UNIVERSITY OF AARHUS
DEPARTMENT OF MATHEMATICS

ISSN: 1397–4076

COMMUNICATION IN COHERENTLY MODELLED HUMAN POPULATIONS

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Preprint Series No.: 4

June 2000

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Communication in coherently modelled human populations

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Contents

To Krista, the guardian spirit of this paper

1. Introduction

In this paper mathematics is used to build a meta-language designed to formalize already existing concepts of the social and political sciences. As a consequence we obtain results based on strict mathematical deductions which are in resonance with the notions already existing within those areas. The structures which form the skeleton of the theory as presented in this paper, are of pure mathematical character. Within these structures the process of communication is modelled. Fortunately, the algebraic and analytic formalism which is used has a built-in direct channel to statistics so that by use of this channel we can utilize statistics to verify theoretical results. However, this process cannot be reversed. The original structure cannot be recovered from the statistical models.

It should be understood that the proposed approach opens up a vast area of possibilities of which this paper covers only a scintilla.

Once the meta-language is ready, it can be used to formalize arbitrarily complicated problems and reach conclusions which would be hard or impossible to obtain by traditional means.

Hence the paper is addressed not that much to researchers using mathematics for direct treatment of some ready empirical data as to researchers in the area of social, political and behavioural sciences who would like to formulate their findings in a language open to both theoretical and experimental treatment.

Confrontation, even to a modest degree, of the already existing empirical data with theoretical predictions based on the results of the present paper requires time and facilities not available at the moment to the authors and is hence not present in our paper.

However, the lack of quantitative results is to some degree compensated by a number of qualitative results the essence of which is collected in Corollaries 4.1, 5.3, 5.5 and 5.9.

To make the presented subjects accessible to readers with insufficient mathematical background, each section starts with a non-mathematical synopsis where the goal, terminology and the results are explained without the use of mathematics. Then follows the mathematical version of the non-mathematical synopsis which requires familiarity with some linear algebra (cf. [2]).

We use the term "communication" to refer to the global flow of information within a population and not to single exchanges between individuals.

People communicate simultaneously on many subjects. In order to extract a part of the global communication we consider "a domain of communication" fixed beforehand by way of the so-called "space of attitudes". The space of attitudes consists of all possible answers to a "questionnaire" covering a chosen scope of interest - here the term questionnaire amounts to a list of statements which can be rejected or accepted by an individual.

When a respondent analyses and eventually selects a statement, the decision is by no means of a deterministic nature. It depends on the actual "state-of-mind" of the respondent. States-of-mind are changing under the influence of the flow of information constantly circulating within the population.

The concept of state-of-mind is more general, better manageable and not less intuitive than the concept of attitude. This is why the latter is eventually subdued by the former. The main reason for this is that though a concrete attitude can be expressed in the form of a number of statements, the slightest deviation in those statements produces a new attitude. This leads to indeterminism which can be managed by using the notion of a state-of-mind.

The set of all attitudes of members of the investigated population shall be called an "opinion". Neither the notion of opinion has a deterministic nature, being the function of a constantly varying flow of information. In this paper we introduce the notion of "profile". We shall consider "profiles" which are social units usually smaller and more specialized than a stratum (cf.[9]) by specific range of interest. Small profiles can be unified to create a new profile with an appropriately extended range of interest.

The collection of states-of-mind of individuals of a profile does not describe by itself the process of communication within the profile. We need a new entity which will be called a state-of-profile and which will be able to reflect consequences of the interaction within the population of the profile, i.e. variations of attitudes of individuals under the influence of exchanging views, listening to the news, reacting on actions of the authorities etc. This interaction has no deterministic description and its consequences have to be intercepted by mechanisms designed in the metatype (cf. 3.1.2, 3.2.1). The experimental background of the formalism we propose primarily consists of polls.

The change of states-of-profile is called a fundamental process. All fundamental processes we consider are built of two types: the process of alteration and the process of consolidation.

To show the power of the introduced formalism, we list a number of theoretical results that seem to be significant and well supported by intuition but hardly verifiable by methods other than those presented in the present paper.

In Section 5 we discuss, and illustrate with examples, different forms of population memory.

We describe the process of education in Section 6. Finally in Section 7 we analyse profiles of political parties.

The so-called "coherent state" of a population will be the one chiefly considered. A formal definition of coherence will be given later, but at the present stage we develop an intuitive comprehension of coherence. Roughly speaking, a coherent population consists of members that have principally the same attitude to a number of selected subjects of communication. States-of-mind of the respondents will vary but will all maintain solidarity with the "pattern state-of-mind" called the *mode*. A branch of striking employees - though their individual views may differ, are all loyal to the main cause. An invaded nation - there is a clear coherence regarding defence of the fatherland.

In a democratic country with a stable economy, the issue "way of life" will usually manifest as a coherent state.

It is important to keep in mind the division of the presented material into two logical categories. In the background lies what we call the *type* which includes the language together with all the statements and procedures concerning the individuals of the population: teaching programs, social improvement programs, election campaigns, etc. all expressed in the language of the type. Then comes what we call the *metatype* which includes all definitions, descriptions and statements concerning the population as a whole. Our goal is to present statements expressed in the language of metatype: the meta-language. The precise description of the type and metatype languages is found at the end of Section 3.2.1.

There exists vast literature where mathematics has the leading role in analysing sociological problems (cf. [1], [10], [3] etc.). The authors believe that in the present paper the situation is quite opposite. The humanistic and empirical content of the paper plays the leading role and actually gives birth to its mathematics. The mathematics is not just being attached to sociological concepts but appears as their consequence.

Remarks and suggestions of our colleague Hans Anton Salomonsen helped us quite a lot in improving the paper. We are grateful for his contribution.

2. The domain of communication

2.1. Non-mathematical synopsis

2.1.1. Attitudes

Consider a population of people who are familiar with a subject which can be described in a series of statements. The statements can be accepted or rejected by members of the population. In what follows we refer to this list of statements as a *questionnaire.* The term "questionnaire" should not be taken literally. For instance, a questionnaire may consist of a set of examination questions; but it can also be an ordinary questionnaire prepared for a poll.

A copy of a completed questionnaire shall be called an *attitude* and each delivery of a filled-in questionnaire is called a *response.* The origin of a response we call a *source* and sometimes a *respondent.* The most common procedure of collecting responses is by way of making a poll.

In general a source is a single person but it can be also an organization. Though our chief concern will be the cases where sources are single persons, it will be convenient to consider other kinds of sources.

The limitation of the usefulness of the notion of attitude is due to its rigidity, i.e. the fact that a slight change in formulation results in creating a new attitude. Saying "it is a good procedure" and saying "it is not a bad procedure" carry trust and mistrust attitudes respectively. Saying "turn it slowly to the right" or just "turn it to the right" may result in repairing or destroying a gadget. It shows that the demand for more precise descriptions of reality will cause an uncontrollable increase of the number of attitudes.

A remedy for this difficulty consists in deriving from the notion of attitude a more advanced but also considerably more efficient and intuitive notion, called the "state-of-mind", which we discuss in Section 2.1.2 below.

2.1.2. States-of-mind

Usually a source does not propagate a single attitude - it has a number of possible attitudes at its disposal: likes and dislikes, preferred and suppressed notions, obligations etc. At each moment of time, the "mind" of a source has preferences as to the choice of an attitude. These preferences we call *states-of-mind.* This elaborates the idea that responses are not of a deterministic nature. A particular state-of-mind of a source tells us to which degree a given attitude is acceptable to the source.

The states-of-mind arising within a given domain of communication shall be called *associated* to this domain.

However, the "probabilistic" description of the states-of-mind is not sufficient to reveal the background interaction between the members of the population and the flow of information within the population. By the "background interaction" we understand the uncontrollable change of states-of-mind caused by subconsciously registered information. To incorporate the interaction problem, we must have a possibility of producing a "superposition" of states-of-mind. This requires using a mathematical description of the notion of a state-of-mind which will be introduced in the mathematical part, Section 2.2.2.

2.1.3. Questions

The questions we consider are those to which the answers can be either "yes" or "no" and which mostly concern a fixed preference in the process of selecting attitudes (questions which would demand more complicated answers can always be converted to a sequence of "yes-no" questions). Answering a question is not a deterministic procedure since it depends on the state-of-mind of the source. Given a question and the state-of-mind of the source, we would like to have means to estimate the chance of getting the answer "yes" or "no". This can be done because questions naturally associate with the so-called orthogonal projections (cf. Section 2.2.3).

2.1.4. Profiles

Suppose that we have a domain of communication with its collection of all states-of-mind. We can select a smaller collection of those states-of-mind connected with a special subject. Such "sub-collections" of states-of-mind will be called *profiles*. A profile constitutes itself a domain of communication with respect to the attitudes, states-of-mind and questions relevant to the profile. This way every domain of communication can be considered as a collection of overlapping and interacting profiles.

One can consider the profile of parliament members of a democratic country. The space of attitudes of this profile will consist of different political convictions and states-of-mind will concern actual political problems. Also the government and its members and officials constitute a profile. Here the set of attitudes will for example contain different policies. The states-of-mind will consist of preferences regarding those policies.

As already mentioned, the same physical population contains many different profiles. Profiles connected with professions are easiest revealed by asking a question to which the answer "yes" selects the states-of-mind of the profession. For example, the question: "do you have a valid certificate qualifying you as a physician?" automatically extracts the profile of medical doctors. Usually an examination filters a profile of a particular profession.

Often when the domain of communication is divided into disjoint profiles, the process is called *stratification* and each profile of such a division is called a *stratum* (cf.[9]). A typical stratification will be the division into the stratum of lower class, the stratum of middle class and the stratum of upper class.

2.1.5. States-of-mind instead of attitudes

Though the notion of state-of-mind may carry a greater load of abstraction than that of attitude, it is indeed a more suitable notion to describe the communication. States-of-mind are also better fitted for doing mathematics. The concept of attitude does not reflect the element of indeterminism present in the exchange of information by use of everyday language. Let us consider the following

Example 2.1. *Two TV-set repairers examine a set that does not work. One of them tries to make it work, without success; the other revives the set. We conclude that in the state-of-mind of the first repairer, the right attitude toward the defect had very low or no preference while the same attitude had high preference in the case of the other. They both have the same training and qualifications but each one chooses his own priorities out of the identical set of attitudes. If faced with a different task, the situation might change so that now the first repairer finds a solution, the second tries without success.*

There can be many different reasons for the malfunction of an electronic device, especially if every two cases differing minimally are counted separately. Each detailed description of a repair process provides an attitude: one turn or two turns of the same screw provide different attitudes no matter whether the difference is or is not significant. This leads to an absurdity unless we find an approach that takes care of all the involved indeterminism. Faced with a malfunctioning TV-set, a repairer does not start checking piece-by-piece the hundreds of different possibilities but "makes up his mind" regarding what has to be done. Hence, instead of the notion of attitude, it is more expedient to start at once considering states-of-mind. We need attitudes mostly to help understanding the origin of the concept of a state-of-mind. From now on we shall use attitudes only occasionally and take states-of-mind and questions as basic ingredients of the concept of a domain of communication. This way we shall find ourselves on considerably firmer ground.

It was explained in Section 2.1.1 that a source should not necessarily be a person but can also be an organization. In such a general case, the notion of attitude becomes still more complicated to work with. Consequently, we build our understanding of communication principally on the concept of state-of-mind, referring to the notion of attitude only when it becomes necessary or convenient.

2.1.6. Energy of propagation

Consider a set-up consisting of a source-profile propagating a message and the target-profile receiving the message. The message consists of the collection of responses from the individuals of the profile - these responses we shall call *emissions*.

We establish the strength of the impact of a single emission by fixing the number of *energy-bits* it carries. This number indicates the "power" of the profile that emits the messages. The impact of the message might cause a change in the state of the target if the total number of energy-bits of the propagation is sufficiently high.

We can have a situation where a single response from a source carries *less* than one energy-bit. An election provides a good example of this situation. Once the total number of votes of an election has been counted, the Members of Parliament are selected on the basis of the appropriate percentages of the votes. If we count the number of votes necessary to elect an MP as one energy-bit, an ordinary vote is just a fraction of the energy-bit. Recording less than one energy-bit in support of a party does not mean that there were no votes from sources supporting the party. But the total number of votes did not sum up to a full energy-bit sufficient to elect an MP representing the party.

A single energy-bit emitted by the government profile should be in resonance with the number of votes sufficient to elect an MP. Consider an example. Let the target-profile be the government. And the source-profile be citizens protesting against an unsatisfactory administration of an issue. Such a protest will often be met with a barrier of ordinances preventing recognition of the claim, and unless the number of protesting respondents is very high, the target will not react by changing its state. As already mentioned, at each propagation a single citizen sends a small fraction of an energy-bit assigned to e.g. an MP or a government official.

Take now the parliament as the source-profile. An energy-bit propagated by a single MP amounts to a great number of electors' votes. Hence a moderate number of energy-bits propagated by the parliament can easily turn out to be sufficiently large to change the state of the target, i.e. to make the government take steps to satisfy the parliament.

A single response from a source can also contain *more* that one energy-bit. Consequently, the total number of energy-bits can be arbitrarily high or low no matter how many sources are propagating pieces of information or statements. To illustrate the case where a vote can carry more than one energy-bit, consider a profile of stockholders. Suppose that we assign one energy-bit to a share of stocks. Then the energy carried by a stockholder's vote will equal the number of shares he owns. In this case a single voter can propagate a large number of energy-bits.

Another example: Consider a group of terrorists as a source - their number is not proportional to the consequences of their actions which are often quite destructive. Here a single propagation contains a high number of energy-bits. Hence, assignations of energy-bits to sources depend on the nature of the involved profiles. In certain cases it can be convenient to take a single vote as one energy-bit.

The formula for the expected number of energy-bits emitted by all the sources of a profile in a fixed state (i.e. the formula for the expected total number of energy-bits) is given in Section 3.2.2.

It is important to understand that all the internal descriptions of the nature of propagation, as for instance preparation of an election, are expressed in the language of the type and have no meaning in the language of the metatype which is used to express the formulas for the expected total number of energy-bits.

A sociologist working within the type uses his intrinsic knowledge of the profile he investigates to construct an appropriate sociological model. Then, to experiment with the model, he moves to the metatype in order to collect experimental data. It is of utmost importance not to mix the knowledge that comes from the type with the data collected by performing the experiment in the metatype: a teacher knows that one of his pupils is talented and knowledgable, but of a shy and nervous disposition, who performs badly at examinations. However, the process of examination amounts to collecting data in an experiment within the metatype. Therefore the results of the examination cannot be influenced by the statements formulated within the type. A student is classified by his or her performance at the examination and not by the intrinsic knowledge of hidden handicaps.

If a substantial disagreement between the experimental and the theoretical results occurs, then either the state-of-profile or the energy-bits were incorrectly assigned.

Later it will turn out that without the concept of energy, the notion of a coherent state which is central in this paper cannot be correctly interpreted (cf. Section 4.2.1).

2.2. The mathematics of the domains of communication

The aim of this section is to provide a mathematical language covering the basic notions of the process of communication such as attitudes, energies, states-of-mind and questions which together provide what we shall call the *domain of communication.* Later on we shall also need the mathematical language to formalize the notions of opinion, state-of-profile and coherence. The basic requirement from the involved mathematics is that it should lead to direct confrontation with the experimental data, mostly results of polls.

2.2.1. The necessary mathematics

The mathematical concept crucial for this paper is the notion of a *vector space* $\mathcal{X}$ with *inner product* $\langle , \rangle$ (also called a *scalar product*) (cf. [2]). The inner product is a function of two variables from $\mathcal{X}$ with real numbers as values. The number $\sqrt{\langle z, z\rangle}$ is called the *length of the vector* z.

A vector x from $\mathcal{X}$ is called a unit vector if $\langle x, x\rangle = 1$. A unit vector x is called a *state* if we disregard its sign. However, in what follows, writing "state" we shall always refer to a single vector x and not to the pair $-x, x$.

To simplify and shorten the notation we briefly write $z_/$ for the state $\frac{z}{\sqrt{\langle z,z\rangle}}$ corresponding to a vector z from $\mathcal{X}$, i.e. $z_/$ denotes the vector z divided by its length.

Given states x and y, the number $\langle x, y\rangle^2$ is called the *correlation* of the states x and y.

Remark 1. *The correlation $\langle x, y\rangle^2$ gives the probability that the numerical results of an experiment performed on an object in the state x will coincide with the results of the same experiment performed on an object in the state y and conversely.*

To any two states x, y there corresponds a unique vector z orthogonal to x such that $y = \alpha x + z$. But then $\alpha = \langle x, y\rangle$ which means that $\langle x, y\rangle^2$ gives the probability of getting identical results of observations in either of the states x and y. States with correlation 0 are called *uncorrelated.* We extend this terminology for arbitrary vectors saying that x and y are *uncorrelated* if $\langle x, y\rangle = 0$ (more customary it would be to say that x and y are *orthogonal*).

Having a number of states $x_1, x_2, \cdots, x_m$ and numbers $\alpha_1, \alpha_2, \cdots, \alpha_m$, the state $(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + x_m \alpha_m)_/$ shall be called a *superposition* of the states $x_1, x_2, \cdots, x_m$.

In the present paper the notion of superposition is an essential component of the formalism describing the process of communication. When considering a population composed of some smaller fractions, which we call profiles, the state of the whole population is the superposition of the states of its different profiles.

More detailed information about the involved mathematics can be found in [2].

The notion of attitude was already introduced and discussed in Section 2.1.1 and requires no more attention.

2.2.2. States-of-mind

The state-of-mind of a source tells about the preferences of the source, i.e. the degree to which a given attitude $\mathfrak{m}_j$, $j = 1, 2, ..., k$, is attractive to a source. The source will propagate the attitude $\mathfrak{m}_j$ with probability p_j attached by the state-of-mind to the attitude $\mathfrak{m}_j$. The sum of all p_j is equal to 1.

The probabilistic description of the state-of-mind given in the non-mathematical synopsis is not sufficient to get hold of the interaction and the flow of information within the population. To repair this we must consider the states-of-mind as vectors.

At first, to each attitude $\mathfrak{m}_j$ we assign a vector $1_{\mathfrak{m}_j}$ called the *eigenstate (-of-mind)* corresponding to this attitude: when asked about preferences, a source in eigenstate $1_{\mathfrak{m}_j}$ will, with probability 1, choose the attitude $\mathfrak{m}_j$ from the set $\{\mathfrak{m}_1, \mathfrak{m}_2, ..., \mathfrak{m}_k\}$ of all attitudes.

In the next step we introduce the vector space $\mathcal{F}$ of all linear combinations

$$x = t_1 1_{\mathfrak{m}_1} + t_2 1_{\mathfrak{m}_2} + \cdots + t_k 1_{\mathfrak{m}_k}, \tag{2.1}$$

of the eigenstates $1_{\mathfrak{m}}$, where t_j are real numbers. We call the vector space $\mathcal{F}$ *associated* with the considered domain of communication. Alternatively, the linear combination (2.1) can be interpreted as a function assigning to each attitude $\mathfrak{m}_j$ a number t_j so that:

the associated space $\mathcal{F}$ is the space of
functions, assigning numbers to attitudes.

The background interaction between the members of the population manifests numerically through the inner product $\langle , \rangle$. We choose $1_{\mathfrak{m}_i}$ to be unit vectors, i.e. $\langle 1_{\mathfrak{m}_i}, 1_{\mathfrak{m}_i} \rangle = 1$ for each i. Then we have to decide which value is to be given to $\left\langle 1_{\mathfrak{m}_j}, 1_{\mathfrak{m}_i} \right\rangle$ for $\mathfrak{m}_j$ different from $\mathfrak{m}_i$. We restrict ourselves to the situation in which attitudes are selected in such a way that $1_{\mathfrak{m}_j}$ are pairwise uncorrelated, i.e. $\left\langle 1_{\mathfrak{m}_j}, 1_{\mathfrak{m}_i} \right\rangle = 0$ for any pair of different i and j. Then, given vectors $x = s_1 1_{\mathfrak{m}_1} + s_2 1_{\mathfrak{m}_2} + \cdots + s_k 1_{\mathfrak{m}_k}$ and $y = t_1 1_{\mathfrak{m}_1} + t_2 1_{\mathfrak{m}_2} + \cdots + t_k 1_{\mathfrak{m}_k}$ from $\mathcal{F}$, we get

$$\langle x, y \rangle = s_1 t_1 + s_1 t_2 + \cdots + s_k t_k. \tag{2.2}$$

States corresponding to this special choice of a vector space with inner product $\mathcal{F}, \langle , \rangle$ shall be called the *states-of-mind* of the sources. Since we assume that our eigenstates $1_{\mathfrak{m}}$ are pairwise uncorrelated, given a state-of-mind x, we get

$$\langle 1_{\mathfrak{m}_1}, x \rangle^2 + \langle 1_{\mathfrak{m}_2}, x \rangle^2 + \cdots + \langle 1_{\mathfrak{m}_k}, x \rangle^2 = 1,$$

where $p_j = \langle 1_j, x \rangle^2$ is the probability that a source in the state x will propagate the attitude $\mathfrak{m}_j$ as if it were in the state $1_{\mathfrak{m}_j}$.

Remark 2. *It should be clearly understood that saying that eigenstates $1_{\mathfrak{m}}$ and $1_{\mathfrak{n}}$ are uncorrelated does not mean that the texts involved in formulating the attitudes $\mathfrak{m}$ and $\mathfrak{n}$ are logically or linguistically uncorrelated. By saying that eigenstates $1_{\mathfrak{m}}$ and $1_{\mathfrak{n}}$ are uncorrelated, we understand that no source in the state $1_{\mathfrak{m}}$ will ever compromise with a source in the state $1_{\mathfrak{n}}$.*

2.2.3. Questions

To connect mathematical arguments with the concept of question, we will be bound to consider a question (Q) as a linguistic formulation of the question coupled with its mathematical image which is an orthogonal projection Q called the projection *attached* to (Q). The parenthesis $(\cdot)$ indicate that the mathematical object Q is coupled with its linguistic counterpart.

The orthogonal projection Q attached to a question (Q) is determined by fixing two sets of states-of-mind, mutually uncorrelated. The first set is denoted by Q_{acc} and called the set of *states-of-acceptance.* It consists of the set of vectors x such that $Qx = x$. If a source is in a state-of-acceptance, it will give the answer "yes" to Q with probability 1. The other set is denoted by Q_{rej} and is called the set of *states-of-rejection.* It consists of the set of vectors x such that $Qx = 0$. If a source is in the state-of-rejection, it will answer "no" to Q with probability 1. Given a state x and a question Q, we define

$$\left.\begin{array}{l}\text{the probability of obtaining the}\\ \text{answer "yes" to the question } Q\\ \text{from a source in state } x\end{array}\right\} = \langle Qx, x\rangle\,.$$

Take a state-of-mind x and consider the projection

$$Q_x y = \langle x, y\rangle\, x, \tag{2.3}$$

where y runs over $\mathcal{F}$. We introduce the question (Q_x) attaching to the projection Q_x the linguistic formulation

"Are you in the state x?".

We call this type of questions the *status-questions.* The only state-of-acceptance for this question is x. Take an arbitrary state y from $\mathcal{F}$. We can see that the probability

$$\langle Q_x y, y\rangle = \langle x, y\rangle^2 \tag{2.4}$$

of the answer "yes" to (Q_x) asked in the state y is equal to the correlation between x and y.

Suppose we have fixed a space of attitudes $\mathfrak{M}$. Given an attitude $\mathfrak{m}$ from $\mathfrak{M}$, consider the question "*do you select precisely the attitude* $\mathfrak{m}$ *out of the collection of all possible attitudes?*". This question we identify with the status-question $(Q_{1_\mathfrak{m}})$, "*are you in the state* $1_\mathfrak{m}$?". Here $1_\mathfrak{m}$ is the only state-of-acceptance i.e. sources that accept $\mathfrak{m}$ with probability 1 are exactly those that answer "yes" to the question with probability 1.

As another example we consider the projection.

$$Qx = \langle 1_{\mathfrak{m}}, x\rangle\, 1_{\mathfrak{m}} + \langle 1_{\mathfrak{n}}, x\rangle\, 1_{\mathfrak{n}},$$

where $\mathfrak{n}$ and $\mathfrak{m}$ are two different attitudes from $\mathfrak{M}$. The question corresponding to this projection should read: are you in the superposition of the states $1_{\mathfrak{m}}$ and $1_{\mathfrak{n}}$? Then, assuming that the source understands only plain attitudes and not their superpositions, the question assumes the form: "*do you favor precisely the attitudes* $\mathfrak{m}$ *and* $\mathfrak{n}$ *out of the collection of all possible attitudes?*" Here we have $Qx = x$, exactly for $x = \langle 1_{\mathfrak{m}}, x\rangle\, 1_{\mathfrak{m}} + \langle 1_{\mathfrak{n}}, x\rangle\, 1_{\mathfrak{n}}$ which means that $x = s1_{\mathfrak{m}} + t1_{\mathfrak{n}}$, with $s^2 + t^2 = 1$ are the states-of-acceptance.

The negation of a question (Q) is the question $(I - Q)$ which consists of the linguistic negation of the question (Q) with attached projection $I - Q$.

2.2.4. Profiles

Consider a domain of communication with the associated space $\mathcal{F}$. Selecting a profile amounts to fixing a subspace $\mathcal{P}$ of the associated space $\mathcal{F}$ or, which is equivalent, selecting a question: "do you belong to the profile $\mathcal{P}$?" accompanied by the appropriate projection Q. Given the space $\mathfrak{M}$ of attitudes, the most natural way of defining a profile is fixing a subspace $\mathfrak{N}$ of the space of all attitudes $\mathfrak{M}$. Then to $\mathcal{P}$ belong all states-of-mind from $\mathcal{F}$ that assign the probability 0 to the attitudes which are not in $\mathfrak{N}$, i.e. $\mathcal{P}$ consists of those x given by (2.1), where all $t_{\mathfrak{m}}$, with $\mathfrak{m}$ not in $\mathfrak{N}$, are equal to zero. It is clear that a profile generates its own domain of communication.

For example, to select a profile of musicians we can use the question (Q_{music}), where the linguistic part of (Q_{music}) is: "do you have some musical education?" and the space of affirmations for Q_{music} is spanned by all the states in which the professional attitude to music is marked. Then we take as the space $\mathcal{P}$ assigned to the profile the space of affirmations of the question.

3. States-of-profile

3.1. Non-mathematical synopsis

3.1.1. Opinions

Let us suppose that using a poll we have gathered information about the actual distribution of attitudes of a profile. It means that we have a collection of attitudes, where the same attitude may appear many times, a single time or not at all. Repetition of the same attitude can have different origins. Either there is a number of persons in the population sharing the same attitude or, if the poll allows it, the same person can appear as source more than once, expressing by this action more than average concern. A respondent attracted by a subject is apt to express his concern more frequently. If he is a stockholder, his responses will be counted by the number of stocks he owns. The described collection of attitudes shall be called an *opinion* of the profile.

3.1.2. States-of-profile

The notion of an opinion of a profile is parallel to the notion of an attitude of an individual source. Hence, on the level of population we must introduce something parallel to the notion of state-of-mind. This shall be called *state-of-profile,* for short just *state*. Similarly, like a state-of-mind assigns a number to every possible attitude, a state-of-profile assigns a number to every possible opinion of a profile. The square of the number assigned to an opinion gives the probability that this is exactly the opinion of the profile.

If a population is a conglomerate of a number of profiles, the state of the whole population is not a plain collection of the states of its parts. There is a background interaction between profiles, and the state of the whole population must be dependent on this interaction. This dependence is taken into account by the process of superposition of states.

3.1.3. Occupation numbers

The concept "occupation number" is very simple. Let us say that we have a profile in a fixed state. We would like to know how a profile in this state will react to an issue in the form of a question (Q), e.g. *Will you vote for X?* We do not count just single answers "yes" but the energy-bits coming with every answer "yes" to the question. It means that a vote will be counted once (as it happens at most elections) only if the energy assigned to a single "yes" is one energy-bit. As already pointed out in Section 2.1.6, the energy assigned to a vote can be less than one energy-bit but it can also be more. Hence the expectation of less than one energy-bit does not mean that there were no answers "yes" (cf. 2.1.6).

The *occupation number* gives the total number of energy-bits coming from sources answering "yes" to the question. In the mathematical part we give a formula for the "expected" occupation number.

3.2. The mathematics of the states-of-profile

3.2.1. Opinions and states-of-profile

Suppose we have fixed a domain of communication and let $\mathfrak{M} = \{\mathfrak{m}_1, \mathfrak{m}_2, ..., \mathfrak{m}_n\}$ be the set of all attitudes. Then $(k_1 \times \mathfrak{m}_1, k_2 \times \mathfrak{m}_2, ..., k_n \times \mathfrak{m}_n)$ is the *opinion* in which the attitude $\mathfrak{m}_j$ appears k_j times for $j = 1, 2, ..., n$. If a particular attitude, say $\mathfrak{m}_j$, does not appear at all, we write $k_j = 0$.

To introduce the states-of-profile, the vector space $\mathcal{F}$ associated with the considered domain of communication must be extended to a larger vector space $\widetilde{\mathcal{F}}$ consisting of all finite sums of "product-vectors" of the form $x_1 x_2 \cdots x_k e^x$ (we disregard the order of appearance of the vectors in the product), where $x_1, x_2, \cdots, x_k$ are arbitrary vectors from $\mathcal{F}$, and the vector e^x, where x is also taken from $\mathcal{F}$, is given by the series

$$e^x = \emptyset + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

and is called a *coherent vector*. The vector $\emptyset = e^0$ is called the *vacuum state*. The vacuum state refers to the situation where no opinion can be detected in the profile.

The inner product in $\widetilde{\mathcal{F}}$ which yields the appropriate statistics is complicated. It can be derived from the property

$$\langle e^x, e^y \rangle = e^{\langle x,y \rangle} \tag{3.1}$$

of the inner product of coherent vectors. The involved mathematics can be found in [5]. However, here we operate only with the results of the mathematical argumentation disregarding the techniques necessary to obtain the results save a few formal proofs given in the Appendix.

The space $\widetilde{\mathcal{F}}$ provided with the inner product fulfilling (3.1) will be called the *grand associated space*. Similarly, starting from the associated space $\mathcal{P}$ of a profile we define the grand associated space $\widetilde{\mathcal{P}}$ of the profile. Take a vector f from $\widetilde{\mathcal{F}}$. Then f is called a state-of-profile if it fulfils the condition $\langle f, f \rangle = 1$.

The notions of associated space and grand associated space can be used to give the precise description of the concepts of type and metatype. We say that all constructions concerning the structure of the associated space $\mathcal{F}, \langle , \rangle$ belong to the type; in other words, the type is characterized by constructions within $\mathcal{F}$, as for instance choice of particular attitudes, states-of-mind etc.

To the metatype belong all the constructions performed within the grand associated space $\widetilde{\mathcal{F}}, \langle , \rangle$. Intrinsic properties of any concrete representation of $\mathcal{F}, \langle , \rangle$ are not allowed be directly interpreted within $\widetilde{\mathcal{F}}, \langle , \rangle$.

As an example we use Socrates' famous antinomy.

Consider a profile X. Suppose that a respondent issues a statement: "all the respondents from the profile X lie". Now suppose that the same respondent declares: "I am a respondent from the profile X". As we know, there is no way out of this logical loop. The concept of profile belongs to the meta-language, hence the respondent expressing a global opinion about the profile X acts within the metatype. His subsequent declaration "I am a respondent from the profile X" belongs to the type. Either he is an observer investigating a special profile or he is being observed as a member of the profile but not both things at the same time.

3.2.2. Occupation numbers and their statistics

Take a state-of-profile f and consider a question (Q). The *occupation number* of (Q) at f is the number of energy-bits from positive responses to (Q) while the profile is in the state f.

To find the probability distribution of the occupation numbers of (Q) at f we proceed as follows. First we extend the projection Q to a "derivation" $d\Gamma Q$ in the grand associated space $\widetilde{\mathcal{F}}$ which is a mapping fulfilling the Leibniz relation

$$d\Gamma Q\,(fg) = (d\Gamma Q f)\,g + f\,(d\Gamma Q g)\,,$$

where f and g are arbitrary vectors from $\widetilde{\mathcal{F}}$. For example, for a coherent vector e^x, where x is taken from $\mathcal{F}$, we get

$$d\Gamma Q e^x = (Qx)\,e^x. \tag{3.2}$$

Then we find projections $Q^{(n)}$ of $\widetilde{\mathcal{F}}$ into itself such that

$$Q^{(n)}\left(Q^{(m)}g\right) = 0 \text{ for } m \neq n$$

and

$$(d\Gamma Q)\,g = \sum_{n=1}^{\infty} nQ^{(n)}g$$

for each g taken from the grand associated space $\widetilde{\mathcal{F}}$. For example,

$$Q^{(n)}e^x = \frac{1}{n!}\,(Qx)\,x^{n-1}.$$

The detailed description of $d\Gamma Q$ and $Q^{(n)}$ is given in [5] and [7].

Having $Q^{(k)}$ and the state-of-profile f, we define the probability that in the state f, the occupation number of energy-bits coming from the answer "yes" is equal to k :

$$\left.\begin{array}{l}\text{the probability that the total number of}\\ \text{energy-bits of answers "yes" to } (Q) \text{ propagated}\\ \text{by a profile in the state-of-profile } f \text{ is equal to } k\end{array}\right\} = \left\langle f, Q^{(k)}f\right\rangle. \tag{3.3}$$

The importance of $d\Gamma Q$ manifests in the following statement:

$$\left.\begin{array}{l}\text{the expected total number of}\\ \text{energy-bits of answers "yes" to } (Q)\\ \text{propagated by a profile in the state } f\end{array}\right\} = \langle f, (d\Gamma Q)\, f\rangle\,. \tag{3.4}$$

For $Q = I$ we have

$$\left.\begin{array}{c}\text{the expected total number of energy-bits}\\ \text{in the state-of-profile } f\end{array}\right\} = \langle f, (d\Gamma I)\, f\rangle\,. \tag{3.5}$$

Now, suppose that we know the total number of sources. Then

$$\left.\begin{array}{l}\text{the expected number of energy-bits}\\ \text{in a single emission of the source}\end{array}\right\} = \frac{\langle f, (d\Gamma I)\, f\rangle}{\textit{total number of sources}}.$$

The number of sources can stay unchanged but their energy levels can change, influencing the number $\langle f, (d\Gamma I)\, f\rangle$.

4. Coherent states

4.1. Non-mathematical synopsis

4.1.1. The concept of coherence

Consider a fixed domain of communication and take a profile within this domain. It can be the total population of a country, a profile of employees in a specific branch, the profile of a branch of employees in the state of striking; it can be the profile of people joined by a common belief or need, the profile of a profession or of identically educated people, the profile of supporters or the profile of antagonists towards a political or economical program, a profile fixed by a particular choice of socio-economical coordinates etc.

In all those cases we have the collection of states-of-mind (Section 2.1.2) from which we select a *mode* state-of-mind with the property that the other states-of-mind of sources of the profile are only small deviations from the mode. In the internal process of communication about the subject reflected by the mode state-of-mind there exists an essential feeling of unity. The sources of such a profile "cohere" relative to a subject or a cause reflected by the mode. We say that the profile is in a *coherent state*.

One more factor is necessary to determine a coherent state: *energy.* The energy of a coherent state is a number which reflects the social, political or economical strength of the state. It registers how much power is injected into the coherence built from a particular state-of-mind taken as its mode.

We can consider profiles characterized by professions: physicians, engineers, masons, musicians, chauffeurs etc. The respondents of such profiles are usually people who have obtained special education and passed appropriate examinations. An examination followed by a certificate works like a filter and afterwards the profile of profession manifests through appropriate professional organizations as trade unions or societies. In the mathematical section we shall see how the corresponding coherent states are defined.

4.1.2. Passive and active polarized states

Consider a population consisting of two profiles, each in a coherent state, and let the state of the whole population be a superposition of these coherent states. We then say that the population is in a *polarized state*. The polarized state is then a state of two interacting coherent profiles.

Polarized states can be divided into two types - one is called *passive* and refer to the situation where profiles of coherent components live together in peace like e.g. a number of Protestant and Catholic communities in Germany. The other type is called *active* and is well illustrated by the example of Protestants and Catholics in Northern Ireland. In the active polarized state the fractions fight each other. The passive and active polarized states can also be observed in smaller communities: villages, small towns or districts, where the residents are divided into two incompatible fractions that live apart in peace if the total polarized state is a passive one; or fight each other if the total polarized state is an active one. The mathematical characterization of passiveness and activeness of polarized states will be given in Section 4.2.2.

By *polarization* we understand the manifestation of a polarized state, the consequence of a conflict between the respective coherent states of the sub-profiles. Examples of polarization are numerous ; slight conflicts of interests between, say, left and right wings in politics, employers and employees polarizing the population. Here the energies of interaction are low. Examples of polarization with high level energy of interaction (a large number of energy-bits per signal) are observed in the so-called independence movements: Basques in Spain; Curds in Turkey; groups trying to establish Islamic rule in a number of states etc. The latter cases demonstrate polarization, where a small coherent group with quite a high energy level can act against the majority of the population of the profile which usually is in an uncorrelated coherent state. We discuss the mathematics of the subject in Section 4.2.2.

4.2. The mathematics of coherence

4.2.1. Definition of a coherent state

Since coherence does not occur on the level of states-of-mind but only on the level of states-of-profiles, we shall briefly say "coherent state" instead of coherent state-of-profile. Take a vector x from the associated space $\mathcal{F}$. The *coherent state* $c(x)$ generated by x is the normalized coherent vector e^x :

$$\begin{aligned} c(x) &= e^x_{/} = e^{-\frac{1}{2}\langle x,x\rangle} e^x \\ c(0) &= \emptyset. \end{aligned} \tag{4.1}$$

The number $\langle x, x\rangle$ will be called the *expected energy* of the coherent state $c(x)$ and the state-of-mind $x_{/}$ will be called the *mode* of the coherent state $c(x)$; the vector x is called the *generating* vector of the coherent state $c(x)$. Hence in the background of a given coherent state lies the mode which is the state-of-mind that provides the right frequencies of occurrence of the attitudes from a fixed list. Given a coherent state, we can approximate the mode for this coherent state as follows. We produce a "super-questionnaire" out of all involved attitudes; then count the frequencies of the choice of particular attitudes and take their square roots as coefficients to the respective eigenstates.

Clearly

$$1 = \langle c(x), c(x)\rangle = \Sigma_{n=0}^{\infty} \frac{e^{-\langle x,x\rangle}}{n!} \langle x, x\rangle^n ,$$

so that

$$\text{the probability that the total energy consists of } n \text{ bits} = \frac{e^{-\langle x,x\rangle}}{n!} \langle x, x\rangle^n .$$

This means that the number of bits is Poisson distributed. As a function of n, this probability increases as long as $\langle x, x\rangle > n$ and then it decreases for $\langle x, x\rangle < n$. The most probable number of bits is $n = \langle x, x\rangle$ which is only symbolic if $\langle x, x\rangle$ is not a natural number. This, of course, coincides with the expected total number of bits,

$$\mathrm{E}c(x) = \Sigma_{j=0}^{\infty} n \frac{e^{-\langle x,x\rangle}}{n!} \langle x, x\rangle^n = \langle x, x\rangle . \tag{4.2}$$

Hence, if $\langle x, x\rangle$ is smaller than 1, we do not expect recording one or more energy-bits. But a single response might represent only a small part of an energy-bit. This means that there can be several responses which, however, did not sum up to a whole energy-bit. Let us return to the example discussed in Section 2.1.6. Every packet containing the required percentage of votes elects an MP. Say the percentage necessary for electing an MP requires n votes. Even a substantial number of votes will not, as long as their number is less than n, elect an MP. This means that an energy-bit emitted by an MP as a source is equivalent to n votes.

Observe that if the number $\langle x-y, x-y\rangle$ is very large, the correlation

$$\langle c(x), c(y)\rangle^2 = e^{-\langle x-y, x-y\rangle} \tag{4.3}$$

is almost 0, i.e. $c(x)$ and $c(y)$ are almost uncorrelated. Hence as a consequence of Remark 1 of Section 2.2.1 and (4.3) above, any experiment performed in one of those states has almost no probable relation to an experiment performed in the other state.

The coherent states are "almost" multiplicative; we have

$$c(x+y) = e^{-\langle x,y\rangle} c(x) c(y).$$

Given a question Q, we get

$$d\Gamma Q c(x) = (Qx) c(x) \tag{4.4}$$

so that (3.4) gives

$$\left.\begin{array}{l}\text{the expected total number}\\ \text{of energy-bits of answers}\\ \text{"yes" to } Q \text{ propagated from}\\ \text{the coherent state } c(x)\end{array}\right\} = \langle d\Gamma Q c(x), c(x)\rangle = \langle x, Qx\rangle. \tag{4.5}$$

In particular, for $Q = I$, the number

$$\langle c(x), (d\Gamma I) c(x)\rangle = \langle x, x\rangle$$

gives the expected total number of energy-bits coming from a profile in the state $c(x)$ which, of course, coincides with (4.2). Now it becomes clear that the energy-bits described in Section 2.1.6 cannot be identified with single emissions from a source. The number $\langle c(x), (dI\Gamma) c(x)\rangle = \langle x, x\rangle$ can be arbitrarily large while the number of sources does not exceed a fixed number N. Hence, if at each emission all the sources of a coherent state send identical packages of energy-bits, the expected number of energy-bits per package must be $\frac{1}{N}\langle x, x\rangle$.

Take a state-of-mind y. Then routine computations (see Appendix 8.1) show that the probability of finding at least j states y in the state $c(x)$ of the population is

$$p\left(y^j\right) = e^{-\langle y,x\rangle^2} \Sigma_{n=j}^{\infty} \frac{1}{n!} \langle y, x\rangle^{2n}. \tag{4.6}$$

Hence, the probability of finding a respondent in the state-of-mind y within a profile in the coherent state $c(x)$ is $e^{-\langle y,x\rangle^2} \Sigma_{n=1}^{\infty} \frac{1}{n!} \langle y, x\rangle^{2n} = e^{-\langle y,x\rangle^2} \left(e^{\langle y,x\rangle^2} - 1\right) = 1 - e^{-\langle y,x\rangle^2}$.

In particular, the probability of finding the state-of-mind $y = \frac{x}{\langle x,x\rangle^{\frac{1}{2}}}$ is $1 - e^{-\langle x,x\rangle}$. This probability is never 1 but it approaches 1 when the expected energy of the coherent state $c(x)$ increases.

4.2.2. Passive and active polarized states

Given coherent states $c(u)$ and $c(v)$, where $u \neq v$, and a number λ, we consider the superposition

$$c_\lambda(u,v) = (c(u) + \lambda c(v))_/ \,. \tag{4.7}$$

The state of the form (4.7) shall be called a *polarized state.* For positive λ the state $c_\lambda(u,v)$ will be called *passive,* and for negative λ *active.* The passive polarized state does not encourage open confrontation between its coherent components. They might disagree but they are not in combat. The parts of an active polarized state are in combat (cf. Section 4.1.2).

The greater is $\langle u-v, u-v\rangle$, the more the states $c(u)$ and $c(v)$ act as uncorrelated, and $c_\lambda(u,v)$ describes a profile split into two groups which are hardly able to cooperate within a common domain of communication.

The situation is called a *polarization* and was described in Section 4.1.2.

With fixed u and v, when λ increases to infinity, the state $c_\lambda(u,v)$ converges to the state $c(v)$, and when λ decreases to zero, it converges to the state $c(u)$. Simple computation gives the expected number of energy-bits of responses "yes" to a question (Q),

$$\begin{aligned} &\langle c_\lambda(u,v), (d\Gamma Q)\, c_\lambda(u,v)\rangle \\ &= \varphi_\lambda\left(\langle Qu,u\rangle + \lambda^2\langle Qv,v\rangle + 2\lambda\langle Qu,v\rangle e^{-\frac{1}{2}\langle u-v,u-v\rangle}\right), \end{aligned} \tag{4.8}$$

where

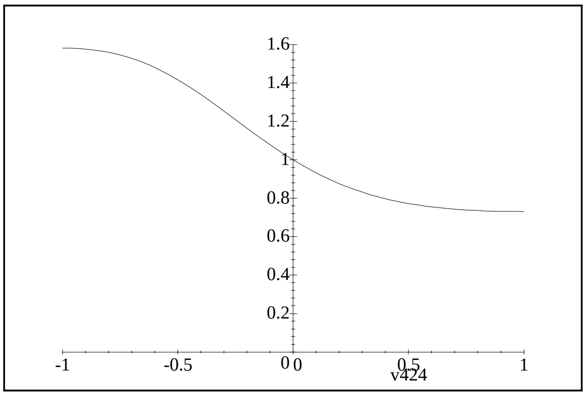
$$\varphi_\lambda = \frac{1}{1+\lambda^2+2\lambda e^{-\frac{1}{2}\langle u-v,u-v\rangle}} \tag{4.9}$$

(since $u \neq v$, we have $1+\lambda^2+2\lambda e^{-\frac{1}{2}\langle u-v,u-v\rangle} \neq 0$) and the term $2\lambda\varphi_\lambda\langle Qu,v\rangle e^{-\frac{1}{2}\langle u-v,u-v\rangle}$ takes care of the energy-bits which appear in consequence of the interaction between Qu and v.

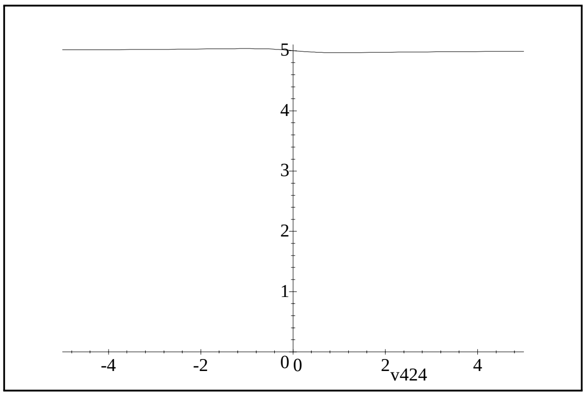
To illustrate the formula (4.8) we set $\langle u,v\rangle = 0$ and calculate

$$\langle c_\lambda(u,v), (d\Gamma I)\, c_\lambda(u,v)\rangle = \frac{(1+\lambda^2)\langle u,u\rangle}{1+\lambda^2+2\lambda e^{-\langle u,u\rangle}},$$

which is the emitted energy. For $\langle u,u\rangle = \langle v,v\rangle = 1$, the emitted energy as a function of the parameter λ is :

$\langle u, u \rangle = 1$: 

As we can see the number of responses is larger for λ negative than for λ positive. This asymmetry, so visible in this example, depends on the value of $\langle u, u \rangle$. When $\langle u, u \rangle$ increases, the asymmetry decreases rapidly and the graph for $\langle u, u \rangle = 5$ is fairly symmetric:

$\langle u, u \rangle = 5$: 

Corollary 4.1. *Sources of an active polarized profile of low energy emit more energy-bits than sources of a passive profile obtained from the identical coherent components.*

Analysing some concrete cases will make it necessary to consider polarization into states which are more composed than plain coherent states - the discussion above provides no more than guidance to future constructions and analysis of more complicated cases.

The active case will be further discussed in Section 5.1.3.

5. Fundamental processes

The pressures coming from the outside may urge a profile to change its state. For example: education will provide better opportunities, acquiring experience in politics may change a common respondent into a politician, learning how the stock market operates presents a chance to change ones socio-economical coordinates, perfection in pursuing some line of research can induce promotion etc.

It may also happen that an outside pressure forcing a change is not welcome by the sources of a profile and provokes resistance which might cause withdrawal of the pressure.

The course of changing a state-of-profile into another state-of-profile is called a *fundamental process.* In the present paper we limit ourselves to consider fundamental processes which arise as the result of a superposition of a number of two basic types of fundamental processes. The mathematics of those basic processes is well-known though not quite elementary. The first process is related to the Weyl transformation (cf. 5.2.1) and the other to the so-called functor Γ (cf.[5]). The respective fundamental processes are those of *alteration* and *consolidation.*

Note that the social forces responsible for changes of states-of-profile are described in the language of the type and as such cannot be expressed in the language of the metatype in which the results of this paper are written (cf. the end of Section 3.2.1).

5.0.3. The problem of time

Evaluation of the period of time used to carry through a fundamental process is not included in the theory. It may amount to days, months or years until a fundamental process is completed. The situation can be modelled on a simple example from elementary physics. Define the state of a solid block of mass m as the "potential energy" of the block, i.e. the product of its mass m times the height h over the surface of the earth. Lifting a block of mass m from the height h to the height $h+1$ changes its state and increases the potential energy to $m\,(h+1)$. But nothing is said as to how the process was accomplished or how long it lasted. At the time of the Pharaohs it could take years while in our era mostly the time to assemble a crane counts.

5.1. Non-mathematical synopsis

5.1.1. The process of alteration

Consider a profile in a fixed state. An external force can change the state into a new one. By changing for example the mode of a coherent state-of-profile (cf. Section 4.1.1), we can move the profile to a new coherent state. Education represents one of such forces. Unfortunately, indoctrination must also be counted as a force that can change states-of-profile and within the meta-language there are no criterions that distinguish between education and indoctrination: we cannot tell what is right and what is wrong without referring to experience, tradition, custom and knowledge expressed by the language of the profile. However, the social forces as for example a parliament or a dictatorship can change informally formulated rules to laws.

The process of alteration can meet resistance of sources of the profile and result in producing a polarized state being a superposition of the original state against the altered state. A number of important examples appear in connection with the notion of polarization (cf. Section 4.1.2). In Example 5.1 we discuss a case of alteration, the aim of which is to remove some features from the mode of a coherence. Though it has to involve some mathematics, the spirit of the example can be understood without its mathematical background.

5.1.2. The process of consolidation

Take a profile in a fixed state. A process of *consolidation* changes a state-of-profile by rearranging the states-of-mind of sources following a special procedure, e.g. by homework, apprenticeship, training, participating in joint activities, contemplation, or any other form of rearranging and securing the available states-of-mind without introducing new information.

As an illustration we can consider Example 2.1. At the starting point, the state of the concerned profile is: unskilled TV-set repairers. Let us assume that one of the two repairers from the example is a skilled one and the other is an apprentice. Then the process described in Example 2.1 amounts to instructing the apprentice. After a sequence of instructions, the apprentice becomes a skilled repairer.

Consolidation manifests in the process of learning. Pupils return home after lessons and do their homework reading notes, prescribed sections in textbooks, solving exercises etc. In this way they consolidate their knowledge. After completing the homework, the state of the class changes into a new consolidated state.

We show that an appropriate consolidation can remove the influence of activity/passivity of a polarization (cf. Proposition 5.5).

Examples concerning teaching will be discussed in Section 6.1.

5.1.3. Reflection of a past experience

Consider a profile in a fixed state. Suppose that an external force tries to change this state. It can be the negotiation of a new law, acceptance of a new kind of food - e.g. originating from gene-manipulated vegetables, an election campaign or something as serious as invasion by a foreign power.

We consider a state-of-profile, perform its alteration by a state-of-mind as described in Section 5.1.1 and take the superposition of the original and the altered states into an active polarized state (cf. 5.10). Suppose that the force causing the alteration fades away. We compute the final state-of-profile after the altering force has completely disappeared. Does the profile return to its original state after the disappearance of the forcing factor or does the conflicts of the past influence the final state?

The answer is quite remarkable and not trivial to verify. After removal of the external force, the polarized profile does not return to its state of origin but becomes a composition of the state of origin with the enforcing state-of-mind. The memory of the experience is reflected in the new state. A population "remembers" its past (cf. Example 5.8, 5.10).

What we have described is a global phenomenon which can, however, be detected by analysing the change in the output of the sources of the profile.

The change of a state by adding a "memory" state-of-mind will manifest in the expected total numbers of energy-bits which will increase by 1. The reflecting factor of the post-conflict state-of-profile will activate some sources which were not emitting before the conflict.

Any historical major event, e.g. the rise and fall of communism in European countries, does not pass without leaving a trace in the state of the involved population.

5.1.4. Disappearance of a conflict-induced polarization

The procedure of taking the superposition and then passing to the limit as described in Section 5.1.3 can also be applied in the situation where a profile in a fixed state is subjected to simultaneous pressure of two opposite forces driven by two different states-of-mind. Acting with a given intensity against each other, they produce an active polarized state: those in favour of acceptance of the first state-of-mind and those in favour of acceptance of the second one. Suppose that in time, forces driven by those states-of-mind disappear. Then, as in the case discussed in Section 5.1.3, the profile does not return to the original state but to a new state which contains the "remembrance" factor of the conflict caused by the forces that disappeared.

A population "remembers" encounters from its immediate past. How long-lasting is the memory depends on the involved energies. A dissolved serious conflict between workers and employers will be remembered for maybe a year but memories of a world war are preserved in history books.

5.2. The mathematics of fundamental processes

5.2.1. The process of alteration, Weyl transformations

The process of alteration is best seen and explained on a profile in a coherent state $c(x)$. Say there exist forces altering the coherent state $c(x)$ to another coherent state $c(y)$. Then, writing $z = y - x$, we can consider z as the vector altering the generating vector x of the given coherent state to a new generating vector $x + z$ of the new coherent state $c(x+z) = c(y)$. This reduces the process of alteration of a coherent state to application of a transformation W_z dependent on a vector z from $\mathcal{P}$ which incorporates both the state-of-mind $z_/$ of the alteration and its expected energy $\langle z, z \rangle$. The described transformation

$$W_z c(x) = c(x+z)$$

of $c(x)$ into $c(x+z)$ is called the *Weyl transformation* and is uniquely extendable to a linear isometry (i.e. states-to-states transformation) of the grand associated space $\widetilde{\mathcal{P}}$ onto itself. In particular, given a state $\alpha(x_1 x_2 \cdots x_k)\, c(x)$ from the grand associated space $\widetilde{\mathcal{P}}$, (p. 16), we have

$$\begin{aligned} & W_z \alpha(x_1 x_2 \cdots x_k)\, c(y) \\ = \;& \alpha(x_1 + \langle z, x_1 \rangle \emptyset)(x_2 + \langle z, x_2 \rangle \emptyset) \cdots (x_k + \langle z, x_k \rangle \emptyset)\, c(z+y) \end{aligned}$$

(cf. [5]).

Due to (4.5) the expected number of responses after applying W_z to $c(x)$ will be

$$\begin{aligned} & \langle d\Gamma I(W_z c(x)), W_z c(x) \rangle \\ = \;& \langle x+z, x+z \rangle = \langle x, x \rangle + \langle z, z \rangle + \langle x, z \rangle + \langle z, x \rangle \\ = \;& \langle x, x \rangle + \langle z, z \rangle + 2\sqrt{\langle x, x \rangle \langle z, z \rangle} \cos v, \end{aligned}$$

where

$$\cos v = \frac{\langle x, z \rangle}{\sqrt{\langle x, x \rangle \langle z, z \rangle}}$$

can be any number from 1 to -1. Consequently, depending on the angle between vectors x and z, the expected number of responses can vary from $\left(\sqrt{\langle x, x \rangle} - \sqrt{\langle z, z \rangle}\right)^2$ to $\left(\sqrt{\langle x, x \rangle} + \sqrt{\langle z, z \rangle}\right)^2$.

Example 5.1. *Take a coherent state of the form $c(x + ny)$, where x is a vector from $\mathcal{F}$ uncorrelated with a state-of-mind y and n is a number. The mode of this* *coherent state depends on two uncorrelated vectors x and y. Suppose we want to alter $c(x+ny)$ by removing the influence of the state-of-mind y (the state-of-mind y can represent some prejudices or some incorrect pieces of information but it can as well represent disagreement with the official policy). We apply the Weyl transformation W_{ky} and obtain a new coherent state $c(x+(n-k)y)$. Only when $n=k$, the state-of-mind y will be wiped off the mode of $c(x+ny)$.*

Application of W_{ky} with a too large number k can result in alienating the whole population which in consequence may convert some originally neutral sources to become sympathizers of the state-of-mind y. For example, some actions of Basque terrorists resulted in a considerable loss of popularity of their cause. The exaggerated terror of a regime against dissidents often increases the popularity of the dissidents' cause.

One should realize that in concrete cases, designing and applying W_z is a complicated process. For example, successful teaching uses the whole spectrum of traditional methods and ad hoc approaches to design and apply W_z.

5.2.2. The process of consolidation

The *consolidation* of a state f from the grand associated space $\widetilde{\mathcal{P}}$ starts on the level of the associated space $\mathcal{P}$. We take a linear transformation U of $\mathcal{P}$ onto itself and assume that it preserves the correlations: $\langle Ux, Uy\rangle = \langle x, y\rangle$ for any pair of vectors x, y from $\mathcal{P}$. Such transformations are called *unitary.* The state Ux is supposed to consolidate the state x in the sense described in Section 5.1.2. In practice the realization of U amounts to a program which can be an apprenticeship, a period of practice or a period of preparation for an examination, making up one's mind about political affiliations etc. The reader should be aware of the fact that the program which is supposed to consolidate a state is designed within the type and only the results of records of emissions (cf. Section 2.1.6) can be confronted with the appropriate theoretically derived results. We say that U *designs* a process of consolidation. We shall need the *operator of consolidation* which is the unique extension of U to a multiplicative unitary transformation $\widetilde{U}$ of the grand associated space $\widetilde{\mathcal{P}}$. The formal definition of $\widetilde{U}$ is given in the Appendix, Section 8.2.

Any question (Q) in the original profile can be reformulated to the question $\left(\widetilde{Q}\right)$ in the consolidated profile in such a way that all statistical properties are preserved. For this purpose the projection $\widetilde{Q}$ is chosen in such a way that the transformations UQ and $\widetilde{Q}U$ are identical. The equivalence of statistics attached to questions (Q) and $\left(\widetilde{Q}\right)$ amounts to the identical expectations

$$\langle d\Gamma Q f, f\rangle = \left\langle \left(d\Gamma \widetilde{Q}\right) \widetilde{U} f, \widetilde{U} f \right\rangle \tag{5.1}$$

of the number of energy-bits of answers "yes" to (Q) and to $\left(\widetilde{Q}\right)$.

We illustrate this procedure on the status-question (Q_u) (cf. 2.3), "are you in the state-of-mind u ?" with the assigned projection $Q_u x = \langle u, x\rangle\, u$, for x from $\mathcal{P}$. The reformulated question $(Q_{\widetilde{u}})$ in the consolidated profile is "are you in the state-of-mind $\widetilde{u}$?", where $\widetilde{u} = Uu$ is the state u after consolidation, and the assigned projection is $\widetilde{Q_u} x = Q_{\widetilde{u}} x = \langle \widetilde{u}, x\rangle\, \widetilde{u}$ for x from $\mathcal{P}$.

5.2.3. Alteration and consolidation in teaching

Consider a class of pupils as a profile in a coherent state $c(x)$, where x is taken from the associated space $\mathcal{P}$ of the profile. Let x reflect the fact that the pupils were taught how to add fractions and perfected this technique to a reasonable degree. Still there is a possibility of finding a pupil producing e.g. "$\frac{1}{3} + 1 = \frac{1+1}{3}$"!

Now the pupils enter the next degree of initiation: learning the abstract algebraic formula for adding fractions: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

It is the teacher who extends the associated space $\mathcal{P}$ to a larger associated space $\mathcal{P}_1$ containing an appropriate state-of-mind z capable of implanting the abstract formula in the heads of the pupils. The teacher and the appropriate pedagogical approach reflect the action of the Weyl transformation W_z producing the alteration $c(x+z)$ of the state $c(x)$.

The state-of-mind $(x+z)_/$ displays both the concrete and the abstract addition of fractions, but pupils in this state-of-mind do not necessarily understand the connection between those two procedures.

Within $\mathcal{P}_1$ there exists a state-of-mind y which provides understanding of the abstract model as well as its connection with the concrete computations. Now consolidation provides a unitary transformation U changing the state-of-mind $(x+z)_/$ into y. In practice, this amounts to solving exercises till the pupils reach states-of-mind sufficiently close to y to pass a qualifying test.

The subject will be continued in Section 6.

5.2.4. Superposition with the vacuum

We have some general results. Given a question (Q), we get

$$\frac{\langle d\Gamma Q f, f\rangle}{\left\langle d\Gamma Q\,(f+\lambda\emptyset)_/\,,(f+\lambda\emptyset)_/\right\rangle} = 1 + 2\,\langle f, \emptyset\rangle + \lambda^2$$

which proves the following

Theorem 5.2. *The number of energy-bits of answers "yes" to a question (Q) asked in the state $(f+\lambda\emptyset)_/$ will diminish compared to the number of energy-bits* *of answers "yes" to the same question asked in the state f. The diminishing rate is* $\frac{1}{(1+2\lambda\langle f,\emptyset\rangle+\lambda^2)}$, $\left(\frac{1}{(1-|\lambda|)^2} \geq \frac{1}{(1+2\lambda\langle f,\emptyset\rangle+\lambda^2)} \geq \frac{1}{(1+|\lambda|)^2}\right)$.

Corollary 5.3. *The superposition of a state from the grand associated space with the vacuum, which is the state that propagates no responses, produces a new state, where some of the sources, which were active before the superposition, stop propagating.*

5.2.5. Removal of polarization by alteration

Consider a polarized state $c_\lambda(u,v)$ given by (4.7), where u and v are uncorrelated. From (4.8) we get the expected total number of energy-bits in the state $c_\lambda(u,v)$,

$$\langle c_\lambda(u,v), (d\Gamma I)\, c_\lambda(u,v)\rangle = \Big(\langle u,u\rangle + \lambda^2 \langle v,v\rangle\Big)\,\varphi_\lambda, \tag{5.2}$$

where φ_λ is given by (4.9). Applying W_{-v} to $c_\lambda(u,v)$ removes (by an outside action) the component $c(v)$ and produces the "altered" state,

$$\left(c(u-v)+\lambda\emptyset\right)_/ = W_{-v}c_\lambda(u,v) = \left(c(u-v)+\lambda c(0)\right)_/. \tag{5.3}$$

The expected total number of energy-bits in this state is

$$\Big\langle d\Gamma I\left(c(u-v)+\lambda\emptyset\right)_/, \left(c(u-v)+\lambda\emptyset\right)_/\Big\rangle = \left(\langle u,u\rangle+\langle v,v\rangle\right)\varphi_\lambda, \tag{5.4}$$

and the ratio of the expected total number (5.4) by the expected total number (5.2) is

$$\mathrm{R}_\lambda = \frac{\langle u,u\rangle + \langle v,v\rangle}{\langle u,u\rangle + \lambda^2\langle v,v\rangle}.$$

We can see that the result of the alteration by use of W_{-v} depends on λ. For $\lambda < 1$, the ratio R_λ indicates that after the alteration the expected total number of energy-bits increases up to $1 + \frac{\langle v,v\rangle}{\langle u,u\rangle}$. There is no change if $\lambda = 1$. But for $\lambda > 1$, the ratio R_λ indicates that after the alteration the expected total number of energy-bits decreases to zero when the numerical value of λ tends to infinity.

The sub-profile in the state $c(u-v)$ of the altered profile in the state $\left(c(u-v)+\lambda\emptyset\right)_/$ of $c_\lambda(u,v)$ relates to the sources which accepted the mode $(u-v)_/$ reflecting the compromise.

These conclusions can be interpreted directly on the alteration $\left(c(u-v)+\lambda\emptyset\right)_/$ of $c_\lambda(u,v)$. The sub-profile in the "state of compromise" $c(u-v)$ collects the sources which accepted the mode $(u-v)_/$ reflecting the compromise. The sources that cannot accept the compromise withdraw to the sub-profile in the vacuum state $\emptyset$ and hence provide no responses. The smaller is λ, the lesser is the number of sources staying mute. Conversely, the larger is λ, the greater is the number of sources staying mute.

Example 5.4. *Suppose that a profile describes a community of two coexisting sub-profiles, the "yellows" and the "blues", with respective attitudes* $\mathfrak{y} = \{$*I do not go along with the blues*$\}$ *and* $\mathfrak{b} = \{$*I do not go along with the yellows*$\}$*. We consider the associated space* $\mathcal{P}_{\mathfrak{y}\&\mathfrak{b}}$ *spanned by uncorrelated vectors* $1_{\mathfrak{y}}$ *and* $1_{\mathfrak{b}}$*. Let the state of this profile be* $c_\lambda\left(1_{\mathfrak{y}}, 1_{\mathfrak{b}}\right)$*. Applying* $W_{-1_{\mathfrak{b}}}$ *to* $c_\lambda\left(1_{\mathfrak{y}}, 1_{\mathfrak{b}}\right)$ *gives the state* $\left(c\left(1_{\mathfrak{y}} - 1_{\mathfrak{b}}\right) + \lambda\emptyset\right)_{/}$ *from the grand associated space* $\widetilde{\mathcal{P}}_{\mathfrak{l}\&\mathfrak{r}}$*. Within the sub-profile in the state* $c\left(1_{\mathfrak{y}} - 1_{\mathfrak{b}}\right)$*, the blues and yellows coexist in a kind of mutual understanding. The blues and yellows that still cannot stand each other are pushed away to the vacuum state and keep quiet.*

In real life, producing an appropriate $W_{-1_{\mathfrak{b}}}$ *is usually a difficult socio-political problem.*

We have described a simplified case. When considering concrete social problems, one will be faced with more composed associated spaces, states-of-profile and Weyl operators.

5.2.6. Consolidation of a polarized state

We shall examine a particular case of consolidation of the polarized state-of-profile $c_\lambda(u,v)$ introduced and discussed in Section 4.2.2 only for cases $\lambda = 1$ and $\lambda = -1$.

Consider the unitary transformation T of $\mathcal{P}$, where

$$\begin{aligned} Tu &= \tfrac{1}{\sqrt{2}}(u+v) \\ Tv &= \tfrac{1}{\sqrt{2}}(-u+v), \end{aligned} \tag{5.5}$$

and $Tw = w$ for all w uncorrelated with both u and v. We have

$$\langle Tu, u\rangle^2 = \langle Tv, u\rangle^2 = \langle Tv, v\rangle^2 = \langle Tu, v\rangle^2 = \frac{1}{2}\langle u, u\rangle^2 \tag{5.6}$$

which means that both Tu and Tv are now correlated with probability $\frac{1}{2}$ to the original vectors u and v. We call Tu and Tv the *compromise* states-of-mind and denote them by $\widetilde{u}$ and $\widetilde{v}$ respectively.

The process of consolidation of $c_\lambda(u,v)$ by T provides the new state-of-profile

$$c_\lambda(\widetilde{u}, \widetilde{v}) = \left(c(\widetilde{u}) + \lambda c(\widetilde{v})\right)_{/}. \tag{5.7}$$

By virtue of (5.1) the expected total number of energy-bits emitted by the profile in the new state $c_\lambda(\widetilde{u}, \widetilde{v})$ and in the old one are the same,

$$\begin{aligned} &\langle c_\lambda(\widetilde{u}, \widetilde{v}), (d\Gamma I)\, c_\lambda(\widetilde{u}, \widetilde{v})\rangle \\ &= \tfrac{1}{1+\lambda e^{-\langle u,u\rangle}}\langle u, u\rangle = \langle c_\lambda(u,v), (d\Gamma I)\, c_\lambda(u,v)\rangle. \end{aligned} \tag{5.8}$$

Similarly the expected numbers of energy-bits of answers "yes" to the status-question (Q_u) of the profile in state $c_\lambda(u,v)$ and to the status question $(Q_{\widetilde{u}})$ of the profile in state $c_\lambda(\widetilde{u},\widetilde{v})$ are equal

$$\begin{aligned} &\left\langle c_\lambda(\widetilde{u},\widetilde{v}), \left(d\Gamma Q_{\widetilde{u}_/}\right) c_\lambda(\widetilde{u},\widetilde{v})\right\rangle \\ &= \tfrac{1}{2}\tfrac{1}{1+\lambda e^{-\langle u,u\rangle}}\langle u,u\rangle = \left\langle c_\lambda(u,v), \left(d\Gamma Q_{u_/}\right) c_\lambda(u,v)\right\rangle . \end{aligned} \tag{5.9}$$

(cf. 5.1).

However, counting "yes" for $Q_{u_/}$ and for $Q_{v_/}$ after consolidation gives

$$\left\langle c_\lambda(\widetilde{u},\widetilde{v}), \left(d\Gamma Q_{u_/}\right) c_\lambda(\widetilde{u},\widetilde{v})\right\rangle = \left\langle c_\lambda(\widetilde{u},\widetilde{v}), \left(d\Gamma Q_{v_/}\right) c_\lambda(\widetilde{u},\widetilde{v})\right\rangle = \frac{1}{2}\langle u,u\rangle$$

which is half of the expected number of energy-bits of the coherent state $c(u) = c(v)$. This can be expressed as follows:

Corollary 5.5. *Take a polarized profile in a state* $c_\lambda(u,v)$, *where* $\lambda = \pm 1$, $\langle u,v\rangle = 0$ *and* $\langle u,u\rangle = \langle v,v\rangle$. *After consolidation by* T, *the effect of activity-passivity disappears. Both fractions of the profile, one initiated by* $c(u)$ *and the other by* $c(v)$, *become more confident of their own beliefs no matter whether the original polarization was a passive or an active one (cf. Corollary 4.1).*

5.2.7. Memory of a past experience

Let w be a state-of-mind and f a state-of-profile. We say that w is *strongly uncorrelated* with f if wf is a state-of-profile, i.e. if $\langle wf, wf\rangle = 1$. Take a real number α and the Weyl transformation $W_{\alpha w}$. We consider the altered state-of-profile $W_{\alpha w}f$ resulting from the enforcement of the state-of-mind w on f.

The state-of-profile $W_{\alpha w}f$ interacts with the original state-of-profile producing a polarized state,

$$g_\alpha = \left((W_{\alpha w}f) - f\right)_/ . \tag{5.10}$$

If f is a coherent state, g_α is an active polarized state (cf. 4.7) where the altered state $W_{\alpha w}f$ interacts with the original state f.

If the pressure from w does not diminish, i.e. if α is far from zero, the profile will polarize into two sub-profiles in combat. If f is a coherent state $c(x)$, we have $\langle c(x+\alpha w), c(x)\rangle = e^{-\frac{1}{2}\alpha^2}$ making the states f and $W_{\alpha w}f$ practically uncorrelated, and the profile splits into two uncorrelated sub-profiles.

If, however, the pressure caused by w diminishes, i.e. α tends to zero, the state g_α tends to a very interesting equilibrium. We have the following theorems (proofs are given in the Appendix).

Theorem 5.6. *Let f, w and g_α be as above. Then for α converging to zero, the states g_α converge in norm to the state wf, i.e. $\langle wf - g_\alpha, wf - g_\alpha \rangle$ converges to zero if α converges to zero.*

Theorem 5.7. *Take a question Q. Once the exposure to w has faded out, we get*

$$\langle (d\Gamma Q)(wf), wf \rangle = \langle w, Qw \rangle + \langle d\Gamma Q f, f \rangle .$$

Before the exposure we had

$$\langle (d\Gamma Q) f, f \rangle = \langle d\Gamma Q f, f \rangle .$$

Hence the expected number of energy-bits of answers "yes" to questions touching the subject of w, i.e. with $\langle w, Qw \rangle \neq 0$, will be higher after the exposure has faded out.

In particular, if $Q = I$, then the expected number of energy-bits will increase by one,

$$\langle d\Gamma I)(wf), wf \rangle = 1 + \langle d\Gamma I f, f \rangle . \tag{5.11}$$

This means that the "memory state-of-mind w" occurring in the state-of-profile wf activates some sources which were indifferent to questions involving w when the profile was in the state f. Now they will react if confronted with such questions.

In the following two examples we present applications of Theorems 5.6 and 5.7.

As already mentioned in Section 2.1.4 we can consider a parliament of a democratic country as a profile - it has its set of attitudes consisting of different political standpoints and the associated space containing states-of-mind of its representants. In particular, these states-of-mind give the probabilities of adherence to the political standpoints from the set of attitudes.

Example 5.8. *Consider the political profile of a democratic country. Suppose that an election is approaching and the election campaign is in progress, advertising a state-of-mind w strongly uncorrelated with the actual state-of-profile f and unavoidably generating controversy among the voters. Suppose that prior to the campaign a poll is made asking the question: "would you participate, if the election was to take place tomorrow?". Let n be the number of answers "yes". We know that the expected number of energy-bits delivered by the answer "yes" is $\langle d\Gamma I f, f \rangle$.*

Subsequently the election is held, and soon after the election an identical poll is made giving the number m of answers "yes". According to Theorem 5.6 the state-of-profile soon after the election is wf and according to (5.11) the number of energy-bits of answers "yes" increases by one energy -bit. Hence, the number m should be greater than n and the difference $m - n$ reflects the expected one energy-bit of increase of the participation.

Then we have the following

Corollary 5.9. *The memories of a controversy activate voters.*

Another example can be taken from the history of the rise and fall of communism in Eastern European countries:

Example 5.10. *In pre-war Poland, the attitude to the concepts of communism was mostly negative but of remote concern. Say the pre-war state-of-profile was f. The introduction of "the communistic state-of mind" w on f gives the state-of-profile $W_{\alpha w}f$, where α marks the intensity of the enforcement. The old state-of-profile f and the new one $W_{\alpha w}f$ are then in combat producing the state-of-profile g_α (cf. 5.10). Consider Poland now returning, after the fall of the communistic regime (α converging to 0), to the status of a democratic republic and the capitalistic economy known from the pre-war period as represented by the state-of-profile f. Now the reaction to questions involving communism will be considerably more violent. This means that while in the 30-ties the issue of communism was of secondary importance, in the 90-ties most citizens would exhibit emotional reactions confronted with this issue. According to (5.11) the reactions to questions involving the concept of communism will increase with one energy-bit compared with the pre-communistic status-quo (under the circumstances one energy-bit will amount to a sizable packet of single votes).*

5.2.8. Disappearance of a conflict-induced polarization

Suppose that, as described in the non-mathematical synopsis, we have two social forces acting against each other within a profile in a state f and manifesting through states-of-mind a and b. Then the state-of-profile subjected to those forces will become the superposition

$$g_\alpha = \left(W_{\alpha a}f - W_{\alpha b}f\right)_/ . \tag{5.12}$$

From Theorem 5.6 we get the following

Corollary 5.11. *Consider a profile in a state f and two states-of-mind a and b such that $(b-a)_/$ and f are strongly uncorrelated as defined in Theorem 5.6. Then for α converging to zero the superposition (5.12) converges in norm to the state $(b-a)_/ f$, i.e.*

$$\left\langle g_\alpha - (b-a)_/ f, g_\alpha - (b-a)_/ f\right\rangle \text{ converges to zero if } \alpha \text{ converges to zero.} \tag{5.13}$$

Proof. We apply Theorem 5.6 to the state

$$W_{-\alpha a} g_\alpha = \frac{f - W_{\alpha(b-a)} f}{\sqrt{\left\langle f - W_{\alpha(b-a)} f, f - W_{\alpha(b-a)} f \right\rangle}}$$

and then show that the factor $W_{-\alpha a}$ can be removed. ■

In particular, a coherent state $c(x)$ subjected to conflicts with a and b, after releasing the pressure will turn into a non-coherent state $\frac{(b-a)}{\sqrt{2(1-\langle a,b \rangle)}} c(x)$.

Here we can as well produce another version of conflict where the limit (5.13) occurs.

Example 5.12. *Assume that a profile of workers of a fixed trade is in a state f. Suppose that a conflict develops between the respective trade union and the employers. Let the trade union promote a state-of-mind a while the employers' union promotes a state-of-mind b, where $\frac{(b-a)}{\sqrt{2(1-\langle a,b \rangle)}}$ and f are strongly uncorrelated. Then according to Corollary 5.11, after the conflict ceases, the state-of-profile will be $\frac{b-a}{\sqrt{2(1-\langle a,b \rangle)}} f$. To a question Q which receives the answer "yes" in the state $\frac{b-a}{\sqrt{2(1-\langle a,b \rangle)}}$, we get from (5.11)*

$$\begin{gathered} \langle (d\Gamma Q)(wf), wf \rangle = 1 + \langle d\Gamma I f, f \rangle, \\ \text{for } w = \frac{b-a}{\sqrt{2(1-\langle a,b \rangle)}}. \end{gathered} \tag{5.14}$$

An experiment can be designed to show how many responses account for the one energy-bit increase in the formula (5.14). To the members of the trade union we distribute copies of a questionnaire which inquires about a neutral subject. Afterwards we distribute another questionnaire, where the neutral subject is mixed with a reference to the previous conflict. This should stimulate the memory and result in an increase of the number of answers. The extra answers measure the one energy-bit increase

6. Teaching-and-learning

The subject of education was mentioned in Section 5.2.3. Now we shall provide a more thorough analysis. Consider a class of pupils. Select one of the subjects they learn and take the textbook they use as a questionnaire. We adjust the notions of attitude and poll to the situation of the class-room as follows. First we introduce the space of attitudes $\mathfrak{M} = \{\mathfrak{m}_1, \mathfrak{m}_2, ..., \mathfrak{m}_k\}$ as the collection of possible interpretations of subjects from the textbook. This way an attitude from $\mathfrak{M}$ realises as a bit of information, not necessarily correct. As usual we associate with $\mathfrak{M}$ the space of states-of-mind. The process of learning will then change the states-of-mind of the pupils. In the place of polls we use tests checking the pupils' accomplishments in learning the subjects in the curriculum.

6.1. A teaching-and-learning model

6.1.1. Non-mathematical synopsis

We shall analyse the process of education of young students who have finished seven-to-eight years of school and are going through the last three-to-four years of secondary education which in Danish/German/Polish terminology is called gymnasium. Let $\mathfrak{M}_{start}$ denote the collection of attitudes covering the basic knowledge of students entering gymnasium. The class of students - a profile of students - is subjected to subsequent alterations and consolidations (cf. Sections 5.1.1 and 5.1.2). The teacher introduces and explains new subjects to the class and the students try to memorize and understand what they have been told. This alters the students' states-of-mind. But pieces of information and the references the students have got require consolidation: the students go home and try to reconstruct what they have learned, consulting their notes and books. Now their set of attitudes and the corresponding space of states-of-mind become enlarged. This process is repeated a number of times until the curriculum of the year is exhausted. Then the students are in the final set of states-of-mind based on the final set of attitudes $\mathfrak{M}_{final}$.

6.1.2. The mathematics of the process of education

We have a class of students with some initial knowledge, i.e. the class is in a coherent state $c\left(x_{start}\right)$. The state x_{start} originates from the initial attitudes $\mathfrak{M}_{start}=\{\mathfrak{m}_1,\mathfrak{m}_2,...,\mathfrak{m}_r\}$ from the set $\mathfrak{M}=\{\mathfrak{m}_1,\mathfrak{m}_2,...,\mathfrak{m}_n\}$ of all attitudes. We start with a class of students that completed the first part of their education and are in the coherent state $c\left(x_{start}\right)$, where

$$x_{start}=\sqrt{\lambda}\left(\sqrt{q_1}1_{\mathfrak{m}_1}+\sqrt{q_2}1_{\mathfrak{m}_2}+\cdots+\sqrt{q_r}1_{\mathfrak{m}_r}\right),$$
$$q_1+q_2+\cdots+q_r=1.$$

Pose any question Q requiring familiarity with attitudes $\mathfrak{m}_{r+1}$ up to $\mathfrak{m}_n$, where familiarity with the first r attitudes is of no help, i.e. $Qx_{start}=0$. Then the expected number of answers is

$$\begin{aligned} & \left\langle c\left(x_{start}\right),d\Gamma Qc\left(x_{start}\right)\right\rangle \\ = & \left\langle c\left(x_{start}\right),c\left(x_{start}\right)d\Gamma Qx_{start}\right\rangle=\left\langle c\left(x_{start}\right),\left(Qx_{start}\right)c\left(x_{start}\right)\right\rangle=0. \end{aligned}$$

The teacher's task is to get the class into a new coherent state $c\,(x_{final})$, where the attitudes $\mathfrak{m}_{r+1}$ up to $\mathfrak{m}_n$ are in the scope of the students' knowledge. Hence at the end of the school year the class should be in the state $c\,(x_{final})$, where

$$x_{final} = \sqrt{\mu}\,(\sqrt{q_1}1_{\mathfrak{m}_1} + \sqrt{q_2}1_{\mathfrak{m}_2} + \cdots + \sqrt{q_n}1_{\mathfrak{m}_n})\,,$$

and the number μ is large enough to secure an acceptable number of responses during the test. And here one would like to have not too small preferences q_j of subjects - students have to know something even about unpopular subjects.

The intermediate stages in the process of teaching should be repetitions of the acquired knowledge. The repetition program is contained in the transformation $c\,(x) \rightarrow c\,(Ax)$ which allows the questions to be reformulated in order to deepen the understanding of the subject.

While teaching a new part of a subject, we do not at once achieve the final goal, the state $c\,(x_{final})$. The process consists in gradual increase of the students' knowledge, mixed with repetitions.

Hence the actual process of teaching reads as follows,

$$\begin{aligned}
&c\,(x_{start}) \rightarrow W_{y_1}c\,(x_{start}) = c\,(x_{start} + z_1) \rightarrow c\,(A_1\,(x_{start} + z_1)) \\
&\rightarrow W_{z_2}c\,(A_1\,(x_{start} + z_1)) = c\,(A_1\,(x_{start} + z_1) + z_2) \\
&\rightarrow c\,(A_2\,(A_1\,(x_{start} + z_1) + z_2)) \rightarrow W_{z_3}c\,(A_2\,(A_1\,(x_{start} + z_1) + z_2)) \\
&= c\,(A_2\,(A_1\,(x_{start} + z_1) + z_2) + z_3) \rightarrow c\,(x_{intermediate}) \rightarrow etc. \\
&\rightarrow c\,(x_{final})\,,
\end{aligned}$$

where the knowledge is divided into small doses and intermediate repetitions: subsequent alterations and consolidations. At each stage one can perform a test and compare the actual result with the theoretical expectation.

7. Selecting a party

7.1. Non-mathematical synopsis

Discussion of the VP-function (cf. [4]) can hardly be incorporated in this paper - it requires a separate publication. We shall restrict ourselves to the less ambitious and simpler task of discussing the parameters connected with the notion of a political party which represents some selected points of view of the citizens.

Consider a set of states-of-mind that represents different political profiles of the citizens. A party image would be a selection of those different states-of-mind in a form of a global "state of the party" which shall be called the party image.

The correlation between the actual state of the population and the party's image expresses the party's popularity and is directly linked with the expected number of votes. This correlation we call *the popularity of the party.* Most people sympathize with more than one party so that they have to make a choice as to where their ballots are to be placed. The size of a fraction of the population supporting a specific party depends on

a) the voters' profile (given as a state-of-profile)

b) the party's image

c) the other parties' images.

It is less obvious how the dependence can be measured. However, using the introduced formalism, we can produce some relevant estimates.

Since we analyse politics in a democratic country, we must count on the presence of more than one party. The actual support will then be influenced by the extent of support to the other parties resulting in the concept of relative support.

7.2. The mathematics of selecting a party

7.2.1. The image of a party, its popularity and relative support

In this section we consider the total population of a democratic country. As the space of attitudes $\mathfrak{M}$ we take the various political attitudes of the citizens, and consider the corresponding associated space $\mathcal{F}$ and the grand associated space $\widetilde{\mathcal{F}}$ containing the states of population.

A political party will be characterised by its *image* which will be a state g from $\widetilde{\mathcal{F}}$.

Let us consider the case of just one party with an image g so that each voter has the choice either to vote for the party or not vote at all. If the state-of-profile of the population is f, then the number

$$u = \langle f, g \rangle^2 ,$$

which is equal to the probability of coincidence between f and g, shall be called the *popularity* of g with the population in the state f. The popularity gives the percentage of the population that will vote for the party were there not competition among parties.

The number u can also be interpreted as the degree to which the population accepts the party's image. And u measures the party's potential support in the population. It is obvious that there are two parallel time evolutions: the population's state f and the party's image g. It is natural to expect that the evolution of opinion is considerably slower than the evolution of an image. The latter can change instantly as the consequence of a single official party proclamation.

Let f from $\widetilde{\mathcal{F}}$ be the state-of-profile of the population and let $g_1, g_2, ..., g_n$ be the images of all parties qualified to participate in elections.

Since the parties compete with each other, their popularity will usually decrease to the *relative support* which for the party g_j will now be given by the number

$$\rho_j = \frac{u_j}{u},$$

where $u = u_1 + u_2 + \cdots + u_n$ and for each j the number u_j denotes the popularity of the party g_j.

7.2.2. Effects of a party's movement towards the centre

We shall examine a hypothetical example of a population with a fixed state f, represented by three parties with images g_l, g_c and g_r respectively. To facilitate computations, we assume that all the states and images are coherent, i.e. of the form $c(z)$, where z is taken from $\mathcal{F}$. Take uncorrelated vectors x and y from $\mathcal{F}$ and assume that the population is in a coherent state $f = c(x)$. Given a coherent image $g = c(y)$, the popularity for g is

$$u = \langle f, g \rangle^2 = e^{-\langle x-y, x-y \rangle}.$$

Suppose that the radical parties move toward the centre. We estimate the influence of this fact on the relative supports of the parties. Consider an extreme case where the image g_c is identical with the state f of the population, i.e. let

$$g_c = f = c(x)$$

so that the popularity of the party in the centre is equal to 1, i.e. the opinion of the population coincides totally with the image of the party.

Let the images of two radical parties be coherent, i.e.

$$g_l = c(x+y)$$

and

$$g_r = c(x-y),$$

where x and y are uncorrelated. We assume $\langle x, x \rangle = \langle y, y \rangle$, so that $x+y$ and $x-y$ are uncorrelated as well. Hence the popularity of g_l and g_r are

$$u_l = e^{-\langle x-(x+y), x-(x+y) \rangle} = e^{-\langle y, y \rangle}$$

and

$$u_r = e^{-\langle x-(x-y), x-(x-y) \rangle} = e^{-\langle y, y \rangle}.$$

Assume that $|y|^2 = -\ln \frac{5}{100}$. Then

$$u_l = u_r = 0,05,$$

and the relative supports are

$$\begin{aligned}\rho_c &= \frac{1}{1,1} = 91\% \\ \rho_l &= \frac{0,05}{1,1} = 4,5\% \\ \rho_r &= \frac{0,05}{1,1} = 4,5\%.\end{aligned}$$

Now we analyse the consequences of the change of the image g_l. Take for example

$$\tilde{g}_l = c\,(x + 0,4y)$$

which moves the party slightly toward the centre. Now the popularity for $\tilde{g}_l$ is

$$|\langle f, \tilde{g}_l\rangle|^2 = e^{-\langle 0,4y, 0,4y\rangle} = 0.02$$

and the popularity of the party becomes

$$\widetilde{u}_l = e^{-0,4^2\langle y,y\rangle} = 0,62.$$

The other popularities stay unchanged but the relative supports now become

$$\begin{aligned}\rho_c &= \frac{1}{1,67} = 60\% \\ \rho_l &= \frac{0,62}{1,67} = 37\% \\ \rho_r &= \frac{0,05}{1,67} = 3,0\%.\end{aligned}$$

Several points deserve attention. The substantial increase from $4,5\%$ to 37% of expected votes for the left wing party on the expense of the centre which is expected to lose one third of the votes; but also the right wing is expected to lose one third of the votes. No wonder that politicians are rather careful not to say incidentally something new that matters!

7.2.3. Two parties

Let the state-of-profile of a population be

$$f = c\,(x)$$

and suppose that the two parties we are going to analyse have images

$$\begin{aligned}g_p &= c\,(x + py) \\ g_r &= c\,(x + ry)\,,\end{aligned}$$

where $\langle x, y \rangle = 0$, x, y are taken from $\mathcal{F}$ and $p, r > 0$ are parameters. The vector y represents the subject on which the parties and the voters disagree. Then the popularities are

$$\begin{aligned} u_p &= e^{-\langle py, py \rangle} \\ u_r &= e^{-\langle ry, ry \rangle} \end{aligned}$$

so that the relative support for the party g_p is

$$\begin{aligned} \rho_p &= \frac{e^{-\langle py, py \rangle}}{e^{-\langle py, py \rangle} + e^{-\langle ry, ry \rangle}} \\ &= \frac{1}{1 + e^{(p^2 - r^2)\langle y, y \rangle^2}}. \end{aligned}$$

We see that the percentage of voters choosing the party g_p depends on the parameters p and r of which only one, namely p, can be controlled by the party.

Hence we must fix several values of r and for each of them separately analyse ρ as the function of p. At first one observes that for all values of r, the function attains its maximum for $p = 0$ which means that the population votes for the party whose image is closest to the population's own state. The second observation is that for $r \geq p+1 \geq 1,5$ we have $r^2 - p^2 \geq 2$ so that the values of ρ_p are very close to 1. Hence, if the population is to make a choice between two extreme cases, it decides on the least extreme. The third rather important observation is that in the case of $p = r$, the balance $\rho_p = \rho_s = 50\%$ becomes more and more unstable when both p and r increase so that very small variations of p can cause substantial changes of ρ_p. This may explain the so-called "landslide victories" where a party's number of votes is way above what was expected. Which happened for instance in Denmark in '73 when the "Fremskridtsparti" won a landslide victory. That could mean that landslide victories are consequences of too big a gap between the parties' images and the state-of-profile of the population.

8. APPENDIX

8.1. Computation of $p\left(y^j\right)$

To illustrate how the mathematics of coherent states works, we now prove the identity (4.6).

Take z in $\mathcal{F}$, uncorrelated with y, such that $x = sy + z$. Then $s = \frac{\langle y, x \rangle}{\langle y, y \rangle}$ and

$$\langle x, x \rangle = s^2 \langle y, y \rangle + \langle z, z \rangle = \langle y, x \rangle^2 + \langle z, z \rangle .$$

We have

$$c(x) = e^{-\frac{1}{2}\langle x,x\rangle}e^{x} = e^{-\frac{1}{2}\langle x,x\rangle}e^{sy+z} = e^{-\frac{1}{2}\langle x,x\rangle}e^{sy}e^{z} = e^{-\frac{1}{2}\langle x,x\rangle}\Sigma_{n=0}^{\infty}\frac{e^{z}(sy)^{n}}{n!}.$$

Since $\langle y,z\rangle = 0$, we have

$$\langle e^{z}y^{m}, e^{z}y^{n}\rangle = e^{\langle z,z\rangle}\langle y^{m}, y^{n}\rangle$$

which means that vectors $e^{z}(sy)^{n}$ are pairwise orthogonal. Moreover, the contribution of y^{j} starts from $n=j$ and the square of the length $p(y^{j})$ of the vector of this contribution is

$$\begin{aligned} p\left(y^{j}\right) &= \left\langle e^{-\frac{1}{2}\langle x,x\rangle}\Sigma_{n=j}^{\infty}\frac{e^{z}(sy)^{n}}{n!}, e^{-\frac{1}{2}\langle x,x\rangle}\Sigma_{n=j}^{\infty}\frac{e^{z}(sy)^{n}}{n!}\right\rangle \\ &= \Sigma_{n=j}^{\infty}\frac{1}{n!^{2}}e^{-\langle x,x\rangle}\langle e^{z}(sy)^{n}, e^{z}(sy)^{n}\rangle = e^{\langle z,z\rangle-\langle x,x\rangle}\Sigma_{n=j}^{\infty}\frac{s^{2n}}{n!} \\ &= e^{\langle z,z\rangle-\langle x,x\rangle}\Sigma_{n=j}^{\infty}\frac{1}{n!}\langle y,x\rangle^{2n} = e^{-\langle y,x\rangle^{2}}\Sigma_{n=j}^{\infty}\frac{1}{n!}\langle y,x\rangle^{2n}. \end{aligned}$$

8.2. Definition of the operator $\widetilde{U}$ of consolidation

We define $\widetilde{U}$ on the generating states $\alpha(x_1x_2\cdots x_k)e^{x}$, where $x, x_1, x_2, \cdots, x_k$ belong to $\mathcal{P}$, setting

$$\widetilde{U}\left(\alpha(x_1x_2\cdots x_k)e^{x}\right) = \alpha(Ux_1)(Ux_2)\cdots(Ux_k)e^{Ux}.$$

Since generating states constitute a total subset of $\widetilde{\mathcal{P}}$, $\widetilde{U}$ extends uniquely over the whole $\widetilde{\mathcal{P}}$. Now, given a state f from $\widetilde{\mathcal{P}}$, its consolidation designed by U will be the state $\widetilde{U}f$.

8.3. Computing the memory

First we observe that

$$\langle w^{*}f, w^{*}f\rangle = \langle f, ww^{*}f\rangle = \langle f, w^{*}(wf)\rangle - \langle f,f\rangle\langle w,w\rangle = \langle wf, wf\rangle - \langle f,f\rangle\langle w,w\rangle$$

which shows that the assumptions $w^{*}f = 0$ and $\langle wf, wf\rangle = \langle f,f\rangle\langle w,w\rangle$ are equivalent.

Proof of Theorem 5.6. Since $\{e^{x} : x \in \mathcal{F}\}$ is a total subset of $\mathcal{F}$ and since on the Hilbert sphere the weak convergence is equivalent with the norm convergence (cf.[6]), it is sufficient to verify the following

Lemma 8.1. *Take $f \in \widetilde{\mathcal{F}}$ and a state-of-mind $w \in \mathcal{F}$ i.e. $\langle w,w\rangle = 1$, and assume that $w^{*}f = 0$. Then for every $z \in \mathcal{F}$ we have*

$$\lim_{\alpha\to 0}\frac{\langle W_{\alpha w}f - f, e^{z}\rangle}{\sqrt{\langle W_{\alpha w}f - f, W_{\alpha w}f - f\rangle}} = \langle wf, e^{z}\rangle.$$

Proof. We have

$$\begin{aligned} & \langle W_{\alpha w} f - f, e^{z} \rangle \\ = & \ \langle f, W_{-\alpha w} e^{z} - e^{z} \rangle = \langle f, e^{-\frac{1}{2}\alpha^2} e^{-\alpha w} e^{\alpha w^*} e^{z} - e^{z} \rangle = \langle f, e^{-\frac{1}{2}\alpha^2} e^{-\alpha w} e^{e^{\alpha w^*} z} - e^{z} \rangle \\ = & \ \langle f, e^{-\frac{1}{2}\alpha^2} e^{-\alpha w} e^{z + \alpha \langle w, z \rangle \emptyset} - e^{z} \rangle = \langle f, \left(e^{\left(-\frac{1}{2}\alpha^2 + \alpha \langle w, z \rangle\right)\emptyset - \alpha w} - \emptyset \right) e^{z} \rangle. \end{aligned}$$

We compute the limit in 0 of

$$\phi(\alpha) = e^{\left(-\frac{1}{2}\alpha^2 + \alpha\langle w, z\rangle\right)\emptyset - \alpha w} - \emptyset = e^{-\frac{1}{2}\alpha^2 + \alpha\langle w, z\rangle} e^{\alpha w} - \emptyset$$

divided by the norm $\sqrt{\langle W_{\alpha w} f - f, W_{\alpha w} f - f \rangle}$. Notice, that $\phi(0) = 0$.

The derivative of $\phi(\alpha)$ is

$$\phi'(\alpha) = (-\alpha + \langle w, z \rangle) e^{-\frac{1}{2}\alpha^2 + \alpha\langle w, z\rangle} e^{\alpha w} + w e^{-\frac{1}{2}\alpha^2 + \alpha\langle w, z\rangle} e^{\alpha w},$$

and we observe that

$$\phi'(0) = \langle w, z \rangle \emptyset + w.$$

Then

$$\langle f, (\langle w, z \rangle \emptyset + w) e^{z} \rangle = \langle w^* f, e^{z} \rangle + \langle w, z \rangle \langle f, e^{z} \rangle = \langle w^* f, e^{z} \rangle + \langle w f, e^{z} \rangle$$

and since $w^* f = 0$, we get the key result,

$$\begin{aligned} & \lim_{\alpha \to 0} \langle W_{\alpha w} f - f, e^{z} \rangle' \\ = & \lim_{\alpha \to 0} \langle f, \left(e^{\left(-\frac{1}{2}\alpha^2 + \alpha\langle w, z\rangle\right)\emptyset - \alpha w} - \emptyset \right) e^{z} \rangle' = \langle w f, e^{z} \rangle. \end{aligned}$$

In order to compute

$$\left(\sqrt{2\left(1 - \langle W_{\alpha w} f, f \rangle\right)} \right)' = \frac{2\left(1 - \langle W_{\alpha w} f, f \rangle\right)'}{\sqrt{2\left(1 - \langle W_{\alpha w} f, f \rangle\right)}} \frac{\langle W_{\alpha w} f, f \rangle'}{\sqrt{2\left(1 - \langle W_{\alpha w} f, f \rangle\right)}}$$

we need

$$\begin{aligned} \langle W_{\alpha w} f, f \rangle' &= \left(e^{\frac{1}{2}\alpha^2} \left\langle e^{\alpha w} f, e^{-\alpha w} f \right\rangle \right)' \\ &= e^{\frac{1}{2}\alpha^2} \left(\alpha + \left\langle w e^{\alpha w} f, e^{-\alpha w} f \right\rangle - \left\langle e^{\alpha w} f, w e^{-\alpha w} f \right\rangle \right) \\ &= e^{\frac{1}{2}\alpha^2} \left(\alpha + \left\langle e^{\alpha w} f, w^* \left(e^{-\alpha w} f \right) \right\rangle - \left\langle w^* \left(e^{\alpha w} f \right), e^{-\alpha w} f \right\rangle \right) \\ &= e^{\frac{1}{2}\alpha^2} \left(\alpha + \left\langle e^{\alpha w} f, \left(w^* e^{-\alpha w} \right) f \right\rangle - \left\langle \left(w^* e^{\alpha w} \right) f, e^{-\alpha w} f \right\rangle \right) \\ &= \alpha e^{\frac{1}{2}\alpha^2}. \end{aligned}$$

Since

$$\langle W_{\alpha w}f,f\rangle'_{\alpha=0}=\langle wf,f\rangle-\langle f,wf\rangle=0,$$

we apply the l'Hospital rule to $\frac{\left(\langle W_{\alpha w}f,f\rangle'\right)^2}{2(1-\langle W_{\alpha w}f,f\rangle)}$ and get

$$-\frac{\left(\left(\langle W_{\alpha w}f,f\rangle'\right)^2\right)'}{\langle W_{\alpha w}f,f\rangle'}=-\frac{2\langle W_{\alpha w}f,f\rangle'\langle W_{\alpha w}f,f\rangle''}{\langle W_{\alpha w}f,f\rangle'}=-2\langle W_{\alpha w}f,f\rangle''.$$

Further,

$$\begin{aligned}\langle W_{\alpha w}f,f\rangle'' &= e^{\frac{1}{2}\alpha^2}\alpha\left(\left(\alpha+\left\langle we^{\alpha w}f,e^{-\alpha w}f\right\rangle-\left\langle e^{\alpha w}f,we^{-\alpha w}f\right\rangle\right)\right)\\ &\quad+e^{\frac{1}{2}\alpha^2}(1+\left\langle w^2e^{\alpha w}f,e^{-\alpha w}f\right\rangle-\left\langle we^{\alpha w}f,we^{-\alpha w}f\right\rangle\\ &\quad-\left\langle we^{\alpha w}f,we^{-\alpha w}f\right\rangle+\left\langle e^{\alpha w}f,w^2e^{-\alpha w}f\right\rangle)\\ &= e^{\frac{1}{2}\alpha^2}\alpha\left(\left(\alpha+\left\langle we^{\alpha w}f,e^{-\alpha w}f\right\rangle-\left\langle e^{\alpha w}f,we^{-\alpha w}f\right\rangle\right)\right)\\ &\quad+e^{\frac{1}{2}\alpha^2}\left(1-2\left\langle we^{\alpha w}f,we^{-\alpha w}f\right\rangle+\left\langle e^{\alpha w}f,w^2e^{-\alpha w}f\right\rangle\right).\end{aligned}$$

Setting $\alpha=0$, we get

$$\begin{aligned}\langle W_{\alpha w}f,f\rangle''_{\alpha=0} &= 1-2\langle wf,wf\rangle+\left\langle f,w^2f\right\rangle=1-2\left(w^*\left(wf\right),f\right)+\langle w^*f,wf\rangle\\ &= 1-2\left(\langle w,w\rangle\langle f,f\rangle\right)=-1+\langle w^*f,w^*f\rangle+\langle w^*f,wf\rangle=1\end{aligned}$$

which gives the desired result ■

Proof of Theorem 5.7. Let $w^*f=0$. We have

$$d\Gamma Q)\left(wf\right)=\left(Qw\right)f+wd\Gamma f,$$

and hence

$$\begin{aligned}\langle wd\Gamma Qf,wf\rangle &= \langle d\Gamma Qf,w^*\left(wf\right)\rangle=\langle d\Gamma f,\left(w^*w\right)f+w\left(w^*f\right)\rangle=\langle w,w\rangle\langle d\Gamma f,f\rangle\\ \langle\left(Qw\right)f,wf\rangle &= \langle w^*\left(Qw\right)f,f\rangle=\langle w,Qw\rangle\langle f,f\rangle+\langle w^*\left(Qw\right)w^*f,f\rangle=\langle w,Qw\rangle\langle f,f\rangle\end{aligned}$$

which gives

$$\begin{aligned}\langle d\Gamma Q)\left(wf\right),wf\rangle &= \langle\left(Qw\right)f+wd\Gamma Qf,wf\rangle=\langle\left(Qw\right)f,wf\rangle+\langle wd\Gamma Qf,wf\rangle\\ &= \langle w,Qw\rangle\langle f,f\rangle+\langle w,w\rangle\langle d\Gamma f,f\rangle\qquad\blacksquare\end{aligned}$$

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