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LIFTINGS OF QUANTUM TILTING MODULES

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1. Introduction

Let $\bar{k}$ be an algebraically closed field of characteristic $p > 0$, $G_{\bar{k}}$ a simple algebraic group over $\bar{k}$, and $B_{\bar{k}}$ (resp. $H_{\bar{k}}$, $N_{\bar{k}}$, $N_{\bar{k}}^-$) a Borel subgroup (resp. a maximal torus in $B_{\bar{k}}$, the maximal unipotent subgroup of $B_{\bar{k}}$, the maximal unipotent subgroup of the opposite Borel subgroup relative to $H_{\bar{k}}$). We denote by X^+, the corresponding set of dominant integral weights. We call Δ-modules or Weyl modules (resp. ∇-modules or induced modules) the family $\{\Delta(\lambda)\}_{\lambda \in X^+}$ (resp. $\{\nabla(\lambda)\}_{\lambda \in X^+}$) of indecomposable rational representations of $G_{\bar{k}}$ satisfying the universal property $Hom_{G_{\bar{k}}}(\Delta(\lambda), M) \simeq M_\lambda^{N_{\bar{k}}}$ (resp. $Hom_{G_{\bar{k}}}(M, \nabla(\lambda)) \simeq (M^*)_\lambda^{N_{\bar{k}}^-}$) for any rational $G_{\bar{k}}$-module M. Here $M_\lambda^{N_{\bar{k}}}$ denotes the space of $N_{\bar{k}}$-invariants of weight λ in M.

A well known theorem, due to Cline-Parshall-Scott-van der Kallen [CPSvdK] and Donkin [D1], asserts that $Ext^1_{G_{\bar{k}}}(\Delta(\lambda), M) = 0$ for all $\lambda \in X^+$ if and only if $Ext^i_{G_{\bar{k}}}(\Delta(\lambda), M) = 0$ for all $\lambda \in X^+$ and all $i \geq 1$ if and only if M admits a ∇-filtration (i.e. there exists a filtration $0 = F_0M \subset F_1M \subset \cdots \subset F_rM = M$, such that each $F_jM/F_{j-1}M$ is a ∇-module).

We prove in this note that we have a similar statement for modules with Δ-filtration up to S-torsion (see 3.1, for the exact definition) over the corresponding quantum group $U_{\mathcal{A}_0}$ defined over $\mathcal{A}_0 = \mathbb{Z}[v, v^{-1}]$, S being a multiplicative subset of $\mathcal{A}_0$. More precisely, under some mild assumptions on M, we obtain that $Ext^1(\Delta_{\mathcal{A}_0}(\lambda), M)$ is S-torsion for all $\lambda \in X^+$ if and only if $Ext^i(\Delta_{\mathcal{A}_0}(\lambda), M)$ is S-torsion for all $\lambda \in X^+$ and all $i \geq 1$ if and only if M has a ∇-filtration up to S-torsion. The category of such modules with ∇-filtration up to S-torsion shares some common properties with modules with ∇-filtration, e.g. stability under direct summands and tensor products (see section 3).

These results can actually be extended to any $\mathbb{Z}[v, v^{-1}]$-algebra. In particular, let us consider the ring $\mathcal{A} = \mathbb{Z}[v]_{(p,v-1)}$ of polynomials over $\mathbb{Z}$ localized at the maximal ideal $(p, v-1)$. Let ξ be a primitive $(p^e)^{\text{th}}$ root of unity for some $e \in \mathbb{N}$ and ϕ_ξ the $(p^e)^{\text{th}}$ cyclotomic polynomial. Denote by $\mathbb{C}_e$, the field of complex numbers viewed as a $\mathbb{Z}[v, v^{-1}]$-algebra by letting v act as ξ. Between some appropriate categories of representations of the quantum groups $U_{\mathcal{A}}$, $U_{\mathbb{C}_e}$ and $U_{\bar{k}}$, there exist functors as follows:

$$\begin{array}{ccc} & \{U_{\mathcal{A}} - \text{rep}\} & \\ & F_{\mathbb{C}_e} \swarrow \quad \searrow F_{\bar{k}} & \\ \{U_{\mathbb{C}_e} - \text{rep}\} & & \{U_{\bar{k}} - \text{rep}\} \end{array}$$

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and, furthermore, according to [APW], the corresponding category over $U_{\bar{k}}$ is equivalent to the category of rational $G_{\bar{k}}$-modules. In [A], the first author proved that indecomposable tilting modules over $U_{\bar{k}}$ (or equivalently over $G_{\bar{k}}$) can be lifted to indecomposable tilting modules over $U_{\mathcal{A}}$. Recall that a tilting module, as defined in [R],[D2], is a module with Δ-filtration and ∇-filtration. The image of an indecomposable tilting module by $F_{\mathbb{C}_e}$ is again a tilting module, however, in general, it is not anymore an indecomposable tilting module. Indeed, an indecomposable tilting module in $\{U_{\mathbb{C}_e} - \text{rep}\}$ cannot usually be lifted to an indecomposable tilting module over $U_{\mathcal{A}}$. In section 4, we prove however, that an indecomposable tilting module over $U_{\mathbb{C}_e}$ can be lifted to an indecomposable "almost" tilting module over $U_{\mathcal{A}}$, namely an indecomposable module with a Δ-filtration up to $\mathcal{A}_0 - (\phi_\xi)$-torsion and a ∇-filtration (or a ∇-filtration up to $\mathcal{A}_0 - (\phi_\xi)$-torsion and a Δ-filtration).

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2. Notations and preliminaries

(2.1) If R is a domain and S is a multiplicatively stable subset of R with $0 \notin S$ then an R-module M is said to be of S-torsion (or, an S-torsion module) if for any $m \in M$, there exists $\alpha \in S$ such that $\alpha.m = 0$. In other words, M is an S-torsion module if and only if $S^{-1}M = 0$.

For any short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, we have that B is an S-torsion module if and only if A and C are S-torsion modules.

(2.2) Let $\mathcal{A}_0 = \mathbb{Z}[v, v^{-1}]$ be the ring of Laurent polynomials over $\mathbb{Z}$. Set also $\mathcal{A} = \mathbb{Z}[v]_{(p,v-1)}$, with p a prime number. Let $\bar{k}$ be the algebraic closure of the residue field of $\mathcal{A}$, i.e of $\mathbb{Z}/p\mathbb{Z}$. If $e \in \mathbb{N}$ and ξ is a primitive $(p^e)^{\text{th}}$-root of unity, we denote by $\mathbb{C}_e$ the field of complex numbers $\mathbb{C}$ viewed as an $\mathcal{A}_0$-algebra by specializing v to ξ. Note that $\mathbb{C}_e$ is the algebraic closure of $\mathbb{Q}[\xi]$ which in turn is the residue field of $\mathcal{A}_0$ localized at (ϕ_ξ), the prime ideal in $\mathcal{A}_0$ generated by the $(p^e)^{\text{th}}$ cyclotomic polynomial. Remark also that, when $l = p^e$ for some $e \in \mathbb{N}$, $\bar{k}$ and $\mathbb{C}_e$ are both $\mathcal{A}$-algebras.

(2.3) Let $(X, Y, <,>, ...)$ be a root datum of finite type $(I, .)$. To each such datum we associate, as in [L1],[L2], a quantum group $U_{\mathcal{A}_0}$ defined over $\mathcal{A}_0$. Similarly, we denote by $U^+_{\mathcal{A}_0}$, $U^0_{\mathcal{A}_0}$, $U^-_{\mathcal{A}_0}$, the subalgebras of $U_{\mathcal{A}_0}$ corresponding respectively to the positive, toroidal, negative part of the triangular decomposition of $U_{\mathcal{A}_0}$. The subalgebra $U^+_{\mathcal{A}_0}$ (resp. $U^-_{\mathcal{A}_0}$) is generated by elements $\{E_i^{(N)}\}_{i\in I, N\geq 1}$ (resp. $\{F_i^{(N)}\}_{i\in I, N\geq 1}$). The subalgebra $U^0_{\mathcal{A}_0}$ is commutative. It is generated by the elements $(K_i^{\pm 1})_{i\in I}$ and some polynomial functions of them, see [L1]. If K is any $\mathcal{A}_0$-algebra, then we define U_K, U_K^+, U_K^0, U_K^- to be $U_{\mathcal{A}_0} \otimes_{\mathcal{A}_0} K$, $U^+_{\mathcal{A}_0} \otimes_{\mathcal{A}_0} K$, $U^0_{\mathcal{A}_0} \otimes_{\mathcal{A}_0} K$, $U^-_{\mathcal{A}_0} \otimes_{\mathcal{A}_0} K$, respectively. We will, however, use the notations $U_{\mathbb{C}_e}$, $U^+_{\mathbb{C}_e}$, $U^-_{\mathbb{C}_e}$ when $K = \mathbb{C}_e$. By [APW, 3.7], we know that $U_{\bar{k}}$ is an associative algebra whose representation theory is analogous to the one of $G_{\bar{k}}$, the semi-simple algebraic group defined on $\bar{k}$ with corresponding Cartan datum $(I, .)$. The functors $F_{\mathbb{C}_e}$, $F_{\bar{k}}$ of the introduction are just: $F_{\mathbb{C}_e}(\_) = \_ \otimes_{\mathcal{A}} \mathbb{C}_e$, $F_{\bar{k}}(\_) = \_ \otimes_{\mathcal{A}} \bar{k}$.

(2.4) We consider the Coxeter group (W, S) associated to the Cartan datum $(I, .)$. (Here S should not be confused with the multiplicative subset considered above). Let w_0 be the longest element of W with respect to the set of generators S. The subset X^+ of the weight lattice X is defined to be

$$X^+ = \{\lambda \in X \mid \langle i, \lambda\rangle \in \mathbb{N} \text{ for all } i \in I\}.$$

The Weyl group W acts on X. For $\lambda \in X$, define $\lambda^* = -w_0(\lambda)$.

(2.5) For any U_K-module M, define $M_\lambda = \{m \in M \mid um = \chi_\lambda(u)m \text{ for all } u \in U_K^0\}$. Here χ_λ denotes the character of U_K^0 determined by λ, see [APW]. We say that M is U_K^0-diagonalizable if and only if $M = \bigoplus_{\lambda\in X} M_\lambda$. Let $\mathcal{C}_K$ be the category of U_K-integral modules of type 1. Objects in this category are modules M, which are U_K^0-diagonalizable, U_K^+ and U_K^--finite (i.e. for any $m \in M$, $E_i^{(N)}.m = 0 = F_i^{(N)}.m$ for all $N \gg 0$). If M is any U_K-module, we denote by FM the largest submodule of M belonging to the category $\mathcal{C}_K$.

(2.6) We define ∇-modules using an induction functor in the category $\mathcal{C}_K$, namely $\nabla_K(\lambda) = FHom_{U_K^-U_K^0}(U_K, \lambda)$ for any $\lambda \in X^+$. Let also, $\Delta_K(\lambda) = \nabla_K(\lambda^*)^*$. These modules are indecomposable modules in the category $\mathcal{C}_K$. Their character is given by the Weyl character formula. Furthermore, the module $\Delta_K(\lambda)$ (resp. $\nabla_K(\lambda)$) satisfy the universal property that for any $M \in \mathcal{C}_K$, $Hom_{\mathcal{C}_K}(\Delta_K(\lambda), M) \simeq M_\lambda^{U_K^+}$ (resp. $Hom_{\mathcal{C}_K}(M, \nabla_K(\lambda)) \simeq (M_\lambda^*)^{U_K^-}$). Finally, we have $Ext^i_{\mathcal{C}_K}(\Delta_K(\lambda), \nabla_K(\mu)) = 0$ for all $\lambda, \mu \in X^+$ and all $i > 0$. These fact are proved in [APW], when $K = \mathcal{A}, \bar{k}, \mathbb{C}_e$. When K is any $\mathcal{A}_0$-algebra, the universal property of Δ, ∇-modules can be obtained as in *loc. cit.* Moreover, Kempf's vanishing theorem is proved in general, i.e over $\mathcal{A}_0$ in [W], see also [Ka].

We say that a U_K-module has a Δ-filtration (resp. a ∇-filtration) if there exists a filtration $0 = F_0M \subset ... \subset F_rM = M$ such that $F_jM/F_{j-1}M$ is isomorphic to $\Delta_K(\lambda_j)$ (resp. $\nabla_K(\lambda_j)$) for some $\lambda_j \in X^+$. Using the standard resolution of K in the category $\mathcal{C}_K$ defined as in [APW, 2.17], it follows as in [Ja, II.4.9-13] that $Ext^i_{\mathcal{C}_K}(\Delta_K(\lambda), \nabla_K(\mu)) = 0$ for all $\lambda, \mu \in X^+$ and all $i > 0$. One also deduce (see [APW]) that a module $M \in \mathcal{C}_K$ which is K-free has a Δ-filtration if and only if $Ext^1_{\mathcal{C}_K}(\Delta(\lambda), M) = 0$ for all $\lambda \in X^+$.

3. Filtrations up to S-torsion

In all this section, K is any $\mathcal{A}_0$-algebra and, unless otherwise stated, S is any multiplicatively stable subset of K with $0 \notin S$.

Definition 3.1:

A U_K-module M is said to have a ∇-filtration (resp. Δ-filtration) up to S-torsion if $S^{-1}M$ has a ∇-filtration (resp. Δ-filtration).

Remark 3.2: These two definitions are dual to one another. In the following we shall mostly be considering modules which have a ∇-filtration up to S-torsion leaving it to the reader to formulate the straightforward dual statements.

Remark 3.3: Note that by our definition any S-torsion module is a module with a ∇-filtration up to S-torsion.

Lemma 3.4:

Let R be any integral ring, Λ an R-algebra and S a multiplicative subset of R. Suppose M is a Λ-module finitely generated over R with S-torsion. The modules $Ext^i_\Lambda(\_, M)$, $Ext^i_\Lambda(M, \_)$ and $Tor_i^\Lambda(\_, M)$ are all S-torsion modules.

Proof: Consider a projective (resp. injective, projective) resolution and apply the functor $Hom_\Lambda(\_, M)$ (resp. $Hom_\Lambda(M, \_)$, $\_ \otimes_\Lambda M$) to it. All the terms of the resulting complex are S-torsion modules by the assumptions. Therefore we get the result. □

Proposition 3.5:

Let M be a U_K-module in the category $\mathcal{C}_K$ which is finitely generated and free over K. The following statements are equivalent:

i) The module M has a ∇-filtration up to S-torsion.

ii) $Ext^i_{\mathcal{C}_K}(\Delta_K(\lambda), M)$ is an S-torsion module for all $i \geq 1$ and all $\lambda \in X^+$.

iii) $Ext^1_{\mathcal{C}_K}(\Delta_K(\lambda), M)$ is an S-torsion module for all $\lambda \in X^+$.

Proof:

We have i) implies ii) by Lemma 3.4 and the vanishing of $Ext^i_{\mathcal{C}_K}(\Delta_K(\lambda), \nabla_K(\mu))$ for all $i \geq 1$ and all $\lambda, \mu \in X^+$. Obviously, ii) implies iii). To see that iii) implies i) note that

$$S^{-1}Ext^1_{\mathcal{C}_K}(\Delta_K(\lambda), M) = Ext^1_{\mathcal{C}_{S^{-1}K}}(\Delta_{S^{-1}K}(\lambda), S^{-1}M).$$

Since M is free over K we have that $S^{-1}M$ is free over $S^{-1}K$. The conclusion now follows from the last sentence in 2.6.

Proposition 3.6:

Any direct summand of a module with ∇-filtration up to S-torsion is a module with ∇-filtration up to S-torsion.

Proof: This is an immediate corollary of proposition 3.5. □

Proposition 3.7:

A tensor product of two modules which both have ∇-filtrations up to S-torsion is a module with ∇-filtration up to S-torsion.

Proof: Let M_1, M_2 be two modules with ∇-filtrations up to S-torsion. The proposition follows by observing that $S^{-1}(M_1 \otimes_K M_2) \simeq S^{-1}M_1 \otimes_{S^{-1}K} S^{-1}M_2$ and using the fact (([M],[P]) that the tensor product of two modules with ∇-filtrations has a ∇-filtration. □

Following 3.1, we introduce the following definition

Definition 3.8:

A tilting module up to S-torsion is a module which has both a ∇-filtration up to S-torsion and a Δ-filtration up to S-torsion.

We have the following immediate corollaries of propositions 3.6, 3.7:

Proposition 3.9:

A module is tilting up to S-torsion if and only if all its direct summands are tilting up to S-torsion. □

Proposition 3.10:

A tensor product of tilting modules up to S-torsion is a tilting module up to S-torsion. □

4. Quantum tilting and tilting up to S-torsion

From now on, let p be a prime number, ξ a primitive $(p^e)^{\text{th}}$-primitive root of unity for some $e \in \mathbb{N}$. Recall our notation $\mathcal{A} = \mathbb{Z}[v]_{(p,v-1)}$. We denote by S_e the multiplicative set $\mathcal{A} - (\phi_\xi) = \{a \in \mathcal{A} \mid a \notin (\phi_\xi)\}$. As usual, $\mathbb{C}_e$ will be considered as an $\mathcal{A}$-module by specializing v to ξ.

A $U_{\bar{k}}$-tilting module can be lifted to a $U_{\mathcal{A}}$ tilting module [A]. In general, it is not the case for a quantum tilting module T. However, we will prove that any quantum tilting module can be lifted to a $U_{\mathcal{A}}$-tilting module up to S_e-torsion.

In particular, we will introduce lifts T^{∇} (resp. T^{Δ}) which have Δ-filtrations up to S_e-torsion (resp. ∇-filtrations up to S_e-torsion) and ∇-filtrations (resp. Δ-filtrations).

Lemma 4.1:

Let M be a finitely generated $\mathcal{A}$-module. Let $\zeta_1, ..., \zeta_n$ be a minimal set of generators ordered in such a way that the images of $\zeta_1, ..., \zeta_d$ in $M \otimes_{\mathcal{A}} \mathbb{C}_e$ constitute a basis of $M \otimes_{\mathcal{A}} \mathbb{C}_e$. The module $M/ < \zeta_1, ..., \zeta_d >$ is of S_e-torsion.

Proof:

We have the short exact sequence

$$0 \longrightarrow < \zeta_1, ..., \zeta_d > \longrightarrow M \longrightarrow M/ < \zeta_1, ..., \zeta_d > \longrightarrow 0.$$

Since $\_ \otimes_{\mathcal{A}} \mathbb{C}_e$ is a left exact functor, $M/ < \zeta_1, ..., \zeta_d > \otimes_{\mathcal{A}} \mathbb{C}_e = 0$, or equivalently, $M/ < \zeta_1, ..., \zeta_d > \otimes_{\mathcal{A}} \mathbb{Q}[\xi] = 0$. Since $\mathbb{Q}[\xi]$ is the residue field of the local ring $S_e^{-1}\mathcal{A}$, by Nakayama lemma, this is equivalent to say that $M/ < \zeta_1, ..., \zeta_d > \otimes_{\mathcal{A}} S_e^{-1}\mathcal{A} = 0$, i.e. $S_e^{-1}M/ < \zeta_1, ..., \zeta_d >= 0$. □

Proposition 4.2:

If M is a $U_{\mathcal{A}}$-module with ∇-filtration up to S_e-torsion (resp. Δ-filtration up to S_e-torsion) then, $M \otimes_{\mathcal{A}} \mathbb{C}_e$ is a $U_{\mathbb{C}_e}$-module with ∇-filtration (resp. Δ-filtration).

Proof: By definition, we have $S_e^{-1}M$ has a ∇-filtration. Thus, $M \otimes_{\mathcal{A}} \mathbb{C}_e = S_e^{-1}M \otimes_{S_e^{-1}\mathcal{A}} \mathbb{C}_e$ has a ∇-filtration. □

Corollary 4.3: *Let M be a $U_{\mathcal{A}}$-tilting module up to S_e-torsion then $M \otimes_{\mathcal{A}} \mathbb{C}_e$ is a $U_{\mathbb{C}_e}$-tilting module.* □

Theorem 4.4:

Let $T_{\mathbb{C}_e}(\lambda)$ be the indecomposable $U_{\mathbb{C}_e}$-tilting module with highest weight λ. There exists an indecomposable lift $T^{\nabla}_{\mathcal{A},\mathbb{C}_e}(\lambda)$, (resp. $T^{\Delta}_{\mathcal{A},\mathbb{C}_e}(\lambda)$) of $T_{\mathbb{C}_e}(\lambda)$ having a Δ-filtration (resp. ∇-filtration) up to S_e-torsion and a ∇-filtration (resp. Δ-filtration).

Proof: We build the module $T^{\Delta}_{\mathcal{A},\mathbb{C}_e}(\lambda)$ inductively. To begin with, set $X_0 = \Delta_{\mathcal{A}}(\lambda)$, and $\lambda_0 = \lambda$. We build inductively some modules X_i which are Δ-filtered and such that $Ext^1_{\mathcal{C}_{\mathcal{A}}}(\Delta_{\mathcal{A}}(\mu), X_i)$ is an S_e-torsion module for all $\mu \geq \lambda_i$, where the sequence (λ_i) is a strictly decreasing sequence of elements in X^+.

Assume that we have built X_i. Take λ_{i+1} to be the largest element in X^+ having the property that $Ext^1_{\mathcal{C}_{\mathcal{A}}}(\Delta_{\mathcal{A}}(\lambda_{i+1}), X_i) \otimes_{\mathcal{A}} \mathbb{C}_e$ is not zero. Let $\zeta_1, \ldots, \zeta_{n_{i+1}}$ be a minimal set of generators of the module $Ext^1_{\mathcal{C}_{\mathcal{A}}}(\Delta_{\mathcal{A}}(\lambda_{i+1}), X_i)$ ordered in such a way that the image of $\zeta_1, \ldots, \zeta_{d_{i+1}}$ in $Ext^1_{\mathcal{C}_{\mathcal{A}}}(\Delta_{\mathcal{A}}(\lambda_{i+1}), X_i) \otimes_{\mathcal{A}} \mathbb{C}_e$ form a basis. Let X_{i+1} be the universal extension of $\Delta_{\mathcal{A}}(\lambda_{i+1})^{d_{i+1}}$ by X_i associated to the element $\zeta_1 \oplus \cdots \oplus \zeta_{d_{i+1}}$.

Using the long exact sequence associated to the short exact sequence

$$0 \longrightarrow X_i \longrightarrow X_{i+1} \longrightarrow \Delta_{\mathcal{A}}(\lambda_{i+1})^{d_{i+1}} \longrightarrow 0$$

we obtain that $Ext^1_{\mathcal{C}_{\mathcal{A}}}(\Delta_{\mathcal{A}}(\mu), X_{i+1}) \simeq Ext^1_{\mathcal{C}_{\mathcal{A}}}(\Delta_{\mathcal{A}}(\mu), X_i)$ for any $\mu > \lambda_{i+1}$ and $Ext^1_{\mathcal{C}_{\mathcal{A}}}(\Delta_{\mathcal{A}}(\lambda_{i+1})^{d_{i+1}}, X_{i+1})$ is isomorphic to the cokernel of the map

$$Hom_{\mathcal{C}_{\mathcal{A}}}(\Delta_{\mathcal{A}}(\lambda_{i+1})^{d_{i+1}}, \Delta_{\mathcal{A}}(\lambda_{i+1})) \longrightarrow Ext^1_{\mathcal{C}_{\mathcal{A}}}(\Delta_{\mathcal{A}}(\lambda_{i+1})^{d_{i+1}}, X_i).$$

This cokernel is by construction equal to $Ext^1_{\mathcal{C}_\mathcal{A}}(\Delta_\mathcal{A}(\lambda_{i+1})^{d_{i+1}}, X_i)/ < \zeta_1, \dots, \zeta_{d_{i+1}} >$. Thus, by Lemma 4.1, $Ext^1_{\mathcal{C}_\mathcal{A}}(\Delta_\mathcal{A}(\lambda_{i+1}), X_{i+1})$ is of S_e-torsion.

This construction terminates when we obtain a module X_m which satisfies that $Ext^1_{\mathcal{C}_\mathcal{A}}(\Delta_\mathcal{A}(\mu), X_i)$ is an S_e-torsion module for all $\mu \geq 0$. By proposition 3.5, the module X_m has a ∇-filtration up to S_e-torsion. Furthermore, it is a Δ-filtered module, free over $\mathcal{A}$. By Corollary 4.3, $X_m \otimes_\mathcal{A} \mathbb{C}_e$ is a $U_{\mathbb{C}_e}$-tilting module. It has the same character as $T_{\mathbb{C}_e}(\lambda)$ by construction. Thus, it is a lift of $T_{\mathbb{C}_e}(\lambda)$. We denote it $T^\Delta_{\mathcal{A},\mathbb{C}_e}(\lambda)$.

We have now only left to prove that $T^\Delta_{\mathcal{A},\mathbb{C}_e}(\lambda)$ is indecomposable. Let $T^\Delta_{\mathcal{A},\mathbb{C}_e}(\lambda) = M_1 \oplus M_2$. We have $T_{\mathbb{C}_e}(\lambda) = (M_1 \otimes_\mathcal{A} \mathbb{C}_e) \oplus (M_2 \otimes_\mathcal{A} \mathbb{C}_e)$. Since $T_{\mathbb{C}_e}(\lambda)$ is indecomposable, we have, say, $M_2 \otimes_\mathcal{A} \mathbb{C}_e = 0$. But since M_2 is a summand of $T^\Delta_{\mathcal{A},\mathbb{C}_e}(\lambda)$ it is a free $\mathcal{A}$-module and we conclude $M_2 = 0$. □

Remark 4.5: It follows from [A] that we have: $T^\Delta_{\mathcal{A},\mathbb{C}_0}(\lambda) = \Delta(\lambda)$ and $T^\nabla_{\mathcal{A},\mathbb{C}_0}(\lambda) = \nabla(\lambda)$. Thus, tilting modules over $U_\mathcal{A}$ are modules which are $T^\Delta_{\mathcal{A},\mathbb{C}_0}$-filtered and $T^\nabla_{\mathcal{A},\mathbb{C}_0}$-filtered. One could ask if it is true more generally that a $U_\mathcal{A}$-tilting module is indeed $T^\Delta_{\mathcal{A},\mathbb{C}_e}$-filtered and $T^\nabla_{\mathcal{A},\mathbb{C}_e}$-filtered.

Proposition 4.6: *Let* $\lambda \in X^+$*.*

i) *There is a natural injection from* $T^\Delta_{\mathcal{A},\mathbb{C}_e}(\lambda)$ *into* $T^\nabla_{\mathcal{A},\mathbb{C}_e}(\lambda)$ *whose cokernel is of* S_e*-torsion.*

ii) *We have ch* $T_{\mathbb{C}_e}(\lambda) =$ *ch* $T_{\bar{k}}(\lambda)$ *if and only if* $T^\Delta_{\mathcal{A},\mathbb{C}_e}(\lambda) = T^\nabla_{\mathcal{A},\mathbb{C}_e}(\lambda)$*.*

Proof: The natural homomorphism from $\Delta_\mathcal{A}(\lambda)$ to $\nabla_\mathcal{A}(\lambda)$ extends first to a homomorphism from $T^\Delta_{\mathcal{A},\mathbb{C}_e}(\lambda)$ to $\nabla_\mathcal{A}(\lambda)$ (because $Ext^1_{\mathcal{C}_\mathcal{A}}(T^\Delta_{\mathcal{A},\mathbb{C}_e}(\lambda)/\Delta_\mathcal{A}(\lambda), \nabla_\mathcal{A}(\lambda)) = 0$) and then this lifts to a homomorphism $\phi : T^\Delta_{\mathcal{A},\mathbb{C}_e}(\lambda) \longrightarrow T^\nabla_{\mathcal{A},\mathbb{C}_e}(\lambda)$ (because $Ext^1_{\mathcal{C}_\mathcal{A}}(T^\Delta_{\mathcal{A},\mathbb{C}_e}(\lambda), Ker(T^\nabla_{\mathcal{A},\mathbb{C}_e}(\lambda) \rightarrow \nabla_\mathcal{A}(\lambda))) = 0$). By construction ϕ is an isomorphism on the λ-weight space. Since by Theorem 4.4 we have $S_e^{-1}T^\Delta_{\mathcal{A},\mathbb{C}_e}(\lambda) \simeq T_{S_e^{-1}\mathcal{A}}(\lambda) \simeq S_e^{-1}T^\nabla_\mathcal{A}(\lambda)$ we see that $S_e^{-1}\phi$ is an isomorphism and i) follows.

To prove ii) note first that if $T^\Delta_{\mathcal{A},\mathbb{C}_e}(\lambda) = T^\nabla_{\mathcal{A},\mathbb{C}_e}(\lambda)$ then this module is Δ-filtered and ∇-filtered. By unicity of the indecomposable tilting module of given highest weight, we have $T^\Delta_{\mathcal{A},\mathbb{C}_e}(\lambda) = T^\nabla_{\mathcal{A},\mathbb{C}_e}(\lambda) = T_\mathcal{A}(\lambda)$, the lifting of the indecomposable tilting module $T_{\bar{k}}(\lambda)$ as constructed in [A]. Therefore, we have: ch $T_{\mathbb{C}_e}(\lambda) =$ ch $T_{\bar{k}}(\lambda)$.

Conversely, suppose ch $T_{\mathbb{C}_e}(\lambda) =$ ch $T_{\bar{k}}(\lambda)$. It implies that in each step of the construction of $T^\Delta_{\mathcal{A},\mathbb{C}_e}(\lambda)$ given in theorem 4.4, we have $d_i = n_i$. Thus

$$Ext^1_{\mathcal{C}_\mathcal{A}}(\Delta_\mathcal{A}(\lambda_{i+1})^{d_{i+1}}, X_i)/ < \zeta_1, \dots, \zeta_{d_{i+1}} >$$

is actually zero. We therefore obtain a module $T^\Delta_{\mathcal{A},\mathbb{C}_e}(\lambda)$ which is both Δ-filtered and ∇-filtered. Again, by unicity of indecomposable tilting module with given highest weight, we obtain: $T^\Delta_{\mathcal{A},\mathbb{C}_e}(\lambda) = T^\nabla_{\mathcal{A},\mathbb{C}_e}(\lambda)$. □

Remark 4.7 There is a natural injection from $T^\Delta_{\mathcal{A},\mathbb{C}_e}(\lambda)$ into $T_\mathcal{A}(\lambda)$. This is seen by arguments analogous to those used in the proof of Proposition 4.6 i) (this time one obtains a map which upon localization at S_e is a (split) injection).

Dually, there is a natural surjection from $T_\mathcal{A}(\lambda)$ into $T^\nabla_{\mathcal{A},\mathbb{C}_e}(\lambda)$.

Remark 4.8 (Reformulation of [A, Conjecture 5.1]):

Let $C_{p^2} = \{\lambda \in X^+ \mid \langle \lambda + \rho, h_0 \rangle \leq p^2\}$. In [A], the first author conjectured that for any $\lambda \in C_{p^2}$, we have: ch $T_{\mathbb{C}_1}(\lambda) =$ ch $T_{\bar{k}}(\lambda)$. By proposition 4.6, this conjecture is equivalent to say that the corresponding lifts from Theorem 4.4 correspond, i.e., $T^{\Delta}_{\mathcal{A},\mathbb{C}_1}(\lambda) = T^{\nabla}_{\mathcal{A},\mathbb{C}_1}(\lambda)$.

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