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THE CIRCLE TRANSFER AND K-THEORY

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THE CIRCLE TRANSFER AND K-THEORY

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1. Introduction

Let L be the canonical line bundle over $\mathbb{C}P^\infty$. The Thom spectrum $\mathbb{C}P^\infty_{-1} = \mathbb{T}\mathrm{h}(-L)$ and its associated infinite loop space $\Omega^\infty(\mathbb{C}P^\infty_{-1})$ (that is, the 0'th space of the associated Ω-spectrum), has appeared in various geometric contexts in topology. For example it plays a central role in connection with the trace invariant determination of Waldhausen's $\mathrm{A}(X)$, and conjecturally it is the group completion of the classifying space of the stable mapping class group, cf. [19], [21].

The classifying space $\Omega^\infty(\Sigma\mathbb{C}P^\infty_{-1})$ appears as the fiber of the dimension shifting transfer map, so

$$\Omega^\infty(\Sigma\mathbb{C}P^\infty_{-1}) \longrightarrow Q(S^1 \wedge \mathbb{C}P^\infty_+) \xrightarrow{\mathrm{trf}} Q_{(0)}(S^0)$$

is a homotopy fibration. The main point of this paper is to compare this fibration at a given prime p to the homotopy fibration

$$U_{(p)} \xrightarrow{\Omega(1-\psi^g)} U_{(p)} \xrightarrow{\Delta} JU_{(p)} \times \mathbb{Z}_{(p)},$$

where $\psi^g : BU \to BU$ is the usual Adams operation.

The complex reflection map $R : S^1 \wedge \mathbb{C}P^\infty_+ \to U$ extends via Bott periodicity to a map $Q(R)$ from $Q(S^1 \wedge \mathbb{C}P^\infty_+)$ to U. The unit of the ring spectrum $J_{(p)}$ with $\Omega^\infty(J_{(p)}) = JU_{(p)} \times \mathbb{Z}_{(p)}$ gives a map e_J from $Q_{(0)}(S^0)$ to $JU_{(p)}$, which induces Adams' complex e-invariant on homotopy.

One might suspect that the diagram

$$\begin{tikzcd}
Q(S^1 \wedge \mathbb{C}P^\infty_+) \arrow[r, "Q(R)"] \arrow[d, "\mathrm{trf}"] & U_{(p)} \arrow[d, "\Delta"] \\
Q_{(0)}(S^0) \arrow[r, "e_J"] & JU_{(p)}
\end{tikzcd}$$

was homotopy commutative. However, in [16], Klein and Rognes point out that the above diagram cannot possible be commutative. In this paper, we show firstly that there are in fact infinite loop maps l and $\tilde{l}$ making the diagram

$$\begin{tikzcd}
\Omega^\infty(\Sigma\mathbb{C}P^\infty_{-1}) \arrow[r] \arrow[d, "\tilde{l}"] & Q(S^1 \wedge \mathbb{C}P^\infty_+) \arrow[r, "\mathrm{trf}"] \arrow[d, "l"] & Q_{(0)}(S^0) \arrow[d, "e_J"] \\
U_{(p)} \arrow[r, "\Omega(1-\psi^g)"] & U_{(p)} \arrow[r, "\Delta"] & JU_{(p)}
\end{tikzcd} \tag{1.1}$$

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homotopy commutative, and secondly that the vertical maps have compatible splittings when localizing at an odd prime p, although not in the category of infinite loop maps. Moreover, both $\tilde{l}$ and l are rational homotopy equivalences. This gives

Theorem 1.1. *For odd primes p there exists a decomposition*

$$\Omega^\infty(\Sigma\mathbb{C}P^\infty_{-1})_{(p)} \simeq U_{(p)} \times X_{(p)},$$

with $\pi_*(X_{(p)}) \otimes \mathbb{Q} = 0$.

In sections 2–5 below we survey work primarily due to J.C. Becker and R.E. Schultz, M.C. Crabb and K. Knapp, but in a form that is convenient for our applications in Section 6 and 7. We have strived to make these sections self contained and accessable to the inexperienced reader.

Many results in this paper are formulated in the stable homotopy category of spectra, and the actual choice of a point set level category is not important; for example the one in [3] will do. The paper is divided in the following sections.

Contents

2. Equivariant transfer maps

In this section G will be a compact Lie group. A based G-space is a based space with a (left) G-action that fixes the base-point, and a G-representation will mean a finite dimensional real G-representation with a G-invariant inner metric. The one-point compactification of a G-representation V is a based G-space denoted S^V.

To a based G-space X, there is an associated G-equivariant infinite loop space

$$Q_G(X) = \operatorname{colim} \operatorname{Map}(S^V, S^V \wedge X),$$

where $\operatorname{Map}(-,-)$ means based maps, and the limit is over a complete set of G-representation V. When G is the trivial group we just write $Q(X)$; this is the usual non-equivariant infinite loop-space associated with X.

Let $Ad(G)$ denote the adjoint representation of G, i.e. the representation on the tangentspace T_1G induced from conjugation.

Theorem 2.1 ([5],[18])**.** *Let X be a based G-CW complex such that the action is free away from the base-point. Then the natural transfer map*

$$\operatorname{trf}^G : Q(S^{Ad(G)} \wedge_G X) \xrightarrow{\sim} Q_G(X)^G,$$

is a homotopy equivalence. ☐

In our application in Section 5 of this result we shall need the construction of trf^G, so we start by recalling it (in a slightly special case). Let K and M be (left) smooth G-manifolds without boundary. We assume that K is compact, and that we are given an equivariant embedding

$$i : K \to M.$$

The normal bundle $N(i) = i^*T(M)/T(K)$ embeds as an invariant tubular neighborhood of $i(K)$:

$$j : N(i) \to M,$$

and we can choose j so that the tangent map induces the identity on quotient bundles:

$$\begin{tikzcd} \frac{T(N(i))|K}{T(K)} \arrow[r, "T(j)"] & \frac{T(M)|i(K)}{T(i(K))} \\ N(i) \arrow[u, "\cong"'] \arrow[r, equal] & N(i), \arrow[u, "\cong"'] \end{tikzcd}$$

where on the left we identify K with the zero section in $N(i)$. The diagram ensures that the isotopy class of j is uniquely determined by the embedding i, cf. [11, 6.2.6].

Let M^c be the one-point compactification of M, let $\mathrm{Th}(N(i))$ be the Thom-space of $N(i)$, and let

$$t : M^c \to \mathrm{Th}(N(i)) \tag{2.1}$$

be the Pontrjagin-Thom map.

Suppose that K is a compact free G-manifold, and let $B = K/G$. The projection $p : K \to B$ is a smooth fiber-bundle, and we have an exact sequence of G-vector-bundles

$$0 \to T^v(K) \to T(K) \to p^*T(B) \to 0,$$

where $T^v(M)$ is the vector bundle of tangents along the fibers. The differential of the action map $g \mapsto gx$ gives for each $x \in K$ an isomorphism $Ad(G) \cong T^v_x(K)$, and thus a trivialization $Ad(G)_K \cong T^v(K)$. (Throughout the paper we write V_X for the product bundle $X \times V \to X$ of a G-space X and a representation V).

For suitably V there is an embedding

$$i_0 = (p, q) : K \to B \times V, \tag{2.2}$$

where p is the projection. The differential embeds each fiber of $T^v(K)$ in V, and defines an embedding of bundles $T^v(K) \to V_K$, whose quotient $V_K/T^v(K)$ is the normal bundle along the fibers. The inclusion

$$V_K \to i_0^*T(B \times V) = p^*T(B) \oplus V_K, \quad v \mapsto (0, v),$$

passes to quotients and defines an isomorphism $V_K/T^v(K) \cong N(i_0)$, and the invariant metric on V induces a canonical isomorphism

$$T^v(K) \oplus N(i_0) \cong V_K.$$

We want a trivial normal bundle, and replace i_0 with the embedding

$$i : K \xrightarrow{i_0} V_B \xrightarrow{j_0} V_B \oplus T^v(K)/G, \tag{2.3}$$

where $T^v(K)/G$ is the quotient bundle over B, and j_0 is the inclusion in the zero section of $T^v(K)/G$. Then we get a canonical isomorphism

$$N(i) \cong T^v(K) \oplus N(i_0) \cong V_K, \tag{2.4}$$

and the G-equivariant Pontrjagin-Thom map

$$t : (K_+ \wedge_G S^{Ad(G)}) \wedge S^V \to K_+ \wedge S^V,$$

with trivial action on $K_+ \wedge_G S^{Ad(G)}$. By adjunction we get

$$\hat{t} : K_+ \wedge_G S^{Ad(G)} \to \mathrm{Map}(S^V, K_+ \wedge S^V)^G \to Q_G(K_+)^G, \tag{2.5}$$

and the equivalence trf^G of Theorem 2.1 is the unique extension to an infinite loop map, using the infinite loop space structure of $Q_G(K_+)^G$.

The homotopy class of $\hat{t}$, and hence that of trf^G, is independent of the choice of i_0. Indeed, the composition of i_0 with the inclusion $V \to V \oplus W$ does not affect $\hat{t}$. Thus two embeddings may be assumed related by an isotopy of the form

$$h : K \times I \to B \times I \times V, \quad h_t(x) = (p(x), t, q_t(x)).$$

Using h as the input for the construction of $\hat{t}$ in (2.5) (with K replaced by $K \times I$) gives the required homotopy.

Remark 2.2. The above construction generalizes to the case of a smooth manifold with boundary M, and gives an equivalence

$$\mathrm{trf}^G : Q(S^{Ad(G)} \wedge_G M/\partial M) \xrightarrow{\simeq} Q_G(M/\partial M)^G.$$

We need a few naturality properties of trf^G, which we now discuss. Let $H \subseteq G$ be a closed subgroup. Then there is a homotopy commutative diagram

$$\begin{array}{ccc} Q(S^{Ad(G)} \wedge_G K_+) & \xrightarrow{\mathrm{trf}^G_H} & Q(S^{Ad(H)} \wedge_H K_+) \\ \big\downarrow{\scriptstyle \mathrm{trf}^G} & & \big\downarrow{\scriptstyle \mathrm{trf}^H} \\ Q_G(K_+)^G & \longrightarrow & Q_H(K_+)^H. \end{array} \tag{2.6}$$

The lower horizontal map is the inclusion of the G fixed points of $Q_G(K_+)$ into its H fixed points composed with the obvious homotopy equivalence $Q_G(K_+)^H \simeq Q_H(K_+)^H$. The upper horizontal map trf^G_H is similar to the construction above: One considers the bundle $p : K/H \to K/G$, and replaces i in (2.3) by the embedding

$$i : K/H \to (K \times_G Ad(G)) \oplus V_{K/G}.$$

Its normal bundle is $(K \times_H Ad(H)) \oplus V_{K/H}$, and the Pontrjagin map induces trf^G_H, cf. [22].

The transfer trf^G is also natural in the variable K in the sense that given a G-map $f : K \to L$, the diagram

$$\begin{array}{ccc} Q(S^{Ad(G)} \wedge_G K_+) & \xrightarrow{f_*} & Q(S^{Ad(G)} \wedge_G L_+) \\ \big\downarrow{\scriptstyle \mathrm{trf}^G} & & \big\downarrow{\scriptstyle \mathrm{trf}^G} \\ Q_H(K_+)^G & \xrightarrow{f_*} & Q_G(L_+)^G \end{array} \tag{2.7}$$

is homotopy commutative. Taken together, (2.6) and (2.7) shows that trf^G is a natural transformation from the category $\mathcal{C}$ with objects consisting of pairs (K, G), and morphisms $(f, i) : (K, G) \to (L, H)$ given by a closed inclusion $i : H \to G$ together with a smooth H-map $f : K \to L$.

Our application of the equivariant transfer will be for $G = S^1$, where $Ad(G) = \mathbb{R}$ with trivial G-action. The standard action of S^1 on S^{2n+1} gives an equivalence

$$\mathrm{trf}^{S^1}_n : Q(S^1 \wedge \mathbb{C}P^n_+) \xrightarrow{\simeq} Q_{S^1}(S^{2n+1})^{S^1}.$$

These transfers are compatible for varying n by (2.7), and one gets a map

$$\mathrm{trf}^{S^1} : Q(S^1 \wedge \mathbb{C}P^\infty_+) \to Q_{S^1}(ES^1_+)^{S^1}, \tag{2.8}$$

whose restriction to $Q(S^1 \wedge \mathbb{C}P^n_+)$ is homotopic to $\mathrm{trf}^{S^1}_n$. In fact, the homotopy class of trf^{S^1} is uniquely determined by Anderson's criteria for the vanishing of derived limits, see [6]

3. Thom spectra and equivariant function spaces

In this section and the next, we restrict attention to the circle group S^1, although things work similarly for any compact Lie group G that admits free G-representations. Let L denote the tautological line bundle over $\mathbb{C}P^\infty$, and L_n its restriction to $\mathbb{C}P^n$; it is a subbundle of the product bundle $\mathbb{C}P^n \times \mathbb{C}^{n+1}$, and we let $L_n^\perp$ denote its orthogonal complement. The restriction of $L_n^\perp$ to $\mathbb{C}P^{n-1}$ is $L_{n-1}^\perp \oplus \mathbb{C}$, and we have induced maps of Thom spaces

$$\epsilon : S^2 \wedge \mathrm{Th}(L_{n-1}^\perp) \to \mathrm{Th}(L_n^\perp). \tag{3.1}$$

This defines the connective (i.e. (-1)-connected) spectrum $\mathbb{T}\mathrm{h}(\mathbb{C} - L)$ with $2n$'th space $\mathrm{Th}(L_n^\perp)$; $\pi_0(\mathbb{T}\mathrm{h}(\mathbb{C}-L)) = \mathbb{Z}$.

The connective Thom spectra $\mathbb{T}\mathrm{h}(\mathbb{C}^k - kL)$ for $k \geq 1$ are defined similarly, and we let $\mathbb{T}\mathrm{h}(kL) = \Sigma^\infty(\mathrm{Th}(kL))$ for $k \geq 0$.

For a spectrum E, we let $\Omega^\infty(E) = \operatorname{colim} \Omega^n(E_n)$. In using this definition we implicitely assume that the structure maps $S^1 \wedge E_n \to E_{n+1}$ are cofibrations; this will always be the case for the spectra we consider. The following is a well-known consequence of Theorem 2.1.

Proposition 3.1. *There are infinite loop space equivalences*

$$\mathrm{Map}(S^{\mathbb{C}^k}, Q_{S^1}(ES^1_+))^{S^1} \simeq \Omega^\infty \Sigma\, \mathbb{T}\mathrm{h}(-kL)$$

for $k \geq 0$.

Proof. For notational convenience we consider only the case $k = 1$; the proof for $k > 1$ is completely analogous. We have

$$\mathrm{Map}(S^{\mathbb{C}}, Q_{S^1}(ES^1_+))^{S^1} = \operatorname{colim} \mathrm{Map}(S^{\mathbb{C}}, Q_{S^1}(S(\mathbb{C}^n)_+))^{S^1}$$

and

$$\Omega^\infty \Sigma\, \mathbb{T}\mathrm{h}(-L) = \operatorname{colim} \Omega^{2n}(S^1 \wedge \mathrm{Th}(L_{n-1}^\perp)).$$

Let $p : S(\mathbb{C}^n) \to \mathbb{C}P^{n-1}$ be the projection. Pulling back the isomorphism $L_{n-1} \oplus L_{n-1}^\perp \cong \mathbb{C}^n_{\mathbb{C}P^{n-1}}$ along p, we get

$$p^*(L_{n-1}^\perp) \oplus \mathbb{C}_{S(\mathbb{C}^n)} \cong |\mathbb{C}^n|_{S(\mathbb{C}^n)},$$

where $|\cdot|$ denotes trivial S^1-action. In $\mathrm{Map}(S^{\mathbb{C}}, Q_{S^1}(S(\mathbb{C}^n)_+))$ the action on $S^{\mathbb{C}}$ is non-trivial, so we cannot use Theorem 2.1 directly. Instead, we get a sequence of equivalences:

$$\begin{aligned}
&\mathrm{Map}(S^{\mathbb{C}}, Q_{S^1}(S(\mathbb{C}^n)_+))^{S^1} = \operatorname{colim}\mathrm{Map}(S^{\mathbb{C}} \wedge S^V, S(\mathbb{C}^n)_+ \wedge S^V)^{S^1} \\
&\xrightarrow{\simeq} \operatorname{colim}\mathrm{Map}(S^{\mathbb{C}} \wedge S^{|\mathbb{C}^n|} \wedge S^V, S(\mathbb{C}^n)_+ \wedge S^{|\mathbb{C}^n|} \wedge S^V)^{S^1} \\
&\simeq \operatorname{colim}\mathrm{Map}(S^{\mathbb{C}} \wedge S^{|\mathbb{C}^n|} \wedge S^V, S^{\mathbb{C}} \wedge \mathrm{Th}(p^*(L_{n-1}^{\perp})) \wedge S^V)^{S^1} \\
&\xleftarrow{\simeq} \operatorname{colim}\mathrm{Map}(S^{|\mathbb{C}^n|} \wedge S^V, \mathrm{Th}(p^*(L_{n-1}^{\perp})) \wedge S^V)^{S^1} \\
&\simeq \Omega^{2n}(Q_{S^1}(\mathrm{Th}(p^*(L_{n-1}^{\perp})))^{S^1}) \\
&\xleftarrow{\mathrm{trf}^{S^1}} \Omega^{2n}(Q(S^1 \wedge \mathrm{Th}(L_{n-1}^{\perp}))),
\end{aligned}$$

where we in the last equivalence have used Theorem 2.1 on the free based S^1-space $p^*(L_{n-1}^{\perp})$, cf. Remark 2.2. The above equivalences are natural with respect to inclusions $S(\mathbb{C}^n) \to S(\mathbb{C}^{n+1})$, and since

$$S^1 \wedge \mathrm{Th}(L_{n-1}^{\perp}) \to Q(S^1 \wedge \mathrm{Th}(L_{n-1}^{\perp}))$$

becomes highly connected with increasing n, we obtain the required equivalence of colimits. $\square$

Given a based S^1-space Y, let $\widetilde{\mathrm{Map}}(S^{\mathbb{C}^n}, Y) = \mathrm{Map}((S^{\mathbb{C}^n}, S^0), (Y, *))$, where $S^0 \subseteq S^{\mathbb{C}^n}$ denotes the fixedpoint set. This is an S^1-subspace of $\mathrm{Map}(S^{\mathbb{C}^n}, Y)$. Define

$$Q^f_{S^1}(Y) = \operatorname{colim}\mathrm{Map}(S^{\mathbb{C}^n}, Y \wedge S^{\mathbb{C}^n})$$

and

$$\widetilde{Q}^f_{S^1}(Y) = \operatorname{colim}\widetilde{\mathrm{Map}}(S^{\mathbb{C}^n}, Y \wedge S^{\mathbb{C}^n}),$$

where the superscript f indicates that the colimit is over the *free* representations $\mathbb{C}^n$ only. Notice that $\widetilde{Q}^f_{S^1}(Y) \to Q^f_{S^1}(Y) \to Q_{S^1}(Y)$ are non-equivariant equivalences.

Lemma 3.2. *There are S^1-equivariant homotopy equivalences*

(i): $Q^f_{S^1}(ES^1_+ \wedge Y) \xrightarrow{\simeq} Q_{S^1}(ES^1_+ \wedge Y)$

(ii): $\widetilde{Q}^f_{S^1}(ES^1_+ \wedge Y) \xrightarrow{\simeq} Q^f_{S^1}(ES^1_+ \wedge Y)$

(iii): $\widetilde{Q}^f_{S^1}(ES^1_+ \wedge Y) \xrightarrow{\simeq} \widetilde{Q}^f_{S^1}(Y)$

Proof. (i) follows from the equivariant suspension theorem [15, II.2.10], and (ii) follows from the equivariant fibration sequence obtained by applying $\mathrm{Map}(-, ES^1_+ \wedge Y)$ to the cofibration sequence $S^0 \to S^{\mathbb{C}^n} \to S^{\mathbb{C}^n}/S^0$.

The map in (iii) is induced from the projection $ES^1_+ \wedge Y \to Y$; we denote it by π. In order to prove that it is an equivalence, we exhibit an explicit homotopy inverse. For fixed n let

$$\gamma_n : \widetilde{\mathrm{Map}}(S^{\mathbb{C}^n}, Y \wedge S^{\mathbb{C}^n}) \to \widetilde{\mathrm{Map}}(S^{\mathbb{C}^n}, S(\mathbb{C}^n)_+ \wedge Y \wedge S^{\mathbb{C}^n})$$

be given by

$$\gamma_n(f)(v) = \begin{cases} (v/|v|, f(v)), & \text{for } v \neq 0, \infty \\ *, & \text{for } v = 0, \infty, \end{cases}$$

This is continuous since $f(0) = f(\infty) = \infty$. Given m and n, the diagram

$$\begin{array}{ccc} \widetilde{\mathrm{Map}}(S^{\mathbb{C}^n}, Y \wedge S^{\mathbb{C}^n}) & \xrightarrow{\gamma_n} & \widetilde{\mathrm{Map}}(S^{\mathbb{C}^n}, S(\mathbb{C}^n)_+ \wedge Y \wedge S^{\mathbb{C}^n}) \\ \downarrow & & \downarrow \\ \widetilde{\mathrm{Map}}(S^{\mathbb{C}^{n+m}}, Y \wedge S^{\mathbb{C}^{n+m}}) & \xrightarrow{\gamma_{n+m}} & \widetilde{\mathrm{Map}}(S^{\mathbb{C}^{n+m}}, S(\mathbb{C}^{n+m})_+ \wedge Y \wedge S^{\mathbb{C}^{n+m}}) \end{array}$$

is homotopy commutative via the homotopy

$$h_t(f)(v, w) = ((v, tw)/|(v, tw)|, f(v), w).$$

Thus the maps γ_n defines a unique homotopy class

$$\gamma : \widetilde{Q}^f_{S^1}(Y) \to \widetilde{Q}^f_{S^1}(ES^1_+ \wedge Y)^{S^1}. \tag{3.2}$$

It is clear that $\pi \circ \gamma = \mathrm{id}$. In order to examine $\gamma \circ \pi$ we let

$$\phi : \widetilde{Q}^f_{S^1}(ES^1_+ \wedge Y) \to \widetilde{Q}^f_{S^1}(ES^1_+ \wedge ES^1_+ \wedge Y)$$

be the direct limit of the maps

$$\phi : \widetilde{\mathrm{Map}}(S^{\mathbb{C}^n}, ES^1_+ \wedge Y \wedge S^{\mathbb{C}^n}) \to \widetilde{\mathrm{Map}}(S^{\mathbb{C}^n}, S(\mathbb{C}^n)_+ \wedge ES^1_+ \wedge Y \wedge S^{\mathbb{C}^n}),$$

given by

$$\phi(f)(v) = \begin{cases} (v/|v|, f(v)), & \text{for } v \neq 0, \infty \\ *, & \text{for } v = 0, \infty. \end{cases}$$

The two projections $p_1, p_2 : ES^1 \times ES^1 \to ES^1$ are S^1-equivariantly homotopic, and thus $\gamma \circ \pi = p_{1*} \circ \phi \sim p_{2*} \circ \phi = \mathrm{id}$. □

The case $k \geq 0$ of the next Theorem is due to Becker and Schultz, [10].

Theorem 3.3. *For $k \in \mathbb{Z}$,*

$$\mathrm{colim}\, \widetilde{\mathrm{Map}}(S^{\mathbb{C}^n}, S^{\mathbb{C}^{n+k}})^{S^1} \simeq \Omega^\infty \Sigma\, \mathbb{T}\mathrm{h}(kL).$$

Furthermore, for $k < 0$, $\widetilde{\mathrm{Map}}(S^{\mathbb{C}^n}, S^{\mathbb{C}^{n+k}})^{S^1} = \mathrm{Map}(S^{\mathbb{C}^n}, S^{\mathbb{C}^{n+k}})^{S^1}$.

Proof. The case $k \geq 0$ follows by letting $Y = S^{\mathbb{C}^k}$ in Lemma 3.2, and applying Theorem 2.1 to the free S^1-space $ES^1_+ \wedge S^{\mathbb{C}^k}$. For $k < 0$, Proposition 3.1 reduces us to proving that

$$\mathrm{colim}\, \widetilde{\mathrm{Map}}(S^{\mathbb{C}^{n-k}}, S^{\mathbb{C}^n})^{S^1} \simeq \mathrm{Map}(S^{\mathbb{C}^{-k}}, Q_{S^1}(ES^1_+))^{S^1}.$$

Letting $Y = S^0$ in Lemma 3.2 and applying $\mathrm{Map}(S^{\mathbb{C}^{-k}}, -)^{S^1}$ gives the result. In order to prove the second statement, it suffices to show that any S^1-equivariant map $S^{\mathbb{C}^n} \to S^{\mathbb{C}^{n+k}}$ sends the fixedpoint set S^0 to the basepoint when $k < 0$. This follows from a standard argument using the mapping degree in S^1-equivariant K-theory, cf. [15, II,5]. □

4. The Becker-Schultz equivalence $F^{S^1} \simeq Q(S^1 \wedge \mathbb{C}P^\infty_+)$

We continue to consider free S^1-representations $\mathbb{C}^n$. Since $S(\mathbb{C}^m \oplus \mathbb{C}^n) \cong S(\mathbb{C}^m) * S(\mathbb{C}^n)$, we may form the colimit

$$F^{S^1} = \mathrm{colim}\, \mathrm{Map}(S(\mathbb{C}^n), S(\mathbb{C}^n))^{S^1},$$

upon taking joins with the identity.

Theorem 4.1. *There is a homotopy equivalence*

$$\lambda : F^{S^1} \to Q_{S^1}(ES^1_+)^{S^1}.$$

This is really a theorem of Becker and Schultz, since upon using Theorem 2.1 it becomes the statement from [10]. We shall follow [24] in deducing Theorem 4.1 from the results in Section 3.

In order to define λ it is convenient to introduce auxiliary spaces and maps as follows. From Section 3 we have the space

$$Q^f_{S^1}(S^0)^{S^1} = \operatorname{colim} \operatorname{Map}(S^{\mathbb{C}^n}, S^{\mathbb{C}^n})^{S^1},$$

with the colimit running over free representations. The fixed set of $S^{\mathbb{C}^n}$ is S^0, and we have the subspaces $F^{S^1}_{(1)}(\infty)$ and $F^{S^1}_{(0)}(\infty)$ of maps with $f|S^0 = \mathrm{id}$ and $f|S^0 = *$, respectively. (Thus $F^{S^1}_{(0)}(\infty) = \widetilde{Q}^f_{S^1}(S^0)$).

Following [24], let

$$d : F^{S^1}_{(1)}(\infty) \times F^{S^1}_{(1)}(\infty) \to F^{S^1}_{(0)}(\infty),$$

be the difference map given by

$$d(f,g)(t,x) = \begin{cases} f(1-2t,x), & \text{for } 0 \le t \le 1/2, \\ g(2t-1,x), & \text{for } 1/2 \le t \le 1, \end{cases} \tag{4.1}$$

where we identify $S^{\mathbb{C}^n}$ with the unreduced suspension $\Sigma S(\mathbb{C}^n) = S^0 * S(\mathbb{C}^n)$, and we use $(1,*) \in S^0 * S(\mathbb{C}^n)$ as base point. Taking $g = \mathrm{id}$ we get homotopy inverse maps

$$F^{S^1}_{(1)}(\infty) \xrightarrow{d(-,\mathrm{id})} F^{S^1}_{(0)}(\infty) \xrightarrow{d(-,\mathrm{id})} F^{S^1}_{(1)}(\infty),$$

so $F^{S^1}_{(1)}(\infty) \simeq F^{S^1}_{(0)}(\infty)$.

Proof of Theorem 4.1. The map λ is the composite of three maps

$$\lambda : F^{S^1} \xrightarrow{\lambda_1} F^{S^1}_{(1)} \xrightarrow{\lambda_2} F^{S^1}_{(0)}(\infty) \xrightarrow{\lambda_3} Q_{S^1}(ES^1_+)^{S^1}.$$

Here λ_1 maps $f : S(\mathbb{C}^n) \to S(\mathbb{C}^n)$ to its unreduced suspention, $\lambda_2 = d(-,\mathrm{id})$, and λ_3 is the map γ from (3.2) (with $Y = S^0$) followed by the inclusion $\widetilde{Q}^f_{S^1}(ES^1_+)^{S^1} \xrightarrow{\simeq} Q_{S^1}(ES^1_+)^{S^1}$. Each of these maps are homotopy equivalences. For λ_3 this was proved in Lemma 3.2, and for λ_2 this follows from the above discussion. For λ_1 one can use that the space $\operatorname{Map}_{(1)}(S^{\mathbb{C}^n}, S^{\mathbb{C}^n})^{S^1}$ is equivalent to the space of bounded self maps of $S(\mathbb{C}^n) \times \mathbb{R}$ and that this space in turn is homotopy equivalent to $\operatorname{Map}(S(\mathbb{C}^n), S(\mathbb{C}^n))^{S^1}$. Alternatively, see Corollary 1.7 of [24]. □

The projection $ES^1 \to *$ induces a map

$$Q_{S^1}(ES^1_+)^{S^1} \to Q_{S^1}(S^0)^{S^1}, \tag{4.2}$$

which is split injective in the homotopy category according to the tom-Dieck splitting ([15], Theorem II.7.7):

$$Q_{S^1}(S^0)^{S^1} \simeq Q(S^0) \times {\prod_{n\ge 0}}' Q_{S^1}(ES^1_+)^{S^1},$$

where Π' is the weak product, and (4.2) corresponds to the inclusion of the component indexed by $n = 0$. Let $[1] \in Q_{S^1}(S^0)^{S^1}$ be the identity, and let

$$\zeta : Q_{S^1}(S^0)^{S^1} \to Q_{S^1}(S^0)^{S^1},\ x \mapsto [1] - x. \tag{4.3}$$

Then, by the definition of λ, we have

Corollary 4.2. *The diagram*

$$\begin{array}{ccc} F^{S^1} & \xrightarrow{\lambda} & Q_{S^1}(ES^1_+)^{S^1} \\ \downarrow & & \downarrow \\ Q_{S^1}(S^0)^{S^1} & \xrightarrow{\zeta} & Q_{S^1}(S^0)^{S^1} \end{array}$$

is homotopy commutative. The horizontal maps are homotopy equivalences, and the vertical maps are split up to homotopy. □

We next consider the multiplicative properties of λ. The product on F^{S^1} is by join of maps. To define a product on $Q_{S^1}(ES^1_+)^{S^1}$, let $p : ES^1 \times ES^1 \to ES^1$ be an S^1-equivariant homotopy inverse to the diagonal inclusion. Then

$$\begin{aligned} \mu : Q_{S^1}(ES^1_+)^{S^1} \times Q_{S^1}(ES^1_+)^{S^1} &\xrightarrow{\wedge} Q_{S^1\times S^1}(ES^1_+ \wedge ES^1_+)^{S^1\times S^1} \\ &\xrightarrow{p_*} Q_{S^1}(ES^1_+)^{S^1} \end{aligned} \tag{4.4}$$

defines the required product; it is is homotopy associative and commutative, and the map in (4.2) is multiplicative (as well as additive).

Proposition 4.3. *The map $\lambda : F^{S^1} \to Q_{S^1}(ES^1_+)^{S^1}$ is quadratic in the sense that*

$$\lambda(x \cdot y) \sim \lambda(x) + \lambda(y) - \mu(\lambda(x), \lambda(y))$$

as maps $F^{S^1} \times F^{S^1} \to Q_{S^1}(ES^1_+)^{S^1}$.

Proof. Apply Corollary 4.2. □

We also have a product

$$\begin{aligned} \bar{\mu} : Q(S^1 \wedge \mathbb{C}P^\infty_+) \times Q(S^1 \wedge \mathbb{C}P^\infty_+) &\xrightarrow{\wedge} Q(S^1 \wedge \mathbb{C}P^\infty_+ \wedge S^1 \wedge \mathbb{C}P^\infty_+) \\ &\xrightarrow{\mathrm{trf}} Q(S^1 \wedge \mathbb{C}P^\infty_+), \end{aligned} \tag{4.5}$$

where trf is the transfer associated with the diagonal inclusion $\Delta : S^1 \to S^1 \times S^1$ (or, more precisely, trf is obtained by functoriality from the morphism

$$(\mathrm{proj}, \Delta) : (S(V) \times S(V), S^1 \times S^1) \to (S(V), S^1)$$

in the category $\mathcal{C}$ introduced i Section 2).

Lemma 4.4. *The products μ and $\bar{\mu}$ correspond under the transfer equivalence*

$$\mathrm{trf}^{S^1} : Q(S^1 \wedge \mathbb{C}P^\infty_+) \xrightarrow{\sim} Q_{S^1}(ES^1_+)^{S^1}.$$

Proof. The smash product in (4.4) corresponds to the smash product in (4.5) since transfer maps are compatible with products, cf. [22, Note 1.14] . The second map in (4.4) corresponds to trf in (4.5) by naturality (2.6). □

The diagram

$$\begin{array}{ccc} F^{S^1} & \xrightarrow{\bar{\lambda}} & Q(S^1 \wedge \mathbb{C}P^\infty_+) \\ \| & & \downarrow{\scriptstyle \mathrm{trf}^{S^1}} \\ F^{S^1} & \xrightarrow{\lambda} & Q_{S^1}(ES^1_+)^{S^1} \end{array}$$

defines the homotopy equivalence from [10]. We note that the analogue of Proposition 4.3 is satisfied for $(\bar{\lambda}, \bar{\mu})$.

For applications in Section 5, we need an explicit expression for λ. As above we identify $S^{\mathbb{C}^n}$ with the unreduced suspension $S^0 * S(\mathbb{C}^n)$ with basepoint $(1, *)$. As target we identify $S^{\mathbb{C}^n}$ with the quotient $D(\mathbb{C}^n)/S(\mathbb{C}^n)$. For definiteness, we choose specific homeomorphisms

$$S^0 * S(\mathbb{C}^n) \to D(\mathbb{C}^n)/S(\mathbb{C}^n), \quad (t, x) \mapsto tx \tag{4.6}$$

$$D(\mathbb{C}^n)/S(\mathbb{C}^n) \to S^{\mathbb{C}^n}, \quad v \mapsto v/(1 - |v|). \tag{4.7}$$

For $f : S(\mathbb{C}^n) \to S(\mathbb{C}^n)$ in F^{S^1},

$$\lambda(f) : S^0 * S(\mathbb{C}^n) \to S(\mathbb{C}^n)_+ \wedge (D(\mathbb{C}^n)/S(\mathbb{C}^n))$$

is (up to homotopy) given by

$$\lambda(f)(t, x) = (x, tx + (1 - t)f(x)). \tag{4.8}$$

Indeed, if $f, g : S(\mathbb{C}^n) \to S(\mathbb{C}^n)$, then

$$d(\lambda_1(f), \lambda_1(g)) : S^0 * S(\mathbb{C}^n) \to D(\mathbb{C}^n)/S(\mathbb{C}^n)$$

is given by

$$(t, x) \mapsto \begin{cases} (1 - 2t)f(x), & \text{for } 0 \le t \le 1/2 \\ (2t - 1)g(x), & \text{for } 1/2 \le t \le 1, \end{cases}$$

and this is clearly homotopic to the map

$$(t, x) \mapsto tg(x) + (1 - t)f(x).$$

This gives (4.8) when $g = \mathrm{id}$.

5. The complex reflection map

Given $z \in S^1$ and a complex line $L \subseteq \mathbb{C}^{n+1}$, let $R(z, L)$ be the unitary transformation of $\mathbb{C}^{n+1}$ that multiplies by z on L and fixes the orthogonal complement $L^\perp$ pointwise. Since $R(1, L) = \mathrm{id}$, one obtains a map

$$R : S^1 \wedge \mathbb{C}P^n_+ \to U(n + 1).$$

Explicitely, if $\langle v, w\rangle_{\mathbb{C}} = \sum v_i \bar{w}_i$ denotes the usual Hermitian product on $\mathbb{C}^{n+1}$ and we think of $\mathbb{C}P^n$ as $S(\mathbb{C}^{n+1})/S^1$, then

$$R(z, [y])(x) = x + (z - 1)\langle x, y\rangle_{\mathbb{C}} \cdot y, \quad x \in \mathbb{C}^{n+1}. \tag{5.1}$$

Passing to colimits gives

$$R : S^1 \wedge \mathbb{C}P^\infty_+ \to \operatorname{colim} U(n) = U,$$

and we have a commutative diagram

$$\begin{array}{ccc} S^1 \wedge \mathbb{C}P^\infty_+ & \longrightarrow & U \\ \Big\downarrow \text{proj} & & \Big\downarrow \text{det} \\ S^1 & = & S^1. \end{array}$$

It follows that the adjoint

$$\widehat{R} : \mathbb{C}P^\infty_+ \to \Omega(U)$$

maps $\mathbb{C}P^\infty$ to the 1-component $\Omega_{(1)}(U)$. Let

$$\mathrm{B} : BU \times \mathbb{Z} \to \Omega(U)$$

be the Bott periodicity map, and $\mathrm{B}_{(1)}$ the restriction to the 1-component.

Proposition 5.1. *The composite*

$$\mathrm{B}_{(1)}^{-1} \circ \widehat{R} : \mathbb{C}P^\infty \to BU \times \{1\}$$

classifies the canonical line bundle L over $\mathbb{C}P^\infty$.

Proof. We consider the inclusions

$$\frac{U(2n)}{U(n) \times U(n)} \to \frac{U(2(n+1))}{U(n+1) \times U(n+1)}, \quad A \mapsto I_1 \oplus A \oplus I_1,$$

with union

$$BU = \bigcup_n \frac{U(2n)}{U(n) \times U(n)}.$$

We also have inclusions

$$i_n : \mathbb{C}P^n \cong \frac{U(1+n)}{U(1) \times U(n)} \to \frac{U(2n)}{U(n) \times U(n)}, \quad A \mapsto I_{n-1} \oplus A,$$

and these define the map $i : \mathbb{C}P^\infty \to BU$, that classifies the canonical line bundle L over $\mathbb{C}P^\infty$ (or $L - 1$, if we think of BU as $BU \times \{0\}$). We need an explicit construction of the Bott map

$$\mathrm{B}_{(0)} : BU \to \Omega(SU) \simeq \Omega_{(0)}(U),$$

and follow H. Cartan and J. Moore, [12]: Given $n \geq 1$, let $\alpha_n \in \Omega_{(-n)}(U(2n))$ be the loop

$$\alpha_n(\theta) = (\exp(-i2\pi\theta) \cdot I_n) \oplus I_n.$$

Then $B_{(0)}$ is the colimit over n of the adjoints of

$$(I/\partial I) \wedge \frac{U(2n)}{U(n) \times U(n)} \to SU(2n), \quad (\theta, A) \mapsto \alpha_n(\theta) \cdot A \cdot \alpha_n(-\theta) \cdot A^{-1}.$$

The statement in the proposition is equivalent to the commutativity (in the homotopy category) of the diagrams

$$\begin{array}{ccccc} \mathbb{C}P^n & \xrightarrow{i} & \dfrac{U(2n)}{U(n) \times U(n)} & \xrightarrow{\mathrm{B}_{(0)}} & \Omega_{(0)}(U(2n)) \\ \Big\| & & & & \Big\downarrow +1 \\ \mathbb{C}P^n & \xrightarrow{\widehat{R}} & \Omega_{(1)}(U(n+1)) & \xrightarrow{\Omega(i)} & \Omega_{(1)}(U(2n)) \end{array} \tag{5.2}$$

for all n. Here the component-shift $+1$ can be realized by taking pointwise multiplication in $U(2n)$ with any fixed loop in $\Omega_{(1)}(U(2n))$. Given $A \in U(1+n)$, we have

$$\mathrm{B}_{(0)}(i[A])(\theta) = I_{n-1} \oplus (\beta_n(\theta) \cdot A \cdot \beta_n(-\theta) \cdot A^{-1}),$$

where $\beta_n \in \Omega_{(-1)}(U(1+n))$ is the loop

$$\beta_n(\theta) = (\exp(-i2\pi\theta) \cdot I_1) \oplus I_n.$$

On the other hand, if v denotes the first column in A, then

$$R(\exp(i2\pi\theta), [v]) = A \cdot \beta_n(-\theta) \cdot A^{-1},$$

and the commutativity of (5.2) follows. $\square$

Let $J^{S^1} : U \to F^{S^1}$ be the map that restricts a unitary transformation to the unit sphere.

Proposition 5.2. *The restriction of the transfer* trf^{S^1} *to* $S^1 \wedge \mathbb{C}P^\infty_+$*:*

$$S^1 \wedge \mathbb{C}P^\infty_+ \longrightarrow Q(S^1 \wedge \mathbb{C}P^\infty_+) \xrightarrow{\mathrm{trf}^{S^1}} Q_{S^1}(ES^1_+)^{S^1},$$

is homotopic to the composition

$$S^1 \wedge \mathbb{C}P^\infty_+ \xrightarrow{R} U \xrightarrow{J^{S^1}} F^{S^1} \xrightarrow{\lambda} Q_{S^1}(ES^1_+)^{S^1}.$$

In preparation of the proof we have the following remark on the Pontrjagin-Thom map considered in Section 2.

Remark 5.3. With notation of Section 2, let $i : K \to M$ be a smooth G-invariant embedding of a closed G-manifold K. A continuous map $t : M^c \to \mathrm{Th}(N(i))$ is homotopic to the Pontrjagin-Thom map provided that

(i): The restriction of t to $i(K)$ is inverse to i, and t preserves the complements:

$$t(M^c - i(K)) \subseteq \mathrm{Th}(N(i)) - K.$$

(ii): t is smooth in a neighborhood of $i(K)$, and the differential induces an isomorphism on quotient bundles, which is fiberwise diffeotopic to the identity:

$$\begin{array}{ccc} \frac{T(M)|i(K)}{T(i(K))} & \xrightarrow{T(t)} & \frac{T(N(i))|K}{T(K)} \\ \uparrow\cong & & \uparrow\cong \\ N(i) & = & N(i). \end{array}$$

For notational convenience, write $V = \mathbb{C}^{n+1}$ and $P(V) = S(V)/S^1$. The fiber bundle $p : S(V) \to P(V)$, induces an exact sequence

$$0 \to T^v S(V) \to TS(V) \to p^*TP(V) \to 0$$

as in Section 2. We may consider V as the real vectorspace $\mathbb{R}^{2(n+1)}$ equipped with the complex structure $\mathbb{J} : V \to V$,

$$\mathbb{J}(x_1, y_1, \ldots, x_{n+1}, y_{n+1}) = (-y_1, x_1, \ldots, -y_{n+1}, x_{n+1}).$$

We have

$$TS(V) = \{(x, v) \in S(V) \times V \,|\, \langle x, v\rangle = 0\}$$

with subbundle

$$T^v S(V) = \{(x, t \cdot \mathbb{J}(x)) \in S(V) \times V \,|\, t \in \mathbb{R}\}.$$

With this notation we get the trivialization

$$Ad(S^1)_{P(V)} \to T^v S(V)/S^1, \quad ([x], t \cdot \mathbb{J}) \mapsto [x, t \cdot \mathbb{J}(x)], \tag{5.3}$$

where $t \in \mathbb{R}$ and (by abuse of notation), $\mathbb{J} = \exp(i\pi/2)$ is a basis for $Ad(S^1) \subseteq \mathbb{C}$.

Proof of Proposition 5.2. With $V = \mathbb{C}^{n+1}$ as above, let f be the restriction of $\lambda \circ J^{S^1} \circ R$ to $S^1 \wedge P(V)_+$,

$$f : S^1 \wedge P(V)_+ \to \mathrm{Map}_*(S^V, S(V)_+ \wedge S^V)^{S^1}.$$

Using the expression (4.8) for λ, f becomes a pointed map with adjoint

$$\hat{f} : S^1 \wedge P(V)_+ \wedge (S^0 * S(V)) \to S(V)_+ \wedge (D(V)/S(V))$$

given by

$$\hat{f}(z, [y], t, x) = (x, x + (1-t)(z-1)\langle x, y\rangle_{\mathbb{C}} \cdot y). \tag{5.4}$$

On the other hand, in constructing the transfer, we may use the obvious embedding

$$i : S(V) \to P(V) \times V \to (T^v S(V)/S^1) \oplus V_{P(V)},$$

and apply the Pontrjagin-Thom construction to get

$$t : ((T^v S(V)/S^1) \oplus V_{P(V)})^c \to \mathrm{Th}(N(i)).$$

We identify S^1 and $S^{Ad(S^1)}$ by stereographic projection from the unit in S^1, and use (5.3) and the homeomorphisms in (4.6), to give an explicit identification of the domain of $\hat{f}$ with the domain of t. Similarly, the trivialization in (2.4) identifies the target of $\hat{f}$ with $\mathrm{Th}(N(i))$. With these identifications, the embedding i corresponds to

$$\tilde{\imath} : S(V) \to S^1 \wedge P(V)_+ \wedge (S^0 * S(V)), \quad x \mapsto (-1, [x], 1/2, x),$$

and the zero section of $N(i)$ becomes

$$\tilde{s} : S(V) \to S(V)_+ \wedge (D(V)/S(V)), \quad x \mapsto (x, 0).$$

We prove that $f \sim t$ by checking that f satisfies the conditions in Remark 5.3. The first condition, the pull-back diagram

$$\begin{array}{ccc} S(V) & \xrightarrow{\tilde{\imath}} & S^1 \wedge P(V)_+ \wedge (S^0 * S(V)) \\ \| & & \downarrow{\scriptstyle \hat{f}} \\ S(V) & \xrightarrow{\tilde{s}} & S(V)_+ \wedge (D(V)/S(V)), \end{array}$$

follows from (5.4). For $x \in S(V)$, we have the monomorphism

$$V \to T_{\{-1\}} S^1 \times T_{[x]} P(V) \times T_{\{1/2\}} I \times T_x S(V)$$

given by

$$v \mapsto (\langle v, \mathbb{J}(x)\rangle \cdot \mathbb{J}, 0, \langle v, x\rangle, v - (\langle v, \mathbb{J}(x)\rangle \cdot \mathbb{J}(x) + \langle v, x\rangle \cdot x)),$$

where we take $\mathbb{J}$ as a basis for $T_{\{-1\}} S^1$. This gives a trivialization $V_{S(V)} \cong N(\tilde{\imath})$, compatible with the trivialization of $N(i)$ from (2.4). We now have a diagram

$$\begin{array}{ccc} N(\tilde{\imath}) & \xrightarrow{T(f)} & N(\tilde{s}) \\ \uparrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ V_{S(V)} & \longrightarrow & V_{S(V)}, \end{array}$$

and using the relationship between the Hermitian product $\langle\ ,\ \rangle_{\mathbb{C}}$ and the real inner product $\langle\ ,\ \rangle$:

$$\langle v, w\rangle_{\mathbb{C}} = \langle v, w\rangle + \langle v, \mathbb{J}(w)\rangle \cdot \mathbb{J}, \quad v, w \in V,$$

we see that the lower map is given by

$$(x, v) \mapsto (x, v + \langle v, x\rangle \cdot x - (1/2)\langle v, \mathbb{J}(x)\rangle \cdot \mathbb{J}(x)).$$

This is fiberwise diffeotopic to the identity via the diffeotopy

$$h_t(x, v) = (x, v + t(\langle v, x\rangle \cdot x - (1/2)\langle v, \mathbb{J}(x)\rangle \cdot \mathbb{J}(x))).$$

This verifies the second condition in Remark 5.3. □

6. Diagram 1.1

Let K denote the complex periodic K-theory spectrum, and write $\widetilde{K}(E) = [E, K]$ for a spectrum E. For a based space X, we have $\widetilde{K}(X) = \widetilde{K}(\Sigma^\infty(X))$. Recall, that L denotes the tautological line bundle over $\mathbb{C}P^\infty$, and L_n the restriction to $\mathbb{C}P^n$. The standard Thom classes $\lambda_{L_n^\perp} \in \widetilde{K}(\mathrm{Th}(L_n^\perp))$, cf. [7], are compatible under the maps in (3.1) and Bott periodicity, so define

$$\lambda_{\mathbb{C}-L} \in \widetilde{K}(\mathbb{T}\mathrm{h}(\mathbb{C} - L)) = \lim \widetilde{K}(\mathrm{Th}(L_n^\perp)),$$

and $K(\mathbb{C}P^\infty) \cong \widetilde{K}(\mathbb{T}\mathrm{h}(\mathbb{C} - L))$ by $\alpha \mapsto \alpha \cdot \lambda_{\mathbb{C}-L}$.

The inclusion of a fiber into $L_n^\perp$ induce a map from S^{2n} to $\mathrm{Th}(L_n^\perp)$ and in turn a map from the sphere spectrum S

$$i : S \to \mathbb{T}\mathrm{h}(\mathbb{C} - L). \tag{6.1}$$

The element $\lambda_{\mathbb{C}-L}$ is an *orientation class* in the sense that $i^*(\lambda_{\mathbb{C}-L}) \in \widetilde{K}(S) = \mathbb{Z}$ is the multiplicative unit. Any other orientation class has the form $(1 + \alpha_0) \cdot \lambda_{\mathbb{C}-L}$ with $\alpha_0 \in \widetilde{K}(\mathbb{C}P^\infty) = x\mathbb{Z}[[x]]$, $x = L - \mathbb{C}$.

The cofiber of the map i in (6.1) is the spectrum $\mathbb{T}\mathrm{h}(\mathbb{C}) = \Sigma^\infty(S^2 \wedge \mathbb{C}P^\infty_+)$, cf. [17], [23], and we now compare the cofiber sequence

$$S \xrightarrow{i} \mathbb{T}\mathrm{h}(\mathbb{C} - L) \xrightarrow{\omega} \mathbb{T}\mathrm{h}(\mathbb{C}) \xrightarrow{\partial} \Sigma S \tag{6.2}$$

to the standard p-local cofiber sequence from K-theory

$$J_{(p)} \xrightarrow{D} K_{(p)} \xrightarrow{1-\Psi^g} K_{(p)} \longrightarrow \Sigma J_{(p)}. \tag{6.3}$$

Here g is a fixed natural number which defines a generator of the units $(\mathbb{Z}/p^2)^*$ when p is odd, and $g = 3$ when $p = 2$. The operation Ψ^g in (6.3) is the Adams operation ψ^g on the 0'th space of K and equal to $(1/g^n)\psi^g$ on the $2n$'th space $K_{(p)2n} = BU_{(p)} \times \mathbb{Z}_{(p)}$. The spectrum $J_{(p)}$ has the structure of a ring spectrum; let $e_J : S \to J_{(p)}$ be its unit. Then $D \circ e_J = e_K$ is the unit of K, and e_J is the unique lift of e_K.

Consider now the following diagram in which $\alpha_0 \in \widetilde{K}_{(p)}(\mathbb{C}P^\infty)$ is given, and $\beta \in K_{(p)}(\mathbb{C}P^\infty)$ is to be determined:

$$\begin{array}{ccccccc} S & \xrightarrow{i} & \mathbb{T}\mathrm{h}(\mathbb{C} - L) & \xrightarrow{\omega} & \mathbb{T}\mathrm{h}(\mathbb{C}) & \xrightarrow{\partial} & \Sigma S \\ \downarrow{\scriptstyle e_J} & & \downarrow{\scriptstyle (1+\alpha_0)\cdot\lambda_{\mathbb{C}-L}} & & \downarrow{\scriptstyle \beta\cdot\lambda_{\mathbb{C}}} & & \downarrow{\scriptstyle \Sigma e_J} \\ J_{(p)} & \xrightarrow{D} & K_{(p)} & \xrightarrow{1-\Psi^g} & K_{(p)} & \xrightarrow{\Delta} & \Sigma J_{(p)}. \end{array} \tag{6.4}$$

Since $(1+\alpha_0)\cdot\lambda_{\mathbb{C}-L}$ is an orientation class, the left hand square homotopy commutes, and since $\pi_{-1}(K_{(p)})=0$, there is a unique element $\beta\in K_{(p)}(\mathbb{C}P^\infty)$ making the ladder commutative.

Lemma 6.1. *Write $\alpha_0=x\alpha_1$, $\alpha_1\in K(\mathbb{C}P^\infty)$. The class $\beta\in K_{(p)}(\mathbb{C}P^\infty)$ in (6.4) is given by*

$$\beta=-\frac{L^{g-2}+2L^{g-3}+\cdots+(g-1)}{L^{g-1}+L^{g-2}+\cdots+1}-(\alpha_1-g\psi^g(\alpha_1)).$$

The proof of Lemma 6.1 is based upon the exponential homomorphism

$$\rho^g: K(X)\to 1+\widetilde{K}_{(p)}(X)\subset K_{(p)}(X),$$

cf. [1]. On an n-dimensional complex vector bundle

$$\rho^g(E)\cdot\lambda_E=(1/g^n)\cdot\psi^g(\lambda_E),$$

and for a line bundle L,

$$\rho^g(L)=(1/g)\cdot(1+L+\cdots+L^{g-1}). \tag{6.5}$$

Proof of Lemma 6.1. The map ω in (6.3) induces a $K_{(p)}(\mathbb{C}P^\infty)$-linear homomorphism

$$\omega^*:\widetilde{K}_{(p)}(\mathbb{T}\mathrm{h}(\mathbb{C}))\to\widetilde{K}_{(p)}(\mathbb{T}\mathrm{h}(\mathbb{C}-L)),$$

with

$$\omega^*(\lambda_{\mathbb{C}})=\omega^*(\lambda_{L\oplus(\mathbb{C}-L)})=\lambda_{-1}(L)\cdot\lambda_{\mathbb{C}-L}.$$

Here $\lambda_{-1}(L)$ denotes the K-theoretic Euler class of L. On a line bundle, $\lambda_{-1}(L)=1-L$, so

$$\omega^*(\beta\cdot\lambda_{\mathbb{C}})=-\beta x\cdot\lambda_{\mathbb{C}-L}. \tag{6.6}$$

On the other hand, an easy calculation using (6.5) shows that

$$\begin{aligned}&(1-\Psi^g)((1+x\alpha_1)\cdot\lambda_{\mathbb{C}-L})\\ &=(1+x\alpha_1-\rho^g(-L)\cdot\psi^g(1+x\alpha_1))\cdot\lambda_{\mathbb{C}-L}\\ &=(L-1)\left(\frac{L^{g-2}+2L^{g-3}+\cdots+(g-1)}{L^{g-1}+L^{g-2}+\cdots+1}+\alpha_1-g\psi^g(\alpha_1)\right)\cdot\lambda_{\mathbb{C}-L}.\end{aligned}$$

Comparing with (6.6) gives the expression for β.

□

If we in Lemma 6.1 let $\alpha_0=L-\mathbb{C}$, then β becomes the class

$$\beta=\frac{(g-1)L^{g-1}+(g-2)L^{g-2}+\cdots+L}{L^{g-1}+L^{g-2}+\cdots+1}. \tag{6.7}$$

This element has an alternative description as follows. Let

$$\tilde\rho=\rho^g-1:\widetilde{K}(X)\to\widetilde{K}_{(p)}(X),$$

and define the operation $\Omega^2(\tilde\rho)$ by commutativity of the diagram

$$\begin{array}{ccc}\widetilde{K}(X) & \xrightarrow{\cdot\lambda_{\mathbb{C}}} & \widetilde{K}(S^2\wedge X)\\ \big\downarrow{\scriptstyle\Omega^2(\tilde\rho)} & & \big\downarrow{\scriptstyle\tilde\rho}\\ \widetilde{K}_{(p)}(X) & \xrightarrow{\cdot\lambda_{\mathbb{C}}} & \widetilde{K}_{(p)}(S^2\wedge X).\end{array}$$

(The isomorphism $\cdot\lambda_{\mathbb{C}}$ is the Bott periodicity operator that defines the structure maps in the spectrum K).

Proposition 6.2. *The β corresponding to $\alpha_0 = L - \mathbb{C}$ in Lemma 6.1 is $\beta = \Omega^2(\tilde{\rho})(L)$.*

Proof. Let H be the canonical line bundle over $\mathbb{C}P^1 = S^2$, so that $\lambda_{\mathbb{C}} = H - \mathbb{C}$. Since ρ^g is exponential,

$$\tilde{\rho}(L \cdot \lambda_{\mathbb{C}}) = \rho^g(HL - L) - 1 = \frac{\rho^g(HL) - \rho^g(L)}{\rho^g(L)}.$$

Now $H^i - \mathbb{C} = i(H - \mathbb{C}) = i\lambda_{\mathbb{C}}$, so

$$\begin{aligned}\rho^g(HL) &= (1/g)((HL)^{g-1} + \cdots + HL + 1)\\ &= \rho^g(L) + (1/g)\lambda_{\mathbb{C}}((g-1)L^{g-1} + \cdots + L),\end{aligned}$$

and thus

$$\tilde{\rho}(L \cdot \lambda_{\mathbb{C}}) = \frac{(g-1)L^{g-1} + (g-2)L^{g-2} + \cdots + L}{L^{g-1} + L^{g-2} + \cdots + 1} \cdot \lambda_{\mathbb{C}}.$$

The result follows by comparison with (6.7). □

The spectra in (6.2) are all connective, whereas the spectra in (6.3) are periodic and thus has non-zero homotopy groups in negative dimensions. Let $ku = K[0, \infty)$ denote the (-1)-connected cover of K, and $bu = K[2, \infty)$ its 1-connected cover. Let also $ju_{(p)} = J_{(p)}[0, \infty)$. The diagram (6.4) then lifts to the p-local diagram of connective spectra.

$$\text{(6.8)}\qquad \begin{array}{ccccccc} S_{(p)} & \xrightarrow{i} & \mathbb{T}\mathrm{h}(\mathbb{C} - L)_{(p)} & \xrightarrow{\omega} & \mathbb{T}\mathrm{h}(\mathbb{C})_{(p)} & \xrightarrow{\partial} & \Sigma S_{(p)} \\ \downarrow{\scriptstyle e_J} & & \downarrow{\scriptstyle L \cdot \lambda_{\mathbb{C}-L}} & & \downarrow{\scriptstyle \Omega^2(\tilde{\rho})(L) \cdot \lambda_{\mathbb{C}}} & & \downarrow{\scriptstyle \Sigma e_J} \\ ju_{(p)} & \xrightarrow{D} & ku_{(p)} & \xrightarrow{1-\Psi^g} & bu_{(p)} & \xrightarrow{\Delta} & \Sigma ju_{(p)}. \end{array}$$

On the 0'th level of the associated Ω spectra, e_J is split surjective by the affirmed Adams conjecture. This is far from being the case for the two middle arrows. In fact they are not even rational equivalences. We shall remedy this fact by rechoosing the middle maps. This requires that one decomposes $K_{(p)}$ into its so called Adams components, cf. [4], [8]. There are idempotent operations E_i on $K_{(p)}(X)$ for $i \in \mathbb{Z}/(p-1)$, so

$$K_{(p)}(X) = \bigoplus_{i=0}^{p-2} K^{[i]}_{(p)}(X),$$

with $K^{[i]}_{(p)}(X) = E_i K_{(p)}(X)$. The coefficient groups are

$$\text{(6.9)}\qquad \widetilde{K}^{[i]}_{(p)}(S^{2n}) = \begin{cases} \mathbb{Z}_{(p)} & \text{for } n \equiv i \bmod (p-1), \\ 0 & \text{otherwise,} \end{cases}$$

and one has the isomorphisms:

(i): $1 - \psi^g : K^{[i]}_{(p)}(X) \xrightarrow{\simeq} K^{[i]}_{(p)}(X)$, if $i \not\equiv 0 \bmod (p-1)$,

(ii): $\tilde{\rho} : K^{[0]}_{(p)}(X) \xrightarrow{\simeq} K^{[0]}_{(p)}(X)$, for X connected.

(The action of $1 - \psi^g$ on $\widetilde{K}(S^{2n})$ is by multiplication with $1 - g^n$, so (i) is immediate. (ii) is more complicated and involves Bernoulli numbers, see [1] or [8]). Since $1 - \Psi^g$

is an equivalence of $K^{[i]}$ when $i \not\equiv 0 \bmod (p-1)$, (6.3) implies the cofibration sequence

$$J_{(p)} \xrightarrow{D^{[0]}} K^{[0]}_{(p)} \xrightarrow{1-\Psi^g} K^{[0]}_{(p)} \xrightarrow{\Delta^{[0]}} \Sigma J_{(p)},$$

and $\Delta^{[0]} \circ E_0 = \Delta$.

We now modify the middle arrows in (6.8) on the non-zero Adams components. Let $l \in \widetilde{K}_{(p)}(\mathbb{T}\mathrm{h}(\mathbb{C})) = \bigoplus \widetilde{K}^{[i]}_{(p)}(\mathbb{T}\mathrm{h}(\mathbb{C}))$ have components

$$l^{[i]} = \begin{cases} \tilde{\rho}^{[0]} \circ E_0(L \cdot \lambda_{\mathbb{C}}) & \text{for } i = 0, \\ (1 - \Psi^g) \circ E_i(L \cdot \lambda_{\mathbb{C}}) & \text{for } i \neq 0. \end{cases} \tag{6.10}$$

Then l is represented by the map

$$S^2 \wedge \mathbb{C}P^\infty_+ \xrightarrow{L \cdot \lambda_{\mathbb{C}}} BU_{(p)} \times \mathbb{Z}_{(p)} \xrightarrow{h} BU_{(p)} \times \mathbb{Z}_{(p)}, \tag{6.11}$$

where h acts by $\tilde{\rho}$ on the 0'th Adams component, and by $1 - \psi^g$ on the other. Notice that h induces a homotopy equivalence on $BU_{(p)} \times \{0\}$.

Similarly, let $\tilde{l} \in \widetilde{K}_{(p)}(\mathbb{T}\mathrm{h}(\mathbb{C} - L)) = \bigoplus \widetilde{K}^{[i]}_{(p)}(\mathbb{T}\mathrm{h}(\mathbb{C} - L))$ have components

$$\tilde{l}^{[i]} = \begin{cases} E_0(L \cdot \lambda_{\mathbb{C}-L}) & \text{for } i = 0, \\ \omega^* E_i(L \cdot \lambda_{\mathbb{C}}) & \text{for } i \neq 0. \end{cases} \tag{6.12}$$

On the spectrum level we pass to connective covers and get maps

$$l : \mathbb{T}\mathrm{h}(\mathbb{C}) \to bu_{(p)}, \qquad \tilde{l} : \mathbb{T}\mathrm{h}(\mathbb{C} - L) \to ku_{(p)}.$$

Theorem 6.3. *There is a commutative diagram of spectrum cofibration sequences*

$$\begin{array}{ccccccc} S & \xrightarrow{i} & \mathbb{T}\mathrm{h}(\mathbb{C} - L) & \xrightarrow{\omega} & \mathbb{T}\mathrm{h}(\mathbb{C}) & \xrightarrow{\partial} & \Sigma S \\ \downarrow{\scriptstyle e_J} & & \downarrow{\scriptstyle \tilde{l}} & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle \Sigma e_J} \\ ju_{(p)} & \xrightarrow{D} & ku_{(p)} & \xrightarrow{1-\Psi^g} & bu_{(p)} & \xrightarrow{\Delta} & \Sigma ju_{(p)}, \end{array} \tag{6.13}$$

and the vertical maps are rational equivalences.

Proof. The commutativity follows from the above remarks and Proposition 6.2. Furthermore, the outer arrows are clearly rational equivalences, and it suffices to check that the same holds for l. The double desuspension

$$\Sigma^{-2}(l) : \Sigma^\infty(\mathbb{C}P^\infty_+) \to ku_{(p)},$$

is the unique extension of the space map

$$\mathbb{C}P^\infty_+ \xrightarrow{L} BU_{(p)} \times \mathbb{Z}_{(p)} \xrightarrow{\Omega^2(h)} BU_{(p)} \times \mathbb{Z}_{(p)}.$$

Since $\Omega^2(h)$ is an H-map and a homotopy equivalence, a straightforward homology calculation implies that $\Sigma^{-2}(l)$ is a rational equivalence. (It also follows from the splitting considered in the next section). $\square$

7. The splitting theorem

The cofibration sequence (6.2) gives upon passage to infinite loop spaces a homotopy fibration sequence

$$\Omega^\infty(\Sigma\,\mathbb{T}\mathrm{h}(-L)) \xrightarrow{\omega} Q(S^1\wedge \mathbb{C}P^\infty_+) \xrightarrow{\partial} Q(S^0). \tag{7.1}$$

Proposition 7.1 ([17])**.** *The map ∂ in (7.1) is homotopic to the composition*

$$\mathrm{trf}: Q(S^1\wedge \mathbb{C}P^\infty_+) \xrightarrow{\mathrm{trf}^{S^1}} Q_{S^1}(ES^1_+)^{S^1} \xrightarrow{\mathrm{inc}} Q_{S^1}(ES^1_+) \simeq Q(S^0),$$

with trf^{S^1} *from Theorem 2.1.*

Proof. This follows from Proposition 3.1 by applying $\mathrm{Map}(-, Q_{S^1}(ES^1_+))^{S^1}$ to the equivariant cofibration sequence $S^1_+ \to S^0 \to S^{\mathbb{C}}$, and using that

$$\mathrm{Map}(S^1_+, Q_{S^1}(ES^1_+))^{S^1} \cong \mathrm{Map}(S^0, Q_{S^1}(ES^1_+)) \simeq Q(S^0).$$

(In fact, one can derive the entire cofibration sequence (6.2) this way). □

Recall that $\Omega^\infty(\Sigma^{-1}ku_{(p)}) \simeq U_{(p)}$ and $\Omega^\infty(\Sigma^{-1}bu_{(p)}) \simeq U_{(p)}$, and that the usual complex image of J space is $JU_{(p)}\times \mathbb{Z}_{(p)} = \Omega^\infty(ju_{(p)})$. Then applying $\Omega^\infty\circ\Sigma^{-1}$ to (6.13) we get following diagram of infinite loop spaces.

$$\begin{array}{ccccc}
\Omega^\infty(\Sigma\,\mathbb{T}\mathrm{h}(-L)) & \xrightarrow{\omega} & Q(S^1\wedge\mathbb{C}P^\infty_+) & \xrightarrow{\mathrm{trf}} & Q(S^0) \\
\downarrow{\scriptstyle \tilde{l}} & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle e} \\
U_{(p)} & \xrightarrow{\Omega(1-\psi^g)} & U_{(p)} & \xrightarrow{\Delta} & JU_{(p)}\times\mathbb{Z}_{(p)}.
\end{array} \tag{7.2}$$

It follows from Proposition 5.1, that l is the infinite loop map extension of

$$S^1\wedge\mathbb{C}P^\infty_+ \xrightarrow{R} U \longrightarrow U_{(p)} \xrightarrow{\Omega(h)} U_{(p)}, \tag{7.3}$$

with h defined in (6.11). Let s be the composite

$$s: U \xrightarrow{J^{S^1}} F^{S^1} \xrightarrow{\bar{\lambda}} Q(S^1\wedge\mathbb{C}P^\infty_+), \tag{7.4}$$

with J^{S^1} the obvious inclusion that restricts a unitary transformation to the unit sphere, and $\bar{\lambda}$ the equivalence of Theorem 4.1.

Theorem 7.2. *The composition $l\circ s: U_{(p)} \to U_{(p)}$ is a homotopy equivalence.*

The next three lemmas provide the proof.

Lemma 7.3. *The maps*

$$[BU,BU] \to \mathrm{Hom}(H_*(BU,\mathbb{Q}), H_*(BU,\mathbb{Q}))$$
$$[U,U] \to \mathrm{Hom}(H_*(U,\mathbb{Q}), H_*(U,\mathbb{Q}))$$

are injections. The same hold for the localized spaces $BU_{(p)}$ and $U_{(p)}$.

Proof. The groups

$$[BU,BU] \cong \lim K(BU(n))$$
$$[U,U] \cong \lim K^{-1}(U(n))$$

are torsion free (see e.g. [7, p.112, p.116]), and since the Chern character is rationally an isomorphism, elements of $[BU,BU]$ and $[U,U]$ are determined by their action on rational cohomology. Dually, such elements are then also determined by their action on rational homology. □

Given a map $f : X \to E$ with E an infinite loop space, we write $Q(f) : Q(X) \to E$ for the infinite loop map that restricts to f on X.

Lemma 7.4 ([13]). *For any map $f : S^1 \wedge \mathbb{C}P^\infty_+ \to U$, the composite map*

$$\phi : F^{S^1} \xrightarrow{\lambda} Q(S^1 \wedge \mathbb{C}P^\infty_+) \xrightarrow{Q(f)} U$$

is an H-map, and similarly if U is replaced by $U_{(p)}$.

Proof. Recall from Proposition 4.3 and the paragraphs following it that $Q(S^1 \wedge \mathbb{C}P^\infty_+)$ has a product $\bar{\mu}$ over which $\bar{\lambda}$ becomes quadratic. Since $Q(f)$ is additive,

$$\phi(x \cdot y) \sim \phi(x) + \phi(y) - Q(f) \circ \bar{\mu}(\bar{\lambda}(x), \bar{\lambda}(y)).$$

In order to see that the term $Q(f) \circ \bar{\mu}(\lambda(x), \lambda(y))$ vanish, we use the factorization of $\bar{\mu}$ from (4.5), $\bar{\mu} = \mathrm{trf} \circ \wedge$. By definition, $\mathrm{trf} \circ Q(f)$ is an infinite loop map, and it suffices to prove that the restriction to $S^2 \wedge \mathbb{C}P^\infty_+ \wedge \mathbb{C}P^\infty_+$ is null homotopic, which is the case since $K^1(\mathbb{C}P^\infty_+ \wedge \mathbb{C}P^\infty_+) = 0$. □

Lemma 7.5. *Let $f : U \to U$ be an H-map. Then the composite map*

$$\phi : U \xrightarrow{s} Q(S^1 \wedge \mathbb{C}P^\infty_+) \xrightarrow{Q(f \circ R)} U$$

is homotopic to f, and similarly in the p-local situation.

Proof. According to Lemma 7.3, it suffices to prove that ϕ and f induce the same map in rational homology. It is well known that $H_*(U, \mathbb{Q})$ is an exterior algebra generated by the image of

$$R_* : H_*(S^1 \wedge \mathbb{C}P^\infty_+, \mathbb{Q}) \to H_*(U, \mathbb{Q}),$$

cf. [25]. Lemma 7.4, implies that f_* and ϕ_* are algebra homomorphisms, so it suffices to prove that the restrictions $f \circ R$ and $\phi \circ R$ are homotopic. This is a consequence of Proposition 5.2: The diagram

$$\begin{tikzcd}
S^1 \wedge \mathbb{C}P^\infty_+ \arrow[r, equal] \arrow[d, "R"] & S^1 \wedge \mathbb{C}P^\infty_+ \arrow[d, "\mathrm{inc}"] \\
U \arrow[r, "s"] & Q(S^1 \wedge \mathbb{C}P^\infty_+)
\end{tikzcd}$$

is homotopy commutative. □

Proof of Proposition 7.2. Since $\Omega(h)$ is an H-map and a homotopy equivalence, the statement in the proposition is a direct consequence of Lemma 7.5. □

The next result is a well-known reformulation of the main result of [2], given the affirmed Adams conjecture.

Proposition 7.6. *The map $e_J : Q_{(0)}(S^0)_{(p)} \to JU_{(p)}$ is split surjective provided p is odd.*

Proof. The affirmed Adams conjecture is the statement that the sequence

$$BU_{(p)} \xrightarrow{1-\psi^g} BU_{(p)} \longrightarrow BSF_{(p)}$$

is null homotopic, see e.g. [9]. This implies the homotopy commutative diagram

$$\begin{tikzcd} U_{(p)} \arrow[r, "\Delta"] \arrow[d, equal] & JU_{(p)} \arrow[r, "D"] \arrow[d, "\alpha_1"] & BU_{(p)} \arrow[r, "1-\psi^g"] \arrow[d, "\alpha_0"] & BU_{(p)} \arrow[d, equal] \\ U_{(p)} \arrow[r] & SF_{(p)} \arrow[r] & SF_{(p)}/U_{(p)} \arrow[r] & BU_{(p)}. \end{tikzcd} \tag{7.5}$$

We compose α_1 with the map $\zeta : SF \to Q_{(0)}(S^0)$ from (4.3) to get

$$\alpha = \zeta \circ \alpha_1 : JU_{(p)} \to Q_{(0)}(S^0)_{(p)}. \tag{7.6}$$

The composite $e_J \circ \alpha$ is a homotopy equivalence, see e.g. [14], [20]. □

Lemma 7.7. *With s and α being the maps defined in (7.4) and (7.6) respectively, the diagram*

$$\begin{tikzcd} U_{(p)} \arrow[r, "\Delta"] \arrow[d, "s"] & JU_{(p)} \arrow[d, "\alpha"] \\ Q(S^1 \wedge \mathbb{C}P^\infty_+)_{(p)} \arrow[r, "\mathrm{trf}"] & Q_{(0)}(S^0)_{(p)} \end{tikzcd}$$

is homotopy commutative.

Proof. Consider the homotopy commutative diagram

$$\begin{tikzcd} U \arrow[r, "J^{S^1}"] \arrow[d, equal] & F^{S^1} \arrow[r, "\bar\lambda"] \arrow[d] & Q(S^1 \wedge \mathbb{C}P^\infty_+) \arrow[d, "\mathrm{trf}"] \\ U \arrow[r, "J"] & SF \arrow[r, "\zeta"] & Q_{(0)}(S^0), \end{tikzcd}$$

where the square follows from Corollary 4.2 upon composing with the inclusion $Q_{S^1}(S^0)^{S^1} \to Q(S^0)$. □

Theorem 7.8. *For odd primes p, the map*

$$\tilde{l} : \Omega^\infty \Sigma\, \mathbb{T}\mathrm{h}(-L)_{(p)} \to U_{(p)}$$

from (7.2) is split surjective up to homotopy.

Proof. From Lemma 7.7 we get the diagram

$$\begin{tikzcd} U_{(p)} \arrow[r, "\Omega(1-\psi^g)"] \arrow[d, "\tilde{l}"] & U_{(p)} \arrow[r] \arrow[d, "s"] & JU_{(p)} \arrow[d, "\alpha"] \\ \Omega^\infty \Sigma\, \mathbb{T}\mathrm{h}(-L)_{(p)} \arrow[r] & Q(S^1 \wedge \mathbb{C}P^\infty_+)_{(p)} \arrow[r, "\mathrm{trf}"] & Q_{(0)}(S^0)_{(p)} \end{tikzcd} \tag{7.7}$$

In view of Theorem 7.2 and Proposition 7.6, this splits diagram (7.2), and in particular $\tilde{l} \circ \tilde{s}$ is an equivalence. □

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