ON THE COSINE-SINE FUNCTIONAL EQUATION ON GROUPS

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1. Introduction

An ambitious project is to obtain the general solution $f, g, h \in C(G)$ of the functional equation

$$\int_K f(xk \cdot y)dk = f(x)g(y) + g(x)f(y) + h(x)h(y),\ \ x, y \in G, \tag{1}$$

where G is a topological group, and $C(G)$ denotes the algebra of all continuous, complex valued functions on G. Furthermore K is a compact, transformation group, acting by automorphisms on G, and $k \cdot x$ denotes the action of $k \in K$ on $x \in G$. In particular the map $(k, x) \mapsto k \cdot x$ of $K \times G$ into G is continuous. Finally dk is the normalized Haar measure on K. This notation will be used throughout the paper.

We give the complete continuous solution to (1) for $K = \mathbf{Z}_2$ acting on a topological abelian group G. That is we solve the functional equation

$$\frac{f(x+y) + f(x+\sigma y)}{2} = f(x)g(y) + g(x)f(y) + h(x)h(y),\ \ x, y \in G, \tag{2}$$

where $\sigma : G \to G$ denotes a continuous involutive automorphism of G. Obvious examples of such automorphisms are $\sigma = I$ and $\sigma = -I$, where I denotes the identity operator. Letting σ be a reflection in a hyperplane in $G = \mathbf{R}^n$ we get an example for which $\sigma \neq \pm I$. It turns out that the solutions are certain exponential polynomials. Chung, Kanappan and Ng's paper [4] deals with the functional equation

$$f(xy) = f(x)g(y) + g(x)f(y) + h(x)h(y),\ \ x, y \in G, \tag{3}$$

that can be viewed as the case of $\sigma = I$ in (2). Our results encompass those of [4] (see Remark 3.4). For $G = \mathbf{R}$ the functional equation (2) describes involved addition formulas for trigonometric and related functions. See also [7].

The classical example $\sigma = -I$ of the equation (2) has been studied extensively for d'Alembert's functional equation ($g = f - h = 0$), the trigonometric functional equations in [5] ($h = 0$ and $g = f = ih$) and the quadratic equation ($h = g - 1 = 0$). The special case of $h = g$ turns up as part of a system of 2 functional equations in ([5]; Formula (3.6)) and in ([11]; Lemma V.3). The case of $g = 1$ is Swiataks equation (see [3] and [13]).

The general form of the solution sets for functional equations of d'Alembert's type, i.e., with the left hand side $(f(x + y) + f(x - y))/2$, can be found in Rukhin [10].

The new of the present paper is that we:

- **i:** Produce the explicit solution formulas for the special functional equations (2) in question.
- **ii:** Do it for any involutive automorphism σ, not just for $\sigma = \pm I$.
- **iii:** Take continuity into account.

We reveal part of the underlying structure in the set up by discussing the general equation (1). The results of the present paper can be compared with the ones of [4] because we formulate them in the same way. It is intriguing to see that many of the methods of [4] carry over to the more general situations (1) and (2). However, our formulas for the solutions of (2) contain certain types of functions that are absent in [4], because they vanish for $\sigma = I$. For example the 4'th order term in Proposition 3.2. So new phenomena show up.

With more than one term on the left hand side the possibility of varying signs exists. We give the complete solution of the functional equation

$$\frac{f(x+y)-f(x+\sigma y)}{2}=f(x)g(y)+g(x)f(y)+h(x)h(y),\ x,y\in G, \tag{4}$$

in Section 6.

2. Main result

The following notation will be used throughout the paper unless explicitly stated otherwise.

Notation $(G,+)$ is an abelian topological group, 0 its identity element. We let $\sigma : G \to G$ be an continuous automorphism of order 2. $\mathcal{A}(G)$ is the vector space of all continuous additive maps from G to $\mathbf{C}$ and $\mathcal{A}^{\pm}(G) := \{A \in \mathcal{A}(G) : A \circ \sigma = \pm A\}$. Furthermore $\mathcal{S}^{-}(G)$ denotes the vector space of all continuous, biadditive, symmetric maps $S : G \times G \to \mathbf{C}$ for which $S(\sigma x, y) = -S(x,y)$ for all $x, y \in G$. If $S^{-} \in \mathcal{S}^{-}(G)$ we let for brevity S^{-} also denote the function $S^{-}(x) := S^{-}(x,x), x \in G$. With $K = \mathbb{Z}_2 = (\pm 1, \cdot)$ equipped with the discrete topology, the action of K on G given by $1 \cdot x = x, \forall x \in G$ and $-1 \cdot x = \sigma x, \forall x \in G$. A *K-spherical function* is a function $\phi \in C(G)$ such that $\phi \neq 0$ and ϕ satifies $\int_K \phi(xk \cdot y)dk = \phi(x)\phi(y)$ for all $x, y \in G$, in the case $K = \mathbb{Z}_2$ the K-spherical functions are given by theorem III.1 in [12]. If f is a function on G and $k \in K$ we define the function $k \cdot f$ by $(k \cdot f)(x) := f(k^{-1} \cdot x)$ for $x \in G$. We let $\mathbf{C}^*$ denote the multiplicative group of nonzero complex numbers.

Our main result is

Theorem 2.1. *Let (f,g,h) be a continuous solution of*

$$\frac{f(x+y)+f(x+\sigma y)}{2}=f(x)g(y)+g(x)f(y)+h(x)h(y), x,y\in G. \tag{5}$$

Then f,g and h have one of the following six forms, and conversely.

(A): $f = h = 0$ *and* $g \in C(G)$.

(B):

$$\begin{Bmatrix} f \\ g \\ h \end{Bmatrix} = \begin{Bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{Bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}, \tag{6}$$

where ϕ_1,ϕ_2 and ϕ_3 are K-spherical functions on G and $a_i, b_i, c_i, i = 1,2,3$ are complex constants satisfying the matrix equation

$$\begin{Bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{Bmatrix} \begin{Bmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{Bmatrix} = \begin{Bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{Bmatrix}. \tag{7}$$

(C):

$$\begin{Bmatrix} f \\ g \\ h \end{Bmatrix} = \begin{Bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{Bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ (m_2 A + (m_2 \circ \sigma)(A \circ \sigma))/2 \end{Bmatrix}, \tag{8}$$

where ϕ_1 is a K-spherical function on G, $m_2 : G \to \mathbf{C}^*$ is a continuous homomorphism for which $m_2 \neq m_2 \circ \sigma$, ϕ_2 is the corresponding K-spherical *function* $\phi_2 = \int_K k \cdot m_2 dk$, $A \in \mathcal{A}(G)$, *and* $a_i, b_i, c_i, i = 1,2,3$ *are complex constants satisfying the matrix equation*

$$\left\{ \begin{array}{ccc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array} \right\} \left\{ \begin{array}{ccc} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{array} \right\} = \left\{ \begin{array}{ccc} a_1 & 0 & 0 \\ 0 & a_2 & a_3 \\ 0 & a_3 & 0 \end{array} \right\}. \tag{9}$$

Furthermore $a_3 = 1$. *If* f, g *and* h *are linearly independent it may also be assumed that* $a_1 = -a_2$.

(D):

$$\left\{ \begin{array}{c} f \\ g \\ h \end{array} \right\} = \left\{ \begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array} \right\} \left\{ \begin{array}{c} \phi_1 \\ m_2 \\ m_2 q \end{array} \right\}, \tag{10}$$

where ϕ_1 *is a* K*-spherical function on* G, $m_2 : G \to \mathbf{C}^*$ *is a continuous homomorphism for which* $m_2 = m_2 \circ \sigma$, $q = A^+ + S^-$ *where* $A^+ \in \mathcal{A}^+(G)$ *and* $S^- \in \mathcal{S}^-(G)$ *and* $a_i, b_i, c_i, i = 1,2,3$ *are complex constants satisfying the matrix equation (9). It may be assumed that* $a_3 = 1$. *If* f,g *and* h *are linearly independent it may also be assumed that* $a_1 = -a_2$.

(E):

$$\left\{ \begin{array}{c} f \\ g \\ h \end{array} \right\} = \left\{ \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{array} \right\} \left\{ \begin{array}{c} (mA_1 + (m \circ \sigma)(A_1 \circ \sigma))/2 \\ (m + m \circ \sigma)/2 \\ (mA + (m \circ \sigma)(A \circ \sigma))/2 \\ (mA^2 + (m \circ \sigma)(A^2 \circ \sigma))/2 \end{array} \right\}, \tag{11}$$

where $m : G \to \mathbf{C}^*$ *is a continuous homomorphism for which* $m \neq m \circ \sigma$, $A, A_1 \in \mathcal{A}(G)$, *and* $a_i, b_i, c_i, i = 1,2,3,4$ *are complex constants satisfying the matrix equation*

$$\left\{ \begin{array}{ccc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{array} \right\} \left\{ \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \\ a_1 & a_2 & a_3 & a_4 \\ c_1 & c_2 & c_3 & c_4 \end{array} \right\} = \left\{ \begin{array}{cccc} 0 & a_1 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ 0 & a_3 & 2a_4 & 0 \\ 0 & a_4 & 0 & 0 \end{array} \right\}. \tag{12}$$

Futhermore $a_1 = 1$ *and* $a_3 = 0$. *If* f,g *and* h *are linearly independent then it may also be assumed that* $a_2 = 0$ *and that* $a_4 = 1/2$.

(F):

$$\left\{ \begin{array}{c} f \\ g \\ h \end{array} \right\} = \left\{ \begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array} \right\} \left\{ \begin{array}{c} mF \\ m \\ mq \end{array} \right\}, \tag{13}$$

where $a_i, b_i, c_i, i = 1,2,3$ *are complex constants satisfying the matrix equation*

$$\left\{ \begin{array}{ccc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array} \right\} \left\{ \begin{array}{ccc} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{array} \right\} = \left\{ \begin{array}{ccc} 0 & a_1 & 0 \\ a_1 & a_2 & a_3 \\ 0 & a_3 & a_1 \end{array} \right\}, \tag{14}$$

$m : G \to \mathbf{C}^*$ *is a continuous homomorphism for which* $m = m \circ \sigma$, *and*

$$F = \frac{1}{2}(A^+)^2 + \frac{1}{6}(A^-)^4 + A^+(A^-)^2 + A_1^+ + S^-, \tag{15}$$

$$q = A^+ + (A^-)^2, \tag{16}$$

where $A^+, A_1^+ \in \mathcal{A}^+(G)$, $A^- \in \mathcal{A}^-(G)$ and $S^- \in \mathcal{S}^-(G)$. We may even assume that

$$\left\{ \begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array} \right\} = \left\{ \begin{array}{ccc} 1 & 0 & 0 \\ -z^2/2 & 1 & z \\ -z & 0 & 1 \end{array} \right\} \text{ for some } z \in \mathbf{C}. \tag{17}$$

Proof. It is elementary to check that the functions listed are solutions. Thus what is left is to show that each continuous solution (f, g, h) of (5) occurs in the list. Apart from the last section the rest of the paper is dedicated to this. □

Remark 2.2. (a) Theorem 2.1 above yields for $\sigma = I$ the main result of [4]. In checking that this is so it is advisable to take the discussion on p. 276 of [4] into account or the remark after our Lemma 3.3.

(b) The matrix equations (7), (9) and (12) occur in [3].

3. Technicalities

Proposition 3.1. *The solutions $q \in C(G)$ of the quadratic equation*

$$\frac{q(x+y) + q(x+\sigma y)}{2} = q(x) + q(y), \; x, y \in G, \tag{18}$$

are the functions of the form $q = A^+ + S^-$, where $A^+ \in \mathcal{A}^+(G)$ and $S^- \in \mathcal{S}^-(G)$.

Proof. This is Corollary III.8 of [12]. □

Proposition 3.2. *The solutions $F, q \in C(G)$ of the system of functional equations*

$$\begin{aligned} \frac{F(x+y) + F(x+\sigma y)}{2} &= F(x) + F(y) + q(x)q(y), \; x, y \in G, \qquad (19) \\ \frac{q(x+y) + q(x+\sigma y)}{2} &= q(x) + q(y), \; x, y \in G, \end{aligned}$$

are

$$\begin{aligned} F &= \frac{1}{2}(A^+)^2 + \frac{1}{6}(A^-)^4 + A^+(A^-)^2 + q', \qquad (20) \\ q &= A^+ + (A^-)^2, \end{aligned}$$

where $A^\pm \in \mathcal{A}^\pm(G)$ and where $q' \in C(G)$ is a solution of the quadratic equation (18).

Proof. It suffices to prove that q has the stated form. Indeed, if so then $f_0 := (A^+)^2/2 + A^+(A^-)^2 + (A^-)^4/6$ is a particular solution of the first equation of (19). Its complete solution is $F = f_0 + q'$ where q' ranges over the solutions of the corresponding homogeneous equation, i.e. of the quadratic equation.

If the function $x \mapsto q(x+t) - q(x)$ is a constant, say $c(t)$, for any $t \in G$ then $c(t) = q(t) - q(0) = q(t)$, implying that q is additive. Substituting σy for y in (18) shows that any solution of the quadratic equation is invariant under σ, hence $q \in \mathcal{A}^+(G)$. So from now on we may assume that there exists a $t \in G$ such that the function $x \mapsto q(x+t) - q(x)$ is not constant. The function $F_t(x) := F(x+t) - F(x) - F(t)$, $x \in G$ satisfies

$$\frac{F_t(x+y) + F_t(x+\sigma y)}{2} = F_t(x) + [q(x+t) - q(x)]q(y), \; x, y \in G, \tag{21}$$

which is a version of the functional equation of symmetric differences. It is solved by theorem IV.1 of [12] according to which there are five cases (a)-(e) to take into account: The cases (a) and (b) do not apply under our assumptions here. (c) gives $q = A^{+} + (A^{-})^2$. In each of the two remaining cases q has the form $q = c(\phi - 1)$ where $c \in \mathbf{C}$ and ϕ is a $\mathbf{Z}_2$-spherical function. But if a function q of this form satisfies the second equation of (19) then it is 0. □

The general result that makes thing work is the following technical lemma. It says that if $f, g, h \in C(G)$ constitute a solution of (22) then g and h satisfy functional equations of the same nature as f in (22). The corresponding result of [4] is there expressed by (3.7) and (3.8).

Lemma 3.3. *In this Lemma G need not be abelian, so we use the multiplicative way of writing the group composition. If $f, g, h \in C(G)$ constitute a solution of*

$$\int_K f(xk \cdot y)dk = f(x)g(y) + g(x)f(y) + h(x)h(y), \ x, y \in G, \tag{22}$$

and $f \neq 0$ then there exists constants $\alpha, \beta, \gamma, \delta \in \mathbf{C}$ such that $\gamma^2 = \alpha + \beta\delta$ and

$$\begin{aligned}\int_K g(xk \cdot y)dk - g(x)g(y) \\ = \alpha f(x)f(y) + \beta[f(x)h(y) + h(x)f(y)] + \gamma h(x)h(y), \ x, y \in G,\end{aligned} \tag{23}$$

$$\begin{aligned}\int_K h(xk \cdot y)dk - g(x)h(y) - h(x)g(y) = \\ \beta f(x)f(y) + \gamma[f(x)h(y) + h(x)f(y)] + \delta h(x)h(y), \ x, y \in G.\end{aligned} \tag{24}$$

Proof. This is Propostition II.5 of [9]. We have included the proof for the readers convinience as [9] is not readily available.

Case A : f and h are linearly independent. Lemma II.2 in [12] implies here that

$$G(x, y)f(z) + H(x, y)h(z) = G(y, z)f(x) + H(y, z)h(x), \ x, y, z \in G, \tag{25}$$

where

$$\begin{aligned} G(x, y) &= \int_K g(xk \cdot y)dk - g(x)g(y), \ x, y \in G, \\ H(x, y) &= \int_K h(xk \cdot y)dk - g(x)h(y) - h(x)g(y), \ x, y \in G. \end{aligned} \tag{26}$$

By (25) we have for any $z_1, z_2 \in G$ that

$$\begin{pmatrix} f(z_1) & h(z_1) \\ f(z_2) & h(z_2) \end{pmatrix} \begin{pmatrix} G(x, y) \\ H(x, y) \end{pmatrix} = \begin{pmatrix} G(y, z_1)f(x) + H(y, z_1)h(x) \\ G(y, z_2)f(x) + H(y, z_2)h(x) \end{pmatrix}. \tag{27}$$

Since f and h are linearly independent there exists $z_1, z_2 \in G$ such that the matrix on the left is invertible (see Lemma 14.1 in [1]), and so

$$\begin{aligned} G(x, y) &= \phi_1(y)f(x) + \psi_1(y)h(x), \\ H(x, y) &= \phi_2(y)f(x) + \psi_2(y)h(x), \end{aligned} \tag{28}$$

for some functions $\phi_1, \phi_2, \psi_1, \psi_2 \in C(G)$. When we substitute this back into (25) we get by the linear independence of f and h that

$$\begin{aligned} \phi_1(y)f(z) + \phi_2(y)h(z) &= \phi_1(z)f(y) + \psi_1(z)h(y), \ y, z \in G, \\ \psi_1(y)f(z) + \psi_2(y)h(z) &= \phi_2(z)f(y) + \psi_2(z)h(y), \ y, z \in G. \end{aligned} \tag{29}$$

Using the linear independence of f and h once more we get that there exist constants $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{C}$ such that $\phi_i = a_i f + b_i h$ and $\psi_i = c_i f + d_i h$ for $i = 1, 2$.

Substituting this back into (29) we find that $b_1 = c_1$, $a_2 = b_1$, $b_2 = d_1$, $c_1 = a_2$, $c_2 = b_2$, $d_1 = c_2$, so that $\phi_1 = \alpha f + \beta h$, $\phi_2 = \psi_1 = \beta f + \gamma h$, and $\psi_2 = \gamma f + \delta h$, where $\alpha = a_1$, $\beta = b_1$, $\gamma = b_2$, and $\delta = d_2$. This means by (28) that G and H have the forms stated in (23) and (24). That $\gamma^2 = \alpha + \beta\delta$ follows from applying Lemma II.2 in [12] to any of the two identities (23) and (24).

Case B: f and h are linearly dependent. Since $f \neq 0$ there exists a constant $c \in \mathbb{C}$ such that $h = \sqrt{2}cf$. The identity (22) then becomes

$$\begin{aligned} \int_K f(xk \cdot y)dk &= f(x)g(y) + g(x)f(y) + 2c^2 f(x)f(y) \\ &= f(x)[g + c^2 f](y) + [g + c^2 f](x)f(y), \ x, y \in G. \end{aligned} \tag{30}$$

From Lemma V.1 in [11] it follows after elementary computations that there exists a constant $\kappa \in \mathbb{C}$ such that

$$\int_K g(xk \cdot y)dk - g(x)g(y) = \kappa^2 f(x)f(y), \ x, y \in G. \tag{31}$$

From (22) we get that

$$\int_K h(xk \cdot y)dk - g(x)h(y) - h(x)g(y) = \sqrt{2}ch(x)h(y), \ x, y \in G, \tag{32}$$

So all that remains to be show is that there exist $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $\gamma^2 = \alpha + \beta\delta$, $\alpha + 2\sqrt{2}c\beta + 2c^2\gamma = \kappa^2$, and $\beta + 2\sqrt{2}c\gamma + 2c^2\delta = \sqrt{2}c$. But this is obvious. □

Remark 3.4. Let K be the trivial group as in [4] and let (f, g, h) be a solution of (22) with $f \neq 0$. Then it follows from (22),(23) and (24) that each of the functions f, g and h satisfies Kannappans condition. Hence we may assume without loss of generality that G is abelian in this case. Unfortunately the argument does not generalize to $K = \mathbf{Z}_2$ so here we assume that G is Abelian.

Proposition 3.5. *Let each of the functions ϕ_1, ϕ_2, ϕ_3 be a K-spherical function on G or the zero function. For $a_i, b_i, c_i \in \mathbf{C}, i = 1, 2, 3$, we define the functions $f := \sum_{i=1}^{3} a_i\phi_i$, $g := \sum_{i=1}^{3} b_i\phi_i$ and $h := \sum_{i=1}^{3} c_i\phi_i$. If the coefficients $a_i, b_i, c_i \in \mathbf{C}, i = 1, 2, 3$ satisfy the matrix equation (7) then the triple (f,g,h) constitutes a solution of the functional equation (22).*

Conversely if the triple (f,g,h) solves (22) and if f,g and h are linearly independent then the coefficients satisfy the matrix equation (7).

Proof. To prove the converse result note that f, g, and h linearly independent implies that ϕ_1, ϕ_2, ϕ_3 are linearly independent and the result follows by direct computations. □

Proposition 3.6. *Let $m_1, m_2 : G \to \mathbf{C}^*$ be continuous homomorphisms and let $A \in \mathcal{A}(G)$. For $a_i, b_i, c_i \in \mathbf{C}, i = 1, 2, 3$ we define the functions*

$$\left\{ \begin{array}{c} f \\ g \\ h \end{array} \right\} = \left\{ \begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array} \right\} \left\{ \begin{array}{c} \phi_1 \\ \phi_2 \\ \int_K k \cdot (m_2 A)dk \end{array} \right\}, \tag{33}$$

where $\phi_i, i = 1, 2$ denotes the K-spherical function $\phi_i := \int_K k \cdot m_i dk$.

If the coefficients satisfy the matrix equation (9) then the triple (f,g,h) constitutes a solution of the functional equation (22).

Conversely if the triple $(f, g, h) \in C(G)$ solves (22) and if f, g and h are linearly independent then the coefficients satisfy the matrix equation (9).

4. THE CASE OF LINEAR INDEPENDENCE

Let us assume that the triple (f, g, h) solves the functional equation (22) and that $f \neq 0$. Explicit calculations based on the identities (22), (23) and (24) reveal that

$$\int_K \left\{ \begin{array}{c} f \\ g \\ h \end{array} \right\} (xk \cdot y) dk = \Phi(y)^t \left\{ \begin{array}{c} f \\ g \\ h \end{array} \right\} (x), \; x, y \in G, \tag{34}$$

where Φ is defined by

$$\Phi = gI + f \left\{ \begin{array}{ccc} 0 & \alpha & \beta \\ 1 & 0 & 0 \\ 0 & \beta & \gamma \end{array} \right\} + h \left\{ \begin{array}{ccc} 0 & \beta & \gamma \\ 0 & 0 & 1 \\ 1 & \gamma & \delta \end{array} \right\}. \tag{35}$$

Elementary computations based on the definition of Φ and the identities (22), (23) and (24) where $\gamma^2 = \alpha + \beta\delta$ show that Φ satisfies the spherical equation

$$\int_K \Phi(xk \cdot y) dk = \Phi(x)\Phi(y) \text{ for all } x, y \in G. \tag{36}$$

Since the right hand sides of (22), (23) and (24) are symmetric in x and y it follows that

$$\int_K F(xk \cdot y) dk = \int_K F(yk \cdot x) dk, \;\; \forall x, y \in G, \; F \in \{f, g, h\}. \tag{37}$$

Hence

$$\Phi(x)\Phi(y) = \int_K \Phi(xk \cdot y) dk = \int_K \Phi(yk \cdot x) dk = \Phi(y)\Phi(x), \; \forall x, y \in G. \tag{38}$$

By linear Algebra this ensures the existence of a 3×3 complex matrix A such that $A^{-1}\Phi(x)A$ is upper triangular for all $x \in G$. Below we find such an A explicitly. If we put $y = e$ in (22) we get that

$$(g(e) - 1)f + f(e)g + h(e)h = 0. \tag{39}$$

If f, g and h are linearly independent this means that $g(e) = 1$ and $f(e) = h(e) = 0$. In particular we find in this case that $\Phi(e) = I$ so Φ is a matrix valued K-spherical function. In the remaining part of this section we shall assume that the triple (f,g,h) constitutes a solution of (22) and that f, g and h are linearly independent.

<u>CASE 1:</u> $\beta = 0$ but $\alpha = \gamma^2 \neq 0$. Here Φ takes the form

$$\Phi = gI + f \left\{ \begin{array}{ccc} 0 & \gamma^2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \gamma \end{array} \right\} + h \left\{ \begin{array}{ccc} 0 & 0 & \gamma \\ 0 & 0 & 1 \\ 1 & \gamma & \delta \end{array} \right\}. \tag{40}$$

Let

$$\lambda_{\pm} := \frac{\delta}{2} \pm \sqrt{\left(\frac{\delta}{2}\right)^2 + 2\gamma}. \tag{41}$$

CASE 1.A: $\lambda_+ \neq \lambda_-$. Here

$$\left\{ \begin{array}{ccc} -\gamma & \gamma & \gamma \\ 1 & 1 & 1 \\ 0 & \lambda_+ & \lambda_- \end{array} \right\}^{-1} \Phi \left\{ \begin{array}{ccc} -\gamma & \gamma & \gamma \\ 1 & 1 & 1 \\ 0 & \lambda_+ & \lambda_- \end{array} \right\} = \tag{42}$$

$$\left\{ \begin{array}{ccc} g-\gamma f & 0 & 0 \\ 0 & g+\gamma f+\lambda_+ h & 0 \\ 0 & 0 & g+\gamma f+\lambda_- h \end{array} \right\}.$$

None of the three functions in the diagonal are zero, because f, g, and h are linearly independent. From (36) we read that $\phi_1 := g - \gamma f$, $\phi_2 := g + \gamma f + \lambda_+ h$, and $\phi_3 := g + \gamma f + \lambda_- h$ are K-spherical functions on G. We find that there exist constants $a_i, b_i, c_i \in \mathbf{C}$ for $i = 1, 2, 3$ such that $f = \sum_{i=1}^3 a_i\phi_i$, $g = \sum_{i=1}^3 b_i\phi_i$, and $h = \sum_{i=1}^3 c_i\phi_i$. It follows from Proposition 3.5 that the coefficients satisfy the matrix equation (7). The solution occurs in (B) of the list of Theorem 2.1.

CASE 1.B: $\lambda_+ = \lambda_-$. This means that $\gamma = -\delta^2/8$. In particular $\delta \neq 0$ since $\gamma \neq 0$. Here we find that

$$\left\{ \begin{array}{ccc} \frac{\delta^2}{8} & -\frac{\delta^2}{8} & 0 \\ 1 & 1 & 0 \\ 0 & \frac{\delta}{2} & 1 \end{array} \right\}^{-1} \Phi \left\{ \begin{array}{ccc} \frac{\delta^2}{8} & -\frac{\delta^2}{8} & 0 \\ 1 & 1 & 0 \\ 0 & \frac{\delta}{2} & 1 \end{array} \right\} = \tag{43}$$

$$\left\{ \begin{array}{ccc} g+\frac{\delta^2}{8}f & 0 & 0 \\ 0 & g-\frac{\delta^2}{8}f+\frac{\delta}{2}h & h \\ 0 & 0 & g-\frac{\delta^2}{8}f+\frac{\delta}{2}h \end{array} \right\}.$$

We see from (36) that $\phi_1 := g + \delta^2 f/8$ and $\phi_2 := g - \delta^2 f/8 + \delta h/2$ are K-spherical functions on G. Furthermore

$$f = \left(\frac{2}{\delta}\right)^2 (\phi_1 - \phi_2 + \frac{\delta}{2}h) \text{ and } g = \frac{1}{2}(\phi_1 + \phi_2 - \frac{\delta}{2}h), \tag{44}$$

and h is a non-zero solution of the functional equation

$$\int_K h(xk \cdot y)dk = \phi_2(x)h(y) + h(x)\phi_2(y),\ x, y \in G. \tag{45}$$

We specialize to $\mathbf{Z}_2$ for a moment. By Theorem III.1 of [12] (or Theorem 3 of [2]) there exists a continuous homomorphism $m_2 : G \to \mathbf{C}^*$ such that $\phi_2 = (m_2 + m_2 \circ \sigma)/2$. By Theorem V.1 of [13] there are two possibilities for h :

CASE 1.B.1: $m_2 \neq m_2 \circ \sigma$. Here

$$h = \frac{m_2 + m_2 \circ \sigma}{2}A^+ + \frac{m_2 - m_2 \circ \sigma}{2}A^- = \frac{m_2 A + (m_2 \circ \sigma)(A \circ \sigma)}{2}, \tag{46}$$

where $A^\pm \in \mathcal{A}^\pm(G)$, $A \in \mathcal{A}(G)$. It follows from Proposition 3.6 that the coefficients satisfy the matrix equation (9). The solution occurs in (C) of the list of Theorem 2.1.

CASE 1.B.2: $m_2 = m_2 \circ \sigma$. Here $h = m_2 q$ where q is a solution of the quadratic equation (18). We find that

$$\left\{ \begin{array}{c} f \\ g \\ h \end{array} \right\} = \left\{ \begin{array}{ccc} (\frac{2}{\delta})^2 & -(\frac{2}{\delta})^2 & 1 \\ \frac{1}{2} & \frac{1}{2} & -\frac{\delta^2}{8} \\ 0 & 0 & \frac{\delta}{2} \end{array} \right\} \left\{ \begin{array}{c} \phi_1 \\ m_2 \\ m_2 q \end{array} \right\}, \tag{47}$$

so that the solution occurs in (D) of the list of Theorem 2.1.

CASE 2: $\alpha = \beta = \gamma = 0$ but $\delta \neq 0$. We find that

$$\left\{\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & \delta & -1/\delta \end{array}\right\}^{-1} \Phi \left\{\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & \delta & -1/\delta \end{array}\right\} = \left\{\begin{array}{ccc} g & 0 & f - h/\delta \\ 0 & g + \delta h & 0 \\ 0 & 0 & g \end{array}\right\}. \quad (48)$$

From here we proceed exactly as in Case 1.B above. The solutions occur in (C) and (D) of the list of Theorem 2.1.

CASE 3: $\alpha = \beta = \gamma = \delta = 0$. We get

$$\left\{\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right\}^{-1} \Phi \left\{\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right\} = \left\{\begin{array}{ccc} g & h & f \\ 0 & g & h \\ 0 & 0 & g \end{array}\right\}, \quad (49)$$

so that g is a K-spherical function. We know from Theorem III.1 of [12] (or Theorem 3 of [2]) that there exists a continuous homomorphism $m : G \to \mathbf{C}^*$ such that $g = (m + m \circ \sigma)/2$.

CASE 3.A: $m \neq m \circ \sigma$. By Proposition III.6 of [12] there exists a continuous homomorphism $M : G \to GL(3, \mathbf{C})$ of the form

$$M = \left\{\begin{array}{ccc} m & \psi & \phi \\ 0 & m & \psi \\ 0 & 0 & m \end{array}\right\} \text{ such that } \left\{\begin{array}{ccc} g & h & f \\ 0 & g & h \\ 0 & 0 & g \end{array}\right\} = \frac{1}{2}(M + M \circ \sigma). \quad (50)$$

The homomorphism property of M means that

$$\begin{aligned} \psi(x+y) &= m(x)\psi(y) + \psi(x)m(y), \quad (51) \\ \phi(x+y) &= \phi(x)m(y) + m(x)\phi(y) + \psi(x)\psi(y). \end{aligned}$$

Dividing by $m(x + y) = m(x)m(y)$ in the above identities we get that

$$\frac{\psi}{m}(x+y) = \frac{\psi}{m}(x) + \frac{\psi}{m}(y), \; x, y \in G, \quad (52)$$

so that $\psi = mA$ where $A \in \mathcal{A}(G)$, and

$$\frac{\phi}{m}(x+y) = \frac{\phi}{m}(x) + \frac{\phi}{m}(y) + A(x)A(y), \; x, y \in G. \quad (53)$$

A particular solution of this inhomogeneous equation is $\frac{\phi}{m} = A^2/2$ so its complete solution is $\frac{\phi}{m} = A_1 + A^2/2$ where $A_1 \in \mathcal{A}(G)$. Now

$$M = m \left\{\begin{array}{ccc} 1 & A & A_1 + \frac{1}{2}A^2 \\ 0 & 1 & A \\ 0 & 0 & 1 \end{array}\right\}, \quad (54)$$

from which we find that

$$\left\{\begin{array}{c} f \\ g \\ h \end{array}\right\} = \left\{\begin{array}{cccc} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right\} \left\{\begin{array}{c} (mA_1 + (m \circ \sigma)(A_1 \circ \sigma))/2 \\ (m + m \circ \sigma)/2 \\ (mA + (m \circ \sigma)(A \circ \sigma))/2 \\ (mA^2 + (m \circ \sigma)(A^2 \circ \sigma))/2 \end{array}\right\}. \quad (55)$$

The solution occurs in (E) in the list of Theorem 2.1.

CASE 3.B: $m = m \circ \sigma$. In this case $g = m = m \circ \sigma$. The original functional equation (22) and the one for h from Lemma 3.3 are in this case:

$$\frac{f(x+y)+f(x+\sigma y)}{2} = f(x)m(y)+m(x)f(y)+h(x)h(y), \; x,y \in G, \quad (56)$$

$$\frac{h(x+y)+h(x+\sigma y)}{2} = h(x)m(y)+m(x)h(y), \; x, \in G. \quad (57)$$

We find that

$$\begin{Bmatrix} f \\ g \\ h \end{Bmatrix} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} mF \\ m \\ mq \end{Bmatrix}, \quad (58)$$

where the functions $F := f/m$ and $q := h/m$ satisfy the equations

$$\frac{F(x+y)+F(x+\sigma y)}{2} = F(x)+F(y)+q(x)q(y), \; x,y \in G, \quad (59)$$

$$\frac{q(x+y)+q(x+\sigma y)}{2} = q(x)+q(y), \; x,y \in G. \quad (60)$$

Proposition 3.2 shows that the solution occurs in (F) of the list Theorem 2.1.

CASE 4: $\beta \neq 0$. For any $z \in \mathbf{C}$ we put

$$G := g - \frac{z^2}{2} f - zh \text{ and } H := h + zf. \quad (61)$$

Note that

$$\begin{Bmatrix} 1 & 0 & 0 \\ \frac{-z^2}{2} & 1 & -z \\ z & 0 & 1 \end{Bmatrix}^{-1} = \begin{Bmatrix} 1 & 0 & 0 \\ \frac{-z^2}{2} & 1 & z \\ -z & 0 & 1 \end{Bmatrix}. \quad (62)$$

Now f, G and H are linearly independent because f, g and h are so. Brute force calculations show that

$$\int_K f(xk \cdot y)dk = f(x)G(y)+G(x)f(y)+H(x)H(y), \quad (63)$$

$$\int_K G(xk \cdot y)dk - G(x)G(y) = \quad (64)$$
$$Af(x)f(y)+B[f(x)H(y)+H(x)f(y)]+CH(x)H(y),$$

$$\int_K H(xk \cdot y)dk - G(x)H(y) - H(x)G(y) = \quad (65)$$
$$Bf(x)f(y)+C[f(x)H(y)+H(x)f(y)]+DH(x)H(y),$$

where

$$A = -\frac{3}{4}z^4 - \delta z^3 + 3\gamma z^2 - 3\beta z + \alpha, \quad (66)$$

$$B = z^3 + \delta z^2 - 2\gamma z + \beta, \quad (67)$$

$$C = -\frac{3}{2}z^2 - \delta z + \gamma, \quad (68)$$

$$D = \delta + 3z. \quad (69)$$

We note that $C^2 = A + BD$ corresponding to the earlier identity $\gamma^2 = \alpha + \beta\delta$. Choosing $z \in \mathbf{C}$ such that $B = 0$ we can apply the earlier results to the new set of functions $\{f, G, H\}$ replacing $\{\alpha, \beta, \gamma, \delta\}$ by $\{A, B, C, D\}$. Thus

$$\left\{ \begin{matrix} f \\ G \\ H \end{matrix} \right\} = \left\{ \begin{matrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \\ c_1 & \dots & c_n \end{matrix} \right\} \left\{ \begin{matrix} \psi_1 \\ \vdots \\ \psi_n \end{matrix} \right\}, \tag{70}$$

corresponding to the various cases above. Now

$$\left\{ \begin{matrix} f \\ g \\ h \end{matrix} \right\} = \left\{ \begin{matrix} 1 & 0 & 0 \\ \frac{-z^2}{2} & 1 & z \\ -z & 0 & 1 \end{matrix} \right\} \left\{ \begin{matrix} f \\ G \\ H \end{matrix} \right\} = \left\{ \begin{matrix} a_1 & \cdots & a_n \\ b_1(z) & \cdots & b_n(z) \\ c_1(z) & \cdots & c_n(z) \end{matrix} \right\} \left\{ \begin{matrix} \psi_1 \\ \vdots \\ \psi_n \end{matrix} \right\}, \tag{71}$$

where $b_i(z) := b_i - \frac{1}{2}z^2 a_i + zc_i$ and $c_i(z) := c_i - za_i$ for $z \in \mathbf{C}$ and $i = 1, \dots, n$. We are through by the following matrix identity:

$$\left\{ \begin{matrix} a_1 & b_1(z) & c_1(z) \\ \vdots & \vdots & \vdots \\ a_n & b_n(z) & c_n(z) \end{matrix} \right\} \left\{ \begin{matrix} b_1(z) & \cdots & b_n(z) \\ a_1 & \cdots & a_n \\ c_1(z) & \cdots & c_n(z) \end{matrix} \right\} = \tag{72}$$

$$\left\{ \begin{matrix} a_1 & b_1 & c_1 \\ \vdots & \vdots & \vdots \\ a_n & b_n & c_n \end{matrix} \right\} \left\{ \begin{matrix} b_1 & \cdots & b_n \\ a_1 & \cdots & a_n \\ c_1 & \cdots & c_n \end{matrix} \right\}.$$

Indeed, in all the cases (B)-(F) of Theorem 2.1 the matrix equation contains only the a_i entries on the right hand side and they are independent of z.

5. The case of linear dependence

This section deals with the remaining case of f, g and h linearly dependent. We divide it into three subcases (A), (B) and (C).

(A) f and h linearly independent, so that $g = \lambda f + \mu h$ for some $\lambda, \mu \in \mathbf{C}$. Substituting this expression for g into the functional equation (1) and introducing $H := h + \mu f$ instead of h we get

$$\int_K f(xk \cdot y)dk = (2\lambda - \mu^2) f(x) f(y) + H(x) H(y), \ x, y \in G. \tag{73}$$

If $2\lambda = \mu^2$ then taking $y = e$ in (73) we find that $f = H(e)H = H(e)h + H(e)\mu f$, contradicting that f and h are linearly independent. So $2\lambda - \mu^2 \neq 0$. Letting $\rho \in \mathbf{C}$ be a square root of $2\lambda - \mu^2$ the equation (73) becomes

$$\int_K F(xk \cdot y) = F(x)F(y) + G(x)G(y), \ x, y \in G, \tag{74}$$

where $F := \rho^2 f$ and $G := \rho H$. The solutions of (74) are written down as Theorem V.5 of [13] for $K = \mathbf{Z}_2$ with n=1. The theorem states that there are only the following possibilities (a)-(e):

(a) $F = G = 0$. This implies here that $f = 0$. However that possibility must be excluded since f and h are assumed to be linearly independent.

(b) There exists a continuous homomorphism $m : G \to \mathbf{C}^*$ and a $c \in \mathbf{C} \setminus \{\pm i\}$ such that

$$\rho H = \frac{c}{1 + c^2} \frac{m + m \circ \sigma}{2} \text{ and } \rho^2 f = \frac{1}{1 + c^2} \frac{m + m \circ \sigma}{2}. \tag{75}$$

It follows that $c\rho f = h + \mu f$. But f and h are assumed to be linearly independent so the possibility (b) must also be excluded.

(c) There exist continuous homomorphisms $m_1, m_2 : G \to \mathbf{C}^*$ and a $c \in \mathbf{C}\setminus\{\pm i\}$ such that

$$\rho H = \frac{c}{1+c^2}\left(\frac{m_1 + m_1\circ\sigma}{2} - \frac{m_2 + m_2\circ\sigma}{2}\right), \tag{76}$$

$$\rho^2 f = \frac{1}{1+c^2}\frac{m_1 + m_1\circ\sigma}{2} + \frac{c^2}{1+c^2}\frac{m_2 + m_2\circ\sigma}{2}. \tag{77}$$

Letting $\phi_1 := (m_1 + m_1\circ\sigma)/2$ and $\phi_2 := (m_2 + m_2\circ\sigma)/2$ we find that

$$\left\{\begin{array}{c} f \\ g \\ h \end{array}\right\} = \frac{1}{\rho^2}\frac{1}{1+c^2}\left\{\begin{array}{cc} 1 & c^2 \\ \lambda + \mu\rho c - \mu^2 & c(\lambda c - \mu\rho - c\mu^2) \\ \rho c - \mu & -c(\rho + \mu c) \end{array}\right\}\left\{\begin{array}{c} \phi_1 \\ \phi_2 \end{array}\right\}. \tag{78}$$

A calculation reveals that this fits into case (B) of Theorem 2.1 when we take

$$\left\{\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array}\right\} = \frac{1}{\rho^2}\frac{1}{1+c^2}\left\{\begin{array}{ccc} 1 & c^2 & 0 \\ \lambda + \mu\rho c - \mu^2 & c(\lambda c - \mu\rho - c\mu^2) & 0 \\ \rho c - \mu & -c(\rho + \mu c) & 0 \end{array}\right\}. \tag{79}$$

(d) There exists a continuous homomorphism $m : G \to \mathbf{C}^*$ for which $m \neq m\circ\sigma$, $A^+ \in \mathcal{A}^+(G)$ and $A^- \in \mathcal{A}(G)^-$ such that

$$\rho H = \frac{m + m\circ\sigma}{2}A^+ + \frac{m - m\circ\sigma}{2}A^-, \tag{80}$$

$$\rho^2 f = \frac{m + m\circ\sigma}{2} \pm i\left[\frac{m + m\circ\sigma}{2}A^+ + \frac{m - m\circ\sigma}{2}A^-\right]. \tag{81}$$

With $A := \pm i\rho^{-2}(A^+ + A^-)$ we find that

$$\left\{\begin{array}{c} f \\ g \\ h \end{array}\right\} = \left\{\begin{array}{ccc} 0 & \rho^{-2} & 1 \\ 0 & (\rho^2 - \mu^2)\rho^{-2}/2 & (\rho \mp i\mu)^2/2 \\ 0 & -\mu\rho^{-2} & \mp i(\rho \mp i\mu) \end{array}\right\}\left\{\begin{array}{c} (m + m\circ\sigma)/2 \\ (m + m\circ\sigma)/2 \\ (mA + (mA)\circ\sigma)/2 \end{array}\right\}. \tag{82}$$

The solution fits into case (C) of Theorem 2.1.

(e) There exists a continuous homomorphism $m : G \to \mathbf{C}^*$ for which $m = m\circ\sigma$ and a solution $q \in C(G)$ of the quadratic equation (18) such that $\rho H = mq$ and $\rho^2 f = m \pm imq$. Here we find that

$$\left\{\begin{array}{c} f \\ g \\ h \end{array}\right\} = \left\{\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array}\right\}\left\{\begin{array}{c} m \\ m \\ mq \end{array}\right\}, \tag{83}$$

where

$$\left\{\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array}\right\} = \left\{\begin{array}{ccc} 0 & \rho^{-2} & 1 \\ 0 & \frac{1}{2}(\rho^2 - \mu^2)\rho^{-2} & -\frac{1}{2}(\mp i\rho - \mu)^2 \\ 0 & -\mu\rho^{-2} & \mp i\rho - \mu \end{array}\right\}. \tag{84}$$

A calculation reveals that this fits into case (D) of Theorem 2.1.

(B) $f = 0$. Here $h = 0$ and $g \in C(G)$, which is the trivial case (A) of Theorem 2.1.

(C) $f \neq 0$ and h are linearly dependent, so that $h = \alpha f$ for some $\alpha \in \mathbf{C}$. Here the functional equation (5) reduces to

$$\frac{f(x+y)+f(x+\sigma y)}{2} = f(x)\left[g(y)+\frac{\alpha^2}{2}f(y)\right]+\left[g(x)+\frac{\alpha^2}{2}f(x)\right]f(y), x,y \in G, \tag{85}$$

which is once again a well known functional equation. The solutions of the equation (85) are written down as Theorem V.4 of [13] for $K = \mathbf{Z}_2$ with n=1. The theorem state that there are only the following possibilities (a)-(e):

(a) $f = 0$. This possibility is excluded by assumption.

(b) There exists a continuous homomorphism $m : G \to \mathbf{C}^*$ and a constant $c \in \mathbf{C}$ such that $g + \alpha^2 f/2 = (m + m \circ \sigma)/4$ and $f = c(m + m \circ \sigma)$.

Letting $\phi_1 = \phi_2 = \phi_3 = (m + m \circ \sigma)/2$ we get

$$\left\{\begin{array}{c} f \\ g \\ h \end{array}\right\} = \left\{\begin{array}{ccc} 2c & 0 & 0 \\ \frac{1}{2}-\alpha^2 c & 0 & 0 \\ 2\alpha c & 0 & 0 \end{array}\right\}\left\{\begin{array}{c} \phi_1 \\ \phi_2 \\ \phi_3 \end{array}\right\}, \tag{86}$$

from which a small calculation reveals that we this is case (B) of Theorem 2.1.

(c) There exist continuous homomorphisms $m_1, m_2 : G \to \mathbf{C}^*$ and a constant $c \in \mathbf{C}$ such that $g + \alpha^2 f/2 = (m_1 + m_1 \circ \sigma + m_2 + m_2 \circ \sigma)/4$ and $f = c[m_1 + m_1 \circ \sigma - (m_2 + m_2 \circ \sigma)]$.

Letting $\phi_1 := (m_1 + m_1 \circ \sigma)/2$ and $\phi_2 = \phi_3 := (m_2 + m_2 \circ \sigma)/2$ we find that

$$\left\{\begin{array}{c} f \\ g \\ h \end{array}\right\} = \left\{\begin{array}{ccc} 2c & -2c & 0 \\ \frac{1}{2}-\alpha^2 c & \frac{1}{2}+\alpha^2 c & 0 \\ 2\alpha c & -2\alpha c & 0 \end{array}\right\}\left\{\begin{array}{c} \phi_1 \\ \phi_2 \\ \phi_3 \end{array}\right\}, \tag{87}$$

from which a small calculation reveals that this is case (B) of Theorem 2.1.

(d) There exists a continuous homomorphism $m : G \to \mathbf{C}^*$ for which $m \neq m \circ \sigma$, $A^+ \in \mathcal{A}^+(G)$ and $A^- \in \mathcal{A}^-(G)$ such that

$$g + \frac{1}{2}\alpha^2 f = \frac{m + m\circ\sigma}{2} \text{ and } f = \frac{m+m\circ\sigma}{2}A^+ + \frac{m - m\circ\sigma}{2}A^-. \tag{88}$$

We find with $A := A^+ + A^-$ that

$$\left\{\begin{array}{c} f \\ g \\ h \end{array}\right\} = \left\{\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & -\alpha^2/2 \\ 0 & 0 & \alpha \end{array}\right\}\left\{\begin{array}{c} (m+m\circ\sigma)/2 \\ (m+m\circ\sigma)/2 \\ (mA+(mA)\circ\sigma)/2 \end{array}\right\}, \tag{89}$$

from which a small calculation reveals that this is case (C) of Theorem 2.1.

(e) There exists a continuous homomorphism $m : G \to \mathbf{C}^*$ for which $m = m \circ \sigma$, and a solution $q \in C(G)$ of the quadratic equation (18) such that $g + \alpha^2 f/2 = m$ and $f = mq$. We find that

$$\left\{\begin{array}{c} f \\ g \\ h \end{array}\right\} = \left\{\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & -\alpha^2/2 \\ 0 & 0 & \alpha \end{array}\right\}\left\{\begin{array}{c} m \\ m \\ mq \end{array}\right\}, \tag{90}$$

from which a small calculation reveals that this is case (D) of Theorem 2.1.

6. The signed equation

Proposition 6.1. *$f, g, h \in C(G)$ is a solution to*

$$\int_K f(x+k\cdot y)\overline{\chi(k)}dk = f(x)g(y) + g(x)f(y) + h(x)h(y), \; x, y \in G, \tag{91}$$

where χ is a continuous homomorphism from K into the circle group $\{z \in \mathbb{C} : |z| = 1\}$ and $\chi \neq 1$, if and only if one of the following three conditions holds:

a): *$f = h = 0$, and $g \in C(G)$.*

b): *$f = \nu$, $g = -\alpha^2\nu/2$, and $h = \alpha\nu$ where $\alpha \in \mathbb{C}$ and $\nu \in C(G)$ is a solution to*

$$\int_K \nu(x+k\cdot y)\overline{\chi(k)}dk = 0, \; x, y \in G. \tag{92}$$

c): *$g = -\mu^2 f/2 + \mu H$, $h = H - \mu f$ where $\mu \in \mathbb{C}$, and $f, H \in C(G)$ is a solution to*

$$\int_K f(x+k\cdot y)\overline{\chi(k)}dk = H(x)H(y), \; x, y \in G. \tag{93}$$

Proof. That the conditions are sufficient is verified by trivial calculations. Now suppose that the triplet f, g, and h is a solution. Suppose $f(k\cdot x) = \chi(k)f(x)$ for all $x \in G$ and for all $k \in K$. Then

$$\begin{aligned}\int_K f(y+k\cdot x)dk &= \int_K f(k^{-1}\cdot x + y)dk = \int_K f(k^{-1}\cdot(x+k\cdot y))dk \\ &= \int_K f(x+k\cdot y)\chi(k^{-1})dk = \int_K f(x+k\cdot y)\overline{\chi(k)}dk \\ &= f(y)g(x) + g(y)f(x) + h(y)h(x) \\ &= \int_K f(y+k\cdot x)\overline{\chi(k)}dk, \end{aligned} \tag{94}$$

where we have used that K is unimodular since it is compact (see Theorem 15.13 and Theorem 15.14 in [6]). Taking $x = e$ we get

$$f(y) = \int_K f(y+k\cdot e)dk = \int_K f(y+k\cdot e)\overline{\chi(k)}dk = f(y)\int_K \overline{\chi(k)}dk = 0, \tag{95}$$

since $\overline{\chi} \neq 1$ (see Lemma 23.19 in [6]). So if $f \neq 0$ we can not have $f(k\cdot x) = \chi(k)f(x)$ for all $x \in G$ and for all $k \in K$. We will use this observation to exclude a number of cases and thereby prove that the only possible solutions are those given by the proposition.

Suppose that f, g, and h are linearly independent. it follows immediately from Theorem II.2 in [11] that $f(k\cdot x) = \chi(k)f(x)$, $\forall x \in G$, $\forall k \in K$. But this is impossible since $f \neq 0$. So f, g, and h have to be linearly dependent.

Case A: f and h are linearly independent. Then $g = \lambda f + \mu h$ for some $\lambda, \mu \in \mathbb{C}$. Define $H = h + \mu f$, then we have

$$\int_K f(x+k\cdot y)\overline{\chi(k)}dk = (2\lambda - \mu^2)f(x)f(y) + H(x)H(y), \; x, y \in G. \tag{96}$$

Take $\rho \in \mathbb{C}$ such that $\rho^2 = 2\lambda - \mu^2$. Suppose $\rho \neq 0$ and define $F = \rho f$. We have

$$\int_K f(x+k\cdot y)\overline{\chi(k)}dk = F(x)F(y) + H(x)H(y), \; x, y \in G. \tag{97}$$

$F = \rho f$ and $H = h + \mu f$ are linearly independent since f and h are linearly independent. Again it follows from Theorem II.2 in [11] that $F(k \cdot x) = \chi(k)F(x)$ and hence $f(k \cdot x) = \chi(k)f(x)$, $\forall x \in G$, $\forall k \in K$. So $f = 0$, but this is impossible so $\rho = 0$. Hence

$$\int_K f(x + k \cdot y)\overline{\chi(k)}dk = H(x)H(y), \ x, y \in G. \tag{98}$$

This is case c in Proposition 6.1.

Case B: f and h are linearly dependent.

Case B1: $f \equiv 0$, then $h \equiv 0$ and $g \in C(G)$ can be arbitrary. This is case a in Proposition 6.1.

Case B2: $f \neq 0$, so $h = \alpha f$ and we have

$$\int_K f(x + k \cdot y)\overline{\chi(k)}dk = f(x)[\frac{\alpha^2}{2}f + g](y) + [\frac{\alpha^2}{2}f + g](x)f(y), \ x, y \in G. \tag{99}$$

Suppose f and $\alpha^2 f/2 + g$ are linearly independent then again it follows from Theorem II.2 in [11] that $f(k \cdot x) = \chi(k)f(x)$, $\forall x \in G$, $\forall k \in K$. So $f \equiv 0$. But this is impossible so f and $\alpha^2 f/2 + g$ have to be linearly dependent. So $\alpha^2 f/2 + g = \lambda^2 f/2$ for some $\lambda \in \mathbb{C}$. Hence we have

$$\int_K f(x + k \cdot y)\overline{\chi(k)}dk = (\lambda f)(x)(\lambda f)(y), \ x, y \in G. \tag{100}$$

Suppose $\lambda \neq 0$ then, using Theorem II.2 in [11] it follows that $f(k \cdot x) = \chi(k)f(x)$, $\forall x \in G$, $\forall k \in K$ so $f = 0$ and this is impossible. So $\lambda = 0$ and we have

$$\int_K f(x + k \cdot y)\overline{\chi(k)}dk = 0, \ x, y \in G. \tag{101}$$

This is case b in Proposition 6.1 This proves the proposition. □

Proposition 6.2. *$f, g, h \in C(G)$ is a solution to*

$$\frac{f(x + y) - f(x + \sigma y)}{2} = f(x)g(y) + g(x)f(y) + h(x)h(y), \ x, y \in G, \tag{102}$$

if and only if one of the following conditions holds:

- **a):** *$f = h = 0$ and $g \in C(G)$.*
- **b):** *$f = \nu$, $g = -\alpha^2\nu/2$, $h = \alpha\nu$, where $\alpha \in \mathbb{C}$ and $\nu(x + y) = \nu(x + \sigma y)$, $\forall x, y \in G$.*
- **c):** *$g = -\mu^2 f/2 + \mu H$, $h = H - \mu f$, where $\mu \in \mathbb{C}$, and where $f = c^2(m + m \circ \sigma)/2 + \nu$ and $H = c(m - m \circ \sigma)/2$ where $c \in \mathbb{C}$ and $\nu(x + y) = \nu(x + \sigma y)$, $\forall x, y \in G$ and $m : G \to \mathbb{C}^*$ is a continuous homomorphism such that $m \neq m \circ \sigma$.*
- **d):** *$g = -\mu^2 f/2 + \mu H$, $h = H - \mu f$, where $\mu \in \mathbb{C}$, and $f = m(A^-)^2/2 + \nu$ and $H = mA^-$, where $A \in \mathcal{A}^-(G)$, and $m : G \to C^*$ is a continuous homomorphism such that $m = m \circ \sigma$, and $\nu(x + y) = \nu(x + \sigma y)$, $\forall x, y \in G$.*

Proof. To check that anything on the list is a solution is trivial. Now suppose that f, g, and h is a solution. We let $K = \mathbb{Z}_2$ act in the usual way on G. We define $\chi : K \mapsto \mathbb{C}^*$ by $\chi(1) = 1$ and $\chi(-1) = -1$. Then we have

$$\begin{aligned} \int_K f(x + k \cdot y)\overline{\chi(k)}dk &= \frac{f(x + y) - f(x + \sigma y)}{2} \\ &= f(x)g(y) + g(x)f(y) + h(x)h(y), \ x, y \in G. \end{aligned} \tag{103}$$

This equation was treated in Proposition 6.1.

If we are in case a in Proposition 6.1 then we are in case a in Proposition 6.2. If we are in case b in Proposition 6.1 then we are in case b in Proposition 6.2.

If we are in case c in Proposition 6.1. Then for some $\mu \in \mathbb{C}$ we have $g = -\mu^2 f/2 + \mu H$ and $h = H - \mu f$ where $f, H \in C(G)$ is a solution to

$$\frac{f(x+y) - f(x+\sigma y)}{2} = H(x)H(y), \ x, y \in G. \tag{104}$$

The equation (104) has been solved in Corollary III.5 in [12].

If we are in case 1 or 2 in Corollary III.5 in [12] then $H = 0$ and $f(x+y) = f(x+\sigma y), \ x, y \in G$. This is case b of Proposition 6.2.

If we are in case 3 in Corollary III.5 in [12] then there exist a continuous homomorphism $m : G \to \mathbb{C}^*$ for which $m \neq m \circ \sigma$, $c \in \mathbb{C} \setminus \{0\}$, $c_1, c_2 \in \mathbb{C}$, and $\nu \in C(G)$ for which $\nu(x+y) = \nu(x+\sigma y)$, $x, y \in G$ such that

$$c_1 \frac{m + m \circ \sigma}{2} + c_2 \frac{m - m \circ \sigma}{2} = H = c \frac{m - m \circ \sigma}{2}, \tag{105}$$

and

$$f = cc_2 \frac{m + m \circ \sigma}{2} + cc_1 \frac{m - m \circ \sigma}{2} + \nu. \tag{106}$$

From (105) it follows that $c_1 = H(e) = 0$, and since $m \neq m \circ \sigma$ it follows from (105) that $c_2 = c$, and we are in case c of Proposition 6.2.

If we are in case 4 of Corollary III.5 in [12] then there exist a continuous homomorphism $m : G \to \mathbb{C}^*$ for which $m = m \circ \sigma$, $c, c_1 \in \mathbb{C}$, $A^- \in \mathcal{A}(G)$, and $\nu \in C(G)$ for which $\nu(x+y) = \nu(x+\sigma y)$, $x, y \in G$, such that

$$cm + c_1 m A^- = H = mA^-, \tag{107}$$

and

$$f = cmA^- + \frac{1}{2} c_1 m (A^-)^2 + \nu. \tag{108}$$

From equation (107) it follows that $c = H(e) = mA^-(e) = 0$, if $A^- \equiv 0$ we can take c_1 to be 1, if $A^- \neq 0$ then c_1 has to be 1. We are in case d in Proposition 6.2. □

Remark 6.3. Note that if G is 2-divisible and $\sigma = -I$ then the condition $\nu(x+y) = \nu(x+\sigma y)$, $x, y \in G$ implies that ν is constant. Where 2-divisible means that for any $x \in G$ there is a $y \in G$ such that $y^2 = x$, this y is not assumed to be unique.

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