U N I V E R S I T Y O F A A R H U S
D E P A R T M E N T O F M A T H E M A T I C S

ISSN: 1397-4076

d'ALEMBERTS FUNCTIONAL EQUATIONS ON METABELIAN GROUPS

By Henrik Stetkær

Preprint Series No.: 3 February 1999

Ny Munkegade, Bldg. 530
8000 Aarhus C, Denmark
http://www.imf.au.dk
institut@imf.au.dk

d'Alembert's functional equations on metabelian groups

Henrik Stetkær

Abstract

We extend d'Alembert's classical functional equation by replacing the domain of definition $\mathbf{R}$ of the solutions by a metabelian group G and simultaneously replacing the group involution by an arbitrary involution of G. We find all complex valued solutions. In particular we show that the continuous solutions have the same form as in the abelian case if G is connected.

Key words: Functional equation, d'Alembert, metabelian.

1991 Mathematics Subject Classification: 39B52

d'Alembert's functional equations on metabelian groups

I. Introduction

The obvious generalization of d'Alembert's classical functional equation

$$g(x+y)+g(x-y)=2g(x)g(y), \quad x,y\in \mathbf{R}, \tag{1}$$

from the group $\mathbf{R}$ to an arbitrary group G is the functional equation

$$g(xy)+g\big(xy^{-1}\big)=2g(x)g(y), \quad x,y\in G. \tag{2}$$

A further generalization that comes out naturally of the study of Wilson's functional equation (Corovei [6]), is

$$g(xy)+g(yx)+g\big(xy^{-1}\big)+g\big(y^{-1}x\big)=4g(x)g(y), \ \ x,y\in G. \tag{3}$$

It is a generalization of (2), because $g(xy)=g(yx)$ for any solution of (2) (see Lemma V.1 below), so that any solution of (2) also is a solution of (3). We will call and (2) and its generalization (6) for *the short d'Alembert functional equation,* and (3) and its generalization (12) for *the long d'Alembert functional equation.*

The purpose is to study and solve these functional equations on metabelian groups, i.e. groups G for which the commutator subgroup $[G,G]$ is contained in the center $Z(G)$ of G. An abelian group is metabelian, but the converse is false which the example of the Heisenberg group

$$H_3 := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x,y,z\in \mathbf{R} \right\} \tag{4}$$

reveals. On the other hand any metabelian group is nilpotent. Corovei [8] studied d'Alembert's and Wilson's functional equations on the so-called P_3-groups. They are also metabelian. The quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ is an example of a P_3-group.

Any function of the form

$$g(x)=\frac{m(x)+m\big(x^{-1}\big)}{2}, \quad x\in G, \tag{5}$$

where $m: G\to \mathbf{C}^*$ is a homomorphism, is a solution of (2). No restrictions on the group are needed for that statement. For abelian groups the converse is true: Any non-zero solution of (2) has this form (Kannappan [14]). The metabelian groups are close to being abelian, so one could hope that all solutions of d'Alembert's functional equations on such groups had the form (5). This is indeed so for special groups (See Corovei [4], Friis [12] and Proposition V.5 below). But solutions of (2) of a different form occur even on the quaternion group (Corovei [4; p. 105±106]), so new phenomena show up in the passage from abelian to metabelian groups.

Our main results are the following:

1) Theorem IV.1 below shows that the long version (3) of d'Alembert functional equation reduces to the short one (2) when G is metabelian, generalizing [8] where this is proved for P_3-groups.
2) Consider the short d'Alembert functional equation

$$g(xy) + g(x\tau(y)) = 2g(x)g(y), \quad x, y \in G, \tag{6}$$

where $\tau : G \to G$ is an arbitrary involution of the metabelian group G. Any non-zero solution g of (6) has either the form $g = (m + m \circ \tau)/2$ where $m : G \to \mathbf{C}^*$ is a homomorphism, or the form

$$g(x) = \begin{cases} m(x) & x \in H \\ 0 & x \in G \setminus H \end{cases} \tag{7}$$

where H is a τ-invariant normal subgroup of G and $m : H \to \mathbf{C}^*$ is a homomorphism such that $m = m \circ \tau$. This is described in Theorem V.4 that generalizes and reformulates the results of [7] in which $\tau(x) = x^{-1}$ for all $x \in G$.
3) Any continuous solution g of (6) on a metabelian connected topological group G has the form $g = (m + m \circ \tau)/2$ where $m : G \to \mathbf{C}^*$ is a continuous homomorphism (Theorem V.6).

There are only few results in the literature about d'Alembert's functional equations on non-abelian groups. Corovei [5], [6], [7] and [8] discuss them on certain nilpotent groups. Friis [12] solves d'Alembert's and Wilson's functional equations on connected nilpotent Lie groups and on semidirect products of two abelian groups like the group of affine transformations of the real line. Formally Kannappan [14] deals with d'Alembert's functional equation (2) on non-abelian groups, but the solutions are assumed to satisfy Kannappan's condition which in essence reduces the considerations to the abelian case, so that the solutions are given by (5). We recall that a function f on a group G is said to satisfy Kannappan's condition if $f(xyz) = f(xzy)$ for all $x, y, z \in G$. Dacić [10] solves (2) under a certain condition on the solution. However, his condition implies Kannappan's condition (see [15]). Aczél, Chung and Ng [1; p. 20±21] show that the solutions of (2) on the group of unit quaternions are not all of the form (5). The group of unit quaternions is isomorphic to the Lie group $SU(2)$ (see section 1.9 of [3]), so it is semisimple and hence of a quite different nature than nilpotent groups. Penney and Rukhin [17] consider square integrable solution of (2) on a locally compact group G. They show under certain conditions on G that the such solutions are zero, unless G is compact.

The present paper is closely related to and inspired by the works by Corovei just mentioned.

There are other ways of extending d'Alembert's classical functional equation (1) from $\mathbf{R}$ to groups than (2) and (3). For example by the connection to the theory of spherical functions described in [18] and [19]. Roughly speaking that corresponds to τ being a homomorphism instead of the anti-homomorphism of the present paper.

We finish this introduction by fixing notation that will be used throughout the paper.

Notation I.1

G denotes a group with center $Z(G)$ and neutral element $e \in G$. The commutator group $[G,G]$ is the subgroup of G generated by the commutators $[x,y] = xyx^{-1}y^{-1}$, $x,y \in G$. We let τ be an involution of G, i.e. a map $\tau : G \to G$ such that $\tau(xy) = \tau(y)\tau(x)$ for all $x,y \in G$ and $\tau(\tau(x)) = x$ for all $x \in G$. The multiplicative group of all non-zero complex numbers is denoted by $\mathbf{C}^*$. For any function $g : G \to \mathbf{C}$ on G we introduce the subgroup $Z(g) := \{u \in G \mid g(xuy) = g(xyu) \text{ for all } x,y \in G\}$.

II. The subgroup Z(g).

In this section G is an arbitrary group, not necessarily metabelian. The subgroup $Z(g)$ (defined in Notation I.1 above) of G plays a central role in this paper. Here we derive some of its properties.

Lemma II.1

Let G be any group and $g : G \to \mathbf{C}$ a function on G. Then $Z(g)$ is a normal subgroup of G containing the center $Z(G)$ of G. It can also be characterized as $Z(g) = \{u \in G \mid g(xuy) = g(uxy) \text{ for all } x,y \in G\}$. Furthermore $Z(g) \subseteq \{x \in G \mid g([x,y]) = g(e) \text{ for all } y \in G\}$. Finally $Z(g \circ \tau) = \tau(Z(g))$ for any involution τ of G, so $Z(g)$ is τ-invariant if $g = g \circ \tau$.

Proof: The first statements are immediate. Furthermore, if $u \in Z(g)$ and $y \in G$ then $g([u,y]) = g\big(uyu^{-1}y^{-1}\big) = g(e)$. Finally, using the definition of $Z(g)$ we find that

$$\begin{aligned} Z(g\circ\tau) &= \{u \in G \mid (g\circ\tau)(xuy) = (g\circ\tau)(xyu),\ \forall\, x,y \in G\} \\ &= \{u \in G \mid g(\tau(y)\tau(u)\tau(x)) = g(\tau(u)\tau(y)\tau(x)),\ \forall\, x,y \in G\} \\ &= \{u \in G \mid g(y\tau(u)x) = g(\tau(u)yx),\ \forall\, x,y \in G\} \\ &= \{u \in G \mid \tau(u) \in Z(g)\} = \tau^{-1}(Z(g)) = \tau(Z(g)). \qquad \square \end{aligned} \tag{8}$$

To say that $Z(g) = G$ is another way of stating that g satisfies Kannappan's condition on G. The definition of $Z(g)$ expresses that when $u \in Z(g)$ occurs as an argument for g then u can be moved around as though it is in the center $Z(G)$. In general $Z(G)$ is a proper subset of $Z(g)$, a trivial example of that being $Z(1) = G$ on a non-abelian group G.

We continue by studying $Z(g)$ under a condition (9) on the function g that is related to a factorization (17) in Theorem III.2.

Proposition II.2

Let $g : G \to \mathbf{C}$ be a function such that $g \neq 0$ and

$$g(xu) = g(x)g(u) \quad \textit{for all } x \in G,\ u \in [G,G]. \tag{9}$$

Then

(a) $Z(g) = \{x \in G \mid g([x,y]) = 1 \textit{ for all } y \in G\}$.
(b) *If* $g\big(xux^{-1}\big) = g(u)$ *for all* $x \in G$ *and* $u \in [G,G]$, *and* $g(u) = g\big(u^{-1}\big)$ *for all* $u \in [G,G]$, *then* $\big\{x^2 \mid x \in G\big\} \subseteq Z(g)$.
(c) $g(x) = 0$ *for all* $x \in G \setminus Z(g)$ *if* $g\big(xyx^{-1}\big) = g(y)$ *for all* $x, y \in G$.
(d) *If* $g\big(xyx^{-1}\big) = g(y)$ *for all* $x, y \in G$ *and simultaneously* $g|Z(g)$ *is a homomorphism of* $Z(g)$ *into* $\mathbf{C}^*$ *then* $Z(g) = \{x \in G \mid g(x) \neq 0\}$.

Proof: Putting $u = e$ in (9) we see that $g(e) = 1$.
(a) If $g([u,y]) = 1$ for all $y \in G$, then

$$g(xyu) = g\big(xuy\big[u, u^{-1}y^{-1}\big]\big) = g(xuy)g\big(\big[u, u^{-1}y^{-1}\big]\big) = g(xuy)1 = g(xuy), \tag{10}$$

so $u \in Z(g)$. The other inclusion is part of Lemma II.1.
(b) For any $x, y \in G$ we find from the formula $[ab, z] = a[b,z]a^{-1}[a,z]$ that

$$\begin{aligned} g\big(\big[x^2, y\big]\big) &= g\big(x[x,y]x^{-1}[x,y]\big) = g\big(x[x,y]x^{-1}\big)g([x,y]) = g([x,y])g([x,y]) \\ &= g\Big([x,y]^{-1}\Big)g([x,y]) = g\Big([x,y]^{-1}[x,y]\Big) = g(e) = 1, \end{aligned} \tag{11}$$

and the statement follows from (a).
(c) Let $x_0 \in G \setminus Z(g)$. Then also $x_0^{-1} \in G \setminus Z(g)$, because $Z(g)$ is a group. So by (a) there exists $y_0 \in G$ such that $g\big(\big[x_0^{-1}, y_0\big]\big) \neq 1$. Using (9) we get that $g(x_0)g\big(\big[x_0^{-1}, y_0\big]\big) = g\big(x_0\big[x_0^{-1}, y_0\big]\big) = g\big(x_0x_0^{-1}y_0x_0y_0^{-1}\big) = g\big(y_0x_0y_0^{-1}\big) = g(x_0)$. But that can only hold if $g(x_0) = 0$ because $g\big(\big[x_0^{-1}, y_0\big]\big) \neq 1$.
(d) Using first (c) and then the homomorphism property we find that $\{x \in G \mid g(x) \neq 0\} \subseteq Z(g) \subseteq \{x \in G \mid g(x) \neq 0\}$. □

III. The long d'Alembert functional equation on any group

We do not impose any conditions on the group G in this section, in which we derive some results about the form of the solutions of the following extension of d'Alembert's functional equation

$$g(xy) + g(yx) + g(x\tau(y)) + g(\tau(y)x) = 4g(x)g(y), \quad x, y \in G, \tag{12}$$

where $\tau : G \to G$ is an involution.

A typical example of the involution τ is the group involution $\tau(g) = g^{-1}$, $g \in G$. Another is the adjoint $A \to A^*$ in the matrix group $GL(n, \mathbf{C})$. A third one is

$$\tau\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \tag{13}$$

on the Heisenberg group H_3. Representations of the Heisenberg group and of related metabelian groups play a basic role in harmonic analysis [11].

Lemma III.1

If $g \neq 0$ is a solution of (12) then $g \circ \tau = g$, $g(e) = 1$,

$$g(x^2) + \frac{g(x\tau(x)) + g(\tau(x)x)}{2} = 2[g(x)]^2 \text{ for all } x \in G, \tag{14}$$

$$2g(x)g(u) = g(xu) + g(x\tau(u)) \text{ for } x \in G,\ u \in Z(g), \quad \text{and} \tag{15}$$

$$g(xu\tau(u)) = g(x)g(u\tau(u)) \text{ for } x \in G,\ u \in Z(g). \tag{16}$$

Proof: We see that $g \circ \tau = g$ when we note that the left hand side of (12) is invariant under interchange of y and $\tau(y)$. Putting $x = e$ in (12) we see that $g(e) = 1$. Putting $y = x$ in (12) we get (14). (15) follows immediately when we put $y = u \in Z(g)$ in (12) and note that $\tau(Z(g)) = Z(g)$ (Lemma II.1). Finally, replacing y by $y\tau(y)$ in (12) we get $g(xy\tau(y)) + g(y\tau(y)x) = 2g(x)g(y\tau(y))$ which for $y = u \in Z(g)$ is the same as (16). □

Theorem III.2

Let $g : G \to \mathbf{C}$ be a solution of the functional equation (12).

(a) *If there exists a $u \in Z(g)$ such that $g(u)^2 \neq g(u\tau(u))$ then g has the form $g = (m + m \circ \tau)/2$ for some homomorphism $m : G \to \mathbf{C}^*$. If G is a topological group and g is continuous then m is also continuous.*

(b) *If $g(u)^2 = g(u\tau(u))$ for all $u \in Z(g)$ then*

$$g(xu) = g(x)g(u) \text{ for all } x \in G \text{ and } u \in Z(g). \tag{17}$$

In particular $g|Z(g)$ is a homomorphism of $Z(g)$ into $\mathbf{C}^*$ if $g \neq 0$.

Corollary III.3

Let $g : G \to \mathbf{C}$ be a non-zero solution of the functional equation (12). If $Z(g) = G$, i.e. if Kannappan's condition holds, then there exists a homomorphism $m : G \to \mathbf{C}^*$ such that $g = (m + m \circ \tau)/2$.

Corollary III.3 is well known for the short equation (6). See for example [14], [2] and [20].

The proof of Theorem III.2 consists of modifications of the corresponding computations of [5]. A key observation is that the arguments of [5] for $u \in Z(G)$ actually work for u in the bigger group $Z(g)$. Another difference is the presence of the general involution τ instead of just the group inversion $x \to x^{-1}$.

Lemma III.4

Let $g \neq 0$ be a solution of (12). For fixed $\alpha \in \mathbf{C}$ and $y \in G$ we define

$$m(x) = m_{\alpha,y}(x) := g(x) + \alpha\left[\frac{g(xy)+g(yx)}{2} - g(x)g(y)\right] \text{ for } x \in G. \tag{18}$$

Then $g = (m + m \circ \tau)/2$. In particular $m \neq 0$.

Proof: Using that $g = g \circ \tau$ we find that

$$\begin{aligned} m(\tau(x)) &= g(\tau(x)) + \alpha\left[\frac{g(\tau(x)y)+g(y\tau(x))}{2} - g(\tau(x))g(y)\right] \\ &= g(x) + \alpha\left[\frac{g(\tau(y)x)+g(x\tau(y))}{2} - g(x)g(y)\right]. \end{aligned} \tag{19}$$

Adding this to $m(x)$ we find from (12) that $m(x) + m(\tau(x)) = 2g(x)$. □

We will next examine whether m is a homomorphism. To do so we shall for fixed $u \in Z(g)$ derive some properties of the function

$$s(x) = s_u(x) := \frac{g(xu)+g(ux)}{2} - g(x)g(u) = g(xu) - g(x)g(u), \quad x \in G. \tag{20}$$

Thinking of g as a cosine function, s as a sine function and $m = m_{\alpha,u} = g + \alpha s$ as an exponential function we find in Lemma III.5 some of the well known addition formulas for the trigonometric functions in a noncommutative setting.

Lemma III.5

Let $g : G \to \mathbf{C}$ be a solution of (12). For any $u \in Z(g)$, $x, y \in G$ and $\alpha \in \mathbf{C}$ we have the following identities for s (defined by (20)), g and $m = g + \alpha s$:

$$\frac{s(xy)+s(yx)}{2} = s(x)g(y) + g(x)s(y), \tag{21}$$

$$s(x)s(y) = \left[\frac{g(xy)+g(yx)}{2} - g(x)g(y)\right]\left[g(u)^2 - g(u\tau(u))\right], \tag{22}$$

$$\begin{aligned} &m(x)m(y) - \frac{m(xy)+m(yx)}{2} \\ &= \left[g(x)g(y) - \frac{g(xy)+g(yx)}{2}\right]\left\{1 - \alpha^2\left[g(u)^2 - g(u\tau(u))\right]\right\}. \end{aligned} \tag{23}$$

Proof: Note that $\tau(u) \in Z(g)$ because $u \in Z(g)$ (Lemma II.1).

First some computations the result of which we shall need below. For $\tau(x) = x^{-1}$ they can be found in [5]. More precisely, we will prove the two formulas:

$$g(xu)g(y) + g(yu)g(x) = \frac{g(xyu)+g(yxu)}{2} + g(u)\left[2g(x)g(y) - \frac{g(xy)+g(yx)}{2}\right], \tag{24}$$

$$g(xu)g(yu) = g(u)\frac{g(xyu)+g(yxu)}{2} + g(u\tau(u))\left[g(x)g(y) - \frac{g(xy)+g(yx)}{2}\right]. \tag{25}$$

(24) is proved by a computation in which we use (12), that $g = g\circ\tau$ and (15):

$$\begin{aligned}
&4[g(xu)g(y)+g(yu)g(x)] = g(xyu)+g(x\tau(y)u)+g(yxu)+g(\tau(y)xu)\\
&+g(yxu)+g(y\tau(x)u)+g(xyu)+g(\tau(x)yu)\\
&=2g(xyu)+2g(yxu)+g(y\tau(x)\tau(u))+g(y\tau(x)u)+g(\tau(x)y\tau(u))+g(\tau(x)yu)\\
&=2g(xyu)+2g(yxu)+2g(y\tau(x))g(u)+2g(\tau(x)y)g(u)\\
&=2\{g(xyu)+g(yxu)+g(u)[g(y\tau(x))+g(\tau(x)y)]\}\\
&=2\{g(xyu)+g(yxu)+g(u)[4g(x)g(y)-g(yx)-g(xy)]\}.
\end{aligned} \tag{26}$$

We prove (25) in a similar way using (12) and (15) plus the identity $g(xu\tau(u)) = g(x)g(u\tau(u))$ that was derived as formula (16) in Lemma III.1:

$$\begin{aligned}
&4g(xu)g(yu) = g(xyuu)+g(x\tau(y)u\tau(u))+g(yxuu)+g(\tau(y)xu\tau(u))\\
&=g(xyuu)+g(xyu\tau(u))-g(xyu\tau(u))+g(yxuu)+g(yxu\tau(u))-g(yxu\tau(u))\\
&+[g(x\tau(y)+g(\tau(y)x))]g(u\tau(u))\\
&=2g(xyu)g(u)+2g(yxu)g(u)+[g(x\tau(y)+g(\tau(y)x))-g(xy)-g(yx)]g(u\tau(u))\\
&=2g(u)[g(xyu)+g(yxu)]+g(u\tau(u))[4g(x)g(y)-2g(xy)-2g(yx)]\\
&=2\{g(u)[g(xyu)+g(yxu)]+g(u\tau(u))[2g(x)g(y)-g(xy)-g(yx)]\}.
\end{aligned} \tag{27}$$

We can now prove (21).

$$\begin{aligned}
&s(xy)+s(yx)-2s(x)g(y)-2g(x)s(y)\\
&=g(xyu)-g(xy)g(u)+g(yxu)-g(yx)g(u)\\
&-2[g(xu)-g(x)g(y)]g(y)-2g(x)[g(yu)-g(y)g(y)]\\
&=g(xyu)+g(yxu)-[g(xy)+g(yx)-4g(x)g(y)]g(u)-2[g(xu)g(y)+g(x)g(yu)].
\end{aligned} \tag{28}$$

When we use the formula (24) on the last term of the right hand side of (28) the entire expression vanishes, proving (21).

The formula (22) is proved by similar computations:

$$\begin{aligned}
&2s(x)s(y) = 2[g(xu)-g(x)g(u)][g(yu)-g(y)g(u)]\\
&=2g(xu)g(yu)-2g(xu)g(y)g(u)-2g(x)g(yu)g(u)+2g(x)g(y)g(u)^2\\
&=2g(xu)g(yu)-2[g(xu)g(y)+g(x)g(yu)]g(u)+2g(x)g(y)g(u)^2.
\end{aligned} \tag{29}$$

Using (25) on the first and (24) on the second term of the right hand side of (29) the expression reduces to (22).

Finally we prove (23) by help of two identities (21) and (22) just derived:

$$
\begin{aligned}
&m(x)m(y) - \frac{m(xy)+m(yx)}{2} \\
&= [g(x)+\alpha s(x)][g(y)+\alpha s(y)] - \frac{1}{2}[g(xy)+\alpha s(xy)+g(yx)+\alpha s(yx)] \\
&= g(x)g(y) - \frac{g(xy)+g(yx)}{2} + \alpha\left\{s(x)g(y)+g(x)s(y) - \frac{s(xy)+s(yx)}{2}\right\} \\
&+ \alpha^2 s(x)s(y) \\
&= g(x)g(y) - \frac{g(xy)+g(yx)}{2} + 0 \\
&+ \alpha^2\left\{\frac{g(xy)+g(yx)}{2} - g(x)g(y)\right\}\left\{g(u)^2 - g(u\tau(u))\right\} \\
&= \left[g(x)g(y) - \frac{g(xy)+g(yx)}{2}\right]\left\{1-\alpha^2\left[g(u)^2 - g(u\tau(u))\right]\right\}.
\end{aligned}
\tag{30}
$$

This finishes the proof of Lemma III.5. □

Proof of Theorem III.2: We use the notation from Lemma III.5.

(a) Let $u \in Z(g)$ be such that $g(u)^2 - g(u\tau(u)) \neq 0$. Then $g \neq 0$. The formula $g = (m + m\circ\tau)/2$ was derived in Lemma III.4. Choosing $\alpha \in \mathbf{C}$ such that $\alpha^2\left[g(u)^2 - g(u\tau(u))\right] = 1$ we get from (23) that

$$
\frac{m(xy)+m(yx)}{2} = m(x)m(y) \quad \text{for all } x, y \in G. \tag{31}
$$

Furthermore $m \neq 0$ according to Lemma III.4. We may now refer to [16] where it is shown that any non-zero solution m of (31) is a homomorphism from G into $\mathbf{C}^*$. That the continuity of m follows from that of $g = m/2 + m\circ\tau/2$ is proved in Proposition V.7.

(b) From the identity (22) we find that $s(x) = s_u(x) = 0$ for all $u \in Z(G)$, i.e. according to (20) that (17) holds. □

IV. Reduction to the short functional equation

The following Theorem IV.1 was derived for P_3-groups in [8]. A P_3-group is metabelian (Combine 3.3 and 3.1 of [9]) so the result below generalizes the one of [8].

Theorem IV.1

On a metabelian group the solutions $g : G \to \mathbf{C}$ of the long d'Alembert functional equation

$$
g(xy) + g(yx) + g(xy^{-1}) + g(y^{-1}x) = 4g(x)g(y), \quad x, y \in G, \tag{32}
$$

are the same as those of the short one

$$
g(xy) + g(xy^{-1}) = 2g(x)g(y), \quad \forall\, x, y \in G. \tag{33}
$$

Proof: We will here prove that $g(xy) = g(yx)$ for all $x, y \in G$ and any solution g of (32). This implies that any solution of (32) is a solution of (33). The other direction is contained in Remark V.2 below. To prove that $g(xy) = g(yx)$ we adopt the ideas of [8; proof of Theorem 6] to the situation at hand.

The result is trivial for $g = 0$ so we may assume that $g \neq 0$. If there exists a $u \in Z(g)$ such that $g(u)^2 \neq 1$ then Theorem III.2 tells us that $g(x) = \big(m(x) + m(x^{-1})\big)/2$ where $m : G \to \mathbf{C}^*$ is a homomorphism. Obviously $g(xy) = g(yx)$ for such a g.

Left is the case of $g(u)^2 = 1$ for all $u \in Z(g)$. According to Theorem III.2(b) we have $g(xu) = g(x)g(u)$ for $x \in G$ and $u \in Z(g)$. Using the assumption about the group being metabelian we get that $y^2 \in Z(g)$ for all $y \in G$ (Proposition II.2(b) and (a)), so that

$$g(xy^2) = g(y^2x) = g(x)g(y^2) \text{ for all } x, y \in G. \tag{34}$$

In particular $g(y^2)g(y^{-2}) = g(e) = 1$.

Replacing x first by xy and then by yx in (32) and subtracting the resulting identities we find after rearranging terms that

$$4[g(xy) - g(yx)]g(y) = g(xy^2) - g(y^2x) + g(y^{-1}xy) - g(yxy^{-1}). \tag{35}$$

Since $y^2 \in Z(g)$ the first two terms on the right hand side cancel one another. Furthermore, since $g(y^2)g(y^{-2}) = 1$ we get

$$g(y^{-1}xy) = g(y^2)g(y^{-1}xy)g(y^{-2}) = g(y^2y^{-1}xyy^{-2}) = g(yxy^{-1}), \tag{36}$$

so also the last two terms cancel one another. Thus $g(y)[g(xy) - g(yx)] = 0$ for all $x, y \in G$. By symmetry $g(x)[g(xy) - g(yx)] = 0$ for all $x, y \in G$. We see that $g(xy) = g(yx)$ at least when $g(x) \neq 0$ or $g(y) \neq 0$. In the final case of $g(x) = g(y) = 0$ we get from the identity (14) that $g(x^2) = g(y^2) = -1$. Using first that $g(z) = g(z^{-1})$ for all $z \in G$ (Lemma III.1) and then (34) we get that

$$g(xy) = g(y^{-1}x^{-1}) = g(y^2)g(y^{-1}x^{-1})g(x^2) = g(y^2y^{-1}x^{-1}x^2) = g(yx). \tag{37}$$

$\square$

V. The short d'Alembert functional equation

In this section we solve the short d'Alembert functional equation

$$g(xy) + g(x\tau(y)) = 2g(x)g(y), \ \ \forall\, x, y \in G, \tag{38}$$

on a metabelian group G for any involution τ of G. In view of Theorem IV.1 we also get the solution of the long d'Alembert functional equation (32) on such a group.

Lemma V.1

Let G be any group. Any solution $g \neq 0$ of the functional equation (38) has for all $x, y \in G$ the properties

$$g \circ \tau = g, \quad g(e) = 1, \quad g(xy) = g(yx), \quad g(x^2) + g(x\tau(x)) = 2[g(x)]^2, \\ g(xy\tau(y)) = g(x)g(y\tau(y)) \text{ and } x\tau(x) \in Z(g). \tag{39}$$

Proof: We see that $g \circ \tau = g$ when we note that the left hand side of (38) is invariant under interchange of y and $\tau(y)$. Putting $x = e$ in the identity (38) we see that $g(e) = 1$. Comparing the left and the right hand sides of the following computation

$$\begin{aligned} g(xy) + g(x\tau(y)) &= 2g(x)g(y) = 2g(y)g(x) = g(yx) + g(y\tau(x)) \\ &= g(yx) + (g \circ \tau)(y\tau(x)) = g(yx) + g(x\tau(y)) \end{aligned} \tag{40}$$

we see that $g(xy) = g(yx)$. Putting $y = x$ in (38) we get that $g(x^2) + g(x\tau(x)) = 2[g(x)]^2$. Replacing y by $y\tau(y)$ in (38) we get $g(xy\tau(y)) = g(x)g(y\tau(y))$. Using this and that $g(xy) = g(yx)$ repeatedly we find for any $x, y \in G$ that

$$g(xyu\tau(u)) = g(xy)g(u\tau(u)) = g(yx)g(u\tau(u)) = g(yxu\tau(u))g(xu\tau(u)y) \tag{41}$$

showing that $u\tau(u) \in Z(g)$ for any $u \in G$. □

Remark V.2

Any solution g of the short d'Alembert functional equation (38) is also a solution of the long one (12), because $g(xy) = g(yx)$ for all $x, y \in G$ (Lemma V.1).

So far we have not used that the group G is supposed to be metabelian. That we do now where we derive some of the properties of the solutions of the short d'Alembert functional equation (38) on a metabelian group.

Proposition V.3

Let $g : G \to \mathbf{C}$ be a non-zero solution of the functional equation (38) on a metabelian group G.

Then $Z(g)$ is a normal, τ-invariant subgroup of G such that $x\tau(x^{-1}) \in Z(g)$ and $x^2 \in Z(g)$ for all $x \in G$. Furthermore $Z(g) = \{x \in G \mid g([x, y]) = 1 \text{ for all } y \in G\}$.

If $Z(g) \neq G$ then $Z(g) = \{x \in G \mid g(x) \neq 0\}$, g is a homomorphism of $Z(g)$ into $\mathbf{C}^*$ and $g(x\tau(x^{-1})) = -1$ for all $x \in G \setminus Z(g)$.

Proof: Let us first assume that there exists a $u \in Z(g)$ such that $g(u)^2 \neq g(u\tau(u))$. Since g is a solution of the short d'Alembert functional equation (38) it is also a solution of the long one (Remark V.2), so we can apply Theorem III.2 to infer that $g = (M + M \circ \tau)/2$ for some homomorphism $M : G \to \mathbf{C}^*$. It follows that $Z(g) = G$. The Proposition is trivially true in this case.

We may thus assume that $g(u)^2 = g(u\tau(u))$ for all $u \in Z(g)$. Going back to Theorem III.2(b) we see that in this case $g(xu) = g(x)g(u)$ for all $x \in G$ and

$u \in Z(g)$. We may apply Proposition II.2 because $[G,G] \subseteq Z(G) \subseteq Z(g)$. From (a) of the Proposition we see that $Z(g) = \{x \in G \mid g([x,y]) = 1 \text{ for all } y \in G\}$. Since $g\big(xyx^{-1}\big) = g(y)$ for all $x, y \in G$ (Lemma V.1) we see, now from (d) of Proposition II.2, that $Z(g) = \{x \in G \mid g(x) \neq 0\}$. So $x \in G \setminus Z(g)$ implies that $g(x) = 0$. From the functional equation (38) we get with $y = x^{-1}$ that $1 + g\big(x\tau\big(x^{-1}\big)\big) = 2g(x)g\big(x^{-1}\big) = 0$, so $g\big(x\tau\big(x^{-1}\big)\big) = -1$ as desired. That in turn implies that $x\tau\big(x^{-1}\big) \in \{y \in G \mid g(y) \neq 0\} = Z(g)$. From Lemma V.1 we get that $\tau(x)x \in Z(g)$. Now, $x^2 = x\tau\big(x^{-1}\big)\tau(x)x \in Z(g)Z(g) = Z(g)$. □

Theorem V.4

Let G be a metabelian group. The non-zero solutions $g : G \to \mathbf{C}$ of the short d'Alembert functional equation (38) are the following:

I) $g = (M + M \circ \tau)/2$ *for some homomorphism* $M : G \to \mathbf{C}^*$.

II) *There exists a normal, τ-invariant subgroup H of G with the property $x\tau\big(x^{-1}\big) \in H$ for all $x \in G$, and a homomorphism $m : H \to \mathbf{C}^*$ with the properties $m = m \circ \tau$ and $m\big(x\tau\big(x^{-1}\big)\big) = -1$ for all $x \in G \setminus H$, such that*

$$g(x) = \begin{cases} m(x) & \text{for } x \in H \\ 0 & \text{for } x \in G \setminus H \end{cases} \tag{42}$$

Let g be a solution as described above under the present case II. Then

(a) $Z(g) = G$ *if and only if m extends to a homomorphism $M : G \to \mathbf{C}^*$. If so then* $g = (M + M \circ \tau)/2$.

(b) *If $Z(g) \neq G$ then* $H = Z(g) = \{x \in G \mid g(x) \neq 0\}$.

Proof: Let g be a non-zero solution of (38). If $Z(g) = G$ we see from Corollary III.3 that we are in case I. So we may assume that $Z(g) \neq G$. Taking $H := Z(g)$ and $m := g|Z(g)$ we see from Proposition V.3 that we have case II.

We shall next prove the converse, i.e. that the formulas of the cases I and II define solutions of (38). If g has the form from case I then clearly g is a solution of the functional equation (38). In case II we prove that $g(xy) + g(x\tau(y)) - 2g(x)g(y) = 0$ for all $x, y \in G$ by going through the 5 different possibilities for x and y.

(α) $x \in H$ and $y \in H$. Easy computations because $g = m$ on H.

(β) $x \in H$ and $y \in G \setminus H$. Here $\tau(y) \in G \setminus H$ because H is τ-invariant. Since H is a subgroup of G we have that $xy \in G \setminus H$ and $x\tau(y) \in G \setminus H$. Now, $g(xy) + g(x\tau(y)) - 2g(x)g(y) = 0 + 0 - 2g(x) \cdot 0 = 0$.

(γ) $x \in G \setminus H$ and $y \in H$. The same arguments as in (β) work also here.

(δ) $x \in G \setminus H$, $y \in G \setminus H$ and $xy \in G \setminus H$. Using the assumption $y^{-1}\tau(y) \in H$ we get $x\tau(y) = xyy^{-1}\tau(y) \in (G \setminus H)H \subseteq G \setminus H$, so $g(xy) + g(x\tau(y)) - 2g(x)g(y) = 0 + 0 - 2 \cdot 0 \cdot 0 = 0$.

(ϵ) $x \in G \setminus H$, $y \in G \setminus H$ and $xy \in H$. Here we use the assumptions $y^{-1}\tau(y) \in H$ and $m\big(y^{-1}\tau(y)\big) = -1$ as follows:

$$\begin{aligned} & g(xy) + g(x\tau(y)) - 2g(x)g(y) = g(xy) + g\big(xyy^{-1}\tau(y)\big) - 2\cdot 0 \cdot 0 \\ & = m(xy) + m\big(xyy^{-1}\tau(y)\big) = m(xy) + m(xy)m\big(y^{-1}\tau(y)\big) \\ & = m(xy)\big[1 + m\big(y^{-1}\tau(y)\big)\big] = m(xy)[1-1] = 0. \end{aligned} \tag{43}$$

To prove the statement II.a let us first assume that $Z(g) = G$. According to Corollary III.3 there exists a homomorphism $M : G \to \mathbf{C}^*$ such that $g = (M + M \circ \tau)/2$. Restricting to H we see that $m = (M + M \circ \tau)/2$ on H. The set of homomorphisms from H into $\mathbf{C}^*$ is linearly independent (see for example Lemma 29.41 of [13]), so we infer that $m = M$ on H.

Assume conversely that m extends to a homomorphism $M : G \to \mathbf{C}^*$. For $x \in H$ we get that $(M + M \circ \tau)(x) = (m + m \circ \tau)(x) = m(x) + m(x) = 2m(x) = 2g(x)$. For $x \in G \setminus H$ we get because $x^{-1}\tau(x) \in H$ that $(M + M \circ \tau)(x) = M(x)\big(M\big(x^{-1}\tau(x)\big) + 1\big) = M(x)\big(m\big(x^{-1}\tau(x)\big) + 1\big) = M(x)(-1+1) = 0 = 2g(x)$. So $g(x) = (M(x) + (M \circ \tau)(x))/2$ for all $x \in G$. But $Z(g) = G$ for any function g of this form.

Finally we get to the statement II.b. From Proposition V.3 we find that $Z(g) = \{x \in G \mid g(x) \neq 0\}$. From (42) we see that $H = \{x \in G \mid g(x) \neq 0\}$, so $H = Z(g)$. $\square$

Proposition V.5 below shows that for 2±divisible metabelian groups the only non-zero solutions of (2) are the ones of the form (5). The Heisenberg group H_3 is an example of such a group. The same holds according to Theorem IV.1 for the solutions of the long functional equation (3). See [4] for a different type of criterion.

Proposition V.5

Let $g : G \to \mathbf{C}$ be a non-zero solution of the short d'Alembert functional equation (38) where G is a metabelian group generated by the squares x^2, $x \in G$.

Then there exists a homomorphism $M : G \to \mathbf{C}^*$ such that $g = (M + M \circ \tau)/2$.

Proof: From Proposition V.3 we get that $x^2 \in Z(g)$ for each $x \in G$ so $Z(g) = G$. We then apply Corollary III.3. $\square$

A result similar to Proposition V.5 holds for continuous solutions of the functional equation (38). An example is the Heisenberg group.

Theorem V.6

Let G be a metabelian, connected topological group. If $g \neq 0$ is a continuous solution of the functional equation (38) then there exists a continuous homomorphism $m : G \to \mathbf{C}^*$ such that $g = (m + m \circ \tau)/2$.

Proof: If $Z(g) \neq G$ then Proposition V.3 says that $Z(g) = \{x \in G \mid g(x) \neq 0\}$. This formula shows that $Z(g)$ is open. By its very definition it is also closed, so $Z(g) = G$

because G is connected. According to Corollary III.3 there then exists a homomorphism $m : G \to \mathbf{C}^*$ such that $g = (m + m \circ \tau)/2$. The continuity of m is a consequence of the following general Proposition V.7. $\square$

Proposition V.7 below extends Theorem 1 of [14].

<u>**Proposition V.7**</u> *Let G be a topological group. Let $\gamma_1, \gamma_2, \cdots, \gamma_n : G \to \mathbf{C}^*$ be different homomorphisms, and let $c_1, c_2, \cdots, c_n$ be non-zero complex numbers. If $g = c_1\gamma_1 + c_2\gamma_2 + \cdots + c_n\gamma_n$ is continuous then each γ_j, $j = 1, 2, \cdots, n$, is continuous.*

<u>Proof:</u> The proof goes by induction on n, the induction hypothesis being that the Proposition is true for sums of n or less terms. The proposition is correct for $n = 1$. Consider a continuous function g of the form $g = c_1\gamma_1 + c_2\gamma_2 + \cdots + c_n\gamma_n + c_{n+1}\gamma_{n+1}$, where $\gamma_1, \gamma_2, \cdots, \gamma_{n+1} : G \to \mathbf{C}^*$ are different homomorphisms and where $c_1, c_2, \cdots, c_{n+1}$ are non-zero complex numbers. It suffices to prove that γ_{n+1} is continuous, for in that case we can apply the induction hypothesis to the continuous function $g - c_{n+1}\gamma_{n+1}$ to get the continuity of the remaining homomorphisms. From the homomorphism property we get for any $x, y \in G$ that

$$g(xy) = c_1\gamma_1(x)\gamma_1(y) + \cdots + c_n\gamma_n(x)\gamma_n(y) + c_{n+1}\gamma_{n+1}(x)\gamma_{n+1}(y). \tag{44}$$

Dividing through by $\gamma_1(y)$ we find that

$$\frac{g(xy)}{\gamma_1(y)} - g(x) = c_2\left(\frac{\gamma_2(y)}{\gamma_1(y)} - 1\right)\gamma_2(x) + \cdots + c_{n+1}\left(\frac{\gamma_{n+1}(y)}{\gamma_1(y)} - 1\right)\gamma_{n+1}(x). \tag{45}$$

Since $\gamma_{n+1} \neq \gamma_1$ there is a $y_1 \in G$ such that $\gamma_{n+1}(y_1)/\gamma_1(y_1) \neq 1$. This means that the coefficient of the last term of (45) is different from zero at $y = y_1$. The induction hypothesis applied to the left hand side $x \to g(xy_1)/\gamma_1(y_1) - g(x)$ of (45) at $y = y_1$ ensures that γ_{n+1} is continuous as desired. $\square$

References

[1] Aczél, J., Chung, J. K. and Ng, C. T.: ™*Symmetricsecond differences in product form on groups.*∫ Topics in mathematical analysis (pp. 1±22) edited by Th. M. Rassias. Ser. Pure Math., 11, World Scientific Publ. Co., Teaneck, NJ, 1989.

[2] Baker, John A., *The stability of the cosine equation.* Proc. Amer. Math. Soc. **80** (1980), 411±416.

[3] Bröcker, T. and Dieck, T.: ™*Representations of Compact Lie Groups.* Springer-Verlag. New York, Berlin, Heidelberg, Tokyo. 1985.

[4] Corovei, I., *The cosine functional equation for nilpotent groups.* Aequationes Math. **15** (1977), 99±106.

[5] Corovei, I., *The functional equation $f(xy) + f(yx) + f(xy^{-1}) + f(y^{-1}x) = 4f(x)f(y)$ for nilpotent groups.* (Romanian. English summary) Bul. Ştiinţ. Inst. Politehn. Cluj-Napoca Ser. Mat.-Fiz.-Mec. Apl. **20** (1977), 25±28.

[6] Corovei, I., *The functional equation* $f(xy) + f\left(xy^{-1}\right) = 4f(x)g(y)$ *for nilpotent groups*. Mathematica (Cluj) **22** (**45**) (1980), 33±41.

[7] Corovei, I., *The d'Alembert functional equation on metabelian groups*. To appear in Aequationes Math.

[8] Corovei, I., *Wilson's functional equation on P_3-groups*. Manuscript 1998. 7 pp.

[9] Curzio, M., Longobardi, P. and Maj, M., *Su di un problema combinatorio in teoria dei gruppi*. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **74** (1983), no. 3, 136±142.

[10] Dacić, R., *The cosine functional equation for groups*. Mat. Vesnik **6** (**21**) (1969), 339±342.

[11] Folland, G. B.: ™*Harmonic analysis in phase space.*∫ Princeton University Press, Princeton, New Jersey 1989.

[12] Friis, P., *Trigonometric Functional Equations on Groups*. Progress report. Department of Mathematics. Aarhus University. December 1998.

[13] Hewitt, E. and Ross, K. A.: ™*Abstract Harmonic Analysis.*∫ *Volume II*. Springer-Verlag. Berlin-Heidelberg-New York 1970.

[14] Kannappan, PL., *The functional equation* $f(xy) + f\left(xy^{-1}\right) = 2f(x)f(y)$ *for groups*. Proc. Amer. Math. Soc. **19** (1968), 69±74.

[15] Kannappan, PL., *A note on cosine functional equation for groups*. Mat. Vesnik **8** (1971), 317±319.

[16] Kannappan, PL., *PL177, PL177S1, PL177S2*. Aequationes Math. **19** (1979), 114±115.

[17] Penney, R. C. and Rukhin, A. L., *D'Alembert's functional equation on groups*. Proc. Amer. Math. Soc. **77** (1979), 73±80.

[18] Stetkær, H., *d'Alembert's equation and spherical functions*. Aequationes Math. **48** (1994), 220±227.

[19] Stetkær, H., *Wilson's functional equations on groups*. Aequationes Math. **49** (1995), 252±275.

[20] Stetkær, H., *Functional equations on abelian groups with involution*. Aequationes Math. **54** (1997), 144±172.

Henrik Stetkær
Department of Mathematics, Aarhus University
Ny Munkegade
DK 8000 Aarhus C
Denmark
E-mail: stetkaer@imf.au.dk