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1. Introduction

The defect, $D(\pi)$, of a factor map $\pi : (Y, \psi) \to (X, \varphi)$ between dynamical systems was defined in [Th1] under the assumption that X is a totally disconnected compact metric space, and it was calculated in a series of specific cases. The defect gives a numerical indication of how far π is from being injective; an indication which is particularly sensitive to the ambiguity of π over periodic orbits of φ. In [Th2] a variational principle for the defect was established:

$$D(\pi) = \sup_{\mu} \int_X \log \# \pi^{-1}(x) \ d\mu(x),$$

where we take the supremum over all φ-invariant Borel probability measures on X. In this paper the definition of the defect is extended to the general case, i.e. we drop the assumption that Y is totally disconnected and define the defect in a way which is analogous to - and generalizes - the case when Y is totally disconnected. We prove the variational principle in the general case and show how almost all the general properties of the defect follow from this principle. In particular, we obtain the subadditivity

$$D(\pi_2 \circ \pi_1) \leq D(\pi_2) + D(\pi_1)$$

for the composition of factor maps between invertible dynamical systems. This property fails dramatically for general non-invertible dynamical systems and this has effects for the notion of finite equivalence between dynamical systems which the defect suggests in a natural way. For this reason we consider a slight variation in the definition of the defect which makes no difference for factor maps between invertible dynamical systems, but results in a smaller number in general. We call this *the reduced defect* of the factor map, and denote it by $D_r(\pi)$. The reduced defect is sub-additive in general and relates directly to the defect via the notion of natural invertible extensions of dynamical systems. Recall that the (inverse of the) natural invertible extension is the invertible dynamical system which arises as the shift acting on the inverse limit of the given space with the given map as bonding maps. A factor map, π, between (non-invertible) dynamical systems induces in a natural way a factor map, $\widehat{\pi}$, between the natural invertible extensions, and it turns out that

$$D_r(\pi) = D(\widehat{\pi}).$$

This makes it possible to transfer general properties of the defect to properties of the reduced defect. For example, we use it to obtain a variational principle for the

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reduced defect of a factor map $\pi : (Y, \psi) \to (X, \varphi)$;

$$D_r(\pi) = \sup_{\mu} \int_X \log A_\pi(x) \ d\mu(x),$$

where we take the supremum over all φ-invariant Borel probability measures on X and A_π is a Borel function $A_\pi : X \to \mathbb{N} \cup \{\infty\}$ canonically associated to π. It turns out that, unlike the defect itself, the reduced defect is always the logarithm of a natural number. It follows therefore that to any factor map between arbitrary dynamical systems there is associated a natural number (or $+\infty$) which carries substantial information about how well the factor map relates the dynamical systems.

As indicated above the subadditivity of the defect (or the reduced defect), combined with the variational principle, leads to a natural generalization of the notion of finite equivalence first introduced by Parry, cf. [P], and used by him to give a classification of irreducible sofic shifts in terms of topological entropy. Namely we say that two invertible dynamical systems, $(X, \varphi), (Y, \psi)$, are *finitely equivalent* when there is an invertible dynamical system (Z, κ) and factor maps $\pi_1 : (Z, \kappa) \to (X, \varphi)$, $\pi_2 : (Z, \kappa) \to (Y, \psi)$ such that $D(\pi_1) + D(\pi_2) < \infty$. This equivalence relation generalizes the notion of finite equivalence of irreducible sofic shifts, and by using the reduced defect instead we obtain a further generalization to arbitrary dynamical systems. In the remaining part of the paper we make a first investigation of this equivalence relation. Specifically, we determine the finite equivalence classes of a series of dynamical systems which are all quite well-understood: Irreducible sofic shifts (two-sided as well as one-sided), hyperbolic toral automorphisms and expansive endomorphisms, periodic maps, homeomorphisms of the circle with an irrational rotation number and minimal rotations of tori. The most important invariant for finite equivalence is the topological entropy, and for some sufficiently restricted classes of dynamical systems (such as hyperbolic automorphisms of tori or expansive endomorphisms of manifolds) it is also the only invariant. But in general it is not. For example we show that two orientation preserving homeomorphisms of the circle $\mathbb{T}$ with irrational rotation numbers, $\alpha, \beta \in \mathbb{R}$, are finitely equivalent if and only if $1, \alpha$ and β are rationally dependent.

2. The defect of factor maps: Definition and the variational principle

Let (X, φ) be a dynamical system[1] acting on a compact space X. Let Y be another compact metric space and $\pi : Y \to X$ be a continuous surjection. Let $\mathcal{U} = \{U_i : i \in I\}$ be a finite cover of Y. For any subset $F \subseteq Y$ we let $C(F, \mathcal{U})$ denote the minimal number of elements in $\mathcal{U}$ needed to cover F, i.e.

$$C(F, \mathcal{U}) = \min\{\#J : J \subseteq I, \ \bigcup_{j \in J} U_j \supseteq F\} .$$

For $x \in X$, set

$$a_k(x, \varphi, \mathcal{U}) = C(\pi^{-1}(x), \mathcal{U}) C(\pi^{-1}(\varphi(x)), \mathcal{U}) C(\pi^{-1}(\varphi^2(x)), \mathcal{U}) \cdots C(\pi^{-1}(\varphi^{k-1}(x)), \mathcal{U}) . \tag{2.1}$$

[1]Here and in the following all dynamical systems are implicitly assumed to act on a compact *metric* space.

When $\mathcal{U}$ is a partition of Y, $a_k(x,\varphi,\mathcal{U}) = q_k(x,\pi(\mathcal{U}))$, where the last quantity was one of the fundamental entities used to define the defect in [Th1]. Set

$$a_k(\varphi,\mathcal{U}) = \sup_{x\in X} a_k(x,\varphi,\mathcal{U}),$$

so that $a_k(\varphi,\mathcal{U}) = q_k(\varphi,\pi(\mathcal{U}))$ when $\mathcal{U}$ is a partition, cf. [Th1]. Then

1) $a_{k+n}(\varphi,\mathcal{U}) \leq a_k(\varphi,\mathcal{U})a_n(\varphi,\mathcal{U})$,
2) $a_k(\varphi,\mathcal{U}) \leq a_k(\varphi,\mathcal{V})$ when $\mathcal{V}$ is a refinement of $\mathcal{U}$.

It follows from 1) that we can consider the limit

$$A(\varphi,\mathcal{U}) = \lim_{n\to\infty} \frac{1}{n} \log a_n(\varphi,\mathcal{U}).$$

We define the defect of π to be

$$D(\pi) = \sup_{\mathcal{U}} A(\varphi,\mathcal{U}),$$

where we take the supremum over all finite open covers $\mathcal{U}$ of Y. It follows from 2) that

$$D(\pi) = \lim_{k\to\infty} A(\varphi,\mathcal{U}_k),$$

for any sequence $\mathcal{U}_k, k \in \mathbb{N}$, of open covers of Y for which the maximal diameter of any set in the cover $\mathcal{U}_k$ goes to zero as k tends to infinity. In particular, $D(\pi)$ agrees with the defect defined in [Th1] when Y is totally disconnected.

Theorem 2.1. *(The variational principle.) The function $x \mapsto \#\pi^{-1}(x)$ is Borel, and*

$$D(\pi) = \sup_{\mu} \int_X \log \#\pi^{-1}(x) \ d\mu(x),$$

where we take the supremum over all φ-invariant Borel probability measures on X. In fact, it suffices to take the supremum over all φ-ergodic Borel probability measures on X.

Proof. Let $\mathcal{U}_n = \{U_i^n : i = 1,2,\cdots,I_n\}$ be a sequence of finite open covers of Y such that

$$\lim_{n\to\infty} \max\{\operatorname{diam} U_i^n : i = 1,2,\cdots,I_n\} \ = \ 0.$$

For each n, set $\tilde{U_1^n} = U_1^n$ and

$$\tilde{U_i^n} = U_i^n \backslash (U_1^n \cup U_2^n \cup \cdots \cup U_{i-1}^n),$$

for $i \geq 2$. Note that $\tilde{U_i^n}$ is an F_σ-set so that $\pi(\tilde{U_i^n})$ is an F_σ-set and hence also a Borel set for all n,i. For each n, define a Borel function $f_n : X \to \mathbb{N}$ by

$$f_n(x) = \#\{i : x \in \pi(\tilde{U_i^n})\} = \sum_{i=1}^{I_n} 1_{\pi(\tilde{U_i^n})}(x).$$

We claim that

$$\#\pi^{-1}(x) = \lim_{n\to\infty} f_n(x) \tag{2.2}$$

for all $x \in X$. To see this, let $k \in \mathbb{N}$ satisfy that $k \leq \#\pi^{-1}(x)$. There are then k distinct elements $y_1, y_2, \cdots, y_k \in \pi^{-1}(x)$. Let $N \in \mathbb{N}$ satisfy that $\max_i \operatorname{diam} \tilde{U_i^n}$ is smaller than any distance between y_k and y_l when $k \neq l$, for all $n \geq N$. Then

$k \leq f_n(x) \leq \#\pi^{-1}(x)$ for all $n \geq N$, proving (2.2). In particular, we see that $x \mapsto \#\pi^{-1}(x)$ is a Borel function. By Fatou's lemma,

$$\int_X \log \#\pi^{-1}(x) \ d\mu(x) \leq \liminf_n \int_X \log f_n \ d\mu \tag{2.3}$$

for any φ-invariant Borel probability measure μ on X. Let $t < \int_X \log \#\pi^{-1}(x) \ d\mu$. It follows from (2.3) that we can choose n so large that

$$t < \int_X \log f_n \ d\mu.$$

Since each $\tilde{U_i^n}$ is an F_σ-set we can find sequences $F_i^1 \subseteq F_i^2 \subseteq F_i^3 \subseteq \cdots$ of closed sets such that $\tilde{U_i^n} = \bigcup_k F_i^k$. Then

$$\lim_{k \to \infty} \max\{\#\{i : x \in \pi(F_i^k)\}, 1\} = f_n(x),$$

non-decreasingly, for all $x \in X$, so Lebesgue's monotone convergence theorem gives us a k such that

$$t < \int_X \log(\max\{\#\{i : x \in \pi(F_i^k)\}, 1\}) \ d\mu(x). \tag{2.4}$$

When $\mathcal{F} = \{F_i : i \in I\}$ is a collection of subsets of X (not necessarily a cover), we set

$$q_k'(x, \mathcal{F}) = \prod_{j=0}^{k-1} \max\{1, \#\{i : \varphi^j(x) \in F_i\}\}$$

for $x \in X, k \in \mathbb{N}$. Set $q_k'(\varphi, \mathcal{F}) = \sup_{x \in X} q_k'(x, \mathcal{F})$. Then the limit $Q'(\varphi, \mathcal{F}) = \lim_{n \to \infty} \frac{1}{n} \log q_n'(\varphi, \mathcal{F})$ exists and is equal to $\inf_n \frac{1}{n} \log q_n'(\varphi, \mathcal{F})$. In particular, there is, for $\epsilon > 0$, an m such that

$$\frac{1}{m} \log q_m'(\varphi, \mathcal{G}) < Q'(\varphi, \mathcal{G}) + \epsilon$$

when $\mathcal{G} = \{\pi(F_i^k) : i = 1, 2, \cdots, I_n\}$. For each i we choose a decreasing sequence $U_i^1 \supseteq U_i^2 \supseteq \cdots$ of open sets in X such that $\overline{U_i^{l+1}} \subseteq U_i^l$ for all l and $\bigcap_l U_i^l = \pi(F_i^k)$. Since

$$\begin{aligned} &\bigcap_l U_{i_1}^l \cap \varphi^{-1}(U_{i_2}^l) \cap \varphi^{-2}(U_{i_3}^l) \cap \cdots \cap \varphi^{-n+1}(U_{i_m}^l) \\ &\qquad = \pi(F_{i_1}^k) \cap \varphi^{-1}(\pi(F_{i_2}^k)) \cap \varphi^{-2}(\pi(F_{i_3}^k)) \cap \cdots \cap \varphi^{-n+1}(\pi(F_{i_m}^k)) \end{aligned}$$

for each tuple $(i_1, i_2, \cdots, i_m) \in I_n^m$, there is an l so large that

$$q_m'(\varphi, \mathcal{U}^l) = q_m'(\varphi, \mathcal{G}),$$

when we set $\mathcal{U}^l = \{U_i^l : i = 1, 2, \cdots, I_n\}$. It follows that

$$Q'(\varphi, \mathcal{U}^l) \leq Q'(\varphi, \mathcal{G}) + \epsilon. \tag{2.5}$$

For each $d \in \mathbb{N}$, let

$$L_d = \bigcup_J \bigcap_{j \in J} U_j^l,$$

where we take the union over all subsets J of $\{1,2,\cdots,I_n\}$ of cardinality $\leq d$. Take continuous functions $g_d : X \to [0,1]$ with support in L_d. We claim that

$$Q'(\varphi,\mathcal{G}) \geq \int_X \log(\max\{\sum_{d=1}^{I_n} g_d(x),1\})\ d\mu(x) - 2\epsilon. \tag{2.6}$$

To prove (2.6) it suffices, since μ is the weak*-limit of a convex combination of φ-ergodic Borel probability measures, to consider the case when μ is φ-ergodic. In that case

$$\int_X \log(\max\{\sum_{d=1}^{I_n} g_d(x),1\})\ d\mu(x) = \lim_{m\to\infty}\frac{1}{m}\sum_{i=0}^{m-1}\log(\max\{\sum_{d=1}^{I_n} g_d(\varphi^i(z)),1\})$$

for μ-almost all $z \in X$. There is therefore a point $z \in X$ such that

$$\frac{1}{m}\sum_{i=0}^{m-1}\log(\max\{\sum_{d=1}^{I_n} g_d(\varphi^i(z)),1\}) \geq \int_X \log(\max\{\sum_{d=1}^{I_n} g_d(x),1\})\ d\mu(x) - \epsilon$$

for all sufficiently large m. Since $\max\{\sum_{d=1}^{I_n} g_d(\varphi^i(z)),1\} \leq \max\{1,\#\{d \in I_n : \varphi^i(z) \in U_d^l\}\}$, we deduce that

$$\frac{1}{m}\log q'_m(\varphi,\mathcal{U}^l) \geq \int_X \log(\max\{\sum_{d=1}^{I_n} g_d(x),1\})\ d\mu(x) - \epsilon$$

for all large enough m. (2.6) follows from this and (2.5). Let $g_d^1 \leq g_d^2 \leq g_d^3 \leq \cdots$ be an increasing sequence of continuous functions such that $\lim_{n\to\infty} g_d^n = 1_{L_d}$. Then

$$\lim_{m\to\infty}\log(\max\{\sum_{d=1}^{I_n} g_d^m(x),1\})\ = \log(\max\{\#\{i : x \in U_i^l\},1\})$$

for all $x \in X$. It follows therefore from (2.4) and (2.6) that

$$\begin{aligned} t - 2\epsilon &< \int_X \log(\max\{\#\{i : x \in \pi(F_i^k)\},1\})\ d\mu(x) - 2\epsilon \\ &\leq \int_X \log(\max\{\#\{i : x \in U_i^l\},1\})\ d\mu(x) - 2\epsilon \leq Q'(\varphi,\mathcal{G}). \end{aligned}$$

Since $F_i^k \cap F_j^k = \emptyset$ when $i \neq j$, we can easily construct an open cover $\mathcal{V} = \{V_i : i = 1,2,\cdots,I_n+1\}$ of Y such that $F_i^k \subseteq V_i \backslash \bigcup_{j\neq i} V_j$ for all $i = 1,2,\cdots,I_n$. Then $C(\pi^{-1}(x),\mathcal{V}) \geq \#\{i : x \in \pi(F_i^k)\}$ for all $x \in X$ and hence

$$A(\varphi,\mathcal{V}) \geq Q'(\varphi,\mathcal{G}) > t - 2\epsilon.$$

It follows that $D(\pi) > t - 2\epsilon$, proving that

$$D(\pi) \geq \sup_\mu \int_X \log\#\pi^{-1}(x)\ d\mu(x)\ .$$

To prove the reversed inequality, let $t \in \mathbb{R}$ be a number such that $t < D(\pi)$. There is an open cover $\mathcal{U}$ of Y such that $A(\varphi,\mathcal{U}) > t$. For each n choose a point $x_n \in X$ such that $\frac{1}{n}\log a_n(x_n,\varphi,\mathcal{U}) > \frac{1}{n}\log a_n(\varphi,\mathcal{U}) - \epsilon$. Then

$$\frac{1}{n}\log a_n(x_n,\varphi,\mathcal{U}) > A(\varphi,\mathcal{U}) - \epsilon > t - \epsilon \tag{2.7}$$

for all n. Let μ_n be the measure

$$\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} \delta_{\varphi^i(x_n)}.$$

There is then a sequence $\{n_j\}$ in $\mathbb{N}$ and a φ-invariant Borel probability measure μ_∞ on X such that $\lim_{j\to\infty} \mu_{n_j} = \mu_\infty$. Let $\delta > 0$ be a Lebesgue number for $\mathcal{U}$. We need the following

Observation 2.2. There is an open cover $\mathcal{V} = \{U_i : i = 1, 2, \cdots, m\}$ of Y such that $\operatorname{diam}(U_i) < \delta$ for all i and such that

$$\mu_n(\pi(\overline{U_1})\backslash\pi(U_1)) = 0,$$

and

$$\mu_n(\pi(\overline{U_j\backslash(U_1 \cup U_2 \cup \cdots \cup U_{j-1})})\backslash\pi(U_j\backslash(U_1 \cup U_2 \cup \cdots \cup U_{j-1}))) = 0$$

for $j = 2, 3, \cdots, m$, and for all $n \in \mathbb{N} \cup \{\infty\}$.

To prove this observation we need some notation. For every set $B \subseteq Y$ and every $\epsilon > 0$, let $B^\epsilon = \{y \in Y : \operatorname{dist}(y, B) < \epsilon\}$. Let $\{S_i : i = 1, 2, \cdots, m\}$ be an open cover of Y such that $\operatorname{diam} S_i < \frac{\delta}{2}$ for all i. Since $\overline{S_1^t} \subseteq S_1^s$ when $t < s$, we see that the sets

$$\pi(\overline{S_1^t})\backslash\pi(S_1^t), \ t \in]0, \frac{\delta}{2}[,$$

are mutually disjoint. There must therefore be an $\epsilon_1 \in]0, \frac{\delta}{2}[$ such that $\mu_n(\pi(\overline{S_1^{\epsilon_1}})\backslash\pi(S_1^{\epsilon_1})) = 0$ for all $n \in \mathbb{N} \cup \{\infty\}$. Note that

$$\pi(\overline{S_2^t\backslash S_1^{\epsilon_1}}) \subseteq \pi(S_2^s\backslash S_1^{\epsilon_1})$$

when $t < s$. We can therefore repeat the above argument to find a $\epsilon_2 \in]0, \frac{\delta}{2}[$ such that

$$\mu_n(\pi(\overline{S_2^{\epsilon_2}\backslash S_1^{\epsilon_1}})\backslash\pi(S_2^{\epsilon_2}\backslash S_1^{\epsilon_1})) = 0$$

for all $n \in \mathbb{N} \cup \{\infty\}$. Continuing in this way we find $\epsilon_j \in]0, \frac{\delta}{2}[, j = 1, 2, \cdots, m$, such that

$$\mu_n(\pi(\overline{S_j^{\epsilon_j}\backslash(S_1^{\epsilon_1} \cup S_2^{\epsilon_2} \cup \cdots \cup S_{j-1}^{\epsilon_{j-1}})})\backslash\pi(S_j^{\epsilon_j}\backslash(S_1^{\epsilon_1} \cup S_2^{\epsilon_2} \cup \cdots \cup S_{j-1}^{\epsilon_{j-1}}))) = 0$$

for all $n \in \mathbb{N} \cup \{\infty\}$. Set $U_j = S_j^{\epsilon_j}, j = 1, 2, \cdots, m$. Then $\mathcal{V} = \{U_i : i = 1, 2, \cdots, m\}$ is an open cover with the desired property.

Set $V_1 = U_1, V_j = U_j\backslash(U_{j-1} \cup U_{j-2} \cup \cdots \cup U_1)$. Then $\mathcal{W} = \{V_i : i = 1, 2, \cdots, m\}$ is a partition of Y which refines $\mathcal{U}$. In particular, $a(x, \varphi, \mathcal{U}) \leq a(x, \varphi, \mathcal{W})$ for all x and hence

$$\frac{1}{n}\log\ a_n(x_n, \varphi, \mathcal{U}) \leq \frac{1}{n}\log\ a_n(x_n, \varphi, \mathcal{W}) \tag{2.8}$$

for all $n \in \mathbb{N}$. Note that

$$\frac{1}{n}\log a_n(x_n, \varphi, \mathcal{W}) = \frac{1}{n}\sum_{j=0}^{n-1} \log\ C(\pi^{-1}(\varphi^j(x_n)), \mathcal{W})$$

$$= \frac{1}{n}\sum_{j=0}^{n-1} \log \#\{l : \varphi^j(x_n) \in \pi(V_l)\} = \int_X \log \#\{l : x \in \pi(V_l)\}\ d\mu_n(x)$$

for all n. The special properties of $\mathcal{V}$ ensure that

$$\int_X \log \#\{l : x \in \pi(V_l)\}\ d\mu_n(x) = \int_X \log \#\{l : x \in \pi(\overline{V_l})\}\ d\mu_n(x)$$

for all n. So when we combine with (2.7) and (2.8) we find that

$$\int_X \log \#\{l : x \in \pi(\overline{V_l})\}\ d\mu_n(x) > t - \epsilon \tag{2.9}$$

for all n. Since $\pi(\overline{V_l})$ is closed for all l there is a decreasing sequence $h_1 \geq h_2 \geq h_3 \geq \cdots$ of continuous functions such that $\lim_{n\to\infty} h_n(x) = \log \#\{l : x \in \pi(\overline{V_l})\}$ for all $x \in X$. For each n we have that

$$t - \epsilon \leq \int_X h_k\ d\mu_n$$

for all k. By restricting to $\{n_j\}$ and taking the limit over j it follows that

$$t - \epsilon \leq \int_X h_k\ d\mu_\infty$$

for all k, and by taking the limit over k, that $t-\epsilon \leq \int_X \log \#\{l : x \in \pi(\overline{V_l})\}\ d\mu_\infty(x) = \int_X \log \#\{l : x \in \pi(V_l)\}\ d\mu_\infty(x)$. Since each V_l is an F_σ-set, an application of Lebesgue's theorem on monotone convergence gives us closed subsets $F_l \subseteq V_l$, $l = 1, 2, \cdots, m$, such that

$$t - 2\epsilon \leq \int_X \log \#\{l : x \in \pi(F_l)\}\ d\mu_\infty(x). \tag{2.10}$$

Being the infimum of continuous functions, the map W given by

$$W(\nu) = \int_X \log \#\{l : x \in \pi(F_l)\}\ d\nu(x)$$

is upper semi-continuous on the compact convex set of φ-invariant Borel probability measures on X and hence it attains it maximum at an extreme point. It follows therefore from (2.10) that there is φ-ergodic measure ν on X such that $t - 2\epsilon \leq \int_X \log \#\{l : x \in \pi(F_l)\}\ d\nu(x)$. Since $\#\{l : x \in \pi(F_l)\} \leq \#\pi^{-1}(x)$ we conclude that

$$D(\pi) \leq \sup\{\int_X \log \#\pi^{-1}(x)\ d\mu(x) : \ \mu \text{ is a } \varphi\text{-ergodic Borel probability measure}\}.$$

□

3. General properties of the defect

With the variational principle established we can now quickly generalize the general properties of the defect from [Th1].

Proposition 3.1. *1) $D(\pi) \leq \log d$ when $\#\pi^{-1}(x) \leq d$ for all $x \in X$, and $D(\pi) = \log d$ when $\#\pi^{-1}(x) = d$ for all $x \in X$.*

2) $D(\pi) \geq \frac{1}{p}\sum_{i=0}^{p-1} \log \#\pi^{-1}(\varphi^i(x))$ when $x \in X$ is p-periodic.

3) When $A_i \subseteq X, i \in I$, is a family of closed φ-invariant subsets such that $\bigcup_{i\in I} A_i = X$, $D(\pi) = \sup_i D(\pi|_{\pi^{-1}(A_i)})$.

4) $D(\pi) = D(\pi|_{\pi^{-1}(\Omega)})$, where Ω is the set of non-wandering points for φ.

5) $D(\pi) = D(\pi|_{\pi^{-1}(\cap_{k=0}^\infty \varphi^k(X))})$.

6) $D(\pi_2 \circ \pi_1) \geq D(\pi_1)$ when $\pi_1 : (Y, \psi) \to (X, \varphi)$ and $\pi_2 : (X, \varphi) \to (Z, \lambda)$ are factor maps, and $D(\pi_2 \circ \pi_1) = D(\pi_1)$ when π_2 is a conjugacy.

Proof. All items follow straightforwardly from Theorem 2.1. □

Theorem 3.2. *Let $\pi_n : Y_n \to X_n,\ \varphi_n : X_n \to X_n, n \in \mathbb{N}$, be continuous maps between compact metric spaces. Assume that each π_n is surjective. Consider the dynamical system $(\prod_{n=1}^\infty X_n, \prod_{n=1}^\infty \varphi_n)$ and the continuous surjection $\prod_{n=1}^\infty \pi_n : \prod_{n=1}^\infty Y_n \to \prod_{n=1}^\infty X_n$. Then*

$$D(\prod_{n=1}^\infty \pi_n) = \sum_{n=1}^\infty D(\pi_n).$$

Proof. Let μ be a $\prod_{n=1}^\infty \varphi_n$-invariant Borel probability measure on $\prod_{n=1}^\infty X_n$ and set $\pi_\infty = \prod_{n=1}^\infty \pi_n$. Let $\rho_k : \prod_{n=1}^\infty X_n \to X_k$ be the projection. Then

$$\begin{aligned}
&\int_{\prod_{n=1}^\infty X_n} \log \#\pi_\infty^{-1}(x) \ d\mu(x) = \int_{\prod_{n=1}^\infty X_n} \sum_{k=1}^\infty \log \#\pi_k^{-1}(\rho_k(x)) \ d\mu(x) \\
&= \sum_{k=1}^\infty \int_{X_k} \log \#\pi_k^{-1}(z) \ d\mu \circ \rho_k^{-1}(z) \leq \sum_{k=1}^\infty D(\pi_k),
\end{aligned}$$

proving that $D(\pi_\infty) \leq \sum_{k=1}^\infty D(\pi_k)$. To obtain the reversed inequality, choose for each k a $t_k \in \mathbb{R}$ such that $t_k < D(\pi_k)$ and a φ_k-invariant Borel probability measure μ_k on X_k such that

$$\int_{X_k} \log \#\pi_k^{-1}(x) \ d\mu_k(x) \geq t_k.$$

Let μ be the product measure $\prod_{k=1}^\infty \mu_k$ on $\prod_{k=1}^\infty X_k$ and note that

$$\int_{\prod_{n=1}^\infty X_n} \log \#\pi_\infty^{-1}(x) \ d\mu(x) = \sum_{k=1}^\infty \int_{X_k} \log \#\pi_k^{-1}(z) \ d\mu_k(z) \geq \sum_{k=1}^\infty t_k.$$

It follows that $D(\pi_\infty) \geq \sum_{k=1}^\infty D(\pi_k)$. □

Consider a commuting diagram

$$\begin{tikzcd}
Y_1 \arrow[d, "\pi_1"] & Y_2 \arrow[l, "\lambda_1^Y"'] \arrow[d, "\pi_2"] & Y_3 \arrow[l, "\lambda_2^Y"'] \arrow[d, "\pi_3"] & Y_4 \arrow[l, "\lambda_3^Y"'] \arrow[d, "\pi_4"] & \cdots \arrow[l, "\lambda_4^Y"'] \\
X_1 \arrow[d, "\varphi_1"'] & X_2 \arrow[l, "\lambda_1^X"] \arrow[d, "\varphi_2"'] & X_3 \arrow[l, "\lambda_2^X"] \arrow[d, "\varphi_3"'] & X_4 \arrow[l, "\lambda_3^X"] \arrow[d, "\varphi_4"'] & \cdots \arrow[l, "\lambda_4^X"] \\
X_1 & X_2 \arrow[l, "\lambda_1^X"] & X_3 \arrow[l, "\lambda_2^X"] & X_4 \arrow[l, "\lambda_3^X"] & \cdots \arrow[l, "\lambda_4^X"]
\end{tikzcd}$$

of compact metric spaces and continuous maps such that each π_n is surjective. The φ_n's give rise to a dynamical system $\varphi_\infty : \varprojlim(X_n, \lambda_n^X) \to \varprojlim(X_n, \lambda_n^X)$ and the π_n's to a continuous surjection $\pi_\infty : Y_\infty = \varprojlim(Y_n, \lambda_n^Y) \to \varprojlim(X_n, \lambda_n^X) = X_\infty$. In fact, when we set $X_{\infty,k} = \bigcap_{j>k} \lambda_k^X \circ \lambda_{k-1}^X \circ \cdots \circ \lambda_j^X(X_{j+1})$ and $Y_{\infty,k} = \bigcap_{j>k} \lambda_k^Y \circ \lambda_{k-1}^Y \circ \cdots \circ \lambda_j^Y(Y_{j+1})$, we have that

$$\pi_k(Y_{\infty,k}) = X_{\infty,k}$$

for all $k \in \mathbb{N}$.

Proposition 3.3.

$$D(\pi_\infty) \leq \liminf_{k\to\infty} D(\pi_k|_{Y_{\infty,k}}).$$

Proof. Let $\rho_i : Y_\infty \to Y_i$ and $\rho'_i : X_\infty \to X_i$ be the projections to the i'th coordinate. By definition of the topology, for every finite open cover $\mathcal{U}$ of Y_∞ there is an $N \in \mathbb{N}$ such that for all $k \geq N$ there is a refinement of $\mathcal{U}$ of the form $\rho_k^{-1}(\mathcal{V})$ where $\mathcal{V}$ is a finite open cover of $\rho_k(Y_\infty) = Y_{\infty,k}$. Let $x \in X_{\infty,k}$. Since $\rho_k(\pi_\infty^{-1}(x)) \subseteq \pi_k^{-1}(\rho'_k(x)) \cap \rho_k(Y_\infty)$, we find that $C(\pi_\infty^{-1}(x), \rho_k^{-1}(\mathcal{V})) \leq C(\pi_k^{-1}(\rho'_k(x)), \mathcal{V})$. It follows that $A(\varphi_\infty, \mathcal{U}) \leq A(\varphi_\infty, \rho_k^{-1}(\mathcal{V})) \leq A(\varphi_k|_{X_{\infty,k}}, \mathcal{V}) \leq D(\pi_k|_{Y_{\infty,k}})$. Since this is true for all $k \geq N$ we find that $A(\varphi_\infty, \mathcal{U}) \leq \inf_{k\geq N} D(\pi_k|_{Y_{\infty,k}}) \leq \liminf_k D(\pi_k|_{Y_{\infty,k}})$. □

In general equality fails in Proposition 3.3, except under appropriate additional assumptions, like condition (A) of Theorem 1.9 in [Th1].

Lemma 3.4. *Let $(Y, \psi), (X, \varphi)$ be dynamical systems on compact metric spaces X and Y. Let $\pi : (Y, \psi) \to (X, \varphi)$ be a factor map. Then*

$$\sup_{x\in X} h(\psi, \pi^{-1}(x)) \leq D(\pi).$$

Proof. Choose $t \in \mathbb{R}$ such that $t < \sup_{x\in X} h(\psi, \pi^{-1}(x))$ and let $x \in X$ be a point such that $t < h(\psi, \pi^{-1}(x))$. There is then an $\epsilon > 0$ such that, in the notation of [B2],

$$t < \overline{s}_{\psi,d}(\epsilon, \pi^{-1}(x)).$$

Let $\mathcal{V} = \{V_i : i \in I\}$ be an open cover of Y by balls of radius $< \frac{\epsilon}{2}$. Consider an $n \in \mathbb{N}$ and let E_n be an (n, ϵ)-separated subset of $\pi^{-1}(x)$ of maximal cardinality. If

$$\prod_{j=0}^{n-1} C(\pi^{-1}(\varphi^j(x)), \mathcal{V})) < \#E_n,$$

there would have to be two different elements, s_1 and s_2, of E_n such that $\psi^j(s_1)$ and $\psi^j(s_2)$ were contained in the same element of $\mathcal{V}$ for all $j = 0, 1, 2, \cdots, n-1$. These two elements would not be (n, ϵ)-separated, contradicting the choice of E_n. So we see that $\prod_{j=0}^{n-1} C(\pi^{-1}(\varphi^j(x)), \mathcal{V}) \geq \#E_n$. Thus $\log a_n(\varphi, \mathcal{V}) \geq \log a_n(x, \varphi, \mathcal{V}) \geq \log \#E_n = \log s_n(\epsilon, \pi^{-1}(x))$. Since n was arbitrary we conclude that

$$D(\pi) \geq A(\varphi, \mathcal{V}) \geq \overline{s}_{\psi,d}(\epsilon, \pi^{-1}(x)) > t.$$

□

Lemma 3.5. *Let $(Y, \psi), (X, \varphi)$ be dynamical systems on compact metric spaces X and Y. Let $\pi : (Y, \psi) \to (X, \varphi)$ be a factor map. Then*

$$h(\psi) \leq h(\varphi) + D(\pi).$$

Proof. Combine Lemma 3.4 with Theorem 17 of [B2]. □

Theorem 3.6. *Let $(Y, \psi), (X, \varphi)$ be dynamical systems on compact metric spaces X and Y. Let $\pi : (Y, \psi) \to (X, \varphi)$ be a factor map. Then*

$$D(\pi) < \infty \;\Rightarrow\; h(\psi) = h(\varphi).$$

Proof. With Lemma 3.5 substituting for Lemma 3.6 of [Th2] and Proposition 3.3 for Remark 1.10 of [Th1] the proof of Theorem 3.5 in [Th2] can be used ad verbatim. □

Lemma 3.7. *Let $\pi : (Y, \psi) \to (X, \varphi)$ be a factor map. Assume that ψ is surjective and φ injective. Let μ be a φ-ergodic Borel probability measure. There is then a natural number $k_\mu^\pi \in \mathbb{N}$ or $k_\mu^\pi = \infty$ and a φ-invariant Borel set $B \subseteq X$ of full measure such that $\#\pi^{-1}(x) = k_\mu^\pi$ for all $x \in B$.*

Proof. Under the present assumptions on φ and ψ the identity $\pi^{-1}(\varphi(x)) = \psi(\pi^{-1}(x))$ is valid and shows that $\varphi^{-1}(\{x \in X : \#\pi^{-1}(x) \geq k\}) \subseteq \{x \in X : \#\pi^{-1}(x) \geq k\}$ for all $k \in \mathbb{N}$. By ergodicity, this implies that

$$\mu(\{x \in X : \#\pi^{-1}(x) \geq k\}) \in \{0, 1\}.$$

Let k^{π}_{μ} be the supremum of all $k \in \mathbb{N}$ for which $\mu(\{x \in X : \#\pi^{-1}(x) \geq k\}) = 1$ and set $B = \bigcap_{k < k^{\pi}_{\mu}} \{x \in X : \#\pi^{-1}(x) > k\}$. □

Theorem 3.8. *Let $\pi : (Y, \psi) \to (X, \varphi)$ be a factor map. Assume that ψ is surjective and φ injective and that $D(\pi) < \infty$. There is then a $k \in \mathbb{N}$ and a φ-ergodic probability measure μ on X such that $\#\pi^{-1}(x) = k$ for μ-almost all x, and*

$$D(\pi) = \log k.$$

Proof. Combine Lemma 3.7 with Theorem 2.1.

□

As in [Th2], the variational principle implies a certain subadditivity of the defect between invertible dynamical systems. This fact will be exploited below.

Theorem 3.9. *(Subadditivity of the defect.) Let $(X, \varphi), (Y, \psi)$ and (Z, λ) be invertible dynamical systems. Let $\pi_1 : (X, \varphi) \to (Y, \psi)$ and $\pi_2 : (Y, \psi) \to (Z, \lambda)$ be factor maps. It follows that*

$$D(\pi_2 \circ \pi_1) \leq D(\pi_1) + D(\pi_2).$$

Proof. The proof of Theorem 3.3 in [Th2] can be adopted ad verbatim. □

It follows from Theorem 3.7 that the defect of a factor map $\pi : (Y, \psi) \to (X, \varphi)$ between invertible dynamical systems is infinite or the logarithm of a natural number. This number has other interpretations: When ν is a ψ-ergodic Borel probability measure and $\mathcal{B}$ and $\mathcal{B}_0$ denote the Borel σ-algebras of Y and X, respectively, the relative entropy $H_\nu(\mathcal{B}|\pi^{-1}(\mathcal{B}_0))$ equals $\int_X \log \#\pi^{-1}(x) \ d\nu \circ \pi^{-1}(x)$ by Lemma 1 of [NP] and hence

$$D(\pi) = \sup_{\nu} H_\nu(\mathcal{B}|\pi^{-1}(\mathcal{B}_0)), \tag{3.1}$$

where we take the supremum over all ψ-ergodic Borel probability measures.

Remark 3.10. Mike Boyle, Doris and Ulf Fiebig have introduced a notion of conditional entropy for factor maps between invertible dynamical systems, [BFF], and shown, among others, that this quantity is related to the defect. As pointed out in Proposition B.4 of [BFF], it follows from the variational principle for the defect that whenever the identity in their variational principle holds, see Theorem 6.6 of [BFF], finite defect implies that the conditional entropy is zero.

The defect is also related, via the crossed product construction (or group-measure space construction), to the Jones index for sub-factors, [J]. See [DT1] and [DT2].

4. The reduced defect

For general (non-invertible) dynamical systems the subadditivity of the defect, Theorem 3.9, fails as shown by example in Remark 3.4 of [Th2]. In fact, as the next example shows, there are factor maps

$$(X,\varphi) \xrightarrow{\pi_1} (Y,\psi) \xrightarrow{\pi_2} (Z,\sigma)$$

such that $D(\pi_1) = 0,\ D(\pi_2) < \infty$ and $D(\pi_2 \circ \pi_1) = \infty$.

Example 4.1. We elaborate first on the examples from Example 2.5 of [Th1] and Remark 3.4 of [Th2] as follows. Let $m \in \mathbb{N}$. Consider finite sets A and B such that $\#A \geq \#B \geq 1$. Define $\varphi : A \cup \{1,2,\cdots,m\} \to A \cup \{1,2,\cdots,m\}$ such that $\varphi(A \cup \{1\}) = \{2\}, \varphi(i) = i+1$, modulo m, $2 \leq i \leq m$, and define $\psi : B \cup \{1,2,\cdots,m\} \to B \cup \{1,2,\cdots,m\}$, such that $\psi(B \cup \{1\}) = \{2\}$ and $\psi(i) = i+1$, modulo m, $2 \leq i \leq m$. Then $(A \cup \{1,2,\cdots,m\},\varphi)$ and $(B \cup \{1,2,\cdots,m\},\psi)$ are both non-invertible dynamical systems. Define $\pi_1 : A \cup \{1,2,\cdots,m\} \to B \cup \{1,2,\cdots,m\}$ such that $\pi_1(A) = B$ and $\pi_1(i) = i$ for all i. Then π_1 is a factor map with defect $D(\pi_1) = 0$, cf. Remark 3.4 of [Th2]. Let $\sigma : \{1,2,\cdots,m\} \to \{1,2,\cdots,m\}$ be cyclic permutation and define $\pi_2 : B \cup \{1,2,\cdots,m\} \to \{1,2,\cdots,m\}$ such that $\pi_2(B \cup \{1\}) = \{1\}$ and $\pi_2(i) = i$ when $i \geq 2$. Then π_2 is a factor map and $D(\pi_2) = \frac{\log(\#B+1)}{m}$ while $D(\pi_2 \circ \pi_1) = \frac{\log(\#A+1)}{m}$, cf. Remark 3.4 of [Th2].

By using the freedom in this construction we can find sequences of factor maps,

$$(X_n,\varphi_n) \xrightarrow{\pi_1^n} (Y_n,\psi_n) \xrightarrow{\pi_2^n} (Z_n,\sigma_n),$$

such that $D(\pi_1^n) = 0$ for all n, $\sum_{n=1}^\infty D(\pi_2^n) < \infty$ and $\sum_{n=1}^\infty D(\pi_2^n \circ \pi_1^n) = \infty$. Then, by Theorem 3.2 above or Theorem 1.11 of [Th2],

$$\left(\prod_{n=1}^\infty X_n, \prod_{n=1}^\infty \varphi_n\right) \xrightarrow{\prod_{n=1}^\infty \pi_1^n} \left(\prod_{n=1}^\infty Y_n, \prod_{n=1}^\infty \psi_n\right) \xrightarrow{\prod_{n=1}^\infty \pi_2^n} \left(\prod_{n=1}^\infty Z_n, \prod_{n=1}^\infty \sigma_n\right)$$

are factor maps such that $D(\prod_{n=1}^\infty \pi_1^n) = 0,\ D(\prod_{n=1}^\infty \pi_2^n) < \infty$, while $D((\prod_{n=1}^\infty \pi_2^n) \circ (\prod_{n=1}^\infty \pi_1^n)) = D(\prod_{n=1}^\infty \pi_2^n \circ \pi_1^n) = \infty$.

To obtain a notion of defect which is also subadditive for factor maps between non-invertible dynamical systems, we take the definition of the defect up for a slight revision. Let (X,φ) and (Y,ψ) be dynamical systems acting on compact metric spaces, and let $\pi : (Y,\psi) \to (X,\varphi)$ be a factor map. Let $\mathcal{U} = \{U_i : i \in I\}$ be a finite open cover of Y. For each $k \in \mathbb{N}$, $H \subseteq X$, set

$$b_k(H,\pi,\mathcal{U}) = $$
$$C(\pi^{-1}(H),\mathcal{U})C(\psi(\pi^{-1}(H)),\mathcal{U})C(\psi^2(\pi^{-1}(H)),\mathcal{U})\cdots C(\psi^{k-1}(\pi^{-1}(H)),\mathcal{U}).$$

For $x \in X$, set $b_k(x,\pi,\mathcal{U}) = b_k(\pi^{-1}(x),\pi,\mathcal{U})$, i.e.

$$b_k(x,\pi,\mathcal{U}) = C(\pi^{-1}(x),\mathcal{U})C(\psi(\pi^{-1}(x)),\mathcal{U})C(\psi^2(\pi^{-1}(x)),\mathcal{U})\cdots C(\psi^{k-1}(\pi^{-1}(x)),\mathcal{U}).$$

(Compare with (2.1).) Note that $b_{k+n}(x,\pi,\mathcal{U}) \leq b_k(x,\pi,\mathcal{U})b_n(\varphi^k(x),\pi,\mathcal{U})$ and that $b_k(x,\pi,\mathcal{U}) \leq b_k(x,\pi,\mathcal{V})$ when $\mathcal{V}$ refines $\mathcal{U}$. We set

$$b_k(\psi,\mathcal{U}) = \sup_{x\in X} b_k(x,\pi,\mathcal{U}).$$

Then

1) $b_{k+n}(\psi,\mathcal{U}) \leq b_k(\psi,\mathcal{U})b_n(\psi,\mathcal{U})$,

2) $b_k(\psi,\mathcal{U}) \leq b_k(\psi,\mathcal{V})$ when $\mathcal{V}$ is a refinement of $\mathcal{U}$.

It follows from 1) that we can consider the limit

$$B(\psi,\mathcal{U}) = \lim_{n\to\infty} \frac{1}{n} \log b_n(\psi,\mathcal{U}).$$

We define *the reduced defect of* π to be

$$D_r(\pi) = \sup_{\mathcal{U}} B(\psi,\mathcal{U}),$$

where we take the supremum over all finite open covers $\mathcal{U}$ of Y. It follows from 2) that

$$D_r(\pi) = \lim_{k\to\infty} B(\psi,\mathcal{U}_k),$$

for any sequence $\mathcal{U}_k, k \in \mathbb{N}$, of open covers of Y for which the maximal diameter of any set in the cover $\mathcal{U}_k$ goes to zero as k tends to infinity.

Lemma 4.2. *1) $D_r(\pi) \leq D(\pi)$,*

2) $D_r(\pi) = D(\pi)$ when ψ is surjective and φ injective.

Proof. In general $\psi^j(\pi^{-1}(x)) \subseteq \pi^{-1}(\varphi^j(x))$ for all j, x, and this gives 1). Under the assumptions of 2) we have that $\psi^j(\pi^{-1}(x)) = \pi^{-1}(\varphi^j(x))$. □

Lemma 4.3. *Let $\pi : (Y,\psi) \to (X,\varphi)$ be a factor map. Then $\pi(\bigcap_{j\in\mathbb{N}} \psi^j(Y)) = \bigcap_{j\in\mathbb{N}} \varphi^j(X)$ and*

$$B(\psi,\mathcal{U}) = B(\psi|_{\bigcap_{j\in\mathbb{N}} \psi^j(Y)},\mathcal{U})$$

for every finite open cover $\mathcal{U}$ of Y. In particular,

$$D_r(\pi) = D_r(\pi|_{\bigcap_{j\in\mathbb{N}} \psi^j(Y)}).$$

Proof. The first statement is straightforward to check. Let $\mathcal{U}$ be a finite open cover of Y. We claim that

$$B(\psi|_{\bigcap_j \psi^j(Y)},\mathcal{U}) = \lim_{k\to\infty} B(\psi|_{\psi^k(Y)},\mathcal{U}). \tag{4.1}$$

To prove (4.1), let $\epsilon > 0$ and choose $n \in \mathbb{N}$ such that

$$\frac{1}{n} \log b_n(\psi|_{\bigcap_j \psi^j(Y)},\mathcal{U}) \leq B(\psi|_{\bigcap_j \psi^j(Y)},\mathcal{U}) + \epsilon.$$

Recall that

$$\begin{aligned} &b_n(\psi|_{\bigcap_j \psi^j(Y)},\mathcal{U}) \\ &= \sup_{x\in\bigcap_j \varphi^j(X)} \prod_{l=0}^{n-1} C(\psi^l(\pi^{-1}(x) \cap \bigcap_j \psi^j(Y)),\mathcal{U}). \end{aligned}$$

An easy compactness argument gives us for each $x \in \bigcap_j \varphi^j(X)$ a $\delta_x > 0$ and a $j_x \in \mathbb{N}$ such that

$$\begin{aligned} &\prod_{l=0}^{n-1} C(\psi^l(\pi^{-1}(x) \cap \bigcap_j \psi^j(Y)),\mathcal{U}) \\ &= \prod_{l=0}^{n-1} C(\psi^l(\pi^{-1}(B_{\delta_x}(x)) \cap \psi^i(Y)),\mathcal{U}) \end{aligned}$$

for all $i \geq j_x$. The cover $B_{\delta_x}(x), x \in \bigcap_j \varphi^j(X)$, of $\bigcap_j \varphi^j(X)$ has a finite subcover $\{B_{\delta_i}(x_i) : i \in I\}$. By (the proof of) Lebesgue's covering lemma, (cf. Theorem 0.20

of [W]), there is $\delta > 0$ so small that every ball $B_\delta(x)$, $x \in \bigcap_j \varphi^j(X)$, is contained in $B_{\delta_r}(x_r)$ for some $r \in I$. Hence

$$\begin{aligned}
&\sup_{x\in\bigcap_j \varphi^j(X)} \prod_{l=0}^{n-1} C(\psi^l(\pi^{-1}(B_\delta(x)) \cap \psi^i(Y)), \mathcal{U}) \\
&\leq \max_r \prod_{l=0}^{n-1} C(\psi^l(\pi^{-1}(B_{\delta_r}(x_r)) \cap \psi^i(Y)), \mathcal{U}) \\
&\leq b_n(\psi|_{\bigcap_j \psi^j(Y)}, \mathcal{U})
\end{aligned} \tag{4.2}$$

for all $i \geq \max_r j_{x_r}$. Choose $k \geq \max_r j_{x_r}$ so large that every element of $\varphi^k(X)$ has distance less than δ to an element of $\bigcap_j \varphi^j(X)$. Then (4.2) shows that

$$\begin{aligned}
&\sup_{x\in\varphi^k(X)} \prod_{l=0}^{n-1} C(\psi^l(\pi^{-1}(x) \cap \psi^k(Y)), \mathcal{U}) \\
&\leq b_n(\psi|_{\bigcap_j \psi^j(Y)}, \mathcal{U}).
\end{aligned}$$

Hence

$$\begin{aligned}
B(\psi|_{\psi^k(Y)}, \mathcal{U}) &\leq \frac{1}{n}\log b_n(\psi|_{\psi^k(Y)}, \mathcal{U}) \\
&\leq \frac{1}{n}\log b_n(\psi|_{\bigcap_j \psi^j(Y)}, \mathcal{U}) \leq B(\psi|_{\bigcap_j \psi^j(Y)}, \mathcal{U}) + \epsilon.
\end{aligned}$$

Since $B(\psi|_{\psi^k(Y)}, \mathcal{U})$ decreases with k and $B(\psi|_{\psi^k(Y)}, \mathcal{U}) \geq B(\psi|_{\bigcap_j \psi^j(Y)}, \mathcal{U})$ for all k, this proves (4.1). To complete the proof, we need only show that

$$B(\psi, \mathcal{U}) = B(\psi|_{\psi^m(Y)}, \mathcal{U}) \tag{4.3}$$

for all $m \in \mathbb{N}$. To establish (4.3) observe that

$$\psi^{m+i}(\pi^{-1}(x)) \subseteq \psi^i(\pi^{-1}(\varphi^m(x)) \cap \psi^m(Y))$$

for all $x \in X$ and all $i \in \mathbb{N}$. It follows that there is an $L \in \mathbb{N}$ which only depends on $\mathcal{U}$ and m such that

$$b_k(x, \pi, \mathcal{U}) \leq L b_{k-m}(\varphi^m(x), \pi|_{\psi^m(Y)}, \mathcal{U})$$

for all $k > m$ and all $x \in X$. Hence $b_k(\psi|_{\psi^m(Y)}, \mathcal{U}) \leq b_k(\psi, \mathcal{U}) \leq L b_{k-m}(\psi|_{\psi^m(Y)}, \mathcal{U})$ all $k > m$, proving (4.3). □

Proposition 4.4. *In the setting of Proposition 3.3,*

$$D_r(\pi_\infty) \leq \liminf_{k\to\infty} D_r(\pi_k|_{Y_{\infty,k}}).$$

Proof. Let $\rho_i : Y_\infty \to Y_i$ and $\rho'_i : X_\infty \to X_i$ be the projections to the i'th coordinate. By definition of the topology, for every finite open cover $\mathcal{U}$ of Y_∞ there is an $N \in \mathbb{N}$ such that for all $k \geq N$ there is a refinement of $\mathcal{U}$ of the form $\rho_k^{-1}(\mathcal{V})$ where $\mathcal{V} = \{V_i : i \in I\}$ is a finite open cover of Y_k. Since

$$\begin{aligned}
&\psi_\infty^j(\pi_\infty^{-1}(x)) \subseteq \bigcup_{j\in J} \rho_k^{-1}(V_j) \\
&\Leftrightarrow \\
&\psi_k^j({\pi_k|_{Y_{\infty,k}}}^{-1}(\rho'_k(x))) \subseteq \bigcup_{j\in J} V_j
\end{aligned} \tag{4.4}$$

for all $x \in X_\infty$, all $J \subseteq I$ and all $j \in \mathbb{N}$, we see that $b_k(\psi_\infty, \rho_k^{-1}(\mathcal{V})) = b_k(\psi_k|_{Y_{\infty,k}}, \mathcal{V})$ for all k. It follows that $B(\psi_\infty, \mathcal{U}) \leq D_r(\pi_k|_{Y_{\infty,k}})$ for all $k \geq N$. Hence $B(\psi_\infty, \mathcal{U}) \leq \liminf_k D_r(\pi_k|_{Y_{\infty,k}})$ and by taking the supremum over $\mathcal{U}$ we get that $D_r(\pi_\infty) \leq \liminf_k D_r(\pi_k|_{Y_{\infty,k}})$.

□

For $\epsilon > 0$, let $B_\epsilon(x)$ denote the closed ball of radius ϵ centered at x. Set

$$b_k^\epsilon(\psi, \mathcal{U}) = \sup_{x \in X} b_k(B_\epsilon(x), \mathcal{U}).$$

Clearly $b_k^\epsilon(\psi, \mathcal{U}) \geq b_k(\psi, \mathcal{U})$ for all $\epsilon > 0$.

Lemma 4.5. *For each k there is an $\epsilon > 0$ for which $b_k^\epsilon(\psi, \mathcal{U}) = b_k(\psi, \mathcal{U})$.*

Proof. For any given $x \in X$ there is clearly an $\epsilon_x > 0$ such that

$$b_k(B_{\epsilon_x}(x), \pi, \mathcal{U}) = b_k(x, \pi, \mathcal{U}).$$

Let $\{B_{\epsilon_{x_i}}(x_i) : i \in I\}$ be a finite subcover of $\{B_{\epsilon_x}(x) : x \in X\}$ and let $\epsilon > 0$ be a Lebesgue number for $\{B_{x_i}(x_i) : i \in I\}$. Let $z \in X$. Then $B_\epsilon(z) \subseteq B_{\epsilon_{x_i}}(x_i)$ for some i and hence

$$b_k(B_\epsilon(z), \pi, \mathcal{U}) \leq b_k(B_{\epsilon_{x_i}}(x_i), \pi, \mathcal{U}) = b_k(x_i, \pi, \mathcal{U}) \leq b_k(\psi, \mathcal{U}).$$

It follows that $b_k^\epsilon(\psi, \mathcal{U}) = b_k(\psi, \mathcal{U})$.

□

Consider a factor map $\pi : (Y, \psi) \to (X, \varphi)$. Then

$$\begin{tikzcd} Y & Y \arrow[l, "\psi"'] \arrow[d, "\pi"] & Y \arrow[l, "\psi"'] \arrow[d, "\pi"] & Y \arrow[l, "\psi"'] \arrow[d, "\pi"] & \cdots \arrow[l, "\psi"'] \\ X & X \arrow[l, "\varphi"] & X \arrow[l, "\varphi"] & X \arrow[l, "\varphi"] & \cdots \arrow[l, "\varphi"] \arrow[from=1-1, to=2-1, "\pi"] \end{tikzcd}$$

commutes and we get therefore a factor map $\widehat{\pi} : (\widehat{Y}, \widehat{\psi}) \to (\widehat{X}, \widehat{\varphi})$, where $\widehat{Y} = \varprojlim(Y, \psi)$ and $\widehat{X} = \varprojlim(X, \varphi)$. $\widehat{\psi}$ is the homeomorphism of $\widehat{Y}$ given by $\widehat{\psi}((y_i)) = (\psi(y_i))$. $\widehat{\varphi}$ is defined similarly. The invertible dynamical system $(\widehat{Y}, \widehat{\psi})$ is *the natural invertible extension* of (Y, ψ), and we call the factor map $\widehat{\pi}$ *the natural extension of* π.

Theorem 4.6. *Let $\pi : (Y, \psi) \to (X, \varphi)$ be a factor map. Then*

$$D_r(\pi) = D(\widehat{\pi}),$$

where $\widehat{\pi} : (\widehat{Y}, \widehat{\psi}) \to (\widehat{X}, \widehat{\varphi})$ is the natural extension of π.

Proof. It follows from Proposition 4.4 that $D_r(\widehat{\pi}) \leq D_r(\pi)$, so by combining with 2) of Lemma 4.2, we have that $D(\widehat{\pi}) \leq D_r(\pi)$. To prove the reversed inequality observe first that $\widehat{\pi} : (\widehat{Y}, \widehat{\psi}) \to (\widehat{X}, \widehat{\varphi})$ is also induced by the commuting diagram

$$\begin{tikzcd} \bigcap_j \psi^j(Y) \arrow[d, "\pi"] & \bigcap_j \psi^j(Y) \arrow[l, "\psi"'] \arrow[d, "\pi"] & \bigcap_j \psi^j(Y) \arrow[l, "\psi"'] \arrow[d, "\pi"] & \bigcap_j \psi^j(Y) \arrow[l, "\psi"'] \arrow[d, "\pi"] & \cdots \arrow[l, "\psi"'] \\ \bigcap_j \varphi^j(X) & \bigcap_j \varphi^j(X) \arrow[l, "\varphi"'] & \bigcap_j \varphi^j(X) \arrow[l, "\varphi"'] & \bigcap_j \varphi^j(X) \arrow[l, "\varphi"'] & \cdots \arrow[l, "\varphi"'] \end{tikzcd}$$

We can therefore, by Lemma 4.3, substitute $\pi : (Y,\psi) \to (X,\varphi)$ by $\pi : (\bigcap_j \psi^j(Y),\psi) \to (\bigcap_j \varphi^j(X),\varphi)$, and hence assume that φ and ψ are both surjective.

Let $t < D_r(\pi)$ and choose a finite open cover $\mathcal{V}$ of Y such that $B(\psi,\mathcal{V}) > t$. Let $\delta > 0$ and choose $m \in \mathbb{N}$ so large that $\frac{1}{m}\log b_m(\widehat{\psi},\rho_1^{-1}(\mathcal{V})) \leq B(\widehat{\psi},\rho_1^{-1}(\mathcal{V})) + \delta$. (Remember that $\rho_k : \widehat{Y} \to Y$ and $\rho'_k : \widehat{X} \to X$ are the projections to the k'th coordinate.) By Lemma 4.5 there is an $\epsilon > 0$ so small that $b_m(\widehat{\psi},\rho_1^{-1}(\mathcal{V})) = b^{\epsilon}_m(\widehat{\psi},\rho_1^{-1}(\mathcal{V}))$. There is a $k \in \mathbb{N}$ so large that the diameter of every subset of $\widehat{X}$ of the form $\rho_k^{-1}(z)$ for some $z \in X$ has diameter less than ϵ. Since $\rho_k(\pi_\infty^{-1}({\rho'_k}^{-1}(z))) = \pi^{-1}(z)$ (because of the surjectivity of ψ), we have that $\rho_k(\psi^j_\infty(\pi_\infty^{-1}({\rho'_k}^{-1}(z))) = \psi^j(\pi^{-1}(z))$ for all $j \in \mathbb{N}$ and all $z \in X$. Since ${\rho'_k}^{-1}(z) \subseteq B_\epsilon(z_\infty)$ for some $z_\infty \in \widehat{X}$ we can combine this with (4.4) to see that

$$b_m(\psi,\mathcal{U}) \leq b^{\epsilon}_m(\widehat{\psi},\rho_k^{-1}(\mathcal{U}))$$

for every finite open cover $\mathcal{U}$ of Y. If we set $\mathcal{U} = \psi^{-k+1}(\mathcal{V})$, we have that $\rho_k^{-1}(\mathcal{U}) = \rho_1^{-1}(\mathcal{V})$ and hence that

$$\begin{aligned} &\frac{1}{m}\log b_m(\psi,\psi^{-k+1}(\mathcal{V})) \leq \frac{1}{m}\log b^{\epsilon}_m(\widehat{\psi},\rho_1^{-1}(\mathcal{V})) \\ &= \frac{1}{m}\log b_m(\widehat{\psi},\rho_1^{-1}(\mathcal{V})) \leq B(\widehat{\psi},\rho_1^{-1}(\mathcal{V})) + \delta. \end{aligned} \tag{4.5}$$

But it is easy to see, directly from the definition, that $B(\psi,\psi^{-k+1}(\mathcal{V})) = B(\psi,\mathcal{V})$, so we see from (4.5) that $B(\psi,\mathcal{V}) \leq B(\widehat{\psi},\rho_1^{-1}(\mathcal{V})) + \delta$. It follows that $D_r(\pi) \leq D_r(\widehat{\pi})$. Since $D_r(\widehat{\pi}) = D(\widehat{\pi})$ by 2) of Lemma 4.2, the proof is complete. $\square$

Corollary 4.7. *(Subadditivity of the reduced defect.) Let $\pi_1 : (Y,\psi) \to (X,\varphi)$ and $\pi_2 : (X,\varphi) \to (Z,\lambda)$ be factor maps. Then*

$$D_r(\pi_2 \circ \pi_1) \leq D_r(\pi_1) + D_r(\pi_2).$$

Proof. Combine Theorem 4.6 with Theorem 3.9. $\square$

Corollary 4.8. *Let $\pi : (Y,\psi) \to (X,\varphi)$ be a factor map. Then*

$$D_r(\pi) < \infty \;\Rightarrow\; h(\psi) = h(\varphi).$$

Proof. Combine Proposition 5.2 of [B1] with Theorem 4.6 and Theorem 3.6. $\square$

There is also a variational principle for the reduced defect. It is, however, somewhat more complicated. Let $\pi : (Y,\psi) \to (X,\varphi)$ be a factor map. For $\epsilon > 0$ and any closed subset $F \subseteq Y$ we let $\#^\epsilon F$ denote the largest number of elements in an ϵ-separated subset of F, i.e. if d denotes the metric of Y,

$$\#^\epsilon F = \max\{n \in \mathbb{N} : \; \exists\{x_1,x_2,\cdots,x_n\} \subseteq F \text{ such that } d(x_i,x_j) \geq \epsilon,\; i \neq j\}.$$

For $F = \emptyset$ we set $\#^\epsilon F = 0$.

Lemma 4.9. *For every $\epsilon > 0$ and every $k \in \mathbb{N}$, the function*

$$x \mapsto \#^\epsilon \psi^k\big(\pi^{-1}(x) \cap \bigcap_j \psi^j(Y)\big)$$

is Borel.

Proof. For $F \subseteq Y$ closed, set

$$\#_\epsilon F = \max\{n \in \mathbb{N} : \ \exists \{x_1, x_2, \cdots, x_n\} \subseteq F \text{ such that } d(x_i, x_j) > \epsilon, \ i \neq j\}.$$

Then

$$\#^\epsilon \psi^k\big(\pi^{-1}(x) \cap \bigcap_j \psi^j(Y)\big) = \inf_q \#_q \psi^k\big(\pi^{-1}(x) \cap \bigcap_j \psi^j(Y)\big)$$

for all $x \in X$, when we take the infimum over all rational $q < \epsilon$. It suffices therefore to show that

$$x \mapsto \#_\epsilon \psi^k\big(\pi^{-1}(x) \cap \bigcap_j \psi^j(Y)\big)$$

is Borel. Let $\mathcal{F} = \{F_1, F_2, \cdots, F_M\}$ be closed subsets of Y. Set

$$B(\mathcal{F})(x) = \#\{j : \psi^k\big(\pi^{-1}(x) \cap \bigcap_l \psi^l(Y)\big) \bigcap F_j \neq \emptyset\},$$

for $x \in X$. Since

$$\#\{j : \psi^k\big(\pi^{-1}(x) \cap \bigcap_l \psi^l(Y)\big) \bigcap F_j \neq \emptyset\} \ = \ \sum_{i=1}^M 1_{\pi(\psi^{-k}(F_i) \cap \bigcap_l \psi^l(Y))}(x),$$

we see that $B(\mathcal{F})$ is Borel. For each $t > 0$ and $y \in Y$, let $B_t(y)$ denote the open ball of radius t centered at y. Let $\{y_i\}$ be a dense sequence in Y. Then

$$\#_\epsilon \psi^k\big(\pi^{-1}(x) \cap \bigcap_l \psi^l(Y)\big) = \sup_{\mathcal{F}} B(\mathcal{F})(x),$$

where we take the supremum over all collections $\mathcal{F}$ of the form

$$\mathcal{F} = \{\overline{B_q(y_{i_j})} : j = 1, 2, \cdots, K\},$$

where $q > 0$ is rational and $d(y_{i_k}, y_{i_j}) > \epsilon + 2q$ for $k \neq j$. Since this is a countable collection of functions, we are done.

□

Since $\#^\epsilon \psi^k\big(\pi^{-1}(x) \cap \bigcap_j \psi^j(Y)\big)$ is decreasing in ϵ, we can define

$$A_\pi(x) = \lim_{\epsilon \to 0}\big[\limsup_k \#^\epsilon \psi^k\big(\pi^{-1}(x) \cap \bigcap_j \psi^j(Y)\big)\big],$$

which is a Borel function $A_\pi : X \to \mathbb{N} \cup \{\infty\}$ by Lemma 4.9. Observe that $A_\pi(x) = 0$ for $x \notin \bigcap_j \varphi^j(X)$, and that A_π does not depend on the metric.

Theorem 4.10. *(The variational principle for the reduced defect.) Let $\pi : (Y, \psi) \to (X, \varphi)$ be a factor map. Then*

$$D_r(\pi) = \sup_\mu \int_X \log A_\pi(x) \ d\mu(x), \tag{4.6}$$

where we take the supremum over all φ-invariant Borel probability measures on X and use the convention $\log 0 = 0$. In fact, it suffices to take the supremum over all φ-ergodic Borel probability measures on X.

Proof. We will prove (4.6) by combining Theorem 4.6 with the variational principle for the defect, Theorem 2.1. Let $\rho_i : \widehat{Y} \to Y$ and $\rho'_i : \widehat{X} \to X$ be the projections to the i'th coordinate. Then

$$\#\widehat{\pi}^{-1}(z) = \liminf_{l \to \infty} \lim_{\epsilon \to 0} \lim_{k \to \infty} \#^\epsilon \psi^{k-l}[\pi^{-1}(\rho'_k(z)) \cap \bigcap_j \psi^j(Y)] \tag{4.7}$$

for all $z \in \widehat{X}$. To see this, observe first that a compactness argument shows that

$$\lim_{k\to\infty} \#^{\epsilon}\psi^{k-l}[\pi^{-1}(\rho_k'(z)) \cap \bigcap_j \psi^j(Y)] = \#^{\epsilon}\rho_l(\widehat{\pi}^{-1}(z)). \tag{4.8}$$

Assume then that $\#\widehat{\pi}^{-1}(z) \geq N$ for some $N \in \mathbb{N}$, and let $y_1, y_2, \cdots, y_N$ be different elements of $\widehat{\pi}^{-1}(z)$. There is then a $K \in \mathbb{N}$ so large that the elements $\rho_i(y_1), \rho_i(y_2), \cdots, \rho_i(y_N)$ are different for all $i \geq K$. For such an i, the elements $\rho_i(y_1), \rho_i(y_2), \cdots, \rho_i(y_N)$ are δ_i-separated for some $\delta_i > 0$ and since $\rho_i(y_j) = \psi^{d-i}(\rho_d(y_j))$ for all $d > i$, we see that $\#^{\delta_i}\psi^{d-i}[\pi^{-1}(\rho_d'(z)) \cap \bigcap_j \psi^j(Y)] \geq N$ for all $d > i$. Hence $\lim_{k\to\infty} \#^{\delta_i}\psi^{k-i}[\pi^{-1}(\rho_k'(z)) \cap \bigcap_j \psi^j(Y))] \geq N$ and consequently

$$\lim_{\epsilon\to 0}\lim_{k\to\infty} \#^{\epsilon}\psi^{k-i}[\pi^{-1}(\rho_k'(z)) \cap \bigcap_j \psi^j(Y))] \geq N$$

for all $i \geq K$. It follows that the righthand side of (4.7) dominates the lefthand side. To prove the reversed inequality, consider an $\epsilon > 0$ and some $l \in \mathbb{N}$. If $\lim_{k\to\infty} \#^{\epsilon}\psi^{k-l}[\pi^{-1}(\rho_k'(z)) \cap \bigcap_j \psi^j(Y)] \geq N$ for some $N \in \mathbb{N}$, there is $K \in \mathbb{N}$ so large that $\#^{\epsilon}\psi^{k-l}[\pi^{-1}(\rho_k'(z)) \cap \bigcap_j \psi^j(Y)] \geq N$ for all $k \geq K$. We can therefore find, for any $k \geq K$, elements $y_1^k, y_2^k, \cdots, y_N^k \in \prod_{j=0}^{\infty} Y$ such that $\psi(\rho_j(y_i^k)) = \rho_{j-1}(y_i^k)$, $\pi(\rho_j(y_i^k)) = \rho_j'(z)$ for all i and all $j \leq k$, and such that the set

$$\{\rho_l(y_1^k), \rho_l(y_2^k), \cdots, \rho_l(y_N^k)\}$$

is ϵ-separated for all k. For some sequence $\{k_j\}$ in $\mathbb{N}$ the limits $\lim_{j\to\infty} y_i^{k_j} = y_i$ will all exist in $\prod_{j=0}^{\infty} Y$ and by construction they will lie not only in $\widehat{Y} = \varprojlim(Y, \psi)$, but actually in $\widehat{\pi}^{-1}(z)$. Furthermore, by construction the set $\{\rho_l(y_1), \rho_l(y_2), \cdots, \rho_l(y_N)\}$ will be ϵ-separated, so it follows that $\#\widehat{\pi}^{-1}(z) \geq N$. This proves (4.7).

There is a bijective correspondance between the $\widehat{\varphi}$-invariant Borel probability measures on $\widehat{X}$ and the φ-invariant Borel probability measures on X such that a φ-invariant Borel probability measure ν on X corresponds to the $\widehat{\varphi}$-invariant Borel probability measure μ on $\widehat{X}$ with the property that $\mu \circ {\rho_k'}^{-1} = \nu$ for all $k \in \mathbb{N}$. For such a μ Lebesgue's theorem on monotone convergence and Fatou's lemma combined with (4.7) gives us that

$$\begin{aligned}&\int_{\widehat{X}} \log \#\widehat{\pi}^{-1}(z)\ d\mu(z) \\ &\leq \liminf_{l\to\infty}\lim_{\epsilon\to 0}\int_{\widehat{X}} \log\big[\lim_{k\to\infty} \#^{\epsilon}\psi^{k-l}\big(\pi^{-1}(\rho_k'(z)) \cap \bigcap_j \psi^j(Y)\big)\big]\ d\mu(z).\end{aligned} \tag{4.9}$$

By compactness of Y the functions $z \mapsto \#^{\epsilon}\psi^{k-l}[\pi^{-1}(\rho_k'(z)) \cap \bigcap_j \psi^j(Y)]$ are uniformly bounded, so by Lebesgue's theorem on dominated convergence we have that

$$\begin{aligned}&\int_{\widehat{X}} \log\big[\lim_{k\to\infty} \#^{\epsilon}\psi^{k-l}\pi^{-1}(\rho_k'(z)) \cap \bigcap_j \psi^j(Y)\big]\ d\mu(z) \\ &= \lim_{k\to\infty}\int_{\widehat{X}} \log\big[\#^{\epsilon}\psi^{k-l}\big(\pi^{-1}(\rho_k'(z)) \cap \bigcap_j \psi^j(Y)\big)\big]\ d\mu(z).\end{aligned} \tag{4.10}$$

Since

$$\int_{\widehat{X}} \log[\#^{\epsilon}\psi^{k-l}\big(\pi^{-1}(\rho_k'(z))\cap\bigcap_j \psi^j(Y)\big)]\ d\mu(z) = \int_X \log[\#^{\epsilon}\psi^{k-l}\big(\pi^{-1}(x)\cap\bigcap_j \psi^j(Y)\big)]\ d\nu(x),$$

where ν is the φ-invariant Borel probability measure on X corresponding to μ, we find that

$$\begin{aligned} &\int_{\widehat{X}} \log\big[\lim_{k\to\infty} \#^{\epsilon}\psi^{k-l}\big(\pi^{-1}(\rho_k'(z)) \cap \bigcap_j \psi^j(Y)\big)\big]\ d\mu(z) \\ &= \lim_{k\to\infty} \int_X \log\big[\#^{\epsilon}\psi^{k-l}\big(\pi^{-1}(x) \cap \bigcap_j \psi^j(Y)\big)\big]\ d\nu(x) \\ &= \lim_{k\to\infty} \int_X \log\big[\#^{\epsilon}\psi^{k}\big(\pi^{-1}(x) \cap \bigcap_j \psi^j(Y)\big)\big]\ d\nu(x). \end{aligned} \tag{4.11}$$

Since there is a uniform bound on $\#^{\epsilon}\psi^k\big(\pi^{-1}(x)\cap\bigcap_j \psi^j(Y)\big)$ and ν is a finite measure, we can use Fatou's lemma to conclude that

$$\begin{aligned} &\lim_{k\to\infty} \int_X \log\big[\#^{\epsilon}\psi^{k}\big(\pi^{-1}(x) \cap \bigcap_j \psi^j(Y)\big)\big]\ d\nu(x) \\ &\leq \int_X \log\big[\limsup_k \#^{\epsilon}\psi^{k}\big(\pi^{-1}(x) \cap \bigcap_j \psi^j(Y)\big)\big]\ d\nu(x). \end{aligned} \tag{4.12}$$

However,

$$\begin{aligned} &\int_X \log\big[\limsup_k \#^{\epsilon}\psi^{k}\big(\pi^{-1}(x) \cap \bigcap_j \psi^j(Y)\big)\big]\ d\nu(x) \\ &= \int_{\widehat{X}} \log\big[\limsup_k \#^{\epsilon}\psi^{k-1}\big(\pi^{-1}(\rho_1'(z)) \cap \bigcap_j \psi^j(Y)\big)\big]\ d\mu(z) \\ &= \int_{\widehat{X}} \log\big[\lim_k \#^{\epsilon}\psi^{k-1}\big(\pi^{-1}(\rho_1'(z)) \cap \bigcap_j \psi^j(Y)\big)\big]\ d\mu(z) \\ &\leq \int_{\widehat{X}} \log \#\widehat{\pi}^{-1}(z)\ d\mu(z) \quad \text{(by (4.8))}. \end{aligned} \tag{4.13}$$

It follows from (4.10)-(4.13) that

$$\begin{aligned} &\int_{\widehat{X}} \log\big[\lim_{k\to\infty} \#^{\epsilon}\psi^{k-l}\big(\pi^{-1}(\rho_k'(z)) \cap \bigcap_j \psi^j(Y)\big)\big]\ d\mu(z) \\ &\leq \int_X \log\big[\limsup_k \#^{\epsilon}\psi^{k}\big(\pi^{-1}(x) \cap \bigcap_j \psi^j(Y)\big)\big]\ d\nu(x) \\ &\leq \int_{\widehat{X}} \log \#\widehat{\pi}^{-1}(z)\ d\mu(z), \end{aligned}$$

for all $\epsilon > 0$ and all $l \in \mathbb{N}$. Combining with (4.9) and using Lebesgue's theorem on monotone convergence, we find that

$$\int_{\widehat{X}} \log \#\widehat{\pi}^{-1}(z)\ d\mu(z) = \int_X \log A_{\pi}(x)\ d\nu(x).$$

Since μ is $\widehat{\varphi}$-ergodic if and only if ν is φ-ergodic the theorem follows now from Theorem 4.6 and Theorem 2.1.

□

Corollary 4.11. *Let $\pi : (Y, \psi) \to (X, \varphi)$ be a factor map such that $D_r(\pi) < \infty$. There is then a $k \in \mathbb{N}$ and a φ-ergodic Borel probability measure μ on X such that $A_\pi(x) = k$ for μ-almost all x, and $D_r(\pi) = \log k$.*

Proof. This follows from Theorem 4.10 in essentially the same way as Theorem 3.8 follows from Theorem 2.1, using that $\#^\epsilon \psi^k(\pi^{-1}(\varphi(x))) \geq \#^\epsilon \psi^{k+1}(\pi^{-1}(x))$.

□

5. Equivalence relations based on the defect

The subadditivity of the defect, Theorem 3.9, forms the basis for at least two equivalence relations among invertible dynamical systems which it seems worthwhile to investigate. They are both inspired by the work of Adler and Marcus in [AM]. The point of departure is the following lemma which is analogous to Proposition (2.14) of [AM].

Lemma 5.1. *Given invertible dynamical systems and factor maps,*

$$\begin{tikzcd}
(X,\varphi) \arrow[dr, "\pi_1"'] & & (Y,\psi) \arrow[dl, "\pi_2"] \\
& (Z,\lambda) &
\end{tikzcd}$$

there is an invertible dynamical system (W, κ) and a commuting diagram,

$$\begin{tikzcd}
& (W,\kappa) \arrow[dl, "\pi_3"'] \arrow[dr, "\pi_4"] & \\
(X,\varphi) \arrow[dr, "\pi_1"'] & & (Y,\psi) \arrow[dl, "\pi_2"] \\
& (Z,\lambda) &
\end{tikzcd}$$

of factor maps such that $D(\pi_3) = D(\pi_2)$ and $D(\pi_4) = D(\pi_1)$.

Proof. Set $W = \{(x, y) \in X \times Y : \pi_1(x) = \pi_2(y)\}$, $\kappa = \varphi \times \psi$, $\pi_3(x, y) = x$, $\pi_4(x, y) = y$. Then $\#\pi_3^{-1}(x) = \#\pi_2^{-1}(\pi_1(x))$, so for any φ-invariant Borel probability measure μ we find that

$$\int_X \log \#\pi_3^{-1}(x) \ d\mu(x) = \int_Z \log \#\pi_2^{-1}(z) \ d\mu \circ \pi_1^{-1}(z) \ .$$

Since any λ-invariant Borel probability measure on Z has the form $\mu \circ \pi_1^{-1}$ for some φ-invariant Borel probability measure μ on X, we conclude from Theorem 2.1 that $D(\pi_3) = D(\pi_2)$. The equality $D(\pi_4) = D(\pi_1)$ follows in the same way. □

Definition 5.2. Two invertible dynamical systems, (X, φ), (Y, ψ), are *finitely equivalent* (resp. *strongly equivalent*) when there is an invertible dynamical system (Z, κ) and factor maps $\pi_1 : (Z, \kappa) \to (X, \varphi)$, $\pi_2 : (Z, \kappa) \to (Y, \psi)$ such that $D(\pi_1) + D(\pi_2) < \infty$ (resp. $D(\pi_1) = D(\pi_2) = 0$).

It follows from Lemma 5.1 and Theorem 3.9 that 'finite equivalence' and 'strong equivalence' are both equivalence relations for invertible dynamical systems.

Let us immediate extend the definition to cover general (non-invertible) dynamical systems.

Definition 5.3. Two dynamical systems, (X,φ), (Y,ψ), are *finitely equivalent* (resp. *strongly equivalent*) when there is a dynamical system (Z,κ) and factor maps $\pi_1 : (Z,\kappa) \to (X,\varphi)$, $\pi_2 : (Z,\kappa) \to (Y,\psi)$ such that $D_r(\pi_1) + D_r(\pi_2) < \infty$ (resp. $D_r(\pi_1) = D_r(\pi_2) = 0$).

Finite equivalence and strong equivalence are equivalence relations thanks to the sub-additivity of the reduced defect, Corollary 4.7, and the variational principle for the reduced defect, Theorem 4.10. The argument is basically the same as in the proof of Lemma 5.1.

Lemma 5.4. *Two invertible dynamical systems are finitely equivalent (resp. strongly equivalent) in the sense of Definition 5.2 if and only if they are finitely equivalent (resp. strongly equivalent) in the sense of Definition 5.3.*

Proof. When

$$\begin{tikzcd} & (W,\kappa) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ (X,\varphi) & & (Y,\psi) \end{tikzcd}$$

is a diagram of factor maps such that $D_r(\pi_1) + D_r(\pi_2) < \infty$ (resp. $D_r(\pi_1) = D_r(\pi_2) = 0$), and (X,φ) and (Y,ψ) are both invertible, by passing to natural invertible extensions we get also a diagram

$$\begin{tikzcd} & (\widehat{W},\widehat{\kappa}) \arrow[dl, "\widehat{\pi_1}"'] \arrow[dr, "\widehat{\pi_2}"] & \\ (X,\varphi) & & (Y,\psi) \end{tikzcd}$$

where $D(\widehat{\pi_i}) = D_r(\pi_i)$, $i = 1,2$, by Theorem 4.6. Hence (X,φ) and (Y,ψ) are finitely equivalent (resp. strongly equivalent) in the sense of Definition 5.2. Since the other implication follows from 2) of Lemma 4.2, the proof is complete.

□

Note that finitely equivalent dynamical systems must have the same topological entropy by Theorem 3.6 and/or Corollary 4.8.

Theorem 5.5. *Let (Σ_1,σ) and (Σ_2,σ) be irreducible twosided sofic shifts. Then the following conditions are equivalent:*

1) $h(\Sigma_1) = h(\Sigma_2)$.
2) (Σ_1,σ) *and* (Σ_2,σ) *have a common finite-to-one extension which is a twosided irreducible subshift of finite type.*
3) (Σ_1,σ) *and* (Σ_2,σ) *are finitely equivalent.*

Proof. 1) $\Leftrightarrow$ 2) follows from Theorem 3.6 and the entropy-classification of irreducible subshifts of finite type, cf. Theorem 8.3.8 of [LM]. 2) $\Rightarrow$ 3) follows from 1) of Proposition 3.1. 3) $\Rightarrow$ 1) follows from Theorem 3.6. □

Corollary 5.6. *Let (X, φ) and (Y, ψ) be boundedly finite-to-one factors of irreducible subshifts of finite type. Then (X, φ) and (Y, ψ) are finitely equivalent if and only if $h(\varphi) = h(\psi)$.*

In particular, we see that hyperbolic toral automorphisms are finitely equivalent if and only if they have the same entropy. Compare [AM].

Proposition 5.7. *Let (X, φ) and (Y, ψ) be invertible dynamical systems. Then the following are equivalent :*

1) *(X, φ) and (Y, ψ) are finitely equivalent.*
2) *(X, φ^n) and (Y, ψ^n) are finitely equivalent for all $n \in \mathbb{Z}$.*
3) *(X, φ^n) and (Y, ψ^n) are finitely equivalent for some $n \in \mathbb{Z}$.*

Proof. 1) $\Rightarrow$ 2) follows from the argument which proved Lemma 3.7 of [Th2], using the variational principle. It suffices therefore to show that 3) $\Rightarrow$ 1). Furthermore, it suffices to consider the case $n > 1$. Let (Z, σ) be a dynamical system and $\pi_1 : (Z, \sigma) \to (X, \varphi^n)$, $\pi_2 : (Z, \sigma) \to (Y, \psi^n)$ factor maps such that $D(\pi_1) + D(\pi_2) < \infty$. Let $Z_0, Z_1, Z_2, \cdots, Z_{n-1}$ be disjoint copies of Z and define $\widetilde{\sigma} : \bigcup_{j=0}^{n-1} Z_j \to \bigcup_{j=0}^{n-1} Z_j$ such that $\widetilde{\sigma}|_{Z_j} : Z_j \to Z_{j+1}$ is the identity when $j < n-1$ and $\widetilde{\sigma}|_{Z_{n-1}} : Z_{n-1} \to Z_0$ is σ. Define $\widetilde{\pi_1} : \bigcup_{j=0}^{n-1} Z_j \to X$ such that $\widetilde{\pi_1}|_{Z_j} = \varphi^j \circ \pi_1$ for all $j = 0, 1, 2, \cdots, n-1$. Then $\widetilde{\pi_1} : (\bigcup_{j=0}^{n-1} Z_j, \widetilde{\sigma}) \to (X, \varphi)$ is a factor map and $\#\widetilde{\pi_1}^{-1}(x) = n \# \pi_1^{-1}(x)$ for all $x \in X$. Hence $D(\widetilde{\pi_1}) = \log n + D(\pi_1)$ by Theorem 2.1. Similarly, we define a factor map $\widetilde{\pi_2} : (\bigcup_{j=0}^{n-1} Z_j, \widetilde{\sigma}) \to (Y, \psi)$ such that $D(\widetilde{\pi_2}) = \log n + D(\pi_2)$. □

Corollary 5.8. *Let (X, φ) and (Y, ψ) be dynamical systems. Assume that X and Y are finite-dimensional spaces. Assume that φ and ψ are periodic, i.e. that $\varphi^k = \mathrm{id}_X$, $\psi^m = \mathrm{id}_Y$ for some $k, m \in \mathbb{N}$. Then (X, φ) and (Y, ψ) are finitely equivalent if and only if there is a compact metric space Z and continuous surjections $\pi_0 : Z \to X$ and $\pi_1 : Z \to Y$ such that $\sup_{x,y} \max\{\#\pi_0^{-1}(x), \#\pi_1^{-1}(y)\} < \infty$.*

Proof. This follows immediately from Proposition 5.7. □

Remark 5.9. In many cases it is easy to see that there is a Z satisfying the requirement in Corollary 5.8. On the other hand, it is also easy to give examples which shows that it does not always exist, so it would be nice to have general criteria for the existence of such a space Z.

Lemma 5.10. *Let (Σ_1, σ) and (Σ_2, σ) be two-sided mixing subshifts of finite type. Assume that $h(\Sigma_1, \sigma) \leq h(\Sigma_2, \sigma)$. Then the disjoint union $(\Sigma_1 \sqcup \Sigma_2, \sigma \sqcup \sigma)$ is finitely equivalent to (Σ_2, σ).*

Proof. Assume first that $h(\Sigma_1, \sigma) = h(\Sigma_2, \sigma)$. Then (Σ_1, σ) and (Σ_2, σ) are finitely equivalent by Lemma 5.5. It follows then easily that $(\Sigma_1 \sqcup \Sigma_2, \sigma \sqcup \sigma)$ is finitely equivalent to $(\Sigma_2 \sqcup \Sigma_2, \sigma \sqcup \sigma)$, which in turn is finitely equivalent to (Σ_2, σ). Assume next that $h(\Sigma_1, \sigma) < h(\Sigma_2, \sigma)$. By a wellknown formula for the topological entropy of a mixing subshift of finite type, cf. [LM], there is then an $N \in \mathbb{N}$ so large that

$$\#\{x \in \Sigma_1 : x \text{ has minimal period } n\} < \#\{x \in \Sigma_2 : x \text{ has minimal period } n\}$$

for all $n \geq N$. Let P be a prime larger than N. Let $\underline{P} = \{1, 2, \cdots, P\}$, and let $q : \underline{P} \to \underline{P}$ be cyclic permutation. Then $(\Sigma_1 \times \underline{P}, \sigma \times q)$ and $(\Sigma_2 \times \underline{P}, \sigma \times q)$ are irreducible subshifts of finite type, with entropy $h(\Sigma_1, \sigma)$ and $h(\Sigma_2, \sigma)$, respectively. In addition

$$\begin{aligned} &\#\{x \in \Sigma_1 \times \underline{P} : x \text{ has minimal period } n \text{ under } \sigma \times q\} \\ &< \#\{x \in \Sigma_2 \times \underline{P} : x \text{ has minimal period } n \text{ under } \sigma \times q\} \end{aligned}$$

for all n. By Krieger's embedding theorem, cf. Theorem 10.1.1 of [LM], there is then an embedding $(\Sigma_1 \times \underline{P}, \sigma \times q) \to (\Sigma_2 \times \underline{P}, \sigma \times q)$. Since $(\Sigma_i \times \underline{P}, \sigma \times q)$ is finitely equivalent to (Σ_i, σ), and

$$(\Sigma_1 \times \underline{P} \sqcup \Sigma_2 \times \underline{P}, \sigma \times q \sqcup \sigma \times q)$$

to $(\Sigma_1 \sqcup \Sigma_2, \sigma \sqcup \sigma)$, we can assume from outset that we are given an embedding $\kappa : (\Sigma_1, \sigma) \to (\Sigma_2, \sigma)$. Define $\pi : (\Sigma_1 \sqcup \Sigma_2, \sigma \sqcup \sigma) \to (\Sigma_2, \sigma)$ such that $\pi|_{\Sigma_1} = \kappa$ and $\pi|_{\Sigma_2}$ is the identity. Then $\#\pi^{-1}(x) \leq 2$ for all $x \in \Sigma_2$ and hence $D(\pi) \leq \log 2$. Thus (Σ_2, σ) is finitely equivalent to $(\Sigma_1 \sqcup \Sigma_2, \sigma \sqcup \sigma)$, as asserted.

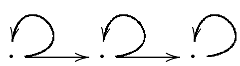
□

Proposition 5.11. *The non-wandering parts of two subshifts of finite type are finitely equivalent if and only if they have the same entropy.*

Proof. For an irreducible subshift of finite type some power has a non-wandering part which is the disjoint union of a finite number of mixing subshifts of finite type. Combine Lemma 5.10 with Theorem 5.5 and Proposition 5.7. □

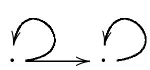
Remark 5.12. I am not sure how sensitive finite equivalence is towards the wandering points of a subshift of finite type. However, entropy is certainly not the only invariant for finite equivalence of general subshifts of finite type. To illustrate the situation in the simplest cases, consider the graphs

A



B

C



D

Of the four edge-shifts, all with zero entropy, given by these graphs, A and B are finitely equivalent, but no pair among B, C and D are finitely equivalent. So it appears as if finite equivalence is sensitive to how many irreducible components are connected, but ignores how they are connected. Presently I do not know how representative these very simple examples are.

Concerning strong equivalence it is clear that (X, φ) and (Y, ψ) can only be strongly equivalent when

$$\#\{x \in X : \varphi^n(x) = x\} = \#\{y \in Y : \psi^n(y) = y\}$$

for all $n \in \mathbb{N}$. In view of Krieger's Embedding Theorem and Boyle's Lower Entropy Factor Theorem, cf. Theorem 10.1.1 and Theorem 10.3.1 of [LM], respectively, it seems reasonable to ask if this condition is sufficient for irreducible subshifts of finite type. In this paper we concentrate the investigations on finite equivalence.

Lemma 5.13. *Let $\lambda \in \mathbb{T}^n$ and let $\underline{\lambda} : \mathbb{T}^n \to \mathbb{T}^n$ be the corresponding rotation, viz. $\underline{\lambda}(z) = \lambda z$. Then $(\mathbb{T}^n, \underline{\lambda})$ is finitely equivalent to $(\mathbb{T}^n, \underline{\lambda}^k)$ for all $k \in \mathbb{Z}$.*

Proof. Define $\pi_1 : \mathbb{T}^n \to \mathbb{T}^n$ by $\pi_1(z) = z^k$ and let π_2 be the identity map of $\mathbb{T}^n$. Then $\pi_1 : (\mathbb{T}^n, \underline{\lambda}) \to (\mathbb{T}^n, \underline{\lambda}^k)$ and $\pi_2 : (\mathbb{T}^n, \underline{\lambda}^k) \to (\mathbb{T}^n, \underline{\lambda}^k)$ are factor maps of defect $D(\pi_1) = \log n|k|$ and $D(\pi_2) = 0$, respectively. It follows from Lemma 5.1 that $(\mathbb{T}^n, \underline{\lambda})$ is finitely equivalent to $(\mathbb{T}^n, \underline{\lambda}^k)$. □

Lemma 5.14. *Let $\lambda, \mu \in \mathbb{T}$, and let $\underline{\lambda} : \mathbb{T} \to \mathbb{T}$ and $\underline{\mu} : \mathbb{T} \to \mathbb{T}$ be the corresponding rotations of the circle. Then $\underline{\lambda}$ and $\underline{\mu}$ are equivalent if and only if there are numbers $k, m \in \mathbb{Z}$ such that $\lambda^m = \mu^k$, i.e. if and only if $\lambda = \mu$ modulo $\mathbb{Q}/\mathbb{Z}$.*

Proof. If such m and k exist it follows from Lemma 5.13 that $\underline{\lambda}$ and $\underline{\mu}$ are equivalent. Conversely, assume that $\underline{\lambda}$ and $\underline{\mu}$ are equivalent. If both λ and μ are rational (i.e. of finite order in the group $\mathbb{T}$), there is nothing to prove, so assume that λ is irrational. Let (X, ψ) be a common finite defect extension of $(\mathbb{T}, \underline{\lambda})$ and $(\mathbb{T}, \underline{\mu})$. Let $\pi : (X, \psi) \to (\mathbb{T}, \underline{\lambda})$ be a factor map of finite defect. It follows from Theorem 3.7 that there is a ψ-ergodic Borel probability measure ν on X such that $\nu \circ \pi^{-1}$ is Lebesgue measure on $\mathbb{T}$ and $\#\pi(x) = k$ for almost all $x \in \mathbb{T}$. Since $(\mathbb{T}, \underline{\mu})$ is a factor of (X, ψ), μ must be an eigenvalue for the unitary $T_\psi : L^2(X, \nu) \to L^2(X, \nu)$ induced by ψ. Let $f : X \to \mathbb{T}$ be the corresponding (continuous) eigenfunction. As is well-known, cf. Lemma 1 of [NP], we can identify the measure space (X, ν) with the space $\mathbb{T} \times \{1, 2, \cdots, k\}$ equipped with the product of Lebesgue measure on $\mathbb{T}$ with the homogeneous probability measure on $\{1, 2, \cdots, k\}$. In this picture $\psi(x, i) = (\lambda x, \sigma_x(i))$, where $\sigma : \mathbb{T} \to \Sigma_k$ is a Borel function taking values in the symmetric group. Hence

$$g(x) = \prod_{i=1}^{k} f(x, i)$$

is a Borel function $g : \mathbb{T} \to \mathbb{T}$ such that $g(\lambda x) = \mu^k g(x)$ for almost all $x \in \mathbb{T}$. Since the spectrum of the unitary on $L^2(\mathbb{T})$ induced by $\underline{\lambda}$ is $\{\lambda^z : z \in \mathbb{Z}\}$, we conclude that $\mu^k = \lambda^m$ for some $m \in \mathbb{Z}$. □

Theorem 5.15. *Two orientation preserving homeomorphisms of the circle with irrational rotation numbers, α and β, are finitely equivalent if and only if there are integers, $n, m \in \mathbb{Z}$, such that $n\alpha - m\beta \in \mathbb{Z}$.*

Proof. By the Poincaré Classification Theorem, cf. Theorem 11.2.7 of [KH], any orientation preserving homeomorphism of the circle with irrational rotation number has the rigid rotation with the same rotation number (mod $\mathbb{Z}$) as a factor under a factor map π for which $\#\pi^{-1}(x) = 1$ for all x outside of a countable subset of the circle. Hence $D(\pi) = 0$ by Theorem 2.1 and we see that any orientation preserving homeomorphism of the circle with irrational rotation number is strongly equivalent to the rigid rotation with the same rotation number. Apply Lemma 5.14. □

Remark 5.16. Theorem 5.15 is not true for orientation preserving homeomorphisms of the circle with rational rotation numbers. Indeed, it is not true that any homeomorphism of the interval $[0,1]$ is finitely equivalent to the identity map. For example, it is not difficult to see that a homeomorphism of $[0,1]$ for which the set of fixed points is infinite can not be finitely equivalent to one for which the set of fixed points is finite. So at least some characteristics of the fixed point set is preserved under finite equivalence, and presently I do not know exactly which. The problem is related to the problem mentioned in Remark 5.9. However, it is true that two orientation preserving homeomorphisms of the circle or two homeomorphisms of the interval are finitely equivalent when they have the same set of periodic points.

The method of proof in Lemma 5.14 can be extended to give a classification of minimal rotations of higher-dimensional tori as follows.

Theorem 5.17. *Let $\underline{\lambda}, \underline{\mu}$ be minimal rotations of the n-torus $\mathbb{T}^n$, $n \geq 1$. Then $(\mathbb{T}^n, \underline{\lambda})$ and $(\mathbb{T}^n, \underline{\mu})$ are finitely equivalent if and only if there is a continuous surjective endomorphism $B : \mathbb{T}^n \to \mathbb{T}^n$ and a natural number $k \in \mathbb{N}$ such that*

$$\underline{\lambda}^k \circ B = B \circ \underline{\mu}. \tag{5.1}$$

Proof. Assume first that B and k exist. Since B is constant-to-one (with a finite constant) we see immediately from (5.1) that B is factor map of finite defect showing that $(\mathbb{T}^n, \underline{\lambda}^k)$ and $(\mathbb{T}^n, \underline{\mu})$ are finitely equivalent. Hence $(\mathbb{T}^n, \underline{\lambda})$ and $(\mathbb{T}^n, \underline{\mu})$ are finitely equivalent by Lemma 5.13.

Assume then that $(\mathbb{T}^n, \underline{\lambda})$ and $(\mathbb{T}^n, \underline{\mu})$ are finitely equivalent, and write $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$, $\mu = (\mu_1, \mu_2, \cdots, \mu_n)$, where $\lambda_i, \mu_i \in \mathbb{T}$ for all i. The argument from the proof of Lemma 5.13 shows that there is a $k \in \mathbb{N}$ such that λ_i^k is in the spectrum of $(\mathbb{T}^n, \underline{\mu})$ for all i. Since the spectrum of $(\mathbb{T}^n, \underline{\mu})$ is the set

$$\{\mu_1^{z_1}\mu_2^{z_2}\cdots\mu_n^{z_n} : z_1, z_2, \cdots, z_n \in \mathbb{Z}\},$$

we conclude that there is a $n \times n$-matrix $A = (A_{ij})$ with $\mathbb{Z}$-entries such that

$$\lambda_i^k = \mu_1^{A_{i1}}\mu_2^{A_{i2}}\cdots\mu_n^{A_{in}} \tag{5.2}$$

for all i. When B denotes the endomorphism of $\mathbb{T}^n$ given by A, (5.2) means that $\lambda^k = B(\mu)$, so the transitivity of $\underline{\lambda}$ (which implies the transitivity and hence the minimality of $\underline{\lambda}^k$) shows that B is surjective. □

Theorem 5.17 can also be formulated as follows: Choose $\alpha, \beta \in \mathbb{R}^n$ such that $p(\alpha) = \lambda$, $p(\beta) = \mu$, where $p : \mathbb{R}^n \to \mathbb{T}^n$ is the canonical surjection. Then $(\mathbb{T}^n, \underline{\lambda})$ and $(\mathbb{T}^n, \underline{\mu})$ are finitely equivalent if and only if there is an invertible $n \times n$-matrix D over $\mathbb{Q}$ such that $D\beta - \alpha \in \mathbb{Q}^n$.

Before we turn to a few non-invertible dynamical systems, let us first observe that all surjective dynamical systems are finitely equivalent to their natural invertible extension. If namely (X, φ) is a dynamical system with φ surjective, the projection to the first coordinate, $p_0 : (\widehat{X}, \widehat{\varphi}) \to (X, \varphi)$, is a factor map which ensure a strong equivalence:

Proposition 5.18. *Let (X,φ) be a dynamical system with φ surjective. Then the reduced defect of the factor map $p_0 : (\widehat{X},\widehat{\varphi}) \to (X,\varphi)$ is 0, and (X,φ) is strongly equivalent to $(\widehat{X},\widehat{\varphi})$.*

Proof. Recall that $\widehat{X}$ is a closed subset of the infinite product $\prod_{i=0}^{\infty} X$, equipped with the metric

$$d_\infty((x_i),(y_i)) = \sum_{i=0}^{\infty} \frac{d(x_i,y_i)}{2^i}.$$

Then $p_0^{-1}(x) \subseteq \prod_{i=0}^{\infty} \varphi^{-i}(x)$ and $\widehat{\varphi}^k(p_0^{-1}(x)) \subseteq \prod_{i=0}^{\infty} \varphi^{k-i}(x)$ for all $x \in X, k \in \mathbb{N}$. Hence the d_∞-diameter of $\widehat{\varphi}^k(p_0^{-1}(x))$ tends to 0 as k tends to infinity, and hence $\limsup_k \#^\epsilon \widehat{\varphi}^k(p_0^{-1}(x)) = 0$ for all x and all $\epsilon > 0$. It follows that $A_{p_0} = 0$ and hence from Theorem 4.10 that $D_r(p_0) = 0$.

□

Remark 5.19. An alternative proof of Proposition 5.18 goes as follows : The natural extension $\widehat{p_0}$ of p_0 is a conjugacy and hence $D_r(p_0) = D(\widehat{p_0}) = 0$ by Theorem 4.6.

It follows from Proposition 5.18 that for any dynamical system (X,φ), the system $(\bigcap_j \varphi^j(X),\varphi)$ will be strongly equivalent to an invertible dynamical system. It seems to be generally agreed that the interesting dynamics of φ takes place in $\bigcap_j \varphi^j(X)$, so we may conclude that up to strong equivalence all interesting dynamics can be realized in invertible dynamical systems.

Theorem 5.20. *Two one-sided irreducible sofic subshifts are finite equivalent if and only if they have the same entropy.*

Proof. The natural extension of a one-sided irreducible sofic subshift is a two-sided irreducible sofic subshift to which it is strongly equivalent by Proposition 5.18. Apply Theorem 5.5. □

Corollary 5.21. *Let (X,φ) and (Y,ψ) be boundedly finite-to-one factors of irreducible one-sided subshifts of finite type. Then (X,φ) and (Y,ψ) are finitely equivalent if and only if $h(\psi) = h(\varphi)$.*

Proof. The necessity of equal entropy follows from Corollary 4.8. So assume that $h(\psi) = h(\varphi)$. The assumptions, combined with Theorem 2.1 and 2) of Lemma 4.2, show that both dynamical systems are finite reduced defect factors of irreducible one-sided subshifts of finite type. Apply Theorem 5.20. □

Corollary 5.22. *Let (X,ψ) and (Y,φ) be expansive endomorphisms of compact finite-dimensional differentiable manifolds, X and Y, respectively. Then (X,ψ) and (Y,φ) are finitely equivalent if and only if $h(\psi) = h(\varphi)$.*

Proof. By Corollary 5.21 it suffices to show that both dynamical systems are finite-to-one factors of irreducible subshifts of finite type. First note that expansive endomorphisms are factors of full shifts by [S] and hence transitive. By Theorem 7.30 of [Ru] it suffices therefore to show that an expansive endomorphism is expanding in the sense of 7.26 of [Ru]. But this is not difficult, and was pointed out already in [CR]. □

Example 5.23. Let $\psi : [0,1] \to [0,1]$ be a unimodular map with positive entropy, and let $\varphi : [0,1] \to [0,1]$ be the tent-map with the same entropy. Using their kneading theory, Milnor and Thurston constructed in [MT] a factor map $h : ([0,1], \psi) \to ([0,1], \varphi)$. The construction is reproduced in [KH], pp. 514-518. We will show here that the reduced defect of h is zero, provided that there are points $a, b \in [0,1]$ such that $a \leq c \leq b$, where c is the critical point, for which the union of pre-images is dense in $[0,1]$, i.e. we will show that

$$\begin{aligned} &\exists a, b \in [0,1], a \leq c \leq b \text{ such that } \bigcup_{m \in \mathbb{N}} \psi^{-m}(a) \text{ and } \bigcup_{m \in \mathbb{N}} \psi^{-m}(b) \text{ are both dense} \\ &\Rightarrow \ D_r(h) = 0. \end{aligned} \tag{5.3}$$

To prove this we first argue that

$$\begin{aligned} &\text{if } x \in [0,1], k \in \mathbb{N}, \text{ and } I \subseteq h^{-1}(x) \text{ is a closed non-empty interval} \\ &\text{such that } \psi^k(I) = I, \text{ then } I \text{ contains only one point.} \end{aligned} \tag{5.4}$$

To prove (5.4) let I be an interval with the specified property, and assume to reach a contraction that I is non-degenerate. Then all the intervals $I, \psi(I), \psi^2(I), \cdots, \psi^{k-1}(I)$ are non-degenerate, so for any $j \in \{1, 2, \cdots, k-1\}$ we can find $x \in \psi^{-n}(a) \cap I$ and $y \in \psi^{-m}(a) \cap \psi^j(I)$ for some $n, m \in \mathbb{N}$. Then $I \ni \psi^{nk}(x) = \psi^k(a) = \psi^{mk}(y) \in \psi^j(I)$, so that $I \cap \psi^j(I) \neq \emptyset$. Since h is constant on I and $\psi^j(I)$ (equal to x on I and to $\varphi^j(x)$ on $\psi^j(I)$), we conclude that h is constant on $I \cup \psi^j(I)$. Since j was arbitrary we deduce that h is constant equal to x on $\bigcup_{j \in \mathbb{N}} \psi^j(I)$, and that x is a fixed point for φ. Since $\bigcup_m \psi^{-m}(a)$ is dense there must be an $l \in \mathbb{N}$ such that $a \in \psi^l(I)$, and we conclude that $h(a) = x$. Similarly, we deduce that $h(b) = x$. Since $a \leq c \leq b$ and h is non-decreasing we deduce that $h(c) = x$. But $h(c) = \frac{1}{2}$ by (4) and (5) of Lemma 15.6.8 of [KH], so we conclude that $\frac{1}{2}$ is a fixed point for the tent-map φ. This is not possible because the entropy of φ is positive. This contraction establishes (5.4).

To prove that $D_r(h) = 0$ it suffices to show that $\lim_{n\to\infty} \operatorname{diam} \psi^n(h^{-1}(x)) = 0$ for all $x \in [0,1]$ by Theorem 4.10. Since h is nondecreasing, $h^{-1}(x)$ is an interval, and the conclusion we seek is automatic if the sets $\psi^j(h^{-1}(x)), j \in \mathbb{N}$, are all mutually disjoint, so we may assume that there are $k < l$ in $\mathbb{N}$ such that $\psi^k(h^{-1}(x)) \cap \psi^l(h^{-1}(x)) \neq \emptyset$. Set $J = h^{-1}(\varphi^k(x))$ and note that $\psi^k(h^{-1}(x)) \subseteq J$. Since h is constant on J and $\psi^l(h^{-1}(x))$, and these sets intersect, we conclude that $\varphi^{l-k}(\varphi^k(x)) = \varphi^k(x)$, so that $\psi^{l-k}(J) \subseteq J$. If follows that $I = \bigcap_{n \in \mathbb{N}} \psi^{(l-k)n}(J)$ is a closed non-empty interval in $J = h^{-1}(\varphi^k(x))$ such that $\psi^{l-k}(I) = I$. It follows therefore from (5.4) that I is a point. Since $J \supseteq \psi^{l-k}(J) \supseteq \psi^{2(l-k)}(J) \supseteq \psi^{3(l-k)} \supseteq \cdots$ we deduce that $\lim_{n\to\infty} \operatorname{diam} \psi^{n(l-k)}(J) = 0$. Since the functions $\psi, \psi^2, \cdots, \psi^{l-k}$ are uniformly continuous it follows that $\lim_{n\to\infty} \operatorname{diam} \psi^n(J) = 0$. Hence $\lim_{n\to\infty} \operatorname{diam} \psi^n(h^{-1}(x)) = \lim_{n\to\infty} \operatorname{diam} \psi^{n-k}(J) = 0$, and we have established (5.3).

Note that this conclusion actually does give us some information which does not follow from the mere existence of the factor map h. In fact, it follows that

$$\#\{x \in [0,1] : \psi^n(x) = x\} = \#\{x \in [0,1] : \varphi^n(x) = x\}$$

for all n. Indeed, there is a bijective (dynamical) correspondance between the periodic orbits of a dynamical system and that of its invertible natural extension, and since $D_r(h) = 0$ we get from Theorem 4.6 that $D(\widehat{h}) = 0$, and we conclude that $\widehat{h}$, and hence also h, sets up a bijective correspondance between the periodic orbits of ψ and that of φ. In fact, with a little more effort it is not difficult to see that h sets up a continuous affine homeomorphism between the ψ-invariant and the φ-invariant Borel probability measures.

I must admit that I am not sure if the existence of a and b as in (5.3) implies that h is actually a conjugacy. Note that the existence of a periodic point p for ψ for which $\bigcup_j \psi^{-j}(p)$ is dense in $[0,1]$ will give us the points a and b we need. Furthermore, an inspection of the above argument shows that if we instead of (5.3) just assume the existence of single point whose pre-images are dense, and that $\lim_{j\to\infty} \psi^j(\operatorname{diam} h^{-1}(*)) = 0$, where $*$ denotes the fixed point ($\neq 0$) for the tent map, then we can again deduce that $D_r(h) = 0$. If, in any of these cases, there is an attractive periodic point for ψ, h will be a bijection between the periodic orbits of ψ and φ, and yet not a conjugacy. [2]

Remark 5.24. Following Williams, [Wi], Franks and Richeson calls two dynamical systems, (Y,ψ) and (X,φ), *shift equivalent* (of lag m), when there are maps $r : (Y,\psi) \to (X,\varphi)$ and $s : (X,\varphi) \to (Y,\psi)$, not neccesarily surjective, but such that $r \circ s = \psi^m$ and $s \circ r = \varphi^m$ for some $m \in \mathbb{N}$, [FR]. For invertible dynamical systems shift equivalence is the same as conjugacy, but not in general. Shift equivalence of (Y,ψ) and (X,φ) implies that $(\bigcap_j \psi^j(Y),\psi)$ and $(\bigcap_j \varphi^j(X),\varphi)$ are strongly equivalent. Indeed, if (Y,ψ) and (X,φ) are shift equivalent, say of lag m, via r and s as above, it follows that $r : (\bigcap_j \psi^j(Y),\psi) \to (\bigcap_j \varphi^j(X),\varphi)$ and $s : (\bigcap_j \varphi^j(X),\varphi) \to (\bigcap_j \psi^j(Y),\psi)$ are factor maps (i.e. surjective) and hence that the natural extensions $\widehat{r} : (\widehat{Y},\widehat{\psi}) \to (\widehat{X},\widehat{\varphi})$ and $\widehat{s} : (\widehat{X},\widehat{\varphi}) \to (\widehat{Y},\widehat{\psi})$ define a shift equivalence (of lag m) between $(\widehat{X},\widehat{\varphi})$ and $(\widehat{Y},\widehat{\psi})$. By using 6) of Proposition 3.1 this implies that $D(\widehat{s}) = 0$ and hence, by Theorem 4.6, that $D_r(s|_{\bigcap_j \varphi^j(X)}) = 0$.

In the preceding we have used the defect to identify what seems to be a natural notion of finite equivalence for general dynamical systems. There is, however, also another more obvious application of the defect to dynamical systems which we would like to mention in closing: A dynamical system (X,φ) can be considered as a factor map from itself to itself; $\varphi : (X,\varphi) \to (X,\varphi)$, at least if φ is surjective. (If φ is not surjective, consider $\varphi : (\bigcap_j \varphi^j(X),\varphi) \to (\bigcap_j \varphi^j(X),\varphi)$ instead.) While the reduced defect is zero in this setting, the defect itself becomes a conjugacy invariant for dynamical systems, which carries information, not on the complexity of the dynamical system, but about 'how non-invertible' the system is. As an invariant it is in some cases more sensitive than the topological entropy. This is illustrated in the last example below.

Example 5.25. Let $n \in \mathbb{N},\ n \geq 3$. For each natural number $k,\ k \geq \frac{n-2}{2}\sqrt{1+n^{-2}}$, there is a (unique) piecewise linear map $\varphi_{n,k} : [0,1] \to [0,1]$ with slope n on all intervals of linearity, with $2k+1$ turning points, such that $\varphi_{n,k}(t) = nt,\ t \in [0,\frac{1}{n}]$, and $\varphi_{n,k}(t) = -nt + n,\ t \in [1-\frac{1}{n}, 1]$.

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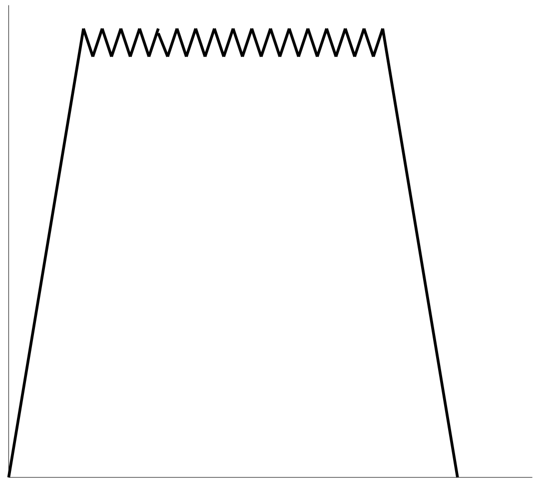
[2] I am grateful to Joao Alves for pointing out that the existence of an attractive periodic point prevents h from being a conjugacy.



The graph of $\varphi_{n,k}$.

Set $c(n,k) = \frac{n-2}{2k}\sqrt{1+n^{-2}}$ which is $1 - \min_{t \in [\frac{1}{n}, 1-\frac{1}{n}]} \varphi_{n,k}(t)$. When $\#\varphi_{n,k}^{-1}(t) > 2$, $\#\varphi_{n,k}^{-1}(\varphi_{n,k}^{j}(t)) = 2$ for all $j \geq 1$ such that $n^j c(n,k) < 1 - c(n,k)$. So if $k > \frac{(n^j+1)(n-2)}{2}\sqrt{1+n^{-2}}$, we find that $D(\varphi_{n,k}) \leq \frac{1}{j+1}\log(2k+2) + \frac{j}{j+1}\log 2$. On the other hand, when $n^{j+1}c(n,k) \geq 1$, we have a $j+1$-periodic point x_0 such that $\#\varphi_{n,k}^{-1}(x_0) = 2k+2$ and hence $D(\varphi_{n,k}) \geq \frac{1}{j+1}\log(2k+2) + \frac{j}{j+1}\log 2$ by 2) of Proposition 3.1. So for arbitrary $j \in \mathbb{N}$ we find that

$$D(\varphi_{n,k}) = \frac{1}{j+1}\log(2k+2) + \frac{j}{j+1}\log 2$$

when $\frac{(n^j+1)(n-2)}{2}\sqrt{1+n^{-2}} < k \leq \frac{n^{j+1}(n-2)}{2}\sqrt{1+n^{-2}}$.

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