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Quasi-Einstein Kähler Metrics*

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Abstract

We write an ansatz for quasi-Einstein Kähler metrics and construct new complete examples. Moreover, we construct new compact generalized quasi-Einstein Kähler metrics on some ruled surfaces, including some of Guan's examples as special cases.

1 Introduction

Let (M, J) be a complex manifold. In this paper we consider pairs (g, V) consisting of a Kähler metric and a real holomorphic vector field V on M, such that JV is an isometry of g and

$$\rho - \lambda\Omega = L_V\Omega, \tag{1}$$

where ρ is the Ricci form, Ω is the Kähler form and λ is some constant. Such structures are called quasi-Einstein Kähler metrics or Kähler Ricci solitons [4, 7, 9, 14]. Quasi-Einstein metrics are solitons for the Hamilton flow [9]

$$\frac{d}{dt}g_t = -r_t + \frac{\overline{s_t}}{n}g_t, \tag{2}$$

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where r_t is the Ricci curvature tensor and $\overline{s_t}$ is the average scalar curvature of g_t. Indeed, if g_0 is quasi-Einstein then $(\Phi_{-t})^* g_0$ solves (2), where $\Phi_t = \exp(tV)$. Friedan [6] studied quasi-Einstein metrics in connection with bosonic σ-models. He showed that the one-loop renormalizability of the model is ensured if and only if the metric of the target space is quasi-Einstein.

If M is a compact manifold, equation (1) implies that $[\rho] = \lambda[\Omega] \in H^2(M, \mathbb{R})$. Thus a necessary condition for M to admit a quasi-Einstein Kähler metric is that $c_1(M) = [\frac{\rho}{2\pi}]$ is either positive, negative, or zero. In short, c_1 must have a sign. If $c_1 \leq 0$, then Calabi, Yau and Aubin [3, 16, 1] showed that there exist Kähler-Einstein metrics. However, for $c_1 > 0$ we do not always have Kähler-Einstein metrics and the quasi-Einstein Kähler metrics serve as suitable generalizations.

For any Kähler metric on a compact manifold, we have that

$$\rho - \rho_H = \sqrt{-1}\partial\overline{\partial}\varphi_\Omega,$$

where ρ_H is the harmonic part of ρ and φ_Ω is called the Ricci potential. In fact, $\varphi_\Omega = -Gs$, where G is the Green's operator of the Laplacian Δ and s is the scalar curvature. In particular, if $[\rho] = \lambda[\Omega]$, then

$$\begin{aligned}
\rho - \lambda\Omega &= i\partial\overline{\partial}\varphi_\Omega \\
&= L_{(\overline{\partial}\varphi_\Omega)^\sharp}\Omega \\
&= L_{\frac{1}{2}grad\varphi_\Omega}\Omega - iL_{J\frac{1}{2}grad\varphi_\Omega}\Omega \\
&= L_{\frac{1}{2}grad\varphi_\Omega}\Omega,
\end{aligned}$$

where the last equality follows from the fact that $\rho - \lambda\Omega$ is real (we use $\sharp$ for raising indices and $\flat$ for lowering indices). Thus if $grad\varphi_\Omega$ is a holomorphic vector field and $[\rho] = \lambda[\Omega]$, then $(g, \frac{1}{2}grad\varphi_\Omega)$ is a Kähler-Ricci soliton.

Conversely, if (g, V) is a Kähler-Ricci soliton on a compact manifold with *positive* first Chern class, then $V = \frac{1}{2}grad\varphi_\Omega$: From equation (1) we have that

$$L_V\Omega = L_{\frac{1}{2}grad\varphi_\Omega}\Omega.$$

Let $W = V - iJV$ be the holomorphic $(1,0)$ vector field corresponding to V. Then

$$L_W\Omega = \sqrt{-1}\partial\overline{\partial}\varphi_\Omega$$

or

$$d(JW)^\flat = \sqrt{-1}\partial\overline{\partial}\varphi_\Omega.$$

Since JW is holomorphic and g is Kähler, we have that

$$\partial(JW)^\flat = \partial(\sqrt{-1}\overline{\partial}\varphi_\Omega).$$

Thus

$$(JW)^\flat = \sqrt{-1}\overline{\partial}\varphi_\Omega + \alpha,$$

where α is a ∂-closed $(0,1)$-form. Moreover, by the Hodge decomposition,

$$(JW)^\flat = \overline{\partial} f + H$$

for some function f and a harmonic form H. But since we assumed that $c_1 > 0$, there are no non-trivial harmonic 1-forms (see page 324 in [2]) and hence

$$\overline{\partial} f = \sqrt{-1}\overline{\partial}\varphi_\Omega + \alpha.$$

This gives

$$\partial\overline{\partial} f = \sqrt{-1}\partial\overline{\partial}\varphi_\Omega.$$

Therefore

$$f = \sqrt{-1}\varphi_\Omega + constant$$

and α must vanish. Thus

$$(JW)^\flat = \sqrt{-1}\overline{\partial}\varphi_\Omega$$

or

$$W = (\overline{\partial}\varphi_\Omega)^\sharp,$$

which is equivalent to

$$V = \frac{1}{2} grad\varphi_\Omega.$$

In particular, $grad\varphi_\Omega$ is a holomorphic real vector field.

The observation above tells us that the existence of quasi-Einstein Kähler metrics with non-trivial vector fields is an obstruction to the existence of Kähler-Einstein metrics: The Futaki invariant of the Kähler class on the vector field V is given as the L^2-norm of V [14]. In the compact case, Tian and Zhu [14] have proved uniqueness (modulo automorphisms) for Kähler-Ricci solitons with a fixed vector field.

In this paper, we construct quasi-Einstein Kähler metrics on complex line bundles (or their compactifications $\mathbb{P}(\mathcal{O}\oplus L)$) over Kähler-Einstein base manifolds B. We impose no restriction on the sign of the scalar curvature s_B of B and our results therefore extend known constructions. However, for $s_B \leq 0$, the first Chern class of $\mathbb{P}(\mathcal{O}\oplus L)$ does not have a sign. This motivates our study of generalized quasi-Einstein Kähler metrics in section 4. Previously, a family of extremal Kähler metrics was obtained [13] on $\mathbb{P}(\mathcal{O}\oplus L)$ over Riemann surfaces of genus at least two. Contrary to the result in [13], our construction of generalized quasi-Einstein Kähler metrics on these ruled surfaces exhausts the Kähler cone.

The structure of this paper is as follows. In section 2, we write an ansatz for quasi-Einstein Kähler metrics with a torus symmetry. In section 3, we find solutions in the case of S^1 symmetry and some additional assumptions. This gives new complete (non-compact) quasi-Einstein Kähler metrics, as well as some already known examples. In section 4, we consider Guan's generalized quasi-Einstein Kähler metrics [7]. We then construct such metrics in every Kähler class on the compactification of holomorphic line bundles over compact Riemann surfaces. This construction includes new compact metrics, as well as some metrics already constructed by Guan and Koiso.

2 An ansatz for quasi-Einstein Kähler metrics

In this section, assuming the existence of a real torus acting through holomorphic isometries on a Kähler manifold, we construct an ansatz for quasi-Einstein Kähler metrics.

2.1 The moment map construction of Kähler metrics

Following [12], we consider the situation of a real torus T^N acting freely on the Kähler manifold M^{2m} through holomorphic isometries.

Proposition 1 [12] *Let (w_{ij}), $i,j=1,\dots,N$ be a positive definite symmetric matrix and $(h_{\mu\nu})$, $\mu,\nu=1,\dots,m-N$ a positive definite hermitian matrix of smooth functions on an open set U in $\mathbb{C}^{m-N}\times\mathbb{R}^N$ with coordinates (ξ^μ, z^i).*

Assume that the 2-form

$$\Omega_h := \frac{\sqrt{-1}}{2} h_{\mu\nu} d\xi^\mu \wedge d\overline{\xi^\nu}$$

is a Kähler form on an open set in $\mathbb{C}^{m-N}$ with corresponding Kähler metric h. Let M be a T^N-bundle over U with connection 1-form $\omega = (\omega_1, \ldots, \omega_N)$. Suppose that

$$\frac{\partial^2 h_{\mu\nu}}{\partial z^i \partial z^j} + 4 \frac{\partial^2 w_{ij}}{\partial \xi^\mu \partial \overline{\xi^\nu}} = 0, \tag{3}$$

$$\frac{\partial w_{ij}}{\partial z^k} = \frac{\partial w_{ik}}{\partial z^j} \tag{4}$$

and assume the torus bundle has curvature

$$F_i = \frac{\sqrt{-1}}{2} \frac{\partial h_{\mu\nu}}{\partial z^i} d\xi^\mu \wedge d\overline{\xi^\nu} + \sqrt{-1} \frac{\partial w_{ij}}{\partial \xi^\mu} dz^j \wedge d\xi^\mu - \sqrt{-1} \frac{\partial w_{ij}}{\partial \overline{\xi^\mu}} dz^j \wedge d\overline{\xi^\mu}. \tag{5}$$

Then

$$g = h + w_{ij} dz^i \, dz^j + w^{ij} \omega_i \omega_j, \tag{6}$$

where $w^{ij} = (w^{-1})_{ij}$, is a Kähler metric on M. Conversely any Kähler metric with a torus acting freely through holomorphic isometries can locally be constructed as above.

Proof: The proof is straightforward and we just make some remarks concerning the second part of the proposition. Suppose that M is a T^N-symmetric Kähler manifold with metric g, Kähler form Ω and complex structure J. Suppose further that $(X_1, \ldots, X_N)$ are Hamiltonian vector fields generated by the torus action. Let $dz^j = -i_{X_j}\Omega$ define the Hamiltonian functions z^j. Then the metric is given as in equation (6), where h is a Kähler metric in the quotient space of each level set of the Hamiltonians. Note that $w^{ij} = g(X_i, X_j)$ and $\omega_i = w_{ij} X_j^\flat$ so $J\omega_i = -w_{ij} dz^j$ and $\Omega = dz^i \wedge \omega_i + \Omega_h$, where Ω_h is the Kähler form of the of the Kähler quotient. As J is integrable, the exterior derivative $d\varphi_i$ of the $(1,0)$ forms $\varphi_i = w_{ij} dz^j + \sqrt{-1}\omega_i$ must have no $(0,2)$ part. Also for g to be Kähler we need $d\Omega = 0$. These conditions are captured by equation (4) and by the equation $d\omega_i = F_i$ with F_i as in (5)[1]. Then equation (3) is just the integrability condition $dF_i = 0$. ∎

[1]To be absolutely precise, the pull-back of F_i with respect to the bundle projection is given by $d\omega_i$.

2.2 Quasi-Einstein Kähler metrics

Now let M^{2m} be a T^N-symmetric Kähler metric as above. Let V be the holomorphic vector field

$$V = -\frac{1}{2}\sum_{i=1}^{N} J\ (c_i X_i),$$

where c_i are constants. We will look for the condition under which the metric is quasi-Einstein Kähler with respect to V and some constant λ. Thus the equation to examine is

$$\rho - \lambda\Omega = L_V\Omega. \tag{7}$$

The Ricci form ρ is given by

$$\rho = -\sqrt{-1}\partial\overline{\partial}u = -\frac{1}{2}dJdu,$$

where $u = \log(\frac{\det h}{\det w})$. The Lie derivative of Ω with respect to V is given by

$$L_V\Omega = d(i_V\Omega) = \frac{1}{2}c_i d(w^{ij}\omega_j).$$

Using equation (5) with $d\omega_i = F_i$, we find that equation (7) is equivalent to the pair of equations

$$\left(c_j + \frac{\partial u}{\partial z^j}\right) w^{ij} = -2\lambda\cdot(z^i + B^i)$$

and

$$4\frac{\partial^2 u}{\partial\xi^\mu\partial\overline{\xi^\nu}} = -2\lambda\cdot\left(h_{\mu\nu} - (z^i + B^i)\frac{\partial h_{\mu\nu}}{\partial z^i}\right),$$

where B^i is some constant. We conclude

Proposition 2 *Let M^{2m} be a T^N-symmetric Kähler manifold as in Proposition 1. Then the metric is a quasi-Einstein Kähler metric, $\rho - \lambda\Omega = L_V\Omega$, with vector field $V = -\frac{1}{2}\sum_{i=1}^N J(c_i X_i)$ if and only if the following equations are satisfied;*

$$\left(c_j + \frac{\partial u}{\partial z^j}\right) w^{ij} = -2\lambda\cdot(z^i + B^i) \tag{8}$$

$$4\frac{\partial^2 u}{\partial \xi^\mu \partial \overline{\xi^\nu}} = -2\lambda \cdot \left(h_{\mu\nu} - (z^i + B^i)\frac{\partial h_{\mu\nu}}{\partial z^i} \right), \tag{9}$$

where B^i is some constant.

Notice that by setting $c_j = 0$ in the above ansatz, we get the Pedersen and Poon ansatz for Kähler-Einstein metrics [12]. In fact, the difference between the two ansätze appears only in equation (8).

In the case $m = 2$ and $N = 1$, we have that $h = e^u w(dx^2 + dy^2)$ with $\xi = x + iy$ and the above equations can be rewritten as

$$\frac{u_z + c}{w} = -2\lambda(z + B) \tag{10}$$

and

$$u_{xx} + u_{yy} + 2\lambda(e^u w - (z + B)(e^u w)_z) = 0. \tag{11}$$

By changing the moment map by a constant, we may assume that $B = 0$. Using equation (10), one can then replace equation (11) by an equation completely decoupled for the u function:

$$u_{xx} + u_{yy} + z^2\left(z^{-2}(e^u)_z\right)_z + c\,z^2\left(z^{-2}e^u\right)_z = 0. \tag{12}$$

Equation (3), i.e., the constraint following from the Kähler property, is given by

$$w_{xx} + w_{yy} + (we^u)_{zz} = 0.$$

However, this equation follows from equations (10) and (12). Thus, as in the Kähler-Einstein case considered in [12], we are left with the single partial differential equation (12). Once we have solved this equation, we easily find w, by using equation (10).

3 A construction of new complete quasi-Einstein Kähler metrics

In this section we consider the case $N = 1$. By solving the differential equations from the previous section in a special case, we find new complete quasi-Einstein Kähler metrics. First, we give the details of the special case in which we solve the equations. Then, we apply the ansatz from the previous section.

3.1 The assumptions

Let (B, g_B) be a $(m-1)$-dimensional compact Kähler manifold with scalar curvature s_B. Assume that the Kähler form Ω_B is such that the deRham class $[\frac{\Omega_B}{2\pi}]$ is contained in the image of $H^2(B,\mathbb{Z}) \to H^2(B,\mathbb{R})$. Let L be a holomorphic line bundle such that $c_1(L) = [\frac{-\Omega_B}{2\pi}]$. On the total space M of $(L-0) \xrightarrow{\pi} B$ we can form an S^1-symmetric Kähler metric

$$g = zg_B + wdz^2 + w^{-1}\omega^2$$

where z, being the coordinate of $(a,b) \subset (0,\infty]$, becomes the moment map of g with the obvious S^1 action on L, w is a positive function depending only on z, and ω is the connection one-form of the connection induced by g on the S^1-bundle

$$(L-0) \stackrel{(\pi,z)}{\rightarrow} B \times (a,b).$$

That is

$$d\omega = \Omega_B.$$

Clearly, equations (3) and (4) are satisfied. The complex structure J on M is given by the complex structure on B and

$$J\omega = -wdz.$$

The Kähler form is given by

$$\Omega = z\Omega_B + dz \wedge \omega.$$

If X is the Hamiltonian vector field generated by the S^1 action, then

$$dz = \Omega(-X,\cdot) = (-JX)^\flat,$$

$$w^{-1} = g(X,X),$$

and

$$\omega = \frac{g(X,\cdot)}{g(X,X)} = wX^\flat.$$

The Ricci form ρ is given by

$$\rho = \rho_B - i\partial\overline{\partial}\log(\frac{z^{m-1}}{w})$$

which implies that the scalar curvature s is given by

$$s = \frac{s_B}{z} - \frac{(\frac{z^{m-1}}{w})_{zz}}{z^{m-1}}.$$

If w^{-1} (by which we mean $1/w$) is such that $w^{-1}(a) = 0$ and $(w^{-1})'(a) = 2$, then we can add a copy of B at $z = a$ and extend the Kähler metric g over the zero-section of the bundle $L \to B$. If moreover $b < \infty$, $w^{-1}(b) = 0$, and $(w^{-1})'(b) = -2$ then we can add another copy of B at $z = b$ and extend g to a Kähler metric on the total space of the $\mathbb{CP}_1$-bundle $\mathbb{P}(\mathcal{O} \oplus L)$. We refer to [10, 11] for the details.

3.2 Applying the ansatz

In this special case, h is the local description of zg_B. That is, if locally

$$\Omega_B = \frac{\sqrt{-1}}{2} q_{\mu\nu} d\xi^\mu \wedge d\overline{\xi^\nu},$$

then $h_{\mu\nu} = z\, q_{\mu\nu}$. Equation (8) is here

$$(c + u_z)\, w^{-1} = -2\lambda \cdot (z + B) \tag{13}$$

and equation (9) is

$$4\frac{\partial^2 u}{\partial \xi^\mu \partial \overline{\xi^\nu}} = 2\lambda B q_{\mu\nu}. \tag{14}$$

Since $u = (m-1)\log z - \log w + \log(\det q)$ and

$$\rho_B = -\sqrt{-1}\frac{\partial^2 \log(\det q)}{\partial \xi^\mu \partial \overline{\xi^\nu}} d\xi^\mu \wedge d\overline{\xi^\nu}$$

we see from equation (14) that g_B must be Kähler-Einstein. From now on, we assume that $\lambda \neq 0$. Since, by the Lefschetz decomposition,

$$\rho_B = \frac{s_B}{2(m-1)} \Omega_B \tag{15}$$

we have that $B = \frac{-s_B}{2\lambda(m-1)}$. Equation (13) now becomes

$$\frac{c + u_z}{w} = -2\lambda z + \frac{s_B}{(m-1)}$$

or

$$(\frac{z^{m-1}}{w})_z + c\frac{z^{m-1}}{w} = -2\lambda z^m + \frac{s_B}{(m-1)}z^{m-1}. \tag{16}$$

The general solution of this ODE is

$$(\frac{z^{m-1}}{w}) = ke^{-cz} + A_m z^m + \cdots + A_1 z + A_0, \tag{17}$$

where the A_is are fixed (given by λ, c, m and s_B) and k is chosen freely.

For $s_B \neq 0$, we have from equation (15) that

$$c_1(L) = \frac{-2(m-1)}{s_B}[\frac{\rho_B}{2\pi}] = \frac{-2(m-1)}{s_B}c_1(K^{-1}),$$

where K^{-1} is the anti-canonical line bundle of B. Since $c_1(L)$ must be an integer class, there are some restrictions on the size of s_B. For $s_B = 0$ we cannot use equation (15) to calculate $c_1(L)$.

3.3 The case $s_B \leq 0$:

Given $a > 0$, suppose we chose (the unique) k such that $w^{-1}(a) = 0$. Then from equation (16), we see that $(w^{-1})'(a) = 2$ if and only if

$$\lambda = \frac{s_B - 2(m-1)}{2a(m-1)}.$$

Assume now that we have chosen k and λ such that the endpoint conditions at $z = a$ are met. For any $b > a$, $w^{-1}(b) = 0$ implies, together with equation (16), that

$$(w^{-1})'(b) = -2\lambda b + \frac{s_B}{(m-1)} = \frac{2b}{a} + \frac{s_B}{(m-1)}(1 - \frac{b}{a}) > 0.$$

Since $(w^{-1})'(a) > 0$, we conclude that there can be no such b. Hence the function w^{-1} is positive for all $z > a$. Thus we have a Kähler quasi-Einstein metric g on the total space of $L \to B$. If c is chosen to be positive, then

$$\lim_{z\to\infty} w^{-1} = A_m z.$$

This implies [12, 8] that g is complete.

Theorem 1 *Let (B, g_B) be a non-positive compact Kähler-Einstein manifold of dimension $(m-1)$. Assume that $[\frac{\Omega_B}{2\pi}]$ is an integer cohomology class. Let L be a holomorphic line bundle on B such that $c_1(L) = [\frac{-\Omega_B}{2\pi}]$. Let X denote the Hamiltonian vector field generating the natural S^1 action on L and let J denote the complex structure on the total space of $L \to B$. Then, for a given $a > 0$ and a given $c > 0$, there exists a complete Kähler metric g on the total space of the bundle $L \to B$ such that the pair $(g, -\frac{1}{2}cJX)$ is quasi-Einstein, satisfying the equation*

$$\rho - \lambda\Omega = L_{-\frac{1}{2}cJX}\Omega,$$

where

$$\lambda = \frac{s_B - 2(m-1)}{2a(m-1)}.$$

Notice that there is no hope of producing quasi-Einstein metrics on the compact manifold $\mathbb{P}(\mathcal{O} \oplus L) \to B$. This can be seen by the fact that c_1 has no sign for $s_B \leq 0$: If $m = 2$ the argument is quite simple. Let C denote the fiber in $\mathbb{P}(\mathcal{O} \oplus L) \to B$ and let E_0 denote the zero-section in $\mathbb{P}(\mathcal{O} \oplus L) \to B$. Then by the adjunction formula for complex surfaces we have that $c_1 \cdot C = 2 > 0$ and $c_1 \cdot E_0 = deg\, L - 2(\mathbf{g}_B - 1)$, where $\mathbf{g}_B$ denotes the genus of B. Thus for $\mathbf{g}_B > 0$ (or equivalently $s_B \leq 0$) we see that c_1 cannot have a sign. In higher dimension, take any compact metric on M of the type described in subsection 3.1. Since $\int_C \rho > 0$ and $\int_{E_0} \rho \wedge \Omega_B^{m-2} < 0$ when $s_B \leq 0$, we conclude that $c_1 = [\frac{\rho}{2\pi}]$ does not have a sign.

In the next section we will consider Guan's generalization of quasi-Einstein Kähler metrics and get existence results for all ruled surfaces of the form $\mathbb{P}(\mathcal{O} \oplus L) \to B$.

3.4 The case $s_B > 0$:

This case has already been considered by others [9, 7, 5]. If L is such that c_1 of the compact complex manifold $M = \mathbb{P}(\mathcal{O} \oplus L) \to B$ is positive, then there exist compact quasi-Einstein Kähler metrics on M of the type described in this section. This follows as a special case of the results in [7][2], which generalizes Koiso's construction [9]. Moreover, Chave and Valent [5]

[2]In a private communication Prof. Guan pointed out that, due to editorial error, Theorem 2 and Corollary 2 in [7] are both missing the assumption that ρ_B (hence s_B) is non-negative.

considered the case $B = \mathbb{CP}_{m-1}$. They found the compact solutions on $M = \mathbb{P}(\mathcal{O} \oplus K^{\frac{p}{m}}) \to B$ for $p = 1, \ldots, m-1$, where K is the canonical line bundle. For $p = m+1, m+2, \ldots$ they found complete non-compact quasi-Einstein Kähler metrics on the total space of $K^{\frac{p}{m}} \to B$. Note that $s_B = \frac{2m(m-1)}{p}$. Using the same arguments that we used in the case $s_B \leq 0$, it is easy to show the following general result.

Theorem 2 *Let (B, g_B) be a positive compact Kähler-Einstein manifold of dimension $(m-1)$. Assume that $s_B = \frac{2m(m-1)}{p}$ with $p = m+1, m+2, \ldots$. Let L be a holomorphic line bundle on B such that $c_1(L) = \frac{p}{m} c_1(K)$, where K is the canonical line bundle. (Such a line bundle L exists when $B = \mathbb{CP}_{m-1}$. In general, we may, at worst, have to assume that $p = 2m, 3m, \ldots$.) Then for a given $a > 0$ and a given $c > 0$, there exists a complete Kähler metric g on the total space of the bundle $L \to B$ such that the pair $(g, -\frac{1}{2}cJX)$ is quasi-Einstein, satisfying the equation*

$$\rho - \lambda\Omega = L_{-\frac{1}{2}cJX}\Omega,$$

where

$$\lambda = \frac{m-p}{pa}.$$

4 Generalized quasi-Einstein Kähler metrics

In [7] Guan defines generalized quasi-Einstein Kähler metrics in any Kähler class on any compact complex manifold. In this section, we introduce the defining equation for such metrics. Then we solve it under the same conditions as in subsection 3.1, with the additional assumptions that $m = 2$, g_B is Kähler-Einstein and that the final metric g is compactified to a Kähler metric on $\mathbb{P}(\mathcal{O} \oplus L) \to B$. In fact, our solutions exhaust the Kähler cone.

4.1 The equation

Let g be a Kähler metric and V be a real holomorphic vector field on a compact manifold M. The pair (g, V) is said to be generalized quasi-Einstein if

$$\rho - \rho_H = L_V \Omega. \tag{18}$$

A generalized quasi-Einstein Kähler metric serves as a generalization of constant scalar curvature Kähler metrics. Indeed, Guan proved that a generalized quasi-Einstein Kähler metrics has constant scalar curvature if and only if the Futaki invariant of the Kähler class vanishes. Thus, these metrics behave very much like the extremal Kähler metrics.

4.2 The equation in a special case

We now make the same assumptions as in subsection 3.1. Moreover we assume that $m = 2$, that is, B is a compact Riemann surface, and that g_B is Kähler-Einstein, that is, $\rho_B = \frac{s_B}{2}\Omega_B$. Finally, we assume that g is such that it can be extended to a smooth Kähler metric on the compact manifold $M = \mathbb{P}(\mathcal{O} \oplus L) \to B$. This means that for a given pair (a, b) (determining the Kähler class [13]), with $0 < a < b$, at the point $z = a$ (resp. b) the function $\frac{1}{w}$ vanishes and has derivative equal to 2 (resp. -2). By rescaling, we may assume that $a = 1$. Since g is a positive definite metric, we have that $\frac{1}{w}$ must be positive on the interval $(1, b)$.

If $V = -\frac{1}{2}cJX$, then equation (18) implies that

$$\begin{aligned}
s - \overline{s} &= 2\langle L_V\Omega, \Omega\rangle \\
&= 2\langle d(i_V\Omega), \Omega\rangle \\
&= c\langle dX^\flat, \Omega\rangle \\
&= c\langle d(w^{-1}\omega), \Omega\rangle \\
&= c\langle dJdz, \Omega\rangle \\
&= 2c\langle \sqrt{-1}\partial\overline{\partial}z, \omega\rangle \\
&= -c\Delta z,
\end{aligned}$$

where $\overline{s}$ denotes the average scalar curvature

$$\overline{s} = \frac{\int_M s d\mu}{\int_M d\mu}.$$

On the other hand, if

$$s - \overline{s} = -c\Delta z,$$

then, by the Hodge decomposition,

$$\Delta G s = -c\Delta z,$$

which implies that

$$\varphi_\Omega = -Gs = cz + constant.$$

Then $grad\varphi_\Omega = cgradz = 2V$ and, since

$$\rho - \rho_H = L_{\frac{1}{2}grad\varphi_\Omega}\Omega$$

is satisfied for any compact Kähler metric, we conclude that (g, V) is generalized quasi-Einstein. The above is summarized in the following lemma.

Lemma 1 *Let B be a compact Riemann surface with Kähler-Einstein metric g_B and let M be the ruled surface $\mathbb{P}(\mathcal{O} \oplus L) \to B$, where L is a holomorphic vector bundle such that $c_1(L) = [\frac{-\Omega_B}{2\pi}]$. For each $b > 1$, that is, for each Kähler class on M [13], consider the Kähler metric from subsection 3.1*

$$g = zg_B + wdz^2 + w^{-1}\omega^2,$$

where w^{-1} is a smooth function, positive on the interval $(1, b)$, such that it satisfies the boundary conditions

$$w^{-1}(1) = w^{-1}(b) = 0 \tag{19}$$

$$(w^{-1})'(1) = 2, \tag{20}$$

and

$$(w^{-1})'(b) = -2. \tag{21}$$

Then the pair $(g, k\frac{1}{2}JX)$ is a generalized quasi-Einstein Kähler metric if and only if w^{-1} is such that

$$s - \overline{s} = k\Delta z. \tag{22}$$

In this case, $kJX = grad\varphi_\Omega$.

4.3 Solving the equation in the special case

Recall that

$$s = \frac{s_B}{z} - \frac{(\frac{z}{w})_{zz}}{z}.$$

Inserting equation (19), (20), and (21) into equation (22), we get

$$\begin{aligned} \tfrac{s_B}{z} - \tfrac{(\frac{z}{w})_{zz}}{z} - 2\tfrac{s_B(b-1)+2(b+1)}{(b^2-1)} &= k\Delta z \\ &= -k\langle d(w^{-1}\omega), \Omega\rangle \\ &= -k((w^{-1})_z + \tfrac{w^{-1}}{z}) \end{aligned}$$

or

$$\left(\frac{z}{w}\right)_{zz} - k\left(\frac{z}{w}\right)_z = -2(\frac{s_B(b-1)+2(b+1)}{(b^2-1)})z + s_B. \tag{23}$$

Integrating equation (23) and inserting equations (19), (20) and (21) we get

$$\left(\frac{z}{w}\right)_z - k\left(\frac{z}{w}\right) = -\frac{s_B(b-1)+2(b+1)}{(b^2-1)}z^2 + s_B z + \frac{2b(b+1) - s_B b(b-1)}{(b^2-1)}. \tag{24}$$

Notice now, that if w solves equation (24) and equation (19) is satisfied, then equations (20) and (21) follow. Thus we have to find a solution of (24) such that $\frac{1}{w}$ vanishes at $z = 1$ and at $z = b$ but is positive in the interval $(1, b)$.

The solution of (24), such that $\left(\frac{z}{w}\right)$ vanishes at $z = 1$, is given by

$$\frac{z}{w}e^{-kz} \equiv G(z,k) = \int_1^z (Au^2 + Bu + C)e^{-ku}\,du,$$

where $P(u) = Au^2 + Bu + C$ is the polynomial on the right hand side of equation (24). Notice that $P(1) = 2$ and $P(b) = -2b$. Let u_0 be a zero of the polynomial $P(u)$ which lies in $(1, b)$ and u_1 the second possible zero. In the case $s_B \neq -2\frac{b+1}{b-1}$, elementary computations show that $P(u) = A(u - u_1)(u - u_0)$, with $A(u - u_1) < 0$ for $u \in (1, b)$. In the case $s_B = -2\frac{b+1}{b-1}$, we have that $Au^2 + Bu + C = B(u - u_0)$, with $B < 0$. It follows that we can write $P(u) = p(u)(u - u_0)$ with $p(u)$ negative for $u \in (1, b)$.

We want to prove that, given a fixed $b > 1$, $G(b, k) = 0$ has a unique non-zero solution in k. Consider the auxiliary function

$$F(k) = G(b,k)e^{u_0 k} = \int_1^b p(u)(u - u_0)e^{-k(u-u_0)}\,du.$$

Its derivative with respect to the variable k is positive. Thus $F(k)$ is monotonic and increasing. Furthermore, $F(-\infty) = -\infty$ and $F(+\infty) = +\infty$. This proves the existence and uniqueness of k.

Lastly, the possibility of $k = 0$ needs to be excluded. Toward this end, we compute

$$G(b,0) = \frac{(b-1)^2}{3}\left(\frac{s_B}{2}\frac{b-1}{b+1} - 2\right).$$

This vanishes only if $s_B = 4\frac{b+1}{b-1} > 4$, which can be excluded by the following reasoning. Recall that if $s_B \neq 0$ then $c_1(L) = \frac{2}{s_B}c_1(K)$, where K is the canonical line bundle on B. Thus if $s_B > 0$ (equivalently, if the genus of B is equal to 0), then $c_1(L) = \frac{-4}{s_B}$. However, since $c_1(L)$ must be an integer [15], the situation $s_B > 4$ is not possible. Since $G(b,0) < 0$, it follows that k is always strictly positive.

At this point, since equations (19), (20), and (21) are satisfied, and $(\frac{z}{w})$ is nothing but the sum of an exponential function and a second degree polynomial, we can show that the function $\left(\frac{z}{w}\right)$ and hence w must be positive between $z = 1$ and $z = b$: clearly $\left(\frac{z}{w}\right)$ has at most three zeroes (since the graph of an exponential function intersects a parabola at, at most, three points). We know from (19) that it vanishes at $z = 1$ and $z = b$. Suppose that a third vanishing point, x, lies between 1 and b. Then one easily sees that $\left(\frac{z}{w}\right)$ is positive on the interval $(1,x)$ and negative on the interval (x,b) - or vice versa. In either case, $\left(\frac{z}{w}\right)_z$ has the same sign at $z = 1$ and at $z = b$. By equations (20) and (21), this is not possible. Therefore $\left(\frac{z}{w}\right) \neq 0$ for $1 < z < b$. Since $\left(\frac{z}{w}\right)$ is positive in a small neighborhood to the right of $z = a$ (alternatively in a small neighborhood to the left of $z = b$), it must be positive everywhere on the interval $(1,b)$.

By rescaling g_B appropriately, we can obtain any (negative) holomorphic line bundle on B.

We conclude with the following theorem.

Theorem 3 *Let $M = \mathbb{P}(\mathcal{O} \oplus L) \to B$, where L is a non-trivial holomorphic line bundle on a compact Riemann surface B. Then any Kähler class on M admits a generalized quasi-Einstein Kähler metric (g,V), where $V = \frac{1}{2}grad\varphi_\Omega$ and $W := V - iJV$ is a multiple of the holomorphic vector field generating the natural $\mathbb{C}^\times$ action on L.*

Remark 1 If the genus of B is less than 2, then the metrics in the above Theorem were constructed by Guan in [7]. In particular, if the genus of B is equal to 0, $L = K^{\frac{1}{2}}$ and b is such that the Kähler class is a multiple of $c_1(M)$, we have the quasi-Einstein Kähler metric on $\mathbb{CP}_2 \sharp \overline{\mathbb{CP}_2}$ constructed by Koiso in [9] (see also [5]).

If the genus of B is at least 2, we notice that, as opposed to the family of extremal Kähler metrics constructed in [13], the metrics do exhaust the Kähler cone.

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