UNIVERSITY OF AARHUS
DEPARTMENT OF MATHEMATICS

ISSN: 1397–4076

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Preprint Series No.: 11

December 2000

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p-FILTRATIONS AND THE STEINBERG MODULE

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Let k be an algebraically closed field of characteristic $p > 0$. Denote by G a connected and simply connected reductive algebraic group over k. Fix a maximal torus T in G and let $X = X(T)$ be the set of characters of T. In X we choose a chamber X^+ and call its elements the dominant weights.

For each $\lambda \in X^+$ we have a simple module $L(\lambda)$, a Weyl module $\Delta(\lambda)$, a dual Weyl module $\nabla(\lambda)$, and an indecomposable tilting module $T(\lambda)$. All these modules have λ as their unique highest weight. Moreover, $L(\lambda)$ is the unique simple quotient (resp. submodule) of $\Delta(\lambda)$ (resp. $\nabla(\lambda)$), and $\Delta(\lambda)$ (resp. $\nabla(\lambda)$) occurs as the first (resp. last) subquotient in a Weyl (resp. good) filtration of $T(\lambda)$.

In this paper we study p-filtrations of G-modules. Let M be a finite dimensional G-module. Recall that a good filtration of M is a sequence of submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M \tag{$*$}$$

such that $M_i/M_{i-1} \simeq \nabla(\lambda_i)$ for some $\lambda_i \in X^+$. If instead the quotients M_i/M_{i-1} have the form $L(\lambda_i^0) \otimes \nabla(\lambda_i^1)^{(p)}$ for some $\lambda_i = \lambda_i^0 + p\lambda_i^1 \in X^+$ with λ_i^0 restricted then we say that $(*)$ is a good p-filtration of M. Here ${}^{(p)}$ denotes Frobenius twist, see 1.3 below.

Let St denote the Steinberg module for G, see 1.5. In a lecture at MSRI on November 14, 1990 S. Donkin formulated the following conjecture.

Conjecture . *(Donkin) Suppose M is a finite dimensional G-module. Then M has a good p-filtration if and only if $M \otimes St$ has a good filtration.*

We prove one half of this conjecture for $p \geq 2h-2$, h being the Coxeter number (see Corollary 2.7) and give several indications (including a proof in the case $G = SL_2(k)$) that the other half also holds. Our starting point is the cohomological criterion (Theorem 2.2) involving tilting modules for a module to have a good filtration. This leads to a criterion of the same type for a module M to have the property that $M \otimes St$ has a good filtration. Moreover, we prove (again for $p \geq 2h-2$) that if M has a good filtration then M has also a good p-filtration. And we establish some results on tensor products on the two categories. These results are consistent with the above conjecture.

For each $r \in \mathbb{N}$ we have a Steinberg module St_r which is analogous to St but where p is replaced by p^r. Also it it straightforward to define good p^r-filtrations. Then we can consider the r-th version of Donkin's conjecture. As we point out, however, it is easy to reduce this case to the $r = 1$ case.

If we replace G by the corresponding quantum group U_q with q being an l-th root of unity in some field K then the representation theory for U_q has lots in common

*Supported in part by the TMR programme "Algebraic Lie Representations", EC Network Contract No. ERB FMRX-CT97/0100.

with the representation theory for G. In particular, the above conjecture has a straightforward quantum analogue. When $\operatorname{char} K = 0$ this conjecture is already known to be true by results of [2]. However, when $\operatorname{char} K = p > 0$ the situation is exacly as in the modular case: We can prove one way of the conjecture when $p \geq 2h - 2$.

The paper is organized as follows. In Section 1 we fix notation and recall some basic facts about the representation theory for G that we need (they can all be found in [11]). Section 2 discusses the various criteria for the existence of good filtrations and also in this section we show how to deduce from this the "only if" part of Donkin's conjecture. In Section 3 we prove several general results on good p-filtrations. These are used to prove that if we tensor a module which has a good filtration with a module which has a good p-filtration, then the tensor product has a good p-filtration. In particular, if a module has a good filtration then it has also a good p-filtration. Further results on tensor products as well as other results related to Donkin's conjecture are found in Section 4. Finally Section 5 treats the quantum case where we have a very similar theory. Also we illustrate here how the p-filtrations of Weyl modules are connected to the conjecture by Lusztig, which says that the restricted irreducible characters for G should coincide (for $p \geq h$) with the corresponding characters in the quantum case.

1. Notations and recollections

1.1. Throughout this paper we shall assume that the reductive group G we consider is almost simple. This means that the root system R associated to (G, T) is irreducible. All the problems we deal with can be easily reduced to this case.

We shall fix a set of positive roots R^+. As usual we denote by ρ half the sum of the positive roots. We let α_0 be the highest short root. Then the Coxeter number h of R is related to ρ and α_0 via the equality $\langle \rho, \alpha_0^\vee \rangle = h - 1$.

1.2. The root system R is contained in the weight lattice $X = X(T)$ and R^+ determines the set of dominant weights X^+,

$$X^+ = \{\lambda \in X \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in R^+\}.$$

Moreover, R^+ induces a partial order on X by

$$\lambda \leq \mu \text{ if and only if } \mu - \lambda = \sum_{\alpha \in R^+} a_\alpha \alpha \text{ for some } a_\alpha \in \mathbb{Z}_{\geq 0}.$$

We shall also need the corresponding "rational" order $\leq_{\mathbb{Q}}$ on X given by

$$\lambda \leq_{\mathbb{Q}} \mu \text{ if and only if } \mu - \lambda = \sum_{\alpha \in R^+} a_\alpha \alpha \text{ for some } a_\alpha \in \mathbb{Q}_{\geq 0}.$$

Let S be the set of simple roots in R^+. Then we define the set of restricted weights X_p by

$$X_p = \{\lambda \in X \mid 0 \leq \langle \lambda, \alpha^\vee \rangle < p \text{ for all } \alpha \in S\}.$$

Each $\lambda \in X$ can then be written $\lambda = \lambda^0 + p\lambda^1$ for unique $\lambda^0 \in X_p$, $\lambda^1 \in X$. We call this the p-adic decomposition of λ.

Finally, the first alcove C in X^+ is defined by

$$C = \{\lambda \in X \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle < p \text{ for all } \alpha \in R^+\}.$$

The "closure" $\overline{C}$ is the corresponding set with equalities allowed. Note that $C \neq \emptyset$ if and only if $p \geq h$.

1.3. Let $F : G \to G$ be the Frobenius homomorphism on G (coming from the p-th power map on k). Considered as a map of group schemes F has an interesting kernel which we denote by G_1. This is a normal subgroup scheme of G and there are close connections between the representation theory of G and that of G_1.

Let V be a G-module. In this paper this will always mean a finite dimensional rational G-module. We obtain a new G-structure $V^{(p)}$ on V by composing the action of G on V by F. In other words, the G-module $V^{(p)}$ coincides with V as k-vector space but the action of G is given by

$$g \cdot v = F(g)(v), \quad g \in G\ , v \in V^{(p)}.$$

Note that G_1 clearly acts trivially on $V^{(p)}$. In fact, a G-module M is trivial as a G_1-module if and only if $M \simeq V^{(p)}$ for some G-module V.

If $\lambda \in X_p$ then $L(\lambda)$ remains irreducible as a G_1-module. Moreover, for general $\lambda \in X^+$ we have the Steinberg tensor product theorem

$$L(\lambda) \simeq L(\lambda^0) \otimes L(\lambda^1)^{(p)}. \tag{1}$$

1.4. When $H \leq G$ is a closed subgroup we denote by Ind_H^G the induction functor from H-modules to G-modules. We let $B \leq G$ be the Borel subgroup in G associated with the negative roots $-R^+$. Then the dual Weyl module with highest weight λ is defined by

$$\nabla(\lambda) = \mathrm{Ind}_B^G \lambda,$$

where the 1-dimensional T-module $\lambda \in X$ is made into a B-module via the natural projection $B \to T$.

We have a corresponding definition in the infinitesimal case (where G and B are replaced by their Frobenius subgroups G_1 and B_1). It is advantageous to include the full torus T, i.e. we define the analogue of $\nabla(\lambda)$ by

$$Z(\lambda) = \mathrm{Ind}_{B_1T}^{G_1T} \lambda, \quad \lambda \in X.$$

It turns out that $Z(\lambda)$ extends to a G_1B-module. In fact, if we set

$$\hat{Z}(\lambda) = \mathrm{Ind}_B^{G_1B} \lambda, \quad \lambda \in X,$$

then $\hat{Z}(\lambda)_{|_{G_1T}} = Z(\lambda)$. Moreover, by transitivity of induction

$$\nabla(\lambda) = \mathrm{Ind}_{G_1B}^G(\hat{Z}(\lambda)). \tag{2}$$

1.5. In this paper the Steinberg module St will play a prominent role. By definition $St = L((p-1)\rho)$. The strong linkage principle implies

$$St = L((p-1)\rho) = \nabla((p-1)\rho) = \Delta((p-1)\rho) = T((p-1)\rho). \tag{1}$$

Moreover, we have as G_1B-modules

$$St = \hat{Z}((p-1)\rho). \tag{2}$$

Finally, St is injective as G_1T-module. This implies in particular that for $\lambda \in X_p$ the injective hull $Q(\lambda)$ of the simple G_1T-module $L(\lambda)$ may be realized inside $St \otimes L((p-1)\rho + w_0\lambda)$. (Here w_0 is the longest element in the Weyl group of R).

For $p \geq 2h-2$ we have a G-module structure on $Q(\lambda)$. In fact,

$$Q(\lambda) = T(2(p-1)\rho + w_0\lambda)_{|_{G_1T}}. \tag{3}$$

Remark . *Donkin has conjectured that (3) should hold for all p, see* [8].

2. Good filtrations and the Steinberg module.

2.1. Let $\lambda, \mu \in X^+$. Then we have

$$H^i(G, \nabla(\lambda) \otimes \nabla(\mu)) = 0 \text{ for all } i > 0. \tag{1}$$

Here $H^i(G, -)$ is the i-th Hochschild cohomology, i.e. the i-th right derived functor of the fixed point functor $M \mapsto M^G$. Alternatively, $H^i(G, -) \simeq \mathrm{Ext}^i_G(k, -)$.

Donkin has proved that in fact (1) characterizes modules which have a good filtration:

Theorem . *(Donkin*[7]*) The following conditions on a G-module M are equivalent*

i) *M has a good filtration.*
ii) *$H^i(G, M \otimes \nabla(\lambda)) = 0$ for all $i > 0$, $\lambda \in X^+$.*

2.2. Recall that a G-module is said to be tilting if it has both a good filtration and a Weyl filtration. It follows from Theorem 2.1 that if M is a G-module with a good filtration and Q is a tilting module then $H^i(G, M \otimes Q) = 0$ for all $i > 0$. We now prove that this is in fact another way of characterizing modules with a good filtration.

Theorem . *(Ringel* [14]*) The following conditions on a G-module M are equivalent*

i) *M has a good filtration.*
ii) *$H^i(G, M \otimes T(\lambda)) = 0$ for all $i > 0$, $\lambda \in X^+$.*

Proof: As observed above Theorem 2.1 immediately gives that i) implies ii). So assume now that ii) holds.

Another appeal to Theorem 2.1 shows that we are done if we prove

$$H^i(G, M \otimes \nabla(\lambda)) = 0 \text{ for } i > 0, \ \lambda \in X^+. \tag{1}$$

If λ is minimal (with respect to the partial order $\leq$ on X, see 1.2) in X^+ then $\nabla(\lambda) = T(\lambda)$ and (1) is clear. In general we have a short exact sequence

$$0 \to N \to T(\lambda) \to \nabla(\lambda) \to 0 \tag{2}$$

where N has a good filtration whose quotients $\nabla(\mu)$ all have $\mu < \lambda$. By induction we therefore have $\mathrm{H}^i(G, M \otimes N) = 0$ for $i > 0$. Hence (1) follows via the long exact cohomology sequence coming from (2).

2.3. Recall that for all $\lambda, \mu \in X^+$ we have [9]

$$\nabla(\lambda) \otimes \nabla(\mu) \text{ has a good filtration.} \tag{1}$$

This will be used repeatedly in the following. Moreover, we shall need

Theorem . *(Donkin* [8]*) Let $\lambda \in X^+$.*

i) *$T(\lambda)$ is injective for G_1 if and only if $\lambda \in (p-1)\rho + X^+$.*
ii) *Assume $p \geq 2h - 2$. Let $\lambda \in (p-1)\rho + X^+$ and write $\lambda = \lambda_0 + p\lambda_1$ with $\lambda_0 \in (p-1)\rho + X_p$, $\lambda_1 \in X^+$. Then $T(\lambda) \simeq T(\lambda_0) \otimes T(\lambda_1)^{(p)}$.*

Proof: We give a proof of i) which is slightly different from the one in [8].

If $\lambda \in (p-1)\rho + X^+$ then $St \otimes T(\lambda - (p-1)\rho)$ is a tilting module by (1) and injective for G_1 by 1.5. It clearly contains $T(\lambda)$ as a summand so we have the "if" part of i).

On the other hand, if $T(\lambda)$ is injective for G_1 for some $\lambda \in X^+$ then as a G_1T-module it has a Z-filtration [11],II. 11.4. One of the factors must be $Z(\lambda)$ and since for each $\alpha \in S$ the weight $\lambda - (p-1)\alpha$ occurs in $Z(\lambda)$ we conclude that $\lambda - (p-1)\alpha$ is also a weight of $T(\lambda)$. Being a G-module the set of weights of $T(\lambda)$ is stable under the Weyl group. It follows that $\langle \lambda, \alpha^\vee \rangle \geq p-1$ for all $\alpha \in S$, i.e. $\lambda \in (p-1)\rho + X^+$.

For the proof of ii) we refer to [8].

Remark . *With the notation as in ii) we have according to (1.5)*

$$T(\lambda_0)_{|G_1T} \simeq Q(2(p-1)\rho + w_0\lambda_0). \tag{1}$$

As remarked in 1.5 Donkin has conjectured that (1) holds without restrictions on p. If this is verified then we may also lift the restriction on p in ii).

2.4. Theorem 2.2 leads to the following characterization of modules which after tensoring with the Steinberg module have a good filtration.

Theorem . *The following conditions on a G-module V are equivalent*

i) *$V \otimes St$ has a good filtration.*
ii) *$V \otimes T(\lambda)$ has a good filtration for all $\lambda \in (p-1)\rho + X^+$.*
iii) *$H^i(G, V \otimes T(\lambda)) = 0$ for all $i > 0,\ \lambda \in (p-1)\rho + X^+$.*

Proof: For $\lambda \in (p-1)\rho + X^+$ we have that $T(\lambda)$ is a summand of $St \otimes T(\lambda - (p-1)\rho)$. Hence i) implies ii) (using 2.3 (1)).

By Theorem 2.1 we see that ii) implies iii). So assume that iii) holds. We shall verify that i) is then true by checking that (see Theorem 2.2)

$$H^i(G, V \otimes St \otimes T(\mu)) = 0 \text{ for all } i > 0,\ \mu \in X^+. \tag{1}$$

But $St \otimes T(\mu)$ is tilting by 2.3 (1) and injective for G_1 by 1.5. Hence when we break it into indecomposable summands

$$St \otimes T(\mu) = \bigoplus_\lambda T(\lambda), \tag{2}$$

all the λ's occurring must belong to $(p-1)\rho + X^+$ according to Theorem 2.3i). So we see that iii) implies (1).

2.5.

Corollary . *Assume $p \geq 2h-2$. Then $L(\nu) \otimes St$ has a good filtration for all $\nu \in X_p$.*

Proof: According to Theorem 2.4 it is enough to check that $H^i(G, L(\nu) \otimes T(\lambda)) = 0$ for all $i > 0,\ \lambda \in (p-1)\rho + X^+$. Since $T(\lambda)$ is injective for G_1 (see Theorem 2.3 i)) we have (by the Lyndon-Hochschild-Serre spectral sequence)

$$H^i(G, L(\nu) \otimes T(\lambda)) \simeq H^i(G/G_1, H^0(G_1, L(\nu) \otimes T(\lambda))). \tag{1}$$

Employing Theorem 2.3 ii) we get

$$H^0(G_1, L(\nu) \otimes T(\lambda)) \simeq H^0(G_1, L(\nu) \otimes T(\lambda_0)) \otimes T(\lambda_1)^{(p)}. \tag{2}$$

Now Remark 2.3 gives that $H^0(G_1, L(\nu) \otimes T(\lambda_0))$ is either 0 or k. In the first case the desired vanishing is clear. In the second case we get by (1) and (2)

$$H^i(G_1, L(\nu) \otimes T(\lambda)) \simeq H^i(G/G_1, T(\lambda_1)^{(p)}) \simeq H^i(G, T(\lambda_1)).$$

The last term is 0 for all $i > 0$ according to Theorem 2.1.

2.6.

Proposition . *Let V be a G-module. The following two conditions are equivalent*

i) *$V \otimes St$ has a good filtration.*

ii) *$V \otimes \nabla(\lambda)^{(p)} \otimes St$ has a good filtration for all $\lambda \in X^+$.*

Proof: Clearly ii) implies i) since $\nabla(0) = k$. Now assume that $V \otimes St$ has a good filtration. Then so does $V \otimes \nabla(p\lambda) \otimes St$ for all $\lambda \in X^+$ (using 2.3(1) again). However, recall from [1] that $\nabla(\lambda)^{(p)} \otimes St \simeq \nabla(p\lambda + (p-1)\rho)$. This implies that $\nabla(\lambda)^{(p)} \otimes St$ is a direct summand of $\nabla(p\lambda) \otimes St$ and ii) follows.

2.7. We can now deduce one half on the Donkin conjecture mentioned in the introduction.

Corollary . *Assume $p \geq 2h - 2$. If the G-module M has a good p-filtration then $M \otimes St$ has a good filtration.*

Proof: It is clearly enough to treat the case where $M = L(\lambda^0) \otimes \nabla(\lambda^1)^{(p)}$ for some $\lambda \in X^+$. In this case the corollary follows by combining Corollary 2.5 and Proposition 2.6.

2.8. All the results in this section have obvious analogues involving Weyl filtrations. Also combining the above results with their dual statements we obtain results on tilting modules. For instance Theorem 2.4 and Corollary 2.6 give

Corollary . *Let M be a G-module. The following conditions on M are equivalent*

i) *$M \otimes St$ is tilting.*

ii) *$M \otimes T(\mu)^{(p)} \otimes St$ is tilting for all $\mu \in X^+$.*

iii) *$M \otimes T(\lambda)$ is tilting for all $\lambda \in (p-1)\rho + X^+$.*

2.9. Let $r \in \mathbb{N}$ and replace p by p^r above. In particular, ${}^{(p^r)}$ means twist by the r-th power of F, and St_r means the r-th Steinberg module $(= L((p^r - 1)\rho))$. Then it is straightforward to generalize the above. In particular we find

Proposition . *Assume $p \geq 2h - 2$. If a G-module M has a good p^r-filtration then $M \otimes St_r$ has a good filtration.*

Proof: As above we immediately reduce to the case where $M = L(\lambda)$ for some $\lambda \in X_{p^r}$. By the Steinberg tensor product theorem we then have $L(\lambda) \simeq L(\lambda^0) \otimes L(\lambda^1)^{(p)}$ and $St_r \simeq St \otimes St_{r-1}^{(p)}$. Therefore $L(\lambda) \otimes St_r \simeq L(\lambda^0) \otimes St \otimes (L(\lambda^1) \otimes St_{r-1})^{(p)}$. By induction on r we may assume that $L(\lambda^1) \otimes St_{r-1}$ has a good filtration. Hence the proposition follows by combining Proposition 2.6 and Corollary 2.7.

3. p-FILTRATIONS.

3.1. Let

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M \tag{1}$$

be a filtration of a G-module M by G-submodules $M_j, j = 0, \cdots, r$. Recall from the introduction that we call (1) *a good p-filtration* if $M_i/M_{i-1} \simeq L(\lambda_i^0) \otimes \nabla(\lambda_i^1)^{(p)}$ for some $\lambda_i \in X^+, i = 1, \cdots, r$. Dually, we say that (1) is *a Weyl p-filtration* if $M_i/M_{i-1} \simeq L(\lambda_i^0) \otimes \Delta(\lambda_i^1)^{(p)}, i = 1, \cdots, r$. Clearly, we have

M has a good p-filtration if and only if M^*has a Weyl p-filtration.

If M has both a good p-filtration and a Weyl p-filtration then we say that M is *p-tilting.*

Remark . *If $\lambda \in X^+$ then clearly $L(\lambda^0) \otimes T(\lambda^1)^{(p)}$ is p-tilting and indecomposable. Each indecomposable tilting module has the form $T(\lambda)$ for some $\lambda \in X^+$. However, not all indecomposable p-tilting modules have the above form. If for instance two restricted simple modules extend non-trivially then the extension is clearly an indecomposable p-tilting module and it has two different composition factors as G_1-module.*

3.2. For any $\lambda \in X^+$ the G-module $L(\lambda^0) \otimes \nabla(\lambda^1)^{(p)}$ has λ as its unique highest weight. Therefore these modules constitute a basis for the Grothendieck group of the category of G-modules. If for a G-module M we let $[M]$ denote its class in the Grothendieck group then there exist unique $c_\lambda(M) \in \mathbb{Z}$, $\lambda \in X^+$ such that

$$[M] = \sum_{\lambda \in X^+} c_\lambda(M)[L(\lambda^0) \otimes \nabla(\lambda^1)^{(p)}]. \tag{1}$$

In particular, if 3.1 (1) is a good p-filtration of M then we have

$$c_\lambda(M) = \#\{i \mid \ M_i/M_{i-1} \simeq L(\lambda^0) \otimes \nabla(\lambda^1)^{(p)}\}. \tag{2}$$

We say that the number r in 3.1(1) is the length of the filtration. The length of a good p-filtration for M is denoted $l_p(M)$. By the above any two good p-filtrations of M have the same length (and up to permutation the same factors).

Similar remarks apply to Weyl p-filtrations. If M is p-tilting then clearly the length of any Weyl filtration is also equal to $l_p(M)$ (and the factors have up to permutation the same highest weights as the factors in a good p-filtration).

3.3. Suppose

$$0 \to M_1 \to M \to M_2 \to 0 \tag{1}$$

is a short exact sequence of G-modules. Obviously, if both M_1 and M_2 have good p-filtrations then so does M. Moreover, we have

Lemma . *If M_1 and M both have good p-filtrations then so does M_2.*

Proof: We consider first the case where $l_p(M_1) = 1$. If also $l_p(M) = 1$ then we must have $M_1 = M$ (because if $L(\lambda^0) \otimes \nabla(\lambda^1)^{(p)} \subset L(\mu^0) \otimes \nabla(\mu^1)^{(p)}$ then $\lambda = \mu$) and there is nothing to prove. If $l_p(M) > 1$ we let F_1 denote the first term in a good p-filtration of M. In case $F_1 = M_1$ the lemma is clear. But if $F_1 \neq M_1$ we have $F_1 \cap M_1 = 0$ and hence an exact sequence

$$0 \to M_1 \to M/F_1 \to M_2/F_1 \to 0.$$

Since $l_p(M/F_1) = l_p(M) - 1$ we conclude by induction that M_2/F_1 has a good p-filtration. Hence so does M_2.

Now consider the case where $l_p(M_1)$ is arbitrary. Again we proceed by induction on $l_p(M)$. This time we let F_1 be the first term in a good p-filtration of M_1. Then we have the following two exact sequences of G-modules

$$0 \to F_1 \to M \to M/F_1 \to 0 \tag{1}$$

and

$$0 \to M_1/F_1 \to M/F_1 \to M_2 \to 0. \tag{2}$$

The first part of the proof gives via (1) that M/F_1 has a good p-filtration. The induction hypothesis and (2) then give that M_2 has a good p-filtration.

3.4.

Lemma . *If M is a G-module whose weights μ all satisfy $\langle \mu, \alpha_0^\vee \rangle \leq p(p-h+1)$ then M is p-tilting.*

Proof: We claim that a composition series for G is both a good p-filtration and a Weyl p-filtration. In fact, if $L(\mu)$ is a composition factor of M then we have

$$p\langle \mu^1 + \rho, \alpha_0^\vee \rangle \leq \langle \mu, \alpha_0^\vee \rangle + p\langle \rho, \alpha_0^\vee \rangle \leq p(p-h+1) + p(h-1) = p^2.$$

This means that $\mu^1 \in \overline{C}$. Hence by the strong linkage principle we have $L(\mu^1) = \nabla(\mu^1) = \Delta(\mu^1)$.

Remark . *The lemma is empty if $p < h-1$. For arbitrary primes it is still true that modules with "small" weights are p-tilting. For instance, if all composition factors of M have restricted weights then M is clearly p-tilting. More generally, if for any composition factor $L(\mu)$ of M we have that μ^1 is minimal (either with respect to the partial order $\leq$ or with respect to the strong linkage relation) in X^+ then M is p-tilting. This follows by the same arguments as the ones used in the proof of the lemma above.*

3.5. Let ω be a fundamental weight. This means that for some $\alpha \in S$ we have $\langle \omega, \alpha^\vee \rangle = 1$ and $\langle \omega, \beta^\vee \rangle = 0$ for all $\beta \in S \backslash \{\alpha\}$.

Lemma . *Assume $p \geq 2h-2$. If the G-module M has a good p-filtration then so does $M \otimes \nabla(\omega)$.*

Proof: Using 2.3 (1) we immediately reduce the lemma to the case where $M = L(\lambda^0)$ for some $\lambda^0 \in X_p$. By Lemma 3.4 it is then enough to check that $\langle \lambda^0 + \omega, \alpha_0^\vee \rangle \leq p(p-h+1)$. But $\langle \lambda^0, \alpha_0^\vee \rangle \leq (p-1)\langle \rho, \alpha_0^\vee \rangle$ and $\langle \omega, \alpha_0^\vee \rangle \leq \langle \rho, \alpha_0^\vee \rangle$. Our assumption on p therefore easily gives the desired inequality.

3.6. By 2.1 (1) we see that if for some G-module M the tensor product $M \otimes St$ has a good filtration then also $M \otimes V \otimes St$ has a good filtration for any G-module V with a good filtration. If Donkin's conjecture from the introduction is true then the same should hold for modules M which have a p-filtration. This is indeed the case:

Theorem . *Assume $p \geq 2h-2$ and let M and V be two G-modules. If M has a good p-filtration and V has a good filtration then $M \otimes V$ has a good p-filtration.*

Proof: It is enough to consider the case where $V = \nabla(\lambda)$, $\lambda \in X^+$. We shall prove that $M \otimes \nabla(\lambda)$ has a good p-filtration by induction on λ with respect to the partial order $\leq_{\mathbb{Q}}$ on X.

For $\lambda = 0$ we have $\nabla(\lambda) = k$ and the claim is obvious. For $\lambda >_{\mathbb{Q}} 0$ we can find a fundamental weight ω such that $\lambda - \omega \in X^+$. Then we have a short exact sequence

$$0 \to C \to \nabla(\lambda - \omega) \otimes \nabla(\omega) \to \nabla(\lambda) \to 0$$

where C has a good filtration with quotients $\nabla(\mu)$ satisfying $\mu < \lambda$. By induction hypothesis $M \otimes C$ as well as $M \otimes \nabla(\lambda - \omega)$ have good p-filtrations. By Lemma 3.5 it follows then that so does $M \otimes \nabla(\lambda - \omega) \otimes \nabla(\omega)$. Conclusion by Lemma 3.3 .

3.7. If in Theorem 3.6 we take $M = k$ we get the following special case.

Corollary . *Assume $p \geq 2h-2$. Then $\nabla(\lambda)$ has a good p-filtration for each $\lambda \in X^+$.*

Remark . *For λ "small" the statement in this corollary is contained in Lemma 3.4. For λ sufficiently large the statement was also known, see e.g.* [10]. *The arguments in this case work for all p. They run as follows:*

Recall from 1.4 that we have $\nabla(\lambda) = \mathrm{Ind}_{G_1B}^G \hat{Z}(\lambda)$. Moreover, if $L(\mu^0) \otimes p\mu^1$ is a G_1B-composition factor of $\hat{Z}(\lambda)$ then $\mathrm{Ind}_{G_1B}^G(L(\mu^0) \otimes p\mu^1) \simeq L(\mu^0) \otimes \nabla(\mu^1)^{(p)}$ (this is 0 unless $\mu^1 \in X^+$). Assume that λ is so big that all these composition factors satisfy $\mu^1 \in X^+$. Then Kempf's vanishing theorem shows that $\mathrm{Ind}_{G_1B}^G$ will take a G_1B-composition series of $\hat{Z}(\lambda)$ into a good p-filtration of $\nabla(\lambda)$.

3.8. All results in this section have straightforward dual analogues involving Weyl p-filtrations. We leave the formulation of these dual results to the reader.

4. On Donkin's conjecture.

4.1. Let $\mathcal{C}$ denote the category of finite dimensional G-modules. Consider the following subcategories in $\mathcal{C}$

$\mathcal{C}^g = \{M \in \mathcal{C} \mid M \text{ has a good filtration}\}$,
$\mathcal{C}^t = \{M \in \mathcal{C} \mid M \text{ is tilting}\}$,
$\mathcal{C}_p^g = \{M \in \mathcal{C} \mid M \text{ has a good } p\text{-filtration}\}$,
$\mathcal{C}_p^t = \{M \in \mathcal{C} \mid M \text{ is } p\text{-tilting}\}$,
$\mathcal{C}_{St}^g = \{M \in \mathcal{C} \mid M \otimes St \text{ has a good filtration}\}$,

and

$\mathcal{C}_{St}^t = \{M \in \mathcal{C} \mid M \otimes St \text{ is tilting}\}$.

Note that by definition

$$\mathcal{C}^t \subset \mathcal{C}^g \text{ and } \mathcal{C}_{St}^t \subset \mathcal{C}_{St}^g. \tag{1}$$

Moreover, we have by 2.3(1)

$$\mathcal{C}^g \subset \mathcal{C}_{St}^g \text{ and } \mathcal{C}^t \subset \mathcal{C}_{St}^t. \tag{2}$$

For $p \geq 2h-2$ we have by Corollary 2.7

$$\mathcal{C}_p^g \subset \mathcal{C}_{St}^g \text{ and } \mathcal{C}_p^t \subset \mathcal{C}_{St}^t; \tag{3}$$

and by Corollary 3.7

$$\mathcal{C}^g \subset \mathcal{C}_p^g \text{ and } \mathcal{C}^t \subset \mathcal{C}_p^t. \tag{4}$$

Donkin's conjecture says that we should have equalities in (3) (for all p). In this section we shall prove that the two categories $\mathcal{C}_p^g$ and $\mathcal{C}_{St}^g$ do indeed share many properties.

4.2.

Proposition . *Assume $p \geq 2h-2$ and let M be a G-module which is semi-simple for G_1. Then*

$$M \in \mathcal{C}_p^g \textit{ if and only if } M \in \mathcal{C}_{St}^g.$$

Proof: By the general result 4.1(3) we only need to prove the "if-part". The assumption that M is semi-simple for G_1 means that we may write

$$M = \bigoplus_{\nu \in X_p} L(\nu) \otimes \mathrm{Hom}_{G_1}(L(\nu), M). \tag{1}$$

We may assume that there is just one summand in (1), i.e. $M = L(\nu) \otimes E^{(p)}$ for some $\nu \in X_p$. Here E is the G-module determined by $E^{(p)} = \mathrm{Hom}_{G_1}(L(\nu), M)$. We shall prove that if $M \otimes St$ has a good filtration then so does E.

By Theorem 2.4 we have $H^i(G, L(\nu) \otimes E^{(p)} \otimes T(\lambda)) = 0$ for all $i > 0$, $\lambda \in (p-1)\rho + X^+$. Arguing as in the proof of Corollary 2.5 we get then

$$H^i(G/G_1, H^0(G_1, L(\nu) \otimes T(\lambda_0)) \otimes E^{(p)} \otimes T(\lambda_1)^{(p)}) = 0 \tag{2}$$

for all $i > 0$, $\lambda_0 \in (p-1)\rho + X_p$, $\lambda_1 \in X^+$. If we choose $\lambda_0 = 2(p-1)\rho - \nu$ then we have (by 1.5 (3)) $H^0(G_1, L(\nu) \otimes T(\lambda_0)) = k$. Hence (2) gives

$$H^i(G, E \otimes T(\lambda_1)) = 0 \tag{3}$$

for all $i > 0$, $\lambda_1 \in X^+$. By Theorem 2.2 this is equivalent to $E \in \mathcal{C}^g$.

4.3. We shall now prove that for $G = SL_2(k)$ we do have equality in 4.1 (2), i.e that Donkin's conjecture is true for $SL_2(k)$. Note that for this group $h = 2$ so that the assumption $p \geq 2h - 2$ always holds.

Proposition . *Let $G = SL_2(k)$ and let p be arbitrary. A G-module M has a good p-filtration if and only if $M \otimes St$ has a good filtration.*

Proof: Let $M \in \mathcal{C}^g_{St}$. Choose λ minimal in $X^+ = \mathbb{N}$ such that $L(\lambda) \subset M$. We claim that we can extend this inclusion to an inclusion $L(\lambda^0) \otimes \nabla(\lambda^1)^{(p)} \subset M$. To see this it is clearly enough to check that $\mathrm{Ext}^1_G(L(\lambda^0) \otimes (\nabla(\lambda^1)/L(\lambda^1))^{(p)}, M)$ is zero. So consider a composition factor $L(\mu)$ of $\nabla(\lambda^1)/L(\lambda^1)$ and let R be the radical of $\Delta(\mu)$. The short exact sequence

$$0 \to R \to \Delta(\mu) \to L(\mu) \to 0$$

gives after twisting by the Frobenius and tensoring by $L(\lambda^0)$ rise to the exact sequence

$$\mathrm{Hom}_G(L(\lambda^0) \otimes R^{(p)}, M) \to \mathrm{Ext}^1_G(L(\lambda^0) \otimes L(\mu)^{(p)}, M) \to \mathrm{Ext}^1_G(L(\lambda^0) \otimes \Delta(\mu)^{(p)}, M).$$

Here the first term is zero by the minimality of λ. We claim that the last term also vanish. To see this we tensor the short exact sequence

$$0 \to k \to St \otimes St \to Q \to 0$$

by M to obtain the exact sequence

$$\mathrm{Hom}_G(L(\lambda^0) \otimes \Delta(\mu)^{(p)}, M \otimes Q) \to \mathrm{Ext}^1_G(L(\lambda^0) \otimes \Delta(\mu)^{(p)}, M)$$

$$\to \mathrm{Ext}^1_G(L(\lambda^0) \otimes \Delta(\mu)^{(p)}, M \otimes St \otimes St).$$

Here the last term is isomorphic to $\mathrm{Ext}^1_G(St \otimes L(\lambda^0) \otimes \Delta(\mu)^{(p)}, M \otimes St)$ and this is zero by the Weyl module versions of Corollary 2.5 and Proposition 2.6. To see that also the first term vanish we note that the weights of Q^* are $\leq 2p - 2$. Since $\mu \leq \lambda^1 - 2$ we see that any composition factor of $Q^* \otimes L(\lambda^0) \otimes \Delta(\mu)^{(p)}$ has highest weight $\leq 2p - 2 + \lambda^0 + p\mu \leq \lambda - 2$ and hence by the minimality of λ we conclude $\mathrm{Hom}_G(Q^* \otimes L(\lambda^0) \otimes \Delta(\mu)^{(p)}, M) = 0$.

So we have proved that $L(\lambda^0) \otimes \nabla(\lambda^1)^{(p)} \subset M$. This is then the first term in the desired good p-filtration of M. An easy induction now finishes the proof.

4.4. Our next result says that the two categories $\mathcal{C}^g_{St}$ and $\mathcal{C}^g_p$ are stable with respect to tensor products. For the first category we can prove this without any restrictions on p whereas for the second category our proof only works for $p \geq 3h-3$.

Note that by 2.3(1) we already know that the category $\mathcal{C}^g$ is stable under tensor products. Hence so is $\mathcal{C}^t$. The same result gives that $\mathcal{C}^g \otimes \mathcal{C}^g_{St} \subset \mathcal{C}^g_{St}$ and $\mathcal{C}^t \otimes \mathcal{C}^t_{St} \subset \mathcal{C}^t_{St}$. For $p \geq 2h-2$ we have also $\mathcal{C}^g \otimes \mathcal{C}^g_p \subset \mathcal{C}^g_p$ by Theorem 3.6.

Proposition . i) *Let $M_1, M_2 \in \mathcal{C}^g_{St}$ then also $M_1 \otimes M_2 \in \mathcal{C}^g_{St}$.*
ii) *Assume $p \geq 3h-3$. If $M_1, M_2 \in \mathcal{C}^g_p$ then $M_1 \otimes M_2 \in \mathcal{C}^g_p$.*

Proof: i). Using 2.3(1) we see that $M_1 \otimes M_2 \otimes St \otimes St \in \mathcal{C}^g$ and also $M_1 \otimes M_2 \otimes St \otimes St \otimes St \in \mathcal{C}^g$. Note that for a general $M \in \mathcal{C}$ the tensor product $M \otimes M^* \otimes M$ contains M as a summand. Since St is selfdual we see that St is a direct summand of $St \otimes St \otimes St$. We conclude that $M_1 \otimes M_2 \otimes St \in \mathcal{C}^g$.

ii) By 2.3(1) we easily reduce to the case where $M_1 = L(\lambda)$ and $M_2 = L(\mu)$ for some $\lambda, \mu \in X_p$. But then any weight ν of $M_1 \otimes M_2$ will satisfy

$$\langle \nu, \alpha_0^\vee \rangle \leq \langle \lambda + \mu, \alpha_0^\vee \rangle \leq \langle 2(p-1)\rho, \alpha_0^\vee \rangle = 2(p-1)(h-1).$$

Our assumption on p is chosen such that the bound in Lemma 3.4 is satisfied.

Remark . *Of course this proposition implies similar statements for the categories $\mathcal{C}^t_{St}$ and $\mathcal{C}^t_p$.*

5. Quantum groups.

5.1. Let U_q denote the quantum group corresponding to G. We shall assume that q is a primitive l-th root of unity in some arbitrary field K. More precisely, $U_q = U_{\mathbb{Z}[v,v^{-1}]} \otimes_{\mathbb{Z}[v,v^{-1}]} K$ where $U_{\mathbb{Z}[v,v^{-1}]}$ is Lusztig's "divided power" quantum group over $\mathbb{Z}[v, v^{-1}]$ and K is made into an algebra over $\mathbb{Z}[v, v^{-1}]$ via $v \mapsto q$. For convenience we assume l to be odd and if G is of type G_2 we also require l to be prime to 3 (see [12] and [4] for how we may handle even l).

We refer to [5] and [6] for general facts about finite dimensional representations of U_q. In analogy with the representations of G described in the introduction we have for each $\lambda \in X^+$ a simple U_q-module $L_q(\lambda)$, a Weyl module $\Delta_q(\lambda)$, a dual Weyl module $\nabla_q(\lambda)$, and an indecomposable tilting module $T_q(\lambda)$. These modules all have λ as their unique highest weight and they are of type **1**. As observed e.g. in [5] once we can handle type **1** modules it is easy to generalize to arbitrary finite dimensional U_q-modules. So in the following we restrict ourselves to modules of type **1**.

5.2. Let $\bar{U}_{\mathbb{Z}}$ denote the Kostant $\mathbb{Z}$-form of the universal enveloping algebra of the complex Lie algebra corresponding to G. Set $\bar{U}_K = \bar{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$. Recall that we then have a "Frobenius homomorphism" [6]

$$\mathcal{F} : U_q \to \bar{U}_K.$$

This allows us to consider each $\bar{U}_K$-module M also as a U_q-module. When equipped with the U_q-structure coming in this way via $\mathcal{F}$ we denote the module $M^{[l]}$.

We shall denote by $\bar{L}(\lambda), \bar{\Delta}(\lambda), \bar{\nabla}(\lambda)$, and $\bar{T}(\lambda)$ the $\bar{U}_K$-modules analogous to the corresponding G-modules. Note that if $K = k$ then $\bar{U}_K$ is the hyperalgebra corresponding to G and all G-modules are in a natural way also $\bar{U}_K$-modules, see [11] II.7.10–11. In this case we therefore have $L(\lambda) = \bar{L}(\lambda)$, etc. In general, we may consider the $\bar{U}_K$-modules as coming from the Chevalley group over K associated with R.

The set of restricted weights (or better l-restricted weights) are now

$$X_l = \{\lambda \in X^+ \mid \langle \lambda, \alpha^\vee \rangle < l \text{ for all } \alpha \in S\},$$

and we have for each $\lambda \in X$ the l-adic decomposition $\lambda = \lambda^0 + l\lambda^1$ with $\lambda^0 \in X_l$ and $\lambda^1 \in X^+$. Then a good l-filtration of a U_q-module M is a sequence of U_q-submodules of M such that their successive quotients have the form $L_q(\lambda_i^0) \otimes \bar{\nabla}(\lambda_i^1)^{[l]}$ for suitable $\lambda_i \in X^+$. We have of course also the dual concept of a Weyl l-filtration and we can combine to define the concept of l-tilting U_q-modules.

5.3. Suppose $\operatorname{char} K = 0$. Then we have for each $\lambda \in X^+$

$$\bar{L}(\lambda) = \bar{\Delta}(\lambda) = \bar{\nabla}(\lambda) = \bar{T}(\lambda) \tag{1}$$

and

$$L_q(\lambda) = L_q(\lambda^0) \otimes \bar{\nabla}(\lambda^1)^{[l]}. \tag{2}$$

¿From this we conclude immediately

Proposition . *When* $\operatorname{char} K = 0$ *a composition series for a finite dimensional U_q-module is both a good and a Weyl l-filtration. Hence in this case all U_q-modules are l-tilting.*

5.4. Assume also in this subsection that $\operatorname{char} K = 0$. We can carry over many of the results and arguments from the previous sections. In fact, because of 5.3(1) the situation is much simpler. Let us record the following

$$\textit{The analogues of Theorems 2.1 and 2.2 still hold.} \tag{1}$$

In the case of Theorem 2.1 this was carried out in [13].

Let $St_q = L_q((l-1)\rho)$. Then the strong linkage principle combined with 5.3(1) imply that

$$St_q \textit{ is injective in the category of finite dimensional } U_q\textit{-modules.} \tag{2}$$

Clearly, this implies that if V is an arbitrary U_q-module then $St_q \otimes V$ is injective. Hence we get from (1) that

$$St \otimes V \textit{is a tilting module for all finite dimensional } U_q\textit{-modules } V. \tag{3}$$

This was proved in [2]. Note that when combined with Proposition 5.3 this shows that the (characteristic zero) quantum analogue of Donkin's conjecture from the introduction holds.

5.5. Before we leave the characteristic zero case we want to point out that the analogue of Theorem 2.3 and Remark 2.3 hold (without restrictions on l) and are in fact easy to deduce:

Let $\lambda \in X$ have l-adic decomposition $\lambda = \lambda^0 + l\lambda^1$ as usual. Then we set

$$\hat{\lambda} = 2(l-1)\rho + w_0\lambda^0 + l\lambda^1.$$

Note that this is a bijection on X. If we restrict it to X^+ we get a bijection $\hat{}: X^+ \to (l-1)\rho + X^+$.

Theorem . *Suppose* $\operatorname{char} K = 0$ *and let $\lambda \in X^+$. Then*

i) *$T_q(\lambda)$ is injective for U_q if and only if $\lambda \in (l-1)\rho + X^+$.*

ii) *$T_q(\hat{\lambda})$ is the injective envelope of $L_q(\lambda)$.*

iii) *$T_q(\hat{\lambda}) \simeq T_q(\hat{\lambda}^0) \otimes \bar{\nabla}(\lambda^1)^{[l]}$.*

Proof: i). A similar proof as for Theorem 2.3 i) applies: St_q is injective for U_q (see 5.4(2)); when $\lambda \in (l-1)\rho + X^+$ the module $T_q(\lambda)$ is a summand of $St_q \otimes T_q(\lambda - (l-1)\rho)$ and is hence also injective; and any injective U_q-module has a Z_q-filtration. Here Z_q denotes induction from $u_q^- U_q^0$ to $u_q U_q^0$ with $U_q = U_q^- U_q^0 U_q^+$ being the usual triangular decomposition of U_q and u_q being the "small quantum group" (i.e. the subalgebra of U_q generated by the $E_i, F_i, K_i^{\pm}$'s).

ii). Being both indecomposable and injective $T_q(\hat{\lambda})$ must be the injective envelope of some $L_q(\mu)$. Now (using 5.4(3)) we see that

$$St_q \otimes L_q(\hat{\lambda} - (l-1)\rho) = T_q(\hat{\lambda}) \oplus (\bigoplus_{\hat{\nu} < \hat{\lambda}} T_q(\hat{\nu})). \tag{1}$$

Of course, ii) is obvious for $\lambda = (l-1)\rho$. Proceeding by induction on λ we may assume that $T(\hat{\nu})$ is the injective envelope of $L_q(\nu)$ for all the $\nu > \lambda$ occurring on the right hand side of (1). Since

$$\mathrm{Hom}_{U_q}(L_q(\lambda^0), St_q \otimes L_q(\hat{\lambda}^0 - (l-1)\rho)) \simeq \mathrm{Hom}_{U_q}(L_q(\lambda^0) \otimes L_q((l-1)\rho - \lambda^0), St_q) \simeq K$$

we see that $\mathrm{Hom}_{U_q}(L_q(\lambda), St_q \otimes L_q(\hat{\lambda} - (l-1)\rho))$ is non-zero. We conclude that $T_q(\hat{\lambda})$ must contain $L_q(\lambda)$.

iii) It follows from i) and 5.3 (3) that $T_q(\hat{\lambda}^0) \otimes \bar{\nabla}(\lambda^1)^{[l]}$ is a tilting module. It clearly has highest weight $\hat{\lambda}$. Hence we only have to check that it has socle equal to $L_q(\lambda)$. But $\mathrm{Hom}_{U_q}(L_q(\mu), T_q(\hat{\lambda}^0) \otimes \bar{\nabla}(\lambda^1)^{[l]}) \simeq \mathrm{Hom}_{U_q}(\bar{\nabla}(\mu^1)^{[l]}, \mathrm{Hom}_{u_q}(L_q(\mu^0), T_q(\hat{\lambda}^0)) \otimes \bar{\nabla}(\lambda^1)^{[l]}) \simeq \delta_{\mu,\lambda} K$. For the last isomorphism we have used ii) to see that $\mathrm{Hom}_{u_q}(L(\mu^0), T_q(\lambda^0)) \subset \mathrm{Hom}_{U_q}(L(\mu^0), T_q(\lambda^0)) \simeq \delta_{\mu^0,\lambda^0} K$.

5.6. Suppose now that $\mathrm{char}\, K = p > 0$. This is called the mixed case in [6] and is related to the representation theory of the corresponding finite Chevalley group in non-defining characteristics.

In this case the analogue of Steinberg's tensor product theorem says ([6])

$$L_q(\lambda) \simeq L_q(\lambda^0) \otimes \bar{L}(\lambda^1)^{[l]}, \quad \lambda \in X^+. \tag{1}$$

Since in general $\bar{L}(\lambda^1) \neq \bar{\nabla}(\lambda^1)$ we no longer have a result like Proposition 5.3. However, the theory developed in Sections 2–4 still carry over: First, Theorems 2.1 and 2.2 have straightforward analogues. In the case of Theorem 2.1 this may again be deduced from [13]. Instead of 5.4 (2) we have

Lemma . *Suppose $p \geq h$. Then St_q is injective in the category of U_q-modules whose weights λ satisfy $\langle \lambda, \alpha_0^\vee \rangle < 2lp - (l+1)(h-1)$.*

Proof: As usual St_q is injective for the small quantum group u_q. By using (1) we get therefore for any $\lambda \in X^+$

$$\mathrm{Ext}^1_{U_q}(L_q(\lambda), St_q) \simeq \mathrm{Ext}^1_{U_q}(\bar{L}(\lambda^1)^{[l]}, \mathrm{Hom}_{u_q}(L_q(\lambda^0), St_q)).$$

This is clearly 0 unless $\lambda^0 = (l-1)\rho$. In that case it equals $\mathrm{Ext}^1_{\bar{U}_K}(\bar{L}(\lambda^1), K)$. By the strong linkage principle the smallest weight for which this Ext-group is non-zero is $\lambda^1 = (p-h+1)\alpha_0$ (this is the weight obtained by reflecting 0 in the first "dominant" hyperplane). We conclude that $\mathrm{Ext}^1_{U_q}(M, St_q) = 0$ for all U_q-modules M whose weights λ satisfy $\langle \lambda, \alpha_0^\vee \rangle < \langle (l-1)\rho + l(p-h+1)\alpha_0, \alpha_0^\vee \rangle = (l-1)(h-1) + 2l(p-h+1)$.

5.7. We continue to assume that char $K = p > 0$. For each $\lambda \in X$ we have a unique irreducible $u_qU_q^0$-module with highest λ, namely $L_q(\lambda^0) \otimes l\lambda^1$. Here we have written $l\lambda^1$ instead of $(\lambda^1)^{[l]}$ for the one dimensional $u_qU_q^{[l]}$-module obtained by composing $\mathcal{F}$ and λ^1.

Denote by $Q_q(\lambda)$ the injective envelope of $L_q(\lambda^0) \otimes l\lambda^1$. Clearly we have an isomorphism of $u_qU_q^0$-modules $Q_q(\lambda) \simeq Q_q(\lambda^0) \otimes l\lambda^1$.

Using the bijection ^ from 5.5 we get

Proposition . *If $p \geq 2h-2$ then we have a $u_qU_q^0$-isomorphism $Q_q(\lambda) \simeq T_q(\hat{\lambda})_{|_{u_qU_q^0}}$ for all $\lambda \in X_l$.*

Proof: The arguments go as in the modular case: $T_q(\hat{\lambda})$ is a U_q-summand of $St_q \otimes T_q(\hat{\lambda} - (l-1)\rho)$ and is therefore injective as a $u_qU_q^0$-module. Moreover, we have $\mathrm{Ext}^1_{U_q}(M, T_q(\hat{\lambda})) \subset \mathrm{Ext}^1_{U_q}(M, St_q \otimes T_q(\hat{\lambda} - (l-1)\rho)) = \mathrm{Ext}^1_{U_q}(M \otimes T_q(\hat{\lambda} - (l-1)\rho)^*, St_q)$. Now $T_q(\hat{\lambda} - (l-1)\rho)^* = T_q((l-1)\rho - \lambda)$ and hence Lemma 5.6 shows that this Ext-group is zero if the weights ν of M satisfy $\langle \nu + (l-1)\rho - \lambda, \alpha_0^\vee \rangle < 2lp - (l+1)(h-1)$, i.e. if $\langle \nu, \alpha_0^\vee \rangle < 2lp - (l+1)(h-1) - (l-1)(h-1)$. This means that $T_q(\hat{\lambda})$ is injective in the category of U_q-modules whose weights ν satisfy $\langle \nu, \alpha_0^\vee \rangle < 2lp - 2l(h-1)$. Note that our assumption on p ensures that $T_q(\hat{\lambda})$ belongs to this category for all $\lambda \in X_l$. Being also indecomposable it follows that in this category $T_q(\hat{\lambda})$ is the injective envelope of $L_q(\lambda)$ (it is easy to check that $L_q(\lambda) \subset T_q(\hat{\lambda})$). It follows that $\mathrm{Hom}_{u_q}(L_q(\mu), T_q(\hat{\lambda})) = 0$ for all $\mu \in X_l \setminus \{\lambda\}$ and that $K \subset \mathrm{Hom}_{u_q}(L_q(\lambda), T_q(\hat{\lambda}))$. To finish the proof we observe that by easy weight considerations we have $\mathrm{Hom}_{u_qU_q^0}(L_q(\lambda), St_q \otimes T_q(\hat{\lambda} - (l-1)\rho)) = K$.

5.8. Just like in the modular case (see Theorem 2.3.ii) and [8]) we deduce from Proposition 5.7

Corollary . *Assume $p \geq 2h-2$. Let $\lambda \in (l-1)\rho + X^+$ and write $\lambda = \lambda_0 + l\lambda_1$ with $\lambda_0 \in (l-1)\rho + X_l$. Then $T_q(\lambda) \simeq T_q(\lambda_0) \otimes \bar{T}(\lambda_1)^{[l]}$.*

Remark . *Note that if $\lambda_1 \in (p-1)\rho + X^+$ then we can use Theorem 2.3.ii) to factorize $\bar{T}(\lambda_1)$ further.*

5.9. The task of formulating (and proving) the analogues of the modular results from 2.5–8 as well as all results in Sections 3–4 is now straightforward. One just have to replace p by l and to add an index q or a $^{-}$ to the appropriate modules. We leave details to the reader.

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