE CHARACTERISTIC

JESPER FUNCH THOMSEN

1. Introduction

Let G be a semisimple linear algebraic group over an algebraically closed field k. The subset of nilpotent elements $\mathcal{N}$ inside the Lie algebra $\mathfrak{g}$ of G is called the nilpotent cone. The group G acts on $\mathcal{N}$ by the adjoint action with only finitely many orbits. A nilpotent variety is the closure of such an orbit.

When the characteristic of the ground field k is zero it was proved by Broer [3] that the subregular nilpotent variety is normal. One of the main ingredients in the proof was a vanishing result concerning line bundles on the cotangent bundle of a flag variety. By a clever induction and use of the Borel-Bott-Weil theorem Broer later [2] generalized the vanishing result, and were in this way able to prove the normality of a broader class of nilpotent varieties.

Recently (see [12]) it was realized that the theory of Frobenius splitting could be used to generalize Broer's original vanishing result in [2] to flag varieties over fields of good characteristics (see Definition 3). In the same paper the normality of the subregular nilpotent variety in good characteristic was obtained.

This paper deals with the generalization of the vanishing and normality results of [3] to positive good characteristics. One obstruction is that the Borel-Bott-Weil theorem only remains true under some restrictions. The vanishing result on line bundles on the cotangent bundle of a flag variety obtained in this paper, is therefore weaker than in the corresponding characteristic zero situation. It is however noticeable that all the normality results from [3] generalize to good characteristics. Contrary to characteristic zero situation some of the normality results obtained for classical groups seem to be unknown. In characteristic zero these results was already contained in [11].

The approach in this paper is very similar to the one in [3]. Only minor changes are needed to make Broer's approach work in positive characteristic. For convenience of the reader we have however tried to make this paper independent of [3].

The author is partially supported by the TMR programme "Algebraic Lie Representations" (ECM Network Contract No. ERB FMRX-CT 97/0100).

We would like to thank Jens Carsten Jantzen for some useful conversations and suggestions. In particular the approach in Section 7 is due to him. We should also say that the approach to Proposition 6 is very similar to an approach shown to us by A. Broer.

2. Notation

Let G be a connected semisimple simply connected linear algebraic group over an algebraically closed field k of characteristic $p > 0$. Let T be a maximal torus and B be a Borel subgroup containing T. By Φ we denote the roots of G with respect to T. If α is a root, we denote by s_α the corresponding reflection inside the Weyl group $W = N_G(T)/T$ of G. The negative roots Φ^- is by definition the set of roots which are T weights of the Lie algebra of B. The set of positive root is denoted by Φ^+, while the set of simple positive roots is denoted by Δ. Any subset I of Δ defines a parabolic subgroup P_I containing B. The unipotent radical of a parabolic subgroup P is denoted by U_P and the Levi part by L_P. The Lie algebra of U_P is denoted by $\mathfrak{u}_P$, but when $P_I = B$ we will also use the notation $\mathfrak{u}$.

The character group of T is denoted by Λ, and elements in here is called weights. There is a natural perfect pairing $\langle , \rangle$ between Λ and the set of cocharacters $X_*(T)$. If α is a root we denote by $\alpha^\vee$ the corresponding coroot. In this setting we have

$$s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \ , \ \alpha \in \Phi \ , \lambda \in \Lambda.$$

A weight λ in Λ is said to be dominant if

$$\langle \lambda, \alpha^\vee \rangle \geq 0 \ , \ \forall \alpha \in \Delta.$$

The set of dominant weights is denoted by Λ^+. On Λ we have an order denoted by $\geq$, and defined by $\lambda \geq \mu$ if and only if $\lambda - \mu$ is a sum of positive roots. For each weight λ the W-orbit of λ contains precisely one dominant weight denoted by λ^+. For a weight λ which can be written as a sum of n simple positive roots, we define the height $ht(\lambda)$ of λ to be n. Every T-character $\lambda \in \Lambda$ can be uniquely extended to a B-character. The corresponding 1-dimensional representation is denoted by k_λ.

If P is a parabolic subgroup and M is a P module we denote by $G \times^P M$ the variety which is the quotient of $G \times M$ under the P action

$$p \cdot (g, m) = (gp^{-1}, p.m) \ , \ p \in P, g \in G, m \in M.$$

If X is a variety (over k) we write $k[X]$ for the global regular functions on X, and $\mathcal{O}_X$ for the sheaf of regular functions on X.

3. Weyl group translates

Let $\lambda \in \Lambda$ be a weight. As mentioned above there exist a unique dominant Weyl group translate λ^+ of λ. In this section we will study the behavior of λ^+ with respect to the order $\geq$.

To λ in Λ we define

$$l(\lambda) = \#\{\alpha \in \Phi^+ : \langle \lambda, \alpha^\vee \rangle < 0\},$$

and say that $l(\lambda)$ is the length of λ. Notice that a weight is dominant exactly when is has length zero.

Lemma 1. *Let λ be a weight and α be a short simple root such that $\langle \lambda, \alpha^\vee \rangle$ is negative. Then*

$$l(s_\alpha \lambda) = l(\lambda) - 1.$$

Proof. For any positive root β we have $\langle s_\alpha \lambda, \beta^\vee \rangle = \langle \lambda, s_\alpha(\beta)^\vee \rangle$. Noticing that s_α acts as a permutation on the set $\Phi^+ \setminus \{\alpha\}$ and that $s_\alpha(\alpha) = -\alpha$, this implies the result. □

Corollary 1. *If $\lambda \in \Lambda$ is a weight then $\lambda^+ \geq \lambda$.*

Proof. Let λ be a weight and assume by induction that the statement is correct for weights of smaller length than λ. We may assume that there exist a simple root α as in Lemma 1. Then

$$\lambda \leq s_\alpha(\lambda) \leq (s_\alpha(\lambda))^+ = \lambda^+,$$

where the second equality follows by induction and Lemma 1. □

Proposition 1. *Let $\lambda \in \Lambda$ be a weight and α be a positive root.*

(i). *If $\langle \lambda, \alpha^\vee \rangle \geq 0$ then $\lambda^+ < (\lambda + \alpha)^+$.*
(ii). *If $\langle \lambda, \alpha^\vee \rangle = -1$ then $\lambda^+ = (\lambda + \alpha)^+$.*
(iii). *If $\langle \lambda, \alpha^\vee \rangle \leq -2$ then $\lambda^+ > (\lambda + \alpha)^+$.*

Proof. If $\langle \lambda, \alpha^\vee \rangle = -1$ it follows that $\lambda + \alpha = s_\alpha(\lambda)$ and therefore that $\lambda + \alpha$ is in the the same Weyl group orbit as λ. This implies (ii). Assume now that $\langle \lambda, \alpha^\vee \rangle \geq 0$. We will prove (i) by induction in $l(\lambda)$. Assume that the statement is correct for all μ with $l(\mu) < l(\lambda)$. If λ is dominant we have

$$\lambda^+ = \lambda < \lambda + \alpha \leq (\lambda + \alpha)^+,$$

where the last relation follows from Corollary 1. We may therefore assume that there exist a simple positive root β such that $\langle \lambda, \beta^\vee \rangle$ is negative. By Lemma 1 we know that $l(s_\beta(\lambda)) < l(\lambda)$. As β is simple we also know that $s_\beta(\alpha)$ is positive. By induction (used on $s_\beta(\lambda)$ and $s_\beta(\alpha)$) we therefore conclude

$$\lambda^+ = (s_\beta(\lambda))^+ < (s_\beta(\lambda) + s_\beta(\alpha))^+ = (\lambda + \alpha)^+.$$

This implies (i). Assume finally that $\langle \lambda, \alpha^\vee \rangle \leq -2$. Then

$$\langle s_\alpha(\lambda + \alpha), \alpha^\vee \rangle = -\langle \lambda + \alpha, \alpha^\vee \rangle = -\langle \lambda, \alpha^\vee \rangle - 2 \geq 0$$

By this and the proof of (i) we conclude that

$$(\lambda + \alpha)^+ = (s_\alpha(\lambda + \alpha))^+ < (s_\alpha(\lambda + \alpha) + \alpha)^+ = (s_\alpha(\lambda))^+ = \lambda^+,$$

which ends the proof. □

4. Minimal dominant weights

In the previous section we saw that the dominant Weyl group translate λ^+ of a weight λ had the property that $\lambda^+ \geq \lambda$. In general λ^+ is not minimal with this property. In fact, for most weights λ there exist a dominant weight $\mu \neq \lambda^+$, such that

$$\lambda \leq \mu \leq \lambda^+.$$

However, as the height of $\lambda^+ - \lambda$ is finite we see that there must exist weights μ minimal among dominant weights with the property $\mu \geq \lambda$. The following arguments show that there is a unique minimal dominant weight μ with $\mu \geq \lambda$.

Lemma 2. *Let $\lambda \in \Lambda$ be a weight and α be a simple positive root such that $\langle \lambda, \alpha^\vee \rangle$ is negative. Then every dominant weight μ with $\mu \geq \lambda$ satisfies that $\mu \geq \alpha + \lambda$.*

Proof. Let λ and α be as described above and let μ be a dominant weight with $\mu \geq \lambda$. Let $\alpha_1, \ldots, \alpha_n$ $(n = ht(\mu - \lambda))$ be a collection of (not necessarily distinct) simple positive roots such that

$$\mu - \lambda = \sum_{i=1}^{n} \alpha_i.$$

It is enough to show that there exist an i such that $\alpha = \alpha_i$. By the choice of α we have

$$\langle \mu - \lambda, \alpha^\vee \rangle = \langle \mu, \alpha^\vee \rangle + (-\langle \lambda, \alpha^\vee \rangle) > 0.$$

Therefore also

$$\langle \sum_{i=1}^{n} \alpha_i, \alpha^\vee \rangle > 0.$$

But if $\alpha_j \neq \alpha$ then $\langle \alpha_j, \alpha^\vee \rangle \leq 0$, which implies that there must exist an i such that $\alpha_i = \alpha$. □

Proposition 2. *To each weight $\lambda \in \Lambda$ there exist a dominant weight λ^* such that*

(i). $\lambda^* \geq \lambda$.
(ii). *If μ is dominant and $\mu \geq \lambda$ then $\mu \geq \lambda^*$.*

Proof. For each weight λ define the number

$$N_\lambda = min\{ht(\mu - \lambda) : \mu \text{ dominant and } \mu \geq \lambda\}.$$

Notice that the minimum is taken over a nonempty set as λ^+ is dominant and $\lambda^+ \geq \lambda$. We will prove the proposition by induction in N_λ. So assume that the result is true for all λ' with $N_{\lambda'} < N_\lambda$. If λ is dominant we may choose $\lambda^* = \lambda$. Assume therefore that λ is not dominant. Then there exist a positive simple root α such that $\langle \lambda, \alpha^\vee \rangle$ is negative. By Lemma 2 it follows that $N_{\lambda+\alpha} < N_\lambda$ and that $\lambda^* := (\lambda + \alpha)^*$ satisfies the desired conditions. □

In view of this result we will in the following use the notation λ^* to denote the minimal dominant weight with $\lambda^* \geq \lambda$.

Corollary 2. *Let $\lambda \in \Lambda$ be a weight and α be a positive simple root such that $\langle \lambda, \alpha^\vee \rangle$ is negative. Then $\lambda^* = (\lambda + \alpha)^*$.*

Proof. Clear by Lemma 2. □

Following [3] we define

Definition 1. If $\lambda \in \Lambda$ is a weight, we denote by $Cht(\lambda)$ the largest integer r for which there exist dominant weights $\mu_0, \mu_1, \dots, \mu_r$ satisfying

$$\lambda^* = \mu_0 < \mu_1 < \cdots < \mu_{r-1} < \mu_r = \lambda^+.$$

Proposition 3. *Let $\lambda \in \Lambda$ be a weight and α be a simple positive root.*
(i). *If $\langle \lambda, \alpha^\vee \rangle = -1$ then $Cht(\lambda) = Cht(\lambda + \alpha)$.*
(ii). *If $\langle \lambda, \alpha^\vee \rangle \leq -2$ then $Cht(\lambda) > Cht(\lambda + \alpha)$.*
(iii). *If $\langle \lambda, \alpha^\vee \rangle \leq 0$ then $Cht(\lambda) \geq Cht(s_\alpha(\lambda))$.*
(iv). *If $\langle \lambda, \alpha^\vee \rangle \leq -2$ then $Cht(\lambda) > Cht(s_\alpha(\lambda) - \alpha)$.*

Proof. For (i) and (ii) use Proposition 1 and Corollary 2.

If $\langle \lambda, \alpha^\vee \rangle \leq 0$ then $\lambda \leq s_\alpha(\lambda) \leq (s_\alpha(\lambda))^*$, and by Proposition 2 we conclude that $\lambda^* \leq (s_\alpha(\lambda))^*$. Furthermore $\lambda^+ = (s_\alpha(\lambda))^+$ which proves the equality in (iii). Assume finally that $\langle \lambda, \alpha^\vee \rangle \leq -2$. Then $\langle \lambda + \alpha, \alpha^\vee \rangle \leq 0$ and thus by (iii) and (ii) we have

$$Cht(s_\alpha(\lambda) - \alpha) = Cht(s_\alpha(\lambda + \alpha)) \leq Cht(\lambda + \alpha) < Cht(\lambda).$$

This ends the proof. □

5. Vanishing

In this section α will denote a simple positive root, and P will denote the minimal parabolic subgroup $P_{\{\alpha\}}$ of G.

Lemma 3. *Let V be a P module and $\lambda \in \Lambda$ be a weight. If n denotes the number $\langle \lambda, \alpha^\vee \rangle$, then*

$$H^i(G/B, V \otimes k_\lambda) = \begin{cases} H^i(G/P, V \otimes H^0(P/B, k_\lambda)) & \text{if } n \geq -1, \\ H^{i-1}(G/P, V \otimes H^1(P/B, k_\lambda)) & \text{if } n \leq -1, \\ 0 & \text{if } n = -1. \end{cases}$$

By definition $H^{i-1}(G/P, V \otimes H^1(P/B, k_\lambda))$ is zero when $i = 0$.

Proof. We want to calculate the cohomology group $H^i(G/B, V \otimes k_\lambda)$ by using the spectral sequence corresponding to the natural map $G/B \to G/P$, $gB \mapsto gP$. This gives us

$$E_2^{p,q} = H^p(G/P, V \otimes H^q(P/B, k_\lambda)) \Rightarrow H^{p+q}(G/B, V \otimes k_\lambda).$$

As at most one of the cohomology groups $H^q(P/B, k_\lambda)$ is nonzero (remember that $P/B \simeq \mathbb{P}^1$) this spectral sequence degenerates. Furthermore $H^1(P/B, k_\lambda)$ is nonzero only if $n < -1$ while $H^0(P/B, k_\lambda)$ is nonzero only if $n > -1$. □

Proposition 4. *Notation as in the Lemma 3. If* $(-p-1) \leq n < 0$, *then*

$$H^i(G/B, V \otimes k_\lambda) \simeq H^{i-1}(G/B, V \otimes k_{s_\alpha(\lambda)-\alpha}),\ \forall i \geq 1.$$

Proof. By Lemma 3 we know that

$$H^i(G/B, V \otimes k_\lambda) \simeq H^{i-1}(G/P, V \otimes H^1(P/B, k_\lambda)).$$

Consider $H^1(P/B, k_\lambda)$. As $(-p-1) \leq n < 0$ we know by the Borel-Bott-Weil theorem (see Prop.II.5.4 in [10]) that

$$H^1(P/B, k_\lambda) \simeq H^0(P/B, k_{s_\alpha(\lambda)-\alpha}),$$

as P-modules. Therefore

$$H^i(G/B, V \otimes k_\lambda) \simeq H^{i-1}(G/P, V \otimes H^0(P/B, k_{s_\alpha(\lambda)-\alpha})).$$

As $\langle s_\alpha(\lambda) - \alpha, \alpha^\vee \rangle \geq -1$ the statement follows by using Lemma 3 on the right side. □

5.1. **Application.** Let $\mathfrak{u}_P$ denote the Lie algebra of U_P, and let $\mathfrak{u}$ denote the Lie algebra of U. Restricting linear functions on $\mathfrak{u}$ to $\mathfrak{u}_P$ gives us a short exact sequence

$$0 \to k_\alpha \to \mathfrak{u}^* \to \mathfrak{u}_P^* \to 0.$$

For each integer $i > 0$ this induces a short exact sequence

$$0 \to S^{i-1}(\mathfrak{u}^*) \otimes k_\alpha \to S^i(\mathfrak{u}^*) \to S^i(\mathfrak{u}_P^*) \to 0, \tag{1}$$

where S^i denotes the i'th symmetric product. Thinking of $S^{-1}(\mathfrak{u}^*)$ as being equal to zero, we may also make sense to (1) when $i = 0$. Summing over all $i \geq 0$ we arrive at the following short exact sequence

$$0 \to S^\bullet(\mathfrak{u}^*) \otimes k_\alpha \to S^\bullet(\mathfrak{u}^*) \to S^\bullet(\mathfrak{u}_P^*) \to 0. \tag{2}$$

To ease the notation we now define

Definition 2. If $\lambda \in \Lambda$ is a weight we define

$$H^i(\lambda) := H^i(G/B, S^\bullet(\mathfrak{u}^*) \otimes k_\lambda).$$

$$H^i_\alpha(\lambda) := H^i(G/B, S^\bullet(\mathfrak{u}_P^*) \otimes k_\lambda).$$

Corollary 3. *Let* λ *be a weight such that* $(-p-1) \leq n < 0$ *where* $n = \langle \lambda, \alpha^\vee \rangle$. *Then*

$$H^i_\alpha(\lambda) = H^{i-1}_\alpha(s_\alpha\lambda - \alpha)\ ,\forall i \geq 1.$$

Proof. This follows from Proposition 4 with $V = S^\bullet(\mathfrak{u}_P^*)$. □

Proposition 5. *Let* λ *be a weight such that* $(-p-1) \leq n < 0$ *where* $n = \langle \lambda, \alpha^\vee \rangle$. *Then*

(i). *If* $n = -1$ *then* $H^i(\lambda) \simeq H^i(\lambda + \alpha)$ *for all* $i \geq 1$.
(ii). *If* $H^i(\lambda + \alpha) = H^{i-1}(s_\alpha(\lambda) - \alpha) = H^i(s_\alpha(\lambda)) = 0$ *for an* $i \geq 1$, *then* $H^i(\lambda) = 0$.

Proof. Assume first that $n = -1$. Then Lemma 3 tells us that $H^i_\alpha(\lambda) = 0$ for all $i \geq 0$. Consider now the long exact sequence of cohomology groups induced by the short exact sequence (2) tensored by k_λ :

$$\begin{aligned} &0 \to H^0(\lambda+\alpha) \to H^0(\lambda) \to H^0_\alpha(\lambda) \to \\ &\to H^1(\lambda+\alpha) \to H^1(\lambda) \to H^1_\alpha(\lambda) \to \cdots \\ &\cdots \to H^i(\lambda+\alpha) \to H^i(\lambda) \to H^i_\alpha(\lambda) \to \cdots \end{aligned} \tag{3}$$

Then (i) follows immediately. Assume now that $n \leq -2$ and that $H^i(\lambda+\alpha) = H^{i-1}(s_\alpha(\lambda)-\alpha) = H^i(s_\alpha(\lambda)) = 0$ with $i \geq 1$. Consider the long exact sequence of cohomology groups corresponding to (2) tensored by $k_{s_\alpha(\lambda)-\alpha}$:

$$\begin{aligned} \cdots \to H^{i-1}(s_\alpha(\lambda)) \to H^{i-1}(s_\alpha(\lambda)-\alpha) \to H^{i-1}_\alpha(s_\alpha(\lambda)-\alpha) \to \cdots \\ \to H^i(s_\alpha(\lambda)) \to \cdots \end{aligned}$$

By the assumptions we conclude that $H^{i-1}_\alpha(s_\alpha(\lambda)-\alpha) = 0$. This implies by Corollary 3 that $H^i_\alpha(\lambda) = 0$. Finally this together with the assumptions implies, by using the exact sequence (3), that $H^i(\lambda) = 0$. □

6. The Vanishing Theorem

In this section we will state a vanishing theorem which will enable us to conclude the normality of certain nilpotent varieties. Compared to the characteristic zero situation in [3], the vanishing result in positive characteristic is less general. Still the positive characteristic vanishing result is sufficient to conclude the same normality results as in characteristic zero. From now on we assume that the characteristic of the ground field is good, which means

Definition 3. If G is almost simple then the characteristic p of the ground field is said to be a *good* prime for G if : $p \geq 2$ for type A, $p \geq 3$ for type B, C and D, $p \geq 5$ for type F_4 ,G_2 ,E_6 and E_7, $p \geq 7$ for type E_8. If G is arbitrary, the characteristic is defined to be good if it is so for all almost simple normal subgroups of G.

To prove the vanishing result we have to restrict our attention to the following subset of the set of weights :

$$C_p = \{\lambda \in \Lambda : \langle \lambda, \beta^\vee \rangle \geq (-p-1)\ ,\ \forall \beta \in \Phi^+\}.$$

The invariance of this set is described in

Lemma 4. *Let λ be an element of C_p and let α be a positive simple root such that $n := \langle \lambda, \alpha^\vee \rangle$ is negative. If m is an integer satisfying $0 \leq m \leq -n$ then*

$$\lambda + m\alpha \in C_p.$$

In particular $(\lambda+\alpha)$, $s_\alpha(\lambda)$ and $(s_\alpha(\lambda)-\alpha)$ belongs to C_p.

Proof. Let β be a positive root, and m be an integer between 0 and $-n$. If $\langle \alpha, \beta^\vee \rangle \geq 0$, then

$$\langle \lambda + m\alpha, \beta^\vee \rangle = \langle \lambda, \beta^\vee \rangle + m \langle \alpha, \beta^\vee \rangle \geq -p - 1.$$

We may therefore assume that $\langle \alpha, \beta^\vee \rangle < 0$. Then $\beta \neq \alpha$ and

$$\begin{aligned} \langle \lambda + m\alpha, \beta^\vee \rangle &\geq \langle \lambda - n\alpha, \beta^\vee \rangle \\ &= \langle s_\alpha(\lambda), \beta^\vee \rangle \\ &= \langle \lambda, s_\alpha(\beta)^\vee \rangle \geq (-p-1). \end{aligned}$$

The last equality follows as $s_\alpha(\beta)$ is a positive root as α is simple and not equal to β. □

Theorem 1. *For every weight λ in C_p we have*

$$H^i(\lambda) = 0 \ , \ \forall i > Cht(\lambda).$$

Proof. By induction we may assume that the statement is correct for all weights λ' in C_p satisfying

$$Cht(\lambda') < Cht(\lambda) \quad \text{or} \quad (Cht(\lambda') = Cht(\lambda) \text{ and } l(\lambda') < l(\lambda)).$$

If λ is dominant then $Cht(\lambda) = 0$ and the result follows from Thm. 2 in [12]. We may therefore assume that there exist a simple root α such that $n := \langle \lambda, \alpha^\vee \rangle$ is negative.

If $n = -1$ then $Cht(\alpha + \lambda) = Cht(\lambda)$ by Proposition 3. At the same time Lemma 1 tells us that $l(\alpha + \lambda) < l(\lambda)$. As $\alpha + \lambda$ is an element of C_p by Lemma 4 we conclude by induction that

$$H^i(\alpha + \lambda) = 0 \ , \ \forall i > Cht(\lambda).$$

Using Proposition 5(i) the result follows.

We may therefore assume that $n \leq -2$. By Proposition 3 we have $Cht(\lambda + \alpha) < Cht(\lambda)$. Consequently induction and Lemma 4 tells us that

$$H^i(\lambda + \alpha) = 0 \ , \ \forall i > Cht(\lambda). \tag{4}$$

Consider now $s_\alpha(\lambda)$. Then Proposition 3(iii), Lemma 1, Lemma 4 and induction tells us

$$H^i(s_\alpha(\lambda)) = 0 \ , \ \forall i > Cht(\lambda). \tag{5}$$

Consider finally $s_\alpha(\lambda) - \alpha$. Then Proposition 3(iv), Lemma 4 and induction tells us

$$H^i(s_\alpha(\lambda) - \alpha) = 0 \ , \ \forall i > Cht(\lambda) - 1. \tag{6}$$

Now (4), (5) and (6) together with Proposition 5 ends the proof. □

7. Pairwise orthogonal simple short roots

From now on we assume that the group G is almost simple.

Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be a set of pairwise orthogonal simple short roots, and let $\alpha := \alpha_1 + \cdots + \alpha_m$. In this section we will calculate $Cht(\alpha)$. As for a large part of this paper, this may also be found in [3]. However in this paper we will here choose a slightly different approach following an argument shown to us by J. C. Jantzen.

Choose a Weyl group invariant bilinear form $(,)$ on the Euclidean space $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, and normalize it so that $(\alpha, \alpha) = 1$ whenever α is a short root. With this normalization of $(,)$ we have

Lemma 5. *If β is a root, then (β, β) is an integer.*

Proof. Let β be any root. As the underlying root system of G is irreducible there exist a short root α such that $(\beta, \alpha) \neq 0$ (see [8] Lemma B, Sect. 10.4). Then Table 1, Sect. 9.4 in [8], tells us that

$$(\beta, \beta) = \frac{(\beta, \beta)}{(\alpha, \alpha)} \in \{1, 2, 3\}.$$

□

Lemma 6. *If λ is a weight in the root lattice $\mathbb{Z}\Phi$, then (λ, λ) is an integer.*

Proof. Write λ as a sum of roots $\lambda = \sum_{i=1}^{n} \beta_i$. Then

$$\begin{aligned}(\lambda, \lambda) &= \sum_{1 \leq i < j \leq n} 2(\beta_i, \beta_j) + \sum_{i=1}^{n} (\beta_i, \beta_i) \\ &= \sum_{1 \leq i < j \leq n} (\beta_j, \beta_j)\langle \beta_i, \beta_j^{\vee} \rangle + \sum_{i=1}^{n} (\beta_i, \beta_i),\end{aligned}$$

which is an integer by Lemma 5. □

Lemma 7. *Let λ and μ be dominant weights with $\mu < \lambda$. Then $(\mu, \mu) < (\lambda, \lambda)$.*

Proof. Consider

$$(\lambda, \lambda) = (\mu, \mu) + (\lambda - \mu, \lambda - \mu) + 2(\lambda - \mu, \mu) > (\mu, \mu) + 2(\lambda - \mu, \mu).$$

As μ is dominant and $\lambda > \mu$ the number $(\lambda - \mu, \mu)$ is nonnegative, and the result follows. □

Proposition 6. $Cht(\alpha) = m - 1$.

Proof. Let γ denote the short dominant root. Then $\gamma = \alpha_i^+$ for all i. In particular $\alpha_i \leq \gamma$ for all i, and as γ may be written uniquely as a sum of simple roots we conclude that $\alpha \leq \gamma$. Therefore $\alpha^* \leq \gamma$. If $\alpha^* \neq \gamma$ then Lemma 7 tells us that $(\alpha^*, \alpha^*) < (\gamma, \gamma) = 1$, as γ was a short root. By Lemma 6 this implies that $(\alpha^*, \alpha^*) = 0$ or that $\alpha^* = 0$. This contradicts that $\alpha \leq \alpha^*$ and we conclude that $\alpha^* = \gamma$.

Consider a sequence $\mu_0, \dots, \mu_r$ of dominant weights with

$$\alpha^* = \mu_0 < \mu_1 < \cdots < \mu_{r-1} < \mu_r = \alpha^+.$$

Then by Lemma 7 we have

$$1 = (\alpha^*, \alpha^*) < (\mu_1, \mu_1) < \cdots < (\mu_r, \mu_r) = (\alpha^+, \alpha^+) = m. \tag{7}$$

Here the last equality follows as α is a sum of pairwise orthogonal short roots, and as $(\alpha^+, \alpha^+) = (\alpha, \alpha)$. Using Lemma 6 we conclude from (7) that $r \leq m-1$, and consequently $Cht(\alpha) \leq m-1$.

To see the opposite equality put $\mu_i = (\alpha_1 + \cdots + \alpha_{i+1})^+$ for $i = 1, \dots, m-1$. Then Proposition 1(i) gives us a sequence of dominant weights :

$$\alpha^* < \mu_1 < \cdots < \mu_{m-2} < \mu_{m-1} = \alpha^+.$$

This implies that $Cht(\alpha) \geq m-1$. □

We will also need that α is contained in C_p, which follows from

Lemma 8. *Let $\alpha_1, \alpha_2, \dots, \alpha_m$ be a set of short pairwise orthogonal simple roots, and let $\alpha := \alpha_1 + \cdots + \alpha_m$ denote their sum. Then*

$$\langle \alpha, \beta^\vee \rangle \geq -3 \ , \ \forall \beta \in \Phi^+.$$

In particular $\alpha \in C_p$.

Proof. Let β be a positive root. We may assume that $\langle \alpha_i, \beta^\vee \rangle < 0$ for all $i = 1, \dots, m$. As each α_i is short this means that $\langle \alpha_i, \beta^\vee \rangle = -1$. So we have to show that $m \leq 3$. Assume therefore that $m = 4$. Then

$$\begin{aligned}(\alpha + 2\beta, \alpha + 2\beta) &= (\alpha, \alpha) + 4(\beta, \beta) + 4(\alpha, \beta) \\ &= 4 + 4(\beta, \beta) + 2(\beta, \beta)\langle \alpha, \beta^\vee \rangle \\ &= 4(1 - (\beta, \beta)) \leq 0.\end{aligned}$$

The last equality follows from Lemma 5. This implies that $\alpha + 2\beta = 0$ which is a contradiction as $\alpha + 2\beta$ is a nontrivial sum of positive roots □

8. Normality

Let from now on $\alpha_1, \dots, \alpha_m$ be a set of pairwise orthogonal short simple roots, and consider the parabolic subgroup $P = P_I$ of G with $I = \{\alpha_1, \dots, \alpha_m\}$. Let $\mathfrak{u}_P$ denote the Lie algebra of U_P and $\mathfrak{u}$ denote the Lie algebra of the unipotent radical U of B. Consider the short exact sequence of B-modules

$$0 \to (\mathfrak{u}/\mathfrak{u}_P)^* \to \mathfrak{u}^* \to \mathfrak{u}_P^* \to 0. \tag{8}$$

As $\alpha_1, \dots, \alpha_m$ are pairwise orthogonal we know that

$$(\mathfrak{u}/\mathfrak{u}_P)^* \simeq k_{\alpha_1} \oplus k_{\alpha_2} \oplus \cdots \oplus k_{\alpha_m}.$$

For each positive integer s we let V_s denote the s'th exterior power of $(\mathfrak{u}/\mathfrak{u}_P)^*$, i.e. :

$$V_s = \wedge^s(\mathfrak{u}/\mathfrak{u}_P)^* = \bigoplus_{1\leq n_1<n_2<\cdots<n_s\leq m} k_{\alpha_{n_1}+\cdots+\alpha_{n_s}}.$$

By Proposition 6 we see that V_s is a direct sum of 1-dimensional B-representations with weights λ with $Cht(\lambda) = s-1$. In particular

Lemma 9. *$H^i(G/B, S^\bullet\mathfrak{u}^* \otimes V_s) = 0$, $i \geq s$.*

Proof. Use Lemma 8 and Theorem 1. □

Let j be a positive integer and consider the Koszul resolution induced by (8) and j (see e.g. [10] II.12.12)

$$\cdots \to S^{j-s}\mathfrak{u}^* \otimes V_s \to \cdots \to S^{j-2}\mathfrak{u}^* \otimes V_2 \to S^{j-1}\mathfrak{u}^* \otimes V_1 \to K_j \to 0.$$

Here K_j denotes the kernel of the surjective map $S^j\mathfrak{u}^* \to S^j\mathfrak{u}_P^*$.

Lemma 10. *$H^i(G/B, K_j) = 0$, $i > 0$.*

Proof. Break the Koszul resolution of K_j up into short exact sequences and use Lemma 9. □

Consider the short exact sequence

$$0 \to K \to S^\bullet\mathfrak{u}^* \to S^\bullet\mathfrak{u}_P^* \to 0, \tag{9}$$

where $K = \oplus_j K_j$. Then Lemma 10 tells us that the induced map

$$H^0(G/B, S^\bullet\mathfrak{u}^*) \to H^0(G/B, S^\bullet\mathfrak{u}_P^*) \simeq H^0(G/P, S^\bullet\mathfrak{u}_P^*), \tag{10}$$

is surjective. The last isomorphism follows by an argument similar to the one used in the proof of Lemma 3.

8.1. **Springer resolution.** Let $\mathcal{N}$ denote the closed subset of nilpotent elements inside the Lie algebra $\mathfrak{g}$ of G. By results of Springer it is known that $\mathcal{N}$ is a normal variety. The projective morphism

$$\begin{aligned} \pi : G \times^B \mathfrak{u} &\to \mathcal{N} \\ (g, x) &\mapsto g \cdot x. \end{aligned} \tag{11}$$

is a resolution of singularities of $\mathcal{N}$ and is called the Springer resolution. As $\mathcal{N}$ is normal and π is a birational projective map we know that $\pi_*(\mathcal{O}_{G\times^B\mathfrak{u}}) = \mathcal{O}_{\mathcal{N}}$. In particular

$$k[\mathcal{N}] \xrightarrow{\pi^*} k[G \times^B \mathfrak{u}] \tag{12}$$

is surjective. Consider the commutative diagram

$$\begin{tikzcd} G \times^P \mathfrak{u}_P \arrow[drr, "\pi_P"] & & G \times^B \mathfrak{u}_P \arrow[ll, "p"'] \arrow[r, hook, "\iota_1"] \arrow[d, "\pi_{|G\times^B\mathfrak{u}_P}"] & G \times^B \mathfrak{u} \arrow[d, "\pi"] \\ & & \overline{G\mathfrak{u}_P} \arrow[r, hook, "\iota_2"] & \mathcal{N} \end{tikzcd}$$

Here ι_1 and ι_2 are inclusion maps, p denotes the natural projection map and $\overline{G\mathfrak{u}_P}$ denotes the closure of $G\mathfrak{u}_P$ inside $\mathcal{N}$. Reformulating (10) we see that

$$k[G \times^B \mathfrak{u}] \xrightarrow{\iota_1^*} k[G \times^B \mathfrak{u}_P] \overset{p^*}{\simeq} k[G \times^P \mathfrak{u}_P]$$

is surjective, and composing this with the surjective map (12) and using the commutativity of the diagram above, we conclude that

$$k[\overline{G\mathfrak{u}_P}] \xrightarrow{\pi_P^*} k[G \times^P \mathfrak{u}_P] \tag{13}$$

is surjective. Furthermore as π_P^* is dominant (13) is in fact an isomorphism.

8.2. **Proofs of main results.**

Proposition 7. *$\overline{G\mathfrak{u}_P}$ is a normal variety.*

Proof. As $G \times^P \mathfrak{u}_P$ is a normal variety the ring $k[G \times^P \mathfrak{u}_P]$ is normal. As π_P^* is an isomorphism this implies that the coordinate ring $k[\overline{G\mathfrak{u}_P}]$ of the affine variety $\overline{G\mathfrak{u}_P}$ is normal. □

That π_P^* is an isomorphism means, as $\overline{G\mathfrak{u}_P}$ is affine, that

$$(\pi_P)_* \mathcal{O}_{G\times^P \mathfrak{u}_P} = \mathcal{O}_{\overline{G\mathfrak{u}_P}}. \tag{14}$$

Using that π_P is projective this implies ([6] Cor.III.11.3) that π_P has connected fibers.

Lemma 11. *The map π_P is birational.*

Proof. As π_P is a dominant morphism between varieties of the same dimension, and as the fibers of π_P are connected, it is enough to show that π_P is separable. By a theorem of Richardson (see [15]) the P module $\mathfrak{u}_P$ has a dense P-orbit. Let $x \in \mathfrak{u}_P$ be an element in this dense P-orbit, and consider the morphism $\iota : G \to G \times^P \mathfrak{u}_P$ given by $\iota(g) = (g, x)$. By the choice of x this map is dominant. As a composite of two field extensions $F_0 \subseteq F_1 \subseteq F_2$ is separable only if $F_0 \subseteq F_1$ is separable, we see that it is enough to show that $\pi_P \circ \iota$ is separable. In other words we have to show that the orbit map $g \mapsto g \cdot x$ of x is separable. Using that the the nilpotent variety $\mathcal{N}$ is isomorphic to the unipotent variety, it is enough to show that the orbit map $g \mapsto g \cdot x'$ is separable for any unipotent element x' in G. Unless G is of type A, this now follows from [17], I 5.1-5.6 and [1], Sect. 9.1. If G is of type A we may compose $\pi_P \circ \iota$ with the natural map $GL_n(k) \mapsto G$ and use a similar argument. □

For convenience of the reader we state the following definition of a rational resolution.

Definition 4. A proper birational map $f : X \to Y$ is a *rational resolution* of Y if

1. Y is normal and X is smooth.

2. $R^i f_*(\mathcal{O}_X) = 0$, $i > 0$.
3. $R^i f_*(\omega_X) = 0$, $i > 0$, where ω_X is the dualizing sheaf of X.

If there exist a rational resolution of Y we say that Y has *rational singularities.*

Lemma 12. *The smooth variety $G \times^P \mathfrak{u}_P$ has trivial dualizing sheaf.*

Proof. By [16] Lemma 4.4 we recognize $\mathfrak{u}_P$ as $(\mathfrak{g}/\mathfrak{p})^*$, where $\mathfrak{p}$ is the Lie algebra of P. The result now follows as $G \times^P (\mathfrak{g}/\mathfrak{p})^*$ is the cotangent bundle over G/P and as such has trivial dualizing sheaf. $\square$

Theorem 2. *The nilpotent variety $\overline{G\mathfrak{u}_P}$ is a normal Gorenstein variety with rational singularities.*

Proof. We claim that π_P is a rational resolution of $\overline{G\mathfrak{u}_P}$. By Proposition 7, Lemma 11 and Lemma 12 this follows if we show

$$R^i(\pi_P)_*(\mathcal{O}_{G\times^P\mathfrak{u}_P}) = 0 \ , i > 0.$$

As $\overline{G\mathfrak{u}_P}$ is affine this is equivalent to

$$H^i(G/P, S^\bullet\mathfrak{u}_P^*) = H^i(G \times^P \mathfrak{u}_P, \mathcal{O}_{G\times^P\mathfrak{u}_P}) = 0 \ , i > 0. \tag{15}$$

Using an argument similarly to the one used in the proof of Lemma 3 we see that $H^i(G/P, S^\bullet\mathfrak{u}_P^*) = H^i(G/B, S^\bullet\mathfrak{u}_P^*)$. Furthermore Lemma 10 and the short exact sequence (9) implies that $H^i(G/B, S^\bullet\mathfrak{u}_P^*) = H^i(G/B, S^\bullet\mathfrak{u}^*)$. Now (15) follows from Theorem 1. That $\overline{G\mathfrak{u}_P}$ is Gorenstein now follows (see [5], p.49-50) from Lemma 12. $\square$

9. Bala-Carter labels and partitions

In the previous section the normality of certain kinds of Richardson nilpotent varieties was proved. These varieties were given by sets of short pairwise orthogonal simple roots. In this section we will determine exactly which nilpotent varieties that arises in this way. For the classical groups we will parameterize these by certain partitions and for the exceptional groups we will use the Bala-Carter label. Much of this is straightforward checking from definitions and tables, and most of this will be left to the reader. Useful references are [7] (Lemma 7.3.) and Table 1-3, 6-8 in [14].

Lemma 13. *Let $\gamma_1, \gamma_2, \dots, \gamma_m, \gamma$ and $\gamma_1, \dots, \gamma_m, \gamma'$ be two sets of pairwise orthogonal short simple roots, and let P and P' denote the corresponding parabolic groups. If $\langle \gamma', \gamma^\vee \rangle < 0$ then $\overline{G\mathfrak{u}_P} = \overline{G\mathfrak{u}_{P'}}$.*

Proof. Let L and L' denote the Levi parts of P and P' respectively. As

$$s_\gamma s_{\gamma'}(\gamma) = \gamma'$$
$$s_\gamma s_{\gamma'}(\gamma_i) = \gamma_i \ , i = 1, \dots, m,$$

it follows that L and L' are conjugated under the Weyl group element $s_\gamma s_{\gamma'}$. By definition this means that P and P' are associated (see [9] p. 84) and the corresponding Richardson orbits are therefore equal. $\square$

Definition 5. We say : (*) two sets $\gamma_1, \dots, \gamma_m, \gamma$ and $\gamma_1, \dots, \gamma_m, \gamma'$ of pairwise orthogonal simple short roots are *equivalent* if γ and γ' are non-orthogonal. In general two sets E and E' of short simple pairwise orthogonal roots is said to be *equivalent* if there exist a sequence $E_0, E_1 \dots, E_r$ of sets of short simple pairwise orthogonal roots such that for each $i = 0, \dots, r$ the set E_i is equivalent to E_{i+1} in the sense of (*).

By Lemma 13 we see that two equivalent sets of pairwise orthogonal simple short roots determine the same nilpotent variety. In the following $\beta_1, \beta_2, \dots, \beta_n$ will denote the simple roots of G taken in the Bourbaki order.

Lemma 14. *The equivalence classes of sets of short pairwise orthogonal simple roots are given by*

1. *If G is of type A_n, B_n, C_n, G_2, F_4, E_6 or E_8 then any two sets of pairwise orthogonal simple short roots of rank m are equivalent.*
2. *If G is of type E_7 then any two sets of pairwise orthogonal simple short roots of rank $m \neq 3$ are equivalent. The sets $\beta_1, \beta_2, \beta_5$ and $\beta_2, \beta_5, \beta_7$ represent the two equivalence classes of rank 3*
3. *In case G is of type D_n we have the following two distinct types of equivalence classes :*
   (a) *Any two sets of short pairwise orthogonal simple roots of the same rank not containing both β_{n-1} and β_n are equivalent, except that $\beta_1, \beta_3, \dots, \beta_{n-3}, \beta_{n-1}$ and $\beta_1, \beta_3, \dots, \beta_{n-3}, \beta_n$ are nonequivalent when n is even.*
   (b) *Any two sets of short pairwise orthogonal simple roots of the same rank containing both β_{n-1} and β_n are equivalent.*

Proof. Straightforward by definition. □

As above we let $\overline{G\mathfrak{u}_P}$ denote the nilpotent variety determined by a set $\alpha_1, \dots, \alpha_m$ of short pairwise orthogonal short simple roots. The following types of nilpotent varieties is of this form.

A_n : In this case nilpotent orbits correspond to partitions of n. For each integer $1 \leq m \leq (n+1)/2$ there exist $\alpha_1, \dots, \alpha_m$ as above, and the corresponding nilpotent variety is given by the partition $[n, n-m]$.

B_n : In this case nilpotent orbits correspond to partitions of $2n+1$ in which even parts appears with even multiplicity. As there is only one short simple root, only the subregular nilpotent variety is contained in Theorem 2. This correspond to the partition $[2n+1]$.

C_n : In this case nilpotent orbits correspond to partitions of $2n$ in which odd parts occur with even multiplicity. For each integer $1 \leq m \leq n/2$ there exist $\alpha_1, \dots, \alpha_m$ as above, and the corresponding nilpotent variety is given by the partition $[2(n-m), 2m]$.

D_n : In this case nilpotent orbits correspond to partitions of $2n$ in which even parts occur with even multiplicity, except that every partition with only even parts (the "very even" partitions) correspond to two orbits. We divide into 3 cases (remember that $\beta_1, \beta_2, \dots \beta_n$ denotes the simple roots of G)

(a) If $\{\beta_{n-1}, \beta_n\} \nsubseteq \{\alpha_1, \dots, \alpha_m\}$: For each integer $1 \le m < n/2$ there exist $\alpha_1, \dots, \alpha_m$ with the above properties, and the corresponding partition is $[2(n-m)-1, 2m+1]$.

(b) If $\{\beta_{n-1}, \beta_n\} \nsubseteq \{\alpha_1, \dots, \alpha_m\}$ and $m = n/2$: There is 2 possible nonequivalent sets of $\alpha_1, \dots, \alpha_m$ with these properties. These correspond to the 2 nilpotent orbits with very even partition $[n^2]$.

(c) Both β_{n-1} and β_n are among $\alpha_1, \dots, \alpha_m$: For each integer $2 \le m \le n/2+1$ a set $\alpha_1, \dots, \alpha_m$ with the above properties exist, and the corresponding partition is $[2(n-m)+1, 2m-3, 1^2]$.

G_2 : In this case $m = 1$ and the only nilpotent variety which is contained in Theorem 2 is the subregular nilpotent variety $G_2(a_1)$.

F_4 : In this case $m = 1$ and the only nilpotent variety which is contained in Theorem 2 is the subregular nilpotent variety $F_4(a_1)$.

E_6 : By Lemma 14 three distinct nilpotent varieties are contained in Theorem 2. Each of these nilpotent varieties is contained in those of higher dimension. In particular the nilpotent variety corresponding to $m = 3$ sits in a sequence of nilpotent varieties of length 4. At the same time Table 1 in [14] tells us that $A_5 + A_1$ (corresponding to Bala-Carter label $E_6(a_3)$) is contained in a nilpotent variety of the form $\overline{G\mathfrak{u}_P}$ with $m = 3$. By Table 6 in [14] this forces D_5 and $E_6(a_1)$ to correspond to $m = 2$ and $m = 1$ respectively.

E_7 : By Lemma 14 there are three or four distinct nilpotent varieties contained in Theorem 2. Each of these nilpotent varieties is contained in those of higher dimension. In particular the nilpotent variety corresponding to $m = 4$ must be contained in a sequence of nilpotent varieties of length 5. At the same time Table 2 in [14] tells us that $E_6(a_1)$ is contained in a nilpotent variety of the form $\overline{G\mathfrak{u}_P}$ with $m = 4$. By Table 7 in [14] this forces $E_6(a_1)$ to be the nilpotent variety corresponding to $m = 4$. We also conclude that $E_7(a_2)$ and $E_7(a_1)$ corresponds to $m = 2$ and $m = 1$ respectively. The case $m = 3$ remains. In this case there is 2 equivalence classes of sets of short simple pairwise orthogonal roots. By Table 7 in [14] they correspond to either E_6 or $D_6 + A_1$ (Bala-Carter label $E_7(a_3)$). By Table 2 in [14] (and dimension reasoning) it follows that E_6 corresponds to $\beta_2, \beta_5, \beta_7$, while the nilpotent variety with Bala-Carter label $E_7(a_3)$ arises from e.g. $\beta_2, \beta_3, \beta_6$.

E_8 : By Lemma 14 four distinct nilpotent varieties are contained in Theorem 2. Each nilpotent variety is contained in those of higher dimension. In particular the nilpotent variety corresponding to $m = 4$ must be contained in a sequence of nilpotent varieties of length 5. At the same time Table 3 in [14] tells us that D_8 (Bala-Carter label $E_8(a_4)$) is contained in a nilpotent variety of the form $\overline{G\mathfrak{u}_P}$ with $m = 4$. By Table 8 in [14] this forces the variety with Bala-Carter label $E_8(a_4)$ to be the nilpotent variety corresponding to $m = 4$. We also conclude that $E_7 + A_1$ (Bala-Carter label $E_8(a_3)$), $E_8(a_2)$ and $E_8(a_1)$ corresponds to $m = 3$, $m = 2$ and $m = 1$ respectively.

Remark 1. The normality of all nilpotent varieties for groups of type A over fields of positive characteristic was proved by Donkin in [4]. Using the theory of Frobenius splitting, this result together with rational singularities was later also obtained by Mehta and van der Kallen [13]. For arbitrary groups (and good characteristics) the subregular nilpotent variety is known [12] to be a normal Gorenstein variety with rational singularities. Besides the overlap with these result it seems that the results in this paper are new.

References

1. A. Borel, *Linear Algebraic Groups*, Springer Verlag, 1991.
2. A. Broer, *Line bundles on the cotangent bundle of the flag variety*, Invent. Math. **113** (1993), 1–20.
3. ______, *Normality of some nilpotent varieties and cohomology of line bundles on the cotangent bundle of the flag variety*, Lie theory and geometry (Boston), Prog. Math., Birkhäuser, 1994, pp. 1–19.
4. S. Donkin, *The normality of closures of conjugacy classes of matrices*, Invent. Math. **101** (1990), 717–736.
5. G. Kempf et al., *Toroidal embeddings*, Lecture Notes in Mathematics, vol. 339, Springer-Verlag, Berlin-New York, 1973.
6. R. Hartshorne, *Algebraic Geometry*, GTM 52, Springer Verlag, 1977.
7. W.H. Hesselink, *Polarisations in the classical groups*, Math. Z. **160** (1978), 217–234.
8. J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, GTM 9, Springer Verlag, 1972.
9. ______, *Conjugacy Classes in Semisimple Algebraic Groups*, Mathematical surveys and monographs, vol. 43, A.M.S., 1995.
10. J. C. Jantzen, *Representations of Algebraic Groups*, Academic Press, 1987.
11. H. Kraft and C. Procesi, *On the geometry of conjugacy classes in classical groups*, Comment. Math. Helv. **57** (1982), 539–602.
12. S. Kumar, N. Lauritzen and J. F. Thomsen, *Frobenius splitting of cotangent bundles of flag varieties*, To appear in Invent. Math.
13. V.B. Mehta and W. van der Kallen, *A simultaneous frobenius splitting for closures of conjugacy classes of nilpotentmatrices.*, Comp. Math. **84** (1992), 211–221.
14. K. Mizuno, *The conjugate classes of unipotent elements of the Chevalley groups E_7 and E_8*, Tokyo J. Math **3** (1980), 391–461.
15. R.W. Richardson, *Conjugacy classes in parabolic subgroups of semisimple algebraic groups*, Bull. London Math. Soc. (1974), 21–24.

16. T.A. Springer, *The unipotent variety of a semisimple group*, Algebraic Geometry (London) (S. Abhyankar, ed.), Oxford University Press, 1969, pp. 373–391.
17. T.A. Springer and R. Steinberg, *Conjugacy classes*, Seminar on Algebraic Groups and Related Finite Groups (A. Borel, ed.), Lecture notes in Mathematics, vol. 131, Springer Verlag, 1970.

Matematisk Institut, Aarhus Universitet, Ny Munkegade, DK-8000 Århus C, Denmark

E-mail address: funch@imf.au.dk