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THE STABLE MAPPING CLASS GROUP AND $Q(\mathbf{C}P^\infty)$

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Abstract. In [T1] it was shown that the classifying space of the stable mapping class groups after plus construction $B\Gamma_\infty^+$ has an infinite loop space structure. This result and the tools developed in [BM] to analyse transfer maps, are used here to show the following splitting theorem.

Let $\Sigma^\infty(\mathbb{C}P^\infty)^\wedge_p \simeq E_0 \vee \cdots \vee E_{p-2}$ be the "Adams-splitting" of the p-completed suspension spectrum of $\mathbb{C}P^\infty$. Then for some infinite loop space W_p

$$(B\Gamma_\infty^+)^\wedge_p \simeq \Omega^\infty(E_0) \times \cdots \times \Omega^\infty(E_{p-3}) \times W_p$$

where $\Omega^\infty E_i$ denotes the infinite loop space associated to the spectrum E_i. The homology of $\Omega^\infty E_i$ is known, and as a corollary one obtains large families of torsion classes in the homology of the stable mapping class group. This splitting also detects all the Miller-Morita-Mumford classes. Indeed, after p-completion for p odd, there is a split surjective map

$$B\Gamma_\infty^+ \longrightarrow BU.$$

The Mumford conjecture asserts that this map is a rational homotopy equivalence.

§1. Introduction and statement of theorems.

For an oriented surface F, we let $\mathrm{Diff}(F)$ denote the topological group of *orientation preserving* diffeomorphisms that keep the boundary ∂F pointwise fixed when $\partial F \neq \emptyset$. The components of $\mathrm{Diff}(F)$ are contractible when the genus of F is greater than one [EE], [ES], so $B\mathrm{Diff}(F) \simeq B\Gamma(F)$ where $\Gamma(F) = \pi_0\mathrm{Diff}(F)$ is the mapping class group of F.

Let $F_{g,1}$ denote a genus g surface with one boundary component. One may add a torus with two boundary circles to $F_{g,1}$ to get an inclusion into $F_{g+1,1}$, and hence a map

$$B\mathrm{Diff}(F_{g,1}) \longrightarrow B\mathrm{Diff}(F_{g+1,1}).$$

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The mapping class group $\Gamma_{g,1} = \Gamma(F_{g,1})$ is perfect for $g \geq 3$ [P], so one may apply Quillen's plus construction. The maps

$$B\mathrm{Diff}(F_{g,1})^+ \longrightarrow B\mathrm{Diff}(F_{g+1,1})^+, \quad B\mathrm{Diff}(F_{g,1})^+ \longrightarrow B\mathrm{Diff}(F_g)^+ \tag{1.1}$$

are $[g/2]$-connected, respectively $[(g-2)/2]$-connected by [H], [I]. The homotopy direct limit of the maps in (1.1) as $g \to \infty$ is denoted $B\Gamma_\infty^+$. Its homology is the stable homology of the mapping class group. By the main result of [T1], $\mathbb{Z} \times B\Gamma_\infty^+$ (and $B\Gamma_\infty^+$) are *infinite loop* spaces, i.e. the 0-th space in a connective Ω-spectrum.

1.1. The map to $Q(\mathbb{C}P_+^\infty)$.

This paper compares $\mathbb{Z} \times B\Gamma_\infty^+$ to the infinite loop space $Q(BS_+^1)$ where $Q(-) = \Omega^\infty\Sigma^\infty(-)$ and where we prefer to write BS^1 instead of $\mathbb{C}P^\infty$ for the infinite complex projective space. Here and elsewhere the subscript $+$ indicates the addition of a disjoint base point. The construction of the infinite loop space structure on $\mathbb{Z} \times B\Gamma_\infty^+$ is described in Section 2 below, where we also produce an infinite loop map

$$\alpha : \mathbb{Z} \times B\Gamma_\infty^+ \longrightarrow A(BS^1)$$

into Waldhausen's A-functor applied to BS^1. The topological Dennis trace is an infinite loop map

$$\mathrm{tr} : A(X) \to Q(\Lambda X_+)$$

into the stable homotopy of the free loop space, and we can compose it with the map from ΛX to X that evaluates a free loop at 1 to get an infinite loop map σ from $A(X)$ to $Q(X_+)$, cf. [BHM], [W1]. There results an infinite loop map

$$\sigma \circ \alpha : \mathbb{Z} \times B\Gamma_\infty^+ \longrightarrow Q(BS_+^1). \tag{1.2}$$

The definition of α is very abstract and not so well suited for calculational purposes. But there is another interpretation of the composition $\sigma \circ \alpha$ that we now describe. We also indicate the relation of (1.2) to the Miller-Morita-Mumford classes $\kappa_i \in H^{2i}(B\Gamma_\infty)$, cf. [Mi], [Mo], [Mu].

Let $F \to E \to B$ be a smooth oriented surface bundle, and T^vE the tangent bundle along the fibres. This is an oriented 2-plane bundle over E, i.e. a complex line bundle. Let

$$I_F : H^{2i+2}(E) \to H^{2i}(B)$$

be the "integration along the fibres" map. One gets characteristic classes

$$\kappa_i = I_F(c_1(T^vE)^{i+1})$$

In the universal situation:

$$F \longrightarrow E(F) \xrightarrow{\pi_F} B\mathrm{Diff}(F), \quad E(F) = E\mathrm{Diff}(F) \times_{\mathrm{Diff}(F)} F, \tag{1.3}$$

where F is some compact surface, one has

$$T^vE(F) = E\mathrm{Diff}(F) \times_{\mathrm{Diff}(F)} TF,$$

and gets classes $\kappa_i(F) \in H^{2i}(B\mathrm{Diff}(F))$. By (1.1) one obtains stable classes

$$\kappa_i \in H^{2i}(B\Gamma^+_\infty), \quad i = 1, 2, \dots$$

They are known to generate a polynomial subalgebra

$$\mathbb{Q}[\kappa_1, \kappa_2, \dots] \subset H^*(B\Gamma^+_\infty; \mathbb{Q}), \tag{1.4}$$

cf. [Mi], [Mo] and it is expected that one has equality in (1.4); this is the Mumford conjecture.

Let us return to the smooth fibre bundle $F \to E \to B$ and suppose for simplicity that F is a closed surface. Choose a smooth embedding of E in some Euclidean space $\mathbb{R}^k$. The normal bundle $N^v(E)$ of the resulting embedding $E \subset B \times \mathbb{R}^k$ is the "normal bundle along the fibres",

$$T^v(E) \oplus N^v(E) \simeq E \times \mathbb{R}^k.$$

Collapsing a tube around E in $B \times \mathbb{R}^k$ induces a map

$$B_+ \wedge S^k \longrightarrow \mathrm{Th}(N^vE) \tag{1.5}$$

into the Thom space. The induced map in cohomology composed with the Thom isomorphism is the integration homomorphism I_F. The maps in (1.5) for varying k define a map

$$\Sigma^\infty(B_+) \longrightarrow \mathrm{Th}(-T^vE)$$

from the suspension spectrum of B_+ into the (-3)-connected Thom spectrum $\mathrm{Th}(-T^vE)$. Let

$$\theta_E : E \longrightarrow BS^1$$

classify the complex line bundle T^vE. There is an induced map of spectra

$$\mathrm{Th}(-T^vE) \longrightarrow \mathrm{Th}(-L)$$

where L is the universal line bundle. The range is the spectrum usually denoted $\mathbb{C}P^\infty_{-1}$ amongst topologists, cf. [R].

Notation 1.1. For a connective spectrum E it is common to denote by $\Omega^\infty E$ the bottom space in the associated Ω-spectrum

$$\Omega^\infty E = \operatorname{hocolim} \Omega^k E_k.$$

If E is not connective, i.e. has non-zero homotopy in negative degrees, $\Omega^\infty E$ is defined to be the bottom space in the connective cover $E[0, \infty)$ where one has killed the homotopy groups of E in negative degrees. We write $Q(S^n \wedge X) = \Omega^\infty(\Sigma^\infty(S^n \wedge X)), n \in \mathbb{Z}$.

One can approximate $B\mathrm{Diff}(F)$ by manifolds and obtain from the above a map

$$\hat{\tau}_F : B\mathrm{Diff}(F) \longrightarrow \Omega^\infty \mathbb{C}P^\infty_{-1}$$

and in turn

$$\hat{\tau}_\infty : B\Gamma^+_\infty \longrightarrow \Omega^\infty \mathbb{C}P^\infty_{-1}. \tag{1.6}$$

If we compose (1.5) with the map on Thom spaces induced by the inclusion

$$w : N^v(E) \longrightarrow N^v(E) \oplus T^v(E),$$

via the zero section of $T^v(E)$, we get a map from $B_+ \wedge S^k$ into $E_+ \wedge S^k$. This is the Becker-Gottlieb transfer map [BG] of the fibre bundle $E \to B$ with fibre F. In the universal situation (1.3), on composition with $\theta_{E(F)} : E(F) \to BS^1$, we get a map

$$\tau_F : B\mathrm{Diff}(F) \longrightarrow Q(E(F)_+) \xrightarrow{Q(\theta_{E(F)+})} Q(BS^1_+), \tag{1.7}$$

This leads to a map

$$\tau_\infty : B\Gamma^+_\infty \longrightarrow Q(BS^1_+) \tag{1.8}$$

such that the diagram

$$\begin{tikzcd} B\Gamma^+_\infty \arrow[r, equal] \arrow[d, "\hat{\tau}_\infty"'] & B\Gamma^+_\infty \arrow[d, "\tau_\infty"'] \\ \Omega^\infty \mathbb{C}P^\infty_{-1} \arrow[r, "\omega_\infty"] & Q(BS^1_+). \end{tikzcd} \tag{1.9}$$

is homotopy commutative. (We do not know if there are phantom maps from $B\Gamma^+_\infty$ into $\Omega^\infty(\mathbb{C}P^\infty_{-1})$ or into $Q(\mathbb{C}P^\infty_+)$, i.e. non-trivial maps that are null homotopic on all finite skeletons. Thus (1.9) is in the weak sense of homotopy commutative on all finite skeletons.)

In Section 2.5, we prove, based on results from [DWW] that the diagram

$$\begin{tikzcd} B\mathrm{Diff}(F_{g,1})^+ \arrow[r, "\gamma_g"] \arrow[d, "\tau_{g,1}"'] & \mathbb{Z} \times B\Gamma^+_\infty \arrow[d, "\sigma\circ\alpha"'] \\ Q(BS^1_+) & Q(BS^1_+) \arrow[l, "+[1]"'] \end{tikzcd} \tag{1.10}$$

is homotopy commutative; here $\tau_{g,1} = \tau_{F_{g,1}}$ maps into the $(1-2g)$-th component, while γ_g maps into the g-th component, $\sigma \circ \alpha$ as defined in (1.2) multiplies by -2 on components, and the arrow labeled $+[1]$ denotes loop sum with the base point in the 1-st component. Since γ_g (when translated to a fixed component) becomes highly connected for $g \to \infty$, the weak homotopy type of $(+[1]) \circ \sigma \circ \alpha$ is determined by the $\tau_{g,1}$, and τ_∞ is weakly homotopy equivalent to $(+[1]) \circ \sigma \circ \alpha$.

1.2. A partial splitting for τ_∞.

The spaces in (1.10) are of finite type (have finitely generated homotopy groups in each degree), so little is lost by replacing them with their profinite or p-completions, cf. [BK, chap. VI]. The universal (p-local) Bockstein operator

$$\beta_{(p)} : B(\mathbb{Q}/\mathbb{Z}_{(p)}) \longrightarrow BS^1_{(p)}$$

(with homotopy fibre $B\mathbb{Q}$) induces an isomorphism on ordinary homology with $\mathbb{Z}/p$ coefficients. Thus for any p-complete spectrum $E_p^\wedge$, the induced map

$$\beta^*_{(p)} : [BS^1, E_p^\wedge] \xrightarrow{\cong} [B(\mathbb{Q}/\mathbb{Z}_{(p)}), E_p^\wedge]$$

is an isomorphism, [BK, chap. VI, prop. 5.4]. But

$$[B(\mathbb{Q}/\mathbb{Z}_{(p)}), E_p^\wedge] \cong \varprojlim [BC_{p^n}, E_p^\wedge]$$

since $\operatorname{hocolim} BC_{p^n} \simeq B(\mathbb{Q}/\mathbb{Z}_{(p)})$ so we have

$$\varprojlim [BC_{p^n}, E_p^\wedge] \cong [BS^1, E_p^\wedge]. \tag{1.11}$$

One can produce maps

$$\rho_F : BC_{p^n} \longrightarrow B\mathrm{Diff}(F) \tag{1.12}$$

by exhibiting a suitable surface F equipped with an action of C_{p^n}, and one can then study $\tau_F \circ \rho_F$. This amounts to a study of the transfer for bundles

$$\pi_n : EC_{p^n} \times_{C_{p^n}} F \longrightarrow BC_{p^n},$$

where we can use results from [BM]: If F admits a non-degenerate vector field X and $S(X)$ denotes its singular set, then π_n contains the covering space

$$EC_{p^n} \times_{C_{p^n}} S(X) \longrightarrow BC_{p^n}$$

and the transfer $\mathrm{trf}(\pi_n)$ is expressible in terms of the covering space transfer, which is easy to calculate.

The maps ρ_F of (1.12), for suitable surfaces F, can be assembled to a map from $\operatorname{hocolim} BC_{p^n}$ into $(B\Gamma^+_\infty)^\wedge_p$, and so by (1.11) to a map from BS^1 into $(B\Gamma^+_\infty)^\wedge_p$. Since the latter is an infinite loop space there is a unique extension to an infinite loop map

$$\mu_p : Q(BS^1) \longrightarrow (B\Gamma^+_\infty)^\wedge_p. \tag{1.13}$$

This is done in Section 3.

$Q(-)$ takes wedge sums to products. Thus

$$Q(BS^1_+) \simeq Q(BS^1) \times Q(S^0).$$

Let $S^0 \subset \mathbb{Z} \times B\Gamma_\infty^+$ be the embedding that sends the non-base point into $(1, *)$. It is proved in [T2] that the composition

$$QS^0 \longrightarrow \mathbb{Z} \times B\Gamma_\infty^+ \xrightarrow{\sigma\circ\alpha} Q(BS^1_+) \xrightarrow{\text{proj}_2} QS^0 \tag{1.14}$$

is multiplication with 2. In this paper we are interested in the other factor

$$B\Gamma_\infty^+ \xrightarrow{\sigma\circ\alpha} Q(BS^1_+) \xrightarrow{\text{proj}_1} Q(BS^1).$$

Consider the map

$$\psi^k : BS^1 \longrightarrow (BS^1)^\wedge_p$$

that represents $k \cdot c_1(L)$ in $H^2(BS^1; \mathbb{Z}_p)$. It extends uniquely (up to homotopy) to a self map of $Q(BS^1)^\wedge_p$ which we again denote by ψ^k. Our main result, proved in Section 3, is

Theorem 1.2. *The composition*

$$proj_1 \circ (\tau_\infty)^\wedge_p \circ \mu_p : Q(BS^1)^\wedge_p \longrightarrow Q(BS^1)^\wedge_p$$

is homotopic to $1 - g\psi^g$, where $g \in \mathbb{Z}_p^\times$ is a topological generator ($g = 3$ if $p = 2$).

The p-complete infinite loop space $Q(BS^1)^\wedge_p$ decomposes into a product of $(p-1)$ infinite loop spaces,

$$Q(BS^1)^\wedge_p \simeq \Omega^\infty E_0 \times \ldots \times \Omega^\infty E_{p-2}. \tag{1.15}$$

There is a corresponding decomposition of $BU^\wedge_p$, considered an infinite loop space via Bott periodicity,

$$BU^\wedge_p \simeq B_0 \times \ldots \times B_{p-2}.$$

In fact even the localized space $BU_{(p)}$ decomposes into $p - 1$ pieces, cf. [A]. The two decompositions correspond under the infinite loop map

$$Q(BS^1) \longrightarrow BU$$

which on $BS^1 \subset Q(BS^1)$ is the reduced canonical line bundle.

In Section 4 we evaluate $1 - g\psi^g$ in $H_*(E_i)$ to see that

$$1 - g\psi^g : \Omega^\infty E_i \longrightarrow \Omega^\infty E_i$$

is a homotopy equivalence when $i \neq p - 2$. This produces the decomposition given in the abstract,

$$(B\Gamma_\infty^+)^\wedge_p \simeq \Omega^\infty E_0 \times \cdots \times \Omega^\infty E_{p-3} \times W_p \tag{1.16}$$

where by (1.14) Q_0S^0, the component of degree zero maps, splits off W_p for p odd. We have thus exhibited large families of torsion classes in the homology of the stable mapping class group, cf. Corollary 4.3 and 4.4.

1.3. The split surjection $B\Gamma_\infty^+ \to BU$ on p-completions.

The map ω_∞ in (1.9) is not a homotopy equivalence. In fact its homotopy fibre is $Q(S^{-2})$, so there is a homotopy fibration

$$(1.17)\qquad Q(S^{-2}) \longrightarrow \Omega^\infty(\mathbb{C}P_{-1}^\infty) \xrightarrow{\omega_\infty} Q(BS_+^1) \simeq Q(BS^1)\times Q(S^0).$$

The 0-th skeleton of $\mathbb{C}P_{-1}^\infty$ is $\Sigma^\infty(S^{-2}\cup_\eta D^0)$ where η is the desuspension of the Hopf map from S^3 to S^2. Since η has order two, when p is odd,

$$(\mathbb{C}P_{-1}^\infty)_{(p)} \simeq \Sigma^\infty(S^0 \vee \mathbb{C}P_{-1}^\infty/S^0)_{(p)}$$

and

$$\Omega^\infty(\mathbb{C}P_{-1}^\infty)_{(p)} \simeq Q(\mathbb{C}P_{-1}^\infty/S^0)_{(p)} \times Q(S^0)_{(p)}.$$

For p odd, ω_∞ in (1.17) is the identity on the $Q(S^0)$ factor.

Let $\mathrm{im}J_p$ denote the homotopy fibre of

$$1-\psi^g : BSU_{(p)} \longrightarrow BSU_{(p)}$$

with g as above. By Bott periodicity

$$\Omega^2 BSU \simeq BU, \quad \Omega^2(\psi^g)\simeq g\psi^g,$$

so there is a fibration sequence

$$\Omega^2\mathrm{im}J_p \longrightarrow BU_{(p)} \xrightarrow{1-g\psi^g} BU_{(p)}.$$

This was compared to (1.17) in [MS]; there is a homotopy commutative diagram

$$(1.18)\qquad \begin{array}{ccccc} Q(S^{-2}) & \longrightarrow & \Omega^\infty\mathbb{C}P_{-1}^\infty & \xrightarrow{\omega_\infty} & Q(BS_+^1) \\ \downarrow & & \downarrow{\scriptstyle l_{-1}} & & \downarrow{\scriptstyle l_0} \\ \Omega^2(\mathrm{im}J_p) & \longrightarrow & BU_{(p)} & \xrightarrow{1-g\psi^g} & BU_{(p)}. \end{array}$$

For odd p the vertical maps are split surjective. The splitting maps are not infinite loop maps, but they are single loop maps by [MS]. In Section 4.2 we prove

Theorem 1.3. *For odd primes p, the composition*

$$l_{-1}\circ\hat{\tau}_\infty : (B\Gamma_\infty^+)_p^\wedge \longrightarrow BU_p^\wedge$$

is split surjective.

The named arrows in (1.18) are all rational homotopy equivalences, but are far from being p-local homotopy equivalences. The Mumford conjecture asserts that $l_{-1}\circ\hat{\tau}_\infty$, or equivalently τ_∞, is a rational equivalence. One may wonder about a p-integral version (p odd). There are two natural candidates. Either

$$(\mathrm{A})\qquad \mu_p : Q(BS_+^1)_p^\wedge \longrightarrow (\mathbb{Z}\times B\Gamma_\infty^+)_p^\wedge$$

or

$$(\mathrm{B})\qquad \hat{\tau}_\infty : (\mathbb{Z}\times B\Gamma_\infty^+)_{(p)} \longrightarrow \Omega^\infty(\mathbb{C}P_{-1}^\infty)_{(p)}$$

could be a homotopy equivalence (p odd).

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§2. Construction of the infinite loop map.

The main task of this section is to construct a map of infinite loop spaces

$$\alpha : \mathbb{Z} \times B\Gamma_\infty^+ \longrightarrow A(BS^1)$$

from $\mathbb{Z} \times B\Gamma_\infty^+$ to the Waldhausen K-theory of BS^1, and hence a map (1.2) of infinite loop spaces to $Q(BS^1_+)$ on compostion with Waldhausen's trace map [W1]

$$\sigma : A(BS^1) \longrightarrow Q(BS^1_+).$$

The construction of α is technically somewhat complicated though the underlying principle is straight forward and can be outlined as follows.

Dwyer, Weiss and Williams [DWW] construct for a fibration $E \to B$, with fibres F_b homotopy equivalent to a compact CW-complex, an A-theory transfer map

$$\chi : B \longrightarrow A_B(E).$$

Here $A_B(E)$ is a fibration over B with fibres $A(F_b)$. The inclusions $F_b \hookrightarrow E$ induce a natural map $A_B(E) \to A(E)$. In the case of interest to us, E is the universal smooth bundle $E(F)$ of a surface F (1.3). Then the classifying map $\theta_{E(F)} : E(F) \to BS^1$ for the vertical tangent bundle induces a map in A-theory, and on composition a map

$$\Theta : A_B(E(F)) \longrightarrow A(E(F)) \xrightarrow{A(\theta_{E(F)})} A(BS^1).$$

The maps $\alpha_F := \Theta \circ \chi$ are compatible with gluing and disjoint union of surfaces, the two operations in the symmetric monoidal category $\mathcal{S}$. that gives rise to the infinite loop space structure of $\mathbb{Z} \times B\Gamma_\infty^+$, cf. [T1] or Section 2.3 below. Hence, the induced map α is a map of infinite loop spaces.

In Section 2.5 we will relate $\sigma \circ \alpha$ to the transfer map $\tau_{g,1} = \tau_{F_{g,1}}$ of (1.7) using results from [DWW].

2.1. Homotopy colimit constructions.

The definition of the A-theory transfer map uses the notion of homotopy colimit. Let $\mathcal{I}$ be a small category and $D : \mathcal{I} \to \mathrm{TOP}$ be a functor from $\mathcal{I}$ to the category of topological spaces. The *homotopy colimit*[1] *of* D is the simplicial space $E\mathcal{I} \times_{\mathcal{I}} D$

[1] When the indexing category $\mathcal{I}$ is the translation category of the natural numbers, we also use in other sections the familiar notation $\operatorname{hocolim} \mathcal{I}(n)$.

with q-simplices $(f_1,\dots,f_q;x)$ where $f_i : a_i \to a_{i-1}$ is a map in $\mathcal{I}$ and $x \in D(a_q)$. The face maps are given by

$$(f_1,\dots,f_q;x) \xrightarrow{\partial_i} \begin{cases} (f_2,\dots,f_q;x), & \text{for } i=0 \\ (f_1,\dots,f_if_{i+1},\dots f_q;x), & \text{for } 0<i<q \\ (f_1,\dots,f_{q-1};D(f_q)(x)), & \text{for } i=q. \end{cases}$$

A natural transformation τ of two functors $D, D' : \mathcal{I} \to \mathrm{TOP}$ induces a map on homotopy colimits

$$\tau : E\mathcal{I} \times_{\mathcal{I}} D \longrightarrow E\mathcal{I} \times_{\mathcal{I}} D'$$

mapping $(f_1,\dots,f_q;x) \mapsto (f_1,\dots,f_q;\tau_{a_q}(x))$. In these terms, the natural forgetful map $E\mathcal{I} \times_{\mathcal{I}} D \to N.\mathcal{I}$ to the nerve of $\mathcal{I}$ is induced by the canonical natural transformation from D to the trivial functor that assigns to each object in $\mathcal{I}$ the one point space $*$.

We need below a slight generalization, namely where $\mathcal{I}$ is replaced by a simplicial index category. Note first the following functoriality in $\mathcal{I}$. Given a functor of index categories $F : \mathcal{I} \to \mathcal{I}'$ the assignment $(f_1,\dots,f_q;x) \mapsto (F(f_1),\dots,F(f_q);x)$ defines a map of homotopy colimits

$$E\mathcal{I} \times_{\mathcal{I}} (D' \circ F) \xrightarrow{F} E\mathcal{I}' \times_{\mathcal{I}'} D'.$$

Definition 2.1. A *simplicial functor* $D. : \mathcal{I}. \to \mathrm{TOP}$ is a collection of functors $D_k : \mathcal{I}_k \to \mathrm{TOP}$ with a collection of natural transformations

$$\delta_i : D_k \to D_{k-1} \circ \partial_i, \qquad \sigma_i : D_{k-1} \to D_k \circ s_i$$

that satisfies the standard simplicial identities.

A simplicial natural transformation $\tau.$ between two simplicial functors $D., D.' : \mathcal{I}. \to \mathrm{TOP}$ is a collection of natural transformation $\tau_k : D_k \to D'_k$ which commute with the face and degeneracy maps.

The *simplicial homotopy colimit* in this situation is the bisimplical space

$$E\mathcal{I}. \times_{\mathcal{I}.} D. = \{E\mathcal{I}_k \times_{\mathcal{I}_k} D_k\}_k.$$

The i-th face map in the k-direction sends $(f_1,\dots,f_k,x) \in E\mathcal{I}_k \times_{\mathcal{I}_k} D_k$ to

$$(\partial_i f_1,\dots,\partial_i f_k, \delta_i x) \in E\mathcal{I}_{k-1} \times_{\mathcal{I}_{k-1}} D_{k-1}$$

and similarly for the degeneracy maps.

Natural transformations of simplicial functors induce maps of associated simplicial homotopy colimits.

2.2. The A-theory transfer map.

¿From a pointed category $\mathcal{P}$ with cofibrations and weak equivalence $w\mathcal{P}$, Waldhausen in [W2] constructs a simplicial category $w\mathrm{S}.\mathcal{P}$. The objects of $w\mathrm{S}_k\mathcal{P}$ are k-flags A, that is sequences of cofibrations

$$* \to A_{01} \to \cdots \to A_{0k} \tag{2.1}$$

with a choice of subquotients $A_{ij} \simeq A_{0j}/A_{0i}$ for $i < j$. A morphism between two flags A and A' is given by a set of compatible weak equivalences $A_{ij} \to A'_{ij}$.

Let $\mathcal{P}(Y)$ denote the Waldhausen category of homotopy finitely dominated spaces over a space Y. Its objects are homotopy finitely domintated retractive spaces X over Y

$$X \underset{s}{\overset{r}{\rightleftarrows}} Y$$

with $rs = 1_Y$ and s a closed embedding with the homotopy extension property.[2] A map $X \to X'$ over Y is a cofibration if the underlying map of spaces is a closed embedding having the homotopy extension property. It is a weak equivalence if the underlying map is. For any (simplicial) category $\mathcal{C}$ we write $|\mathcal{C}|$ for the realization of the (bi)simplicial set $N.\mathcal{C}$. Waldhausen defines

$$A(Y) := \Omega|w\mathrm{S}.\mathcal{P}(Y)|.$$

A map $Y \sqcup Y' \to Z$ defines a functor

$$w\mathcal{P}(Y) \times w\mathcal{P}(Y') \xrightarrow{\simeq} w\mathcal{P}(Y \sqcup Y') \longrightarrow w\mathcal{P}(Z)$$

where the first functor is the isomorphism $(X, X') \mapsto X \sqcup X'$. In particular the map $1 \sqcup 1 : Y \sqcup Y \to Y$ defines a sum operation on $w\mathcal{P}(Y)$ and $w\mathrm{S}.\mathcal{P}(Y)$, and hence gives rise to a Γ-space (in the sense of Segal [S1]). Since $\mathrm{S}_1\mathcal{P} = \mathcal{P}$ and $\mathrm{S}_0\mathcal{P} = *$ the map induced by the natural inclusion $[0,1] \times |w\mathcal{P}(Y)| \to |w\mathrm{S}.\mathcal{P}(Y)|$, factors over the reduced suspension and its adjoint is the *A-theory group completion map*

$$|w\mathcal{P}(Y)| \longrightarrow A(Y). \tag{2.2}$$

Now let $B = |\mathcal{C}|$ for a small category $\mathcal{C}$ and $\Pi : \mathcal{C} \to \mathrm{TOP}^{hf}$ a functor to the category of homotopy finite spaces such that all morphisms are mapped to homotopy equivalences. Then the natural projection

$$E = |E\mathcal{C} \times_{\mathcal{C}} \Pi| \xrightarrow{\pi} B$$

[2] X is homotopy finitely dominated over Y if it fits into a diagram $X' \to W \to X$ of retractive spaces over Y with W homotopy finite over Y and the composite map a weak homotopy equivalence. W is homotopy finite over Y if there is a weak homotopy equivalence $Z \to W$ where Z is a space over Y which is a finite CW-complex relative to Y.

is a quasi-fibration with fibres $\Pi(a)$, cf. [S1]. Let D^0, D^1, D^2 be three functors from $\mathcal{C}$ to the category CAT of small categories,

$$\begin{aligned} & D^0 : a \mapsto *, && \text{the trivial category,} \\ (2.3) \qquad & D^1 : a \mapsto \mathcal{C}/a, && \text{the category of objects over } a \text{ in } \mathcal{C}, \\ & D^2 : a \mapsto w\mathcal{P}(\Pi(a)). && \end{aligned}$$

They define functors $|D^i|$ from $\mathcal{C}$ to TOP, and there are natural transformations $\tau_1 : |D^1| \to |D^0|$, $\tau_2 : |D^1| \to |D^2|$ induced from the trivial functor $\mathcal{C}/a \to *$, and from the functor $\mathcal{C}/a \to w\mathcal{P}(a)$ which to $\phi : a_1 \to a$ assigns the retractive space

$$\Pi(a_1) \sqcup \Pi(a) \underset{\text{incl}}{\overset{\Pi(\phi)\sqcup 1}{\rightleftarrows}} \Pi(a).$$

The *A-theory transfer* is now defined (up to homotopy) as

$$(2.4) \qquad \chi : B = E\mathcal{C} \times_{\mathcal{C}} |D^0| \underset{\simeq}{\xleftarrow{\tau_1}} E\mathcal{C} \times_{\mathcal{C}} |D^1| \xrightarrow{\tau_2} E\mathcal{C} \times_{\mathcal{C}} |D^2|.$$

Here τ_1 is a homotopy equivalence because $\mathcal{C}/a$ has a terminal object and hence $|\mathcal{C}/a| \simeq *$. We may think of χ as a section of the quasi-fibration $|w\mathcal{P}|_B(E) \to B$ with fibres $|w\mathcal{P}(\Pi(a))|$.

2.3. The infinite loop space structure of $B\Gamma_\infty^+$.

Next we recall the infinite loop space structure on $B\Gamma_\infty^+$ following [T2]. Let $\mathcal{K}$ be the cobordism category with objects $\{0, 1, 2, \dots\}$ representing the empty manifold, a copy of the circle, two copies of the circle, ...; the morphisms $\mathcal{K}(n, m)$ are defined as follows: Consider the following (generating) morphisms with separate labeling of incoming and outgoing boundary circles:

(i) a fixed smooth disk, $D \in \mathcal{K}(0, 1)$;
(ii) a fixed smooth pair of pants, $P \in \mathcal{K}(2, 1)$;
(iii) a fixed smooth torus with two disks removed, $T \in \mathcal{K}(1, 1)$;
(iv) for each permutation σ on n letters, $n \geq 0$, a morphism $C_\sigma \in \mathcal{K}(n, n)$ (one may think of these as n zero length cylinders with incoming labels $i = 1, \dots, n$ and outgoing labels $\sigma(i)$).

D, P and T come equipped with a fixed smooth collar of their boundary circles. Elements in $\mathcal{K}(n, m)$ will be those labelled cobordisms from n to m copies of a circle obtained from these generating morphisms by a finite number of applications of the following two operations:

(I) Gluing: for $F_1 \in \mathcal{K}(n_1, n_2)$ and $F_2 \in \mathcal{K}(n_2, n_3)$, let $F_2 \circ F_1 \in \mathcal{K}(n_1, n_3)$ be the cobordism obtained by gluing the two sets of n_2 boundary circles according to their labeling, using the parametrization induced by the collars;
(II) Disjoint union: for $F_i \in \mathcal{K}(n_i, m_i)$, let $F_1 \sqcup F_2 \in \mathcal{K}(n_1 + n_2, m_1 + m_2)$ be the disjoint union where F_1 retains its labels while the labels of the source and target boundary components of F_2 are shifted by n_1 and m_1 respectively.

The cobordism category $\mathcal{K}$ is a symmetric monoidal category with composition (I) and product (II).

We will now construct the simplicial category $\mathcal{S}.$: The simplicial set ob $\mathcal{S}.$ of objects of $\mathcal{S}.$ is the nerve of $\mathcal{K}$, ob $\mathcal{S}. = N.\mathcal{K}$. Thus the objects of $\mathcal{S}_k$ are the composable k-tuples of cobordisms

$$\bar{F} = (F_1, F_2, \ldots, F_k), \qquad F_i \in \mathcal{K}(n_i, n_{i-1}).$$

A morphism between $\bar{F}$ and $\bar{F}'$ in $\mathcal{S}_k$ is a k-tuple

$$\bar{\phi} = (\phi_1, \phi_2, \ldots, \phi_k), \qquad \phi_i \in \mathrm{Diff}(F_i, F_i{}')$$

of oriented diffeomorphisms from F_i to F_i' which fix the collars. ($\mathcal{S}_0$ is the category with objects $0, 1, 2, \ldots$ and identity morphisms). The face and degeneracy maps of ob $\mathcal{S}. = N.\mathcal{K}$ are extended to the morphisms in the obvious way. Disjoint union defines a funtor $\mathcal{S}. \times \mathcal{S}. \to \mathcal{S}.$ and gives rise to a Γ-space with underlying space $|\mathcal{S}.|$ (in the sense of [S1]). More precisely, $|\mathcal{S}.|$ is the realization of the bisimplicial set $N.\mathcal{S}.$. Interchanging simplicial directions, $N.\mathcal{S}.$ can also be seen to be the nerve of the simplicial category with constant object set $\{0, 1, 2, \ldots\}$ and morphism set $N.\mathcal{S}_1$, cf. [T2]. Disjoint union defines a symmetric monoidal structure on this simplicial category and hence one can construct a symmetric monoidal Γ-category from it (see May [Ma; construction 10]).

Theorem 2.2 ([T1], [T2]). $\mathbb{Z} \times B\Gamma_\infty^+ \simeq \Omega|\mathcal{S}.|$.

We now construct the analogue of the group completion map for $\mathcal{S}.$. As $\mathcal{S}_0$ has many objects, some care has to be taken. Let $F \in \mathcal{K}(n, m)$ and $\mathcal{S}_1(F)$ be the full subcategory of $\mathcal{S}_1$ with one object F. The inclusion $\mathcal{S}_1(F) \hookrightarrow \mathcal{S}_1$ induces, just as in (2.2), a natural map

$$\hat{\gamma}_F : |\mathcal{S}_1(F)| \longrightarrow \Omega^{n,m}|\mathcal{S}.| \tag{2.5}$$

to the space of paths in $|\mathcal{S}.|$ from n to m. If we choose 1 to be the base point in $|\mathcal{S}.|$, a choice of cobordisms $F' \in \mathcal{K}(n, 1)$ and $F'' \in \mathcal{K}(m, 1)$ (and hence of paths from n to 1 and m to 1 in $|\mathcal{S}.|$) defines up to homotopy a map from $|\mathcal{S}_1(F)|$ to $\Omega^{1,1}|\mathcal{S}.| = \Omega|\mathcal{S}.|$. The component of the image depends on F' and F''. Indeed, by [T0], the component is given by

$$\frac{1}{2}(\chi(F') - \chi(F) - \chi(F''))$$

where χ denotes the Euler characteristic. Concretely, let $F = F_{g,1} \in \mathcal{K}(0, 1)$ and let $F' \in \mathcal{K}(0, 1)$ be the disk. Then this defines a *group completion map* into the g-th component

$$\gamma_g : |\mathcal{S}_1(F_{g,1})| \simeq B\mathrm{Diff}(F_{g,1}) \longrightarrow \Omega|\mathcal{S}.| \simeq \mathbb{Z} \times B\Gamma_\infty^+. \tag{2.6}$$

Remark on topologies. The morphism sets $\mathcal{S}_k(\bar{F}, \bar{F}')$ have a natural topology, coming from the compact-open topology on diffeomorphism spaces. Above $|\mathcal{S}.|$ means the realization which takes this topology into account. In order to apply the A-theory transfer map, however, $\mathcal{S}_k$ has to be replaced by a suitable discrete simplicial category. This can be done as follows.

Let $\mathcal{C}$ be a topological category with a discrete set of objects. Define the category $\sin.\mathcal{C}$ to be the simplicial category whose objects in each degree is ob $\mathcal{C}$. The morphisms between $c, d \in \sin_k\mathcal{C}$ is the set of singular k-simplices $\mathrm{Map}(\triangle^k, \mathcal{C}(c,d))$. We note that

$$\sin.(\mathcal{C} \times \mathcal{D}) = \sin.\mathcal{C} \times \sin.\mathcal{D}, \quad |\sin.\mathcal{C}| \simeq |\mathcal{C}|.$$

Hence, sin. takes topological, symmetric categories to simplicial, symmetric categories without changing the (weak) homotopy type of the associated classifying spaces. In the following section we shall assume that our topological categories have been replaced by their simplicial version. But in order not to make the notation too complicated we are not going to display this extra simplicial direction – we pretend $\mathcal{S}.$ is discrete rather than working with the bisimplicial category $\sin.\mathcal{S}.$.

2.4. Construction of α as a map of infinite loop spaces.

The definition of α is given in two steps: We first define the analogue of the A-theory transfer χ, and then the map Θ induced by the classifying map of the vertical tangent bundle.

Consider the functor $\Pi_k : \mathcal{S}_k \to \mathrm{TOP}^{hf}$ defined on objects by

$$\Pi_k : (F_1, \dots, F_k) \mapsto F_1 \sqcup \dots \sqcup F_k$$

for $k > 0$, and $\Pi_0(n) = *_n$, a different one-point space for each $n \geq 0$. On a connected component $(\mathcal{S}_k)_{[\bar{F}]}$ containing the object $\bar{F}$, Π_k takes all morphisms to homotopy equivalences. Apply now the A-theory transfer map (2.4) to the quasi-fibration

$$|E\mathcal{S}_k \times_{\mathcal{S}_k} \Pi_k|_{[\bar{F}]} \xrightarrow{\pi} |\mathcal{S}_k|_{[\bar{F}]}.$$

This yields maps

$$\chi_{k,[\bar{F}]} : |\mathcal{S}_k|_{[\bar{F}]} \longrightarrow |E\mathcal{S}_k \times_{\mathcal{S}_k} |w\mathcal{P} \circ \Pi_k||_{[\bar{F}]}. \tag{2.7}$$

Proposition 2.3. *The maps in (2.7) define a map of Γ-spaces*

$$\chi : |\mathcal{S}.| \longrightarrow |E\mathcal{S}. \times_{\mathcal{S}.} |w\mathcal{P} \circ \Pi.||.$$

Proof. The functors D^0, D^1 and $D^2 = w\mathcal{P} \circ \Pi_k$ used in the definition of $\chi_{k,[\bar{F}]}$ are compatible with gluing of surfaces and hence extend to simplicial functors $D.^0, D.^1, D.^2 : \mathcal{S}. \to \mathrm{CAT}$ in the sense of Definition 2.1. This is trivially true for $D.^0$

and easily checked for $D.^1$. For $D.^2$ the natural transformations $\delta_i : D^2_k \to D^2_{k-1} \circ \partial_i$ are induced by the natural functor

$$w\mathcal{P}(F_1) \times w\mathcal{P}(F_2) \cong w\mathcal{P}(F_1 \sqcup F_2) \longrightarrow w\mathcal{P}(F_1 \circ F_2).$$

The simplicial identities are satisfied, so the maps $\chi_{k,[\bar{F}]}$ fit together to define the map χ.

Disjoint union not only defines a functor $\sqcup : \mathcal{S}. \times \mathcal{S}. \to \mathcal{S}.$ but also a natural transformation between the functors $w\mathcal{P} \circ \sqcup \circ (\Pi. \times \Pi.)$ and $w\mathcal{P} \circ \Pi. \circ \sqcup$ from $\mathcal{S}. \times \mathcal{S}.$ to CAT in the sense of Definition 2.1. Using this, one can construct a Γ-space along the lines of [Ma; construction 10] with $|E\mathcal{S}. \times_{\mathcal{S}.} |D.^2||$ as underlying space. The $\chi_{k,[\bar{F}]}$'s are also compatible with disjoint union so that χ extends to a map of Γ-spaces. □

We next define the map $\Theta : |E\mathcal{S}. \times_{\mathcal{S}.} |w\mathcal{P} \circ \Pi.|| \to |w\mathrm{S}.\mathcal{P}(BS^1)|$ which extends to a map of Γ-spaces. First note that

$$|E\mathcal{S}_1 \times_{\mathcal{S}_1} \Pi_1|_{[F]} \simeq |E\mathcal{S}_1(F) \times_{\mathcal{S}_1(F)} F| = E(F),$$

the universal bundle from (1.3). Second, the bundle classifying map $\theta(F) : E(F) \to BS^1$ has the following interpretation. Let

$$\theta_{[F]} : |E\mathcal{S}_1 \times_{\mathcal{S}_1} \Pi_1|_{[F]} \longrightarrow B\mathrm{GL}_1(\mathbb{C}) \simeq BS^1$$

be the map defined on simplices by

$$\theta_{[F]}(\sigma_1, \ldots, \sigma_q; x) = (d\sigma_1|_{\sigma_2 \ldots \sigma_q(x)}, \ldots, d\sigma_q|_x). \tag{2.8}$$

Here $d\sigma|_x$ denotes the derivative of the diffeomorphisms σ evaluated at the point x, and we have identified the classifying space of the category of 2-dimensional oriented vector spaces with $B\mathrm{GL}_1(\mathbb{C})$.

Now define $\Theta_{[\bar{F}]}$ as the composition:

$$\begin{array}{c} |E\mathcal{S}_k \times_{\mathcal{S}_k} |w\mathcal{P} \circ \Pi_k||_{[\bar{F}]} \\ \downarrow \\ |w\mathcal{P}(|E\mathcal{S}_k \times_{\mathcal{S}_k} \Pi_k|)|_{([\bar{F}])} \\ \downarrow \\ |w\mathrm{S}_k\mathcal{P}(BS^1)|. \end{array} \tag{2.9}$$

The individual maps are defined as follows. The inclusion of the fibres into the total space always defines a map $F_B(E) \to F(E)$ for any homotopy equivalence preserving functor F from spaces to spaces and any quasi-fibration $E = |E\mathcal{C} \times_{\mathcal{C}}$

$\Pi| \to B = |\mathcal{C}|$. This explains the first map with $F = |w\mathcal{P}(-)|$. Next note that $|E\mathcal{S}_k \times_{\mathcal{S}_k} \Pi_k| = Y_1 \sqcup \cdots \sqcup Y_k$ where

$$Y_i = |E\mathcal{S}_k \times_{\mathcal{S}_k} \Pi_k^i|_{[F_i]}$$

is the space corresponding to the i-th component; here $\Pi_k^i(\bar{F}') := F'_i$. The second map is induced by the functor taking the k-tuple $(X_1, \ldots, X_k)$ of retractive spaces

$$X_i \underset{s_i}{\overset{r_i}{\rightleftarrows}} Y_i$$

to the flag A of the form (2.1) with subquotients

$$A_{ij} := BS^1 \sqcup_{(Y_{i+1} \sqcup \cdots \sqcup Y_j)} (X_{i+1} \sqcup \cdots \sqcup X_j).$$

In other words, A_{ij} is the pushout for the maps $\tilde{\theta}_{[F_{i+1}]} \sqcup \cdots \sqcup \tilde{\theta}_{[F_j]}$ and $s_{i+1} \sqcup \cdots \sqcup s_j$, where

$$\tilde{\theta}_{[F_i]} : Y_i = |E\mathcal{S}_k \times_{\mathcal{S}_k} \Pi_k^i|_{[F_i]} \xrightarrow{\mathrm{pr}_i} |E\mathcal{S}_1 \times_{\mathcal{S}_1} \Pi_1|_{[F_i]} \xrightarrow{\theta_{[F_i]}} BS^1.$$

Weak equivalences of k-tuples of retractive spaces are clearly taken to maps of flags.

Proposition 2.4. *The maps (2.9) define a map of* Γ*-spaces*

$$\Theta : |E\mathcal{S}. \times_{\mathcal{S}.} |w\mathcal{P} \circ \Pi.|| \longrightarrow |wS.\mathcal{P}(BS^1)|.$$

Proof. Again, it is straight forward to check that the maps $\Theta_{[\bar{F}]}$ are compatible with gluing and hence yield the map Θ.

Similarly, the maps $\Theta_{[F]}$ are compatible with disjoint union of surfaces. In the target space this corresponds to the functor

$$w\mathcal{P}(BS^1) \times w\mathcal{P}(BS^1) \longrightarrow w\mathcal{P}(BS^1)$$

induced by $1 \sqcup 1 : BS^1 \sqcup BS^1 \to BS^1$ which defines the sum operation in $w\mathrm{S}.\mathcal{P}(BS^1)$. A functor that takes the product of a symmetric category to the sum operation of another category induces a map of associated Γ-spaces. □

Remark. We are not claiming that the intermediate space in (2.9) gives rise to a Γ-space.

Disjoint union allows us to define a functor $G : \Gamma \to \mathrm{TOP}$ where Γ is the category of pointed sets, such that $G(1) = |w\mathcal{P}(|E\mathcal{S}. \times_{\mathcal{S}.} \Pi.|)|$ and $G(k) \simeq |w\mathcal{P}(|E\mathcal{S}.^k \times_{\mathcal{S}.^k} \Pi.^k|)|$ (along the lines of construction 10 in [Ma]). However, $G(k)$ is not homotopic to $G(1)^k$, and therefore G is not a Γ-space (in the sense of [S1]). But since we are only interested in the composition in (2.9), this is of no consequence.

As $|\mathcal{S}.|$ and $|w\mathrm{S}.P(BS^1)|$ are connected, the two propositions together prove that

$$\Theta \circ \chi : |\mathcal{S}.| \longrightarrow |w\mathrm{S}.\mathcal{P}(BS^1)|$$

is a map of infinite loop spaces. Looping this once, we have proved the main goal of this section.

Theorem 2.5. *The map $\alpha := \Omega(\Theta \circ \chi) : \mathbb{Z} \times B\Gamma_\infty^+ \to A(BS^1)$ is a map of infinite loop spaces.* □

2.5. Comparing $\sigma \circ \alpha$ with $\tau_{g,1}$.

Let F be a surface in $\mathcal{S}_1$. Consider the universal F-bundle (1.3)

$$E(F) = E\mathrm{Diff}(F) \times_{\mathrm{Diff}(F)} F \xrightarrow{\pi_F} B = B\mathrm{Diff}(F),$$

and let χ_F denote the composition

$$\chi_F : B \xrightarrow{\chi} A_B E(F) \longrightarrow A(E(F)).$$

Here χ is defined just as in (2.4) with the functor $|D_2| = |w\mathcal{P} \circ \Pi|$ replaced by the functor $A \circ \Pi = \Omega|w\mathrm{S}.\mathcal{P} \circ \Pi|$.

Lemma 2.6. *The following diagram commutes up to homotopy.*

$$\begin{tikzcd}
B\mathrm{Diff}(F) \arrow[r, "\chi_F"] \arrow[d, equal] & A(E(F)) \arrow[r, "A(\theta_{E(F)})"] \arrow[d, "\sigma"'] & A(BS^1) \arrow[d, "\sigma"'] \\
B\mathrm{Diff}(F) \arrow[r, "\mathrm{trf}(\pi_F)"] & Q(E(F)_+) \arrow[r, "Q(\theta_{E(F)_+})"] & Q(BS^1_+)
\end{tikzcd}$$

Proof. One of the main results in [DWW] asserts that for a smooth fibre bundle like $E(F)$, χ_F factors as the Becker-Gottlieb transfer $\mathrm{trf}(\pi_F) : B\mathrm{Diff}(F) \to Q(E(F)_+)$ composed with Waldhausen's standard map $i : Q(E(F)_+) \to A(E(F))$. Hence, the first square commutes up to homotopy for σ is a homotopy left inverse of i, cf. [W1]. The second square commutes by naturallity of the trace map σ. □

The bottom row in the above diagram is the definition of τ_F (1.7). We thus have

$$\tau_F \simeq \sigma \circ A(\theta_{E(F)}) \circ \chi_F. \tag{2.10}$$

Now take $F = F_{g,1}$. By definition (2.5), $\hat{\gamma}_{F_{g,1}}$ is compatible with the A-theory group completion map in the sense that the following diagram commutes up to homotopy

$$\begin{tikzcd}
B\mathrm{Diff}(F_{g,1}) \arrow[r, "\hat{\gamma}_{F_{g,1}}"] \arrow[d, "\chi_{F_{g,1}}"'] & \Omega^{0,1}|S.| \arrow[d, "\Omega^{0,1}(\Theta\circ\chi)"'] \\
A(E(F_{g,1})) \arrow[r, "A(\theta_{E(F_{g,1})})"] & A(BS^1).
\end{tikzcd}$$

Identifying the path space with the loop space as in the definition of γ_g (2.6), yields the homotopy commutative diagram

$$\begin{tikzcd}
\Omega^{0,1}|S.| \arrow[r, "\simeq"] \arrow[d, "\Omega^{0,1}(\Theta\circ\chi)"'] & \Omega|S.| \arrow[d, "\Omega^{1,1}(\Theta\circ\chi)=\alpha"'] \\
A(BS^1) & A(BS^1) \arrow[l, "+[1]"']
\end{tikzcd}$$

where $+[1]$ denotes loop sum with the base point in the component of 1. Hence,

$$A(\theta_{E(F_{g,1})}) \circ \chi_{F_{g,1}} \simeq (+[1]) \circ \alpha \circ \gamma_g.$$

This together with (2.10) proves the commutativity of Diagram (1.10). That is

Theorem 2.7. $\tau_{g,1} \simeq (+[1]) \circ \sigma \circ \alpha \circ \gamma_g : BDiff(F_{g,1}) \to Q(BS^1_+)$. □

Remark 2.8. Commutativity of the left square in Lemma 2.6 is folklore and is supposed to hold not just for smooth fibrations but also for any homotopy fibration with homotopy finite fibre. Unfortunately, it seems that a proof would be rather involved, and technical details have never been written down. The results of [DWW] used here are much deeper and they do give a quick argument. They also imply that α factors through $Q(BS^1_+)$.

§3. Transfers and diffeomorphisms of surfaces.

This section constructs surfaces equipped with an action of a cyclic group C_q and exhibits C_q-invariant vector fields on them. We then apply the "parametrised Poincaré-Hopf" theorem from [BM] to get information about the transfer of the universal smooth bundle (1.3).

3.1. Branched covers.

Let Σ be a (fixed) connected surface. We consider divisors $D = \Sigma_{i=0}^k n_i p_i \in C_0(\Sigma; \mathbb{Z}/q)$ with

$$(n_i, q) = 1 \quad \text{and} \quad \Sigma_{i=0}^k n_i \equiv 0 \pmod q. \tag{3.1}$$

Given D, we construct an associated connected surface F with a smooth C_q-action, and

$$F^{C_q} = \{p_0, ..., p_k\}, \quad F/C_q = \Sigma.$$

Let $\mathbb{C}(n)$ denote the complex plane with C_q-action $t \cdot z = e^{2\pi i n/q} \cdot z$ where $t \in C_q$ is a generator. The tangent representation at p_i of the surface F will be

$$T_{p_i} F = \mathbb{C}(m_i), \quad m_i n_i \equiv 1 \pmod q. \tag{3.2}$$

To construct F, consider the complement Σ^* of a small open tube $N\{p_0, ..., p_k\}$ of the branch points. We have the Poincaré duality diagram

$$\begin{array}{ccc}
H^1(\Sigma^*; \mathbb{Z}/q) & \xrightarrow{\delta^*} & H^2(\Sigma, \Sigma^*; \mathbb{Z}/q) \\
\simeq\big\downarrow & & \simeq\big\downarrow \\
H_1(\Sigma, \{p_0, ..., p_k\}; \mathbb{Z}/q) & \xrightarrow{\partial_*} & H_0(\{p_0, ..., p_k\}; \mathbb{Z}/q),
\end{array}$$

and note by excision that

$$H^2(\Sigma, \Sigma^*; \mathbb{Z}/q) \simeq \bigoplus_{i=0}^{k} H^2(D^2_{p_i}, S^1_{p_i}; \mathbb{Z}/q).$$

The second condition in (3.1) implies a class $\kappa_D \in H^1(\Sigma^*; \mathbb{Z}/q)$ with $\partial_*(\kappa_D \cap [\Sigma]) = D$. We view κ_D as a map from Σ^* to BC_q and let $F^* \to \Sigma^*$ be the induced principal C_q-cover.

Let $S^1(m)$ (resp. $D^2(m)$) be the unit C_q-sphere (resp. -disk) of $\mathbb{C}(m)$, and let $\Delta_n : S^1 \to S^1$ be the n-th power map $\Delta_n(z) = z^n$. The restriction of the C_q-cover $F^* \to \Sigma^*$ to the i-th boundary component $S^1_{p_i}$ is by construction the pull-back

$$\begin{array}{ccc} \partial_i F^* & \longrightarrow & S^1(1) \\ \downarrow & & \downarrow{\scriptstyle \Delta_q} \\ S^1_{p_i} & \xrightarrow{\Delta_{n_i}} & S^1, \end{array}$$

so $\partial_i F^* = \{(w,z) | w^{n_i} = z^q\}$. This is a circle by associating to $u \in S^1$ the pair (u^q, u^{n_i}). Thus $\partial_i F^* = S^1(m_i)$ as a C_q-space, where $m_i n_i \equiv 1 \pmod q$.

Definition 3.1. *The C_q-branched cover of Σ associated with the divisor D is the surface*

$$F = F^* \sqcup_\partial (D(m_0) \sqcup \ldots \sqcup D(m_k)).$$

Lemma 3.2. *There exists a non-degenerate C_q-invariant vector field X on F whose singular set $S(X)$ contains the branch points $\{p_0, ..., p_k\}$ with local indices*

$$ind_{p_i}(X) = +1.$$

Proof. We choose a Morse function $f : \Sigma \to \mathbb{R}$ such that $\{p_0, ..., p_k\}$ are local maxima or local minima. Let $\bar{X}$ be its gradient vector field. Its singular set includes the branch points, so it lifts to an equivariant vector field X on F.

The local index of $\bar{X}$ at p_i is $+1$, since the Morse index at p_i is ± 2, and thus $\mathrm{ind}_{p_i}(X)$ is also $+1$. (If the Morse index for f at p_i had been ± 1, then $\mathrm{ind}_{p_i}(\bar{X}) = -1$ and $\mathrm{ind}_{p_i}(X) = 1 - 2q$ so X would have been degenerate at p_i). For $p \in S(X) \setminus \{p_0, ..., p_k\}$, $\mathrm{ind}_p(X) = \mathrm{ind}_{\pi(p)}(\bar{X})$. This completes the proof. □

3.2. Transfers and vector fields.

Consider a smooth fibre bundle with closed oriented fibre,

$$F \longrightarrow E \xrightarrow{\pi} B \tag{3.3}$$

with finite (or compact Lie) structure group. Then $E = P \times_G F$ where $P \to B$ is the associated principal bundle. We assume G acts on F preserving the orientation. Suppose given a non-degenerate vector field X on F that is G-invariant. Its singular set $S(X)$ is a finite G-set, so there is a finite covering space contained in (3.3):

$$
\begin{tikzcd}
P \times_G S(X) \arrow[r, "\mathrm{incl}"] \arrow[d, "\pi_S"'] & P \times_G F \arrow[d, "\pi"'] \\
B \arrow[r, equal] & B.
\end{tikzcd}
\tag{3.4}
$$

Theorem 2.10 from [BM] asserts a relationship between the Becker-Gottlieb transfers $\mathrm{trf}(\pi_S)$ and $\mathrm{trf}(\pi)$:

Theorem 3.3 ([BM]). *There is a homotopy commutative diagram*

$$
\begin{tikzcd}
B \arrow[r, "\mathrm{trf}(\pi)"] \arrow[d, "\mathrm{trf}(\pi_S)"'] & Q(P \times_G F_+) \\
Q(P \times_G S(X)_+) \arrow[r, "\mathrm{IND}(X)"] & Q(P \times_G S(X)_+). \arrow[u, "Q(\mathrm{incl}_+)"']
\end{tikzcd}
$$

□

We recall the definition of $\mathrm{IND}(X)$. Choose a G-invariant metric on F, a G-embedding of F into a representation space V, and a complement η to the vector bundle $P \times_G V$. This is possible since B is compact. For $\sigma \in S(X)$, the differential dX_σ may be considered as an automorphism of the tangent space $T_\sigma F$, and gives rise to a G-bundle automorphism dX of $\tau_F|_{S(X)}$. We add the identity on $\nu_F|_{S(X)}$ and apply $P \times_G (-)$ to get a bundle automorphism

$$P \times_G dX : P \times_G (S(X) \times V) \longrightarrow P \times_G (S(X) \times V)$$

over $P \times_G S(X)$. Let $\dot\eta$ denote the fibrewise one point compactification of η. Then

$$(P \times_G S(X) \times S^V) \wedge_{P \times_G S(X)} \pi_S^*(\dot\eta) \simeq (P \times_G S(X)_+) \wedge S^m$$

where on the left we take the fibrewise smash product. The fibrewise one point compactification of $P \times_G dX$ induces a homotopy automorphism of the above space. Looping down m times and letting $m \to \infty$ we obtain the map

$$\mathrm{IND}(X) : Q(P \times_G S(X)_+) \longrightarrow Q(P \times_G S(X)_+).$$

Our next lemma identifies $\mathrm{IND}(X)$ in more computable terms. First some preparations are necessary.

Let W be a representation space for G and $f : S^W \to S^W$ a G-homotopy equivalence. Its G-homotopy class is determined by the set of degrees $\{\deg f^H | H \subseteq$

$G\}$. These degrees (all equal to ± 1) define a unit $d(f)$ of the Burnside ring $A(G)$, cf. [tDP]. Conversely, any $d \in A(G)^\times$ is equal to $d(f)$ for a suitable W.

Given a principal G-bundle $E \to B$ with compact base space there is a map

$$A(G)^\times \longrightarrow [Q(B_+), Q(B_+)]_{\Omega^\infty} \simeq [B_+, Q(B_+)] \tag{3.5}$$

into the homotopy invertible infinite loop maps. It maps $d(f)$ into the element determined by

$$E \times_G f : E \times_G S^W \longrightarrow E \times_G S^W$$

upon taking fibrewise smash product with $\dot{\xi}$ as above, where ξ is a complement to $E \times_G W$.

In the situation of Theorem 3.3, let $\sigma \in S(X)$ have isotropy group G_σ. The one point compactification of

$$dX_\sigma : T_\sigma F \longrightarrow T_\sigma F$$

defines an element $\chi(X, \sigma) \in A(G_\sigma)^\times$, i.e.

$$\chi(X, \sigma) = \{det(dX_\sigma^H) | H \subseteq G_\sigma\} \in A(G_\sigma)^\times. \tag{3.6}$$

We decompose $S(X)$ into its G-orbits,

$$P \times_G S(X) = \coprod P \times_{G_\sigma} \{\sigma\}; \quad \sigma \in S(X)/G.$$

Since $Q(-)$ converts wedge sums into products,

$$Q(P \times_G S(X)_+) = \prod Q(P/G_{\sigma+}); \quad \sigma \in S(X)/G. \tag{3.7}$$

The image of $\chi(X, \sigma)$ in $[Q(P/G_{\sigma+}), Q(P/G_{\sigma+})]$ under the map (3.5) is again denoted by $\chi(X, \sigma)$.

Lemma 3.4. *Under the identification (3.7),*

$$IND(X) \simeq \prod \chi(X, \sigma); \quad \sigma \in S(X)/G.$$

□

Remark. In [BM], Theorem 2.10, the map $\mathrm{IND}(X)$ was falsely asserted to be $\prod \mathrm{ind}_\sigma(X)$ rather than the more complicated expression in Lemma 3.4. The mistake – pointed out in [MP] – occurs at the top of page 141 in [BM]. Indeed the homotopy $\hat{J}_t$ need not be proper. We note that $\mathrm{ind}_\sigma(X) = \chi(X, \sigma)$ if and only if $det(dX_\sigma^H), H \subseteq G_\sigma$ is independent of H. This happens always if G_σ is of odd order, since $A(G_\sigma)^\times = \{\pm 1\}$.

In our application, (3.3) is the bundle

$$EC_q \times_{C_q} F \longrightarrow BC_q$$

where F is the branched cover from Definition 3.1, and X is the vector field from Lemma 3.2. In this case $\chi(X,\sigma) = \mathrm{ind}_\sigma(X)$. Indeed, in this case the action on F is free off the fixed set and it suffices to check that

$$dX_\sigma : S^{T_\sigma F} \longrightarrow S^{T_\sigma F}$$

has equal degrees on fixed sets of the isotropy subgroup $(C_q)_\sigma$. This is clear since $(C_q)_\sigma \neq 1$ only for $\sigma \in \{p_0, \ldots, p_k\}$ where dX_σ has degree $+1$. Moreover, the C_q fixed set is S^0, and dX_σ maps it by the identity.

Recall from (1.7) and (1.8) that we seek information about the composition

$$\tau_F : B\mathrm{Diff}(F) \stackrel{\mathrm{trf}(\pi_F)}{\longrightarrow} Q(E(F)_+) \stackrel{Q(\theta_{E(F)_+})}{\longrightarrow} Q(BS^1_+).$$

Let F be the C_q-surface of Definition 3.1, and

$$\rho_F : BC_q \longrightarrow B\mathrm{Diff}(F)$$

the associated map. Our next result calculates $\tau_F \circ \rho_F$ in terms of the maps

$$\begin{aligned} \hat{\psi}^{m_i} &: BC_q \stackrel{j}{\longrightarrow} BS^1 \stackrel{\psi^{m_i}}{\longrightarrow} BS^1 \stackrel{\mathrm{incl}}{\longrightarrow} Q(BS^1_+) \\ \hat{\tau}_q &: BC_q \stackrel{\tau_q}{\longrightarrow} Q((EC_q)_+) \stackrel{\simeq}{\longrightarrow} Q(S^0) \hookrightarrow Q(BS^1_+) \end{aligned}$$

where j is the standard map and τ_q is the transfer of the canonical covering $EC_q \to BC_q$.

Theorem 3.5. *The composition $\tau_F \circ \rho_F$ is equal to*

$$\Sigma_{i=0}^k \hat{\psi}^{m_i} + \Sigma\, \varepsilon_j \hat{\tau}_q$$

in $[BC_{q+}, Q(BS^1_+)]$, where ε_j is a sign and the second sum runs over the number of free C_q-orbits in the singular set of the vector field from Lemma 3.2.

Proof. Consider the pull-back diagram

$$\begin{array}{ccc} EC_q \times_{C_q} F & \stackrel{\hat{\rho}_F}{\longrightarrow} & E(F) \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi_F}\big\downarrow \\ BC_q & \stackrel{\rho_F}{\longrightarrow} & B\mathrm{Diff}(F). \end{array}$$

Transfers are natural under pull-backs, so

$$\mathrm{trf}(\pi_F) \circ \rho_F = Q(\hat{\rho}_{F+}) \circ \mathrm{trf}(\pi),$$

and we can apply Theorem 3.3 to study π. We have

$$S(X) = \{p_0, ..., p_k\} \sqcup S' \times C_q, \qquad EC_q \times_{C_q} S(X) = (\coprod_{i=0}^k BC_q) \sqcup (S' \times EC_q),$$

and hence

$$Q(EC_q \times_{C_q} S(X)_+) = (\prod_{i=0}^{k} Q(BC_{q+})) \times (\prod_{j\in S'} Q(EC_{q+})).$$

The transfer for a sum of covering spaces is the product of the individual transfers. So the transfer for

$$\pi_S : EC_q \times_{C_q} S(X) \longrightarrow BC_q$$

is the product of $(k+1)$ copies of the inclusion $\iota : BC_q \to Q(BC_{q+})$ and $|S'|$ copies of the standard transfer $\tau_q : BC_q \to Q(EC_{q+})$. By (3.2)

$$\hat\rho_F^*(T^v E(F)) = EC_q \times_{C_q} TF$$

restricts to $EC_q \times_{C_q} \mathbb{C}(m_i)$ over $EC_q \times_{C_q} \{p_i\}$, and this bundle is classified by $\psi^{m_i} \circ j : BC_q \to BS^1$. An application of Theorem 3.3 and Lemma 3.4 completes the proof. $\square$

3.3. The splitting map μ_p.

We fix a base surface Σ, a prime p and a p-adic divisor,

(3.7) $$D = 1 \cdot p_0 + mp_1 + \ldots + mp_k \in C_0(\Sigma; \mathbb{Z}_p)$$

with $m \in \mathbb{Z}_p^\times$, $1 + km \equiv 0$ and $-k \in \mathbb{Z}_p^\times$ a topological generator of $\mathbb{Z}_p^\times$. (If $p = 2$, take $k \equiv 5 \pmod 8$.) For each prime power $q = p^n$, let $F = F(n)$ be the closed surface from Definition 3.1. It has genus

$$g(n) = g(\Sigma) \cdot p^n + \frac{1}{2}(p^n - 1)(k-1)$$

by the Riemann-Hurwitz formula. We remove an open disc from $F(n)$ to get the surface $F(n)_1$ with one boundary circle. There is a commutative diagram

(3.8) $$\begin{array}{ccc} BC_{p^n} & \xrightarrow{\rho_n} & B\mathrm{Diff}(F(n))^+ \\ {\scriptstyle \mathrm{incl}}\uparrow & & \uparrow \\ (BC_{p^n})^{[g(n)/2]} & \xrightarrow{\rho_n} & B\mathrm{Diff}(F(n)_1)^+ \end{array}$$

since the right-hand vertical map is $[g(n)/2]$-connected by [H], [I]; $X^{[r]}$ denotes the r-skeleton of X. We can compose the bottom horizontal map in (3.8) with the map from $B\mathrm{Diff}(F(n)_1)^+$ to $B\Gamma_\infty^+$ to obtain homotopy classes

(3.9) $$[\rho_n] \in [BC_{p^n}^{[g(n)/2]}, B\Gamma_\infty^+].$$

We do not know (at present) that those elements fit together to define an element of the inverse limit, and hence give a homotopy class

$$\rho_\infty : BC_{p^\infty} \to B\Gamma_\infty^+,$$

with a possible extension over the universal Bockstein $BC_{p^\infty} \to BS^1_{(p)}$. However, we can get around this difficulty when we complete our spaces.

Recall the notation that $\hat\psi^l : BC_{p^n} \to Q(BS^1)$ is the standard map into BS^1 followed by ψ^l and the inclusion into $Q(BS^1)$.

Theorem 3.6. *There exists a map $\mu_p : BS^1 \to (B\Gamma_\infty^+)_p^\wedge$ such that*

$$[\tau_\infty \circ \mu_p] = \hat{1} + k\hat{\psi}^{-k} \in [BS^1, Q(BS^1)_p^\wedge].$$

Proof. Theorem 2.9 and Theorem 3.5 show that

$$[\tau_\infty \circ \rho_n] = \hat{1} + k\hat{\psi}^{-k} \in [BC_{p^n}^{[g(n)/2]}, Q(BS^1)].$$

Therefore the subgroup G_n of $[BC_{p^n}^{[g(n)/2]}, (B\Gamma_\infty^+)_p^\wedge]$, given by

$$G_n = \{[f] \mid [\tau_\infty \circ f] = \hat{1} + k\hat{\psi}^{-k}\}$$

is non-empty. It is also compact, since $B\Gamma_\infty^+$ is of finite type (has finitely generated homotopy groups in all dimensions), and Tychonov's theorem implies that

$$\varprojlim G_n \neq \emptyset.$$

Let $(\rho_n) \in \varprojlim G_n$. Since $g(n) \to \infty$ for $n \to \infty$,

$$\varprojlim [BC_{p^n}^{[g(n)/2]}, (B\Gamma_\infty^+)_p^\wedge] = \varprojlim [BC_{p^n}, (B\Gamma_\infty^+)_p^\wedge].$$

The element $\mu_p := (\rho_n) \in \varprojlim G_n$ gives the required map, cf. (1.11). $\square$

Remark. We used above the homotopy commutative diagram

$$\begin{array}{ccc} B\mathrm{Diff}(F_1) & \xrightarrow{\mathrm{trf}(\pi_1)} & Q(E(F_1)_+) \\ \downarrow & & \downarrow {\scriptstyle +[1]} \\ B\mathrm{Diff}(F) & \xrightarrow{\mathrm{trf}(\pi)} & Q(E(F)_+) \end{array}$$

where $+[1]$ denotes shift of components by 1, and

$$\pi_1 : E(F_1) = E\mathrm{Diff}(F_1) \times_{\mathrm{Diff}(F_1)} F_1 \longrightarrow B\mathrm{Diff}(F_1)$$
$$\pi : E(F) = E\mathrm{Diff}(F) \times_{\mathrm{Diff}(F)} F \longrightarrow B\mathrm{Diff}(F)$$

($F_1 = F \setminus \overset{\circ}{D}{}^2$). Since transfers are natural under pull-backs, the commutativity amounts to comparing the transfers of π_1 and π. The second bundle can be viewed as the fibrewise union of $E(F_1)$ and the trivial disc bundle. It is now easy to make the comparison. Details are left to the reader.

§4. Proof of the splitting theorems.

4.1. The splitting of $Q(BS^1)_p^\wedge$ and the proof of (1.16).

Consider the (reduced) cohomology theory associated to $Q(BS^1)_p^\wedge$,

$$E^i(X) = [X, Q(S^i \wedge BS^1)_p^\wedge].$$

For $k \in \mathbb{Z}_p^\times$, $Q(1 \wedge \psi^k)$ defines a natural endomorphisms of $E^i(X)$ that commutes with suspension. Let $\omega : \mathbb{Z}/p^\times \to \mathbb{Z}_p^\times$ be the Teichmüller character that splits the natural projection $\mathbb{Z}^\times \to \mathbb{Z}/p^\times$. We get a natural action of $\mathbb{Z}_p[\mathbb{Z}/p^\times]$ on $E^*(X)$. The ring $\mathbb{Z}_p[\mathbb{Z}/p^\times]$ is semisimple and decomposes into a sum of $(p-1)$ copies of $\mathbb{Z}_p$, and there is an induced isomorphism of cohomology theories

$$E^*(X) \cong E_0^*(X) \oplus \ldots \oplus E_{p-2}^*(X) \tag{4.1}$$

More precisely, if l generates $\mathbb{Z}/p^\times$ then

$$e_i = \frac{1}{p-1} \sum_{\nu=0}^{p-1} \omega(l^{-\nu i}) \psi^{\omega(l^\nu)}, \quad i = 0, ..., p-2 \tag{4.2}$$

are orthogonal idempotents of $\mathbb{Z}_p[\mathbb{Z}/p^\times]$, and

$$E_i^*(X) = e_i E^*(X).$$

There is a similar splitting of p-complete (or even p-local) K-theory, often called the Adam's splitting [A]. We get from (4.1), and its K-theory analogue, the induced splitting of infinite loop spaces

$$\begin{aligned} Q(BS^1)_p^\wedge &\simeq \Omega^\infty E_0 \times \Omega^\infty E_1 \times \ldots \times \Omega^\infty E_{p-2} \\ BU_p^\wedge &\simeq B_0 \times B_1 \times \ldots \times B_{p-2}. \end{aligned} \tag{4.3}$$

The map from $Q(BS^1)$ into BU, induced from the reduced canonical line bundle $[L-1] \in \tilde{K}(BS^1)$, gives infinite loop maps from $\Omega^\infty E_i$ to B_i, whose homotopy fibre is rationally homotopy equivalent to a point.

Lemma 4.1. *Let p be an odd prime and $g \in \mathbb{Z}_p^\times$ be a topological generator. Then*

$$1 - g\psi^g : \Omega^\infty E_i \longrightarrow \Omega^\infty E_i$$

is a homotopy equivalence for $i = 0, 1, ..., p-3$.

Proof. It suffices to prove that the induced map on spectrum homology

$$(1 - g\psi^g)_* : H_*(E_i) \longrightarrow H_*(E_i)$$

is an isomorphism for each $i \neq p-2$. The homology of E_i is one copy of $\mathbb{Z}_p$ in each degree $2n$ with $n \equiv i (\operatorname{mod} p-1)$ and zero in other degrees. Indeed, the wedge product $E_0 \vee \ldots \vee E_{p-2}$ is the p-complete suspension spectrum of BS^1, so

$$H_*(E_0 \vee \cdots \vee E_{p-2}) = H_*((\Sigma^\infty BS^1)^\wedge_p) = H_*(\Sigma^\infty BS^1; \mathbb{Z}_p),$$

a copy of $\mathbb{Z}_p$ in each even degree. On the other hand, it follows from (4.2) that

$$\psi^{\omega(l)} \circ e_i = \omega(l^i) e_i,$$

so that $\psi^{\omega(l)}$ induces multiplication by $\omega(l^i)$ on $H_{2n}(E_i)$ for all n. But $\psi^{\omega(l)}$ induces multiplication by $\omega(l)^n$ on $H_{2n}((\Sigma^\infty BS^1)^\wedge_p)$. Thus $\omega(l^n) = \omega(l^i)$ on $H_{2n}(E_i)$, and $n \equiv i (\operatorname{mod}\ p-1)$.

The map $1 - g\psi^g$ induces multiplication by $1 - g^{n+1}$ on $H_{2n}((\Sigma^\infty BS^1)^\wedge_p)$ and $1 - g^{n+1}$ is a p-adic unit precisely when $n \not\equiv -1 (\operatorname{mod} p-1)$. $\square$

¿From Theorem 3.6 and the previous lemma we get

Theorem 4.2. *The map*

$$(B\Gamma^+_\infty)^\wedge_p \xrightarrow{\tau_\infty} Q(BS^1)^\wedge_p \xrightarrow{\text{proj}} \Omega^\infty E_0 \times \ldots \times \Omega^\infty E_{p-3}.$$

is split surjective as a map of infinite loop spaces. $\square$

Combining with the main result of [T2], cf. (1.15), we get

Corollary 4.3. *For odd primes p there is an infinite loop space W'_p such that*

$$(\mathbb{Z} \times B\Gamma^+_\infty)^\wedge_p \simeq Q(S^0)^\wedge_p \times \Omega^\infty E_0 \times \cdots \times \Omega^\infty E_{p-3} \times W'_p.$$

$\square$

We remark that $Q(S^0)^\wedge_p \times \Omega^\infty E_0$ is the 0-th component of the splitting of $Q(BS^1_+)^\wedge_p$ induced by the idempotents e_i of (4.2), so that the space $Q(S^0)^\wedge_p \times \Omega^\infty E_0 \times \cdots \times \Omega^\infty E_{p-3}$ classifies the functor

$$X \longmapsto (1 - e_{p-2}) \cdot ([X, Q(BS^1_+)] \otimes \mathbb{Z}_p). \tag{4.5}$$

The homology $H_*(Q(BS^1_+); \mathbb{Z}/p)$ is completely known. The original source is [DL], but [CLM] is a better reference. We briefly recall the result.

Consider sequences $I = (\varepsilon_1, s_1, \ldots, \varepsilon_k, s_k)$ with

$$\varepsilon_j \in \{0,1\}, \quad s_j \geq e_j \quad \text{and} \quad ps_j - e_j \geq s_{j-1}.$$

Define

$$
\begin{aligned}
e(I) &= 2s_1 - \varepsilon_1 - \sum_{j=2}^{k} (2s_j(p-1) - \varepsilon_j) \\
b(I) &= \varepsilon_1 \\
d(I) &= \sum_{j=1}^{k} (2s_j(p-1) - \varepsilon_j)
\end{aligned} \tag{4.6}
$$

Let X be an infinite loop space. For each I there is a homology operation

$$Q^I : H_q(X; \mathbb{F}_p) \longrightarrow H_{q+d(I)}(X; \mathbb{F}_p)$$

which can be non-zero only if $e(I) + b(I) \geq q$. Here $\mathbb{F}_p = \mathbb{Z}/p$.

The homology of $Q(BS^1_+)$ can be described in terms of the homology operations, applied to $H_*(BS^1; \mathbb{F}_p) \subset H_*(Q(BS^1_+); \mathbb{F}_p)$. Indeed, let

$$T = \{Q^I \iota_{2q} \mid q \geq 0,\ e(I) + b(I) > 2q\}. \tag{4.7}$$

The group of components $\pi_0 Q(BS^1_+) \simeq \pi_0 QS^0$ is the infinite cyclic group $\mathbb{Z}$, and all components are homotopy equivalent. The homology of the component $Q_0(BS^1_+) = Q(BS^1) \times Q_0(S^0)$ turns out to be the free commutative algebra on the graded set T, i.e. a tensor product of the polynomial algebra on the even dimensional generators and exterior algebra on the odd dimensional generators. The homology of the full space is then

$$H_*(Q(BS^1_+); \mathbb{F}_p) = \mathrm{FreeCommAlg}(T) \otimes \mathbb{F}_p[\mathbb{Z}]. \tag{4.8}$$

In view of the Corollary 4.3 and (4.5) we have

$$(1 - e_{p-2})\, H_*(Q(BS^1); \mathbb{F}_p) \subset H_*(B\Gamma^+_\infty; \mathbb{F}_p).$$

The left hand side is the free commutative algebra on

$$T' = \{Q^I \iota_{2q} \in T \,|\, q \neq -1 (\bmod p-1)\}.$$

So we get

Corollary 4.4. *FreeCommAlg*$(T') \subset H_*(B\Gamma^+_\infty; \mathbb{F}_p)$. □

4.2. The proof of Theorem 1.3.

The proof given at the end of the section requires some preliminaries. We shall be brief, and refer the reader to [MS] for more details.

The infinite unitary group U may be considered as a subspace of the space F_{S^1} of S^1-equivariant homotopy equivalences of the free S^1-spheres S^{2n-1} as $n \to \infty$. There is a homotopy equivalence

$$\zeta : F_{S^1} \longrightarrow Q(S^1 \wedge BS^1_+),$$

cf. [BS], [MS]. This yields a map from U to $Q(S^1 \wedge BS^1_+)$ such that the diagram

$$\begin{array}{ccc} S^1 \wedge BS^1_+ & = & S^1 \wedge BS^1_+ \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle \mathrm{incl}} \\ U & \xrightarrow{\zeta} & Q(S^1 \wedge BS^1_+) \end{array} \tag{4.9}$$

is homotopy commutative. Here R is the complex rotation map, adjoint to the map from BS^1 to $\Omega(U) \simeq \mathbb{Z} \times BU$ that represents the canonical line bundle. One may extend R to an infinite loop map, again denoted by R, from $Q(S^1 \wedge BS^1)$ to U, and (4.9) implies that the composition

$$U \xrightarrow{\zeta} Q(S^1 \wedge BS^1_+) \xrightarrow{R} U \tag{4.10}$$

is the identity up to homotopy. The splitting of $Q(S^1 \wedge BS^1_+)$ induced from (4.10) is not well-connected to the S^1-transfer

$$\mathrm{trf} : \Sigma^\infty(S^1 \wedge BS^1_+) \longrightarrow \Sigma^\infty(S^0).$$

Indeed, the fibre of trf is the suspension $\Sigma\mathbb{C}P^\infty_{-1}$ of the spectrum $\mathbb{C}P^\infty_{-1}$, but the above splitting does not induce a splitting of $\Omega^\infty(\Sigma\mathbb{C}P^\infty_{-1})$. However from [MS, Theorem 2.4.5] we do have a commutative diagram

$$\begin{array}{ccccc} \Omega^\infty(\Sigma\mathbb{C}P^\infty_{-1})_{(p)} & \xrightarrow{\omega_\infty} & Q(S^1 \wedge BS^1_+)_{(p)} & \xrightarrow{\mathrm{trf}} & QS^0_{(p)} \\ {\scriptstyle l^1_{-1}}\downarrow & & {\scriptstyle l^1_0}\downarrow & & {\scriptstyle e}\downarrow \\ U_{(p)} & \xrightarrow{\Omega(1-\psi^g)} & U_{(p)} & \xrightarrow{\triangle} & \mathbb{Z} \times \mathrm{im}J_{(p)}, \end{array} \tag{4.11}$$

where e is the unit of the connective ring spectrum $\mathbb{Z} \times \mathrm{im}J_{(p)}$, and l^1_{-1}, l^1_0 are infinite loop maps which are split surjective but by maps that are *not* infinite loop maps.

We remarked in Section 1.2 that after localization at an *odd* prime, $Q(S^1)_{(p)}$ splits off both $\Omega^\infty(\Sigma\mathbb{C}P^\infty_{-1})_{(p)}$ and $\Omega^\infty(S^1 \wedge BS^1_+)_{(p)}$, so that the upper fibration sequence in (4.11) may be replaced by its reduced version

$$\Omega^\infty(\Sigma(\mathbb{C}P^\infty_{-1}/S^0))_{(p)} \xrightarrow{\bar\omega_\infty} Q(S^1 \wedge BS^1)_{(p)} \xrightarrow{\mathrm{trf}} QS^0_{(p)}.$$

Looping down this reduced version of (4.11) we get the fibration diagram

$$(4.12)\qquad \begin{array}{ccccc} \Omega^\infty(\mathbb{C}P^\infty_{-1}/S^0)_{(p)} & \xrightarrow{\bar{\omega}_\infty} & Q(BS^1)_{(p)} & \xrightarrow{\Omega \mathrm{trf}} & Q(S^{-1})_{(p)} \\ \downarrow{\scriptstyle l_{-1}} & & \downarrow{\scriptstyle l_0} & & \downarrow{\scriptstyle \Omega e} \\ BU_{(p)} & \xrightarrow{1-g\psi^g} & BU_{(p)} & \xrightarrow{\Omega\triangle} & \Omega \mathrm{im} J_{(p)}. \end{array}$$

¿From the reduced version of (4.10) we get upon looping once the homotopy commutative diagram

$$\begin{array}{ccc} BS^1 & \xrightarrow{[L-1]} & BU \\ {\scriptstyle \mathrm{incl}}\downarrow & & \| \\ Q(BS^1) & \xrightarrow{\Omega\bar{R}} & BU, \end{array}$$

where $[L-1]$ is the reduced canonical line bundle. Similarly (4.10) implies a map $\Omega\bar{\zeta}$ so that the composite

$$BU \xrightarrow{\Omega\bar{\zeta}} Q(BS^1) \xrightarrow{\Omega\bar{R}} BU$$

is the identity. Here BU is the 0-th component of $\mathbb{Z}\times BU \simeq \Omega U$ or equivalently $BU \simeq \Omega SU$. It follows easily that

$$(4.13)\qquad \begin{array}{ccc} BS^1 & = & BS^1 \\ {\scriptstyle \mathrm{incl}}\downarrow & & \downarrow{\scriptstyle [L-1]} \\ Q(BS^1) & \xleftarrow{\Omega\bar{\zeta}} & BU \end{array}$$

is homotopy commutative.

We are now ready to present the proof of Theorem 1.3 which asserts that

$$l_{-1}\circ\hat{\tau}_\infty : (B\Gamma^+_\infty)^\wedge_p \longrightarrow BU^\wedge_p$$

is a split surjection for all odd primes p.

Proof. Theorem 3.4 (with $g=-k$ a topological generator of $\mathbb{Z}_p^\times$) and the above yields a homotopy commutative diagram

$$(4.14)\qquad \begin{array}{ccccc} (B\Gamma^+_\infty)^\wedge_p & \xrightarrow{\hat{\tau}_\infty} & \Omega^\infty(\mathbb{C}P^\infty_{-1}/S^0)^\wedge_p & \xrightarrow{l_{-1}} & BU^\wedge_p \\ {\scriptstyle \mu_p}\uparrow & & \downarrow{\scriptstyle \bar{\omega}_\infty} & & \downarrow{\scriptstyle 1-g\psi^g} \\ Q(BS^1)^\wedge_p & \xrightarrow{1-g\psi^g} & Q(BS^1)^\wedge_p & \xrightarrow{l_0} & BU^\wedge_p. \end{array}$$

¿From (4.13) we see that $\Omega\bar{\zeta}\circ(1-g\psi^g)\simeq(1-g\psi^g)\circ\Omega\bar{\zeta}$, and (4.14) yields the diagram

$$\begin{array}{ccccc} BU^\wedge_p & \xrightarrow{\mu_p\circ\Omega(\zeta)} & (B\Gamma^+_\infty)^\wedge_p & \xrightarrow{l_{-1}\circ\hat{\tau}_\infty} & BU^\wedge_p \\ {\scriptstyle 1-g\psi^g}\downarrow & & \downarrow{\scriptstyle \bar{\omega}_\infty\circ\hat{\tau}_\infty} & & \downarrow{\scriptstyle 1-g\psi^g} \\ BU^\wedge_p & \xrightarrow{\Omega(\zeta)} & Q(BS^1) & \xrightarrow{l_0} & BU^\wedge_p. \end{array}$$

The lower composition is a homotopy equivalence, and the diagram implies that the upper composite is a homotopy equivalence. This completes the proof. □

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