D'ALEMBERT'S AND WILSON'S EQUATIONS ON LIE GROUPS

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1. Introduction

In this paper we discuss certain topics in the theory of functional equations on non-abelian groups. Our first aim is to study d'Alembert's equation

$$g(xy) + g(xy^{-1}) = 2g(x)g(y),\ x, y \in G, \tag{1}$$

where G is a group and g is a complex valued function on G, and the following generalization that comes out naturally of the study of Wilson's functional equation (see Corovei [3])

$$g(xy) + g(yx) + g(xy^{-1}) + g(y^{-1}x) = 4g(x)g(y),\ x, y \in G. \tag{2}$$

Secondly we will study Wilson's equation

$$f(xy) + f(xy^{-1}) = 2f(x)g(y),\ x, y \in G. \tag{3}$$

Finally we will solve Jensen's equation

$$f(xy) + f(xy^{-1}) = 2f(x),\ x, y \in G, \tag{4}$$

on a semidirect product of groups.

Notation: The following notation will be used throughout the article. G denotes a group with e as neutral element and $Z(G)$ its centre. $\mathbb{C}^*$ denotes the multiplicative group of non-zero complex numbers. If $m : G \mapsto \mathbb{C}^*$ is a homomorphism, then $\check{m} : G \mapsto \mathbb{C}^*$ is the homomorphism given by $\check{m}(x) = m(x^{-1})$, $x \in G$. A group G is said to be 2-divisible, if for any $x \in G$ there exists $y \in G$ such that $y^2 = x$. y is not assumed to be unique. An involution τ of G is a map $\tau : G \mapsto G$ such that $\tau(xy) = \tau(y)\tau(x)$, $\forall x, y \in G$ and $\tau(\tau(x)) = x$ for all $x \in G$.

If $m : G \to \mathbb{C}^*$ is a homomorphism, then

$$g(x) = \frac{m + \check{m}}{2}(x),\ x \in G, \tag{5}$$

is a solution to (1). No restrictions on the group are needed for that statement. For abelian groups the converse is true: Any nonzero solution of (1) has this form (Kannappan [6]). This is true for certain other groups as well (Corovei [2] and Stetkær [10]). We are going to show it for still another class of groups.

Our main results are the following:

1. We show that any solution g to (1) and (2) is of the form (5) when G is a connected nilpotent Lie group (see Theorem 2.6 and Corollary 2.8).
2. We give all solutions to Wilson's equation on connected nilpotent Lie groups, provided that it is not the degenerate version of Wilson's equation where $g \equiv 1$, i.e. Jensen's equation (see Theorem 3.4).

(3) We give the solution to Jensen's equation on a semidirect product of two groups, where we suppose that the normalized solutions to Jensen's equation on the groups, which enter in the formation of the semidirect product, are homomorphisms. This is the case if they are abelian (see Theorem 4.1).

Ng has solved Jensen's equation for all free groups and $GL_n(\mathbb{Z})$ for $n \geq 3$ (see Ng [8]). Ng has also studied the following version of Jensen's equation

$$f(xy) + f(y^{-1}x) = 2f(x), \ \forall x \in G. \tag{6}$$

We are not going to pursue this, but we will compare our results to his on the Heisenberg group (see Example 4.2). The difference is somewhat surprising.

The parts of the present paper concerned with d'Alembert's and Wilson's functional equations are closely related to and inspired by Corovei [2] and [3]. However there is a shift of emphasis from insisting on that all elements have odd order to looking at 2-divisibility as we do. Apart from the trivial group, connected Lie groups contain elements of infinite order, so it is essentially a phenomenon for discrete groups that all elements have odd order. We manage to treat the connected nilpotent Lie groups which play an important role in Analysis. This is one reason that the results are interesting. These groups are 2-divisible. Since 2-divisibility was what made Corovei's proofs work, some of his proofs are copied with only modest changes. But our results are more general (see Remark 2.7 and Remark 3.5).

From Lemma 1 in [1] we know that a solution $f : G \mapsto \mathbb{C}$ of Jensen's equation with $f(e) = 0$ and $f(xy) = f(yx)$ for all $x, y \in G$ is a homomorphism. But not all solutions f with $f(e) = 0$ are homomorphisms. We have a counterexample when G is the Heisenberg group (see Example 4.2), the simplest connected nilpotent Lie group which is not abelian.

2. d'Alembert's equation on nilpotent connected Lie groups

In this section we will solve (2). It is a generalization of (1), because $g(xy) = g(yx)$ for any solution g of (1) (see Remark V.2 in [10]) so that any solution of (1) is also a solution of (2).

Lemma 2.1. *Let $g : G \to \mathbb{C}$ be a non-zero solution of the equation*

$$g(xy) + g(yx) + g(x\tau(y)) + g(\tau(y)x) = 4g(x)g(y), \ x, y \in G, \tag{7}$$

where τ is an involution of G. Then $g(e) = 1$, $g \circ \tau = g$, and

$$g(x^2) + \frac{g(x\tau(x)) + g(\tau(x)x)}{2} = 2g(x)^2, \ x \in G. \tag{8}$$

Proof. See Lemma III.1 of [10]. □

Theorem 2.2. *Let g be a solution of the following extension of d'Alembert's functional equation*

$$g(xy) + g(yx) + g(x\tau(y)) + g(\tau(y)x) = 4g(x)g(y), \ \forall x, y \in G, \tag{9}$$

where $\tau : G \mapsto G$ is an involution and $Z(g) = \{u \in G : g(xuy) = g(xyu), \ \forall x, y \in G\}$.

a: *If there exists $u \in Z(g)$ such that $g(u)^2 \neq g(u\tau(u))$ then g has the form*

$$g = \frac{m + m \circ \tau}{2}, \tag{10}$$

where $m : G \to \mathbb{C}^*$ is a homomorphism.

b: *If $g(u)^2 = g(u\tau(u))$ for all $u \in Z(g)$ then*

$$g(xu) = g(x)g(u), \ \forall x \in G, \ \forall u \in Z(g). \tag{11}$$

Proof. See Theorem III.2 of [10]. □

Lemma 2.3. *Let H be a 2-divisible subgroup of G. Let g be a solution of*

$$g(xy) + g(yx) + g(xy^{-1}) + g(y^{-1}x) = 4g(x)g(y), \ x, y \in G. \tag{12}$$

If $g^2 \equiv 1$ on H then $g \equiv 1$ on H.

Proof. For any $x \in H$ there exists $y \in H$ such that $y^2 = x$. Using Lemma 2.1 we get $g(x) = g(y^2) = 2g(y)^2 - g(e) = 1$. □

Lemma 2.4. *If $g : G \to \mathbb{C}$ is a solution to*

$$g(xy) + g(yx) + g(xy^{-1}) + g(y^{-1}x) = 4g(x)g(y), \ x, y \in G, \tag{13}$$

and $g(u) = 1, \ \forall u \in Z(G)$, then one can define $F : G/Z(G) \to \mathbb{C}$ by $F(\bar{x}) = g(x), \ \forall x \in G$, where $\bar{x} = xZ(G)$. Furthermore F satisfies the equation

$$F(\bar{x}\bar{y}) + F(\bar{y}\bar{x}) + F(\bar{x}\bar{y}^{-1}) + F(\bar{y}^{-1}\bar{x}) = 4F(\bar{x})F(\bar{y}), \ \bar{x}, \bar{y} \in G/Z(G). \tag{14}$$

Proof. See Lemma 4 of [2]. □

Lemma 2.5. *If G is a connected nilpotent Lie group, then G is 2-divisible. Furthermore $Z(G)$ and $G/Z(G)$ are connected nilpotent Lie groups and hence also 2-divisible.*

Proof. Let $\mathcal{G}$ be the Lie algebra of G. Since G is connected and nilpotent, the exponential map $\exp : \mathcal{G} \to G$ is onto (see Corollary VI 4.4 of [5] (p. 269)). Let $x \in G$, there exists $X \in \mathcal{G}$ such that $\exp(X) = x$. $\mathcal{G}$ being a vector space, put $y = \exp(\frac{1}{2}X)$, then

$$y^2 = \exp(\frac{1}{2}X)\exp(\frac{1}{2}X) = \exp(X) = x. \tag{15}$$

So G is 2-divisible. Z(G) is a closed subgroup of G and hence a Lie group in it's own right. Furthermore Z(G) is connected (see Corollary 3.6.4 of [11]). Being nilpotent Z(G) is 2-divisible. Since Z(G) is a closed normal subgroup of G, it follows that $G/Z(G)$ is a Lie group (see Theorem 2.9.6 of [11]). The natural map $\pi : G \to G/Z(G)$ given by $\pi(g) = gZ(G)$ is continuous, so $G/Z(G)$ is connected. Hence $G/Z(G)$ is a connected nilpotent Lie group. □

For Lie groups the following theorem extends Proposition V.5 of [10].

Theorem 2.6. *If G is a nilpotent connected Lie group, then $g : G \to \mathbb{C}$ is a non-zero solution to*

$$g(xy) + g(yx) + g(xy^{-1}) + g(y^{-1}x) = 4g(x)g(y), \ x, y \in G, \tag{16}$$

if and only if g has the form

$$g = \frac{m + \check{m}}{2}, \tag{17}$$

where $m : G \to \mathbb{C}^*$ is a homomorphism.

Proof. Standard technique. Let $\{e\} = Z_0 < \cdots < Z_n = G$ be an ascending central series for G, with $Z_{i+1}/Z_i = Z(G/Z_i)$. We will prove the result by induction on n. If $n = 0, 1$ then $Z(G) = G$, hence G is abelian, and g therefore satisfies the equation

$$g(xy) + g(xy^{-1}) = 2g(x)g(y), \ x, y \in G. \tag{18}$$

That is d'Alembert's equation on an abelian group, where it is known that g has the stated form. Let $n \in \mathbb{N}$ and assume that the result is true for all nilpotent connected Lie groups with ascending central series of length n. Let G be a nilpotent connected Lie group with ascending central series $\{e\} = Z_0 < Z_1 < \cdots < Z_n < Z_{n+1} = G$, where $Z_{i+1}/Z_i = Z(G/Z_i)$. If there exists $u \in Z(G)$ such that $g(u)^2 \neq 1$, then it follows from the previous theorem that g has the stated form. So we can assume that $g(u)^2 = 1, \quad \forall u \in Z(G)$. Since Z(G) is 2-divisible it follows that $g(u) = 1, \ \forall u \in Z(G)$. By the previous lemma $G/Z(G)$ is a nilpotent connected Lie group. Furthermore $Z_1/Z_1 < \cdots < Z_{n+1}/Z_1 = G/Z_1$ is an ascending central series for $G/Z_1 = G/Z(G)$ with $(Z_{i+1}/Z_1)/(Z_i/Z_1) = Z((G/Z_1)/(Z_i/Z_1))$. We have shown above that we can define $F : G/Z_1 \to \mathbb{C}$ by $F(\bar{x}) = g(x)$, where $\bar{x} = xZ_1$, and furthermore $F : G/Z_1 \to \mathbb{C}$ is a solution to

$$F(\bar{x}\bar{y}) + F(\bar{y}\bar{x}) + F(\bar{x}\bar{y}^{-1}) + F(\bar{y}^{-1}\bar{x}) = 4F(\bar{x})F(\bar{y}), \ \bar{x}, \bar{y} \in G/Z(G). \tag{19}$$

By assumption, there exists a homomorphism $M : G/Z_1 \to \mathbb{C}^*$ such that $F = (M + \check{M})/2$. Define a homomorphism $m : G \to \mathbb{C}^*$ by $m(x) = M(\bar{x})$, then $\check{m}(x) = \check{M}(\bar{x})$. So we have that

$$g(x) = F(\bar{x}) = \frac{M + \check{M}}{2}(\bar{x}) = \frac{m + \check{m}}{2}(x), \ x \in G. \tag{20}$$

The theorem now follows by induction on n. □

Remark 2.7. (a) Instead of $g : G \to \mathbb{C}$ we could consider $g : G \to K$ where K is any quadratically closed field with characteristic different from 2.

(b) Instead of nilpotent connected Lie groups, we could consider any class $\mathcal{C}$ of nilpotent groups G, for which $G \in \mathcal{C}$ implies $Z(G)$ is 2-divisible and $G/Z(G) \in \mathcal{C}$. Note that if we take $\mathcal{C}$ to be all nilpotent groups where the order of all elements are odd, then $\mathcal{C}$ fulfils the requirement. So if we formulate the theorem for classes $\mathcal{C}$ with the above mentioned properties, instead of for connected nilpotent Lie groups, then it contains as a special case Theorem 2 in [2].

Corollary 2.8. *If G is a connected nilpotent Lie group, and K be a quadratically closed field with characteristic different from 2. Then $g : G \to K$ is a nonzero solution of d'Alembert's equation*

$$g(xy) + g(xy^{-1}) = 2g(x)g(y), \ x, y \in G, \tag{21}$$

if and only if g has the form $g = (m + \check{m})/2$, where $m : G \to K^*$ is a homomorphism. Furthermore suppose $K = \mathbb{C}$, then g is continuous if and only if m is continuous.

Proof. Let $g : G \to K$ be a non-zero solution of d'Alembert's equation. Then $g(xy) = g(yx) \ \forall x, y \in G$. Hence g satisfies the equation

$$g(xy) + g(yx) + g(xy^{-1}) + g(y^{-1}x) = 4g(x)g(y), \ x, y \in G. \tag{22}$$

Hence by the previous theorem g has the form $g = (m + \check{m})/2$, where $m : G \to K^*$ is a homomorphism. The converse result is trivial. Suppose $K = \mathbb{C}$. If m is continous then obviously so is g. If g is continuous then it follows from Theorem 1 in [6] or Proposition V.7 of [10] that m is continuous. □

3. Wilson's equation on connected nilpotent Lie groups

The following lemma is a slight extension of Lemma 1 of [3] in that the group inversion has been replaced by a general involution.

Lemma 3.1. *Let the pair $f, g : G \to \mathbb{C}$ be a solution of Wilson's equation*

$$f(xy) + f(x\tau(y)) = 2f(x)g(y), \ x, y \in G. \tag{23}$$

where $\tau : G \mapsto G$ is an involution. If f is not identically zero then g satisfies the following equation

$$g(xy) + g(yx) + g(x\tau(y)) + g(\tau(y)x) = 4g(x)g(y), \ x, y \in G. \tag{24}$$

Proof.

$$\begin{aligned} 8f(x)g(y)g(z) &= 4f(x)g(y)g(z) + 4f(x)g(z)g(y) \qquad (25) \\ &= 2f(xy)g(z) + 2f(x\tau(y))g(z) + 2f(xz)g(y) + 2f(x\tau(z))g(y) \\ &= f(xyz) + f(xy\tau(z)) + f(x\tau(y)z) + f(x\tau(y)\tau(z)) \\ &\quad + f(xzy) + f(xz\tau(y)) + f(x\tau(z)y) + f(x\tau(z)\tau(y)) \\ &= 2f(x)[g(yz) + g(zy) + g(y\tau(z)) + g(\tau(z)y)], \ x, y, z \in G \end{aligned}$$

Since f is assumed not to be identically zero, the result follows. □

The following theorem is a slight extension of Theorem 1 in [3], again because the group inversion has been replaced by a general involution τ.

Theorem 3.2. *Let G be a group. Suppose that the pair f and g is a solution to Wilson's equation,*

$$f(xy) + f(x\tau(y)) = 2f(x)g(y), \ x, y \in G, \tag{26}$$

where τ is an involution and f is nonzero. Suppose furthermore that there exists $u \in Z(G)$ such that $g(u)^2 \neq g(u\tau(u))$, then f and g has the form,

$$f = A\frac{m + m \circ \tau}{2} + B\frac{m - m \circ \tau}{2}, \ \ g = \frac{m + m \circ \tau}{2}, \tag{27}$$

where m is a homomorphism of G into $\mathbb{C}^*$, and $A, B \in \mathbb{C}$ are constants.

Proof. Since f is non-zero it follows from the previous lemma that g satisfies the following equation

$$g(xy) + g(yx) + g(x\tau(y)) + g(\tau(y)x) = 4g(x)g(y), \ x, y \in G. \tag{28}$$

Since we assume that there exists $u_0 \in Z(G)$ such that $g(u_0)^2 \neq g(u_0\tau(u_0))$ then it follows from Theorem 2.2 that $g = (m + m\circ\tau)/2$. Now $g(u_0)^2 \neq g(u_0\tau(u_0))$ implies that $m(u_0) \neq m(\tau(u_0))$. Now fix $x_0 \in G$ for the moment and consider the smallest abelian subgroup G_{x_0} of G which contains $Z(G)$ and x_0 ($G_{x_0} = \{x_0^n z : n \in \mathbb{Z}, z \in Z(G)\}$). We obviously have

$$f(xy) + f(x\tau(y)) = 2f(x)g(y), \ x, y \in G_{x_0}, \tag{29}$$

It follows from Theorem III.4 in [9] that f's restriction to G_{x_0} has the form

$$f(x) = c_1(x_0)\frac{m + m \circ \tau}{2}(x) + c_2(x_0)\frac{m - m \circ \tau}{2}(x), \ x \in G_{x_0}, \tag{30}$$

where $c_1(x_0), c_2(x_0) \in \mathbb{C}$ are constants. Putting $x = e$ and $x = u_0$ we find that

$$c_1(x_0) = f(e), \tag{31}$$

and

$$c_2(x_0) = \frac{2}{m(u_0) - m(\tau(u_0))}(f(u_0) - f(e)g(u_0)). \tag{32}$$

So the constants $A = c_1(x_0)$, $B = c_2(x_0) \in \mathbb{C}$ do not depend on our particular choice of x_0. So for arbitrary $x_0 \in G$ we have

$$f(x_0) = A\frac{m + m \circ \tau}{2} + B\frac{m - m \circ \tau}{2}. \tag{33}$$

□

Lemma 3.3. *Let G be a 2-divisible group. Let the pair $f, g : G \to \mathbb{C}$ be a solution to Wilson's equation*

$$f(xy) + f(xy^{-1}) = 2f(x)g(y), \ \forall x, y \in G, \tag{34}$$

where f is non-zero. Suppose that $\forall u \in Z(G) : g(u) = 1$ and $\exists x \in G : g(x) \neq 1$. Then the functions $F_1, G_1 : G/Z(G) \to \mathbb{C}$ can be defined by $F_1(\bar{x}) = f(x), \ \forall x \in G$, and $G_1(\bar{x}) = g(x), \ \forall x \in G$, where $\bar{x} = xZ(G)$. The functions F_1 and G_1 fulfil the equation

$$F_1(\bar{x}\bar{y}) + F_1(\bar{x}\bar{y}^{-1}) = 2F_1(\bar{x})G_1(\bar{y}), \ \bar{x}, \bar{y} \in G/Z(G). \tag{35}$$

Proof. This is like the proof of Lemma 2 of [3] with minor modifications. We already know from Lemma 2.4 that $G_1(\bar{x}) = g(x), \ \forall x \in G$ is a valid definition, since $g(xu) = g(x), \ \forall x \in G, \ \forall u \in Z(G)$.

We split f into its even and odd parts $f(x) = f_1(x) + f_2(x)$ where $f_1(x^{-1}) = f_1(x)$ and $f_2(x^{-1}) = -f_2(x)$.

$$2f(e)g(y) = f(y) + f(y^{-1}) = 2f_1(y). \tag{36}$$

So $f_1(x) = Ag(x)$ where $A = f(e)$. Now

$$f_1(xy) + f_2(xy) + f_1(xy^{-1}) + f_2(xy^{-1}) = 2[f_1(x) + f_2(x)]g(y) \tag{37}$$

implies that

$$f_2(xy) + f_2(xy^{-1}) = 2[Ag(x) + f_2(x)]g(y) - A[g(xy) + g(xy^{-1})]. \tag{38}$$

Exchange x and y in this equation.

$$f_2(yx) + f_2(yx^{-1}) = 2[Ag(y) + f_2(y)]g(x) - A[g(yx) + g(yx^{-1})]. \tag{39}$$

Note that $f_2(yx^{-1}) = -f_2((yx^{-1})^{-1}) = -f_2(xy^{-1})$, and $g(xy^{-1}) = g((xy^{-1})^{-1}) = g(yx^{-1})$. Adding the previous two equations, and using these two facts we get

$$f_2(xy) + f_2(yx) = 2f_2(x)g(y) + 2f_2(y)g(x) + A[g(y^{-1}x) - g(xy^{-1})]. \tag{40}$$

Taking $y = x$ in the following identity

$$f(xyu) + f(xy^{-1}u^{-1}) = 2f(x)g(uy) = 2f(x)g(y) = f(xy) + f(xy^{-1}), \tag{41}$$

we find that

$$f(x^2u) + f(u^{-1}) = f(x^2) + A, \ \forall x \in G, \ \forall u \in Z(G). \tag{42}$$

Since G is 2-divisible we get

$$f(xu) + f(u^{-1}) = f(x) + A, \ \forall x \in G, \ \forall u \in Z(G). \tag{43}$$

We know that

$$f_1(xu) = Ag(xu) = Ag(x) = f_1(x), \ \forall x \in G, \ \forall u \in Z(G). \tag{44}$$

So we get

$$\begin{aligned} f_2(xu) &= f(xu) - f_1(xu) = f(xu) - f(x) + f_2(x) \\ &= f_2(x) + A - Ag(u^{-1}) - f_2(u^{-1}) = f_2(x) + f_2(u). \end{aligned} \tag{45}$$

From (40)) we have that

$$\begin{aligned} f_2(xu) + f_2(ux) &= 2f_2(x)g(u) + 2g(x)f_2(u) + A[g(u^{-1}x) - g(xu^{-1})] \\ &= 2f_2(x) + 2g(x)f_2(u). \end{aligned} \tag{46}$$

Hence

$$f_2(x) + f_2(u) = f_2(xu) = f_2(x) + g(x)f_2(u). \tag{47}$$

So we get

$$0 = f_2(u)[g(x) - 1], \ \ \forall x \in G, \ \ \forall u \in Z(G). \tag{48}$$

Since there exists $x \in G$ such that $g(x) \neq 1$ we deduce that $f_2(u) = 0, \ \ \forall u \in Z(G)$. So

$$f_2(xu) = f_2(x), \ \forall x \in G, \ \forall u \in Z(G). \tag{49}$$

Hence

$$f(xu) = f_1(xu) + f_2(xu) = f_1(x) + f_2(x) = f(x), \ \ \forall x \in G, \ \ \forall u \in Z(G). \tag{50}$$

So $F_1(\bar{x}) = f(x)$ is a valid definition. It is trivial to check that

$$F_1(\bar{x}\bar{y}) + F_1(\bar{x}\bar{y}^{-1}) = 2F_1(\bar{x})G_1(\bar{y}), \ \bar{x}, \bar{y} \in G/Z(G). \tag{51}$$

□

Theorem 3.4. *Let G be a connected nilpotent Lie group. Let $f, g : G \to \mathbb{C}$ be a solution to Wilson's equation, where f is non-zero. Suppose that there exists $x \in G$ such that $g(x) \neq 1$. Then f and g have the form:*

$$f = A\frac{m + \check{m}}{2} + B\frac{m - \check{m}}{2}, \quad g = \frac{m + \check{m}}{2} \tag{52}$$

where $A, B \in \mathbb{C}$ are constants, and $m : G \to \mathbb{C}^*$ is a homomorphism. Conversely if f and g have this form where A, B are arbitrary constants, then the pair f, g is a solution to Wilsons equation.

Proof. The last claim is a trivial calculation. The proof of the fact that the solutions must have this form is standard technique. Let $\{e\} = Z_0 < \cdots < Z_n = G$ be an ascending central series for G, with $Z_{i+1}/Z_i = Z(G/Z_i)$. We will prove the result by induction on n. If $n = 1$ then $Z(G) = G$ and it follows by the previous theorem that f and g has the stated form. Let $n \in \mathbb{N}$ and suppose that the result is true for any connected nilpotent Lie group with an ascending central series af length n. Let G be a connected nilpotent Lie group with an ascending central series of length $n+1$, $\{e\} = Z_0 < \cdots < Z_n < Z_{n+1} = G$, with $Z_{i+1}/Z_i = Z(G/Z_i)$, $Z_1 = Z(G)$. If there exists $u \in Z(G)$ such that $g(u) \neq 1$, then the result follows by the Theorem 3.2. So suppose $g(u) = 1, \ \ \forall u \in Z(G)$. By the previous lemma we can define $F_1, G_1 : G/Z_1 \to \mathbb{C}$ by $F_1(\bar{x}) = f(x)$ and $G_1(\bar{x}) = g(x)$, furthermore F_1 and G_1 satisfy the equation

$$F_1(\bar{x}\bar{y}) + F_1(\bar{x}\bar{y}^{-1}) = 2F_1(\bar{x})G_1(\bar{y}), \ \bar{x}, \bar{y} \in G/Z_1. \tag{53}$$

G/Z_1 is a connected nilpotent Lie group with an ascending central series $Z_1/Z_1 < \cdots < Z_{n+1}/Z_1 = G/Z_1$ with $(Z_{i+1}/Z_1)/(Z_i/Z_1) = Z((G/Z_1)/(Z_i/Z_1))$. By assumption there exist a homomorphism $M : G/Z_1 \to \mathbb{C}^*$ and constants $A, B \in \mathbb{C}$ such that

$$F_1 = A\frac{M + \check{M}}{2} + B\frac{M - \check{M}}{2}, \quad G_1 = \frac{M + \check{M}}{2}. \tag{54}$$

Define $m : G \to \mathbb{C}^*$ by $m(x) = M(\bar{x})$, m is a homomorphism and $\check{m}(x) = \check{M}(\bar{x})$.

$$f(x) = F_1(\bar{x}) = A\frac{M + \check{M}}{2}(\bar{x}) + B\frac{M - \check{M}}{2}(\bar{x}) = A\frac{m + \check{m}}{2}(x) + B\frac{m - \check{m}}{2}(x), \tag{55}$$

and

$$g(x) = G_1(\bar{x}) = \frac{M + \check{M}}{2}(\bar{x}) = \frac{m + \check{m}}{2}(x). \tag{56}$$

The theorem follows by induction on n. □

Remark 3.5. (a) Instead of $f, g : G \to \mathbb{C}$ we could consider $f, g : G \to K$ where K is any quadratically closed field with characteristic different from 2.

(b) Instead of nilpotent connected Lie groups, we could consider a class $\mathcal{C}$ of nilpotent groups G, for which $G \in \mathcal{C}$ implies G and $Z(G)$ are 2-divisible and $G/Z(G) \in \mathcal{C}$. Note that if we take $\mathcal{C}$ to be all nilpotent groups where all elements are of odd order then $\mathcal{C}$ fulfils the requirement. So if we formulate the theorem in terms of classes $\mathcal{C}$ with the above mentioned properties, instead of for connected nilpotent Lie groups, then it contains as a special case Theorem 2 of [3].

4. Jensen's equation on a semidirect product of two groups

Let G be a semidirect product of G_1 and G_2. So we assume that G_1 is a transformation group of G_2 acting by homomorphisms, that is $a \cdot (xy) = (a \cdot x)(a \cdot y)$, $\forall a \in G_1$, $\forall x, y \in G_2$, and that the group operation in $G = G_2 \times G_1$, is given by

$$(x, a)(y, b) = (x(a \cdot y), ab), \ \forall (x, a), (y, b) \in G_2 \times G_1. \tag{57}$$

We let e_i denote the neutral element of G_i for $i = 1, 2$. Then $e = (e_1, e_2)$. Note that $(x, a)^{-1} = (a^{-1} \cdot x^{-1}, a^{-1})$, $\forall (x, a) \in G_2 \times G_1$. The idea is to reduce the study of functional equations on G to the study of functional equations on the subgroups G_1 and G_2. Clearly constant functions on G and homomorphisms of G into $\mathbb{C}^*$ are solutions to Jensen's equation on G. If f is a solution to Jensen's equation on G, then so is $f - f(e)$, so we may assume that $f(e) = 0$. If G is abelian and f is a solution of Jensen's equation on G such that $f(e) = 0$, then f is a homomorphism of G into $\mathbb{C}^*$ (see Lemma 1 in [1]).

Theorem 4.1. *Assume that G_1 and G_2 satisfy the following. If $f_i : G_i \mapsto \mathbb{C}$ satisfies*

$$f_i(cd) + f_i(cd^{-1}) = 2f_i(c) \ \forall c, d \in G_i \ \text{ and } \ f_i(e_i) = 0, \tag{58}$$

then $f_i \in Hom(G_i, \mathbb{C})$ $i = 1, 2$. Then $f : G \mapsto \mathbb{C}$ is a solution to Jensens equation on G

$$f((x, a)(y, b)) + f((x, a)(y, b)^{-1}) = 2f(x, a), \ \forall (x, a), (y, b) \in G, \tag{59}$$

such that $f(e) = 0$ if and only if

$$f(x, a) = A_1(a) + A_2(x) + A_2(a^{-1} \cdot x), \ \forall (x, a) \in G, \tag{60}$$

where $A_i \in Hom(G_i, \mathbb{C})$, $i = 1, 2$ *and*

$$A_2((ab) \cdot x) = A_2(a \cdot x) + A_2(b \cdot x) - A_2(x),\ \forall (x,a) \in G_2 \times G_1. \tag{61}$$

Suppose f *is of this form. Then* $f \in Hom(G, \mathbb{C})$ *if and only if* $A_2(a \cdot x) = A_2(x),\ \forall (x,a) \in G$.

Proof. Assume that $f : G \mapsto \mathbb{C}$ is a solution to Jensen's equation with $f(e) = 0$. Then

$$2f(x,a) = f(x(a \cdot y), ab) + f(x(ab^{-1} \cdot y^{-1}), ab^{-1}),\ \forall (x,a),(y,b) \in G. \tag{62}$$

Putting $b = e_1$ and fixing $a \in G_1$ in (62) we have

$$2f_a(x) = 2f(x,a) = f_a(x(a \cdot y)) + f_a(x(a \cdot y)^{-1}),\ \forall x, y \in G_2. \tag{63}$$

Since $a\cdot : G_2 \mapsto G_2$ is a bijection, it follows that $f_a : G_2 \mapsto \mathbb{C}$ is a solution to Jensen's equation on G_2. Hence

$$f(x,a) = f_a(x) = A_a(x) + f_a(e_2) = A_a(x) + f(e_2, a),\ \forall (x,a) \in G, \tag{64}$$

where $A_a : G_2 \mapsto \mathbb{C}$ is additive. Put $y = e_2$ and fix $x \in G_2$ in (62)

$$2f^x(a) = 2f(x,a) = f^x(ab) + f^x(ab^{-1}),\ \forall a, b \in G_1. \tag{65}$$

Hence $f^x : G_1 \mapsto \mathbb{C}$ is a solution to Jensen's equation on G_1.

$$f^x(a) = A^x(a) + f^x(e_1) = A^x(a) + f(x, e_1),\ \forall (x,a) \in G, \tag{66}$$

where $A^x : G_1 \mapsto \mathbb{C}$ is additive.

$$f(x, e_1) = A_{e_1}(x) + f(e_2, e_1) = A_{e_1}(x),\ \forall x \in G_2, \tag{67}$$

and

$$f(e_2, a) = f^{e_2}(a) = A^{e_2}(a) + f(e_2, e_1) = A^{e_2}(a),\ \forall a \in G_1. \tag{68}$$

Hence we have

$$f(x,a) = A_a(x) + A^{e_2}(a) = A^x(a) + A_{e_1}(x),\ \forall (x,a) \in G. \tag{69}$$

Note that

$$A^{xy}(a) = A_a(xy) + A^{e_2}(a) - A_{e_1}(xy) = A^x(a) + A^y(a) - A^{e_2}(a). \tag{70}$$

Now

$$\begin{aligned} 2f(x,a) &= f((x,a)(x,a)) + f((x,a)(x,a)^{-1}) = f(x(a \cdot x), a^2) \\ &= A^{x(a \cdot x)}(a^2) + A_{e_1}(x(a \cdot x)) \\ &= 2(A^x(a) + A_{e_1}(x)) - A_{e_1}(x) + 2(A^{a \cdot x}(a) - A^{e_2}(a)) + A_{e_1}(a \cdot x) \\ &= 2f(x,a) - A_{e_1}(x) + 2A_a(a \cdot x) - A_{e_1}(a \cdot x),\ \forall (x,a) \in G. \end{aligned}$$

So

$$A_a(a \cdot x) = \frac{1}{2}(A_{e_1}(x) + A_{e_1}(a \cdot x)),\ \forall (x,a) \in G. \tag{71}$$

Substitute $a^{-1} \cdot x$ for x, that gives us

$$A_a(x) = \frac{1}{2}(A_{e_1}(x) + A_{e_1}(a^{-1} \cdot x)),\ \forall (x,a) \in G. \tag{72}$$

So

$$f(x,a) = A_a(x) + A^{e_2}(a) = A^{e_2}(a) + B(x) + B(a^{-1} \cdot x), \forall (x,a) \in G, \tag{73}$$

where $B = \frac{1}{2}A_{e_1}$ is an additive function on G_2. $f(x,a) = A^{e_2}(a)+B(x)+B(a^{-1}\cdot x)$ is a solution to Jensen's equation if and only if $g(x,a) = B(x) + B(a^{-1} \cdot x)$ is a solution to Jensen's equation. The computation

$$\begin{aligned} 2B(x) + 2B(a^{-1}\cdot x) &= 2g(x,a) = g(x(a\cdot y), ab) + g(x(ab^{-1}\cdot y^{-1}), ab^{-1}) \\ &= B(x(a\cdot y)) + B((ab)^{-1}\cdot(x(a\cdot y))) \\ &\quad +B(x(ab^{-1}\cdot y^{-1})) + B((ab^{-1})^{-1}\cdot(x(ab^{-1}\cdot y^{-1}))) \\ &= 2B(x) + B(a\cdot y) + B(b^{-1}a^{-1}\cdot x) + B(b^{-1}\cdot y) \\ &\quad +B(ab^{-1}\cdot y^{-1}) + B(ba^{-1}\cdot x) + B(y^{-1}), \end{aligned} \tag{74}$$

shows that g is a solution to Jensen's equation if and only if

$$\begin{aligned} 2B(a^{-1}\cdot x) &= B(a\cdot y) + B(b^{-1}a^{-1}\cdot x) + B(b^{-1}\cdot y) \\ &\quad +B(ab^{-1}\cdot y^{-1}) + B(ba^{-1}\cdot x) - B(y),\ \forall (x,a),(y,b) \in G. \end{aligned} \tag{75}$$

Put $x = e_2$ in (75) to get

$$0 = B(a\cdot y) + B(b^{-1}\cdot y) - B(ab^{-1}\cdot y) - B(y). \tag{76}$$

Put $a = b$ in (76) to get

$$2B(y) = B(a\cdot y) + B(a^{-1}\cdot y). \tag{77}$$

In particular we have

$$2B(a^{-1}\cdot x) = B(ba^{-1}\cdot x) + B(b^{-1}a^{-1}\cdot x). \tag{78}$$

It is now obvious that if conversely (76) holds, i.e. if

$$0 = B(a\cdot y) + B(b^{-1}\cdot y) - B(ab^{-1}\cdot y) - B(y),\ \ \forall a,b \in G_1,\ \ \forall y \in G_2, \tag{79}$$

then g is a solution to Jensen's equation. The condition (76) is equivalent to

$$0 = B(a\cdot y) + B(b\cdot y) - B(ab\cdot y) - B(y),\ \ \forall a,b \in G_1,\ \ \forall y \in G_2. \tag{80}$$

Now all that remains is to determine when f is additive. This is the case if and only if g is additive.

$$\begin{aligned} g((x,a)(y,b)) - g(x,a) - g(y,b) &= B(x(a\cdot y)) + B((ab)^{-1}\cdot(x(a\cdot y))) \qquad (81) \\ &\quad -B(x) - B(a^{-1}\cdot x) - B(y) - B(b^{-1}\cdot y) \\ &= B(b^{-1}a^{-1}\cdot x) - B(a^{-1}\cdot x) + B(a\cdot y) - B(y). \end{aligned}$$

Suppose that g is additive

$$0 = B(b^{-1}a^{-1}\cdot x) - B(a^{-1}\cdot x) + B(a\cdot y) - B(y). \tag{82}$$

Put $x = e_2$

$$B(a\cdot y) = B(y),\ \forall a \in G_1,\ \forall y \in G_2. \tag{83}$$

Conversely if this condition is fulfilled then g is additive.

□

Example 4.2. The Heisenberg-group $H_3 = \mathbb{R}^2 \times_s \mathbb{R}$, where the action of $\mathbb{R}$ on $\mathbb{R}^2$ is given by $x\cdot(y,z) = (y, z+xy)$. Here $e = ((0,0),0)$.

Proposition 4.3. *f is a solution to Jensen's equation on H_3 with $f(e) = 0$ if and only if and only if*

$$f((y, z), x) = A_1(x) + A_2(y) + 2A_3(z - \frac{1}{2}xy),\ \forall x, y, z \in \mathbb{R}, \tag{84}$$

where $A_i \in Hom(\mathbb{R}, \mathbb{C})$ are arbitrary, $i = 1, 2, 3$.

Suppose f is a solution to Jensen's equation. Then $f \in Hom(H_3, \mathbb{C})$ if and only if $A_3 \equiv 0$.

Proof. Let $B : \mathbb{R}^2 \mapsto \mathbb{C}$ be any additive function on $\mathbb{R}^2$. It is a simple calculation to check that

$$B((x_1+x_2)\cdot(y, z))+B(y, z) = B(x_1\cdot(y, z))+B(x_2\cdot(y, z)),\ \forall x_1, x_2, y, z \in \mathbb{R}. \tag{85}$$

So it follows immediately from Theorem 4.1 that the solutions to Jensen's equation on H_3 are of the form

$$f((y, z), x) = A_1(x) + B(y, z) + B(y, z - xy), \forall x, y, z \in \mathbb{R}, \tag{86}$$

where $B \in Hom(\mathbb{R}^2, \mathbb{C})$ is arbitrary. For $B \in Hom(\mathbb{R}^2, \mathbb{C})$ there exist $A_2, A_3 \in Hom(\mathbb{R}, \mathbb{C})$ such that $B(y, z) = \frac{1}{2}A_2(y) + A_3(z),\ \forall y, z \in \mathbb{R}$. When is f additive? We know from Theorem 4.1 that it is the case if and only if

$$\frac{1}{2}A_2(y)+A_3(z+xy) = B(x\cdot(y, z)) = B(y, z) = \frac{1}{2}A_2(y)+A_3(z),\ \forall x, y, z \in \mathbb{R}, \tag{87}$$

that is if and only if $A_3 \equiv 0$. □

Note in particular that $f((y, z)x) = 2z - xy$ is a solution to Jensen's equation on the Heisenberg group with $f(e) = 0$ which is not a homomorphism. So this example show that genuine differences occur, from the abelian case, when we attempt to solve Jensen's equation on non-abelian groups. Furthermore H_3 is a connected nilpotent Lie group, so the example also shows that contrary to what Theorem 3.4 might lead one to suspect, the solutions to the degenerate Wilson's equation, i.e. Jensen's equation, need not be of the classical form, even on connected nilpotent Lie groups. By the term the classical form we mean homomorphisms, which is the form of the solutions in the case where G is abelian. In contrast Ng has shown that all solutions to (6) with $f(e) = 0$ are homomorphisms for certain groups including the Heisenberg group (private communication and presented in his talk at the 37th ISFE).

Example 4.4. The (ax+b)-group $G = \mathbb{R} \times_s \mathbb{R}_+$ where the action of $\mathbb{R}_+$ on $\mathbb{R}$ is given by $a \cdot y = ay$. Here $e = (0, 1)$.

Proposition 4.5. *$f : G \mapsto \mathbb{C}$ is a solution to Jensen's equation with $f(e) = 0$ if and only if $f(x, a) = A(a),\ \forall a \in \mathbb{R}_+,\ \forall x \in \mathbb{R}$, where $A : \mathbb{R}_+ \mapsto \mathbb{C}$ is additive.*

Proof. Assume that $B : \mathbb{R} \mapsto \mathbb{C}$ is additive and

$$B((ab) \cdot x) + B(x) = B(a \cdot x) + B(b \cdot x),\ \forall x \in \mathbb{R},\ \forall a, b \in \mathbb{R}_+. \tag{88}$$

Put $a = b = 2$

$$B(4x) = B(2 \cdot x) + B(2 \cdot x) = B(4x) + B(x),\ \forall x \in \mathbb{R}. \tag{89}$$

Hence $B(x) = 0,\ \forall x \in \mathbb{R}$. The proposition now follows immediately from Theorem 4.1. □

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