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ON THE KK-THEORY AND THE E-THEORY OF AMALGAMATED FREE PRODUCTS OF C^*-ALGEBRAS

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Abstract. We establish six terms exact sequences relating the KK-theory groups and the E-theory groups of an amalgamated free product C^*-algebra, $A_1 *_B A_2$, to the respective groups of the three constituents, A_1, A_2 and B. In the KK-theory case we assume the existence of conditional expectations from A_k onto B or that A_1, A_2 and B are all nuclear, and in the E-theory case that there exist sequences $R^k_n : A_k \to B, n \in \mathbb{N}$, of completely positive contractions such that $\lim_{n\to\infty} R^k_n(b) = b$ for all $b \in B, k = 1, 2$. This condition is fullfilled e.g. when B is nuclear or sits as a hereditary C^*-subalgebra of the A_k's.

1. Introduction

Cuntz and Germain have conjectured the existence of two short exact sequences which should relate the KK-groups of an amalgamated free product $A_1 *_B A_2$ to the KK-groups of A_1, A_2 and B. See Remark 2 of [C1], Conjecture 0.1 of [G2] and Conjecture 3.11 of [G3] where the conjecture is formulated in varying generality. In [C1] Cuntz proved the conjecture when there are retractions from the A_k's onto B, in [G1] Germain proved it when $B = \mathbb{C}$ sits unitally inside the A_k's which were assumed to be 'K-pointed', cf. Definition 5.1 of [G1], a condition which he subsequently, in [G2], weakened to K-nuclearity (in the sense of Skandalis, [S]). Finally, in [G3] he announced a proof of the conjecture under certain technical assumptions ('relative K-nuclearity') which among other things require the existence of conditional expectations $P_k : A_k \to B$. In another direction the conjecture was established in increasing generality for examples coming from groups or actions by groups in [C2], [L], [N]. In this direction the ultimate result seems to be that of Pimsner, [Pi], who obtained results which, among other, verify the conjecture when G_1 and G_2 are countable discrete groups containing a common subgroup H, $A_k = A \ltimes G_k, k = 1, 2$, and $B = A \ltimes H$ for some actions of G_1 and G_2 on A which agree on H. However, in the general case the conjecture remained open even when $B = \mathbb{C}$.

In this paper we establish the conjectured six terms exact sequences when there are conditional expectations $A_k \to B, k = 1, 2$, or A_1, A_2 and B are all nuclear. In principle the method we use for this is the same as that of Germain. In [G2] and [G3] Germain wrote down a $*$-homomorphism $\varphi : C \to S(A_1 *_B A_2)$ between the mapping cone C for the inclusion $B \to A_1 \oplus A_2$ and the suspension $S(A_1 *_B A_2)$ of $A_1 *_B A_2$, and made the observation that the conjecture is equivalent to the KK-invertibility of φ. He was then able to invert φ in KK-theory when $B = \mathbb{C}$ under the assumption on A_1 and A_2 mentioned above. The method of proof that we shall use is in principle the same, but the goal - to invert φ in KK-theory - is achieved by completely different means. In fact, we shall obtain the proof by working with extension groups in much

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the same way as in the work of L. Brown, [Br], who obtained partial results which inspired Cuntz in the formulation of the conjecture, cf. Remark 2 of [C1]. By working with extensions we shall establish enough of the desired exact sequences to deduce that Germains homomorphism is invertible in KK-theory. To do this we use two important ingredients which were not available when Brown did his work, namely Boca's result on free products of completely positive unital maps, [Bo], and the automatic existence of absorbing trivial extensions together with the related duality results for KK-theory obtained in full generality by the author in [Th1].

A major part of the paper is an attack on the analogous conjecture in the E-theory of Connes and Higson, [CH], and we obtain the desired six terms exact sequences under even weaker conditions in this setting, as described in the abstract. The approach we take for this is new: Provided B is properly embedded in both A_k's, meaning that an approximate unit in B is also an approximate unit in A_k, there is an exact sequence

$$0 \longrightarrow S(A_1 *_B A_2) \longrightarrow \operatorname{cone}(A_1) *_{SB} \operatorname{cone}(A_2) \longrightarrow A_1 * A_2 \longrightarrow 0, \qquad (1.1)$$

where $A_1 * A_2$ is the unrestricted free product. As shown by Cuntz, [C3], $A_1 * A_2$ is KK-equivalent to $A_1 \oplus A_2$. Based on methods and results from [DE] and [Th2] we show here that $\operatorname{cone}(A_1) *_{SB} \operatorname{cone}(A_2)$ is equivalent to B in E-theory provided there are sequences of completely positive contractions $R^n_k : A_k \to B$ such that $\lim_{n\to\infty} R^n_k(b) = b$ for all $b \in B, k = 1, 2$. The desired exact sequences then come up as the E-theory exact sequences arising from (1.1). Note that the extension (1.1) is actually semi-split (this follows from Boca's result, [Bo]), so it is not inconceivable that this extension can be used to obtain the result in KK-theory rather than E-theory. However, the methods that we use here to show that $\operatorname{cone}(A_1) *_{SB} \operatorname{cone}(A_2)$ is equivalent to B works only in E-theory. In a final section we point out a serious limitation of our methods which explains why they stop short off a proof in the general case.

2. On absorbing extensions and asymptotic homomorphisms

In this section we gather a series of lemmas. Only the first two are needed for our results in KK-theory. Let A and D be separable C^*-algebras, D stable. Let $M(D)$ denote the multiplier algebra of D. Since D is stable there are isometries $V_1, V_2 \in M(D)$ such that $V_1V_1^* + V_2V_2^* = 1$ and $V_1^*V_2 = 0$ and we can define the orthogonal sum $a \oplus b$ of two elements $a, b \in M(D)$ to be $V_1aV_1^* + V_2bV_2^*$. Similarly, we can add maps, $\varphi, \psi : A \to M(D)$, orthogonally; viz. $(\varphi \oplus \psi)(a) = V_1\varphi(a)V_1^* + V_2\psi(a)V_2^*$. We call a $*$-homomorphism $\varphi : A \to M(D)$ absorbing when the following holds:

When $\pi : A \to M(D)$ is a $*$-homomorphism, there is a sequence of unitaries $\{U_n\} \subseteq M(D)$ such that $\lim_{n\to\infty} U_n(\varphi \oplus \pi)(a)U_n^* - \varphi(a) = 0$ for all $a \in A$.

See Theorem 2.5 of [Th1] for alternative characterizations of absorbing $*$-homomorphisms which we shall use quite freely. By Theorem 2.7 of [Th1] there always exists an absorbing $*$-homomorphism.

Lemma 2.1. *Let A, D be separable C^*-algebras, D stable and $B \subseteq A$ a C^*-subalgebra of A. Assume that that there is a sequence of completely positive contractions $R_n : A \to B$ such that $\lim_n R_n(b) = b, b \in B$. If $\pi : A \to M(D)$ is an absorbing $*$-homomorphism, then $\pi|_B : B \to M(D)$ is an absorbing $*$-homomorphism.*

Proof. By Theorem 2.5 of [Th1] we must show that the unitization $(\pi|_B)^+ : B^+ \to M(D)$ of $\pi|_B$ is unitally absorbing. We check that condition 1) of Theorem 2.1 of [Th1] is satisfied. Consider therefore a completely positive contraction $\varphi : B^+ \to D$. Then $\varphi \circ R_k^+ : A^+ \to D$ is also a completely positive contraction and since $\pi^+ : A^+ \to M(D)$ is unitally absorbing, we know that there is a sequence $\{W_n^k\} \subseteq M(D)$ such that $\lim_{n\to\infty} \|\varphi \circ R_k^+(a) - W_n^{k*}\pi^+(a)W_n^k\| = 0$ for all $a \in A^+$ and $\lim_{n\to\infty} \|W_n^{k*}d\| = 0$ for all $d \in D$. Since $\lim_{k\to\infty} R_k^+(b) = b$ for all $b \in B^+ \subseteq A^+$, it follows that also $(\pi|_B)^+ = \pi^+|_{B^+}$ satisfies condition 1). □

Lemma 2.2. *Let A, D be separable C^*-algebras, D stable and $B \subseteq A$ a C^*-subalgebra of A. Assume that B is nuclear. If $\pi : A \to M(D)$ is an absorbing $*$-homomorphism, then $\pi|_B : B \to M(D)$ is an absorbing $*$-homomorphism.*

Proof. Since B is nuclear there are sequences $S_n : B \to F_n,\ T_n : F_n \to B, n \in \mathbb{N}$, of completely positive contractions, where the F_n's are finite dimensional C^*-algebras, such that $\lim_{n\to\infty} T_n \circ S_n(b) = b$ for all $b \in B$. By Arvesons extension theorem, [A1], there is for each n a completely positive contraction, $V_n : A \to F_n$, extending S_n. Set $R_n = T_n \circ V_n$ and apply Lemma 2.1. □

Throughout the rest of the paper A_1, A_2, B, D are separable C^*-algebras with D stable, and $i_k : B \to A_k, k = 1, 2$, are embeddings.

Lemma 2.3. *Assume that there are sequences $R_n^k : A_k \to B, n = 1, 2, 3, \cdots$, of completely positive contractions such that $\lim_{n\to\infty} R_n^k(i_k(b)) = b$ for all $b \in B, k = 1, 2$. Let $\pi_k : A_k \to M(D), k = 1, 2$, be saturated and absorbing $*$-homomorphisms. It follows that there is a normcontinuous path $\{u_t\}_{t\in[1,\infty)}$ of unitaries in $M(D)$ such that $u_t\pi_1 \circ i_1(b)u_t^* - \pi_2 \circ i_2(b) \in D$ for all $t \in [1, \infty), b \in B$, and $\lim_{t\to\infty} u_t\pi_1 \circ i_1(b)u_t^* - \pi_2 \circ i_2(b) = 0$ for all $b \in B$.*

Proof. Recall from [Th2] that π_k being saturated means that the infinite direct sum $0 \oplus \pi_k \oplus \pi_k \oplus \pi_k \oplus \cdots$ is unitarily equivalent to π_k. It follows from Lemma 2.1 that $\pi_k \circ i_k : B \to M(D), k = 1, 2$, are both absorbing (and saturated). From the uniqueness of absorbing $*$-homomorphisms it follows that $\pi_1 \circ i_1 \oplus (\pi_2 \circ i_2)_\infty \sim \pi_1 \circ i_1$ and $(\pi_1 \circ i_1)_\infty \oplus \pi_2 \circ i_2 \sim \pi_2 \circ i_2$, in the notation of [DE]. It follows therefore from Lemma 2.4 of [DE] that there is a normcontinuous path $\{w_t\}_{t\in[1,\infty)}$ of unitaries in $M(D)$ such that $w_t(\pi_1 \circ i_1)_\infty(b)w_t^* - (\pi_2 \circ i_2)_\infty(b) \in D$ for all t, b, and $\lim_{t\to\infty} w_t(\pi_1 \circ i_1)_\infty(b)w_t^* - (\pi_2 \circ i_2)_\infty(b) = 0$ for all $b \in B$. Since π_k is saturated, π_k is unitarily equivalent to $(\pi_k)_\infty$, so the conclusion follows. □

In the following we shall consider the suspensions SA_1, SA_2, SB and the cones $\mathrm{cone}(A_1), \mathrm{cone}(A_2), \mathrm{cone}(B)$. The embeddings $i_k : B \to A_k, k = 1, 2$, induce embeddings between some of these algebras (e.g. $SB \to \mathrm{cone}(A_k)$) in a natural way, and in order to avoid too heavy notation we shall denote a map induced by $i_k : B \to A_k$ by i_k again. It will always be clear from the context which domain and target is meant.

Lemma 2.4. *Assume that there are sequences $R_n^k : A_k \to B, n = 1, 2, 3, \cdots$, of completely positive contractions such that $\lim_{n\to\infty} R_n^k(i_k(b)) = b$ for all $b \in B, k = 1, 2$. There exist absorbing and saturated $*$-homomorphisms $\alpha_k : \mathrm{cone}(A_k) \to M(D), k = 1, 2$, such that $\alpha_k \circ i_k : \mathrm{cone}(B) \to M(D), k = 1, 2$, are both absorbing and saturated, and there exist normcontinuous paths, $\{p_t\}_{t\in[0,\infty)}$, $\{w_t\}_{t\in[0,\infty)}$, of elements in $M(D)$ such that the w_t's are unitaries, and*

1) $0 \leq p_t \leq 1, \quad t \in [0, \infty)$,
2) $p_t \alpha_k(\operatorname{cone}(A_k)) \subseteq D \ , \quad t \in [0,\infty), \ k = 1,2$,
3) $(p_t^2 - p_t)\alpha_k(\operatorname{cone}(A_k)) \ = \ \{0\} \ , \quad t \in [0,\infty)$,
4) $\lim_{t\to\infty} p_t d \ = \ d \ , \quad d \in D$,
5) $\lim_{t\to\infty} \|p_t\alpha_k(a) - \alpha_k(a)p_t\| \ = \ 0 \ , \quad a \in \operatorname{cone}(A_k), \ k = 1,2$,
6) $p_0 = 0, \ p_n^2 = p_n, \ n = 1,2,3,\cdots$,
7) $\lim_{t\to\infty} w_t\alpha_1 \circ i_1(b)w_t^* - \alpha_2 \circ i_2(b) = 0$ *for all* $b \in \operatorname{cone}(B)$,
8) $\lim_{t\to\infty} p_t w_t - w_t p_t = 0$.

Except for 7) and 8), Lemma 2.4 follows from Theorem 3.7 of [Th2]. To obtain 7) and 8), which will be crucial for us here, we must elaborate on the proof of Theorem 3.7 of [Th2] as follows.

Lemma 2.5. *Let D be a separable C^*-algebra. Let $K_1 \subseteq K_2 \subseteq M(D)$ and $F \subseteq D$ be compact subsets. Let $\delta > 0$ and assume that $p \in M(D)$ is a projection such that $[p,m] \in D, \ m \in K_2$, and*

$$\|mp - pm\| < \delta \quad , \quad m \in K_1. \tag{2.1}$$

Let $0 \leq z \leq 1$ be a strictly positive element in $(1-p)D(1-p)$ and let $\epsilon_1, \ \epsilon_2 \in \,]0,1[$ be given. There is then a continuous function $h : [0,1] \to [0,1]$ such that h is zero in a neighbourhood of 0, $h(t) = 1, \ t \geq \epsilon_1$,

$$\sup_{t\in[0,1]} \|[m, p + h(tz)]\| \ < \ 5\delta \ , \qquad m \in K_1 \ , \tag{2.2}$$

$$\|[m, p + h(z)]\| \ < \ \epsilon_2 \ , \qquad m \in K_2 \ , \tag{2.3}$$

and

$$\|pd + h(z)d - d\| \ < \ \epsilon_2 \ , \ d \in F \ . \tag{2.4}$$

Proof. Let Λ denote the convex set of continuous functions $H : [0,1] \to [0,1]$ such that H is zero in a neighbourhood of 0 and $H(t) = 1, \ t \geq \epsilon_1$. For each $x \in K_2$ define a multiplier $\tilde{x}$ of $\operatorname{cone}((1-p)D(1-p))$ by $(\tilde{x}f)(t) \ = \ (1-p)x(1-p)f(t) \ , \ t \in [0,1]$, and define $\tilde{H} \in \operatorname{cone}((1-p)D(1-p))$ by $\tilde{H}(t) \ = \ H(tz)$. Since $t \mapsto tz$ is a strictly positive element of $\operatorname{cone}((1-p)D(1-p))$, $\{(\tilde{H}, \ p + H(z))\}_{H\in\Lambda}$ is a convex net in $\operatorname{cone}((1-p)D(1-p)) \oplus M(D)$ such that

$$\lim_{H\in\Lambda} (\tilde{H}, \ p + H(z))X = X$$

for all $X \in \operatorname{cone}((1-p)D(1-p)) \oplus D$. Since $[m,p] \in D$ for all $m \in K_2$ we can therefore use the arguments from the proof of the existence of quasi-central approximate units to find a $h \in \Lambda$ such that $\|[(\tilde{x}, y), (\tilde{h}, p + h(z))]\| \ < \min\{\delta, \epsilon_2\}, \ x \in K_1, \ y \in K_2$, and $\|pd + h(z)d - d\| < \epsilon_2, \ d \in F$. In particular (2.3) and (2.4) hold and we have that

$$\sup_{t\in[0,1]} \|[(1-p)x(1-p), h(tz)]\| \ < \ \delta, \ x \in K_1 \ . \tag{2.5}$$

Since $[x, h(tz)] = [(1-p)x(1-p), h(tz)] + [(1-p)xp, h(tz)] + [px(1-p), h(tz)]$, we get (2.2) by combining (2.5) with (2.1).

$\square$

Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space. We can then define $g : [0,\infty[\to [0,2]$ by

$$g(s) \ = \ \sup\{\|[a, \sqrt{x}]\| \ : \ a, x \in \mathcal{B}(\mathcal{H}), \ \|a\| \leq 1, \ 0 \leq x \leq 1, \ \|[a,x]\| \leq s\} \ .$$

By the lemma on page 332 of [A2], g is continuous at 0, i.e. $\lim_{s\to 0} g(s) = 0$. g will feature in the next lemma. In that lemma we introduce the notation 0_n for the zero in the n-by-n matrices over a C^*-algebra.

Lemma 2.6. *Let A and D be separable C^*-algebras with A contractible. Let $\varphi_t : A \to A,\ t \in [0,1]$, be a homotopy of endomorphisms of A such that $\varphi_0 = \mathrm{id}$ and $\varphi_1 = 0$, and let $L \subseteq M(D)$ be a fixed subset. Let $F_0 \subseteq F_1 \subseteq A$ and $K \subseteq D,\ G_0 \subseteq G_1 \subseteq L$ be compact subsets. Let $\pi : A \to M(D)$ be a $*$-homomorphism and $p \in M(D)$ a projection such that $p\pi(A) \subseteq D,\ [p,m] \in D,\ m \in L,\ \|p\pi(\varphi_t(a)) - \pi(\varphi_t(a))p\| < \kappa,\ a \in F_0,\ t \in [0,1]$, and $\|pm - mp\| < \kappa,\ m \in G_0$, for some $\kappa > 0$.*

For any $\epsilon > 0$ there is an $n \in \mathbb{N}$, a $*$-homomorphism $\pi_1 : A \to M(M_n(D))$ of the form

$$\pi_1(a) = \mathrm{diag}(\pi(\varphi_{s_1}(a)), \pi(\varphi_{s_2}(a)), \cdots, \pi(\varphi_{s_n}(a)))$$

for some $s_1, s_2, \cdots, s_n \in [0,1],\ s_1 = 0, s_n = 1$, and a normcontinuous path $p_t,\ t \in [0,1]$, of elements $p_t \in M(M_{n+1}(D))$ such that

1) $0 \le p_t \le 1,\ t \in [0,1]$,
2) $(p_t^2 - p_t)\begin{pmatrix} \pi(a) & \\ & \pi_1(a) \end{pmatrix} = 0 , \quad a \in A,\ t \in [0,1]$,
3) $p_t \begin{pmatrix} \pi(a) & \\ & \pi_1(a) \end{pmatrix} \in M_{n+1}(D) , \quad a \in A,\ t \in [0,1]$,
4) $\|p_t \begin{pmatrix} \pi(a) & \\ & \pi_1(a) \end{pmatrix} - \begin{pmatrix} \pi(a) & \\ & \pi_1(a) \end{pmatrix} p_t\| \le 6g(20\kappa) + 3\kappa , \quad a \in F_0,\ t \in [0,1]$,
5) $\begin{pmatrix} p & \\ & 0_n \end{pmatrix} \le p_t,\ t \in [0,1]$,
6) $\|p_1 \begin{pmatrix} \pi(\varphi_t(a)) & \\ & \pi_1(\varphi_t(a)) \end{pmatrix} - \begin{pmatrix} \pi(\varphi_t(a)) & \\ & \pi_1(\varphi_t(a)) \end{pmatrix} p_1\| \le \epsilon , \quad a \in F_1,\ t \in [0,1]$,
7) $\|p_1 \begin{pmatrix} d & \\ & 0_n \end{pmatrix} - \begin{pmatrix} d & \\ & 0_n \end{pmatrix}\| \le \epsilon ,\ d \in K$,
8) $p_1 = p_1^2,\ p_0 = \begin{pmatrix} p & \\ & 0_n \end{pmatrix}$,
9) $[(1_{M_{n+1}(\mathbb{C})} \otimes m), p_t] \in M_{n+1}(D),\ m \in L,\ t \in [0,1]$,
10) $\|[(1_{M_{n+1}(\mathbb{C})} \otimes m), p_t]\| \le 6g(20\kappa) + 3\kappa ,\ m \in G_0,\ t \in [0,1]$,
11) $\|[p_1, (1_{M_{n+1}(\mathbb{C})} \otimes m)]\| \le \epsilon ,\ m \in G_1$.

Proof. The proof is an elaboration of Voiculescus proof of Proposition 3 in [V]. Let $\delta > 0$ be so small that $6g(4\delta)+3\delta < \epsilon,\ \delta < \kappa$ and $\delta + \sqrt{\|d\|\delta} < \epsilon$ for all $d \in K$. Choose n so large that $t, s \in [0,1],\ |s-t| \le (n-1)^{-1} \Rightarrow \|\varphi_t(a) - \varphi_s(a)\| < \delta,\ a \in F_1$. Let $0 \le z \le 1$ be a strictly positive element in $(1-p)D(1-p)$. It follows from Lemma 2.5 that there are continuous functions $g_i : [0,1] \to [0,1],\ i = 0, 1, \cdots, n-1$, which are all zero in a neighbourhood of 0 such that $g_j g_{j-1} = g_{j-1},\ j = 1, 2, \cdots, n-1$, and such that the elements $x_j = p + g_j(z)$ and $x_j^t = p + g_j(tz)$ satisfy that

$$\|x_j m - m x_j\| < \delta,\ m \in G_1, \tag{2.6}$$

$$\|x_j \pi \circ \varphi_s(a) - \pi \circ \varphi_s(a) x_j\| < \delta , a \in F_1, \tag{2.7}$$

$$\|x_j^t \pi \circ \varphi_s(a) - \pi \circ \varphi_s(a) x_j^t\| < 5\kappa,\ a \in F_0, \tag{2.8}$$

and

$$\|x_j^t m - m x_j^t\| < 5\kappa,\ m \in G_0, \tag{2.9}$$

for all s, t and all $j = 0, 1, 2, \cdots, n-1$, and $\|x_0 d - d\| \le \delta,\ d \in K$. Set $s_j = \frac{j-1}{n-1},\ j = 1, 2, \cdots, n$, and $\pi_1 = \mathrm{diag}(\pi \circ \varphi_{s_1}, \pi \circ \varphi_{s_2}, \cdots, \pi \circ \varphi_{s_n})$. Let

$$p_t = \begin{pmatrix} p & & \\ & 0_{n-1} & \\ & & 2t(1-p) \end{pmatrix} , \quad t \in [0, \frac{1}{2}] .$$

Then 1)-5), 9) and 10) hold trivially for $t \in [0, \frac{1}{2}]$. Note that $x_i^t x_{i-1}^t = x_{i-1}^t$, $i = 1, \cdots, n-1$. Set $X_t^0 = x_0^{2t-1}$, $X_t^j = x_j^{2t-1} - x_{j-1}^{2t-1}$, $j = 1, 2, \cdots, n-1$, and $X_t^n = 1 - x_{n-1}^{2t-1}$, $t \in [\frac{1}{2}, 1]$. Define $T_t \in M(M_{n+1}(D))$, $t \in [\frac{1}{2}, 1]$, by

$$T_t = \begin{pmatrix} \sqrt{X_t^0} & 0 & \dots & 0 \\ \sqrt{X_t^1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{X_t^n} & 0 & \dots & 0 \end{pmatrix} .$$

Then $T_t T_t^*$ is a projection since $T_t^* T_t$ clearly is. Since $T_{\frac{1}{2}} T_{\frac{1}{2}}^* = p_{\frac{1}{2}}$ we can extend p_t, $t \in [0, \frac{1}{2}]$, to a continuous path in $M(M_{n+1}(D))$ by setting $p_t = T_t T_t^*$, $t \in [\frac{1}{2}, 1]$. Then 1) and 2) clearly hold and 3) follows from the observation that

$$\begin{pmatrix} \pi(a) & \\ & \pi_1(a) \end{pmatrix} T_t \subseteq M_{n+1}(D) \quad , \quad a \in A,\ t \in [\frac{1}{2}, 1] .$$

It follows from (2.7) and (2.8), by using that $T_t T_t^*$ is tri-diagonal as in the proof of Proposition 3 in [V], that

$$\|[p_1, \begin{pmatrix} \pi(\varphi_s(a)) & \\ & \pi_1(\varphi_s(a)) \end{pmatrix}]\| \leq 6g(4\delta) + 3\delta \leq \epsilon,\ a \in F_1\ s \in [0, 1],$$

and

$$\|[p_t, \begin{pmatrix} \pi(a) & \\ & \pi_1(a) \end{pmatrix}]\| \leq 6g(20\kappa) + 3\kappa\ ,\ a \in F_0\ ,\ t \in [\frac{1}{2}, 1]\ ,$$

i.e. 4) and 6) hold. 10) and 11) follow in the same way. 5) is trivial when $t \in [0, \frac{1}{2}]$ and for $t > \frac{1}{2}$ it follows from the observation that

$$\begin{pmatrix} p & \\ & 0_n \end{pmatrix} T_t = \begin{pmatrix} p & \\ & 0_n \end{pmatrix} \ , \quad \begin{pmatrix} p & \\ & 0_n \end{pmatrix} T_t^* = \begin{pmatrix} p & \\ & 0_n \end{pmatrix} .$$

It is straightforward to check that $\|p_1 \begin{pmatrix} d & \\ & 0_n \end{pmatrix} - \begin{pmatrix} d & \\ & 0_n \end{pmatrix}\| \leq \|X_1^0 d - d + \sqrt{X_1^1}\sqrt{X_1^0} d\| \leq \delta + \sqrt{\|d\|\delta}$ when $d \in K$, and 7) holds. 8) is trivial and 9) is a consequence of the construction of p_t and the assumption that $[m, p] \in D$, $m \in L$.

□

Proof. (of Lemma 2.4) We apply first Lemma 2.3 to obtain saturated and absorbing $*$-homomorphisms $\Theta_k : \mathrm{cone}(A_k) \to M(D), k = 1, 2$, and a normcontinuous path $\{u_t\}_{t \in [1,\infty)}$ of unitaries in $M(D)$ such that $\lim_{t\to\infty} u_t \Theta_1 \circ i_1(b) u_t^* - \Theta_2 \circ i_2(b) = 0$ for all $b \in \mathrm{cone}(B)$. Define $\Theta : \mathrm{cone}(A_1) \oplus \mathrm{cone}(A_2) \to M(D)$ by $\Theta(a_1, a_2) = \Theta_1(a_1) \oplus \Theta_2(a_2)$. There is then a normcontinuous path $\{v_t\}_{t \in [0,\infty)}$ of unitaries in $M(D)$ such that $v_t \Theta(i_1(b), 0) v_t^* - \Theta(0, i_2(b)) \in D$ for all $t \in [0, \infty), b \in \mathrm{cone}(B)$, and $\lim_{t\to\infty} v_t \Theta(i_1(b), 0) v_t^* - \Theta(0, i_2(b)) = 0$ for all $b \in \mathrm{cone}(B)$. Let $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$ and $G_1 \subseteq G_2 \subseteq G_3 \subseteq \cdots$ be sequences of finite sets with dense union in $\mathrm{cone}(A_1) \oplus \mathrm{cone}(A_2)$ and D, respectively. By using Lemma 2.6 with $L = \{v_s : s \in [0, \infty)\}$ we can construct a sequence $1 = n_0 < n_1 < n_2 < \cdots$ of natural numbers, paths $p_i(t)$, $t \in [i, i+1]$, in $M_{n_i}(M(D))$, $i = 0, 1, 2, \cdots$, and $*$-homomorphisms $\widetilde{\pi}_i : \mathrm{cone}(A_1) \oplus \mathrm{cone}(A_2) \to M_{n_i - n_{i-1}}(M(D))$, $i = 1, 2, \cdots$, such that $\pi_0 = \Theta$ and $\pi_i = \pi_{i-1} \oplus \widetilde{\pi}_i : \mathrm{cone}(A_1) \oplus \mathrm{cone}(A_2) \to M_{n_i}(M(D))$, $i = 1, 2, \cdots$, satisfy

1) $0 \leq p_i(t) \leq 1$, $t \in [i, i+1]$, $i = 0, 1, 2, \cdots$,
2) $\|p_i(t)\pi_i(a) - \pi_i(a)p_i(t)\| \leq \frac{1}{i}$, $a \in F_i$, $t \in [i, i+1]$, $i = 0, 1, 2, \cdots$,
3) $p_i(t)\pi_i(\mathrm{cone}(A_1) \oplus \mathrm{cone}(A_2)) \subseteq M_{n_i}(D)$, $t \in [i, i+1]$, $i = 0, 1, 2, \cdots$,

4) $\|p_{i+1}(t)\begin{pmatrix} d & \\ & 0_{n_i-n_{i-1}}\end{pmatrix} - \begin{pmatrix} d & \\ & 0_{n_i-n_{i-1}}\end{pmatrix}\| \leq \frac{1}{i}$ when all the entries of $d \in M_{n_{i-1}}(D)$ come from G_i, $t \in [i+1, i+2]$, $i = 1,2,3,\cdots$,
5) $(p_i(t)^2 - p_i(t))\pi_i(\mathrm{cone}(A_1) \oplus \mathrm{cone}(A_2)) = \{0\}$, $t \in [i, i+1]$, $i = 0,1,2,\cdots$,
6) $p_i(i+1) = p_i(i+1)^2$, $p_i(i) = \begin{pmatrix} p_{i-1}(i) & 0 \\ 0 & 0_{n_i-n_{i-1}}\end{pmatrix}$, $i = 1,2,3,\cdots$,
7) $\|p_i(t)(1_{M_{n_i}(\mathbb{C})} \otimes v_s) - (1_{M_{n_i}(\mathbb{C})} \otimes v_s)p_i(t)\| \leq \frac{1}{i}$, $t \in [i, i+1], s \in [0, i+1], i = 0,1,2,\cdots$,
8) $[p_i(t), (1_{M_{n_i}(\mathbb{C})} \otimes v_s)] \in M_{n_i}(D)$, $t \in [0,1]$, $s \in [0,\infty)$,

and $p_0 = 0$. Note that thanks to the way π_1 is constructed in Lemma 2.6 we find that $\widetilde{\pi_i}$ has the form $\widetilde{\pi_i} = \pi_{i-1} \oplus \varphi_i \oplus 0_{n_{i-1}}$ for some $*$-homomorphism $\varphi_i : \mathrm{cone}(A_1) \oplus \mathrm{cone}(A_2) \to M_{n_i - 2n_{i-1}}(M(D))$, and that

$$\lim_{t\to\infty} \sup_i \|(1_{M_{n_i}(\mathbb{C})} \otimes v_t)\pi_i(i_1(b), 0)(1_{M_{n_i}(\mathbb{C})} \otimes v_t^*) - \pi_i(0, i_2(b))\| = 0$$

for all $b \in \mathrm{cone}(B)$. Now define $\varphi' : \mathrm{cone}(A_1) \oplus \mathrm{cone}(A_2) \to \mathbb{L}_D(l_2(D))$ by $\varphi'(a) = \mathrm{diag}(\Theta(a), \widetilde{\pi_1}(a), \widetilde{\pi_2}(a), \widetilde{\pi_3}(a), \cdots)$, and set

$$p'_t = \begin{pmatrix} p_i(t) & \\ & 0_\infty \end{pmatrix} , \quad t \in [i, i+1] , \ i = 0,1,2,\cdots ,$$

and $w'_t = \mathrm{diag}(v_t, v_t, v_t, \cdots)$, $t \in [0,\infty)$. φ' is unitarily equivalent to a $*$-homomorphism $\pi : \mathrm{cone}(A_1) \oplus \mathrm{cone}(A_2) \to M(D)$ since $l_2(D) \simeq D$ as Hilbert D-modules. Via the isomorphism $l_2(D) \simeq D$, p' and w' become paths in $M(D)$ which satisfy 1)-8) relative to $\alpha_1(a) = \pi(a, 0)$ and $\alpha_2(a) = \pi(0, a)$ in the statement of the lemma. α_k and $\alpha_k \circ i_k, k = 1,2$, are all absorbing because Θ_1 and Θ_2 are, and they are also all saturated because each π_i occurs infinitely often as a direct summand in the sequence $\widetilde{\pi_1}, \widetilde{\pi_2}, \widetilde{\pi_3}, \cdots$. □

Lemma 2.7. *Assume that there are sequences $R^k_n : A_k \to B, n = 1,2,3,\cdots, k = 1,2$, of completely positive contractions such that $\lim_{n\to\infty} R^k_n(i_k(b)) = b$ for all $x \in B, k = 1,2$. For any pair of asymptotic homomorphisms $\varphi, \psi : \mathrm{cone}(B) \to D$, there are asymptotic homomorphisms $\mu^k : \mathrm{cone}(A_k) \to D, k = 1,2$, such that $\lim_{t\to\infty} \mu^1_t \circ i_1(x) - \mu^2_t \circ i_2(x) = 0, x \in \mathrm{cone}(B)$, and a normcontinuous path of unitaries $\{W_t\}_{t\in[1,\infty)}$ in $M_2(D)^+$ such that*

$$\lim_{t\to\infty} W_t \begin{pmatrix} \varphi_t(x) & \\ & \mu^1_t \circ i_1(x)\end{pmatrix} W_t^* - \begin{pmatrix} \psi_t(x) & \\ & \mu^1_t \circ i_1(x)\end{pmatrix} = 0$$

for all $x \in \mathrm{cone}(B)$.

Proof. Let $\tilde{\psi}, \tilde{\varphi} : \mathrm{cone}(B) \to C_b([1,\infty), D)/C_0([1,\infty), D)$ be the $*$-homomorphisms arising from ψ and φ, respectively. By Lemma 2.6 of [Th2] there is a stable separable C^*-algebra $D_0 \subseteq C_b([1,\infty), D)/C_0([1,\infty), D)$ such that $\tilde{\psi}(\mathrm{cone}(B)) \cup \tilde{\varphi}(\mathrm{cone}(B)) \subseteq D_0$. For any C^*-algebra X, define $p : \mathrm{cone}(X) \to \mathrm{cone}(X)$ and $s : \mathrm{cone}(X) \to SX \subseteq \mathrm{cone}(X)$ by $p(f)(t) = f(\frac{t}{2})$ and

$$s(f)(t) = \begin{cases} f(2t), & t \in [0, \frac{1}{2}] \\ f(2-2t), & t \in [\frac{1}{2}, 1] \end{cases} .$$

Note that $\tilde{\psi} \circ p|_{SB}$ and $\tilde{\varphi} \circ p|_{SB}$ both represent zero in $[[SB, D_0]]_{cp}$. Let p_t, w_t, α_1 and α_2 be as in Lemma 2.4, relative to D_0. By Theorem 4.1 in [Th2] there is an increasing continuous function $r : [1,\infty) \to [1,\infty)$ with $\lim_{t\to\infty} r(t) = \infty$ and a normcontinuous path $\{S_t\}_{t\in[1,\infty)}$ of unitaries in $M_2(D_0)^+$ such that

$$\lim_{t\to\infty} S_t \begin{pmatrix} \tilde{\varphi}\circ p(x) & \\ & p_{r(t)}\alpha_1\circ i_1(x)p_{r(t)} \end{pmatrix} S_t^* - \begin{pmatrix} \tilde{\psi}\circ p(x) & \\ & p_{r(t)}\alpha_1\circ i_1(x)p_{r(t)} \end{pmatrix} = 0$$

for all $x \in SB$. Set

$$T_t = \begin{pmatrix} 1 & \\ & w_{r(t)} \end{pmatrix} S_t \begin{pmatrix} 1 & \\ & w_{r(t)}^* \end{pmatrix}$$

and $l_t^1(\cdot) = w_{r(t)}p_{r(t)}\alpha_1(\cdot)p_{r(t)}w_{r(t)}^*$, $l_t^2(\cdot) = p_{r(t)}\alpha_2(\cdot)p_{r(t)}$. Then $l^k : \mathrm{cone}(A_k) \to D_0, k = 1,2$, are asymptotic homomorphisms such that $\lim_{t\to\infty} l_t^1\circ i_1(b) - l_t^2\circ i_2(b) = 0$ for all $b \in \mathrm{cone}(B)$ and

$$\lim_{t\to\infty} T_t \begin{pmatrix} \tilde{\varphi}\circ p(b) & \\ & l_t^1\circ i_1(b) \end{pmatrix} T_t^* - \begin{pmatrix} \tilde{\psi}\circ p(b) & \\ & l_t^1\circ i_1(b) \end{pmatrix} = 0$$

for all $b \in SB$. Note that $\{T_t\}_{t\in[1,\infty)}$ is a normcontinuous path of unitaries in $M_2(D_0)^+$. Let $\chi : C_b([1,\infty), D^+)/C_0([1,\infty), D^+) \to C_b([1,\infty), D^+)$ be a continuous right-inverse for the quotient map $q : C_b([1,\infty), D^+) \to C_b([1,\infty), D^+)/C_0([1,\infty), D^+)$. Standard arguments give us a normcontinuous path of unitaries $\{V_t\}$ in $M_2(D)^+$ such that $(\mathrm{id}_{M_2} \otimes q)(V_t) = T_t$. Set $\nu_t^k(x) = \chi(l^k_{s^{-1}(t)}(x))(t)$ for a sufficiently rapidly increasing continuous bijection $s : [1,\infty) \to [1,\infty)$. If s increases fast enough, this will give us asymptotic homomorphisms $\nu^k : \mathrm{cone}(A_k) \to D, k = 1,2$, such that $\lim_{t\to\infty} \nu_t^1 \circ i_1(x) - \nu_t^2 \circ i_2(x) = 0, x \in \mathrm{cone}(B)$, and a normcontinuous path of unitaries $W_t = V_{s^{-1}(t)}(t)$ in $M_2(D)^+$ such that

$$\lim_{t\to\infty} W_t \begin{pmatrix} \varphi_t\circ p(x) & \\ & \nu_t^1\circ i_1(x) \end{pmatrix} W_t^* - \begin{pmatrix} \psi_t\circ p(x) & \\ & \nu_t^1\circ i_1(x) \end{pmatrix} = 0$$

for all $x \in SB$. Set $\mu^k = \nu^k \circ s : \mathrm{cone}(A_k) \to D, k = 1,2$. Then $\mu^k \circ i_k = \nu^k \circ s \circ i_k = \nu^k \circ i_k \circ s$ and hence $\lim_{t\to\infty} \mu_t^1 \circ i_1(x) - \mu_t^2 \circ i_2(x) = 0, x \in \mathrm{cone}(B)$, and

$$\lim_{t\to\infty} W_t \begin{pmatrix} \varphi_t(x) & \\ & \mu_t^1\circ i_1(x) \end{pmatrix} W_t^* - \begin{pmatrix} \psi_t(x) & \\ & \mu_t^1\circ i_1(x) \end{pmatrix} = 0$$

for all $x \in \mathrm{cone}(B)$. □

3. Results in KK-theory

Assume now that A_1, A_2, B and $i_k : B \to A_k, k = 1,2$, are all unital. Let $j_k : A_k \to A_1 *_B A_2, k = 1,2$, be the natural maps. The basic assumption in this section is that there is an absorbing $*$-homomorphism $\alpha : A_1 *_B A_2 \to M(D)$ such that $\alpha \circ j_k$ and $\alpha \circ j_k \circ i_k, k = 1,2$, are all absorbing. (Of course, $\alpha \circ j_1 \circ i_1 = \alpha \circ j_2 \circ i_2$.) This is the case when either,

a) there are surjective conditional expectations $P_k : A_k \to B, k = 1,2$,

or

b) A_1, A_2 and B are all nuclear.

Indeed, in case a) it follows from [Bo] that there are also conditional expectations $\mathrm{id}_{A_1} *_B P_2 : A_1 *_B A_2 \to A_1$ and $P_1 *_B \mathrm{id}_{A_2} : A_1 *_B A_2 \to A_2$. Hence by Lemma 2.1 any absorbing $*$-homomorphism $\alpha : A_1 *_B A_2 \to M(D)$ will have the desired property. And α exists by Theorem 2.7 of [Th1]. In case b) it suffices to use Lemma 2.2 instead, plus the non-trivial fact that $j_k : A_k \to A_1 *_B A_2, k = 1,2$, are injective, see Theorem 3.1 of [Bl] or Theorem 4.2 of [P2].

This α will be fixed throughout this section. To simplify notation we set $\alpha_k = \alpha \circ j_k, k = 1, 2$. Set

$$\begin{aligned}
\mathcal{A}_k &= \{x \in M(D) : x\alpha_k(a) - \alpha_k(a)x \in D,\ a \in A_k\},\\
\mathcal{B}_k &= \{x \in \mathcal{A}_k : x\alpha_k(a) \in D,\ a \in A_k\},\\
\mathcal{A} &= \{x \in M(D) : x\alpha_1 \circ i_1(b) - \alpha_1 \circ i_1(b)x \in D,\ b \in B\},\\
\mathcal{B} &= \{x \in \mathcal{A} : x\alpha_1 \circ i_1(b) \in D,\ b \in B\}.
\end{aligned}$$

Obviously, $\mathcal{A}_k \subseteq \mathcal{A}, k = 1, 2$, and since A_1, A_2, B share the same unit, we see that $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}$. Hence $\mathcal{A}_k/\mathcal{B}_k \subseteq \mathcal{A}/\mathcal{B}, k = 1, 2$. By Theorem 3.2 of [Th1] we can make the following identifications

$$\begin{aligned}
KK(A_k, D) &= K_1(\mathcal{A}_k/\mathcal{B}_k),\ k = 1, 2,\\
KK(B, D) &= K_1(\mathcal{A}/\mathcal{B}).
\end{aligned}$$

In the following, when given a $*$-homomorphism $\varphi : E \to F$ between C^*-algebras, we will denote the $*$-homomorphism $E \to M_n(F)$, given by

$$E \ni e \mapsto \operatorname{diag}(\varphi(e), \varphi(e), \cdots, \varphi(e)),$$

by $[1_n \otimes \varphi]$. Let $q_D : M(D) \to Q(D) = M(D)/D$ be the quotient map. We define a map $\rho : KK(B, D) \to \operatorname{Ext}^{-1}(A_1 *_B A_2, D)$ in the following way. Let u be a unitary $M_n(\mathcal{A}/\mathcal{B})$ for some n, and let $\tilde{u} \in M_n(\mathcal{A})$ be a lift of u. Then $\tilde{u}[1_n \otimes \alpha_1 \circ i_1](b)\tilde{u}^* - [1_n \otimes \alpha_2 \circ i_2](b) \in M_n(D)$ for all $b \in B$ and $\tilde{u}\tilde{u}^*[1_n \otimes \alpha_1](a) = \tilde{u}^*\tilde{u}[1_n \otimes \alpha_1](a) = [1_n \otimes \alpha_1](a)$ modulo $M_n(D)$ for all $a \in A_1$, so

$$\rho(u) = (q_{M_n(D)} \circ \operatorname{Ad} \tilde{u} \circ [1_n \otimes \alpha_1]) *_B (q_{M_n(D)} \circ [1_n \otimes \alpha_2])$$

is a well-defined extension, $\rho(u) : A_1 *_B A_2 \to Q(M_n(D)) \simeq Q(D)$.

Lemma 3.1. *$\rho(u)$ is an invertible extension. In fact, $\begin{pmatrix} \rho(u) & \\ & \rho(u^*) \end{pmatrix}$ is a split extension.*

Proof. Choose first a unitary lift $w \in M_{2n}(\mathcal{A})$ of $\begin{pmatrix} u & \\ & u^* \end{pmatrix}$. Then

$$\begin{pmatrix} \rho(u) & \\ & \rho(u^*) \end{pmatrix} = (q_{M_{2n}(D)} \circ \operatorname{Ad} w \circ [1_{2n} \otimes \alpha_1]) *_B (q_{M_{2n}(D)} \circ [1_{2n} \otimes \alpha_2]).$$

As is wellknown, there is a continuous path of unitaries in $M_{2n}(\mathcal{A})$ connecting w to a unitary $w_0 \in M_{2n}(\mathcal{B})^+$. Set $w_1 = ww_0^* \in M_{2n}(\mathcal{A})$, and observe that

$$\begin{pmatrix} \rho(u) & \\ & \rho(u^*) \end{pmatrix} = (q_{M_{2n}(D)} \circ \operatorname{Ad} w_1 \circ [1_{2n} \otimes \alpha_1]) *_B (q_{M_{2n}(D)} \circ [1_{2n} \otimes \alpha_2]).$$

Note that $\operatorname{Ad} w_1$ leaves $[1_{2n} \otimes \alpha_1 \circ i_1]^+(B^+) + M_{2n}(D)$ globally invariant and that the path of unitaries in $M_{2n}(\mathcal{A})$ connecting w_1 to 1 shows that the automorphism of $[1_{2n} \otimes \alpha_1 \circ i_1]^+(B^+) + M_{2n}(D)$ given by $\operatorname{Ad} w_1$ is homotopic to the identity in the uniform normtopology. Consequently this automorphism is inner by Corollary 8.7.8 of [P1], i.e. there is a unitary $T \in [1_{2n} \otimes \alpha_1 \circ i_1]^+(B^+) + M_{2n}(D)$ such that $w_1 x w_1^* = TxT^*$ for all $x \in [1_{2n} \otimes \alpha_1 \circ i_1]^+(B^+) + M_{2n}(D)$. Write $T = [1_{2n} \otimes \alpha_1 \circ i_1]^+(S) + d$, where $S \in B^+$ and $d \in M_{2n}(D)$. Since $\alpha_1 \circ i_1$ is absorbing, $q_{M_{2n}(D)} \circ [1_{2n} \otimes \alpha_1 \circ i_1]^+$ is injective, so we conclude that S is a unitary. Furthermore, since

$$\begin{aligned}
&q_{M_{2n}(D)} \circ [1_{2n} \otimes \alpha_1 \circ i_1](b)\\
&= q_{M_{2n}(D)} \circ \operatorname{Ad} w_1 \circ [1_{2n} \otimes \alpha_1 \circ i_1](b) = q_{M_{2n}(D)} \circ [1_{2n} \otimes \alpha_1 \circ i_1](SbS^*)
\end{aligned}$$

for all $b \in B$, we conclude that $SbS^* = b$ for all $b \in B$. We can therefore define an automorphism $\mathbf{\Phi} = (\operatorname{Ad} S) *_B \mathrm{id}_{A_2}$ of $A_1 *_B A_2$ such that $\mathbf{\Phi} \circ j_1(x) = j_1(i_1^+(S)^* x i_1^+(S))$, $x \in A_1$, and $\mathbf{\Phi} \circ j_2(y) = j_2(y), y \in A_2$. Then

$$\begin{aligned}
&\begin{pmatrix} \rho(u) & \\ & \rho(u^*) \end{pmatrix} \circ \mathbf{\Phi} \\
&= (q_{M_{2n}(D)} \circ \operatorname{Ad}(w_1[1_{2n} \otimes \alpha_1 \circ i_1]^+(S)^*) \circ [1_{2n} \otimes \alpha_1]) *_B (q_{M_{2n}(D)} \circ [1_{2n} \otimes \alpha_2]) \\
&= (q_{M_{2n}(D)} \circ \operatorname{Ad}(w_1 T^*) \circ [1_{2n} \otimes \alpha_1]) *_B (q_{M_{2n}(D)} \circ [1_{2n} \otimes \alpha_2])
\end{aligned}$$

which admits the lift $(\operatorname{Ad}(w_1T^*) \circ [1_{2n} \otimes \alpha_1]) *_B [1_{2n} \otimes \alpha_2]) : A_1 *_B A_2 \to M_{2n}(M(D))$. It follows that $\begin{pmatrix} \rho(u) & \\ & \rho(u^*) \end{pmatrix}$ admits the lift $((\operatorname{Ad}(w_1T^*) \circ [1_{2n} \otimes \alpha_1]) *_B [1_{2n} \otimes \alpha_2]) \circ \mathbf{\Phi}^{-1}$.

□

Given Lemma 3.1 it is clear that the construction gives us a homomorphism $\rho : KK(B, D) \to \operatorname{Ext}^{-1}(A_1 *_B A_2, D)$.

Lemma 3.2.

$$\begin{tikzcd}
KK(A_1, D) \oplus KK(A_2, D) \arrow[r, "i_1^* - i_2^*"] & KK(B, D) \arrow[r, "\rho"] & \operatorname{Ext}^{-1}(A_1 *_B A_2, D) \arrow[d, "(j_1^*, j_2^*)"] \\
& \operatorname{Ext}^{-1}(B, D) & \operatorname{Ext}^{-1}(A_1, D) \oplus \operatorname{Ext}^{-1}(A_2, D) \arrow[l, "i_1^* - i_2^*"']
\end{tikzcd}$$

is exact.

Proof. Exactness at $KK(B, D)$: Consider elements $v_k \in M_n(\mathcal{A}_k)$ that are unitaries modulo $M_n(\mathcal{B}_k)$. Then

$$(q_{M_n(D)} \circ \operatorname{Ad} v_2^* v_1 \circ [1_n \otimes \alpha_1]) *_B (q_{M_n(D)} \circ [1_n \otimes \alpha_2])$$

is unitarily equivalent to

$$\begin{aligned}
&(q_{M_n(D)} \circ \operatorname{Ad} v_1 \circ [1_n \otimes \alpha_1]) *_B (q_{M_n(D)} \circ \operatorname{Ad} v_2 \circ [1_n \otimes \alpha_2]) = \\
&(q_{M_n(D)} \circ [1_n \otimes \alpha_1]) *_B (q_{M_n(D)} \circ [1_n \otimes \alpha_2]) = q_{M_n(D)} \circ [1_n \otimes \alpha],
\end{aligned}$$

which is a split extension. This shows that $\rho \circ (i_1^* - i_2^*) = 0$. Consider then a unitary $u \in M_n(\mathcal{A}/\mathcal{B})$ and assume that $[\rho(u)] = 0$ in $\operatorname{Ext}^{-1}(A_1 *_B A_2, D)$. Since α is absorbing this implies that

$$\operatorname{Ad} q_{M_{n+1}(D)}(W) \circ \begin{pmatrix} \rho(u) & \\ & q_D \circ \alpha \end{pmatrix} = \begin{pmatrix} q_{M_n(D)} \circ [1_n \otimes \alpha] & \\ & q_D \circ \alpha \end{pmatrix}$$

for some unitary $W \in M_{n+1}(M(D))$. Alternatively,

$$\operatorname{Ad} q_{M_{n+1}(D)}(W) \circ \begin{pmatrix} q_{M_n(D)} \circ \operatorname{Ad} \tilde{u} \circ [1_n \otimes \alpha_1] & \\ & q_D \circ \alpha_1 \end{pmatrix} = \begin{pmatrix} q_{M_n(D)} \circ [1_n \otimes \alpha_1] & \\ & q_D \circ \alpha_1 \end{pmatrix}$$

and

$$\operatorname{Ad} q_{M_{n+1}(D)}(W) \circ \begin{pmatrix} q_{M_n(D)} \circ [1_n \otimes \alpha_2] & \\ & q_D \circ \alpha_2 \end{pmatrix} = \begin{pmatrix} q_{M_n(D)} \circ [1_n \otimes \alpha_2] & \\ & q_D \circ \alpha_2 \end{pmatrix}$$

Hence W^* and $W \begin{pmatrix} \tilde{u} & \\ & 1 \end{pmatrix}$ represent unitaries in $M_{n+1}(\mathcal{A}_2/\mathcal{B}_2)$ and $M_{n+1}(\mathcal{A}_1/\mathcal{B}_1)$, respectively, and since the product of their images in $M_{n+1}(\mathcal{A}/\mathcal{B})$ is $\begin{pmatrix} u & \\ & 1 \end{pmatrix}$, we conclude that $[u]$ is in the range of $i_1^* - i_2^*$.

Exactness at $\operatorname{Ext}^{-1}(A_1 *_B A_2, D)$: It is obvious that the composition $(j_1^*, j_2^*) \circ \rho$ is zero, so consider an extension - a priori not necessarily invertible - $\varphi : A_1 *_B A_2 \to$

$Q(D)$ with the property that $\varphi \circ j_k, k = 1,2$, are both split. Since α_k is absorbing there are unitaries $S_k \in M_2(M(D))$ such that

$$\operatorname{Ad} q_{M_2(D)}(S_k) \circ \begin{pmatrix} \varphi\circ j_k & \\ & q_D\circ\alpha_k \end{pmatrix} = \begin{pmatrix} q_D\circ\alpha_k & \\ & q_D\circ\alpha_k \end{pmatrix},$$

$k = 1,2$. It follows that $S_2S_1^* \in M_2(\mathcal{A})$ and that $\begin{pmatrix} \varphi & \\ & q_D\circ\alpha \end{pmatrix}$ is unitarily equivalent to $(\operatorname{Ad} q_{M_2(D)}(S_2S_1^*) \circ [1_2 \otimes \alpha_1]) *_B [1_2 \otimes \alpha_2]$ which is clearly in the range of ρ. In particular, φ is invertible afterall, and we have exactness at $\operatorname{Ext}^{-1}(A_1 *_B A_2, D)$.

Exactness at $\operatorname{Ext}^{-1}(A_1, D) \oplus \operatorname{Ext}^{-1}(A_2, D)$: It is trivial that $(i_1^* - i_2^*)\circ(j_1^*, j_2^*) = 0$, so consider a pair of invertible extensions $\varphi_k : A_k \to Q(B), k = 1,2$, with the property that $i_1^*[\varphi_1] = i_2^*[\varphi_2]$. There is then a unitary $S \in M_2(M(D))$ such that

$$\operatorname{Ad} q_{M_2(D)}(S) \circ \begin{pmatrix} \varphi_1\circ i_1 & \\ & q_D\circ\alpha_1\circ i_1 \end{pmatrix} = \begin{pmatrix} \varphi_2\circ i_2 & \\ & q_D\circ\alpha_2\circ i_2 \end{pmatrix}.$$

After adding $q_D \circ \alpha_k$ to φ_k we may assume that $\varphi_1 \circ i_1 = \varphi_2 \circ i_2$. Similarly, if $\psi_k : A_k \to Q(D)$ represents the inverse of φ_k in $\operatorname{Ext}^{-1}(A_k, D)$, $k = 1,2$, we may assume that $\psi_1 \circ i_1 = \psi_2 \circ i_2$. We can then consider the two extensions $\varphi_1 *_B \varphi_2$, $\psi_1 *_B \psi_2 : A_1 *_B A_2 \to Q(B)$ whose sum $\mu = (\varphi_1 *_B \varphi_2) \oplus (\psi_1 *_B \psi_2)$ has the property that $\mu \circ j_k : A_k \to Q(D), k = 1,2$, both split. By the arguments in the last paragraph we conclude that μ and hence also $\varphi_1 *_B \varphi_2$ is an invertible extension. Since $\varphi_k = (\varphi_1 *_B \varphi_2) \circ j_k, k = 1,2$, the proof is complete. □

Theorem 3.3. *Let A_1, A_2, B be separable C^*-algebras. Assume that $i_k : B \to A_k, k = 1,2$, are embeddings, and that there are surjective conditional expectations $P_k : A_k \to i_k(B), k = 1,2$, or that A_1, A_2 and B are all nuclear. Let $j_k : A_k \to A_1 *_B A_2, k = 1,2$, be the natural maps. For any separable C^*-algebra D there are six terms exact sequences*

$$\begin{tikzcd}
KK(D,B) \arrow[r, "(i_{1*},i_{2*})"] & KK(D,A_1)\oplus KK(D,A_2) \arrow[r, "j_{1*}-j_{2*}"] & KK(D,A_1 *_B A_2) \arrow[d] \\
KK(SD,A_1 *_B A_2) \arrow[u] & KK(SD,A_1)\oplus KK(SD,A_2) \arrow[l, "j_{1*}-j_{2*}"'] & KK(SD,B) \arrow[l, "(i_{1*},i_{2*})"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
KK(B,D) \arrow[d] & KK(A_1,D)\oplus KK(A_2,D) \arrow[l, "i_1{}^*-i_2{}^*"'] & KK(A_1 *_B A_2,D) \arrow[l, "(j_1^*,j_2^*)"'] \\
KK(A_1 *_B A_2,SD) \arrow[r, "(j_1^*,j_2^*)"'] & KK(A_1,SD)\oplus KK(A_2,SD) \arrow[r, "i_1{}^*-i_2{}^*"'] & KK(B,SD) \arrow[u]
\end{tikzcd}.$$

Proof. Consider first the case where A_1, A_2 and B share the same unit, and let $\varphi : C \to S(A_1 *_B A_2)$ be Germain's $*$-homomorphism, cf. [G3], where C is the mapping cone for the embedding $B \to A_1 \oplus A_2$. φ relates the Puppe exact sequence of Theorem 1 in [CS] to the exact sequence in Lemma 3.2 in such a way that we can conclude from the five lemma that $\varphi^* : KK(S(A_1 *_B A_2), D) \to KK(C, D)$ is an isomorphism. Since D is arbitrary here, standard KK-theory arguments show that $[\varphi] \in KK(C, S(A_1 *_B A_2))$ is invertible. By using that $(A_1 *_B A_2)^+ = A_1^+ *_{B^+} A_2^+$ it follows straightforwardly that $[\varphi]$ is also invertible in the general (non-unital) case. As pointed out by Germain in [G3], this completes the proof. □

The possibilities of our approach are not completely exhausted; if we assume that D is nuclear we can obtain the second of the six terms exact sequences in Theorem 3.3 without any conditions on $i_k : B \to A_k, k = 1, 2$.

Theorem 3.4. *Let A_1, A_2, B be separable C^*-algebras. Assume that $i_k : B \to A_k, k = 1, 2$, are embeddings, and let $j_k : A_k \to A_1 *_B A_2, k = 1, 2$, be the natural maps. For any separable nuclear C^*-algebra D the following six terms sequence is exact:*

$$\begin{tikzcd}
KK(A_1, D) \oplus KK(A_2, D) \arrow[r, "i_1^* - i_2^*"] & KK(B, D) \arrow[r, "\rho"] & \mathrm{Ext}^{-1}(A_1 *_B A_2, D) \arrow[d, "(j_1^*, j_2^*)"] \\
KK(A_1 *_B A_2, D) \arrow[u, "(j_1^*, j_2^*)"] & \mathrm{Ext}^{-1}(B, D) \arrow[l, "\beta\circ\rho\circ\alpha"'] & \mathrm{Ext}^{-1}(A_1, D) \oplus \mathrm{Ext}^{-1}(A_2, D) \arrow[l, "i_1^* - i_2^*"']
\end{tikzcd},$$

where $\alpha : \mathrm{Ext}^{-1}(-, D) \to KK(-, SD)$ and $\beta : \mathrm{Ext}^{-1}(-, SD) \to KK(-, D)$ are Kasparov's natural transformations.

Proof. By adjoining units we may assume that A_1, A_2 and B share the same unit, and we may assume that D is stable. By Theorem 5 of [K] the nuclearity of D implies that any absorbing $*$-homomorphism $\pi : A_1 *_B A_2 \to M(D)$ restricts to absorbing $*$-homomorphisms on A_1, A_2 and B. Consequently the proof of Lemma 3.2 works to give us the stated six terms exact sequence. □

In particular, Theorem 3.4 calculates of the K-homology of an arbitrary amalgamated free product of separable C^*-algebras.

The crucial Lemma 3.2 in this section is in some sense merely an updated version of the result in [Br]. Brown's result contains also the statement that $\mathrm{Ext}(A_1 *_B A_2)$ is a group when $\mathrm{Ext}(A_k), k = 1, 2$, are groups and B is finite dimensional. This part of Brown's result can now be improved as follows.

Proposition 3.5. *Let A_1, A_2, B, D be separable C^*-algebras, D stable. Assume that $i_k : B \to A_k, k = 1, 2$, are embeddings, and that $\mathrm{Ext}(A_k, D), k = 1, 2$, are both groups. If either*
a) there are surjective conditional expectations $A_k \to B, k = 1, 2$,
or
b) A_1, A_2 and B are nuclear,
or
c) D is nuclear,
it follows that also $\mathrm{Ext}(A_1 *_B A_2, D)$ is a group.

Proof. The assumptions ensure that there is an absorbing $*$-homomorphism $\beta : A_1 *_B A_2 \to M(D)$ such that $\beta \circ j_k, k = 1, 2$, and $\beta \circ j_1 \circ i_1 = \beta \circ j_2 \circ i_2$ are all absorbing; in case a) this follows from Lemma 2.1, in case b) from Lemma 2.2 and in case c) from Theorem 5 of [K]. Therefore the arguments from the proof of Lemma 3.2 give that every extension of $A_1 *_B A_2$ by D is invertible.

□

As a particular case of c) in Proposition 3.5 we get that $\mathrm{Ext}(A_1 *_B A_2)$ is always a group when $\mathrm{Ext}(A_k), k = 1, 2$, both are. It is wellknown that the assumption that $\mathrm{Ext}(A_k, D), k = 1, 2$, are groups is redundant in case b).

4. An appropriate picture of the E-theory groups

Let A, D be separable C^*-algebras, D stable. In this section an *E-pair* for (A, D) will be a pair (W, φ), where $\varphi : \mathrm{cone}(A) \to D$ is an asymptotic homomorphism and $W = \{W_t\}_{t\in[1,\infty)}$ is a strictly continuous path of unitaries in $M(D)$ such that

$$\lim_{t\to\infty} \|W_t\varphi_t(a) - \varphi_t(a)W_t\| = 0 \tag{4.1}$$

for all $a \in SA$. The pair (W, φ) is *degenerate* when (4.1) holds for all $a \in \mathrm{cone}(A)$. We let $X_0(A, D)$ denote the set of homotopy classes of E-pairs, where a homotopy is given by an E-pair for $(A, C[0, 1] \otimes D)$. The direct sum of E-pairs, performed with the aid of any pair V_1, V_2 of isometries in $M(D)$ such that $V_1^*V_2 = 0$ and $V_1V_1^* + V_2V_2^* = 1$, makes obviously $X_0(A, D)$ into an abelian semigroup. The subsemigroup of $X_0(A, D)$ consisting of the elements of $X_0(A, D)$ that can be represented by a degenerate E-pair will be denoted by $X_{00}(A, D)$. The quotient semigroup $X(A, D) = X_0(A, D)/X_{00}(A, D)$ is then an abelian group; a standard rotation argument shows that $(W, \varphi) \oplus (W^*, \varphi)$ is homotopic to a degenerate E-pair. Define a map $\kappa : X(A, D) \to [[S^2A, D]]$ in the following way. Given an E-pair (W, φ) we can define an asymptotic homomorphism $W \otimes \varphi : C(\mathbb{T}) \otimes SA \to D$ such that

$$\lim_{t\to\infty} (W \otimes \varphi)_t(g \otimes f) - g(W_t)\varphi_t(f) = 0$$

for all $g \in C(\mathbb{T}), f \in SA$. We set $\kappa[W, \varphi] = i^*[W \otimes \varphi]$, where $i : S^2A \to C(\mathbb{T}) \otimes SA$ is the canonical embedding. To show that κ is an isomorphism, we need two lemmas.

Lemma 4.1. *Let $\varphi : C(\mathbb{T}) \otimes A \to D$ be an asymptotic homomorphism. There is then a strictly continuous path $W = \{W_t\}_{t\in[1,\infty)}$ of unitaries in $M_2(M(D))$ such that*

$$\lim_{t\to\infty} g(W_t) \begin{pmatrix} \varphi_t(1_{C(\mathbb{T})}\otimes a) & \\ & 0 \end{pmatrix} - \begin{pmatrix} \varphi_t(g\otimes a) & \\ & 0 \end{pmatrix} = 0$$

for all $g \in C(\mathbb{T}), a \in A$.

Proof. Let $\varphi_1 : C(\mathbb{T}) \otimes A \to C_b([1, \infty), D)/C_0([1, \infty), D)$ be the $*$-homomorphism defined from φ in the usual way. Set $H = \varphi_1(C(\mathbb{T}) \otimes A)$, and $H_0 = q^{-1}(H) \subseteq C_b([1, \infty), D)$, where $q : C_b([1, \infty), D) \to C_b([1, \infty), D)/C_0([1, \infty), D)$ is the quotient map. Since φ_1 and q extend to surjections $\overline{\varphi_1} : M(C(\mathbb{T}) \otimes A) \to M(H)$ and $\overline{q} : M(H_0) \to M(H)$, we can find a unitary lift $W \in M_2(M(H_0))$ of $\begin{pmatrix} \overline{\varphi_1}(z) & \\ & \overline{\varphi_1}(z^*) \end{pmatrix}$, where $z \in M(C(\mathbb{T}) \otimes A)$ is the element given by the identity function on $\mathbb{T}$. Since $M(M_2(H_0)) \subseteq M(M_2(C_0([1, \infty), D)))$, W is given by a strictly continuous path of unitaries in $M_2(M(D))$ with the desired property. □

Let $c : SA \to C(\mathbb{T}) \otimes SA$ be the $*$-homomorphism $c(f) = 1_{C(\mathbb{T})} \otimes f$.

Lemma 4.2. *Let $\psi : C(\mathbb{T}) \otimes SA \to D$ be an asymptotic homomorphism such that $c^*[\psi] = 0$ in $[[SA, D]]$. It follows that there are asymptotic homomorphisms $\psi'' : C(\mathbb{T}) \otimes \mathrm{cone}(A) \to D$, $\psi' : \mathrm{cone}(A) \to D$ and a strictly continuous path $\{W_t\}_{t\in[1,\infty)}$ of unitaries in $M(D)$ such that*

$$\lim_{t\to\infty} [\psi_t \oplus \psi''_t](g \otimes f) - g(W_t)\psi'_t(f) = 0$$

for all $g \in C(\mathbb{T}), f \in SA$.

Proof. Since $c^*[\psi] = 0$ it follows from Theorem 4.2 of [Th2] that there is an asymptotic homomorphism $\nu : \mathrm{cone}(A) \to D$ and a normcontinuous path of unitaries $\{U_t\}_{t\in[1,\infty)} \subseteq M_2(D)^+$ such that

$$\lim_{t\to\infty} U_t \begin{pmatrix} \psi_t(1_{C(\mathbb{T})}\otimes f) & \\ & \nu_t(f) \end{pmatrix} U_t^* - \begin{pmatrix} 0 & \\ & \nu_t(f) \end{pmatrix} = 0$$

for all $f \in SA$. Let $ev : C(\mathbb{T}) \otimes SA \to SA$ be the $*$-homomorphism obtained by evaluation at some point in $\mathbb{T}$. It follows from Lemma 4.1 that there is a strictly continuous path $\{W_t\}_{t\in[1,\infty)}$ of unitaries in $M(D)$ such that

$$\lim_{t\to\infty} g(W_t)(\psi_t \oplus \nu_t \circ ev \oplus 0)(1_{C(\mathbb{T})} \otimes f) - (\psi_t \oplus \nu_t \circ ev \oplus 0)(g \otimes f) = 0,$$

$g \in C(\mathbb{T}), f \in SA$. Set $\psi'' = \nu \circ ev \oplus 0$ and $\psi' = \mathrm{Ad}(U \oplus 1)^* \circ (0 \oplus \nu \oplus 0)$. □

To use these two lemmas to define a map $\delta : [[S^2A, D]] \to X(A, D)$, we remind the reader that $[[S-, D]] = E(S-, D)$, cf. [DL]. In particular, the contravariant functor $[[S-, D]]$ is split-exact, and this will be used now. Let $\psi : S^2A \to D$ be an asymptotic homomorphism. There is then an asymptotic homomorphism $\varphi : C(\mathbb{T}) \otimes SA \to D$ such that $c^*[\varphi] = 0$ and $[\psi] = [\varphi \circ i]$ in $[[S^2A, B]]$. Since $c^*[\varphi] = 0$, Lemma 4.2 gives us asymptotic homomorphisms $\varphi'' : C(\mathbb{T}) \otimes \mathrm{cone}(A) \to D$, $\varphi' : \mathrm{cone}(A) \to D$ and a strictly continuous path $\{W_t\}_{t\in[1,\infty)}$ of unitaries in $M(D)$ such that

$$\lim_{t\to\infty} [\varphi_t \oplus \varphi''_t](g \otimes f) - g(W_t)\varphi'_t(f) = 0$$

for all $g \in C(\mathbb{T}), f \in SA$. Then (W, φ') is an E-pair and we claim that we can define δ such that $\delta[\psi] = [W, \varphi']$. To see this, the only non-trivial point is to show that the class of (W, φ') is independent of the choices made. So assume that $\lambda'' : C(\mathbb{T}) \otimes \mathrm{cone}(A) \to D$, $\lambda' : \mathrm{cone}(A) \to D$ are asymptotic homomorphisms and $\{S_t\}_{t\in[1,\infty)}$ a strictly continuous path of unitaries in $M(D)$ such that

$$\lim_{t\to\infty} [\varphi_t \oplus \lambda''_t](g \otimes f) - g(S_t)\lambda'_t(f) = 0$$

for all $g \in C(\mathbb{T}), f \in SA$. By Lemma 4.1 there are strictly continuous pathes, $\{Y_t\}_{t\in[1,\infty)}, \{X_t\}_{t\in[1,\infty)}$ of unitaries in $M(D)$ such that

$$\lim_{t\to\infty} g(Y_t)\varphi''_t(1_{C(\mathbb{T})} \otimes f) - \varphi''_t(g \otimes f) = 0,$$
$$\lim_{t\to\infty} g(X_t)\lambda''_t(1_{C(\mathbb{T})} \otimes f) - \lambda''_t(g \otimes f) = 0$$

for all $g \in C(\mathbb{T}), f \in \mathrm{cone}(A)$. Since $(Y, \varphi'' \circ c)$ and $(X, \lambda'' \circ c)$ are degenerate E-pairs, $[W, \varphi'] = [W, \varphi'] + [X, \lambda'' \circ c] = [W \oplus X, \varphi' \oplus \lambda'' \circ c]$ and $[S, \lambda'] = [S, \lambda'] + [Y, \varphi'' \circ c] = [S \oplus Y, \lambda' \oplus \varphi'' \circ c]$ in $X(A, D)$. Conjugating the pair $(S \oplus Y, \lambda' \oplus \varphi'' \circ c)$ by a unitary, we see that $[W, \varphi'] = [W^1, \varphi^1]$ and $[S, \lambda'] = [W^2, \varphi^2]$, where the E-pairs (W^1, φ^1) and (W^2, φ^2) are related such that

$$\lim_{t\to\infty} g(W_t^1)\varphi_t^1(f) - g(W_t^2)\varphi_t^2(f) = 0$$

for all $g \in C(\mathbb{T}), f \in SA$. In particular, $\lim_{t\to\infty} \varphi_t^1(f) - \varphi_t^2(f) = 0$ for all $f \in SA$, so a standard rotation argument shows that $(W^1 \oplus W^{2*}, \varphi^1 \oplus \varphi^2)$ is homotopic to $(W^{2*}W^1 \oplus 1, \varphi^1 \oplus \varphi^2)$. This shows that $[W^1, \varphi^1] - [W^2, \varphi^2]$ is represented by an E-pair (V, μ) where $\lim_{t\to\infty} g(V_t)\mu_t(f) - g(1)\mu_t(f) = 0$ for all $g \in C(\mathbb{T}), f \in SA$. By rotation we find that $[V, \mu] + [1, 0] = [V, 0] + [1, \mu] = 0$ in $X(A, D)$. Hence $[W, \varphi'] = [S, \lambda']$ and we conclude that δ is well-defined. Since δ is clearly an inverse for κ, we have obtained the following proposition.

Proposition 4.3. *$\kappa : X(A,D) \to [[S^2A, D]]$ is an isomorphism with inverse δ.*

5. $\operatorname{cone}(A_1) *_{SB} \operatorname{cone}(A_2)$ is equivalent to B in E-theory

Assume now that $i_k : B \to A_k, k = 1,2$, are proper embeddings, i.e. that $i_k(B)A_k$ spans a dense subspace in A_k, and that there are sequences $R_n^k : A_k \to B, n = 1,2,3,\cdots, k = 1,2$, of completely positive contractions such that $\lim_{n\to\infty} R_n^k(i_k(x)) = x$ for all $x \in B, k = 1,2$. By Theorem 5.5 of [P2] we have natural isomorphisms

$$S(A_1 *_B A_2) = SA_1 *_{SB} SA_2, \quad \operatorname{cone}(A_1 *_B A_2) = \operatorname{cone}(A_1) *_{\operatorname{cone}(B)} \operatorname{cone}(A_2).$$

We can then define a map $X(B,D) \to [[\operatorname{cone}(A_1) *_{SB} \operatorname{cone}(A_2), D]]$ in the following way. Let (W,φ) be an E-pair. It follows from Lemma 2.7 that there are asymptotic homomorphisms $\mu^k : \operatorname{cone}(A_k) \to D$ and $\nu^k : \operatorname{cone}(A_k) \to D, k = 1,2$, such that $\lim_{t\to\infty} \mu_t^1 \circ i_1(x) - \mu_t^2 \circ i_2(x) = \lim_{t\to\infty} \nu_t^1 \circ i_1(x) - \nu_t^2 \circ i_2(x) = 0$ and

$$\lim_{t\to\infty} \varphi_t(x) \oplus \mu_t^1 \circ i_1(x) - \nu_t^1 \circ i_1(x) = 0$$

for all $x \in \operatorname{cone}(B)$. Since $\lim_{t\to\infty} \operatorname{Ad}(W_t \oplus 1) \circ \nu_t^1 \circ i_1(x) - \nu_t^2 \circ i_2(x) = 0$ for all $x \in SB$, there is an asymptotic homomorphism

$$(\operatorname{Ad}(W \oplus 1) \circ \nu^1) *_{SB} \nu^2 : \operatorname{cone}(A_1) *_{SB} \operatorname{cone}(A_2) \to D$$

such that $\lim_{t\to\infty}((\operatorname{Ad}(W \oplus 1) \circ \nu^1) *_{SB} \nu^2)_t \circ j_1(x) - \operatorname{Ad}(W_t \oplus 1) \circ \nu_t^1 \circ j_1(x) = 0$ for all $x \in \operatorname{cone}(A_1)$ and $\lim_{t\to\infty}((\operatorname{Ad}(W \oplus 1) \circ \nu^1) *_{SB} \nu^2)_t \circ j_2(x) - \nu_t^2 \circ j_2(x) = 0$ for all $x \in \operatorname{cone}(A_2)$. We claim that we can define a map $\rho : X(B,D) \to [[\operatorname{cone}(A_1) *_{SB} \operatorname{cone}(A_2), D]]$ such that $\rho[W,\varphi] = [(\operatorname{Ad}(W \oplus 1) \circ \nu^1) *_{SB} \nu^2]$. If $\kappa^k : \operatorname{cone}(A_k) \to D$ and $l^k : \operatorname{cone}(A_k) \to D, k = 1,2$, are other asymptotic homomorphisms such that $\lim_{t\to\infty} l_t^1 \circ i_1(x) - l_t^2 \circ i_2(x) = \lim_{t\to\infty} \kappa_t^1 \circ i_1(x) - \kappa_t^2 \circ i_2(x) = 0$ and

$$\lim_{t\to\infty} \varphi_t(x) \oplus \kappa_t^1 \circ i_1(x) - l_t^1 \circ i_1(x) = 0$$

for all $x \in \operatorname{cone}(B)$, observe first that $\kappa^1 *_{SB} \kappa^2, l^1 *_{SB} l^2$ and $\mu^1 *_{SB} \mu^2$ are all restrictions of asymptotic homomorphisms defined on $\operatorname{cone}(A_1 *_B A_2) = \operatorname{cone}(A_1) *_{\operatorname{cone}(B)} \operatorname{cone}(A_2)$, and hence null-homotopic. Consequently

$$[(\operatorname{Ad}(W\oplus 1)\circ\nu^1)*_{SB}\nu^2] = [((\operatorname{Ad}(W\oplus 1)\circ\nu^1)*_{SB}\nu^2)\oplus(\kappa^1*_{SB}\kappa^2)\oplus(l^1*_{SB}l^2)\oplus(\mu^1*_{SB}\mu^2)].$$

Note that there is a unitary $S \in M(D)$ which commutes both with $W\oplus 1\oplus 1\oplus 1\oplus 1\oplus 1$ and $1 \oplus 1 \oplus 1 \oplus W \oplus 1 \oplus 1$, and has the property that

$$\lim_{t\to\infty}(((\operatorname{Ad}(W \oplus 1) \circ \nu^1) *_{SB} \nu^2) \oplus (\kappa^1 *_{SB} \kappa^2) \oplus (l^1 *_{SB} l^2) \oplus (\mu^1 *_{SB} \mu^2))_t \circ j_k \circ i_k(x)$$
$$- \operatorname{Ad} S \circ (\varphi \oplus \kappa^1 \oplus \mu^1 \oplus \varphi \oplus \kappa^1 \oplus \mu^1)_t \circ i_k(x) \ = \ 0$$

for all $x \in SB, k = 1,2$. By first 'rotating' $W\oplus 1\oplus 1\oplus 1\oplus 1\oplus 1$ to $1\oplus 1\oplus 1\oplus W\oplus 1\oplus 1$ and then connecting S to 1 through a strictly continuous path of unitaries in $M(D)$, we get a homotopy connecting

$$((\operatorname{Ad}(W \oplus 1) \circ \nu^1) *_{SB} \nu^2) \oplus (\kappa^1 *_{SB} \kappa^2) \oplus (l^1 *_{SB} l^2) \oplus (\mu^1 *_{SB} \mu^2)$$

to

$$(\nu^1 *_{SB} \nu^2) \oplus (\kappa^1 *_{SB} \kappa^2) \oplus ((\operatorname{Ad}(W \oplus 1) \circ l^1) *_{SB} l^2) \oplus (\mu^1 *_{SB} \mu^2).$$

Hence $[(\operatorname{Ad}(W \oplus 1) \circ \nu^1) *_{SB} \nu^2] = [(\operatorname{Ad}(W \oplus 1) \circ l^1) *_{SB} l^2]$, and we conclude that ρ is a well-defined homomorphism. Note, however, that at this point we do not know that $[[\operatorname{cone}(A_1) *_{SB} \operatorname{cone}(A_2), D]]$ is a group.

To obtain an inverse to ρ, consider an asymptotic homomorphism $\psi : \mathrm{cone}(A_1) *_{SB} \mathrm{cone}(A_2) \to D$. By Lemma 2.7 there are asymptotic homomorphisms $\nu^k : \mathrm{cone}(A_k) \to D, k = 1, 2$, such that $\lim_{t\to\infty} \nu^1_t \circ i_1(x) - \nu^2_t \circ i_2(x) = 0$ for all $x \in \mathrm{cone}(B)$ and a normcontinuous path of unitaries $\{W_t\}$ in D^+ such that

$$\lim_{t\to\infty} W_t(\psi_t \circ j_1 \circ i_1(x) \oplus \nu^1_t \circ i_1(x))W^*_t - \psi_t \circ j_2 \circ i_2(x) \oplus \nu^2_t \circ i_2(x) = 0$$

for all $x \in \mathrm{cone}(B)$. Then $(W^*, \psi \circ j_2 \circ i_2 \oplus \nu^2 \circ i_2)$ is an E-pair, and we claim that $\delta[\psi] = [W^*, \psi\circ j_2\circ i_2\oplus\nu^2\circ i_2]$ is a welldefined map $\delta : [[\mathrm{cone}(A_1) *_{SB} \mathrm{cone}(A_2), D]] \to X(B, D)$. To see this, let $\mu^k : \mathrm{cone}(A_k) \to D, k = 1, 2$, be another pair of asymptotic homomorphisms and $\{S_t\}$ another normcontinuous path of unitaries in D^+ such that

$$\lim_{t\to\infty} S_t(\psi_t \circ j_1 \circ i_1(x) \oplus \mu^1_t \circ i_1(x))S^*_t - \psi_t \circ j_2 \circ i_2(x) \oplus \mu^2_t \circ i_2(x) = 0$$

for all $x \in \mathrm{cone}(B)$. There is a unitary in $M(D)$ conjugating $(S, \psi \circ j_2 \circ i_2 \oplus \mu^2 \circ i_2) \oplus (1, \nu^2 \circ i_2)$ to $(T, \psi \circ j_2 \circ i_2 \oplus \nu^2 \circ i_2 \oplus \mu^2 \circ i_2)$, where $\{T_t\}$ is a normcontinuous path of unitaries in D^+ such that

$$\lim_{t\to\infty} T_t(\psi_t\circ j_1\circ i_1(x)\oplus\nu^1_t\circ i_1(x)\oplus\mu^1_t\circ i_1(x))T^*_t - \psi_t\circ j_2\circ i_2(x)\oplus\nu^2_t\circ i_2(x)\oplus\mu^2_t\circ i_2(x) = 0$$

for all $x \in \mathrm{cone}(B)$. Then a standard rotation argument shows that

$$\begin{aligned}
&[W^*, \psi \circ j_2 \circ i_2 \oplus \nu^2 \circ i_2] + [S, \psi \circ j_2 \circ i_2 \oplus \mu^2 \circ i_2]\\
&= [W^*, \psi \circ j_2 \circ i_2 \oplus \nu^2 \circ i_2] + [1, \mu^2 \circ i_2] + [S, \psi \circ j_2 \circ i_2 \oplus \mu^2 \circ i_2] + [1, \nu^2 \circ i_2]\\
&= [T(W^* \oplus 1), \psi \circ j_2 \circ i_2 \oplus \nu^2 \circ i_2 \oplus \mu^2 \circ i_2] + [1, \psi \circ j_2 \circ i_2 \oplus \nu^2 \circ i_2 \oplus \mu^2 \circ i_2]\\
&= 0
\end{aligned}$$

in $X(B, D)$. Since $X(A, B)$ is a group we deduce that

$$[W^*, \psi \circ j_2 \circ i_2 \oplus \nu^2 \circ i_2] = [S^*, \psi \circ j_2 \circ i_2 \oplus \mu^2 \circ i_2],$$

proving that δ is well-defined. It is straightforward to see that δ is an inverse to ρ so we have proved the following

Lemma 5.1. $[[\mathrm{cone}(A_1)*_{SB}\mathrm{cone}(A_2), D]]$ *is a group, and* $\rho : X(B, D) \to [[\mathrm{cone}(A_1)*_{SB} \mathrm{cone}(A_2), D]]$ *is an isomorphism.*

Note that it follows from Lemma 5.1 and [DL] that $E(\mathrm{cone}(A_1)*_{SB}\mathrm{cone}(A_2), D) = [[\mathrm{cone}(A_1) *_{SB} \mathrm{cone}(A_2), D]]$.

In the following we let $\mathcal{K}$ denote the C^*-algebra of compact operators on an infinite dimensional separable Hilbert space.

Theorem 5.2. *Assume that* $i_k : B \to A_k, k = 1, 2$, *are proper embeddings, and that there are sequences* $R^k_n : A_k \to B, n = 1, 2, 3, \cdots, k = 1, 2$, *of completely positive contractions such that* $\lim_{n\to\infty} R^k_n(i_k(x)) = x$ *for all* $x \in B, k = 1, 2$. *There is an asymptotic homomorphism* $\mathbf{\Phi} : \mathrm{cone}(A_1) *_{SB} \mathrm{cone}(A_2) \to B \otimes \mathcal{K}$ *which is invertible in E-theory.*

Proof. By Proposition 4.3 and Lemma 5.1, $\rho \circ \kappa^{-1} : [[S^2B, D]] \to [[\mathrm{cone}(A_1) *_{SB} \mathrm{cone}(A_2), D]]$ is an isomorphism. Let $\varphi : S^2B \to B \otimes \mathcal{K}$ be the asymptotic homomorphism obtained by applying the Connes-Higson construction to the Toeplitz-extension tensored with B. Let $\mathbf{\Phi}$ be an asymptotic homomorphism such that $[\mathbf{\Phi}] = \rho \circ \kappa^{-1}[\varphi]$ in $[[\mathrm{cone}(A_1) *_{SB} \mathrm{cone}(A_2), B \otimes \mathcal{K}]]$. Standard KK- and E-theory arguments show that $\mathbf{\Phi}$ must be invertible in E-theory because of Lemma 5.1. □

6. Results in E-theory

Theorem 6.1. *Let A_1, A_2, B be separable C^*-algebras. Assume that $i_k : B \to A_k, k = 1, 2$, are embeddings, and that there are sequences $R_n^k : A_k \to B, n = 1, 2, 3, \cdots, k = 1, 2$, of completely positive contractions such that $\lim_{n\to\infty} R_n^k(i_k(x)) = x$ for all $x \in B, k = 1, 2$. Let $j_k : A_k \to A_1 *_B A_2, k = 1, 2$, be the natural maps. For any separable C^*-algebra D there are six terms exact sequences*

$$\begin{tikzcd}
E(D,B) \arrow[r, "(i_{1*},i_{2*})"] & E(D,A_1)\oplus E(D,A_2) \arrow[r, "j_{1*}-j_{2*}"] & E(D,A_1 *_B A_2) \arrow[d] \\
E(SD,A_1 *_B A_2) \arrow[u] & E(SD,A_1)\oplus E(SD,A_2) \arrow[l, "j_{1*}-j_{2*}"] & E(SD,B) \arrow[l, "(i_{1*},i_{2*})"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
E(B,D) \arrow[d] & E(A_1,D)\oplus E(A_2,D) \arrow[l, "i_1{}^*-i_2{}^*"'] & E(A_1 *_B A_2,D) \arrow[l, "(j_1^*,j_2^*)"'] \\
E(A_1 *_B A_2,SD) \arrow[r, "(j_1^*,j_2^*)"'] & E(A_1,SD)\oplus E(A_2,SD) \arrow[r, "i_1{}^*-i_2{}^*"'] & E(B,SD) \arrow[u]
\end{tikzcd} .$$

Proof. Let A_1^+, A_2^+, B^+ denote the C^*-algebras obtained by adjoining units to A_1, A_2 and B. Let X denote the kernel of the natural map $\mathrm{cone}(A_1^+) *_{S(B^+)} \mathrm{cone}(A_2^+) \to \mathrm{cone}(\mathbb{C}) *_{S\mathbb{C}} \mathrm{cone}(\mathbb{C})$ and Y the kernel of the natural map $A_1^+ * A_2^+ \to \mathbb{C} * \mathbb{C}$. There is then a commuting diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S(A_1 *_B A_2) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S(A_1^+ *_{B^+} A_2^+) \arrow[r] \arrow[d] & \mathrm{cone}(A_1^+) *_{S(B^+)} \mathrm{cone}(A_2^+) \arrow[r] \arrow[d] & A_1^+ * A_2^+ \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S\mathbb{C} \arrow[r] \arrow[d] & \mathrm{cone}(\mathbb{C}) *_{S\mathbb{C}} \mathrm{cone}(\mathbb{C}) \arrow[r] \arrow[d] & \mathbb{C} * \mathbb{C} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{6.1}$$

By [C2] $A_1^+ * A_2^+$ and $\mathbb{C} * \mathbb{C}$ are KK-equivalent to $A_1^+ \oplus A_2^+$ and $\mathbb{C} \oplus \mathbb{C}$, respectively, so we find that Y is equivalent to $A_1 \oplus A_2$ in E-theory. In the same way it follows from Theorem 5.2 that X is equivalent to B in E-theory. The two six terms exact sequences of the theorem now arise by writing down the two six terms exact sequences of E-theory coming from the first row in (6.1), substituting $A_1 \oplus A_2$ for Y and B for X, and finally identifying the resulting maps in the diagram. We leave this to the reader. □

7. Conclusion

As pointed out by Germain in [G3] the six terms exact sequences of Theorem 3.3 imply that $S(A_1 *_B A_2)$ is KK-equivalent to the mapping cone C of the embedding $B \to A_1 \oplus A_2$ (and vice versa, essentially). It follows therefore from Theorem 3.3 that $S(A_1 *_B A_2)$ is equivalent to C in E-theory under the assumptions of that theorem, and the six terms exact sequences of Theorem 6.1 follow from this by writing down the E-theory Puppe sequences for the inclusion $B \to A_1 \oplus A_2$. In other words, Theorem 6.1 is a consequence of Theorem 3.3 when there are conditional expectations from the A_k's onto B, and when A_1, A_2 and B are all nuclear. But the assumptions of Theorem 6.1 are much weaker than this; it suffices for example that B is nuclear, or that B sits as a hereditary C^*-subalgebra of the A_k's. Nonetheless it would be nice to be able to remove the condition in Theorem 6.1 altogether, and also in Theorem 3.3 for that matter. Let us therefore conclude by pointing out that the methods we have used can not, without some serious adjustments, give the six terms exact sequences of Theorem 6.1 in full generality.

It is clear that the assumption of Theorem 6.1 was used above to guarantee that some absorbing $*$-homomorphism $\operatorname{cone}(B) \to M(D)$ can be extended to a $*$-homomorphism $\operatorname{cone}(A_k) \to M(D)$. Such an extension will not exist in general. To see this observe that if $B \subseteq A$ are separable C^*-algebras and D is a stable separable C^*-algebra, then there can only be a $*$-homomorphism $\pi : A \to M(D)$ such that $\pi|_B : B \to M(D)$ is absorbing when

$$\{\varphi|_B : \ \varphi : A \to D \text{ is a completely positive contraction}\}$$

is dense for the topology of pointwise normconvergence among all the completely positive contractions $B \to D$, see [Th1]. (In fact, this condition is also sufficient.) Now consider a separable exact C^*-algebra B for which $\operatorname{Ext}(\operatorname{cone}(B))$ is not a group - such C^*-algebras exist in abundance by [Ki1]. Then $B \subseteq A$ for some nuclear separable C^*-algebra A; in fact one can take $A = O_2$, cf. [Ki2]. That $\operatorname{Ext}(\operatorname{cone}(B))$ is not a group means that there is a $*$-homomorphism $\chi : \operatorname{cone}(B) \to Q$ ($=$ the Calkin algebra) which does not lift to a completely positive map $\operatorname{cone}(B) \to \mathbb{B}(l_2)$. Consider $D = \chi(\operatorname{cone}(B)) \otimes \mathcal{K}$ which is certainly a separable stable C^*-algebra. If $\varphi_n : \operatorname{cone}(A) \to D, n \in \mathbb{N}$, is a sequence of completely positive contractions such that $\lim_{n\to\infty} \varphi_n(b) = \chi(b) \otimes e_{11}, \ b \in \operatorname{cone}(B)$, we would clearly also have a sequence of completely positive contractions $\psi_n : \operatorname{cone}(A) \to Q$ such that $\lim_{n\to\infty} \psi_n(b) = \chi(b)$ for all $b \in \operatorname{cone}(B)$. Since $\operatorname{cone}(A)$ is nuclear each ψ_n would be liftable in the sense of [A2] and hence this would force χ to be liftable by Theorem 6 in [A2], contradicting the choice of it. It follows that for such an inclusion $B \subseteq A$ the approach we have taken to prove Theorem 6.1 does not suffice to prove the general conjecture in E-theory. The method we use to prove Theorem 3.3 requires even more; namely that we can extend some absorbing $*$-homomorphism out of B, not only to A_1 and A_2, but all the way to $A_1 *_B A_2$. The obstacle we have just identified is therefore even more serious in regards to Theorem 3.3.

References

[A1] W. Arveson, *Subalgebras of C^*-algebras*, Acta Math. **123** (1969), 141-224.

[A2] ——, *Notes on extensions*, Duke Math. J. **44** (1977), 329-355.

[Bl] B. Blackadar, *Weak expectations and nuclear C^*-algebras*, Indiana Univ. Math. J. **27** (1978), 1021-1026.

[Bo] F. Boca, *Free products of completely positive maps and spectral sets*, J. Func. Anal. **97** (1991), 251-263.

[Br] L. Brown, *Ext of certain free product C^*-algebras*, J. Operator Theory **6** (1981), 135-141.

[CH] A. Connes and N. Higson, *Deformations, morphisms asymptotiques et K-theorie bivariante*, C.R. Acad. Sci. Paris, Sér. I Math. **313** (1990), 101-106.

[C1] J. Cuntz, *The K-groups for free products of C^*-algebras*, Proc. Symp. Pure Math. **38**, Part 1, A.M.S. (1982).

[C2] ———, *K-theoretic amenability for discrete groups*, J. Reine angew. Math. **344** (1983), 180-195.

[C3] ———, *A new look at KK-theory*, K-theory **1** (1987), 31-51.

[CS] J. Cuntz and G. Skandalis, *Mapping cones and exact sequences in KK-theory*, J. Operator Theory **15** (1986), 163-180.

[DE] M. Dadarlat and S. Eilers, *Asymptotic unitary equivalence in KK-theory*, Preprint 1999.

[DL] M. Dadarlat and T. Loring, *K-homology, Asymptotic Representations, and Unsuspended E-theory*, J. Func. Analysis **126** (1994), 367-383.

[G1] E. Germain, *KK-theory of reduced free product C^*-algebras*, Duke Math. J. **82** (1996), 707-723.

[G2] ———, *KK-theory of the full free product of unital C^*-algebras*, J. Reine angew. Math. **485** (1997), 1-10.

[G3] ———, *Amalgamated Free Product C^*-Algebras and KK-Theory*, Fields Institute Communications **12** (1997), 89-103.

[K] G.G. Kasparov, *Hilbert C^*-Modules: Theorems of Stinespring and Voiculescu*, J. Operator Theory **4** (1980), 133-150.

[Ki1] E. Kirchberg, *On non-semi-split extensions, tensor products and exactness of group C^*-algebras*, Invent. Math. **112** (1993), 449-489.

[Ki2] ———, *On subalgebras of the CAR-algebra*, J. Functional Analysis **129** (1995), 35-63.

[L] E. Lance, *K-theory for certain group C^*-algebras*, Acta Math. **151** (1883), 209-230.

[N] T. Natsume, *On $K_*(C^*(SL_2(\mathbb{Z})))$*, J. Operator Theory **13** (1985), 103-118.

[P1] G.K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, London, 1979.

[P2] ———, *Pullback and Pushout Constructions in C^*-Algebra Theory*, J. Func. Anal. **167** (1999), 243-344.

[Pi] M. V. Pimsner, *KK-groups of crossed products by groups acting on trees*, Invent. Math. **86** (1986), 603-634.

[S] G. Skandalis, *Une notion de nucléarité en K-theorie (d'aprés J. Cuntz)* K-theory **1**, (1987), 549-573.

[Th1] K. Thomsen, *On absorbing extensions*, Proc. Amer. Math. Soc. **129** (2001), 1409-1417.

[Th2] ———, *Homotopy invariance for bifunctors defined from asymptotic homomorphisms*, Preprint 1999.

[V] D. Voiculescu, *A note on quasi-diagonal C^*-algebras and homotopy*, Duke Math. J. **62** (1991), 267-271.

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