UNIVERSITY OF AARHUS
DEPARTMENT OF MATHEMATICS

ISSN: 1397–4076

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Preprint Series No.: 20

November 1999

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FROBENIUS SPLITTING OF HILBERT SCHEMES OF POINTS ON SURFACES

SHRAWAN KUMAR, JESPER FUNCH THOMSEN

Abstract. Let X be a quasiprojective smooth surface defined over an algebraically closed field of positive characteristic. In this note we show that if X is Frobenius split then so is the Hilbert scheme $\mathrm{Hilb}^n(X)$ of n points in X. In particular, we get the higher cohomology vanishing for ample line bundles on $\mathrm{Hilb}^n(X)$ when X is projective and Frobenius split.

Introduction

Let X be a quasiprojective smooth surface defined over an algebraically closed field k of positive characteristic p. For an integer $n \geq 1$, let $X^{(n)}$ be the n-th symmetric product of X and let $X^{[n]}$ denote the Hilbert scheme of n points in X (parametrizing the zero dimensional closed subschemes of X of length n). Recall that $X^{[n]}$ is smooth and there is a birational 'Hilbert-Chow' morphism $\psi : X^{[n]} \to X^{(n)}$, which to each zero dimensional closed subscheme in X of length n associates its support (with multiplicities). Let $X_*^{(n)}$ denote the open locus of $X^{(n)}$ corresponding to the set of n-tuples with at least $n-1$ distinct points and let $X_*^{[n]}$ denote its inverse image under ψ. We show that $\psi : X_*^{[n]} \to X_*^{(n)}$ is a crepant resolution if $p > 2$, in the sense that $X_*^{(n)}$ is Gorenstein such that its dualizing line bundle $\omega_{X_*^{(n)}}$ pulls back to the canonical bundle $\omega_{X_*^{[n]}}$ on $X_*^{[n]}$ under ψ (cf. Theorem 1). In fact, if $p > n$, $\psi : X^{[n]} \to X^{(n)}$ itself is a crepant resolution (cf. Corollary 1). (This generalizes the corresponding result in char. 0 due to Beauville.) We make crucial use of our Theorem 1 to prove the following main result of this paper:

Let X be as above and $p > 2$. Then, for any $n \geq 1$, the Hilbert scheme $X^{[n]}$ is Frobenius split (cf. Theorem 2). In particular, if X, in addition, is projective and L is an ample line bundle on $X^{[n]}$, then L has vanishing higher cohomology (cf. Corollary 2).

The contents of the paper are as follows: Section 1 is devoted to recalling the definition of Hilbert schemes, and Section 2 is devoted to the basic definitions of Frobenius splitting. Sections 3 and 4 are devoted to proving that ψ is a crepant resolution. We prove our main theorem (Theorem 2) in Section 5.

We thank M. Brion, S. Hansen and V. Mehta for some helpful conversations. Part of this work was done while the first author was visiting Ecole Normale Supérieure, Paris, hospitality of which is gratefully acknowledged. The second author is pleased to thank UNC for its hospitality during his visit. The second author was partially supported by the TMR programme "Algebraic Lie Representations" (ECM Network Contract No. ERB FMRX-CT 97/0100).

1. Hilbert schemes of points

Let X be a quasiprojective variety defined over an algebraically closed field k. Fix an integer $n \geq 1$. The Hilbert scheme $X^{[n]} = \mathrm{Hilb}^n(X)$ of n points in X parameterizes zero dimensional closed subschemes of X of length n. The scheme $\mathrm{Hilb}^n(X)$ is quasiprojective and in fact projective when X is so.

1.1. **Symmetric products.** Let $X^n = X \times \cdots \times X$ denote the n-fold product of X, and let S_n denote the symmetric group on n letters. Then S_n acts on X^n by permuting the factors. As X^n is quasiprojective and S_n is finite, the geometric quotient of this action exists (cf. [9], Chap. III, §14). The quotient is denoted by $X^{(n)}$ and is called the n-th *symmetric product* of X. Let $\Phi : X^n \to X^{(n)}$ denote the quotient map.

Points in $X^{(n)}$ correspond to unordered tuples of (not necessarily distinct) n points in X. The open subset of $X^{(n)}$ consisting of the tuples of n distinct points is denoted by $X^{(n)}_{**}$. If X is smooth, the variety $X^{(n)}$ is smooth along $X^{(n)}_{**}$ and moreover it is singular along the complement of $X^{(n)}_{**}$ if $\dim X \geq 2$ (cf. [3], §2). Clearly, the codimension of $X^{(n)} \setminus X^{(n)}_{**}$ in $X^{(n)}$ is equal to $\dim X$. Let $X^{(n)}_{*}$ denote the open locus of $X^{(n)}$ corresponding to the set of n-tuples with at least $n-1$ distinct points.

1.2. **Hilbert-Chow morphism ([2], §2).** Let $X^{[n]}_{red}$ denote the underlying reduced subscheme of $X^{[n]}$. The *Hilbert-Chow morphism* is the map $\psi : X^{[n]}_{red} \to X^{(n)}$, which to each zero dimensional closed subscheme in X of length n associates its support (with multiplicities). The Hilbert-Chow morphism is birational, being an isomorphism over the open set $X^{(n)}_{**}$.

When X is a smooth surface, the Hilbert scheme $X^{[n]}$ is also smooth (in particular reduced). Hence, in this case, ψ is a desingularization of the symmetric product $X^{(n)}$.

2. Frobenius splitting - basic definitions

Let $\pi : X \to \mathrm{Spec}(k)$ be a scheme defined over an algebraically closed field k of positive characteristic p. The *absolute Frobenius morphism* on X is the identity on point spaces and raising to the p-th power locally on functions. The absolute Frobenius morphism is *not* a morphism of

k-schemes. Let X' be the scheme obtained from X by base change with the absolute Frobenius morphism on $\mathrm{Spec}(k)$, i.e., the underlying topological space of X' is that of X with the same structure sheaf $\mathcal{O}_X$ of rings, only the underlying k-algebra structure on $\mathcal{O}_{X'}$ is twisted as $\lambda \odot f = \lambda^{1/p} f$, for $\lambda \in k$ and $f \in \mathcal{O}_{X'}$. Using this description of X', *the relative Frobenius morphism* $F : X \to X'$ is defined in the same way as the absolute Frobenius morphism and it is a morphism of k-schemes.

2.1. **Frobenius splitting** [8]. Recall that a variety X is called *Frobenius split* if the homomorphism $\mathcal{O}_{X'} \to F_*\mathcal{O}_X$ of $\mathcal{O}_{X'}$-modules is split. A homomorphism $\sigma : F_*\mathcal{O}_X \to \mathcal{O}_{X'}$ is a splitting of $\mathcal{O}_{X'} \to F_*\mathcal{O}_X$ (called a *Frobenius splitting*) if and only if $\sigma(1) = 1$.

When X is a smooth variety with canonical bundle ω_X, there is a natural isomorphism of $\mathcal{O}_{X'}$-modules ([8]):

$$F_*(\omega_X^{1-p}) \cong \mathcal{H}om_{\mathcal{O}_{X'}}(F_*\mathcal{O}_X, \mathcal{O}_{X'}).$$

In this way global sections of ω_X^{1-p} correspond to homomorphisms $F_*\mathcal{O}_X \to \mathcal{O}_{X'}$. A section of ω_X^{1-p} which corresponds to a Frobenius splitting in this way, is called a *splitting section*. Checking whether a section of ω_X^{1-p} is a splitting section can be done locally. More precisely, we have the following result.

Lemma 1 ([8]). *Let U be an open dense subset of a smooth variety X. If a section $s \in \mathrm{H}^0(X, \omega_X^{1-p})$ restricts to a splitting section $s|_U \in \mathrm{H}^0(U, \omega_U^{1-p})$ on U, then s is a splitting section.*

An immediate consequence of the definition of Frobenius splitting is

Lemma 2 ([8]). *Let X be a Frobenius split variety and let L be a line bundle on X such that $H^i(X, L^m) = 0$ for all large m (for a fixed i). Then $H^i(X, L) = 0$.*

Proof. This follows from the fact that if X is Frobenius split and L is a line bundle on X, then there is an injective map

$$\mathrm{H}^i(X, L) \hookrightarrow \mathrm{H}^i(X, L^p)$$

of abelian groups. □

In particular, Lemma 2 implies that ample line bundles on projective Frobenius split varieties have vanishing higher cohomology.

3. Ramification

In this section we have the following setup. By $H = \{e, \sigma\}$ we denote the group of order 2, acting nontrivially on a smooth quasiprojective variety Y over a field of characteristic $p \neq 2$. We denote the quotient of Y under this action by X with the corresponding quotient map π. We will assume, in addition, that X is smooth.

Lemma 3. *Assume that $Y = \mathrm{Spec}(B)$ and $X = \mathrm{Spec}(A)$ are affine. Let E denote an irreducible subvariety of X of codimension 1 corresponding to a prime ideal $\mathfrak{p}$ in A. Let $s \in A$ generate $\mathfrak{p}$ in the local ring $A_{\mathfrak{p}}$. If π is bijective over E, then there exist a unique prime ideal $\mathfrak{p}'$ in B over $\mathfrak{p}$. Furthermore, if v denotes the valuation on the discrete valuation ring $B_{\mathfrak{p}'}$, then $v(s) = 2$.*

Proof. Assume that $\mathfrak{p}'$ and $\mathfrak{p}''$ are two different prime ideals in B lying over the prime ideal $\mathfrak{p}$ in A. Let E' and E'' denote the corresponding subvarieties of Y. Then $\sigma(E') = E''$. Choose $y \in E' \setminus E''$. Since $\sigma(y) \in E''$, $\sigma(y) \neq y$. But y and $\sigma(y)$ both map to the same point in E, which is a contradiction. This proves the first part of the statement.

Let E' denote the irreducible subvariety of Y corresponding to the prime ideal $\mathfrak{p}'$ in B lying over $\mathfrak{p}$. Let $t \in B$ be an element generating the maximal ideal in the local ring $B_{\mathfrak{p}'}$. Choose $b, b' \in B \setminus \mathfrak{p}'$ such that

$$s = t^{v(s)} \frac{b}{b'}.$$

As the product $\sigma(t)t$ is H-invariant, we can find $a, a' \in A \setminus \mathfrak{p}$ and a positive integer l such that

$$\sigma(t)t = s^l \frac{a}{a'}.$$

Hence, we get

$$s^2 = \sigma(s)s = (\sigma(t)t)^{v(s)} \frac{\sigma(b)b}{\sigma(b')b'} = s^{lv(s)} (\frac{a}{a'})^{v(s)} \frac{\sigma(b)b}{\sigma(b')b'},$$

from which we obtain $lv(s) = 2$ (observe that $\sigma(b)b$ and $\sigma(b')b' \in A \setminus \mathfrak{p}$). Assume, if possible, that $v(s) = 1$. Then replacing t by $tb\sigma(b')$ and s by $sb'\sigma(b')$, we can assume that $s = t$. Take a nonzero $f \in B$ such that $\sigma(f) = -f$ (e.g. $f = g - \sigma(g)$ for an element g not invariant under H). Since H is acting trivially on E', it acts trivially on $B/\mathfrak{p}'$ and hence f belongs to $\mathfrak{p}'$ (here we are using the assumption that $p \neq 2$).

Write

$$f = t^{v(f)} \frac{c}{c'},$$

for $c, c' \in B \setminus \mathfrak{p}'$. Applying σ we get,

$$\sigma(f) = t^{v(f)} \frac{\sigma(c)}{\sigma(c')}.$$

But, by choice, $\sigma(f) = -f$ and hence $\sigma(c\sigma(c')) = -(c\sigma(c'))$. In particular, $c\sigma(c') \in \mathfrak{p}'$. A contradiction, proving that $v(s) = 2$. ☐

Proposition 1. *Let E be an irreducible reduced divisor of X and assume that π is bijective over E. Then there exist a unique irreducible reduced divisor E' of Y mapping onto E. Furthermore, $\pi^*(\mathcal{O}(E)) = \mathcal{O}(2E')$.*

Proof. That there exists a unique (reduced and irreducible) divisor E' in Y mapping onto E follows from the corresponding local statement in Lemma 3. Let s be a section of $\mathcal{O}(E)$ with scheme theoretic divisor of zeros $(s)_0$ equal to E. We want to show that $\pi^*(s)$ has divisor of zeros equal to $2E'$. But this can be checked locally, and the local statement follows from Lemma 3. $\square$

Remark 1. The above proposition is false, in general, for $p = 2$ and so is the next lemma.

The following lemma is well known.

Lemma 4. *Let V be a closed H-invariant subvariety of Y. Then $\pi(V)$ (with the reduced closed subscheme structure) is the quotient of V by H. (For this lemma, it is not necessary to assume Y or X to be smooth.)*

The following is an analogue of Hurwitz theorem.

Proposition 2. *Let $E = \{y \in Y : \sigma(y) = y\}$ denote the fixed point (reduced) subvariety of the action of H on Y. If E is a (closed) irreducible divisor in Y, then $\pi^*(\omega_X) = \omega_Y \otimes \mathcal{O}(-E)$.*

Proof. Let $(d\pi)^n : \pi^*(\omega_X) \to \omega_Y$ denote the n-th (where $n := dim(Y)$) exterior power of the differential $d\pi : \pi^*(\Omega_X) \to \Omega_Y$ of π, and let ρ denote the corresponding global section of the line bundle $\omega_Y \otimes \pi^*(\omega_X)^{-1}$. We want to show that the scheme theoretic divisor of zeros $(\rho)_0$ of ρ is equal to E:

Let U denote the complement of E in Y. Then U is an open subset of Y on which H acts freely. The restriction of the quotient map π to U is hence étale. In particular, the support of $(\rho)_0$ must be contained in E. As E is irreducible and $(\rho)_0$ is effective, there exists a non-negative integer l such that $(\rho)_0 = lE$. We have to show that $l = 1$: This can be done locally around a point in E, so we may assume that X and Y are affine with coordinate rings $A \subset B$ respectively.

By Lemma 4, the image $\pi(E)$ (with the reduced closed subscheme structure) is isomorphic to E. We may therefore think of E as a closed (irreducible) subvariety of both X and Y (of codim. 1). Let $\mathfrak{p}$ (resp. $\mathfrak{p}'$) denote the prime ideal of height 1 in A (resp. B) corresponding to E. Choose $s \in A$ (resp. $t \in B$) generating $\mathfrak{p}$ (resp. $\mathfrak{p}'$) in the local ring $A_{\mathfrak{p}}$ (resp. $B_{\mathfrak{p}'}$).

By Lemma 3, we know that there exist $b, b' \in B \setminus \mathfrak{p}'$ such that

$$s = t^2 \frac{b}{b'}.$$

Replacing s by $sb'\sigma(b')$ and b by $b\sigma(b')$, we may assume that $b' = 1$. Hence $s = t^2 b$. Now choose a point z in E such that

- E is smooth at z.
- $b(z) \neq 0$.

- $\mathfrak{p}$ (resp. $\mathfrak{p}'$) is generated by s (resp. t) in the local ring A_{m_z} (resp. $B_{m'_z}$), where m_z (resp. m'_z) is the maximal ideal corresponding to z in X (resp. Y).

(Since all these three conditions are separately valid on dense open sets in E, such a z indeed exists.) As E (by the choice of z) is smooth at z, the local ring $A_{m_z}/\mathfrak{p} = B_{m'_z}/\mathfrak{p}'$ is regular. We can therefore choose elements $s_2, \ldots, s_n \in A$ generating the maximal ideal in this local ring. Hence $ds \wedge ds_2 \wedge \cdots \wedge ds_n$ (resp. $dt \wedge ds_2 \wedge \cdots \wedge ds_n$) is a generator of $\pi^*(\omega_X)$ (resp. ω_Y) at z. Let $c \in B_{m'_z}$ be the element such that

$$db \wedge ds_2 \wedge \cdots \wedge ds_n = c \cdot (dt \wedge ds_2 \wedge \cdots \wedge ds_n).$$

Then

$$ds \wedge ds_2 \wedge \cdots \wedge ds_n = t(ct + 2b) \cdot (dt \wedge ds_2 \wedge \cdots \wedge ds_n). \tag{1}$$

Noticing that $ct + 2b$ is a unit in $B_{m'_z}$ (by the choice of z), it follows that $l = 1$ (since l is the exponent of t on the right side of Equation (1) above). $\square$

Remark 2. All the results in this section are apparently known, but we did not find an appropriate reference. Also one can formulate and prove the analogues of all these results for H replaced by any finite group G, provided that p is coprime to the order of G.

4. Crepant resolutions

In this section X will denote a smooth quasiprojective surface over an algebraically closed field k of char. $p \neq 2$. For any positive integer n, as in §1, let $X^{[n]}$ denote the Hilbert scheme of n points in X and $\psi : X^{[n]} \to X^{(n)}$ the Hilbert-Chow morphism. Whenever Z is a smooth variety, we denote by ω_Z the canonical bundle on Z.

4.1. **A fibre diagram.** As in §1, let $\Phi : X^n \to X^{(n)}$ denote the quotient map. Restrictions of Φ and ψ to $X^n_* := \Phi^{-1}(X^{(n)}_*)$ and $X^{[n]}_* := \psi^{-1}(X^{(n)}_*)$ respectively, yields the fibre product diagram:

$$\begin{array}{ccc} \tilde{X}^n_* & \xrightarrow{\tilde{\psi}} & X^n_* \\ {\scriptstyle\tilde{\Phi}}\downarrow & & \downarrow{\scriptstyle\Phi} \\ X^{[n]}_* & \xrightarrow{\psi} & X^{(n)}_* \end{array}$$

It is well known that $\tilde{X}^n_*$ is the blow-up of X^n_* along the big diagonals $\Delta_{ij} := \{(x_1, \ldots, x_n) \in X^n_* : x_i = x_j\}$ $(i < j)$, and that the map $\tilde{\Phi}$ is the quotient map by the induced S_n-action (cf. [3], Lemma 4.4). Let $\tilde{E}_{ij}$ denote the exceptional (reduced) divisor in $\tilde{X}^n_*$ corresponding to the diagonal Δ_{ij}, and let $\tilde{E}$ denote the union of the $\tilde{E}_{ij}$. Let $X^{[n]}_{**}$ denote the open subset $\psi^{-1}(X^{(n)}_{**})$ in $X^{[n]}_*$, and let E denote the complement of

$X_{**}^{[n]}$ in $X_*^{[n]}$ with the reduced scheme structure. The variety E is called the *exceptional locus* of $X_*^{[n]}$. Clearly, E is the image of $\tilde{E}_{ij}$ under $\tilde{\Phi}$ for any $i < j$. In particular, E is an irreducible variety.

4.2. **Factorization of $\tilde{\Phi}$.** As mentioned above, the map $\tilde{\Phi} : \tilde{X}_*^n \to X_*^{[n]}$ is the quotient of a certain S_n-action on $\tilde{X}_*^n$. We may divide this quotient into two parts. Let A_n be the alternating (normal) subgroup of S_n, and let H denote the quotient S_n/A_n. Let $\tilde{X}_*^{[n]}$ denote the quotient of $\tilde{X}_*^n$ by A_n, and let $\tilde{\Phi}_1$ denote the corresponding quotient map. Clearly $X_*^{[n]}$ is then the quotient of $\tilde{X}_*^{[n]}$ by H, and we denote the corresponding quotient map by $\tilde{\Phi}_2$. Then $\tilde{\Phi} = \tilde{\Phi}_2 \circ \tilde{\Phi}_1$.

4.2.1. *Description of $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$.* It is easily seen that A_n is acting freely on X_*^n and hence also on $\tilde{X}_*^n$. As $\tilde{X}_*^n$ is smooth, this implies that the quotient $\tilde{X}_*^{[n]}$ is also smooth, and that the quotient map is étale. In particular, we get

Lemma 5. $\tilde{\Phi}_1^*(\omega_{\tilde{X}_*^{[n]}}) = \omega_{\tilde{X}_*^n}$.

All the divisors $\tilde{E}_{ij}$ map to the same divisor E' in $\tilde{X}_*^{[n]}$. Clearly H acts trivially on E', hence it follows from Lemma 4 that (reduced) E' is isomorphic to E. We will however keep the notation E' to emphasize that E' is thought of as a subvariety of $\tilde{X}_*^{[n]}$. By Proposition 1, we get

Lemma 6. $\tilde{\Phi}_2^*(\mathcal{O}(E)) = \mathcal{O}(2E')$.

We also need the following similar result.

Lemma 7. $\tilde{\Phi}_1^*(\mathcal{O}(E')) = \mathcal{O}(\tilde{E})$.

Proof. This follows easily since $\tilde{\Phi}_1$ is an étale map (in particular, a smooth morphism) and the set theoretic inverse image of E' under $\tilde{\Phi}_1$ is exactly equal to $\tilde{E}$. ☐

Finally, we need the following result which follows immediately from Proposition 2.

Lemma 8. $\tilde{\Phi}_2^*(\omega_{X_*^{[n]}}) = \omega_{\tilde{X}_*^{[n]}} \otimes \mathcal{O}(-E')$.

4.3. **Crepant resolution.** In this section we will prove the following crucial result.

Theorem 1. *Let char. $k \neq 2$. Then $\psi : X_*^{[n]} \to X_*^{(n)}$ is a crepant resolution, meaning that $X_*^{(n)}$ is Gorenstein such that its dualizing line bundle $\omega_{X_*^{(n)}}$ pulls back to the canonical bundle $\omega_{X_*^{[n]}}$ on $X_*^{[n]}$ under ψ.*

First we need the following preparatory lemmas.

Recall that two cycles $Z = \sum m_i Z_i$ and $Y = \sum n_j Y_j$ in an irreducible scheme X are said to meet *properly* if $codim(Z_i \cap Y_j) = codim(Z_i) + codim(Y_j)$, whenever m_i and n_j are non-zero (cf. [4], §11.4).

Lemma 9. *Let L be a line bundle on any quasiprojective smooth variety X defined over an algebraically closed field k, and $p_1, p_2, \ldots, p_n$ be a finite set of points in X. Then there exist an open subset U in X containing $p_1, p_2, \ldots, p_n$ such that the restriction of L to U is trivial.*

Proof. As any line bundle on a smooth variety is the quotient of two effective line bundles, we may assume that L is effective. Let s be a global section of L, and let $(s)_0$ denote the divisor of zeros of s. By the Moving Lemma ([4], §11.4), there exist a divisor Z rationally equivalent to $(s)_0$ such that Z meets properly with $\sum p_i$. In other words, the complement U of the support of Z contains $p_1, \ldots, p_n$. Since rationally equivalent divisors give rise to isomorphic line bundles (cf. [4], Example 2.1.1), $L_{|U}$ is trivial. This proves the lemma. □

Let now X be a smooth quasiprojective even dimensional variety of dimension m, and let ω denote the canonical bundle on X. Then the canonical bundle on X^n is isomorphic to $\omega_n := \otimes_{i=1}^n p_i^*(\omega)$, where p_i is the projection $X^n \to X$ on the i-th factor. We regard ω_n as a S_n-equivariant sheaf on X^n in the obvious way. The sheaf $\omega_n^{S_n}$ of S_n-invariant sections of ω_n can then naturally be thought of as a sheaf on $X^{(n)}$. We claim

Lemma 10. *The sheaf $\omega_n^{S_n}$ is a line bundle on $X^{(n)}$.*

Proof. Let $p = (p_1, \ldots, p_n)$ be a point of $X^{(n)}$. By the above Lemma, there exist an open subset U of X containing $p_1, \ldots, p_n$ and such that the line bundle $\omega_{|U}$ is trivial. As the fibre over p (under the quotient map) is contained in U^n and as the assertion of the lemma is local, we may assume that $X = U$. In particular, we can assume that ω is trivial.

Let dX be a generating global section of ω. Then $dX_n = \boxtimes_{i=1}^n dX$ is a generating global section of ω_n. As dX is an even form, the section dX_n is S_n-invariant, and hence also a global generating section of $\omega_n^{S_n}$. This proves that $\omega_n^{S_n}$ is a line bundle. □

Lemma 11. *As above, let X be a smooth quasiprojective even dimensional variety. Then, there exists a unique line bundle L on $X^{(n)}$ which restricts to the canonical bundle on $X_{**}^{(n)}$.*

In particular, if char. $k \neq 2$, $X_*^{(n)}$ is Gorenstein with the dualizing line bundle $L_{|X_*^{(n)}}$. (We denote $L_{|X_*^{(n)}}$ by $\omega_{X_*^{(n)}}$.)

Proof. Taking $L = \omega_n^{S_n}$, the existence of line bundle L follows from the above lemma. Since the map Φ restricted to X_{**}^n is étale, the canonical bundle $\omega_{X_{**}^{(n)}}$ pulls back to the canonical bundle $\omega_{X_{**}^n}$. Hence, L restricts to the canonical bundle on $X_{**}^{(n)}$. The uniqueness of L follows since the codimension of $X^{(n)} \setminus X_{**}^{(n)}$ in the normal variety $X^{(n)}$ is $m \geq 2$.

Since A_n acts freely on (smooth) X_*^n, the quotient $\tilde{X}_*^{(n)}$ is smooth (and hence Cohen-Macaulay). Further, $X_*^{(n)} = \tilde{X}_*^{(n)}/H$ and hence it is Cohen-Macaulay (since $p \neq 2$). Now, the assertion that $X_*^{(n)}$ is Gorenstein, follows from [6], Lemma (2.7). □

Remark 3. Let X be a normal and Gorenstein variety X of even dimension over an algebraically closed field of char. p. Then $X^{(n)}$ is Gorenstein (and normal) provided $p > n$. (This is a result due to Aramova [1].) To prove this, apply the 'descent' lemma (cf., e.g., [7]) to the canonical bundle ω_{X^n} of the S_n-variety X^n to get a line bundle L on $X^{(n)}$. Moreover, $L_{|U_{**}^{(n)}}$ is the canonical bundle (where $U \subset X$ is the smooth locus), since $\Phi_{|U^n}$ is an étale map. But the complement of $U_{**}^{(n)}$ in $X^{(n)}$ has codim. ≥ 2 and $X^{(n)}$ is Cohen-Macaulay. Hence, by [6], Lemma (2.7), L is the dualizing line bundle of $X^{(n)}$. This proves that $X^{(n)}$ is Gorenstein.

From now on, we revert to the assumption that X is a smooth quasiprojective surface and $p \neq 2$.

Lemma 12. *Let $\omega_{X_*^{(n)}}$ be the dualizing line bundle on $X_*^{(n)}$ guaranteed by the above lemma. Then there exist an integer t such that*

$$\psi^*(\omega_{X_*^{(n)}}) \backsimeq \omega_{X_*^{[n]}} \otimes \mathcal{O}(tE).$$

Proof. As ψ is an isomorphism over $X_{**}^{(n)}$ and the restriction of $\omega_{X_*^{(n)}}$ to $X_{**}^{(n)}$ is isomorphic to the canonical bundle, we see that

$$(\psi^*(\omega_{X_*^{(n)}}))_{|X_{**}^{[n]}} = \omega_{X_{**}^{[n]}}.$$

As E is irreducible, this clearly implies the result. □

Lemma 13. *The canonical bundle on $\tilde{X}_*^n$ is given by*

$$\omega_{\tilde{X}_*^n} = \tilde{\psi}^*(\omega_{X_*^n}) \otimes \mathcal{O}(\tilde{E}),$$

where $\omega_{X_*^n}$ denotes the canonical bundle on X_*^n.

Proof. Follows from [5], Exercise II.8.5. □

Now we can prove Theorem 1.

Proof. (*of Theorem 1*) Choose $t \in \mathbb{Z}$ with the property as given in Lemma 12. We need to show that $t = 0$: By Lemmas 12, 5 - 8, we know that

$$\begin{aligned} \tilde{\Phi}^*(\psi^*(\omega_{X_*^{(n)}})) &= \tilde{\Phi}^*(\omega_{X_*^{[n]}} \otimes \mathcal{O}(tE)) \\ &= \tilde{\Phi}_1^*(\omega_{\tilde{X}_*^{[n]}} \otimes \mathcal{O}((2t-1)E')) \\ &= \omega_{\tilde{X}_*^n} \otimes \mathcal{O}((2t-1)\tilde{E}). \end{aligned} \tag{2}$$

We want to compare this with an alternative way of calculating the left side of the equation above. Since $\psi \circ \tilde{\Phi} = \Phi \circ \tilde{\psi}$,

$$\tilde{\Phi}^*(\psi^*(\omega_{X_*^{(n)}})) = \tilde{\psi}^*(\Phi^*(\omega_{X_*^{(n)}})). \tag{3}$$

As Φ is étale over $X_{**}^{(n)}$, the canonical bundle on $X_{**}^{(n)}$ pulls back to the canonical bundle on X_{**}^n. In particular, $\Phi^*(\omega_{X_*^{(n)}})$ restricts to the canonical bundle on X_{**}^n. But the complement of X_{**}^n in X_*^n has codimension 2, which forces $\Phi^*(\omega_{X_*^{(n)}})$ to be the canonical bundle on X_*^n (as X_*^n is smooth, in particular, normal). By Lemma 13, we therefore get

$$\tilde{\psi}^*(\Phi^*(\omega_{X_*^{(n)}})) = \omega_{\tilde{X}_*^n} \otimes \mathcal{O}(-\tilde{E}). \tag{4}$$

Combining $(2) - (4)$, we get $(2t-1) = -1$ (since $\mathcal{O}(\tilde{E})$ is a nontorsion element of Pic $\tilde{X}_*^n$), which forces t to be equal to zero as desired. □

The following result in char. 0 is due to Beauville.

Corollary 1. *Let char. $k > n$. Then $X^{(n)}$ is Gorenstein and $\psi : X^{[n]} \to X^{(n)}$ is a crepant resolution.*

Proof. The assertion that $X^{(n)}$ is Gorenstein follows by the same argument as for $X_*^{(n)}$ (cf. the proof of Lemma 11) . Now, since the codim. of $X^{[n]} \setminus X_*^{[n]}$ in $X^{[n]}$ is at least two, the corollary follows from the above theorem. □

5. Frobenius splitting of Hilbert schemes

Let X be a quasiprojective smooth surface over an algebraically closed field k of positive char. p. In this section we will prove that $X^{[n]}$ is Frobenius split if X is Frobenius split. First we need

Lemma 14. *Let Y be a quasiprojective Frobenius split variety over k. Then the n-th symmetric product $Y^{(n)}$ of Y is Frobenius split.*

Proof. Let $\sigma : F_*\mathcal{O}_Y \to \mathcal{O}_{Y'}$ be a Frobenius splitting of Y. Then $\sigma^{\boxtimes n} : F_*\mathcal{O}_{Y^n} \to \mathcal{O}_{(Y^n)'}$ is a Frobenius splitting of the n-fold product of Y. As $\sigma^{\boxtimes n}$ is equivariant with respect to the natural actions of the symmetric group S_n, it takes S_n-invariant functions on Y^n to S_n-invariant functions on $(Y^n)'$. As $\mathcal{O}_{Y^{(n)}}$ is the subsheaf of $\mathcal{O}_{Y^n}$ consisting of S_n-invariant functions, $\sigma^{\boxtimes n}$ induces a Frobenius splitting of $Y^{(n)}$. □

Theorem 2. *Let X be a quasiprojective Frobenius split smooth surface over an algebraically closed field k of char. $p > 2$. Then, for any $n \geq 1$, the Hilbert scheme $X^{[n]}$ of n points in X is Frobenius split.*

Proof. By Lemma 14, the n-th symmetric product $X^{(n)}$ is Frobenius split. In particular, $X_{**}^{(n)}$ is Frobenius split. Let σ' be a splitting section of $\omega^{1-p}_{X_{**}^{(n)}}$ on $X_{**}^{(n)}$. Thinking of σ' as a section of $\omega^{1-p}_{X_*^{(n)}}$ over $X_{**}^{(n)}$, as $X_*^{(n)}$ is normal and codim. of $X_*^{(n)} \setminus X_{**}^{(n)}$ in $X_*^{(n)}$ is two, we can extend σ'

to a global section σ of $\omega^{1-p}_{X_*^{(n)}}$ over $X_*^{(n)}$ (cf. Lemma 11). Consider the section $\tilde{\sigma} = \psi^*(\sigma)$ of $\psi^*(\omega^{1-p}_{X_*^{(n)}}) = \omega^{1-p}_{X_*^{[n]}}$ over $X_*^{[n]}$ (cf. Theorem 1), and extend it to a section $\hat{\sigma}$ of $\omega^{1-p}_{X^{[n]}}$ over $X^{[n]}$. (This is possible since $X^{[n]}$ is smooth, in particular, normal and the codim. of $X^{[n]} \setminus X_*^{[n]}$ in $X^{[n]}$ is at least two.)

We claim that $\hat{\sigma}$ is a splitting section of $\omega^{1-p}_{X^{[n]}}$ over $X^{[n]}$. To see this, it is enough to prove that the restriction $\hat{\sigma}'$ of $\hat{\sigma}$ to $X_{**}^{[n]}$ is a splitting section over $X_{**}^{[n]}$. But $X_{**}^{[n]}$ is isomorphic to $X_{**}^{(n)}$ under ψ, and moreover $\hat{\sigma}'$ corresponds to σ' under this isomorphism. As σ', by definition, Frobenius splits $X_{**}^{(n)}$, the result follows. □

Corollary 2. *Let X be a smooth projective Frobenius split surface over a field of characteristic $p > 2$, and let L be an ample line bundle on the Hilbert scheme $X^{[n]}$. Then L has vanishing higher cohomology.*

Remark 4. (a) One can use Corollary 2 and the Semicontinuity Theorem to get a similar vanishing result in characteristic 0.

(b) As mentioned by V. Mehta, the known list of Frobenius split smooth surfaces includes

(1) Projective examples : toric surfaces (in particular $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$), minimal rational surfaces, ordinary K3 surfaces and ordinary abelian surfaces. Furthermore, if s is a splitting section of a smooth surface X which vanishes to order $(p-1)$ along a point x on X, then the blow-up of X along x is also Frobenius split.
(2) Affine examples : any smooth affine surface is Frobenius split. (In fact, any smooth affine variety is Frobenius split.)

It is furthermore known that any projective surface, with Kodaira dimension ≥ 1, is not Frobenius split. Also non-ordinary K3 and abelian surfaces are not Frobenius split.

(c) If the punctual Hilbert scheme $H^{[n]}$ (i.e. the fibre of the Hilbert-Chow morphism ψ at $(x, \cdots, x)$ for some $x \in X$) has $H^i(H^{[n]}, \mathcal{O}_{H^{[n]}}) = 0$ for all $i > 0$, then (under the assumptions of Theorem 2) ψ is a rational resolution. In particular, for any smooth quasi projective surface X (not necessarily Frobenius split), $X^{(n)}$ would be Cohen-Macaulay.

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