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ON THE K-THEORY OF LOCAL FIELDS

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Lars Hesselholt* and Ib Madsen**

Contents

Introduction

In this paper we establish a connection between the Quillen K-theory of certain local fields and the de Rham-Witt complex of their rings of integers with logarithmic poles at the maximal ideal. The fields K we consider are complete discrete valuation fields of characteristic zero with perfect residue field k of characteristic $p > 2$. When K contains the p^vth roots of unity, the relationship between K-theory and the de Rham-Witt complex can be described by a sequence

$$\cdots \to K_*(K,\mathbb{Z}/p^v) \to W\,\omega^*_{(A,M)} \otimes S_{\mathbb{Z}/p^v}(\mu_{p^v}) \xrightarrow{1-F} W\,\omega^*_{(A,M)} \otimes S_{\mathbb{Z}/p^v}(\mu_{p^v}) \xrightarrow{\partial} \cdots$$

which is exact in degrees ≥ 1. Here $A = \mathcal{O}_K$ is the valuation ring and $W\,\omega^*_{(A,M)}$ is the de Rham-Witt complex of A with log poles at the maximal ideal. The factor $S_{\mathbb{Z}/p^v}(\mu_{p^v})$ is the symmetric algebra of μ_{p^v} considered as a $\mathbb{Z}/p^v$-module located in degree two. Using this sequence, we evaluate the K-theory with $\mathbb{Z}/p^v$-coefficients of K. The result, which is valid also if K does not contain the p^vth roots of unity, verifies the Lichtenbaum-Quillen conjecture for K, [**21**], [**30**]:

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THEOREM A. *There are natural isomorphisms for $s \geq 1$,*

$$K_{2s}(K, \mathbb{Z}/p^v) = H^0(K, \mu_{p^v}^{\otimes s}) \oplus H^2(K, \mu_{p^v}^{\otimes(s+1)}),$$
$$K_{2s-1}(K, \mathbb{Z}/p^v) = H^1(K, \mu_{p^v}^{\otimes s}).$$

The Galois cohomology on the right can be effectively calculated when k is finite, or equivalently, when K is a finite extension of $\mathbb{Q}_p$, [**34**]. For m prime to p,

$$K_i(K, \mathbb{Z}/m) = K_i(k, \mathbb{Z}/m) \oplus K_{i-1}(k, \mathbb{Z}/m)$$

by Gabber-Suslin, [**36**], and for k finite, the K-groups on the right are known by Quillen, [**28**].

For any linear category with cofibrations and weak equivalences in sense of [**40**], one has the cyclotomic trace

$$\mathrm{tr}\colon K(\mathcal{C}) \to \mathrm{TC}(\mathcal{C}; p)$$

from K-theory to topological cyclic homology, [**6**]. It coincides in the case of the exact category of finitely generated projective modules over a ring with the original definition in [**3**]. The exact sequence above and theorem A are based upon calculation of $\mathrm{TC}_*(\mathcal{C}; p, \mathbb{Z}/p^v)$ for certain categories associated with the field K. Let $A = \mathcal{O}_K$ be the valuation ring in K, and let $\mathcal{P}_A$ be the category of finitely generated projective A-modules. We consider three categories with cofibrations and weak equivalences: the category $C^b_z(\mathcal{P}_A)$ of bounded complexes in $\mathcal{P}_A$ with homology isomorphisms as weak equivalences, the subcategory with cofibrations and weak equivalences $C^b_z(\mathcal{P}_A)^q$ of complexes whose homology is torsion, and the category $C^b_q(\mathcal{P}_A)$ of bounded complexes in $\mathcal{P}_A$ with rational homology isomorphisms as weak equivalences. One then has a cofibration sequence of K-theory spectra

$$K(C^b_z(\mathcal{P}_A)^q) \xrightarrow{i^!} K(C^b_z(\mathcal{P}_A)) \xrightarrow{j} K(C^b_q(\mathcal{P}_A)) \xrightarrow{\partial} \Sigma K(C^b_z(\mathcal{P}_A)^q),$$

and using Waldhausen's approximation theorem, the terms in this sequence may be identified with the K-theory of the exact categories $\mathcal{P}_k$, $\mathcal{P}_A$ and $\mathcal{P}_K$. The associated long-exact sequence of homotopy groups is the localization sequence of [**29**],

$$\dots \to K_i(k) \xrightarrow{i^!} K_i(A) \xrightarrow{j_*} K_i(K) \xrightarrow{\partial} K_{i-1}(k) \to \dots$$

The map ∂ is a split surjection by [**12**]. We show in §1 below that one has a similar cofibration sequence of topological cyclic homology spectra

$$\mathrm{TC}(C^b_z(\mathcal{P}_A)^q; p) \xrightarrow{i^!} \mathrm{TC}(C^b_z(\mathcal{P}_A); p) \xrightarrow{j} \mathrm{TC}(C^b_q(\mathcal{P}_A); p) \xrightarrow{\partial} \Sigma\, \mathrm{TC}(C^b_z(\mathcal{P}_A)^q; p),$$

and again Waldhausen's approximation theorem allows us to identify the first two terms on the left with the topological cyclic homology of the exact categories $\mathcal{P}_k$ and $\mathcal{P}_A$. But the third term is different from the topological cyclic homology of $\mathcal{P}_K$. We write

$$\mathrm{TC}(A|K; p) = \mathrm{TC}(C^b_q(\mathcal{P}_A); p),$$

and we then have a map of cofibration sequences

$$\begin{array}{ccccccc}
K(k) & \xrightarrow{i!} & K(A) & \xrightarrow{j_*} & K(K) & \xrightarrow{\partial} & \Sigma K(k) \\
\downarrow{\scriptstyle \mathrm{tr}} & & \downarrow{\scriptstyle \mathrm{tr}} & & \downarrow{\scriptstyle \mathrm{tr}} & & \downarrow{\scriptstyle \mathrm{tr}} \\
\mathrm{TC}(k;p) & \xrightarrow{i!} & \mathrm{TC}(A;p) & \xrightarrow{j_*} & \mathrm{TC}(A|K;p) & \xrightarrow{\partial} & \Sigma\, \mathrm{TC}(k;p).
\end{array}$$

By [**16**, theorem D], the first two vertical maps from the left induce isomorphism of homotopy groups with $\mathbb{Z}/p^v$-coefficients in degrees ≥ 0. It follows that the remaining two vertical maps induce isomorphism of homotopy groups with $\mathbb{Z}/p^v$-cofficients in degrees ≥ 1,

$$\mathrm{tr}\colon K_i(K, \mathbb{Z}/p^v) \xrightarrow{\sim} \mathrm{TC}_i(A|K; p, \mathbb{Z}/p^v), \quad i \geq 1.$$

It is the right hand side we evaluate.

The spectrum $\mathrm{TC}(\mathcal{C}; p)$ is defined as the homotopy fixed points of an operator called Frobenius on another spectrum $\mathrm{TR}(\mathcal{C}; p)$, so there is a natural cofibration sequence

$$\mathrm{TC}(\mathcal{C}; p) \to \mathrm{TR}(\mathcal{C}; p) \xrightarrow{1-F} \mathrm{TR}(\mathcal{C}; p) \to \Sigma\,\mathrm{TC}(\mathcal{C}; p).$$

The spectrum $\mathrm{TR}(\mathcal{C}; p)$, in turn, is the homotopy limit of a pro-spectrum $\mathrm{TR}^{\cdot}(\mathcal{C}; p)$, its homotopy groups given by the Milnor sequence

$$0 \to \varprojlim_R{}^1 \mathrm{TR}^{\cdot}_{s+1}(\mathcal{C}; p) \to \mathrm{TR}_s(\mathcal{C}; p) \to \varprojlim_R \mathrm{TR}^{\cdot}_s(\mathcal{C}; p) \to 0,$$

and there are maps of pro-spectra

$$F\colon\ \mathrm{TR}^n(\mathcal{C}; p) \to \mathrm{TR}^{n-1}(\mathcal{C}; p),$$
$$V\colon\ \mathrm{TR}^{n-1}(\mathcal{C}; p) \to \mathrm{TR}^n(\mathcal{C}; p).$$

The spectrum $\mathrm{TR}^1(\mathcal{C}; p)$ is the topological Hochschild homology $T(\mathcal{C})$. It has an action by the circle group $\mathbb{T}$ and the higher levels in the pro-system by definition are the fixed sets of the cyclic subgroups of $\mathbb{T}$ of p-power order,

$$\mathrm{TR}^n(\mathcal{C}; p) = T(\mathcal{C})^{C_{p^{n-1}}}.$$

The map F is the obvious inclusion and V is the accompanying transfer. The structure map R in the pro-system is harder to define and uses the so-called cyclotomic structure of $T(\mathcal{C})$, see §1 below.

The homotopy groups $\mathrm{TR}^{\cdot}_*(A|K; p)$ of this pro-spectrum with its various operators have a rich algebraic structure which we now describe. The description involves the notion of a log differential graded ring from [**19**]. A log ring (R, M) is a ring R with a pre-log structure, defined as a map of monoids

$$\alpha\colon M \to (R, \cdot\,),$$

and a log differential graded ring (E^*, M) is a differential graded ring E^*, a pre-log structure $\alpha\colon M \to E^0$ and a map of monoids $d\log\colon M \to (E^1, +)$ which satisfies $d \circ d\log = 0$ and $d\alpha(a) = \alpha(a) d\log a$ for all $a \in M$. There is a universal log differential graded ring with underlying log ring (R, M): the de Rham complex with log poles $\omega^*_{(R,M)}$.

The groups $\mathrm{TR}^1_*(A|K; p)$ form a log differential graded ring whose underlying log ring is $A = \mathcal{O}_K$ with the *canonical* pre-log structure given by the inclusion

$$\alpha\colon M = A \cap K^\times \to A.$$

We show that the canonical map

$$\omega^*_{(A,M)} \to \mathrm{TR}^1_*(A|K; p)$$

is an isomorphism in degrees ≤ 2 and that the left hand side is uniquely divisible in degrees ≥ 2. We do not know a *natural* description of the higher homotopy groups, but we do for the homotopy groups with $\mathbb{Z}/p$-coefficients. The Bockstein

$$\mathrm{TR}_2^1(A|K;p,\mathbb{Z}/p) \xrightarrow{\sim} {}_p\mathrm{TR}_1^1(A|K;p)$$

is an isomorphism, and we let κ be the element on the left which corresponds to the class $d\log p$ on the right. The abstract structure of the groups $\mathrm{TR}_*^1(A;p)$ was determined in [**22**]. We use this calculation in §2 below to show

THEOREM B. *There is a natural isomorphism of log differential graded rings*

$$\omega_{(A,M)}^* \otimes_{\mathbb{Z}} S_{\mathbb{F}_p}\{\kappa\} \xrightarrow{\sim} \mathrm{TR}_*^1(A|K;p,\mathbb{Z}/p),$$

where $d\kappa = \kappa d\log p$.

The higher levels $\mathrm{TR}_*^n(A|K;p)$ are also log differential graded rings. The underlying log ring is the ring of Witt vectors $W_n(A)$ with the pre-log structure

$$M \xrightarrow{\alpha} A \to W_n(A),$$

where the right hand map is the multiplicative section $\underline{a}_n = (a, 0, \dots, 0)$. The maps R, F and V extend the restriction, Frobenius and Verschiebung of Witt vectors. Moreover,

$$F\colon \mathrm{TR}_*^n(A|K;p) \to \mathrm{TR}_*^{n-1}(A|K;p)$$

is a map of pro-log graded rings, which satisfies

$$\begin{aligned} F d\log_n a &= d\log_{n-1} a, && \text{for all } a \in M = A \cap K^\times, \\ F d\underline{a}_n &= \underline{a}_{n-1}^{p-1} d\underline{a}_{n-1}, && \text{for all } a \in A, \end{aligned}$$

and V is a map of pro-graded modules over the pro-graded ring $\mathrm{TR}_*^{\boldsymbol\cdot}(A|K:p)$,

$$V\colon F^*\,\mathrm{TR}_*^{n-1}(A|K;p) \to \mathrm{TR}_*^n(A|K;p).$$

Finally,

$$\begin{aligned} FdV &= d, \\ FV &= p. \end{aligned}$$

The algebraic structure described here makes sense for any log ring (R,M), and we show that there exists a universal example: the de Rham-Witt pro-complex with log poles $W.\,\omega_{(R,M)}^*$. For log rings of characteristic $p > 0$, a different construction has been given by Hyodo-Kato, [**17**].

We show in §3 below that the canonical map

$$W.\,\omega_{(A,M)}^* \to \mathrm{TR}_*^{\boldsymbol\cdot}(A|K;p)$$

is an isomorphism in degrees ≤ 2 and that the left hand side is uniquely divisible in degrees ≥ 2. Suppose that $\mu_{p^v} \subset K$. We then have a map

$$S_{\mathbb{Z}/p^v}(\mu_{p^v}) \to \mathrm{TR}_*^{\boldsymbol\cdot}(A|K;p,\mathbb{Z}/p^v)$$

which takes $\zeta \in \mu_{p^v}$ to the associated Bott element defined as the unique element with image $d\log_{\boldsymbol\cdot}\zeta$ under the Bockstein

$$\mathrm{TR}_2^{\boldsymbol\cdot}(A|K;p,\mathbb{Z}/p^v) \xrightarrow{\sim} {}_{p^v}\mathrm{TR}_1^{\boldsymbol\cdot}(A|K;p).$$

The following is the main theorem of this paper.

Theorem C. *Suppose that $\mu_{p^v} \subset K$. Then the canonical map*

$$W.\,\omega^*_{(A,M)} \otimes_{\mathbb{Z}} S_{\mathbb{Z}/p^v}(\mu_{p^v}) \xrightarrow{\sim} \mathrm{TR}^{\cdot}_*(A|K;p,\mathbb{Z}/p^v)$$

is a pro-isomorphism.

It is in order to explain the structure of the groups in the theorem. We limit ourselves to the case $v = 1$, and let $E^*_{\cdot}$ stand for either side of the statement above. The group E^i_n has a natural descending filtration of length n given by

$$\mathrm{Fil}^s\, E^i_n = V^s E^i_{n-s} + dV^s E^{i-1}_{n-s} \subset E^i_n, \quad 0 \le s < n.$$

There is a natural k-vector space structure on E^i_n, and for all $0 \le s < n$ and all $i \ge 0$,

$$\dim_k \mathrm{gr}^s\, E^i_n = e_K,$$

the absolute ramification index of K. In particular, the domain and range of the map in the statement are abstractly isomorphic.

The main theorem implies that for $s \ge 0$,

$$\begin{aligned} \mathrm{TC}_{2s}(A|K;p,\mathbb{Z}/p^v) &= H^0(K,\mu_{p^v}^{\otimes s}) \oplus H^2(K,\mu_{p^v}^{\otimes(s+1)}), \\ \mathrm{TC}_{2s+1}(A|K;p,\mathbb{Z}/p^v) &= H^1(K,\mu_{p^v}^{\otimes(s+1)}), \end{aligned}$$

and thus in turn theorem A.

It is also easy to see that the canonical map

$$K_*(K,\mathbb{Z}/p^v) \to K_*^{\text{ét}}(K,\mathbb{Z}/p^v)$$

is an isomorphism in degrees ≥ 1. Here the right hand side is the Dwyer-Friedlander étale K-theory of K with $\mathbb{Z}/p^v$-coefficients. This may be defined as the homotopy groups with $\mathbb{Z}/p^v$-coefficients of the spectrum

$$K^{\text{ét}}(K) = \underset{\overrightarrow{L/K}}{\mathrm{holim}}\, \mathbb{H}^{\cdot}(G_{L/K}, K(L)),$$

where the homotopy colimit runs over the finite Galois extensions L/K contained in an algebraic closure $\bar{K}/K$, and where the spectrum $\mathbb{H}^{\cdot}(G_{L/K},K(L))$ is the group cohomology spectrum or homotopy fixed point spectrum of $G_{L/K}$ acting on $K(L)$. There is a spectral sequence

$$E^2_{s,t} = H^{-s}(K,\mu_{p^v}^{\otimes(t/2)}) \Rightarrow K^{\text{ét}}_{s+t}(K,\mathbb{Z}/p^v),$$

where the identification of the E^2-term is a consequence of the celebrated theorem of Suslin, [**35**], that

$$K_t(\bar{K},\mathbb{Z}/p^v) = \mu_{p^v}^{\otimes(t/2)}.$$

For K a finite extension of $\mathbb{Q}_p$, the p-adic homotopy type of the $K^{\text{ét}}(K)$ was calculated in [**7**]. Let $F\Psi^r$ be the homotopy fiber

$$F\Psi^r \to \mathbb{Z} \times BU \xrightarrow{\Psi^r - 1} BU.$$

It follows from this calculation and from the isomorphism above that

THEOREM D. *If K is a finite extension of $\mathbb{Q}_p$, then after p-completion*

$$\mathbb{Z} \times BGL(K)^+ \simeq F\Psi^{g^{p^{a-1}d}} \times BF\Psi^{g^{p^{a-1}d}} \times U^{|K\,:\,\mathbb{Q}_p|},$$

where $d = (p-1)/|K(\mu_p) : K|$, $a = \max\{v \mid \mu_{p^v} \subset K(\mu_p)\}$, and where $g \in \mathbb{Z}_p^\times$ is a topological generator.

The proof of theorem C is given in §6 below. It is based on the calculation in §5 of the Tate spectra for the cyclic groups C_{p^n} acting on the topological Hocschild spectrum $T(A|K)$: Given a finite group G and G-spectrum X, one has the Tate spectrum $\hat{\mathbb{H}}(G, X)$ of [**9**], [**10**]. Its homotopy groups are approximated by a spectral sequence

$$E^2_{s,t} = \hat{H}^{-s}(G, \pi_t X) \Rightarrow \pi_{s+t}\hat{\mathbb{H}}(G, X),$$

which converges conditionally in the sense of [**1**]. In §4 below we give a slightly different construction of this spectral sequence which is better suited for studying multiplicative properties. The cyclotomic structure of $T(A|K)$ gives rise a map

$$\hat{\Gamma}_K \colon \mathrm{TR}^n(A|K;p) \to \hat{\mathbb{H}}(C_{p^n}, T(A|K)),$$

and we show in §5 that this map induces an isomorphism of homotopy groups with $\mathbb{Z}/p^v$-coefficients in degrees ≥ 0. We then evaluate the Tate spectral sequence for the right hand side.

Throughtout this paper, A will be a complete discrete valuation ring with field of fractions K of characteristic zero and perfect residue field k of characteristic $p > 2$. All rings are assumed commutative and unital without further notice. Occasionally, we will write $\bar{\pi}_*(-)$ for homotopy groups with $\mathbb{Z}/p$-coefficients.

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1. Topological Hochschild homology and localization

1.1. The stable homotopy category is a triangulated category and a closed symmetric monoidal category, and the two structures are compatible. By a spectrum we will mean an object of this category, and by a ring spectrum we will mean a monoid in this category. The purpose of this paragraph is to produce the following. Let $\mathcal{C}$ be a *linear* category with cofibrations and weak equivalences. We define a pro-spectrum $\mathrm{TR}^{\cdot}(\mathcal{C};p)$ together with maps of pro-spectra

$$F\colon \mathrm{TR}^n(\mathcal{C};p) \rightleftarrows \mathrm{TR}^{n-1}(\mathcal{C};p)\colon V$$

$$\mu\colon S^1_+ \wedge \mathrm{TR}^n(\mathcal{C};p) \to \mathrm{TR}^n(\mathcal{C};p).$$

The spectrum $\mathrm{TR}^1(\mathcal{C};p)$ is the topological Hochschild spectrum of $\mathcal{C}$. The cyclotomic trace is a map of pro-spectra

$$\mathrm{tr}\colon K(\mathcal{C}) \to \mathrm{TR}^{\cdot}(\mathcal{C};p),$$

where the algebraic K-theory spectrum on the left is regarded as a constant pro-spectrum.

Suppose that the category $\mathcal{C}$ has a strict symmetric monoidal structure such that the tensor product is bi-exact. Then there is a natural product on $\mathrm{TR}^{\cdot}(\mathcal{C};p)$ which makes it a commutative pro-ring spectrum. Similarly, $K(\mathcal{C})$ is naturally a commutative ring spectrum and the maps F and tr are maps of ring-spectra.

The pro-spectrum $\mathrm{TR}^{\cdot}(\mathcal{C};p)$ has a preferred homotopy limit $\mathrm{TR}(\mathcal{C};p)$, and there are preferred lifts to the homotopy limit of the maps F, V and μ. Its homotopy groups are related to those of the pro-system by the Milnor sequence

$$0 \to \varprojlim_R{}^1 \pi_{s+1}\,\mathrm{TR}^{\cdot}(\mathcal{C};p) \to \pi_s\,\mathrm{TR}(\mathcal{C};p) \to \varprojlim_R \pi_s\,\mathrm{TC}^{\cdot}(\mathcal{C};p) \to 0.$$

There is a natural cofibration sequence

$$\mathrm{TC}(\mathcal{C};p) \to \mathrm{TR}(\mathcal{C};p) \xrightarrow{R-F} \mathrm{TR}(\mathcal{C};p) \to \Sigma\,\mathrm{TC}(\mathcal{C};p),$$

where $\mathrm{TC}(\mathcal{C};p)$ is the topological cyclic homology spectrum of $\mathcal{C}$. The cyclotomic trace has a preferred lift to a map

$$\mathrm{tr}\colon K(\mathcal{C}) \to \mathrm{TC}(\mathcal{C};p),$$

and in the case where $\mathcal{C}$ has a bi-exact strict symmetric monoidal product, the natural product on $\mathrm{TR}^{\cdot}(\mathcal{C};p)$ have preferred lifts to natural products on $\mathrm{TR}(\mathcal{C};p)$ and $\mathrm{TC}(\mathcal{C};p)$, and the maps F and tr are ring maps.

Let G be a compact Lie group. One then has the G-stable category which is a triangulated category with a compatible closed symmetric monoidal structure. The objects of this category are called G-spectra, and the monoids for the smash product are called ring G-spectra. Let $H \subset G$ be a closed subgroup and let $W_H G = N_G H/H$ be the Weil group. There is a forgetful functor which to a G-spectrum X assigns the underlying H-spectrum $U_H X$. We also write $|X|$ for $U_{\{1\}}X$. It comes with a natural map of spectra

$$\mu_X\colon G_+ \wedge |X| \to |X|.$$

One also has the H-fixed point functor which to a G-spectrum X assigns the $W_H G$-spectrum X^H. If $H \subset K \subset G$ are two closed subgroups, there is a map of spectra

$$\iota_H^K\colon |X^K| \to |X^H|,$$

and if $|K:H|$ if finite, a map in the opposite direction

$$\tau_H^K\colon |X^H| \to |X^K|.$$

If X is a ring G-spectrum then $U_H X$ is an ring H-spectrum and X^H is a ring $W_G H$-spectrum.

Let $\mathbb{T}$ be the circle group, and let $C_r \subset \mathbb{T}$ be the cyclic subgroup of order r. We then have the canonical isomorphism of groups

$$\rho_r\colon \mathbb{T} \xrightarrow{\sim} \mathbb{T}/C_r = W_{\mathbb{T}} C_r$$

given by the rth root. It induces an isomorphism of the $\mathbb{T}/C_r$-stable category and the $\mathbb{T}$-stable category which to a $\mathbb{T}/C_r$-spectrum Y assigns the $\mathbb{T}$-spectrum $\rho_r^* Y$. Moreover, there is a transitive system of natural isomorphisms of spectra

$$\varphi_r \colon |\rho_r^* Y| \xrightarrow{\sim} |Y|,$$

and the following digrams commute

$$\begin{tikzcd}
\mathbb{T} \wedge |\rho_r^* Y| \arrow[r, "\mu"] \arrow[d, "\rho\wedge\varphi_r"] & |\rho_r^* Y| \arrow[d, "\varphi_r"] \\
\mathbb{T}/C_{r+} \wedge |Y| \arrow[r, "\mu"] & |Y|.
\end{tikzcd}$$

We define a $\mathbb{T}$-spectrum $T(\mathcal{C})$ such that

$$\mathrm{TR}^n(\mathcal{C};p) = |\rho_{p^{n-1}}^* T(\mathcal{C})^{C_{p^{n-1}}}|$$

with the maps F and V given by the composites

$$F = \varphi_{p^{n-2}}^{-1} \iota_{C_{p^{n-2}}}^{C_{p^{n-1}}} \varphi_{p^{n-1}} \colon |\rho_{p^{n-1}}^* T(\mathcal{C})^{C_{p^{n-1}}}| \to |\rho_{p^{n-2}}^* T(\mathcal{C})^{C_{p^{n-2}}}|$$
$$V = \varphi_{p^{n-1}}^{-1} \tau_{C_{p^{n-2}}}^{C_{p^{n-1}}} \varphi_{p^{n-2}} \colon |\rho_{p^{n-2}}^* T(\mathcal{C})^{C_{p^{n-2}}}| \to |\rho_{p^{n-1}}^* T(\mathcal{C})^{C_{p^{n-1}}}|$$

and the map μ is given by

$$\mu = \mu_{\rho_{p^{n-1}}^* T(\mathcal{C})^{C_{p^{n-1}}}} \colon \mathbb{T}_+ \wedge |\rho_{p^{n-1}}^* T(\mathcal{C})^{C_{p^{n-1}}}| \to |\rho_{p^{n-1}}^* T(\mathcal{C})^{C_{p^{n-1}}}|.$$

There is a natural map

$$K(\mathcal{C}) \to T(\mathcal{C})^{\mathbb{T}},$$

and the cyclotomic trace is then the composite of this map and $\varphi_{p^{n-1}}^{-1} \iota_{C_{p^{n-1}}}^{\mathbb{T}}$. The definition of the structure maps in the pro-system $\mathrm{TR}^{\cdot}(\mathcal{C};p)$ is more complicated and uses the *cyclotomic* structure on $T(\mathcal{C})$ which we now explain.

There is a cofibration sequence of $\mathbb{T}$-CW-complexes

$$E_+ \to S^0 \to \tilde{E} \to \Sigma E_+,$$

where E is a free contractible $\mathbb{T}$-space, and where the left hand map collapses E to the non-base point of S^0. It induces upon smashing with a $\mathbb{T}$-spectrum T a cofibration sequence of $\mathbb{T}$-spectra

$$E_+ \wedge T \to T \to \tilde{E} \wedge T \to \Sigma E_+ \wedge T,$$

and hence the following basic cofibration sequence of spectra

$$|\rho_{p^v}(E_+ \wedge T)^{C_{p^v}}| \to |\rho_{p^v} T^{C_{p^v}}| \to |\rho_{p^v}(\tilde{E} \wedge T)^{C_{p^v}}| \to \Sigma |\rho_{p^v}(E_+ \wedge T)^{C_{p^v}}|,$$

natural in T. The left hand term is written $\mathbb{H}.(C_{p^v}, T)$ and called the group homology spectrum or Borel spectrum. Its homotopy groups are approximated by a strongly convergent first quadrant homology type spectral sequence

$$E^2_{s,t} = H_s(C_{p^v}, \pi_t T) \Rightarrow \pi_{s+t} \mathbb{H}.(C_{p^v}, T).$$

The cyclotomic structure on $T(\mathcal{C})$ means that there is a natural map of $\mathbb{T}$-spectra

$$r \colon \rho_p^*(\tilde{E} \wedge T(\mathcal{C}))^{C_p} \to T(\mathcal{C})$$

such that $U_{C_{p^s}} r$ is an isomorphism of C_{p^s}-spectra, for all $s \geq 0$. More generally, since

$$\rho_{p^v}^*(\tilde{E} \wedge T(\mathcal{C}))^{C_{p^v}} = \rho_{p^{v-1}}^*(\rho_p^*(\tilde{E} \wedge T(\mathcal{C}))^{C_p})^{C_{p^{v-1}}},$$

the map r induces a map of $\mathbb{T}$-spectra

$$r_{v+1}\colon \rho_{p^v}^*(\tilde{E} \wedge T(\mathcal{C}))^{C_{p^v}} \to \rho_{p^{v-1}}^* T(\mathcal{C})^{C_{p^{v-1}}}$$

such that $U_{C_{p^s}} r_{v+1}$ is an isomorphism of C_{p^s}-spectra, for all $s \geq 0$. The map

$$R\colon\ \mathrm{TR}^n(\mathcal{C};p) \to \mathrm{TR}^{n-1}(\mathcal{C};p)$$

is then defined as the composite

$$|\rho_{p^{n-1}}^* T(\mathcal{C})^{C_{p^{n-1}}}| \to |\rho_{p^{n-1}}^*(\tilde{E} \wedge T(\mathcal{C}))^{C_{p^{n-1}}}| \xrightarrow[\sim]{r_n} |\rho_{p^{n-2}}^* T(\mathcal{C})^{C_{p^{n-2}}}|,$$

where the right hand map is the middle map in the cofibration sequence above. We thus have a natural cofibration sequence of spectra

$$\mathbb{H}_{\cdot}(C_{p^{n-1}}, T(\mathcal{C})) \xrightarrow{N} \mathrm{TR}^n(\mathcal{C};p) \xrightarrow{R} \mathrm{TR}^{n-1}(\mathcal{C};p) \xrightarrow{\partial} \Sigma\mathbb{H}_{\cdot}(C_{p^{n-1}}, T(\mathcal{C})).$$

When $\mathcal{C}$ has a bi-exact strict symmetric monoidal product, the map r is a map of ring $\mathbb{T}$-spectra, and hence R is a map of ring spectra. The cofibration sequence above is a sequence of $\mathrm{TR}^n(\mathcal{C};p)$-module spectra and maps.

For any $\mathbb{T}$-spectrum X, one has the function spectrum $F(E_+, X)$, and the projection $E_+ \to S^0$ defines a natural map

$$\gamma\colon X \to F(E_+, X).$$

This map induces an isomorphism of group homology spectra. One defines the group cohomology spectrum and Tate spectrum,

$$\mathbb{H}^{\cdot}(C_{p^v}, X) = |\rho_{p^v} F(E_+, X)^{C_{p^v}}|,$$
$$\hat{\mathbb{H}}\,(C_{p^v}, X) = |\rho_{p^v}(\tilde{E} \wedge F(E_+, X))^{C_{p^v}}|,$$

whose homotopy groups are approximated by homology type spectral sequences

$$E^2_{s,t} = H^{-s}(C_{p^v}, \pi_t X) \Rightarrow \pi_{s+t}\mathbb{H}^{\cdot}(C_{p^v}, X),$$
$$\hat{E}^2_{s,t} = \hat{H}^{-s}(C_{p^v}, \pi_t X) \Rightarrow \pi_{s+t}\hat{\mathbb{H}}\,(C_{p^v}, X),$$

both of which are conditionally convergent in the sense of [**1**]. The latter sequence, called the Tate spectral sequence, will be considered in great detail in paragraph 4 below. Taking $T = F(E_+, X)$ in the basic cofibration sequence above, we get the Tate cofibration sequence of spectra

$$\mathbb{H}_{\cdot}(C_{p^v}, X) \xrightarrow{N^h} \mathbb{H}^{\cdot}(C_{p^v}, X) \xrightarrow{R^h} \hat{\mathbb{H}}(C_{p^v}, X) \xrightarrow{\partial^h} \Sigma\mathbb{H}_{\cdot}(C_{p^v}, X).$$

Finally, when $X = T(\mathcal{C})$, the map

$$\gamma\colon T(\mathcal{C}) \to F(E_+, T(\mathcal{C}))$$

induces a map of cofibration sequences

$$\begin{array}{ccccccc}
\mathbb{H}_{\cdot}(C_{p^v}, T(\mathcal{C})) & \xrightarrow{N} & \mathrm{TR}^{v+1}(\mathcal{C};p) & \xrightarrow{R} & \mathrm{TR}^v(\mathcal{C};p) & \xrightarrow{\partial} & \Sigma\mathbb{H}_{\cdot}(C_{p^v}, T(\mathcal{C})) \\
\| & & \downarrow{\scriptstyle\Gamma} & & \downarrow{\scriptstyle\hat{\Gamma}} & & \| \\
\mathbb{H}_{\cdot}(C_{p^v}, T(\mathcal{C})) & \xrightarrow{N^h} & \mathbb{H}^{\cdot}(C_{p^v}, T(\mathcal{C})) & \xrightarrow{R^h} & \hat{\mathbb{H}}(C_{p^v}, T(\mathcal{C})) & \xrightarrow{\partial^h} & \Sigma\mathbb{H}_{\cdot}(C_{p^v}, T(\mathcal{C})),
\end{array}$$

in which all maps commute with the action maps μ. Moreover, if $\mathcal{C}$ is strict symmetric monoidal with bi-exact tensor product, the four spectra in the middle square are all ring spectra and R, R^h, Γ and $\hat{\Gamma}$ are maps of ring spectra. In this case, the diagram is a diagram of $\mathrm{TR}^{v+1}(\mathcal{C};p)$-module spectra.

1.2. In order to construct the $\mathbb{T}$-spectrum $T(\mathcal{C})$ we need a model category for the $\mathbb{T}$-stable category. The model category we use is the category of symmetric spectra of orthogonal $\mathbb{T}$-spectra. We first recall the topological Hochschild space $\mathrm{THH}(\mathcal{C})$.

A linear category $\mathcal{C}$ is naturally enriched over the symmetric monoidal category of symmetric spectra. The symmetric spectrum of maps from c to d, $\underline{\mathrm{Hom}}_{\mathcal{C}}(c,d)$, is the Eilenberg-MacLane spectrum for the abelian group $\mathrm{Hom}_{\mathcal{C}}(c,d)$ concentrated in degree zero. In more detail, if X is a pointed simplicial set, then

$$\mathbb{Z}(X) = \mathbb{Z}\langle X\rangle/\mathbb{Z}\langle x_0\rangle$$

is a simplicial abelian group whose homology is the reduced singular homology of X. Here $\mathbb{Z}\langle X\rangle$ denotes the degree-wise free abelian group generated by X. Let S^i be the i-fold smash product of the standard simplicial circle $S^1 = \Delta[1]/\partial\Delta[1]$. Then the spaces $\{|\mathbb{Z}(S^i)|\}_{i\geq 0}$ is a symmetric ring spectrum with the homotopy type of an Eilenberg-MacLane spectrum for $\mathbb{Z}$ concentrated in degree zero, and we define

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(c,d)_i = |\,\mathrm{Hom}_{\mathcal{C}}(c,d)\otimes\mathbb{Z}(S^i)|.$$

This gives the stated enrichment.

Let I be the category with objects the finite sets

$$\underline{i} = \{1,2,\dots,i\}, \qquad i\geq 1,$$

and the empty set $\underline{0}$, and morphisms all injective maps. It is a strict monoidal category under concatenation of sets and maps. There is a functor $V_k(\mathcal{C};X)$ from I^{k+1} to the category of pointed spaces which on objects is given by

$$V_k(\mathcal{C};X)(\underline{i_0},\dots,\underline{i_k}) = \bigvee_{c_0,\dots,c_k\in\operatorname{ob}\mathcal{C}} \underline{\mathrm{Hom}}_{\mathcal{C}}(c_0,c_k)_{i_0}\wedge\dots\wedge\underline{\mathrm{Hom}}_{\mathcal{C}}(c_k,c_{k-1})_{i_k}\wedge X.$$

It induces a functor $G_k(\mathcal{C};X)$ from I^{k+1} to pointed spaces with

$$G_k(C;X)(\underline{i_0},\dots,\underline{i_k}) = F(S^{i_0}\wedge\dots\wedge S^{i_k}, V(\mathcal{C};X)(\underline{i_0},\dots,\underline{i_k})),$$

and we define

$$\mathrm{THH}_k(\mathcal{C}) = \operatorname*{holim}_{\overrightarrow{I^{k+1}}} G_k(\mathcal{C};S^0).$$

This is naturally the the space of k-simplices in a cyclic space, and by definition

$$\mathrm{THH}(\mathcal{C}) = |[k]\mapsto \mathrm{THH}_k(\mathcal{C})|.$$

It is a $\mathbb{T}$-space with $\mathbb{T}$-fixed set $\operatorname{ob}\mathcal{C}$.

More generally, let (n) be the finite ordered set $\{1,2,\dots,n\}$. The product category $I^{(n)}$ is a strict monoidal category under component wise concatenation of sets and maps. Concatenation of sets and maps according to the ordering of (n) also defines a functor

$$\sqcup_n\colon I^{(n)}\to I,$$

but this does *not* preserve the monoidal structure. We let $G_k^{(n)}(\mathcal{C};X)$ be the functor from $(I^{(n)})^{k+1}$ to the category of pointed spaces given by

$$G_k^{(n)}(\mathcal{C};X) = G_k(\mathcal{C};X) \circ (\sqcup_n)^{k+1},$$

and define

$$\mathrm{THH}_k^{(n)}(\mathcal{C};X) = \underset{(I^{(n)})^{k+1}}{\underrightarrow{\mathrm{holim}}}\ G_k^{(n)}(\mathcal{C};X).$$

Again this is the space of k-simplices in a cyclic space, and we define

$$\mathrm{THH}^{(n)}(\mathcal{C};X) = |[k] \mapsto \mathrm{THH}_k^{(n)}(\mathcal{C};X)|.$$

It is a $\Sigma_n \times \mathbb{T}$-space, whose $\mathbb{T}$-fixed set is $\mathrm{ob}\,\mathcal{C} \wedge X$.

There is a natural product

$$\mathrm{THH}^{(m)}(\mathcal{C};X) \wedge \mathrm{THH}^{(n)}(\mathcal{D};Y) \to \mathrm{THH}^{(m+n)}(\mathcal{C}\otimes\mathcal{D}; X \wedge Y),$$

which is $\Sigma_m \times \Sigma_n \times \mathbb{T}$-equivariant when $\mathbb{T}$ acts diagonally on the left. Here the category $\mathcal{C}\otimes\mathcal{D}$ has as objects all pairs (c,d) with $c \in \mathrm{ob}\,\mathcal{C}$ and $d \in \mathcal{D}$, and

$$\mathrm{Hom}_{\mathcal{C}\otimes\mathcal{D}}((c,d),(c',d')) = \mathrm{Hom}_{\mathcal{C}}(c,c') \otimes \mathrm{Hom}_{\mathcal{D}}(d,d').$$

For any category $\mathcal{C}$, the nerve category $\mathbf{N}.\mathcal{C}$ is the simplicial category with k-simplicies the functor category

$$\mathbf{N}_k\mathcal{C} = \mathcal{C}^{[k]},$$

where $[k]$ is the partially ordered set $\{0,1,\dots,k\}$ viewed as a category. An order preserving map $\theta\colon [k] \to [l]$ may be viewed as a functor and hence induces a functor

$$\theta^*\colon \mathbf{N}_l\mathcal{C} \to \mathbf{N}_k\mathcal{C}.$$

The objects of $\mathbf{N}.\mathcal{C}$ is the nerve $\mathcal{C}$, $N.\mathcal{C}$. Clearly, the nerve category is a functor from categories to simplicial categories.

Suppose now that $\mathcal{C}$ is a category with cofibrations and weak equivalences and define

$$\mathbf{N}^w_.\mathcal{C} \subset \mathbf{N}.\mathcal{C}$$

to be the *full* simplicial subcategory with

$$\mathrm{ob}\,\mathbf{N}^w_.\mathcal{C} = N.w\mathcal{C}.$$

There is a natural structure of simplicial category with weak equivalences on $\mathbf{N}^w_.\mathcal{C}$: $\mathrm{co}\,\mathbf{N}^w_.\mathcal{C}$ and $w\mathbf{N}^w_.\mathcal{C}$ are the simplicial subcategories which contain all objects but where morphisms are natural transformations through cofibrations and weak equivalences in $\mathcal{C}$, respectively. With these definitions there is a natural isomorphism of bi-simplicial categories with cofibrations and weak equivalences

$$\mathbf{N}.S.\mathcal{C} \cong S.\mathbf{N}.\mathcal{C}, \tag{1.2.1}$$

where $S.\mathcal{C}$ is Waldhausen's construction, [**40**].

Let (n) be the finite ordered set $\{1,2,\dots,n\}$ and let V be a finite dimentional orthogonal $\mathbb{T}$-representation. We define the (n,V)th space in the symmetric orthogonal $\mathbb{T}$-spectrum $T(\mathcal{C})$ by

$$T(\mathcal{C})_{n,V} = |\,\mathrm{THH}^{(n)}(\mathbf{N}^w_.S^{(n)}\mathcal{C}; S^V)|. \tag{1.2.2}$$

There are two $\mathbb{T}$-actions on the this space: one which comes from the topological Hocschild space, and another induced from the $\mathbb{T}$-action on S^V. We give $T(\mathcal{C})_{n,V}$

the diagonal $\mathbb{T}$-action. There are also two Σ_n-actions: one which comes from the Σ_n-action on the topological Hocschild space, and another induced from the permutation of the simplicial directions in the n-simplicial category $S^{(n)}_{\cdot}\mathcal{C}$, compare [**8**, 6.1]. We also give $T(\mathcal{C})_{n,V}$ the diagonal Σ_n-action. In particular, the $(0,0)$th space is the cyclic bar construction

$$T(\mathcal{C})_{0,0} = |N^{\mathrm{cy}}(\mathbf{N}^w_{\cdot}\mathcal{C})|.$$

The $\mathbb{T}$-fixed sets are

$$(T(\mathcal{C})_{n,V})^{\mathbb{T}} = |\,\mathrm{ob}\,\mathbf{N}^w_{\cdot} S^{(n)}_{\cdot}\mathcal{C} \wedge S^{V^{\mathbb{T}}},|$$

which indeed is a model for $K(\mathcal{C})$. Moreover, by a construction similar to that of [**16**, §2], there are $\mathbb{T}$-equivariant maps

$$\rho_p^*(T(\mathcal{C})_{n,V})^{C_p} \to T(\mathcal{C})_{n,\rho_p^* V^{C_p}},$$

and one can prove that for fixed n, the object of the $\mathbb{T}$-stable category defined by the orthogonal spectrum $V \mapsto T(\mathcal{C})_{n,V}$ has a cyclotomic structure.

Suppose that $\mathcal{C}$ is a strict symmetric monoidal category and that the tensor product is bi-exact. There is then an induced $\Sigma_m \times \Sigma_n$-equivariant product

$$S^{(m)}_{\cdot}\mathcal{C} \otimes S^{(n)}_{\cdot}\mathcal{C} \to S^{(m+n)}_{\cdot}\mathcal{C},$$

and hence

$$T(\mathcal{C})_{m,V} \wedge T(\mathcal{C})_{n,W} \to T(\mathcal{C})_{m+n,V\oplus W}.$$

This product makes $T(\mathcal{C})$ a monoid in the symmetric monoidal category of symmetric spectra of orthogonal $\mathbb{T}$-spectra.

1.3. We need to recall some of the properties of this construction. It is convenient to work in a more general setting.

Let Φ be a functor from a category of categories with cofibrations and weak equivalences to the category of pointed spaces. If $\mathcal{C}.$ is a simplicial category with cofibrations and weak equivalences, we define

$$\Phi(\mathcal{C}.) = |[n] \mapsto \Phi(\mathcal{C}_n)|.$$

We shall assume that Φ satisfies the following axioms:

(i) the trivial category with cofibrations and weak equivalences is mapped to a one-point space.

(ii) for any pair $\mathcal{C}$ and $\mathcal{D}$ of categories with cofibrations and weak equivalences, the canonical map

$$\Phi(\mathcal{C}\times\mathcal{D}) \xrightarrow{\sim} \Phi(\mathcal{C}) \times \Phi(\mathcal{D})$$

is a weak equivalence.

(iii) (realization lemma) if $f.\colon \mathcal{C}. \to \mathcal{D}.$ is a map of simplicial categories with cofibrations and weak equivalences, and if for all n, $\Phi(f_n)\colon \Phi(\mathcal{C}_n) \to \Phi(\mathcal{D}_n)$ is a weak equivalence, then

$$\Phi(f.)\colon \Phi(\mathcal{C}.) \to \Phi(\mathcal{D}.)$$

is a weak equivalence.

The functors which we will consider later will in fact only depend on the underlying category. But since all proofs works for the more general Φ, we state the theorems in this generality.

We next recall some generalities. Let

$$f, g\colon \mathcal{C}. \to \mathcal{D}.$$

be two exact simplicial functors. An *exact* simplicial homotopy from f to g is an exact simplicial functor

$$h\colon \Delta[1]. \times \mathcal{C}. \to \mathcal{D}.$$

such that $h \circ (d^1 \times \mathrm{id}) = f$ and $h \circ (d^0 \times \mathrm{id}) = g$. Here $\Delta[n].$ is viewed as a discrete simplicial category with its unique structure of a simplicial category with cofibrations and weak equivalences. An exact simplicial functor $f\colon \mathcal{C}. \to \mathcal{D}.$ is an exact simplicial homotopy equivalence if there exists an exact simplicial functor $g\colon \mathcal{D}. \to \mathcal{C}.$ and exact simplicial homotopies of the two composites to the respective identity simplicial functors.

LEMMA 1.3.1. *An exact simplicial homotopy $\Delta[1]. \times \mathcal{C}. \to \mathcal{D}.$ induces a homotopy*

$$\Delta[1] \times \Phi(\mathcal{C}.) \to \Phi(\mathcal{D}.).$$

Hence Φ takes exact simplicial homotopy equivalences to homotopy equivalences.

PROOF. There is a natural transformation

$$\Delta[1]_k \times \Phi(\mathcal{C}_k) \to \Phi(\Delta[1]_k \times \mathcal{C}_k).$$

Indeed, $\Delta[1]_k \times \Phi(\mathcal{C}_k)$ and $\Delta[1]_k \times \mathcal{C}_k$ are coproducts in the category of spaces and the category of categories with cofibrations and weak equivalences, respectively, indexed by the set $\Delta[1]_k$. The map exists by the universal property of coproducts. □

LEMMA 1.3.2. *An exact functor of categories with cofibrations and weak equivalences $f\colon \mathcal{C} \to \mathcal{D}$ induces an exact simplicial functor $\mathbf{N}.f\colon \mathbf{N}^w_.\mathcal{C} \to \mathbf{N}^w_.\mathcal{D}$. A natural transformation through weak equivalences of $\mathcal{D}$ between two such functors f and g induces an exact simplicial homotopy between $\mathbf{N}.f$ and $\mathbf{N}.g$.*

PROOF. The first statement is clear. We view the partially ordered set $[1]$ as a category with cofibrations and weak equivalences where the non-identity map is a weak equivalence but not a cofibration. Then the natural transformation defines an exact functor $[1] \times \mathcal{C} \to \mathcal{D}$, and the required exact simplicial homotopy is given by the composite

$$\Delta[1]. \times \mathbf{N}^w_.\mathcal{C} \to \mathbf{N}^w_.[1] \times \mathbf{N}^w_.\mathcal{C} \to \mathbf{N}^w_.([1] \times \mathcal{C}) \to \mathbf{N}^w_.\mathcal{D},$$

where the first and the middle arrow are the canonical simplicial functors, and the last is induced from the natural transformation. (Note that $\mathbf{N}^w_.[n]$ is not a discrete category.) □

LEMMA 1.3.3. ([**40**, lemma 1.4.1]) *Let $f, g\colon \mathcal{C} \to \mathcal{D}$ be a pair of exact functors of categories with cofibrations. A natural isomorphism from f to g induces an exact simplicial homotopy*

$$\Delta[1]. \times S.\mathcal{C} \to S.\mathcal{D}$$

from $S.f$ to $S.g$.

PROOF. We recall the proof. The natural transformation from f to g amounts to a functor $F\colon \mathcal{C} \times [1] \to \mathcal{D}$. Recall that $S_n\mathcal{C}$ is a sub-category of the functor category $\mathcal{C}^{\mathrm{Ar}[n]}$. The homotopy is then given by

$$(a\colon [n] \to [1]) \mapsto ((A\colon \mathrm{Ar}[n] \to \mathcal{C}) \mapsto (A'\colon \mathrm{Ar}[n] \to \mathcal{D})),$$

where A' is defined by the composition

$$\mathrm{Ar}[n] \xrightarrow{(A,a_*)} \mathcal{C} \times \mathrm{Ar}[1] \xrightarrow{\mathrm{id}\times p} \mathcal{C} \times [1] \xrightarrow{F} \mathcal{D}$$

and $p\colon \mathrm{Ar}[1] \to [1]$ is given by $(0 \to 0) \mapsto 0$, $(0 \to 1) \mapsto 1$, and $(1 \to 1) \mapsto 1$. The requirement that the natural transformation be through isomorphisms is needed in showing that the maps above are compatible with the zeroth face. $\square$

COROLLARY 1.3.4. *Let $\mathcal{C}$ be a category with cofibrations. Then the map induced from the degeneracies in the nerve direction induces a weak equivalence*

$$\Phi(S.\mathcal{C}) \xrightarrow{\sim} \Phi(\mathbf{N}^i_. S.\mathcal{C}).$$

PROOF. For each k, the iterated degeneracy functor

$$s\colon \mathcal{C} = \mathbf{N}^i_0\mathcal{C} \to \mathbf{N}^i_k\mathcal{C},$$

has the retraction

$$\theta^*\colon \mathbf{N}^i_k\mathcal{C} \to \mathcal{C},$$

where $\theta\colon [0] \to [k]$ is given by $\theta(0) = 0$. Moreover, there is a natural isomorphism $\mathrm{id} \xrightarrow{\approx} \theta^*$, and hence by the lemma,

$$S.s\colon S.\mathcal{C} \to S.\mathbf{N}^i_k\mathcal{C} = \mathbf{N}^i_k S.\mathcal{C}$$

is an exact simplicial homotopy equivalence. The corollary follows from lemma 1.3.1 and the realization lemma. $\square$

1.4. The proof of the additivity theorem given by McCarthy in [**27**] has the advantage that it immediately generalizes to the present situation. We recall the proof here.

Concatenation of sets and maps defines a functor

$$\sqcup\colon \mathbf{\Delta} \times \mathbf{\Delta} \to \mathbf{\Delta},$$

aand there are natural transformations

$$\mathrm{pr}_1 \xleftarrow{\epsilon_L} \sqcup \xrightarrow{\epsilon_R} \mathrm{pr}_2$$

given by the canonical inclusions. If X is a simplicial object in a category $\mathcal{C}$, the functors $\sqcup$, pr_1 and pr_2 give rise to three bi-simplicial objects in $\mathcal{C}$ which we denote $\mathbf{P}X$, XL and XR, respectively. Moreover, the natural transformations above induce natural bi-simplicial maps

$$XL \xleftarrow{\epsilon_L} \mathbf{P}X \xrightarrow{\epsilon_R} XR.$$

In general, there are no bi-simplicial sections of these functors. But for fixed $m \geq 0$, the simplicial map

$$\epsilon_L\colon \mathbf{P}X_{m,\cdot} \to X_m$$

has a section

$$\iota_L\colon X_m \to \mathbf{P}X_{m,\cdot}.$$

induced from the natural transformation $[m] \sqcup [n] \to [m]$ which collapses $[n]$ on the point $m \in [m]$. Similarly, for fixed $n \geq 0$, the map $\epsilon_R \colon \mathbf{P}X_{\cdot,n} \to X_n$ has a section $\iota_R \colon X_n \to \mathbf{P}X_{\cdot,n}$ induced from the natural transformation $[m] \sqcup [n] \to [n]$ which collapses $[m]$ on the point $0 \in [n]$.

LEMMA 1.4.1. *The composite simplicial maps*

$$\mathbf{P}X_{m,\cdot} \xrightarrow{\epsilon_L} X_m \xrightarrow{\iota_L} \mathbf{P}X_{m,\cdot}$$
$$\mathbf{P}X_{\cdot,n} \xrightarrow{\epsilon_R} X_n \xrightarrow{\iota_R} \mathbf{P}X_{\cdot,n}$$

are naturally simplicially homotopic to the identity.

PROOF. The proof is similar to [**40**, lemma 1.5.1]. The second of the composite maps of the statement is induced from the map

$$[m] \sqcup [n] \to [m] \sqcup [n]$$

which is the identity on $[n]$ and collapses $[m]$ on the point $0 \in [n]$. The homotopy of this map to the identity is given by the natural transformation

$$(a \colon [m] \to [1]) \mapsto (\varphi_a^* \colon \mathbf{P}X_{m,n} \to \mathbf{P}X_{m,n})$$

induced from

$$(a \colon [m] \to [1]) \mapsto (\varphi_a \colon [m] \sqcup [n] \to [m] \sqcup [n]),$$

where φ_a is the identity on $[n]$ and

$$\varphi_a(j) = \begin{cases} j & \text{, if } a(j) = 0, \\ m+1 & \text{, if } a(j) = 1. \end{cases}$$

Suppose that $a \colon [m] \to [1]$ and $\theta \colon [m'] \to [m]$ are maps in $\mathbf{\Delta}$, and let $a' = a \circ \theta$. Then by definition, the diagram

$$\begin{tikzcd} {[m'] \sqcup [n]} \arrow[r, "\varphi_{a'}"] \arrow[d, "\theta \sqcup \mathrm{id}"] & {[m'] \sqcup [n]} \arrow[d, "\theta \sqcup \mathrm{id}"] \\ {[m] \sqcup [n]} \arrow[r, "\varphi_a"] & {[m] \sqcup [n]} \end{tikzcd}$$

commutes. This shows that the homotopy is indeed a simplicial map. □

Let $f \colon \mathcal{C} \to \mathcal{D}$ be an exact functor of categories with cofibrations and weak equivalences. Following [**27**] we define the bi-simplicial category $S.f|\mathcal{D}$ by the pull back diagram

$$\begin{tikzcd} S.f|\mathcal{D} \arrow[r] \arrow[d, "\epsilon_L"] & \mathbf{P}S.\mathcal{D} \arrow[d, "\epsilon_L"] \\ (S.\mathcal{C})L \arrow[r, "S\mathbf{\Delta}"] & (S.\mathcal{D})L \end{tikzcd}$$

Neglecting choices of quotients, an object of the category of (m, n)-simplices of $S.f|\mathcal{D}$ consists of a pair of flags

$$C_1 \rightarrowtail \cdots \rightarrowtail C_m$$
$$D_1 \rightarrowtail \cdots \rightarrowtail D_m \rightarrowtail S_0 \rightarrowtail \cdots \rightarrowtail S_n$$

in $\mathcal{C}$ and $\mathcal{D}$, respectively, such that $f(C_i) = D_i$, $1 \leq i \leq m$.

It follows from [**40**, 1.1-1.2] that $S.f|\mathcal{D}$ is a bi-simplicial category with cofibrations and weak equivalences in a natural way, and that the functors in the diagram above are exact. The section $\iota_L \colon S_m\mathcal{D} \to (\mathbf{P}S.\mathcal{D})_{m,\cdot}$ induces a section

$$\iota_L \colon S_m\mathcal{C} \to (S.f|\mathcal{D})_{m,\cdot},$$

and the homotopy of lemma 1.4.1 induces a homotopy of the composite simplicial functor

$$(S.f|\mathcal{D})_{m,\cdot} \xrightarrow{\epsilon_L} S_m\mathcal{C} \xrightarrow{\iota_L} (S.f|\mathcal{D})_{m,\cdot}$$

to the identity functor. Moreover, one easily checks that the section ι_L is exact and that the given homotopy of $\iota_L \circ \epsilon_L$ to the identity is through exact functors.

There are natural maps of pull-back diagrams of simplicial categories

$$\begin{tikzcd}
(\mathbf{P}S.\mathcal{D})_{\cdot,n} \arrow[r, "\epsilon_L"] \arrow[d, "\epsilon_R"] & S.\mathcal{D} \arrow[d] & S.\mathcal{C} \arrow[l, "f"'] \arrow[d] \\
S_m\mathcal{D} \arrow[r] \arrow[d, "\iota_R"] & * \arrow[d] & * \arrow[l] \arrow[d] \\
(\mathbf{P}S.\mathcal{D})_{\cdot,n} \arrow[r, "\epsilon_L"] & S.\mathcal{D} & S.\mathcal{C} \arrow[l, "f"']
\end{tikzcd}$$

which define natural simplicial exact functors

$$(S.f|\mathcal{D})_{\cdot,n} \xrightarrow{\epsilon_R} S_n\mathcal{D} \xrightarrow{\iota_R} (S.f|\mathcal{D})_{\cdot,n}. \tag{1.4.2}$$

In general, the composite map is not homotopic to the identity.

PROPOSITION 1.4.3. (McCarthy) *Let $f \colon \mathcal{C} \to \mathcal{D}$ be an exact functor of categories with cofibrations and weak equivalences and suppose that for all n, there exists an exact simplicial homotopy of the composite functor* (1.4.2) *to the identity. Then*

$$\Phi(S.f) \colon \Phi(S.\mathcal{C}) \to \Phi(S.\mathcal{D})$$

is a weak equivalence.

PROOF. There is a commutative diagram,

$$\begin{tikzcd}
\Phi((S.\mathcal{D})R) \arrow[d, equal] & \Phi(S.f|\mathcal{D}) \arrow[l, "\epsilon_R"', "\sim"] \arrow[r, "\epsilon_L", "\sim"'] \arrow[d, "f"] & \Phi((S.\mathcal{C})L) \arrow[d, "f"] \\
\Phi((S.\mathcal{D})R) & \Phi(\mathbf{P}S.\mathcal{D}) \arrow[l, "\epsilon_R"', "\sim"] \arrow[r, "\epsilon_L", "\sim"'] & \Phi((S.\mathcal{D})L),
\end{tikzcd}$$

and the horizontal maps are all weak equivalences. Let us show in detail that the upper left hand map is a weak equivalence. By assumption, the simplicial exact functor

$$\epsilon_R \colon (S.f|\mathcal{D})_{\cdot,n} \to S_n\mathcal{D}$$

is a simplicial homotopy equivalence, for all $n \geq 0$. The homotopy inverse is ι_R. It follows from lemma 1.3.1 that for all $n \geq 0$,

$$\epsilon_R \colon \Phi((S.f|\mathcal{D})_{\cdot,n}) \xrightarrow{\sim} \Phi(S_n\mathcal{D})$$

is a homotopy equivalence. Since we assume that the realization lemma holds for Φ, it follows that

$$\epsilon_R \colon \Phi(S.f|\mathcal{D}) \xrightarrow{\sim} \Phi((S.\mathcal{D})R)$$

is a weak equivalence. A similar argument shows that the remaining horizontal arrows are weak equivalences. □

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be categories with cofibrations and weak equivalences and suppose that $\mathcal{A}$ and $\mathcal{B}$ are subcategories of $\mathcal{C}$ and that the inclusion functors are exact. Following [**40**, p.335], let $E(\mathcal{A},\mathcal{C},\mathcal{B})$ be the category with cofibrations and weak equivalences given by the pull-back diagram

$$\begin{tikzcd} E(\mathcal{A},\mathcal{C},\mathcal{B}) \arrow[r, "{(s,t,q)}"] \arrow[d] & \mathcal{A}\times\mathcal{C}\times\mathcal{B} \arrow[d] \\ S_2\mathcal{C} \arrow[r, "{(d_2,d_1,d_0)}"] & \mathcal{C}\times\mathcal{C}\times\mathcal{C}. \end{tikzcd}$$

In other words, $E(\mathcal{A},\mathcal{C},\mathcal{B})$ is the category of cofibration sequences in $\mathcal{C}$ of the form

$$A \rightarrowtail C \twoheadrightarrow B, \qquad A\in\mathcal{A},\ B\in\mathcal{B}.$$

The exact functors s, t and q take this sequence to A, C and B, respectively. The extension of the additivity theorem to the present situation is due to McCarthy, [**27**].

THEOREM 1.4.4. (Additivity theorem) *The following equivalent assertions hold:*

(1) *The exact functors s and q induce a weak equivalence*

$$\Phi(\mathbf{N}^w_{\cdot}S.E(\mathcal{A},\mathcal{C},\mathcal{B})) \xrightarrow{\sim} \Phi(\mathbf{N}^w_{\cdot}S.\mathcal{A})\times\Phi(\mathbf{N}^w_{\cdot}S.\mathcal{B}).$$

(2) *The exact functors s and q induce a weak equivalence*

$$\Phi(\mathbf{N}^w_{\cdot}S.E(\mathcal{C},\mathcal{C},\mathcal{C})) \xrightarrow{\sim} \Phi(\mathbf{N}^w_{\cdot}S.\mathcal{C})\times\Phi(\mathbf{N}^w_{\cdot}S.\mathcal{C}).$$

(3) *The functors t and $s\vee q$ induce homotopic maps*

$$\Phi(\mathbf{N}^w_{\cdot}S.E(\mathcal{C},\mathcal{C},\mathcal{C})) \to \Phi(\mathbf{N}^w_{\cdot}S.\mathcal{C}).$$

(4) *Let $F' \rightarrowtail F \twoheadrightarrow F''$ be a cofibration sequence of exact functors $\mathcal{C}\to\mathcal{D}$. Then the exact functors F and $F'\vee F''$ induce homotopic maps*

$$\Phi(\mathbf{N}^w_{\cdot}S.\mathcal{C}) \to \Phi(\mathbf{N}^w_{\cdot}S.\mathcal{D}).$$

PROOF. We refer to [**40**, proposition 1.3.2] for the proof that the four statements are equivalent. We also employ the trick used there that the bi-simplicial categories with cofibrations and weak equivalences $\mathbf{N}^w_{\cdot}S.\mathcal{C}$ and $S.\mathbf{N}^w_{\cdot}\mathcal{C}$ are canonically isomorphic. It is therefore enough to show that for any category with cofibrations $\mathcal{C}$, the functors s and q induce a weak equivalence

$$\Phi(S.E(\mathcal{C},\mathcal{C},\mathcal{C})) \to \Phi(S.\mathcal{C})\times\Phi(S.\mathcal{C}).$$

To this end, we follow McCarthy, [**27**], and apply proposition 1.4.3 to the exact functor

$$(d_2,d_0)\colon S_2\mathcal{C}\to\mathcal{C}\times\mathcal{C}.$$

The required homotopy of $\iota_R\circ\epsilon_R$ to the identity is a composite of two homotopies. In [**27**, 3.5.1], McCarthy gives explicite formulas for these homotopies. □

COROLLARY 1.4.5. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be an exact functor of categories with cofibrations and weak equivalences. Then the commutative square*

$$\begin{array}{ccc} \Phi(\mathbf{N}^w_{\cdot} S.\mathcal{C}) & \longrightarrow & \Phi(\mathbf{N}^w_{\cdot} S.S.(\mathrm{id}\colon \mathcal{C} \to \mathcal{C})) \\ \downarrow & & \downarrow \\ \Phi(\mathbf{N}^w_{\cdot} S.\mathcal{D}) & \longrightarrow & \Phi(\mathbf{N}^w_{\cdot} S.S.(f\colon \mathcal{C} \to \mathcal{D})) \end{array}$$

is homotopy cartesian, and there is a canonical contraction of the upper right hand term. □

We call a map $f\colon X \to Y$ of $\mathbb{T}$-spaces is called an $\mathcal{F}$-equivalence, if for all $r \geq 1$, the the induced map of C_r-fixed points is a weak equivalence of spaces.

PROPOSITION 1.4.6. *Let $\mathcal{C}$ be a linear category with cofibrations and weak equivalences, and let $T(\mathcal{C})$ be its topological Hochschild spectrum. The the for all $m, n \geq 1$ and all orhtogonal $\mathbb{T}$-representations W and V, the spectrum structure maps*

$$T(\mathcal{C})_{n,V} \xrightarrow{\sim} F(S^m \wedge S^W, T(\mathcal{C})_{m+n,W\oplus V})$$

are $\mathcal{F}$-equivalences. In particular, the spectrum $|T(\mathcal{C})^{C_r}|$ and the pointed space $\Omega\,\mathrm{THH}(\mathbf{N}^w_{\cdot} S.\mathcal{C})^{C_r}$ have canonically isomorphic homotopy groups.

PROOF. We factor the map in the statement as

$$T(\mathcal{C})_{n,V} \to F(S^m, T(\mathcal{C})_{m+n,V}) \to F(S^m, F(S^W, T(\mathcal{C})_{m+n,W\oplus V})).$$

Since S^m is C_r-fixed the map of C_r-fixed sets induced from the first map may be identified with the map

$$(T(\mathcal{C})_{n,V})^{C_r} \to \Omega^m (T(\mathcal{C})_{m+n,V})^{C_r},$$

and by definition, this is the map

$$\mathrm{THH}^{(n)}(\mathbf{N}^w_{\cdot} S^{(n)}_{\cdot}\mathcal{C}; S^V)^{C_r} \to \Omega^m\,\mathrm{THH}^{(m+n)}(\mathbf{N}^w_{\cdot} S^{(m+n)}_{\cdot}\mathcal{C}; S^V)^{C_r}.$$

By the approximation lemma [**2**, theorem 1.6], we can replace the functor $\mathrm{THH}^{(k)}(-;-)$ by the common functor $\mathrm{THH}(-;-)$, and the claim now follows corollary 1.4.5 of the additivity theorem applied to the functor

$$\Phi(\mathcal{C}) = \mathrm{THH}(\mathcal{C}; S^V)^{C_r},$$

compare [**40**, ?]. It remains to show that

$$(T(\mathcal{C})_{m+n,V})^{C_r} \to F(S^W, T(\mathcal{C})_{m+n,W\oplus V}))^{C_r}$$

is a weak equivalences. This follows from the proof of [**16**, proposition 2.4]. □

1.5. In this section, we extend Waldhausen's fibration theorem to the present situation. We follow the original proof in [**40**], where also the notion of a cylinder functor is defined.

LEMMA 1.5.1. *Suppose that $\mathcal{C}$ has a cylinder functor, and that $w\mathcal{C}$ satisfies the cylinder axiom and the saturation axiom. Then*

$$\Phi(\mathbf{N}^{\bar{w}}_{\cdot}\mathcal{C}) \xrightarrow{\sim} \Phi(\mathbf{N}^w_{\cdot}\mathcal{C})$$

is a weak equivalence.

Proof. The proof is analogous to the proof of [**40**, lemma 1.6.3], but we need the proof of Quillen's theorem A and not just the statement. We consider the bi-simplicial category $\mathbf{T}(\mathcal{C})$ whose category of (p,q)-simplices has objects pairs of diagrams in $\mathcal{C}$ of the form

$$(A_q \to \cdots \to A_0, A_0 \to B_0 \to \cdots \to B_p),$$

and morphisms all natural transformations of such pairs of diagrams. We let

$$\mathbf{T}^{\bar{w},w}(\mathcal{C}) \subset \mathbf{T}(\mathcal{C})$$

be the *full* subcategory with objects the pairs of diagrams with the left hand diagram in $\bar{w}\mathcal{C}$ and the right hand diagram in $w\mathcal{C}$. There are bi-simplicial functors

$$\mathbf{N}^{\bar{w}}(\mathcal{C}^{\mathrm{op}})R \xleftarrow{p_1} \mathbf{T}^{\bar{w},w}(\mathcal{C}) \xrightarrow{p_2} \mathbf{N}^{w}(\mathcal{C})L,$$

and applying Φ in each simplicial bi-degree, we get corresponding maps of bi-simplicial spaces. We show that both maps induce weak equivalences after realization.

For fixed q, the simplicial functor

$$p_1\colon \mathbf{T}^{\bar{w},w}_{\cdot,q}(\mathcal{C}) \to \mathbf{N}^{\bar{w}}_q(\mathcal{C}^{\mathrm{op}})$$

is a simplicial homotopy equivalence, and hence induces a homotopy equivalence upon realization. It follows that

$$\Phi(p_1)\colon \Phi(\mathbf{T}(\mathcal{C})) \xrightarrow{\sim} \Phi(\mathbf{N}^{\bar{w}}_{\cdot}(\mathcal{C}^{\mathrm{op}}))$$

is a weak equivalence of spaces.

Similarly, we claim that for fixed p, the simplicial functor

$$p_2\colon \mathbf{T}^{\bar{w},w}_{p,\cdot}(\mathcal{C}) \to \mathbf{N}^{w}_p(\mathcal{C})$$

is a simplicial homotopy equivalence. The homotopy inverse σ maps

$$(B_0 \to \cdots \to B_p) \mapsto (B_0 \xrightarrow{\mathrm{id}} \dots \xrightarrow{\mathrm{id}} B_0, B_0 \xrightarrow{\mathrm{id}} B_0 \to \cdots \to B_p).$$

Following the proof of [**40**, lemma 1.6.3] we also consider the simplicial functor

$$t\colon \mathbf{T}^{\bar{w},w}_{p,\cdot}(\mathcal{C}) \to \mathbf{T}^{\bar{w},w}_{p,\cdot}(\mathcal{C})$$

which maps

$$\begin{aligned}&(A_q \to \cdots \to A_0, A_0 \to B_0 \to \dots B_p)\\ &\mapsto (T(A_q \to B_0) \to \cdots \to T(A_0 \to B_0), T(A_0 \to B_0) \xrightarrow{p} B_0 \to \cdots \to B_p),\end{aligned}$$

where T is the cylinder functor. There are exact simplicial homotopies from $\sigma \circ p_2$ to t and from the identify functor to t. Hence

$$\Phi(p_2)\colon \Phi(\mathbf{T}^{\bar{w},w}(\mathcal{C})) \xrightarrow{\sim} \Phi(\mathbf{N}^{w}(\mathcal{C}))$$

is a weak equivalence of spaces.

Finally, consider the diagram of bi-simplicial categories

$$\begin{tikzcd}
\mathbf{N}^{\bar{w}}(\mathcal{C}^{\mathrm{op}})R \arrow[d, "i"] & \mathbf{T}^{\bar{w},w}(\mathcal{C}) \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "i'"] & \mathbf{N}^{w}(\mathcal{C})L \arrow[d, "i"] \\
\mathbf{N}^{w}(\mathcal{C}^{\mathrm{op}})R & \mathbf{T}^{w,w}(\mathcal{C}) \arrow[l, "p_1"'] \arrow[r, "p_2"] & \mathbf{N}^{w}(\mathcal{C})L,
\end{tikzcd}$$

where i' is the obvious inclusion functor. Applying Φ the horizontal functors all induce weak equivalences. The lemma follows. $\square$

Let $\mathcal{C}$ be a category with cofibrations and two categories of weak equivalences $v\mathcal{C}$ and $w\mathcal{C}$, and write

$$\mathbf{N}^{v,w}_{\cdot}\mathcal{C} = \mathbf{N}^{v}_{\cdot}(\mathbf{N}^{w}_{\cdot}\mathcal{C}) \cong \mathbf{N}^{w}_{\cdot}(\mathbf{N}^{v}_{\cdot}\mathcal{C}).$$

This is a bi-simplicial category with cofibrations which again has two categories of weak equivalences.

LEMMA 1.5.2. (Swallowing lemma) *If $v\mathcal{C} \subset w\mathcal{C}$ then*

$$\Phi((\mathbf{N}^{w}_{\cdot}\mathcal{C})) = \Phi((\mathbf{N}^{w}_{\cdot}\mathcal{C})R) \xrightarrow{\sim} \Phi(\mathbf{N}^{v,w}_{\cdot}\mathcal{C})$$

is a homotopy equivalence with a canonical homotopy inverse.

PROOF. We claim that for fixed m, the iterated degeneracy in the v-direction,

$$\mathbf{N}^{w}_{\cdot}\mathcal{C} \to \mathbf{N}^{w}_{\cdot}(\mathbf{N}^{v}_{m}\mathcal{C}),$$

is an exact simplicial homotopy equivalence. Given this, the lemma follows from the lemma 1.3.1 and the realization lemma. The iterated degeneracy above is induced from the (exact) iterated degeneracy map $\mathcal{C} \to \mathbf{N}^{v}_{m}\mathcal{C}$ in the simplicial category $\mathbf{N}^{v}_{\cdot}\mathcal{C}$. This map has a retraction given by the (exact) iterated face map which takes $c_0 \to \cdots \to c_m$ to c_0. The other composite takes $c_0 \to \cdots \to c_m$ to the appropriate sequence of identity maps on c_0. There is a natural transformation from this functor to the identity functor, given by

$$\begin{array}{ccccccc}
c_0 & = & c_0 & = & \cdots & = & c_0 \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f_1} & & & & \downarrow{\scriptstyle f_m \circ \cdots \circ f_1} \\
c_0 & \xrightarrow{f_1} & c_1 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_m} & c_m.
\end{array}$$

The natural transformation is through arrows in $v\mathcal{C}$, and hence in $w\mathcal{C}$. The claim now follows from lemma 1.3.2. $\square$

THEOREM 1.5.3. (*Fibration theorem*) *Let $\mathcal{C}$ be a category with cofibrations equipped and two categories of weak equivalences $v\mathcal{C} \subset w\mathcal{C}$, and let $\mathcal{C}^w$ be the full subcategory with cofibrations of $\mathcal{C}$ given by the objects A which satisfy that $* \to A$ is in $w\mathcal{C}$. Suppose that $\mathcal{C}$ has a cylinder functor, and that $w\mathcal{C}$ satisfies the cylinder axiom, the saturation axiom, and the extension axiom. Then*

$$\begin{array}{ccc}
\Phi(\mathbf{N}^{v}_{\cdot}S_{\cdot}\mathcal{C}^w) & \longrightarrow & \Phi(\mathbf{N}^{w}_{\cdot}S_{\cdot}\mathcal{C}^w) \\
\downarrow & & \downarrow \\
\Phi(\mathbf{N}^{v}_{\cdot}S_{\cdot}\mathcal{C}) & \longrightarrow & \Phi(\mathbf{N}^{w}_{\cdot}S_{\cdot}\mathcal{C})
\end{array}$$

is a homotopy cartesian square of pointed simplicial sets, and there is a canonical contraction of the upper right hand term.

PROOF. Following the proof of [**40**, theorem 1.6.4], we consider the diagram

$$\begin{tikzcd}
\Phi(\mathbf{N}^v S.\mathcal{C}^w) \arrow[r] \arrow[d] & \Phi(\mathbf{N}^{v,\bar{w}} S.\mathcal{C}^w) \arrow[r, "\sim"] \arrow[d] & \Phi(\mathbf{N}^{v,w} S.\mathcal{C}^w) \arrow[r, "\sim"] \arrow[d] & \Phi(\mathbf{N}^w S.\mathcal{C}^w) \arrow[d] \\
\Phi(\mathbf{N}^v S.\mathcal{C}) \arrow[r] & \Phi(\mathbf{N}^{v,\bar{w}} S.\mathcal{C}) \arrow[r, "\sim"] & \Phi(\mathbf{N}^{v,w} S.\mathcal{C}) \arrow[r, "\sim"] & \Phi(\mathbf{N}^w S.\mathcal{C}).
\end{tikzcd}$$

The horizontal maps in the middle square are weak equivalences by lemma 1.5.1 and the horizontal maps in the right hand square are weak equivalences by the swallowing lemma. The left hand square may be rewritten as

$$\begin{tikzcd}
\Phi(\mathbf{N}^v S.\mathcal{C}^w) \arrow[r] \arrow[d] & \Phi(\mathbf{N}^v S.S.(\mathcal{C}^w \to \mathcal{C}^w)) \arrow[d] \\
\Phi(\mathbf{N}^v S.\mathcal{C}) \arrow[r] & \Phi(\mathbf{N}^v S.S.(\mathcal{C}^w \to \mathcal{C}))
\end{tikzcd}$$

It is therefore homotopy cartesian by the corollary 1.4.5 of the additivity theorem. Finally, since $w\mathcal{C}^w$ has an initial object, lemma 1.3.2 gives a contracting exact simplicial homotopy of $\mathbf{N}^w_{\cdot}\mathcal{C}^w$. Hence $\Phi(S.\mathbf{N}^w_{\cdot}\mathcal{C}^w) \cong \Phi(\mathbf{N}^w_{\cdot} S.\mathcal{C}^w)$ is contractible by lemma 1.3.1. □

1.6. Let $\mathcal{A}$ be an abelian category. We view $\mathcal{A}$ as a category with cofibrations and weak equivalences by choosing a null-object and taking the monomorphisms as the cofibrations and the isomorphisms as the weak equivalences. Let $\mathcal{E}$ be an additive category embedded as a full subcategory of $\mathcal{A}$, and assume that for every exact sequence in $\mathcal{A}$,

$$0 \to A' \to A \to A'' \to 0,$$

if A' and A'' are in $\mathcal{E}$ then A is in $\mathcal{E}$, and if A and A'' are in $\mathcal{E}$ then A' is in $\mathcal{E}$. We then view $\mathcal{E}$ as a subcategory with cofibrations and weak equivalences of $\mathcal{A}$.

The category $C^b(\mathcal{A})$ of bounded complexes in $\mathcal{A}$ is a category with cofibrations and weak equivalences, where the cofibrations are the degree wise monomorphisms and the weak equivalences are the quasi-isomorphisms. We view the category $C^b(\mathcal{E})$ of bounded complexes in $\mathcal{E}$ as a subcategory with cofibrations and weak equivalences of $C^b(\mathcal{A})$. The inclusion

$$\mathcal{E} \to C^b(\mathcal{E})$$

of $\mathcal{E}$ as the subcategory of complexes concentrated in degree zero, is an exact functor.

THEOREM 1.6.1. *With $\mathcal{E}$ as above, the map induced from the inclusion*

$$\Phi(\mathbf{N}^i_{\cdot} S.\mathcal{E}) \xrightarrow{\sim} \Phi(\mathbf{N}^w_{\cdot} S.C^b(\mathcal{E}))$$

is a weak equivalence.

Proof. We follow the proof of [**38**, theorem 1.11.7]. Since the category $C^b(\mathcal{E})$ has a cylinder functor which satisfies the cylinder axiom with respect to quasi-isomorphisms, the fibration theorem shows that the right hand square in the diagram

$$\begin{tikzcd}
\Phi(\mathbf{N}^i_. S.\mathcal{E}^i) \arrow[r] \arrow[d] & \Phi(\mathbf{N}^i_. S.C^b(\mathcal{E})^w) \arrow[r] \arrow[d] & \Phi(\mathbf{N}^w_. S.C^b(\mathcal{E})^w) \arrow[d] \\
\Phi(\mathbf{N}^i_. S.\mathcal{E}) \arrow[r] & \Phi(\mathbf{N}^i_. S.C^b(\mathcal{E})) \arrow[r] & \Phi(\mathbf{N}^w_. S.C^b(\mathcal{E}))
\end{tikzcd}$$

is homotopy cartesian. Moreover, the composite of the maps in the lower row is equal to the map of the statement, and the upper left hand and upper right hand terms are contractible. Hence the theorem is equivalent to showing that the left hand square is homotopy cartesian.

Let $\mathcal{C}_a^b$ be the full subcategory of $C^b(\mathcal{E})$ consisting of the complexes E_* with $E_i = 0$ for $i > b$ and $i < a$. Then $C^b(\mathcal{E})$ is the colimit of the categories $\mathcal{C}_a^b$ as a and b tends to $-\infty$ and $+\infty$, respectively. We consider $\mathcal{C}_a^b$ as a subcategory with cofibrations of $C^b(\mathcal{E})$.

We first show that there is a weak equivalence

$$\Phi(\mathbf{N}^i_. S.\mathcal{C}_a^b) \to \prod_{a \le s \le b} \Phi(\mathbf{N}^i_. S.\mathcal{E}), \qquad E_* \mapsto (E_b, E_{b-1}, \dots, E_a).$$

The map is an isomorphism for $b = a$. If $b > a$, the functor

$$e \colon \mathcal{C}_a^b \to E(\mathcal{C}_a^a, \mathcal{C}_a^b, \mathcal{C}_{a+1}^b),$$

which takes E_* to the extension

$$\sigma_{\le a} E_* \rightarrowtail E_* \twoheadrightarrow \sigma_{>a} E_*,$$

is an exact equivalence of categories. The inverse, given by the total-object functor, is also exact. Hence, the induced map

$$\Phi(\mathbf{N}^i_. S.\mathcal{C}_a^b) \xrightarrow{\sim} \Phi(\mathbf{N}^i_. S.E(\mathcal{C}_a^a, \mathcal{C}_a^b, \mathcal{C}_{a+1}^b)),$$

is a homotopy equivalence by lemma 1.3.2. McCarthy's additivity theorem 1.4.4 then shows that

$$(s,q) \colon \Phi(\mathbf{N}^i_. S.E(\mathcal{C}_a^a, \mathcal{C}_a^b, \mathcal{C}_{a+1}^b)) \xrightarrow{\sim} \Phi(\mathbf{N}^i_. S.\mathcal{C}_a^a) \times \Phi(\mathbf{N}^i_. S.\mathcal{C}_{a+1}^b),$$

so in all, we have a weak equivalence

$$\Phi(\mathbf{N}^i_. S.\mathcal{C}_a^b) \xrightarrow{\sim} \Phi(\mathbf{N}^i_. S.\mathcal{E}) \times \Phi(\mathbf{N}^i_. S.\mathcal{C}_{a+1}^b), \qquad E_* \mapsto (E_a, \sigma_{>a} E_*).$$

It now follows by easy induction that the map in question is a weak equivalence.

Next, we claim that

$$\Phi(\mathbf{N}^i_. S.\mathcal{C}_a^{bw}) \to \prod_{a \le s < b} \Phi(\mathbf{N}^i_. S.\mathcal{E}), \qquad E_* \mapsto (B_{b-1}, B_{b-2}, \dots, B_a),$$

where $B_i \subset E_i$ are the boundaries, is a weak equivalence. Note that the exactness of the functors $E_* \mapsto B_i$ uses that the complex E_* is acyclic. If $a = b-1$ the functor $E_* \mapsto B_{b-1}$ is an equivalence of categories with exact inverse functor. Therefore, in this case, the claim follows from lemma 1.3.2. If $b - 1 > a$, we consider the functor

$$\mathcal{C}_a^{bw} \to E(\mathcal{C}_{b-1}^{bw}, \mathcal{C}_a^{bw}, \mathcal{C}_a^{(b-1)w})$$

which takes the acyclic complex E_* to the extension

$$\tau_{\geq b-1}E_* \rightarrowtail E_* \twoheadrightarrow \tau_{<b-1}E_*.$$

The functor is exact, since we only consider acyclic complexes, and it is an equivalence of categories with exact inverse given by the total-object functor. Hence the induced map

$$\Phi(\mathbf{N}^i_{\cdot}S.\mathcal{C}^{bw}_a) \xrightarrow{\sim} \Phi(\mathbf{N}^i_{\cdot}S.E(\mathcal{C}^{bw}_{b-1}, \mathcal{C}^{bw}_a, \mathcal{C}^{(b-1)w}_a))$$

is a homotopy equivalence by lemma 1.3.2. The additivity theorem now shows that

$$\Phi(\mathbf{N}^i_{\cdot}S.\mathcal{C}^{bw}_a) \xrightarrow{\sim} \Phi(\mathbf{N}^i_{\cdot}S.\mathcal{E}) \times \Phi(\mathbf{N}^i_{\cdot}S.\mathcal{C}^{b-1}_a), \qquad E_* \mapsto (B_{b-1}, \tau_{<b-1}E_*),$$

is a weak equivalence, and the claim follows by induction.

One of the standard corollaries of the additivity theorem shows that there is a homotopy commutative diagram

$$\begin{tikzcd}
\Phi(\mathbf{N}^i_{\cdot}S.\mathcal{C}^{bw}_a) \arrow[r, "\sim"] \arrow[d] & \prod_{a\leq s<b} \Phi(\mathbf{N}^i_{\cdot}S.\mathcal{E}) \arrow[d] \\
\Phi(\mathbf{N}^i_{\cdot}S.\mathcal{C}^{b}_a) \arrow[r, "\sim"] & \prod_{a\leq s\leq b} \Phi(\mathbf{N}^i_{\cdot}S.\mathcal{E})
\end{tikzcd}$$

where the horizontal maps are the equivalences established above, and where the right hand vertical map is given by

$$(x_s) \mapsto (x_s + x_{s-1}).$$

It follows that the diagram

$$\begin{tikzcd}
\Phi(\mathbf{N}^i_{\cdot}S.\mathcal{C}^{0w}_0) \arrow[r] \arrow[d] & \Phi(\mathbf{N}^i_{\cdot}S.\mathcal{C}^{bw}_a) \arrow[d] \\
\Phi(\mathbf{N}^i_{\cdot}S.\mathcal{C}^{0}_0) \arrow[r] & \Phi(\mathbf{N}^i_{\cdot}S.\mathcal{C}^{b}_a)),
\end{tikzcd}$$

where the maps are induced by the canonical inclusions, is homotopy cartesian. Indeed, the map of horizontal homotopy fibers may be identified with the map

$$\prod_{a\leq s<b} \Omega\Phi(\mathbf{N}^i_{\cdot}S.\mathcal{E}) \to \prod_{a\leq s\leq b, s\neq 0} \Omega\Phi(\mathbf{N}^i_{\cdot}S.\mathcal{E})$$

given by

$$(x_s) \mapsto (x_s + x_{s-1}),$$

and this is clearly a homotopy equivalence. Taking the homotopy colimit over a and b, we see that the left hand square in the diagram at the beginning of the proof is homotopy cartesian. $\square$

1.7. In this section we recall the equivalence criterion of Dundas-McCarthy for topological Hochschild homology. This is based on the following

PROPOSITION 1.7.1. ([**6**, proposition 2.2.3]) *The map induced from the inclusion of the zero-skeleton*

$$\underset{\overrightarrow{n}}{\operatorname{holim}}\,\Omega^{n-k}\,\mathrm{THH}_0(\mathcal{S}^{(n)}_{\cdot}\mathcal{C}) \to \underset{\overrightarrow{n}}{\operatorname{holim}}\,\Omega^{n-k}\,\mathrm{THH}(\mathcal{S}^{(n)}_{\cdot}\mathcal{C})$$

is a weak equivalences of pointed spaces. □

Given a linear category $\mathcal{C}$ and a finite pointed set X, one defines the endomorphism category $\mathbf{End}_X(\mathcal{C})$, where the objects are pairs (c, v) with $c \in \mathrm{ob}\,\mathcal{C}$ and $v \in \mathrm{Hom}_{\mathcal{C}}(c, c \otimes \mathbb{Z}(X))$. A morphism in $\mathbf{End}_X(\mathcal{C})$ from (c, v) to (d, w) is a morphism $f\colon c \to d$ in $\mathcal{C}$ which makes the diagram

$$\begin{tikzcd} c \arrow[r, "v"] \arrow[d, "f"] & c \otimes \mathbb{Z}(X) \arrow[d, "f\otimes\mathrm{id}"] \\ d \arrow[r, "w"] & d \otimes \mathbb{Z}(X) \end{tikzcd}$$

commute. In particular, if X has one point, $\mathbf{End}_X(\mathcal{C})$ is equivalent $\mathcal{C}$, and if X has two points, $\mathbf{End}_X(\mathcal{C})$ is equivalent to the category of endomorphisms $\mathbf{End}(\mathcal{C})$.

PROPOSITION 1.7.2. ([**6**, proposition 2.3.3]) *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor of linear categories with cofibrations and weak equivalences, and suppose that for every finite set X, the map $|\mathrm{ob}\,\mathbf{N}^w_{\cdot} S.\mathbf{End}_X(F)|$ is a weak equivalence. Then F induces a weak equivalence*

$$\mathrm{THH}(\mathbf{N}^w_{\cdot} S.\mathcal{C}) \xrightarrow{\sim} \mathrm{THH}(\mathbf{N}^w_{\cdot} S.\mathcal{D}).$$

PROOF. The proof has two parts. We first show that it is enough to show that F induces a weak equivalence

$$|V(S^j_{\cdot}\mathbf{N}^w_{\cdot}\mathcal{C}, \underline{i})| \xrightarrow{\sim} |V(S^j_{\cdot}\mathbf{N}^w_{\cdot}\mathcal{D}, \underline{i})|$$

for j large. We then show that this spells out to the condition listed in the statement.

By the previous proposition, the canonical maps

$$\mathrm{THH}(S.\mathcal{A}) \xrightarrow{\sim} \underset{\overrightarrow{j}}{\operatorname{holim}}\,\Omega^{j-1}\,\mathrm{THH}(S^j_{\cdot}\mathcal{A}) \xleftarrow{\sim} \underset{\overrightarrow{j}}{\operatorname{holim}}\,\Omega^{j-1}\,\mathrm{THH}_0(S^j_{\cdot}\mathcal{A})$$

are weak equivalences for any linear category $\mathcal{A}$. It thus suffices to show that F induces a weak equivalence

$$\mathrm{THH}_0(S^j_{\cdot}\mathbf{N}^w_{\cdot}\mathcal{C}) \xrightarrow{\sim} \mathrm{THH}_0(S^j_{\cdot}\mathbf{N}^w_{\cdot}\mathcal{D}),$$

for j large. Writing out definitions, this map is

$$|\underset{\overrightarrow{I}}{\operatorname{holim}}\,F(S^i, V(S^j_{\cdot}\mathbf{N}^w_{\cdot}\mathcal{C}, \underline{i}))| \to |\underset{\overrightarrow{I}}{\operatorname{holim}}\,F(S^i, V(S^j_{\cdot}\mathbf{N}^w_{\cdot}\mathcal{D}, \underline{i}))|,$$

and since homotopy colimits commute with realization and preserve weak equivalences (of well-pointed spaces), it suffices to show that F induces a weak equivalence

$$|F(S^i, V(S^j_{\cdot}\mathbf{N}^w_{\cdot}\mathcal{C}, \underline{i}))| \to |F(S^i, V(S^j_{\cdot}\mathbf{N}^w_{\cdot}\mathcal{D}, \underline{i}))|$$

for j large and for all i. We consider the following diagram

$$\begin{array}{ccc} |F(S^i, V(S^j_\cdot \mathbf{N}^w_\cdot \mathcal{C}, \underline{i}))| & \longrightarrow & |F(S^i, V(S^j_\cdot \mathbf{N}^w_\cdot \mathcal{D}, \underline{i}))| \\ \downarrow & & \downarrow \\ F(S^i, |V(S^j_\cdot \mathbf{N}^w_\cdot \mathcal{C}, \underline{i})|) & \longrightarrow & F(S^i, |V(S^j_\cdot \mathbf{N}^w_\cdot \mathcal{D}, \underline{i})|). \end{array}$$

The simplicial space $V(S^j_\cdot \mathbf{N}^w_\cdot \mathcal{C}, \underline{i})$ is $(i-1)$-connected in each simplicial degree. It is also *good* in the sense that the degeneracy maps are (Serre) cofibrations. But then the vertical maps in the diagram above are weak equivalences by [**26**, theorem 12.3]. This finishes the first part of the proof.

For any linear category $\mathcal{A}$, we have a cofibration sequence

$$\mathrm{ob}\,\mathcal{A} \rightarrowtail \coprod_{a \in \mathrm{ob}\,\mathcal{A}} \underline{\mathrm{Hom}}_{\mathcal{A}}(a,a)_i \twoheadrightarrow \bigvee_{a \in \mathrm{ob}\,\mathcal{A}} \underline{\mathrm{Hom}}_{\mathcal{A}}(a,a)_i,$$

and the right hand side by definition is $V(\mathcal{A}, \underline{i})$. Here, we remember,

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(a,a)_i = |\,\mathrm{Hom}_{\mathcal{A}}(a,a) \otimes \mathbb{Z}(S^i)|,$$

and since

$$\coprod_{a \in \mathrm{ob}\,\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(a,a) \otimes \mathbb{Z}(S^i) = \mathrm{ob}\,\mathbf{End}_{S^i}(\mathcal{A}),$$

the cofibration sequence above takes the form

$$\mathrm{ob}\,\mathcal{A} \rightarrowtail |\,\mathrm{ob}\,\mathbf{End}_{S^i}(\mathcal{A})| \twoheadrightarrow V(\mathcal{A}, i).$$

In the case at hand, we get a map of cofibration sequences

$$\begin{array}{ccccc} |\,\mathrm{ob}\,S^j_\cdot \mathbf{N}^w_\cdot \mathcal{C}| & \rightarrowtail & |\,\mathrm{ob}\,\mathbf{End}_{S^i}(S^j_\cdot \mathbf{N}^w_\cdot \mathcal{C})| & \twoheadrightarrow & |V(S^j_\cdot \mathbf{N}^w_\cdot \mathcal{C}, i)| \\ \downarrow & & \downarrow & & \downarrow \\ |\,\mathrm{ob}\,S^j_\cdot \mathbf{N}^w_\cdot \mathcal{D}| & \rightarrowtail & |\,\mathrm{ob}\,\mathbf{End}_{S^i}(S^j_\cdot \mathbf{N}^w_\cdot \mathcal{D})| & \twoheadrightarrow & |V(S^j_\cdot \mathbf{N}^w_\cdot \mathcal{D}, i)|. \end{array}$$

The spaces in this diagram are all $(j-1)$-connected. Hence for $j > 1$, the right hand vertical map will be a weak equivalence if the left hand and middle vertical maps are weak equivalences. To see that this is the case, we rewrite

$$\mathbf{End}_{S^i}(S^j_\cdot \mathbf{N}^w_\cdot \mathcal{C}) \cong \mathbf{N}^w_\cdot S^j_\cdot \mathbf{End}_{S^i}(\mathcal{C}).$$

By our assumptions, the map

$$|\,\mathrm{ob}\,\mathbf{N}^w_\cdot S.\mathbf{End}_{S^i}(\mathcal{C})| \xrightarrow{\sim} |\,\mathrm{ob}\,\mathbf{N}^w_\cdot S.\mathbf{End}_{S^i}(\mathcal{C})|$$

is a weak equivalence. And the addditivity theorem implies that the vertical maps in the diagram

$$\begin{array}{ccc} |\,\mathrm{ob}\,\mathbf{N}^w_\cdot S.\mathbf{End}_{S^i}(\mathcal{C})| & \xrightarrow{\sim} & |\,\mathrm{ob}\,\mathbf{N}^w_\cdot S.\mathbf{End}_{S^i}(\mathcal{D})| \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ \Omega^{j-1}|\,\mathrm{ob}\,\mathbf{N}^w_\cdot S^j_\cdot \mathbf{End}_{S^i}(\mathcal{C})| & \longrightarrow & \Omega^{j-1}|\,\mathrm{ob}\,\mathbf{N}^w_\cdot S^j_\cdot \mathbf{End}_{S^i}(\mathcal{D})| \end{array}$$

are weak equivalences. Hence the lower horizontal map is a weak equivalence. But this map is the $(j-1)$st loop of the middle vertical map in the diagram of cofibration sequences above. And since the domain and range of the latter map are

$(j-1)$-connected, it is a weak equivalence. The proof that the left hand map in the diagram of cofibration sequences above is a weak equivalence is similar. $\square$

1.8. Let A be a Dedekind ring with fraction field K, and let $\mathcal{M}_A$ denote the category of finitely generated A-modules. We consider two categories with cofibrations with and weak equivalences, $C^b_z(\mathcal{M}_A)$ and $C^b_q(\mathcal{M}_A)$, both of which have the category of bounded complexes in $\mathcal{M}_A$ with degree-wise monomorphisms as their underlying category with cofibrations. The weak equivalences are the category $zC^b(\mathcal{M}_A)$ of quasi-isomorphisms and the category $qC^b(\mathcal{M}_A)$ of chain maps which become quasi-isomorphisms in $C^b(\mathcal{M}_K)$, respectively. Similarly, we let $C^b_z(\mathcal{P}_A)$ and $C^b_q(\mathcal{P}_A)$ be the category of bounded complexes of finitely generated projective A-modules considered as a subcategory with cofibrations and weak equivalences of $C^b_z(\mathcal{M}_A)$ and $C^b_q(\mathcal{M}_A)$, respectively.

THEOREM 1.8.1. *The inclusion functor induces a weak equivalence*

$$\mathrm{THH}(\mathbf{N}^z_\cdot S.\mathcal{C}^b(\mathcal{M}_A{}^q)) \xrightarrow{\sim} \mathrm{THH}(\mathbf{N}^z_\cdot S.\mathcal{C}^b(\mathcal{M}_A)^q).$$

PROOF. We show that the assumptions of the Dundas-McCarthy equivalence criterion 1.7.2 are satisfied. The proof relies on Waldhausen's approximation theorem, [**40**, theorem 1.6.7], but in a formulation due to Thomason, [**38**, theorem 1.9.8], which is particularly suited for the situation at hand.

If X is a finite pointed set, we let $A\{X\}$ denote the ring of non-commutative polynomials in the variables $X-\{x_0\}$ with coefficients in A. The $A\{X\}$ is an associative unital A-algebra, and we let $\mathcal{M}_{A,X}$ denote the category of $A\{X\}$-modules which are finitely generated as A-modules. Then there are canonical isomorphisms of categories

$$\begin{aligned}\mathbf{End}_X C^b(\mathcal{M}_A) &\cong C^b(\mathcal{M}_{A,X}),\\ \mathbf{End}_X C^b(\mathcal{M}_A)^q &\cong C^b(\mathcal{M}_{A,X})^q,\\ \mathbf{End}_X C^b(\mathcal{M}^q_A) &\cong C^b(\mathcal{M}^q_{A,X}),\end{aligned}$$

where $C^b(\mathcal{M}_{A,X})^q \subset C^b(\mathcal{M}_{A,X})$ is the subcategory of chain maps whose image under the forgetful functor

$$C^b(\mathcal{M}_{A,X}) \to C^b(\mathcal{M}_A)$$

lies in $C^b(\mathcal{M}_A)^q$, and similarly for $\mathcal{M}^q_{A,X}$. We must show that the inclusion functor induces a weak equivalence

$$|\operatorname{ob}\mathbf{N}^z_\cdot S.C^b(\mathcal{M}^q_{A,X})| \xrightarrow{\sim} |\operatorname{ob}\mathbf{N}^z_\cdot S.C^b(\mathcal{M}_{A,X})^q|,$$

and use [**38**, theorem 1.9.8]. The categories $C^b(\mathcal{M}^q_{A,X})$ and $C^b(\mathcal{M}_{A,X})^q$ are both complicial bi-Waldhausen categories in the sense of [**38**, 1.2.4], which are closed under the formation of canonical homotopy pushouts and homotopy pullbacks in the sense of [**38**, 1.96]. The inclusion functor

$$F\colon C^b(\mathcal{M}^q_{A,X}) \to C^b(\mathcal{M}_{A,X})^q$$

is a complicial exact functor in the sense of [**38**, 1.2.16]. We must verify the conditions [**38**, 1.9.7.0–1.9.7.3]. These conditions are easily verified with the exception of condition 1.9.7.1 which reads: for every object B of $C^b(\mathcal{M}_{A,X})^q$, there exists an object A of $C^b(\mathcal{M}^q_{A,X})$ and a map $FA \xrightarrow{\sim} B$ in $zC^b(\mathcal{M}_{A,X})^q$. This follows from the following lemma. $\square$

LEMMA 1.8.2. *Let A be a Dedekind ring and let $f\colon A \to B$ be a ring homomorphism. Let C_* be a bounded complex of left B-modules and suppose that f^*C_* is a complex of finitely generated A-modules whose homology is torsion. Then there exists a quasi-isomorphism*

$$C_* \xrightarrow{\sim} D_*$$

where D_* is a bounded complex of left B-modules such that f^*D_* is a complex of finitely generated torsion A-modules.

PROOF. Suppose that f^*C_i is a torsion module for $i > n$. We construct a quasi-isomorphism $C_* \xrightarrow{\sim} C'_*$ to a complex C'_* with $f^*C'_i$ torsion for $i > n-1$. The lemma then follows by simple induction.

We will show that there exists a submodule $I \subset C_n$ which intersects Z_n trivially and such that $f^*(C_n/I)$ is torsion. The first of these properties may also be expressed as a map of exact sequences

$$\begin{tikzcd} Z_n \arrow[r, tail] \arrow[d, equal] & C_n \arrow[r, two heads] \arrow[d, two heads] & B_{n-1} \arrow[d, two heads] \\ Z_n \arrow[r, tail] & C_n/I \arrow[r, two heads] & B_{n-1}/dI. \end{tikzcd}$$

Given this, we define

$$C'_i = \begin{cases} C_n/I, & \text{for } i = n, \\ C_{n-1}/dI, & \text{for } i = n-1, \\ C_i, & \text{else,} \end{cases}$$

with the differential determined by the requirement that the natural projection $C_* \to C'_*$ be a chain map. It is clear that this chain map is then a quasi-isomorphism.

To construct the submodule $I \subset C_n$, we consider the extension

$$Z_n \rightarrowtail C_n \twoheadrightarrow B_{n-1}.$$

Since f^*B_{n-1} is a finitely generated A-module, we can find $a \in A$ such that $af^*(B_{n-1}) \subset f^*B_{n-1}$ is a free A-module. We form the pull-back extension

$$Z_n \rightarrowtail C_n \times_{B_{n-1}} f(a)B_{n-1} \twoheadrightarrow f(a)B_{n-1}.$$

The sequences $Z_{n+1} \rightarrowtail C_{n+1} \twoheadrightarrow B_n$ and $B_n \rightarrowtail Z_n \twoheadrightarrow H_n$ show that f^*Z_n is torsion; let $a' \in A$ be an annihilator. Then $f(a')$ annihilates $\mathrm{Ext}_B(M, Z_n)$, for any B-module M. In particular, the composite

$$\mathrm{Ext}_B(f(a)B_{n-1}, Z_n) \xrightarrow{f(a')} \mathrm{Ext}(f(a'a)B_{n-1}, Z_n) \xrightarrow{\iota^*} \mathrm{Ext}_B(f(a)B_{n-1}, Z_n)$$

is zero. But since $f^*(f(a)B_{n-1})$ is a free A-module, the right hand map is an isomorphism. Hence the left hand map is zero. It follows that the pull-back extension

$$Z_n \rightarrowtail C_n \times_{B_{n-1}} f(a'a)B_{n-1} \twoheadrightarrow f(a'a)B_{n-1}$$

is trivial. Let σ be a section of the projection on the right and let $I = \sigma(f(a'a)B_{n-1})$. By construction, $f^*(B_{n-1}/I)$ is torsion, and hence so is $f^*(C_n/I)$. $\square$

Proposition 1.8.3. *Let A be a Dedekind ring. Then map induced from the inclusion functor*

$$\mathrm{THH}(\mathbf{N}^z_{\cdot}S^{(n)}_{\cdot}C^b(\mathcal{P}_A)^q) \xrightarrow{\sim} \mathrm{THH}(\mathbf{N}^z_{\cdot}S^{(n)}_{\cdot}C^b(\mathcal{M}_A)^q)$$

is a weak equivalence of pointed spaces, for all $n \geq 1$.

Proof. Let $\mathcal{M}_{A,X}$ be as in the proof of theorem 1.8.1, and let $\mathcal{P}_{A,X}$ be the full subcategory of $A\{X\}$-modules which are finitely generated projective as A-modules. Then

$$\mathbf{End}_X C^b(\mathcal{M}_A)^q \cong C^b(\mathcal{M}_{A,X})^q,$$
$$\mathbf{End}_X C^b(\mathcal{P}_A)^q \cong C^b(\mathcal{P}_{A,X})^q,$$

and we thus have to show that the inclusion functor

$$F\colon C^b(\mathcal{P}_{A,X})^q \to C^b(\mathcal{M}_{A,X})^q$$

induces a weak equivalence

$$|\operatorname{ob}\mathbf{N}^z_{\cdot}S.C^b(\mathcal{P}_{A,X})^q| \xrightarrow{\sim} |\operatorname{ob}\mathbf{N}^z_{\cdot}S.C^b(\mathcal{M}_{A,X})^q|.$$

Again, we use [**38**, theorem 1.9.8], where the non-trivial thing to check is condition 1.9.7.1: for every object C_* of $C^b(\mathcal{M}_{A,X})^q$, there exists an object P_* of $C^b(\mathcal{P}_{A,X})^q$ and a map

$$FP_* \xrightarrow{\sim} C_*$$

in $zC^b(\mathcal{M}_{A,X})^q$. But this follows from [**5**, p. 363]. Indeed, let $\epsilon\colon P_{*,*} \to C_*$ be a resolution in the sense of loc.cit. of C_* regarded as a complex of A-modules. We can assume without loss of generality that each $P_{i,j}$ is a finitely A-module, and since A is regular, we may further assume that $P_{i,j}$ is zero for all but finitely many (i,j). Moreover, there exists automatically an $A\{X\}$-module structure on $P_{*,*}$ such that ϵ is $A\{X\}$-linear. Therefore, the total complex $P_* = \operatorname{Tot}(P_{*,*})$ is in $C^b(\mathcal{P}_{A,X})$ and

$$F\operatorname{Tot}(\epsilon)\colon FP_* = F\operatorname{Tot}(P_{*,*}) \xrightarrow{\sim} C_*$$

is in $zC^b(\mathcal{M}_{A,X})$. Hence P_* is in $C^b(\mathcal{P}_{A,X})^q$ as required. $\square$

Definition 1.8.4. Let A be a discrete valuation ring with field of fractions K and residue field k. We define ring $\mathbb{T}$-spectra

$$T(A|K) = T(C^b_q(\mathcal{P}_A)),$$
$$T(A) = T(C^b_z(\mathcal{P}_A)),$$
$$T(k) = T(C^b_z(\mathcal{P}_A)^q)$$

and we write $\mathrm{TR}^{\cdot}(A|K;p)$, $\mathrm{TR}^{\cdot}(A;p)$ and $\mathrm{TR}^{\cdot}(k;p)$ for the associated pro-ring spectra.

It follows from theorem 1.6.1 that the inclusion

$$T(\mathcal{P}_A) \to T(C^b_z(\mathcal{P}_A)) = T(A)$$

is an $\mathcal{F}$-equivalence, and hence we have an isomorphism of pro-spectra

$$\mathrm{TR}^{\cdot}(A;p) \simeq \mathrm{TR}^{\cdot}(\mathcal{P}_A;p). \tag{1.8.5}$$

Here the exact category $\mathcal{P}_A$ is considered a category with cofibrations the admissable monomorphisms and weak equivalences the isomorphisms. Similarly, the inclusion functors induce $\mathcal{F}$-equivalences

$$T(\mathcal{P}_k) \to T(C^b_z(\mathcal{P}_k)) \to T(C^b_z(\mathcal{M}^q_A)).$$

For the right hand map this is devisage, [**6**]. We proved in theorem 1.8.1 and proposition 1.8.3 above that also

$$T(C^b_z(\mathcal{M}^q_A)) \to T(C^b_z(\mathcal{M}_A)^q) \leftarrow T(C^b_z(\mathcal{P}_A)^q) = T(k)$$

are $\mathcal{F}$-equivalences, and hence there is an isomorphism of pro-spectra

$$\mathrm{TR}^{\boldsymbol{\cdot}}(k;p) \simeq \mathrm{TR}^{\boldsymbol{\cdot}}(\mathcal{P}_k;p), \tag{1.8.6}$$

which is natural in A.

THEOREM 1.8.7. *Let A be a discrete valuation ring with field of fractions K and residue field k. Then there is a natural cofibration sequence of pro-spectra*

$$\mathrm{TR}^{\boldsymbol{\cdot}}(k;p) \xrightarrow{i^!} \mathrm{TR}^{\boldsymbol{\cdot}}(A;p) \xrightarrow{j_*} \mathrm{TR}^{\boldsymbol{\cdot}}(A|K;p) \xrightarrow{\partial} \Sigma\,\mathrm{TR}^{\boldsymbol{\cdot}}(k;p),$$

and the maps in the sequence are all $\mathrm{TR}^{\boldsymbol{\cdot}}(A;p)$-module maps and commute with the maps F, V and μ. Moreover, their preferred homotopy limits form a cofibration sequence of spectra.

PROOF. We have a commutative square of symmetric orthogonal $\mathbb{T}$-spectra

$$\begin{tikzcd} T(C^b_z(\mathcal{P}_A)^q) \arrow[r] \arrow[d] & T(C^b_q(\mathcal{P}_A)^q) \arrow[d] \\ T(C^b_z(\mathcal{P}_A)) \arrow[r] & T(C^b_q(\mathcal{P}_A)), \end{tikzcd}$$

and the fibration theorem shows that the underlying square of symmetric othogonal spectra is homotopy cartesian. It follows that there is natural sequence of spectra

$$\mathrm{TR}^n(k;p) \xrightarrow{i^!} \mathrm{TR}^n(A;p) \xrightarrow{j_*} \mathrm{TR}^n(A|K;p) \xrightarrow{\partial} \Sigma\,\mathrm{TR}^n(k;p),$$

compatible with R, F, V and μ, and that this sequence is a cofibration sequence when $n = 0$. It follows by an induction argument based on the fundamental cofibration sequence

$$\mathbb{H}_{\boldsymbol{\cdot}}(C_{p^{n-1}}, T(\mathcal{D})) \xrightarrow{N} \mathrm{TR}^n(\mathcal{D};p) \xrightarrow{R} \mathrm{TR}^{n-1}(\mathcal{D};p) \to \Sigma\mathbb{H}_{\boldsymbol{\cdot}}(C_{p^{n-1}}, T(\mathcal{D}))$$

that the sequence above is cofibration sequence for all $n \geq 0$. □

ADDENDUM 1.8.8. *Let A be a discrete valuation ring with field of fractions K and residue field k. Then there is a natural map of cofibration sequences*

$$\begin{tikzcd} K(k) \arrow[r, "i^!"] \arrow[d, "\mathrm{tr}"] & K(A) \arrow[r, "j_*"] \arrow[d, "\mathrm{tr}"] & K(K) \arrow[r, "\partial"] \arrow[d, "\mathrm{tr}"] & \Sigma K(k) \arrow[d, "\mathrm{tr}"] \\ \mathrm{TC}(k;p) \arrow[r, "i^!"] & \mathrm{TC}(A;p) \arrow[r, "j_*"] & \mathrm{TC}(A|K;p) \arrow[r, "\partial"] & \Sigma\,\mathrm{TC}(k;p) \end{tikzcd}$$

and the vertical maps are all maps of ring spectra.

2. The homotopy groups of $T(A|K)$

2.1. In this paragraph, we evaluate the homotopy groups modulo p of the topological Hochschild spectrum $T(A|K)$ introduced in the preceeding paragraph. We first recall the notion of differentials with logarithmic poles. The standard reference for this material is [**19**].

A pre-log structure on a ring R is a map of monoids

$$\alpha\colon M \to R,$$

where R is considered a monoid under multiplication. By a log ring we mean a ring with a pre-log structure. A derivation of a log ring (R, M) into an R-module E is a pair of maps

$$(D, D\log)\colon (R, M) \to E,$$

where $D\colon R \to E$ is a derivation and $D\log\colon M \to E$ a map of monoids, such that for all $a \in M$,

$$\alpha(a) D\log a = D\alpha(a).$$

A log differential graded ring (E^*, M) is a triple consisting of a differential graded ring E^*, a pre-log structure $\alpha\colon M \to E^0$, and a derivation $(D, D\log)\colon (E^0, M) \to E^1$ such that D is equal to the differential $d\colon E^0 \to E^1$ and such that $d \circ D\log = 0$.

There is a universal example of a derivation of a log ring (R, M) given by the R-module

$$\omega_{(R,M)} = \big(\Omega_R \oplus (R \otimes_{\mathbb{Z}} G(M))\big)/\big\langle d\alpha(a) - \alpha(a) \otimes a \;\big|\; a \in M\big\rangle,$$

where $G(M)$ is the Grothendieck group of M. The structure maps are

$$d\colon R \to \omega_{(R,M)}, \qquad da = da \oplus 0,$$
$$d\log\colon M \to \omega_{(R,M)}, \qquad d\log a = 0 \oplus (1 \otimes a).$$

The exterior algebra

$$\omega^*_{(R,M)} = \Lambda^*_R(\omega_{(R,M)})$$

endowed with the usual differential is the universal log differential graded ring whose underlying log ring is (R, M).

When A is a discrete valuation ring with field of fractions K and residue field k, we have the canonical pre-log structure given by the inclusion

$$\alpha\colon M = A \cap K^\times \to A.$$

The Poincaré residue homomorphism is the natural map

$$\operatorname{res}\colon \omega_{(A,M)} \to A/\mathfrak{m}, \qquad \operatorname{res}(a d\log b) = av(b) + \mathfrak{m},$$

where $v\colon K^\times \to \mathbb{Z}$ is the valuation.

Proposition 2.1.1. *There is a natural short exact sequence*

$$0 \to \Omega_{A/\mathbb{Z}} \to \omega_{(A,M)} \to k \to 0.$$

Proof. For $a \in A \cap K^\times$, $av(a) \in \mathfrak{m}$ which shows that the composition of the two maps in the statement is zero. Only the exactness in the middle needs proof. Let π be a uniformizer and let $a d\log b$ be an element of $\omega_{(A,M)}$. If we write $b = \pi^i u$ with $u \in A^\times$, then

$$a d\log b = i a d\log \pi + a u^{-1} du.$$

Suppose that $\operatorname{res}(ad\log b) = ia + \mathfrak{m}A$ is trivial. Then $ia \in \mathfrak{p}A$, and hence $ia\pi^{-1} \in A$ and $iad\log\pi = ia\pi^{-1}d\pi$. It follows that $ad\log b \in \Omega_{A/\mathbb{Z}}$. $\square$

Let $W = W(k)$ be the ring of Witt vectors in k, and let $M_0 = W^\times \to W$ be the trivial log-structure on W. We define

$$\omega_{(A,M)/(W,M_0)} = \big(\Omega_{A/W} \oplus (A \otimes_{\mathbb{Z}} K^\times)\big) / \big\langle da - a \otimes a \bigm| a \in A \cap K^\times \big\rangle,$$

and an argument similar to the proof of the above proposition shows that there is a natural exact sequence

$$0 \to \Omega_{A/W} \to \omega_{(A,M)/(W,M_0)} \to k \to 0. \tag{2.1.2}$$

Lemma 2.1.3. *Let $\pi \in A$ be a uniformizer with characteristic polynomial $\phi(\pi)$. Then as an A-module, $\omega_{(A,M)/(W,M_0)}$ is generated by $d\log\pi$ with annihilator $(\phi'(\pi)\pi)$.*

Proof. Every element of $K^\times$ is of the form $\pi^i u$, where $i \in \mathbb{Z}$ and $u \in A^\times$. Hence the formula

$$d\log(\pi^i u) = id\log\pi + u^{-1}du$$

shows that as an A-module $\omega_{(A,M)/(W(k),M_0)}$ is generated by the element $d\log\pi$. The relation identifies

$$\phi'(\pi)\pi d\log\pi = d(\phi(\pi)) = 0,$$

so the annihilator ideal is generated $\phi'(\pi)\pi$. $\square$

Lemma 2.1.4. *There is a natural exact sequence*

$$A \otimes_{W(k)} \Omega^i_{W(k)} \to \omega^i_{(A,M)} \to \omega^i_{(A,M)/(W(k),M_0)} \to 0,$$

and the left hand group is uniquely divisible.

Proof. We first prove the statement for $i = 1$. The map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_{A/\mathbb{Z}} & \longrightarrow & \omega_{(A,M)} & \longrightarrow & k & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \Omega_{A/W} & \longrightarrow & \omega_{(A,M)/(W,M_0)} & \longrightarrow & k & \longrightarrow & 0 \end{array}$$

and the standard exact sequence

$$A \otimes_W \otimes \Omega_{W/\mathbb{Z}} \to \Omega_{A/\mathbb{Z}} \to \Omega_{A/W} \to 0$$

yields the sequence of the statement. We show that Ω_W is a uniquely p-divisible group. In effect, $\mathrm{HH}_i(W)$ is uniquely p-divisible, for all $i > 0$. For W is torsion free and $W/pW = k$, so the coefficient long-exact sequence takes the form

$$\cdots \to \mathrm{HH}_{i+1}(k) \to \mathrm{HH}_i(W) \xrightarrow{p} \mathrm{HH}_i(W) \to \mathrm{HH}_i(k) \to \cdots$$

and for a perfect field of positive characteristic, $\mathrm{HH}_i(k) = 0$, for $i > 0$. See e.g. [**16**, lemma 5.5]. This proves the lemma for $i = 1$. In particular, the maximal divisible sub-A-module of $\omega_{(A,M)}$ is equal to the image of $A \otimes_W \Omega_{W/\mathbb{Z}}$, and $\omega_{(A,M)}$ is the sum of this divisible module D and the cyclic torsion A-module $\omega_{(A,M)/(W,M_0)}$. It follows that for $i > 1$,

$$\omega^i_{(A,M)} = \Lambda^i_A D,$$

and this in turn is the image of left hand map of the statement. $\square$

COROLLARY 2.1.5. *The p-torsion submodule of $\omega_{(A,M)}$ is*

$$_p\omega_{(A,M)} = A/pA\langle d\log p\rangle.$$

PROOF. It follows from lemma 2.1.4 that

$$_p\omega_{(A,M)} \xrightarrow{\sim} {}_p\omega_{(A,M)/(W(k),M_0)}$$

is an isomorphism. By the previous lemma, if π be a uniformizer with minimal polynomial $\phi(\pi) \in W(k)[\pi]$, then

$$\omega_{(A,M)/(W(k),M_0)} = A/(\pi\phi'(\pi))\langle d\log\pi\rangle.$$

We write $\phi(\pi) = \pi^{e_K} + p\theta(\pi)$. Then $\theta(\pi)$ is a unit in A and $p = \pi^{e_K}\theta(\pi)^{-1}$. Hence, on the one hand

$$\pi\phi'(\pi) = e_K\pi^{e_K} + p\pi\theta'(\pi) = (e_K - \pi\theta'(\pi)\theta(\pi)^{-1})\pi^{e_K},$$

and on the other hand,

$$d\log p = d\log(\pi^{e_K}\theta(\pi)^{-1}) = (e_K - \pi\theta'(\pi)\theta(\pi)^{-1})d\log\pi.$$

The claim follows. □

Let L be a finite extension of K and let B be the integral closure of A in L. Then the following diagram commutes

$$\begin{tikzcd} \omega_{(A,M_A)/(W,M_0)} \arrow[r, "\mathrm{res}_A"] \arrow[d, "i_*"] & A/\mathfrak{m}_A \arrow[d, "e_{L/K}\cdot i"] \\ \omega_{(B,M_B)/(W,M_0)} \arrow[r, "\mathrm{res}_B"] & B/\mathfrak{m}_B, \end{tikzcd} \tag{2.1.6}$$

where $e_{L/K}$ is the ramification index of L/K. Recall that L/K is *unramified* if and only if the canonical map

$$B \otimes_A \Omega_{A/W} \to \Omega_{B/W}$$

is an isomorphism.

LEMMA 2.1.7. *The extension L/K is a tamely ramified if and only if the canonical map*

$$B \otimes_A \omega_{(A,M_A)/(W,M_0)} \to \omega_{(B,M_B)/(W,M_0)}$$

is an isomorphism.

PROOF. Suppose that L/K is tamely ramified. If L/K is unramified, the lemma follows from the natural exact sequence

$$0 \to \Omega_{A/W} \to \omega_{(A,M)/(W,M_0)} \to A/\mathfrak{m}_A \to 0$$

and the isomorphism mentioned before the lemma. So replacing K by the maximal subfield of L which is unramified over K, we may assume that the extension is totally ramified. Then there exists $\pi_A \in A$ such that

$$L = K(\pi_A^{1/e_{L/K}}).$$

Indeed, if π_A and π_B are uniformizers of A and B over W, then

$$\pi_A = u\pi_B^{e_{L/K}},$$

where $u \in B^\times$ is a unit. But the sequence

$$1 \to U_B^1 \to B^\times \xrightarrow{r} k^\times \to 1$$

is split by the composition of the Teichmüller character

$$\tau\colon k^\times \to W^\times$$

and the inclusion $W^\times \to B^\times$. Therefore, replacing π_A by $\tau(r(u))^{-1}\pi_A$, we can assume that the unit u lies in the subgroup U_B^1 of units in B which are congruent to 1 mod $\mathfrak{m}_B$. But every element of U_B^1 has an $e_{L/K}$th root, so replacing π_B by $u^{1/e_{L/K}}\pi_B$ we may assume that $u = 1$.

Let π_A and π_B be uniformizers of A and B over W such that $\pi_A = \pi_B^{e_{L/K}}$, and let $\phi_A(\pi_A)$ be the minimal polynomial of π_A. Then

$$\phi_B(\pi_B) = \phi_A(\pi_B^{e_{L/K}})$$

is the minimal polynomial of π_B. The A-module $\omega_{(A,M_A)/(W,M_0)}$ is generated by $d\log\pi_A$ with annihilator $(\phi_A'(\pi_A)\pi_A)$, and similarly, the B-module $\omega_{(B,M_B/(W,M_0)}$ is generated by $d\log\pi_B$ with annihilator $(\phi_B'(\pi_B)\pi_B)$. But

$$d\log\pi_A = d\log(\pi_B^{e_{L/K}}) = e_{L/K}\, d\log\pi_B$$

and

$$\phi_B'(\pi_B)\pi_B = \phi_A'(\pi_B^{e_{L/K}}) \cdot e_{L/K}\pi_B^{e_{L/K}} = e_{L/K}\phi_A'(\pi_A)\pi_A,$$

so the claim follows since $e_{L/K}$ is a unit. It is also clear from this argument that the map of the statement cannot be an isomorphism if the extension L/K is wildly ramified. $\square$

2.2. Let $\mathcal{C}$ be a category with cofibrations $\operatorname{co}\mathcal{C}$ and weak equivalences $w\mathcal{C}$. The Waldhausen K-theory of $\mathcal{C}$ is the symmetric spectrum $K(\mathcal{C})$ whose nth space is

$$K(\mathcal{C})_n = |N.wS.^{(n)}\mathcal{C}|.$$

Let X be an object of $\mathcal{C}$. The endomorphisms of X in the category of weak equivalences $w\mathcal{C}$ is a monoid $\operatorname{Aut}(X)$, the homotopy automorphisms of X. There is a natural map in the homotopy category of symmetric spectra

$$B\operatorname{Aut}(X) \to K(\mathcal{C}), \tag{2.2.1}$$

which we now recall. The inclusion functor $\operatorname{Aut}(X) \to w\mathcal{C}$ induces

$$N.\operatorname{Aut}(X) \to N.w\mathcal{C},$$

and hence a map of symmetric spectra

$$(B\operatorname{Aut}(X))_+ \to K(\mathcal{C}).$$

Moreover, there is a natural sum diagram in the homotopy category

$$B\operatorname{Aut}(X) \underset{i_1}{\overset{p_1}{\leftrightarrows}} (B\operatorname{Aut}(X))_+ \underset{i_2}{\overset{p_2}{\rightleftarrows}} S^0.$$

where p_1 maps the extra base point to the base point of $B\operatorname{Aut}(X)$, p_2 collapses $B\operatorname{Aut}(X)$ to the non base point of S^0, and i_2 maps the non base point of S^0 to the base point of $B\operatorname{Aut}(X)$. Finally, if σ is any section of p_1, then $i_1 = (\mathrm{id} - i_2p_2)\sigma$.

Let $\mathcal{C} = C^b(\mathcal{P}_A)$ be the category of $\mathbb{Z}$-graded bounded complexes of finitely generated projective A-modules with weak equivalences $q\mathcal{C}$ the chain maps $f\colon C \to$

D for which $K \otimes_A f\colon K \otimes_A C \to K \otimes_A D$ is a quasi-isomorphism. Viewing A a complex concentrated in degree zero,

$$\mathrm{Aut}(A) = A \cap K^\times = M,$$

and 2.2.1 then induces a map

$$M \to \pi_1 BM \to \pi_1 K(\mathcal{C}),$$

which is a group completion. Composing with the cyclotomic trace, we get

$$d\log_n\colon M \to \pi_1 \mathrm{TR}^n(A|K;p). \tag{2.2.2}$$

This map may also be described as the composite

$$\begin{aligned} M_+ \wedge S^{l+1} &\xrightarrow{\mathrm{id}\wedge\sigma} (M \times S^1)_+ \wedge S^l \to |\operatorname{ob}\mathbf{N}^q_\cdot\mathcal{C}| \wedge S^l \to S^l \wedge |N^{\mathrm{cy}}_{\wedge,\cdot}(\mathbf{N}^q_\cdot\mathcal{C})| \wedge S^l \\ &\to |N^{\mathrm{cy}}_{\wedge,\cdot}(\mathbf{N}^q_\cdot\mathcal{C})|^{C_r} \wedge S^l \xrightarrow{\lambda_{l,0}} \mathrm{TR}^n(A|K;p)_l. \end{aligned}$$

The Teichmüller character

$$\underline{}_n\colon A \to \pi_0 \mathrm{TR}^n(A|K;p) \tag{2.2.3}$$

is defined to be the composite

$$\begin{aligned} A \to N^{\mathrm{cy}}_{\wedge,0}(\mathbf{N}^q_0\mathcal{C}) \to |N^{\mathrm{cy}}_{\wedge,\cdot}(\mathbf{N}^q_\cdot\mathcal{C})| &\xrightarrow{\Delta_r} |\operatorname{sd}_r N^{\mathrm{cy}}_{\wedge,\cdot}(\mathbf{N}^q_\cdot\mathcal{C})^{C_r}| \\ &\xrightarrow{\cong} |\operatorname{sd}_r N^{\mathrm{cy}}_{\wedge,\cdot}(\mathbf{N}^q_\cdot\mathcal{C})|^{C_r} \xrightarrow{D_r} |N^{\mathrm{cy}}_{\wedge,\cdot}(\mathbf{N}^q_\cdot\mathcal{C})|^{C_r} = \mathrm{TR}^n(A|K;p)_0, \end{aligned}$$

where the first map takes a to $A \xrightarrow{a} A$, and where $r = p^{n-1}$.

PROPOSITION 2.2.4. $d\underline{a}_n = \underline{a}_n d\log_n a$.

PROOF. The map

$$\underline{}_n d\log_n\colon M \to \pi_1 \mathrm{TR}^n(A|K;p) \tag{2.2.5}$$

is given by the composite

$$\begin{tikzcd}
M_+ \wedge S^{k+l+1} \arrow[d, "\Delta\wedge\sigma"] & \\
M_+ \wedge (M \times S^1)_+ \wedge S^{k+l} \arrow[d] & \\
|N^{\mathrm{cy}}_{\wedge,\cdot}(\mathbf{N}^q_\cdot\mathcal{C})| \wedge |N^{\mathrm{cy}}_{\wedge,\cdot}(\mathbf{N}^q_\cdot\mathcal{C})| \wedge S^{k+l} \arrow[r, "\mu_{0,0}\wedge\mathrm{id}"] \arrow[d] & |N^{\mathrm{cy}}_{\wedge,\cdot}(\mathbf{N}^q_\cdot\mathcal{C})| \wedge S^{k+l} \arrow[d] \\
|N^{\mathrm{cy}}_{\wedge,\cdot}(\mathbf{N}^q_\cdot\mathcal{C})|^{C_r} \wedge |N^{\mathrm{cy}}_{\wedge,\cdot}(\mathbf{N}^q_\cdot\mathcal{C})|^{C_r} \wedge S^{k+l} \arrow[r, "\mu_{0,0}\wedge\mathrm{id}"] \arrow[d] & |N^{\mathrm{cy}}_{\wedge,\cdot}(\mathbf{N}^q_\cdot\mathcal{C})|^{C_r} \wedge S^{k+l} \arrow[dd, "\lambda_{k+l}"] \\
|N^{\mathrm{cy}}_{\wedge,\cdot}(\mathbf{N}^q_\cdot\mathcal{C})|^{C_r} \wedge S^k \wedge |N^{\mathrm{cy}}_{\wedge,\cdot}(\mathbf{N}^q_\cdot\mathcal{C})|^{C_r} \wedge S^l \arrow[d, "\lambda_k\wedge\lambda_l"] & \\
\mathrm{TR}^n(A|K;p)_k \wedge \mathrm{TR}^n(A|K;p)_l \arrow[r, "\mu_{k,l}"] & \mathrm{TR}^n(A|K;p)_{k+l}.
\end{tikzcd}$$

We wish to compare this to the map

(2.2.6) $$d_{-n} \colon A \to \pi_1 \operatorname{TR}^n(A|K;p)$$

given by the composite

$$\begin{tikzcd}
M_+ \wedge S^{l+1} \arrow[r, "\mathrm{id} \wedge \sigma"] & S^1_+ \wedge M_+ \wedge S^l \arrow[d] & \\
& S^1_+ \wedge |N^{\mathrm{cy}}_{\wedge,\cdot}(\mathbf{N}^q_\cdot\mathcal{C})| \wedge S^l \arrow[r, "\omega"] \arrow[d] & |N^{\mathrm{cy}}_{\wedge,\cdot}(\mathbf{N}^q_\cdot\mathcal{C})| \wedge S^l \arrow[d] \\
& S^1_+ \wedge \rho^*_{C_r}|N^{\mathrm{cy}}_{\wedge,\cdot}(\mathbf{N}^q_\cdot\mathcal{C})|^{C_r} \wedge S^l \arrow[r, "\omega"] \arrow[d, "\lambda_{0,l}"] & \rho^*_{C_r}|N^{\mathrm{cy}}_{\wedge,\cdot}(\mathbf{N}^q_\cdot\mathcal{C})|^{C_r} \wedge S^l \arrow[d, "\lambda_{0,l}"] \\
& S^1_+ \wedge \operatorname{TR}^n(A|K;p)_l \arrow[r, "\omega"] & \operatorname{TR}^n(A|K;p)_l.
\end{tikzcd}$$

Comparing the two diagrams above, we see that it suffices to show that the diagram

(2.2.7) $$\begin{tikzcd}
M_+ \wedge S^1_+ \arrow[r, "\Delta\wedge\mathrm{id}"] \arrow[d, "\mathrm{tw}"] & M_+ \wedge (M \times S^1)_+ \arrow[r] & |N^{\mathrm{cy}}_{\wedge,\cdot}(\mathbf{N}^q_\cdot\mathcal{C})| \wedge |N^{\mathrm{cy}}_{\wedge,\cdot}(\mathbf{N}^q_\cdot\mathcal{C})| \arrow[d, "\mu_{0,0}"] \\
S^1_+ \wedge M_+ \arrow[r] & S^1_+ \wedge |N^{\mathrm{cy}}_{\wedge,\cdot}(\mathbf{N}^q_\cdot\mathcal{C})| \arrow[r, "\omega"] & |N^{\mathrm{cy}}_{\wedge,\cdot}(\mathbf{N}^q_\cdot\mathcal{C})|
\end{tikzcd}$$

is homotopy commutative. Since M is discrete, this may be checked separately for each $a \in M$. The composite of the upper horizontal maps and the right hand vertical map restricted to $\{a\} \times S^1$ traces out the loop in the realization given by the 1-simplex

(2.2.8) $$\begin{array}{ccccc}
A & \xrightarrow{a} & A & \xrightarrow{1} & A \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a} \\
A & \xrightarrow{a} & A & \xrightarrow{1} & A.
\end{array}$$

Similarly, the composite of the left hand vertical map and the lower horizontal maps, when restricted to $\{a\} \times S^1$, traces out the loop given by the 1-simplex

(2.2.9) $$\begin{array}{ccccc}
A & \xrightarrow{1} & A & \xrightarrow{a} & A \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\
A & \xrightarrow{1} & A & \xrightarrow{a} & A.
\end{array}$$

Note that both loops are based at the vertex $A \xrightarrow{a} A$. We must show that these loops are homotopic through loops based at $A \xrightarrow{a} A$. First, the 2-simplex

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "1"] & A \arrow[r, "1"] \arrow[d, "1"] & A \arrow[r, "1"] \arrow[d, "1"] & A \arrow[d, "1"] \\
A \arrow[r, "a"] \arrow[d, "a"] & A \arrow[r, "1"] \arrow[d, "1"] & A \arrow[r, "1"] \arrow[d, "a"] & A \arrow[d, "a"] \\
A \arrow[r, "1"] & A \arrow[r, "a"] & A \arrow[r, "1"] & A
\end{tikzcd}$$

defines a homotopy through loops based at $A \xrightarrow{a} A$ between the loop given by (2.2.8) and the loop given by the 1-simplex

(2.2.10)
$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "a"] & A \arrow[r, "1"] \arrow[d, "1"] & A \arrow[d, "a"] \\
A \arrow[r, "1"] & A \arrow[r, "a"] & A.
\end{tikzcd}$$

Second, the 2-simplex

$$\begin{tikzcd}
A \arrow[r, "1"] \arrow[d, "1"] & A \arrow[r, "1"] \arrow[d, "1"] & A \arrow[r, "a"] \arrow[d, "a"] & A \arrow[d, "1"] \\
A \arrow[r, "1"] \arrow[d, "1"] & A \arrow[r, "a"] \arrow[d, "1"] & A \arrow[r, "1"] \arrow[d, "1"] & A \arrow[d, "1"] \\
A \arrow[r, "1"] & A \arrow[r, "a"] & A \arrow[r, "1"] & A
\end{tikzcd}$$

defines a homotopy though loops based at $A \xrightarrow{a} A$ between the loops given by (2.2.10) and (2.2.9). Thus (2.2.7) homotopy commutes. $\square$

COROLLARY 2.2.11. *The homotopy groups $(\pi_* T(A|K), M)$ form a differential graded ring with a log structure.* $\square$

PROPOSITION 2.2.12. *The sequence*

$$0 \to \pi_1 T(A) \to \pi_1 T(A|K) \to \pi_0 T(k) \to 0$$

is canonically isomorphic to the sequence of proposition 2.1.1.

PROOF. Since $\pi_1 T(k) = \Omega_{k/\mathbb{Z}}$ vanishes the sequence is exact. The B-operator induces a canonical homomorphism form $\Omega_{A/\mathbb{Z}}$ to $\pi_1 T(A)$, which is an isomorphism since $\pi_1 T(A) = \mathrm{HH}_1(A)$. The trace map

$$K^\times \to \pi_1 T(A|K)$$

is a map of abelian groups, which we extend to a map of A-modules

$$A \otimes_{\mathbb{Z}} K^\times \to \pi_1 T(A|K)$$

to get a homomorphism

$$\Omega_{A/\mathbb{Z}} \oplus (A \otimes K^\times) \to \pi_1 T(A|K).$$

On the first summand it is the composition of the canonical isomorphism with the inclusion $\pi_1 T(A) \to \pi_1 T(A|K)$. By corollary 2.2.11 it factors to define a map

$$\omega_{(A,M)} \to \pi_1 T(A|K).$$

Finally, it is clear that this map is compatible with the canonical maps above. It is therefore an isomorphism by the five lemma. $\square$

The homotopy groups modulo p of the topological Hochschild spectrum $T(A)$ were evaluated in [**22**]. The statement of the result is different depending on whether A/W is wildly or tamely ramified. In the wild case, a choice of uniformizer $\pi \in A$ specifies an isomorphism of differential graded k-algebras

$$A/pA \otimes_k \Lambda_k\{d\pi\} \otimes_k S\{a_K\} \xrightarrow{\sim} \bar{\pi}_* T(A),$$

where on the left, $da_K = 0$. (In [**22**], a_K and $d\pi$ were denoted α_2 and α_1, respectively.) The class a_K is characterized by its image under the primary Bockstein. Indeed, $\pi_2 T(A)$ is divisible, so the Bockstein induces an isomorphism

$$\beta\colon \bar{\pi}_2 T(A) \xrightarrow{\sim} {}_p\pi_1 T(A).$$

Let $\phi_K(\pi) = \pi^{e_K} + p\theta_K(\pi)$ be the minimal polynomial of π. Then $a_K \in \bar{\pi}_2 T(A)$ is the unique element with

$$\beta(a_K) = (\phi_K'(\pi)/p)d\pi = -(e_K\pi^{-1}\theta_K(\pi) - \theta_K'(\pi))d\pi.$$

The group $\pi_2 T(\mathbb{Z}_p)$ is uniquely divisible and $\pi_1 T(\mathbb{F}_p)$ is trivial. Hence $\pi_2 T(\mathbb{Z}_p|\mathbb{Q}_p)$ is uniquely divisible. Therefore, the Bockstein induces an isomorphism

$$\beta\colon \bar{\pi}_2 T(\mathbb{Z}_p|\mathbb{Q}_p) \xrightarrow{\sim} {}_p\pi_1 T(\mathbb{Z}_p|\mathbb{Q}_p).$$

We define $\kappa \in \bar{\pi}_2 T(\mathbb{Z}_p|\mathbb{Q}_p)$ to be the class which corresponds to the generator $d\log p$ on the right. We now prove theorem B of the introduction:

THEOREM 2.2.13. *There is a canonical isomorphism of log differential graded k-algebras*

$$\omega^*_{(A,M)} \otimes S_{\mathbb{F}_p}\{\kappa\} \xrightarrow{\sim} \bar{\pi}_* T(A|K),$$

where $d\kappa = (d\log p)\kappa$.

PROOF. Suppose first that K/K_0 is wildly ramified and consider the diagram

$$\begin{tikzcd} \bar{\pi}_2 T(A) \arrow[r, "\beta"] \arrow[d, "i_*"] & {}_p\pi_1 T(A) \arrow[d, "i_*"] \\ \bar{\pi}_2 T(A|K) \arrow[r, "\beta"] & {}_p\pi_1 T(A|K). \end{tikzcd}$$

It is proved in [**22**] that $\pi_2(T(A), \mathbb{Z}_p)$ vanishes, and hence the sequence

$$\pi_2(T(A), \mathbb{Z}_p) \to \pi_2(T(A|K), \mathbb{Z}_p) \to \pi_1 T(k)$$

shows that so does $\pi_2(T(A|K), \mathbb{Z}_p)$. Hence the horizontal maps in the diagram above are both isomorphisms. The right hand vertical map may be identified with the left hand map in the sequence

$$0 \to {}_p\Omega_{A/\mathbb{Z}} \to {}_p\omega_{(A,M)} \xrightarrow{\text{res}} k.$$

The generator $d\log p$ of the middle term has

$$\operatorname{res}(d\log p) = v_K(p) = e_K,$$

which vanishes since K/K_0 is wildly ramified. The vertical maps in the diagram therefore are isomorphism. Moreover, since $\pi^{e_K} + p\theta_K(\pi) = 0$,

$$\beta(\kappa) = d\log p = (e_K - \theta_K(\pi)^{-1}\theta'_K(\pi)\pi)d\log\pi = i_*(-\beta(\theta_K(\pi)^{-1}a_K)).$$

We have here used that $d\log(-x) = d\log x$ which follows from corollary 2.2.11 since $d\log(-1) = -d(-1) = 0$. The calculation gives that

$$\kappa = i_*(-\theta_K(\pi)^{-1}a_K),$$

and since $da_K = 0$, we find

$$\begin{aligned} d\kappa &= d(-i_*(\theta_K(\pi)^{-1}a_K)) = i_*(d(-\theta_K(\pi)^{-1}a_K)) \\ &= i_*(\theta_K(\pi)^{-2}\theta'_K(\pi)d\pi \cdot a_K) = d\log p \cdot \kappa \end{aligned}$$

as stated. Since κ and $d\log p$ are in $\bar{\pi}_*T(K_0)$, this formula is valid for any finite extension K/K_0. This proves the existence of the stated map of differential graded k-algebras

$$\omega^*_{(A,M)} \otimes S_{\mathbb{F}_p}\{\kappa\} \to \bar{\pi}_*T(A|K).$$

Assume again that K/K_0 is wildly ramified. We claim that the transfer

$$j^!\colon \bar{\pi}_{2n}T(k) \to \bar{\pi}_{2n}T(A)$$

is trivial in even degrees. This is true for $n = 0$, so in particular $j^!(1) = 0$. Now by Frobenius reciprocity, the composite

$$\bar{\pi}_*T(A) \xrightarrow{j_*} \bar{\pi}_*T(k) \xrightarrow{j^!} \bar{\pi}_*T(A)$$

is given by multiplication by $j^!(j_*(1)) = j^!(1)$ and hence is zero. Since the left hand map is surjective in even dimensions, the claim follows. It follows that we have isomorphisms

$$i_*\colon \bar{\pi}_{2n}T(A) \xrightarrow{\sim} \bar{\pi}_{2*}T(A|K)$$

in even dimensions, and four term exact sequences

$$\bar{\pi}_{2n+1}T(k) \rightarrowtail^{j^!} \bar{\pi}_{2n+1}T(A) \xrightarrow{i_*} \bar{\pi}_{2n+1}T(A|K) \xrightarrow{\partial} \bar{\pi}_{2n}T(k)$$

in odd degrees. The diagrams

$$\begin{tikzcd} \bar{\pi}_0T(A) \arrow[r, "\sim"] \arrow[d, "\sim"', "(\theta(\pi)^{-1}a_K)^n"] & \bar{\pi}_0T(A|K) \arrow[d, "\kappa^r"] \\ \bar{\pi}_{2n}T(A) \arrow[r, "\sim"] & \bar{\pi}_{2n}T(A|K) \end{tikzcd}$$

and

$$\begin{tikzcd} \bar{\pi}_1T(k) \arrow[r, tail] \arrow[d, "\sim"', "(\theta(\pi)^{-1}a_K)^n"] & \bar{\pi}_1T(A) \arrow[r] \arrow[d, "\sim"', "(\theta(\pi)^{-1}a_K)^n"] & \bar{\pi}_1T(A|K) \arrow[r, two heads] \arrow[d, "\kappa^r"] & \bar{\pi}_0T(k) \arrow[d, "\sim"', "(\theta(\pi)^{-1}a_K)^n"] \\ \bar{\pi}_{2n+1}T(k) \arrow[r, tail] & \bar{\pi}_{2n+1}T(A) \arrow[r] & \bar{\pi}_{2n+1}T(A|K) \arrow[r, two heads] & \bar{\pi}_{2n}T(k) \end{tikzcd}$$

then proves the theorem in the wildly ramified case.

Finally, suppose that K/K_0 is tamely ramified. Let L/K be a totally wildly ramified extension and let B/A be the integral closure of A in L. We then have a commutative diagram

$$\begin{array}{ccc} \omega^*_{(A,M_A)} \otimes S\{\kappa\} & \longrightarrow & \bar{\pi}_* T(A|K) \\ \downarrow & & \downarrow \\ \omega^*_{(B,M_B)} \otimes S\{\kappa\} & \xrightarrow{\sim} & \bar{\pi}_* T(B|\,L), \end{array}$$

and the lower horizontal map is an isomorphism. It is easy to see that there exists L/K for which the left hand vertical map is a monomorphism. For example, one can take $L = K[\pi_B]/(\pi_B^{e_{B/A}} - \pi_A\pi_B - \pi_A)$. It follows that the upper horizontal map is a monomorphism. A dimension counting argument then shows that it is an isomorphism. □

LEMMA 2.2.14. *The canonical maps*

$$\Omega^*_K = \omega^*_{(K,K^\times)} \xleftarrow{\sim} \omega^*_{(A,M)} \otimes \mathbb{Q} \xrightarrow{\sim} \pi_* T(A|K) \otimes \mathbb{Q}$$

are isomorphisms.

PROOF. We first treat the left hand map. For a fraction $s^{-1}a$ with $a \in A$ and $s \in A - \mathfrak{p}$, so

$$d(s^{-1}a) = s^{-1}da - s^{-2}ads$$

which shows that the canonical map

$$A \otimes_A \Omega_{A/\mathbb{Z}} \xrightarrow{\sim} \Omega_{K/\mathbb{Z}},$$

and hence also

$$K \otimes_A (\Omega_{A\mathbb{Z}} \oplus (A \otimes_{\mathbb{Z}} K^\times)) \xrightarrow{\sim} \Omega_{K/\mathbb{Z}} \oplus (K \otimes_{\mathbb{Z}} K^\times),$$

is an isomorphism. Similarly, the formula

$$d(s^{-1}a) - s^{-1}a \otimes s^{-1}a = s^{-1}(da - a \otimes a) - s^{-2}a(ds - s \otimes s)$$

shows that as submodules of the K-vector space $\Omega_{K/\mathbb{Z}} \oplus (K \otimes_{\mathbb{Z}} K^\times)$,

$$K\langle da - a \otimes a \mid a \in A \cap K^\times\rangle = K\langle dq - q \otimes q \mid q \in K^\times\rangle.$$

Since $\pi_* T(k)$ is torsion, the map $T(A) \to T(A|K)$ is a rational equivalence. Also, the linearization map $T(A) \to \mathrm{HH}(A)$ is a rational equivalence. It thus remains to prove that the canonical map

$$\Omega^*_{K/\mathbb{Z}} \to \mathrm{HH}_*(K)$$

is an isomorphism. This in turn follows from the Hochschild-Kostant-Rosenberg theorem and from the fact that every field can be written as a separable algebraic extension of the fraction field of a filtered colimit of smooth algebras over the prime field. □

2.3. In the remainder of this paragraph, we will examine the descent properties of the functor $T(A|K)$. We first show the following positive result:

THEOREM 2.3.1. *If L/K is a tamely ramified Galois extension, then the canonical map*

$$T(A|K) \to \mathbb{H}^{\cdot}(G_{L/K}, T(B|\,L))$$

becomes a weak equivalence upon p-completion.

PROOF. We first show that for all $t \geq 0$, the $G_{L/K}$-module $\bar{\pi}_t T(B|\,L)$ is isomorphic to B/pB. If $t = 2i$ is even this follows from the natural isomorphism

$$\kappa^i \colon B/pB \xrightarrow{\sim} \bar{\pi}_t T(B|\,L)$$

and does not use that L/K is tamely ramified. For $t = 2i+1$ odd, we have the natural isomorphism

$$\kappa^i \colon \omega_{(B,M_B,W,M_0)}/p \xrightarrow{\sim} \bar{\pi}_{2*+1} T(B|\,L),$$

so it is enough to consider the module $\omega_{(B,M_B/W,M_0)}/p$. As an A-module $\omega_{(A,M_A)/(W,M_0)}$ is generated by $d\log \pi_A$ with annihilator $(\phi'_A(\pi_A)\pi_A)$, and since p divides the annihilator ideal,

$$\omega_{(A,M_A)/(W,M_0)}/p = A/pA\langle d\log \pi_A\rangle.$$

Since B/A is flat, lemma 2.1.7 shows that

$$\omega_{(B,M_B)/(W,M_0)}/p = B/pB\langle d\log \pi_A\rangle,$$

and as a $G_{L/K}$-module this is B/pB.

A classical theorem of Noether states that as a $G_{L/K}$-module B is isomorphic to $A[G_{L/K}]$ if and only if L/K is tamely ramified. Hence, the spectral sequence

$$E^2_{s,t} = H^{-s}(G_{L/K}, \bar{\pi}_t T(B|\,L)) \Rightarrow \bar{\pi}_{s+t} \mathbb{H}^{\cdot}(G_{L/K}, T(B|\,L))$$

collapses to an isomorphism

$$\bar{\pi}_* T(A|K) \xrightarrow{\sim} \bar{\pi}_* \mathbb{H}^{\cdot}(G_{L/K}, T(B|\,L)).$$

Finally, a map of spectra becomes an equivalence after p-completion if and only if it induces isomorphism on homotopy groups with $\mathbb{Z}/p\mathbb{Z}$-coefficients. $\square$

It is in order to examine the canonical map

$$\gamma_{L/K} \colon T(A|K) \to \mathbb{H}^{\cdot}(G_{L/K}, T(B|\,L))$$

for Galois extensions L/K in general, and hence the canonical map

$$\gamma_K \colon T(A|K) \to \underset{\overrightarrow{L}}{\operatorname{holim}}\, \mathbb{H}^{\cdot}(G_{L/K}, T(B|\,L)),$$

where the homotopy limit runs over finite extensions L/K contained in an algebraic closure K_s.

There are spectral sequences

$$E^2_{s,t} = H^{-s}(G_{L/K}, \pi_t T(B|\,L)) \Rightarrow \pi_{s+t} \mathbb{H}^{\cdot}(G_{L/K}, T(B|\,L)),$$
$$E^2_{s,t} = H^{-s}(G_K, \pi_t T(A_s|K_s)) \Rightarrow \pi_{s+t}\, \underset{\overrightarrow{L}}{\operatorname{holim}}\, \mathbb{H}^{\cdot}(G_{L/K}, T(B|\,L)),$$

where in the latter the E^2-term is given by the continuous cohomology of the profinite group G_K with coefficients in the discrete module $\pi_t T(A_s|K_s)$ defined by

$$T(A_s|K_s) = \operatorname*{holim}_{\overrightarrow{L}} T(B|\,L).$$

There are similar spectral sequences for the homotopy groups with $\mathbb{Z}/p\mathbb{Z}$-coefficients. Let $A \subset K$ be the valuation ring, and let $B \subset L$ and $A_s \subset K_s$ be the integral closure of A in L and K_s, respectively. Then B is a $G_{L/K}$-module and A_s is a discrete G_K-module. There is a natural isomorphism

$$A_s/pA_s \otimes S\{\kappa\} \xrightarrow{\sim} \bar{\pi}_* T(A_s|K_s),$$

where $\kappa \in \bar{\pi}_2 T(\mathbb{Z}_p|\mathbb{Q}_p)$ and hence is G_K-fixed. Since the group G_K has p-cohomological dimension 2, the spectral sequence above degenerates to a natural exact sequence

$$0 \to H^2(G_K, A_s/pA_s) \to \bar{\pi}_{2i} \operatorname*{holim}_{\overrightarrow{L}} \mathbb{H}^{\cdot}(G_{L/K}, T(B|\,L)) \to H^0(G_K, A_s/pA_s) \to 0$$

and a natural isomorphism

$$H^1(G_K, A_s/pA_s) \xrightarrow{\sim} \bar{\pi}_{2i+1} \operatorname*{holim}_{\overrightarrow{L}} \mathbb{H}^{\cdot}(G_{L/K}, T(B|\,L)),$$

for all $i \geq 0$. We note that the canonical map

$$A/pA \to H^0(G_{L/K}, B/pB)$$

is injective. For $A \to B$ is injective and the cokernel is a free A-module. The same is true with A_s in place of B.

Now suppose that $\gamma_{L/K}$ is an equivalence for all K and all L/K. Then γ_K is also an equivalence. The commutative diagram

$$\begin{tikzcd}
0 \to H^2(G_K, A_s/pA_s) \arrow[r] & \bar{\pi}_{2i}\operatorname{holim}_{\rightarrow} \mathbb{H}^{\cdot}(G_{L/K}, T(B|\,L)) \arrow[r] & H^0(G_K, A_s/pA_s) \to 0 \\
 & \bar{\pi}_{2i} T(A|K) \arrow[u, "\gamma_K", "\sim"'] \arrow[r, "\sim"] & A/pA \arrow[u]
\end{tikzcd}$$

shows that the canonical map

$$A/pA \to H^0(G_K, A_s/pA_s)$$

is surjective and hence an isomorphism. Since this is true for any K, the horizontal maps in the diagram

$$\begin{tikzcd}
A/pA \arrow[r] \arrow[d] & H^0(G_K, A_s/pA_s) \arrow[d, "\sim"] \\
H^0(G_{L/K}, B/pB) \arrow[r] & H^0(G_{L/K}, H^0(G_L, A_s/pA_s))
\end{tikzcd}$$

are both isomorphism, and hence

$$A/pA \to H^0(G_{L/K}, B/pB)$$

is an isomorphism. We will show in the example below that this is not the case.

Suppose that L/K is wildly ramified, and consider the natural filtration of the $G_{L/K}$-module B/pB by the powers of the maximal ideal $\mathfrak{m}_B \subset B$. The filtration has length $e_B = e_{B/W(k)}$ and the product defines a natural isomorphism

$$(\mathfrak{m}_B/\mathfrak{m}_B^2)^{\otimes i} \xrightarrow{\sim} \mathfrak{m}_B^i/\mathfrak{m}_B^{i+1},$$

for $i \geq 1$. When $i = 0$ the right hand side is the residue field $k_B = B/\mathfrak{m}_B$, and we then take this as our definition of the left hand side. The filtration gives rise to a cohomology type spectral sequence with

$$E_1^{s,t} = H^{s+t}(G_{L/K}, (\mathfrak{m}_B/\mathfrak{m}_B^2)^{\otimes s}) \Rightarrow H^{s+t}(G_{L/K}, B/pB)$$

and concentrated on the lines $0 \leq s < e$ in the right half plane.

EXAMPLE 2.3.2. Let $K = \mathbb{Q}_p$ and $L = \mathbb{Q}_p(\mu_{p^n})$. The extension is totally ramified, so $k_B = k_A$ is a trivial module, and we have a canonical isomorphism

$$\mu_p \xrightarrow{\sim} \mathfrak{m}_B/\mathfrak{m}_B^2, \qquad z \mapsto z^{p^{-(n-1)}} - 1.$$

The canonical map

$$G_{L/K} \xrightarrow{\sim} \mathrm{Aut}(\mu_{p^n})$$

is an isomorphism and the action on μ_p is induced from the natural inclusion $\mu_p \subset \mu_{p^n}$. Hence the cohomology of $\mu_p^{\otimes s}$ is trivial unless $s \equiv 0 \pmod{p-1}$ in which case we have isomorphisms

$$H^n(G_{L/K}, \mu_p^{\otimes s}) \cong \mathbb{Z}/p\mathbb{Z}, \qquad n \geq 0,$$

which depend on a choice of a primitive pth root of one. Since $A/pA = k_A = k_B$, the composition of the map in question

$$A/pA \to H^0(G_{L/K}, B/pB)$$

with the edge homomorphism

$$H^0(G_{L/K}, B/pB) \to H^0(G_{L/K}, k_B) = E_1^{0,0}$$

is an isomorphism. So it suffices to show that for some $1 \leq s < e$, $E_\infty^{s,-s}$ is non-trivial. But if s is the largest integer which is divisible by $p-1$ and less than or equal to $e-1$, then

$$E_\infty^{s,-s} = E_1^{s,-s} \cong \mathbb{Z}/p\mathbb{Z}$$

for degree reasons.

3. The de Rham-Witt pro-complex and $\mathrm{TR}_*^{\cdot}(A|K;p)$

3.1. In this paragraph, we evaluate the integral homotopy groups $\mathrm{TR}_i^{\cdot}(A|K;p)$, for $i \leq 2$. We first consider Witt vectors.

Let p be a prime, let R be a ring, and let $W(R)$ be the ring of p-typical Witt vectors. The ghost map

$$w\colon W(R) \to R^{\mathbb{N}_0}$$

which maps a vector $(a_0, a_1, \dots)$ to the sequence $(w_0, w_1, \dots)$, where

$$w_s = a_0^{p^s} + pa_1^{p^{s-1}} + \dots + p^s a_s,$$

is a ring homomorphism. It is injective if R has no p-torsion. Moreover, if R possesses a ring endomorphism ϕ with the property that for all $a \in R$, $a^p \equiv \phi(a)$

(mod pR), then the image of the ghost map may be characterized as the set of sequences $(w_0, w_1, \dots)$ for which

$$w_s \equiv \phi(w_{s-1}) \qquad (\text{modulo } p^s R),$$

for all $s \geq 1$. If $R = \mathbb{Z}[X_\alpha]$, the ring homomorphism which maps X_α to X_α^p is such an endomorphism. Let $\underline{\ }\colon R \to W(R)$ be the multiplicative section $\underline{a} = (a, 0, 0, \dots)$.

LEMMA 3.1.1. *If $p > 2$ then $\underline{p} + V(1) \equiv 0$ and $\underline{-1} \equiv -1$ modulo $pW(R)$.*

PROOF. By naturality, we may assume that $R = \mathbb{Z}$. We have

$$w(\underline{p} + V(\underline{1})) = p(1, 1 + p^{p-1}, 1 + p^{p^2-1}, 1 + p^{p^3-1}, \dots),$$

and therefore it is enough to show that the sequence

$$(1, 1 + p^{p-1}, 1 + p^{p^2-1}, 1 + p^{p^3-1}, \dots)$$

is in the image of the ghost map. This in turn follows, by what was said earlier, from the congruences

$$1 + p^{p^s-1} \equiv 1 + p^{p^{s-1}-1} \qquad (\text{mod } p^s),$$

valid, when $p > 2$, for all $s \geq 2$ as required, but fail for $p = 2$ and $s = 2$. The second congruence is proved in a similar manner. $\square$

In general, $\underline{x + y}$ and $\underline{x} + \underline{y}$ are not equivalent modulo $pW(A)$. However, we have the following

LEMMA 3.1.2. *For all $x, y \in R$,*

$$(\underline{x + y})^p \equiv (\underline{x} + \underline{y})^p \equiv \underline{x}^p + \underline{y}^p$$

modulo $pW(R)$.

PROOF. The right hand congruence is valid in any ring. To prove the left hand congruence, we place ourselves in the universal case $R = \mathbb{Z}[x, y]$. The ghost map

$$w\colon W(R) \to R^{\mathbb{N}_0}$$

is an injection and maps the Witt vector $\underline{x}^p + \underline{y}^p - (\underline{x + y})^p$ to the tuple

$$(x^p + y^p - (x + y)^p, \dots, x^{p^{n+1}} + y^{p^{n+1}} - (x + y)^{p^{n+1}}, \dots).$$

As an element of $R^{\mathbb{N}_0}$ this is divisible by p. We must show that the quotient is in the image of the ghost map. By the criterion recalled above, we must show that

$$(x^{p^{n+1}} + y^{p^{n+1}} - (x + y)^{p^{n+1}})/p \equiv (x^{p^{n+1}} + y^{p^{n+1}} - (x^p + y^p)^{p^n})/p \quad (\text{mod } p^n),$$

or equivalently, that

$$(x + y)^{p^{n+1}} \equiv (x^p + y^p)^{p^n} \quad (\text{mod } p^{n+1}).$$

But this follows from

$$(x + y)^p \equiv x^p + y^p \quad (\text{mod } p)$$

and from the fact, valid in any commutative ring, that $a \equiv b$ (mod p) implies $a^{p^n} \equiv b^{p^n}$ (mod p^{n+1}). Indeed, one easily sees that $a \equiv b$ (mod p^k) implies that $a^p \equiv b^p$ (mod p^{k+1}), and the desired formula then follows by simple induction. $\square$

When A is a complete discrete valuation ring, the ring $W_n(A)/p$ is a k-algebra via the ring homomorphism

$$k = W(k)/p \xrightarrow{\Delta} W(W(k))/p \xrightarrow{W(\eta)} W(A)/p \xrightarrow{R} W_n(A)/p.$$

Here $\Delta\colon W(k) \to W(W(k))$ is the universal p-typical λ-operation, [**13**]. Here the identification on the left holds for k perfect since then $VW(k) = FVW(k)$, and since one always has $FV = p$. Let π be a uniformizer with minimal polynomial

$$\pi^{e_K} + p\theta_K(\pi).$$

We introduce the modified Verschiebung

$$V_\pi\colon W_{n-1}(A) \to W_n(A), \quad V_\pi(x) = \theta_K(\underline{\pi}_n)V(x),$$

which satisfies

$$FV_\pi(x) = p\theta_K(\underline{\pi})^p x.$$

PROPOSITION 3.1.3. *The k-algebra $W_n(A)/p$ is generated by the elements $V_\pi^s(\underline{\pi}^i)$ with $0 \le s < n$ and $i \ge 0$ subject to the relations*

$$\begin{aligned} V_\pi^s(\underline{\pi}^i) \cdot V_\pi^t(\underline{\pi}^j) &= p^s V_\pi^t(\theta(\underline{\pi})^{p^{t-s}(\frac{p^{s+1}-1}{p-1}-1)}\underline{\pi}^{p^{t-s}i+j}), \\ V_\pi^s(\underline{\pi}^{e_K+i}) &= V_\pi^{s+1}(\underline{\pi}^{pi}). \end{aligned}$$

PROOF. As a k-vector space, $W_n(A)/p$ is generated by the monomials in the variables $V^s(\underline{\pi}^i)$ with $0 \le s < n$ and $i \ge 0$. Indeed, if $a \in k$ then

$$V^s(\underline{a\pi}^i) = \underline{\varphi^{-s}(a)}V^s(\underline{\pi}^i).$$

Since $\theta_K(\underline{\pi})$ is a unit, we may instead use the elements $V_\pi^s(\underline{\pi}^i)$ as our generators. If $s \le t$,

$$\begin{aligned} V_\pi^s(\underline{\pi}^i)V_\pi^t(\underline{\pi}^j) &= V_\pi^s(\underline{\pi}^i)V_\pi^s(V_\pi^{t-s}(\underline{\pi}^j)) = V_\pi^s(F^sV_\pi^s(\underline{\pi}^i)V_\pi^{t-s}(\underline{\pi}^j)) \\ &= p^sV_\pi^s(\theta_K(\underline{\pi})^{\frac{p^{s+1}-1}{p-1}-1}\underline{\pi}^iV_\pi^{t-s}(\underline{\pi}^j)) \\ &= p^sV_\pi^t(\theta_K(\underline{\pi})^{p^{t-s}(\frac{p^{s+1}-1}{p-1}-1)}\underline{\pi}^{p^{t-s}i+j}), \end{aligned}$$

which proves the first relation. Next, lemmas 3.1.1 and 3.1.2 shows that

$$\begin{aligned} \underline{\pi}^{e_K} &= \underline{-p}\cdot\underline{\theta_K(\pi)} \equiv -\underline{p}\cdot\underline{\theta_K(\pi)} \equiv V(1)\underline{\theta_K(\pi)} \\ &= V((\underline{\theta_K(\pi)})^p) \equiv V(\theta_K^{(1)}(\underline{\pi}^p)) = V(1)\theta_K(\underline{\pi}) = V_\pi(1). \end{aligned}$$

The second relation is an immediate consequence of the relation. It remains to prove that there are no further relations. Since $W_n(A)$ is torsion free, the sequences

$$0 \to A/p \xrightarrow{V^{n-1}} W_n(A)/p \xrightarrow{R} W_{n-1}(A)/p \to 0$$

are exact and show that $W_n(A)/p$ is an ne_K-dimensional k-vector space. The relations of the statement implies that

$$\mathrm{gr}_V^s\, W_n(A)/p = k\langle V_\pi^s(\underline{\pi}^i) \bigm| 0 \le i < e_K\rangle,$$

which is an e_K-dimensional k-vector space. Thus there can be no further relations among the $V_\pi^s(\underline{\pi}^i)$. □

3.2. A pre-log structure $\alpha\colon M \to R$ on a ring R induces one on $W_n(R)$ upon composition with the multiplicative section $\underline{}_n\colon R \to W_n(R)$. We write $(W_n(R), M)$ for this log ring.

DEFINITION 3.2.1. A log Witt functor consists of the following data:

(i) a log ring (R, M);

(ii) a pro-log differential graded ring $(E^*_{\cdot}, M)$ whose underlying pro-log ring is $(W_{\cdot}(R), M)$;

(iii) a map of pro-log graded rings

$$F\colon E^*_n \to E^*_{n-1},$$

which extends the Frobenius $F\colon W_n(R) \to W_{n-1}(R)$, and such that

$$\begin{aligned} F d\log_n a &= d\log_{n-1} a, && \text{for all } a \in M, \\ F d\underline{a}_n &= \underline{a}_{n-1}^{p-1} d\underline{a}_{n-1}, && \text{for all } a \in R; \end{aligned}$$

(iv) a map of pro-graded modules over the pro-graded ring $E^*_{\cdot}$,

$$V\colon F^* E^*_n \to E^*_{n+1},$$

such that $FV = p$ and $FdV = d$.

A map of log Witt functors is a map of pro-log differential graded rings which commutes with the maps F and V.

The following relations are valid in any log Witt functor

$$dF = pFd, \quad Vd = pdV, \quad V(xdy) = V(x)dV(y). \tag{3.2.2}$$

Indeed, $V(xdy) = V(xFdV(y)) = V(x)dV(y)$, and

$$\begin{aligned} dF(x) &= FdVF(x) = Fd(V(1)x) = FdV(1)F(x) + FV(1)F(dx) \\ &= d(1)F(x) + pFd(x) = pFd(x), \\ Vd(x) &= V(1)dV(x) = d(V(1)V(x)) - dV(1)V(x) \\ &= dV(xFV(1) - V(xd(1))) = pdV(x). \end{aligned}$$

PROPOSITION 3.2.3. *The forgetful functor from the category of log Witt functors to the category of log rings has a left adjoint,*

$$(R, M) \mapsto W.\,\omega^*_{(R,M)}$$

Moreover, the canonical map $\lambda\colon \omega^*_{(W_{\cdot}(R),M)} \to W.\omega^*_{(R,M)}$ *is surjective.*

PROOF. We use the Freyd adjoint functor theorem to prove the existence of the left adjoint. Let $(E^*_{\cdot}, M)$ be a log Witt functor whose underlying log ring is (R, M). Then, in particular, there is a canonical map

$$\omega^*_{(W_{\cdot}(R),M)} \to E^*_{\cdot}$$

of pro-log differential graded rings. The image of this map, $\mathrm{im}^*_{E,\cdot}$, is a pro-log differential graded ring whose underlying pro-log ring is $(W_{\cdot}(R), M)$. We claim that, in fact, $\mathrm{im}^*_{E,\cdot}$ has a natural structure of a sub-log Witt functor of $(E^*_{\cdot}, M)$. Granting this for the moment, we pick a representative $(E^*_{\cdot}, M)$ for each isomorphism class of log Witt functors of the form $\mathrm{im}^*_{E,\cdot}$. Each $\mathrm{im}^*_{E,\cdot}$ is a quotient of $\omega^*_{(W\Delta R),M)}$, so these representatives form a set. Hence, we may form the product

$$(D^*_\cdot, M) = \prod_{(E\Delta M)} \mathrm{im}^*_{E,\cdot},$$

and $W.\omega^*_{(R,M)}$ is then denined as the equalizer of all endomorphisms of $(D^*_\cdot, M)$. Indeed, it is easy to see that this equalizer is universal among log Witt functor with underlying log ring (R, M), [**23**, p. 116]. We also note that the canonical map

$$\omega^*_{(W\Delta R),M)} \to W\,.\omega^*_{(R,M)}$$

is surjective. Indeed, its image is a log Witt functor $\mathrm{im}^*_{W,\cdot}$, so there is a canonical map $W.\,\omega^*_{(R,M)} \to \mathrm{im}^*_{W,\cdot}$. The composite

$$W.\,\omega^*_{(R,M)} \to \mathrm{im}^*_{W,\cdot} \to W.\,\omega^*_{(R,M)}$$

is a map of log Witt functors, and since $W.\,\omega^*_{(R,M)}$ is universal, this can only be the identity map.

It remains to prove that

$$F(\mathrm{im}^*_{E,n}) \subset \mathrm{im}^*_{E,n-1},$$
$$V(\mathrm{im}^*_{E,n}) \subset \mathrm{im}^*_{E,n+1}\,.$$

As a graded ring, $\mathrm{im}^*_{E,n}$ is generated by $a \in E^0_n$ and $da \in E^1_n$ with $a \in W_n(R)$, and by $d\log_n x \in E^1_n$ with $x \in M$. Since F is multiplicative, it suffices to check that the image of these generators under F are in $\mathrm{im}^*_{E,n-1}$. This is clear for the elements a and $d\log_n x$, and since every element $a \in W_n(R)$ can be written uniquely as

$$a = \sum_{i=0}^{n-1} V^i \underline{a_i}_{n-i},$$

we see that

$$Fda = \underline{a_0}^{p-1}_{n-1} d\underline{a_0}_{n-1} + \sum_{j=1}^{n-2} dV^j \underline{a_j}_{n-1-j},$$

which is in $\mathrm{im}^*_{E,n-1}$. Finally, the formulas

$$V(xdy) = V(xFdV(y)) = V(x)dV(y),$$
$$V(xd\log \underline{y}_n) = V(xFd\log \underline{y}_{n+1}) = V(x)d\log \underline{y}_{n+1},$$

shows that $V(\mathrm{im}^*_{E,n}) \subset \mathrm{im}^*_{E,n+1}$. □

The filtration of a log Witt functor by the differential graded ideals

$$\mathrm{Fil}^s E^i_n = V^s E^i_{n-s} + dV^s E^{i-1}_{n-s} \subset E^i_n$$

is called the standard filtration. It satisfies

$$F(\mathrm{Fil}^s E^i_n) \subset \mathrm{Fil}^{s-1} E^i_{n-1},$$
$$V(\mathrm{Fil}^s E^i_n) \subset \mathrm{Fil}^{s+1} E^i_{n+1},$$

but in general is not multiplicative.

Lemma 3.2.4. *For any log ring (R, M), the map induced from the restriction maps,*

$$W_n\,\omega^i_{(R,M)}/\operatorname{Fil}^s W_n\,\omega^i_{(R,M)} \xrightarrow{\sim} W_s\,\omega^i_{(R,M)},$$

is an isomorphism.

Proof. For a fixed value of $n - s$, the filtration quotients

$$'W_s\,\omega^i_{(R,M)} = W_n\,\omega^i_{(R,M)}/\operatorname{Fil}^s W_n\,\omega^i_{(R,M)}$$

form a log Witt functor whose underlying log ring is (R, M). We show that it has the universal property. Let $(E^*_{\cdot}, M)$ be a log Witt functor whose underlying log ring is (R, M). Then there exists a map of log Witt functors

$$'W_{\cdot}\,\omega^*_{(R,M)} \to E^*_{\cdot}.$$

Indeed, the standard filtration is natural, so we have maps

$$W_n\,\omega^i_{(R,M)}/\operatorname{Fil}^s W_n\,\omega^i_{(R,M)} \to E^i_n/\operatorname{Fil}^s E^i_n \to E^i_s,$$

where the right hand map is induced from the restriction maps in $E^*_{\cdot}$. We must show that this map of log Witt functors is unique. To prove this, it will suffice to show that the canonical map

$$\omega^i_{(W_s(R),M)} \to {'W_n}\,\omega^i_{(R,M)}$$

is surjective. But this follows from the commutativity of the diagram

$$\begin{tikzcd} \omega^i_{(W_n(R),M)} \arrow[r, two heads] \arrow[d, two heads] & W_n\,\omega^i_{(R,M)} \arrow[d, two heads] \\ \omega^i_{(W_s(R),M)} \arrow[r] & 'W_s\,\omega^i_{(R,M)} \end{tikzcd}$$

since the top horizontal and right hand vertical maps are surjective. $\square$

We define a map $F^{n-1}d\colon W_n(R) \to \omega^1_{(R,M)}$ by the formula

$$F^{n-1}d(a) = a_0^{p^{n-1}-1}da_0 + a_1^{p^{n-2}-1}da_1 + \cdots + da_{n-1},$$

where $a = (a_0, \ldots, a_{n-1})$. One easily verifies that $F^{n-1}d$ is a derivation of $W_n(R)$ into the $W_n(R)$-module $(F^{n-1})^*\omega^1_{(R,M)}$ and that the following relation holds:

$$dF^{n-1} = p^{n-1}F^{n-1}d.$$

It follows immediately from the derivation property that the formula

$$a \cdot (\omega_1 \oplus \omega_2) = F^{n-1}(a)\omega_1 \oplus (F^{n-1}(a)\omega_2 - F^{n-1}da \cdot \omega_1)$$

defines a $W_n(R)$-module structure on $\omega^{i-1}_{(R,M)} \oplus \omega^i_{(R,M)}$. And the relation shows that the image of the map

$$\omega^{i-1}_{(R,M)} \to \omega^{i-1}_{(R,M)} \oplus \omega^i_{(R,M)}, \quad \omega \mapsto p^{n-1}\omega \oplus -d\omega,$$

is a sub-$W_n(R)$-module. We denote the quotient $W_n(R)$-module by

$${}_hW_n\,\omega^i_{(R,M)}.$$

Lemma 3.2.5. *There is a natural exact sequence of $W_n(R)$-modules*

$$\begin{aligned}&(F^{n-1})^*{}_{p^{n-1}}\omega^{i-1}_{(R,M)} \xrightarrow{d} (F^{n-1})^*\omega^i_{(R,M)}\\ &\to {}_hW_n\,\omega^i_{(R,M)} \to (F^{n-1})^*(\omega^{i-1}_{(R,M)}/p^{n-1}\omega^{i-1}_{(R,M)}) \to 0.\end{aligned}$$

Proof. Indeed, as an abelian group, ${}_hW_n\,\omega^i_{(R,M)}$ is equal to the push out

$$\begin{tikzcd} \omega^{i-1}_{(R,M)} \arrow[r, "d"] \arrow[d, "p^{n-1}"] & \omega^i_{(R,M)} \arrow[d] \\ \omega^{i-1}_{(R,M)} \arrow[r] & {}_hW_n\,\omega^i_{(R,M)} \end{tikzcd}$$

so the underlying sequence of abelian groups is exact. One readily verifies that the various maps are $W_n(R)$-linear. $\square$

Remark 3.2.6. It is easy to see that the canonical map

$$\lambda : \omega^i_{(R,M)} \to W_1\,\omega^i_{(R,M)}$$

is an isomorphism for $i \le 1$. Indeed, one can construct a log Witt functor $(E^*_\cdot, M)$ such that $E^i_1 = \omega^i_{(R,M)}$, $i \le 1$, as follows: In degree zero, $E^0_\cdot = W_\cdot(R)$, in degree one, $E^1_\cdot = \omega^1_{(R,M)}$ with the identity map as structure map, and $E^i_\cdot = 0$ for $i > 1$. The differential $E^0_n \to E^1_n$ is given by the map $F^{n-1}d\colon W_n(R) \to \omega^1_{(R,M)}$, $F\colon E^1_n \to E^1_{n-1}$ is the identity map and $V\colon E^1_{n-1} \to E^1_n$ is multiplication by p. We expect that the map is an isomorphism for all i.

Proposition 3.2.7. *For any log ring (R, M), there is a natural exact sequence of $W_n(R)$-modules,*

$${}_hW_n\,\omega^i_{(R,M)} \xrightarrow{N} W_n\,\omega^i_{(R,M)} \xrightarrow{R} W_{n-1}\,\omega^i_{(R,M)} \to 0,$$

where $N(\omega_1 \oplus \omega_2) = dV^{n-1}\lambda(\omega_1) + V^{n-1}\lambda(\omega_2)$.

Proof. The defining properties of a log Witt functor shows that for all $a \in W_n(R)$,

$$\lambda(F^{n-1}da) = F^{n-1}d\lambda(a).$$

Hence N is $W_n(R)$-linear. Since the image of N is equal to $\mathrm{Fil}^{n-1} W_n\,\omega^i_{(R,M)}$, the statement follows from lemma 3.2.4. $\square$

Corollary 3.2.8. *When (A, M) is a complete discrete valuation ring of mixed characteristic with the canonical log structure then for all $n \ge 1$ and $i \ge 2$, $W_n\,\omega^i_{(A,M)}$ is a uniquely divisible group.*

Proof. Recall from lemma 2.1.4 that $\omega^i_{(A,M)}$ is a divisible group for $i \ge 2$. It follows that ${}_hW_n\,\omega^i_{(A,M)}$ is divisible for $i \ge 3$, and an induction argument based on proposition 3.2.7 then shows that so is $W_n\,\omega^i_{(A,M)}$. The group ${}_hW_n\,\omega^2_{(A,M)}$ is a direct sum of a uniquely divisible group and the group $\omega_{(A,M)}/p^{n-1}\omega_{(A,M)}$. Hence $W_n\,\omega^2_{(A,M)}$ is a direct sum of a uniquely divisible group and a finitely generated torsion $W(k)$-module. It is therefore enough to show that the modulo p reduction $\bar{W}_n\,\omega^2_{(A,M)}$ is trivial. Inductively, it suffices to show that the map

$$dV_{n-1}\colon \bar{\omega}^1_{(A,M)} \to \bar{W}_n\omega^2_{(A,M)}$$

is trivial. The map is k-linear, and the domain is generated as a k-vector space by the elements $\pi^i d\log\pi$, $0 \le i < e$. Now the relation

$$\underline{\pi}_n^e + \theta(\underline{\pi}_n)V(1),$$

valid in $\bar{W}_n(A)$, shows that $V^{n-1}(\pi^i d\log\pi) = V^{n-1}(\pi^i)d\log\underline{\pi}_n$ is either trivial or in the span of elements of the form $\underline{\pi}_n^j d\log\underline{\pi}_n$. But these elements have vanishing differential. □

3.3. It follows from the results proved in [**14**, §1] and from proposition 2.2.4 above that $\mathrm{TR}^{\cdot}_*(A|K;p)$ is a log Witt functor. We consider the canonical map

$$W.\,\omega^*_{(A,M)} \to \mathrm{TR}^{\cdot}_*(A|K;p).$$

The homotopy groups of the homotopy orbit spectra,

$${}_h\mathrm{TR}^n_*(A|K;p) = \pi_*\mathbb{H}.(C_{p^{n-1}}, T(A|K)),$$

are differential graded modules over $\mathrm{TR}^n_*(A|K;p)$, and there are maps of $\mathrm{TR}^n_*(A|K;p)$-modules

$$F\colon {}_h\mathrm{TR}^n_*(A|K;p) \to F^*({}_h\mathrm{TR}^{n-1}_*(A|K;p)),$$
$$V\colon F^*({}_h\mathrm{TR}^{n-1}_*(A|K;p)) \to {}_h\mathrm{TR}^n_*(A|K;p),$$

which satisfy

$$FdV = d,$$
$$FV = p.$$

Moreover, there is a natural spectral sequence of $W_n(A)$-modules,

$$E^2_{s,t} = H_s(C_{p^{n-1}}, (F^{n-1})^*\,\pi_t T(A|K)) \Rightarrow {}_h\mathrm{TR}^n_{s+t}(A|K;p). \tag{3.3.1}$$

Lemma 3.3.2. *Let $\rho\colon \omega^i_{(A,M)} \to \pi_i T(A|K)$ be the canonical map. Then the map*

$$\begin{aligned} {}_hW_n\,\omega^i_{(A,M)} &\to {}_h\mathrm{TR}^n_i(A|K;p),\\ \omega_1 \oplus \omega_2 &\mapsto dV^{n-1}\rho(\omega_1) + V^{n-1}\rho(\omega_2), \end{aligned}$$

is a map of $W_n(A)$-modules. It is an isomorphism, for $i \le 1$, and for $i = 2$, there is an exact sequence

$$0 \to (F^{n-1})^*(A/p^{n-1}A) \to {}_hW_n\,\omega^2_{(A,M)} \to {}_h\mathrm{TR}^n_2(A|K;p) \to 0,$$

where the map on the left takes a to $dV^{n-1}(da)$.

Proof. If $a \in W_n(A)$, $\omega_1 \in \omega^{i-1}_{(A,M)}$ and $\omega_2 \in \omega^i_{(A,M)}$, then

$$\begin{aligned} a\cdot dV^{n-1}\rho(\omega_1) &= d(a\cdot V^{n-1}\rho(\omega_1)) - da\cdot V^{n-1}\rho(\omega_1)\\ &= dV^{n-1}(F^{n-1}a\cdot\rho(\omega_1)) - F^{n-1}da\cdot\rho(\omega_1))\\ &= dV^{n-1}\rho(F^{n-1}a\cdot\omega_1) - V^{n-1}\rho(F^{n-1}da\cdot\omega_1)),\\ a\cdot V^{n-1}\rho(\omega_2) &= V^{n-1}(F^{n-1}a\cdot\rho(\omega_2))\\ &= V^{n-1}\rho(F^{n-1}a\cdot\omega_2), \end{aligned}$$

which shows that the map of the statement is indeed a map of $W_n(A)$-modules.

The map ρ is an isomorphism for $i \leq 2$. So the spectral sequence gives an isomorphism of $W_n(A)$-modules

$$\iota_0 \colon (F^{n-1})^* A \xrightarrow{\sim} {}_h\mathrm{TR}_0^n(A|K;p)$$

and a natural exact sequence of $W_n(A)$-modules

$$0 \to (F^{n-1})^* \omega^1_{(A,M)} \xrightarrow{\iota_1} {}_h\mathrm{TR}_1^n(A|K;p) \to (F^{n-1})^*(A/p^{n-1}A) \to 0.$$

The sequence of lemma 3.2.5 maps to the sequence above, and the map of the left hand term is an isomorphism. It remains to show that the same holds for the map of the right hand terms. This map is induced from the composite

$$A \to {}_hW_n\,\omega^1_{(A,M)} \to {}_h\,\mathrm{TR}_1^n(A|K;p) \to A/p^{n-1}A$$

which in turn may be identified with the map

$$H_0(C_{p^{n-1}}, A) \to H_1(C_{p^{n-1}}, A)$$

given by multiplication by the fundamental class $[S^1/C_{p^{n-1}}]$. This map is an epimorphism with kernel $p^{n-1}A$, and the lemma follows for $i = 1$. The statement for $i = 2$ is proved in an entirely similar manner. □

REMARK 3.3.3. For $i \leq 1$, the proof above does not use that A is a Dedekind ring beyond the definition of $T(A|K)$. In effect, the same proof gives an isomorphism

$${}_hW_n\,\Omega^1_A \xrightarrow{\sim} \pi_1\mathbb{H}_{\cdot}(C_{p^{n-1}}, T(A))$$

for any ring A (with the trivial pre-log structure).

LEMMA 3.3.4. *For all $i \geq 0$, the Frobenius*

$$F\colon\ \mathrm{TR}^n_{2i+1}(A|K;p) \to \mathrm{TR}^{n-1}_{2i+1}(A|K;p)$$

is surjective.

PROOF. For $i > 0$, the group $\mathrm{TR}_i^n(A|K;p)$ is a sum of a uniquely divisible group and a p-torsion group of bounded height. Indeed, this is true when $n = 1$, and the general case then follows by an induction argument based on the cofibration sequence

$${}_h\mathrm{TR}^n(A|K;p) \xrightarrow{N} \mathrm{TR}^n(A|K;p) \xrightarrow{R} \mathrm{TR}^{n-1}(A|K;p)$$

and the spectral sequence (3.3.1). Since $FV = p$, the Frobenius induces a surjection of uniquely divisible summands. It is therefore enough to prove that the statement of the lemma holds after p-completion. To this end, we show that the canonical map

$$\pi_{2i+1}(\mathbb{H}^{\cdot}(\mathbb{T}, T(A|K)), \mathbb{Z}_p) \to \pi_{2i+1}(\mathbb{H}^{\cdot}(C_{p^n}, T(A|K)), \mathbb{Z}_p)$$

is surjective. Consider the spectral sequences

$$E^2_{s,t}(\mathbb{T}) = H^{-s}(BS^1, \pi_t(T(A|K), \mathbb{Z}_p)) \Rightarrow \pi_{s+t}(\mathbb{H}^{\cdot}(\mathbb{T}, T(A|K)), \mathbb{Z}_p),$$
$$E^2_{s,t}(C_{p^n}) = H^{-s}(BC_{p^n}, \pi_t(T(A|K), \mathbb{Z}_p)) \Rightarrow \pi_{s+t}(\mathbb{H}^{\cdot}(C_{p^n}, T(A|K)), \mathbb{Z}_p).$$

Both of these are strongy convergent second quadrant homology type spectral sequences. That is, the associated filtration $\mathrm{Fil}^s\,\pi_*(\mathbb{H}^{\cdot}(\mathbb{T}, T(A|K)), \mathbb{Z}_p)$ of the actual homotopy groups $\pi_*(\mathbb{H}^{\cdot}(\mathbb{T}, T(A|K)), \mathbb{Z}_p)$ is such that

$$\mathrm{gr}^s\,\pi_{s+t}(\mathbb{H}^{\cdot}(\mathbb{T}, T(A|K)), \mathbb{Z}_p) = E^\infty_{s,t}(\mathbb{T}),$$

and the canonical map

$$\pi_*(\mathbb{H}^\cdot(\mathbb{T}, T(A|K)), \mathbb{Z}_p) \xrightarrow{\sim} \varprojlim_s \pi_*(\mathbb{H}^\cdot(\mathbb{T}, T(A|K)), \mathbb{Z}_p) / \operatorname{Fil}^s \pi_*(\mathbb{H}^\cdot(\mathbb{T}, T(A|K)), \mathbb{Z}_p)$$

is an isomorphism. (The structure maps in this limit system are surjections, so the derived limit vanishes.) Similar remarks hold for the spectral sequence $E^r(C_{p^n})$. It will therefore suffice to show that

$$\operatorname{gr}^* \pi_{2i+1}(\mathbb{H}^\cdot(\mathbb{T}, T(A|K)), \mathbb{Z}_p) \to \operatorname{gr}^* \pi_{2i+1}(\mathbb{H}^\cdot(C_{p^n}, T(A|K)), \mathbb{Z}_p)$$

is a surjection for $i \geq 0$.

On the E^2-terms of the spectral sequences, the map in question corresponds to the map on cohomology induced from the inclusion $C_{p^n} \to S^1$. It is thus surjective for s even. Moreover, $\pi_*(T(A|K), \mathbb{Z}_p)$ is concentrated in odd degrees with the exception of $\pi_0(T(A|K), \mathbb{Z}_p)$, and hence, the non-zero differentials in the spectral sequence $E^r(\mathbb{T})$ must originate on the line $t = 0$. It follows that for s even and $t > 0$, the map

$$E^r_{s,t}(\mathbb{T}) \to E^r_{s,t}(C_{p^n})$$

is surjective for all $2 \leq r \leq \infty$. (Since these groups do not support non-zero differentials, they are stable for $r > s$.) But in the spectral sequence $E^r(C_{p^n})$, only the groups $E^r_{s,t}$ with s even and $t > 0$ can contribute to $\pi_{2i+1}(\mathbb{H}^\cdot(C_{p^n}, T(A|K)), \mathbb{Z}_p)$. This shows that the map

$$\operatorname{gr}^* \pi_{2i+1}(\mathbb{H}^\cdot(\mathbb{T}, T(A|K)), \mathbb{Z}_p) \to \operatorname{gr}^* \pi_{2i+1}(\mathbb{H}^\cdot(C_{p^n}, T(A|K)), \mathbb{Z}_p)$$

is indeed surjective, and hence the lemma follows. $\square$

Since $\omega^2_{(A,M)}$ is a uniquely divisible group, the spectral sequence (3.3.1) gives an exact sequence of $W_n(A)$-modules

$$(F^{n-1})^*(A/p^{n-1}) \xrightarrow{d} (F^{n-1})^*(\omega^1_{(A,M)}/p^{n-1}) \to {}_h\mathrm{TR}^n_2(A|K; p, \mathbb{Z}_p) \to 0,$$

and d is $W_n(A)$-linear since $dF^{n-1} = p^{n-1}F^{n-1}d$. If π is a uniformizer, then $d\log\pi$ represents a class in the cokernel. We denote this class by $[d\log\pi]_n$.

Lemma 3.3.5. *The map of $W_n(A)$-modules*

$$F\colon {}_h\mathrm{TR}^n_2(A|K; p, \mathbb{Z}_p) \to {}_h\mathrm{TR}^{n-1}_2(A|K; p, \mathbb{Z}_p)$$

is a surjection whose kernel is generated by $p^{n-2}[d\log\pi]_n$.

Proof. The exact sequence above shows that the map of the statement is a surjection that the kernel is a quotient of the cokernel of the following map

$$(F^{n-1})^*(p^{n-2}A/p^{n-1}A) \xrightarrow{d} (F^{n-1})^*(p^{n-2}\omega^1_{(A,M)}/p^{n-1}\omega^1_{(A,M)}).$$

It is therefore enough to show that this cokernel is generated by $p^{n-2}[d\log\pi]_n$. We consider the polynomial ring $P = W(k)[x]$ with the pre-log structure $\alpha\colon \mathbb{N}_0 \to P$ given by $\alpha(i) = x^i$. The map of $W(k)$-algebras $\epsilon\colon P \to A$, $\epsilon(x) = \pi$, preserves the pre-log structure and induces a surjection $\omega_{(P,\mathbb{N}_0)} \twoheadrightarrow \omega_{(A,M)}$. It follows that the map $p^i\omega^1_{(P,\mathbb{N}_0)} \twoheadrightarrow p^i\omega^1_{(A,M)}$ is a surjection, for $i \geq 0$, and therefore, it suffices to show that the cokernel of the map

$$(F^{n-1})^*(p^{n-2}P/p^{n-1}P) \xrightarrow{d} (F^{n-1})^*(p^{n-2}\omega^1_{(P,\mathbb{N}_0)}/p^{n-1}\omega^1_{(P,\mathbb{N}_0)})$$

is generated as a $W_n(P)$-module by the canonical image of $p^{n-2}d\log x$. Now as a P-module, the quotient $p^{n-2}\omega_{(P,\mathbb{N}_0)}/p^{n-1}\omega_{(P,\mathbb{N}_0)}$ is generated by $p^{n-2}d\log x$, and hence the $W_n(P)$-module $(F^{n-1})^*(p^{n-2}\omega_{(P,\mathbb{N}_0)}/p^{n-1}\omega_{(P,\mathbb{N}_0)})$ is generated by the elements $p^{n-2}d\log x$ and $p^{n-2}x^{p^i}d\log x$, $0 \le i < n-1$. But the last $n-1$ generators are all in the image of the map d:

$$p^{n-2}x^{p^i}d\log x = p^{n-2-i}d(x^{p^i}).$$

Hence the cokernel of d is generated by $p^{n-2}d\log x$, and the lemma follows. □

PROPOSITION 3.3.6. *The sequences*

$$0 \to {}_h\mathrm{TR}_i^n(A|K;p) \xrightarrow{N} \mathrm{TR}_i^n(A|K;p) \xrightarrow{R} \mathrm{TR}_i^{n-1}(A|K;p) \to 0$$

are exact, for $i \le 1$, and exact modulo the Serre subcategory of torsion $W(k)$-modules, for $i = 2$. Moreover, $\mathrm{TR}_2^n(A|K;p)$ is uniquely divisible.

PROOF. The statement for $i = 0$ is [**16**, proposition 3.3], so the statement for $i = 1$ is equivalent to showing that the norm map is injective. This is clear on maximal divisible subgroups, so it suffices to show that $\mathrm{TR}_2^n(A|K;p)$ is uniquely divisible. We show inductively that the p-adic homotopy group $\mathrm{TR}_2^n(A|K;p,\mathbb{Z}_p)$ vanishes, the basic case $n = 1$ being established earlier. We must show that the boundary map

$$\partial_{K,n}\colon \mathrm{TR}_3^{n-1}(A|K;p,\mathbb{Z}_p) \to {}_h\mathrm{TR}_2^n(A|K;p,\mathbb{Z}_p)$$

is surjective.

We first consider the case $n = 2$. In the diagram of $W_2(A)$-modules

$$\begin{tikzcd}
\mathrm{TR}_3^1(A|K;p,\mathbb{Z}_p) \arrow[r, "\partial_{K,2}"] \arrow[d, two heads, "\delta"] & {}_h\mathrm{TR}_2^2(A|K;p,\mathbb{Z}_p) \arrow[d, "\delta"] \\
\mathrm{TR}_2^1(k;p) \arrow[r, two heads, "\partial_k"] & {}_h\mathrm{TR}_1^2(k;p),
\end{tikzcd}$$

the lower horizontal map and the left hand vertical map are both surjections. Indeed, for the former, this was proved in [**16**], and for the latter, it follows from the fact, proved in [**22**], that $\mathrm{TR}_2^1(A;p,\mathbb{Z}_p)$ is trivial. The upper right hand group Q is a quotient of the $W_2(A)$-module $M = F^*(\omega^1_{(A,M)}/p)$. We claim that M is annihilated by the ideal $I = VW_2(A) + pW_2(A)$. Indeed, as an abelian group M is p-torsion and $FV = p$. It follows that also Q is annihilated by I, and we can therefore view it as a module over the quotient ring $W_2(A)/I$. This ring is isomorphic to A/pA, the isomorphism given by

$$W_2(A)/I \xrightarrow{\sim} A/pA, \qquad a + I \mapsto R(a) + pA,$$

and we let $g\colon A/pA \to W_2(A)/I$ denote the inverse. As an A/pA-module, Q is generated by the class $[d\log\pi]_2$. The image of this class under the right hand vertical map is a generator ι_1 of the $W_2(A)$-module ${}_h\mathrm{TR}_1^2(k;p)$, which is isomorphic to k. We now pick $\alpha \in \mathrm{TR}_3^1(A|K;p,\mathbb{Z}_p)$ such that $\delta(\partial_{K,2}(\alpha)) = \iota_1$. The difference $\beta = \partial_{K,2}(\alpha) - [d\log\pi]_2$ is then in the kernel of the δ, and we can therefore write

$$\beta = g(x\pi)\cdot[d\log\pi]_2,$$

for some $x \in A/pA$. We then have

$$g(1+x\pi) \cdot [d\log\pi]_2 = \partial_{K,2}(\alpha),$$

and since $(1+x\pi) \in (A/pA)^\times$,

$$[d\log\pi]_2 = (g(1+x\pi)^{-1}) \cdot \partial_{K,2}(\alpha).$$

We would like to know that the map of units

$$W_2(A)^\times \to (W_2(A)/I)^\times$$

is a surjection. This will follow if we know that the I-adic topology on $W_2(A)$ is complete and separated. But the formula

$$V(x) \cdot V(y) = V(FV(x)y) = V(pxy) = pV(xy)$$

implies that the I-adic and p-adic topologies on $W_2(A)$ coincide, and the p-adic topology is complete and separated. So we can find a unit $u \in W_2(A)^\times$ such that $u + I = g(1+x\pi)$. Since $\partial_{K,2}$ is $W_2(A)$-linear, we have

$$[d\log\pi]_2 = u^{-1}\partial_{K,2}(\alpha) = \partial_{K,2}(u^{-1}\alpha),$$

which concludes the proof for $n = 2$.

We proceed by induction, and consider the diagram

$$\begin{tikzcd}
\mathrm{TR}_3^{n-1}(A|K;p,\mathbb{Z}_p) \arrow[r, "\partial_{K,n}"] \arrow[d, two heads, "F"] & {}_h\mathrm{TR}_2^n(A|K;p,\mathbb{Z}_p) \arrow[r, "N"] \arrow[d, "F"] & \mathrm{TR}_2^n(A|K;p,\mathbb{Z}_p) \arrow[d, "F"] \\
\mathrm{TR}_3^{n-2}(A|K;p,\mathbb{Z}_p) \arrow[r, two heads, "\partial_{K,n-1}"] & {}_h\mathrm{TR}_2^{n-1}(A|K;p,\mathbb{Z}_p) \arrow[r, "N"] & \mathrm{TR}_2^{n-1}(A|K;p,\mathbb{Z}_p).
\end{tikzcd}$$

Inductively, the map $\partial_{K,n-1}$ is surjective, and the left hand vertical map F is surjective by the lemma. Moreover, the kernel of the middle vertical map is generated as a $W_n(A)$-module by the class $p^{n-2}[d\log\pi]_n$. It therefore suffices to show that this class is in the image of $\partial_{K,n}$ in the top row, and this, in turn, will follow if we show that the class $[d\log\pi]_n$ is in the image of $\partial_{K,n}$. To see this, we pick $\alpha \in \mathrm{TR}_3^n(A|K;p),\mathbb{Z}_p)$ such that $\partial_{K,n-1}(F(\alpha)) = [d\log\pi]_{n-1}$. Then $\beta = \partial_{K,n}(\alpha) - [d\log\pi]_n$ is in the kernel of the middle vertical map, so we can write $\beta = x \cdot p^{n-2} d\log\pi$, for some $x \in W_n(A)$. But then

$$(1 + p^{n-2}x)[d\log\pi]_n = \partial_{K,n}(\alpha),$$

and hence

$$[d\log\pi]_n = (1+p^{n-2}x)^{-1}\partial_{K,n}(\alpha) = \partial_{K,n}((1+p^{n-2}x)^{-1}\alpha),$$

where the inverse exists since the p-adic topology on $W_n(A)$ is complete and separated. The proof is complete. □

ADDENDUM 3.3.7. *The group $\mathrm{TR}_2^n(A;p)$ is uniquely divisible for all n.*

PROOF. It suffices to show that $\mathrm{TR}_2^n(A;p,\mathbb{Z}_p)$ is trivial. We prove this by induction, the basic case $n = 1$ being proved in [**22**]. Since $\mathrm{TR}_2^n(A|K;p,\mathbb{Z}_p)$ vanishes, we have an exact sequence

$$\mathrm{TR}_3^n(A|K;p,\mathbb{Z}_p) \xrightarrow{\delta_n} \mathrm{TR}_2^n(k;p) \to \mathrm{TR}_2^n(A;p,\mathbb{Z}_p) \to 0,$$

and we thus must prove that the map δ_n is surjective. We consider the diagram

$$\begin{tikzcd} \mathrm{TR}_3^n(A|K;p,\mathbb{Z}_p) \arrow[r, "\delta_n"] \arrow[d, "F"] & \mathrm{TR}_2^n(k;p) \arrow[d, two heads, "F"] \\ \mathrm{TR}_3^{n-1}(A|K;p,\mathbb{Z}_p) \arrow[r, two heads, "\delta_{n-1}"] & \mathrm{TR}_2^{n-1}(k;p), \end{tikzcd}$$

where the map δ_{n-1} is surjective by induction. It was proved in [**16**] that the left hand vertical map F is a surjection whose kernel is equal to the image of the map

$$V^{n-1}\colon\ \mathrm{TR}_2^1(k;p) \to \mathrm{TR}_2^n(k;p).$$

Since the square

$$\begin{tikzcd} \mathrm{TR}_2^1(A|K;p,\mathbb{Z}_p) \arrow[r, two heads, "\delta_1"] \arrow[d, "V^{n-1}"] & \mathrm{TR}_2^1(k;p) \arrow[d, "V^{n-1}"] \\ \mathrm{TR}_2^n(A|K;p,\mathbb{Z}_p) \arrow[r, "\delta_n"] & \mathrm{TR}_2^n(k;p) \end{tikzcd}$$

commutes and the top horizontal map is a surjection, the proof of the induction step is complete. □

Theorem 3.3.8. *The canonical map*

$$W.\,\omega^i_{(A,M)} \to \mathrm{TR}^{\cdot}_i(A|K;p)$$

is an isomorphism, for $i \le 2$.

Proof. The statement for $i=0$, which has already been used, was proved in [**16**, theorem F]. The proof for $i=1,2$ is by induction, the basic case,

$$\omega^i_{(A,M)} \xrightarrow{\sim} \pi_i T(A|K)$$

being proved earlier. In the induction step, we use the exact sequences of lemma 3.2.7 and proposition 3.3.6,

$$\begin{tikzcd} & {}_hW_n\,\omega^i_{(A,M)} \arrow[r] \arrow[d] & W_n\,\omega^i_{(A,M)} \arrow[r, "R"] \arrow[d] & W_{n-1}\,\omega^i_{(A,M)} \arrow[r] \arrow[d, "\sim"] & 0 \\ 0 \arrow[r] & {}_h\mathrm{TR}_i^n(A|K;p) \arrow[r, "N"] & \mathrm{TR}_i^n(A|K;p) \arrow[r, "R"] & \mathrm{TR}_i^{n-1}(A|K;p) \arrow[r] & 0, \end{tikzcd}$$

where for $i=2$, the lower sequence is only exact modulo the Serre subcategory of torsion $W(k)$-modules. When $i=1$, the left hand vertical map is an isomorphism by lemma 3.3.2, and hence the statement follows in this case. When $i=2$, the left hand vertical map is an epimorphism with torsion kernel. Since the domain and range of the middle and right hand vertical maps are both divisible groups, the statement follows. □

Addendum 3.3.9. *The connecting homomorphism*

$$\partial\colon\ \mathrm{TR}_2^1(A|K;p,\mathbb{Z}/p) \to {}_h\,\mathrm{TR}_1^2(A|K;p,\mathbb{Z}/p)$$

maps κ to $dV(1) - V(d\log p)$.

Proof. An application of lemma 3.3.10 below to the diagram obtained as the smash product of the coefficient cofibration sequence

$$S^0 \xrightarrow{p} S^0 \to M_p \xrightarrow{\partial} S^1$$

and the fundamental cofibration sequence

$${}_h\mathrm{TR}^n(A|K;p) \xrightarrow{N} \mathrm{TR}^n(A|K;p) \xrightarrow{R} \mathrm{TR}^{n-1}(A|K;p) \xrightarrow{\partial} \Sigma({}_h\mathrm{TR}^n(A|K;p))$$

shows that the connecting homomorphism of the statement is equal to the opposite of the connecting homomorphism associated with the diagram

$$\begin{tikzcd}
0 \arrow[r] & {}_h\,\mathrm{TR}_1^2(A|K;p) \arrow[r, "N"] \arrow[d, "p"] & \mathrm{TR}_1^2(A|K;p) \arrow[r, "R"] \arrow[d, "p"] & \mathrm{TR}_1^1(A|K;p) \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & {}_h\,\mathrm{TR}_1^2(A|K;p) \arrow[r, "N"] & \mathrm{TR}_1^2(A|K;p) \arrow[r, "R"] & \mathrm{TR}_1^1(A|K;p) \arrow[r] & 0.
\end{tikzcd}$$

And by theorem 3.3.8, this diagram is canonically isomorphic to the diagram

$$\begin{tikzcd}
0 \arrow[r] & {}_hW_2\omega^1_{(A,M)} \arrow[r, "N"] \arrow[d, "p"] & W_2\omega^1_{(A,M)} \arrow[r, "R"] \arrow[d, "p"] & W_1\omega^1_{(A,M)} \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & {}_hW_2\,\omega^1_{(A,M)} \arrow[r, "N"] & W_2\,\omega^1_{(A,M)} \arrow[r, "R"] & W_1\,\omega^1_{(A,M)} \arrow[r] & 0.
\end{tikzcd}$$

The Bockstein maps κ to $d\log p \in W_1\,\omega^1_{(A,M)}$, which is the image under the restriction of $d\log \underline{p}_2 \in W_2\,\omega^1_{(A,M)}$. Using ghost coordinates, one verifies easily that

$$\underline{p}_2 + V(1) = p(1 + p^{p^{p-2}}V(1)),$$

and hence

$$\begin{aligned}
pd\log \underline{p}_2 &= (1 + p^{p-2}V(1))^{-1}(\underline{p}_2 d\log \underline{p}_2 + V(1)d\log \underline{p}_2) \\
&= (1 - p^{p-2}V(1) + \dots)(d\underline{p}_2 + V(d\log p)).
\end{aligned}$$

Since $dV(1)$ is p-torsion, $d\underline{p}_2 = -dV(1)$, and hence

$$pd\log \underline{p}_2 = V(d\log p) - dV(1).$$

The statement follows. $\square$

Lemma 3.3.10. *Given a 3×3-diagram of cofibration sequences*

$$\begin{tikzcd}
E_{11} \arrow[r, "f_{11}"] \arrow[d, "g_{11}"] & E_{12} \arrow[r, "f_{12}"] \arrow[d, "g_{12}"] & E_{13} \arrow[r, "f_{13}"] \arrow[d, "g_{13}"] & \Sigma E_{11} \arrow[d, "\Sigma g_{11}"] \\
E_{21} \arrow[r, "f_{21}"] \arrow[d, "g_{21}"] & E_{22} \arrow[r, "f_{22}"] \arrow[d, "g_{22}"] & E_{23} \arrow[r, "f_{23}"] \arrow[d, "g_{23}"] & \Sigma E_{21} \arrow[d, "\Sigma g_{21}"] \\
E_{31} \arrow[r, "f_{31}"] \arrow[d, "g_{31}"] & E_{32} \arrow[r, "f_{32}"] \arrow[d, "g_{32}"] & E_{33} \arrow[r, "f_{33}"] \arrow[d, "g_{33}"] \arrow[dr, phantom, "(-1)"] & \Sigma E_{31} \arrow[d, "\Sigma g_{11}"] \\
\Sigma E_{11} \arrow[r, "\Sigma f_{11}"] & \Sigma E_{12} \arrow[r, "\Sigma f_{12}"] & \Sigma E_{13} \arrow[r, "\Sigma f_{13}"] & \Sigma^2 E_{11}
\end{tikzcd}$$

and classes $e_{ij} \in \pi_* E_{ij}$ such that $g_{33}(e_{33}) = \Sigma f_{12}(e_{12})$ and $f_{33}(e_{33}) = \Sigma g_{21}(e_{21})$. Then the sum $f_{21}(e_{21}) + g_{12}(e_{12})$ is in the image of $\pi_* E_{11} \to \pi_* E_{22}$. $\square$

REMARK 3.3.11. One may show that the canonical map

$$W.\Omega^1_R \xrightarrow{\sim} \mathrm{TR}^{\cdot}_1(R;p)$$

is an epimorphism for any ring R (with the trivial pre-log structure). In fact, we do not know of a ring R for which this map is not an isomorphism.

3.4. We evaluate the differential graded k-algebra $W_n\,\omega^*_{(A,M)} \otimes \mathbb{F}_p$. Let π be a uniformizer with minimal polynomial

$$\pi^{e_K} + p\theta_K(\pi).$$

The modified Verschiebung

$$V_\pi \colon W_{n-1}(A) \to W_n(A), \quad V_\pi(x) = \theta_K(\underline{\pi}_n)V(x)$$

satisfies

$$\begin{aligned} FV_\pi(x) &= p\theta_K(\underline{\pi})^p x, \\ FdV_\pi(x) &= \theta_K(\underline{\pi})^p dx. \end{aligned}$$

PROPOSITION 3.4.1. *The differential graded k-algebra $E^* = W_n\,\omega^*_{(A,M)} \otimes_{\mathbb{Z}} \mathbb{F}_p$ is concentrated in degrees 0 and 1 and satisfies:*

(i) *the k-algebra E^0_n is generated by the elements $V^s_\pi(\underline{\pi}^i)$ with $0 \le s < n$ and $i \ge 0$ subject to the relations:*

$$\begin{aligned} V^s_\pi(\underline{\pi}^i) \cdot V^t_\pi(\underline{\pi}^j) &= p^s V^t_\pi(\theta(\underline{\pi})^{p^{t-s}(\frac{p^{s+1}-1}{p-1}-1)}\underline{\pi}^{p^{t-s}i+j}), \\ V^s_\pi(\underline{\pi}^{e_K+i}) &= V^{s+1}_\pi(\underline{\pi}^{pi}); \end{aligned}$$

(ii) *the k-vector space E^1_n is generated by the elements $dV^s_\pi(\underline{\pi}^i)$ and $V^s_\pi(\underline{\pi}^i d\log\pi)$ with $0 \le s < n$ and $i \ge 0$ subject to the relations that for $v_p(i - pe_K/(p-1)) \ge s$,*

$$dV^s_\pi(\underline{\pi}^i) = p^{-s}(i - pe_K/(p-1)) \cdot V^s_\pi(\underline{\pi}^i d\log\pi),$$

and for $v_p(i - pe_K/(p-1)) < s$, $V^s_\pi(\underline{\pi}^i d\log\pi) = 0$;

(iii) the E^0_n-module structure on E^1_n is given by

$$V^s_\pi(\underline{\pi}^i)dV^t_\pi(\underline{\pi}^j) = \begin{cases} p^s dV^t_\pi(\theta_K(\underline{\pi})^{p^{t-s}(\frac{p^{s+1}-1}{p-1}-1)}\underline{\pi}^{p^{t-s}i+j}) & \text{if } s \le t, \\ \quad -iV^t_\pi(\theta_K(\underline{\pi})^{p^{t-s}(\frac{p^{s+1}-1}{p-1}-1)}\underline{\pi}^{p^{t-s}i+j}d\log\pi), & \\ jV^s(\theta_K(\underline{\pi})^{p^{s-t}(\frac{p^{t+1}-1}{p-1}-1)}\underline{\pi}^{i+p^{s-t}j}d\log\pi), & \text{if } s \ge t, \end{cases}$$

$$V^s_\pi(\underline{\pi}^i)V^t_\pi(\underline{\pi}^j d\log\pi) = \begin{cases} p^s V^t_\pi(\theta_K(\underline{\pi})^{p^{t-s}(\frac{p^{s+1}-1}{p-1}-1)}\underline{\pi}^{p^{t-s}i+j}d\log\pi), & \text{if } s \le t, \\ p^t V^s_\pi(\theta_K(\underline{\pi})^{p^{s-t}(\frac{p^{t+1}-1}{p-1}-1)}\underline{\pi}^{i+p^{s-t}j}d\log\pi), & \text{if } s \ge t. \end{cases}$$

PROOF. As a graded k-vector space, E^*_n is generated by the monomials in the variables $V^s(\underline{\pi}^i)$, $dV^s(\underline{\pi}^i)$ and $V^s(\underline{\pi}^i d\log\pi)$ with $0 \le s < n$ and $i \ge 0$. Indeed, if $a \in k$ then

$$V^s(a\underline{\pi}^i) = \varphi^{-s}(a)V^s(\underline{\pi}^i)$$

and the operator F applied to any of the elements above is expressible as a linear combination of these elements. Since $\theta_K(\underline{\pi})$ is a unit, we may instead use the elements $V^s_\pi(\underline{\pi}^i)$, $dV^s_\pi(\underline{\pi}^i)$ and $V^s_\pi(\underline{\pi}^i d\log\pi)$ as our generators. Part (i) was proved in proposition 3.1.3 above.

Let $t = v_p(i - pe_K/(p-1))$. For $t \geq s$, $p^{-s}(i + pe_K(p^s - 1)/(p-1))$ is an integer, and iterated use of the second relation in (i) shows that

$$V_\pi^s(\underline{\pi}^i) = \underline{\pi}^{p^{-s}(i + pe_K \frac{p^s - 1}{p-1})}.$$

The first relation in (ii) easily follows. Moreover, if $t < s$ then up to a unit,

$$V_\pi^t(\underline{\pi}^i d\log \pi) = dV_\pi^t(\underline{\pi}^i),$$

and hence

$$V_\pi^s(\underline{\pi}^i d\log \pi) = V_\pi^{s-t} dV_\pi^t(\underline{\pi}^i) = 0.$$

Finally, differentiating the first relation in (i), we get

$$dV_\pi^s(\underline{\pi}^i) \cdot V_\pi^t(\underline{\pi}) + V_\pi^s(\underline{\pi}^i) \cdot dV_\pi^t(\underline{\pi}) = p^s dV_\pi^t(\theta(\underline{\pi})^{p^{t-s}(\frac{p^{s+1}-1}{p-1} - 1)} \underline{\pi}^{p^{t-s}i + j}),$$

and rewriting the first term on the left

$$V_\pi^t(F^t dV_\pi^s(\underline{\pi}^i)\underline{\pi}^j) = iV_\pi^t(\theta_K(\underline{\pi})^{p^{t-s}(\frac{p^{s+1}-1}{p-1} - 1)} \underline{\pi}^{p^{t-s}i + j} d\log \pi),$$

the first case in (iii) follows. The remaining cases are proved similarly. It remains to prove that (ii) give all relations in E_n^1. From (ii) is follows that

$$\begin{aligned} \operatorname{gr}^s E_n^1 = \; & k\langle dV_\pi^s(\underline{\pi}^i) \mid 0 \leq i < e_K, \, v_p(i - \tfrac{pe_K}{p-1}) \leq s \rangle \oplus \\ & k\langle V_\pi^s(\underline{\pi}^i d\log \pi) \mid 0 \leq i < e_K, \, v_p(i - \tfrac{pe_K}{p-1}) > s \rangle, \end{aligned}$$

which implies that that E_n^1 is an ne_K-dimensional k-vector space. We will prove in paragraph 6 that this is indeed the case, and hence there are no further relations. $\square$

4. Tate cohomology and the Tate spectrum

4.1. Let k be a commutative group ring and let G be a finite group. By *complexes* we mean $\mathbb{Z}$-graded chain complexes of left kG-modules with differential of degree -1. If X and Y are two complexes, the tensor product $X \otimes Y$ is given by

$$(X \otimes Y)_n = \bigoplus_{p+q=n} X_p \otimes Y_q; \qquad d(x \otimes y) = dx \otimes y + (-1)^{|x|} x \otimes dy,$$

and the complex of k-homomorphism $\operatorname{Hom}(X, Y)$ is given by

$$\operatorname{Hom}(X, Y)_n = \prod_{p \in \mathbb{Z}} \operatorname{Hom}(X_p, Y_{n+p}); \qquad d(f(x)) = (df)(x) + (-1)^{|f|} f(dx).$$

We recall that $Z_0 \operatorname{Hom}(X, Y)$ is the set of chain maps from X to Y and that $H_0 \operatorname{Hom}(X, Y)$ is the set of chain homotopy classes of chain maps from X to Y. The adjunction and twist isomorphisms are

$$\begin{aligned} &\phi\colon \operatorname{Hom}(X \otimes Y, Z) \to \operatorname{Hom}(X, \operatorname{Hom}(Y, Z)); \qquad \phi(f)(x)(y) = f(x \otimes y), \\ &\gamma\colon X \otimes Y \to Y \otimes X, \qquad \gamma(x \otimes y) = (-1)^{|x||y|} y \otimes x. \end{aligned}$$

The mapping cone C_f of a chain map $f\colon X \to Y$ is the complex with

$$(C_f)_n = Y_n \oplus X_{n-1}, \qquad d(y, x) = (y - f(x), -dx),$$

and the cokernel of the inclusion $i\colon Y \to C_f$ is the suspension ΣX,

$$(\Sigma X)_n = X_{n-1}, \qquad d_{\Sigma X}(x) = -d_X(x).$$

Let $\partial\colon C_f \to \Sigma X$ be the canonical projection. Then the category of complexes and chain homotopy classes of chain maps is a triangulated category with the distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{\partial} \Sigma X.$$

We recall that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a short exact sequence of complexes then the projection $p\colon C_f \to Z$, $p(y, x') = g(y)$, is a quasi-isomorphism and the composite

$$H_n Z \xleftarrow[\sim]{p_*} H_n C_f \xrightarrow{\partial_*} H_n \Sigma X = H_{n-1} X$$

coincides with the connecting homomorphism. The triangulation is compatible with the closed structure in the sense that

$$\Sigma(X \otimes Y) = \Sigma X \otimes Y$$

and that if W is a complex and $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a triangle, then so are

$$X \otimes W \xrightarrow{f\otimes 1} Y \otimes W \xrightarrow{g\otimes 1} Z \otimes W \xrightarrow{h\otimes 1} \Sigma X \otimes W$$

$$W \otimes X \xrightarrow{1\otimes f} W \otimes Y \xrightarrow{1\otimes g} W \otimes Z \xrightarrow{e\circ(1\otimes h)} \Sigma W \otimes X,$$

where $e\colon W \otimes \Sigma X \to \Sigma W \otimes X$ is the canonical map, $e(w \otimes x) = (-1)^{|w|}(w \otimes x)$. Given $f\colon X \to Y$ and a complex W, we define an isomorphism

$$\rho\colon W \otimes C_f \xrightarrow{\sim} C_{W\otimes f}, \qquad \rho(w \otimes (y, x')) = (w \otimes y, (-1)^{|w|} w \otimes x'). \tag{4.1.1}$$

Let $N_G \in kG$ be the norm element defined as the sum of all the elements of G. For every left kG-module M, mulplication by N_G defines a map

$$N_G\colon M_G \to M^G,$$

where $M_G = k \otimes_G M$ and $M^G = \mathrm{Hom}(k, M)^G$ are the coinvariants and invariants of M, respectively. We note that for left kG-modules M and N,

$$(M \otimes N)_G = c^* M \otimes_{kG} N,$$

where $c^* M$ denotes the right kG-module with $m \cdot g = g^{-1} m$.

We define the Tate cohomology of G with coefficients in the left G-module M as follows. Let $\epsilon : P \to k$ be a resolution of k by finitely generated projective left kG-modules and let $\tilde{P}$ be the mapping cone of ϵ.

Definition 4.1.2. $\hat{H}^*(G; M) = H_{-*}((\tilde{P} \otimes \mathrm{Hom}(P, M))^G)$.

The triangle

$$P \xrightarrow{\epsilon} k \xrightarrow{i} \tilde{P} \xrightarrow{\partial} \Sigma P \tag{4.1.3}$$

and the quasi-isomorphism

$$c^* P \otimes_G M \xleftarrow{\sim} (P \otimes M)_G \xrightarrow{N} (P \otimes M)^G \xrightarrow{\epsilon} (P \otimes \mathrm{Hom}(P, M))^G$$

identifies

$$\hat{H}^i(G; M) \cong \begin{cases} H^i(G; M) & \text{if } i \geq 1 \\ H_{-i-1}(G; M) & \text{if } i \leq -1 \end{cases}$$

and gives the exact sequence

$$0 \to \hat{H}^{-1}(G; M) \to H_0(G; M) \xrightarrow{N} H^0(G; M) \to \hat{H}^0(G; M) \to 0.$$

In particular, the above definition agrees with the original one in terms of complete resolutions. More explicitly, suppose that $\epsilon\colon \hat{P} \to k$ is a complete resolution and let P and P^- be the complexes whose non-zero terms are $P_i = \hat{P}_i$, if $i \geq 0$, and $P_i^- = \hat{P}_i$, if $i < 0$, respectively. Then $\epsilon\colon P \to k$ is a resolution of k by finitely generated projective modules and there is a canonical triangle

$$P^- \to \hat{P} \to P \to \Sigma P^-.$$

LEMMA 4.1.4. *Each of the canonical maps*

$$Hom(\hat{P}, M)^G \to (\tilde{P} \otimes \mathrm{Hom}(\hat{P}, M))^G \to (\tilde{P} \otimes \mathrm{Hom}(P, M))^G$$

are quasi-isomorphisms.

PROOF. The triangle (4.1.3) identifies the mapping cones of the maps of the statement with the complexes $(\Sigma P \otimes \mathrm{Hom}(\hat{P}, M))^G$ and $(\Sigma\tilde{P} \otimes \mathrm{Hom}(P, M))^G$, respectively. Both are total complexes of double complexes. The filtration after the first tensor factor define spectral sequences which converge strongly to the homology of the total complexes. The E^1-terms are

$$\begin{aligned} E^1_{s,t} &= H_{s+t-1}((P_s \otimes \mathrm{Hom}(\hat{P}, M))^G) \cong \hat{H}^{-s-t-1}(G; P_s \otimes M), \\ E^1_{s,t} &= H_{s+t}((\tilde{P} \otimes \mathrm{Hom}(P_{-s}, M))^G). \end{aligned}$$

The first E^1 vanishes because $P_s \otimes M$ is weakly projective, and the second because $\mathrm{Hom}(P_{-s}, M)$ is flat and $\tilde{P}$ is acyclic. □

The cup product on group cohomology may be extended to a product

$$\hat{H}^*(G; M) \otimes \hat{H}^*(G; M') \to \hat{H}^*(G; M \otimes M') \tag{4.1.5}$$

in following way. The tensor product $P \otimes P$ is a projective resolution of $k \otimes k$, so we can choose a lifting $P \to P \otimes P$ of the canoncal isomorphism $k \to k \otimes k$. We also choose a chain map $\tilde{P} \otimes \tilde{P} \to \tilde{P}$ which extends $k \otimes k \to k$. The product (4.1.5) is then the map on homology induced from

$$\begin{aligned} (\tilde{P} \otimes \mathrm{Hom}(P, M))^G \otimes (\tilde{P} \otimes \mathrm{Hom}(P, M'))^G &\to (\tilde{P} \otimes \tilde{P} \otimes \mathrm{Hom}(P \otimes P, M \otimes M'))^G \\ &\to (\tilde{P} \otimes \mathrm{Hom}(P, M \otimes M'))^G, \end{aligned}$$

where the first map is the canonical map and the second is induced from the chosen quasi-isomorphisms. Since any two choices of liftings are chain homotopic, the product is well defined and makes $\hat{H}^*(G; k)$ a graded commutative associated ring and $\hat{H}^*(G; M)$ a graded module over this ring.

4.2. Let C be a cyclic group of order r and let $g \in C$ be a generator. We let $\epsilon\colon W \to k$ be the standard resolution which in degree $s \geq 0$ is a free kC-module on a single generator x_s with differential

$$dx_s = \begin{cases} Nx_{s-1}, & s \text{ even}, \\ (g-1)x_{s-1}, & s \text{ odd}, \end{cases}$$

and with augmentation $\epsilon(x_0) = 1$. Then $\tilde{W}$ is the complex which in degree $s > 0$ is a free kC-module on the generator $y_s = (0, x_{s-1})$ and in degree $s = 0$ is a trivial

kC-module on the generator $e = (1, 0)$. The differential is

$$dy_s = \begin{cases} -(g-1)y_{s-1}, & s \text{ even}, \\ -Ny_{s-1}, & s > 1 \text{ odd}. \\ -e & s = 1 \end{cases}$$

The dual of x_s is the element $x_s^* \in \operatorname{Hom}(W_s, k)$ given by $x_s^*(g^i x_s) = \delta_{i,0}$. We note that $g^i \cdot x_n^* = (g^i x_n)^*$ and that the map $(g^i)^* \colon W_s^* \to W_s^*$ maps $x_s^* \mapsto g^{-i} x_s^*$. Thus

$$dx_s^* = \begin{cases} (g^{-1}-1)x_{s+1}^*, & s \text{ even}, \\ Nx_{s+1}^*, & s \text{ odd}. \end{cases}$$

LEMMA 4.2.1. *Suppose that the order of C is odd and congruent to zero in k. Then as a graded ring*

$$\hat{H}^*(C; k) = \Lambda_k\{u\} \otimes S_k\{t, t^{-1}\}$$

where t and u are the classes of $e \otimes Nx_2^*$ and $e \otimes Nx_1^*$, respectively. Moreover, the classes ut^{-1} and t^{-1} are represented by the elements $-Ny_1 \otimes Nx_0^*$ and $Ny_2 \otimes Nx_0^*$, respectively.

PROOF. We first evaluate the homology of the complex

$$(\tilde{W} \otimes \operatorname{Hom}(W, k))^C = (\tilde{W} \otimes DW)^C.$$

This is the total complex of a double complex, and the filtration after the first tensor factor gives rise to a spectral sequence which converges strongly to the homology of the total complex. We have

$$E_{s,t}^1 = H_{s+t}(\tilde{W}_s \otimes DW)^C \xrightarrow{\sim} H_{s+t}(\operatorname{Hom}(W, \tilde{W}_s)^C),$$

which vanishes unless one of s and t are zero. Hence $E_{s,t}^2 = E_{s,t}^\infty$ and it is easy to see that if either s or t is zero, this is a free k-module of rank one generated by the classes of $e \otimes Nx_{-t}^*$ and $Ny_s \otimes Nx_0^*$, respectively. Note that these elements are also cycles in the total complex.

To evaluate the multiplicative structure, we choose liftings

$$\Psi \colon W \to W \otimes W$$

$$\Phi \colon \tilde{W} \otimes \tilde{W} \to \tilde{W}$$

of the canonical maps $k \to k \otimes k$ and $k \otimes k \to k$, respectively:

$$\Psi_{m,n}(g^s x_{m+n}) = \begin{cases} \displaystyle\sum_{s \le p < q < s} g^p x_m \otimes g^q x_n & m \text{ and } n \text{ odd} \\ g^s x_m \otimes g^{s+1} x_n & m \text{ odd, } n \text{ even} \\ g^s x_m \otimes g^s x_n & m \text{ even} \end{cases}$$

and

$$\Phi_{m,n}(g^p y_m \otimes g^q y_n) = \begin{cases} \displaystyle\sum_{p \le s < q < p} g^s y_{m+n} & m \text{ and } n \text{ odd} \\ \delta_{p,q+1} g^p y_{m+n} & m \text{ odd, } n \text{ even} \\ \delta_{p,q} g^p y_{m+n} & m \text{ even,} \end{cases}$$

where in the first line the sum ranges over the g^s between g^p and g^{q-1}, both included, in the cyclic ordering of C specified by the generator g. The sum is zero if and only if $p = q$. The map Ψ induces a product map on the dual DW given by the composite

$$\Psi^* \colon DW \otimes DW \xrightarrow{\nu} D(W \otimes W) \xrightarrow{D\Psi} DW,$$

or

$$\Psi^*_{m,n}(g^{-p}x^*_m \otimes g^{-q}x^*_n) = \begin{cases} -\sum_{p \le s < q < p} g^{-s}x^*_{m+n} & m \text{ and } n \text{ odd} \\ \delta_{p,q+1}g^{-p}x^*_{m+n} & m \text{ odd, } n \text{ even} \\ \delta_{p,q}g^{-p}x^*_{m+n} & m \text{ even.} \end{cases}$$

We find that

$$(e \otimes Nx^*_m) \cdot (e \otimes Nx^*_n) = \begin{cases} -\dfrac{r(r-1)}{2}\, e \otimes Nx^*_{m+n} & m \text{ and } n \text{ odd} \\ e \otimes Nx^*_{m+n} & \text{else} \end{cases}$$

and

$$(Ny_m \otimes Nx^*_0) \cdot (Ny_n \otimes Nx^*_0) = \begin{cases} \dfrac{r(r-1)}{2}\, Ny_{m+n} \otimes Nx^*_0 & m \text{ and } n \text{ odd} \\ Ny_{m+n} \otimes Nx^*_0 & \text{else.} \end{cases}$$

Moreover, the product

$$(e \otimes Nx^*_2) \cdot (Ny_2 \otimes Nx^*_0) = Ny_2 \otimes Nx^*_2$$

is homologous to $e \otimes Nx^*_0$, which represents the multiplicative unit in the cohomology ring. Indeed,

$$d(\Delta(N)(y_1 \otimes x^*_0) + \Delta(N)(y_2 \otimes x^*_1)) = -e \otimes Nx^*_0 + Ny_2 \otimes Nx^*_2.$$

Hence $Ny_2 \otimes Nx^*_0$ represents the class t^{-1}. Finally, for any element $\alpha \in kC$,

$$(1 \otimes \alpha)\Delta(N) = (\bar{\alpha} \otimes 1)\Delta(N),$$

where $\bar{\alpha} = c(\alpha)$ is the antipode. Therefore, if $\alpha \in kC$ is such that $(g-1)\alpha = r - N$, e.g. $\alpha = 1 + 2g + \cdots + rg^{r-1}$, then

$$\begin{aligned} &d((\alpha \otimes 1)\Delta(N)(y_2 \otimes x^*_0)) \\ &\quad = -((g-1) \otimes 1)(\alpha \otimes 1)\Delta(N)(y_1 \otimes x^*_0) \\ &\qquad - (1 \otimes (\bar{g}-1))(1 \otimes \bar{\alpha})\Delta(N)(y_2 \otimes x^*_1) \\ &\quad = Ny_1 \otimes Nx^*_0 + Ny_2 \otimes Nx^*_1 - r\Delta(N)(y_1 \otimes x^*_0 + y_2 \otimes x^*_1), \end{aligned}$$

and hence the element $Ny_1 \otimes Nx^*_0$ represents the class $-ut^{-1}$ in the cohomology ring. $\square$

When k is a perfect field of odd characteristic p and C_{p^n} a cyclic group of order p^n, we get

$$\hat{H}^*(C_{p^n}; k) = \Lambda_k\{u\} \otimes S_k\{t, t^{-1}\},$$

with the classes $u = u_n$ and t defined as above. In the Bockstein spectral sequence the first non-zero differential is

$$\beta_n u_n = 1.$$

4.3. The stable category of G-spectra is a closed triangulated category. We fix some conventions, compatible with the ones from section 4.1. As a model for the G-stable category we use G-CW-spectra.

The cone of a spectrum X is the smash product

$$CX = [0,1] \wedge X,$$

where we use 1 as the base point of the interval. If X is a CW-spectrum, we give CX the product CW-structure, where $[0,1]$ is a a CW-complex as usual with the 1-cell oriented from 0 to 1. The mapping cone of a map of CW-spectra $f\colon X \to Y$ is the pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\iota_0"] & Y \arrow[d, "i_2"] \\ CX \arrow[r] & C_f \end{tikzcd}$$

with the CW-structure determined from that on Y and CX. Collapsing the image of i_2 to the base point defines the map

$$\partial\colon C_f \to S^1 \wedge X = \Sigma X,$$

where $S^1 = [0,1]/\partial[0,1]$ with the induced CW-structure. The distiguished triangles are then the sequences of the form

$$X \xrightarrow{f} Y \xrightarrow{i_2} C_f \xrightarrow{\partial} \Sigma X.$$

With these definitions, the cellular chain functor throws the distinguished triangles of CW-spectra on the distinguished triangles of chain complexes defined in 4.1. Moreover, the isomorphism

$$W \wedge CX \xrightarrow{\gamma\wedge 1} C(W \wedge X)$$

induces an isomorphism

$$\rho\colon W \wedge C_f \xrightarrow{\sim} C_{W\wedge f},$$

which again is carried to the corresponding isomorphism of chain complexes by the cellular chain functor.

For spectra X and Y, we have the external product

$$\wedge\colon \pi_s X \otimes \pi_t Y \to \pi_{s+t}(X \wedge Y) \tag{4.3.1}$$

and define the map

$$\vee\colon \pi_{s+t} F(X,Y) \to \mathrm{Hom}(\pi_{-s} X, \pi_t Y) \tag{4.3.2}$$

as the adjoint of the composite

$$\pi_{s+t} F(X,Y) \otimes \pi_{-s} X \xrightarrow{\wedge} \pi_t(F(X,Y) \wedge X) \xrightarrow{\mathrm{ev}} \pi_t Y.$$

Let X be a CW-spectrum with an increasing filtration $\{X_s\}$ by sub-CW-spectra. Then the exact couple

$$D_{s-1,t+1} \xrightarrow{i} D_{s,t} \xrightarrow{j} E_{s,t} \xrightarrow{\partial} D_{s-1,t}$$

with

$$\begin{aligned} D_{s,t}(X) &= \pi_{s+t} X_s \\ E_{s,t}(X) &= \pi_{s+t}(X_s/X_{s-1}) \end{aligned} \tag{4.3.3}$$

gives rise to a spectral sequence whose abutment is the homotopy groups of X. The spectral sequence converges conditionally in the sense of [**1**] if $\bigcup X_s = X$ and $\operatorname{holim}_{\longleftarrow} X_s \simeq *$.

We next recall products following [**24**]. By a pairing from two exact couples (D, E) and (D', E') to a third exact couple (D'', E'') one understands the following structure: pairings

$$\begin{aligned} D_{s,t} \otimes D'_{s',t'} &\to D''_{s+s',t+t'} \\ E_{s,t} \otimes E'_{s',t'} &\to E''_{s+s',t+t'} \end{aligned} \tag{4.3.4}$$

which satisfies the following conditions

(i) for all $y \in D_{s,t}$ and $y' \in D'_{s',t'}$,

$$j''(yy') = j(y)j'(y'), \qquad i(y)y' = i''(yy') = yi'(y');$$

(ii) for all $y \in D_{s,t}$, $x \in E_{s,t}$, $y' \in D'_{s',t'}$ and $x' \in E'_{s',t'}$,

$$\partial''(j(y)x') = (-)^{|y|} y \partial'(x'), \qquad \partial''(xj'(y')) = \partial(x)y';$$

(iii) for all $x \in E_{s,t}$, $y \in D_{s-n-1,t+n}$, $x' \in E'_{s',t'}$ and $y' \in D'_{s'-n-1,t'+1}$ with $\partial(x) = i^n(y)$ and $\partial'(x') = i^n(y')$, there exists $y'' \in D''_{s+s'-n-1,t+t'+n}$ such that

$$i^n(y'') = k(xx'), \qquad j(y'') = j(y)x' + (-1)^{s+t} xj(y').$$

We recall from [**24**] that such a pairing leads to pairings of the associated spectral sequences, that is, pairings

$$E^r_{s,t} \otimes E'^r_{s',t'} \to E''^r_{s+s',t+t'},$$

for all $r \geq 1$, which satisfies the Leibnitz rule

$$d^r(xx') = d^r xx' + (-1)^{|x|} x d^r x'.$$

Here and above $|x|$ denotes the total degree of x.

We return to the spectral sequence associated with a CW-spectrum filtered by sub-CW-spectra. If X and X' are two CW-spectra with such filtrations, we give the smash product $X \wedge X'$ the usual product filtration

$$(X \wedge X')_n = \bigcup_{s+s'=n} X_s \wedge X'_{s'}.$$

with filtration quotients

$$(X \wedge X')_n / (X \wedge X')_{n-1} = \bigvee_{s+s'=n} X_s / X_{s-1} \wedge X_{s'} / X_{s'-1}.$$

The external product (4.3.1) and the inclusions

$$\begin{gathered} X_s \wedge X'_{s'} \to (X \wedge X')_{s+s'} \\ X_s / X_{s-1} \wedge X'_{s'} / X'_{s'-1} \to (X \wedge X')_{s+s'} / (X \wedge X')_{s+s'-1} \end{gathered}$$

then gives rise to pairings

$$\begin{aligned} D_{s,t}(X) \otimes D_{s',t'}(X') &\to D_{s+s',t+t'}(X \wedge X'), \\ E_{s,t}(X) \otimes E_{s',t'}(X') &\to E_{s+s',t+t'}(X \wedge X'). \end{aligned} \tag{4.3.5}$$

Indeed, it is straightforward to verify that the requirements listed above are satisfied such that one has an external paring of spectral sequences. For example to check condition (iii), we note that by assumption the elements x and x' are represented by maps

$$x\colon (D, \partial D) \to (X_s, X_{s-n-1}), \quad x'\colon (D', \partial D') \to (X'_{s'}, X'_{s'-n-1}),$$

where D and D' are disks of dimension $s+t$ and $s'+t'$, respectively. Then y'' is the homotopy class of the composition

$$\partial D \times D \cup D \times \partial D \to X_{s-n-1} \wedge X'_{s'} \cup X_s \times X'_{s'-n-1} \to (X \wedge X')_{s+s'-n-1}.$$

A filtration preserving product map $X \wedge X \to X$ gives rise to an internal product in the spectral sequence and all differentials will act as derivations for this product. If the product on X is associative, commutative or unital, the same holds for the internal product in the spectral sequence. Here commutativity in the spectral sequence is up to the usual sign.

4.4. Let G be a finite group and let E be a free contractible G-CW-complex with finitely many cells in each dimension. We define $\tilde{E}$ to be the mapping cone of the projection $\mathrm{pr}\colon E_+ \to S^0$ which collapses E to the non-base point of S^0. Thus we have the distinguished triangle

$$E_+ \xrightarrow{\mathrm{pr}} S^0 \to \tilde{E} \xrightarrow{\partial} \Sigma E_+.$$

We let P and $\tilde{P}$ be the cellular complexes of E and $\tilde{E}$ with coefficients in a commutative ground ring k. We recall that taking cellular chains of the triangle above gives the distinguished triangle

$$P \xrightarrow{\mathrm{pr}_*} k \to \tilde{P} \to \Sigma P$$

in the category of chain complexes.

The Tate spectrum of a G-spectrum T was defined in section 1.1 to be

$$\hat{\mathbb{H}}(G; T) = (\tilde{E} \wedge F(E_+, T))^G.$$

Given two G-spectra T and T' indexed on $\mathcal{U}$, we define a pairing

$$\hat{\mathbb{H}}(G; T) \wedge \hat{\mathbb{H}}(G; T') \to \hat{\mathbb{H}}(G; T \wedge T') \tag{4.4.1}$$

as follows. Choose a cellular G-homotopy equivalence $E_+ \to E_+ \wedge E_+$ and a cellular G-homotopy equivalence $\tilde{E} \wedge \tilde{E} \to \tilde{E}$ which extends the canonical isomorphism $S^0 \wedge S^0 \to S^0$. Any two equivalences are G-homotopic. The pairing is then given by

$$\begin{aligned}(\tilde{E} \wedge F(E_+, T))^G \wedge (\tilde{E} \wedge F(E_+, T))^G &\to (\tilde{E} \wedge \tilde{E} \wedge F(E_+ \wedge E_+, T \wedge T'))^G \\ &\to (\tilde{E} \wedge F(E_+, T \wedge T'))^G,\end{aligned}$$

where the first map is the canonical map and the second is induced from the chosen G-equivalences. If T is a G-ring spectrum, the composition of the external product with the map of Tate spectra induced from the product map on T, makes $\hat{\mathbb{H}}(G; T)$ a ring spectrum. Moreover, this is a homotopy associative, homotopy commutative or unital ring spectrum if T is G-homotopy associative, G-homotopy commutative or unital, respectively.

The CW-filtrations of E and $\tilde{E}$ gives rise to a double filtration of the Tate spectrum. Define

$$\begin{aligned} X_{r,s} &= \tilde{E}_r \wedge F(E/E_{-s-1}, T) \\ Y_{r,s} &= \tilde{E}_r/\tilde{E}_{r-1} \wedge F(E/E_{-s-1}, T) \\ Z_{r,s} &= \tilde{E}_r \wedge F(E_{-s}/E_{-s-1}, T) \\ W_{r,s} &= \tilde{E}_r/\tilde{E}_{r-1} \wedge F(E_{-s}/E_{-s-1}, T) \end{aligned}$$

Given a G-spectrum X, we let $\Gamma X \xrightarrow{\sim} X$ be a functorial G-CW-substitute for X. In order to turn the filtration above into a filtration by G-CW-subspectra, we let

$$\bar{X}_{r,s} = \underrightarrow{\operatorname{holim}}\, \Gamma X_{r',s'},$$

where the homotopy colimit runs over all $0 \le r' \le r$ and $s' \le s \le 0$. There are canonical weak equivalences $\bar{X}_{r,s} \xrightarrow{\sim} X_{r,s}$ and $\bar{X}_{r,s}$ is a sub-G-CW-spectrum of the G-CW-spectrum $\bar{X} = \bar{X}_{\infty,0}$. The fixed set of $\bar{X}$ is equivalent to the Tate spectrum. We then let

$$\begin{aligned} \bar{Y}_{r,s} &= \bar{X}_{r,s}/\bar{X}_{r-1,s} \\ \bar{Z}_{r,s} &= \bar{X}_{r,s}/\bar{X}_{r,s-1} \\ \bar{W}_{r,s} &= \bar{X}_{r,s}/\bar{X}_{r-1,s} \cup \bar{X}_{r,s-1} \end{aligned}$$

and define

$$\bar{X}_n = \bigcup_{r+s \le n} \bar{X}_{r,s} \subset \bar{X}.$$

The exact couple 4.3.3 associated with the filtration $\{\bar{X}_n\}$ defines a spectral sequence that approximates the homotopy groups of the Tate spectrum.

LEMMA 4.4.2. *There is a canonical isomorphism of complexes*

$$E^1_{*,t} \cong (\tilde{P} \otimes \operatorname{Hom}(P, \pi_t T))^G$$

and hence $E^2_{s,t} \cong \hat{H}^s(G; \pi_t T)$.

PROOF. The inclusions $\bar{X}_{r,s} \to \bar{X}_{r+s}$ induces an isomorphism

$$\bigvee_{r+s=n} \bar{W}_{r,s} \xrightarrow{\sim} \bar{X}_n/\bar{X}_{n-1}$$

and the boundary map

$$\bar{X}_n/\bar{X}_{n-1} \to \Sigma \bar{X}_{n-1} \to \Sigma(\bar{X}_{n-1}/\bar{X}_{n-2})$$

maps the summand $\bar{W}_{r,s}$ to the summands $\Sigma \bar{W}_{r-1,s}$ and $\Sigma \bar{W}_{r,s-1}$ by the maps

$$\begin{aligned} \partial' &\colon \bar{W}_{r,s} \to \Sigma \bar{Y}_{r,s-1} \to \Sigma \bar{W}_{r,s-1}, \\ \partial'' &\colon \bar{W}_{r,s} \to \Sigma \bar{Z}_{r-1,s} \to \Sigma \bar{W}_{r-1,s} \end{aligned}$$

respectively. We identify

$$\pi_{r+s+t} \bar{W}_{r,s} \cong (\tilde{P}_r \otimes \operatorname{Hom}(P_{-s}, \pi_t T))^G \tag{4.4.3}$$

in the following way. For any pair of G-spectra X and Y, we have the canonical map

$$\pi_*((X \wedge Y)^G) \to (\pi_*(X \wedge Y))^G,$$

and this is an isomorphism if, say, X is a wedge of free G-cells. The isomorphism (4.4.3) is then the composite the inverse of this map, when $X = \tilde{E}_r/\tilde{E}_{r-1}$ and $Y = F(E_{-s}/E_{-s-1}, T)$, and of the map of G-fixed sets induced from the composite

$$\begin{aligned}\pi_{r+s+t}(\tilde{E}_r/\tilde{E}_{r-1} \wedge F(E_{-s}/E_{-s-1}, T)) &\xleftarrow{\wedge} \pi_r(\tilde{E}_r/\tilde{E}_{r-1}) \otimes \pi_{s+t}F(E_{-s}/E_{-s-1}, T) \\ &\xrightarrow{h\otimes\vee} H_r(\tilde{E}_r/\tilde{E}_{r-1}) \otimes \operatorname{Hom}(\pi_{-s}(E_{-s}/E_{-s-1}), \pi_t T) \\ &\xleftarrow{1\otimes h^*} H_r(\tilde{E}_r/\tilde{E}_{r-1}) \otimes \operatorname{Hom}(H_{-s}(E_{-s}/E_{-s-1}), \pi_t T).\end{aligned}$$

Here h is the Hurewitz homomorphism.

Finally, one can show that under the identification 4.4.3, $\pi_*(\partial')$ and $\pi_*(\partial'')$ correspond to the differentials in the algebraic double complex. □

The pairing (4.4.1) induces a pairing $\bar{X}(T) \wedge \bar{X}(T') \to \bar{X}(T \wedge T')$, and since the equivalences $E_+ \to E_+ \wedge E_+$ and $\tilde{E} \wedge \tilde{E} \to \tilde{E}$ were chosen cellular, this pairing preserves the filtration by the subspectra $\{\bar{X}_n\}$. Accordingly, the product maps (4.3.5) give rise to a pairing of spectral sequences.

Proposition 4.4.4. *Let T and T' be two G-spectra indexed on $\mathcal{U}$. Then the pairing of Tate spectra* (4.4.1) *induces a pairing of the associated spectral sequences. On E^2-terms, this pairing corresponds to the pairing on Tate cohomology*

$$H^*(G; \pi_*T) \otimes H^*(G; \pi_*T') \to H^*(G; \pi_*(T \wedge T'))$$

under the isomorphism of lemma 4.4.2. In particular, if T is a G-homotopy associative G-ring spectrum, then $E^2 \cong \hat{H}^*(G; \pi_*T)$ as a bi-graded ring.

Proof. The equivalences $E_+ \to E_+ \wedge E_+$ and $\tilde{E} \wedge \tilde{E} \to \tilde{E}$ induces chain maps $P \to P \otimes P$ and $\tilde{P} \otimes \tilde{P} \to \tilde{P}$ which lifts the canonical maps $k \to k \otimes k$ and $k \otimes k \to k$, respectively. Now suppose T and T' are two G-spectra indexed on $\mathcal{U}$ and consider the spectral sequences corresponding to the filtrations $\{(\bar{X}(T) \wedge \bar{X}(T'))_n\}$ and $\{\bar{X}(T \wedge T')_n\}$. An argument analogous to the proof of the preceeding lemma identifies the E^1-terms of the associated spectral sequences with the complexes

$$(\tilde{P} \otimes \operatorname{Hom}(P, \pi_*T) \otimes \tilde{P} \otimes \operatorname{Hom}(P, \pi_*T'))^G$$

and

$$(\tilde{P} \otimes \operatorname{Hom}(P, \pi_*(T \wedge T')))^G,$$

respectively. We claim that under these identifications, the pairing

$$\bar{X}(T) \wedge \bar{X}(T') \to \bar{X}(T \wedge T')$$

corresponds to the composition

$$\begin{aligned}&(\tilde{P} \otimes \operatorname{Hom}(P, \pi_*T)^G \otimes \tilde{P} \otimes \operatorname{Hom}(P, \pi_*T'))^G \\ &\quad \to (\tilde{P} \otimes \tilde{P} \otimes \operatorname{Hom}(P \otimes P, \pi_*T \otimes \pi_*T'))^G \to (\tilde{P} \otimes \operatorname{Hom}(P, \pi_*(T \otimes T')))^G,\end{aligned}$$

where the first map is canonical map of chain complexes (which involves sign changes) and the second map is induced from the maps $P \to P \otimes P$ and $\tilde{P} \otimes \tilde{P} \to \tilde{P}$ and from the exterior product (4.3.1). This is straightforward to check. Similarly, under the isomorphism of lemma 4.4.2 and the analogous isomorphism above, the pairing (4.3.5) corresponds to the canonical map (no sign changes)

$$(\tilde{P} \otimes \operatorname{Hom}(P, \pi_* T))^G \otimes (\tilde{P} \otimes \operatorname{Hom}(P, \pi_* T'))^G$$
$$\to (\tilde{P} \otimes \operatorname{Hom}(P, \pi_* T) \otimes \tilde{P} \otimes \operatorname{Hom}(P, \pi_* T'))^G.$$

But this was our definition of the paring in Tate cohomology, see (4.1.5). □

Finally, we show that the spectral sequence considered here is canonically isomophic to the spectral sequence obtained from Greenlees' 'filtration' of $\tilde{E}$. In the G-stable category we have the Spanier-Whitehead duals $\tilde{E}_r = D(\tilde{E}_{-r})$, $r < 0$, which gives an upside-down sequence of maps which we splice together with the skeleton filtration together by means of the canonical maps

$$\tilde{E}_{-1} = D(\tilde{E}_1) \to D(S^0) \simeq S^0 = \tilde{E}_0$$

to obtain the $\mathbb{Z}$-graded 'filtration' of $\tilde{E}$,

(4.4.5) $$\cdots \to \tilde{E}_{s-1} \to \tilde{E}_s \tilde{E}_{s+1} \to \dots$$

This gives rise to a complete resolution $\epsilon \colon \hat{P} \to k$ as follows. As a complex

$$(\Sigma\hat{P})_s = H_s(\tilde{E}_s \cup C\tilde{E}_{s-1})$$

with differential

$$H_s(\tilde{E}_s \cup C\tilde{E}_{s-1}) \to H_s(\Sigma\tilde{E}_{s-1}) \xleftarrow{\Sigma} H_{s-1}(\tilde{E}_{s-1}) \to H_{s-1}(E_{s-1} \cup C\tilde{E}_{s-2}),$$

and the structure map $\epsilon \colon \hat{P} \to k$ is given by the composite

$$\hat{P}_0 = H_1(\tilde{E}_1 \cup C\tilde{E}_0) \to H_1(\Sigma E_0) \xleftarrow{\Sigma} H_0(E_0) = k.$$

The map of triangles

$$\begin{tikzcd}
P \arrow[r, "\epsilon"] \arrow[d, equal] & k \arrow[r] \arrow[d, "\eta"] & \tilde{P} \arrow[r] \arrow[d] & \Sigma P \arrow[d, equal] \\
P \arrow[r] & \Sigma P^- \arrow[r] & \Sigma\hat{P} \arrow[r] & \Sigma P
\end{tikzcd}$$

defines a quasi-isomorphism of the mapping cones of the two middle vertical maps.

In the definitions of the spectra $\bar{X}_{r,s}$ and $\bar{X}_n$, we may allow r to vary over all integers. Then let

$$\bar{X}'_{r,s} = \operatorname*{holim}_{\longrightarrow} X_{r,s}, \qquad \bar{X}'_n = \bigcup_{r+s=n} \bar{X}'_{r,s},$$

where the limit is over all $r' \leq r$ and $s' \leq s \leq 0$, and where r is allowed to take negative values. For non-negative values of r, the natural inclusion $\bar{X}_{r,s} \xrightarrow{\sim} \bar{X}'_{r,s}$ is a weak equivalence. We have maps of filtrations

$$\{\bar{X}_n\}_{n\in\mathbb{Z}} \to \{\bar{X}'_n\}_{n\in\mathbb{Z}} \leftarrow \{\bar{X}'_{r,0}\}_{r\in\mathbb{Z}},$$

where the filtration on the right is Greenlees' filtration. The proof of lemma 4.4.2 extends verbatim to show that the induced maps of E^1-terms of the associated spectral sequences are

$$(\tilde{P} \otimes \operatorname{Hom}(P, M))^G \to (\Sigma\hat{P} \otimes \operatorname{Hom}(P, M))^G \leftarrow (\Sigma\hat{P} \otimes \operatorname{Hom}(k, M))^G.$$

Both maps are quasi-isomorphisms by an argument similar to the proof of lemma 4.1.4. Hence the maps of spectral sequences in question are isomorphisms from E^2 on. So the spectral sequence considered here is canonically isomorphic to the spectral sequence obtained from Greenlees' filtration, [**9**], [**10**].

4.5. Let again C be a cyclic group of order r and let g be a generator. As our model for E, we choose

$$E = S(\mathbb{C}^\infty),$$

where the generator g acts on $\mathbb{C}$ by multiplication by $e^{2\pi i/r}$. We give E the usual C-CW-structure with one free cell in each dimension. The skeleta are

$$E_n = \begin{cases} S(\mathbb{C}^d) & n = 2d-1 \text{ odd} \\ S(\mathbb{C}^d) * (C \cdot 1) & n = 2d \text{ even}, \end{cases} \tag{4.5.1}$$

where in the latter case, we identify the join with its image under the canonical homeomorphism $S(\mathbb{C}^n) * S(\mathbb{C}) \cong S(\mathbb{C}^n \oplus \mathbb{C})$. The attaching maps

$$\alpha_n \colon D^n \times C \to E_n$$

are defined in even dimensions by the composite

$$D^{2d} \times C \xrightarrow{\xi} D(\mathbb{C}^d) \times C \xrightarrow{\pi} S(\mathbb{C}^d) * (C \cdot 1),$$

where ξ maps $(z, g^s) \mapsto (g^s \cdot z, g^s)$ and π is the canonical projection. We define

$$\alpha_1(x, g^s) = g^s \cdot e^{\pi i(x+1)/r}$$

and let α_{2d+1} be the composite

$$D^{2d} \times D^1 \times C \xrightarrow{\xi} D(\mathbb{C}^d) \times D^1 \times C \xrightarrow{1\times\alpha_1} D(\mathbb{C}^d) \times S(\mathbb{C}) \xrightarrow{\pi} S(\mathbb{C}^d) * S(\mathbb{C}).$$

We give $D(\mathbb{C}^d)$ the complex orientation and $D^1 = D(\mathbb{R}) = [-1, 1]$ the standard orientation from -1 to 1. We may then identify the cellular complex of E with the standard complex W by the isomorphism

$$W \xrightarrow{\sim} C_*(E) \tag{4.5.2}$$

which maps the generator $x_n \in W_n$ to the image of the fundamental class under the composite

$$H_n(D^n, S^{n-1}) \xrightarrow{\iota_0} H_n(D^n \times C, S^{n-1} \times C) \xrightarrow{\alpha_n} H_n(E_n, E_{n-1}).$$

Here $\iota_0 \colon D^n \to D^n \times C$ maps $z \to (z, 1)$.

The C-CW-structure on E induces one on $\tilde{E}$ and the isomorphism (4.5.2) induces an isomorphism of chain complexes

$$\tilde{W} \xrightarrow{\sim} C_*(\tilde{E}). \tag{4.5.3}$$

We define a homeomorphism

$$\tilde{E} \xrightarrow{\sim} S^{\mathbb{C}^\infty} \tag{4.5.4}$$

by the map

$$CS(\mathbb{C}^\infty)_+ \cup S^0 \to D(\mathbb{C}^\infty)/S(\mathbb{C}^\infty)$$

which sends $t \wedge z \mapsto tz$. Note that under this homeomorphism, the orientation of the cells in $\tilde{E}$ corresponds to the complex orientation of $S^{\mathbb{C}^\infty}$. In particular, the composite

$$H_2(S^{\mathbb{C}}) \xleftarrow{\sim} H_2(\tilde{E}_2) \xrightarrow{\mathrm{pr}_*} H_2(\tilde{E}_2, \tilde{E}_1) \xleftarrow{\sim} \tilde{W}_2$$

maps the fundamental class $[S^{\mathbb{C}}]$ to the class Ny_2.

We consider the B-operator. Let $\mathbb{T}$ be the space $S(\mathbb{C})$ of complex numbers of length one considered as a Lie group and identify $C \subset \mathbb{T}$ with the subgroup of rth roots of unity. Our model for EC is then also a model for $E\mathbb{T}$, and moreover, the action

$$\mu \colon \mathbb{T} \times E \to E$$

is C-cellular, when we give $\mathbb{T}$ the C-CW-structure of $S(\mathbb{C}) = E_1$. The induced action on $\tilde{E}$, we remember, is given by the composite

$$\mathbb{T}_+ \wedge \tilde{E} = \mathbb{T}_+ \wedge C_{\mathrm{pr}} \xrightarrow{\rho} C_{\mathbb{T}_+ \wedge \mathrm{pr}} \xrightarrow{C_\mu} C_{\mathrm{pr}} = \tilde{E},$$

where $\mathrm{pr} \colon E_+ \to S^0$ is the projection. The cellular complex of $\mathbb{T}$ is a differential graded Hopf algebra $\Lambda = \mathbb{Z}C \otimes \Lambda\{B\}$. The differential maps B to $(g-1)\cdot 1$, B is primitive, the coproduct on $g \in C$ is $g \otimes g$, and the antipode is $c(B) = -B$. The maps induced from actions

$$\Lambda \otimes W \to W, \qquad \Lambda \otimes \tilde{W} \to \tilde{W}, \tag{4.5.5}$$

are given by

$$B \cdot x_s = \begin{cases} x_{s+1} & s \text{ even} \\ 0 & s \text{ odd} \end{cases} \qquad B \cdot y_s = \begin{cases} 0 & s \text{ even} \\ -y_{s+1} & s \text{ odd} \end{cases}$$

For any $\mathbb{T}$-space X, let $|X|$ denote the underlying non-equivariant space. The C-CW-filtration of $\mathbb{T}$ and double filtration of $(\tilde{E} \wedge F(E_+, T))^C$ gives rise a triple filtration of the smash product. If we turn these into single filtrations, as we did earlier, then the action is a filtration preserving map

$$|\mathbb{T}|_+ \wedge i^*(\tilde{E} \wedge F(E_+, T)) \to i^*(\tilde{E} \wedge F(E_+, T)),$$

which is $\mathbb{T}$-equivariant because $\mathbb{T}$ is commutative. In particular, it restricts to a filtration preserving map of C-fixed sets, and this in turn induces a filtration preserving map

$$\mathbb{T}/C_+ \wedge i^*(E_+ \wedge T)^C \to i^*(E_+ \wedge T)^C.$$

We evaluate the map of the spectral sequences associated with these filtrations. Under the canonical identifications, the map of E^1-terms induced from the action, is then given by

$$\Lambda_C \otimes (\tilde{W} \otimes \mathrm{Hom}(W, \pi_* T))^C \to (\tilde{W} \otimes \mathrm{Hom}(W, \pi_* T))^C,$$

where $\Lambda_C = \Lambda\{B\}$ and $\pi_* T$ is a trivial Λ-module. When T is a $\mathbb{T}$-ring spectrum, the class $B \otimes Nx_0^*$ is an infinite cycle in the spectral sequence on the left. Hence the spectral sequence on the right becomes a spectral sequence of Λ_C-algebras. The class $B \in H_1(\mathbb{T}_+)$ is the Hurewicz image of $\sigma \in \pi_1^S(\mathbb{T}_+)$, and exterior multiplication by σ composed with the action

$$\mathbb{T}_+ \wedge \hat{\mathbb{H}}(C, T) \to \hat{\mathbb{H}}(C, T)$$

induces the differential $d \colon \pi_* \hat{\mathbb{H}}(C, T) \to \pi_{*+1} \hat{\mathbb{H}}(C, T)$.

Proposition 4.5.6. *Let T be a $\mathbb{T}$-ring spectrum. Then the spectral sequence*

$$E^2 = \hat{H}^*(C; \pi_* T) \Rightarrow \pi_* \hat{\mathbb{H}}(C; T)$$

is a spectral sequence of Λ_C-algebras. If $a \in \pi_* \hat{\mathbb{H}}(C;T)$ is represented by an infinite cycle $z \in E^1_{s,t}$ such that $B \cdot z \in E^1_{s,t}$ is non-zero, then $B \cdot z$ is an infinite cycle and represents the class of $Ba \in \pi_* \hat{\mathbb{H}}(C;T)$. $\square$

Let k be a perfect field of characteristic $p > 0$ and let $T(k)$ be the topological Hochschild spectrum of k. Then

$$\bar{\pi}_* T(k) = \Lambda_k\{\epsilon\} \otimes S_k\{\sigma\},$$

where the classes $\epsilon \in \bar{\pi}_1 T(k)$ and $\sigma \in \bar{\pi}_2 T(k)$ are characterized by $\beta\epsilon = 1$ and $d\epsilon = \sigma$. Here, we remember, $\bar{\pi}_* = \pi_*(-;\mathbb{Z}/p\mathbb{Z})$.

Corollary 4.5.7. *The image of the classes ϵ and σ under the map induced from*

$$\hat{\Gamma}\colon T(k) \to \hat{\mathbb{H}}(C_p; T(k))$$

are represented by the infinite cycles $ut^{-1} \otimes 1 \in E^2_{1,0}$ and $t^{-1} \otimes 1 \in E^2_{2,0}$, respectively.

Proof. Recall from section 1.1 that $\hat{\Gamma}$ is defined as the composite

$$T(k) \underset{\sim}{\xleftarrow{r}} \rho^*_{C_p}(\tilde{E} \wedge T)^{C_p} \to \rho^*_{C_p}(\tilde{E} \wedge F(E_+, T))^{C_p}.$$

Both maps are $\mathbb{T}$-equivariant, so $\hat{\Gamma}$ commutes with the B-operator. It of course also commutes with the Bockstein operator. Now

$$B \cdot (-Ny_1 \otimes Nx_0^*) = -N(B \cdot y_1) \otimes Nx_0^* + Ny_1 \otimes N(B \cdot x_0^*) = Ny_2 \otimes Nx_0^*,$$

so by the proposition, $d(ut^{-1}) = t^{-1}$. Since also $\beta(ut^{-1}) = 1$, we are done. $\square$

Proposition 4.5.8. *Let T be a $\mathbb{T}$-spectrum and suppose the order of C is divisible by p. Then the d^2-differential in the Tate spectral sequence*

$$\hat{E}^2(C,T) = \hat{H}^*(C;\mathbb{F}_p) \otimes \pi_*(T,\mathbb{Z}/p) \Rightarrow \pi_*(\hat{\mathbb{H}}(C,T),\mathbb{Z}/p)$$

is given by

$$d^2(\gamma \otimes \tau) = \gamma t \otimes d\tau.$$

Here t is the generator of $\hat{H}^2(C,\mathbb{F}_p)$ from lemma 4.2.1, and $d\colon \pi_*(T,\mathbb{Z}/p) \to \pi_{*+1}(T,\mathbb{Z}/p)$ is the B-operator.

Proof. We consider the $\mathbb{T}$-Tate spectrum

$$\hat{\mathbb{H}}(\mathbb{T},T) = (\tilde{E} \wedge F(E_+,T))^{\mathbb{T}},$$

where again $E = S(\mathbb{C}^\infty)$. There is a spectral sequence

$$\hat{E}^2(\mathbb{T},T) = S\{t^{\pm 1}\} \otimes \pi_*(T,\mathbb{Z}/p) \Rightarrow \pi_*(\hat{\mathbb{H}}(\mathbb{T},T),\mathbb{Z}/p),$$

where t has bi-degree $(-2,0)$, and it was proved in [**14**, lemma 1.4.2] that the d^2-differential in this spectral sequence is given the formula of the statement. There is a natural map of spectral sequences

$$\hat{E}^*(\mathbb{T},T) \to \hat{E}^*(C,T),$$

which on E^2-terms is given by the obvious inclusion. Now every C-spectrum T is a module C-spectrum over the sphere C-spectrum $\mathbb{S}_C$, and it will therefore be enough to know that the class $u_1 \otimes 1$ is a d^2-cycle in the spectral sequence $E^*(C,\mathbb{S}_C)$. But $\pi_1(\mathbb{S}_C,\mathbb{Z}/p)$ vanishes for p odd, and hence $d^2(u_1 \otimes 1)$ is zero for degree reasons. $\square$

5. The Tate spectral sequence for $T(A|K)$

5.1. Let L be a finite and totally ramified extension K, and let B be the integral closure of A in L. Then B is a complete discrete valuation ring with fraction field L and residue field k. Let π_K and π_L be uniformizers of A and B, respectively. Then the minimal polynomial of π_L over K has the form

$$\phi_{L/K}(\pi_L) = \pi_L^{e_{L/K}} + \pi_K \theta_{L/K}(\pi_L),$$

where $\theta_{L/K}(x)$ is a polynomial over A of degree $< e_{L/K}$ and $\theta_{L/K}(0) \in A^\times$. Moreover, the canonical map

$$A[\pi_L]/(\phi_L(\pi_{L/K})) \xrightarrow{\sim} B$$

is an isomorphism. When $K = K_0$ is the fraction field of $W(k)$, we will always use $\pi_{K_0} = p$ and write write $\theta_L(\pi_L)$ instead of $\theta_{L/K_0}(\pi_L)$.

LEMMA 5.1.1. *Suppose that $\mu_p \subset K$. Then a choice of uniformizer $\pi_K \in A$ and of a generator $\zeta \in \mu_p$ determines a unique polynomial $u_K(x) \in A[x]$ of degree $< e_K$ such that*

$$u(\pi_K)^{p-1} = \theta_K(\pi_K).$$

Moreover in $\omega_{(A,M)}$,

$$d\log\zeta = -\pi_K^{e/(p-1)} u(\pi_K)^{-1} d\log p.$$

PROOF. Let $f, g \in \mathbb{Z}_p[\![x]\!]$ be the power series given by

$$f(x) = px + x^p,$$
$$g(x) = (1+x)^p - 1.$$

Then there exists a *unique* power series $\varphi(x) \in \mathbb{Z}_p[\![x]\!]$ such that

$$f(\varphi(x)) = \varphi(g(x)),$$
$$\varphi(x) \equiv x \mod (x^2),$$

see e.g. [**31**, §3, proposition 5]. If $\zeta \in \mu_p$ is a generator then $\varphi(\zeta - 1) \in A$ is a $(p-1)$st root of

$$-p = \pi_K^e \theta_K(\pi_K)^{-1},$$

and we then define $u_K(x)$ to be the unique polynomial of degree $< e_K$ such that

$$\varphi(\zeta - 1) = \pi^{e/(p-1)} u_K(\pi_K)^{-1}.$$

To prove the second statement, note that

$$\begin{aligned} d\varphi(\zeta-1) &= \varphi(\zeta-1) d\log\varphi(\zeta-1) \\ &= \pi_K^{e/(p-1)} u_K(\pi_K)^{-1} \cdot (p-1)^{-1} d\log(-p) \\ &= -\pi_K^{e/(p-1)} u_K(\pi_K)^{-1} d\log p. \end{aligned}$$

Here we have used that $d\log(-p) = d\log p$ and that the common class is p-torsion. It thus suffices to show that

$$d\varphi(\zeta - 1) = d\log\zeta.$$

By naturality, we may suppose that

$$K = \mathbb{Q}_p(\mu_p) = \mathbb{Q}_p((-p)^{1/(p-1)}),$$

where as a uniformizer, we may take $\pi_K = \zeta - 1$. In this case, $\omega_{(A,M)}$ is annihilated by π_K^{p-1}, and since

$$d\varphi(\zeta - 1) = \varphi'(\zeta - 1)\zeta d\log\zeta,$$

it will suffice to show that

$$\varphi'(x) \equiv \frac{1}{1+x} \mod (x^{p-1}),$$

or equivalently,

$$\varphi(x) \equiv \log(1+x) \mod (x^p).$$

But this follows from the uniqueness of $\varphi(x)$ and from the calculation

$$\log(1 + g(x)) = \log((1+x)^p) = p\log(1+x) = f(\log(1+x)),$$

which takes place in $\mathbb{Z}_p[x]/(x^p)$. $\square$

ADDENDUM 5.1.2. *Let L/K be a finite and totally ramified extension. Then the inclusion of valuation rings, $i\colon A \to B$, maps*

$$i(u_K(\pi_K)) = (-\theta_{L/K}(\pi_L))^{-e_K/(p-1)}u_L(\pi_L).$$

PROOF. Since $i(\varphi(\zeta - 1)) = \varphi(\zeta - 1)$, the definition of $u_K(\pi_K)$ and $u_L(\pi_L)$ gives

$$i_*(\pi_K^{e_K/(p-1)}u_K(\pi_K)^{-1}) = \pi_L^{e_L/(p-1)}u_L(\pi_L)^{-1}.$$

On the other hand,

$$i_*(\pi_K^{e_K/(p-1)}u_K(\pi_K)^{-1}) = (-\theta_{L/K}(\pi_L)^{-1}\pi_L^{e_{L/K}})^{e_K/(p-1)}i_*(u_K(\pi_K)^{-1}),$$

and the stated formula now follows since $e_{L/K}e_K = e_L$ and since $\pi_L^{e_L/(p-1)}$ is a non-zero-divisor in B. $\square$

5.2. We recall the Cartier operator. If k is a ring, if R is a k-algebra, and if $k \to k'$ is a ring homomorphism, the base change of R along $k \to k'$ is the tensor product $R' = k' \otimes_k R$ viewed as a k'-algebra by multiplication in the first tensor factor. In this situation, the canonical map

$$R' \otimes_R \Omega^*_{R/k} \xrightarrow{\sim} \Omega^*_{R'/k'} \tag{5.2.1}$$

is an isomorphism, [**25**, p. 198].

If k is a ring of characteristic $p > 0$, we consider the base change of R along the Frobenius $\varphi\colon k \to k$. This is again a k-algebra, which we denote $R^{(1)}$. The canonical map

$$W\colon R \to R^{(1)}, \quad W(a) = 1 \otimes a,$$

is a φ-linear ring homomorphism. The relative Frobenius of R is the k-algebra homomorphism

$$F_{R/k}\colon R^{(1)} \to R, \quad F_{R/k}(x \otimes a) = xa^p.$$

The absolute Frobenius on R, given by $F_R(a) = a^p$, now factors as the composite

$$R \xrightarrow{W} R^{(1)} \xrightarrow{F_{R/k}} R.$$

If we write $R = k[x_\alpha]/(f_\beta(x_\alpha))$, then $R^{(1)} = k[x_\alpha]/(f_\beta^{(1)}(x_\alpha))$, where $f_\beta^{(1)}(x_\alpha)$ is the Frobenius twist

$$f_\beta^{(1)}(x_\alpha) = \varphi_k[x_\alpha](f_\beta(x_\alpha)),$$

the map W is induced from $\varphi_k[x_\alpha]$, and the relative Frobenius from the k-algebra map which sends x_α to x_α^p.

In general, the de Rham complex $\Omega^*_{R/k}$ is a graded R-algebra with a k-linear derivation. But when k is of characteristic $p > 0$, we may view $\Omega^*_{R/k}$ as a differential graded $R^{(1)}$-algebra via the relative Frobenius $F_{R/k}\colon R^{(1)} \to R$, and hence, the cohomology ring $H^*(\Omega^*_{R/k})$ is naturally an $R^{(1)}$-algebra. Let

$$C_R^{-1}\colon \Omega^*_{R/k} \to W^* H^*(\Omega^*_{R/k})$$

be the map of graded R-algebras given by

$$C_R^{-1}(a) = a^p, \qquad C_R^{-1}(da) = a^{p-1}da + d(R).$$

This map is well-defined since

$$a^{p-1}da + b^{p-1}db - (a+b)^{p-1}d(a+b) = d\Big(\frac{a^p + b^p - (a+b)^p}{p}\Big).$$

The map C_R^{-1} adjoins to a map of graded $R^{(1)}$-algebras

$$R^{(1)} \otimes_R \Omega^*_{R/k} \to H^*(\Omega_{R/k})$$

which composed with the inverse of the canonical isomorphism (5.2.1) yields the (relative) inverse Cartier operator

$$C_{R/k}^{-1}\colon \Omega^*_{R^{(1)}/k} \to H^*(\Omega^*_{R/k}).$$

In degree zero, $C_{R/k}^{-1}$ is induced by the relative Frobenius $F_{R/k}$, and in degree one has

$$C_{R/k}^{-1}(W_*(da)) = a^{p-1}da + d(R).$$

The map $C_{R/k}^{-1}$ is an isomorphism if R/k is smooth. Indeed, since the statement is étale local, we may assume that $R = k[x_1, \dots, x_n]$ is a polynomial algebra, and it is easy in this case to evaluate both sides, compare [**20**, 7.2]. The inverse of $C_{R/k}^{-1}$ is called the (relative) Cartier operator and denoted $C_{R/k}$. It satisfies that for $u \in R^\times$,

$$C_{R/k}(u^{-1}du) = W_*(u^{-1}du).$$

Indeed,

$$\begin{aligned} C_{R/k}(u^{-1}du) &= C_{R/k}(u^{-p}u^{p-1}du) = C_{R/k}(u^{-p})C_{R/k}(u^{p-1}du) \\ &= W(u^{-1})W_*(du) = W_*(u^{-1}du). \end{aligned}$$

Lemma 5.2.2. *Suppose $\theta(x) \in k[\![x]\!]^\times$ and write $\theta'(x)/\theta(x) = a_0 + a_1 x + a_2 x^2 + \dots$. Then $a_{pi-1} = a_{i-1}^p$.*

Proof. We may assume that $\theta(x)$ is a polynomial with $\theta(0) \in k^\times$. The algebra $R = k[x][\theta(x)^{-1}]$ is smooth over k and in $\Omega_{R/k}$,

$$C_{R/k}(\theta(x)^{-1}d\theta(x)) = W_*(\theta(x)^{-1}d\theta(x)).$$

Hence the images of the two differentials in $\Omega_{k[\![x]\!]/k}$ also agree. This is the statement of the lemma. □

5.3. Let $\pi_K \in A$ be a uniformizer. Then as a differential graded k-algebra

$$\bar{\pi}_* T(A|K) \cong S\{\kappa, \pi_K\}/(\pi_K^{e_K}) \otimes \Lambda\{d \log \pi_K\},$$

where

$$\begin{aligned} d\pi_K &= \pi_K d \log \pi_K \\ d\kappa &= \kappa d \log p = (e_K d \log \pi_K - d \log \theta_K(\pi_K)). \end{aligned}$$

We now suppose that $\mu_p \subset K$ and choose a generator $\zeta \in \mu_p$. Let $u_K(x)$ be the polynomial given by lemma 5.1.1, and define $\alpha_K \in \bar{\pi}_2 T(A|K)$ by

$$\alpha_K = u_K(\pi_K)^{-1} \kappa.$$

Then

$$\bar{\pi}_* T(A|K) = S\{\alpha_K, \pi_K\}/(\pi_K^{e_K}) \otimes \Lambda\{d \log \pi_K\},$$

and

$$\begin{aligned} d\pi_K &= \pi_K d \log \pi_K \\ d\alpha_K &= e_K \alpha_K d \log \pi_K. \end{aligned}$$

Indeed,

$$\begin{aligned} d(\alpha_K) &= -u_K(\pi_K)^{-1} d \log u_K(\pi_K) \cdot \kappa + u_K(\pi_K)^{-1} \cdot \kappa d \log p \\ &= -\alpha_K d \log u_K(\pi_K) + \alpha_K (e_K d \log \pi_K - (p-1) d \log u_K(\pi_K)) \\ &= e_K \alpha_K d \log \pi_K. \end{aligned}$$

The Bockstein homomorphism

$$\beta_1 \colon \bar{\pi}_2 T(A|K) \to \pi_1 T(A|K),$$

we remember, is injective, so we can define the Bott element $b \in \bar{\pi}_2 T(A|K)$ by the requirement that $\beta_1(b) = d \log \zeta$, where $\zeta \in \mu_p$ is the chosen generator. Then

(5.3.1) $$b = -\pi_K^{e_K/(p-1)} \alpha_K.$$

Indeed, by lemma 5.1.1,

$$\beta_1(b) = d \log \zeta = -\pi_K^{e_K/(p-1)} u_K(\pi_K)^{-1} d \log p = \beta_1(-\pi_K^{e_K/(p-1)} \alpha_K).$$

Let L/K be a finite and totally ramified extension, and let $i \colon A \to B$ be the inclusion of valuation rings. Then the map

(5.3.2) $$i_* \colon \bar{\pi}_* T(A|K) \to \bar{\pi}_* T(B|L)$$

is given by

$$\begin{aligned} i_*(\pi_K) &= -\theta_{L/K}(\pi_L)^{-1} \pi_L^{e_{L/K}}, \\ i_*(d \log \pi_K) &= e_{L/K} d \log \pi_K - d \log \theta_{L/K}(\pi_L), \\ i_*(\alpha_K) &= (-\theta_{L/K}(\pi_L))^{e_K/(p-1)} \alpha_L. \end{aligned}$$

Indeed, the first two equalities follows immediately from the definition of $\theta_{L/K}(\pi_L)$, and the last equality follows form addendum 5.1.2.

5.4. Suppose that $v_p(e_K) > 0$ such that we may identify

$$j_*\colon \bar{\pi}_* T(A) \to \bar{\pi}_* T(A|K)$$

with the canonical map

$$\Omega_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} S_{\mathbb{F}_p}\{\kappa\} \to \omega_{(A,M)} \otimes_{\mathbb{Z}} S_{\mathbb{F}_p}\{\kappa\},$$

where the class κ is determined by the requirement that the integral Bockstein takes the value

$$\beta(\kappa) = d\log p = \Big(-\frac{e_K \pi_K^{e_K-1}}{p\theta_K(\pi_K)} - \frac{\theta_K'(\pi_K)}{\theta_K(\pi_K)}\Big)d\pi_K.$$

Since the differential $d(\theta_K(\pi_K)\kappa)$ must vanish, we see that in $\bar{\pi}_* T(A)$,

$$d\kappa = \kappa \cdot \Big(-\frac{\theta_K'(\pi_K)}{\theta_K(\pi_K)} d\pi_K\Big),$$

which in general is different from $\kappa d\log p$.

The linearization map

$$L\colon \bar{\pi}_* T(A) \to \bar{\pi}_* \operatorname{HH}(A)$$

may be identified with the canonical map

$$\Omega_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} S_{\mathbb{F}_p}\{\kappa\} \to \Omega_{A/\mathbb{Z}} \otimes_{\mathbb{Z}} \Gamma_{\mathbb{F}_p}\{\kappa\}.$$

To see this, we recall the calculation of $\bar{\pi}_* \operatorname{HH}(A)$ from [**11**]. Here, $\operatorname{HH}_*(A/W)$ is calculated as the homology of the differential graded W-algebra

$$C_*(A/W) = A \otimes_W \Lambda_W\{d\pi_K\} \otimes_W \Gamma_W\{c_K\}$$

with the differential given by $b(\gamma_s(c_K)) = \gamma_{s-1}(c_K)\phi_K'(\pi_K)d\pi_K$ and $b(d\pi_K) = 0$. Hence for $v_p(e_K) > 0$,

$$\bar{\pi}_* \operatorname{HH}(A) = A/pA \otimes \Lambda\{d\pi_K\} \otimes \Gamma\{c_K\},$$

and the Bockstein $\beta\colon \bar{\pi}_* \operatorname{HH}(A) \to \operatorname{HH}_{*-1}(A/W)$ maps

$$\beta(c_K) = \frac{\phi_K'(\pi_K)}{p} d\pi_K = \Big(\frac{e_K}{p}\pi_K^{e_K-1} + \theta_K'(\pi_K)\Big)d\pi_K.$$

This shows that

$$\kappa = -\theta_K(\pi_K)^{-1} c_K.$$

We proceed to evaluate the map induced from the reduction

$$\rho_*\colon \bar{\pi}_* T(A) \to \bar{\pi}_* T(A/pA),$$

or equivalently, to evaluate the class $\rho_*(\kappa)$. The equivalence

$$T(k) \wedge N^{\mathrm{cy}}(\Pi_e) \xrightarrow{\sim} T(A/pA)$$

gives rise to a canonical isomorphism of graded $\pi_* T(k)$-algebras

$$\pi_* T(k) \otimes \pi_* \operatorname{HH}(A/pA) \xrightarrow{\sim} \pi_* T(A/pA)$$

whose composition with the linearization map

$$L\colon \pi_* T(A/pA) \to \pi_* \operatorname{HH}(A/pA)$$

is equal to the map induced from the augmentation $\pi_* T(k) \to k$.

For any $H\mathbb{F}_p$-module spectrum X, we have a canonical isomorphism

$$\pi_* X \otimes \Lambda\{\epsilon\} \xrightarrow{\sim} \bar{\pi}_* X,$$

where $\epsilon \in \bar{\pi}_1 H\mathbb{F}_p$ is the unique class with $\beta(\epsilon) = 1$. We thus have a sum diagram

$$\begin{tikzcd} \pi_* X \arrow[r, "i"] & \bar{\pi}_* X \arrow[l, bend right, "r"'] \arrow[r, "\beta"] & \pi_{*-1} X \arrow[l, bend right, "s"'] \end{tikzcd}$$

where β is the integral Bockstein. The section s is given by multiplication by ϵ, and r is the induced retraction of the inclusion i. This applies in particular to $T(A/pA)$ and $\mathrm{HH}(A/pA)$.

It follows that we have a canonical isomorphism

$$\pi_* T(k) \otimes \pi_* \mathrm{HH}(A/pA) \otimes \Lambda\{\epsilon\} \xrightarrow{\sim} \bar{\pi}_* T(A/pA),$$

and that under this isomorphism

$$\rho_*(\kappa) = x \otimes 1 \otimes 1 + 1 \otimes y \otimes 1 - 1 \otimes z \otimes \epsilon,$$

where x, y and z are the images of κ under the composites

$$\begin{aligned} &\bar{\pi}_* T(A) \to \bar{\pi}_* T(k) \xrightarrow{r} \pi_* T(k), \\ &\bar{\pi}_* T(A) \xrightarrow{L} \bar{\pi}_* \mathrm{HH}(A) \xrightarrow{\rho_*} \bar{\pi}_* \mathrm{HH}(A/pA) \xrightarrow{r} \pi_* \mathrm{HH}(A/pA), \\ &\bar{\pi}_* T(A) \xrightarrow{\beta} \pi_* T(A) \xrightarrow{\rho_*} \pi_* T(A/pA) \xrightarrow{L} \pi_* \mathrm{HH}(A/pA), \end{aligned}$$

respectively. By what was said above, we have

$$\begin{aligned} y &= -\theta_K(\pi_K)^{-1} c_K, \\ z &= -\Big(\frac{e_K \pi_K^{e_K - 1}}{p\theta_K(\pi_K)} + \frac{\theta_K'(\pi_K)}{\theta_K(\pi_K)}\Big) d\pi_K, \end{aligned}$$

and we shall need to know that $x = \sigma$. This is equivalent to the statement that in the spectral sequence used in [**22**],

$$E^2 = \pi_* T(A, \mathrm{Tor}_*^A(A/pA, A/pA)) \Rightarrow \pi_* T(A, A/pA),$$

the element $\sigma - \theta_K(\pi_K)^{-1} c_K$ is a cycle.

LEMMA 5.4.1. *The reduction $\bar{\pi}_* T(A) \to \bar{\pi}_* T(k)$ maps κ to σ.*

PROOF. It follows from addendum 3.3.9 that the top horizontal map in the diagram

$$\begin{tikzcd} \bar{\pi}_2 T(A) \arrow[r, "\partial_A"] \arrow[d] & \bar{\pi}_1 T(A)_{hC_p} \arrow[d] \\ \bar{\pi}_2 T(k) \arrow[r, "\partial_k"] & \bar{\pi}_1 T(k)_{hC_p} \end{tikzcd}$$

maps κ to $dV(1) - V(d\log p)$. This class, in turn, is mapped to $dV(1)$ by the reduction, and it thus remains to show that $\partial_k(\sigma) = dV(1)$. To this end, we consider the diagram

$$\begin{tikzcd} \bar{\pi}_1 T(k) \arrow[r, "\partial_k"] \arrow[d, "d"] \arrow[dr, phantom, "(-1)"] & \bar{\pi}_0 T(k)_{hC_p} \arrow[d, "d"] \\ \bar{\pi}_2 T(k) \arrow[r, "\partial_k"] & \bar{\pi}_1 T(k)_{hC_p}, \end{tikzcd}$$

which commutes up to a sign. An argument similar to the proof of addendum 3.3.9 shows that $\partial_k(\epsilon) = -V(1)$. But $\sigma = d\epsilon$, and hence $\partial_k(\sigma) = dV(1)$. $\square$

The differential structure on $\bar{\pi}_* T(A/pA)$ is given by

$$d\epsilon = \sigma,$$
$$dc_K = \frac{e_K}{p}\pi_K^{e_K-1} d\pi_K \cdot \sigma,$$

and by the rule $d(\gamma_s(c_K)) = \gamma_{s-1}(c_K)dc_K$. It follows that the image of the element $t\theta_K(\pi_K)\kappa$ under the map of spectral sequences

$$\rho_* \colon \hat{E}^*(C_{p^n}, A) \to \hat{E}^*(C_{p^n}, A/pA)$$

is homologous to the element $-tc_K$. Indeed,

$$\rho_*(t\theta_K(\pi_K)\kappa) + tc_K = t\theta_K(\pi_K)\sigma + t\theta'_K(\pi_K)d\pi_K \cdot \epsilon = d^2(\theta_K(\pi_K)\epsilon).$$

PROPOSITION 5.4.2. *Suppose that* $v_p(e_K) > n$. *Then the image of* ${\underline{\pi_K}}_n^{e_K/p^n}$ *under the map*

$$\hat{\Gamma}_{A/pA} \colon \bar{\pi}_* T(A/pA) \to \bar{\pi}_* \hat{\mathbb{H}}(C_{p^n}, T(A/pA))$$

is represented in the spectral sequence $\hat{E}^*(C_{p^n}, A/pA)$ *by the cycle* $-tc_K$.

PROOF. We have a natural decomposition

$$\bigvee_{s \geq 0} T(A/pA, s) \xrightarrow{\sim} T(A/pA),$$

and the class $\hat{\Gamma}_{A/pA}({\underline{\pi_K}})_n^{e_K/p^n}$ lies in the summand $\hat{\mathbb{H}}(C_{p^n}, T(A/pA, e_K))$. It was proved in [**15**] that there is a cofibration sequence

$$T(k) \wedge S^1_+ \xrightarrow{\mathrm{pr}} T(k) \wedge S^1/C_{e+} \xrightarrow{i} T(A/pA, e_K) \xrightarrow{\partial} \Sigma T(k) \wedge S^1_+. \tag{5.4.3}$$

The homotopy groups modolu p are given by

$$\bar{\pi}_*(T(k) \wedge S^1_+) = \bar{\pi}_* T(k) \otimes k\langle \pi_K^e, d(\pi_K^{e_K})\rangle,$$
$$\bar{\pi}_*(T(k) \wedge S^1/C_{e+}) = \bar{\pi}_* T(k) \otimes k\langle \pi_K^e, \pi_K^{e_K-1} d\pi_K\rangle,$$
$$\bar{\pi}_*(T(A/pA, e_K) = \bar{\pi}_* T(k) \otimes k\langle \pi_K^{e_K-1} d\pi_K, c_K\rangle,$$

with the maps induced from the cofibration sequence being the obvious ones except that $\partial_*(c_K) = d(\pi_K^{e_K})$. To see this, recall that $c_K \in \tilde{H}_2(N^{\mathrm{cy}}(\Pi_{e_K}, e_K), \mathbb{Z}/p\mathbb{Z})$ is the unique class whose integral Bockstein is $(e_K/p)\pi_K^{e_K-1} d\pi_K$. But the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}\langle d(\pi_K^{e_K})\rangle \arrow[r] \arrow[d, "p"] & \mathbb{Z}\langle \pi_K^{e_K-1} d\pi_K\rangle \arrow[r] \arrow[d, "p"] & \mathbb{Z}/e_K\mathbb{Z}\langle \pi_K^{e_K-1} d\pi_K\rangle \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & \mathbb{Z}\langle d(\pi_K^{e_K})\rangle \arrow[r] & \mathbb{Z}\langle \pi_K^{e_K-1} d\pi_K\rangle \arrow[r] & \mathbb{Z}/e_K\mathbb{Z}\langle \pi_K^{e_K-1} d\pi_K\rangle \arrow[r] & 0
\end{tikzcd}$$

shows that the connecting homomorphism

$$\partial \colon \tilde{H}_2(N^{\mathrm{cy}}(\Pi_{e_K}, e_K), \mathbb{Z}/p\mathbb{Z}) \to \tilde{H}_1(S^1_+, \mathbb{Z}/p\mathbb{Z})$$

maps the element whose integral Bockstein is $(e_K/p)\pi_K^{e_K-1} d\pi_K$ to $d(\pi_K^{e_K})$. The differential on c_K vanishes since $v_p(e_K) > 1$. The map i induces a weak equivalence

$$\hat{\mathbb{H}}(C_{p^n}, T(k) \wedge S^1/C_{e+}) \xrightarrow{\sim} \hat{\mathbb{H}}(C_{p^n}, T(A/pA, e_K)),$$

and it is easy to see that $i_*^{-1}(\hat{\Gamma}_{A/pA}({\underline{\pi_K}}_n^{e_K/p^n}))$ is represented in the spectral sequence

$$E^2 = \Lambda\{u_n, \epsilon\} \otimes S\{t^{\pm 1}, \sigma\} \otimes k\langle \pi_K^{e_K}, \pi_K^{e_K-1} d\pi_K\rangle \Rightarrow \bar{\pi}_* \hat{\mathbb{H}}(C_{p^n}, T(k) \wedge S^1/C_{e+})$$

by the cycle $\pi_K^{e_K}$. Let

$$X \to Y \to Z \to \Sigma X$$

denote the cofibration sequence obtained by smashing (5.4.3) by the Moore spectrum M_p. We consider the 3×3-diagram of cofibration sequences

$$\begin{tikzcd}
\Sigma^{-2} Y \arrow[r, "f_{11}"] \arrow[d, "g_{11}"] & F_{S^1}(S^3_+, Y) \arrow[r, "f_{12}"] \arrow[d, "g_{12}"] & Y \arrow[r, "f_{13}"] \arrow[d, "g_{13}"] & \Sigma^{-1} Y \arrow[d, "\Sigma g_{11}"] \\
\Sigma^{-2} Z \arrow[r, "f_{21}"] \arrow[d, "g_{21}"] & F_{S^1}(S^3_+, Z) \arrow[r, "f_{22}"] \arrow[d, "g_{22}"] & Z \arrow[r, "f_{23}"] \arrow[d, "g_{23}"] & \Sigma^{-1} Z \arrow[d, "\Sigma g_{21}"] \\
\Sigma^{-1} X \arrow[r, "f_{31}"] \arrow[d, "g_{31}"] & F_{S^1}(S^3_+, \Sigma X) \arrow[r, "f_{32}"] \arrow[d, "g_{32}"] & \Sigma X \arrow[r, "f_{33}"] \arrow[d, "g_{33}"] \arrow[dr, phantom, "(-1)"] & X \arrow[d, "\Sigma g_{31}"] \\
\Sigma^{-1} Y \arrow[r, "\Sigma f_{11}"] & F_{S^1}(S^3_+, \Sigma Y) \arrow[r, "\Sigma f_{12}"] & \Sigma Y \arrow[r, "\Sigma f_{13}"] & Y,
\end{tikzcd}$$

where the horizontal boundary maps f_{i3} are given by the differential d. We now apply lemma 3.3.10 with

$$\begin{aligned}
e_{33} &= \pi_K^{e_K} \in \pi_1(\Sigma X), \\
e_{12} &= \pi_K^{e_K} \in \pi_0(F_{S^1}(S^3_+, Y)), \\
e_{21} &= c_K \in \pi_0(\Sigma^{-2} Z)
\end{aligned}$$

and get that $f_{21}(c_K) + g_{12}(\pi_K^{e_K})$ is the in image of

$$g_{12} f_{11} \colon \pi_0(\Sigma^{-2} Y) \to \pi_0(F_{S^1}(S^3_+, Z)).$$

The domain of this map is a one-dimensional k-vector space generated by the class $\sigma = f_{13}(\epsilon)$, so the map is zero. The proposition follows. □

Corollary 5.4.4. *Suppose that $v_p(e_K) > n$. Then the image of $\underline{\pi_K}_n^{e_K/p^n}$ under the map*

$$\hat{\Gamma}_A \colon \bar{\pi}_* T(A) \to \bar{\pi}_* \hat{\mathbb{H}}(C_{p^n}, T(A))$$

is represented in the spectral sequence $\hat{E}^*(C_{p^n}, A)$ by the cycle $t\theta_K(\pi_K)\kappa$.

Proof. We have already seen that the map

$$\rho_* \colon \hat{E}^3_{s,-s}(C_{p^n}, A) \to \hat{E}^3_{s,-s}(C_{p^n}, A/pA)$$

takes $t\theta_K(\pi_K)\kappa$ to $-tc_K$. Moreover, this map is a monomorphism for $-2 \le s \le 0$ and $\hat{E}^3(C_{p^n}, A/pA) = E^\infty(C_{p^n}, A/pA)$. Hence, $\hat{\Gamma}_A(\underline{\pi_K}_n^{e_K/p^n})$ cannot be represented by an element of $\hat{E}^3_{s,-s}(C_{p^n}, A)$ with $-1 \le s \le 0$ but must be represented by the element $t\theta_K(\pi_K)\kappa$ as stated. □

5.5. The Tate spectral sequence

$$\hat{E}^2(C_{p^n}, K) = \hat{H}^{-s}(C_{p^n}, (\varphi^n)^* \bar{\pi}_t T(A|K)) \Rightarrow \bar{\pi}_* \hat{\mathbb{H}}(C_{p^n}, T(A|K)),$$

is a spectral sequence of bi-graded k-algebras, when the abutment is given the canonical k-algebra structure. Since k is perfect, we have

$$\hat{E}^2(C_{p^n}, K) = \Lambda\{u_n, d\log \pi_K\} \otimes S\{\pi_K, \kappa, t^{\pm 1}\}/(\pi_K^{e_K}),$$

where the canonical generators u_n and t were defined earlier. Suppose that $\mu_p \subset K$. We choose a generator $\zeta \in \mu_p$ and let $u_K(x)$ be the polynomial from lemma 5.1.1. It is convenient to consider the algebra generators

$$\begin{aligned} \alpha_K &= u_K^{(-n)}(\pi_K)^{-1}\kappa, \\ \tau_K &= u_K^{(-n)}(\pi_K)^p\, t, \end{aligned}$$

where, we remember, $u^{(s)}(x)$ denotes the s-fold Frobenius twist of $u(x)$. We note the relations

$$\begin{aligned} \tau_K\alpha_K &= \theta_K^{(-n)}(\pi_K)t\kappa, \\ \tau_K\alpha_K^p &= t\kappa^p. \end{aligned}$$

The E^2-term then takes the form

$$\hat{E}^2(C_{p^n}, K) = \Lambda\{u_n, d\log\pi_K\} \otimes S\{\pi_K, \alpha_K, \tau_K^{\pm 1}\}/(\pi^{e_K}).$$

For integers a, r, d with $0 \leq r < e_K$ and $d \geq 0$, define

$$\{a, r, d\}_K = (pa - d)e_K/(p-1) + r.$$

Then $\{a, e_{L/K}r, d\}_L = e_{L/K}\{a, r, d\}_K$ and the map induced from the inclusion,

$$i_*\colon \hat{E}^2(C_{p^n}, K) \to \hat{E}^2(C_{p^n}, L)$$

is given by

$$\begin{aligned} i_*(\tau_K^a\pi_K^r\alpha_K^d) &= (-\theta_{L/K}(\pi_L))^{-\{a,r,d\}_K}\tau_L^a\pi_L^{e_{L/K}r}\alpha_K^d, \\ i_*(d\log\pi_K) &= (e_{L/K} - \frac{\theta'_{L/K}(\pi_L)\pi_L}{\theta_{L/K}(\pi_L)})d\log\pi_L. \end{aligned}$$

We also write $\tau_{K_0} = t$ and $\alpha_{K_0} = \kappa$.

THEOREM 5.5.1. *Suppose either $\mu_p \subset K$ or $K = K_0$. Then the non-zero differentials in the spectral sequence*

$$\begin{aligned} \hat{E}^2(C_{p^n}, K) &= \Lambda\{u_n, d\log\pi_K\} \otimes S\{\pi_K, \alpha_K, \tau_K^{\pm 1}\}/(\pi_K^{e_K}) \\ &\Rightarrow \bar{\pi}_*\hat{\mathbb{H}}(C_{p^n}, T(A|K)) \end{aligned}$$

are given by

$$\begin{aligned} d^{2(\frac{p^{v+1}-1}{p-1})}(\tau_K^a\pi_K^r\alpha_K^d) &= \lambda\cdot(\tau_K\alpha_K)^{\frac{p^{v+1}-1}{p-1}-1}\tau_K\, d\log\pi_K\cdot\tau_K^a\pi_K^r\alpha_K^d, \\ d^{2(\frac{p^{n+1}-1}{p-1})-1}(u_n) &= \mu\cdot(\tau_K\alpha_K)^{\frac{p^{n+1}-1}{p-1}-1}\tau_K, \end{aligned}$$

where in the first line $v = v_p(\{a, r, d\}_K)$. The units $\lambda = \lambda_K(a, r, d)$ and $\mu = \mu_n$ are given by $\lambda = p^{-v}\{a, r, d\}_K u^{(v+1)}(\underline{\pi_K}_{v+1})^{-1}\lambda_v$ and $\mu = u^{(n+1)}(\underline{\pi_K}_{n+1})\mu_n$, where $\underline{\pi_K}_s$ denotes the class $\pi_K^q(\tau_K\alpha_K)^m$ with $p^s = me_K + q$ and $0 \leq q < e_K$, and where μ_n and λ_v are units of $\mathbb{F}_p$, independent of K.

The proof of theorem 5.5.1 occupies the rest of this paragraph. It will be necessary to know to the structure of the E^r-terms, given the differential structure of theorem 5.5.1.

LEMMA 5.5.2. *Suppose $\mu_p \subset K$ or $K = K_0$, and assume that theorem 5.5.1 is true for K when $n = m$. Let $E^q = E^q(C_{p^m}, K)$. Then for $0 \le s < m$,*

$$E^{2(\frac{p^{s+1}-1}{p-1})} = \bigoplus_{v=1}^{s-1} \Lambda\{u_m\} \otimes k\langle \tau_K^a \pi_K^r \alpha_K^d d\log\pi_K \mid v_p\{a,r,d\}_K = v, d < \tfrac{p^{v+1}-1}{p-1} - 1\rangle$$
$$\oplus\ \Lambda\{u_m, d\log\pi_K\} \otimes k\langle \tau_K^a \pi_K^r \alpha_K^d \mid v_p\{a,r,d\}_K \ge s\rangle,$$
$$E^\infty = \bigoplus_{v=1}^{m-1} \Lambda\{u_m\} \otimes k\langle \tau_K^a \pi_K^r \alpha_K^d d\log\pi_K \mid v_p\{a,r,d\}_K = v,\ d < \tfrac{p^{v+1}-1}{p-1} - 1\rangle$$
$$\oplus\ \Lambda\{d\log\pi_K\} \otimes k\langle \tau_K^a \pi_K^r \alpha_K^d \mid v_p\{a,r,d\}_K \ge m,\ d < \tfrac{p^{m+1}-1}{p-1} - 1\rangle,$$

where $0 \le r < e_K$, $d \in \mathbb{N}_0$ and $a \in \mathbb{Z}$, and $\{a,r,d\}_K = (pa-d)e_K/(p-1) + r$.

PROOF. The class $\lambda = \lambda(a,r,d)$ is a unit in the $E^{2(p^{v+1}-1)/(p-1)}$-term of the spectral sequence, and it can therefore be ignored when evaluating the spectral sequence. Assuming the result for s and that $s+1 < n$, theorem 5.5.1 implies that

$$E^{2(\frac{p^{s+2}-1}{p-1})} = E^{2(\frac{p^{s+1}-1}{p-1})+1},$$

and inductively, $E^{2(\frac{p^s-1}{p-1})}$ is given by the statement of the lemma. Indeed, this is clear in the basic case $s = 0$. The differential $d^{2(p^{s+1}-1)/(p-1)}$ only affects the last summand on the right hand side of the statement and does not involve the tensor factor $\Lambda\{u_m\}$. If we rewrite

$$\begin{aligned}&\Lambda\{d\log\pi_K\} \otimes k\langle \tau_K^a \pi_K^r \alpha_K^d \mid v_p\{a,r,d\}_K \ge s\rangle = \\ &\quad k\langle \tau_K^a \pi_K^r \alpha_K^d \mid v_p\{a,r,d\} = s\rangle \oplus \\ &\quad k\langle \tau_K^a \pi_K^r \alpha_K^d d\log\pi_K \mid v_p\{a,r,d\}_K = s,\ d \ge \tfrac{p^{s+1}-1}{p-1} - 1\rangle \oplus \\ &\quad k\langle \tau_K^a \pi_K^r \alpha_K^d d\log\pi_K \mid v_p\{a,r,d\}_K = s,\ d < \tfrac{p^{s+1}-1}{p-1} - 1\rangle \oplus \\ &\quad k\langle \tau_K^a \pi_K^r \alpha_K^d \mid v_p\{a,r,d\}_K \ge s+1\rangle,\end{aligned}$$

the differential $d^{2(p^{s+1}-1)/(p-1)}$ clearly leaves the last two summands invariant. We claim that this differentials maps the first summand isomorphically only the second summand. Indeed,

$$(\tau_K\alpha_K)^{\frac{p^{s+1}-1}{p-1}-1}\tau_K = \tau_K^{p^{s+1}}(\tau_K\alpha_K^p)^{\frac{p^s-1}{p-1}}$$

and $v_p\{p^{s+1}, 0, 0\}_K > s$.

Assuming that theorem 5.5.1 holds for K with $n = m$, we have

$$E^{2(\frac{p^m-1}{p-1})+1} = E^{2(\frac{p^{m+1}-1}{p-1})-1},$$

and the common value has already been determined. The differential

$$d^{2(\frac{p^{m+1}-1}{p-1})-1}u_m = \mu_m \cdot (\tau_K\alpha_K)^{\frac{p^{m+1}-1}{p-1}-1}\tau_K$$

vanishes on all but the last summand, which we rewrite

$$\begin{aligned}
\Lambda\{u_m\} \otimes k\langle \tau_K^a \pi_K^r \alpha_K^d \mid v_p\{a,r,d\}_K \geq m \rangle = \\
& k\langle \tau_K^a \pi_K^r \alpha_K^d \mid v_p\{a,r,d\}_K \geq m,\ d \geq \tfrac{p^{m+1}-1}{p-1} - 1 \rangle \oplus \\
& k\langle u_m \tau_K^a \pi_K^r \alpha_K^d \mid v_p\{a,r,d\}_K \geq m \rangle \oplus \\
& k\langle \tau_K^a \pi_K^r \alpha_K^d \mid v_p\{a,r,d\}_K \geq m,\ d < \tfrac{p^{m+1}-1}{p-1} - 1 \rangle.
\end{aligned}$$

The differential $d^{2(p^{m+1}-1)/(p-1)-1}$ maps the second summand isomorphically onto the first summand and leaves the last summand unchanged. □

Proposition 5.5.3. *Suppose that theorem 5.5.1 is valid for $n \leq m$, for $K = K_0$ and for all K with $v_p(e_K) > m$. Then the theorem holds for $n \leq m$, for all K.*

Proof. The proof is by induction on m. We fix a field K with $v_p(e_K) \leq m$ and assume, inductively, that theorem 5.5.1 is true for K when $n < m$. Making use the map

$$F\colon \hat{\mathbb{H}}(C_{p^m}, T(A|K)) \to \hat{\mathbb{H}}(C_{p^{m-1}}, T(A|K))$$

the only undetermined differentials are the $d^{2q}(\tau_K^a \pi_K^r \alpha_K^d)$ where $v_p\{a,r,d\}_K \geq m-1$ and $q \geq (p^m-1)/(p-1)$. Moreover, $E^{2(p^m-1)/(p-1)}(C_{p^m}, K)$ is given by (the proof of) lemma 5.5.2.

Let L/K be a totally ramified extension,

$$L = K[\pi_L]/(\pi_L^{e_{L/K}} + \pi_L \theta_{L/K}(\pi_L)),$$

and recall that the map induced from the inclusion,

$$i_*\colon E^{2q}(C_{p^{m+1}}, K) \to E^{2q}(C_{p^{m+1}}, L),$$

is given by

$$\begin{aligned}
i_*(\tau_K^a \pi_K^r \alpha_K^d) &= (\theta_{L/K}(\pi_L))^{-\{a,r,d\}_K} \tau_L^a \pi_L^{e_{L/K} r} \alpha_L, \\
i_*(d\log \pi_K) &= (e_{L/K} - \frac{\theta'_{L/K}(\pi_L)\pi_L}{\theta_{L/K}(\pi_L)}) d\log \pi_L.
\end{aligned}$$

Suppose that $v_p(e_L) > m$. Then by assumption, the differentials in $\hat{E}^*(C_{p^m}, L)$ are given by theorem 5.5.1, and we may thus calculate

$$d^{2(\frac{p^m-1}{p-1})}(i_*(\tau_K^a \pi_K^r \alpha_K^d)) = d^{2(\frac{p^m-1}{p-1})}((-\theta_{L/K}(\pi_L))^{-\{a,r,d\}_K} \cdot \tau_L^a \pi_L^{e_{L/K} r} \alpha_L^d),$$

where $v_p\{a,r,d\}_K \geq m-1$. The formula $\{a, e_{L/K} r, d\}_L = e_{L/K}\{a,r,d\}_K$ and our assumption that $v_p(e_{L/K}) \geq 1$ implies that $v_p\{a, e_{L/K}, d\}_L \geq m$. It follows that $\tau_L^a \pi_L^{e_{L/K} r} \alpha_L^d$ is an infinite cycle in $E^{2(p^m-1)/(p-1)}(C_{p^m}, L)$, and hence

$$d^{2(\frac{p^m-1}{p-1})}(i_*(\tau_K^a \pi_K^r \alpha_K^d)) = d^{2(\frac{p^m-1}{p-1})}((-\theta_{L/K}(\pi_L))^{-\{a,r,d\}_K}) \cdot \tau_L^a \pi_L^{e_{L/K} r} \alpha_L^d.$$

The differential on the right vanishes if $v_p\{a,r,d\}_K > m-1$. If $v_p\{a,r,d\}_K = m-1$, we write $-\{a,r,d\}_K = p^{m-1}l$. Then

$$\begin{aligned}
d^{2(\frac{p^m-1}{p-1})}((-\theta_{L/K}(\pi_L))^{-\{a,r,d\}_K}) &= d^{2(\frac{p^m-1}{p-1})}((-\theta^{(m-1)}_{L/K}(\pi_L^{p^{m-1}}))^l) \\
&= l\cdot(\theta^{(m-1)}_{L/K}(\pi_L^{p^{m-1}}))^{l-1}\cdot(-(\theta^{(m-1)}_{L/K})'(\pi_L^{p^{m-1}}))\cdot d^{2(\frac{p^m-1}{p-1})}(\pi_L^{p^{m-1}}) \\
&= -l\cdot(\theta^{(m-1)}_{L/K}(\pi_L^{p^{m-1}}))^l\cdot\frac{(\theta^{(m-1)}_{L/K})'(\pi_L^{p^{m-1}})\pi_L^{p^{m-1}}}{\theta^{(m-1)}_{L/K}(\pi_L^{p^{m-1}})}\cdot t^{p^{m-1}}v_1^{\frac{p^{m-1}-1}{p-1}}d\log\pi_L \\
&= -l\cdot(-\theta_{L/K}(\pi_L))^{-\{a,r,d\}_K}\cdot t^{p^{m-1}}v_1^{\frac{p^m-1}{p-1}}\cdot i_*(d\log\pi_K),
\end{aligned}$$

where the last identification follows from lemma 5.2.2. It follows that

$$\begin{aligned}
d^{2(\frac{p^m-1}{p-1})}(i_*(\tau_K^a\pi_K^r\alpha_K^d)) &= i_*(p^{-(m-1)}\{a,r,d\}_K\cdot t^{p^{m-1}}v_1^{\frac{p^{m-1}-1}{p-1}}d\log\pi_K\cdot\tau_K^a\pi_K^r\alpha_K^d) \\
&= i_*(\lambda\cdot(\tau_K\alpha_K)^{\frac{p^m-1}{p-1}-1}\tau_K d\log\pi_K\cdot\tau_K^a\pi_K^r\alpha_K^d),
\end{aligned}$$

where $\lambda=\lambda_K(a,r,d)$ as defined in the statement of theorem 5.5.1. The domain and range of the map

$$i_*\colon E^{2(\frac{p^m-1}{p-1})}(C_{p^m},K)\to E^{2(\frac{p^m-1}{p-1})}(C_{p^m},L)$$

are given by lemma 5.5.2. We claim that the extension L/K can be chosen such that this map is injective. Indeed, if we let $\theta_{L/K}(x)=x+1$ then

$$i_*(d\log\pi_K) = -\frac{\pi_L^{p^{m-1}}}{\pi_L^{p^{m-1}}+1}d\log\pi_L,$$

and hence, up to a unit,

$$i_*(\tau_K^a\pi_K^r\alpha_K^d) = \tau_L^a\pi_L^{e_{L/K}r+p^{m-1}}\alpha_L^d d\log\pi_L.$$

In order that i_* be injective, we therefore need that $e_{L/K}r+p^{m-1}<e_L$. Since $r\le e_K-1$ and $e_L=e_{L/K}e_K$ this is equivalent to the requirement that $e_{L/K}\ge p^{m-1}$. We also need $v_p(e_L)>m$, so if we let $\theta_{L/K}(x)=x+1$ and $e_{L/K}=p^{m+1}$, theorem 5.5.1 will be valid for L by assumption and i_* injective. It follows that

$$d^{2(\frac{p^m-1}{p-1})}(\tau_K^a\pi_K^r\alpha_K^d) = \lambda\cdot(\tau_K\alpha_K)^{\frac{p^m-1}{p-1}-1}\tau_K d\log\pi_K\cdot\tau_K^a\pi_K^r\alpha_K^d$$

as desired. A similar argument shows that $d^{2q}(\tau_K^a\pi_K^r\alpha_K^d)=0$ when $(p^m-1)/(p-1)<q<(p^{m+1}-1)/(p-1)$, and finally, the differential on u_m follows by comparison with the spectral sequence for $K=K_0$. □

5.6. We are reduced to proving theorem 5.5.1 for $K=K_0$ and for $K\supset\mu_p$ with $v_p(e_K)>n$. We begin by constructing a number of infinite cycles. Recall the map of ring spectra

$$\hat\Gamma_K\colon T(A|K)^{C_{p^{n-1}}}\to\hat{\mathbb{H}}(C_{p^n},T(A|K)).$$

LEMMA 5.6.1. *For all K, the element $d\log\pi_K\in\hat E^2(C_{p^n},K)$ is an infinite cycle and represents the homotopy class $\hat\Gamma_K(d\log\underline{\pi_K}_n)$.*

Proof. We consider the diagram

$$\begin{tikzcd}
T(A|K)^{C_{p^{n-1}}} \arrow[d, "\hat\Gamma_K"] & T(A|K)^{C_{p^n}} \arrow[l, "R"'] \arrow[d, "\Gamma_K"] \arrow[dr, "F^n"] & \\
\hat{\mathbb{H}}(C_{p^n}, T(A|K)) & \mathbb{H}^{\cdot}(C_{p^n}, T(A|K)) \arrow[l, "R^h"'] \arrow[r] & T(A|K).
\end{tikzcd}$$

The modulo p homotopy groups of the lower middle term are approximated by the spectral sequence

$$E^2(C_{p^n}, K) = \Lambda\{u_n, d\log\pi_K\} \otimes S\{\pi_K, t, \kappa\} \to \bar\pi_* \mathbb{H}^{\cdot}(C_{p^n}, T(A|K)),$$

and the lower left hand horizontal map induces the obvious embedding on E^2-terms. The lower right hand horizontal map is given by the edge homomorphism of this spectral sequence. The maps R and F^n take the class $d\log\underline{\pi_K}_{n+1} \in \pi_0 T(A|K)^{C_{p^n}}$ to the classes $d\log\underline{\pi_K}_n \in \pi_0 T(A|K)^{C_{p^{n-1}}}$ and $d\log\pi_K \in \pi_0 T(A|K)$, respectively. It follows that $d\log\pi_K \in E^2(C_{p^n}, K)$ survives the spectral sequence and represents the homotopy class $\Gamma_K(d\log\underline{\pi_K}_{n+1}) \in \pi_0\mathbb{H}^{\cdot}(C_{p^n}, T(A|K))$. □

Proposition 5.6.2. *Suppose that $\mu_p \subset K$ and let $n < v_p(e_K)$. Then the elements $\pi_K^{p^n}$ and $\tau_K\alpha_K$ of $\hat E^2(C_{p^n}, K)$ are infinite cycles which represent the homotopy classes $\hat\Gamma_K(\underline{\pi_K}_n)$ and $\hat\Gamma_K(\underline{\pi_K}_n^{e_K/p^n})$, respectively.*

Proof. We use the diagram

$$\begin{tikzcd}
T(A|K)^{C_{p^{n-1}}} \arrow[d, "\hat\Gamma_K"] & T(A)^{C_{p^{n-1}}} \arrow[l, "j_*"'] \arrow[r, "\mathrm{pr}_*"] \arrow[d, "\hat\Gamma_A"] & T(A/pA)^{C_{p^{n-1}}} \arrow[d, "\hat\Gamma_{\bar A}"] \\
\hat{\mathbb{H}}(C_{p^n}, T(A|K)) & \hat{\mathbb{H}}(C_{p^n}, T(A)) \arrow[l, "j_*"'] \arrow[r, "\mathrm{pr}_*"] & \hat{\mathbb{H}}(C_{p^n}, T(A/pA)),
\end{tikzcd}$$

and the explicit calculation in section 5.4 below of the right hand map, to show that the images of the homotopy classes $\underline{\pi_K}_n, \underline{\pi_K}_n^{e/p^n} \in \pi_0 T(A|K)^{C_{p^{n-1}}}$ under the left hand vertical map are represented in the spectral sequence by the elements $\pi_K^{p^n}$ and $\tau_K\alpha_K$, respectively. In particular, these elements are infinite cycles. □

We shall see later that the proposition 5.6.2 is true, more generally, for $n \le v_p(e_K)$.

Corollary 5.6.3. *The element $t\kappa^p = \tau_K\alpha_K^p \in \hat E^2(C_{p^n}, K)$ is an infinite cycle and represents the image of the canonical generator $v_1 \in \pi_{2(p-1)}(S^0)$ under the unit map $\eta\colon S^0 \to \hat{\mathbb{H}}(C_{p^n}, T(A|K))$.*

Proof. Suppose that $\mu_p \subset K$, let $\zeta \in \mu_p$ be a generator and let $b \in \bar\pi_2 T(A|K)^{C_{p^{n-1}}}$ be the corresponding Bott element. Since $\mathrm{Aut}(\mu_p)$ has order $p-1$, the product $b^{p-1} \in \bar\pi_{2(p-1)} T(A|K)^{C_{p^{n-1}}}$ is independent of the choice of generator and is equal to the image of the canonical generator $v_1 \in \bar\pi_{2(p-1)}(S^0)$ under the unit map $\eta\colon S^0 \to T(A|K)^{C_{p^{n-1}}}$. We show that b^{p-1} is represented in the spectral sequence

$$\begin{aligned} E^2(C_{p^n}, K) &= \Lambda\{u_n, d\log\pi_K\} \otimes S\{\pi_K, \tau_K, \alpha_K\}/(\pi_K^{e_K}) \\ &\Rightarrow \bar\pi_*\mathbb{H}^{\cdot}(C_{p^n}, T(A|K)) \end{aligned}$$

by the element $\tau_K \alpha_K^p = t\kappa^p$. The Bott element is given by

$$b = -\pi_K^{e_K/(p-1)} \alpha_K.$$

Indeed, this formula is valid in $\bar{\pi}_2 T(A|K)$ by (5.3.1). If $v_p(e_K) > n$, we may write this as

$$b = -(\pi_K^{p^n})^{e_K/p^n(p-1)} \alpha_K,$$

and hence

$$b^{p-1} = (\pi_K^{p^n})^{e_K/p^n} \alpha_K^{p-1}.$$

The desired formula then follows from the multiplicative extension

$$(\pi_K^{p^n})^{e_K/p^n} = \tau_K \alpha_K$$

of proposition 5.6.2. Since $\tau_K \alpha_K^p = t\kappa^p$ is in the image from $\hat{E}^*(C_{p^n}, T(W|K_0))$, the formula is valid for all K. $\square$

PROPOSITION 5.6.4. *Theorem 5.5.1 holds for $n = 1$.*

PROOF. The d^2-differentials, given by Connes' operator, are generated from

$$d^2 \pi_K = td\log \pi_K \cdot \pi_K,$$
$$d^2 \kappa = td\log p \cdot \kappa.$$

When $K = K_0$, we have

$$\hat{E}^3(C_p, K_0) = \Lambda\{u_1, d\log p\} \otimes S\{t^{\pm 1}, \kappa^p\},$$

and for degree reasons, the first possible differential is

$$d^{2p+1} u_1 = \mu_1 \cdot t^{p+1} \kappa^p.$$

Comparing with $\hat{E}^*(S^1, K_0)$, we see that $d^{2p+1} t$ is trivial, and hence so is $d^{2p+1}(t^p)$. Thus

$$d^{2p+1}(t^{-p} u_1) = \mu_1 \cdot t\kappa^p.$$

If this differential was trivial, $t\kappa^p$ would survive the spectral sequence and represent the homotopy class $v_1 \cdot 1$. But $\hat{\mathbb{H}}(C_p, T(W|K_0))$ is a module spectrum over the generalized Eilenberg-MacLane spectrum $T(W)$ and is therefore itself a generalized Eilenberg-MacLane spectrum. So multiplication by v_1 on $\bar{\pi}_* \hat{\mathbb{H}}(C_p, T(W|K_0))$ is identically zero, and therefore, the differential on u_1 must be non-zero, i.e. $\mu_1 \in \mathbb{F}_p^\times$. The spectral sequence collapses.

If $\mu_p \subset K$ and $v_p(e_K) > 1$, we get

$$\hat{E}^3(C_p, K) = \Lambda\{u_1, d\log \pi_K\} \otimes S\{\pi_K^p, \alpha_K, \tau_K^{\pm 1}\}/(\pi_K^e).$$

Since $t \in \hat{E}^2(C_p, K_0)$ is an infinite cycle, then so is its image $u_K^{(1)}(\pi_K^p)^{-1} \tau_K \in \hat{E}^2(C_p, K)$. And since also π_K^p is an infinite cycle, and since $u_K(x) \in k[\![x]\!]$ is a unit, it follows that τ_K is an infinite cycle. Now by proposition 5.6.2, $\tau_K \alpha_K$ is an infinite cycle, and hence so is α_K. Therefore the remaining non-zero differentials are generated from the differential on u_1. Again the spectral sequence collapses.

We have proved theorem 5.5.1 for $n = 1$, for $K = K_0$ and for all K with $v_p(e_K) \geq 1$. By proposition 5.5.3 it is therefore valid for $n = 1$, for all K. $\square$

THEOREM 5.6.5. *For all K, and for $i \geq 0$, the map*

$$\hat{\Gamma}_K \colon \bar{\pi}_i T(A|K) \xrightarrow{\sim} \bar{\pi}_i \hat{\mathbb{H}}(C_p, T(A|K))$$

is an isomorphism.

PROOF. Since both the domain and range of $\hat{\Gamma}_K$ satisfies tame descent, it is enough to prove the statement when $\mu_p \subset K$. If $\mu_p \subset K$ and $v_p(e_K) > 0$ or if $K = K_0$,

$$E^\infty(C_p, K) = \Lambda\{d\log \pi_K\} \otimes S\{\pi_K^p, \alpha_K, \tau_K^{\pm 1}\}/(\pi_K^e, \alpha_K^p),$$

and moreover, proposition 5.6.2 shows that there is a multiplicative extension

$$(\pi_K^p)^{e/p} = \tau_K \alpha_K$$

in passing from $E^\infty(C_p, K)$ to the actual homotopy groups. Therefore, as a k-algebra

$$\bar{\pi}_* \hat{\mathbb{H}}(C_p, T(A|K)) = \Lambda\{\hat{\Gamma}_K(d\log \pi_K)\} \otimes S\{\hat{\Gamma}_K(\pi_K), \tilde{\tau}_K^{\pm 1}\}/(\hat{\Gamma}_K(\pi_K)^{e_K}),$$

where $\tilde{\tau}_K$ is a homotopy class lifting the element τ_K of the spectral sequence. It follows that $\bar{\pi}_* T(A|K)$ and the non-negatively graded part of $\bar{\pi}_* \hat{\mathbb{H}}(C_p, T(A|K))$ are abstractly isomorphic k-algebras, and that the map $\hat{\Gamma}_K$ is an isomorphism for $i = 0$ and $i = 1$. To show that $\hat{\Gamma}_K$ is an isomorphism, for $i \geq 0$, it will therefore suffice to show that

$$\hat{\Gamma}_{K_0} \colon \bar{\pi}_2 T(W|K_0) \xrightarrow{\sim} \bar{\pi}_2 \hat{\mathbb{H}}(C_p, T(W|K_0))$$

is an isomorphism. To this end, we consider the diagram

$$\begin{tikzcd}
\bar{\pi}_2 T(W|K_0) \arrow[r, "\beta_1", "\sim"'] \arrow[d, "\hat{\Gamma}_{K_0}"] & \bar{\pi}_1 T(W|K_0) \arrow[d, "\hat{\Gamma}_{K_0}", "\sim"'] \\
\bar{\pi}_2 \hat{\mathbb{H}}(C_p, T(W|K_0)) \arrow[r, "\beta_1"] & \bar{\pi}_1 \hat{\mathbb{H}}(C_p, T(W|K_0)),
\end{tikzcd}$$

where the upper horizontal map and right hand vertical maps are isomorphisms. Since all groups in the diagram are one-dimensional k-vector spaces, the left hand vertical map and lower horizontal map must also be isomorphisms. This shows that (5.6.5) is an isomorphism if $\mu_p \subset K$ and $v_p(e_K) > 0$ or if $K = K_0$.

If $\mu_p \subset K$ and $v_p(e_K) = 0$,

$$E^\infty(C_p, K) = \Lambda\{d\log \pi_K\} \otimes k\langle \tau_K^a \pi_K^r \alpha_K^d \mid v_p\{a, r, d\}_K \geq 1,\ d < p\rangle,$$

where $0 \leq r < e_K$, $d \in \mathbb{N}_0$ and $a \in \mathbb{Z}$. Again, the domain and range of $\hat{\Gamma}_K$ are abstractly isomorphic k-vector spaces. We choose an extension L/K such that $v_p(e_L) > 0$ and such that

$$i_* \colon \bar{\pi}_* T(A|K) \to \bar{\pi}_* T(B|L)$$

is a monomorphism. The diagram

$$\begin{tikzcd}
\bar{\pi}_i T(A|K) \arrow[r, tail, "i_*"] \arrow[d, "\hat{\Gamma}_K"] & \bar{\pi}_* T(B|L) \arrow[d, "\hat{\Gamma}_L", "\sim"'] \\
\bar{\pi}_* \hat{\mathbb{H}}(C_p, T(A|K)) \arrow[r, "i_*"] & \bar{\pi}_* \hat{\mathbb{H}}(C_p, T(B|L))
\end{tikzcd}$$

shows that for $i \geq 0$, $\hat{\Gamma}_K$ is a monomorphism and hence an isomorphism. $\square$

Given theorem 5.6.5, a theorem of Tsalidis, [**39**], shows that the following more general statement holds.

ADDENDUM 5.6.6. *For all K, for all $n \geq 1$, and for all $i \geq 0$, the map*

$$\hat{\Gamma}_K : \bar{\pi}_i T(A|K)^{C_{p^{n-1}}} \xrightarrow{\sim} \bar{\pi}_i \hat{\mathbb{H}}(C_{p^n}, T(A|K)),$$

is an isomorphism. $\square$

5.7. We now prove theorem 5.5.1 for $K = K_0$. This may be derived from the spectral sequence for $\bar{\pi}_* \hat{\mathbb{H}}(C_{p^n}, T(W))$, which is known from [**4**]. We give, however, a more direct and simpler proof.

LEMMA 5.7.1. *The element $(t\kappa)^{p^n} \in \hat{E}^2(C_{p^n}, K_0)$ is an infinite cycle and represents the homotopy class $V(1)$.*

PROOF. Let K/K_0 be an extension and recall that in $\bar{W}_n(A)$ one has the relation

$$\underline{\pi_K}_n^{e_K} = \theta_K(\underline{\pi_K}_n)V(1).$$

By proposition 5.6.2 we may choose the extension such that $\tau_K \alpha_K \in \hat{E}^2(C_{p^n}, K)$ is an infinite cycle representing the class $\hat{\Gamma}_K(\underline{\pi_K}_n^{e_K/p^n})$. Recall that in

$$\hat{E}^2(C_{p^n}, K) = \Lambda\{u_n\} \otimes S\{t^{\pm 1}\} \otimes (\varphi^n)^* \bar{\pi}_* T(A|K)$$

we have

$$\tau_K \alpha_K = \theta_K^{(-n)}(\pi_K) t\kappa$$

and hence

$$(\tau_K \alpha_K)^{p^n} = \theta_K(\pi_K^{p^n})(t\kappa)^{p^n}.$$

The left hand side represents $\hat{\Gamma}_K(\underline{\pi_K}_n^{e_K})$, and on the right hand, $\pi_K^{p^n}$ is an infinite cycle representing $\hat{\Gamma}_K(\underline{\pi_K}_n)$. Since $\theta_K(x)$ is a unit, it follows that also $(t\kappa)^{p^n}$ is an infinite cycle which represents $V(1)$ as claimed. $\square$

THEOREM 5.7.2. *In the spectral sequence*

$$\hat{E}^2(C_{p^n}, K_0) = \Lambda\{u_n, d\log p\} \otimes S\{t^{\pm 1}, \kappa\} \Rightarrow \bar{\pi}_* \hat{\mathbb{H}}(C_{p^n}, T(W|K_0)),$$

the differentials are multiplicatively generated from

$$d^{2(\frac{p^{v+1}-1}{p-1})} t^{-p^{v-1}} = \lambda_v \cdot (t\kappa)^{p^v} d\log p \cdot v_1^{\frac{p^{v-1}-1}{p-1}}, \quad 1 \leq v < n,$$
$$d^{2(\frac{p^{n+1}-1}{p-1})-1}(u_n t^{-p^n}) = \mu_n \cdot v_1^{\frac{p^n-1}{p-1}},$$

where λ_v and μ_n are units of $\mathbb{F}_p$, and from the fact that $t\kappa^p$ and $d\log p$ are infinite cycles. Moreover, the cycles $(t\kappa)^{p^{s+1}} d\log p$, $1 \leq s < n$, represents the homotopy classes $dV^{n-s}(1)$.

PROOF. The proof, of course, is by induction on n starting from the case $n = 1$ which was proved in proposition 5.6.4. So assume the statement for $n-1$. We first show that the classes $dV^{n-s}(1)$, $1 \leq s < n$, are represented by the elements $(t\kappa)^{p^{s+1}} d\log p$. For $s < n-1$ this follows inductively from the formula

$$FdV^{n-s}(1) = dV^{n-1-s}(1)$$

and from the fact that for degree reasons, $(t\kappa)^{p^{s+1}} d\log p$ is an infinite cycle. When $s = n-1$, one uses that in $\bar{W}_n\omega^1_{(W,M)}$,

$$dV(1) = V(d\log_{n-1} p) = V(1) d\log_n p,$$

which by lemma 5.7.1 is represented by $(t\kappa)^{p^n} d\log p$.

The maps F and V between $\hat{E}^*(C_{p^n}, K_0)$ and $\hat{E}^*(C_{p^{n-1}}, K_0)$ give the stated differential on t^{-p^i} for $i < n-2$ and show that u_n is at least a $d^{2(p^n-1)/(p-1)}$-cycle. By lemma 5.5.2,

$$(5.7.3) \qquad \begin{aligned} E^{2(\frac{p^n-1}{p-1})} = & \bigoplus_{v=1}^{n-2} \Lambda\{u_n\} \otimes k\langle t^i d\log p \mid v_p(i) = v-1 \rangle \otimes S\{v_1\}/(v_1^{\frac{p^v-1}{p-1}}) \\ & \oplus \ \Lambda\{u_n, d\log p\} \otimes S\{t^{\pm p^{n-2}}, v_1\}. \end{aligned}$$

We show that the elements $t^a\kappa^b d\log p$ are infinite cycles. Since they are in the image of the

$$E^2(S^1, K_0) \to E^2(C_{p^n}, K_0)$$

is suffices to show that they are infinite cycles in

$$E^2(S^1, K_0) = S\{t^{\pm 1}\} \otimes \bar{\pi}_* T(W|K_0).$$

The reduction $\pi_* T(W|K_0) \to \bar{\pi}_* T(W|K_0)$ is an epimorphism in odd degrees, so the elements $t^a\kappa^b d\log p$ lift to the integral spectral sequence

$$\mathbb{E}^2(S^1, K_0) = S\{t^{pm1}\} \otimes \pi_* T(W|K_0) \Rightarrow \pi_* \hat{\mathbb{H}}(\mathbb{T}, T(W|K_0)).$$

Since $\pi_* T(W|K_0)$ is rational in even degrees the non-zero differentials in this spectral sequence must all originate on the base line. Hence the elements $t^a\kappa^b d\log p$ are infinite cycles as stated. It follows, in addition, that the elements in the top summands in (5.7.3) are d^r-cycles as long as $d^r u_n = 0$, and moreover, these elements cannot be hit by a differential for degree reasons. Hence the differentials on u_n and $t^{p^{n-2}}$ leaves the top summands of (5.7.3) invariant.

The first possible differential is

$$d^{2(\frac{p^n-1}{p-1})}(t^{-p^{n-2}}) = \lambda_{n-1} \cdot (t\kappa)^{p^{n-1}} d\log p \cdot v_1^{\frac{p^{n-2}-1}{p-1}},$$

where $\lambda_{n-1} \in \mathbb{F}_p$. We treat the cases $n = 2$ and $n > 2$ separately. If $n = 2$, a k-basis of $\bar{\pi}_1 \hat{\mathbb{H}}(C_{p^2}, T(W|K_0))$ is given by the classes $d\log_2 p$ and $dV(1)$. These classes are represented by $d\log p$ and $(t\kappa)^{p^2} d\log p$, respectively and the cycle $(t\kappa)^p d\log p$, therefore, must be hit by a differential. This can only happen if the stated differential on t^{-1} is non-zero, i.e. $\lambda_1 \in \mathbb{F}_p$ is a unit. When $n > 2$ we consider the class $dV^2(1)$ which in the spectral sequence is represented by $(t\kappa)^{p^{n-1}} d\log p$. Inductively, multiplication by $v_1^{(p^{n-2}-1)/(p-1)}$ annihilates $\bar{\pi}_* \hat{\mathbb{H}}(C_{p^{n-2}}, T(W|K_0))$, and hence also the class $dV^2(1)$. The cycle $(t\kappa)^{p^{n-1}} d\log p \cdot v_1^{(p^{n-2}-1)/(p-1)}$ therefore must be hit by a differential, and this can only happen if the differential on $t^{-p^{n-2}}$ is non-zero, i.e. if $\lambda_{n-1} \in \mathbb{F}_p$ is a unit.

For degree reasons the next possible differential is

$$d^{2(\frac{p^{n+1}-1}{p-1})-1}(u_n t^{-p^n}) = \mu_n \cdot v_1^{\frac{p^n-1}{p-1}}$$

and it will thus suffice to show that multiplication by $v_1^{(p^n-1)/(p-1)}$ is identically zero on $\bar{\pi}_* T(W|K_0)^{C_{p^{n-1}}}$. To this end, we use that the map

$$\Gamma_K \colon \bar{\pi}_i T(W|K_0)^{C_{p^{n-1}}} \to \bar{\pi}_i \mathbb{H}^{\cdot}(C_{p^{n-1}}, T(W|K_0))$$

is an isomorphism, for $i \geq 0$. The target of this map is given by the spectral sequence

$$E^2(C_{p^{n-1}}, K_0) = \Lambda\{u_{n-1}, d\log p\} \otimes S\{t, \kappa\} \Rightarrow \bar{\pi}_* \mathbb{H}^{\cdot}(C_{p^{n-1}}, T(W|K_0)).$$

The E^2-term of this spectral sequence is equal to the left half plane of the spectral sequence $\hat{E}^2(C_{p^{n-1}}, K_0)$, and the differentials are obtained from the differentials of the latter sequence. These differentials are known inductively. In particular,

$$d^{2(\frac{p^n-1}{p-1})}(u_{n-1}\kappa^{p^n}) = \mu_{n-1} \cdot v_1^{\frac{p^n-1}{p-1}}.$$

It follows that multiplication by $v_1^{(p^n-1)/(p-1)}$ on $\bar{\pi}_* \mathbb{H}^{\cdot}(C_{p^{n-1}}, T(W|K_0))$ is identically zero, and hence the stated differential on u_n is non-zero, i.e. $\mu_n \in \mathbb{F}_p$ is a unit. The spectral sequence now collapses for degree reasons. □

5.8. It remains to prove that theorem 5.5.1 holds when $\mu_p \subset K$ and $n < v_p(e_K)$. In this case, 5.6.2 and 5.6.3 show that $\tau_K \alpha_K$ and $\tau_K \alpha_K^p$ are infinite cycles. Hence if $d^r \alpha_K$ is non-trivial then so is $d^r(\alpha_K^p)$ contradicting that d^r is a derivation. Thus α_K and τ_K are infinite cycles, and theorem 5.5.1 then amounts to the statement that the differentials in $\hat{E}^*(C_{p^n}, K)$ are multiplicatively generated from

$$d^{2(\frac{p^{v+1}-1}{p-1})}(\pi_K^{p^v}) = \lambda_v \cdot (t\kappa)^{\frac{p^{v+1}-1}{p-1}-1} t\, d\log \pi_K \cdot \pi_K^{p^v}, \qquad 0 \leq v < n,$$

and from the differential on u_n.

LEMMA 5.8.1. *Suppose that $\mu_p \subset K$. If in addition $u_K'(0)$ is non-zero, then theorem 5.5.1 holds for K and $n < v_p(e_K)$.*

PROOF. Since τ_K is an infinite cycle, so is $\tau_K^{p^{v-1}}$, for all $v \geq 1$. Now

$$\tau_K^{p^{v-1}} = (u_K^{(1)}(\pi_K^p)^{-1} t)^{p^{v-1}} = u_K^{(v)}(\pi_K^{p^v})^{-1} t^{p^{v-1}},$$

and since $\tau_K^{p^{v-1}}$ is an infinite cycle and d^r a derivation, we get

$$\frac{(u_K^{(v)})'(\pi_K^{p^v})}{u_K^{(v)}(\pi_K^{p^v})} d^r(\pi_K^{p^v}) = d^r(t^{p^{v-1}}).$$

The assumption that $u_K'(0)$ is non-zero implies that the first factor on the left is a unit in $E^r(C_{p^n}, K)$. We may therefore calculate the differential on $\pi_K^{p^v}$ from the known differential on $t^{p^{v-1}}$. We see that $d^r(\pi_K^{p^v})$ vanishes for $r < 2(p^{v+1}-1)/(p-1)$. When $r = 2(p^{v+1}-1)/(p-1)$,

$$\begin{aligned} d^r(t^{p^{v-1}}) &= -\lambda_v \cdot (t\kappa)^{\frac{p^{v+1}-1}{p-1}-1} t \cdot d\log p \\ &= -\lambda_v \cdot (t\kappa)^{\frac{p^{v+1}-1}{p-1}-1} t \cdot \frac{(\theta_K^{(v)})'(\pi_K^{p^v})\pi_K^{p^v}}{\theta_K^{(v)}(\pi_K^{p^v})} d\log \pi_K, \\ &= \lambda_v \cdot (t\kappa)^{\frac{p^{v+1}-1}{p-1}-1} t \cdot \frac{(u_K^{(v)})'(\pi_K^{p^v})\pi_K^{p^v}}{u_K^{(v)}(\pi_K^{p^v})} d\log \pi_K, \end{aligned}$$

which shows that

$$d^r(\pi_K^{p^v}) = \lambda_v \cdot (t\kappa)^{\frac{p^{v+1}-1}{p-1}-1} td\log\pi_K \cdot \pi_K^{p^v}$$

as desired. $\square$

We now place ourselves in the universal situation and consider the ring spectrum

$$T(W|K_0) \wedge N^{\mathrm{cy}}(\Pi_\infty),$$

where Π_∞ is the pointed monoid $\{0, 1, \pi, \pi^2, \dots\}$. We have

$$\bar{\pi}_*(T(W|K_0) \wedge N^{\mathrm{cy}}(\Pi_\infty)) = \Lambda\{d\log p, d\pi\} \otimes S\{\kappa, \pi\}.$$

Given K and a choice of uniformizer π_K, we get a map of ring spectra

$$\rho_K \colon T(W|K_0) \wedge N^{\mathrm{cy}}(\Pi_\infty) \to T(A|K),$$

which on modulo p homotopy groups is given by

$$\begin{aligned} \rho_{K*}(\pi) &= \pi_K, \\ \rho_{K*}(d\pi) &= \pi_K d\log\pi_K, \\ \rho_{K*}(\kappa) &= \kappa, \\ \rho_{K*}(d\log p) &= (e_K + \frac{u_K'(\pi_K)\pi_K}{u_K(\pi_K)})d\log\pi_K. \end{aligned}$$

Proposition 5.8.2. *In the spectral sequence*

$$\begin{aligned} \hat{E}^2(C_{p^n}, K_0, \pi) &= \Lambda\{u_n, d\log p, d\pi\} \otimes S\{t^{\pm 1}, \kappa, \pi\} \\ &\Rightarrow \bar{\pi}_*\hat{\mathbb{H}}(C_{p^n}, T(W|K_0) \wedge N^{\mathrm{cy}}(\Pi_\infty)), \end{aligned}$$

the non-zero differentials are generated multiplicatively from

$$\begin{aligned} d^{2(\frac{p^{v+1}-1}{p-1})}(\pi^{p^v}) &= \lambda_v \cdot (t\kappa)^{\frac{p^{v+1}-1}{p-1}-1} t \cdot \pi^{p^v-1}d\pi, && 0 \le v < n, \\ d^{2(\frac{p^{v+1}-1}{p-1})}(t^{p^{v-1}}) &= -\lambda_v \cdot (t\kappa)^{\frac{p^{v+1}-1}{p-1}-1} td\log p \cdot t^{p^{v-1}}, && 1 \le v < n, \\ d^{2(\frac{p^{n+1}-1}{p-1})-1}(u_n) &= \mu_n \cdot (t\kappa)^{\frac{p^{n+1}-1}{p-1}-1} t \end{aligned}$$

with $t\kappa^p$, $d\log p$, π^{p^n} and $\pi^{p^n-1}d\pi$ being infinite cycles.

Proof. We choose an extension K/K_0 with $\mu_p \subset K$ and $v_p(e_K) > n$, and such that $u_K'(0)$ is non-zero. Lemma 5.8.1 then shows that for $0 \le v < n$,

$$d^{2(\frac{p^{v+1}-1}{p-1})}(\pi_K^{p^v}) = \lambda_v \cdot (t\kappa)^{\frac{p^{v+1}-1}{p-1}-1} td\log\pi_K \cdot \pi_K^{p^v}.$$

There is an S^1-equivariant decomposition

$$N^{\mathrm{cy}}(\Pi_\infty) \cong \bigvee_{s \ge 0} N^{\mathrm{cy}}(\Pi_\infty, s),$$

and the spectral sequence decomposes accordingly,

$$E^r(C_{p^n}, K_0, \pi) = \bigoplus_{s \ge 0} E^r(C_{p^n}, K_0, \pi, s).$$

Here, $E^r(C_{p^n}, K_0, \pi, 0) = E^r(C_{p^n}, K_0)$, and for $s \ge 1$,

$$\begin{aligned} \hat{E}^2(C_{p^n}, K_0, \pi, s) &= \hat{E}^2(C_{p^n}, K_0) \otimes k\langle \pi^s, \pi^{s-1}d\pi \rangle \\ &= \Lambda\{u_n, d\log p\} \otimes S\{t^{\pm 1}, \kappa\} \otimes k\langle \pi^s, \pi^{s-1}d\pi \rangle. \end{aligned}$$

In particular, for all K with $v_p(e_K) \geq v$,

$$\rho_{K*}\colon \hat{E}^2(C_{p^n}, K_0, \pi, p^v) \to \hat{E}^2(C_{p^n}, K)$$

is a monomorphism. It follows by induction on r that

$$\rho_{K*}\colon E^r(C_{p^n}, K_0, \pi, p^v) \to E^r(C_{p^n}, K)$$

is a monomorphism and that the differentials on the p-powers of t and on π^{p^v} are as stated. $\square$

COROLLARY 5.8.3. *Theorem 5.5.1 holds when $\mu_p \subset K$ and $n < v_p(e_K)$.*

PROOF. This follows immediately from proposition and from the fact that the spectral sequence $\hat{E}^*(C_{p^n}, K)$ is a module spectral sequence over $\hat{E}^*(C_{p^n}, K_0, \pi)$. $\square$

6. The pro-system $\mathrm{TR}^{\cdot}_*(A|K; p, \mathbb{Z}/p^v)$

6.1. In this paragraph, we prove the main theorem of this work. Suppose that K contains the pth roots of unity. Then the canonical map

$$\Sigma^\infty B\mu_{p+} \to K(K),$$

and the fact that for p odd, the Bockstein

$$\bar{\pi}_2(\Sigma^\infty B\mu_{p+}) \xrightarrow{\sim} \pi_1(\Sigma^\infty B\mu_{p+}) = \mu_p$$

is an isomorphism, gives rise to a map

$$\mu_p \to \bar{K}_2(K) = \bar{\pi}_2 K(K).$$

Composing with the cyclotomic trace, we get a map of μ_p to $\bar{\mathrm{TR}}^{\cdot}_2(A|K; p)$. In all, we have a canonical map

$$W.\,\omega_{(A,M)} \otimes S_{\mathbb{F}_p}(\mu_p) \to \bar{\mathrm{TR}}^{\cdot}_*(A|K; p).$$

This is a map of Witt functors with a pre-log structure where on the left hand side, the maps R, F and V act as the identity on $S_{\mathbb{F}_p}(\mu_p)$, and the differential on $S_{\mathbb{F}_p}(\mu_p)$ is trivial.

We consider the composite map

$$W_n\,\omega^*_{(A,M)} \otimes S_{\mathbb{F}_p}(\mu_p) \to \bar{\mathrm{TR}}^n_*(A|K; p) \to \bar{\pi}_* \hat{\mathbb{H}}(C_{p^n}, T(A|K)).$$

The left hand map is an isomorphism in degrees 0 and 1 by theorem 3.3.8, and the right hand map is an isomorphism in all non-negative degrees by addendum 5.6.6. The range of the composite map is given by the spectral sequence $\hat{E}^*(C_{p^n}, K)$ whose structure was determined in the previous paragraph. The result is that

$$\begin{aligned}\hat{E}^\infty(C_{p^n}, K) = \bigoplus_{v=1}^{n-1} & k\langle u_n^\epsilon \tau_K^a \pi_K^r \alpha_K^d \, d\log \pi_K \,|\, v_p\{a, r, d\}_K = v,\ d < \tfrac{p^{v+1}-1}{p-1} - 1\rangle \\ & \oplus\ k\langle \tau_K^a \pi_K^r \alpha_K^d (d\log \pi_K)^\epsilon \,|\, v_p\{a, r, d\}_K \geq n,\ d < \tfrac{p^{n+1}-1}{p-1} - 1\rangle,\end{aligned}$$

where $a \in \mathbb{Z}$, $d \in \mathbb{N}_0$, $\epsilon \in \{0, 1\}$, and $0 \leq r < e_K$, and where

$$\{a, r, d\}_K = (pa - d)e_K/(p-1) + r.$$

The basis for $\hat{E}^{\infty}(C_{p^n}, K)$ as a k-vector space exhibited here will be called the standard basis.

LEMMA 6.1.1. *An element of the standard basis of $\hat{E}^{\infty}(C_{p^n}, K)$ represents a homotopy class in the image of the composite*

$$W_n\, \omega^*_{(A,M)} \otimes S_{\mathbb{F}_p}(\mu_p) \to \bar{\mathrm{TR}}^n_*(A|K;p) \to \bar{\pi}_* \hat{\mathbb{H}}(C_{p^n}, T(A|K))$$

if and only if $\{a, r, d\}_K \geq 0$.

PROOF. The map of the statement is an isomorphism in degrees 0 and 1, and indeed, here $\{a, r, d\}_K$ is automatically non-negative since $a = d$. We must thus show that for all $q \geq 0$ and $\epsilon = 0, 1$, the map

$$\bigoplus_{s \leq 0} \hat{E}^{\infty}_{s,\epsilon - s}(C_{p^n}, K) \to \bigoplus_{s \leq 0} \hat{E}^{\infty}_{s,2q+\epsilon - s}(C_{p^n}, K)$$

induced by multiplication by the qth power of the Bott element is a surjection onto the stated subspace. If we write $q = q_1(p-1) + q_0$ with $0 \leq q_0 < p - 1$, then in the spectral sequence

$$b^q = \pm \tau_K^{q_1} \pi_K^{q_0 e_K/(p-1)} \alpha_K^{q_1 p + q_0},$$

and the statement now follows easily from lemma 5.6.2 by passing to an extension L/K for which $n \leq v_p(e_{L/K})$ and

$$i_* \colon E^{\infty}(C_{p^n}, K) \to E^{\infty}(C_{p^n}, L)$$

a monomorphism. If, for example, a homotopy class is represented in the spectral sequence by the element $\tau_K^a \pi_K^r \alpha_K^d$ then the product of that homotopy class and the qth power of the Bott element is represented by the element $\tau_K^{a'} \pi_K^{r'} \alpha_K^{d'}$ with $\{a', r', d'\}_K = \{a, r, d\}_K$ and $d' - a' = d - a + q$. (The product, of course, may be zero.) $\square$

THEOREM 6.1.2. *Suppose that K contains the pth roots of unity. Then the canonical map*

$$W_{\cdot}\, \omega^*_{(A,M)} \otimes S_{\mathbb{F}_p}(\mu_p) \xrightarrow{\sim} \bar{\mathrm{TR}}^{\cdot}_*(A|K;p)$$

is a pro-isomorphism.

PROOF. Let $E^*_{\cdot}$ denote the pro-system on either side of the map in the statement. The standard filtration, given by

$$\mathrm{Fil}^s E^*_n = V^s E^*_{n-1} + dV^s E^*_{n-1},$$

is a descending filtration with $s \geq 0$. The filtration has length n in level n, i.e. $\mathrm{Fil}^n E^*_n$ is trivial. The map of the statement clearly preserves the filtration. We show that for all $q \geq 0$, there exists $N \geq 1$ such that for all $n \geq 1$ and $0 \leq s < n - N$, the canonical map

$$\mathrm{gr}^s(W_n\, \omega^*_{(A,M)} \otimes S_{\mathbb{F}_p}(\mu_p))_i \to \mathrm{gr}^s\, \bar{\mathrm{TR}}^n_i(A|K;p)$$

is an isomorphism when $0 \leq s < n - N$. Since the structure maps in the pro-systems preserve the standard filtration, the theorem follows.

We have already proved that the map of the statement is an isomorphism in degrees 0 and 1. Hence, it suffices to show that for all $q \geq 0$, there exists $N \geq 1$

such that for all $n \geq 1$, $0 \leq s < n - N$ and $\epsilon = 0, 1$, multiplication by qth power of the Bott element induces an isomorphism

$$\mathrm{gr}^s \,\bar{\mathrm{TR}}^n_\epsilon(A|K;p) \xrightarrow{\sim} \mathrm{gr}^s \,\bar{\mathrm{TR}}^n_{2q+\epsilon}(A|K;p).$$

We claim that any $N \geq 1$ with $(pq+1)e_K/(p-1) < p^N$ will do.

For surjectivity we use the above lemma. Since $d \geq 0$ and $q = a + d$, we have $a \geq -q$, and hence

$$\begin{aligned} \{a, r, d\}_K &= (pa - d)e_K/(p-1) + r = ae_K - qe_K/(p-1) \\ &\geq -pqe_K/(p-1) + r \geq -pqe_K/(p-1) > -p^N. \end{aligned}$$

Therefore, if $v_p\{a, r, d\}_K \geq N$ we have $\{a, r, d\}_K \geq 0$. It follows that multiplication by the qth power of the Bott element induces a surjection of all summands in $E^\infty(C_{p^n}, K)$ except for the summands with $v < N$. But these summands all represent homotopy classes of filtration greater than or equal to $n - N$.

To prove injectivity, we first note that for an element of the standard basis of $\hat{E}^\infty(C_{p^n}, K)$ in total degree $2q + \epsilon$, the requirement that

$$0 \leq d < \frac{p^{v+1} - 1}{p - 1} - 1$$

is equivalent to the requirement that

$$-\frac{pqe_K}{p-1} \leq \{a, r, d\}_K < -\frac{pqe_K}{p-1} + e_K \frac{p^{v+1} - 1}{p - 1}.$$

We show that $v_p\{a, r, d\}_K = v \geq N$ and $\{a, r, d\}_K < e_K(p^{v+1} - 1)/(p-1)$ implies that

$$\{a, r, d\}_K < -\frac{pqe_K}{p-1} + e_K \frac{p^{v+1} - 1}{p - 1}.$$

Note that

$$p^{v+1}e_K/(p-1) - e_K \frac{p^{v+1} - 1}{p - 1} = e_K/(p-1) < p^v,$$

so it suffices to know that

$$\frac{pqe_K}{p-1} < p^v - e_K/(p-1).$$

But this is our assumption on N. This shows that the map induced by multiplication by the qth power of the Bott element induces a monomorphism of all summands in $\hat{E}^\infty(C_{p^n}, K)$ except for the summands with $v < N$. The theorem follows. □

Corollary 6.1.3. *The group* $\mathrm{TR}_{2i}(A|K;p)$ *is uniquely divisible, for* $i > 0$.

Proof. The theorem determines the Bockstein structure on $\bar{\mathrm{TR}}_*(A|K;p)$. For all Bocksteins vanish on $W\omega^*_{(A,M)}$ and

$$\beta_v(\zeta^i) = \zeta^{i-1} d \log \zeta$$

where $v = v_p(i)$. By theorem 3.3.8, $\mathrm{TR}_2(A|K;p)$ is uniquely divisible, so every element of $\bar{\pi}_2 \,\mathrm{TR}(A|K;p)$ is mapped non-trivially by some Bockstein. But then so is every element of $\bar{\mathrm{TR}}_{2i}(A|K;p)$, $i > 0$. □

ADDENDUM 6.1.4. *Suppose that K contains the p^vth roots of unity. Then the map*

$$W.\,\omega^*_{(A,M)} \otimes S_{\mathbb{Z}/p^v}(\mu_{p^v}) \xrightarrow{\sim} \mathrm{TR}^{\cdot}_*(A|K;p,\mathbb{Z}/p^v)$$

is a pro-isomorphism.

PROOF. The proof is by induction on v. Assuming the statement for $v-1$, we show that the Bockstein homomomorphisms

$$\beta_{v-1}\colon\ \mathrm{TR}^{\cdot}_i(A|K;p,\mathbb{Z}/p^{v-1}) \to \mathrm{TR}^{\cdot}_{i-1}(A|K;p,\mathbb{Z}/p)$$

are (Mittag-Leffler) zero, for all $i \geq 0$. The induction step follows readily. Suppose first that $i = 2s+1$ is odd. Inductively, we have pro-isomorphisms

$$W.\,\omega^1_{(A,M)} \otimes \mu^{\otimes s}_{p^{v-1}} \xrightarrow{\sim} \mathrm{TR}^{\cdot}_{2s+1}(A|K;p,\mathbb{Z}/p^{v-1}).$$

The Bockstein is a derivation, and it vanishes on $W.\omega^*_{(A,M)}$. Since the product of odd dimensional classes is zero, the Bockstein vanishes for i odd. When $i = 2s$ even, we consider the commutative diagram

$$\begin{array}{ccc} W.(A) \otimes \mu^{\otimes s}_{p^{v-1}} & \xrightarrow{\sim} & \mathrm{TR}^{\cdot}_i(A|K;p,\mathbb{Z}/p^{v-1}) \\ \Big\downarrow{\scriptstyle \partial} & & \Big\downarrow{\scriptstyle \beta_{v-1}} \\ W.\,\omega^1_{(A,M)} \otimes \mu_p^{\otimes(s-1)} & \xrightarrow{\sim} & \mathrm{TR}^{\cdot}_i(A|K;p,\mathbb{Z}/p). \end{array}$$

The left hand vertical map is given by

$$\partial(x \otimes \zeta_1 \otimes \cdots \otimes \zeta_s) = x \cdot d\log\zeta_1 \otimes \zeta_2 \otimes \cdots \otimes \zeta_s,$$

and since $\zeta_1 \in \mu_{p^{v-1}}$ has a pth root, $d\log\zeta_1$ is divisible by p. It follows that the left hand vertical map is zero, and hence so is the Bockstein homomorphism on the right. □

The last result implies the following algebraic result. It would be desirable to also have an algebraic proof of this fact.

COROLLARY 6.1.5. *If $\mu_{p^v} \subset K$ then the map*

$$W.(A) \otimes \mu_{p^v} \xrightarrow{\sim} {}_{p^v}W.\,\omega^1_{(A,M)},$$

which takes $x \otimes \zeta$ to $x d\log.\,\zeta$, is a pro-isomorphism. □

THEOREM 6.1.6. *There are natural isomorphisms*

$$\begin{aligned} \bar{\mathrm{TC}}_{2s}(A|K;p) &= H^0(K,\mu_p^{\otimes s}) \oplus H^2(K,\mu_p^{\otimes(s+1)}), \\ \bar{\mathrm{TC}}_{2s+1}(A|K;p) &= H^1(K,\mu_p^{\otimes(s+1)}), \end{aligned}$$

valid for $s \geq 0$.

PROOF. Since the extension $K(\mu_p)/K$ is tamely ramified, we may assume that $\mu_p \subset K$. Indeed, it follows from theorem 2.3.1 that the canonical map

$$\bar{\mathrm{TC}}_*(A|K;p) \xrightarrow{\sim} (\bar{\mathrm{TC}}_*(A(\mu_p)|K(\mu_p);p))^{\mathrm{Gal}(K(\mu_p)/K)}$$

is an isomorphism, and the analogous statement holds for $H^*(K,\mu_p^{\otimes s})$. When $\mu_p \subset K$, theorem 6.1.2 shows that the canonical map

$$\bar{\mathrm{TC}}_i(A|K;p) \otimes \mu_p^{\otimes s} \xrightarrow{\sim} \bar{\mathrm{TC}}_{i+2s}(A|K;p)$$

is an isomorphism, for all $s, i \geq 0$. It will therefore suffice to prove the statement in degrees 0 and 1.

In degree one, the cyclotomic trace induces an isomorphism

$$K^\times / K^{\times p} = K_1(K, \mathbb{Z}/p) \xrightarrow{\sim} \mathrm{TC}_1(A|K; p, \mathbb{Z}/p),$$

and by Kummer theory, the left hand side is $H^1(K, \mu_p)$. In degree zero, we use that theorem 1.8.7 gives an exact sequence

$$0 \to \mathrm{TC}_0(A; p, \mathbb{Z}/p) \to \mathrm{TC}_0(A|K; p, \mathbb{Z}/p) \to \mathrm{TC}_{-1}(k; p, \mathbb{Z}/p) \to 0.$$

In this sequence, the left hand term is naturally isomorphic to $\mathbb{Z}/p$, and the left hand map has a natural retraction given by

$$\mathrm{TC}_0(A|K; p, \mathbb{Z}/p) \to \mathrm{TR}_0(A|K; p, \mathbb{Z}/p)^F = \mathbb{Z}/p.$$

It remains to show that the right hand term in the sequence is naturally isomorphic to $H^2(K, \mu_p)$. We recall from [**32**, p. 186] that there is a natural short exact sequence

$$0 \to H^2(k, \mu_p) \to H^2(K, \mu_p) \to H^1(k, \mathbb{F}_p) \to 0.$$

Moreover, since k is perfect, the left hand term vanishes. The normal basis theorem, $H^i(k, \mathbb{G}_a)$ vanishes for $i > 0$, and hence the cohomology sequence associated with the sequence

$$0 \to \mathbb{F}_p \to \mathbb{G}_a \xrightarrow{\varphi - 1} \mathbb{G}_a \to 0$$

gives a natural isomorphism

$$k_\varphi \xrightarrow{\sim} H^1(k, \mathbb{F}_p).$$

Since k is perfect, the restriction $W(k) \to k$ induces a natural isomorphism

$$\mathrm{TC}_{-1}(k; p, \mathbb{Z}/p) = W(k)_F / pW(k)_F \xrightarrow{\sim} k_\varphi$$

which proves the claim. $\square$

REMARK 6.1.7. When $\mu_p \subset K$, we can also give the following non-canonical description of the groups $\mathrm{TC}_*(A|K; p, \mathbb{Z}/p)$. Let $\zeta \in \mu_p$ be a generator, let b be the corresponding Bott element, and let $\pi \in A$ be a uniformizer. Then for $s \geq 0$,

$$\begin{aligned} \mathrm{TC}_{2s}(A|K; p, \mathbb{Z}/p) &= \mathbb{F}_p\langle b^s \rangle \oplus k_\varphi \langle \partial(d\log \pi \cdot b^s) \rangle, \\ \mathrm{TC}_{2s+1}(A|K; p, \mathbb{Z}/p) &= \mathbb{F}_p \langle d\log \pi \cdot b^s \rangle \oplus k_\varphi \langle \partial(b^{s+1}) \rangle \oplus k^{e_K}, \end{aligned}$$

where k_φ is the cokernel of $1 - \varphi\colon k \to k$ and e_K is the ramification index. The summand k^{e_K} in the second line is in the kernel of

$$1 - F\colon \mathrm{TR}_{2s+1}(A|K; p, \mathbb{Z}/p) \to \mathrm{TR}_{2s+1}(A|K; p, \mathbb{Z}/p),$$

with the inclusion

$$\eta\colon k^{e_K} = \bigoplus_{i=0}^{e_K - 1} k \to \mathrm{TR}_{2s+1}(A|K; p, \mathbb{Z}/p)$$

given by

$$\eta_i(a) = \sum_{v \geq 0} a^{p^{-v}(\frac{p^{v+1}-1}{p-1})} u_K(\underline{\pi})^{-p} dV_\pi^v(\underline{\pi}^i) \cdot b^s + \sum_{v > 0} F^v(a u_K(\underline{\pi})^{-p} d(\underline{\pi}^i)) \cdot b^s.$$

The sum on the right is finite and the sum on the left converges.

We shall need a special case of the Thomason-Godement construction of the hyper cohomology spectrum associated with a presheaf of spectra on a site, [**8**, §3]. Suppose that F is a functor which to every finite subextension L/K in an algebraic closure $\bar{K}/K$ assigns a spectrum $F(L)$. For the purpose of this paper, we shall write

$$F^{\text{ét}}(K) = \underset{\overrightarrow{L/K}}{\operatorname{holim}}\, \mathbb{H}^{\cdot}(G_{L/K}, F(L)). \tag{6.1.8}$$

There is a natural strongly convergent spectral sequence

$$E^2_{s,t} = H^{-s}(K, \varinjlim_{L/K} \pi_t F(L)) \Rightarrow \pi_{s+t} F^{\text{ét}}(K), \tag{6.1.9}$$

which is obtained by passing to the limit from the spectral sequences for the group cohomology spectra

$$E^2_{s,t} = H^{-s}(G_{L/K}, \pi_t F(L)) \Rightarrow \pi_{s+t}\mathbb{H}^{\cdot}(G_{L/K}, F(L)).$$

Indeed, filtered colimits are exact so we get a spectral sequence with abutment

$$\varinjlim_{L/K} \pi_* \mathbb{H}^{\cdot}(G_{L/K}, F(L)) \xrightarrow{\sim} \pi_* F^{\text{ét}}(K),$$

and the identification of the E^2-term follows from the isomorphism

$$\begin{aligned} \varinjlim_{L/K} H^*(G_{L/K}, \pi_* F(L)) &\xrightarrow{\sim} \varinjlim_{L/K} H^*(G_{L/K}, (\varinjlim_{N/L} \pi_* F(N))^{G_L}) \\ &= H^*(K, \varinjlim_{N/K} \pi_* F(N)). \end{aligned}$$

This isomorphism, which can be found in [**33**, §2 proposition 8], is a special case of the general fact that on a site with enough points, the Godement construction of a presheaf calculates the sheaf cohomology of the associated sheaf.

THEOREM 6.1.10. *The canonical map*

$$\gamma_K \colon K_*(K, \mathbb{Z}/p^v) \to K_*^{\text{ét}}(K, \mathbb{Z}/p^v)$$

is an isomorphism in degrees ≥ 1.

PROOF. It suffices to consider the case $v = 1$. In the diagram

$$\begin{tikzcd} K(K) \arrow[r, "\gamma_K"] \arrow[d, "\mathrm{tr}"] & K^{\text{ét}}(K) \arrow[d, "\mathrm{tr}"] \\ \mathrm{TC}(A|K) \arrow[r, "\gamma_K"] & \mathrm{TC}^{\text{ét}}(A|K; p), \end{tikzcd}$$

the left hand vertical map induces an isomorphism on homotopy groups with $\mathbb{Z}/p$-coefficients in degrees ≥ 1. We use theorem 6.1.6 to prove that the right hand vertical map induces an isomorphism on homotopy groups with $\mathbb{Z}/p$-coefficients and that the lower horizontal map induces an isomorphism on homotopy groups with $\mathbb{Z}/p$-coefficients in degrees ≥ 0. This proves the theorem.

We first prove the statement for the map induced from the cyclotomic trace

$$K^{\text{ét}}(K) \to \mathrm{TC}^{\text{ét}}(A|K; p).$$

The spectral sequence (6.1.9) for K-theory with $\mathbb{Z}/p$-coefficients takes the form

$$E^2_{s,t} = H^{-s}(K, \mu_p^{\otimes(t/2)}) \Rightarrow K^{\text{ét}}_{s+t}(K, \mathbb{Z}/p).$$

Indeed, this follows from the simple fact that K-theory commutes with filtered colimits and from the celebrated theorem of Suslin [**35**] that

$$K_t(\bar{K}, \mathbb{Z}/p) = \mu_p^{\otimes(t/2)}.$$

Similarly, it follows immediately from theorem 6.1.6 that also the spectral sequence (6.1.9) for topological cyclic takes the form

$$E^2_{s,t} = H^{-s}(K, \mu_p^{\otimes(t/2)}) \Rightarrow \mathrm{TC}^{\text{ét}}_{s+t}(A|K; p, \mathbb{Z}/p).$$

Finally, it is clear that the cyclotomic trace induces an isomorphism of E^2-terms.

It remains to show that the map

$$\gamma_K \colon \mathrm{TC}_i(A|K; p, \mathbb{Z}/p) \to \mathrm{TC}^{\text{ét}}_i(A|K; p, \mathbb{Z}/p)$$

is an isomorphism for $i \geq 0$. The domain and range of γ_K are abstractly isomorphic in this range, so we just need to show that γ_K is an isomorphism for $i \geq 0$. By theorem 2.3.1 we may assume that $\mu_p \subset K$ and that the residue field k is algebraically closed. When $\mu_p \subset K$, we have a commutative square

$$\begin{array}{ccc}
\mathrm{TC}_i(A|K; p, \mathbb{Z}/p) \otimes \mu_p^{\otimes s} & \xrightarrow{\gamma_K \otimes \mathrm{id}} & \mathrm{TC}^{\text{ét}}_i(A|K; p, \mathbb{Z}/p) \otimes \mu_p^{\otimes s} \\
\downarrow \wr & & \downarrow \wr \\
\mathrm{TC}_{i+2s}(A|K; p, \mathbb{Z}/p) & \xrightarrow{\gamma_K \otimes \mathrm{id}} & \mathrm{TC}^{\text{ét}}_{i+2s}(A|K; p, \mathbb{Z}/p),
\end{array}$$

and the vertical maps are isomorphism for $i, s \geq 0$. Hence, it suffices to show that γ_K is an isomorphism in degrees 0 and 1. And when k is algebraically closed, the term $H^2(K, \mu_p)$ in degree zero vanishes. Thus the edge homomorphism of the spectral sequence (6.1.9),

$$\epsilon_K \colon \mathrm{TC}^{\text{ét}}_i(A|K; p, \mathbb{Z}/p) \to H^0(K, \mu_p^{i/2}),$$

is an isomorphism in degree zero, and since the composite

$$\mathrm{TC}_0(A|K; p, \mathbb{Z}/p) \xrightarrow{\gamma_K} \mathrm{TC}^{\text{ét}}_0(A|K; p, \mathbb{Z}/p) \xrightarrow{\epsilon_K} H^0(K, \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism, then so is γ_K. In degree one, we use the spectral sequence (6.1.9) for topological cyclic homology with $\mathbb{Q}_p/\mathbb{Z}_p$-coefficients. As a G_K-module

$$\varinjlim_{L/K} \mathrm{TC}_1(B|\,L; p, \mathbb{Q}_p/\mathbb{Z}_p) \xleftarrow{\sim} \varinjlim_{L/K} K_1(L, \mathbb{Q}_p/\mathbb{Z}_p) \xrightarrow{\sim} K_1(\bar{K}, \mathbb{Q}_p/\mathbb{Z}_p) = \mu_{p^\infty},$$

and the composite

$$\mathrm{TC}_1(A|K; p, \mathbb{Q}_p/\mathbb{Z}_p) \xrightarrow{\gamma_K} \mathrm{TC}^{\text{ét}}_1(A|K; p, \mathbb{Q}_p/\mathbb{Z}_p) \xrightarrow{\epsilon_K} H^0(K, \mu_{p^\infty})$$

is an isomorphism. It follows that γ_K is an isomorphism in degree one. $\square$

6.2. It has long been known that for K-theory with $\mathbb{Z}/p$-coefficients, Galois descent and Bott periodicity are equivalent, [**37**]. Thus in view of theorem 6.1.2 it seems reasonable to expect that the canonical map

$$\mathrm{TR}(A|K;p) \to \mathrm{TR}^{\text{ét}}(A|K;p))$$

induces an isomorphism of homotopy groups with $\mathbb{Z}/p$-coefficients in degrees ≥ 0. We discuss how this may occur.

Suppose that $\mu_p \subset K$. There is a natural exact sequence of pro-$W.(A)$-modules

$$0 \to W.(A) \otimes \mu_p \to W.\,\omega^1_{(A,M)} \xrightarrow{p} W.\,\omega^1_{(A,M)},$$

and the right map is surjective for the étale topology on $\operatorname{Spec} K$. A choice of primitive pth root of unity identifies the left hand term with $W.(A)/p$. The Galois group $G_{L/K}$ acts on $W.\,\omega^*_{(B,M)}$ and on $\mathrm{TR}^{\cdot}_*(B|\,L;p)$. In other words, we have étale pro-sheaves $W.\,\omega^*$ and $\bar{\mathcal{TR}}^{\cdot}_*$ on $\operatorname{Spec} K$. In analog with (6.1.9), we have a spectral sequence

$$E^2_{s,t} = H^{-s}_{\text{cont}}(K, \varinjlim_{L/K} \bar{\mathrm{TR}}^{\cdot}_t(B|\,L;p)) \Rightarrow \bar{\pi}_{s+t}(\operatorname{holim}_{\overleftarrow{n}}(\mathrm{TR}^n)^{\text{ét}}(A|K;p)),$$

where the E^2-term is given by the continuous cohomology in the sense of Jannsen, [**18**]. By theorem 6.1.2, we can replace the pro-system $\bar{\mathcal{TR}}^{\cdot}$ by the pro-system $W.\omega^* \otimes S_{\mathbb{F}_p}(\mu_p)$. Here μ_p is a trivial $G_{L/K}$-module, since $\mu_p \subset K$. We expect that

$$H^i_{\text{cont}}(K, W.\,\omega^1) = \begin{cases} W.\,\omega^1_{(A,M)}, & i = 0, \\ 0, & i > 0. \end{cases}$$

Indeed, if this was true, the short exact sequence of étale pro-sheaves on $\operatorname{Spec} K$,

$$0 \to W.(-) \otimes \mu_p \to W.\,\omega^1 \xrightarrow{p} W.\,\omega^1 \to 0,$$

shows that

$$H^i_{\text{cont}}(K, W.(-)/p) = \begin{cases} W(A)/p, & i = 0, \\ W\,\omega^1_{(A,M_A)}/p, & i = 1, \\ 0, & i \geq 2, \end{cases}$$

which implies descent in non-negative degrees for $\mathrm{TR}(A|K;p)$.

The expected values of the cohomology groups above might appear somewhat surprising since the canonical map

$$W_n(A)/p \to H^0(K, W_n(-)/p) \tag{6.2.1}$$

is not an isomorphism for any n. It is injective, but has a big cokernel, even for $n = 1$. However, this can be understood as follows. The sequence the pro-sheaves

$$0 \to W.(-)/p \xrightarrow{V^n} W.(-)/p \to W_n(-)/p \to 0$$

yields a long-exact sequence

$$\begin{aligned} 0 &\to W(A)/p \xrightarrow{V^n} W(A)/p \to H^0(K, W_n(-)/p) \\ &\to W\,\omega^1_{(A,M)}/p \xrightarrow{V^n} W\omega^1_{(A,M)}/p \to H^1(K, W_n(-)/p) \to 0, \end{aligned}$$

which identifies the cokernel of (6.2.1) with the kernel of the map V^n in the lower line. This map is zero for n large and its image is a finite dimensional k-vector space for all n. This explains the large cokernels in (6.2.1).

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