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HOMOTOPY INVARIANCE FOR BIFUNCTORS DEFINED FROM ASYMPTOTIC HOMOMORPHISMS

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1. Introduction

One of fundamental features in the Brown-Douglas-Fillmore theory of extensions is that the equivalence relation used to define the extension groups turns out to be homotopy invariant, see Theorem 2.14 of [BDF]. Similarly much of the power of Kasparovs generalization of the BDF-theory, cf. [K1]-[K3], comes from the fact that there are several equivalence relations on the fundamental objects, and that only one of these relations is obviously homotopy invariant. The others are then shown to be homotopy invariant, and in fact to define the same relation, by means of the Kasparov product. This variety of apparently different equivalence relations is missing in the variant of KK-theory, called E-theory, which was introduced by Connes and Higson in [CH]. The equivalence relation employed in the general E-theory framework has so far only been homotopy. But recently the efforts towards classifying certain classes of C^*-algebras have met with the problem that while the objects of E-theory, i.e. the asymptotic homomorphisms, seem much more amenable to classification than the graded Hilbert $A-B$-modules of Kasparov, the equivalence relation - namely homotopy - is not. The most striking solution of this occurs in the classification of purely infinite simple nuclear C^*-algebras by Kirchberg and Phillips where a major part of the proof consists of realizing E-theory, for their particular class of C^*-algebras, as asymptotic homomorphisms modulo an equivalence relation which is (apparently) much stronger than homotopy, see [Ki], [Ph], [A]. Similar considerations and results can be found in the work of Lin, [Li1], [Li2] and Dadarlat and Eilers, [DE].

The project of the present work is to transfer to asymptotic homomorphism the two most important equivalence relations which were used by Brown, Douglas, Fillmore and Kasparov and which are not obviously homotopy invariant. To describe what these relations become in E-theory we formulate one of our main results :

Theorem 1.1. *Let A and B be separable C^*-algebras, B stable, and let $\varphi = (\varphi_t)_{t\in[1,\infty)}$, $\psi = (\psi_t)_{t\in[1,\infty)}$: $SA \to B$ be asymptotic homomorphisms. Then the following are equivalent :*

1) *$[\varphi] = [\psi]$ in $[[SA, B]]$ (i.e. φ and ψ are homotopic).*
2) *There is a family $\Phi^\lambda : SA \to B$, $\lambda \in [0,1]$, of asymptotic homomorphisms such that $\Phi^0 = \varphi, \Phi^1 = \psi$, and the family of maps, $[0,1] \ni \lambda \mapsto \Phi^\lambda_t(a)$, $t \in [1,\infty)$, is equicontinuous for all $a \in SA$.*
3) *There is an asymptotic homomorphism $\mu = (\mu_t)_{t\in[1,\infty)} : \mathrm{cone}(A) \to B$ and a norm-continuous path U_t, $t \in [1,\infty)$, of unitaries in $M_2(B)^+$ such that*

$$\lim_{t\to\infty} U_t \begin{pmatrix} \varphi_t(a) & \\ & \mu_t(a) \end{pmatrix} U_t^* - \begin{pmatrix} \psi_t(a) & \\ & \mu_t(a) \end{pmatrix} = 0$$

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for all $a \in SA$.

Here the equivalence relation described in 2) is the analog of operator homotopy while the equivalence relation described in 3) corresponds to unitary equivalence modulo addition by degenerate elements.

By Therem 4.2 of [H-LT] it is possible to realize KK-theory by using asymptotic homomorphisms where the individual maps are completely positive linear contractions. It is therefore interesting that we can improve condition 3) for such completely positive asymptotic homomorphisms in the following way : For given separable C^*-algebras A and B, with B stable, there is a completely positive asymptotic homomorphism $\lambda = (\lambda_t)_{t\in[1,\infty)}$: cone$(A) \to B$ with the property that two completely positive asymptotic homomorphisms $\varphi = (\varphi_t)_{t\in[1,\infty)}$, $\psi = (\psi_t)_{t\in[1,\infty)}$: $SA \to B$ are homotopic (as completely positive asymptotic homomorphisms) if and only if there is a norm-continuous path U_t, $t \in [1,\infty)$, of unitaries in $M_2(B)^+$ and a continuous function $r : [1,\infty) \to [1,\infty)$ such that $\lim_{t\to\infty} r(t) = \infty$, and

$$\lim_{t\to\infty} U_t \begin{pmatrix} \varphi_t(a) & \\ & \lambda_{r(t)}(a) \end{pmatrix} U_t^* - \begin{pmatrix} \psi_t(a) & \\ & \lambda_{r(t)}(a) \end{pmatrix} = 0$$

for all $a \in SA$.

Recently the author explained how naturally discrete asymptotic homomorphisms fit into E-theory and KK-theory, [Th1]. For this reason we prove the analogues for discrete asymptotic homomorphisms of the results we have just described for E-theory and KK-theory. See Theorem 2.11 and Theorem 4.3. As an application of the main results we are able to give a description of E-theory which shows, perhaps surprisingly, that E-theory is a specialization of KK-theory : For separable C^*-algebras A and B there is a natural isomorphism

$$E(A,B) \simeq KK(A, C_b([1,\infty), B\otimes\mathcal{K})/C_0([1,\infty), B\otimes\mathcal{K})) .$$

The proof of this depends in a crucial way on the use of discrete asymptotic homomorphisms.

Acknowledgement. Some of our results have non-empty overlap with results obtained by Dadarlat and Eilers in [DE]. One of the key ideas in the proof of our main results - the one which produces an approximate inner automorphism out of a trivial KK-element - I learned from their work. This idea was first introduced by Huaxin Lin in [Li1]. I am grateful to all three, Dadarlat, Eilers and Lin, for keeping me informed about their work.

2. E-theory as homotopy classes of $*$-homomorphisms

Let X be a locally compact, σ-compact Hausdorff space which is *not* compact. For any C^*-algebra A, let $C_b(X,A)$ denote the C^*-algebra of bounded continuous functions from X to A and let $C_0(X,A)$ be the ideal in $C_b(X,A)$ consisting of the functions vanishing at infinity. This gives us an extension

$$0 \longrightarrow C_0(X,A) \longrightarrow C_b(X,A) \xrightarrow{q_A} C_b(X,A)/C_0(X,A) \longrightarrow 0 .$$

When $\varphi : A \to B$ is a $*$-homomorphism we get induced $*$-homomorphisms $\overline{\varphi}$: $C_b(X,A) \to C_b(X,B)$ and $\underline{\varphi}$: $C_b(X,A)/C_0(X,A) \to C_b(X,B)/C_0(X,B)$ in the obvious way. In the following we will consider an extension

$$0 \longrightarrow J \xrightarrow{j} A \xrightarrow{p} B \longrightarrow 0 \qquad (2.1)$$

of C^*-algebras. A starting point for us here is the following observation, which may be considered a folklore fact.

Lemma 2.1. *The sequence*

$$0 \longrightarrow C_b(X,J)/C_0(X,J) \xrightarrow{\underline{j}} C_b(X,A)/C_0(X,A) \xrightarrow{\underline{p}} C_b(X,B)/C_0(X,B) \longrightarrow 0$$

is exact.

Given a C^*-algebra D we set $ID = C[0,1] \otimes D$ and $\mathrm{cone}(D) = \{f \in ID : f(0) = 0\}$. Recall that the mapping cone of $\underline{p}$ is the C^*-algebra

$$C_{\underline{p}} = \{(z,f) \in C_b(X,A)/C_0(X,A) \oplus \mathrm{cone}(C_b(X,B)/C_0(X,B)) : \underline{p}(z) = f(1)\} .$$

There is a canonical imbedding $C_b(X,J)/C_0(X,J) \subseteq C_{\underline{p}}$ given by $z \mapsto (\underline{j}(z), 0)$. For any pair of C^*-algebras A and B we let $[A,B]$ denote the set of homotopy classes of $*$-homomorphisms from A to B.

Proposition 2.2. *Assume that the extension (2.1) splits. Let D be a separable C^*-algebra. Then the canonical imbedding $C_b(X,J)/C_0(X,J) \subseteq C_{\underline{p}}$ induces a bijection $[D, C_b(X,J)/C_0(X,J)] \simeq [D, C_{\underline{p}}]$.*

To prove this set

$$T_{\underline{p}} = \{(z,f) \in C_b(X,A)/C_0(X,A) \oplus I(C_b(X,B)/C_0(X,B)) : \underline{p}(z) = f(1)\} .$$

Note that $C_b(X,J)/C_0(X,J) \subseteq C_{\underline{p}} \subseteq T_{\underline{p}}$. In the following we will suppress $\underline{j}$ in the notation and consider $C_b(X,J)/C_0(X,J)$ as a C^*-subalgebra of $C_b(X,A)/C_0(X,A)$.

Lemma 2.3. *Assume that the extension (2.1) splits. Let $\mathcal{A} \subseteq T_{\underline{p}}$ be a separable C^*-subalgebra. There is then a $*$-homomorphism $\psi : \mathcal{A} \to C_b(X,A)/C_0(X,A)$ such that*

i) *$\psi(a) = a$ for all $a \in \mathcal{A} \bigcap C_b(X,J)/C_0(X,J)$,*
ii) *$\underline{p} \circ \psi(z,f) = f(0)$ for all $(z,f) \in \mathcal{A}$, and*
iii) *$\psi(z,g) = z$ for all $(z,g) \in \mathcal{A}$ with $g \in I(C_b(X,B)/C_0(X,B))$ a constant $C_b(X,B)/C_0(X,B)$-valued function.*

Proof. Let $\sigma : B \to A$ be a $*$-homomorphism such that $p \circ \sigma = \mathrm{id}_B$. By enlarging $\mathcal{A}$ if necessary we may assume that $(z,f) \in \mathcal{A} \Rightarrow (\underline{\sigma}(f(0)), f(0)) \in \mathcal{A}$. There is a separable C^*-subalgebra $\mathcal{B} \subseteq C_b(X,B)/C_0(X,B)$ such that $(z,f) \in \mathcal{A},\ t \in [0,1] \Rightarrow f(t) \in \mathcal{B}$. By using the Connes-Higson construction, cf. [CH], we can then define an asymptotic homomorphism $\rho' = (\rho'_t)_{t \in [1,\infty)} : \mathrm{cone}(\mathcal{B}) \to C_b(X,J)/C_0(X,J)$ such that $\lim_{t\to\infty} \|\rho'_t(f \otimes b) - f(u_t)\underline{\sigma}(b)\| = 0$ when $f \in C[0,1],\ f(0) = 0,\ b \in \mathcal{B}$, where $\{u_t : t \in [1,\infty)\}$ is a continuous quasi-central approximate unit for the ideal $\mathcal{C} \bigcap C_b(X,J)/C_0(X,J)$ in $\mathcal{C}$ and $\mathcal{C} \subseteq C_b(X,A)/C_0(X,A)$ is the (separable) C^*-algebra generated by $\underline{\sigma}(\mathcal{B})$. Note that by construction ρ' will be equicontinuous in the sense that the following holds :

Observation 2.4. For every $a \in \mathrm{cone}(\mathcal{B})$ and $\epsilon > 0$ there is a $\delta > 0$ such that $\sup_{t \in [1,\infty)} \|\rho'_t(a) - \rho'_t(b)\| < \epsilon$ when $b \in \mathrm{cone}(\mathcal{B})$ and $\|a - b\| < \delta$.

We can assume that $\rho'_t(0) = 0$. Now define $\rho_t : I\mathcal{B} \to C_b(X,A)/C_0(X,A)$ by $\rho_t(g) = \rho'_t(g - g(0)) + \underline{\sigma}(g(0))$. Then $\lim_{t\to\infty} \|\rho_t(h \otimes b) - h(u_t)\underline{\sigma}(b)\| = 0$ for all $h \in C[0,1],\ b \in \mathcal{B}$. In particular, $\rho = (\rho_t)_{t \in [1,\infty)} : I\mathcal{B} \to C_b(X,A)/C_0(X,A)$

is an asymptotic homomorphism which is equicontinuous since ρ' is. Note that $\underline{p}(\rho_t(g)) = g(0)$, $g \in I\mathcal{B}$. As in the proof of Proposition 3.2 of [DL] we may then define $\varphi'_t : \mathcal{A} \to C_b(X,A)/C_0(X,A)$ by $\varphi'_t(z,f) = z - \underline{\sigma}(f(1)) + \rho_t(f)$. As demonstrated in [DL] this gives us an asymptotic homomorphism $\varphi' = (\varphi'_t)_{t\in[1,\infty)} : \mathcal{A} \to C_b(X,A)/C_0(X,A)$ such that $\varphi'_t(a) = a$ for all $a \in \mathcal{A} \bigcap C_b(X,J)/C_0(X,J)$. Note that φ' is equicontinuous since ρ is and that

$$\underline{p}(\varphi'_t(z,f)) = \underline{p}(\rho_t(f)) = f(0) \tag{2.2}$$

for all $(z,f) \in \mathcal{A}$ and all t. Furthermore, observe that when $(z,g) \in \mathcal{A}$ and $g \in I(C_b(X,B)/C_0(X,B))$ is a constant function,

$$\lim_{t\to\infty} \varphi'_t(z,g) = \lim_{t\to\infty} z - \underline{\sigma}(g(1)) + \rho_t(g) = z - \underline{\sigma}(g(1)) + \underline{\sigma}(g(1)) = z\ . \tag{2.3}$$

Finally, set $\varphi_t(z,f) = \varphi'_t(z - \underline{\sigma}(f(0)), f - f(0)) + \underline{\sigma}(f(0))$. By using that $(z,f) = (z - \underline{\sigma}(f(0)), f - f(0)) + (\underline{\sigma}(f(0)), f(0))$ it folllows from (2.3) that $\lim_{t\to\infty} \|\varphi_t(a) - \varphi'_t(a)\| = 0$ for all $a \in \mathcal{A}$, and hence $\varphi = (\varphi_t)_{t\in[1,\infty)}$ is an asymptotic homomorphism. Since φ' is equicontinuous, so is φ. In addition (2.2) implies that $\varphi'_t(C_{\underline{p}} \cap \mathcal{A}) \subseteq C_b(X,J)/C_0(X,J)$ and hence $\underline{p}(\varphi_t(z,f)) = \underline{p}(\underline{\sigma}(f(0))) = f(0)$ for all $(z,f) \in \mathcal{A}$. And (2.3) implies that

$$\lim_{t\to\infty} \varphi_t(z,g) = \lim_{t\to\infty} \varphi'_t(z - \underline{\sigma}(g(0)), 0) + \underline{\sigma}(g(0)) = z \tag{2.4}$$

for all $(z,g) \in \mathcal{A}$ with g constant. The reason that we have exchanged φ' with φ is that the latter satisfies

$$\varphi_t(z,f) = \varphi_t(z - \underline{\sigma}(f(0)), f - f(0)) + \underline{\sigma}(f(0)) \tag{2.5}$$

for all $(z,f) \in \mathcal{A}$ and all t. Let $\{d_1, d_2, d_3, \cdots\} \subseteq \mathcal{A}$ be a dense sequence, and let $S : C_b(X,A)/C_0(X,A) \to C_b(X,A)$ be a continuous section for the quotient map. It has been observed by Loring in [L] that we may choose S such that $\|S(z)\| \leq 2\|z\|$ for all $z \in C_b(X,A)/C_0(X,A)$ and such that $S(C_b(X,J)/C_0(X,J)) \subseteq C_b(X,J)$. (See in particular the remark following Theorem 2 of [L].) Let $\{U_i\}_{i=1}^{\infty}$ be a locally finite open covering of X such that $\overline{U_i}$ is compact for all i. For each $n \in \mathbb{N}$ we can find $m_n \in \mathbb{N}$ so large that $\|S(\varphi_t(a))(x)S(\varphi_t(b))(x) - S(\varphi_t(ab))(x)\| \leq 2\|\varphi_t(a)\varphi_t(b) - \varphi_t(ab)\| + \frac{1}{n}$, $\|S(\varphi_t(a))(x) + S(\varphi_t(b))(x) - S(\varphi_t(a+b))(x)\| \leq 2\|\varphi_t(a) + \varphi_t(b) - \varphi_t(a+b)\| + \frac{1}{n}$ and $\|S(\varphi_t(a^*))(x) - S(\varphi_t(a))(x)^*\| \leq 2\|\varphi_t(a^*) - \varphi_t(a)^*\| + \frac{1}{n}$ for all $t \in [1,n]$, $a, b \in \{d_1, d_2, \cdots, d_n\}$ and all $x \notin \bigcup_{i=1}^{m_n} \overline{U_i}$. Finally, by using (2.5) we can arrange that

$$\|S(\varphi_t(z,f))(x) - S(\varphi_t(z - \underline{\sigma}(f(0)), f - f(0)))(x) - S(\underline{\sigma}(f(0)))(x)\| \leq \frac{1}{n} \tag{2.6}$$

for all $t \in [1,n]$, $(z,f) \in \{d_1, d_2, \cdots, d_n\}$ and all $x \notin \bigcup_{i=1}^{m_n} \overline{U_i}$. We can assume that $1 < m_n < m_{n+1}$ for all n. Set $m_0 = 1$ and let $g_i : U_i \to [0,\infty)$ be the constant function $g_i(x) = n-1$ for each $i \in \{m_{n-1}+1, m_{n-1}+2, \cdots, m_n\}$, $n = 1,2,3,\cdots$. Let $\{h_i\}$ be a partition of unity subordinate to $\{U_i\}$ and define $g : X \to [0,\infty)$ by $g(x) = \sum_i h_i(x)g_i(x)$. Set $K_j = \overline{\bigcup_{i=1}^{m_j} U_i}$. Then

$$\lim_{j\to\infty} \sup_{x\notin K_j} \|S(\varphi_{g(x)}(a))(x)S(\varphi_{g(x)}(b))(x) - S(\varphi_{g(x)}(ab))(x)\| = 0\ , \tag{2.7}$$

$$\lim_{j\to\infty} \sup_{x\notin K_j} \|S(\varphi_{g(x)}(a))(x) + S(\varphi_{g(x)}(a))(x) - S(\varphi_{g(x)}(a+b))(x)\| = 0\ , \tag{2.8}$$

and

$$\lim_{j\to\infty} \sup_{x\notin K_j} \|S(\varphi_{g(x)}(a^*))(x) - S(\varphi_{g(x)}(a))(x)^*\| = 0 \tag{2.9}$$

for all $a, b \in \{d_1, d_2, \cdots\}$. From (2.6) we see that

$$\lim_{j\to\infty} \sup_{x\notin K_j} \|S(\varphi_{g(x)}(z,f))(x) - S(\varphi_{g(x)}(z - \underline{\sigma}(f(0)), f - f(0)))(x) - S(\underline{\sigma}(f(0)))(x)\| = 0 \tag{2.10}$$

for all $(z,f) \in \{d_1, d_2, \cdots\}$. For each $d \in \mathcal{A}$, define $h_d \in C_b(X,A)$ by $h_d(x) = S(\varphi_{g(x)}(d))(x)$, $x \in X$, and set $\psi(d) = q_A(h_d) \in C_b(X,A)/C_0(X,A)$. Since φ is equicontinuous it follows that $\psi(d)$ depends continuously on d. Therefore (2.7)-(2.9) imply that ψ is a $*$-homomorphism. If $a \in \mathcal{A} \bigcap C_b(X,J)/C_0(X,J)$ we have that $\varphi_t(a) = a$ for all $t \in \mathbb{R}$ and hence that $\psi(a) = a$. (2.4) shows that $\psi(z,g) = z$ when $(z,g) \in \mathcal{A}$ and $g \in I(C_b(X,B)/C_0(X,B))$ is constant. To prove that also ii) in the statement holds we use (2.10) and that $S(C_b(X,J)/C_0(X,J)) \subseteq C_b(X,J)$. This gives us that

$$\begin{aligned} &\underline{p}(\psi(z,f)) = \underline{p} \circ q_A(h_{(z,f)}) = q_B(\overline{p}(h_{(z-\underline{\sigma}(f(0)), f-f(0))} + S(\underline{\sigma}(f(0))))) \\ &\quad = q_B(\overline{p}(S(\underline{\sigma}(f(0))))) = \underline{p}(\underline{\sigma}(f(0))) = f(0) \end{aligned}$$

when $(z,f) \in \{d_1, d_2, \cdots\}$ since $S \circ \varphi_{g(x)}(C_{\underline{p}} \cap \mathcal{A}) \subseteq C_b(X,J)$ for all $x \in X$. ii) follows by continuity. □

Lemma 2.5. *Assume that the extension (2.1) splits. Let $\mathcal{A} \subseteq C_{\underline{p}}$ be a separable C^*-subalgebra. There is then a family $\Phi_s : \mathcal{A} \to C_{\underline{p}}$, $s \in [0,1]$, of $*$-homomorphisms such that*

a) $[0,1] \ni s \mapsto \Phi_s(a)$ *is continuous for all* $a \in \mathcal{A}$,
b) $\Phi_0(a) \in C_b(X,J)/C_0(X,J)$ *for all* $a \in \mathcal{A}$,
c) $\Phi_0(a) = a$ *for all* $a \in \mathcal{A} \bigcap C_b(X,J)/C_0(X,J)$,
d) Φ_1 *is the identity on* $\mathcal{A}$.

Proof. For each $s \in [0,1]$ define $*$-homomorphisms $\theta_s, \eta_s : \operatorname{cone}(C_b(X,B)/C_0(X,B)) \to I(C_b(X,B)/C_0(X,B))$ by $\theta_s(f)(r) = f(sr)$, and $\eta_s(f)(r) = f(s + (1-s)r)$, $r \in [0,1]$. Note that $(z, \eta_s(f)) \in T_{\underline{p}}$ for all $s \in [0,1]$ and all $(z,f) \in C_{\underline{p}}$. Let $\mathcal{B}$ be a separable C^*-subalgebra of $T_{\underline{p}}$ containing $(z, \eta_s(f))$ for all $(z,f) \in \mathcal{A}$ and all $s \in [0,1]$. Lemma 2.3 gives us a $*$-homomorphism $\psi : \mathcal{B} \to C_b(X,A)/C_0(X,A)$ satisfying i), ii) and iii). Set $\Phi_s(z,f) = (\psi(z, \eta_s(f)), \theta_s(f))$, $(z,f) \in \mathcal{A}$. It is straightforward to check that $\{\Phi_s\}$ has the stated properties. □

Proposition 2.2 follows immediately from Lemma 2.5.

In the following we let $\mathcal{K}$ denote the C^*-algebra of compact operators on an infinite dimensional separable Hilbert space. For any C^*-algebra B we let $\mathcal{M}(B)$ denote the multiplier algebra of B.

Lemma 2.6. *Let D be a separable C^*-subalgebra of $C_b(X, B \otimes \mathcal{K})/C_0(X, B \otimes \mathcal{K})$. There is then a stable separable C^*-algebra E such that $D \subseteq E \subseteq C_b(X, B \otimes \mathcal{K})/C_0(X, B \otimes \mathcal{K})$.*

Proof. The crucial point is the following

Observation 2.7. Let $f \in C_b(X, B \otimes \mathcal{K})$ be a positive element. There is then an element $z \in C_b(X, B \otimes \mathcal{K})/C_0(X, B \otimes \mathcal{K})$ such that $z^*z = q_{B\otimes\mathcal{K}}(f)$ and $zz^*q_{B\otimes\mathcal{K}}(f) = 0$.

To prove this observation, let $W_1 \subseteq W_2 \subseteq W_3 \subseteq \cdots$ be a sequence of relatively compact open sets in X such that $\bigcup_{n=1}^{\infty} W_n = X$. Let $\{U_i\}_{i=1}^{\infty}$ be a locally finite covering of X by relatively compact open sets. Choose $1 < m_1 < m_2 < m_3 < \cdots$ in $\mathbb{N}$ such that $\overline{W_n} \subseteq \bigcup_{i=1}^{m_n} U_i$. Set $V_i = U_i, i = 1, 2, \cdots, m_1$, and $V_i = U_i \backslash \overline{W_n}$, $i = m_n + 1, m_n + 2, \cdots, m_{n+1}$, $n \in \mathbb{N}$. Then $\{V_i\}$ is also a locally finite covering of X by relatively compact open sets, and

$$\overline{W_n} \bigcap V_i = \emptyset , \quad i > m_n , \ n \in \mathbb{N} . \tag{2.11}$$

Consider $\mathcal{K}$ as a C^*-subalgebra of $\mathcal{M}(B \otimes \mathcal{K})$ via the embedding $x \mapsto 1_B \otimes x$. Standard arguments give us projections $p_1 \leq p_2 \leq p_3 \leq \cdots$ in $\mathcal{K} \subseteq \mathcal{M}(B \otimes \mathcal{K})$ such that

$$\sup_{x \in \overline{V_j}} \|p_j f(x) - f(x)\| \leq \frac{1}{j} . \tag{2.12}$$

Let $\{h_i\}$ be a partition of unity subordinate to $\{V_i\}$ and define $g \in C_b(X, B \otimes \mathcal{K})$ by $g(x) = \sum_{i=1}^{\infty} h_i(x) p_i f(x) p_i$. Then $q_{B\otimes\mathcal{K}}(g) = q_{B\otimes\mathcal{K}}(f)$ by (2.12). Note that $p_{m_k} g(x) = g(x)$, $x \in \overline{W_k}$, by (2.11), and that there are projections $q_1 \leq q_2 \leq q_3 \leq \cdots$ in $\mathcal{K}$ such that $q_i g(x) = g(x)$, $x \in \overline{V_i}$ (use that $\overline{V_i} \subseteq W_l$ for all sufficiently large l). Choose partial isometries $\{v_j\}_{j=1}^{\infty} \in \mathcal{M}(B \otimes \mathcal{K})$ recursively such that $v_j v_j^* p_{m_n} = 0$, $v_j^* v_j = q_j$ for all $j \leq m_n, n \in \mathbb{N}$, and $v_i^* v_j = 0$ when $i \neq j$. Define $h \in C_b(X, B \otimes \mathcal{K})$ by $h(x) = \sum_{i=1}^{\infty} \sqrt{h_i(x)} v_i \sqrt{g(x)}$. Then $h^*h = g$ and $hh^*g = 0$. Setting $z = q_{B\otimes\mathcal{K}}(h)$ we have established the observation. It follows that we can find a sequence $D \subseteq D_1 \subseteq D_2 \subseteq \cdots$ of separable C^*-subalgebras of $C_b(X, B \otimes \mathcal{K})/C_0(X, B \otimes \mathcal{K})$ and for each n have a dense sequence $\{g_1, g_2, \cdots\}$ in the positive part of D_n and elements $\{v_1, v_2, \cdots\}$ in D_{n+1} such that $v_k^* v_k = g_k$ and $v_k v_k^* g_k = 0$ for all k. Set $E = \overline{\bigcup_{n=1}^{\infty} D_n}$ which is a separable C^*-subalgebra of $C_b(X, B \otimes \mathcal{K})/C_0(X, B \otimes \mathcal{K})$ containing D. If $a \in E$ is a positive element and $\epsilon > 0$ there are elements $b, x \in E$, $b \geq 0$, such that $\|a - b\| < \epsilon$, $x^*x = b$ and $xx^*b = 0$. By Proposition 2.2 and Theorem 2.1 of [HR] we conclude that E is stable.

□

When D, B are C^*-algebras $[D, B \otimes \mathcal{K}]$ is an abelian semigroup. We make now the following assumption on D :

$$[D, E \otimes \mathcal{K}] \text{ is a group for any separable } C^*\text{-algebra } E . \tag{2.13}$$

Under this assumption, and when D is separable, it follows from Lemma 2.6 that $[D, C_b(X, B \otimes \mathcal{K})/C_0(X, B \otimes \mathcal{K})]$ has the structure of an abelian group, and we can define a functor, F_X, from the category of C^*-algebras to the category of abelian groups such that

$$F_X(B) = [D, C_b(X, B \otimes \mathcal{K})/C_0(X, B \otimes \mathcal{K})]$$

and $\psi_* : F_X(A) \to F_X(B)$ is given by $\psi_*[\varphi] = [\underline{\psi \otimes \mathrm{id}_{\mathcal{K}}} \circ \varphi]$, when $\psi : A \to B$ and $\varphi : D \to C_b(X, A \otimes \mathcal{K})/C_0(X, A \otimes \mathcal{K})$ are $*$-homomorphisms.

Proposition 2.8. *F_X is a split-exact and stable functor.*

Proof. The split-exactness of F_X follows by combining Proposition 2.2 with Theorem 3.8 of [R]. By use of Lemma 2.6 the stability of F_X can be proved by adopting the well-known argument for the stability of the functor $[D, - \otimes \mathcal{K}]$. We leave this to the reader. □

Theorem 2.9. *Let D be a separable C^*-algebra such that (2.13) holds. For any C^*-algebra B, $[[D, B \otimes \mathcal{K}]]$ is a group and the canonical map*

$$[D, C_b([1,\infty), B \otimes \mathcal{K})/C_0([1,\infty), B \otimes \mathcal{K})] \;\to\; [[D, B \otimes \mathcal{K}]]$$

is an isomorphism.

Proof. For any C^*-algebra A and $\lambda \in [0,1]$, let $\pi_\lambda : IA \to A$ be the $*$-homomorphism $IA \ni f \mapsto f(\lambda)$. The map $[D, C_b([1,\infty), B\otimes\mathcal{K})/C_0([1,\infty), B\otimes\mathcal{K})] \to [[D, B\otimes\mathcal{K}]]$ is clearly surjective so it suffices to show that it is also injective. Thus we must show that if $\Phi : D \to C_b([1,\infty), IB\otimes\mathcal{K})/C_0([1,\infty), IB\otimes\mathcal{K})$ is a $*$-homomorphism, then $\underline{\pi_0 \otimes \mathrm{id}_{\mathcal{K}}} \circ \Phi$ and $\underline{\pi_1 \otimes \mathrm{id}_{\mathcal{K}}} \circ \Phi$ are homotopic. Equivalently, we must show that the functor $F_{[1,\infty)}$ is homotopy invariant. By Proposition 2.8 this follows from Theorem 3.2.2 of [H]. □

Corollary 2.10. *Let A and B be C^*-algebras with A separable. Let $\varphi = (\varphi_t)_{t\in[1,\infty)}$, $\psi = (\psi_t)_{t\in[1,\infty)} : SA \to B \otimes \mathcal{K}$ be asymptotic homomorphisms. Then $[\varphi] = [\psi]$ in $[[SA, B\otimes\mathcal{K}]]$ if and only if there is a family $\Phi^\lambda = (\Phi^\lambda_t)_{t\in[1,\infty)} : SA \to B \otimes \mathcal{K}$, $\lambda \in [0,1]$, of asymptotic homomorphisms such that $\Phi^0 = \varphi$, $\Phi^1 = \psi$, and*

$$[0,1] \;\ni\; \lambda \;\mapsto\; \Phi^\lambda_t(a)\;, \quad t \in [1,\infty)\;,$$

is an equicontinuous family of maps from $[0,1]$ to $B \otimes \mathcal{K}$ for all $a \in SA$.

Proof. As is wellknown SA satisfies (2.13) so Theorem 2.9 applies. □

By choosing $X = \mathbb{N}$ in Proposition 2.8 we get analogues of Theorem 2.9 (and its corollaries) for discrete asymptotic homomorphisms. To state the result in this case we denote $C_b(\mathbb{N}, A)$ by $\prod_1^\infty A$ and $C_0(\mathbb{N}, A)$ by $\oplus_1^\infty A$. Recall from [Th1] that $[[A,B]]_{\mathbb{N}}$ denotes the homotopy classes of discrete asymptotic homomorphisms $\varphi = (\varphi_n)_{n\in\mathbb{N}} : A \to B$.

Theorem 2.11. *Let D be a separable C^*-algebra such that (2.13) holds. For any C^*-algebra B, $[[D, B\otimes\mathcal{K}]]_{\mathbb{N}}$ is a group and the canonical map*

$$[D, \prod_1^\infty B \otimes \mathcal{K} / \oplus_1^\infty B \otimes \mathcal{K}] \;\to\; [[D, B\otimes\mathcal{K}]]_{\mathbb{N}}$$

is an isomorphism.

3. On absorbing extensions of a suspended C^*-algebra

Lemma 3.1. *Let $A \subseteq D$ and B be C^*-algebras, D separable, B σ-unital. Assume that there is a sequence $\{m_k\}$ in $\mathcal{M}(D)$ such that $0 \le m_k \le m_{k+1} \le 1$, $m_k D \subseteq A$ and $m_k a = a m_k$ for all $k \in \mathbb{N}$, $a \in A$, and such that $\lim_{k\to\infty} m_k a = a$ for all $a \in A$. Let $\varphi : A \to \mathcal{M}(B)$ be a completely positive contraction. For every finite set $F \subseteq A$ and every $\epsilon > 0$ there is a completely positive contraction $\psi : D \to \mathcal{M}(B)$ such that $\psi(a) - \varphi(a) \in B$ for all $a \in A$ and $\|\varphi(x) - \psi(x)\| < \epsilon$ for all $x \in F$.*

Proof. Let X be a compact subset of positive elements in A such that every element $f \in F$ has the form $f = x_1 - x_2 + i(x_3 - x_4)$ for some $x_1, x_2, x_3, x_4 \in X$, and such that the span of X is dense in A. Let $t = (8\sum_{k=1}^{\infty} k2^{-\frac{k+1}{2}})^{-1}$ and set $m_0 = 0$ and $d_k = (m_k - m_{k-1})^{\frac{1}{2}},\ k \in \mathbb{N}$. By passing to a subsequence we may assume that

$$\|d_k^2 x\| \ \leq \ t\epsilon 2^{-k},\ k \geq 2,\ x \in X\ . \tag{3.1}$$

Let $\{b_k\}$ be a countable approximate unit in B such that $\lim_{k\to\infty} \|b_k\varphi(a) - \varphi(a)b_k\| = 0$ for all $a \in A$. Set $b_0 = 0$ and $f_k = (b_k - b_{k-1})^{\frac{1}{2}},\ k \in \mathbb{N}$. By passing to a subsequence of $\{b_k\}$ we can arrange that

$$\|f_l\varphi(xd_i^2)f_l \ - \ \varphi(xd_i^2)f_l^2\| \ \leq t\epsilon 2^{-l} \tag{3.2}$$

when $x \in X$ and $i \leq l$. It follows from (3.1) and (3.2) that

$$\|f_l\varphi(d_ixd_i)f_l \ - \ \varphi(xd_i^2)f_l^2\| \ \leq \ 2t\epsilon 2^{-\frac{m}{2}}\ , \quad x \in X,\ i + l = m \geq 2\ . \tag{3.3}$$

Set $\psi_k(d) = \sum_{l+i=k+1} f_l\varphi(d_idd_i)f_l,\ d \in D$. Let $x = \sum_{k=n}^{m} \psi_k(d)$ and $y = \sum_{i,j=n}^{m} f_i\varphi(d_jdd_j)f_i$. For $d \geq 0$ we have the estimate

$$\|\sum_{k=n}^{m} \psi_k(d)b\|^4 \leq \|x\|^2\|b^*xb\|^2 \leq \|x\|^2\|b^*yb\|^2 \leq \|d\|^4\|b\|^2\|\sum_{i=n}^{m} b^*f_i^2b\| \tag{3.4}$$

for all $b \in B$. (3.4) shows that $\sum_{k=1}^{\infty} \psi_k(d)$ converges in the strict topology for all positive $d \in D$, and hence in fact for all $d \in D$. The resulting map, $\psi(d) = \sum_{k=1}^{\infty} \psi_k(d)$, is then a completely positive contraction. Set $\varphi_k(a) = \sum_{l+i=k+1} \varphi(ad_i^2)f_l^2\ ,\ a \in A$. It follows from (3.3) that

$$\|\psi_k(a) \ - \ \varphi_k(a)\| \ \leq \ 2kt\epsilon 2^{-\frac{k+1}{2}} \tag{3.5}$$

for all $k \in \mathbb{N}$ and all $a \in X$. Hence $\sum_{k=1}^{\infty} \varphi_k(a)b$ converges for all $a \in X$ and all $b \in B$. In fact, it follows from (3.3) that

$$\sum_{k=1}^{\infty} \varphi_k(a)b = \lim_{m\to\infty} \sum_{i,j=1}^{m} \varphi(ad_i^2)f_j^2b = \varphi(a)b$$

for all $a \in X,\ b \in B$. Note that for all $a \in X$

$$\|\varphi(a) - \psi(a)\| \leq \sum_{k=1}^{\infty} \|\psi_k(a) \ - \ \varphi_k(a)\| \leq \sum_{k=1}^{\infty} 2kt\epsilon 2^{-\frac{k+1}{2}} \ = \ \frac{\epsilon}{4}\ .$$

Since

$$\lim_{m\to\infty} \|\varphi(a) - \psi(a) - (\sum_{k=1}^{m} \varphi_k(a) - \psi_k(a))\| \leq \lim_{m\to\infty} \sum_{k=m+1}^{\infty} \|\varphi_k(a) - \psi_k(a)\| = 0$$

for $a \in X$ by (3.5), we see that $\varphi(a) - \psi(a)$ is the norm-limit of $\{\sum_{k=1}^{m} \varphi_k(a) - \psi_k(a)\}_{m=1}^{\infty} \subseteq B$ proving that $\varphi(a) - \psi(a) \ \in \ B$ for all $a \in X$, and hence in fact for all $a \in A$.

□

The preceding lemma is a generalization of Lemma 10 from [K2] which it reduces to when $A = D$ (except that no group action is considered), and the proof is an elaboration of Kasparovs argument. The point of the version above is that it covers the case where A is a suspended C^*-algebra, i.e. $A = SA_1$, and D is the cone of A_1.

Given a Hilbert B-module E we let $\mathcal{L}_B(E)$ denote the C^*-algebra of adjoinable operators on E. The ideal of 'compact' operators in $\mathcal{L}_B(E)$ is denoted by $\mathcal{K}_B(E)$. In the special case where $E = B$ there are well-known identifications $\mathcal{L}_B(B) = \mathcal{M}(B)$ and $\mathcal{K}_B(B) = B$ which we shall use freely. Given a C^*-algebra A we denote by A^+ the C^*-algebra obtained by adding a unit to A. Any linear completely positive contraction $\varphi : A \to \mathcal{M}(B)$ admits a unique linear extension $\varphi^+ : A^+ \to \mathcal{M}(B)$ such that $\varphi^+(1) = 1$. φ^+ is automatically a completely positive contraction, and is automatically a $*$-homomorphism when φ is.

Lemma 3.2. *Let A and B be separable C^*-algebras with B stable. If $\pi : \mathrm{cone}(A) \to \mathcal{M}(B)$ is an absorbing $*$-homomorphism then so is $\pi|_{SA} : SA \to \mathcal{M}(B)$.*

Proof. It follows from Lemma 3.1 that $\pi^+|_{(SA)^+} : (SA)^+ \to \mathcal{M}(B)$ satisfies condition 2) of Theorem 2.1 in [Th2]. □

Assuming that B is stable we can choose a sequence $S_i, \ i = 1, 2, \cdots$, of isometries in $\mathcal{M}(B)$ with orthogonal ranges such that $\sum_{i=1}^{\infty} S_i S_i^* = 1$, where the sum converges in the strict topology. If $\pi : A \to \mathcal{M}(B)$ is a $*$-homomorphism we can then form a new $*$-homomorphism $\pi^{\infty} \oplus 0^{\infty} : A \to \mathcal{M}(B)$ which is given by $(\pi^{\infty} \oplus 0^{\infty})(a) = \sum_{i=1}^{\infty} S_{2i}\pi(a)S_{2i}^*$.

Definition 3.3. A $*$-homomorphism $\pi : A \to \mathcal{M}(B)$ is *saturated* when π is unitarily equivalent to $\pi^{\infty} \oplus 0^{\infty}$.

Lemma 3.4. *Let A and B be separable C^*-algebras with B stable. Let $\pi : A \to \mathcal{M}(B)$ be a saturated and absorbing $*$-homomorphism. Let X be a compact metrizable space with base-point $x_0 \in X$ and set $C_0(X) = \{f \in C(X) : f(x_0) = 0\}$. Define $1_{C_0(X)} \otimes \pi : A \to \mathcal{M}(C_0(X) \otimes B)$ by $(1_{C_0(X)} \otimes \pi(a)f)(x) = \pi(a)f(x), \ x \in X, \ f \in C_0(X) \otimes B$. Then $1_{C_0(X)} \otimes \pi$ is absorbing.*

Proof. By Theorem 2.1 of [Th2] it suffices to consider a completely positive contraction $\varphi : A^+ \to C_0(X) \otimes B$, finite subsets $F \subseteq A^+$, $G \subseteq C_0(X) \otimes B$ and $\epsilon > 0$, and construct $L \in \mathcal{M}(C_0(X) \otimes B)$ such that $\|L^* g\| < \epsilon, \ g \in G$, and $\|\varphi(a) - L^*(1_{C_0(X)} \otimes \pi)^+(a)L\| < \epsilon$ for all $a \in F$. There is a finite set $x_1, x_2, x_3, \cdots, x_n$ in $X \backslash \{x_0\}$ and a partition of unity $\{h_i : i = 1, 2, \cdots, n\}$ in $C(X)$ such that $\|\varphi(a) - \sum_{i=1}^{n} h_i \varphi(a)(x_i)\| < \frac{\epsilon}{2}, \ a \in F$. Since π is saturated there is a sequence of isometries $T_i, \ i \in \mathbb{N}$, in $\mathcal{M}(B)$ such that $T_i^* \pi^+(A^+) T_j = \{0\}, \ i \neq j$, $T_i^* \pi^+(a) T_i = \pi^+(a)$ for all i, a and $\lim_{k \to \infty} \|T_k^* b\| = 0$ for all $b \in B$. Since $\{g(x) : x \in X, \ g \in G\}$ is a compact subset of B and π^+ is unitally absorbing, it follows from Theorem 2.1 of [Th2] that we can find elements $V_1, V_2, \cdots, V_n \in \mathcal{M}(B)$ such that $\|V_i^* \pi^+(a) V_i - \varphi(a)(x_i)\| < \frac{\epsilon}{2}, \ a \in F \ , \ i = 1, 2, \cdots, n$. Set $W_i = T_{K+i} V_i, \ i = 1, 2, \cdots, n$. If K is large enough we have that $\|W_i^* \pi^+(a) W_i - \varphi(a)(x_i)\| < \frac{\epsilon}{2}, \ a \in F$, $W_i^* \pi^+(A^+) W_j = \{0\}, \ i \neq j$, and $\|W_i^* g(x)\| < \frac{\epsilon}{n}, \ g \in G, \ x \in X$. Define the desired L by $(Lf)(x) = \sum_{i=1}^{n} \sqrt{h_i(x)} W_i f(x)$. □

In the following we will let 1_m and 0_m denote the unit and the zero element of $M_m(\mathcal{M}(B))$, respectively. We will identify $M_m(\mathcal{M}(B))$ and $\mathcal{M}(M_m(B))$.

Lemma 3.5. *Let D and B be C^*-algebras, B separable. Let $\pi : D \to \mathcal{M}(B)$ be a $*$-homomorphism and $p \in \mathcal{M}(B)$ a projection such that $p\pi(D) \subseteq B$. Assume that $F \subseteq D$ is a finite set and $\delta > 0$ is such that*

$$\|\pi(a)p - p\pi(a)\| < \delta \quad , \quad a \in F \ . \tag{3.6}$$

Let $F_1 \subseteq D$ and $G \subseteq B$ be finite sets. Let $0 \leq z \leq 1$ be a strictly positive element in $(1-p)B(1-p)$ and let $\epsilon_1,\ \epsilon_2 \in]0,1[$ be given. There is then a continuous function $g : [0,1] \to [0,1]$ such that g is zero in a neighbourhood of 0, $g(t) = 1,\ t \geq \epsilon_1$,

$$\sup_{t\in[0,1]} \|[\pi(d), p + g(tz)]\| \ < \ 5\delta \ , \qquad d \in F \ , \tag{3.7}$$

$$\|[\pi(d), p + g(z)]\| \ < \ \epsilon_2 \ , \qquad d \in F_1 \ , \tag{3.8}$$

and

$$\|pb + g(z)b - b\| \ < \ \epsilon_2 \ , \ b \in G \ . \tag{3.9}$$

Proof. Let Λ denote the convex set of continuous functions $g : [0,1] \to [0,1]$ such that g is zero in a neighbourhood of 0 and $g(t) = 1,\ t \geq \epsilon_1$. For each $x \in F$ define a multiplier $\tilde{x}$ of cone$((1-p)B(1-p))$ by $(\tilde{x}f)(t) \ = \ (1-p)\pi(x)(1-p)f(t)\ ,\ t \in [0,1]$, and define $\tilde{g} \in$ cone$((1-p)B(1-p))$ by $\tilde{g}(t) \ = \ g(tz)$. Then $(\tilde{g},\ g(z)),\ g \in \Lambda$, form a convex approximate unit in cone$((1-p)B(1-p)) \oplus (1-p)B(1-p)$. Since $\pi(D)p \subseteq B$ we can use the argument from the proof of the existence of quasi-central approximate units to find a $g \in \Lambda$ such that $\|[(\tilde{x}, \pi(y)), (\tilde{g}, p + g(z))]\| \ < \min\{\delta, \epsilon_2\},\ x \in F,\ y \in F_1$, and $\|pb + g(z)b - b\| < \epsilon_2,\ b \in G$. In particular (3.8) and (3.9) hold and we have that

$$\sup_{t\in[0,1]} \|[(1-p)\pi(x)(1-p), g(tz)]\| \ < \ \delta,\ x \in F \ . \tag{3.10}$$

Since $[\pi(x), g(tz)] = [(1-p)\pi(x)(1-p), g(tz)] + [(1-p)\pi(x)p, g(tz)] + [p\pi(x)(1-p), g(tz)]$, we get (3.7) by combining (3.10) with (3.6). □

Let $\mathcal{H}$ be an infinite-dimensional separable C^*-algebra. We can then define $g : [0,\infty[\to [0,2]$ by

$$g(s) \ = \ \sup\{\|[a, \sqrt{x}]\| \ : \ a, x \in \mathcal{B}(\mathcal{H}),\ \|a\| \leq 1,\ 0 \leq x \leq 1,\ \|[a,x]\| \leq s\} \ .$$

By the lemma on page 332 of [Ar], g is continuous at 0, i.e. $\lim_{s\to 0} g(s) \ = \ 0$. g will feature in the next lemma.

Lemma 3.6. *Let D and B be separable C^*-algebras with D contractible. Let $\varphi_t : D \to D,\ t \in [0,1]$, be a homotopy of endomorphisms of D such that $\varphi_0 = \mathrm{id}$ and $\varphi_1 = 0$. Let $F_0 \subseteq F_1 \subseteq D$ and $G_1 \subseteq B$ be finite subsets. Let $\pi : D \to \mathcal{M}(B)$ be a $*$-homomorphism and $p \in \mathcal{M}(B)$ a projection such that $p\pi(D) \subseteq B$ and $\|p\pi(\varphi_t(a)) - \pi(\varphi_t(a))p\| < \kappa,\ a \in F_0,\ t \in [0,1]$, for some $\kappa > 0$.*

For any $\epsilon > 0$ there is then a $n \in \mathbb{N}$, a $*$-homomorphism $\pi_1 : D \to \mathcal{M}(M_n(B))$ and a continuous path $p_t,\ t \in [0,1]$, of elements $p_t \in \mathcal{M}(M_{n+1}(B))$ such that

1) $0 \leq p_t \leq 1,\ t \in [0,1]$,
2) $(p_t^2 - p_t)\begin{pmatrix} \pi(a) & \\ & \pi_1(a) \end{pmatrix} \ = \ 0 \ , \quad a \in D,\ t \in [0,1]$,
3) $p_t \begin{pmatrix} \pi(a) & \\ & \pi_1(a) \end{pmatrix} \ \in \ M_{n+1}(B) \ , \quad a \in D,\ t \in [0,1]$,
4) $\|p_t \begin{pmatrix} \pi(a) & \\ & \pi_1(a) \end{pmatrix} \ - \ \begin{pmatrix} \pi(a) & \\ & \pi_1(a) \end{pmatrix} p_t\| \ \leq \ 6g(20\kappa) + 3\kappa \ , \quad a \in F_0,\ t \in [0,1]$,
5) $\begin{pmatrix} p & \\ & 0_n \end{pmatrix} \ \leq \ p_t,\ t \in [0,1]$,
6) $\|p_1 \begin{pmatrix} \pi(\varphi_t(a)) & \\ & \pi_1(\varphi_t(a)) \end{pmatrix} - \begin{pmatrix} \pi(\varphi_t(a)) & \\ & \pi_1(\varphi_t(a)) \end{pmatrix} p_1\| \leq \epsilon \ , \quad a \in F_1,\ t \in [0,1]$,
7) $\|p_1 \begin{pmatrix} b & \\ & 0_n \end{pmatrix} \ - \ \begin{pmatrix} b & \\ & 0_n \end{pmatrix}\| < \epsilon \ , \ b \in G_1$,
8) $p_1 = p_1^2,\ p_0 = p$.

Proof. The proof is an elaboration of Voiculescus proof of Proposition 3 in [V]. Let $\delta > 0$ be so small that $6g(4\delta) + 3\delta < \frac{\epsilon}{3}$, $\delta < \kappa$ and $\delta + \sqrt{\|b\|\delta} < \epsilon$ for all $b \in G_1$. Choose first a finite $\frac{\epsilon}{3}$-dense subset F of $\{\varphi_t(a) : t \in [0,1],\ a \in F_1\}$, and then a n so large that $t, s \in [0,1]$, $|s-t| \leq 1/n \ \Rightarrow\ \|\varphi_t(a) - \varphi_s(a)\| < \delta$, $a \in F$. Let $0 \leq z \leq 1$ be a strictly positive element in $(1-p)B(1-p)$. It follows from Lemma 3.5 that there are continuous functions $g_i : [0,1] \to [0,1]$, $i = 0, 1, \cdots, n-1$, which are all zero in a neighbourhood of 0 such that $g_j g_{j-1} = g_{j-1}$, $j = 1, 2, \cdots, n-1$, and such that the elements $x_j = p + g_j(z)$ and $x_j^t = p + g_j(tz)$ satisfy that

$$\|x_j \pi \circ \varphi_{\frac{j}{n}}(a) - \pi \circ \varphi_{\frac{j}{n}}(a) x_j\| \ \leq\ \delta \ , \tag{3.11}$$

$j = 0, 1, 2, \cdots, n-1$, $a \in F$, $\|x_0 b - b\| \leq \delta$, $b \in G_1$, and

$$\|x_j^t \pi \circ \varphi_{\frac{j}{n}}(a) - \pi \circ \varphi_{\frac{j}{n}}(a) x_j^t\| \ <\ 5\kappa \ , \tag{3.12}$$

$j = 0, 1, 2, \cdots, n-1$, $a \in F_0$, $t \in [0,1]$. Set $\pi_1 = \operatorname{diag}(\pi \circ \varphi_{\frac{1}{n}}, \pi \circ \varphi_{\frac{2}{n}}, \cdots, \pi \circ \varphi_1)$ and

$$p_t \ =\ \begin{pmatrix} p & & \\ & 0_{n-1} & \\ & & 2t(1_1 - p) \end{pmatrix} \quad , \quad t \in [0, \frac{1}{2}] \ .$$

Then 1)-5) hold trivially for $t \in [0, \frac{1}{2}]$. Note that $x_i^t x_{i-1}^t = x_{i-1}^t$, $i = 1, \cdots, n-1$. Set $X_t^0 = x_0^{2t-1}$, $X_t^j \ =\ x_j^{2t-1} - x_{j-1}^{2t-1}$, $j = 1, 2, \cdots, n-1$, and $X_t^n = 1_1 - x_{n-1}^{2t-1}$, $t \in [\frac{1}{2}, 1]$. Define $T_t \in \ \mathcal{M}(M_{n+1}(B))$, $t \in [\frac{1}{2}, 1]$, by

$$T_t \ =\ \begin{pmatrix} \sqrt{X_t^0} & 0 & \ldots & 0 \\ \sqrt{X_t^1} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{X_t^n} & 0 & \ldots & 0 \end{pmatrix} \ .$$

Then $T_t T_t^*$ is a projection since $T_t^* T_t$ clearly is. Since $T_{\frac{1}{2}} T_{\frac{1}{2}}^* \ =\ p_{\frac{1}{2}}$ we can extend p_t, $t \in [0, \frac{1}{2}]$, to a continuous path in $\mathcal{M}(M_{n+1}(B))$ by setting $p_t \ =\ T_t T_t^*$, $t \in [\frac{1}{2}, 1]$. Then 1) and 2) clearly hold and 3) follows from the observation that

$$\begin{pmatrix} \pi(a) & \\ & \pi_1(a) \end{pmatrix} T_t \ \subseteq\ M_{n+1}(B) \quad , \quad a \in D, \ t \in [\frac{1}{2}, 1] \ .$$

It follows from (3.11) and (3.12), by using that $T_t T_t^*$ is tri-diagonal as in the proof of Proposition 3 in [V], that

$$\|[p_1, \begin{pmatrix} \pi(a) & \\ & \pi_1(a) \end{pmatrix}]\| \leq 6g(4\delta) + 3\delta \leq \frac{\epsilon}{3}, \ a \in F \ ,$$

and

$$\|[p_t, \begin{pmatrix} \pi(a) & \\ & \pi_1(a) \end{pmatrix}]\| \ \leq\ 6g(20\kappa) + 3\kappa \ , \ a \in F_0 \ , \ t \in [\frac{1}{2}, 1] \ ,$$

i.e. 4) and 6) hold. 5) is trivial when $t \in [0, \frac{1}{2}]$ and for $t > \frac{1}{2}$ it follows from the observation that

$$\begin{pmatrix} p & \\ & 0_n \end{pmatrix} T_t \ =\ \begin{pmatrix} p & \\ & 0_n \end{pmatrix} \ , \quad \begin{pmatrix} p & \\ & 0_n \end{pmatrix} T_t^* \ =\ \begin{pmatrix} p & \\ & 0_n \end{pmatrix} \ .$$

It is straightforward to check that $\|p_1 \begin{pmatrix} b & \\ & 0_n \end{pmatrix} - \begin{pmatrix} b & \\ & 0_n \end{pmatrix}\| \leq \|X_1^0 b - b + \sqrt{X_1^1}\sqrt{X_1^0} b\| \leq \delta + \sqrt{\|b\|\delta}$ when $b \in G_1$, and 7) holds. 8) is trivial. □

Theorem 3.7. *Let A and B be separable C^*-algebras, B stable. There exists a saturated and absorbing $*$-homomorphism $\pi : \mathrm{cone}(A) \to \mathcal{M}(B)$ such that also $\pi|_{SA} : SA \to \mathcal{M}(B)$ is saturated and absorbing, and a continuous path $p_t,\ t \in [0,\infty)$, of elements in $\mathcal{M}(B)$ such that*

1) $0 \leq p_t \leq 1, \quad t \in [0,\infty)$,
2) $p_t\pi(\mathrm{cone}(A)) \subseteq B\ , \quad t \in [0,\infty)$,
3) $(p_t^2 - p_t)\pi(\mathrm{cone}(A)) \ = \ \{0\}\ , \quad t \in [0,\infty)$,
4) $\lim_{t\to\infty} p_t b \ = \ b\ , \quad b \in B$,
5) $\lim_{t\to\infty} \|p_t\pi(a) - \pi(a)p_t\| \ = \ 0\ , \quad a \in \mathrm{cone}(A)$,
6) $p_0 = 0,\ p_n^2 = p_n,\ n = 1,2,3,\cdots$.

Proof. By [Th2] and Lemma 3.2 there is an absorbing $*$-homomorphism $SA \to \mathcal{M}(B)$ which is the restriction of an absorbing $*$-homomorphism $\Theta : \mathrm{cone}(A) \to \mathcal{M}(B)$. Let $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$ and $G_1 \subseteq G_2 \subseteq G_3 \subseteq \cdots$ be sequences of finite sets with dense union in $\mathrm{cone}(A)$ and B, respectively. By using Lemma 3.6 we can construct a sequence $1 = n_0 < n_1 < n_2 < \cdots$ of natural numbers, paths $p_i(t),\ t \in [i, i+1]$, in $M_{n_i}(\mathcal{M}(B)),\ i = 0,1,2,\cdots$, and $*$-homomorphisms $\widetilde{\pi}_i\ :\ \mathrm{cone}(A) \to M_{n_i - n_{i-1}}(\mathcal{M}(B)),\ i = 1,2,\cdots$, such that $\pi_0 = \Theta$ and $\pi_i = \pi_{i-1} \oplus \widetilde{\pi}_i\ :\ \mathrm{cone}(A) \to M_{n_i}(\mathcal{M}(B)),\ i = 1,2,\cdots$, satisfy

1) $0 \leq p_i(t) \leq 1,\ t \in [i,i+1],\ i = 0,1,2,\cdots$,
2) $\|p_i(t)\pi_i(a) \ - \ \pi_i(a)p_i(t)\| \ \leq \ \frac{1}{i},\ a \in F_i\ ,\ t \in [i,i+1],\ i = 0,1,2,\cdots$,
3) $p_i(t)\pi_i(\mathrm{cone}(A)) \subseteq M_{n_i}(B),\ t \in [i,i+1],\ i = 1,2,\cdots$,
4) $\left\|p_{i+1}(t)\begin{pmatrix} b & \\ & 0_{n_i - n_{i-1}}\end{pmatrix} - \begin{pmatrix} b & \\ & 0_{n_i - n_{i-1}}\end{pmatrix}\right\| \ \leq \ \frac{1}{i}$ when all the entries of $b \in M_{n_{i-1}}(B)$ come from $G_i,\ t \in [i,i+1],\ i = 1,2,3,\cdots$,
5) $(p_i(t)^2 - p_i(t))\pi_i(\mathrm{cone}(A)) \ = \ \{0\}\ ,\ t \in [i,i+1],\ i = 0,1,2,\cdots$,
6) $p_i(i) = p_i(i)^2 = \begin{pmatrix} p_{i-1}(i) & 0 \\ 0 & 0_{n_i - n_{i-1}}\end{pmatrix},\ i = 1,2,3,\cdots$,

and $p_0 = 0$. Note that we can arrange that $\widetilde{\pi}_i$ has the form $\widetilde{\pi}_i = \pi_{i-1} \oplus 0 \oplus \varphi_i$ for some $*$-homomorphism $\varphi_i : \mathrm{cone}(A) \to M_{n_i - 2n_{i-1} - 1}(\mathcal{M}(B))$. Now define φ' : $\mathrm{cone}(A) \to \mathcal{L}_B(l_2(B))$ by $\varphi'(d) \ = \ \mathrm{diag}(\Theta(d), \widetilde{\pi}_1(d), \widetilde{\pi}_2(d), \widetilde{\pi}_3(d), \cdots)$, and set

$$p'_t \ = \begin{pmatrix} p_i(t) & \\ & 0_\infty \end{pmatrix}\ , \quad t \in [i, i+1]\ ,\ i = 0,1,2,\cdots .$$

φ' is unitarily equivalent to a $*$-homomorphism $\pi\ :\ \mathrm{cone}(A) \to \mathcal{M}(B)$ since $l_2(B) \simeq B$ as Hilbert B-modules. Note that both π and $\pi|_{SA} : SA \to \mathcal{M}(B)$ are absorbing because Θ has these properties. Furthermore both π and $\pi|_{SA}$ are saturated since each π_i as well as 0 occur as direct summands in $\widetilde{\pi}_k$ for infinitely many k's. Via the isomorphism $l_2(B) \simeq B$, p' becomes a path $p_t,\ t \in [0,\infty)$, in $\mathcal{M}(B)$ which satisfy 1)-6) in the statement of the theorem. □

Corollary 3.8. *Let $\Theta : SA \to \mathcal{M}(B)$ be an absorbing $*$-homomorphism. It follows that there is a sequence $\{q_n\}$ of projections in $\mathcal{M}(B)$ such that*

1) $q_n\Theta(SA) \subseteq B,\ n \in \mathbb{N}$,
2) $\lim_{n\to\infty} q_n\Theta(a) - \Theta(a)q_n \ = \ 0, \quad a \in SA$,
3) $\lim_{n\to\infty} q_n b = b, \quad b \in B$.

Proof. By Theorem 3.7 there is an absorbing $*$-homomorphism $\pi : SA \to \mathcal{M}(B)$ and a sequence $\{q'_n\}$ of projections in $\mathcal{M}(B)$ which satisfy 1)-3) relative to π. But

Θ is also absorbing so there is a unitary $U \in \mathcal{M}(B)$ such that $U\pi(a)U^* - \Theta(a) \in B$ for all $a \in SA$. Set $q_n = Uq_n'U^*$. □

4. Homotopy invariance

Let A and B be separable C^*-algebras, B stable. By Theorem 3.7 there is an absorbing and saturated $*$-homomorphism π : cone$(A) \to \mathcal{M}(B)$ such that $\pi|_{SA} : SA \to \mathcal{M}(B)$ is also absorbing and saturated, and a continuous path p_t, $t \in [0, \infty)$, in $\mathcal{M}(B)$ such that 1)-6) of Theorem 3.7 hold. We can then define a completely positive asymptotic homomorphism $\lambda = (\lambda_t)_{t\in[1,\infty)}$: cone$(A) \to B$ by $\lambda_t(a) = p_t\pi(a)p_t$. This asymptotic homomorphism will feature in the following theorem.

Theorem 4.1. *Let A and B be separable C^*-algebras, B stable. Let $\varphi = (\varphi_t)_{t\in[1,\infty)}$, $\psi = (\psi_t)_{t\in[1,\infty)} : SA \to B$ be completely positive asymptotic homomorphisms. Then the following are equivalent :*

1) $[\varphi] = [\psi]$ *in* $[[SA, B]]_{cp}$.
2) *There is a completely positive asymptotic homomorphism $\mu = (\mu_t)_{t\in[1,\infty)} : SA \to B$ and a strictly continuous path $\{U_t\}_{t\in[1,\infty)}$ of unitaries in $\mathcal{M}(M_2(B))$ such that*
$$\lim_{t\to\infty} U_t \begin{pmatrix} \varphi_t(a) & \\ & \mu_t(a) \end{pmatrix} U_t^* - \begin{pmatrix} \psi_t(a) & \\ & \mu_t(a) \end{pmatrix} = 0$$
for all $a \in SA$.
3) *There is a norm-continuous path $\{S_t\}_{t\in[1,\infty)}$ of unitaries in $M_2(B)^+$ and an increasing continuous function $r : [1, \infty) \to [1, \infty)$ with $\lim_{t\to\infty} r(t) = \infty$ such that*
$$\lim_{t\to\infty} S_t \begin{pmatrix} \varphi_t(a) & \\ & \lambda_{r(t)}(a) \end{pmatrix} S_t^* - \begin{pmatrix} \psi_t(a) & \\ & \lambda_{r(t)}(a) \end{pmatrix} = 0$$
for all $a \in SA$.

Proof. Since 3) $\Rightarrow$ 2) is trivial it suffices to prove 1) $\Rightarrow$ 3) and 2) $\Rightarrow$ 1). First 1) $\Rightarrow$ 3) : Define $\hat{\varphi}$, $\hat{\psi}$: $SA \to \mathcal{M}(C_0(0,\infty) \otimes B)$ by
$$(\hat{\varphi}(a)f)(t) = \begin{cases} \varphi_t(a)f(t) , & t \in (1, \infty) \\ t\varphi_1(a)f(t), & t \in (0, 1] \end{cases} , \quad f \in C_0(1, \infty) \otimes B ,$$
and similarly for $\hat{\psi}$. Let $q : \mathcal{M}(C_0(0,\infty)\otimes B) \to \mathcal{M}(C_0(0,\infty)\otimes B)/C_0(0,\infty)\otimes B$ be the quotient map. Then $q\circ\hat{\varphi}$ and $q\circ\hat{\psi}$ define invertible (or semi-split) extensions of SA by $C_0(0,\infty)\otimes B$ which define the same element of $\mathrm{Ext}^{-1}(SA, C_0(0,\infty)\otimes B)$ since φ and ψ are homotopic as completely positive asymptotic homomorphisms. Such a homotopy gives namely rise to a diagram of semi-split extensions as in Theorem 3.3.14 of [K-JT]. Set $\tilde{\pi} = 1_{C_0(0,\infty)} \otimes \pi$, cf. Lemma 3.4. Since $[q \circ \hat{\varphi}]$ and $[q \circ \hat{\psi}]$ are equal in $\mathrm{Ext}^{-1}(SA, C_0(0,\infty) \otimes B)$ and $\tilde{\pi}$ is absorbing, it follows from Kasparovs theory that there is a unitary $U \in \mathcal{M}(C_0(0,\infty) \otimes M_2(B))$ such that
$$U \begin{pmatrix} \hat{\varphi}(a) & \\ & \tilde{\pi}(a) \end{pmatrix} U^* - \begin{pmatrix} \hat{\psi}(a) & \\ & \tilde{\pi}(a) \end{pmatrix} \in C_0(0,\infty) \otimes M_2(B) \tag{4.1}$$
for all $a \in SA$. U defines a strictly continuous path, $\{U_t\}_{t\in(0,\infty)}$, of unitaries in $\mathcal{M}(M_2(B))$ such that
$$\lim_{t\to\infty} U_t \begin{pmatrix} \varphi_t(a) & \\ & \pi(a) \end{pmatrix} U_t^* - \begin{pmatrix} \psi_t(a) & \\ & \pi(a) \end{pmatrix} = 0$$

for all $a \in SA$. Let $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$ be a sequence of finite subsets with dense union in SA and fix $n \in \mathbb{N}$ for a while. Then $U_t,\ t \in (0,n]$, defines a unitary W_n in $\mathcal{M}(C_0(0,n] \otimes M_2(B))$. Consider $\tilde{\pi}$ as a $*$-homomorphism $SA \to \mathcal{M}(C_0(0,n] \otimes B)$ in the obvious way and observe that (4.1) implies that

$$W_n \begin{pmatrix} 0 & \\ & \tilde{\pi}(a) \end{pmatrix} W_n^* - \begin{pmatrix} 0 & \\ & \tilde{\pi}(a) \end{pmatrix} \in C_0(0,n] \otimes M_2(B)$$

for all $a \in SA$. Hence $\left(\begin{pmatrix} 0 & \\ & \tilde{\pi} \end{pmatrix}, \begin{pmatrix} 0 & \\ & \tilde{\pi} \end{pmatrix}, W_n\right)$ defines an element of $KK(SA, C_0(0,n] \otimes B)$ in the Cuntz-Higson picture. But $KK(SA, C_0(0,n] \otimes B) = 0$ because $C_0(0,n] \otimes B$ is contractible. Since $\mu = \begin{pmatrix} 0 & \\ & \tilde{\pi} \end{pmatrix}$ is absorbing it follows from the general Paschke-Valette-Skandalis duality theorem, Theorem 3.2 of [Th2], that there is a m and a continuous path of unitaries in

$$\{x \in \mathcal{M}(C_0(0,n] \otimes M_{2m}(B)) \ : \ [x, \mu^m(a)] \in C_0(0,n] \otimes M_{2m}(B) \ , \ a \in SA\}$$

connecting $\begin{pmatrix} W_n & \\ & 1_{2m-2} \end{pmatrix}$ to a unitary v of the form $v = 1_{2m} + z$ where $z\mu^m(SA) \subseteq C_0(0,n] \otimes M_{2m}(B)$. Here $\mu^m(a) = \operatorname{diag}(\mu(a), \mu(a), \cdots, \mu(a))$ where $\mu(a)$ is repeated m times. But $\tilde{\pi}$ is saturated since π is and hence μ^{m-1} is unitarily equivalent to μ. There is therefore an isomorphism $\chi : \mathcal{M}(C_0(0,n] \otimes M_{2m}(B)) \to \mathcal{M}(C_0(0,n] \otimes M_4(B))$ such that $\chi(C_0(0,n] \otimes M_{2m}(B)) = C_0(0,n] \otimes M_4(B)$, $\chi \circ \mu^m = \mu^2$ and $\chi \begin{pmatrix} W_n & \\ & 1_{2m-2} \end{pmatrix} = \begin{pmatrix} W_n & \\ & 1_2 \end{pmatrix}$. We may therefore assume that $m = 2$. Since $z\mu^2(SA) \subseteq C_0(0,n] \otimes M_4(B)$, the standard homotopy of unitaries connecting $\begin{pmatrix} v & \\ & v^* \end{pmatrix}$ to 1_8 is contained in the C^*-algebra

$$\mathcal{A} = \{x \in \mathcal{M}(C_0(0,n] \otimes M_8(B)) \ : \ [x, \begin{pmatrix} \mu^2(a) & \\ & 0_4 \end{pmatrix}] \in C_0(0,n] \otimes M_8(B),\ a \in SA\} \ .$$

In combination with the first path of unitaries this gives us a path of unitaries in $\mathcal{A}$ connecting $V_n = \operatorname{diag}(W_n, 1_2, W_n^*, 1_2)$ to 1_8. By composing with the restriction map $C_0(0,n] \otimes M_8(B) \to C[1,n] \otimes M_8(B)$ we can consider $\Psi = \mu^2 \oplus 0_4 : SA \to \mathcal{M}(C[1,n] \otimes M_8(B))$. Set

$$D = \{x \in \mathcal{M}(M_8(C[1,n] \otimes B)) \ : \ [x, \Psi(a)] \in M_8(C[1,n] \otimes B) \ , \ a \in SA\} \ .$$

Let E_n be the unital C^*-subalgebra of $\mathcal{M}(M_8(C[1,n] \otimes B))$ generated by $C[1,n]$, $\Psi(SA)$ and $M_8(C[1,n] \otimes B)$. Set $\Phi = (0 \oplus \pi)^2 \oplus 0_4 : SA \to \mathcal{M}(M_8(B))$. Then $E_n = C[1,n] \otimes E$ where E is the unital C^*-subalgebra of $\mathcal{M}(M_8(B))$ generated by 1_8, $\Phi(SA)$ and $M_8(B)$. Note that we can consider V_n as an element of D. The unitary path we have constructed shows that V_n is homotopic to 1_8 in the unitary group of D. Conjugation by V_n defines an automorphism α_n of E_n such that $\alpha_n(\operatorname{diag}(b, \tilde{\pi}(a), 0_1, \tilde{\pi}(a), 0_4)) = V_n \operatorname{diag}(b, \tilde{\pi}(a), 0_1, \tilde{\pi}(a), 0_4) V_n^*$ for all $b \in C[1,n] \otimes B,\ a \in SA$. The path of unitaries connecting V_n to 1_8 in the unitary group of D gives us a path of automorphisms of E_n connecting α_n to id_{E_n}. The automorphisms in the path act trivially on $C[0,1] \subseteq E_n$ so the path is given by a map $L : [1,n] \times [0,1] \to \operatorname{Aut} E$ such that $L(t,0) = \mathrm{id}$ and $L(t,1) = \operatorname{Ad} S_t$, where $S_t = \operatorname{diag}(U_t, 1_2, U_t^*, 1_2)$, for all $t \in [1,n]$. L is jointly continuous with respect to the topology of norm-convergence on elements of E and $s \mapsto L(t,s),\ t \in [1,n]$, is equicontinuous in the norm-topology on $\operatorname{Aut} E$. In the same way we find a map $L_0 : [1,n+1] \times [0,1] \to \operatorname{Aut} E$ with the same continuity properties such that $L_0(t,0) = \mathrm{id}$ and $L_0(t,1) = \operatorname{Ad} S_t$ for all $t \in [1,n+1]$. We can then extend L to a map $L : [1,n+1] \times [0,1] \to \operatorname{Aut} E$ by setting $L(t,s) = L_0(t,s) \circ L_0(n,s)^{-1} \circ L(n,s)$ when $t \in [n,n+1]$. The extended map is continuous in the same way as L. Proceding inductively in this way we obtain a map $L : [1,\infty) \times [0,1] \to \operatorname{Aut} E$ such that $L(t,0) = \mathrm{id}$ and $L(t,1) = \operatorname{Ad} S_t$

for all $t \in [1, \infty)$. On compact subsets of $[1, \infty)$ the continuity properties remain unchanged. Choose continuous functions $f_i : [1, \infty) \to [0, 1]$, $i = 0, 1, 2, \cdots$, such that

1) $0 = f_0(t) \leq f_i(t) \leq f_{i+1}(t)$, $t \in [1, \infty)$, $i \in \mathbb{N}$,
2) for all $n \in \mathbb{N}$ there is a $N_n \in \mathbb{N}$ such that $f_i(t) = 1$ for all $t \in [1, n]$, $i \geq N_n$,
3) $\|L(t, f_i(t)) - L(t, f_{i+1}(t))\| < \frac{1}{2}$, $t \in [1, \infty)$, $i \in \mathbb{N}$.

Set $\delta(t, i) = \mathrm{Log}[L(t, f_{i-1}(t))^{-1} \circ L(t, f_i(t))]$, $i = 1, 2, 3, \cdots$, and note that by a result of Kadison and Ringrose, [KR], or 8.7.7 of [Pe] , $\delta(t, i)$ is a derivation of E for all (t, i). Note that $\|\delta(t, i)\|$ is uniformly bounded in t and i. We find that $L(t, 1) = e^{\delta(t,1)} \circ e^{\delta(t,2)} \circ e^{\delta(t,3)} \circ \cdots$ for all t, where there on compact subsets of $[1, \infty)$ only occur finitely many non-trivial automorphisms in the composition. For each n, i, define a bounded derivation $\delta_{n,i}$ of $C[1, n] \otimes E$ by setting $\delta_{n,i}(f)(t) = \delta(t, i)(f(t))$, $f \in C[1, n] \otimes E$. For $a \in SA$, define $\tilde{a} \in C_b[1, \infty) \otimes E$ by $\tilde{a}(t) = \mathrm{diag}(\varphi_t(a), \tilde{\pi}(a), 0_1, \tilde{\pi}(a), 0_4)$. Define $F'_n \subseteq C[1, n] \otimes E$ by $F'_n = \{\tilde{a}|_{[1,n]} : a \in F_n\}$. Let $\{\epsilon_n\}$ be a decreasing sequence of positive numbers. By applying Lemma 8.6.12 of [Pe] to the $\delta_{n,i}$ we find elements $h^n_1, h^n_2, h^n_3, \cdots$ in $C[1, n] \otimes E$ such that $\|\delta(t, i)(\tilde{a}(t)) - \sqrt{-1}[h^n_i(t), \tilde{a}(t)]\| < \epsilon_n$, $t \in [1, n]$, $a \in F'_n$, for all i and such that, for each n, $h^n_k \neq 0$ for only finitely many k's. Choose functions $f^n, g^n : [n - \frac{1}{2}, n + \frac{1}{2}] \to [0, 1]$ such that $f^n(n - \frac{1}{2}) = 1$, $g^n(n + \frac{1}{2}) = 1$, f is supported on $[n - \frac{1}{2}, n]$ and $f^n(t) + g^n(t) = 1$, $t \in [n - \frac{1}{2}, n + \frac{1}{2}]$. Define $h_i : [1, \infty) \to E$ such that $h_i(t) = f^n(t) h^n_i(t) + g^n(t) h^{n+1}_i(t)$, $t \in [n - \frac{1}{2}, n + \frac{1}{2}]$. It follows that $\|\delta(t, i)(\tilde{a}(t)) - \sqrt{-1}[h_i(t), \tilde{a}(t)]\| < \epsilon_n$, $t \geq n + \frac{1}{2}$, $a \in F_n$. Define $W_t \in E$ by $W_t = e^{ih_1(t)} e^{ih_2(t)} e^{ih_3(t)} \cdots$ for all $t \in [1, \infty)$. Again there is only finitely many non-trivial terms for t in a compact subset of $[1, \infty)$. If the ϵ_n's are chosen small enough this will give os a norm-continuous path $\{W_t\}_{t \in [1,\infty)}$ of unitaries in E such that

$$\lim_{t \to \infty} W_t \begin{pmatrix} \varphi_t(a) & & & & \\ & \tilde{\pi}(a) & & & \\ & & 0_1 & & \\ & & & \tilde{\pi}(a) & \\ & & & & 0_4 \end{pmatrix} W_t^* - \begin{pmatrix} \psi_t(a) & & & & \\ & \tilde{\pi}(a) & & & \\ & & 0_1 & & \\ & & & \tilde{\pi}(a) & \\ & & & & 0_4 \end{pmatrix} = 0$$

for all $a \in SA$. Being saturated π is unitarily equivalent to $\pi \oplus 0_1 \oplus \pi \oplus 0_4$, so there is a unitary $T \in \mathcal{L}_B(B^7, B)$ such that $T \,\mathrm{diag}(\pi(a), 0_1, \pi(a), 0_4) T^* = \pi(a)$, $a \in SA$. Set $W = 1 \oplus T \in \mathcal{L}_B(B^8, B \oplus B)$. Then $\mathrm{Ad}\, W(E) = E_0$ where E_0 is the C^*-subalgebra of $\mathcal{M}(M_2(B))$ generated by 1_2, $M_2(B)$ and $\begin{pmatrix} 0 & \\ & \pi(SA) \end{pmatrix}$. Set $V_t = W W_t W^* \in E_0$. Note that there is a unique decomposition $V_t = \lambda_t 1_2 - a_t$, where $\lambda_t \in \mathbb{C}$, $|\lambda_t| = 1$, and $a_t \in \begin{pmatrix} 0 & \\ & \pi(SA) \end{pmatrix} + M_2(B)$. Set

$$X_{s,t} = \lambda_t^{-1} \begin{pmatrix} 1 & \\ & p_s \end{pmatrix} V_t \begin{pmatrix} 1 & \\ & p_s \end{pmatrix} + \begin{pmatrix} 0 & \\ & 1 - p_s^2 \end{pmatrix} .$$

Because $t \mapsto V_t$ is norm-continuous, the properties of $\{p_t\}$, specifically 1), 3), 4) and 5) of Theorem 3.7, imply that we can choose $m_n \in [1, \infty)$ such that

$$\sup_{t \in [1,n]} \left\| \left[\begin{pmatrix} 1 & \\ & p_s \end{pmatrix} , V_t \right] \right\| < \frac{1}{n}$$

and

$$\sup_{t \in [1,n]} \left(\|X_{s,t} X_{s,t}^* - 1_2\| + \|X_{s,t}^* X_{s,t} - 1_2\| \right) < \frac{1}{n}$$

for all $s \geq m_n$. We can arrange that $m_n < m_{n+1}$ for all $n \in \mathbb{N}$. Define a continuous function $r : [1, \infty) \to [1, \infty)$ such that $r(n) = m_{n+1}$ and r is linear between n and $n + 1$ for all n. Then

$$\begin{aligned}
&\|X_{r(t),t}\begin{pmatrix}\varphi_t(a) & \\ & p_{r(t)}\pi(a)p_{r(t)}\end{pmatrix}X^*_{r(t),t} - \begin{pmatrix}\psi_t(a) & \\ & p_{r(t)}\pi(a)p_{r(t)}\end{pmatrix}\| \\
&\leq \|V_t\begin{pmatrix}\varphi_t(a) & \\ & p_{r(t)}\pi(a)p_{r(t)}\end{pmatrix}V_t^* - \begin{pmatrix}\psi_t(a) & \\ & p_{r(t)}\pi(a)p_{r(t)}\end{pmatrix}\| \\
&\leq 2\|a\|\|[\begin{pmatrix}1 & \\ & p_{r(t)}\end{pmatrix}, V_t]\| + \|V_t\begin{pmatrix}\varphi_t(a) & \\ & \pi(a)\end{pmatrix}V_t^* - \begin{pmatrix}\psi_t(a) & \\ & \pi(a)\end{pmatrix}\|
\end{aligned}$$

tends to zero as t tends to infinity for all $a \in SA$. It follows that $X_{r(t),t}(X^*_{r(t),t}X_{r(t),t})^{-\frac{1}{2}}$ is a norm-continuous path $\{S_t\}_{t\in[1,\infty)}$ of unitaries in $M_2(B)^+$ with the desired properties.

2) $\Rightarrow$ 1) : By introducing the composition product $\bullet$ for the homotopy classes of completely positive asymptotic homomorphisms, 2) implies that $[\mathcal{U}] \bullet ([\varphi] + [\mu]) = [\psi] + [\mu]$, where $\mathcal{U} : B \to B$ is the asymptotic homomorphism $\mathcal{U}_t(b) = U_t b U_t^*$. It suffices therefore to show that $[\mathcal{U}] = [\mathrm{id}_B]$ in $[[B,B]]_{cp}$. This is done by connecting U_t to 1 via the path $V_\lambda U_t V_\lambda^* + (1 - V_\lambda V_\lambda^*)$, where V_λ is the path of isometries from Lemma 1.3.6 of [K-JT].

□

Theorem 4.2. *Let A and B be separable C^*-algebras, B stable. Let $\varphi = (\varphi_t)_{t\in[1,\infty)}$, $\psi = (\psi_t)_{t\in[1,\infty)}$: $SA \to B$ be asymptotic homomorphisms. Then the following are equivalent :*

1) *$[\varphi] = [\psi]$ in $[[SA,B]]$.*
2) *There is an asymptotic homomorphism $\nu = (\nu_t)_{t\in[1,\infty)}$: $\mathrm{cone}(A) \to B$ and a norm-continuous path $\{U_t\}_{t\in[1,\infty)}$ of unitaries in $M_2(B)^+$ such that*

$$\lim_{t\to\infty} U_t\begin{pmatrix}\varphi_t(a) & \\ & \nu_t(a)\end{pmatrix}U_t^* - \begin{pmatrix}\psi_t(a) & \\ & \nu_t(a)\end{pmatrix} = 0$$

for all $a \in SA$.

Proof. The implication 2) $\Rightarrow$ 1) is proved in the same way as the corresponding implication in the proof of Theorem 4.1. We prove 1) $\Rightarrow$ 2) : By Theorem 2.9 and Lemma 2.6 there is a separable and stable C^*-subalgebra D of $C_b([1,\infty),B)/C_0([1,\infty),B)$ such that the $*$-homomorphisms $\hat{\varphi}$, $\hat{\psi} : SA \to C_b([1,\infty),B)/C_0([1,\infty),B)$ defined from φ and ψ take values in D and are homotopic in $\mathrm{Hom}(SA,D)$. By Theorem 4.1 there is a completely positive asymptotic homomorphism $\mu : \mathrm{cone}(A) \to D$ and a norm-continuous path $\{S_t\}_{t\in[1,\infty)}$ in $M_2(D)^+$ such that

$$\lim_{t\to\infty} S_t\begin{pmatrix}\hat{\varphi}(a) & \\ & \mu_t(a)\end{pmatrix}S_t^* - \begin{pmatrix}\hat{\psi}(a) & \\ & \mu_t(a)\end{pmatrix} = 0$$

for all $a \in SA$. Let χ be a continuous right-inverse for the quotient map $C_b([1,\infty),B) \to C_b([1,\infty),B)/C_0([1,\infty),B)$. Lift S to a norm-continuous path $W = \{W_t\}$ of unitaries in $C_b([1,\infty),M_2(B)^+)$ and note that if $r : [1,\infty) \to [1,\infty)$ is a continuous and sufficiently slowly increasing function with $\lim_{t\to\infty} r(t) = \infty$ then $\nu = (\chi \circ \mu_{r(t)}(\cdot)(t))_{t\in[1,\infty)}$ is an asymptotic homomorphism $\nu : \mathrm{cone}(A) \to B$ such that

$$\lim_{t\to\infty} W_{r(t)}(t)\begin{pmatrix}\varphi_t(a) & \\ & \nu_t(a)\end{pmatrix}W_{r(t)}(t)^* - \begin{pmatrix}\psi_t(a) & \\ & \nu_t(a)\end{pmatrix} = 0$$

for all $a \in SA$. Since $t \mapsto W_{r(t)}(t)$ is norm-continuous we are done. □

Theorem 4.3. *Let A and B be separable C^*-algebras, B stable. Let $\varphi = (\varphi_n)_{n\in\mathbb{N}}$, $\psi = (\psi_n)_{n\in\mathbb{N}} : SA \to B$ be discrete asymptotic homomorphisms. Then the following are equivalent :*

1) $[\varphi] = [\psi]$ *in* $[[SA, B]]_{\mathbb{N}}$.
2) *There is a discrete asymptotic homomorphism* $\nu = (\nu_n)_{n\in\mathbb{N}} : \operatorname{cone}(A) \to B$ *and a sequence* $\{U_n\}_{n\in\mathbb{N}}$ *of unitaries in* $M_2(B)^+$ *such that*

$$\lim_{n\to\infty} U_n \begin{pmatrix} \varphi_n(a) & \\ & \nu_n(a) \end{pmatrix} U_n^* - \begin{pmatrix} \psi_n(a) & \\ & \nu_n(a) \end{pmatrix} = 0$$

for all $a \in SA$.

Proof. The implication 2) $\Rightarrow$ 1) follows as above. The proof of 1) $\Rightarrow$ 2) is the same as the proof of the same implication in the proof of Theorem 4.2, the only difference is that one works with $\prod_1^\infty B / \oplus_1^\infty B$ instead of $C_b([1,\infty), B)/C_0([1,\infty), B)$. □

5. A description of E-theory in terms of KK-theory

In this section we will use the results of the previous sections to show that

$$E(A, B) \simeq KK(A, C_b([1,\infty), B \otimes \mathcal{K})/C_0([1,\infty), B \otimes \mathcal{K}))$$

when A and B are separable C^*-algebras. Since the second variable of the KK-functor in this statement is not even σ-unital we must point out that we use the following definition regarding the KK-theory of a non-separable C^*-algebra D:

$$KK(A, D) = \lim_{T} KK(A, T) ,$$

where the limit is taken over the net of separable C^*-subalgebras T of D ordered by inclusion.

It follows from Theorem 2.11 that if two discrete asymptotic homomorphisms, $\varphi = (\varphi_n)_{n\in\mathbb{N}}$, $\psi = (\psi_n)_{n\in\mathbb{N}} : SA \to B$, define the same element in $D(A, B)$, the two $*$-homomorphisms, $\hat{\varphi}$, $\hat{\psi} : SA \to \prod_1^\infty B / \oplus_1^\infty B$, which they define are homotopic. In particular it follows that the recipe $[\varphi] \mapsto [\hat{\varphi}]$ defines a homomorphism $\Phi : [[SA, B]]_{\mathbb{N}} \to KK(SA, \prod_1^\infty B / \oplus_1^\infty B)$.

Theorem 5.1. *Let A and B be separable C^*-algebras with B is stable. Then $\Phi : [[SA, B]]_{\mathbb{N}} \to KK(SA, \prod_1^\infty B / \oplus_1^\infty B)$ is an isomorphism.*

Proof. Injectivity of Φ : If $\Phi[\varphi] = \Phi[\psi]$ it follows that there is a separable C^*-subalgebra D of $\prod_1^\infty B / \oplus_1^\infty B$ such that $\hat{\varphi}(SA) \cup \hat{\psi}(SA) \subseteq D$ and $[\hat{\varphi}] = [\hat{\psi}]$ in $KK(SA, D)$. By Lemma 2.6 we may assume that D is stable. As pointed out in [MT] it follows from [H-LT] and [DL] that $KK(SA, D) = [[SA, D]]_{cp}$. So we conclude from Theorem 4.1 that there is a sequence $\{V_n\}_{n\in\mathbb{N}}$ of unitaries in $M_2(D)^+$ and a discrete completely positive asymptotic homomorphism $\mu : SA \to D$ such that

$$\lim_{n\to\infty} V_n \begin{pmatrix} \hat{\varphi}(a) & \\ & \mu_n(a) \end{pmatrix} V_n^* - \begin{pmatrix} \hat{\psi}(a) & \\ & \mu_n(a) \end{pmatrix} = 0$$

for all $a \in SA$. As in the proof of 1) $\Rightarrow$ 2) in Theorem 4.2 we obtain a sequence of unitaries $\{U_n\}_{n\in\mathbb{N}} \subseteq M_2(B)^+$ and a discrete asymptotic homomorphism $\nu : SA \to B$ such that

$$\lim_{n\to\infty} U_n \begin{pmatrix} \varphi_n(a) & \\ & \nu_n(a) \end{pmatrix} U_n^* - \begin{pmatrix} \psi_n(a) & \\ & \nu_n(a) \end{pmatrix} = 0$$

for all $a \in SA$. Hence $[\varphi] = [\psi]$ in $[[SA, B]]_{\mathbb{N}}$ by Theorem 4.3.

Surjectivity of Φ : We must show that each element of $KK(SA, \prod_1^\infty B/\oplus_1^\infty B)$ is represented by a $*$-homomorphism. To this end it suffices by Lemma 2.6 to consider a separable stable C^*-subalgebra $D \subseteq \prod_1^\infty B/\oplus_1^\infty B$ and show that for any element $x \in KK(SA, D)$ there is a separable C^*-algebra D_1 such that $D \subseteq D_1 \subseteq \prod_1^\infty B/\oplus_1^\infty B$ and such that the image of x in $KK(SA, D_1)$ is represented by a $*$-homomorphism. To this end we use the Cuntz-Higson picture of KK-theory. There is then a pair $\varphi_1, \varphi_2 : SA \to \mathcal{M}(D)$ of $*$-homomorphisms such that $\varphi_1(a) - \varphi_2(a) \in D$ for all $a \in SA$ and such that $[\varphi_1, \varphi_2] = x$ in $KK(SA, D)$. By [Th2] there is an absorbing $*$-homomorphism $\pi : SA \to \mathcal{M}(D)$ and by adding the pair (π, π) to (φ_1, φ_2) we may assume that φ_1 is absorbing. It follows from Corollary 3.9 that there is a sequence of projections $\{e_n\} \subseteq \mathcal{M}(D)$ such that $\lim_{n\to\infty} e_n d = d,\ d \in D,\ e_n\varphi_i(SA) \subseteq D$ for all n and $\lim_{n\to\infty} e_n\varphi_i(z) - \varphi_i(z)e_n = 0$ for all $z \in SA,\ i = 1, 2$. Let $\chi : \prod_1^\infty B/\oplus_1^\infty B \to \prod_1^\infty B$ be a continuous right-inverse for the quotient map $\prod_1^\infty B \to \prod_1^\infty B/\oplus_1^\infty B$. Let $\{z_i\}$ and $\{d_i\}$ be dense sequences in SA and D, respectively. Set $g_i = \chi(d_i)$. For each $n \in \mathbb{N}$ there is a $N_n \in \mathbb{N}$ so large that

$$\|\chi(e_i\varphi_l(z_{j_1})e_i)(m) + \chi(e_i\varphi_l(z_{j_2})e_i)(m) - \chi(e_i\varphi_l(z_{j_1} + z_{j_2})e_i)(m)\| \leq \frac{1}{n}, \tag{5.1}$$

$$\begin{aligned} &\|\chi(e_i\varphi_l(z_{j_1})e_i)(m)\chi(e_i\varphi_l(z_{j_2})e_i)(m) - \chi(e_i\varphi_l(z_{j_1}z_{j_2})e_i)(m)\| \\ &\leq \|e_i\varphi_l(z_{j_1})e_i\varphi_l(z_{j_2})e_i - e_i\varphi_l(z_{j_1}z_{j_2})e_i\| + \frac{1}{n}, \end{aligned} \tag{5.2}$$

$$\|\chi(e_i\varphi_l(z_{j_1})e_i)(m)^* - \chi(e_i\varphi_l(z_{j_1}^*)e_i)(m)\| \leq \frac{1}{n}, \tag{5.3}$$

$$\begin{aligned} &\|\chi(e_i\varphi_l(z_{j_1})e_i)(m)g_k(m) - \chi(\varphi_l(z_{j_1}))(m)g_k(m)\| \\ &\leq \|e_i\varphi_l(z_{j_1})e_i d_k - \varphi_l(z_{j_1})d_k\| + \frac{1}{n}, \end{aligned} \tag{5.4}$$

$$\begin{aligned} &\|\chi(e_i\varphi_1(z_{j_1})e_i)(m) - \chi(e_i\varphi_2(z_{j_1})e_i)(m) - \chi(\varphi_1(z_{j_1}) - \varphi_2(z_{j_1}))(m)\| \\ &\leq \|e_i\varphi_1(z_{j_1})e_i - e_i\varphi_2(z_{j_1})e_i - (\varphi_1(z_{j_1}) - \varphi_2(z_{j_1}))\| + \frac{1}{n}, \end{aligned} \tag{5.5}$$

for all $i, j_1, j_2, k \in \{1, 2, \cdots, n\}$ and all $m \geq N_n,\ l = 1, 2$. We may assume that $N_n < N_{n+1}$ for all n. Now define $r : \mathbb{N} \to \mathbb{N}$ such that $r(i) = 1,\ i \leq N_1,\ r(i) = n,\ i \in \{N_n + 1, N_n + 2, \cdots, N_{n+1}\},\ n \in \mathbb{N}$. It follows from (5.1)-(5.3) that $\alpha_n^l(z) = \chi(e_{r(n)}\varphi_l(z)e_{r(n)})(n)$ defines a discrete asymptotic homomorphism $\{\alpha_n^l\}_{n\in\mathbb{N}} : SA \to B,\ l = 1, 2$. Let $\alpha_l : SA \to \prod_1^\infty B/\oplus_1^\infty B$ be the $*$-homomorphism defined from $\{\alpha_n^l\}_{n\in\mathbb{N}}$. It follows from (5.4) that $\alpha_l(z)b = \varphi_l(z)b$ for all $z \in SA,\ b \in D,\ l = 1, 2$, and from (5.5) that $\varphi_1(z) - \varphi_2(z) = \alpha_1(z) - \alpha_2(z),\ z \in SA$. By Lemma 2.6 we can find a separable stable C^*-subalgebra D_1 of $\prod_1^\infty B/\oplus_1^\infty B$ such that $D \cup \alpha_1(SA) \cup \alpha_2(SA) \subseteq D_1$. Then the image of x in $KK(SA, D_1)$ is represented by the pair (α_1, α_2). A standard homotopy argument shows that this pair defines the same element of $KK(SA, D_1)$ as the $*$-homomorphism $V_1\alpha_1(\cdot)V_1^* + V_2\alpha_2 \circ \gamma(\cdot)V_2^*$ where V_1 and V_2 are isometries in $\mathcal{M}(D_1)$ such that $V_1V_1^* + V_2V_2^* = 1$ and $\gamma : SA \to SA$ is the automorphism $\gamma(f)(t) = f(1-t),\ t \in [0, 1]$. □

By using Theorem 2.9 instead of Theorem 2.11 we may define a homomorphism $\Psi : [[SA, B]] \to KK(SA, C_b([1, \infty), B)/C_0([1, \infty), B))$ in the same way as Φ was defined above.

Theorem 5.2. *Let A and B be separable C^*-algebras with B is stable. Then $\Psi : [[SA, B]] \to KK(SA, C_b([1,\infty), B)/C_0([1,\infty), B))$ is an isomorphism.*

Proof. Note that

$$KK(SA, C_b([1,\infty), B)/C_0([1,\infty), B)) \;=\; [q(SA), C_b([1,\infty), B)/C_0([1,\infty), B)]$$

by [Cu] and that $[q(SA), C_b([1,\infty), B)/C_0([1,\infty), B)] \;=\; [[q(SA), B]]$ by Theorem 2.11 and Proposition 1.4 of [Cu]. By using Lemma 5.5 and Lemma 5.6 of [Th1] this gives us the following commuting diagram of abelian groups

$$\begin{array}{ccccccccc}
[[SA, SB]]_{\mathbb{N}} & \longrightarrow & [[SA, SB]]_{\mathbb{N}} & \longrightarrow & [[SA, B]] & \longrightarrow & [[SA, B]]_{\mathbb{N}} & \longrightarrow & [[SA, B]]_{\mathbb{N}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
[[q(SA), SB]]_{\mathbb{N}} & \longrightarrow & [[q(SA), SB]]_{\mathbb{N}} & \longrightarrow & [[q(SA), B]] & \longrightarrow & [[q(SA), B]]_{\mathbb{N}} & \longrightarrow & [[q(SA), B]]_{\mathbb{N}}
\end{array}$$

where the rows are exact and the vertical maps are induced by Φ, except the one in the middle which is induced by Ψ. Hence the result follows from Theorem 5.1 and the five lemma. □

Corollary 5.3. *Let A and B be separable C^*-algebras. There is a natural isomorphism $E(A, B) \;\simeq\; KK(A, C_b([1,\infty), B \otimes \mathcal{K})/C_0([1,\infty), B \otimes \mathcal{K}))$.*

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