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FILTRATIONS ON G_1T-MODULES

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FILTRATIONS ON G_1T-MODULES

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Let G be an almost simple and simply connected algebraic group defined and split over the prime field $\mathbb{F}_p$. Choose a split maximal torus T in G and a Borel subgroup B containing T. We denote the kernel of the Frobenius homomorphism on G (resp. B) by G_1 (resp. B_1).

Recall that the representation theory for G is closely related to the corresponding theory for G_1T, see [Ja2]. This paper will mainly be concerned with the latter theory. More precisely, we study filtrations in the G_1T-setup analogous to those introduced by the first author in [An3] involving tilting modules for G.

To explain our results we need a little more notation. Let X denote the character group of T and set R (resp. R^+, resp. S) equal to the root system for (G, T) (resp. the set of positive roots relative to B^+, the Borel subgroup opposite to B, resp. the set of simple roots in R^+). Write W for the Weyl group.

For each $\lambda \in X$ we have a standard G_1T-module $Z(\lambda)$ with highest weight λ. It is sometimes called a baby Verma module and it is defined as the G_1T- module induced by the 1-dimensional B_1T-module λ. In fact, for each $w \in W$ we have such a standard G_1T-module $Z^w(\lambda)$ (obtained by replacing B by wBw^{-1}). By using certain deformations of these modules it was shown in [AJS, Section 6] how one may construct Jantzen filtrations of $Z^w(\lambda)$ and prove the corresponding sum formulas.

In this paper we shall use the same deformation theory to construct for each projective G_1T-module Q a filtration of the vector space $F_\lambda(Q) = \mathrm{Hom}_{G_1T}(Z(\lambda)^\tau, Q)$. Here $^\tau$ denotes contravariant dual (see 1.6). Our construction is completely analogous to the one used in [An3] once one notices that a tilting module for G_1T is the same as a projective module.

There are certain natural homomorphisms between standard G_1T-modules which play a crucial role in the theory. We prove that our filtrations may be described in terms of the canonical homomorphism $Z(\lambda) \to Z(\lambda)^\tau$. This description allows us via standard arguments to prove a sum formula for our filtration. Moreover, it gives in turn a direct relation between our filtrations for $F_\lambda(Q(\mu))$ and the occurrence of $L(\mu)$ in the Jantzen filtration of $Z(\lambda)$. Here $L(\mu)$ is the simple G_1T-module with highest weight μ and $Q(\mu)$ is its projective cover.

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The Verma modules for a complex semi-simple Lie algebra are simple when their highest weights are anti-dominant. Likewise Weyl modules for G are simple when their highest weights belong to the bottom dominant alcove. However, a standard G_1T-module with a p-regular highest weight is never simple. This presents a major difficulty in developing a G_1T-analogue of the Gabber-Joseph theory for Verma-modules, see [GJ] (and its Weyl module version [An3]), namely there is no starting point for an induction. However, our way of constructing filtrations in this paper presents us with a natural first case: the projective cover with highest weight in the bottom alcove (relative to any special point). We describe our filtrations completely in that case and then go on via translation functors to the other alcoves. We show (exactly as in [An3]) that if the filtrations behave as expected with respect to the wall crossing functors, then the Lusztig conjecture for the character of irreducible G_1T-modules (or G-modules) holds. Comparing with [AK] we can then also deduce a refinement of Humphreys-Brauer reciprocity.

We conclude the paper by considering the following related cases: the 'small' quantum groups, the ordinary quantum groups (at complex roots of unity) and the semi-simple algebraic groups (in characteristic p).

The paper is organized as follows. In Section 1 we give the set up and prove a few refinements of the deformation theory of G_1T-modules from [AJS]. Section 2 deals with projective objects in our categories and then Section 3 contains the above mentioned results on our filtration of the Hom-spaces between standard modules and projective modules. In Section 4 we study the translation functors relative to our situation. We use these results in Section 5 first to give an explicit description of our filtrations in the case where the projective module is indecomposable with highest weight in the bottom dominant alcove and then to analyse the behavior of our filtrations under 'wall-crossings'. Finally, in Section 6 we look at the above mentioned related cases.

1. Standard modules

In this section we recall from [AJS, Sections 2-6] the parts of the deformation theory for G_1T-modules that we need.

1.1. Fix an algebraically closed field k of characteristic $p > 0$ and consider G as an algebraic group over k. Let $\mathfrak{g}$ denote the Lie algebra of G. This is a p-Lie algebra whose p-operation we denote $x \mapsto x^{[p]}, x \in \mathfrak{g}$.

We have $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ where $\mathfrak{h}$ is the Lie algebra of T, $\mathfrak{n}^-$ (resp. $\mathfrak{n}^+$) is the Lie algebra of the unipotent radical of B (resp. B^+). As in [AJS] we then set I equal to the ideal in the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ generated by $\{x^p - x^{[p]} \mid x \in \mathfrak{n}^- \cup \mathfrak{n}^+\}$ and define $U = U(\mathfrak{g})/I$.

Denote by U^- (resp. U^0, resp. U^+) the image in U of $U(\mathfrak{n}^-)$ (resp. $U(\mathfrak{h})$, resp. $U(\mathfrak{n}^+)$). Then we have $U = U^-U^0U^+$.

1.2. The algebra U has a natural X-grading and the subspaces U^0 and U^0U^+ are both graded subalgebras. If , is a noetherian commutative U^0-algebra then we denote by $\mathcal{C}_\Gamma$ the category introduced in [AJS]. The objects in $\mathcal{C}_\Gamma$ are certain X-graded $U \otimes_k ,$ -modules.

The corresponding category of $U^0U^+ \otimes_k ,$ -modules (resp. $U^0 \otimes_k ,$ -modules) which in [AJS] is denoted $\mathcal{C}'_\Gamma$ (resp. $\mathcal{C}''_\Gamma$) will here be denoted $\mathcal{C}^{\geq 0}_\Gamma$ (resp. $\mathcal{C}^0_\Gamma$).

We then have the induction functor $Z_\Gamma : \mathcal{C}^{\geq 0}_\Gamma \to \mathcal{C}_\Gamma$ defined by

$$Z_\Gamma(M) = U \otimes_{U^0U^+} M, \quad M \in \mathcal{C}^{\geq 0}_\Gamma$$

as in [AJS]. More generally, if we for $w \in W$ let $T_w \in \mathrm{End}_{k-alg}(U)$ denote the endomorphism of U given as conjugation by w (or more precisely by a representative in $N_G(T)$ of w) then we have a category $\mathcal{C}^{\geq 0,w}_\Gamma$ consisting of certain X-graded $T_w(U^0U^+) \otimes_k ,$ -modules corresponding to $\mathcal{C}^{\geq 0}_\Gamma$. The induction functor from $\mathcal{C}^{\geq 0,w}_\Gamma$ to $\mathcal{C}_\Gamma$ is denoted Z^w_Γ.

These induction functors behave well under base change: If ,$'$ is a , -algebra then in $\mathcal{C}^{\geq 0}_{\Gamma'}$ we have

(1) $$Z^w_\Gamma(M) \otimes_\Gamma ,' \simeq Z^w_{\Gamma'}(M \otimes_\Gamma ,')$$

for all $M \in \mathcal{C}^{\geq 0,w}_\Gamma$.

1.3. Let as usual $e^\nu \in \mathbb{Z}[X]$ be the element corresponding to $\nu \in X$. If $M \in \mathcal{C}_\Gamma$ is free as a , -module then we set

$$\mathrm{ch}\, M = \sum_{\nu \in X} (\mathrm{rk}_\Gamma\, M_\nu) e^\nu \in \mathbb{Z}[X].$$

Each $\lambda \in X$ defines an element in $\mathcal{C}_\Gamma^{\geq 0}$, namely the $U^0U^+ \otimes_k$, -module which as X-graded module is equal to , concentrated in degree λ and on which the U^+-action is trivial. Likewise, λ defines for any $w \in W$ an object in $\mathcal{C}_\Gamma^{\geq, w}$ which is free of rank 1 over , .

For $\lambda \in X, w \in W$ we set (as in [AJS]) $\lambda\langle w\rangle = \lambda + (p-1)(w\rho - \rho)$, where ρ as usual denotes half the sum of the positive roots. Then

(1) $$\operatorname{ch} Z_\Gamma^w(\lambda\langle w\rangle) = e^\lambda \prod_{\alpha \in R^+} \frac{1 - e^{-p\alpha}}{1 - e^{-\alpha}} = \operatorname{ch} Z_\Gamma(\lambda).$$

1.4. In this paper we shall mainly consider three choices for , . The first is , $= k$. In this case we drop subscripts and write e.g., $\mathcal{C}$ instead of $\mathcal{C}_k$ and Z instead of Z_k. Recall that $\mathcal{C}$ may be identified with the category of finite dimensional G_1T-modules and Z_k with the coinduction functor from B_1^+T-modules to G_1T-modules.

The other two choices for , are , $= k[t]_{(t)}$ (which we will denote $\hat{k}$ from now on) and , $= k(t)$ (which we will denote $\tilde{k}$), i.e., the local ring obtained by localizing the polynomial ring $k[t]$ in one variable at the maximal ideal generated by t and the fraction field of $k[t]$. The U^0-algebra structures on these algebras are given by [AJS,6.5]. In these cases we write $\hat{\mathcal{C}}$ (resp. $\tilde{\mathcal{C}}$) instead of $\mathcal{C}_{k[t]_{(t)}}$ (resp. $\mathcal{C}_{k(t)}$), $\hat{Z}$ (resp. $\tilde{Z}$) instead of $Z_{k[t]_{(t)}}$ (resp. $Z_{k(t)}$). Most of what we do in this paper remain valid when we replace k by any field of characteristic p.

1.5. Let , be a field. For each $\lambda \in X$ the standard module $Z_\Gamma(\lambda)$ has a unique simple quotient which we denote $L_\Gamma(\lambda)$. In other words, if for $M \in \mathcal{C}_\Gamma$ we let $\operatorname{hd}(M)$ denote the head of M, i.e. the maximal semi-simple quotient of M, then

(1) $\operatorname{hd} Z_\Gamma(\lambda) = L_\Gamma(\lambda), \quad \lambda \in X$.
(2) $\{L_\Gamma(\lambda) \mid \lambda \in X\}$ is a full set of simple objects in $\mathcal{C}_\Gamma$.
(3) If , $= \tilde{k}$, then all $\tilde{Z}(\lambda)$ are irreducible and in fact all objects of $\mathcal{C}_\Gamma$ are semi-simple, i.e., direct sums of $\tilde{Z}(\lambda)$'s.

Recall also that if , $= k$, then $L_k(\lambda)$ may be identified with the irreducible G_1T-module $L(\lambda)$ of highest weight λ. If moreover λ is restricted, i.e. $0 \leq \langle\lambda, \alpha\rangle < p$ for all $\alpha \in S$, then $L(\lambda)$ identifies with the irreducible G-module with highest weight λ.

Remark . *The above results* (1) - (3) *hold also when we replace* $1 \in W$ *by an arbitrary* $w \in W$ *(with appropriate change of weight in* (1)*, see* [AJS]*).*

1.6. For $\alpha \in S$ we let $F_\alpha \in \mathfrak{n}^-, E_\alpha \in \mathfrak{n}^+$ and $H_\alpha = [E_\alpha, F_\alpha] \in \mathfrak{h}$ denote Chevalley generators (elements of a Chevalley basis for $\mathfrak{g}$). There is an involutative k-algebra antiautomorphism τ of U given by

$$\tau(F_\alpha) = E_\alpha, \quad \tau(E_\alpha) = F_\alpha, \quad \tau(H_\alpha) = H_\alpha, \quad \alpha \in S.$$

This allows us to define the (contravariant) dual of an object M in $\mathcal{C}_\Gamma$ as $M^\tau = \mathrm{Hom}_\Gamma(M, ,)$ with U-action defined by $uf : m \mapsto f(\tau(u)m), m \in M,\ f \in M^\tau,\ u \in U$. The X-grading on M^τ is given by $(M^\tau)_\lambda = \{f \in M^\tau \mid f(M_\nu) = 0 \text{ for } \nu \neq \lambda\}$.

If $M = \mathcal{C}_\Gamma$ is free as a , -module, we see

$$(M^\tau)^\tau \simeq M \text{ in } \mathcal{C}_\Gamma \tag{1}$$

and for any , -algebra , $'$

$$M^\tau \otimes_\Gamma ,' \simeq (M \otimes_\Gamma ,')^\tau \text{ in } \mathcal{C}_\Gamma. \tag{2}$$

If also $M' \in \mathcal{C}_\Gamma$ is free over , then

$$\mathrm{Ext}^i_{\mathcal{C}_\Gamma}(M, M') \simeq \mathrm{Ext}^i_{\mathcal{C}_\Gamma}(M'^\tau, M^\tau). \tag{3}$$

Note that $\mathrm{ch}\, M = \mathrm{ch}\, M^\tau$. In particular, it follows that if , is a field, then we have for all $\lambda \in X$

$$L_\Gamma(\lambda)^\tau \simeq L_\Gamma(\lambda) \text{ in } \mathcal{C}_\Gamma. \tag{4}$$

1.7. Let $\lambda \in X$ and recall the notation $\lambda\langle w\rangle = \lambda + (p-1)(w\rho - \rho)$ from 1.3. Then by [AJS, 4.7] we have for all $w, w' \in W$

$$\mathrm{Hom}_{\mathcal{C}_\Gamma}(Z^w_\Gamma(\lambda\langle w\rangle), Z^{w'}_\Gamma(\lambda\langle w'\rangle)) \simeq , . \tag{1}$$

Moreover, if w_0 denotes the longest element in W, then by slight generalizations of [AJS, 4.10 and 4.12]

$$Z^w_\Gamma(\lambda\langle w\rangle)^\tau \simeq Z^{ww_0}_\Gamma(\lambda\langle ww_0\rangle) \text{ in } \mathcal{C}_\Gamma \tag{2}$$

and if $\nu \in X$

$$\mathrm{Ext}^i_{\mathcal{C}_\Gamma}(Z^w_\Gamma(\lambda\langle w\rangle), Z^{ww_0}_\Gamma(\nu)) \simeq \begin{cases} , & \text{if } i = 0 \text{ and } \nu = \lambda\langle ww_0\rangle \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

In the case where , $= \hat{k}$ we may add

$$\mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}^w(\lambda\langle w\rangle), \hat{Z}^y(\nu\langle y\rangle)) = \delta_{\lambda,\nu}\hat{k} \tag{4}$$

for all $\lambda, \nu \in X,\ w, y \in W$. This is seen by noticing that the left hand side is torsion free as a $\hat{k}$-module and hence free. Now tensor by $\tilde{k}$ and apply 1.5 (3).

1.8. Let $\lambda \in X$ and $x, y \in W$. Suppose $x^{-1}y = s_1 \cdots s_r$ is a reduced expression with $s_i = s_{\alpha_i}$ for some $\alpha_i \in S$, $i = 1, \cdots, r$. Set $y_i = xs_1 \cdots s_{i-1}, \quad i = 1, \cdots, r+1$.

By 1.7(1) we have a generator $\varphi^\lambda_{\Gamma,i} \in \mathrm{Hom}_{\mathcal{C}_\Gamma}(Z^{y_i}_\Gamma(\lambda\langle y_i\rangle), Z^{y_{i+1}}_\Gamma(\lambda\langle y_{i+1}\rangle))$ (unique up to unit in ,), resp. ${}'\varphi^\lambda_{\Gamma,i} \in \mathrm{Hom}_{\mathcal{C}_\Gamma}(Z^{y_{i+1}}_\Gamma(\lambda\langle y_{i+1}\rangle), Z^{y_i}_\Gamma(\lambda\langle y_i\rangle))$. By [AJS, 5.13] we have

(1) $\varphi^\lambda_{\Gamma,i}$ is bijective on the λ-weight space if $y_i\alpha_i \in R^+$.

If , $'$ is a , -algebra then we have up to units in , $'$

(2) $\varphi^\lambda_{\Gamma,i} \otimes_\Gamma \Gamma' = \varphi^\lambda_{\Gamma',i}$ and ${}'\varphi^\lambda_{\Gamma,i} \otimes_\Gamma \Gamma' = {}'\varphi^\lambda_{\Gamma',i}$ (see 1.2(1)).

If $\Gamma = \tilde{k}$, then $\varphi^\lambda_{\Gamma,i}$ and ${}'\varphi^\lambda_{\Gamma,i}$ are isomorphisms for all i. If in (2) we take $\Gamma = \hat{k}$ and $\Gamma' = \tilde{k}$, we deduce that (writing $\hat{\varphi}^\lambda_i$ short for $\varphi^\lambda_{\hat{k},i}$)

(3) $\hat{\varphi}^\lambda_i$ is injective for $i = 1, \cdots, r$.

1.9.

Lemma . *Up to units in $\hat{k}$ we have*

(i) ${}'\hat{\varphi}^\lambda_i \circ \hat{\varphi}^\lambda_i = \begin{cases} t \ \mathrm{id}_{\hat{Z}^{y_i}(\lambda\langle y_i\rangle)} & \text{if } \langle \lambda + \rho, y_i\alpha_i^\vee\rangle \not\equiv 0 \pmod p \\ \mathrm{id}_{\hat{Z}^{y_i}(\lambda\langle y_i\rangle)} & \text{otherwise.} \end{cases}$

(ii) $\hat{\varphi}^\lambda_i \circ {}'\hat{\varphi}^\lambda_i = \begin{cases} t \ \mathrm{id}_{\hat{Z}^{y_{i+1}}(\lambda\langle y_{i+1}\rangle)} & \text{if } \langle \lambda + \rho, y_i\alpha_i^\vee\rangle \not\equiv 0 \pmod p \\ \mathrm{id}_{\hat{Z}^{y_{i+1}}(\lambda\langle y_{i+1}\rangle)} & \text{otherwise.} \end{cases}$

(iii) *If $\langle \lambda + \rho, y_i\alpha_i^\vee\rangle \equiv 0 \pmod p$ then both $\hat{\varphi}^\lambda_i$ and ${}'\hat{\varphi}^\lambda_i$ are isomorphisms.*

Proof: By 1.7(1) it is enough to evaluate ${}'\hat{\varphi}^\lambda_i \circ \hat{\varphi}^\lambda_i$ on a single non-zero element. We choose $v_0 \in \hat{Z}^{y_i}(\lambda\langle y_i\rangle)_{\lambda\langle y_i\rangle} \setminus \{0\}$ and find by [AJS, 5.6]

(1)
$$
{}'\hat{\varphi}^\lambda_i \circ \hat{\varphi}^\lambda_i(v_0) = \binom{\langle \rho, y_i\alpha_i^\vee\rangle t + \langle \lambda\langle y_i\rangle, y_i\alpha_i^\vee\rangle}{p-1} v_0
$$

up to a unit in $\hat{k}$. Hence (i) follows.

The proof of (ii) is similar and (iii) is an immediate consequence of (i) and (ii).

1.10. We still keep the notation from 1.8. Set

$$
\hat{\varphi}^\lambda_{x,y} = \hat{\varphi}^\lambda_r \circ \cdots \circ \hat{\varphi}^\lambda_1 \text{ and } {}'\hat{\varphi}^\lambda_{x,y} = {}'\hat{\varphi}^\lambda_1 \circ \cdots \circ {}'\hat{\varphi}^\lambda_r.
$$

Then $\hat{\varphi}^\lambda_{x,y} \in \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}^x(\lambda\langle x\rangle), \hat{Z}^y(\lambda\langle y\rangle))$ and ${}'\hat{\varphi}^\lambda_{x,y} \in \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}^y(\lambda\langle y\rangle), \hat{Z}^x(\lambda\langle x\rangle))$. This makes sense for any $x, y \in W$. Note that since $y_i^{-1}y_{i+1} = s_i$, we have $\hat{\varphi}^\lambda_{y_i,y_{i+1}} = \hat{\varphi}^\lambda_i$ (up to a unit in $\hat{k}$). Likewise $\hat{\varphi}^\lambda_{y_{i+1},y_i} = {}'\hat{\varphi}^\lambda_i$.

Lemma . *Assume $\langle \lambda+\rho, y_i\alpha_i^\vee\rangle \not\equiv 0 \pmod p$. If m_j denotes a generator of $\hat{Z}^{y_j}(\lambda\langle y_j\rangle)_\lambda$, $j = i, i+1$, then we have up to units in $\hat{k}$*

$$
\hat{\varphi}^\lambda_{y_i,y_{i+1}} m_i = \hat{\varphi}^\lambda_i m_i = \begin{cases} m_{i+1} & \text{if } y_i\alpha_i > 0 \\ t m_{i+1} & \text{otherwise.} \end{cases}
$$

In particular, $\hat{\varphi}^\lambda_i$ (resp. ${}'\hat{\varphi}^\lambda_i$) is an isomorphism on the λ-weight space iff $y_i\alpha_i \in R^+$ (resp. $-R^+$).

Proof: If $y_i\alpha_i > 0$ the statements hold by 1.8(1). So assume $y_i\alpha_i < 0$. Then $y_{i+1}\alpha_i > 0$ and $\hat{\varphi}^\lambda_{y_{i+1},y_i}$ is an isomorphism on the λ-weight space. Therefore we have up to units in $\hat{k}$

$$'\hat{\varphi}^\lambda_i m_{i+1} = \hat{\varphi}^\lambda_{y_{i+1},y_i} m_{i+1} = m_i.$$

Hence $\hat{\varphi}^\lambda_i m_i = t m_{i+1}$ (because of Lemma 1.9).

1.11. Let m_x (resp. m_y) denote a generator of $\hat{Z}^x(\lambda\langle x\rangle)_\lambda$ (resp. $\hat{Z}^y(\lambda\langle y\rangle)_\lambda$). Set

$$N(x,y,\lambda) = \#\{\alpha \in R^+ \mid x^{-1}\alpha < 0, y^{-1}\alpha > 0 \text{ and } \langle \lambda+\rho, \alpha\rangle \not\equiv 0 \ (\operatorname{mod} p)\}.$$

Note that $N(1,y,\lambda) = 0 = N(y,w_0,\lambda)$ for all $y \in W$. If λ is p-regular (i.e. $\langle \lambda + \rho, \alpha^\vee\rangle \not\equiv 0 \ (\operatorname{mod} p)$ for all $\alpha \in R$) then $N(x,1,\lambda) = l(x) = N(w_0, w_0x, \lambda)$ for all $x \in W$. Here l denotes the length function on W.

Proposition . *The homomorphism $\hat{\varphi}^\lambda_{x,y}$ is (up to a unit in $\hat{k}$) independent of the choice of reduced expression for $x^{-1}y$. On the λ-weight space we have (up to a unit in $\hat{k}$)*

$$\hat{\varphi}^\lambda_{x,y}(m_x) = t^{N(x,y,\lambda)} m_y.$$

Proof: By 1.7(1) it is enough to prove the second statement. By Lemma 1.10 we get $\varphi^\lambda_{x,y} m_x = t^n m_y$ where $n = \#\{i \mid y_i\alpha_i < 0 \text{ and } \langle\lambda+\rho, y_i\alpha_i\rangle \not\equiv 0 \ (\operatorname{mod} p)\}$. But

$$\{\alpha \in R^+ \mid y^{-1}x\alpha < 0\} = \{s_1\cdots s_{i-1}(\alpha_i) \mid i = 1,\cdots,r\}$$

and $xs_1\cdots s_{i-1}(\alpha_i) = y_i\alpha_i$, i.e., $n = N(x,y,\lambda)$.

1.12. **Corollary.** *If $l(y) = l(x) + l(x^{-1}y)$ (resp. $l(y) = l(x) - l(x^{-1}y)$) then the homomorphism $\hat{\varphi}^\lambda_{x,y}$ is an isomorphism on the λ-weight (resp. $\lambda\langle w_0\rangle$-weight) space. In either case $\hat{\varphi}^\lambda_{x,y}$ therefore generates $\operatorname{Hom}_{\hat{\mathcal{C}}}(\hat{Z}^x(\lambda\langle x\rangle), \hat{Z}^y(\lambda\langle y\rangle))$. In particular, $\hat{\varphi}^\lambda_{1,w_0}$ is an isomorphism on the λ-weight space. By symmetry so is $\hat{\varphi}^\lambda_{w,ww_0}$ on the $\lambda\langle w\rangle$-weight space for all $w \in W$.*

Proof: If $l(y) = l(x) + l(x^{-1}y)$ then $N(x,y,\lambda) = 0$ and the corollary follows immediately from Proposition 1.11. At the same time we see that $\hat{\varphi}^\lambda_{1,w_0} : \hat{Z}(\lambda) \to \hat{Z}^{w_0}(\lambda\langle w_0\rangle)$ is an isomorphism on the λ-weight space (and in fact under the given condition $\hat{\varphi}^\lambda_{x,y}$ is a factor in $\hat{\varphi}^\lambda_{1,w_0}$).

By symmetry $\hat{\varphi}_{w_0,1} : \hat{Z}^{w_0}(\lambda\langle w_0\rangle) \to \hat{Z}(\lambda)$ is an isomorphism on the $\lambda\langle w_0\rangle$-weight space. If $l(y) = l(x) - l(x^{-1}y)$ then $\hat{\varphi}_{w_0,1}$ factors through $\hat{\varphi}^\lambda_{x,y}$ and the corollary follows in this case as well.

Recall from 1.7(3) that $\operatorname{Hom}_{\hat{\mathcal{C}}}(\hat{Z}^x(\lambda\langle x\rangle), \hat{Z}^y(\lambda\langle y\rangle)) \simeq \hat{k}$. Hence the second statement in the corollary is clear.

2. Projectives

In [AJS, Section, 3-4] one finds a treatment of the projective objects in $\mathcal{C}_\Gamma$, , a general noetherian commutative U^0-algebra. We shall basically only need the cases , $= k, \hat{k}$ and $\tilde{k}$ and in this section we have collected the results we need about projectives in these cases.

2.1. Let $M \in \mathcal{C}_\Gamma$ and $w \in W$. We say that M admits a Z^w-filtration if there exists a filtration
$$M = M^0 \supset M^1 \supset \cdots \supset M^r = 0$$
of M in $\mathcal{C}_\Gamma$ with $M^{i-1}/M^i \simeq Z^w_\Gamma(\lambda_i)$ for some $\lambda_i \in X,\ i = 1, \cdots, r$.

We set then for $\lambda \in X$
$$(M : Z^w_\Gamma(\lambda)) = \#\{i \mid \lambda_i = \lambda\},$$
the multiplicity of $Z^w_\Gamma(\lambda)$ in the Z^w-filtration of M. Note that these numbers are the same for any two Z^w-filtrations of M.

By [AJS, 2.14] we have

(1) $$\text{if } \operatorname{Ext}^1_{\mathcal{C}_\Gamma}(Z^w_\Gamma(\lambda), Z^w_\Gamma(\lambda')) \neq 0 \text{ then } \lambda - \lambda' \in \sum_{\alpha \in S} \mathbb{N} w\alpha$$

This implies that in a Z^w-filtration of M as above we can always arrange that

(2) $$\text{if } \lambda_j - \lambda_i \in \sum_{\alpha \in S} \mathbb{N} w\alpha, \text{ then } j > i.$$

We recall finally from [AJS, 2.16] that

(3) all projective modules in $\mathcal{C}_\Gamma$ admit a Z^w-filtration.

2.2. Consider now the case , $= \tilde{k}$. As noted in 1.5(3) the category $\tilde{\mathcal{C}}$ is semi-simple. In fact, each $M \in \tilde{\mathcal{C}}$ splits into a direct sum of $\tilde{Z}(\lambda)$'s. Hence M has certainly a Z-filtration (and a Z^w-filtration for all $w \in W$).

We may also phrase the semi-simplicity of $\tilde{\mathcal{C}}$ by saying that all objects in $\tilde{\mathcal{C}}$ are projective (as well as injective).

2.3. Next we take , $= k$. As pointed out in 1.4 we may identify $\mathcal{C}$ with the category of finite dimensional G_1T-modules. Hence by [Ja2, II, 9.3] we have for $M \in \mathcal{C}$

(1) M is projective iff M is injective.

Moreover, we have the following criteria for injectivity in $\mathcal{C}$

(2) M is injective in $\mathcal{C}$ iff M admits a Z^w-filtration for all $w \in W$.

This follows from [Ja2, 11.2 and 11.4]. For an alternative proof see 2.5 below.

Let us for each $\lambda \in X$ denote by $Q(\lambda)$ the injective hull of $L(\lambda)$. An easy application of 1.7(3) gives the reciprocity formula

(3) $(Q(\lambda) : Z^w(\nu\langle w \rangle)) = (Q(\lambda) : Z(\nu)) = [Z(\nu) : L(\lambda)]$, $\nu \in X$, $w \in W$.

Here $[M : L(\lambda)]$ denotes the multiplicity of $L(\lambda)$ as a composition factor in $M \in \mathcal{C}$.

It follows from (3) and 1.3(1) that

(4) $Q(\lambda)$ has lowest weight $\lambda - 2(p-1)\rho = \lambda\langle w_0 \rangle$ and this weight occurs with multiplicity 1.

Note also that λ is minimal among the λ_i's for which $Z(\lambda_i)$'s occurs in a Z-filtration of $Q(\lambda)$. Hence by 2.1(2) we see that $Z(\lambda)$ and therefore also $L(\lambda)$ are quotients of $Q(\lambda)$. Combining this with (1) we conclude

(5) $Q(\lambda)$ is the projective cover of $L(\lambda)$

and

(6) $Q(\lambda)^\tau \simeq Q(\lambda)$ in $\mathcal{C}$.

2.4. Finally we take $\lambda = \hat{k}$. Let again $\lambda \in X$. Then according to [AJS, 4.19] we have

(1) There is a (unique up to isomorphism) projective object $\hat{Q}(\lambda) \in \hat{\mathcal{C}}$ which satisfies $\hat{Q}(\lambda) \otimes_{\hat{k}} k \simeq Q(\lambda)$.

By 2.1(3) we see that $\hat{Q}(\lambda)$ has a Z^w-filtration for all w. Clearly,

(2) $(\hat{Q}(\lambda) : \hat{Z}^w(\nu)) = (Q(\lambda) : Z^w(\nu))$ for all $\nu \in X$, $w \in W$.

By 1.6(2) and 2.3(6) we get $\hat{Q}(\lambda)^\tau \otimes k \simeq Q(\lambda)$. If we dualize a Z^{w_0}-filtration of $\hat{Q}(\lambda)$ then we obtain a Z-filtration of $\hat{Q}(\lambda)^\tau$. Hence [AJS, 3.5] implies that $\hat{Q}(\lambda)^\tau$ is projective in $\hat{\mathcal{C}}$ and we conclude

(3) $\hat{Q}(\lambda)^\tau \simeq \hat{Q}(\lambda)$ in $\hat{\mathcal{C}}$.

Let us also record the following fact, see [AJS, 4.19].

(4) Any projective module in $\hat{\mathcal{C}}$ is a direct sum of certain $\hat{Q}(\lambda)$.

2.5. **Proposition.** *Let $\hat{Q} \in \hat{\mathcal{C}}$. Then $\hat{Q}$ is projective iff $\hat{Q}$ has a Z^w-filtration for all $w \in W$.*

Proof: The only if part follows from 2.1(3). Moreover, by [AJS, 3.5] we see that it is enough to prove

(1) if $Q \in \mathcal{C}$ has a Z- and a Z^{w_0}-filtration, then Q is projective.

Now this follows from 2.3(2). Here is an alternative argument: Choose $\lambda \in X$ minimal with $(Q : Z(\lambda)) \neq 0$. Then we have two surjections π and π_λ (see 2.1(2))

$$\begin{tikzcd}
 & 0 & \\
Q \arrow[r, "\pi"] \arrow[dr, dashed, shift left] & Z(\lambda) \arrow[u] \arrow[r] & 0 \\
 & Q(\lambda) \arrow[u, "\pi_\lambda"'] \arrow[ul, dashed, shift left] &
\end{tikzcd}$$

We claim that the dotted maps in this diagram exist making the triangle commutative. Since $\ker \pi_\lambda$ has a Z-filtration (by the construction of π) and since Q has a Z^{w_0}-filtration it follows from 1.7(3) that $\mathrm{Ext}^1_{\mathcal{C}}(Q, \ker \pi_\lambda) = 0$. This produces the desired homomorphism $Q \to Q(\lambda)$. The other map is obtained by a similar argument (or simply by using that $Q(\lambda)$ is projective).

Since $Q(\lambda)$ is indecomposable, it follows from this diagram that it is a direct summand of Q. The existence of Z- as well as Z^{w_0}-filtrations is inherited by summands ([AJS, 2.16]) and hence we may continue this until we have a decomposition of Q into a direct sum of certain $Q(\nu), \nu \in X$. So (1) is proved.

Remark . *By analogy with the representation theory for G it is natural to define a tilting module in $\mathcal{C}_\Gamma$ to be a module which admits both a Z-filtration and a Z^{w_0}-filtration. Using this terminology the above results can be stated as follows*

(2) *In $\mathcal{C}$ and $\hat{\mathcal{C}}$ a module is tilting iff it is projective.*

2.6. If $\nu \in X$ is a special point, i.e., $\nu + \rho \in pX$, then we have for all $w \in W$ the following isomorphism in $\mathcal{C}$

(1) $$L(\nu) \simeq Z(\nu) \simeq Z^w(\nu\langle w \rangle) \simeq Q(\nu).$$

In $\hat{\mathcal{C}}$ we have

(2) $$\hat{Z}(\nu) \simeq \hat{Z}^w(\nu\langle w \rangle) \simeq \hat{Q}(\nu).$$

In particular, if $\nu = (p-1)\rho$ then $L(\nu)$ is also a simple module for G. This is called the Steinberg module and denoted by St. Note that any other special point may be written as $(p-1)\rho + p\nu_1$ for some $\nu_1 \in X$. Then we have in $\mathcal{C}$

(3) $$L((p-1)\rho + p\nu_1) \simeq \mathrm{St} \otimes p\nu_1 \simeq Q((p-1)\rho + p\nu_1).$$

3. Filtrations

Let $\lambda \in X$ and write $Z^\tau(\lambda)$ short for $Z^{w_0}(\lambda\langle w_0\rangle)$. In this section we introduce (following [An3]) a filtration of the vector spaces $\mathrm{Hom}_{\mathcal{C}}(Z^\tau(\lambda), Q), Q$ a projective object in $\mathcal{C}$. We prove a sum formula for this filtration and relate it to the Jantzen filtration of $Z(\lambda)$, see [AJS, 6.6].

3.1. In the following we shall work with a fixed generator $c = c_\lambda \in \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}^\tau(\lambda), Z(\lambda))$ (recall from 1.7 that this module is free of rank 1 over $\hat{k}$.

For any projective module $Q \in \mathcal{C}$ there is a unique "lift" to $\hat{\mathcal{C}}$, i.e. there exists a unique (up to isomorphism) projective module $\hat{Q} \in \hat{\mathcal{C}}$ with $\hat{Q} \otimes_{\hat{k}} k \simeq Q$, see 2.4 (1) and (4). We set

(1) $$\hat{F}_\lambda(Q) = \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}^\tau(\lambda), \hat{Q}) \text{ and } \hat{E}_\lambda(Q) = \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Q}, \hat{Z}(\lambda)).$$

Then it follows from 1.7 (2), (3) and 2.1 (3) that

(2) $$\hat{F}_\lambda(Q) \text{ and } \hat{E}_\lambda(Q) \text{ are both free over } \hat{k} \text{ of rank } (Q : Z(\lambda)).$$

Moreover, if we set $F_\lambda(Q) = \mathrm{Hom}_{\mathcal{C}}(Z^\tau(\lambda), Q)$ and $E_\lambda(Q) = \mathrm{Hom}_{\mathcal{C}}(Q, Z(\lambda))$ then

(3) $$F_\lambda(Q) \simeq \hat{F}_\lambda(Q) \otimes_{\hat{k}} k \text{ and } E_\lambda(Q) \simeq \hat{E}_\lambda(Q) \otimes_{\hat{k}} k.$$

Define now a filtration $(\hat{F}_\lambda(Q)^j)_{j\geq 0}$ of $\hat{F}_\lambda(Q)$ by setting

(4) $$\hat{F}_\lambda(Q)^j = \{\varphi \in \hat{F}_\lambda(Q) \mid \psi \circ \varphi \in \hat{k}t^j c \text{ for all } \psi \in \hat{E}_\lambda(Q)\}$$

and let $F_\lambda(Q)^j$ denote the image of $\hat{F}_\lambda(Q)^j$ in $F_\lambda(Q)$. Then

(5) $$F_\lambda(Q)^j \simeq (\hat{F}_\lambda(Q)^j + t\hat{F}_\lambda(Q))/t\hat{F}_\lambda(Q) \simeq \hat{F}_\lambda(Q)^j / t\hat{F}_\lambda(Q)^{j-1}.$$

Finally, we let $F_\lambda(Q)_j$ denote the j-th quotient in the filtration $F_\lambda(Q) = F_\lambda(Q)^0 \supset F_\lambda(Q)^1 \supset \cdots \supset F_\lambda(Q)^r = 0$, i.e.,

(6)
$$F_\lambda(Q)_j = F_\lambda(Q)^j / F_\lambda(Q)^{j+1} \simeq \hat{F}_\lambda(Q)^j / (\hat{F}_\lambda(Q)^{j+1} + t\hat{F}_\lambda(Q)^{j-1}), \quad j = 0, \cdots, r-1.$$

Remark . *If in the above set-up we replace $Z^\tau(\lambda)$ by $Z^{ww_0}(\lambda\langle ww_0\rangle) \simeq Z^w(\lambda\langle w\rangle)^\tau$ and $Z(\lambda)$ by $Z^w(\lambda\langle w\rangle)$ for some $w \in W$ then we obtain analogous filtrations of $F^w_\lambda(Q) = \mathrm{Hom}_{\mathcal{C}}(Z^{ww_0}(\lambda\langle ww_0\rangle), Q)$. As will be evident all the following results could be stated (and proved in the same way) for these filtrations.*

3.2. With notation as in 3.1 consider the pairing

$$a_\lambda : \hat{F}_\lambda(Q) \times \hat{E}_\lambda(Q) \to \hat{k}$$

given by $\psi \circ \varphi = a_\lambda(\varphi, \psi)c, \quad \psi \in \hat{E}_\lambda(Q), \varphi \in \hat{F}_\lambda(Q)$. When tensored with $\tilde{k}$ this pairing becomes non-degenerate, i.e. the associated $\hat{k}$-homomorphism

$$\theta_\lambda : \hat{F}_\lambda(Q) \to \hat{E}_\lambda(Q)^\vee = \mathrm{Hom}_{\hat{k}}(\hat{E}_\lambda(Q), \hat{k})$$

defined by

$$\theta_\lambda(\varphi) : \psi \mapsto a_\lambda(\varphi, \psi)$$

becomes an isomorphism when tensored by $\tilde{k}$. Standard arguments (e.g. [Ja2, II.8.18]) then give (with $\nu_t : \hat{k} \to \mathbb{Z}$ denoting the t-adic valuation).

Lemma . *There exist bases $\{f_1, \cdots, f_{n_\lambda}\}$ of $\hat{F}_\lambda(Q)$ and $\{e_1, \cdots, e_{n_\lambda}\}$ of $\hat{E}_\lambda(Q)^\vee$ and integers $m_\lambda(1), \cdots, m_\lambda(n_\lambda) \in \mathbb{N}$ such that*

$$\theta_\lambda(f_i) = t^{m_\lambda(i)} e_i, \quad i = 1, \cdots, n_\lambda.$$

Moreover,

$$\sum_{j \geq 1} \dim F_\lambda(Q)^j = \nu_t(\det \theta_\lambda) = \sum_{i=1}^{n_\lambda} m_\lambda(i).$$

3.3. According to 2.1 there is a filtration of $\hat{Q}$

$$\hat{Q} = \hat{Q}^0 \supset \hat{Q}^1 \supset \cdots \supset \hat{Q}^r = 0$$

in $\hat{\mathcal{C}}$ with

$$\hat{Q}^{i-1}/\hat{Q}^i \simeq \hat{Z}^\tau(\lambda_i)^{n_i}, \quad n_i = (Q : Z(\lambda_i)) \tag{1}$$

and

$$\text{if } \lambda_i < \lambda_j, \text{ then } i > j. \tag{2}$$

This gives us for each $i = 1, \cdots, r$ a short exact sequence

$$0 \to \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}^\tau(\lambda_i), \hat{Q}^{i-1}) \to \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}^\tau(\lambda_i), \hat{Z}^\tau(\lambda_i)^{n_i}) \to \mathrm{Ext}^1_{\hat{\mathcal{C}}}(\hat{Z}^\tau(\lambda_i), \hat{Q}^i) \to 0. \tag{3}$$

Moreover, the inclusion $\hat{Q}^{i-1} \hookrightarrow \hat{Q}$ induces an isomorphism $\mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}^\tau(\lambda_i), \hat{Q}^{i-1}) \simeq \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}^\tau(\lambda_i), \hat{Q}) = \hat{F}_{\lambda_i}(Q)$ because $\hat{Q}/\hat{Q}^{i-1}$ is filtered by $\hat{Z}^\tau(\lambda_j)$, $j = i+1, \cdots, r$.

We let

$$\phi_i : \hat{F}_{\lambda_i}(\hat{Q}) \to \mathrm{End}_{\hat{\mathcal{C}}}(\hat{Z}^\tau(\lambda_i))^{n_i} \simeq \hat{k}^{n_i}$$

denote the $\hat{k}$-homomorphism resulting from this identification of the first term in (3). Then (3) says that $\mathrm{coker}\, \phi_i = \mathrm{Ext}^1_{\hat{\mathcal{C}}}(\hat{Z}^\tau(\lambda_i), \hat{Q}^i)$. As in [An4, 1.6-7] we deduce (denoting by l_t the length function on $\hat{k}$-modules of finite type).

Lemma .

$$\hat{F}_{\lambda_i}(Q)^j = \{\varphi \in \hat{F}_{\lambda_i}(Q) \mid \phi_i(\varphi) \in t^j \hat{k}^{n_i}\}$$

and

$$\sum_{j \geq 1} \dim F_{\lambda_i}(Q)^j = l_t(\mathrm{Ext}^1_{\hat{\mathcal{C}}}(\hat{Z}^\tau(\lambda_i), \hat{Q}^i)).$$

3.4. Let us recall how the Jantzen filtration from [AJS, 6.6] is constructed.

Choose a generator $c' = c'_\lambda \in \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}(\lambda), \hat{Z}^\tau(\lambda))$. Then Lemma 1.9 implies

$$c \circ c' = t^{N(\lambda)} \mathrm{id}_{\hat{Z}(\lambda)} \text{ and } c' \circ c = t^{N(\lambda)} \mathrm{id}_{\hat{Z}^\tau(\lambda)} \tag{1}$$

where $N(\lambda) = \#\{\alpha \in R^+ \mid \langle \lambda + \rho, \alpha^v ee \rangle \not\equiv 0 (\mathrm{mod}\, p)\}$.

The Jantzen filtration on $\hat{Z}^\tau(\lambda)$ (resp. $\hat{Z}(\lambda)$) is given by

$$\hat{Z}^\tau(\lambda)^j = \{v \in \hat{Z}^\tau(\lambda) \mid cv \in t^j \hat{Z}(\lambda)\}$$

(resp.

$$\hat{Z}(\lambda)^j = \{v' \in \hat{Z}(\lambda) \mid c'v' \in t^j \hat{Z}^\tau(\lambda)\}).$$

Exactly as in 3.2 one sees that there exist bases $\{v_1, \cdots, v_n\}$ of $\hat{Z}^\tau(\lambda)$ and $\{v_1', \cdots, v_n'\}$ of $\hat{Z}(\lambda)$ and integers $a_1, \cdots, a_n \in \mathbb{N}$ such that

$$cv_i = t^{a_i} v_i' \text{ and } c'v_i' = t^{N(\lambda)-a_i} v_i, \tag{2}$$

$i = 1, \cdots, n$. If we denote by $Z^\tau(\lambda)^j$ (resp. $Z(\lambda)^j$) the image of $\hat{Z}^\tau(\lambda)^j$ in $Z^\tau(\lambda)$ (resp. of $\hat{Z}(\lambda)^j$ in $Z(\lambda)$), then we get (setting $\bar{v}_i = v_i \otimes 1$ and $\bar{v}_i' = v_i' \otimes 1$)

$$Z^\tau(\lambda)^j = \sum_{\substack{i \\ a_i \geq j}} k\bar{v}_i \text{ and } Z(\lambda)^j = \sum_{\substack{i \\ N(\lambda)-a_i \geq j}} k\bar{v}_i'. \tag{3}$$

Observing that $\bar{v}_i$ and $\bar{v}_i'$ have the same weight we deduce from (3)

$$\operatorname{ch} Z^\tau(\lambda)^j + \operatorname{ch} Z(\lambda)^{N(\lambda)-j+1} = \operatorname{ch} Z(\lambda) \tag{4}$$

3.5. Recall the notation $\hat{F}_\lambda^{w_0}(Q)$ from Remark 3.1. With $N(\lambda)$ as in 3.4 (1) we have

Lemma . *There exist $\hat{k}$-bases $\{\psi_1, \cdots, \psi_r\}$ of $\hat{E}_\lambda(Q)$ and $\{\psi_1', \cdots, \psi_r'\}$ of $\hat{F}_\lambda^{w_0}(Q)$ such that (up to units in $\hat{k}$)*

$$\psi_i \circ \psi_j' = \delta_{ij} t^{N(\lambda)} \mathrm{id}_{\hat{Z}(\lambda)},$$

$i, j = 1, \cdots, r$.

Proof: With notation as in 3.3 we see from 1.7(3) that if $\lambda = \lambda_i$, then we have isomorphisms

$$\hat{F}_\lambda^{w_0}(Q) \xleftarrow{\sim} \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}(\lambda), \hat{Q}^{i-1}) \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}(\lambda), \hat{Z}^\tau(\lambda_i)^{n_i})$$

induced by the inclusion $\hat{Q}^{i-1} \hookrightarrow \hat{Q}$ and the projection $\hat{Q}^{i-1} \to \hat{Q}^{i-1}/\hat{Q}^i \simeq \hat{Z}^\tau(\lambda_i)^{n_i}$, respectively. Let $\psi_j' \in \hat{F}_\lambda^{w_0}(Q)$ be the element which under these isomorphisms corresponds to the composite of c' with the j-th inclusion $i_j : \hat{Z}^\tau(\lambda) \to \hat{Z}^\tau(\lambda_i)^{n_i}$.

We have similar isomorphisms

$$\hat{E}_\lambda(Q) \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Q}^{i-1}, \hat{Z}(\lambda)) \xleftarrow{\sim} \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}^\tau(\lambda_i)^{n_i}, \hat{Z}(\lambda))$$

and we let $\psi_j \in \hat{E}_\lambda(Q)$ denote the element corresponding to the composite of the j-th projection $\pi_j : \hat{Z}^\tau(\lambda_i)^{n_i} \to \hat{Z}(\lambda)$ with c.

Then we get

$$\psi_s \circ \psi_t' = (c \circ \pi_s) \circ (i_t \circ c') = \delta_{st} c \circ c',$$

$s, t = 1, \cdots, n_i$. The lemma therefore follows from 3.4(1).

3.6. We can now deduce yet another characterization of our filtration

Proposition . *For each $j \in \mathbb{N}$ we have*

$$\hat{F}_\lambda(Q)^j = \{\varphi \in \hat{F}_\lambda(Q) \mid \varphi \circ c' \in t^j \hat{F}_\lambda^{w_0}(Q)\}.$$

Proof: Let $\varphi \in \hat{F}_\lambda(Q)$ and $\psi \in \hat{E}_\lambda(Q)$. In the notation of 3.2 we have $\psi \circ \varphi = a_\lambda(\varphi, \psi)c$ and hence 3.4(1) implies

(1) $$\psi \circ \varphi \circ \ c' = a_\lambda(\varphi, \psi) t^{N(\lambda)} \mathrm{id}_{\hat{Z}(\lambda)}.$$

Using the basis from Lemma 3.5 we may write $\varphi \circ c' = \sum_{s=1}^r b_s \psi'_s$ for some $b_s \in \hat{k}$. Then Lemma 3.5 and (1) give us $b_s = a_\lambda(\varphi, \psi_s)$ for all s. Hence $\varphi \circ c' \in t^j \hat{F}_\lambda^{w_0}(Q)$ iff $t^j \mid a_\lambda(\varphi, \psi_s)$ for all s, i.e. iff $\varphi \in \hat{F}_\lambda(Q)^j$.

3.7. **Proposition.** *Let $\nu \in X$. Then*

$$\sum_{j \geq 1} \dim F_\lambda(Q(\nu))^j = \sum_{j \geq 1} [Z(\lambda)^j : L(\nu)].$$

Proof: Let $w_0 = s_1 \cdots s_N$ be a reduced expression of w_0. In this proof we then use the abbreviations

$$\hat{Z}^i = \hat{Z}^{s_1 \cdots s_i}(\lambda\langle s_1 \cdots s_i\rangle) \text{ and } Z^i = Z^{s_1 \cdots s_i}(\lambda\langle s_1 \cdots s_i\rangle).$$

By Corollary 1.12 we may take the generator $c' \in \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}^0, \hat{Z}^N)$ as the composite

$$\hat{Z}^0 \xrightarrow{\hat{\varphi}_1} \hat{Z}^1 \xrightarrow{\hat{\varphi}_2} \cdots \xrightarrow{\hat{\varphi}_N} \hat{Z}^N$$

where $\hat{\varphi}_i = \hat{\varphi}^\lambda_{s_1 \cdots s_{i-1}, s_1 \cdots s_i}$.

Set $Q = Q(\nu)$ and let $\hat{Q} = \hat{Q}(\nu) \in \hat{\mathcal{C}}$ (see 2.4(1)). Denote by $\hat{\phi}_i : \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}^i, \hat{Q}) \to \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}^{i-1}, \hat{Q})$ the map induced by $\hat{\varphi}_i$ and set $\hat{\phi} = \hat{\phi}_1 \circ \cdots \circ \hat{\phi}_N$. Then Proposition 3.6 says

$$\hat{F}_\lambda(Q)^j = \{\varphi \in \hat{F}_\lambda(Q) \mid \hat{\phi}(\varphi) \in t^j \hat{F}_\lambda^{w_0}(Q)\}$$

Hence the standard arguments (compare 3.2) give

(1) $$\sum_{j \geq 1} \dim F_\lambda(Q)^j = \nu_t(\det \hat{\phi}) = \sum_{i=1}^N \nu_t(\det \hat{\phi}_i)$$

Set $\hat{C} = \operatorname{coker} \hat{\varphi}_i$. Then by Lemma 1.9 we have

(2) $$t\hat{C}_i = 0 \text{ for all } i.$$

Hence also

(3) $$t \ \mathrm{Ext}^1_{\hat{\mathcal{C}}}(\hat{C}_i, \hat{Q}) = 0 \text{ for all } i.$$

Note that by 1.7(3) and Proposition 2.5 we have $\mathrm{Ext}^1_{\hat{\mathcal{C}}}(\hat{Z}^i, \hat{Q}) = 0$. Therefore we have the exact sequence

$$(4) \qquad 0 \to \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}^i, \hat{Q}) \xrightarrow{\hat{\phi}_i} \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}^{i-1}, \hat{Q}) \to \mathrm{Ext}^1_{\hat{\mathcal{C}}}(\hat{C}_i, \hat{Q}) \to 0$$

and so (3) implies

$$(5) \qquad \nu_t(\det \hat{\phi}_i) = l_t(\mathrm{coker}\, \hat{\phi}_i) = \dim_k(\mathrm{Ext}^1_{\hat{\mathcal{C}}}(\hat{C}_i, \hat{Q}) \otimes_{\hat{k}} k).$$

Set $\varphi_i = \hat{\varphi}_i \otimes_{\hat{k}} k$ and $C_i = \hat{C}_i \otimes_{\hat{k}} k$. Then $C_i = \mathrm{coker}\, \varphi_i$ and the injectivity of Q in $\mathcal{C}$ gives the exactness of the top row in the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(C_i,Q) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(Z^i,Q) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(Z^{i-1},Q) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(\ker \varphi_i,Q) \arrow[r] & 0 \\
& & \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}^i,\hat{Q})\otimes_{\hat{k}} k \arrow[u, "\sim"] \arrow[r] & \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}^{i-1},\hat{Q})\otimes_{\hat{k}} k \arrow[u, "\sim"] \arrow[r] & \mathrm{Ext}^1_{\hat{\mathcal{C}}}(\hat{C}_i,Q)\otimes_{\hat{k}} k \arrow[r] & 0
\end{tikzcd}$$

Here the bottom sequence is obtained by tensoring (4) with k and the vertical isomorphisms result from 1.7(3) and 2.5. This diagram shows that $\mathrm{Ext}^1_{\hat{\mathcal{C}}}(\hat{C}_i, Q) \otimes_{\hat{k}} k$ may be identified with $\mathrm{Hom}_{\mathcal{C}}(\ker \varphi_k, Q)$ and that this has the same dimension as $\mathrm{Hom}_{\mathcal{C}}(C_i, Q)$. But this dimension equals $[C_i : L(\nu)]$.

The standard arguments applied to the Jantzen filtration of $Z^0 = Z(\lambda)$ (see 3.4) give

$$(6) \qquad \sum_{j \geq 1} \mathrm{ch}\ Z(\lambda)^j = \sum_{i=1}^{N} \mathrm{ch}\ C_i.$$

Hence combining (1), (5), (6) and the above we deduce

$$\sum_{j \geq 1} \dim F_\lambda(Q)^j = \sum_{i=1}^{N} [C_i : L(\nu)] = \sum_{j \geq 1} [Z(\lambda)^j : L(\nu)]$$

and the proposition is proved.

3.8. **Theorem.** *If $Q \in \mathcal{C}$ is projective, then*

$$\dim F_\lambda(Q)^j = \dim \mathrm{Hom}_{\mathcal{C}}(Z(\lambda)^j, Q)$$

for all $j \in \mathbb{N}$.

Proof: We shall first prove

$$(1) \qquad \text{if } \varphi \in F_\lambda(Q)^j, \text{ then } \varphi(Z^\tau(\lambda)^{N(\lambda)+1-j}) = 0.$$

To see this, note that by definition $\varphi \in F_\lambda(Q)^j$ is the image of some $\hat{\varphi} \in \hat{F}_\lambda(Q)^j$. According to Proposition 3.6 we have $\hat{\varphi} \circ c' \in t^j \hat{F}_\lambda^{w_0}(Q)$. On the other hand, we have from 3.4 (2) that $\hat{\varphi} \circ c'(v_i') = t^{N(\lambda)-a_i} \hat{\varphi}(v_i)$, $i = 1, \cdots, n$. We conclude that

$$(2) \qquad \text{if } N(\lambda) - a_i \leq j, \text{ then } \hat{\varphi}(v_i) \in t^{j-N(\lambda)+a_i} \hat{Q}.$$

However, by 3.4(3) we see that $Z^\tau(\lambda)^{N(\lambda)+1-j}$ is spanned by those $\bar{v}_i$ where $a_i \geq N(\lambda)+1-j$. Hence by (2) we see that if $\bar{v}_i \in Z^\tau(\lambda)^{N(\lambda)+1-j}$, then $\hat{\varphi}(v_i) \in t\hat{Q}$, i.e., $\varphi(\bar{v}_i) = 0$ and (1) is proved.

Now (1) says that $F_\lambda(Q)^j \subseteq \mathrm{Hom}_{\mathcal{C}}(Z^\tau(\lambda)/Z^\tau(\lambda)^{N(\lambda)+1-j}, Q)$. Since Q is injective, the dimension of this Hom-space only depends on the character of $Z^\tau(\lambda)/Z^\tau(\lambda)^{N(\lambda)+1-j}$. Therefore by 3.4(4) we deduce

$$\dim F_\lambda(Q)^j \leq \dim \mathrm{Hom}_{\mathcal{C}}(Z(\lambda)^j, Q).$$

This being true for all j we obtain equality by Proposition 3.7 (the case $j = 0$ is trivial).

3.9. If in Theorem 3.8 we take $Q = Q(\nu)$ for some $\nu \in X$ then the result says

$$\dim F_\lambda(Q(\nu))^j = [Z(\lambda)^j : L(\nu)] \tag{1}$$

for all j.

The length of the Jantzen filtration of $Z(\lambda)$ is $N(\lambda)$. Hence it follows from Theorem 3.8 that

$$F_\lambda(Q)^{N(\lambda)+1} = 0 \tag{2}$$

for all projective modules $Q \in \mathcal{C}$. (This could also be seen directly from Proposition 3.6).

3.10. We can use Theorem 3.8 to translate the results of [AJS, Proposition 6.6] into statements about the filtration $(F_\lambda(Q)^j)$. To formulate these we need the following notation. If $Q \in \mathcal{C}$ is projective we denote for $\nu \in X$ by $(Q : Q(\nu))$ the number of times $Q(\nu)$ occurs as summand in Q. We set $R^+(\lambda) = \{\alpha \in R^+ \mid \langle \lambda+\rho, \alpha^\vee \rangle \not\equiv 0 (\mathrm{mod}\, p)\}$ and for $\alpha \in R^+(\lambda)$ we let $n_\alpha \in \{1, \cdots, p-1\}$ denote the residue of $\langle \lambda+\rho, \alpha^v ee \rangle \mod p$.

Corollary . *For each projective module $Q \in \mathcal{C}$ we have*

i) $\dim F_\lambda(Q)_0 = (Q : Q(\lambda))$.
ii) $\dim F_\lambda(Q)_{N(\lambda)} = (Q : Q(\lambda'))$ *where* $L(\lambda') = \mathrm{soc}\ Z(\lambda)$.
iii) $\sum_{j\geq 1} \dim F_\lambda(Q)^j =$
$\sum_{\alpha \in R^+(\lambda)} (\sum_{i \geq 0} (Q : Z(\lambda - (ip+n_\alpha)\alpha)) - \sum_{i\geq 1}(Q : Z(\lambda - ip\alpha)))$.

ith

4. Translations

In this section we shall prove some results on translation which we need in order to compare our filtrations for two adjacent weights. For the definition and basic properties of translation functors we refer to [AJS, 6-7]. Starting in 4.2 we assume $p \geq h$ (in order to ensure the existence of p-regular weights). This means that the U^0-algebra structure on $\hat{k}$ (and on k) may be given by mapping all H_α to $0, \alpha \in R^+$.

4.1. Let B denote the U^0-algebra from [AJS, 5.3] (this should not be confused with the Borel subgroup of G also denoted B in the introduction). Consider a B-algebra , . When $\lambda \in X$ we let $\mathcal{C}_\Gamma(\lambda)$ denote the block in $\mathcal{C}_\Gamma$ corresponding to the orbit $W_p.\lambda$.

Suppose $\lambda, \mu \in X$ are in the closure of the same alcove. Then we have translation functors (see [AJS, 6])

$$T_\lambda^\mu : \mathcal{C}_\Gamma(\lambda) \to \mathcal{C}_\Gamma(\mu) \text{ and } T_\mu^\lambda : \mathcal{C}_\Gamma(\mu) \to \mathcal{C}_\Gamma(\lambda).$$

These two functors are adjoint. If $M \in \mathcal{C}_\Gamma(\lambda)$ and $N \in \mathcal{C}_\Gamma(\mu)$, then we denote the corresponding functorial isomorphisms

$$\mathrm{adj}_1 : \mathrm{Hom}_{\mathcal{C}_\Gamma}(M, T_\mu^\lambda N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}_\Gamma}(T_\lambda^\mu M, N)$$

and

$$\mathrm{adj}_2 : \mathrm{Hom}_{\mathcal{C}_\Gamma}(N, T_\lambda^\mu M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}_\Gamma}(T_\mu^\lambda N, M).$$

We shall rely heavily on the following two facts, see [AJS, 7.5].

(1) Translation functors commute with base change, e.g. $(T_\lambda^\mu M) \otimes_\Gamma , ' \simeq T_\lambda^\mu (M \otimes_\Gamma , ')$, $M \in \mathcal{C}_\Gamma(\lambda)$, , $\to$, $'$ any , -algebra.
(2) The module $T_\lambda^\mu Z_\Gamma(\lambda)$ admits a Z_Γ-filtration with quotients $Z_\Gamma(x.\mu)$, $x \in \mathrm{Stab}_{W_p}(\lambda)/\mathrm{Stab}_{W_p}(\lambda) \cap \mathrm{Stab}_{W_p}(\mu)$, each occurring with multiplicity 1.

Here $\mathrm{Stab}_{W_p}(\nu) = \{x \in W_p \mid x.\nu = \nu\}$, $\nu \in X$.

4.2. Assume for the rest of this section that λ is p-regular, i.e. $\mathrm{Stab}_{W_p}(\lambda) = 1$ and that μ is semi-p-regular, i.e. $|\mathrm{Stab}_{W_p}(\mu)| = 2$. Denote by $s \in W_p$ the reflection with $s.\mu = \mu$. There exist $\beta \in R^+$ and $n_\beta \in \mathbb{Z}$ such that $s = s_{\beta, n_\beta}$, i.e.,

$$s.\nu = s_\beta.\nu + n_\beta p \beta, \quad \nu \in X.$$

We assume that $\lambda' = s.\lambda < \lambda$. This means $n_\beta p < \langle \lambda + \rho, \beta^\vee \rangle < (n_\beta + 1)p$.

With these assumptions we get from 4.1 (2)

(1) $T_\lambda^\mu Z_\Gamma(\lambda) \simeq Z_\Gamma(\mu) \simeq T_\lambda^\mu Z_\Gamma(\lambda')$.

(2) There is an exact sequence in $\mathcal{C}_\Gamma(\lambda)$

$$0 \to Z_\Gamma(\lambda) \to T_\mu^\lambda Z_\Gamma(\mu) \to Z_\Gamma(\lambda') \to 0$$

4.3. Let $\pi : B \to$, denote the structure homomorphism. Then we have (with the conventions as in 4.2 and with H_β as in [AJS])

Proposition . i) $\mathrm{Ext}^j_{\mathcal{C}_\Gamma}(Z_\Gamma(\lambda'), Z_\Gamma(\lambda)) = 0$ *for* $j > 1$.
ii) $\mathrm{Ext}^1_{\mathcal{C}_\Gamma}(Z_\Gamma(\lambda'), Z_\Gamma(\lambda)) \simeq$, $/\pi(H_\beta)$, .

Proof: Using adjointness we get from 4.2 (1)

$$\mathrm{Ext}^j_{\mathcal{C}_\Gamma}(Z_\Gamma(\lambda'), T_\mu^\lambda Z_\Gamma(\mu)) \simeq \mathrm{Ext}^j_{\mathcal{C}_\Gamma}(Z_\Gamma(\mu), Z_\Gamma(\mu)) \tag{1}$$

for all $j \geq 0$. But for $j > 0$ we have for any $\nu, \eta \in X$ that $\mathrm{Ext}^j_{\mathcal{C}_\Gamma}(Z_\Gamma(\nu), Z_\Gamma(\eta)) = 0$ unless $\eta > \nu$. In fact, we recorded this already in 2.1(1) when $j = 1$. It follows for $j > 1$ by taking a projective cover $Q \in \mathcal{C}_\Gamma$ of $Z_\Gamma(\nu)$ with $[Q : Z_\Gamma(\nu')] = 0$ for $\nu' \not\geq \nu$, see [AJS, Remark 2.13]. Hence 4.2 (2) and (1) give

$$\mathrm{Ext}^j_{\mathcal{C}_\Gamma}(Z_\Gamma(\lambda'), Z_\Gamma(\lambda)) \simeq \mathrm{Ext}^{j-1}_{\mathcal{C}_\Gamma}(Z_\Gamma(\lambda'), Z_\Gamma(\lambda')) = 0$$

for $j > 1$. This proves i).

The vanishing in i) implies by the usual base change arguments that

$$\mathrm{Ext}^1_{\mathcal{C}_\Gamma}(Z_\Gamma(\lambda'), Z_\Gamma(\lambda)) \simeq \mathrm{Ext}^1_{\mathcal{C}_B}(Z_B(\lambda'), Z_B(\lambda)) \otimes_B \, , \, .$$

Therefore, ii) follows from [AJS, 8.6].

4.4. We shall now study in more details the case when , $= \hat{k}$. Using the isomorphism in 4.2(1) we define generators of the following Hom-spaces

$$\begin{aligned}
\hat{i} &= \mathrm{adj}_1^{-1}(\mathrm{id}_{\hat{Z}(\mu)}) && \in \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}(\lambda), T_\mu^\lambda \hat{Z}(\mu)) \\
\hat{p} &= \mathrm{adj}_2(\mathrm{id}_{\hat{Z}(\mu)}) && \in \mathrm{Hom}_{\hat{\mathcal{C}}}(T_\mu^\lambda \hat{Z}(\mu), \hat{Z}(\lambda')) \\
\hat{r} &= \mathrm{adj}_2(\mathrm{id}_{\hat{Z}(\mu)}) && \in \mathrm{Hom}_{\hat{\mathcal{C}}}(T_\mu^\lambda \hat{Z}(\mu), \hat{Z}(\lambda)) \\
\hat{s} &= \mathrm{adj}_1^{-1}(\mathrm{id}_{\hat{Z}(\mu)}) && \in \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}(\lambda'), T_\mu^\lambda \hat{Z}(\mu))
\end{aligned}$$

Then we have

Lemma . i) *The sequence* $0 \to \hat{Z}(\lambda) \xrightarrow{\hat{i}} T_\mu^\lambda \hat{Z}(\mu) \xrightarrow{\hat{p}} \hat{Z}(\lambda') \to 0$ *is exact.*
ii) *We have the following identities (valid up to units in* $\hat{k}$*)*

$$\hat{r} \circ \hat{i} = t \ \mathrm{id}_{\hat{Z}(\lambda)}, \quad \hat{p} \circ \hat{s} = t \ \mathrm{id}_{\hat{Z}(\lambda')} \ \textit{and} \ \hat{i} \circ \hat{r} + \hat{s} \circ \hat{p} = t \ \mathrm{id}_{T_\mu^\lambda \hat{Z}(\mu)} .$$

Proof: i) By 4.2 (2) the modules in question form an exact sequence. Since $\hat{i}$ and $\hat{p}$ are generators of the respective Hom-spaces, it follows easily that they coincide (up to units in $\hat{k}$) with the homomorphisms appearing in any such sequence.

ii) Proposition 4.3 ii) and the arguments in the proof of Proposition 4.3 give the exact sequence

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}(\lambda'), T_\mu^\lambda \hat{Z}(\mu)) & \xrightarrow{\bar{p}} & \mathrm{End}_{\hat{\mathcal{C}}}(\hat{Z}(\lambda')) & \longrightarrow & \hat{k}/t\hat{k} & \longrightarrow & 0 \\
\wr\big| & & \wr\big| & & \wr\big| & & \\
\mathrm{End}_{\hat{\mathcal{C}}}(\hat{Z}(\mu)) & & \hat{k} & & k & & \\
\wr\big| & & & & & & \\
\hat{k} & & & & & &
\end{array}$$

Here $\bar{p}$ takes $f \in \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}(\lambda'), T_\mu^\lambda \hat{Z}(\mu))$ into $\hat{p} \circ f$. This diagram shows that the generator $\hat{s}$ is mapped to $t \in \hat{k}$, i.e., $\hat{p} \circ \hat{s} = t\ \mathrm{id}_{\hat{Z}(\mu)}$. The other identities are proved in the same way (see also [AJS, 8.12]).

4.5. The results in 4.1-4 have straightforward analogues with $Z^w, w \in W$ replacing Z. In particular, we have for $w = w_0$ a short exact sequence

$$0 \to \hat{Z}^{w_0}(\lambda'\langle w_0\rangle) \xrightarrow{\hat{i}_{w_0}} T_\mu^\lambda \hat{Z}^{w_0}(\mu\langle w_0\rangle) \xrightarrow{\hat{p}_{w_0}} \hat{Z}^{w_0}(\lambda\langle w_0\rangle) \to 0$$

with $\hat{i}_{w_0} = \mathrm{adj}_1^{-1}(\mathrm{id}_{\hat{Z}^{w_0}(\mu\langle w_0\rangle)})$ and $\hat{p}_{w_0} = \mathrm{adj}_2(\mathrm{id}_{\hat{Z}^{w_0}(\mu\langle w_0\rangle)})$.

We have also generators $\hat{r}_{w_0}$ and $\hat{s}_{w_0}$ analogous to $\hat{r}$ and $\hat{s}$ and there are identities completely similar to those in Lemma 4.4.

4.6. Recall from 1.10 that if $\nu \in X$ then we have natural homomorphisms $\hat{\varphi}^\nu_{1,w_0} : \hat{Z}(\nu) \to \hat{Z}^{w_0}(\nu\langle w_0\rangle)$ and $\hat{\varphi}^\nu_{w_0,1} : \hat{Z}^{w_0}(\nu\langle w_0\rangle) \to \hat{Z}(\nu)$. According to Corollary 1.12 these homomorphisms generate the Hom-spaces to which they belong. The following lemma gives their behaviour under translation.

Lemma . *With λ, λ' and μ as in 4.2 we have up to units in $\hat{k}$*

i) $T_\lambda^\mu \hat{\varphi}^{\lambda'}_{1,w_0} = \hat{\varphi}^\mu_{1,w_0}$ *and* $T_\lambda^\mu \hat{\varphi}^\lambda_{w_0,1} = \hat{\varphi}^\mu_{w_0,1}$.
ii) $T_\lambda^\mu \hat{\varphi}^{\lambda'}_{w_0,1} = t\hat{\varphi}^\mu_{w_0,1}$ *and* $T_\lambda^\mu \hat{\varphi}^\lambda_{1,w_0} = t\hat{\varphi}^\mu_{1,w_0}$.

Proof: i) Recall that the image of $\varphi^{\lambda'}_{1,w_0} = \hat{\varphi}^{\lambda'}_{1,w_0} \otimes_{\hat{k}} k : Z(\lambda') \to Z^{w_0}(\lambda'\langle w_0\rangle)$ is $L(\lambda')$ and that $T_\lambda^\mu L(\lambda') = L(\mu)$. It follows that $T_\lambda^\mu \varphi^{\lambda'}_{1,w_0} \neq 0$ and hence that $T_\lambda^\mu \hat{\varphi}^{\lambda'}_{1,w_0}$ generate $\mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}(\lambda'), \hat{Z}^{w_0}(\lambda'\langle w_0\rangle))$. This proves the first equality. The second equality is proved completely similarly.

ii) By the identities in 1.9 we see that

$$\hat{\varphi}^\lambda_{1,w_0} \circ \hat{\varphi}^\lambda_{w_0,1} = t^N \mathrm{id}_{\hat{Z}(\lambda)} \tag{1}$$

and

$$\hat{\varphi}^\mu_{1,w_0} \circ \hat{\varphi}^\mu_{w_0,1} = t^{N-1} \mathrm{id}_{\hat{Z}(\mu)} \tag{2}$$

(Here $N = |R^+|$). Applying T^μ_λ to (1) and using i) we get $(T^\mu_\lambda \hat{\varphi}^\lambda_{1,w_0}) \circ \hat{\varphi}^\mu_{1,w_0} = t^N \operatorname{id}_{\hat{Z}(\mu)}$. When we compare this with (2) we get the second equality. The first is obtained in the same way.

4.7. **Proposition.** *With notation as above we have the following diagram in $\hat{\mathcal{C}}$*

$$\begin{tikzcd}
0 \arrow[r] & \hat{Z}^{w_0}(\lambda'\langle w_0\rangle) \arrow[r, shift left, "\hat{i}_{w_0}"] & T^\lambda_\mu \hat{Z}^{w_0}(\mu\langle w_0\rangle) \arrow[l, shift left, "\hat{r}_{w_0}"] \arrow[r, shift left, "\hat{p}_{w_0}"] \arrow[d, shift right, "T^\lambda_\mu \hat{\varphi}^\mu_{w_0,1}"'] & \hat{Z}^{w_0}(\lambda\langle w_0\rangle) \arrow[l, shift left, "\hat{s}_{w_0}"] \arrow[r] & 0 \\
0 \arrow[r] & \hat{Z}(\lambda) \arrow[r, shift left, "\hat{i}"] & T^\lambda_\mu \hat{Z}(\mu) \arrow[l, shift left, "\hat{r}"] \arrow[r, shift left, "\hat{s}"] \arrow[u, shift right, "T^\lambda_\mu \hat{\varphi}^\mu_{1,w_0}"'] & \hat{Z}(\lambda') \arrow[l, shift left, "\hat{p}"] \arrow[r] & 0
\end{tikzcd}$$

in which the rows (reading from left to right) are exact. Moreover, the following relations hold

i) $\hat{s}_{w_0} \circ \hat{\varphi}^\lambda_{1,w_0} = t(T^\lambda_\mu \hat{\varphi}^\mu_{1,w_0} \circ \hat{i})$ *and* $\hat{i}_{w_0} \circ \hat{\varphi}^{\lambda'}_{1,w_0} = T^\lambda_\mu \hat{\varphi}^\mu_{1,w_0} \circ \hat{s}$.
ii) $\hat{i} \circ \hat{\varphi}^\lambda_{w_0,1} = T^\lambda_\mu \hat{\varphi}^\mu_{w_0,1} \circ \hat{s}_{w_0}$ *and* $\hat{s} \circ \hat{\varphi}^{\lambda'}_{w_0,1} = t(T^\lambda_\mu \hat{\varphi}^\mu_{w_0,1} \circ \hat{i}_{w_0})$
iii) $\hat{\varphi}^\lambda_{1,w_0} \circ \hat{r} = t(\hat{p}_{w_0} \circ T^\lambda_\mu \hat{\varphi}^\mu_{1,w_0})$ *and* $\hat{\varphi}^{\lambda'}_{1,w_0} \circ \hat{p} = \hat{r}_{w_0} \circ T^\lambda_\mu \hat{\varphi}^\mu_{1,w_0}$
iv) $\hat{\varphi}^{\lambda'}_{w_0,1} \circ \hat{p}_{w_0} = \hat{r} \circ T^\lambda_\mu \varphi^\mu_{w_0,1}$ *and* $\hat{\varphi}^\lambda_{w_0,1} \circ \hat{r}_{w_0} = t(\hat{p} \circ T^\lambda_\mu \hat{\varphi}^\mu_{w_0,1})$

Proof: i) We have already established the exactness of the two rows. The first identity in i) is obtained by using the functoriality of $\operatorname{adj}_1^{-1}$ (see [AJS, 7.6]) and Lemma 4.6 ii):

$$\hat{s}_{w_0} \circ \hat{\varphi}^\lambda_{1,w_0} = \operatorname{adj}_1^{-1}(\operatorname{id}) \circ \hat{\varphi}^\lambda_{1,w_0} = \operatorname{adj}_1^{-1}(\operatorname{id} \circ T^\mu_\lambda \hat{\varphi}^\lambda_{1,w_0}) =$$
$$t \operatorname{adj}_1^{-1}(\hat{\varphi}^\mu_{1,w_0} \circ \operatorname{id}) = t(T^\lambda_\mu \hat{\varphi}^\mu_{1,w_0} \circ \operatorname{adj}_1^{-1}(\operatorname{id})) = t(T^\lambda_\mu \hat{\varphi}^\mu_{1,w_0} \circ \hat{i}).$$

The second equality in i) is obtained in the same way (using this time the first half of Lemma 4.6 i)).

ii)-iv) are proved analogously by using also the functoriality of adj_2 and Lemma 4.6 ii).

4.8. Finally, we shall need the following result

Proposition . *If p is odd, then the composite*

$$\operatorname{adj}_1(\operatorname{id}_{T^\lambda_\mu \hat{Z}(\mu)}) \circ \operatorname{adj}_2^{-1}(\operatorname{id}_{T^\lambda_\mu \hat{Z}(\mu)}) : \hat{Z}(\mu) \to \hat{Z}(\mu)$$

is an isomorphism.

Proof: By adding a multiple of $p\rho$ if necessary we may assume that μ is dominant. Then we denote by $V(\mu)$ the Weyl module with highest weight μ and we let $\theta : Z(\mu) \to V(\mu)$ denote the corresponding natural G_1T-homomorphism which is an isomorphism on the μ-weight spaces.

Consider now the commutative diagram in $\mathcal{C}$

$$\begin{array}{ccccc} Z(\mu) & \xrightarrow{\mathrm{adj}_2^{-1}(\mathrm{id}_{T_\mu^\lambda Z(\mu)})} & T_\lambda^\mu T_\mu^\lambda Z(\mu) & \xrightarrow{\mathrm{adj}_1(\mathrm{id}_{T_\mu^\lambda Z(\mu)})} & Z(\mu) \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle T_\lambda^\mu T_\mu^\lambda \theta} & & \downarrow{\scriptstyle\theta} \\ V(\mu) & \xrightarrow{\mathrm{adj}_2^{-1}(\mathrm{id}_{T_\mu^\lambda V(\mu)})} & T_\lambda^\mu T_\mu^\lambda V(\mu) & \xrightarrow{\mathrm{adj}_1(\mathrm{id}_{T_\mu^\lambda V(\mu)})} & V(\mu). \end{array}$$

In the bottom row the translation functors as well as the adjointness are taken in the category of G-modules. According to [Ja2, II. 9.19] translation functors commute with the forgetful functors from this category into our category $\mathcal{C}$.

The composite of the maps in the bottom row is an isomorphism, see [An3, 2.2 and 2.8]. It follows that the composite in the top row is an isomorphism on the μ-weight spaces and therefore on all of $Z(\mu)$. The proposition now follows via the Nakayama lemma.

Remark . *Our assumption $p > 2$ is only relevant for type A_1 (since we are assuming $p \geq h$). For type A_1 and $p = 2$ a direct computation shows that then the composite in the proposition is in fact 0.*

5. Filtrations and associated polynomials

In this section we shall study the polynomials attached to our filtrations. Their values at 0 determine the decomposition of a projective module into its indecomposable summands.

We conjecture that the polynomials attached to an indecomposable projective module with p-regular highest weight coincide with the Kazhdan-Lusztig polynomials associated with the corresponding elements in the affine Weyl group. In the case where the highest weight of our projective module is in the "first" alcove we are able to explicitly determine the polynomials attached to our filtration, see Proposition 5.3. The result verifies the conjecture in this case. Then we try to proceed via "wall-crossing". If our filtrations behave as expected (see Conjecture 5.4 and compare with [An3, Conjecture 3.1]) then we deduce that the conjecture is true in general. We also demonstrate that it would give a refinement of the Brauer-Humphreys reciprocity for indecomposable projective G_1T-modules. Moreover, it would imply that the Jantzen filtration of a standard module coincides with its Loewy series.

5.1. Let $Q \in \mathcal{C}$ be projective and suppose $\lambda \in X$. Then we set

$$f_\lambda(Q) = \sum_{j \geq 0} \dim F_\lambda(Q)_j q^j \in \mathbb{Z}[q]. \tag{1}$$

In case Q is indecomposable, i.e., $Q = Q(\nu)$ for some $\nu \in X$, we write $f_{\lambda,\nu}$ instead of $f_\lambda(Q(\nu))$.

A general projective module $Q \in \mathcal{C}$ decomposes into a direct sum of various $Q(\nu)$'s. We denote the number of summands in Q isomorphic to $Q(\nu)$ by $(Q : Q(\nu))$.

Proposition . i) $f_\lambda(Q)(0) = (Q : Q(\lambda))$.
ii) *For all* $\nu, \eta \in X$ *we have* $f_{\lambda,\nu} = f_{\lambda+p\eta,\nu+p\eta}$.

Proof: i) was proved in 3.10.i), and ii) is clear from the definition since $Z(\lambda+p\eta) \simeq Z(\lambda) \otimes p\eta$ and $Q(\nu + p\eta) \simeq Q(\nu) \otimes p\eta$.

5.2. Recall from [Lu] the concept of distance between two alcoves. If A and C are alcoves in X we denote by $d(A, C)$ the distance from A to C and we let $Q_{A,C}$ be the Kazhdan-Lusztig polynomial attached to the pair (A, C) as in loc. cit.

Assume from now on that λ is p-regular (as in 4.2). This requires $p \geq h$ which we assume for the rest of this section. We also assume p to be odd. Then each alcove A contains a unique element $\lambda_A \in W_p.\lambda$ and we write $f_{A,C}$ instead of f_{λ_A,λ_C}. If $w \in W_p$ we let Aw denote the alcove containing $w^{-1}.\lambda_A$

Conjecture . Suppose A and C are alcoves in X. Then we have

$$Q_{A,C}(q) = q^{\frac{1}{2}d(A,C)} f_{Aw_0,Cw_0}\left(q^{-\frac{1}{2}}\right).$$

Remark . *Equivalently, we should have the following relation to the polynomials* $p_{A,C}$ *considered by W. Soergel in* [So, 4.4]

$$p_{A,C}(q) = q^{\frac{3}{2}d(A,C)} f_{Aw_0,Cw_0}\left(q^{-\frac{1}{2}}\right).$$

5.3. The following result shows that Conjecture 5.2 holds when C is the bottom dominant alcove.

Proposition . *Suppose* $\nu \in X$ *belongs to the top antidominant alcove, i.e.* $-p < \langle \nu+\rho, \beta^\vee\rangle < 0$ *for all* $\beta \in R^+$. *Then*

$$f_{\lambda,\nu} = \begin{cases} q^{l(w)} & \text{if } \lambda = w.\nu \text{ for some } w \in W; \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Recall from 2.6 that $Q(-\rho) = Z(-\rho)$. It follows easily that $Q(\nu) = T_{-\rho}^{\nu} Q(-\rho)$ and hence by 4.1 (2)

$$(Q(\nu) : Z(\lambda)) = \begin{cases} 1 & \text{if } \lambda \in W.\nu; \\ 0 & \text{otherwise.} \end{cases}$$

So we may assume $\lambda = w.\nu$ for some $w \in W$. As λ is minimal in $W.\nu$ with respect to the ordering on X determined by $w(R^+)$ we see that $\hat{Z}^{ww_0}(\lambda\langle ww_0\rangle)$ occurs at the bottom of a Z^{ww_0}-filtration of $\hat{Q}(\nu)$. The corresponding inclusion $\hat{Z}^{ww_0}(\lambda\langle ww_0\rangle) \hookrightarrow \hat{Q}(\nu)$ induces two isomorphisms

$$\mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{Z}(\lambda)^\tau, \hat{Z}^{ww_0}(\lambda\langle ww_0\rangle)) \simeq F_\lambda(\hat{Q}(\nu)).$$

and

$$\mathrm{Hom}_{\mathcal{C}}(\hat{Z}(\lambda), \hat{Z}^{ww_0}(\lambda\langle ww_0\rangle)) \simeq F_\lambda^{w_0}(\hat{Q}(\nu)).$$

This says that we may identify a generator of $\hat{F}_\lambda(Q(\nu))$ (resp. $\hat{F}_\lambda^{w_0}(Q(\nu))$) with $\hat{\varphi}^\lambda_{w_0,ww_0}$ (resp. $\hat{\varphi}^\lambda_{1,ww_0}$). We have $\hat{\varphi}^\lambda_{w_0,ww_0} \circ c' = t^{l(w)}\hat{\varphi}^\lambda_{1,ww_0}$ (see Lemma 1.9). Hence Proposition 3.6 gives the desired conclusion.

5.4. We would like to determine the polynomials $f_{A,C}$ for any pair of alcoves A, C. The first step is taken in Proposition 5.3, and we shall now try to proceed by induction on C using the translation functors from Section 4. Since our strategy is very similar to the one used in [An3] we shall leave most of the proofs to the reader.

Suppose λ, λ' and μ are as in 4.2. Let $\hat{Q} \in \hat{\mathcal{C}}(\lambda)$ be projective. Then the exact sequence in 4.6 gives rise to the exact sequence

$$0 \to \hat{F}_\lambda(Q) \xrightarrow{\bar{p}} \hat{F}_\mu(T_\lambda^\mu Q) \xrightarrow{\bar{i}} \hat{F}_{\lambda'}(Q) \to 0. \tag{1}$$

Here $\bar{p}$ and $\bar{i}$ are the homomorphisms induced by $\hat{p}_{w_0}$ and $\hat{i}_{w_0}$, respectively. Explicitly,

$$\bar{p}(\varphi) = adj_2^{-1}(\varphi \circ \hat{p}_{w_0}), \qquad \varphi \in \hat{F}_\lambda(Q)$$

and

$$\bar{i}(\psi) = \hat{i}_{w_0} \circ adj_2(\psi), \qquad \psi \in \hat{F}_\mu(T_\lambda^\mu Q).$$

Recall that s is the reflection in W_p which fixes μ. If $\nu \in W_p.\lambda$, then we write νs for the mirror image of ν in the s-wall, i.e., if $\nu = w.\lambda$, then $\nu s = ws.\lambda$.

Conjecture . Assume $\hat{Q}$ contains no summands of the form $Q(\nu)$ with $\nu < \nu s$. Then for all $j \geq 0$

i) $\hat{F}_{\lambda'}(Q)^{j+1} = \bar{i}(\hat{F}_\mu(T_\lambda^\mu Q)^j)$.
ii) $\hat{F}_\lambda(Q)^j = t\hat{F}(Q)^j + \bar{p}^{-1}(\hat{F}_\mu(T_\lambda^\mu Q)^j)$.

Remarks.

i) *This conjecture gives the expected behavior of our filtrations under the translation functor T_λ^μ. As we shall see below (Proposition 5.6) the behavior under T_μ^λ is easy to prove.*
ii) *Of course we could also phrase the conjecture in terms of the corresponding filtrations of k-spaces. Then the expected behavior is:*
The exact sequence

$$0 \to F_\lambda(Q) \to F_\mu(T_\lambda^\mu Q) \to F_{\lambda'}(Q) \to 0 \tag{2}$$

(obtained from (1) *by tensoring with k) induces for each $j \geq 0$ a short exact sequence*

$$0 \to F_\lambda(Q)^j \to F_\mu(T_\lambda^\mu Q)^j \to F_{\lambda'}(Q)^{j+1} \to 0.$$

5.5. The following four results may be viewed as evidence for or partial verification of Conjecture 5.4. They are deduced via the results in Section 4 exactly as the analogous statements in [An3]. The notation and assumptions are as in 5.4.

$$t\hat{F}_{\lambda'}(Q)^j \subseteq \bar{i}(\hat{F}_\mu(T_\lambda^\mu Q)^j) \subseteq \hat{F}_{\lambda'}(Q)^j \text{ for all } j. \tag{1}$$

$$\hat{F}_\lambda(Q)^{j+1} \subseteq \bar{p}^{-1}(\hat{F}_\mu(T_\lambda^\mu Q)^j) \subseteq \hat{F}_\lambda(Q)^j \text{ for all } j. \tag{2}$$

$$\text{If } \hat{F}_\lambda(Q) = 0 \text{ then } \hat{F}_{\lambda'}(Q)^{j+1} = \bar{i}(\hat{F}_\mu(T_\lambda^\mu Q)^j) \text{ for all } j. \tag{3}$$

$$\text{If } \hat{F}_{\lambda'}(Q) = 0 \text{ then } \bar{p}^{-1}(\hat{F}_\mu(T_\lambda^\mu Q)^j) = \hat{F}_\lambda(Q)^j \text{ for all } j. \tag{4}$$

5.6. Arguing as in [An3, 2.5-7] we obtain

Proposition . *Let $Q' \in \mathcal{C}(\mu)$ be projective. Then*

$$\hat{F}_\lambda(T_\mu^\lambda Q')^{j+1} \simeq \hat{F}_\mu(Q')^j \simeq \hat{F}_{\lambda'}(T_\mu^\lambda Q')^j$$

for all $j \geq 0$.

5.7. Combining 5.4 and 5.6 we can now deduce how our filtrations are expected to behave with respect to the "wall-crossing" functor $\Theta_s = T_\mu^\lambda T_\lambda^\mu$.

Proposition . *Let $Q \in \mathcal{C}(\lambda)$ be projective and assume Q contains no summands of the form $Q(\nu)$ with $\nu < \nu s$. If Conjecture 5.4 holds then we have for any $\eta \in W_p.\lambda$*

$$f_\eta(\Theta_s Q) = \begin{cases} q f_\eta(Q) + f_{\eta s}(Q) & \text{if } \eta > \eta s \\ q^{-1} f_\eta(Q) + f_{\eta s}(Q) & \text{if } \eta < \eta s \end{cases}$$

Proof: Argue as in [An3, 3.5].

5.8. **Corollary.** *Let $\nu \in W_p.\lambda$ and suppose $\nu > \nu s$. If Conjecture 5.4 holds for $Q(\nu)$ then*

$$\Theta_s Q(\nu) = Q(\nu s) \oplus \Big(\bigoplus_{\substack{\nu s < \eta < \eta s \\ \eta}} f'_{\eta,\nu}(0) Q(\eta) \Big).$$

(Here f' denotes the formal derivative of $f \in \mathbb{Z}[q]$).

Proof: To get the decomposition of $\Theta_s Q(\nu)$ into indecomposable summands we have by Proposition 5.1 i) just to compute $f_\eta(\Theta_s Q(\nu))(0)$ for all $\eta \in W_p.\lambda$. This is easy once we have Proposition 5.7.

Remark . *Any $\eta \in X$ may be written uniquely $\eta = \eta^0 + p\eta^1$ with $\eta^0 \in X_1 = \{\nu \in X \mid 0 \leq \langle \nu, \alpha^\vee \rangle < p, \alpha \in S\}$ and $\eta^1 \in X$. Set then $\hat{\eta} = w_0.\eta^0 + p(\eta^1 + 2\rho)$.*

It follows from [AJS, 16.14] *that the η's occurring in the corollary satisfy $\nu s \uparrow \eta \uparrow \hat{\eta} \uparrow \widehat{\nu s}$. Here $\uparrow$ denotes the strong linkage relation, see loc. cit.*

5.9. Another consequence of Proposition 5.7 is that it verifies Conjecture 5.2. In other words, Conjecture 5.4 implies Conjecture 5.2. In fact, we saw already (in Proposition 5.3) that Conjecture 5.2 holds when C is the bottom dominant alcove. For general C we then just compare Corollary 5.8 with the inductive formula [Lu, 10.7] for the Kazhdan-Lusztig polynomials.

In particular, we conclude from 5.2 that the "leading coefficient" $\mu(A, C)$ (see loc.cit) of $Q_{A,C}$ is given by

(1) $$\mu(A, C) = f'_{Aw_0, Cw_0}(0).$$

Hence we may reformulate Corollary 5.8 as follows:

(2) If $\lambda_A < \lambda_A s$ then $\Theta_s Q(\lambda_A) = Q(\lambda_A s) \oplus \Big(\bigoplus_{\lambda_C < \lambda_C s} \mu(Cw_0, Aw_0) Q(\lambda_C) \Big)$.

This is Cline's reformulation of Lusztig's conjecture, see [Cl, 3.2].

5.10. Note that Conjecture 5.2 gives the following identity when we evaluate at 1

(1) $$Q_{A,C}(1) = f_{Aw_0, Cw_0}(1) = [Z(\lambda_{Aw_0}) : L(\lambda_{Cw_0})].$$

This is the G_1T-version of Lusztig's conjecture. We proved in [AK, 6.3] that this conjecture implies the following interpretation of the Kazhdan-Lusztig polynomials

(2) $$Q_{A,C} = \sum_{j \geq 0} q^{\frac{1}{2}(d(A,C)-j)} [\mathrm{rad}_j\, Z(\lambda_{Aw_0}) : L(\lambda_{Cw_0})].$$

Here rad_j denotes the j'th level in the radical series.

When we compare with 5.2 we get the following G_1T-version of [An3, 3.5].

Corollary . *Suppose Conjecture 5.2 holds. Then we have for all p-regular weights* $\nu \in X$

$$\operatorname{rad}^j Z(\nu) = Z(\nu)^j, \quad j \geq 0$$

Proof: By (2) and Conjecture 5.2 we get

$$[\operatorname{rad}_j Z(\lambda_A) : L(\lambda_C)] = \dim F_{\lambda_A}(Q(\lambda_C))_j \tag{3}$$

Remark . *The identity* (1) *may be thought of as a refinement of Brauer-Humphreys reciprocity.*

6. Some related cases

In this section we shall consider some cases which are closely related to the theory developed in the previous sections.

The first case (treated in 6.1) is the "small" quantum group associated with R (this is case II in [AJS]). Here the theory carries over almost verbatim.

The second and third cases are the "big" quantum group at a complex root of unity (treated in 6.2-11) and the algebraic group G itself (treated in 6.12-13). These cases were already studied in [An3] (which in turn inspired many of the results in this paper). In particular, the theory in Sections 4-5 was developed in loc.cit. Here we shall see that the results in Section 3 have at least some partial analogues in these cases as well.

6.1. Replace the ground field considered so far by $k = \mathbb{Q}(\zeta)$, where ζ is a primitive p-th root of unity. We assume that $p \in \mathbb{N}$ is odd and if R is of type G_2, then we also require p not to be divisible by 3.

Denote by U_2 the quantum group over k associated to R, see [AJS]. This is a k-algebra with generators $\{E_\alpha, F_\alpha, K_\alpha, K_\alpha^{-1} \mid \alpha \in S\}$ and certain relations. As in [AJS] we then define $U = U_2/I$ where I is the two-sided ideal in U_2 generated by $\{E_\beta^p, F_\beta^p \mid \beta \in R^+\}$. Here E_β and F_β are some (fixed) choice of root vectors, see loc.cit.

Then U has a triangular decomposition, $U = U^-U^0U^+$ with U^- (resp. U^0, resp. U^+) generated by the images in U of $\{F_\alpha \mid \alpha \in S\}$ (resp. $\{K_\alpha, K_\alpha^{-1} \mid \alpha \in S\}$, resp. $\{E_\alpha \mid \alpha \in S\}$). We have $U^0 \simeq k[\{K_\alpha, K_\alpha^{-1} \mid \alpha \in S\}]$, the Laurent polynomial ring in the variables $\{K_\alpha \mid \alpha \in S\}$.

To stay in analogy with the notation of the previous sections we set also in this case $\hat{k} = k[t]_{(t)}$ and $\tilde{k} = k(t)$. Then $\hat{k}$ (and $\tilde{k}$) is a U^0-algebra with structure homomorphism $\pi : U^0 \to \hat{k}$ given by $\pi(K_\alpha) = t + 1$ for all $\alpha \in S$, see [AJS, 6.5].

We now have categories $\mathcal{C}, \hat{\mathcal{C}}$ and $\tilde{\mathcal{C}}$, standard modules $Z(\lambda), Z^w(\lambda)$ and their counterparts in $\hat{\mathcal{C}}$ and $\tilde{\mathcal{C}}$ as well as projective modules, see [AJS]. The theory from Sections 1-5 carry over with almost no change at all. The only place where there is a small difference is in the proof of 1.9. The relevant formula replacing 1.9 (1) reads (using analogous notation, see also [AJS, 5.6])

$$'\hat{\varphi}_i^\lambda \circ \hat{\varphi}_i^\lambda(v_0) = \pi\left(\begin{bmatrix} K_{y_i\alpha_i}; \langle \lambda\langle y_i\rangle, y_i\alpha_i^\vee\rangle \\ p-1 \end{bmatrix}\right) v_0$$

where for $c \in \mathbb{Z}, t \in \mathbb{N}$ and $\beta \in R^+$ we have $\left[\begin{smallmatrix} K_\beta; c \\ t \end{smallmatrix}\right] = \prod_{j=1}^t \frac{K_\beta \zeta^{d_\beta(c+1-j)} - K_\beta^{-1}\zeta^{-d_\beta(c+1-j)}}{\zeta^{d_\beta j} - \zeta^{-d_\beta j}}$ as in [AJS, 5.1]. Inserting $\pi(K_\alpha) = t + 1$, we easily check that up to units in $\hat{k}$ we get

$$\pi\left(\begin{bmatrix} K_{y_i\alpha_i}; \langle\lambda\langle y_i\rangle, y_i\alpha_i^\vee\rangle \\ p-1 \end{bmatrix}\right) = \begin{cases} t & \text{if } \langle\lambda+\rho, y_i\alpha_i^\vee\rangle \not\equiv 0 \bmod p; \\ 1 & \text{otherwise .} \end{cases}$$

6.2. Let p, ζ and k be as in 6.1.

Consider now the quantum group $U_\zeta = U_{\mathbb{Z}[q,q^{-1}]} \otimes_{\mathbb{Z}[q,q^{-1}]} k$ obtained from the Lusztig $\mathbb{Z}[q,q^{-1}]$-form $U_{\mathbb{Z}[q,q^{-1}]}$ of the generic quantum group associated with R by specializing q to ζ. We let $\mathcal{C} = \mathcal{C}_k$ denote the category of finite dimensional U_ζ-modules of type $\mathbf{1}$.

For each $\lambda \in X^+$ we have (as "standard" modules in $\mathcal{C}$) a Weyl module $\Delta_\zeta(\lambda)$ and an induced module $\nabla_\zeta(\lambda)$ with highest weight λ. The simple module $L_\zeta(\lambda)$ is the head of $\Delta_\zeta(\lambda)$ and the socle of $\nabla_\zeta(\lambda)$. In addition we have an indecomposable tilting module $T_\zeta(\lambda)$ also having highest weight λ. (Recall that $M \in \mathcal{C}$ is tilting if M has both a Δ_ζ- and a ∇_ζ-filtration).

In order to stay close to the notation used in the previous sections we set $\hat{k} = \mathbb{Q}[q]_{(\phi_p)}$, the localization of the polynomial ring $\mathbb{Q}[q]$ at the prime ideal generated by the p-th cyclotomic polynomial ϕ_p. Then k is the residue field of $\hat{k}$ (via $q \mapsto \zeta$) and $\tilde{k} = \mathbb{Q}(q)$ is the fraction field.

We have categories $\hat{\mathcal{C}}$ and $\tilde{\mathcal{C}}$ similar to $\mathcal{C}$. They consist of (integrable) modules for the corresponding quantum groups over $\hat{k}$ and $\tilde{k}$, respectively. Moreover, the standard modules $\Delta_\zeta(\lambda)$ and $\nabla_\zeta(\lambda)$ lift to $\hat{k}$-free modules in $\hat{\mathcal{C}}$ which we denote $\hat{\Delta}(\lambda)$ and $\hat{\nabla}(\lambda)$. The tilting module $T_\zeta(\lambda)$ also lift to a tilting module $\hat{T}(\lambda) \in \hat{\mathcal{C}}$ (see [An5]).

6.3. In the notation of [An3] we have (for $\lambda \in X^+$)

$$\nabla_\zeta(\lambda) = H^0_\zeta(\lambda) \quad \text{and} \quad \Delta_\zeta(\lambda) = H^N_\zeta(w_0.\lambda).$$

Here $H^0_\zeta : \mathcal{C}^{\geq 0} \to \mathcal{C}$ denotes the induction functor and $H^j_q, j \geq 0$ its derived functors, see [APW]. We have similar functors over $\hat{k}$ which we shall denote $\hat{H}^0$ and $\hat{H}^j$. Then (loc.cit.)

$$\hat{\nabla}(\lambda) = \hat{H}^0(\lambda) \quad \text{and} \quad \hat{\Delta}(\lambda) = \hat{H}^N(w_0.\lambda).$$

More generally, we consider for $w \in W$ the module $\hat{H}^{l(w)}(w.\lambda) \in \hat{\mathcal{C}}$. This is not always a free $\hat{k}$-module so we factor out the torsion submodule $\hat{H}^{l(w)}(w.\lambda)_{\mathrm{tor}}$ and set

$$\hat{\nabla}^w(\lambda) = \hat{H}^{l(w)}(w.\lambda)/\hat{H}^{l(w)}(w.\lambda)_{\mathrm{tor}}.$$

Then $\hat{\nabla}^w(\lambda)$ is free of the same rank as $\hat{\nabla}(\lambda)$. In fact, by the quantized Bott theorem [APW] we have

$$\hat{H}^{l(w)}(w.\lambda) \otimes_{\hat{k}} \tilde{k} \simeq \hat{H}^0(\lambda) \otimes_{\hat{k}} \tilde{k}.$$

The character of $\hat{\nabla}^w(\lambda)$ is then given by the Weyl character formula.

Note that in this notation we have

$$\hat{\Delta}(\lambda) = \hat{\nabla}^{w_0}(\lambda).$$

6.4. We shall now prove that we have some results similar to those in 1.7-9. So let $\lambda \in X^+$ and fix $w \in W$. Suppose $s_i w > w$ for some simple reflection s_i. Then

Lemma . *There exist homomorphisms* $\hat{\varphi}_{i,w}^{\lambda} \in \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{\nabla}^{s_i w}(\lambda), \hat{\nabla}^{w}(\lambda))$ *and* $'\hat{\varphi}_{i,w}^{\lambda} \in \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{\nabla}^{w}(\lambda),\ \hat{\nabla}^{s_i w}(\lambda))$ *such that*

(i)

$$'\hat{\varphi}_{i,w}^{\lambda} \circ \hat{\varphi}_{i,w}^{\lambda} = \begin{cases} \phi_p \,\mathrm{id}_{\hat{\nabla}^{s_i w}(\lambda)} & \text{if } \langle w(\lambda+\rho), \alpha_i^{\vee}\rangle > p \text{ and } \langle w(\lambda+\rho), \alpha_i^{\vee}\rangle \not\equiv 0 (\mathrm{mod}\, p); \\ \mathrm{id}_{\hat{\nabla}^{s_i w}(\lambda)} & \text{otherwise.} \end{cases}$$

(ii)

$$\hat{\varphi}_{i,w}^{\lambda} \circ {}'\hat{\varphi}_{i,w}^{\lambda} = \begin{cases} \phi_p \,\mathrm{id}_{\hat{\nabla}^{w}(\lambda)} & \text{if } \langle w(\lambda+\rho), \alpha_i^{\vee}\rangle > p \text{ and } \langle w(\lambda+\rho), \alpha_i^{\vee}\rangle \not\equiv 0 (\mathrm{mod}\, p); \\ \mathrm{id}_{\hat{\nabla}^{w}(\lambda)} & \text{otherwise.} \end{cases}$$

Proof:

Let $U_\zeta(i)$ denote the subalgebra of U_ζ generated by U_q^-, U_q^0 and $\{E_i^{(r)} \mid r \in \mathbb{N}\}$. Then we have a "rank 1"-induction functor $H_i^0 : \mathcal{C}^{\geq 0} \to \mathcal{C}(i)$, where $\mathcal{C}(i)$ is the category of integrable $U_\zeta(i)$-modules. We let H_i^1 denote the first derived functor and have corresponding functors $\hat{H}_i^j : \hat{\mathcal{C}}^{\geq 0} \to \hat{\mathcal{C}}(i), j = 0, 1$.

If $\nu \in X^+$ has $m = \langle \nu, \alpha_i^{\vee}\rangle \geq 0$, then the result in [APW, Section 4] show that $\hat{H}_i^0(\nu)$ and $\hat{H}_i^1(s_i.\nu)$ are both $\hat{k}$-free of rank $m+1$ and that there exist natural homomorphisms in $\hat{\mathcal{C}}(i)$

$$\hat{\psi}_i^{\nu} : \hat{H}_i^1(s_i.\nu) \to \hat{H}_i^0(\nu) \text{ and } '\hat{\psi}_i^{\nu} : \hat{H}_i^0(\nu) \to \hat{H}_i^1(s_i.\nu)$$

which in suitably chosen bases $\{v_j'\}$ for $\hat{H}_i^1(s_i.\nu)$ and $\{v_j\}$ for $\hat{H}_i^0(\nu)$ are given by (using notation as in *loc. cit.* for the Gaussian integers and binomial coefficients)

$$\hat{\psi}_i^{\nu}(v_j') = \begin{bmatrix} m \\ j \end{bmatrix}_{d_i} v_j \text{ and } '\hat{\psi}_i^{\nu}(v_j) = [m-j]_{d_i}![j]_{d_i}!v_j',$$

$j = 0, 1, \cdots, m$. (In fact all this works also over $\mathbb{Z}[q, q^{-1}]$).

Noting that ϕ_p divides $[r]_{d_i}$ iff p divides r, we see that $'\hat{\psi}_i^{\nu}$ is divisible by

$$\begin{cases} \phi_p^{m_1} & \text{if } m < p \text{ or } m \equiv -1 (\mathrm{mod}\, p); \\ \phi_p^{m_1 - 1} & \text{otherwise,} \end{cases}$$

where $m_1 \in \mathbb{N}$ is determined by $m_1 p \leq m < (m_1+1)p$. If we carry out this division and still denote the resulting homomorphism $'\hat{\psi}_i^{\nu}$ then the composite of $\hat{\psi}_i^{\nu}$ and $'\hat{\psi}_i^{\nu}$ (in either order) is multiplication by a scalar $a \in \hat{k}$ which up to a unit in $\hat{k}$ is given by

$$a = \begin{cases} \phi_p & \text{if } m \geq p \text{ and } m \not\equiv -1 (\mathrm{mod}\, p); \\ 1 & \text{otherwise.} \end{cases}$$

Now $\hat{\psi}_i^\nu$ (resp. $'\hat{\psi}_i^\nu$) induces for each j a homomorphism $\hat{H}^{j+1}(s_i.\nu) \to \hat{H}^j(\nu)$ (resp. $\hat{H}^j(\nu) \to \hat{H}^{j+1}(s_i.\nu)$). The composite (in either order) of these induced homomorphisms is again multiplication by a.

Taking $\nu = w.\lambda$ and $j = l(w)$ we obtain the lemma.

6.5. Let $w_0 = s_{i_1} \cdots s_{i_N}$ be a reduced expression. By tracing the effect on the 1-dimensional λ-weight spaces we see that the composite

$$\hat{\Delta}(\lambda) = \hat{\nabla}^{w_0}(\lambda) \xrightarrow{\hat{\varphi}^\lambda_{i_1, s_{i_1} w_0}} \hat{\nabla}^{s_{i_1} w_0}(\lambda) \longrightarrow \cdots \xrightarrow{\hat{\varphi}^\lambda_{i_N, 1}} \hat{\nabla}(\lambda)$$

is, in fact, an isomorphism on the λ-weight spaces, and hence it generates the $\hat{k}$-module $\mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{\Delta}(\lambda), \hat{\nabla}(\lambda)) \simeq \hat{k}$. We denote this generator $c = c_\lambda$.

Similarly, we set $c' = c'_\lambda$ equal to the composite

$$\hat{\nabla}(\lambda) \xrightarrow{'\hat{\varphi}^\lambda_{i_N, 1}} \hat{\nabla}^{s_{i_N}}(\lambda) \longrightarrow \cdots \xrightarrow{'\hat{\varphi}^\lambda_{i_1, s_{i_1} w_0}} \hat{\nabla}^{w_0}(\lambda) = \hat{\Delta}(\lambda)$$

Then Lemma 6.4 shows

$$(1) \qquad c \circ c' = \phi_p^{N'(\lambda)} \mathrm{id}_{\hat{\nabla}(\lambda)} \text{ and } c' \circ c = \phi_p^{N'(\lambda)} \mathrm{id}_{\hat{\Delta}(\lambda)},$$

where $N'(\lambda) = \#\{\beta \in R^+ \mid \langle \lambda + \rho, \beta^\vee \rangle > p \text{ and } \langle \lambda + \rho, \beta^\vee \rangle \not\equiv 0 (\mathrm{mod}\, p)\}$. Recall from 3.4 the notation $N(\lambda)$. Note that $N'(\lambda) = N(\lambda)$ if and only if $\langle \lambda + \rho, \beta^\vee \rangle \geq p$ for all $\beta \in R^+$, i.e. $\lambda - (p-1)\rho \in X^+$. In that case it is known that (see [Ja1, Section 6])

$$(2) \qquad \text{if } \lambda - (p-1)\rho \in X^+, \text{ then } \mathrm{soc}(\Delta_\zeta(\lambda)) = L_\zeta(\hat{\lambda}) = \Delta_\zeta(\lambda)^{N(\lambda)},$$

where $\hat{\lambda}$ is defined as in Remark 5.8 and $(\Delta_\zeta(\lambda)^j)_{j \geq 0}$ denotes the Jantzen filtration of $\Delta_\zeta(\lambda)$. It follows from (1) and (2) that $c' \otimes_{\hat{k}} k \neq 0$ when $\lambda - (p-1)\rho \in X^+$. Hence for such λ we have that c'_λ is a generator of $\mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{\nabla}(\lambda), \hat{\Delta}(\lambda))$. In general, we have

Lemma . *Set $N''(\lambda) = \max\{j \mid \Delta(\lambda)^j \neq 0\}$. Then $N''(\lambda) \leq N'(\lambda)$ and $c'_\lambda = \phi_p^{N'(\lambda) - N''(\lambda)} c''_\lambda$ for some generator $c''_\lambda \in \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{\nabla}(\lambda), \hat{\Delta}(\lambda))$.*

Proof: Choose bases $\{v_i\}$ for $\hat{\Delta}(\lambda)$ and $\{v'_i\}$ for $\hat{\nabla}(\lambda)$ such that

$$c_\lambda(v_i) = \phi_p^{a_i} v'_i$$

for some $a_i \in \mathbb{N}$. Then (1) shows that

$$c'_\lambda(v'_i) = \phi_p^{N'(\lambda) - a_i} v_i.$$

By definition $N''(\lambda) = \max\{a_i\}$. Hence $N''(\lambda) \leq N'(\lambda)$ and there exists some $c''_\lambda \in \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{\nabla}(\lambda), \hat{\Delta}(\lambda))$ with $c'_\lambda = \phi_p^{N'(\lambda) - N''(\lambda)} c''_\lambda$. The above shows that there exists some i for which $c''_\lambda(v'_i) = v_i$. It follows that c''_λ is a generator of $\mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{\nabla}(\lambda), \hat{\Delta}(\lambda))$.

Remark . *The number $N''(\lambda)$ in this lemma is the length of the Jantzen filtration of $\Delta(\lambda)$. No general formula is known for this length. As observed above we have $N''(\lambda) = N(\lambda)$ if $\lambda - (p-1)\rho \in X^+$. However, if λ is close to one of the walls of X^+*

it is easy to find an example where $N''(\lambda) < N'(\lambda)$: Suppose R is of type A_2 and let $\lambda = p\omega$ where ω is one of the fundamental weights. Then $N'(\lambda) = 2 > 1 = N''(\lambda)$.

6.6. Recall that for any $\lambda, \nu \in X^+$ we have [An4]

$$\mathrm{Ext}^i_{\hat{\mathcal{C}}}(\hat{\Delta}(\lambda), \hat{\nabla}(\nu)) = \begin{cases} \hat{k} & \text{if } i = 0 \text{ and } \lambda = \nu; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

If we invert the order of Δ and ∇ then the corresponding result is no longer true. However, we have the following weak version

Proposition . *Let $\nu \in X^+$ and $\lambda \in (p-1)\rho + X^+$.*

$$\mathrm{Hom}_{\mathcal{C}}(\nabla_\zeta(\lambda), \Delta_\zeta(\nu)) = \begin{cases} k & \text{if } \nu = \lambda; \\ 0 & \text{if } \nu \not\leq \lambda. \end{cases} \tag{i}$$

$$\mathrm{Ext}^1_{\hat{\mathcal{C}}}(\hat{\nabla}(\lambda), \hat{\Delta}(\nu)) = 0 \textit{ unless } \nu < \lambda. \tag{ii}$$

Proof: Since $T_\zeta(\lambda)$ maps surjectively onto $\nabla_\zeta(\lambda)$ we have $\mathrm{Hom}_{\mathcal{C}}(\nabla_\zeta(\lambda), \Delta_\zeta(\nu)) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(T_\zeta(\lambda), \Delta_\zeta(\nu))$. Since $\lambda \in (p-1)\rho + X^+$, we know from [An2] that $T_\zeta(\lambda)$ is the projective cover of $L_\zeta(\hat{\lambda})$ (in the notation of Remark 5.8). Hence

$$\dim \mathrm{Hom}_{\mathcal{C}}(T_\zeta(\lambda), \Delta_\zeta(\nu)) = [\Delta_\zeta(\nu) : L_\zeta(\hat{\lambda})] = [\nabla_\zeta(\nu) : L_\zeta(\hat{\lambda})] = \dim \mathrm{Hom}_{\mathcal{C}}(T_\zeta(\lambda), \nabla_\zeta(\nu))$$

Using (1), we see that this is equal to $(T_\zeta(\lambda) : \Delta_\zeta(\nu))$. But this number is 1 if $\nu = \lambda$ and 0 unless $\nu \leq \lambda$. So (i) follows.

By the universal coefficient theorem we have a short exact sequence

$$0 \to \mathrm{Hom}_{\hat{\mathcal{C}}}(\hat{\nabla}(\lambda), \hat{\Delta}(\nu)) \otimes_{\hat{k}} k \to \mathrm{Hom}_{\mathcal{C}}(\nabla_\zeta(\lambda), \Delta_\zeta(\nu)) \to \mathrm{Tor}_1^{\hat{k}}(\mathrm{Ext}^1_{\hat{\mathcal{C}}}(\hat{\nabla}(\lambda), \hat{\Delta}(\nu)), k) \to 0.$$

For $\nu \not\leq \lambda$ the middle term in this sequence is 0 by i). Hence $\mathrm{Tor}_1^{\hat{k}}(\mathrm{Ext}^1_{\hat{\mathcal{C}}}(\hat{\nabla}(\lambda), \hat{\Delta}(\nu)), k) = 0$ and since $\mathrm{Ext}^1_{\hat{\mathcal{C}}}(\hat{\nabla}(\lambda), \hat{\Delta}(\nu))$ is a torsion module, it must be 0. If $\nu = \lambda$ then the two first terms in the sequence are both equal to k and we get the same conclusion.

Remark . i) *As the dual of $\nabla_\zeta(\lambda)$ is $\Delta_\zeta(\lambda)$, it follows from this proposition that $\mathrm{Hom}_{\mathcal{C}}(\nabla_\zeta(\lambda), \Delta_\zeta(\nu)) = \delta_{\lambda,\nu} k$ and $\mathrm{Ext}^1_{\hat{\mathcal{C}}}(\hat{\nabla}(\lambda), \hat{\Delta}(\nu)) = 0$ if both λ and ν are in $(p-1)\rho + X^+$.*

ii) *The proposition fails without the assumption $\lambda \in (p-1)\rho + X^+$. In fact, take $\lambda = 0$ and let $\nu = 2(p-1)\rho$. Then $\nabla_\zeta(\lambda) = k = \mathrm{soc}(\Delta_\zeta(\nu))$ and clearly $\mathrm{Hom}_{\mathcal{C}}(\nabla_\zeta(\lambda), \Delta_\zeta(\nu)) = k$. The proof of (ii) shows that also $\mathrm{Ext}^1_{\hat{\mathcal{C}}}(\hat{\nabla}(\lambda), \hat{\Delta}(\nu)) \neq 0$. If $p \geq h$, we can find many other counter examples: Take for instance λ in the bottom dominant alcove and let ν be its mirror image in the next alcove.*

iii) *The analogue of (ii) is false in $\mathcal{C}$. For instance one may check by direct computation that for type A_2 one has $\mathrm{Ext}^1_{\mathcal{C}}(\nabla_\zeta(p\rho), \Delta_\zeta(p\rho)) = k^2$.*

6.7. Let $\hat{Q} \in \hat{\mathcal{C}}$ be a tilting module. Using 6.6(1) and Proposition 6.6 we get via the same arguments as in 3.5

Lemma . *For each $\lambda \in (p-1)\rho+X^+$ there exist bases $\{\psi_1, \psi_2, \cdots, \psi_n\}$ of* $\operatorname{Hom}_{\hat{\mathcal{C}}}(\hat{Q}, \hat{\nabla}(\lambda))$ *and $\{\psi'_1, \psi'_2, \cdots, \psi'_n\}$ of* $\operatorname{Hom}_{\hat{\mathcal{C}}}(\hat{\nabla}(\lambda), \hat{Q})$ *such that*

$$\psi_i \circ \psi'_j = \phi_p^{N(\lambda)} \delta_{i,j} \operatorname{id}_{\hat{\nabla}(\lambda)} .$$

Remark . *This lemma also fails without the assumption $\lambda \in (p-1)\rho + X^+$. In general the best we can say is that the image of the natural pairing*

$$\begin{array}{ccc} \operatorname{Hom}_{\hat{\mathcal{C}}}(\hat{\nabla}(\lambda), \hat{Q}) \times \operatorname{Hom}_{\hat{\mathcal{C}}}(\hat{Q}, \hat{\nabla}(\lambda)) & \longrightarrow & \operatorname{End}_{\hat{\mathcal{C}}}(\hat{\nabla}(\lambda)) \simeq \hat{k} \\ (\hat{\varphi}, \hat{\psi}) & \longmapsto & \hat{\psi} \circ \hat{\varphi} \end{array}$$

is contained in $\phi_p^{N''(\lambda)} \hat{k}$.

Note, however, that the image of the pairing may be strictly less than $\phi_p^{N''(\lambda)} \hat{k}$. Take for instance λ in the bottom dominant alcove. Then we have equality only if $\hat{Q}$ has a component equal to $\hat{T}_\zeta(\lambda)$.

6.8. Let again $\hat{Q} \in \hat{\mathcal{C}}$ be a tilting module and let $\lambda \in X^+$. Recall [An3] that we have a filtration of $\hat{F}_\lambda(Q) = \operatorname{Hom}_{\hat{\mathcal{C}}}(\hat{\Delta}(\lambda), \hat{Q})$ defined by

$$\hat{F}_\lambda(Q)^j = \{\varphi \in \hat{F}_\lambda(Q) \mid \psi \circ \varphi \in \phi_p^j \hat{k} c_\lambda \text{ for all } \psi \in \operatorname{Hom}_{\hat{\mathcal{C}}}(\hat{Q}, \hat{\nabla}(\lambda))\}$$

In analogy with 3.6 we also have a filtration defined by

$$'\hat{F}_\lambda(Q)^j = \{\varphi \in \hat{F}_\lambda(Q) \mid \varphi \circ c''_\lambda \in \phi_p^j \operatorname{Hom}_{\mathcal{C}}(\hat{\nabla}(\lambda), \hat{Q})\}.$$

Proposition . i) *$\hat{F}_\lambda(Q)^j \supseteq {}'\hat{F}_\lambda(Q)^j$ for all $j \in \mathbb{N}$.*
ii) *If $\lambda \in (p-1)\rho + X^+$, then $\hat{F}_\lambda(Q)^j = {}'\hat{F}_\lambda(Q)^j$ for all $j \in \mathbb{N}$.*

Proof: i) Let $\varphi \in {}'\hat{F}_\lambda(Q)^j$ and $\psi \in \operatorname{Hom}_{\hat{\mathcal{C}}}(\hat{Q}, \hat{\nabla}(\lambda))$. Then Remark 6.7 implies that $\psi \circ \varphi \circ c''_\lambda$ is divisible by $\phi_p^{N''(\lambda)+j}$. On the other hand (see 6.5), $\psi \circ \varphi = a c_\lambda$ for some $a \in \hat{k}$. By 6.5(1) we conclude that a is divisible by ϕ_p^j, i.e., $\varphi \in \hat{F}_\lambda(Q)^j$.

ii) In this case Lemma 6.7 applies and we may argue as in 3.6.

6.9. Let us now compare with the Jantzen filtration of Weyl modules. Suppose Q is a tilting module in $\mathcal{C}$ and let $\hat{Q}$ be a lift of Q to $\hat{\mathcal{C}}$. Let $\lambda \in X^+$ and consider $\varphi \in {}'\hat{F}_\lambda(Q)^j$. Then $\varphi \circ c''_\lambda = \phi_p^j \psi$ for some $\psi \in \operatorname{Hom}_{\hat{\mathcal{C}}}(\hat{\nabla}(\lambda), \hat{Q})$. If $x \in \hat{\Delta}(\lambda)^{N''(\lambda)+1-j}$ then we have (see 6.5 (1))

$$\phi_p^{N''(\lambda)} x = c''_\lambda \circ c_\lambda(x) = \phi_p^{N''(\lambda)+1-j} c''_\lambda(y)$$

for some $y \in \hat{\nabla}(\lambda)$. It follows that

$$\phi_p^{N''(\lambda)} \varphi(x) = \phi_p^{N''(\lambda)+1-j} \varphi \circ c''_\lambda(y) = \phi_p^{N''(\lambda)+1} \psi(y),$$

i.e. $\varphi(x) \in \phi_p \hat{Q}$. We have therefore proved

Proposition . $'F_\lambda(Q)^j \subseteq \mathrm{Hom}_{\mathcal{C}}(\Delta_\zeta(\lambda)/\Delta_\zeta(\lambda)^{N''(\lambda)+1-j}, Q)$ *for all* j.

6.10. Suppose $Q \in \mathcal{C}$ is projective. According to [An2] this is the case if and only if Q is a direct sum of $T_\zeta(\nu)$ with $\nu \in (p-1)\rho + X^+$.

Proposition . *Assume* $p \geq h$. *Let* $\lambda \in (p-1)\rho + X^+$. *Then*

$$\sum_{j\geq 1} \dim F_\lambda(Q)^j = \sum_{j \geq 1} \dim \mathrm{Hom}_{\mathcal{C}}(\Delta_\zeta(\lambda)/\Delta_\zeta(\lambda)^{N(\lambda)+1-j}, Q)$$

Proof: By the observation above we may assume that $Q = T_\zeta(\nu)$ for some $\nu \in (p-1)\rho + X^+$. Set $\xi = \hat{\nu}$. Then we have $[Q : \chi(\eta)] = [\chi(\eta) : L_\zeta(\xi)]$ for all $\eta \in X$ and the sum formula in [An4] shows that the left hand side in the lemma equals

$$-\sum_{\alpha \in R^+} \sum_m [\chi(\lambda - mp\alpha) : L_\zeta(\xi)].$$

The second sum runs over those $m \in \mathbb{Z}$ which satisfy $mp < 0$ or $mp > \langle \lambda + \rho, \alpha^\vee \rangle$.

On the other hand the sum formula (see [APW]) for the Jantzen filtration of $\Delta_\zeta(\lambda)$ shows that the right hand side in the lemma equals

$$N(\lambda)[\chi(\lambda) : L_\zeta(\xi)] + \sum_{\alpha\in R^+} \sum_m [\chi(\lambda - mp\alpha) : L_\zeta(\xi)].$$

Here the second sum is over those $m \in \mathbb{N}$ for which $mp < \langle \lambda + \rho, \alpha^\vee \rangle$. Hence we are done if we prove

$$\sum_{m \in \mathbb{Z}} [\chi(\lambda - mp\alpha) : L_\zeta(\xi)] = 0. \tag{1}$$

We claim that (1) actually holds for all $\lambda \in X$. To see this observe first that since there exists $w \in W$ such that $w(\alpha)$ is simple and $\chi(\lambda - mp\alpha) = (-1)^{l(w)}\chi(w.\lambda - mpw(\alpha))$ it is enough to consider the case where α is simple. Note also that if $\langle \lambda + \rho, \alpha^\vee \rangle \in p\mathbb{Z}$, then the terms in (1) cancel pairwise. So we assume from now on that $\langle \lambda + \rho, \alpha^\vee \rangle \notin p\mathbb{Z}$.

For each $m \in \mathbb{N}$ we let $\lambda_m \in \{\lambda, s_\alpha.\lambda\} + p\mathbb{Z}\alpha$ be determined by $mp < \langle \lambda_m + \rho, \alpha^\vee \rangle < (m+1)p$. Then (1) is equivalent to

$$\sum_{m \geq 0} (-1)^m [\chi(\lambda_m) : L_\zeta(\xi)] = 0. \tag{2}$$

Assume now that $\alpha = \alpha_i$ and let us use the notation from 6.4. We have an exact sequence of $U_\zeta(i)$-modules

$$\cdots \to H_i^0(\lambda_2) \to H_i^0(\lambda_1) \to H_i^0(\lambda_0) \to 0. \tag{3}$$

Let for each $n \geq 0$ $L_i(\lambda_n) = \ker(H_i^0(\lambda_n) \to H_i^0(\lambda_{n-1}))$. (This is in fact the simple $U_\zeta(i)$-module with highest weight λ_n). Since $\chi(\lambda_m) = \chi(H_i^0(\lambda_m))$ for all m, we get from (3)

$$(4) \qquad \sum_{m=0}^{n}(-1)^m\chi(\lambda_m) = (-1)^n\chi(L_i(\lambda_n)).$$

Pick now n so big that if $m > n$ then $[H^j_\zeta(\lambda_m) : L_\zeta(\xi)] = 0$ for all j. Then $\sum_{m\geq 0}(-1)^m[\chi(\lambda_m) : L_\zeta(\xi)] = \sum_{m=0}^n(-1)^m[\chi(\lambda_m) : L_\zeta(\xi)]$ and by (4) this equals $(-1)^n[\chi(L_i(\lambda_n)) : L_\zeta(\xi)]$. If this number is non-zero, then there exists j such that $[H^j_\zeta(L_i(\lambda_n)) : L_\zeta(\xi)] \neq 0$. But the short exact sequence

$$0 \to L_i(\lambda_{n+1}) \to H^0_i(\lambda_{n+1}) \to L_i(\lambda_n) \to 0$$

then implies that also $[H^{j+1}_\zeta(L_i(\lambda_{n+1})) : L_\zeta(\xi)] \neq 0$. Continuing in this way we get $[H^{j+r}_\zeta(L_i(\lambda_{n+r})) : L_\zeta(\xi)] \neq 0$ for all $r \geq 0$. This contradicts the fact that $H^s_\zeta = 0$ for $s > N$ and we are done.

6.11. Putting together Propositions 6.8.ii), 6.9 and 6.10 we get the following analogue of Theorem 3.8

Corollary . *Assume $p \geq h$ and let $\lambda \in (p-1)\rho + X^+$. Then for any projective module $Q \in \mathcal{C}$ we have*

$$F_\lambda(Q)^j = \mathrm{Hom}_{\mathcal{C}}(\Delta_\zeta(\lambda)/\Delta_\zeta(\lambda)^{N(\lambda)+1-j}, Q)$$

for all $j \in \mathbb{N}$.

6.12. We now turn to the representation theory of G (see Section 1). So k is an algebraically closed field of characteristic $p > 0$ and $\mathcal{C} = \mathcal{C}_k$ is the category of finite dimensional G-modules. For $\lambda \in X^+$ we have a Weyl module $\Delta(\lambda)$, an induced module $\nabla(\lambda)$, and an indecomposable tilting module $T(\lambda)$ in $\mathcal{C}$, all having highest weight λ.

Set $\hat{k} = \mathbb{Z}_{(p)}$, the localization of $\mathbb{Z}$ at the prime p and let $\hat{G}$ denote the Chevalley group over $\hat{k}$ corresponding to G. Then $\hat{\mathcal{C}}$ is the category of $\hat{G}$-modules which are finitely generated over $\hat{k}$. Just as in 6.2 we have "lifts" $\hat{\Delta}(\lambda), \hat{\nabla}(\lambda)$ and $\hat{T}(\lambda)$ in $\hat{\mathcal{C}}$ of the above modules in $\mathcal{C}$. Moreover, we define in analogy with 6.3 for each $w \in W$ the module

$$\hat{\nabla}^w(\lambda) = \hat{H}^{l(w)}(w.\lambda)/\hat{H}^{l(w)}(w.\lambda)_{\mathrm{tor}} \in \hat{\mathcal{C}}.$$

Here $\hat{H}^j(\nu)$ may be thought of as the j-th cohomology of the line bundle associated with $\nu \in X$ on the flag variety for $\hat{G}$.

6.13. Let $\mathcal{C}_2$ be the subcategory in $\mathcal{C}$ consisting of those modules whose weights ν satisfy $\langle \nu + \rho, \beta^\vee\rangle \leq p^2$ for all $\beta \in R^+$. Define $\hat{\mathcal{C}}_2$ as the analogous subcategory of $\hat{\mathcal{C}}$. Then for $p \geq 2(h-1)$ all the results in 6.4-11 carry over to $\mathcal{C}_2$ and $\hat{\mathcal{C}}_2$. We simply have to replace ϕ_p everywhere by p.

Remark . *It is of course possible (and interesting) to study the tilting modules in* $\mathcal{C}$ *which are not in* $\mathcal{C}_2$ *by using the methods in this paper. However, in that case higher powers of* p *will occur in the analogue of Lemma* 6.4. *This complicates matters considerably and we shall not pursue this here.*

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