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REMARKS ON DETERMINANT LINE BUNDLES, CHERN-SIMONS FORMS AND INVARIANTS

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REMARKS ON DETERMINANT LINE BUNDLES, CHERN-SIMONS FORMS AND INVARIANTS[1]

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Abstract. We study generalized determinant line bundles for families of principal bundles and connections. We explore the connections of this line bundle and give conditions for the uniqueness of such. Furthermore we construct for families of bundles and connections over manifolds with boundary, a generalized Chern-Simons invariant as a section of a determinant line bundle.

0. Introduction

Determinant line bundles were first constructed for families of Riemann surfaces by D. Quillen in [Q]. This was generalized to higher dimensions by D. Freed (see e.g. [F1], [F2]) and in Freed-Dai [DF] it was used to define a generalization of the Atiyah-Patodi-Singer η-invariant for families of Riemannian manifolds with boundary as a section of the inverse line bundle associated to the family of boundaries. All these constructions were analytical ones involving kernel/cokernels of differential operators.

In this paper we shall study a geometric construction of determinant line bundles going back to T. R. Ramadas, I. M. Singer and J. Weitsman (see [RSW]) for the case of families of connections in trivial $SU(2)$ bundles over closed surfaces and for more general families of principal bundles to the work by L. Bonora, P. Cotta-Ramusino, M. Rinaldi and J. Stasheff (see [BCRS]; see also Brylinski [B]).

The construction in section 1 requires a smooth, closed, even-dimensional manifold X and a family of principal bundles over X, each with a connection, which constitute a fibre bundle over Z. Together with an invariant polynomial this enables us to calculate transition functions of a line bundle $\mathcal{L}$ over Z. In section 2 we explore the connections of this line bundle. If the family A_z of connections can be extended into a connection in the Z-direction also, there is a canonical connection of the line bundle. We describe these connections and determine when they are independent of the extensions.

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Section 3 deals with generalizations of the Chern-Simons invariant. This was originally defined by S. S. Chern and J. Simons in [Cher-S] (see also Cheeger-Simons [Chee-S] or [DK]) for a single connection in a principal bundle over a closed manifold. In analogy with the construction by Freed and Dai of an η-invariant section of the determinant line bundle mentioned above, we define, for a family of bundles and connections over a family of manifolds with boundary, a natural section cs ("Chern-Simons section") of the inverse line bundle which is an exponentiated version of a generalized Chern-Simons invariant. Finally in Section 4 we make a calculation of the line bundle, its connection and Chern-Simons section in the simplest possible case of flat connections over genus g surfaces. It appears that the connection in this case is flat and that the section is parallel.

Notation. The sign convention in this paper has been chosen in accordance with Bott and Tu (see [BT]). If α is a form on X and β is a form on Y then integration over the fibre in $X \times Y \to X$ is defined as

$$\int_Y \alpha \wedge \beta = \left(\int_Y \beta \right) \alpha. \tag{0.1}$$

This implies that differentiation commutes with integration over the fibre, i.e. $d_X \int_Y \alpha \wedge \beta = \int_Y (d_X \alpha) \wedge \beta$.

1. Geometric Construction of a Determinant Line Bundle

First let us recall the construction of a line bundle as in [BCRS] for the following data.

Geometric Data 1.1.

(1) A smooth, closed, oriented manifold X of dimension $2k-2$
(2) A Lie group G, a principal G-bundle $\mathcal{P} \to X$ and an invariant polynomial $P \in I_0^k(G)$
(3) A fixed connection A_0 of $\mathcal{P}$
(4) A fibre bundle $\mathcal{P} \to E \to Z$, where each fibre has a connection and the transition functions are gauge transformations homotopic to the identity

A few explanatory remarks here would seem to be in order:

i) The set of invariant polynomials of degree k, that is, of the G-invariant symmetric, multilinear functions in k variables on the Lie algebra $\mathfrak{g}$, is denoted $I^k(G)$; the subset $I_0^k(G)$ of $I^k(G)$ consists of the polynomials whose image under the Chern-Weil homomorphism is an integral cohomology class. (See [Cher-S].)

ii) Note the following consequence of the geometric data 1.1. Let U_i and U_j be open subsets of Z over which E is trivial. Let $A_i(z)$ be the pull-back of the connection of the fibre E_z to $\mathcal{P}$ via the trivialization

$$U_i \times \mathcal{P} \to E|_{U_i} \tag{1.2}$$

and $A_j(z)$ correspondingly. If g_{ij} denotes the transition function, then on $U_i \cap U_j$

$$A_j(z) = A_i(z)^{g_{ij}(z)}$$

is the gauge transformed connection of $A_i(z)$ by $g_{ij}(z)$.

Let $\mathcal{G}$ denote the group of gauge transformations of $\mathcal{P}$. If $\mathcal{G}$ is not connected it must be replaced by the connected component of the identity.

Theorem 1.3. *The geometric data 1.1 define a complex line bundle $\mathcal{L} \to Z$ with a Hermitian metric.*

For the proof we need the following preparations: For the geometric data 1.1 and a set of trivializations $\varphi_i : U_i \times \mathcal{P} \to E|U_i$ with transition functions g_{ij} as above we wish to construct transition functions $\theta_{ij} : U_i \cap U_j \to U(1)$ to get a line bundle. Let

$$\tilde{g}_{ij} : U_i \cap U_j \times I \to \mathcal{G} \tag{1.4}$$

be a homotopy from g_{ij} to the identity such that $\tilde{g}_{ij}(z, 0) = \mathrm{id}$ and $\tilde{g}_{ij}(z, 1) = g_{ij}$. $\tilde{g}_{ij}$ can be considered as a gauge transformation in the bundle $\mathcal{P} \times I \to X \times I$.

In general, for two connections A_0 and A_1 in a principal bundle $\mathbf{P} \to M$ over a manifold M, let $\mathbf{A}$ be the convex combination

$$\mathbf{A}(p, s) = (1-s)A_0(p) + sA_1(p), \qquad p \in \mathbf{P},\ s \in [0,1] \tag{1.5}$$

This is a connection in $\mathbf{P} \times I \to M \times I$. Let $P \in I_0^k(G)$ be an invariant polynomial of degree k and let $F_{\mathbf{A}}$ be the curvature of the connection $\mathbf{A}$. Then $P(F_{\mathbf{A}})$ is horizontal and the lift of a basic form, which will also be denoted by $P(F_{\mathbf{A}})$. We define the differential $(2k-1)$-form on M:

Definition 1.6.

$$TP(A_0, A_1) = 2\pi \int_{s=0}^{1} P(F_{\mathbf{A}}).$$

Note that

$$TP(A_1, A_0) = -TP(A_0, A_1). \tag{1.7}$$

In the following we apply this to the bundle $\mathcal{P} \times I \to X \times I$.

Lemma 1.8. *The form $TP(A_i, A_i^{\tilde{g}_{ij}})$ is closed.*

Proof. Let $i_0(x,t) = (x,t,0)$ and $i_1(x,t) = (x,t,1)$ be the inclusions $i_\nu : X \times I \to X \times I \times I$, $\nu = 0, 1$, and let $A_{ij}(x,t,s) = (1-s)A_i(x) + sA_i(x)^{\tilde{g}_{ij}(x,t)}$. We have the equation†

$$d\int_{s=0}^{1} P(F_{A_{ij}}) - \int_{s=0}^{1} dP(F_{A_{ij}}) = i_0^* P(F_{A_{ij}}) - i_1^* P(F_{A_{ij}}).$$

†The signs may look unusual but are in agreement with the sign convention of [BT].

P is an invariant polynomial applied to a curvature form, so $P(F_{A_{ij}})$ is closed. Hence

$$\begin{aligned} dTP(A_i, A_i^{\tilde{g}_{ij}}) &= i_0^* P(F_{A_{ij}}) - i_1^* P(F_{A_{ij}}) \\ &= P(F_{A_i}) - P(g_{ij}^* F_{A_i}) \\ &= P(F_{A_i}) - g_{ij}^* P(F_{A_i}) \\ &= 0, \end{aligned}$$

since $P(F_{A_i})$ is a basic form and the gauge transformation g_{ij} acts as the identity on the base. $\square$

Define the function θ_{ij} on $U_i \cap U_j$ relative to a choice of a fixed connection A_0 in $\mathcal{P} \to X$.

Definition 1.9.

$$\theta_{ij}(z) = \exp\left(i \int_{X \times I} TP(A_0, A_i^{\tilde{g}_{ij}(z,t)}(z,x)) \right).$$

In future calculations we shall omit the parameters z, x, and t.

Lemma 1.10. *The function θ_{ij} is independent of the homotopy $\tilde{g}_{ij}$.*

Proof. Let $\tilde{g}_{ij}^1$ and $\tilde{g}_{ij}^2$ be two homotopies of g_{ij} to the identity and let θ_{ij}^1 and θ_{ij}^2 be calculated by means of these two respectively. Then

$$\begin{aligned} (\theta_{ij}^1)^{-1} \theta_{ij}^2 &= \exp i \left(- \int_{X \times I} TP(A_0, A_i^{\tilde{g}_{ij}^1}) + \int_{X \times I} TP(A_0, A_i^{\tilde{g}_{ij}^2}) \right) \\ &= \exp\left(i \int_{X \times S^1} TP(A_0, A_i^g) \right) \\ &= \exp\left(2\pi i \int_{X \times S^1} \int_{s=0}^1 P(F_{(1-s)A_0 + sA_i^g}) \right). \end{aligned}$$

Here, g is the "homotopy" on S^1 made up of the two contributions $\tilde{g}_{ij}^1$ and $\tilde{g}_{ij}^2$. Then g is a gauge transformation and hence an automorphism of the bundle $\mathcal{P} \times S^1 \to X \times S^1$. Indeed,

$$\begin{array}{ccc} \mathcal{P} \times S^1 & \xrightarrow{g^{-1}} & \mathcal{P} \times S^1 \\ \downarrow & & \downarrow \\ X \times S^1 & \xrightarrow{\mathrm{id}} & X \times S^1 \end{array}$$

commutes. Now consider the mapping torus of $\mathcal{P} \times S^1$ given by

$$T = (\mathcal{P} \times S^1) \times I / \sim$$

where we identify $((p,t),0) \sim (g^{-1} \cdot (p,t), 1)$. T is diffeomorphic to $\mathcal{P} \times S^1 \times S^1$. The convex combination $(1-s)A_0 + sA_i^g$ is a connection in the bundle $\mathcal{P} \times S^1 \times I$, but by the construction of the mapping torus, it becomes a connection in T. In fact, the connection in $\mathcal{P} \times S^1 \times I$ at $((p,t),0)$ is A_0, and the connection at $(g^{-1} \cdot (p,t), 1)$ is A_i^g. The connection at $((p,t),0)$ should equal the pull-back along the identification map of the connection at $(g^{-1} \cdot (p,t), 1)$. But the pull-back of A_0^g along $g^{-1}\cdot$ is just $(g^{-1})^* A_0^g = A_0^{gg^{-1}} = A_0$. With this in mind we can make a replacement of the integral from above: If we let $\tilde{A}$ be the connection on T obtained from $(1-s)A_0 + sA_i^g$ under the mapping torus construction,

$$\int_{X \times S^1} \int_{s=0}^{1} P(F_{(1-s)A_0 + sA_i^g}) = \int_T P(F_{\tilde{A}}). \tag{1.11}$$

This is the integral of an invariant polynomial $P \in I_0^k(G)$ over a closed manifold, and the result is an integer. Multiplying by $2\pi i$ and taking exp concludes the proof. □

Lemma 1.12. *Let U_i, U_j and U_k be three open subsets of Z with nonempty intersection. Then θ_{ij}, θ_{jk} and θ_{ik} satisfy the cocycle condition, $\theta_{ij}\theta_{jk} = \theta_{ik}$.*

Proof. For $z \in U_i \cap U_j \cap U_k$,

$$\theta_{ij}\theta_{jk} = \exp i \left(\int_{X \times I} TP(A_0, A_i^{\tilde{g}_{ij}}) + \int_{X \times I} TP(A_0, A_j^{\tilde{g}_{jk}}) \right).$$

Since $A_i^{\tilde{g}_{ij}(z,1)} = A_j$ and $A_j^{\tilde{g}_{jk}(z,0)} = A_j$, the integrands agree at the endpoints of the intervals, and after a slight reparametrization,

$$\tilde{g}_{ik}(z,t) = \begin{cases} \tilde{g}_{ij}(z,2t) & \text{if } 0 \le t \le \frac{1}{2} \\ g_{ij}\tilde{g}_{jk}(z, 2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

is a homotopy from the identity to g_{jk}. Hence the sum of the two integrals above is

$$\int_{X \times I} TP(A_0, A_i^{\tilde{g}_{ik}})$$

and the result follows by lemma 1.8. □

Proof of Theorem 1.3.. The functions in definition 1.9 are well-defined by lemma 1.10 and satisfy the cocycle condition by lemma 1.12. Hence the $\{\theta_{ij}\}$'s are transition functions of a line bundle $\mathcal{L}$. Note that the transition functions are $U(1)$-valued, and hence $\mathcal{L}$ can be equipped with a Hermitian metric. This concludes the proof of theorem 1.3. □

Proposition 1.13. *The isomorphism class of $\mathcal{L}$ is independent of the choice of A_0.*

Proof. For the proof of this we need to rewrite the transition functions θ_{ij}. Let Δ^2 be the two-simplex

$$\Delta^2 = \{s_0, s_1, s_2 \in \mathbb{R} \mid s_0 + s_1 + s_2 = 1,\ s_i \geq 0 \text{ for } i = 0, 1, 2\}.$$

For three connections A_0, A_1, and A_2 of the principal bundle $\mathcal{P} \to X$ the set of convex combinations $\{s_0 A_0 + s_1 A_1 + s_2 A_2 \mid (s_0, s_1, s_2) \in \Delta^2\}$ will be a connection of the bundle $\mathcal{P} \times \Delta^2 \to X \times \Delta^2$. Define the differential $(2k-2)$-form on $X^\dagger$

$$TP(A_0, A_1, A_2) = 2\pi \int_{\Delta^2} P\big(F_{s_0 A_0 + s_1 A_1 + s_2 A_2}\big).$$

An easy calculation shows that

$$dTP(A_0, A_1, A_2) = -\sum_{i=0}^{2} (-1)^i TP(.., \hat{A}_i, ..), \tag{1.14}$$

where $TP(.., \hat{A}_i, ..)$ is the $(2k-1)$-form from definition 1.6 with A_i omitted. Inserting the three connections A_0, A_i and $A_i^{\tilde{g}_{ij}}$ into dTP and integrating over $X \times I$ we get

$$\int_{X \times I} dTP\big(A_0, A_i, A_i^{\tilde{g}_{ij}}\big) = \int_{X \times I} TP\big(A_0, A_i^{\tilde{g}_{ij}}\big) - \int_{X \times I} TP\big(A_i, A_i^{\tilde{g}_{ij}}\big),$$

since $\int_{X \times I} TP(A_0, A_i) = 0$ from a dimension argument. On the other hand,

$$\int_{X \times I} dTP\big(A_0, A_i, A_i^{\tilde{g}_{ij}}\big) = \int_X TP\big(A_0, A_i, A_j\big),$$

and it follows that

$$\int_{X \times I} TP\big(A_0, A_i^{\tilde{g}_{ij}}\big) = \int_{X \times I} TP\big(A_i, A_i^{\tilde{g}_{ij}}\big) + \int_X TP\big(A_0, A_i, A_j\big). \tag{1.15}$$

Let Δ^3 denote the three-simplex and for four connections A_0, A_1, A_2, A_3 define the $(2k-3)$-form on X

$$TP(A_0, A_1, A_2, A_3) = 2\pi \int_{\Delta^3} P\big(F_{s_0 A_0 + s_1 A_1 + s_2 A_2 + s_3 A_3}\big)$$

†The form $P(F_{s_0 A_0 + s_1 A_1 + s_2 A_2})$ is horizontal and can be identified with a basic form.

as above, but in one dimension higher. We have

$$dTP(A_0, A_1, A_2, A_3) = -\sum_{i=0}^{3}(-1)^i TP(.., \hat{A}_i, ..), \tag{1.16}$$

where the TP's in the sum are the $(2k-2)$-forms from above. Also note that $\int_X dTP(A_0, A_1, A_2, A_3) = 0$. Now let A_0 and A_1 be two fixed connections of $\mathcal{P}$, and for $\nu = 0, 1$ let

$$\theta_{ij}^{\nu} = \exp i \int_{X\times I} TP(A_0, A_i^{\tilde{g}_{ij}}).$$

Rewriting the integrals as in (1.15) and applying (1.16) yields

$$\begin{aligned}
\theta_{ij}^0 (\theta_{ij}^1)^{-1} &= \exp i \left(\int_X TP(A_0, A_i, A_j) - \int_X TP(A_1, A_i, A_j) \right) \\
&= \exp i \left(\int_X TP(A_0, A_1, A_i) - \int_X TP(A_0, A_1, A_j) \right),
\end{aligned}$$

but this is a coboundary and hence the $\{\theta_{ij}^0\}$'s and the $\{\theta_{ij}^1\}$'s define isomorphic line bundles. □

Remark 1.17. Note that the proof provides explicit isomorphisms.

Proposition 1.18. *Let X be a closed surface, let $\mathcal{P} = X \times G$ be the product bundle and $P = -C_2 = -\frac{1}{8\pi^2}\operatorname{Tr}$, minus the second Chern polynomial. Then the transition functions θ_{ij} define the same line bundle as the one of Ramadas, Singer and Weitsman in* [RSW].

Proof. First recall that the line bundle of Ramadas, Singer and Weitsman is constructed by means of a 3-manifold Y that has X as boundary and by defining the Chern-Simons functional

$$CS(\bar{A}) = \frac{1}{4\pi}\int_Y \operatorname{Tr}\left(\iota^*(\bar{A}d\bar{A} - \frac{2}{3}\bar{A}\bar{A}\bar{A})\right) \qquad \bmod 2\pi\mathbb{Z}. \tag{1.19}$$

Then a cocycle is defined on $\mathcal{A}\times\mathcal{G}$, where $\mathcal{A}$ is the space of connections of $X \times SU(2)$ and $\mathcal{G}$ is the group of gauge transformations, by

$$\Theta(A, g) = \exp i(CS(\bar{A}^{\bar{g}}) - CS(\bar{A}))$$

where $\bar{A}$ is an extension of A into Y and $\bar{g}$ is an extension of g. This gives a line bundle on the manifold $\mathcal{A}_F^s/\mathcal{G}$, the set of flat, irreducible connections of $X \times SU(2)$ modulo $\mathcal{G}$. Given a covering $\{U_i\}$ and transition functions $\{g_{ij}\}$, the transition functions in the line bundle are given by $\Theta(A_i, g_{ij})$ and we shall show that this equals θ_{ij}.

Recalling that the curvature of the connection A is given by $F_A = A \wedge A + dA$ we get

$$
\begin{aligned}
TP\big(A_0, A_i^{\tilde{g}_{ij}}\big) &= -\frac{i}{4\pi} \int_{s=0}^{1} \mathrm{Tr}\big(F^2_{sA_0+(1-s)A_i^{\tilde{g}_{ij}}}\big) \\
&= -\frac{i}{4\pi} \int_{s=0}^{1} \mathrm{Tr}\big((sA_0 + (1-s)A_i^{\tilde{g}_{ij}})^2 + d(sA_0 + (1-s)A_i^{\tilde{g}_{ij}})\big)^2
\end{aligned}
$$

Calculation of this integral yields

$$
-\big(A_i^{\tilde{g}_{ij}} \wedge dA_i^{\tilde{g}_{ij}} + \frac{2}{3}\big(A_i^{\tilde{g}_{ij}}\big)^3\big) + \big(A_0 \wedge dA_0 + \frac{2}{3}A_0^3\big) - d(A_0 \wedge A_i^{\tilde{g}_{ij}}),
$$

and when integrated over $X \times I$ the terms in parentheses vanish, as does the term $d(A_0 \wedge A_i^{\tilde{g}_{ij}})$. In fact we get the terms $A_0 \wedge A_i$ and $A_0 \wedge A_j$ with opposite signs from the two ends of the cylinder, and since A_i and A_j are gauge equivalent the two terms cancel out each other when pulled back to the base. What is left is then

$$
\theta_{ij} = \exp \frac{1}{4\pi} \int_{X \times I} \mathrm{Tr}\big(A_i^{\tilde{g}_{ij}} \wedge dA_i^{\tilde{g}_{ij}} + \frac{2}{3}(A_i^{\tilde{g}_{ij}})^3\big). \tag{1.20}
$$

Now let W be the closed manifold $Y \cup (X \times I) \cup (-Y)$, where $-Y$ denotes Y with the opposite orientation. The integrands of (1.19) and (1.20) agree at the boundaries of the constituents of W, and letting B denote the connection

$$
B = \begin{cases} \bar{A}_i & \text{on } Y \\ A_i^{\tilde{g}_{ij}} & \text{on } X \times I \\ \bar{A}_j & \text{on } -Y \end{cases},
$$

we have

$$
\Theta(A_i, g_{ij})^{-1}\theta_{ij} = \exp i \left(\frac{1}{4\pi} \int_W \mathrm{Tr}\left(\iota^*(BdB + \frac{2}{3}B^3)\right)\right).
$$

This contains an integral of a Chern-Simons form over a closed manifold. Hence the contents of the parentheses is an integer multiple of 2π, and the whole expression equals 1, which completes the proof. □

In [RSW], Ramadas, Singer, and Weitsman show that the line bundle $\mathcal{L}$ defined by the transition functions in definition 1.9 is isomorphic to the Quillen determinant line bundle $\mathcal{L}_D$ that arises from the family $\{\bar{\partial}_A | A \in \mathcal{A}_F^s\}$.

Remark 1.21. A slightly more general version of the line bundle is obtained if we consider *two* fibre bundles E and F as in the geometric data 1.1 with families of connections A_z and B_z respectively, both with local trivializations as in (1.2) and with homotopies $\tilde{g}_{ij}^A$ and $\tilde{g}_{ij}^B$ respectively as in (1.4). Define transition functions $\theta_{ij}^{AB} = \exp i \int_{X\times I} TP\big(B_i^{\tilde{g}_{ij}^B}, A_i^{\tilde{g}_{ij}^A}\big)$. It is an easy calculation to show that θ_{ij}^{AB} differ from the product $\theta_{ij}^A(\theta_{ij}^B)^{-1}$ by a coboundary and hence define the same line bundle. Here θ_{ij}^A and θ_{ij}^B denote the transition functions of the A-family and the B-family respectively. In other words, given the two families A and B we get a line bundle $\mathcal{L}^{AB}$. If the B-family is constant and equal to A_0 we get the same line bundle as the one defined by the transition functions in definition 1.9. If H is a third fibre bundle with connections C we get three relative line bundles, $\mathcal{L}^{AB}$, $\mathcal{L}^{BC}$, and $\mathcal{L}^{AC}$, and it is not difficult to show that there is an isomorphism $\mathcal{L}^{AB} \otimes \mathcal{L}^{BC} \cong \mathcal{L}^{AC}$.

2. Connections in the Line Bundle

In this section we shall describe connections of the line bundle $\mathcal{L}$ constructed in the previous section.

Theorem 2.1. *Given the geometric data* 1.1*, let A be a connection of the principal bundle $E \to Z \times X$ which fibre-wise restricts to the connection of each fibre of the bundle $E \to Z$. Then the induced line bundle $\mathcal{L}$ with transition functions θ_{ij} from definition 1.9 has a canonical Hermitian connection whose curvature is*

$$2\pi i \int_X P(F_A).$$

Remark 2.2. Thus, for a connection in $\mathcal{L}$ we need an extension of the family $\{A(z)\}$ to a connection also in the Z direction. However, corollary 2.10 below gives conditions (e.g. when $A(z)$ is flat for all $z \in Z$) insuring the connection to be independent of choice of extension. Notice also that such an extension always exists. This is easily seen by considering the fibre bundle E as a principal G-bundle, which locally, has the form $G \to U_i \times \mathcal{P} \to U_i \times X$. Pulling back $A_i(z)$ via $U_i \times \mathcal{P} \to \mathcal{P}$ and using partition of unity yields a connection of $E \to Z \times X$.

In the following we shall see how a connection in $E \to Z \times X$ gives rise to a canonical connection of the line bundle $\mathcal{L}$. Again choose a fixed connection A_0 in $\mathcal{P} \to X$, and let θ_{ij} be the transition functions for $\mathcal{L}$ given by definition 1.9. The local connection one-forms are now given as follows.

Definition 2.3. Let ω_i be the 1-form on $U_i \subseteq Z$

$$\omega_i = -i \int_X TP(A_0, A_i).$$

Lemma 2.4.

$$\theta_{ij}^{-1} d\theta_{ij} = i\left(-\int_X TP(A_0, A_i) + \int_X TP(A_0, A_j)\right).$$

Proof. By (1.15),

$$\theta_{ij} = \exp i\left(\int_{X\times I} TP(A_i, A_i^{\tilde{g}_{ij}}) + \int_X TP(A_0, A_i, A_j)\right),$$

where A_i and A_j are the pull-backs of the connections in $E_{|U_i} \to U_i$ and $E_{|U_j} \to U_j$. It is obvious that

$$\begin{aligned} d\theta_{ij} &= d\exp i\left(\int_{X\times I} TP(A_i, A_i^{\tilde{g}_{ij}}) + \int_X TP(A_0, A_i, A_j)\right) \\ &= i\theta_{ij} d\left(\int_{X\times I} TP(A_i, A_i^{\tilde{g}_{ij}}) + \int_X TP(A_0, A_i, A_j)\right) \end{aligned}$$

and so

$$\theta_{ij}^{-1} d\theta_{ij} = i\left(d\int_{X\times I} TP(A_i, A_i^{\tilde{g}_{ij}}) + d\int_X TP(A_0, A_i, A_j)\right). \tag{2.5}$$

Note that this is a differential form on $U_i \cap U_j$ and of course depends on $z \in U_i \cap U_j$. The two terms are treated separately. First,

$$d_Z \int_{X\times I} TP(A_i, A_i^{\tilde{g}_{ij}}) = \int_{X\times I} dTP(A_i, A_i^{\tilde{g}_{ij}}) - \int_{X\times I} d_{X\times I} TP(A_i, A_i^{\tilde{g}_{ij}}).$$

The integral $\int_{X\times I} dTP(A_i, A_i^{\tilde{g}_{ij}})$ is zero by lemma 1.8. The second term evaluates to

$$\int_{X\times I} d_{X\times I} TP(A_i, A_i^{\tilde{g}_{ij}}) = -\int_X TP(A_i, A_j).$$

The second integral in (2.5) is treated in the same way, i.e.

$$d_Z \int_X TP(A_0, A_i, A_j) = \int_X dTP(A_0, A_i, A_j) - \int_X d_X TP(A_0, A_i, A_j).$$

Stokes' theorem shows that the second term in this expression is zero, and by (1.14),

$$\int_X dTP(A_0, A_i, A_j) = \int_X \left(-TP(A_0, A_i) + TP(A_0, A_j) - TP(A_i, A_j)\right).$$

This leads to

$$\theta_{ij}^{-1} d\theta_{ij} = i\left(-\int_X TP(A_0, A_i) + \int_X TP(A_0, A_j)\right).$$

□

Note that ω_i is purely imaginary, and hence the connection is Hermitian.

Next, we calculate the curvature of ω_i. Since $\mathcal{L}$ is a line bundle, $\omega \wedge \omega = 0$, and the curvature of ω is just $d\omega$. It suffices to show that

$$d\omega_i = 2\pi i \int_X P(F_{A_i}).$$

This is a direct calculation. According to the sign convention, integration along the fibre commutes with the differential d_Z.

$$\begin{aligned}
d\omega_i &= -id \int_X TP(A_0, A_i) \\
&= -2\pi i d_Z \int_X \int_{s=0}^1 P(F_{(1-s)A_0+sA_i}) \\
&= -2\pi i \int_X \int_{s=0}^1 d_Z P(F_{(1-s)A_0+sA_i})) \\
&= -2\pi i \int_X \int_{s=0}^1 (d - d_X - d_s) P(F_{(1-s)A_0+sA_i}) \\
&= -2\pi i \int_X \int_{s=0}^1 dP(F_{(1-s)A_0+sA_i}) + 2\pi i \int_X d_X \int_{s=0}^1 P(F_{(1-s)A_0+sA_i}) \\
&\quad + 2\pi i \int_X \int_{s=0}^1 d_s P(F_{(1-s)A_0+sA_i}) \\
&= 2\pi i \int_X P(F_{A_i}) - 2\pi i \int_X P(F_{A_0}) \\
&= 2\pi i \int_X P(F_{A_i}),
\end{aligned}$$

since the terms containing d and d_X vanish; the form $2\pi i \int_X P(F_{A_0})$ vanishes, since it is independent of $z \in Z$.

This concludes the proof of theorem 2.1. □

We shall now investigate how the connection of the line bundle depends on the connection A of the principal bundle $E \to Z \times X$. Given a connection A^E in E, A^E can be written locally as

$$A_i^E = A_i + B_i, \tag{2.6}$$

where A_i contains all terms involving derivations in $\mathcal{P}$-direction (dp's) and B_i contains all terms involving derivations in Z-direction (dz's). If two different connections A_1^E and A_2^E in E induce the same family $\{A_i(z)\}_{z\in U_i}$ in $\mathcal{P}$ then, locally

$$A_{1,i}^E = A_i + B_{1,i}, \qquad A_{2,i}^E = A_i + B_{2,i} \tag{2.7}$$

because both $A_{1,i}^E$ and $A_{2,i}^E$ restrict to $A_i(z)$ for fixed z.

Theorem 2.8. *Let A_1 and A_2 be two connections of the bundle $E \to Z \times X$. Let ω_1 and ω_2 be two connections in the associated line bundle as defined in definition 2.3. Assume that both A_1 and A_2 restrict to $A(z)$ for each $z \in Z$. Then*

$$\omega_2 - \omega_1 = -k \int_X P(F_A^{k-1} \wedge \beta).$$

Here k is the degree of P and $\beta = A_2 - A_1$ is a horizontal 1-form in $E \to Z$ so that the integral only involves the curvatures $F_{A(z)}$ along the fibres.

Proof. Consider a subset $U_i \subseteq Z$ such that the local considerations from above apply, i.e. ω_i can be calculated explicitly by the expression in definition 2.3. In the proof the index i is left out. The first step is to show that $\omega_2 - \omega_1 = -i \int_X TP(A_1, A_2)$. It has already been shown that

$$d_Z \int_X TP(A_0, A_1, A_2) = -\int_X TP(A_0, A_1) + \int_X TP(A_0, A_2) - \int_X TP(A_1, A_2).$$

Hence it suffices to show that $d_Z \int_X TP(A_0, A_1, A_2) = 0$. Write $A_1 = A + B_1$ and $A_2 = A + B_2$ and consider

$$\int_X TP(A_0, A_1, A_2) = 2\pi \int_X \int_{\Delta^2} P(F_{s_0 A_0 + s_1 A_1 + s_2 A_2})$$

which is a function on Z and therefore can be calculated pointwise. Let $\mathbf{A} = s_0 A_0 + s_1 A_1 + s_2 A_2$ and write $F_{\mathbf{A}} = d\mathbf{A} + \mathbf{A} \wedge \mathbf{A}$ as

$$F_{\mathbf{A}} = \sigma + \phi + \psi, \tag{2.9}$$

where σ contains all terms involving ds's, ϕ contains all terms involving dx's only, and ψ contains all terms involving dz's. Now, a term in the integral

$$\int_X \int_{\Delta^2} P(F_{\mathbf{A}} \wedge \ldots \wedge F_{\mathbf{A}})$$

contributes with something non-vanishing when it contains exactly $(2k-2)$ dx's and two ds's, i.e. such terms contain no ψ's. Hence, one can replace both A_1 and A_2 by A and calculate:

$$d_Z \int_X TP(A_0, A_1, A_2) = d_Z \int_X TP(A_0, A, A) = 0.$$

This concludes the first step.

The second step deals with $\int_X TP(A_1, A_2)$. Recall that $TP(A_1, A_2)$ is given by $2\pi \int_{s=0}^{1} P(F_{(1-s)A_1+sA_2})$. Splitting up A_1 and A_2 as in (2.7) gives

$$(1-s)A_1 + sA_2 = A + B_1 + s(B_2 - B_1).$$

Then a calculation yields

$$F_{(1-s)A_1+sA_2} = F_A + ds \wedge \beta + \gamma,$$

where $\beta = B_2 - B_1$ and γ do not contain any terms involving ds or dx's. Since P is an invariant polynomial of degree k,

$$\begin{aligned}
\int_{s=0}^{1} P(F_{(1-s)A_1+sA_2}) &= \int_{s=0}^{1} P\big(F_{(1-s)A_1+sA_2} \wedge ... \wedge F_{(1-s)A_1+sA_2}\big) \\
&= \int_{s=0}^{1} P\big((F_A + \beta + \gamma) \wedge ... \wedge (F_A + \beta + \gamma)\big) \\
&= \int_{s=0}^{1} P\big(F_A^{k-1} \wedge \beta + F_A^{k-2} \wedge \beta \wedge F_A + ... + \beta \wedge F_A^{k-1}\big) \\
&\quad + \int_{s=0}^{1} R(s, x, z) \\
&= k \int_{s=0}^{1} P(F_A^{k-1} \wedge \beta) + \int_{s=0}^{1} R(s, x, z) \\
&= -kP(F_A^{k-1} \wedge \beta) + \int_{s=0}^{1} R(s, x, z),
\end{aligned}$$

where R contains all terms which are not on the form $F_A^{k-1} \wedge \beta$. Note that all the terms containing $2k-2$ derivations in the X-direction have been accounted for since such forms must contain $(k-1)$ times F_A. The form $R(s, x, z)$ contains no such terms and hence

$$\int_X \int_{s=0}^{1} R(s, x, z) = 0.$$

This concludes the proof. $\square$

Corollary 2.10. *Assume that $A_1 = A + B_1$ and $A_2 = A + B_2$ are two connections in E which agree along the fibres of $E \to Z$. If $F_{A_z}^{k-1} = 0$ for each z, then A_1 and A_2 induce the same connection ω_i in the associated line bundle. In particular this happens when each A_z is flat.* $\square$

Remark 2.11. This construction covers the one of D. Freed (see [F1]) for a family of Riemannian manifolds at least in the case of varying metrics on a fixed manifold

X. In this case let $\mathcal{P}_z = F(X)$ be the oriented orthogonal frame bundle of X with the Levi-Civita connection A_z of the metric of the fibre, and let P be the $\hat{A}$-polynomial. The curvature of the line bundle is $2\pi i \int_X \hat{A}(F_A)$.

Example 2.12. The case of [RSW]. Let X be a closed surface and $\mathcal{P} = X \times SU(2)$. To compare our connection and its curvature to the case in [RSW] consider ω as a 1-form on $\mathcal{A}$. Let $A \in \mathcal{A}$, and $\alpha \in T_A\mathcal{A}$. Let $\gamma : (-\epsilon, \epsilon) \to \mathcal{A}$ be a curve such that $\gamma(0) = A$, and $\gamma'(0) = \alpha$. Then

$$\omega(\alpha) = \omega\big(\gamma_*(\frac{d}{dt})\big) = \gamma^*\omega(\frac{d}{dt}).$$

Since $\mathcal{A}$ is an affine space we can let $\gamma(t) = A + t\alpha$. We wish to compare ω to the form $\hat{\omega}_{RSW}$ from [RSW] given by $\hat{\omega}_{\mathrm{RSW}}(\alpha) = \frac{i}{4\pi}\int_X \mathrm{Tr}(A \wedge \alpha)$. Write A as $A = A_0 + B$, where A_0 is a flat connection. Then

(1) $TP(A_0, A) = \frac{1}{4\pi}\mathrm{Tr}(Bd_{A_0}B + \frac{2}{3}B^3)$
(2) $\omega(\alpha) = \frac{i}{4\pi}\int_X \mathrm{Tr}(B \wedge \alpha)$,

where in this case $P = -C_2 = -\frac{1}{8\pi^2}\mathrm{Tr}$ is minus the second Chern polynomial. The curvature is obtained from this and yields

$$d\omega(\alpha, \beta) = \frac{i}{2\pi}\int_X \mathrm{Tr}(\alpha \wedge \beta)$$

in agreement with [RSW].

Remark 2.13. The connection one-form of the relative line bundle described in Remark (1.27) is given by $\omega_i^{AB} = -i\int_X TP(B_i, A_i)$.

3. The Chern-Simons Invariant

We shall now extend the definition of the Chern-Simons invariant to a family of bundles and connections over a family of odd-dimensional manifolds with boundary. In this situation the Chern-Simons invariant determines a section of the inverse line bundle $\mathcal{L}^{-1}$, where $\mathcal{L}$ is the line bundle constructed in section 1 for the family of boundaries. In the case of a single bundle $\bar{\mathcal{P}}$ with connection $\bar{A}$ over an odd-dimensional manifold Y with boundary X the Chern-Simons invariant of $\bar{A}$ must be defined relative to some "boundary conditions". For these we take once and for all a fixed manifold Y_0 with $\partial Y_0 = X$ together with a principal bundle $\bar{\mathcal{P}}_0 \to Y_0$ with connection $\bar{A}_{00}$ extending our background connection A_0 on $\mathcal{P}$ over X. In the special case of $\mathcal{P} = X \times G$ we can take $\bar{\mathcal{P}}_0 = Y_0 \times G$ and A_{00} the Maurer-Cartan connection. With these data we can now define the *relative* Chern-Simons invariant $cs(\bar{A}, \bar{A}_{00})$ for $P \in I_0^*(G)$ as follows. Consider the "glued" manifold $W = Y \cup (X \times I) \cup (-Y_0)$ with G-bundle $\bar{\mathcal{P}} \cup (X \times I \times G) \cup \bar{\mathcal{P}}_0$ and connection $\bar{B}$ given by

$$\bar{B} = \begin{cases} \bar{A} & \text{on } Y \\ (1-t)A + tA_0 & \text{on } X \times I \times G \\ \bar{A}_{00} & \text{on } -Y_0 \end{cases}$$

Then we put

$$cs(\bar{A}, \bar{A}_{00}) = \exp\left(2\pi i \langle S_P(\bar{B}), [W]\rangle\right), \tag{3.1}$$

where $S_P(\bar{B})$ is the secondary characteristic class for the connection $\bar{B}$ as defined by Cheeger-Chern-Simons [Cher-S] or [Chee-S] (see also [DK]).

Returning to the case of a family we thus have the following general setup with the above "boundary conditions" as point (5):

Geometric Data 3.2.

(1)-(4) as in the geometric data 1.1

(5) A smooth, compact, oriented, odd-dimensional manifold Y_0 with $\partial Y_0 = X$ and a principal G-bundle $\bar{\mathcal{P}}_0 \to Y_0$ which extends $\mathcal{P}$, i.e. $\bar{\mathcal{P}}_0|_X = \mathcal{P}$, and a connection $\bar{A}_{00}$ which extends A_0.

(6) A smooth, compact, oriented, odd-dimensional manifold Y with $\partial Y = X$ and a principal G-bundle $\bar{\mathcal{P}} \to Y$ which extends $\mathcal{P}$, i.e. $\bar{\mathcal{P}}|_X = \mathcal{P}$,

(7) A fibre bundle $\bar{\mathcal{P}} \to \bar{E} \to Z$, where each fibre has a connection $\bar{A}_z$ which extends A_z and such that the transition functions are gauge transformations homotopic to the identity

Theorem 3.3. *The geometric data 3.2 determine a global section cs of the inverse line bundle $\mathcal{L}^{-1} \to Z$. Furthermore, for an extension if $\bar{A}$ to a connection in the G-bundle $\bar{E} \to Z \times Y$,the covariant derivative of cs with respect to the connection of section 2 is*

$$\nabla_{\bar{A}}(cs) = 2\pi i \int_Y P(F_{\bar{A}}) \otimes cs.$$

Corollary 3.4. *Given the geometric data 3.2 the associated line bundle $\mathcal{L}$ is trivial.*

For the proof of theorem 3.3 we choose a covering $\{U_i\}$ of Z and local trivializations of $\bar{E}$

$$\bar{\varphi}_i : \bar{E}_{|U_i} \to U_i \times \bar{\mathcal{P}}. \tag{3.5}$$

This gives rise to a connection $\bar{A}_i(z) = \left(\bar{\varphi}_i^{-1}\right)^* \bar{A}_z$ in $\bar{\mathcal{P}}$ for each $z \in Z$. The trivializations give transition functions over $U_{i'} \cap U_{j'}$:

$$\bar{g}_{ij} : U_i \cap U_j \to \bar{\mathcal{G}}, \tag{3.6}$$

where $\bar{\mathcal{G}}$ is the group of gauge transformations of $\bar{\mathcal{P}}$. Of course these trivializations restrict to local trivializations of the boundary. Also choose a connection $\bar{A}_0$ in $\bar{P} \to Y$ extending A_0.

Define a section of the inverse line bundle $\mathcal{L}^{-1} = \mathcal{L}^*$ as follows. Over U_i, the section is defined by

$$cs_i(z) = cs(\bar{A}_0, \bar{A}_{00}) \cdot \exp i \left(- \int_Y TP(\bar{A}_0, \bar{A}_i) \right). \tag{3.7}$$

Lemma 3.8. *The local sections defined in (3.7) patch together to a global section of $\mathcal{L}^*$, independent of choice of $\bar{A}_0$.*

Proof. On $U_i \cap U_j$ the transition function θ_{ij} "from U_i to U_j" (cf. definition 1.9) is given by:

$$\theta_{ij}(z) = \exp i \left(\int_{X \times I} TP(A_i(z), A_i^{\tilde{g}_{ij}}(z)) + \int_X TP(A_0, A_i(z), A_j(z)) \right).$$

Hence the transition functions θ^*_{ij} of $\mathcal{L}^*$ are $\theta^*_{ij} = \theta_{ij}^{-1} = \theta_{ji}$, or

$$\theta^*_{ij} = \exp i \left(- \int_{X \times I} TP(A_i, A_i^{\tilde{g}_{ij}}) - \int_X TP(A_0, A_i, A_j) \right).$$

On $U_i \cap U_j$ we must show the compatibility condition $cs_j = cs_i \theta^*_{ij}$. To see this first consider $c_0^{-1} cs_i \theta^*_{ij}$, where $c_0 = cs(\bar{A}_0, \bar{A}_{00})$. Then

$$\begin{aligned}
& c_0^{-1} cs_i(z) \theta^*_{ij}(z) \\
&\quad = \exp i \left(- \int_Y TP(\bar{A}_0, \bar{A}_i(z)) - \int_{X \times I} TP(A_i(z), A_i^{\tilde{g}_{ij}}(z)) \right. \\
&\qquad\quad \left. - \int_X TP(A_0, A_i(z), A_j(z)) \right) \\
&\quad = \exp i \left(- \int_Y TP(\bar{A}_0, \bar{A}_j(z)) + \int_Y TP(\bar{A}_i, \bar{A}_j) - \int_{X \times I} TP(A_i(z), A_i^{\tilde{g}_{ij}}(z)) \right),
\end{aligned}$$

since

$$- \int_X TP(A_0, A_i, A_j) = \int_Y TP(\bar{A}_0, \bar{A}_i) - \int_Y TP(\bar{A}_0, \bar{A}_j) + \int_Y TP(\bar{A}_i, \bar{A}_j).$$

Claim.

$$\int_{X \times I} TP(A_i(z), A_i^{\tilde{g}_{ij}}(z)) = \int_Y TP(\bar{A}_i(z), \bar{A}_j(z)) \mod 2\pi\mathbb{Z}.$$

To show this we observe that

$$- \int_Y TP(\bar{A}_i(z), \bar{A}_j(z)) = \int_{-Y} TP(\bar{A}_i(z), \bar{A}_j(z)),$$

where $-Y$ denotes Y with the opposite orientation. Recall that by antisymmetry, $\int_Y TP(\bar{A}_i(z), \bar{A}_i(z)) = 0$. Then consider the closed $(2k-1)$-manifold $W = Y \cup_{X \times \{0\}} (X \times I) \cup_{X \times \{1\}} (-Y)$, where a connection B can be defined as

$$B(z) = \begin{cases} \bar{A}_i(z) & \text{on } Y \\ A_i^{\tilde{g}_{ij}}(z) & \text{on } X \times I \\ \bar{A}_j(z) & \text{on } -Y \end{cases}$$

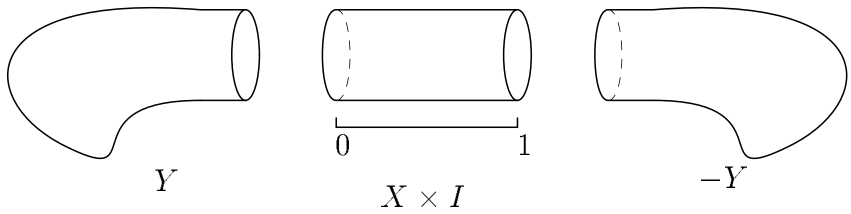

FIGURE 1. $W = Y \cup_{X\times\{0\}} (X \times I) \cup_{X\times\{1\}} (-Y)$

The problem now reduces to showing that

$$\int_W TP(\bar{A}_i(z), B(z)) = 0 \qquad \mod 2\pi\mathbb{Z}. \tag{3.9}$$

By earlier remarks there exists a gauge transformation $\bar{g}_{ij}$ on $\bar{\mathcal{P}}$ such that $\bar{A}_i^{\bar{g}_{ij}} = \bar{A}_j$. Then there is a gauge transformation $\mathbf{g}$ given by

$$h = \begin{cases} \text{id} & \text{on } Y \\ \tilde{g}_{ij} & \text{on } X \times I \\ \bar{g}_{ij} & \text{on } -Y \end{cases}$$

such that $B = \bar{A}_i^h$. With this the integral in (3.9) reads

$$\int_W TP(\bar{A}_i(z), \bar{A}_i^h(z)),$$

or, with the definition of TP

$$2\pi \int_W \int_{s=0}^1 P\big(F_{(1-s)\bar{A}_i(z)+s\bar{A}_i^h(z)}\big). \tag{3.10}$$

Now, at the ends of the manifold $W \times I$, the differential form under the integral are $P(F_{\bar{A}_i(z)})$ and $P(F_{\bar{A}_i^h(z)})$ respectively, but since we are dealing with forms on the base, these two forms agree. Again we apply the mapping torus of $\bar{\mathcal{P}}$. This is $\bar{\mathcal{P}} \times I/\sim$, where $(p,1) \sim (h \cdot p, 0)$, which on the base is just $W \times I/\sim$ with $(w,1) \sim (w,0)$. In other words the integral in (3.10) can be rewritten as

$$\int_{W\times S^1} P\big(F_{(1-s)\bar{A}_i(z)+s\bar{A}_i^h(z)}\big),$$

which is integer valued. This concludes the proof of the compatibility condition. Next we show that the section does not depend on the extension $\bar{A}_0$ of the connection

A_0. Let A_0 be a connection in $\mathcal{P}$. Let $\bar{A}_0$ and $\bar{A}_0'$ be two connections in $\bar{\mathcal{P}}$ extending A_0. Let cs_i and cs_i' be defined over U_i by

$$cs_i(z) = c_0 \exp i \int_Y TP(\bar{A}_0, \bar{A}_i(z)), \quad cs_i'(z) = c_0' \exp i \int_Y TP(\bar{A}_0', \bar{A}_i(z))$$

with $c_0 = cs(\bar{A}_0, \bar{A}_{00})$, $c_0' = cs(\bar{A}_0', \bar{A}_{00})$. Then

$$\begin{aligned}
\left(c_0^{-1} cs_i(z)\right)^{-1} c_0' cs_i'(z) &= \exp i \left(\int_Y TP(\bar{A}_0, \bar{A}_i(z)) - \int_Y TP(\bar{A}_0', \bar{A}_i(z)) \right) \\
&= \exp i \left(\int_Y TP(\bar{A}_0, \bar{A}_0') + \int_X TP(A_0, A_0, A_i(z)) \right) \\
&= \exp i \int_Y TP(\bar{A}_0, \bar{A}_0').
\end{aligned}$$

On the other hand, by (3.1) we have

$$\begin{aligned}
c_0 (c_0')^{-1} = &\exp\left(2\pi i \langle S_P(\bar{B}) - S_P(\bar{B}'), [W]\rangle\right) \\
&\exp\left(-i \int_W TP(\bar{B}, \bar{B}')\right) = \exp\left(-i \int_Y TP(\bar{A}_0, \bar{A}_0')\right).
\end{aligned}$$

by [DK] (2.10). This proves that cs is a well-defined section. Finally we calculate the covariant derivative of the section with respect to the connection obtained in section 2. Locally we have, using the formula for cs_i with $c_0 = cs(\bar{A}_0, \bar{A}_{00})$:

$$\begin{aligned}
dcs_i &= c_0 d \exp\left(-i \int_Y TP(\bar{A}_0, \bar{A}_i)\right) \\
&= -i d_Z \int_Y TP(\bar{A}_0, \bar{A}_i) \otimes cs_i \\
&= -2\pi i \int_Y \int_{s=0}^1 d_Z P(F_{(1-s)\bar{A}_0 + s\bar{A}_i}) \otimes cs_i \\
&= -2\pi i \int_Y \int_{s=0}^1 (d - d_Y - d_s) P(F_{(1-s)\bar{A}_0 + s\bar{A}_i}) \otimes cs_i \\
&= 0 - i \int_X TP(A_0, A_i) \otimes s_i - 2\pi i \int_Y P(F_{\bar{A}_0}) \otimes cs_i + 2\pi i \int_Y P(F_{\bar{A}_i}) \otimes cs_i.
\end{aligned}$$

Here the form $2\pi i \int_Y P(F_{\bar{A}_0})$ is zero, since it does not depend on $z \in Z$. On the other hand, the one-form ω_i that determines the connection of $\mathcal{L}$ was given by

$$\omega_i = -i \int_X TP(A_0, A_i),$$

cf. definition 2.3. Hence the connection of $\mathcal{L}^*$ is determined by the one-form

$$\omega_i^* = i \int_X TP(A_0, A_i). \tag{3.11}$$

It follows that

$$\nabla(cs_i) = \omega_i^* \otimes cs_i + dcs_i = 2\pi i \int_Y P(F_{\bar{A}_i}) \otimes cs_i,$$

and hence, globally

$$\nabla(cs) = 2\pi i \int_Y P(F_{\bar{A}}) \otimes cs.$$

□

Remark 3.12. The Chern-Simons section of the relative line bundle from remark 1.27 is given locally by $cs_i^{AB}(z) = \exp(-i \int_Y TP(\bar{B}_i(z), \bar{A}_i(z))$, and its covariant derivative is $\nabla^{AB}(cs_i^{AB}) = 2\pi i(\int_Y P(F_{\bar{A}_i}) - \int_Y P(F_{\bar{B}_i})) \otimes cs_i^{AB}$.

4. Application to handle bodies

As an example we apply our results to a family of flat connections over a genus g surface X. We shall show the following:

Proposition 4.1. *Let X be a genus g surface and Y the corresponding handle body such that $\partial Y = X$. Let $\rho_z : \pi_1(Y) \to SU(2)$ be a family of representations of the fundamental group of Y indexed by a manifold Z and choose one of these as boundary condition. Then these data define a line bundle $\mathcal{L} \to Z$ with a canonical flat connection and an everywhere non-zero Chern-Simons section which is parallel.*

Proof. Let Y be the "massive interior" of X such that $\partial Y = X$. The homotopy type of Y is the same as g circles, so the fundamental group is the free abelian group with g generators.

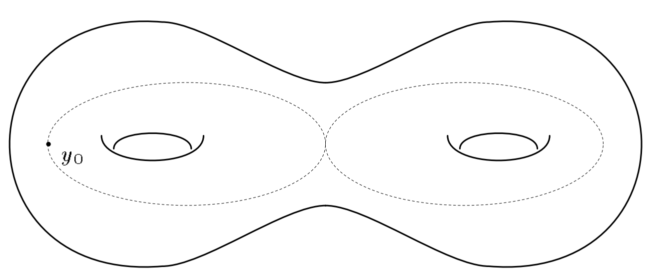

FIGURE 2. A genus 2 handle body with generators of the fundamental group

Let $\tilde{Y}$ be the fundamental covering of Y and $\pi : \tilde{Y} \to Y$ the projection. Let

$$\rho : \pi_1(Y) \to SU(2)$$

be a representation of the fundamental group of Y. Let

$$\tilde{Y} \times_\rho SU(2)$$

be the quotient by the equivalence relation $(\tilde{y} \cdot \xi, g) \sim (\tilde{y}, \rho(\xi)g)$ for $\xi \in \pi_1(Y)$. This is a principal $SU(2)$-bundle over Y; it is well-known that it is trivial. A set of such representations $\{\rho_z\}|_{z \in Z}$ determine a family of (trivial) principal $SU(2)$-bundles $\mathcal{P}_z$ and hence a line bundle $\mathcal{L} \to Z$, cf. theorem 1.3. Now let U be a plane disc with g holes and note that there is a deformation retraction

$$r : Y \to U.$$

Then the above family of flat principal bundles is induced from U via r. It follows from dimension reasons that the integrals in the formulae of theorem 2.1 and theorem 3.3 vanish, i.e., the connection has zero curvature, and the Chern-Simons section is parallel. □

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