UNIVERSITY OF AARHUS
DEPARTMENT OF MATHEMATICS

ISSN: 1397–4076

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Preprint Series No.: 23

December 1999

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KK-EQUIVALENCE AND CONTINUOUS BUNDLES OF C^*-ALGEBRAS

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1. Introduction

Through the work directed at classifiying C^*-algebras it has become apparent that the KK-theory of Kasparov, including the E-theory of Connes and Higson, offers much more than a convenient setup for dealing with topological K-theory. Although KK-theory was first invented and developed, from the late seventies through the eighties, as a tool to attach topological questions (the Novikov conjecture) and calculate the K-theory of group C^*-algebras (the Baum-Connes conjecture), it has found some of its most profound applications in the classification program of the nineties. And although it is clear that KK-theory as a carrier of information about the structure of C^*-algebras can not in general stand alone when we seek to classify C^*-algebras up to isomorphism, it must necessarily play a major role in such efforts. As a consequence of this understanding it has become important to decide in which way the structures of two C^*-algebras are related when they are KK-equivalent and/or equivalent in E-theory. Dadarlat has obtained one answer to this question in [D]: Two separable C^*-algebras are equivalent in E-theory if and only if their stable suspensions are shape equivalent. The purpose of the present paper is to use recent results of the author on KK-theory and E-theory to give an alternative answer, and this time for KK-theory.

First of all we must decide what it means for a C^*-algebra to be KK-contractible, i.e. KK-equivalent to 0. We do this first for E-theory in Section 2 and then modify the approach to handle KK-theory in Section 3. The central notion in the description of which C^*-algebras are KK-contractible is called semi-contractibility. A C^*-algebra is semi-contractible when the identity map of the algebra can be connected to 0 by a continuous path of completely positive contractions such that the maps in the path are almost multiplicative up to an arbitrary small toleration on any given finite subset. It turns out that a separable and stable C^*-algebra A is KK-contractible if and only it is the quotient of a semi-contractible C^*-algebra by a semi-contractible ideal. In order to identify semi-contractible C^*-algebras in later parts of the paper we give a description of them involving generalized inductive limits in Section 4. The notion of a generalized inductive system of C^*-algebras was introduced by Blackadar and Kirchberg in [BK] and the notion is a cornerstone in the approach here. The idea behind such systems comes clearly from the approximate intertwining of Elliott,[E], but it is also inspired by the E-theory of Connes and Higson, and it is only natural that we can use it in Section 5 to transfer KK-theory information, incoded in two completely positive asymptotic homomorphisms, into an isomorphism between two

Version: December 6, 1999.

C^*-algebras which are closely related to the two given KK-equivalent separable C^*-algebras, A and B. The result is that we obtain a continuous bundle of C^*-algebras which connects $SA \otimes \mathcal{K}$ to $SB \otimes \mathcal{K}$ and has several quite special properties. [1]

In the last section we glue the bundle from Section 5 together with other bundles, notably a bundle considered by Elliott, Natsume and Nest in [ENN], to obtain our main result which is that two separable C^*-algebras A and B are KK-equivalent if and only if there is a separable continuous bundle of C^*-algebras over $[0,1]$ which is piecewise trivial with no more than six points of non-triviality such that the kernel of each fiber map is KK-contractible. The bundle can be chosen such that all the fiber maps admit completely positive sections and it follows therefore that all the fiber algebras are KK-equivalent to the bundle C^*-algebra. As an immediate corollary we get that A and B are KK-equivalent if and only if there is a separable C^*-algebra D and two surjective $*$-homomorphisms $\varphi : D \to A$ and $\psi : D \to B$, both of which admit a completely positive section and have KK-contractible kernels. Any KK-equivalence between separable C^*-algebras can therefore be realized by the Kasparov product of a surjective $*$-homomorphism with the inverse of a surjective $*$-homomorphism.

2. Locally contractible C^*-algebras and triviality in E-theory

Definition 2.1. A C^*-algebra D is called *locally contractible* when the following holds : For every finite set $F \subseteq D$ and every $\epsilon > 0$ there is a pointwise norm continuous family of homogeneous maps, $\delta_s : D \to D, s \in [0,1]$, such that $\delta_0 = 0, \delta_1 = \mathrm{id}_D$, $\|\delta_s(a^*) - \delta_s(a)^*\| < \epsilon$, $\|\delta_s(a) + \delta_s(b) - \delta_s(a+b)\| < \epsilon$, $\|\delta_s(a)\delta_s(b) - \delta_s(ab)\| < \epsilon$ and $\|\delta_s(a)\| < \|a\| + \epsilon$ for all $s \in [0,1]$ and all $a, b \in F$.

Recall from [MT] that an extension of C^*-algebras

$$0 \longrightarrow J \longrightarrow E \xrightarrow{p} A \longrightarrow 0$$

is *asymptotically split* when there is an asymptotic homomorphism $\pi_t : A \to E,\ t \in [1,\infty)$, such that $p \circ \pi_t = \mathrm{id}_A$ for all t. If one can choose $\pi = \{\pi_t\}_{t\in[1,\infty)}$ to be a completely positive asymptotic homomorphism, we say that the extension is *completely positive asymptotically split.*

Theorem 2.2. *Let A be a separable C^*-algebra. Then the following conditions are equivalent :*

a) *A is contractible in E-theory (i.e. $[[SA \otimes \mathcal{K}, SA \otimes \mathcal{K}]] = 0$).*
b) *The canonical extension*

$$0 \longrightarrow S^2A \otimes \mathcal{K} \longrightarrow \mathrm{cone}(SA \otimes \mathcal{K}) \longrightarrow SA \otimes \mathcal{K} \longrightarrow 0$$

is asymptotically split.
c) *$SA \otimes \mathcal{K}$ is locally contractible.*
d) *There is an extension*

$$0 \longrightarrow J \longrightarrow E \longrightarrow A \otimes \mathcal{K} \longrightarrow 0 \tag{2.1}$$

of separable C^*-algebras where J is locally contractible and E is contractible.
e) *There is an extension (**??**) of separable C^*-algebras where J and E are locally contractible.*

[1] I have chosen to follow the lead from [KW] and use the word 'bundle' instead of 'field'.

For the proof of this we need to go from information about discrete asymptotic homomorphisms to information about genuine asymptotic homomorphisms. Let A and B be arbitrary separable C^*-algebras. As in [Th1] we denote by $[[A,B]]_{\mathbb{N}}$ the homotopy classes of discrete asymptotic homomorphisms from A to B. The shift σ, given by $\sigma(\varphi)_n = \varphi_{n+1}$, defines an automorphism of the group $[[SA,SB]]_{\mathbb{N}}$. Let $[[SA,SB]]^{\sigma}_{\mathbb{N}}$ denote the fixed point group of σ. By Lemma 5.6 of [Th1] there is a short exact sequence

$$0 \longrightarrow [[SA,SB]]^0 \longrightarrow [[SA,SB]] \longrightarrow [[SA,SB]]^{\sigma}_{\mathbb{N}} \longrightarrow 0. \qquad (2.2)$$

Here $[[SA,SB]]^0$ is the subgroup of $[[SA,SB]]$ consisting of the elements which can be represented by an asymptotic homomorphism $\varphi = \{\varphi_t\}$ which is sequentially trivial in the sense that the sequence $\varphi_1, \varphi_2, \varphi_3, \cdots$ converges pointwise to zero, i.e. $\lim_{n\to\infty} \varphi_n(x) = 0$ for all $x \in SA$. The surjective map $[[SA,SB]] \to [[SA,SB]]^{\sigma}_{\mathbb{N}}$ is obtained by restricting the parameters of the asymptotic homomorphisms from $\mathbb{R}$ to $\mathbb{N}$.

Besides the extension (2.1) from [Th1] we need the observation that the composition product of two elements from $[[-,-]]^0$ is always zero :

Lemma 2.3. *Let $\psi = \{\psi\}_{t\in\mathbb{R}} : B \to C$ and $\varphi = \{\varphi_t\}_{t\in\mathbb{R}} : A \to B$ be asymptotic homomorphisms which are sequentially trivial, i.e. satisfy that $\lim_{n\to\infty} \psi_n(b) = 0,\ b \in B$, and $\lim_{n\to\infty} \varphi_n(a) = 0,\ a \in A$. It follows that $[\psi] \bullet [\varphi] = 0$ in $[[A,C]]$.*

Proof. Choose equicontinuous sequentially trivial asymptotic homomorphisms $\psi' : B \to C$ and $\varphi' : A \to B$ such that $\lim_{t\to\infty} \|\psi_t(b) - \psi'_t(b)\| = 0, b \in B$, and $\lim_{t\to\infty} \|\varphi_t(a) - \varphi'_t(a)\| = 0, a \in A$. Let D be a countable dense subset of A. By definition of $\bullet$ there is a parametrization $r : [1,\infty) \to [1,\infty)$ such that $[\psi] \bullet [\varphi]$ is represented by any equicontinuous asymptotic homomorphism λ which satisfies that $\lim_{t\to\infty} \lambda_t(a) - \psi'_{s(t)} \circ \varphi'_t(a) = 0, a \in D$, for some parametrization $s \geq r$. We leave the reader to construct a parametrization $s \geq r$ with the property that $s(t) \in \mathbb{N}$ for all t outside a neighbourhood of $\mathbb{N} \subseteq [1,\infty)$ and such that $\lim_{t\to\infty} \psi'_{s(t)} \circ \varphi'_t(a) = 0$ for all $a \in D$. By equicontinuity of λ this implies that $\lim_{t\to\infty} \lambda_t(a) = 0$ for all $a \in A$. ☐

We can now give the proof of Theorem 2.2.

Proof. a) $\Rightarrow$ b) : Set $B = A \otimes \mathcal{K}$. Since $[\mathrm{id}_{SB}] = 0$ in $[[SB,SB]]$, it follows from Theorem 1.1 of [Th2] that there is an asymptotic homomorphism $\mu = \{\mu_t\}_{t\in[1,\infty)} : \mathrm{cone}(B) \to SB$ and a norm continuous path, $U_t, t \in [1,\infty)$, of unitaries in $M_2(SB)^+$ such that

$$\lim_{t\to\infty} \begin{pmatrix} b & \\ & \mu_t(b) \end{pmatrix} - U_t \begin{pmatrix} 0 & \\ & \mu_t(b) \end{pmatrix} U_t^* = 0$$

for all $b \in SB$. Let V_1, V_2 be isometries in the multiplier algebra $M(SB)$ of SB such that $V_1V_1^* + V_2V_2^* = 1$ and consider a strictly continuous path $W_s, s \in [0,1]$, of isometries in $M(SB)$ such that $W_0 = 1$ and $W_1 = V_1$. Let T_t be the image of U_t under the isomorphism $M_2(M(SB)) \to M(SB)$ obtained from V_1, V_2. Let $\psi_s : \mathrm{cone}(B) \to \mathrm{cone}(B), s \in [0,1]$, be the canonical trivialization of $\mathrm{cone}(B)$, i.e.

$\psi_s(f)(t) = f(st)$. Define $\delta^t_s : SB \to SB, s \in [0,4]$, by

$$\delta^t_s(a) = \begin{cases} W_s a W_s^*, & s \in [0,1] \\ V_1 a V_1^* + V_2 \mu_t(\psi_{s-1}(a)) V_2^*, & s \in [1,2] \\ (3-s)[V_1 a V_1^* + V_2 \mu_t(a) V_2^*] + (s-2) T_t V_2 \mu_t(a) V_2^* T_t^*, & s \in [2,3] \\ T_t V_2 \mu_t(\psi_{4-s}(a)) V_2^* T_t^*, & s \in [3,4] \end{cases}$$

Define $\pi_t : SB \to \mathrm{cone}(SB)$ by $\pi_t(x)(s) = \delta^t_{4-4s}(x)$.

b) $\Rightarrow$ c) : Let $\pi = \{\pi_t\}_{t \in [1,\infty)} : SA \otimes \mathcal{K} \to \mathrm{cone}(SA \otimes \mathcal{K})$ be an asymptotic homomorphism such that $\pi_t(x)(1) = x$ for all t and all x. Fix a $t \geq 1$. If a finite subset $F \subseteq SA \otimes \mathcal{K}$ and $\epsilon > 0$ are given, define $\delta_s : SA \otimes \mathcal{K} \to SA \otimes \mathcal{K}$ by $\delta_s(x) = \pi_t(x)(s)$. If t is large enough $\{\delta_s\}_{s \in [0,1]}$ will meet the requirements of Definition 2.1.

c) $\Rightarrow$ d) : The canonical extension

$$0 \longrightarrow SA \otimes \mathcal{K} \longrightarrow \mathrm{cone}(A \otimes \mathcal{K}) \longrightarrow A \otimes \mathcal{K} \longrightarrow 0$$

has the stated properties.

d) $\Rightarrow$ e): This is trivial.

e) $\Rightarrow$ a) : Thanks to excision in E-theory it suffices to show that a separable locally contractible C^*-algebra D is contractible in E-theory. We first show that $SD \otimes \mathcal{K}$ is locally contractible when D is. Let $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$ be finite subsets with dense union in D. Since D is locally contractible we can construct homogeneous maps $\delta_n : D \to \mathrm{cone}(D)$ such that $\delta_n(d)(1) = d$ for all $n \in \mathbb{N}$, $d \in D$, and $\|\delta^n(a)\delta^n(b) - \delta^n(ab)\| \leq \frac{1}{n}$, $\|\delta^n(a^*) - \delta^n(a)^*\| \leq \frac{1}{n}$, $\|\delta^n(a+b) - \delta^n(a) - \delta^n(b)\| \leq \frac{1}{n}$, $\|\delta^n(a) - \delta^n(b)\| \leq \|a-b\| + \frac{1}{n}$ for all $a,b \in F_n$. The sequence $\{\delta^n\}$ defines a $*$-homomorphism $\Delta : D \to \prod_n \mathrm{cone}(D)/\oplus_n \mathrm{cone}(D)$ in the obvious way. By the Bartle-Graves selection theorem there is a continuous and homogeneous lift $\psi : D \to \prod_n \mathrm{cone}(D)$ of Δ. Set $\psi_n(d) = \psi(d)(n)$. Then $\{\psi_n\} : D \to \mathrm{cone}(D)$ is an equicontinuous family of maps forming a discrete asymptotic homomorphism such that $\lim_{n\to\infty} \|\psi_n(d)(1) - d\| = 0$ for all $d \in D$. The tensor product construction from [CH] gives us now an equicontinuous discrete asymptotic homomorphism $\psi' : D \otimes S\mathcal{K} \to \mathrm{cone}(D) \otimes S\mathcal{K}$ such that $\lim_{n\to\infty} \|\psi'_n(d \otimes x) - \psi_n(d) \otimes x\| = 0$ for $d \in D, x \in S\mathcal{K}$. In particular it follows that $\lim_{n\to\infty}(ev_1 \otimes \mathrm{id}_{S\mathcal{K}}) \circ \psi'(z) = z$ when $ev_1 : \mathrm{cone}(D) \to D$ denotes evaluation at 1. It follows then readily that $SD \otimes \mathcal{K} \simeq D \otimes S\mathcal{K}$ is locally contractible. Write $SD \otimes \mathcal{K} = \overline{\bigcup_n F_n}$ where $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$ are finite subsets. Since $SD \otimes \mathcal{K}$ is locally contractible we can construct a pointwise norm continuous path $\delta_s, s \in [1,\infty)$, of homogeneous maps $\delta_s : SD \otimes \mathcal{K} \to SD \otimes \mathcal{K}$ such that $\delta_n = \mathrm{id}_{SD\otimes\mathcal{K}}$ and $\delta_{n+\frac{1}{2}} = 0$ for all $n \in \mathbb{N}$, and $\|\delta_s(a^*) - \delta_s(a)^*\| < \frac{1}{n}$, $\|\delta_s(a) + \delta_s(b) - \delta_s(a+b)\| < \frac{1}{n}$, $\|\delta_s(a)\delta_s(b) - \delta_s(ab)\| < \frac{1}{n}$, $\|\delta_s(a) - \delta_s(b)\| < \|a-b\| + \frac{1}{n}, s \in [n, n+1]$, $a,b \in F_n$. There is then an asymptotic homomorphism $\overline{\delta} = \{\overline{\delta}_s\}_{s \in [1,\infty)} : SD \otimes \mathcal{K} \to SD \otimes \mathcal{K}$ such that $\lim_{s\to\infty} \|\overline{\delta}_s(a) - \delta_s(a)\| = 0$ for all $a \in SD \otimes \mathcal{K}$. In particular, $[\overline{\delta}|_{\mathbb{N}}] = [\mathrm{id}_{SD\otimes\mathcal{K}}]$ in $[[SD \otimes \mathcal{K}, SD \otimes \mathcal{K}]]^\sigma_{\mathbb{N}}$. It follows then from (2.1) that $[\overline{\delta}] - [\mathrm{id}_{SD\otimes\mathcal{K}}] \in [[SD \otimes \mathcal{K}, SD \otimes \mathcal{K}]]^0$. Since $\lim_{n\to\infty} \overline{\delta}_{n+\frac{1}{2}}(a) = 0$ for all $a \in SD \otimes \mathcal{K}$ we have also that $[\overline{\delta}|_{\mathbb{N}}] = 0$ in $[[SD \otimes \mathcal{K}, SD \otimes \mathcal{K}]]^\sigma_{\mathbb{N}}$. Consequently $[\mathrm{id}_{SD\otimes\mathcal{K}}] \in [[SD\otimes\mathcal{K}, SD\otimes\mathcal{K}]]^0$ by (2.1) and hence $[\mathrm{id}_{SD\otimes\mathcal{K}}] = [\mathrm{id}_{SD\otimes\mathcal{K}}] \bullet [\mathrm{id}_{SD\otimes\mathcal{K}}] = 0$ by Lemma 2.3.

□

It follows from Theorem 2.2 that the class of separable E-contractible C^*-algebras is the least class of separable C^*-algebras which contains the locally contractible C^*-algebras and is closed under stabilization and under the formation of quotients A/I where both A and the ideal I are in the class.

3. Semi-contractible C^*-algebras and triviality in KK-theory

Definition 3.1. A C^*-algebra D is called *semi-contractible* when the following holds: For every finite set $F \subseteq D$ and every $\epsilon > 0$ there is a pointwise norm continuous family of completely positive contractions, $\delta_s : D \to D, s \in [0,1]$, such that $\delta_0 = 0,\ \delta_1 = \mathrm{id}_D$ and $\|\delta_s(a)\delta_s(b) - \delta_s(ab)\| < \epsilon$ for all $s \in [0,1]$ and all $a, b \in F$.

The results of [Th1] which were used in the last section all have analogues for completely positive asymptotic homomorphisms which were also presented in [Th1]. It is therefore easy to use the same arguments to prove the following result.

Theorem 3.2. *Let A be a separable C^*-algebra. Then the following conditions are equivalent :*

a) *A is contractible in KK-theory (i.e. $KK(A,A) = 0$).*
b) *The canonical extension*

$$0 \longrightarrow S^2A \otimes \mathcal{K} \longrightarrow \mathrm{cone}(SA \otimes \mathcal{K}) \longrightarrow SA \otimes \mathcal{K} \longrightarrow 0$$

is completely positive asymptotically split.

c) *$SA \otimes \mathcal{K}$ is semi-contractible.*
d) *There is a semi-split extension*

$$0 \longrightarrow J \longrightarrow E \longrightarrow A \otimes \mathcal{K} \longrightarrow 0 \tag{3.1}$$

of separable C^*-algebras where J is semi-contractible and E is contractible.

e) *There is a semi-split extension (**??**) of separable C^*-algebras where J and E are semi-contractible.*

It follows from Theorem 3.2 that the class of separable KK-contractible C^*-algebras is the least class of separable C^*-algebras which contains the semi-contractible C^*-algebras and is closed under stabilization and under the formation of quotients A/I when both A and the ideal I are in the class, and there is a completely positive section for the quotient map $A \to A/I$.

Clearly,

$$\begin{aligned} \{\text{contractible } C^*\text{-algebras}\} &\subseteq \{\text{semi-contractible } C^*\text{-algebras}\} \\ &\subseteq \{\text{locally contractible } C^*\text{-algebras}\}. \end{aligned}$$

The following examples show that both inclusions are strict, also when we restrict attention to separable C^*-algebras that are stable suspensions.

Example 3.3. We give here an example of a class of separable C^*-algebras E which are KK-contractible, and whose stable suspension $SE \otimes \mathcal{K}$ are not contractible. [2] Let B be a unital separable infinite dimensional simple C^*-algebra which is KK-equivalent to an abelian C^*-algebra A. The KK-equivalence is represented by a

[2] I am grateful to Mikael Rørdam for pointing examples of this kind out to me.

semi-split extension of SA by $B \otimes \mathcal{K}$ as an element of $\operatorname{Ext}^{-1}(SA, B)$. By stabilizing and suspending the extension becomes

$$0 \longrightarrow SB \otimes \mathcal{K} \longrightarrow SE \otimes \mathcal{K} \longrightarrow S^2A \otimes \mathcal{K} \longrightarrow 0.$$

The algebra E is KK-contractible. This is because the connecting maps of the six-term exact sequence arising by applying the functor $KK(SE, -)$ to the extension is given by taking the Kasparov product with the KK-equivalence which the extension represents and hence are isomorphisms. Another way to see this is to use the UCT-theorem of Rosenberg and Schochet,[RS]. However, $SE \otimes \mathcal{K}$ is not contractible because the properties of B ensure that $\operatorname{Hom}(SB, \mathcal{K}) = 0$ so that also $\operatorname{Hom}(SB \otimes \mathcal{K}, S^2A \otimes \mathcal{K}) = 0$. Consequently, any $*$-endomorphism of $SE \otimes \mathcal{K}$ must leave $SB \otimes \mathcal{K} \subseteq SE \otimes \mathcal{K}$ globally invariant. So when $SB \otimes \mathcal{K}$ is not contractible (as can easily be arranged by requiring $K_*(B) \neq 0$), $SE \otimes \mathcal{K}$ will not be contractible.

Example 3.4. In [S] Skandalis gave an example of a separable C^*-algebra A which is trivial in E-theory, but not in KK-theory. Hence by Theorem 3.2 and Theorem 2.2 $SA \otimes \mathcal{K}$ is locally contractible, but not semi-contractible.

4. The structure of semi-contractible C^*-algebras

To construct and study semi-contractible C^*-algebras we need the notion of a generalized inductive system of C^*-algebras and the inductive limit of a such a system. This was defined by Blackadar and Kirchberg in [BK] and we shall use their terminology and results. Given a contractible C^*-algebra D, a *trivialization* of D will be a pointwise norm continuous path, $\psi_s, s \in [0,1]$, of endomorphisms of D such that $\psi_0 = 0$ and $\psi_1 = \mathrm{id}_D$.

Definition 4.1. A separable C^*-algebra B is called approximately contractible when there is a sequence

$$B_1 \xrightarrow{\varphi_1} B_2 \xrightarrow{\varphi_2} B_3 \xrightarrow{\varphi_3} \cdots \tag{4.1}$$

of contractible C^*-algebras B_n with trivializations $\psi^n_s, s \in [0,1]$, and completely positive contractions $\varphi_n : B_n \to B_{n+1},\ p_n : B_{n+1} \to B_n$, such that

1) for $k \in \mathbb{N}$, $a, b \in B_k$ and $\epsilon > 0$, there is a $N \in \mathbb{N}$ such that

$$\|\varphi_{m,n}(\psi^n_s \circ \varphi_{n,k}(a))\varphi_{m,n}(\psi^n_s \circ \varphi_{n,k}(b)) - \varphi_{m,n}(\psi^n_s(\varphi_{n,k}(a)\varphi_{n,k}(b)))\| < \epsilon$$

for all $s \in [0,1]$ and all $N \leq n \leq m$,

2) $p_{k+1} \circ \varphi_{k+1} \circ \varphi_k = \varphi_k$ for all k,

and $B \simeq \varinjlim(B_n, \varphi_{n,k})$.

Here $\varphi_{n,k}$ is the composite map $\varphi_{n-1} \circ \varphi_{n-1} \circ \cdots \circ \varphi_k : B_k \to B_n$ when $n > k$. (Observe that we use an index convention which the is the reversed of the one used in [BK].) Note that condition 1) of Definition 4.1 ensures that the sequence (4.1) is a generalized inductive system in the sense of Blackadar and Kirchberg, cf. Definition 2.1.1 of [BK].

Proposition 4.2. *Let D be a separable C^*-algebra. The following conditions are equivalent :*

a) *D semi-contractible.*

b) *$D \simeq \varinjlim(\mathrm{cone}(D), \varphi_{n,k})$, where $\varphi_n : \mathrm{cone}(D) \to \mathrm{cone}(D)$ is a sequence of completely positive contractions such that 1) of Definition 4.1 holds relative to the canonical trivialization of $\mathrm{cone}(D)$, and there are completely positive contractions $p_k : \mathrm{cone}(D) \to \mathrm{cone}(D)$ such that 2) of Definition 4.1 holds.*

c) *D is approximately contractible.*

Proof. a) $\Rightarrow$ b) : Let $\psi_s, s \in [0,1]$, be the canonical trivialization of $\mathrm{cone}(D)$. Let $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$ be a sequence of finite sets with dense union in $\mathrm{cone}(D)$. Since D is semi-contractible we can construct, recursively, a sequence δ^n of paths $\delta^n_s : D \to D, s \in [0,1]$, of completely positive contractions such that $\|\delta^n_s(a)\delta^n_s(b) - \delta^n_s(ab)\| < \frac{1}{n}$ for all $s \in [0,1]$ and all $a, b \in \{\delta^j_s(F_n) : s \in [0,1],\ j < n\}$. Define $\chi_n : D \to \mathrm{cone}(D)$ by $\chi_n(d)(s) = \delta^n_s(d), s \in [0,1]$, and $\mu : \mathrm{cone}(D) \to D$ by $\mu(f) = f(1)$. Set $\varphi_n = \chi_{n+1} \circ \mu$ and note that the diagram

$$\begin{tikzcd}
D \arrow[r, equal] \arrow[d, "\chi_1"'] & D \arrow[r, equal] \arrow[d, "\chi_2"'] & D \arrow[r, equal] \arrow[d, "\chi_3"'] & \cdots \\
\mathrm{cone}(D) \arrow[r, "\varphi_1"'] \arrow[ur, "\mu"] & \mathrm{cone}(D) \arrow[r, "\varphi_2"'] \arrow[ur, "\mu"] & \mathrm{cone}(D) \arrow[r, "\varphi_3"'] \arrow[ur, "\mu"] & \cdots
\end{tikzcd}$$

commutes. Since $\varphi_{n,k} = \chi_n \circ \mu, n > k$, it follows easily that $(\mathrm{cone}(D), \varphi_{n,k})$ satisfies 1) of Definition 4.1 and the above diagram shows that $D \simeq \varinjlim(\mathrm{cone}(D), \varphi_{n,k})$, cf. [BK]. Define $p_k : \mathrm{cone}(B) \to \mathrm{cone}(B)$ by $p_k(g)(s) = \delta^k_s(g(1))$. Then the p_k's are completely positive contractions such that 2) of Definition 4.1 holds.

b) $\Rightarrow$ c) : This is trivial.

c) $\Rightarrow$ a) : Let $D \simeq \varinjlim(B_n, \varphi_{n,k})$ where the B_n's are contractible C^*-algebras with trivializations $\psi^n_s, s \in [0,1]$, and completely positive contractions $p_k : B_{k+1} \to B_k$ such that 1) and 2) of Definition 4.1 hold. Set $p_{n,m} = p_n \circ p_{n-1} \circ \cdots \circ p_{m-1}$ and let $q : \prod_i B_i \to \prod_i B_i / \oplus_i B_i$ be the quotient map. Let $x = (x_1, x_2, x_3, \cdots) \in \prod_i B_i$ be an element such that $q(x) \in \varinjlim(B_n, \varphi_{n,k})$. We assert that $\lim_{m\to\infty} p_{n,m}(x_m)$ exists in B_n. By an obvious $\frac{\epsilon}{2}$-argument we may assume that $q(x) = \varphi_{\infty,l}(y)$ for some $y \in B_l,\ l > n$. In this case we see from 2) of Definition 4.1 that $\lim_{m\to\infty} p_{n,m}(x_m) = p_{n,l+1} \circ \varphi_l(y)$. It follows that there is a completely positive contraction $\overline{p}_n : \varinjlim(B_n, \varphi_{n,k}) \to B_n$ such that $\overline{p}_n \circ \varphi_{\infty,l}(y) = p_{n,l+1} \circ \varphi_l(y),\ y \in B_l, l > n$. For $l < n$ we find that

$$\overline{p}_n \circ \varphi_{\infty,l}(x) = \lim_{m\to\infty} p_{n,m} \circ \varphi_{m,l}(x) = \varphi_{n,l}(x), \tag{4.2}$$

$x \in B_l$. It follows that $\lim_{n\to\infty} \varphi_{\infty,n} \circ \overline{p}_n(x) = x$ for all $x \in \varinjlim(B_n, \varphi_{n,k})$. Furthermore, it follows from (4.2), the density of $\bigcup_l \varphi_{\infty,l}(B_l)$ in $\varinjlim(B_n, \varphi_{n,k})$ and 1) of Definition 4.1 that when F is a finite subset of $\varinjlim(B_n, \varphi_{n,k})$ and $\epsilon > 0$, then there is a n so large that $\delta^n_s = \varphi_{\infty,n} \circ \psi^n_s \circ \overline{p}_n, s \in [0,1]$, is a pointwise norm continuous path of completely positive contractions on $\varinjlim(B_n, \varphi_{n,k})$ such that $\delta^n_0 = 0,\ \|\delta^n_1(a) - a\| < \epsilon$ and $\|\delta^n_s(ab) - \delta^n_s(a)\delta^n_s(b)\| < \epsilon$ for all $a, b \in F$. It follows that $\varinjlim(B_n, \varphi_{n,k})$ is semi-contractible.

$\square$

5. Generalized inductive limits and continuous bundles

Let $A_1, A_2, A_3, \cdots$ be a sequence of C^*-algebras. For each n let $\varphi^n_t : A_n \to A_{n+1}, t \in [0,1]$, be a pointwise norm continuous path of completely positive contractions. For $n < m$, set $\varphi^{m,n}_t = \varphi^{m-1}_t \circ \varphi^{m-2}_t \circ \cdots \circ \varphi^n_t : A_n \to A_m$. Assume that the following holds :

For $k \in \mathbb{N}$, $a, b \in A_k$ and $\epsilon > 0$, there is a $M \in \mathbb{N}$ such that

$$\sup_{t\in[0,1]} \|\varphi^{n,m}_t(\varphi^{m,k}_t(a)\varphi^{m,k}_t(b)) - \varphi^{n,k}_t(a)\varphi^{n,k}_t(b)\| < \epsilon \tag{5.1}$$

for all $M \leq m < n$.

For any C^*-algebra A, set $IA = C[0,1] \otimes A$. Given the above, define $\varphi_{m,n} : IA_n \to IA_m$ by

$$\varphi_{m,n}(f)(t) = \varphi^{m,n}_t(f(t)) \ .$$

Then $(IA_n, \varphi_{n,m})$ is a generalized inductive system of C^*-algebras in the sense of [BK] and we can consider the corresponding inductive limit C^*-algebras $\varinjlim(IA_n, \varphi_{m,n})$.

Lemma 5.1. 1) *For each $t \in [0,1]$ there is a surjective $*$-homomorphism*

$$\pi_t : \varinjlim(IA_n, \varphi_{m,n}) \to \varinjlim(A_n, \varphi^{m,n}_t) \ .$$

2) $\ker \pi_t = \varinjlim(I_tA_n, \varphi_{m,n})$, *where* $I_tA_n = \{f \in IA_n : f(t) = 0\}$.
3) *For every* $x \in \varinjlim(IA_n, \varphi_{m,n})$, $\|x\| = \sup_{t\in[0,1]} \|\pi_t(x)\|$.
4) $\varinjlim(IA_n, \varphi_{m,n})$ *is a* $C[0,1]$*-module such that* $\pi_t(fx) = f(t)\pi_t(x)$, $f \in C[0,1]$, $x \in \varinjlim(IA_n, \varphi_{m,n})$.

Proof. 1) Let $e_t : IA_n \to A_n$ denote evaluation at $t \in [0,1]$. Then $e_t \circ \varphi_{m,n} = \varphi^{m,n}_t \circ e_t$ and we get a $*$-homomorphism $\pi_t : \varinjlim(IA_n, \varphi_{m,n}) \to \varinjlim(A_n, \varphi^{m,n}_t)$ by 2.3 of [BK]. π_t is surjective since each e_t is.

2) Clearly, $\varinjlim(I_tA_n, \varphi_{m,n}) \subseteq \ker \pi_t$. Let $x \in \ker \pi_t$, $\epsilon > 0$. There is a $k \in \mathbb{N}$ and an element $f \in IA_k$ such that $\|x - \varphi_{\infty,k}(f)\| < \epsilon$. Then $\|\pi_t(\varphi_{\infty,k}(f))\| < \epsilon$ which implies that there is a $m \geq k$ such that $\|\varphi_{m,k}(f)(t)\| = \|\varphi^{m,k}_t(f(t))\| < \epsilon$. There is therefore an element $g \in I_tA_m$ such that $\|g - \varphi_{m,k}(f)\| < \epsilon$. It follows that $\varphi_{\infty,m}(g) \in \varinjlim(I_tA_n, \varphi_{m,n})$ and $\|x - \varphi_{\infty,m}(g)\| < 2\epsilon$.

3) It suffices to show that $\pi_t(x) = 0 \ \forall t \in [0,1] \Rightarrow x = 0$. So let $f \in IA_n$ and assume that $\|\pi_t(\varphi_{\infty,n}(f))\| < \epsilon$ for all $t \in [0,1]$. For a fixed $t \in [0,1]$ there is then a $k > n$ and an open neighbourhood U_t of t such that $\|\varphi^{k,n}_s(f(s))\| < \epsilon$ for all $s \in U_t$. Since $\varphi^{m,k}_s$ is a contraction we find that $\|\varphi^{m,n}_s(f(s))\| < \epsilon$ for all $s \in U_t$ and all $m \geq k$. By compactness of $[0,1]$ we can then find a $N \in \mathbb{N}$ such that $\|\varphi_{m,n}(f)\| < \epsilon$ for all $m \geq N$, proving that $\|\varphi_{\infty,n}(f)\| < \epsilon$.

4) follows immediately from the observation that $\varphi_{m,n}(fa) = f\varphi_{m,n}(a)$, $a \in IA_n$, $f \in C[0,1]$, $m \geq n$.

□

It follows from Lemma 5.1 that $\varinjlim(IA_n, \varphi_{m,n})$ is a bundle of C^*-algebra over $[0,1]$ in the sense of [KW]. The bundle is always upper semi-continuous, but to obtain a continuous bundle we need to add an additional assumption :

For $n \in \mathbb{N}$, $a \in A_n$ and $\epsilon > 0$, there is a $m > n$ such that

$$\|\varphi_t^{m,n}(a)\| - \epsilon < \|\varphi_t^{k,n}(a)\| \tag{5.2}$$

for all $k \geq m$ and all $t \in [0,1]$.

Lemma 5.2. *Assume that (5.2) holds. Then* $(\varinjlim(IA_n, \varphi_{m,n}), [0,1], \pi)$ *is a continuous bundle of* C^**-algebras.*

Proof. By Lemma 5.1 it only remains to establish the continuity of $t \mapsto \|\pi_t(x)\|$ for an arbitrary element $x \in \varinjlim(IA_n, \varphi_{m,n})$. Let $\epsilon > 0$. There is a $n \in \mathbb{N}$ and an element $g \in IA_n$ such that $\|x - \varphi_{\infty,n}(g)\| < \epsilon$. Then $\|\pi_t(\varphi_{\infty,n}(g))\| = \|\varphi_t^{\infty,n}(g(t))\|$ for all $t \in [0,1]$. Since $\{\|\varphi_t^{k,n}(a)\|\}_{k>n}$ decreases towards $\|\varphi_t^{\infty,n}(a)\|$ for all $a \in A_n$ and since $\{g(t) : t \in [0,1]\}$ is a compact subset of A_n, it follows from (5.2) that there is a $m > n$ such that $|\|\varphi_t^{\infty,n}(g(t))\| - \|\varphi_t^{m,n}(g(t))\|| < \epsilon$ for all $t \in [0,1]$. Then $|\|\pi_t(x)\| - \|\varphi_t^{m,n}(g(t))\|| < 2\epsilon$ for all t and $t \mapsto \|\varphi_t^{m,n}(g(t))\|$ is continuous, so we are done. □

Definition 5.3. Two continuous bundles of C^*-algebras, $(\mathcal{A}, X, \pi)$ and $(\mathcal{A}', Y, \pi')$, are *weakly isomorphic* when there is a homeomorphism $\chi : Y \to X$ and a $*$-isomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$ such that $\varphi(fa) = f \circ \chi\varphi(a)$, $f \in C(X)$, $a \in \mathcal{A}$. When $X = Y$ and χ can be taken to be the identity map we say that the bundles are *isomorphic*.

Let $(\mathcal{A}, [0,1], \pi)$ be a continuous bundle of C^*-algebras. When $U \subseteq [0,1]$ is a relatively open subset we set

$$\mathcal{A}_U = \overline{C_0(U)\mathcal{A}} = \{x \in \mathcal{A} : \pi_t(x) = 0, \ t \notin U\},$$

which is a closed twosided ideal in $\mathcal{A}$. When $X = U \cap F$ where U and F are relatively open and closed in $[0,1]$, respectively, we set $\mathcal{A}_X = \mathcal{A}_U / \mathcal{A}_{U \cap F^c}$. For each $t \in X$ the map $\pi_t : \mathcal{A} \to \mathcal{A}_t$ induces a surjective $*$-homomorphism $\mathcal{A}_X \to \mathcal{A}_t$ which we again denote by π_t. In this way $(\mathcal{A}_X, X, \pi)$ becomes a continous bundle of C^*-algebras over X. Up to isomorphism this construction does not depend on the way X is realized as the intersection of a closed and an open subset of $[0,1]$.

Definition 5.4. Let $(\mathcal{A}, [0,1], \pi)$ be a continuous bundle of C^*-algebras. A point $\alpha \in [0,1]$ is called a point of triviality for the bundle when there is an open neighborhood U of α in $[0,1]$ such that $(\mathcal{A}_U, U, \pi)$ is a trivial bundle. A point $\alpha \in]0,1[$ is called a point of right-sided (resp. left-sided) non-triviality when there is an $\epsilon > 0$ such that $(\mathcal{A}_{]\alpha-\epsilon,\alpha]},]\alpha - \epsilon, \alpha], \pi)$ and $(\mathcal{A}_{]\alpha,\alpha+\epsilon[},]\alpha, \alpha + \epsilon[, \pi)$ (resp. $(\mathcal{A}_{]\alpha-\epsilon,\alpha[},]\alpha - \epsilon, \alpha[, \pi)$ and $(\mathcal{A}_{[\alpha,\alpha+\epsilon[}, [\alpha, \alpha + \epsilon[, \pi))$ are trivial bundles. A point of *non-triviality* is a point which is either a right-sided or a left-sided point of non-triviality. [3]

Definition 5.5. A continuous bundle of C^*-algebras, $(\mathcal{A}, [0,1], \pi)$, is called *piecewise trivial* when there is a finite set of points $x_1 < x_2 < \cdots < x_k$ in $]0,1[$ each of which is a point of non-triviality, while all points of $[0,1]\backslash\{x_1, x_2, \cdots, x_k\}$ are points of triviality for the bundle.

[3] I apologize for the fact that with this terminology an interior point of triviality is also a point of non-triviality, and that a point of non-triviality may in fact be a point of triviality. Note that we need not consider points of two-sided non-triviality.

Definition 5.6. An extension of C^*-algebras

$$0 \longrightarrow J \longrightarrow E \xrightarrow{p} A \longrightarrow 0$$

is *discrete asymptotically semi-split* when there is a discrete completely positive asymptotic homomorphism $\chi_n : A \to E, n \in \mathbb{N}$, such that $\lim_{n\to\infty} p \circ \chi_n(a) = a$ for all $a \in A$. A bundle of C^*-algebras $(\mathcal{A}, [0,1], \pi)$ is called discretely asymptotically semi-split (resp. semi-split) when

$$0 \longrightarrow \ker \pi_t \longrightarrow \mathcal{A} \xrightarrow{\pi_t} \mathcal{A}_t \longrightarrow 0 \tag{5.3}$$

is discrete asymptotically semi-split (resp. semi-split) for all $t \in [0,1]$.

Note that it follows from Theorem 6 of [A] that an extension (and hence also a bundle of C^*-algebras) which is discrete asymptotically semi-split is also semi-split.

Now we strengthen the assumptions on the given sequence φ^n_t in order to use Lemma 5.2 to produce continuous bundles which are piecewise trivial and discrete asymptotically semi-split, and at the same time arrange that the kernels of the fiber maps are all semi-contractible. For $m > n$ and $\underline{t} = (t_1, t_2, t_3, \cdots) \in [0,1]^\infty$, set

$$\varphi^{m,n}_{\underline{t}} = \varphi^{m-1}_{t_{m-1}} \circ \varphi^{m-2}_{t_{m-2}} \circ \cdots \circ \varphi^n_{t_n} \ .$$

We can then consider the following properties of which the two first are stronger than (5.1) and (5.2), respectively.

For $k \in \mathbb{N}$, $a, b \in A_k$ and $\epsilon > 0$, there is a $M \in \mathbb{N}$ such that

$$\sup_{\underline{t} \in [0,1]^\infty} \|\varphi^{n,m}_{\underline{t}}(\varphi^{m,k}_{\underline{t}}(a)\varphi^{m,k}_{\underline{t}}(b)) - \varphi^{n,k}_{\underline{t}}(a)\varphi^{n,k}_{\underline{t}}(b)\| < \epsilon \tag{5.4}$$

for all $M \leq m < n$.

For $n \in \mathbb{N}$, $a \in A_n$ and $\epsilon > 0$, there is a $m > n$ such that

$$\|\varphi^{m,n}_{\underline{t}}(a)\| - \epsilon < \|\varphi^{k,n}_{\underline{t}}(a)\| \tag{5.5}$$

for all $k \geq m$ and all $\underline{t} \in [0,1]^\infty$.

There are completely positive contractions $p_k : A_{k+1} \to A_k$ such that

$$p_{k+1} \circ \varphi^{k+1}_s \circ \varphi^k_t = \varphi^k_t \tag{5.6}$$

for all $s, t \in [0,1]$, $s \geq t$, and all k.

Proposition 5.7. *Assume that (5.4), (5.5) and (5.6) hold and let $(\mathcal{A}, [0,1], \pi)$ be the continuous bundle from Lemma 5.2. There is then a piecewise trivial and discrete asymptotically semi-split continuous bundle $(\mathcal{A}', [0,1], \pi')$ with only one point of non-triviality (a right-sided non-triviality) such that $\pi_0(\mathcal{A}) \simeq \pi'_0(\mathcal{A}')$, $\pi_1(\mathcal{A}) \simeq \pi'_1(\mathcal{A}')$, and such that $\ker \pi_t$ is semi-contractible for all $t \in [0,1]$.*

Proof. For each $n \in \mathbb{N}$ let $h_n : [0,1] \to [0,1]$ be the function

$$h_n(t) = \begin{cases} 0, & t \in [0, \frac{1}{2}] \\ 2nt - n, & t \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{2n}] \\ 1, & t \in [\frac{1}{2} + \frac{1}{2n}, 1] \end{cases}$$

Set $\psi^n_t = \varphi^n_{h_n(t)}$. It follows from (5.4) and (5.5) that the sequence ψ^n_t also satisfies (5.4) and (5.5), and in particular also (5.1) and (5.2). Thus $(\varinjlim(IA_n, \psi_{m,n}), [0,1], \pi)$ is a continuous bundle of C^*-algebras by Lemma 5.2. In order not to confuse it with the original bundle we denote it by $(\mathcal{A}', [0,1], \pi')$. It follows from 2) of Lemma 5.1 that $\ker \pi'_t$ is the inductive limit of the sequence

$$I_tA_1 \xrightarrow{\psi_1} I_tA_2 \xrightarrow{\psi_2} I_tA_3 \xrightarrow{\psi_3} \cdots$$

By using that $h_{n+1} \geq h_n$, it follows from (5.6) that the completely positive contractions $\tilde{p}_k : I_tA_{k+1} \to I_tA_k$ given by $\tilde{p}_k(g)(s) = p_k(g(s))$ satisfy that $\tilde{p}_{k+1} \circ \psi_{k+1} \circ \psi_k = \psi_k$. Hence the above sequence satisfies condition 2) of Definition 4.1. To see that also condition 1) of Definition 4.1 holds, observe that we can define a trivialization $\{\psi_\lambda\}$ of I_tA_n of the form $\psi_\lambda(f)(s) = f(H_\lambda(s))$, where the H_λ's are appropriately chosen functions $H_\lambda : [0,1] \to [0,1]$. Therefore condition 1) of Definition 4.1 follows from (5.4). Consequently $\ker \pi'_t$ is approximately contractible, and hence semi-contractible by Proposition 4.2. Since $h_n(0) = 0,\ h_n(1) = 1$ for all n, it is clear that $\pi_0(\mathcal{A}) \simeq \pi'_0(\mathcal{A}'),\ \pi_1(\mathcal{A}) \simeq \pi'_1(\mathcal{A}')$. To prove triviality over $[0, \frac{1}{2}]$ and $]\frac{1}{2}, 1]$ note first that $\overline{C_0(]\frac{1}{2},1])\mathcal{A}} = \varinjlim(C_0(]\frac{1}{2},1], A_n), \psi^{m,n})$. It follows that $\mathcal{A}_{[0,\frac{1}{2}]} = \mathcal{A}/\mathcal{A}_{]\frac{1}{2},1]} \simeq \varinjlim(C([0,\frac{1}{2}], A_n), \psi^{m,n})$ and since $\psi^{m,n}_t = \varphi^{m,n}_0, t \in [0, \frac{1}{2}]$, we find that $\mathcal{A}_{[0,\frac{1}{2}]} \simeq C([0,\frac{1}{2}], D)$ where $D = \varinjlim(A_n, \varphi^{m,n}_0)$, also as $C[0,\frac{1}{2}]$-modules. To prove triviality over $]\frac{1}{2}, 1]$, consider a $n \in \mathbb{N}$ and an element $a \in C_0(]\frac{1}{2},1], A_n)$. It is then clear that there is an $m \geq n$ such that

$$\sup_{t \in]\frac{1}{2},1]} \|\psi^{k,m}_t \circ \psi^{m,n}_t(a(t)) - \varphi^{k,m}_1 \circ \psi^{m,n}_t(a(t))\| < \epsilon$$

and

$$\sup_{t \in]\frac{1}{2},1]} \|\psi^{k,m}_t \circ \varphi^{m,n}_1(a(t)) - \varphi^{k,m}_1 \circ \varphi^{m,n}_1(a(t))\| < \epsilon$$

for all $k \geq m$. It follows that the identity maps on $C_0(]\frac{1}{2},1], A_n)$ serves to give us an approximate intertwining in the sense of [BK],2.3, and hence we see that $\mathcal{A}_{]\frac{1}{2},1]} \simeq C_0(]\frac{1}{2},1], B)$, also as $C_0(]\frac{1}{2},1])$-modules, where $B = \varinjlim(A_n, \varphi^{m,n}_1)$.

It remains to prove that the extensions (5.3) are all discrete asymptotically semisplit. Fix a $t \in [0,1]$. By (5.6) there is a sequence of completely positive contractions $p_k : A_{k+1} \to A_k$ such that $p_{k+1} \circ \psi^{k+1}_t \circ \psi^k_t = \psi^k_t$ for all k. As observed in the proof of Proposition 4.2 this gives us a sequence of completely positive contractions $\overline{p}_n : \mathcal{A}_t = \varinjlim(A_n, \psi^{m,n}_t) \to A_n$ such that $\overline{p}_n \circ \psi^{\infty,l}_t = \psi^{n,l}_t$ when $n > l$. Let $c_n : A_n \to IA_n$ be the embedding which identifies an element of A_n with the corresponding constant A_n-valued function on $[0,1]$ and define $\chi_n : \mathcal{A}_t \to \mathcal{A}$ by $\chi_n(x) = \psi_{\infty,n} \circ c_n \circ \overline{p}_n(x)$. To see that $\{\chi_n\}$ is a discrete asymptotic homomorphism, let $x, y \in A_l$. For any $\epsilon > 0$ there is then a $m > l$ so large that $\|\psi^{\infty,l}_t(x)\psi^{\infty,l}_t(y) - \psi^{\infty,m}_t(\psi^{m,l}_t(x)\psi^{m,l}_t(x))\| < \epsilon$. Set $x' = \psi^{m,l}_t(x),\ y' = \psi^{m,l}_t(y)$. For $n > m$ we have that

$$\begin{aligned}
&\|\chi_n(\psi^{\infty,l}_t(x))\chi_n(\psi^{\infty,l}_t(y)) - \chi_n(\psi^{\infty,l}_t(x)\psi^{\infty,l}_t(y))\| \\
&\leq \limsup_{k\to\infty} \sup_{s\in[0,1]} \|\psi^{k,n}_s \circ \psi^{n,m}_t(x')\psi^{k,n}_s \circ \psi^{n,m}_t(y') - \psi^{k,n}_s \circ \psi^{n,m}_t(x'y')\| + \epsilon \\
&\leq \limsup_{k\to\infty} \sup_{\underline{t}\in[0,1]^\infty} \|\varphi^{k,n}_{\underline{t}}(\varphi^{n,m}_{\underline{t}}(x'))\varphi^{k,n}_{\underline{t}}(\varphi^{n,m}_{\underline{t}}(y')) - \varphi^{k,m}_{\underline{t}}(x'y')\| + \epsilon.
\end{aligned}$$

It follows from (5.4) that the last expression is $\leq 2\epsilon$ if just n is large enough and hence $\{\chi_n\}$ is a discrete asymptotic homomorphism. Since $\pi_t \circ \chi_n \circ \psi_t^{\infty,l}(x) = \psi_t^{\infty,n} \circ \psi_t^{n,l}(x) = \psi_t^{\infty,l}(x)$ for $x \in A_l,\ l < n$, we see that $\lim_{n\to\infty} \pi_t \circ \chi_n(z) = z$ for all $z \in \mathcal{A}_t$. This shows that (5.3) is discrete asymptotically semi-split.

□

Observe that if all the A_n's are separable and/or nuclear C^*-algebras it follows that the continuous bundles obtained here, in Lemma 5.2 and in Proposition 5.7, are separable and/or nuclear (in the sense that the bundle C^*-algebra is separable and/or nuclear.) For the nuclearity part of this assertion, use Proposition 5.1.3 of [BK].

6. From KK-equivalence to continuous bundles

In this section we consider two separable C^*-algebras A and B. For simplicitiy of notation we assume first that both are stable. Recall from Theorem 4.1 of [Th2] that there is a completely positive asymptotic homomorphism $\nu^A : \mathrm{cone}(A) \to SA$ with the property that when $\psi, \varphi : SA \to SA$ are completely positive asymptotic homomorphisms such that $[\varphi] = [\psi]$ in $[[SA, SA]]_{cp}$, then there is a norm continuous path $U_t, t \in [1,\infty)$, of unitaries in $(SA)^+$ and an increasing continuous function $r : [1,\infty) \to [1,\infty)$ such that

$$\lim_{t\to\infty} \begin{pmatrix} \psi_t(a) & \\ & \nu^A_{r(t)}(a) \end{pmatrix} - \begin{pmatrix} \varphi_t(a) & \\ & \nu^A_{r(t)}(a) \end{pmatrix} = 0$$

for all $a \in SA$.

Lemma 6.1. 1) $\lim_{t\to\infty} \|\nu_t^A(a)\| = \|a\|$ *for all* $\mathrm{cone}(A)$.

2) *Every element of* $[[SA, B]]_{cp}$ *is represented by a completely positive asymptotic homomorphism* $\varphi = (\varphi_t)_{t\in[1,\infty)} : SA \to B$ *with the property that* $\lim_{t\to\infty} \|\varphi_t(a)\| = \|a\|$ *for all* $a \in SA$.

Proof. We prove 1) and 2) in one stroke. Let $\lambda : \mathrm{cone}(A) \to B$ be the completely positive asymptotic homomorphism which features in Theorem 4.1 of [Th2]. Since $[\varphi \oplus \lambda] = [\varphi]$ in $[[SA, B]]_{cp}$ it suffices to show that $\lim_{t\to\infty} \|\lambda_t(a)\| = \|a\|$ for all $a \in \mathrm{cone}(A)$. (With $B = SA$ this will prove 1).) λ has the form $\lambda_t(a) = p_t\pi(a)p_t$, where $\pi : \mathrm{cone}(A) \to M(B)$ is an absorbing $*$-homomorphism and $(p_t)_{t\in[1,\infty)}$ is a norm continuous path of positive elements in $M(B)$ with properties described in Theorem 3.7 of [Th2]. Since π is absorbing and there is an injective $*$-homomorphism $\mathrm{cone}(A) \to M(B)$ (because B is stable) it follows that π is injective. Let $a \in \mathrm{cone}(A)$ and $\epsilon > 0$. There is then an element $b \in B$, $\|b\| \leq 1$, such that $\|a\| - \epsilon < \|\pi(a)b\|$. Since $\lim_{t\to\infty} p_t b = b$ and $\lim_{t\to\infty} p_t\pi(a) - \pi(a)p_t = 0$ it follows that $\|\lambda_t(a)\| \geq \|p_t\pi(a)p_t b\| > \|a\| - \epsilon$ for all large t. □

Remark 6.2. It follows from Lemma 6.1 and [L] that for *any* pair of separable C^*-algebras it holds that the stable suspension of any one of them is a deformation of the stable suspension of the other.

A map $\mu : D \to C$ is said to be δ-*multiplicative* on a subset $F \subseteq D$ when $\|\mu(a)\mu(b) - \mu(ab)\| < \delta$ for all $a, b \in F$.

Lemma 6.3. *Let* $\varphi : SA \to SB$ *and* $\psi : SB \to SA$ *be completely positive asymptotic homomorphisms such that* $[\psi] \bullet [\varphi] = [\mathrm{id}_{SA}]$ *in* $[[SA, SA]]_{cp}$. *It follows* *that there is a continuous function $r : [1,\infty) \to [1,\infty)$ with the following property: When $F \subseteq SA$ and $G \subseteq \operatorname{cone}(A)$ are compact subsets, $\delta > 0$ and $t_0 \in [1,\infty)$, there is a $t_1 \geq t_0$, and for each $s \geq r(t_1)$ a unitary $T_s \in M_2(SA)^+$ and a completely positive contraction $\mu_s : \operatorname{cone}(A) \to SA$ such that μ_s is δ-multiplicative on G, $\|\mu_s(a)\| \geq \|a\| - \delta, a \in G$, and*

$$\|T_s \begin{pmatrix} \psi_s \circ \varphi_{t_1}(a) & \\ & \mu_s(a) \end{pmatrix} T_s^* - \begin{pmatrix} a & \\ & \mu_s(a) \end{pmatrix} \| < \delta$$

for all $a \in F$.

Proof. By definition of the composition product there is a continuous function $r : [1,\infty) \to [1,\infty)$ such that $[\psi] \bullet [\varphi]$ is represented by $\{\psi_{r'(t)} \circ \varphi_t\}_{t \in [1,\infty)}$ for any continuous function $r' : [1,\infty) \to [1,\infty)$ such that $r' \geq r$. We may assume that r is convex. We claim that r then has the stated property, and we prove it by contradiction. So assume F, G, δ and $t_0 \in [1,\infty)$ is a quadruple for which the stated property fails. There is then a sequence $t_0 < t_1 < t_2 < t_3 < \cdots$ in $[1,\infty)$ such that $\lim_{i\to\infty} t_i = \infty$ and for each i a $s_i \geq r(t_i)$ with the property that

$$\sup_{a \in F} \|T \begin{pmatrix} \psi_{s_i} \circ \varphi_{t_i}(a) & \\ & \mu(a) \end{pmatrix} T^* - \begin{pmatrix} a & \\ & \mu(a) \end{pmatrix} \| \geq \delta \tag{6.1}$$

for every unitary $T \in M_2(SA)^+$ and every completely positive contraction $\mu : \operatorname{cone}(A) \to SA$ which is δ-multiplicative on G and satisfies that $\|\mu(a)\| \geq \|a\| - \delta$, $a \in G$. For $s \in [t_i, t_{i+1}]$, write $s = \alpha t_{i+1} + (1-\alpha)t_i$, where $\alpha \in [0,1]$, and set $\lambda_s = \psi_{\alpha s_{i+1} + (1-\alpha)s_i} \circ \varphi_{\alpha t_{i+1} + (1-\alpha)t_i}$. Since $\alpha s_{i+1} + (1-\alpha)s_i \geq \alpha r(t_{i+1}) + (1-\alpha) r(t_i) \geq r(\alpha t_{i+1} + (1-\alpha)t_i)$ it follows that $\lambda = (\lambda_s)_{s \in [t_1,\infty)}$ is a completely positive asymptotic homomorphism. In fact $\lambda_t = \psi_{v(t)} \circ \varphi_t$, $t \geq t_1$, where $v : [t_1,\infty) \to [s_1,\infty)$ is a continuous function such that $v(t) \geq r(t)$, $t \geq t_1$, and hence $[\lambda] = [\psi] \bullet [\varphi]$ in $[[SA, SA]]_{cp}$. By Theorem 4.1 of [Th2] there is a completely positive asymptotic homomorphism $\nu = (\nu_t)_{t \in [1,\infty)} : \operatorname{cone}(A) \to SA$ and a path of unitaries $(T_t) \in M_2(SA)^+$ such that

$$\lim_{t\to\infty} T_t \begin{pmatrix} \lambda_t(a) & \\ & \nu_t(a) \end{pmatrix} T_t^* - \begin{pmatrix} a & \\ & \nu_t(a) \end{pmatrix} = 0$$

for all $a \in SA$. By 1) of Lemma 6.1, ν can be chosen such that $\lim_{t\to\infty} \|\nu_t(a)\| = \|a\|$ for all $a \in \operatorname{cone}(A)$. In particular, it follows that for i large enough, ν_{t_i} is δ-multiplicative on G, $\|\nu_{t_i}(a)\| \geq \|a\| - \delta$, $a \in G$, and

$$\sup_{a \in F} \|T_{t_i} \begin{pmatrix} \psi_{s_i} \circ \varphi_{t_i}(a) & \\ & \nu_{t_i}(a) \end{pmatrix} T_{t_i}^* - \begin{pmatrix} a & \\ & \nu_{t_i}(a) \end{pmatrix} \| < \delta,$$

contradicting (6.1). □

Now assume that A and B are KK-equivalent. This means that there are completely positive asymptotic homomorphisms $\varphi = \{\varphi_t\} : SA \to SB$ and $\psi = \{\psi_t\} : SB \to SA$ such that $[\varphi] \bullet [\psi] = [\mathrm{id}_{SB}]$ in $[[SB, SB]]_{cp}$ and $[\psi] \bullet [\varphi] = [\mathrm{id}_{SA}]$ in $[[SA, SA]]_{cp}$. By 2) of Lemma 6.1 we may assume that $\lim_{t\to\infty} \|\varphi_t(a)\| = \|a\|$, $a \in SA$, and $\lim_{t\to\infty} \|\psi_t(b)\| = \|b\|$, $b \in SB$. Choose a continuous function $r : [1,\infty) \to [1,\infty)$ having the property described in Lemma 6.3 and a continuous function $s : [1,\infty) \to [1,\infty)$ with the same property relative to (ψ, φ) instead of (φ, ψ). To proceed we need to introduce some notation and terminology. Set $D = A \oplus B$. We will consider $SA, SB, \operatorname{cone}(A)$ and $\operatorname{cone}(B)$ as C^*-subalgebras of $\operatorname{cone}(D)$. Observe that $\operatorname{cone}(A), \operatorname{cone}(B), SA, SB$ and $\operatorname{cone}(D)$ are stable since A and B are. When E is any of these stable C^*-algebras we choose isometries, V_1, V_2, in $M(E)$ such that

$V_1V_1^*+V_2V_2^* = 1$. For $e_1, e_2 \in E$ we can then define $e_1 \oplus e_2 = V_1e_1V_1^* + V_2e_2V_2^* \in E$. A completely positive contraction $SA \oplus \operatorname{cone}(D) \to SB \oplus \operatorname{cone}(D)$ is said to be of *φ-type* when it has the form $(a,d) \mapsto (V(\varphi_t(a) \oplus \mu(d))V^*, a \oplus d)$ for some $t \in [1,\infty)$, some unitary $V \in (SB)^+$ and some completely positive contraction $\mu : \operatorname{cone}(D) \to SB$. Similarly, we define a completely positive contraction $SB \oplus \operatorname{cone}(D) \to SA \oplus \operatorname{cone}(D)$ to be of *ψ-type* when it has the form $(b,d) \mapsto (W(\psi_s(b) \oplus \nu(d))W^*, b \oplus d)$ for some $s \in [1,\infty)$, some unitary $W \in (SA)^+$ and some completely positive contraction $\nu : \operatorname{cone}(D) \to SA$. A map $\mu : SA \oplus \operatorname{cone}(D) \to SA \oplus \operatorname{cone}(D)$ is called *an almost identity map* when it has the form $\mu(a,d) = (a \oplus \nu(a,d), \chi(a,d) \oplus d)$ where $\chi : SA \oplus \operatorname{cone}(D) \to \operatorname{cone}(D)$ and $\nu : \operatorname{cone}(A) \oplus \operatorname{cone}(D) \to SA$ are completely positive contractions. For each $s \in [0,1]$, let $h_s : [0,1] \to [0,1]$ be the continuous function such that $h_s(t) = t, \ t \in [0,s]$, and $h_s(t) = s, \ t \in [s,1]$. Define $\psi_s^D : \operatorname{cone}(D) \to \operatorname{cone}(D)$ by $\psi_s^D(f)(t) = f(h_s(t))$. Note that with a natural choice of isometries V_1, V_2 to define $\oplus$ we have that $\psi_t^D(a \oplus d) = \psi_t^D(a) \oplus \psi_t^D(d)$. Given an almost identity map μ as above we can use $\{\psi_s^D\}_{s\in[0,1]}$ and $\{\psi_s^A\}_{s\in[0,1]}$ to define maps $\mu^{(s)}$, $s \in [0,1]$, by

$$\mu^{(s)}(a,d) = (a \oplus \nu(\psi_s^A(a), \psi_s^D(d)), \psi_s^D(\chi(a, \psi_s^D(d))) \oplus \psi_s^D(d)) \ .$$

Given C^*-algebras X, Y and Z and compact subsets $F \subseteq X, \ G \subseteq Y, \ H \subseteq Z$, a diagram of the form

$$\begin{tikzcd}
(X,F) \arrow[r, "\psi"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\epsilon"] & (Y,G) \arrow[dl, "\lambda"] \\
(Z,H) &
\end{tikzcd}$$

will mean that $\epsilon > 0$, that φ, ψ and λ are completely positive contractions such that ψ and φ are ϵ-multiplicative on F, λ is ϵ-multiplicative on G, $\varphi(F) \cup \lambda(G) \subseteq H$, $\psi(F) \subseteq G$, and $\|\varphi(x) - \lambda \circ \psi(x)\| < \epsilon, \ x \in F$.

Let $q_1 : SA \oplus \operatorname{cone}(D) \to SA, \ q_1 : SB \oplus \operatorname{cone}(D) \to SB$ be the projections to the first coordinate and $q_2 : SA \oplus \operatorname{cone}(D) \to \operatorname{cone}(D), \ q_2 : SB \oplus \operatorname{cone}(D) \to \operatorname{cone}(D)$ the projections to the second.

Lemma 6.4. *Let $\epsilon_1 > \epsilon_2 > \epsilon_3 > \cdots$ be a sequence in $]0,1[$ and let $F_1^0 \subseteq F_2^0 \subseteq F_3^0 \subseteq \cdots, \ G_1^0 \subseteq G_2^0 \subseteq G_3^0 \subseteq \cdots$ be compact subsets of $SA \oplus \operatorname{cone}(D)$ and $SB \oplus \operatorname{cone}(D)$, respectively. There is then an infinite diagram*

$$
\begin{tikzcd}
(SA \oplus \mathrm{cone}(D), F_1) \arrow[r, "\Phi_1"] \arrow[d, "\mu_1"'] \arrow[dr, phantom, "\epsilon_1" near start] \arrow[dr, phantom, "\epsilon_1" near end] & (SB \oplus \mathrm{cone}(D), G_1) \arrow[dl, "\Psi_1"] \arrow[d, "\nu_1"] \\
(SA \oplus \mathrm{cone}(D), F_2) \arrow[r, "\Phi_2"] \arrow[d, "\mu_2"'] \arrow[dr, phantom, "\epsilon_2" near start] \arrow[dr, phantom, "\epsilon_2" near end] & (SB \oplus \mathrm{cone}(D), G_2) \arrow[dl, "\Psi_2"] \arrow[d, "\nu_2"] \\
(SA \oplus \mathrm{cone}(D), F_3) \arrow[r, "\Phi_3"] \arrow[d, "\mu_3"'] \arrow[dr, phantom, "\epsilon_3" near start] \arrow[dr, phantom, "\epsilon_3" near end] & (SB \oplus \mathrm{cone}(D), G_3) \arrow[dl, "\Psi_3"] \arrow[d, "\nu_3"] \\
\vdots & \vdots
\end{tikzcd}
$$

where the Φ_i's are of φ-type, the Ψ_i's of ψ-type, the μ_i's and ν_i's are almost identity maps and

$$F_n^0 \cup \bigcup_{j<n} \bigcup_{(t_1,\cdots,t_{n-j})\in[0,1]^{n-j}} \mu_{n-1}^{(t_1)} \circ \mu_{n-2}^{(t_2)} \circ \cdots \circ \mu_j^{(t_{n-j})}(F_n^0) \subseteq F_n,$$

$$G_n^0 \cup \bigcup_{j<n} \bigcup_{(t_1,\cdots,t_{n-j})\in[0,1]^{n-j}} \nu_{n-1}^{(t_1)} \circ \nu_{n-2}^{(t_2)} \circ \cdots \circ \nu_j^{(t_{n-j})}(G_n^0) \subseteq G_n$$

for all n. Furthermore, we arrange that μ_n^s and ν_n^s are ϵ_n-multiplicative on F_n and G_n, respectively, and

$$\|q_1 \circ \mu_n^{(s)}(x)\| \geq \|\mu_n^{(s)}(x)\| - \epsilon_n, \ x \in F_n,$$

$$\|q_1 \circ \nu_n^{(s)}(x)\| \geq \|\nu_n^{(s)}(x)\| - \epsilon_n, \ x \in G_n,$$

for all n and all $s \in [0,1]$.

Proof. To simply notation, set

$$G_n' = G_n^0 \cup \bigcup_{j<n} \bigcup_{(t_1,\cdots,t_{n-j})\in[0,1]^{n-j}} \nu_{n-1}^{(t_1)} \circ \nu_{n-2}^{(t_2)} \circ \cdots \circ \nu_j^{(t_{n-j})}(G_n^0)$$

and

$$F_n' = F_n^0 \cup \bigcup_{j<n} \bigcup_{(t_1,\cdots,t_{n-j})\in[0,1]^{n-j}} \mu_{n-1}^{(t_1)} \circ \mu_{n-2}^{(t_2)} \circ \cdots \circ \mu_j^{(t_{n-j})}(F_n^0)$$

We proceed by induction. So assume that we have constructed everything up to n, i.e. that we have a finite diagram as above ending with

$$(SA \oplus \mathrm{cone}(D), F_n) \xrightarrow{\Phi_n} (SB \oplus \mathrm{cone}(D), G_n)$$

Here Φ_n is of φ-type, i.e. is given by $(a,d) \mapsto (V(\varphi_k(a) \oplus \mu(d))V^*, a \oplus d)$ for some $k \in [1,\infty)$, some unitary $V \in (SB)^+$ and some completely positive contraction $\mu : \mathrm{cone}(D) \to SB$. In addition we shall assume that

A) μ is $\frac{\epsilon_n}{2}$-multiplicative on $\psi_t^D(q_2(F_n)), t \in [0,1]$,

B) for all $y \geq r(k)$ there is a unitary $T_y \in (SA)^+$ and a completely positive contraction $\delta : \operatorname{cone}(D) \to SA$ which is $\frac{\epsilon_n}{2}$-multiplicative on $\{\psi_t^D(a \oplus d) : (a,d) \in F_n\} \cup q_2(F_n) \cup \psi_t^D(q_2(G_n)), t \in [0,1]$, and satisfies that $\|\delta(\psi_t^D(a \oplus d))\| \geq \|\psi_t^D(d)\| - \epsilon_n, \ (a,d) \in F_n, \ t \in [0,1]$, and $\|T_y(\psi_y \circ \varphi_k(a) \oplus \delta(a \oplus d))T_y^* - a \oplus \delta(a \oplus d)\| < \frac{\epsilon_n}{2}$ for all $(a,d) \in F_n$.

This is allright if we make sure that Φ_{n+1} satisfies the corresponding $n+1$-conditions, and we will. Choose $y \geq r(k)$ so large that there is a unitary $W \in (SA)^+$ such that

$$\|W\psi_y(\varphi_k(a)) \oplus \psi_y(\mu(d))W^* - \psi_y(V(\varphi_k(a) \oplus \mu(d))V^*))\| < \frac{\epsilon_n}{2} \tag{6.2}$$

for all $(a,d) \in F_n$. This is possible because $V \in (SB)^+$ and ψ is an asymptotic homomorphism. By B) there is a unitary $U \in (SA)^+$ and a completely positive contraction $\nu : \operatorname{cone}(D) \to SA$ which is $\frac{\epsilon_n}{2}$-multiplicative on $\{\psi_t^D(a \oplus d) : (a,d) \in F_n\} \cup q_2(F_n) \cup \psi_t^D(q_2(G_n)), t \in [0,1]$, and satisfies that $\|\nu(\psi_t^D(a \oplus d))\| \geq \|\psi_t^D(d)\| - \epsilon_n, \ (a,d) \in F_n, \ t \in [0,1]$, and $\|U(\psi_y \circ \varphi_k(a) \oplus \nu(a \oplus d))U^* - a \oplus \nu(a \oplus d)\| < \frac{\epsilon_n}{2}$ for all $(a,d) \in F_n$. Set

$$X = \bigcup_{j<n} \bigcup_{(t_1,\cdots,t_{n-j}) \in [0,1]^{n-j}} \mu_{n-1}^{(t_1)} \circ \mu_{n-2}^{(t_2)} \circ \cdots \circ \mu_j^{(t_{n-j})}(F_{n+1}^0) \ ,$$

and

$$X' = \{q_1 \circ \Phi_n(a, \psi_t^D(d)) \oplus a \oplus \psi_t^D(d) : \ (a,d) \in X, \ t \in [0,1]\}.$$

Then X and X' are both compact subsets of $SA \oplus \operatorname{cone}(D)$ and $\operatorname{cone}(D)$, respectively. By Lemma 6.3 we can arrange, by increasing y further, that

C) For all $z \geq s(y)$ there is a unitary $T_z \in (SB)^+$ and a completely positive contraction $\delta' : \operatorname{cone}(D) \to SB$ which is $\frac{\epsilon_{n+1}}{2}$-multiplicative on $\psi_t^D(\{b \oplus d : (b,d) \in G_n\} \cup \{q_1 \circ \Phi_n(a,d) \oplus a \oplus d : \ (a,d) \in F_n\} \cup q_2(G_n) \cup X' \cup q_2(F_{n+1}^0)), \ t \in [0,1]$, and satisfies that $\|\delta'(\psi_t^D(b \oplus d))\| \geq \|\psi_t^D(d)\| - \epsilon_{n+1}$, for $(b,d) \in G_n, \ t \in [0,1]$, and

$$\|T_z(\varphi_z \circ \psi_y(b) \oplus \delta'(b \oplus d))T_z^* - b \oplus \delta'(b \oplus d)\| < \frac{\epsilon_n}{2}$$

for all $(b,d) \in G_n$.

In addition we can arrange that ψ_y is $\frac{\epsilon_n}{2}$-multiplicative on $q_1(G_n) \cup \mu(\psi_t^D(q_2(F_n))), \ t \in [0,1]$. Observe that there is a unitary $T \in (SA)^+$ such that

$$\|T(\psi_y \circ \varphi_k(a) \oplus \psi_y \circ \mu(d) \oplus \nu(a \oplus d))T^* - a \oplus \psi_y \circ \mu(d) \oplus \nu(a \oplus d)\| < \frac{\epsilon_n}{2} \tag{6.3}$$

for all $(a,d) \in F_n$. Define $\Psi_n : SB \oplus \operatorname{cone}(D) \to SA \oplus \operatorname{cone}(D)$ by

$$\Psi_n(b,d) \ = \ (T(W^*\psi_y(b)W \oplus \nu(d))T^*, b \oplus d) \ ,$$

and $\mu_n : SA \oplus \operatorname{cone}(D) \to SA \oplus \operatorname{cone}(D)$ by

$$\mu_n(a,d) \ = \ (a \oplus \psi_y \circ \mu(d) \oplus \nu(a \oplus d), V(\varphi_k(a) \oplus \mu(d))V^* \oplus a \oplus d) \ .$$

Then Ψ_n is of ψ-type and μ_n is an almost identity map. It is straightforward to see that Ψ_n and μ_n are ϵ_n-multiplicative on G_n and F_n, respectively, so (6.3) and (6.2) give us the diagram

$$\begin{tikzcd}
(SA \oplus \mathrm{cone}(B), F_n) \arrow[r, "\Phi_n"] \arrow[d, "\mu_n"'] \arrow[dr, phantom, "\epsilon_n"] & (SB \oplus \mathrm{cone}(A), G_n) \arrow[dl, "\Psi_n"] \\
(SA \oplus \mathrm{cone}(B), F_{n+1}) &
\end{tikzcd}$$

where $F_{n+1} = \Psi_n(G_n) \cup \mu_n(F_n) \cup \bigcup_{t\in[0,1]} \mu_n^{(t)}(X) \cup F^0_{n+1}$. To check that the additional requirements on μ_n, concerning $\mu_n^{(s)}$, $s \in [0,1]$, are also satisfied, observe first of all that $F'_{n+1} \subseteq F_{n+1}$. Combine the fact that ψ_y is $\frac{\epsilon_n}{2}$-multipliciative on $\mu(\psi^B_t(q_2(F_n)))$ with A) to see that $(a,d) \mapsto \psi_y \circ \mu(\psi^D_t(d))$ is ϵ_n-multiplicative on F_n for all t. It follows from A) that $(a,d) \mapsto V(\varphi_k(a) \oplus \mu(\psi^D_t(d)))V^*$ is ϵ_n-multiplicative on F_n for all t, and one of the requirements on ν was that $(a,d) \mapsto \nu(\psi^D_t(a \oplus d))$ is ϵ_n-multiplicative on F_n for all t. By putting all of this together we see that $\mu_n^{(s)}$ is ϵ_n-multiplicative on F_n for all $s \in [0,1]$. Another requirement on ν was that $\|\nu(\psi^D_t(a\oplus d))\| \geq \|\psi^D_t(d)\| - \epsilon_n$ for all t and all $(a,d) \in F_n$. It follows that $\|q_1 \circ \mu_n^{(s)}(x)\| \geq \|\mu_n^{(s)}(x)\| - \epsilon_n$ for all $x \in F_n$ and all $s \in [0,1]$. We claim that

A') ν is ϵ_n-multiplicative on $\psi^D_t(q_2(G_n)), t \in [0,1]$,

B') for all $z \geq s(y)$ there is a unitary $T_z \in (SB)^+$ and a completely positive contraction $\delta' : \mathrm{cone}(D) \to SB$ which is $\frac{\epsilon_{n+1}}{2}$-multiplicative on $\{\psi^D_t(b \oplus d) : (b,d) \in G_n\} \cup q_2(G_n) \cup \psi^D_t(q_2(F_{n+1})), t \in [0,1]$, and satisfies that $\|\delta'(\psi^D_t(b \oplus d))\| \geq \|\psi^D_t(d)\| - \epsilon_{n+1}$, $(b,d) \in G_n$, and
$$\|T_z(\varphi_z \circ \psi_y(b) \oplus \delta'(b\oplus d))T_z^* - b \oplus \delta'(b \oplus d)\| < \frac{\epsilon_{n+1}}{2}$$
for all $(b,d) \in G_n$.

A') was one of the requirements on ν. To see that B') holds, observe that $q_2(\Psi_n(G_n)) = \{b \oplus d : \ (b,d) \in G_n\}$, that $q_2(\mu_n(F_n)) = \{q_1 \circ \Phi_n(a,d) \oplus a \oplus d : \ (a,d) \in F_n\}$ and that $q_2(\mu_n^{(t)}(X)) \subseteq \psi^D_t(X')$. Therefore B') follows from C).

Now we can exchange the role of φ and ψ and construct in the same way a diagram

$$\begin{tikzcd}
& (SB \oplus \mathrm{cone}(D), G_n) \arrow[dl, "\Psi_n"'] \arrow[d, "\nu_n"] \arrow[dl, phantom, shift left=6ex, "\epsilon_n"] \\
(SA \oplus \mathrm{cone}(D), F_{n+1}) \arrow[r, "\Phi_{n+1}"] & (SB \oplus \mathrm{cone}(D), G_{n+1})
\end{tikzcd}$$

where $G'_{n+1} \subseteq G_{n+1}$. Furthermore by using Lemma 6.3 as it was used to obtain C) above we can arrange that Φ_{n+1} satisfies the $n+1$-version of B). The $n+1$-version of A) follow from the constructions by use of B'). In this way we obtain the desired diagram by induction. $\square$

Let $F^0_1 \subseteq F^0_2 \subseteq F^0_3 \subseteq \cdots$ and $G^0_1 \subseteq G^0_2 \subseteq G^0_3 \subseteq \cdots$ be sequences of finite sets with dense union in $SA \oplus \mathrm{cone}(D)$ and $SB \oplus \mathrm{cone}(D)$, respectively. By combining Lemma 6.4 with 2.4 of [BK] we get sequences of almost identity maps, $\varphi_n : SA \oplus$

$\text{cone}(D) \to SA \oplus \text{cone}(D)$ and $\psi_n : SB \oplus \text{cone}(D) \to SB \oplus \text{cone}(D)$, such that $\varinjlim(SA \oplus \text{cone}(D), \varphi_{m,n}) \simeq \varinjlim(SB \oplus \text{cone}(D), \psi_{m,n})$ and

I) $\varphi_n^{(t)}$ is 2^{-n}-multiplicative on F_n, $t \in [0,1]$,

II) $\|q_1 \circ \varphi_n^{(t)}(x)\| \geq \|\varphi_n^{(t)}(x)\| - 2^{-n}, \ x \in F_n, \ t \in [0,1]$,

where

$$F_n = F_n^0 \cup \bigcup_{j<n} \bigcup_{(t_1,\cdots,t_{n-j}) \in [0,1]^{n-j}} \varphi_{n-1}^{(t_1)} \circ \varphi_{n-2}^{(t_2)} \circ \cdots \circ \varphi_j^{(t_{n-j})}(F_n^0).$$

We arrange also that (ψ_n, G_n) satisfy the analogues of I) and II). In order to make connection with the last section, set $\varphi_t^n = \varphi_n^{(t)}$. We check that (5.4) holds : Let $a, b \in SA \oplus \text{cone}(D)$, $k \in \mathbb{N}$ and $\epsilon > 0$ be given. It follows from the density of $\bigcup_m F_m^0$ in $SA \oplus \text{cone}(D)$ that there is a $M \in \mathbb{N}$ and elements $x, y \in F_M^0$ such that

$$\big| \|\varphi(\lambda(x)\lambda(y)) - \varphi(\lambda(x))\varphi(\lambda(y))\| - \|\varphi(\lambda(a)\lambda(b)) - \varphi(\lambda(a))\varphi(\lambda(b))\| \big| < \frac{\epsilon}{2} \tag{6.4}$$

for all linear contractions $\varphi, \lambda : SA \oplus \text{cone}(D) \to SA \oplus \text{cone}(D)$. By increasing M we may assume $M > k$ and that $2^{-M+1} < \frac{\epsilon}{2}$. It follows from I) that

$$\|\varphi_{\underline{t}}^{n,m}(\varphi_{\underline{t}}^{m,k}(x)\varphi_{\underline{t}}^{m,k}(y)) - \varphi_{\underline{t}}^{n,m}(\varphi_{\underline{t}}^{m,k}(x))\varphi_{\underline{t}}^{n,m}(\varphi_{\underline{t}}^{m,k}(y))\| \leq \sum_{j=m}^{n} 2^{-j} < \frac{\epsilon}{2} \tag{6.5}$$

for all $n > m \geq M$ and all $\underline{t} \in [0,1]^\infty$. By combining (6.4) and (6.5) we get (5.4).

To prove that (5.5) holds observe that by the nature of an almost identity map and II) we have that $\|\varphi_{\underline{t}}^{n+1,n}(x)\| \geq \|\varphi_{\underline{t}}^{m,n}(x)\| \geq \|\varphi_{\underline{t}}^{n+1,n}(x)\| - 2^{-n}$ for all $x \in F_n$, all $\underline{t} \in [0,1]^\infty$ and all $m > n$. So if $a \in SA \oplus \text{cone}(D)$, $n \in \mathbb{N}$ and $\epsilon > 0$ are given we choose a $m > n$ and an element $x \in F_m^0$ such that $2^{-m} < \frac{\epsilon}{3}$ and $\|a - x\| < \frac{\epsilon}{3}$. It follows then that

$$\|\varphi_{\underline{t}}^{k,n}(a)\| \geq \|\varphi_{\underline{t}}^{k,n}(x)\| - \frac{\epsilon}{3} \geq \|\varphi_{\underline{t}}^{m+1,n}(x)\| - \frac{2\epsilon}{3} > \|\varphi_{\underline{t}}^{m+1,n}(a)\| - \epsilon$$

for all $\underline{t} \in [0,1]^\infty$ and all $k > m$. Finally, we must consider (5.6). It is apparent from the definition of an almost identity map that there is a completely positive contraction $p : SA \oplus \text{cone}(D) \to SA \oplus \text{cone}(D)$ with the property that $p \circ \varphi_t^n(a,d) = (a, \psi_t^D(d))$, $t \in [0,1]$, $(a,d) \in SA \oplus \text{cone}(D)$. Since $\psi_s^D \circ \psi_t^D = \psi_t^D, s \geq t$, it follows from the form of φ_t^n, that $p \circ \varphi_s^{n+1} \circ \varphi_t^n = \varphi_t^n$ for $s \geq t$. We can therefore use p as p_k for all k in (5.6).

Having established both (5.4), (5.5) and (5.6), Proposition 5.7 gives us a continuous bundle of C^*-algebras, $(\mathcal{A}, [0,1], \pi)$, which is discrete asymptotically semi-split and piecewise trivial with only one point of non-triviality such that $\ker \pi_t$ is semi-contractible for all $t \in [0,1]$ and such that the fibers at 0 and 1 are $\varinjlim(SA \oplus \text{cone}(D), \varphi_{m,n}^{(0)})$ and $\varinjlim(SA \oplus \text{cone}(D), \varphi_{m,n})$, respectively. Note that $\varinjlim(SA \oplus \text{cone}(D), \varphi_{n,m}^{(0)}) \simeq SA$ since SA is stable and $\varphi_n^{(0)}$ has the form $\varphi_n^{(0)}(a,d) = (VaV^*, 0)$ for the same isometry $V \in M(SA)$. Therefore, when we apply the same procedure to $SB \oplus \text{cone}(D)$, we get all together the following result.

Theorem 6.5. *Let A and B be KK-equivalent separable C^*-algebras. It follows that there are separable continuous bundles of C^*-algebras, $(\mathcal{A}, [0,1], \pi)$ and $(\mathcal{A}', [0,1], \pi')$, which are discrete asymptotically semi-split and piecewise trivial with only one point of non-triviality such that $\ker \pi_t$ and $\ker \pi'_t$ are semi-contractible for all $t \in [0,1]$, $\pi_0(\mathcal{A}) \simeq SA \otimes \mathcal{K}$, $\pi'_0(\mathcal{A}') \simeq SB \otimes \mathcal{K}$ and $\pi_1(\mathcal{A}) \simeq \pi'_1(\mathcal{A}')$.*

7. Concatenation of bundles

Definition 7.1. Let A be a C^*-algebra. A *finite semi-split decomposition series* for A consists of a series of ideals

$$A = I_n \supseteq I_{n-1} \supseteq I_{n-2} \supseteq \cdots \supseteq I_1 \supseteq I_0 = \{0\}$$

such that the corresponding extensions $0 \to I_{n-1} \to I_n \to I_n/I_{n-1} \to 0$ are semi-split for $n = 2, 3, \cdots, n$. n is called the length of the decomposition series and the C^*-algebras I_i/I_{i-1}, $i = 1, 2, \cdots, n$, will be called the succesive quotients of the decomposition series. When $n = 2$ we say that A is a semi-split extension of I_2/I_1 by I_1.

Lemma 7.2. *Let*

$$0 \longrightarrow J \longrightarrow E \xrightarrow{p} A \longrightarrow 0$$

be a semi-split extension and $J = J_n \supseteq J_{n-1} \supseteq J_{n-2} \supseteq \cdots \supseteq J_1 \supseteq J_0 = \{0\}$, $A = A_m \supseteq A_{m-1} \supseteq A_{m-2} \supseteq \cdots \supseteq A_1 \supseteq A_0 = \{0\}$ *finite semi-split decomposition series for* J *and* A, *respectively. Set* $J_{n+i} = p^{-1}(A_i), i = 1, 2, \cdots, m$. *Then*

$$E = J_{m+n} \supseteq J_{m+n-1} \supseteq \cdots \supseteq J_1 \supseteq J_0 = \{0\}$$

is a semi-split decomposition series for E *such that* $J_k/J_{k-1} \simeq A_{k-n}/A_{k-n-1}$ *for* $k > n$.

Proof. Left to the reader. □

Lemma 7.3. *Let* $(\mathcal{A}, [0,1], \pi)$ *and* $(\mathcal{A}', [0,1], \pi')$ *be piecewise trivial and semi-split continuous bundles such that* $\mathcal{A}_1 \simeq \mathcal{A}'_0$. *There is then a piecewise trivial and semi-split continuous bundle* $(\mathcal{B}, [0,1], \pi'')$ *with the following properties.*

1) $(\mathcal{B}_{[0,\frac{1}{2}]}, [0, \frac{1}{2}], \pi'')$ *and* $(\mathcal{B}_{[\frac{1}{2},1]}, [\frac{1}{2}, 1], \pi'')$ *are weakly isomorphic to* $(\mathcal{A}, [0,1], \pi)$ *and* $(\mathcal{A}', [0,1], \pi')$, *respectively,*
2) *for* $t \in [0, \frac{1}{2}]$ *there a* $s \in [0,1]$ *and a semi-split extension*

$$0 \longrightarrow \ker \pi'_0 \longrightarrow \ker \pi''_t \longrightarrow \ker \pi_s \longrightarrow 0,$$

and for $t \in [\frac{1}{2}, 1]$ *there is a* $s \in [0,1]$ *and a semi-split extension*

$$0 \longrightarrow \ker \pi_1 \longrightarrow \ker \pi''_t \longrightarrow \ker \pi'_s \longrightarrow 0.$$

Proof. Let $\alpha : \mathcal{A}_1 \to \mathcal{A}'_0$ be a $*$-isomorphism. The C^*-algebra $\mathcal{B} = \{(a_1, a_2) \in \mathcal{A} \oplus \mathcal{A}' : \alpha \circ \pi_1(a_1) = \pi'_0(a_2)\}$ is the bundle C^*-algebra for a bundle with the described properties. In particular, the part about the extensions being semi-split follows from the triviality of the bundles in a neighbourhood of the endpoints. □

Theorem 7.4. *Let* A *and* B *be* KK*-equivalent separable* C^**-algebras. It follows that there is a separable continuous bundle of* C^**-algebras,* $(\mathcal{A}, [0,1], \pi)$, *which is semi-split and piecewise trivial with no more than 2 points of non-triviality such that* $\pi_0(\mathcal{A}) \simeq SA \otimes \mathcal{K}$, $\pi_1(\mathcal{A}) \simeq SB \otimes \mathcal{K}$, *and* $\ker \pi_t$ *is a semi-split extension of semi-contractible* C^**-algebras for all* $t \in [0,1]$.

Proof. Concatenate the two bundles from Theorem 6.5 by using Lemma 6.8 and apply Lemma 6.7. □

Definition 7.5. A separable C^*-algebra D is called *crossed-contractible* when there is a C^*-dynamical system $(A, \mathbb{R}, \alpha)$ with A contractible such that $D \simeq A \times_\alpha \mathbb{R}$.

It follows from [FS] that a crossed-contractible C^*-algebra is KK-contractible. It is easy to see that a crossed-contractible C^*-algebra need not be locally contractible.

Theorem 7.6. *(Rieffel* [R]*, Elliott, Natsume, Nest* [ENN]*) Let A be a separable C^*-algebra. There is then a separable semi-split and piecewise trivial continuous bundle of C^*-algebras, $(\mathcal{A}, [0,1], \pi)$, with no more than one point of non-triviality such that $\pi_0(\mathcal{A}) \simeq A \otimes \mathcal{K}$, $\pi_1(\mathcal{A}) \simeq S^2A$, and $\ker \pi_t$ is crossed-contractible for all $t \in [0,1]$.*

Proof. For $\lambda \in [0,1]$, define an action $\alpha^\lambda : \mathbb{R} \to \operatorname{Aut} C_0(\mathbb{R}) \otimes A$ by $\alpha^\lambda_s(f)(t) = f(t - (1-\lambda)s)$. For $\lambda \in [1,2]$, set $\alpha^\lambda_s = \mathrm{id}$ for all s. Define an action $\alpha : \mathbb{R} \to \operatorname{Aut}[C[0,2] \otimes C_0(\mathbb{R}) \otimes A]$ by $\alpha_s(g)(\lambda) = \alpha^\lambda_s(g(\lambda))$, $\lambda \in [0,2]$. It follows from [R] that $[C[0,2] \otimes C_0(\mathbb{R}) \otimes A] \times_\alpha \mathbb{R}$ is the bundle C^*-algebra for a bundle over $[0,2]$ such that $\ker \pi_t$ is crossed-contractible for all $t \in [0,2]$. It is well-known that the fiber over 0 is $A \otimes \mathcal{K}$ and the fiber over 2 is S^2A. As pointed out in [ENN] the bundle is trivial over $[0,1[$ and it is clearly trivial over $[1,2]$. The fact that the bundle is semi-split follows from the Choi-Effros lifting theorem, [CE], by observing that the bundle in the general case is obtained from the nuclear bundle which results from the special case where $A = \mathbb{C}$ by tensoring with A. $\square$

By concatenation the bundle from Theorem 6.11 with the bundles from Theorem 6.9, applied to SA and SB, we get the following result.

Theorem 7.7. *Let A and B be separable and stable KK-equivalent C^*-algebras. There is then a separable continuous bundle of C^*-algebras, $(\mathcal{A}, [0,1], \pi)$, which is semi-split and piecewise trivial with no more than four points of non-triviality, and for each $t \in [0,1]$ there is a finite semi-split decomposition series for $\ker \pi_t$ of length four whose succesive quotients are either semi-contractible or crossed-contractible.*

As a final step we can also remove the stabilizations by introducing a little longer decompostion series and slightly more general succesive quotients. When $\mu : A \to B$ is a $*$-homomorphism between separable C^*-algebras which is an isomorphism in KK, the mapping cone $\{(a,f) \in A \oplus \operatorname{cone}(B) : \ f(1) = \mu(a)\}$ is KK-contractible. For our purposes here we need only consider the very special case where $B = A \otimes \mathcal{K}$ and the $*$-homomorphism is the canonical stabilizing $*$-homomorphism $s : A \to A \otimes \mathcal{K}$. Since s is injective we can consider A as a C^*-subalgebra of $A \otimes \mathcal{K}$ and the mapping cone can be described as

$$C_A = \{f \in \operatorname{cone}(A \otimes \mathcal{K}) : \ f(1) \in A\} .$$

Set $\mathcal{A} = \{f \in C([0,1], A \otimes \mathcal{K}) : \ f(t) \in A, \ t \geq \frac{1}{2}\}$ and let π_t be evaluation at $t \in [0,1]$. Then $(\mathcal{A}, [0,1], \pi)$ is a semi-split and piecewise trivial continuous bundle of C^*-algebras connecting $A \otimes \mathcal{K}$ to A. For each $t \in [0,1]$, $\ker \pi_t$ is a semi-split extension, either of a contractible C^*-algebra by a contractible C^*-algebra, or of C_A by a contractible C^*-algebra.

Theorem 7.8. *Let A and B be separable C^*-algebras. Then A and B are KK-equivalent if and only if there is a separable semi-split and piecewise trivial continuous bundle of C^*-algebras, $(\mathcal{A}, [0,1], \pi)$, with no more than six points of non-triviality, such that $\pi_0(\mathcal{A}) \simeq A$, $\pi_1(\mathcal{A}) \simeq B$, and for each $t \in [0,1]$ there is a finite semi-split decomposition series for $\ker \pi_t$ of length eight whose succesive quotients are either semi-contractible, crossed-contractible or isomorphic to C_A or C_B.*

Proof. Concatenate the bundles just described, one for A and one for B, with the bundle from Theorem 6.12. Apply Lemma 6.7 and Lemma 6.8 to prove the 'only if' part. For the 'if' part observe that since the bundle is semi-split the extensions 5.3 are semi-split. Furthermore, $\ker \pi_t$ is contractible in KK-theory since its decomposition series is semi-split and the succesive quotients are all KK-contractible; the semi-contractible quotients by Theorem 3.2. Since KK-theory is half-exact with respect to semi-split extensions by [**?**], [CS], it follows that π_t is a KK-equivalence and that A_t is KK-equivalent to $\mathcal{A}$ for all t. □

Corollary 7.9. *Let A and B be separable C^*-algebras. Then A and B are KK-equivalent if and only if there is a separable C^*-algebra D and surjective semi-split $*$-homomorphisms $\varphi : D \to A$ and $\psi : D \to B$ such that $\ker \varphi$ and $\ker \psi$ are KK-contractible.*

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