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ASYMPTOTICALLY SPLIT EXTENSIONS AND E-THEORY

VLADIMIR MANUILOV AND KLAUS THOMSEN

Abstract. We show that the E-theory of Connes and Higson can be formulated in terms of C^*-extensions in a way quite similar to the way in which the KK-theory of Kasparov can. The essential difference is that the role played by split extensions are taken by extensions which split asymptotically, i.e. for which the quotient map admits right-inverses which form an asymptotic homomorphism. An extension is called semi-invertible when it can be made asymptotically split by adding another extension to it. Our main result establishes a bijective correspondance between homotopy classes of asymptotic homomorphisms from SA to $B \otimes \mathcal{K}$ and homotopy classes of semi-invertible extensions of S^2A by $B \otimes \mathcal{K}$.

1. Introduction

Connes and Higson introduced in [1] a construction which produces an asymptotic homomorphism out of an extension of C^*-algebras. The Connes-Higson construction is the backbone of E-theory and gives us a way to study C^*-extensions via asymptotic homomorphisms. Such a translation can be quite powerful within the territory of KK-theory where the C^*-extensions are semi-split, i.e. admit a completely positive contraction as a right-inverse for the quotient map. It is namely known that the Connes-Higson construction sets up a bijection between homotopy classes of semi-split extensions and completely positive asymptotic homomorphisms. This bijection is particularly useful because completely positive asymptotic homomorphisms are easier to handle than general ones, and because the powerful homotopy invariance results of Kasparov allows one to translate homotopy information to more algebraic information about the C^*-extensions. This well-behaved correspondance between semi-split C^*-extensions and homotopy classes of completely positive asymptotic homomorphism was used in [5] to obtain a better understanding of the short exact sequence of the UCT-theorem by identifying the kernel of the map from $KK(A, B) = \mathrm{Ext}^{-1}(SA, B)$ to $KL(A, B)$ as the group arising from the weakly quasi-diagonal extensions of SA by $B \otimes \mathcal{K}$. The present paper originated in the desire to extend the nice relation between C^*-extensions and asymptotic homomorphisms beyond the case of semi-split extensions. The key problem in this connection is (at least for the moment) to decide if the Connes-Higson construction is injective in general. In other words, the problem is to decide if two C^*-extensions - with stable and maybe suspended ideals - which give rise to homotopic asymptotic homomorphisms must themselves be homotopic. From [3] we know that this is the case when both extensions are suspensions and the result of the present paper shows that it is also the case when both extensions are what we call semi-invertible and the quotient C^*-algebra is a double suspension. But in general we still don't know the answer. Nonetheless, we shall show here that there is a way to faithfully represent E-theory by use of C^*-extensions which does not require infinitely many suspensions as in [3]

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or longer decomposition series as in [2]. To describe this, let A and B be separable C^*-algebras and assume for simplicity that B is stable. We call an extension of A by B *asymptotically split* when there is a family $(\pi_t)_{t\in[1,\infty)}$ of right-inverses for the quotient map such that $(\pi_t)_{t\in[1,\infty)}$ is an asymptotic homomorphism. An extension is then *semi-invertible* when it can be made asymptotically split by adding another extension to it. We prove that

1) Every asymptotic homomorphism $S^2A \to B$ is homotopic to the Connes-Higson construction of a semi-invertible extension of SA by B.
2) Two semi-invertible extensions of S^2A by B are homotopic (as semi-invertible extensions) if and only if the Connes-Higson construction applied to them give homotopic asymptotic homomorphisms.

These results show that the E-theory of Connes and Higson can be formulated in terms of C^*-extensions in a way quite similar to the way in which the KK-theory of Kasparov can. The essential difference is that the role played by split extensions should be taken by asymptotically split extensions. It is our hope that this parallel between the way KK-theory and E-theory can be described in terms of C^*-extensions can be strenghtened even further. In particular it would be nice if some of the suspensions occuring 1) and 2) could be removed and if one could substitute homotopy with a more algebraic relation in the description of E-theory.

2. Asymptotically split extensions and $\mathrm{Ext}^{-1/2}$

In the following A and B are separable C^*-algebras, B stable. Let $M(B)$ denote the multiplier algebra of B, $Q(B) = M(B)/B$ the corresponding corona algebra and $q_B : M(B) \to Q(B)$ the quotient map. We shall identify the set of extensions of A by B with $\mathrm{Hom}(A, Q(B))$. Two extensions $\varphi,\ \psi : A \to Q(B)$ are *unitarily equivalent* when there is a unitary $w \in M(B)$ such that $\mathrm{Ad}\, q_B(w) \circ \varphi = \psi$. As is wellknown the set of unitary equivalence classes of extensions of A by B form a semi-group and we denote this semi-group by $\mathrm{Ext}(A, B)$. An extension $\varphi : A \to Q(B)$ will be called *asymptotically split* when there is an asymptotic homomorphism $\pi = \{\pi_t\}_{t\in[1,\infty)} : A \to M(B)$ such that $q_B \circ \pi_t = \varphi$ for all t. We say that an extension $\varphi : A \to Q(B)$ is *semi-invertible* when there is an extension ψ such that $\varphi \oplus \psi : A \to Q(B)$ is asymptotically split. Two semi-invertible extensions are called *stably equivalent* when they become unitarily equivalent after addition by asymptotically split extensions. This is an equivalence relation on the subset of semi-invertible extensions in $\mathrm{Hom}(A, Q(B))$ and the corresponding equivalence classes form an abelian group which we denote by $\mathrm{Ext}^{-1/2}(A, B)$. $\mathrm{Ext}^{-1/2}$ is a bifunctor which is contravariant in the first variable, A, and covariant with respect to quasi-unital $*$-homomorphisms in the second variable, B. It is easy to see that the Connes-Higson construction, [1], annihilates asymptotically split extensions and therefore gives rise to a group homomorphism

$$CH : \mathrm{Ext}^{-1/2}(A, B) \to [[SA, B]] .$$

To describe our main result about this map we introduce the notion of homotopy between semi-invertible extensions. Two semi-invertible extensions

$$0 \longrightarrow B \longrightarrow E_1 \longrightarrow A \longrightarrow 0$$

and

$$0 \longrightarrow B \longrightarrow E_2 \longrightarrow A \longrightarrow 0$$

are *homotopic* when there is a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] & E_1 \arrow[r] & A \arrow[r] & 0 \\
0 \arrow[r] & C[0,1] \otimes B \arrow[r] \arrow[u, "\pi_1"] \arrow[d, "\pi_0"'] & E \arrow[r] \arrow[u] \arrow[d] & A \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & E_2 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

of semi-invertible extensions. The $*$-homomorphisms $\pi_0,\ \pi_1 : C[0,1] \otimes B \to B$ are here the surjections obtained from evalution at the endpoints of $[0,1]$.

The main tool in this paper is the map E introduced in [5]. We recall the construction here. Given an asymptotic homomorphism $\varphi = \{\varphi_t\}_{t\in[1,\infty)} : A \to B$ we choose a discretization $\{\varphi_{t_i}\}_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} t_i = \infty$ and $\lim_{i\to\infty} \sup_{t\in[t_i,t_{i+1}]} \|\varphi_t(a) - \varphi_{t_i}(a)\| = 0$ for all $a \in A$. To define from such a discretization a map $\Phi : A \to \mathcal{L}_B(l_2(\mathbb{Z}) \otimes B)$ we introduce the standard matrix units e_{ij}, $i,j \in \mathbb{Z}$, which act on the Hilbert B-module $l_2(\mathbb{Z}) \otimes B$ in the obvious way. Then

$$\Phi(a) = \sum_{i\geq 1} \varphi_{t_i}(a) e_{ii}$$

defines a map $\Phi : A \to \mathcal{L}_B(l_2(\mathbb{Z}) \otimes B)$. We identify $\mathcal{K} \otimes B$ with the B-compact operators in $\mathcal{L}_B(l_2(\mathbb{Z}) \otimes B)$ and observe that Φ is a $*$-homomorphism modulo $\mathcal{K} \otimes B$. Furthermore, $\Phi(a)$ commutes modulo $\mathcal{K}\otimes B$ with the two-sided shift $T = \sum_{j\in\mathbb{Z}} e_{j,j+1}$. So we get in this way a $*$-homomorphism

$$E(\varphi) : A \to Q(\mathcal{K} \otimes B) = \mathcal{L}_B(l_2(\mathbb{Z}) \otimes B)/\mathcal{K} \otimes B$$

such that

$$E(\varphi)(f \otimes a) = f(\underline{T})\underline{\Phi(a)}\ ,\ f \in C(\mathbb{T}),\ a \in A\ .$$

Here and in the following we denote by $\underline{S}$ the image in $Q(\mathcal{K} \otimes B) = \mathcal{L}_B(l_2(\mathbb{Z}) \otimes B)/\mathcal{K} \otimes B$ of an element $S \in \mathcal{L}_B(l_2(\mathbb{Z}) \otimes B)$.

Lemma 2.1. *$E(\varphi) \in \mathrm{Ext}^{-1/2}(C(\mathbb{T}) \otimes A, \mathcal{K} \otimes B)$.*

Proof. Let $-E(\varphi) : C(\mathbb{T}) \otimes A \to Q(\mathcal{K} \otimes B)$ be the extension which results when we in the construction of $E(\varphi)$ use

$$\Psi(a) = \sum_{i\leq 0} \varphi_{t_{-i+1}}(a) e_{ii}$$

instead of Φ. Then $-E(\varphi) \oplus E(\varphi)$ is unitary equivalent to an extension $\psi : C(\mathbb{T}) \otimes A \to Q(\mathcal{K} \otimes B)$ such that $\psi(a) = \underline{\pi_t(a)}$ for all $t \in [1,\infty)$, where $\pi_t : C(\mathbb{T}) \otimes A \to \mathcal{L}_B(l_2(\mathbb{Z}) \otimes B)$, $t \in [1,\infty)$, is an asymptotic homomorphism obtained by convex interpolation of maps π_n, $n \in \mathbb{N}$, with the property that

$$\pi_n(f \otimes a) - f(T)\Big(\sum_{|i|\leq n} \varphi_{t_n}(a) e_{ii} + \sum_{i>n} \varphi_{t_i}(a) e_{ii} + \sum_{i<-n} \varphi_{t_{-i+1}}(a) e_{ii}\Big) \in \mathcal{K} \otimes B$$

and

$$\lim_{n\to\infty} \pi_n(f\otimes a) - f(T)(\sum_{|i|\leq n} \varphi_{t_n}(a)e_{ii} + \sum_{i>n} \varphi_{t_i}(a)e_{ii} + \sum_{i<-n} \varphi_{t_{-i+1}}(a)e_{ii}) = 0 ,$$

$f \in C(\mathbb{T}),\ a \in A.$ □

Let $\mathrm{Ext}^{-1/2}(A,B)_h$ denote the abelian group of homotopy classes of semi-invertible extensions of A by B. $\mathrm{Ext}^{-1/2}(A,B)_h$ is then a quotient of $\mathrm{Ext}^{-1/2}(A,B)$. By homotopy invariance of the Connes-Higson construction we get a map $CH : \mathrm{Ext}^{-1/2}(A,B)_h \to [[SA,B]]$. Thanks to Lemma 2.1 we get from the above construction a well-defined map

$$E : [[A,B]] \to \mathrm{Ext}^{-1/2}(C(\mathbb{T})\otimes A, B)_h ,$$

cf. [5]. By pulling back along the canonical inclusion $SA \subseteq C(\mathbb{T})\otimes A$ we can also consider E as a map $E : [[A,B]] \to \mathrm{Ext}^{-1/2}(SA,B)_h$. Our main result can now be formulated as follows.

Theorem 2.2. *a)* $CH : \mathrm{Ext}^{-1/2}(SA,B) \to [[S^2A,B]]$ *is surjective.*

b) $E : [[SA,B]] \to \mathrm{Ext}^{-1/2}(S^2A,B)_h$ *is an isomorphism.*

The proof of a) does not require new constructions and follows essentially from [5]. Indeed, observe that by Lemma 5.5 of [5] there is a commuting diagram

$$\begin{tikzcd} \mathrm{Ext}^{-1/2}(S^2A,B)_h \arrow[dr, "CH"] & \\ [[SA,B]] \arrow[u, "E"] \arrow[r, "\chi"'] & [[S^3A,B]] \end{tikzcd} \tag{2.1}$$

where χ is the map obtained by taking the composition product with the asymptotic homomorphism $S^3A \to SA\otimes\mathcal{K}$ which is the suspension of the asymptotic homomorphism obtained by applying the Connes-Higson construction to the (reduced) Toeplits extension tensored with A, cf. [5]. Since χ is an isomorphism, it follows that $CH : \mathrm{Ext}^{-1/2}(S^2A,B) \to [[S^3A,B]]$ is surjective. But the inverse in E-theory of the asymptotic homomorphism defining χ is a genuine $*$-homomorphism $\mu : SA \to S^3A\otimes M_2$ and the naturality of the Connes-Higson construction gives us a commuting diagram

$$\begin{tikzcd} \mathrm{Ext}^{-1/2}(SA,B) \arrow[d, "CH"'] & \mathrm{Ext}^{-1/2}(S^3A\otimes M_2,B) \arrow[l, "\mu^*"'] \arrow[d, "CH"] \\ [[S^2A,B]] & [[S^4A\otimes M_2,B]] \arrow[l, "\mu^*"] \end{tikzcd}$$

We see that this proves a) of Theorem 2.2.

To complete the proof Theorem 2.2 it now suffices to show that the CH-map of diagram (2.1) is injective. The rest of the paper is devoted to this.

3. Proof of b) of Theorem 2.2

Given two commuting unitaries S, T in a C^*-algebra, we define a projection $P(S,T)$ in the 2×2 matrices over the C^*-algebra generated by S and T in the following way. Let $s, c_0, c_1 : [0,1] \to \mathbb{R}$ be the functions

$$c_0(t) = |\cos(\pi t)| 1_{[0,\frac{1}{2}]}(t)\ ,\ c_1(t) = |\cos(\pi t)| 1_{(\frac{1}{2},1]}(t)\ ,\ s(t) = \sin(\pi t)\ .$$

Set $\tilde{g} = sc_0$, $\tilde{h} = sc_1$ and $\tilde{f} = s^2$. Since $\tilde{f}$, $\tilde{g}$ and $\tilde{h}$ are continuous and 1-periodic they give rise to continuous function, f, g, h, on $\mathbb{T}$. Set

$$P(S,T) \ = \ \begin{pmatrix} f(S) & g(S) + h(S)T \\ h(S)T^* + g(S) & 1 - f(S) \end{pmatrix}\ ,$$

cf. [4]. In particular, this gives us a projection $P \in C(\mathbb{T}^2) \otimes M_2$ when we apply the recipe to the canonical generating unitaries of $C(\mathbb{T}^2)$. Note that P is an element of $M_2((SC(\mathbb{T}))^+) \subseteq M_2(C(\mathbb{T}^2))$. In general, $P(S,T)$ is in the range of $\mathrm{id}_{M_2} \otimes \lambda$ where $\lambda : (SC(\mathbb{T}))^+ \to C^*(S,T)$ is the unital $*$-homomorphism with

$$\lambda((1 - e^{2\pi i x}) \otimes 1) \ = \ 1 - S\ ,\ \lambda((1 - e^{2\pi i x}) \otimes e^{2\pi i y}) = T - ST\ .$$

Consider also the projection

$$P_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M_2 \subseteq C(\mathbb{T}^2) \otimes M_2\ .$$

We can then define a map

$$\mathrm{Bott}_A : \mathrm{Ext}^{-1/2}(C(\mathbb{T}^2) \otimes A, B)_h \to \mathrm{Ext}^{-1/2}(A, B)_h$$

such that

$$[\varphi] \mapsto [(\mathrm{id}_{M_2} \otimes \varphi) \circ b_A] - [(\mathrm{id}_{M_2} \otimes \varphi) \circ b_0]$$

where $b_A, b_0 : A \to M_2(C(\mathbb{T}^2)) \otimes A$ are the maps $b_A(a) = P \otimes a$ and $b_0(a) = P_0 \otimes a$, respectively. The main part of the proof will be to establish the following.

Proposition 3.1. *Let $i : SA \to C(\mathbb{T}) \otimes A$ be the canonical embedding, $e : C(\mathbb{T}) \otimes A \to A$ the map obtained from evaluation at $1 \in \mathbb{T}$ and $c : A \to C(\mathbb{T}) \otimes A$ the $*$-homomorphism which identifies A with the constant A-valued functions over $\mathbb{T}$. Then*

$$-\mathrm{Bott}_{SA} \circ E \circ CH([\psi] - e^* \circ c^*[\psi]) = i^*[\psi]$$

in $\mathrm{Ext}^{-1/2}(SA, B)_h$ for every semi-invertible extension $\psi \in \mathrm{Hom}(C(\mathbb{T}) \otimes A, Q(B))$.

To begin the proof of Proposition 3.1, observe that $c^*([\psi] - e^* \circ c^*([\psi])) = 0$ in $\mathrm{Ext}^{-1/2}(A, B)$. We can therefore add an asymptotically split extension χ to $c^*(\psi - e^* \circ c^*(\psi))$ such that the resulting extension is asymptotically split. It follows that

$$\psi' = \psi - e^* \circ c^*(\psi) + e^*(\chi)$$

is a semi-invertible extension of $C(\mathbb{T}) \otimes A$ by B such that $i^*[\psi'] = i^*[\psi]$ and $c^*(\psi')$ is an asymptotically split extension of A by B. Since $CH[e^*(\chi)] = (Se)^*(CH[\chi]) = 0$ because χ is asymptotically split, it suffices (by using ψ' instead of ψ) to consider a semi-invertible extension $\psi \in \mathrm{Hom}(C(\mathbb{T}) \otimes A, Q(B))$ with the property that $c^*(\psi)$ asymptotically splits, and show that $\mathrm{Bott}_{SA} \circ E \circ CH[\psi] = i^*[\psi]$. So let ψ be such an extension and set $\varphi = \psi \circ i$.

Lemma 3.2. *Let $e^{2\pi ix}$ denote the identity function of the circle $\mathbb{T}$. There is a unitary $U \in M(M_2(B))$ such that*

$$\begin{pmatrix} \psi(e^{2\pi ix}f\otimes a) & \\ & 0 \end{pmatrix} = q_{M_2(B)}(U)\begin{pmatrix} \psi(f\otimes a) & \\ & 0 \end{pmatrix}$$

for all $f \in C(\mathbb{T})$, $a \in A$.

Proof. We use the well-known fact that a surjective $*$-homomorphism between separable C^*-algebras admits a surjective unital extension to a $*$-homomorphism between the multiplier algebras. The $*$-homomorphism ψ extends to a unital $*$-homomorphism $\hat{\psi} : M(C(\mathbb{T}) \otimes A) \to M(\psi(C(\mathbb{T}) \otimes A))$. Then $V = \hat{\psi}(e^{2\pi ix} \otimes 1_A)$ is a unitary in $M(\psi(C(\mathbb{T})\otimes A))$. ($1_A$ means here the unit in $M(A)$ and hence $e^{2\pi ix}\otimes 1_A$ is really just the identity function of $\mathbb{T}$ considered as a unitary multiplier of $C(\mathbb{T}) \otimes A$.) Set $D = q_B^{-1}(\psi(C(\mathbb{T}) \otimes A)) \subseteq M(B)$. Since q_B maps D onto $\psi(C(\mathbb{T}) \otimes A))$ it extends to a surjective unital $*$-homomorphism $\widehat{q_B} : M(D) \to M(\psi(C(\mathbb{T}) \otimes A))$. Since $\begin{pmatrix} V & \\ & V^* \end{pmatrix}$ is in the connected component of 1 in $M_2(M(\psi(C(\mathbb{T})\otimes A)))$ there is a unitary $U \in M_2(M(D))$ such that

$$\mathrm{id}_{M_2} \otimes \widehat{q_B}(U) = \begin{pmatrix} V & \\ & V^* \end{pmatrix} .$$

Note that $M(D) \subseteq M(B)$ since B is an essential ideal in D. We can therefore regard U as a unitary in $M_2(M(B))$. It is then clear that U has the stated property. □

It follows from Lemma 3.2 that after adding 0 to ψ and φ, we may assume that there is a unitary $w \in M(B)$ such that

$$q_B(w)\psi(f \otimes a) = \psi(e^{2\pi ix}f \otimes a) , \ f \in C(\mathbb{T}), \ a \in A . \tag{3.1}$$

Let $\{\pi_t\}_{t\in[1,\infty)} : A \to M(B)$ be an asymptotic homomorphism such that $\psi(1 \otimes a) = q_B(\pi_t(a))$ for all a and t.

Lemma 3.3. *Let $\{u_t\}_{t\in[1,\infty)}$ be a continuous approximate unit for B such that $\lim_{t\to\infty} u_t\pi_1(a) - \pi_1(a)u_t = 0$ for all $a \in A$. There is then an increasing continuous function $r : [1,\infty) \to [1,\infty)$ such that $r(t) \geq t$ for all $t \in [1,\infty)$ and $\lim_{t\to\infty} f(u_{r(t)})\pi_1(a) - f(u_{r(t)})\pi_t(a) = 0$ for all $a \in A, \ f \in C_0(0,1)$.*

Proof. By the Bartle-Graves selection theorem there is a continuous function $\chi : A \to M(B)$ such that $\chi(a) - \pi_1(a) \in B$ for all A. The same selection theorem also provides us with an equicontinuous asymptotic homomorphism $\pi' = (\pi'_t) : A \to M(B)$ such that $\lim_{t\to\infty} \pi_t(a) - \pi'_t(a) = 0$ for all $a \in A$. Let $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$ be a sequence of finite subsets with dense union in A. By using that $\{\pi_t(a) - \chi(a) : t \in [1,n], \ a \in F_n\}$ is a compact subset of B for all n, it is then straightforward to construct an r with $r(t) \geq t$ such that $\lim_{t\to\infty} u_{r(t)}\pi_t(a) - u_{r(t)}\chi(a) = 0$ for all $a \in \bigcup_n F_n$. It follows that $\lim_{t\to\infty} u_{r(t)}\pi'_t(a) - u_{r(t)}\chi(a) = 0$ for all $a \in \bigcup_n F_n$, and by continuity of χ and equicontinuity of $\{\pi'_t\}$ it follows that this actually holds for all $a \in A$. But then $\lim_{t\to\infty} u_{r(t)}\pi_t(a) - u_{r(t)}\pi_1(a) = 0$ since $\chi(a) - \pi_1(a) \in B$ for all $a \in A$. The fact that $\lim_{t\to\infty} f(u_{r(t)})\pi_t(a) - f(u_{r(t)})\pi_1(a) = 0$ for all $a \in A, \ f \in C_0(0,1)$, then follows from Weierstrass' theorem. □

It follows from Lemma 3.3 that $CH[\psi] \in [[SC(\mathbb{T}) \otimes A, B]]$ is represented by an asymptotic homomorphism $CH(\psi)$ such that

$$\lim_{t\to\infty} CH(\psi)_t(f \otimes g \otimes a) - f(u_{r(t)})g(w)\pi_t(a) = 0$$

for all $a \in A$, $g \in C(\mathbb{T})$, $f \in C_0(0,1)$. By choosing $\{u_t\}$ in Lemma 3.3 appropriately we may assume that $\lim_{t\to\infty} u_{r(t)}\pi_t(a) - \pi_t(a)u_{r(t)} = 0$, $\lim_{t\to\infty}(1-u_{r(t)})(w\pi_t(a) - \pi_t(a)w) = 0$ for all $a \in A$, and $\lim_{t\to\infty} u_{r(t)}w - wu_{r(t)} = 0$. We can therefore find a discretization $CH(\psi)_{t_i}, i \in \mathbb{N}$, of $CH(\psi)$ such that

1) $\lim_{i\to\infty} \pi_{t_i}(a) - \pi_{t_{i+1}}(a) = 0$ for all $a \in A$,
2) $\lim_{i\to\infty} u_{r(t_i)} - u_{r(t_{i+1})} = 0$,
3) $\lim_{i\to\infty} wu_{r(t_i)} - u_{r(t_i)}w = 0$,
4) $\lim_{i\to\infty}(1-u_{r(t_i)})(w\pi_{t_i}(a) - \pi_{t_i}(a)w) = 0$ for all $a \in A$.

To simplify notation, set $\pi_n = \pi_{t_n}$ and $u_n = u_{r(t_n)}$. Set $\pi_n = \pi_{-n}$ when $n < 0$ and $\pi_0 = \pi_1$. We find that $E \circ CH[\psi] \in \mathrm{Ext}^{-1/2}(SC(\mathbb{T}^2)\otimes A, B)_h$ is represented by a $*$-homomorphism $\underline{\Phi}$, where $\Phi : SC(\mathbb{T}^2)\otimes A \to \mathcal{L}_B(l_2(\mathbb{Z})\otimes B)$ is a map such that

$$\Phi(f\otimes g\otimes h\otimes a) = (\sum_{n\geq 0} f(u_n)e_{nn})(\sum_{n\in\mathbb{Z}} g(w)e_{nn})h(T)(\sum_{n\in\mathbb{Z}} \pi_n(a)e_{nn})$$

modulo $\mathcal{K}\otimes B$ for all $f \in C_0(0,1)$, $g,h \in C(\mathbb{T})$, $a \in A$.

Set $W = \sum_{n\in\mathbb{Z}} we_{nn}$, $U = \sum_{n\geq 0} u_n$. Then W, T and U commute modulo $\mathcal{K}\otimes B$. Define $\underline{\pi} : A \to Q(\mathcal{K}\otimes B)$ such that $\underline{\pi}(a) = \underline{\sum_n \pi_n(a)e_{nn}}$. Then $\underline{\pi}$ is a $*$-homomorphism which commutes with $\underline{U}$ and $\underline{T}$.

Let $Q \in M_2(Q(B))$ be the projection

$$Q = \begin{pmatrix} s^2(\underline{U}) & sc_0(\underline{U}) + sc_1(\underline{U})\underline{W} \\ sc_1(\underline{U})\underline{W}^* + sc_0(\underline{U}) & (c_0+c_1)^2(\underline{U}) \end{pmatrix} .$$

Lemma 3.4. – $\mathrm{Bott}_{SA} \circ E \circ CH[\psi]$ *is represented in* $\mathrm{Ext}^{-1/2}(SA,B)_h$ *by an extension* $\lambda : SA \to M_2(Q(B))$ *such that*

$$\lambda((1-e^{2\pi ix})\otimes a) \;=\; Q\begin{pmatrix} 1-\underline{T} & \\ & 1-\underline{T}\end{pmatrix}\begin{pmatrix}\underline{\pi}(a) & \\ & \underline{\pi}(a)\end{pmatrix} ,$$

$a \in A$.

Proof. To simplify notation, set

$$\tilde{U} = \begin{pmatrix} (1-e^{2\pi ix})(\underline{U}) & \\ & (1-e^{2\pi ix})(\underline{U})\end{pmatrix} .$$

By definition $\mathrm{Bott}_{SA} \circ E \circ CH[\psi] = [\lambda_+] - [\lambda_-]$ where $\lambda_\pm : SA \to M_2(Q(B))$ are $*$-homomorphisms such that

$$\lambda_+((1-e^{2\pi ix})\otimes a) \;=\; P(\underline{T},\underline{W})\tilde{U}\begin{pmatrix}\underline{\pi}(a) & \\ & \underline{\pi}(a)\end{pmatrix}$$

and

$$\lambda_-((1-e^{2\pi ix})\otimes a) \;=\; P_0\tilde{U}\begin{pmatrix}\underline{\pi}(a) & \\ & \underline{\pi}(a)\end{pmatrix} .$$

Set

$$X = 1 - (1 - P(\underline{T},\underline{W})\tilde{U})(1 - P_0\tilde{U}^*) .$$

Then $[\lambda_+] - [\lambda_-] = [\lambda']$ where $\lambda' : SA \to M_2(Q(B))$ is given by

$$\lambda'((1-e^{2\pi ix})\otimes a) = X\begin{pmatrix}\underline{\pi}(a) & \\ & \underline{\pi}(a)\end{pmatrix} .$$

Note that X is an element in the 2×2 matrices over the C^*-algebra generated by $1-\underline{T}$, $\underline{W}$ and $(1-e^{2\pi ix})(\underline{U})$. In fact, if we define $\Lambda : S\otimes C(\mathbb{T})\otimes S \to Q(B)$ such that

$$\Lambda((1-e^{2\pi ix})\otimes e^{2\pi iy}\otimes(1-e^{2\pi iz})) = (1-e^{2\pi ix})(\underline{U})\underline{W}(1-\underline{T}),$$

there is a quasi-unitary in $d \in M_2(S \otimes C(\mathbb{T}) \otimes S)$ such that $\mathrm{id}_{M_2} \otimes \Lambda(d) = X$. (Here S is shorthand for the C^*-algebra $C_0(0,1)$. Also we remind the reader that a quasi-unitary is an element d of a C^*-algebra D such that $1 - d$ is unitary in D^+. Alternatively, it is a normal element with spectrum in $\{1 - z : z \in \mathbb{T}\}$.) Then

$$\mathrm{id}_{M_2} \otimes \Lambda \otimes \left(\begin{smallmatrix} \pi & \\ & \underline{\pi} \end{smallmatrix} \right) : S \otimes C(\mathbb{T}) \otimes S \otimes A \to M_2(Q(B))$$

is semi-invertible, with the inverse given by the $*$-homomorphism which results when one replaces U with $\sum_{n<0} u_{-n}$ in the definition of Λ. Define $\nu : SA \to M_2(S \otimes C(\mathbb{T}) \otimes S \otimes A)$ such that $\nu((1 - e^{2\pi i x}) \otimes a) = d \otimes a$ and note that $\lambda' = (\mathrm{id}_{M_2} \otimes \Lambda \otimes \left(\begin{smallmatrix} \pi & \\ & \underline{\pi} \end{smallmatrix} \right)) \circ \nu$. Let α be the automorphism of $M_2(S \otimes C(\mathbb{T}) \otimes S \otimes A)$ which exchanges the two suspensions. Then

$$-[(\mathrm{id}_{M_2} \otimes \Lambda \otimes \left(\begin{smallmatrix} \pi & \\ & \underline{\pi} \end{smallmatrix} \right))] = [(\mathrm{id}_{M_2} \otimes \Lambda \otimes \left(\begin{smallmatrix} \pi & \\ & \underline{\pi} \end{smallmatrix} \right)) \circ \alpha]$$

in $\mathrm{Ext}^{-1/2}(S \otimes C(\mathbb{T}) \otimes S \otimes A, B)_h$. It follows that

$$-[\lambda'] = [(\mathrm{id}_{M_2} \otimes \Lambda \otimes \left(\begin{smallmatrix} \pi & \\ & \underline{\pi} \end{smallmatrix} \right)) \circ \alpha \circ \nu]$$

in $\mathrm{Ext}^{-1/2}(SA, B)_h$. Set

$$Y = 1 - (1 - Q \begin{pmatrix} 1-\underline{T} & \\ & 1-\underline{T} \end{pmatrix})(1 - P_0 \begin{pmatrix} 1-\underline{T}^* & \\ & 1-\underline{T}^* \end{pmatrix}) ,$$

and note that $(\mathrm{id}_{M_2} \otimes \Lambda \otimes \left(\begin{smallmatrix} \pi & \\ & \underline{\pi} \end{smallmatrix} \right)) \circ \alpha \circ \nu = \mu$ where $\mu : SA \to M_2(Q(B))$ is such that

$$\mu((1 - e^{2\pi i x}) \otimes a) = Y \begin{pmatrix} \pi(a) & \\ & \underline{\pi}(a) \end{pmatrix} .$$

It follows that $[\mu] = [\lambda] - [\mu']$, where $\mu'((1 - e^{2\pi i x}) \otimes a) = (1 - \underline{T})\underline{\pi}(a)$. It is easily seen that μ' is asymptotically split. Therefore $[\mu] = [\lambda]$.

□

Set

$$X = \begin{pmatrix} s(\underline{U}) & -c_0(\underline{U}) - c_1(\underline{U})\underline{W} \\ c_0(\underline{U}) + c_1(\underline{U})\underline{W}^* & s(\underline{U}) \end{pmatrix} .$$

and

$$Z = \begin{pmatrix} i\underline{W_+} & 0 \\ 0 & -i\underline{W_+} \end{pmatrix}$$

where $W_+ = \sum_{n \geq 0} w e_{nn} + \sum_{n<0} e_{nn} \in \mathcal{L}_B(l_2(\mathbb{Z}) \otimes B)$. Then Z and X are unitaries in $M_2(Q(B))$. Let $T_0 : l_2(\mathbb{Z}) \otimes B \to l_2(\mathbb{Z}) \otimes B$ be the unitary

$$T_0 = \sum_{n \in \mathbb{Z} \setminus \{-1\}} e_{n,n+1} + w e_{-1,0} .$$

We can then define an extension $\lambda_1 : SA \to Q(B)$ such that

$$\lambda_1((1 - e^{2\pi i x}) \otimes a) = (1 - \underline{T_0})\underline{\pi}(a) .$$

Lemma 3.5. *Let $\lambda : SA \to M_2(Q(B))$ be the extension of Lemma 3.4. Then*

$$\mathrm{Ad}\, X^* \circ \lambda = \mathrm{Ad}\, Z \circ \begin{pmatrix} \lambda_1 & \\ & 0 \end{pmatrix} .$$

Proof. Note that λ and $\left(\begin{smallmatrix} \lambda_1 & \\ & 0 \end{smallmatrix} \right)$ both extend to unital $*$-homomorphisms $C(\mathbb{T}) \otimes A \to M_2(Q(B))$ defined such that they send $1 \otimes a$ to $\begin{pmatrix} \pi(a) & \\ & \underline{\pi}(a) \end{pmatrix}$, $a \in A$. By considering these extensions we see that it suffices to show that

$$X^* \begin{pmatrix} \underline{\pi}(a) & \\ & \underline{\pi}(a) \end{pmatrix} X = Z \begin{pmatrix} \underline{\pi}(a) & \\ & \underline{\pi}(a) \end{pmatrix} Z^* , \tag{3.2}$$

and

$$
\begin{aligned}
&X^*(Q \begin{pmatrix} \underline{T} & \\ & \underline{T}^* \end{pmatrix} + Q^{\perp}) \begin{pmatrix} \underline{\pi}(a) & \\ & \underline{\pi}(a) \end{pmatrix} X = \\
&Z \begin{pmatrix} \underline{T_0} & \\ & 1 \end{pmatrix} \begin{pmatrix} \underline{\pi}(a) & \\ & \underline{\pi}(a) \end{pmatrix} Z^* = \\
&Z(\begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \begin{pmatrix} \underline{T_0} & \\ & \underline{T_0}^* \end{pmatrix} + \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}) \begin{pmatrix} \underline{\pi}(a) & \\ & \underline{\pi}(a) \end{pmatrix} Z^*.
\end{aligned}
\tag{3.3}
$$

To simplify the verification, observe that $W_+T_0 = TW_+$ from which it follows that $Z \begin{pmatrix} \underline{T_0} & \\ & \underline{T_0}^* \end{pmatrix} Z^* = \begin{pmatrix} \underline{T} & \\ & \underline{T}^* \end{pmatrix}$. Since X clearly commutes with $\begin{pmatrix} \underline{T} & \\ & \underline{T}^* \end{pmatrix}$ and Z with $\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$ we see that (3.3) will follow from (3.2) and

$$
X^*QX = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} . \tag{3.4}
$$

Thus we need only check (3.2) and (3.4), and we leave that to the reader.

$\square$

All in all we now have that $-\operatorname{Bott}_{SA} \circ E \circ CH[\psi] = [\lambda_1]$ in $\operatorname{Ext}^{-1/2}(SA, B)_h$. Define $\kappa : SA \to Q(B)$ by $\kappa((1 - e^{2\pi i x}) \otimes a) = (1 - \underline{T})\underline{\pi}(a)$. κ is asymptotically split and hence $[\lambda_1] = [\lambda_1] - [\kappa]$. Since $[\lambda_1] - [\kappa] = [\mu]$ where $\mu : SA \to Q(B)$ is given by $\mu((1 - e^{2\pi i x}) \otimes a) = (1 - T_0T^*)\underline{\pi}(a)$ and since $T_0T^* = \sum_{n \neq -1} e_{nn} + we_{-1,-1}$, we see that $[\lambda_1] = [\varphi]$. This completes the proof of Proposition 3.1.

Corollary 3.6. $CH : \operatorname{Ext}^{-1/2}(SA, B)_h \to [[S^2A, B]]$ *is injective on* $i^*(\operatorname{Ext}^{-1/2}(C(\mathbb{T}) \otimes A, B)_h)$.

Proof. Let $\psi \in \operatorname{Ext}^{-1/2}(C(\mathbb{T}) \otimes A, B)$ and assume that $CH(i^*[\psi]) = 0$. By the naturality of the Connes-Higson construction this implies that

$$
(Si)^*(CH[\psi] - (Se)^* \circ (Sc)^*(CH[\psi])) = CH(i^*(\psi)) = 0
$$

in $[[SC(\mathbb{T} \otimes A, B]]$. But the split exactness of the functor $[[S-, B]]$ implies then that $0 = CH[\psi] - (Se)^* \circ (Sc)^*(CH[\psi]) = CH([\psi] - e^* \circ c^*[\psi])$. And then $i^*[\psi] = 0$ by Proposition 3.1. $\square$

Lemma 3.7. $(Si)^* : \operatorname{Ext}^{-1/2}(SC(\mathbb{T}) \otimes A, B)_h \to \operatorname{Ext}^{-1/2}(S^2A, B)_h$ *is surjective.*

Proof. To prove this we shall identify $S^2 = C_0(\mathbb{R}^2)$ with $C_0(D)$ where $D = \mathbb{R}^2 \backslash \{(0, y) \in \mathbb{R}^2 \ : y \geq 0\}$ and $SC(\mathbb{T})$ with $C_0(\mathbb{R}^2 \backslash \{0\})$. It is easy to see that there is a continuous map $F : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(0, -)$ is a homeomorphism $\mu = F(0, -) : \mathbb{R}^2 \to D$, $F(1, z) = z$, $z \in \mathbb{R}^2$, and $F^{-1}(K)$ is compact for every compact subset K of D. It follows that $f \mapsto f \circ \mu^{-1}$ is an endomorphism of $C_0(D)$ which is homotopic to $\operatorname{id}_{C_0(D)}$. Hence if $\varphi \in \operatorname{Hom}(S^2A, Q(B))$ is a semi-invertible extension, $[\varphi] = [\chi]$ in $\operatorname{Ext}^{-1/2}(S^2A, B)$ where $\chi(f) = \varphi(f \circ \mu^{-1})$. Define $\psi : SC(\mathbb{T}) \to Q(B)$ by $\psi(g) = \varphi(g \circ \mu^{-1})$. Then $(Si)^*[\psi] = [\varphi]$. $\square$

Lemma 3.7 and Corollary 3.6 in combination prove that the CH-map of diagram (2.1) is injective. This completes the proof of b) of Theorem 2.2.

References

[1] Connes A., Higson N. Déformations, morphismes asymptotiques et K-théorie bivariante. *C. R. Acad. Sci. Paris Sér. I Math.* **311** (1990), 101–106.

[2] Cuntz J. A general construction of bivariant K-theories on the category of C^*-algebras. Operator algebras and operator theory (Shanghai, 1997), 31-43, Contemp. Math., 228, Amer. Math. Soc., Providence, RI, 1998.

[3] Houghton-Larsen T., Thomsen K. Universal (co)homology theories. *K-theory* **16** (1999), 1-27.

[4] Loring T. A. K-theory and asymptotically commuting matrices. *Canad. J. Math.* **40** (1988), 197–216.

[5] Manuilov V. M., Thomsen K. Quasidiagonal extensions and sequentially trivial asymptotic homomorphisms. Preprint.

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