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QUASI-HOMOGENEOUS PURELY  
ELLIPTIC COMPLETE  
INTERSECTION SINGULARITIES**

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# A CHARACTERIZATION OF QUASI-HOMOGENEOUS PURELY ELLIPTIC COMPLETE INTERSECTION SINGULARITIES

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ABSTRACT. It is well-known that quasi-homogeneity is characterized by equality of the Milnor and Tjurina numbers for isolated complex analytic hypersurface singularities and for low-dimensional isolated complete intersection singularities. In this paper we prove that this characterization extends to isolated purely elliptic complete intersection singularities, with bounds on neither the embedding codimension nor the dimension of the singularity.

## INTRODUCTION

The famous theorem of K. Saito [Sa] from 1971 states that an isolated hypersurface singularity is quasi-homogeneous (i.e. has a good  $\mathbb{C}^*$ -action) if and only if the Milnor number equals the Tjurina number. The question of characterizing quasi-homogeneity for other types of singularities has subsequently occupied many authors (see e.g. [Wal], [G2], [Wahl2], [HM], [XY], [V] to mention but a few), and Saito's result has been extended to irreducible curve singularities by G.M. Greuel [G2, 2.6(3)] and isolated complete intersection surface singularities by J. Wahl [Wahl2] using rather different techniques.

In this paper we shall focus on higher-dimensional isolated complete intersection singularities. The Milnor number and Tjurina number are defined for such singularities, and it is natural to ask if Saito's result generalizes? The main theorem below give an affirmative answer for *purely elliptic* singularities (1.4) which form the border line between rational and 'general' singularities (they are Du Bois but non-rational or, in different terms, log-canonical but non-canonical). In contrast with earlier results it is interesting to note that within the class of purely elliptic singularities there are no bounds on dimension and embedding codimension.

Throughout the paper a *singularity* will mean a germ  $(X, x)$  of a  $\mathbb{C}$ -analytic space. For an isolated complete intersection singularity (ICIS for short) one defines the Milnor number  $\mu = \mu(X, x)$  as the number of spheres appearing in the vanishing cohomology, and the Tjurina number  $\tau = \tau(X, x)$  is the  $\mathbb{C}$ -dimension of the space  $T_{X,x}^1$  of first-order deformations of  $(X, x)$ . See [L] for details.

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**Main Theorem.** *Let  $(X, x)$  be a purely elliptic ICIS of dimension  $n \geq 2$ . Then  $(X, x)$  is quasi-homogeneous if and only if  $\mu = \tau$ .*

The ‘only if’-part is due to Greuel [G1, Satz 0.3] and holds for any ICIS of positive dimension.

To find sufficient conditions for a purely elliptic ICIS  $(X, x)$  to be quasi-homogeneous we start in section 1 with a discussion on the *Hodge type*  $(0, s)$  of  $(X, x)$ ,  $0 \leq s \leq n - 1$ , as defined by S. Ishii [I]. This invariant may be defined from the mixed Hodge theory on the  $(n - 1)$ -cohomology of the link of  $(X, x)$  and measures the ‘purity’ of a certain component of the Hodge-graduation. It will be clear (1.9) that Hodge type  $(0, n - 1)$  is a necessary condition for quasi-homogeneity, but it is not quite sufficient when  $n \geq 3$  (1.11).

E. Looijenga and J. Steenbrink gave in [LS] a formula expressing for any ICIS  $(X, x)$  with  $n \geq 2$  the difference  $\mu - \tau$  as a sum of non-negative terms computed from the Hodge theory of the link and cohomological data of a choice of resolution. We extract from these data a non-negative analytic invariant  $\chi = \chi(X, x)$  satisfying  $\mu - \tau \geq \chi$ . When  $(X, x)$  is purely elliptic the only possible values of  $\chi$  are 0 and 1, and we prove that  $\chi = 0$  precisely when  $(X, x)$  is quasi-homogeneous (Thm 2.6). This result is sharper than (and clearly implies) the ‘if’-part of Main Theorem since the difference  $\mu - \tau$  generally is much greater than  $\chi$ .

The proof that  $\chi = 0$  implies quasi-homogeneity owes much to the ideas and techniques in the papers [I] and [Wahl2] by S. Ishii and J. Wahl. For any good resolution  $(Y, D) \rightarrow (X, x)$  of a purely elliptic ICIS  $(X, x)$  satisfying  $\chi = 0$  there is a distinguished ‘essential’ exceptional irreducible component  $D_J \leq D$ . We produce a vector field (i.e. a derivation)  $\xi$  on  $(X, x)$  that lifts to a meromorphic vector field on  $Y$  with a well-described logarithmic vanishing along  $D_J$ , whose action on the Zariski tangent space is non-nilpotent. By a theorem of Scheia and Wiebe [SW] this ensures quasi-homogeneity except in a more delicate special case which is treated separately.

## 1. PURELY ELLIPTIC SINGULARITIES

**1.1.** Let  $(X, x)$  be a normal singularity of dimension  $n \geq 2$ , and let  $(Y, D) \xrightarrow{\pi} (X, x)$  be a good resolution, i.e.  $Y$  is a nonsingular space,  $D = \pi^{-1}(x)_{\text{red}}$  is a divisor with normal crossings, and  $\pi$  induces an isomorphism of  $Y \setminus D$  and  $X \setminus \{x\}$ . We will assume that each irreducible component of  $D$  is non-singular.

Choose representatives  $X, Y$  such that  $X$  is a good contractible Stein representative of the singularity  $(X, x)$  and  $Y = \pi^{-1}(X)$ . Put  $U = X \setminus \{x\}$  (which we identify via  $\pi$  with  $Y \setminus D$ ). Then  $U$  is homotopy equivalent to the link of  $(X, x)$ . Let  $\Omega_Y^p$  (resp.  $\Omega_U^p$ ) denote the sheaf of holomorphic  $p$ -forms on  $Y$  (resp.  $U$ ).

Recall that the singularity  $(X, x)$  is *rational* if  $H^i(\mathcal{O}_Y) = 0$  for  $i > 0$ , while  $(X, x)$  is *Du Bois* [St1, 3.6] if the natural map  $H^i(\mathcal{O}_Y) \xrightarrow{q_i} H^i(\mathcal{O}_D)$  is an isomorphism for  $i > 0$ . By [St1, Lemma 2.14]  $q_i$  is surjective for all  $i$ , hence rational singularities are also Du

Bois. Moreover,  $(X, x)$  is Du Bois if and only if  $H^i(\mathcal{O}_Y(-D)) = 0$  for  $i > 0$ .

**1.2.** Now assume that  $(X, x)$  is Gorenstein. By Serre-duality and the theorem of Grauert and Riemenschneider  $H_D^i(\mathcal{O}_Y) = 0$  for  $i \neq n$ , so  $H^i(\mathcal{O}_Y) = H^i(\mathcal{O}_U) = 0$  for  $i \neq 0, n-1$ . Therefore,  $(X, x)$  is rational (resp. Du Bois) if and only if  $H^{n-1}(\mathcal{O}_Y) = 0$  (resp.  $H^{n-1}(\mathcal{O}_Y(-D)) = 0$ ). The composite  $H_D^{n-1}(\mathcal{O}_Y(-D)) \hookrightarrow H^{n-1}(\mathcal{O}_Y(-D)) \hookrightarrow H^{n-1}(\mathcal{O}_Y)$  factors through  $H_D^{n-1}(\mathcal{O}_Y) = 0$ , hence  $H_D^{n-1}(\mathcal{O}_Y(-D)) = 0$ , and dually,  $H^1(\Omega_Y^n(D)) = 0$ . It is now immediate that  $(X, x)$  is Du Bois if and only if  $H^0(\Omega_Y^n(D)) \rightarrow H^0(\Omega_U^n)$  is surjective. Similarly,  $(X, x)$  is rational if and only if  $H^0(\Omega_Y^n) \rightarrow H^0(\Omega_U^n)$  is surjective.

**1.3.** Du Bois and rational singularities are also characterized in terms of their discrepancies. Write  $K_Y = \pi^*K_X + D_I - D_J$ , where  $D_I$  and  $D_J$  are effective exceptional divisors having no irreducible components in common. (The notation is due to Ishii [I, Def. 3.3] who calls  $D_J$  the *essential divisor* of  $\pi$ .) A choice  $\omega \in H^0(\Omega_Y^n)$  of generator (as a cyclic  $H^0(\mathcal{O}_X)$ -module) gives rise to an isomorphism

$$\omega : \mathcal{O}_Y(D_I - D_J) \xrightarrow{\cong} \Omega_Y^n. \quad (1.3.1)$$

By the above,  $(X, x)$  is rational (resp. Du Bois) if and only if  $D_J = 0$  (resp.  $D_J \leq D$ ). Note that the condition  $D_J = 0$  also defines *canonical* singularities, while  $D_J \leq D$  defines *log-canonical* singularities.

**Definition 1.4** [Wat, 3.16]. A normal Gorenstein singularity  $(X, x)$  is called purely elliptic if it is Du Bois but not rational (equivalently, the essential divisor  $D_J$  is reduced and non-trivial).

**1.5.** Now recall that the cohomology groups of  $U$  (for any normal singularity  $(X, x)$ ) carry canonical mixed Hodge structures [St1, Sect. 1]. We let  $h^{i,j}$  denote the  $(i, j)$ 'th mixed Hodge number of  $H^{n-1}(U, \mathbb{C})$ , i.e.

$$h^{i,j} = \dim \mathrm{Gr}_F^i \mathrm{Gr}_{i+j}^W H^{n-1}(U, \mathbb{C}). \quad (1.5.1)$$

By the semipurity theorem [St1, Cor. 1.12],  $h^{i,j} = 0$  unless  $i, j \geq 0$  and  $i + j \leq n - 1$ .

From [LS] we obtain isomorphisms

$$\mathrm{Gr}_F^p H^{p+q}(U, \mathbb{C}) \xrightarrow{\cong} H^q(\Omega_Y^p(\log D) \otimes \mathcal{O}_D), \quad (1.5.2)$$

where  $\Omega_Y^p(\log D)$  is the sheaf of holomorphic  $p$ -forms on  $Y$  with logarithmic poles along  $D$ . In particular,

$$\mathrm{Gr}_F^0 H^{n-1}(U, \mathbb{C}) \xrightarrow{\cong} H^{n-1}(\mathcal{O}_D).$$

When  $(X, x)$  is purely elliptic, it follows easily from 1.3 that  $h^{n-1}(\mathcal{O}_D) = 1$ . Consequently, exactly one of the numbers  $h^{0,i}$  is equal to 1, while the rest vanish.

**Definition 1.6** [I, Sect. 4]. A purely elliptic singularity  $(X, x)$  is of Hodge type  $(0, s)$ , if  $h^{0,s} = 1$ .

Clearly,  $0 \leq s \leq n - 1$  when  $n$  is the dimension of  $(X, x)$  and  $(0, s)$  its Hodge type.

This definition is equivalent to the one given by Ishii (see [I, Def. 4.1] and the following remark).

We will give two characterizations of  $(X, x)$  being of Hodge type  $(0, n - 1)$ . The first is essentially proved in [I, Thm. 4.3] and the proof is taken from there.

**Lemma 1.7.** *Assume that  $(X, x)$  is purely elliptic of dimension  $n$ . Then  $(X, x)$  is of Hodge type  $(0, n - 1)$  if and only if the essential divisor  $D_J$  is irreducible.*

*Proof.* If  $D_J$  is irreducible, then  $D_J$  is non-singular and the mixed Hodge structure on  $H^{n-1}(D_J, \mathbb{C})$  is pure of weight  $n - 1$ . Since  $\mathrm{Gr}_F^0 H^{n-1}(D, \mathbb{C}) \rightarrow \mathrm{Gr}_F^0 H^{n-1}(D_J, \mathbb{C})$  is surjective,  $(X, x)$  is of Hodge type  $(0, n - 1)$  in this case.

Conversely, assume  $(X, x)$  is of Hodge type  $(0, n - 1)$ . If  $D_J = D_1 + D_2$  for non-trivial effective divisors  $D_1, D_2$ , then by Mayer-Vietoris,

$$\begin{aligned} \mathrm{Gr}_F^0 \mathrm{Gr}_{n-1}^W H^{n-2}(D_1 \cap D_2, \mathbb{C}) &\rightarrow \mathrm{Gr}_F^0 \mathrm{Gr}_{n-1}^W H^{n-1}(D, \mathbb{C}) \rightarrow \\ &\mathrm{Gr}_F^0 \mathrm{Gr}_{n-1}^W H^{n-1}(D_1, \mathbb{C}) \oplus \mathrm{Gr}_F^0 \mathrm{Gr}_{n-1}^W H^{n-1}(D_2, \mathbb{C}) \end{aligned}$$

is exact. Now,

$$\mathrm{Gr}_F^0 H^{n-1}(D_i, \mathbb{C}) = H^{n-1}(\mathcal{O}_{D_i}) = 0, \quad i = 1, 2$$

by [I, Cor 3.9], while  $\mathrm{Gr}_{n-1}^W H^{n-2}(D_1 \cap D_2, \mathbb{C}) = 0$  since  $D_1 \cap D_2$  is complete of dimension  $< n - 1$ . Thus,  $\mathrm{Gr}_F^0 \mathrm{Gr}_{n-1}^W H^{n-1}(D, \mathbb{C}) = 0$ . But this contradicts our assumption on the Hodge type of  $(X, x)$ , so  $D_J$  is indeed irreducible.  $\square$

**Lemma 1.8.** *Assume that  $(X, x)$  is purely elliptic of dimension  $n$ . Then*

$$h^0(\Omega_Y^{n-1}(\log D) \otimes \mathcal{O}_D) \leq 1$$

*with equality if and only if  $(X, x)$  is of Hodge type  $(0, n - 1)$ .*

*Proof.* By (1.5.2),  $h^0(\Omega_Y^{n-1}(\log D) \otimes \mathcal{O}_D)$  is equal to  $\dim \mathrm{Gr}_F^{n-1} H^{n-1}(U, \mathbb{C})$ , hence by semipurity to  $h^{n-1,0} = h^{0,n-1}$ . The result is now obvious from the definition of Hodge type and the discussion prior to it.  $\square$

**1.9 Examples of purely elliptic singularities.** As we shall be mostly concerned with purely elliptic  $n$ -dimensional ( $n \geq 2$ ) singularities of Hodge type  $(0, n - 1)$ , let us give a brief recipe of how to produce examples.

Let  $f : \mathbb{C}^{n+t} \rightarrow \mathbb{C}^t$  be a polynomial map whose vanishing defines the germ  $(X, 0)$  of an ICIS at 0. Assume that  $f$  is weighted homogeneous with respect to the standard coordinates  $x_1, \dots, x_{n+t}$  (resp.  $y_1, \dots, y_t$ ) of strictly positive integral weights  $w_1, \dots, w_{n+t}$

(resp.  $d_1, \dots, d_t$ ) in the source (resp. target). Then  $\omega$  (1.3.1) may be chosen weighted homogeneously as

$$\omega = \frac{dx_1 \wedge \cdots \wedge dx_{n+t}}{df_1 \wedge \cdots \wedge df_t}. \quad (1.9.1)$$

Thus,  $\omega$  is of weight  $Q = (w_1 + \cdots + w_{n+t}) - (d_1 + \cdots + d_t)$ . Let  $B \xrightarrow{\pi} \mathbb{C}^{n+t}$  be the weighted blow-up of the source. Then the strict transform  $Y$  of  $X$  is a V-manifold, i.e. has only (cyclic) quotient singularities, and the exceptional divisor  $D \subset Y$  has V-normal crossings. From [St2, 2.6 and Cor. 3.4] it is not hard to see that the invariants  $H^{n-1}(\mathcal{O}_Y)$  and  $H^{n-1}(\mathcal{O}_Y(-D))$  of 1.1 can be computed from this situation (that is, we don't need a full resolution; a V-resolution suffices). An easy computation now shows that  $(X, 0)$  is rational (resp. Du Bois) if and only if  $Q > 0$  (resp.  $Q \geq 0$ ), thus  $(X, 0)$  is purely elliptic precisely when  $Q = 0$ . In this case,  $(X, 0)$  is of Hodge type  $(0, n-1)$  since  $D$  is a complete V-manifold.

**1.10.** Now define a new map-germ  $f' = f + g : (\mathbb{C}^{n+t}, 0) \rightarrow (\mathbb{C}^t, 0)$  by adding to  $f$  a holomorphic map  $g = (g_1, \dots, g_t)$  for which all monomials in  $g_i$  are of weight  $> d_i$  for all  $i = 1, \dots, t$ . Let  $(X', 0)$  be its vanishing. Then the weighted blow-up  $\pi$  from before gives a V-resolution  $Y' \rightarrow X'$  in a neighbourhood of 0, and the exceptional divisor  $D' \subset Y'$  is isomorphic to  $D$  as analytic spaces. It follows from the reasoning above that  $(X', 0)$  is purely elliptic of Hodge type  $(0, n-1)$  if  $Q = 0$ .

**1.11.** From weight considerations in the space  $T_{X,0}^1$  of first order deformations of  $(X, 0)$  (see for instance [PW, Prop. 10.5.16]) it follows that any deformation of a purely elliptic surface singularity of the above type is trivial ( $(X, 0)$  and  $(X', 0)$  are isomorphic for all choices of  $g$ ) and indeed, a purely elliptic Gorenstein surface singularity is quasi-homogeneous if and only if it is of Hodge type  $(0, 1)$ ; see 2.8. For dimensions  $n \geq 3$  however, starting with a weighted homogeneous  $f$  with  $Q = 0$ , the generic choice of  $g$  will produce a purely elliptic  $(X', 0)$  of Hodge type  $(0, n-1)$  which is *not* quasi-homogeneous (cf. 2.15).

## 2. QUASI-HOMOGENEITY OF PURELY ELLIPTIC ICIS'S

**2.1.** Throughout this section  $(X, x)$  will be an ICIS of dimension  $n \geq 2$ . Then the Milnor number  $\mu$  and the Tjurina number  $\tau$  are well-defined. Our main concern is to find conditions that ensure quasi-homogeneity in the purely elliptic case. By 1.9, a necessary condition for  $(X, x)$  to be quasi-homogeneous is that it is of Hodge type  $(0, n-1)$ , but by 1.11 this is generally not sufficient. We shall see that the extra condition needed is that the natural map

$$\beta : H^0(\Omega_Y^{n-1}(\log D)) \rightarrow H^0(\Omega_Y^{n-1}(\log D) \otimes \mathcal{O}_D) = \mathrm{Gr}_F^{n-1} H^{n-1}(U, \mathbb{C}) \quad (2.1.1)$$

be surjective. (Here and in the following, we apply the definitions in 1.1.)

This may be motivated as follows. It is a well-known fact that  $(n-1)$ -cohomology of  $U$  may be identified with the  $(n-1)$ -hypercohomology  $\mathbb{H}^{n-1}(\Omega_Y^1(\log D))$  of the complex  $\Omega_Y^1(\log D)$ , and that the *filtration bête* would compute the Hodge filtration if  $Y$  were complete. Now that  $Y$  is not complete, the *filtration bête* is affected by the analytic behaviour of  $Y$  ‘at infinity’ and will not in general give the Hodge filtration; but it does when  $(X, x)$  quasi-homogeneous [V, Thm. 3.3], and  $\beta$  will then be surjective.

**2.2.** We shall make use of the following formula due to E. Looijenga and J. Steenbrink [LS], valid (as most of what we have to say in 2.2 – 2.4) for any ICIS of dimension at least 2,

$$\mu - \tau = \sum_{i=0}^{n-2} h^{i,0} + a_1 + a_2 + a_3. \quad (2.2.1)$$

The  $h^{i,j}$  were defined in (1.5.1), while the  $a_i$  are non-negative integers defined relative to a choice of good resolution. It is not very instructive to include the definitions of the  $a_i$  here. Most important for our purpose is the fact that  $a_3$  dominates the dimension of  $\text{Coker } \beta$ . Hence, the assumption  $\mu = \tau$  implies that  $\beta$  is surjective and that  $\text{Gr}_F^0 H^{n-1}(U, \mathbb{C})$  is pure of weight  $n-1$  (Thus, in the purely elliptic case  $(X, x)$  is of Hodge type  $(0, n-1)$ ).

**2.3.** But we can be more specific. From [V, Lemma 2.7] we get the formula

$$a_2 + a_3 = e_2^{n,0} + \dim \text{Coker } \beta. \quad (2.3.1)$$

Here  $e_2^{n,0}$  is the dimension of the cokernel of the natural map

$$H^0(\Omega_Y^{n-1}(\log D)(-D)) \xrightarrow{d} H^0(\Omega_Y^n). \quad (2.3.2)$$

For later use we note that there is a spectral sequence

$$E_1^{p,q} = H^q(\Omega_Y^p(\log D)(-D)) \implies \mathbb{H}^{p+q}(\Omega_Y^1(\log D)(-D)) = 0 \quad (2.3.3)$$

with  $E_1^{p,q} = 0$  if  $p+q > n$  ([St2, Thm. 2(b)], [V, 2.2, 2.4]). Clearly,  $e_2^{n,0} = \dim E_2^{n,0}$ .

**2.4.** Let us define  $\chi = \chi(X, x) = \sum_{i=0}^{n-2} h^{i,0} + \dim \text{Coker } \beta$ . It follows from [V, Prop. 4.2] that  $\chi$  is an analytic invariant for any ICIS  $(X, x)$ . Note that the codomain of  $\beta$  has dimension  $h^{n-1,0} = h^{0,n-1}$  by semipurity, so  $\chi = \dim \text{Gr}_F^0 H^{n-1}(U, \mathbb{C}) - \text{rk } \beta$ .

The following lemma is now immediate once we observe (by consulting [LS]) that  $a_1 = 0$  when  $(X, x)$  is Du Bois.

**Lemma 2.5.** *Let  $(X, x)$  be a purely elliptic ICIS of dimension  $n \geq 2$ . Then*

- (1)  $\mu - \tau = \chi + e_2^{n,0}$ .
- (2)  $\chi = 1 - \text{rk } \beta$ .
- (3) *The only possible values of  $\chi$  are 0 and 1, and  $\chi = 0$  if and only if  $(X, x)$  is of Hodge type  $(0, n-1)$  and  $\beta$  is surjective.*

The Main Theorem is a corollary to the following.



**Theorem 2.6.** *Let  $(X, x)$  be a purely elliptic ICIS of dimension  $n \geq 2$ . Then the following are equivalent*

- (1)  $(X, x)$  is quasi-homogeneous.
- (2)  $\mu = \tau$ .
- (3)  $\chi = 0$ .
- (4)  $(X, x)$  is of Hodge type  $(0, n-1)$  and  $\beta$  is surjective.
- (5)  $\beta$  is non-trivial

**2.7.** As mentioned earlier, (1)  $\implies$  (2) is due to Greuel [G1, Satz 0.3]. The implications (2)  $\implies$  (3)  $\iff$  (4)  $\iff$  (5) follow from Lemma 2.5. It remains to give a proof of (4)  $\implies$  (1). This will be done in 2.9 – 2.14 below.

**2.8.** It is well-known that a purely elliptic surface ICIS (i.e.  $n = 2$ ) is either a cusp (if of Hodge type  $(0, 0)$ ) or simply elliptic (if of Hodge type  $(0, 1)$ ) [Wat, Thm. 3.20]. With Theorem 2.6 we can prove the slightly weaker statement that  $(X, x)$  is quasi-homogeneous if and only if it is of Hodge type  $(0, 1)$ . This follows immediately from the formula

$$\mu - \tau = \chi = h^{1,0} \quad (2.8.1)$$

which holds for any Du Bois ICIS of dimension 2 since  $a_2 + a_3 = e_2^{2,0} + \dim \text{Coker } \beta = 0$ . (Indeed,  $E_1^{0,1} = 0$  by the Du Bois property, hence  $E_2^{2,0} = E_\infty^{2,0} = 0$  and  $E_1^{1,1} = E_\infty^{1,1} = 0$ . Since  $\text{Coker } \beta$  embeds into the Serre-dual  $H_D^1(\Omega_Y^1(\log D))$  of  $E_1^{1,1}$ , the claim follows).

**2.9 Proof of (4)  $\implies$  (1).** As usual, let  $\Theta_Y(\log D) = \mathcal{H}om(\Omega_Y^1(\log D), \mathcal{O}_Y)$  denote the sheaf of analytic vector fields on  $D$  with logarithmic vanishing along  $D$ . By (1.3.1), contraction of  $\pi^*\omega$  yields an isomorphism

$$\Theta_Y(\log D)(D_I - D_J + D) \xrightarrow{\cong} \Omega_Y^{n-1}(\log D). \quad (2.9.1)$$

Assuming condition (4) in Theorem 2.6 (as we do throughout), surjectivity of  $\beta$  implies via (2.9.1) that

$$\begin{aligned} 0 \rightarrow H^0(\Theta_Y(\log D)(D_I - D_J)) &\rightarrow H^0(\Theta_Y(\log D)(D_I - D_J + D)) \\ &\rightarrow H^0(\Theta_Y(\log D)(D_I - D_J + D) \otimes \mathcal{O}_D) \rightarrow 0, \end{aligned} \quad (2.9.2)$$

is exact. Here  $H^0(\Theta_Y(\log D)(D_I - D_J + D) \otimes \mathcal{O}_D) \cong \mathbb{C}$  by Lemma 1.8 since  $(X, x)$  is of Hodge type  $(0, n-1)$ .

**2.10.** By Lemma 1.7,  $D_J$  is an irreducible exceptional component. From [Wahl1, 1.10.2] we then get an injective morphism

$$\iota : \mathcal{O}_{D_J} \hookrightarrow \Theta_Y(\log D) \otimes \mathcal{O}_{D_J}$$

defined as follows. If  $(y_1, \dots, y_n)$  are local coordinates on  $Y$  centered at some point of  $D_J$  with  $y_1 = 0$  a local equation for  $D_J$ , then  $\iota$  maps  $1 \in \mathcal{O}_{D_J}$  to  $y_1 \partial_{y_1}$ .

It is easily checked that the identity map of  $\Theta_Y(\log D)$  produces a well-defined morphism

$$\Theta_Y(\log D) \otimes \mathcal{O}_{D_J} \rightarrow \Theta_Y(\log D) \otimes \mathcal{O}_D(D + D_I - D_J),$$

whose composition with  $\iota$  gives a morphism

$$\phi : \mathcal{O}_{D_J} \rightarrow \Theta_Y(\log D) \otimes \mathcal{O}_D(D + D_I - D_J).$$

By the local expression  $1 \mapsto y_1 \partial_{y_1}$  and the fact that  $D + D_I - D_J$  is trivial along  $D_J$ ,  $\phi$  must be injective. Hence,

$$H^0(\mathcal{O}_{D_J}) \xrightarrow{\phi} H^0(\Theta_Y(\log D) \otimes \mathcal{O}_D(D + D_I - D_J))$$

is an isomorphism of 1-dimensional vector spaces.

**2.11.** Now choose  $\xi_Y \in H^0(\Theta_Y(\log D)(D_I - D_J + D))$  such that  $\xi_Y$  projects to  $\phi(1)$  in  $H^0(\Theta_Y(\log D)(D_I - D_J + D) \otimes \mathcal{O}_D)$  (cf. (2.9.2)). It is quite standard that  $H^0(\Theta_U) = H^0(\Theta_X)$ , thus  $\xi_Y$  maps to a vector field  $\xi_X$  on  $X$ . This field preserves the maximal ideal  $\mathfrak{m}_{X,x}$  of  $\mathcal{O}_{X,x}$  and induces a derivation on  $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ .

**2.12.** We proceed as in [Wahl2, 3.12]. In local coordinates as before, centered at a point of  $D_J$  which is regular for  $D$ ,  $\xi_Y$  is of the form

$$\xi_Y = y_1 \partial_{y_1} + y_1 \xi'_Y$$

for some  $\xi'_Y$  which is logarithmic along  $D_J$ . Hence,  $\xi'_Y$  preserves for all  $k > 0$  the ideal  $(y_1^k)$  generated by  $y_1^k$ . Let  $k$  be the greatest positive integer satisfying  $\pi^* \mathfrak{m}_{X,x} \subset (y_1^k)$  and pick  $g \in \mathfrak{m}_{X,x}$  for which  $g \circ \pi$  belongs to  $(y_1^k)$  but not to  $(y_1^{k+1})$ . Write  $g \circ \pi$  locally as  $y_1^k h$  for some  $h$ . Then

$$\xi_X(g) \circ \pi = \xi_Y(y_1^k h) = k y_1^k h + y_1^{k+1} \partial_{y_1}(h) + y_1 \xi'_Y(y_1^k h),$$

which modulo  $(y_1^{k+1})$  gives  $\xi_X(g) \circ \pi \equiv k(g \circ \pi)$ . Since  $\pi^* \mathfrak{m}_{X,x}^2 \subset (y_1^{k+1})$ , this shows that  $\xi_X^p(g)$  is non-trivial in  $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$  for all powers  $p$ . Thus,  $\xi_X$  acts non-nilpotently on the Zariski tangent space of  $(X, x)$  at  $x$ .

We can now apply [SW, Thm. 4.1] to conclude that  $(X, x)$  is quasi-homogeneous, except possibly when  $(X, x)$  is a hypersurface singularity of multiplicity 2 (note that a positive-dimensional  $\mathbb{C}$ -analytic ICIS is reduced).

**2.13.** For the remaining case we must work a bit harder. Assume that  $(X, x)$  is the hypersurface defined by an analytic map-germ  $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$  of multiplicity at least 2. Let  $x_1, \dots, x_{n+1}$  be local coordinates on  $\mathbb{C}^{n+1}$  centered at 0, and let  $J(f) = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_{n+1}})$  denote the Jacobian ideal. Since  $f$  has an isolated singularity we may choose  $k \geq 1$  so that  $\mathfrak{m}^k \subset J(f)$ , where  $\mathfrak{m}$  denotes the maximal ideal in the local ring

$\mathcal{O}_{n+1,0}$  of the germ  $(\mathbb{C}^{n+1}, 0)$ . Put  $R = \mathcal{O}_{n+1,0}/\mathfrak{m}^{nk+2}$ . Then  $\xi_X$  from 2.11 lifts to some vector field  $\xi_{\mathbb{C}^{n+1}}$  on  $(\mathbb{C}^{n+1}, 0)$  which in turn induces a  $\mathbb{C}$ -derivation  $\xi_R$  on  $R$ . Let  $\xi'_R : R \rightarrow R$  denote the semi-simple part of the operator  $\xi_R$ . Then  $\xi'_R$  is itself a derivation, polynomial in  $\xi_R$ . Clearly  $\xi_R$  preserves the ideal  $(\bar{f}) \subset R$  generated by the projection  $\bar{f}$  of  $f$ , hence so does  $\xi'_R$ , and we may choose a unit  $\bar{u} \in R$  so that  $\bar{u}\bar{f}$  is an eigenvector for  $\xi'_R$  with eigenvalue  $\lambda \in \mathbb{C}$ , say. Let  $u \in \mathcal{O}_{n+1,0}$  denote a lift of  $u$ , and let  $\xi'_{\mathbb{C}^{n+1}}$  be a vector field on  $(\mathbb{C}^{n+1}, 0)$  lifting  $\xi'_R$ . Then  $\xi'_{\mathbb{C}^{n+1}}(uf) - \lambda uf \in \mathfrak{m}^{nk+2} \subset \mathfrak{m}^2 \cdot J(f)^n$ . By the Briançon-Skoda Theorem  $(uf)^n \in J(uf)$ , from which  $J(f)^n \subset J(uf)$  (indeed,  $J(f) = J(u^{-1}(uf))$  is generated by  $\frac{\partial(uf)}{\partial x_i} + uf \frac{\partial \log u^{-1}}{\partial x_i}$  for  $i = 1, \dots, n+1$ ), whence  $\xi'_{\mathbb{C}^{n+1}}(uf) - \lambda uf \in \mathfrak{m}^2 J(uf)$ . It follows that there exists yet another vector field  $\xi''_{\mathbb{C}^{n+1}}$  on  $(\mathbb{C}^{n+1}, 0)$  satisfying  $\xi''_{\mathbb{C}^{n+1}}(uf) = \lambda uf$  and agreeing with  $\xi'_{\mathbb{C}^{n+1}}$  up to order 2.

By Saito's theorem [Sa],  $(X, x)$  is quasi-homogeneous if  $\lambda \neq 0$ .

**2.14.** Assume  $\lambda = 0$ . We will show that this leads to a contradiction. Upon replacing  $f$  with  $uf$  as defining function for  $(X, x)$  we may assume  $\xi''_{\mathbb{C}^{n+1}}(f) = 0$ . Let  $\xi''_X$  denote the induced vector field on  $X$ . Then by [L, Lemma 9.6], contraction of  $\omega$  (1.3) by  $\xi''_{\mathbb{C}^{n+1}}$  yields an element  $\sigma$  in  $H^0(\Omega_X^{n-1})$ . Obviously  $\pi^*\sigma \in H^0(\Omega_Y^{n-1})$ , but since  $\pi$  is constant along  $D$  and  $\Omega_Y^{n-1}(\log D)(-D)$  are the forms in  $\Omega_Y^{n-1}$  whose restriction to each component of  $D$  is trivial, then  $\pi^*\sigma \in H^0(\Omega_Y^{n-1}(\log D)(-D))$ . We conclude from (2.9.1) that  $\xi''_X$  lifts to a vector field  $\xi''_Y \in H^0(\Theta_Y(\log D)(D_I - D_J))$ . Now we may repeat the arguments of 2.11 and 2.12 with  $\xi_Y - \xi''_Y$  in place of  $\xi_Y$  and conclude that  $\xi_X - \xi''_X$  acts non-nilpotently on  $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ . But this contradicts the fact that the action of  $\xi''_X$  by construction is the semi-simple part of the action of  $\xi_X$ . Thus,  $\lambda \neq 0$  as desired.

This concludes the proof of Theorem 2.6.  $\square$

**2.15 Remark.** To see why Scheia and Wiebe's theorem fails for hypersurfaces of multiplicity 2, let  $(X, x)$  be the germ defined in  $(\mathbb{C}^6, 0)$  by  $(u, v, w, x, y, z) \mapsto uv + w^4 + x^4 + y^4 + z^4 + w^2 x^2 y^2 z^2$ . Then  $(X, x)$  is purely elliptic (by 1.10) and not quasi-homogeneous ( $\mu = \tau + 1 = 81$ ). Nevertheless,  $u\partial_u - v\partial_v$  induces a vector field on  $(X, x)$  whose action on  $\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$  is non-nilpotent.

**2.16 Vector fields on a purely elliptic ICIS.** We end with the following description of the vector fields on a purely elliptic ICIS  $(X, x)$ . This should be contrasted with the equivariance of the minimal good resolution of a surface singularity [Wahl1].

**Proposition 2.17.** *If  $(X, x)$  is a Du Bois ICIS, then*

$$H^0(\Theta_X) = H^0(\Theta_Y(\log D)(D_I - D_J + D)).$$

*In particular, any vector field on a purely elliptic  $(X, x)$  lifts to a meromorphic vector field on  $Y$  with logarithmic vanishing along the essential divisor  $D_J$ .*

*Proof.* The terms  $E_1^{p,q}$  of (2.3.3) vanish for  $p+q > n$ , and  $E_1^{0,n-1} = 0$  since  $(X, x)$  is Du Bois. It follows that  $E_1^{1,n-1} = E_\infty^{1,n-1} = 0$ . By Serre-duality,  $H_D^1(\Omega_Y^{n-1}(\log D)) = 0$

so  $H^0(\Omega_Y^{n-1}(\log D)) \rightarrow H^0(\Omega_U^{n-1})$  is an isomorphism. By (2.9.1) this translates to

$$H^0(\Theta_Y(\log D)(D_I - D_J + D)) = H^0(\Theta_U) = H^0(\Theta_X),$$

as claimed. □

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