UNIVERSITY OF AARHUS
DEPARTMENT OF MATHEMATICS

ISSN: 1397–4076

ON JENSEN'S FUNCTIONAL EQUATION ON GROUPS

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Preprint Series No.: 3

Juni 2001

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Abstract

We prove that the complex valued solutions of Jensen's functional equation $f(xy) + f\left(xy^{-1}\right) = 2f(x)$ on a group G are functions on the quotient group $G/[G,[G,G]]$ and give explicit solution formulas in a setting that includes many examples. We show furthermore that the vector space of odd solutions modulo the subspace of the homomorphisms of G into $\mathbf{C}$ is isomorphic to the vector space of homomorphisms of $[G,G]/[G,[G,G]]$ into $\mathbf{C}$.

Mathematics Subject Classification 2000: 39B32 and 39B52

Key words: Jensen's functional equation, group, semidirect product, general linear group.

1. Introduction

By *Jensen's functional equation* on the group G we mean the functional equation

$$f(xy) + f\left(xy^{-1}\right) = 2f(x), \quad x, y \in G, \tag{1}$$

where $f : G \to \mathbf{C}$ is an unknown function to be determined.

Any solution f of (1) may be written as $f = f(e) + F$ (e being the neutral element of G), where F is an odd solution, i.e. a solution such that $F(e) = 0$ or equivalently $F\left(x^{-1}\right) = -F(x)$ for all $x \in G$. We shall therefore restrict the discussion to odd solutions of Jensen's functional equation. We will denote the complex vector space of odd solutions of (1) on G by $S(G, \mathbf{C})$.

$\operatorname{Hom}(G, \mathbf{C})$, the set of homomorphisms of G into $(\mathbf{C}, +)$, is a subspace of $S(G, \mathbf{C})$. For abelian groups, and some non-abelian as well, $S(G, \mathbf{C}) = \operatorname{Hom}(G, \mathbf{C})$, but in general the two spaces are different.

The most exhaustive study of Jensen's functional equation on groups has been accomplished by Ng [9], [10]. His two papers contain useful reduction formulas and relations for functions in $S(G, \mathbf{C})$ for any group G. Our paper builds on these relations. By help of them Ng solved (1) on free groups and applied the results to solve it on $GL_n(R)$ for all $n \geq 3$, where R is $\mathbf{Z}$, $\mathbf{R}$, a quadratically closed field or a finite field. He observed that there for the free group on two generators are odd solutions of (1) that are not homomorphisms ([9; Corollary 8]), and that $S(GL_n(R), \mathbf{C}) = \operatorname{Hom}(GL_n(R), \mathbf{C})$.

Corovei [5] proved that $S(G, \mathbf{C}) = \operatorname{Hom}(G, \mathbf{C})$, if each commutator in G has finite order.

Friis [6] found explicit formulas for the solutions of Jensen's functional equation on certain semidirect products of groups, like the Heisenberg group and the $(ax + b)$-group. In particular he found that the Heisenberg group possesses odd solutions of (1) that are not homomorphisms, while the $(ax + b)$-group does not.

The purpose of the present paper is to develop a coherent theory for Jensen's functional equation on groups that includes most of the results just mentioned. More precisely, we will

(a) find explicit formulas for solutions of Jensen's functional equation in a reasonably general setting.
(b) parametrize $S(G, \mathbf{C})$ modulo $\operatorname{Hom}(G, \mathbf{C})$.
(c) give sufficient conditions on G to ensure that $S(G, \mathbf{C}) = \operatorname{Hom}(G, \mathbf{C})$, for example to ensure that $S(G, \mathbf{C}) = \{0\}$.

The present paper differs from the previous ones by its discovery of the central roles played by the subgroup $[G, [G, G]]$ and the commutator group $[G, G]$.

We show that any solution f of Jensen's functional equation (1) on G is a function on the quotient group $G/[G, [G, G]]$ (Theorem 2.2), in the sense that $f(x) = f(xy)$ for all $x \in G$ and $y \in [G, [G, G]]$. Of course, this simplifies many computations. It also means that the study of (1) reduces to the study of (1) on metabelian groups, because the quotient group $G/[G, [G, G]]$ always is metabelian.

The commutator subgroup $[G,G]$ enters because the restriction map $f \to f|_{[G,G]}$, $f \in S(G,\mathbf{C})$, factorizes to an isomorphism of $S(G,\mathbf{C})/\operatorname{Hom}(G,\mathbf{C})$ onto $\operatorname{Hom}([G,G]/[G,[G,G]],\mathbf{C})$ (Theorem 3.2), covering (b) above.

Conversely: For a certain class of groups G we find for any homomorphism $\phi \in \operatorname{Hom}([G,G]/[G,[G,G]],\mathbf{C})$ an explicit formula for a solution $f_\phi \in S(G,\mathbf{C})$ such that $f_\phi = \phi$ on $[G,G]$ (Theorem 4.1). This covers (a) above. The class contains for example the Heisenberg group and the free groups.

(c) is done in Section 6. A typical result is that $S(G,\mathbf{C}) = \operatorname{Hom}(G,\mathbf{C})$ if $[G,[G,G]] = [G,G]$ (Proposition 6.3(b)). This takes care of for example the $(ax+b)$-group and $GL(n,\mathbf{R})$ for $n \geq 2$.

Finally we apply our theory to a number of different groups to demonstrate how our formulas can be put into practice (Sections 5 and 7).

We finish this introduction by fixing notation that will be used throughout the paper.

Notation 1.1

G is a group, $e \in G$ its neutral element and $Z(G)$ its center.

For $x \in G$ we let $\langle x \rangle$ denote the subgroup generated by x.

If A and B are subsets of G we let $[A,B]$ denote the subgroup of G generated by the elements $[a,b] := aba^{-1}b^{-1}$, where $a \in A$ and $b \in B$. The commutator subgroup $[G,G]$ of G and the subgroup $[G,[G,G]]$ are normal subgroups of G.

The group G is said to be *metabelian*, if $[G,G] \subseteq Z(G)$, or equivalently if $[G,[G,G]] = \{e\}$. The quotient group $G/[G,[G,G]]$ is metabelian for any group G. The Heisenberg group from Example 5.1 below is metabelian, but not abelian.

When we say that a complex-valued function on G is a *homomorphism*, we mean that it is a homomorphism of G into the additive group $(\mathbf{C},+)$.

Let H be a subgroup of G and let $\pi : G \to G/H$ be the canonical projection. We say that a function $f : G \to \mathbf{C}$ is a function on G/H if it can be written in the form $f = \widetilde{f} \circ \pi$ for some function $\widetilde{f} : G/H \to \mathbf{C}$.

We let $\mathbf{R}_\times$ denote the multiplicative group of all non-zero real numbers, and $\mathbf{R}_\times^+$ the subgroup of all positive real numbers.

2. Formulas and relations

In this section we prove that solutions of Jensen's functional equation on G are functions on $G/[G,[G,G]]$, and we write down some important formulas and relations for commutators and for odd solutions of Jensen's functional equation.

As mentioned in the introduction the normal subgroup $[G,[G,G]]$ will play an important role. To compute modulo $[G,[G,G]]$ is of course the same as working in the metabelian group $G/[G,[G,G]]$. Lemma 2.1 below is very useful for manipulations of commutators modulo $[G,[G,G]]$. The lemma is rather obvious, but it helps to have it stated.

Lemma 2.1

(a) $[x,y] = [y,x]^{-1}$ *for all* $x,y \in G$.
(b) *Commutators belong to* $Z(G)$ *modulo* $[G,[G,G]]$*: For all* $x,y,u \in G$ *we have* $u[x,y] = [x,y]u$ *modulo* $[G,[G,G]]$.
(c) *The commutator product* $[\cdot,\cdot] : G\times G \to G$ *is bi-multiplicative in the following sense:*

$$[u,xy] = [u,x][u,y] \ \text{ and } \ [xy,u] = [x,u][y,u], \quad \forall\, x,y,u \in G, \tag{2}$$

modulo $[G,[G,G]]$. *In particular*

$$[x^m, y^n] = [x,y]^{mn} \ \text{ mod } [G,[G,G]] \text{ for all } x,y \in G. \tag{3}$$

(d) *Cyclic permutations of elements in a commutator do not change it modulo* $[G,[G,G]]$. *In other words,* $[x,y] = \left[y^{-1},x\right] = \left[x^{-1},y^{-1}\right] = \left[y,x^{-1}\right]$ *for all* $x,y \in G$ *modulo* $[G,[G,G]]$.

Proof: (a) and (b) are trivial. (c) The second identity of (2) is an immediate consequence of the first one and (b), because $[v,w]^{-1} = [w,v]$ for all $v,w \in G$. So it suffices to prove the first one. But this one follows from (b) and the derivation formula $[u,xy] = [u,x]x[u,y]x^{-1}$. (d) follows from (a) and (c). □

The statement (a) of Theorem 2.2 below is a special case of (2.2) of [9], while formula (4) is (2.3) of [9] with $z = x^{-1}y^{-1}$. The formula (5) can be found as the formula (2.9) of [10]. Indeed, the function B of [10] is the difference between the left and right hand sides of (5). We give direct proofs of these relations. The property (2.5) of solutions from [9; Theorem 2] is crucial for our proof of Theorem 2.2(d).

Theorem 2.2

Let $f \in S(G,\mathbf{C})$. *Then*

(a) $f(x^n) = nf(x)$ *for all* $x \in G$ *and* $n \in \mathbf{Z}$.
(b) *For all* $x,y \in G$ *we have*

$$f(xy) = f(x) + f(y) + \frac{1}{2} f([x,y]). \tag{4}$$

(c) $f([x,yz]) = f([x,y]) + f([x,z])$ *for all* $x,y,z \in G$.
(d) f *is a function on* $G/[G,[G,G]]$ *vanishing identically on* $[G,[G,G]]$.
(e) *The formula (4) generalizes: For any* $x_1,\cdots,x_N \in G$ *and* $N = 1,2,\cdots$ *we have*

$$f(x_1\cdots x_N) = \sum_{i=1}^{N} f(x_i) + \frac{1}{2} f\left(\prod_{1\le i<j\le N} [x_i,x_j]\right), \tag{5}$$

where the last term on the right shall be interpreted as 0 when $N = 1$.
(f) $\phi := f_{|[G,G]}$ *is a function on* $[G,G]/[G,[G,G]]$ *such that* $\phi \in$ *Hom*$([G,G],\mathbf{C})$. *Furthermore* ϕ *vanishes identically on* $[G,[G,G]]$.

Proof: (a) Since f is odd it suffices to prove (a) for $n \geq 0$. It is trivially true for $n = 0$ and $n = 1$, and for $n = 2$ we find that $f\big(x^2\big) = f\big(x^2\big) + f(e) = f(xx) + f\big(xx^{-1}\big) = 2f(x)$. We proceed by induction: Assuming the formula true for $n \geq 2$ we compute

$$\begin{aligned} f\big(x^{n+1}\big) &= f(x^n x) + f\big(x^n x^{-1}\big) - f\big(x^n x^{-1}\big) = 2f(x^n) - f\big(x^{n-1}\big) \\ &= 2nf(x) - (n-1)f(x) = (n+1)f(x), \end{aligned} \tag{6}$$

which is the induction step.

(b) Replacing x by ab and y by $a^{-1}b^{-1}$ in (1) we get $f([a,b]) + f\big(ab^2a\big) = 2f(ab)$. Replacing x by a and y by b^2a in (1) we get $f\big(ab^2a\big) + f\big(b^{-2}\big) = 2f(a)$. Subtracting the second identity from the first we get $f([a,b]) - f\big(b^{-2}\big) = 2f(ab) - 2f(a)$, which reduces to (b) in view of (a).

(c) Using (4) on both sides of (c) we see that (c) is equivalent to $2f(xyz) = 2f(xy) + 2f(xz) + 2f(yz) - 2f(x) - 2f(y) - 2f(z)$. But this is the identity (2.5) of [9]. This proves (c).

Here we interrupt the proof of Theorem 2.2, because we need the following Lemma 2.3. The lemma is formulated for a subgroup of G, which need not be all of G. The reason for this is that we later (Theorem 3.2) shall extend solutions defined on certain subgroups of G to all of G, so we will need results about solutions on subgroups.

Lemma 2.3

Let $f_0 \in S(G_0, \mathbf{C})$ where G_0 is a subgroup of G such that $G_0 \supseteq [G,[G,G]]$. Assume that $f_0([x,[y,z]]) = 0$ for all $x,y,z \in G$. Then

(a) *f_0 is a function on $G_0/[G,[G,G]]$.*
(b) *f_0 vanishes on all of $[G,[G,G]]$.*
(c) *f_0 is a homomorphism of $G_0 \cap [G,G]$ into $\mathbf{C}$.*

Proof: We will deduce that f_0 is identically 0 on the entire subgroup $[G,[G,G]]$.

The assumption says that the following statement is true for $n = 1$:

Statement: $f_0([x, w_1 \cdots w_n]) = 0$ for all $x \in G$ and all w_j of the form $w_j = [y_j, z_j]$, where $y_1, \cdots, y_n, z_1, \cdots, z_n \in G$.

We prove the statement for all n by induction. So we assume it true for n, and let $w_{n+1} = [y_{n+1}, z_{n+1}]$, where $y_{n+1}, z_{n+1} \in G$. Let us recall the identity $[x, yz] = [x,y][i(y)(x), i(y)(z)]$, where $i(x)(y) = xyx^{-1}$ for all $x,y,z \in G$. Using that and the formula (4) we find with the abbreviation $w' := w_1 \cdots w_n$ that

$$\begin{aligned} &f_0([x, w_1 \cdots w_n w_{n+1}]) = f_0\big(\big[x, w' w_{n+1}\big]\big) \\ &= f_0\big(\big[x, w'\big]\big[i\big(w'\big)(x), i\big(w'\big)(w_{n+1})\big]\big) \\ &= f_0\big(\big[x, w'\big]\big) + f_0\big(\big[i\big(w'\big)(x), i\big(w'\big)(w_{n+1})\big]\big) \\ &+ \frac{1}{2} f_0\big(\big[\big[x, w'\big], \big[i\big(w'\big)(x), i\big(w'\big)(w_{n+1})\big]\big]\big). \end{aligned} \tag{7}$$

Here the first term on the right vanishes by the induction hypothesis and the third term because $f_0([x,[y,z]]) = 0$ for all $x, y, z \in G$ by assumption. Thus

$$\begin{aligned} & f_0([x, w_1 \cdots w_n w_{n+1}]) = f_0\big(\big[i\big(w'\big)(x), i\big(w'\big)(w_{n+1})\big]\big) \\ & = f_0\big(\big[i\big(w'\big)(x), i\big(w'\big)([y_{n+1}, z_{n+1}])\big]\big) \\ & = f_0\big(\big[i\big(w'\big)(x), \big(\big[i\big(w'\big)(y_{n+1}), i\big(w'\big)(z_{n+1})\big]\big)\big]\big), \end{aligned} \tag{8}$$

which also vanishes by the assumption of the Lemma. This finishes the proof of the induction step and hence proves the statement.

Due to the fact that f_0 is odd we may conclude:

$$f_0\Big([x,w]^{\pm 1}\Big) = 0 \ \text{ for } x \in G,\ w \in [G,G]. \tag{9}$$

To get that $f_0 = 0$ on $[G,[G,G]]$ it is left to prove that f_0 vanishes on products of the form $[x_1, w_1]^{\pm 1}[x_2, w_2]^{\pm 1} \cdots [x_n, w_n]^{\pm 1}$, where $x_1, \cdots, x_n \in G$ and $w_1, \cdots, w_n \in [G,G]$. We prove this by induction on n. The statement (9) is the case of $n = 1$. Assume that the statement is true for n factors, and consider $f_0(a[x,w])$, where a has the form $a = [x_1, w_1]^{\pm 1}[x_2, w_2]^{\pm 1} \cdots [x_n, w_n]^{\pm 1}$ with $x_1, \cdots, x_n \in G$ and $w_1, \cdots, w_n \in [G,G]$, and where $x \in G$ and $w \in [G,G]$. Using the formula (4) we find that $f_0(a[x,w]) = f_0(a) + f_0([x,w]) + f_0([a,[x,w]])/2$. The first term $f_0(a) = 0$ due to the induction hypothesis. The second term $f_0([x,w]) = 0$ according to (9). The third term $f_0([a,[x,w]])/2 = 0$ by the assumption of the Lemma. Hence $f_0(a[x,w]) = 0$. In a similar way we prove that $f_0\Big(a[x,w]^{-1}\Big) = 0$. Indeed,

$$\begin{aligned} & f_0\Big(a[x,w]^{-1}\Big) = f_0(a) + f_0\Big([x,w]^{-1}\Big) + \frac{1}{2} f_0\Big(\Big[a, [x,w]^{-1}\Big]\Big) \\ & = 0 - f_0([x,w]) + \frac{1}{2} f_0([a,[w,x]]) = 0 - 0 + 0 = 0. \end{aligned} \tag{10}$$

We have thus proved that $f_0 = 0$ on the group $[G,[G,G]]$.

If $x_0 \in G_0$ and $z \in [G,[G,G]]$ then we get from the formula (4) that

$$f_0(x_0 z) = f_0(x_0) + f_0(z) + \frac{1}{2} f_0([x_0, z]) = f_0(x_0) + 0 + 0 = f_0(x_0), \tag{11}$$

that shows that f_0 is a function on $G_0/[G,[G,G]]$.

The remaining statement (c) is easy to prove by help of the formula (4), so we skip its proof. □

<u>Continuation of the proof of Theorem 2.2:</u> (d) The property (c) says that $f([x,\cdot]) : G \to \mathbf{C}$ is a homomorphism for each $x \in G$. Hence it vanishes on commutators, i.e. $f([x,[y,z]]) = 0$ for all $y, z \in G$. We infer from Lemma 2.3 that f is a function on $G/[G,[G,G]]$, vanishing on all of $[G,[G,G]]$.

(f) Consequently $\phi = f_{|[G,G]}$ is a function on $[G,G]/[G,[G,G]]$, vanishing on $[G,[G,G]]$. Since ϕ is an odd solution of Jensen's functional equation on an abelian group, viz. on the group $[G,G]/[G,[G,G]]$, it is a homomorphism on that group. So $\phi \in \operatorname{Hom}([G,G]/[G,[G,G]],\mathbf{C}) \subseteq \operatorname{Hom}([G,G],\mathbf{C})$.

(e) It is left to prove the formula (5), which we do by induction on N. It is clear for $N=1$, so assume it true for an $N \geq 1$. Then by (4) and the induction hypothesis we find that

$$\begin{aligned} &f(x_1\cdots x_N x_{N+1}) = f(x_1\cdots x_N) + f(x_{N+1}) + \frac{1}{2}\phi([x_1\cdots x_N, x_{N+1}]) \\ &= \sum_{i=1}^{N} f(x_i) + \frac{1}{2}\phi\left(\prod_{1\leq i<j\leq N}[x_i,x_j]\right) + f(x_{N+1}) + \frac{1}{2}\phi([x_1\cdots x_N, x_{N+1}]). \end{aligned} \tag{12}$$

Using Lemma 2.1(c) on the last term on the right we find that

$$\begin{aligned} f(x_1\cdots x_N x_{N+1}) &= \sum_{i=1}^{N+1} f(x_i) \\ &+ \frac{1}{2}\phi\left(\prod_{1\leq i<j\leq N}[x_i,x_j]\right) + \frac{1}{2}\phi\left(\prod_{1\leq i\leq N}[x_i,x_{N+1}]\right), \end{aligned} \tag{13}$$

which finishes the induction, because ϕ is a homomorphism. □

It will be convenient to record the following fact, because it will be used a couple of times during proofs:

Lemma 2.4

If each element of G has finite order, then $S(G,\mathbf{C}) = \{0\}$.

Proof: Any $f \in S(G,\mathbf{C})$ satisfies $f(x^n) = nf(x)$ for all $x \in G$ and all $n \in \mathbf{Z}$ (Theorem 2.2(a)). Since $f(x^n)$ is bounded for any $x \in G$, hence so is $nf(x)$. But then $f(x) = 0$. □

We see from Theorem 2.2 that the study of Jensen's functional equation on groups (at least in principle) reduces to the study of Jensen's functional equation on metabelian groups, because the quotient group $G/[G,[G,G]]$ always is metabelian. To take an example, if G is the free group on 2 generators, then $G/[G,[G,G]]$ is isomorphic to the metabelian group $\mathbb{H}_3(\mathbf{Z})$ (The Heisenberg group with integer entries) from Example 5.1.

Many computations are easy for a function f on $G/[G,[G,G]]$, because commutators occurring in arguments of f may be moved freely around and formulas more generally be manipulated according to Lemma 2.1 without any change of the value of f. We have for example that $f(u[x,y]v) = f(uv[x,y])$ and $f([x,yz]) = f([x,y][x,z])$ for all $x,y,z,u,v \in G$.

Theorem 2.2 explains some of the properties of solutions of Jensen's functional equation. Here are two examples: Lemma 2.1(d) implies that the value of a solution on a commutator does not change under cyclic permutations of the elements; this result was derived by Corovei during his proof of Theorem 6 of [5]. Also the identity (2.12) of [9] is an easy consequence of Lemma 2.1, once it is noted that $f([x,y]) = 2A(x,y)$ and that f is a homomorphism on $[G,G]$ because $\phi \in$ Hom$([G,G], \mathbf{C})$.

3. The parametrization of the solutions

It is clear that $S(G,\mathbf{C})$ is a complex vector space, and that Hom$(G,\mathbf{C})$ is a subspace. In this section Theorem 3.2 shows that $S(G,\mathbf{C})/$ Hom$(G,\mathbf{C})$ may be parametrized by those homomorphisms of $[G,G]$ into $\mathbf{C}$ that vanish on all of $[G,[G,G]]$.

A decisive step in the proof is to show that there exists a solution of Jensen's functional equation on the entire group G with certain prescribed values on the commutator subgroup $[G,G]$. We shall find such a solution by successively extending it from $[G,G]$ to larger and larger subgroups of G. The extension steps will employ Lemma 3.1 below.

In the lemma it is assumed that the subgroup G_0 is normal. This will automatically be satisfied in our applications of the lemma, because there $G_0 \supseteq [G,G]$.

Lemma 3.1

Let G_0 be a normal subgroup of G containing $[G,[G,G]]$. Let H_1 be a subgroup of G which is abelian modulo $[G,[G,G]]$, and let G_1 denote the subgroup of G generated by H_1 and G_0.

Let $f_0 \in S(G_0,\mathbf{C})$ be such that $f_0 = 0$ on $[G,[G,G]]$. Let $\phi_1 \in$ Hom$(H_1,\mathbf{C})$ be such that $\phi_1 = f_0$ on $H_1 \cap G_0$.

Then there exists exactly one $f_1 \in S(G_1,\mathbf{C})$ such that $f_1 = f_0$ on G_0 and $f_1 = \phi_1$ on H_1. It is given by

$$f_1(x_1x_0) = \phi_1(x_1) + f_0(x_0) + \frac{1}{2}f_0([x_1,x_0]) \quad \textit{for} \ \ x_1 \in H_1, \ x_0 \in G_0. \qquad (14)$$

Proof: Any element in G_1 can be written in the form x_1x_0 for some $x_1 \in H_1$ and $x_0 \in G_0$, because G_0 is normal, so it makes sense to say that f_1 is given by (14) on G_1.

If f_1 is a solution such that $f_1 = f_0$ on G_0 and $f_1 = \phi_1$ on H_1, then we get from (4) for $x_1 \in H_1$ and $x_0 \in G_0$ that

$$f_1(x_1x_0) = f_1(x_1) + f_1(x_0) + \frac{1}{2}f_1([x_1,x_0]) = \phi_1(x_1) + f_0(x_0) + \frac{1}{2}f_0([x_1,x_0]), \quad (15)$$

which shows the uniqueness of the solution f_1.

We will next verify that the function f_1, given by (14), is well-defined on G_1. So assuming that $x_1x_0 = y_1y_0$ where $x_1, y_1 \in H_1$ and $x_0, y_0 \in G_0$ we shall deduce that

$$\Delta := \phi_1(x_1) - \phi_1(y_1) + f_0(x_0) - f_0(y_0) + \frac{1}{2}f_0([x_1,x_0]) - \frac{1}{2}f_0([y_1,y_0]) = 0. \quad (16)$$

Since $y_1^{-1}x_1 = y_0x_0^{-1} \in H_1 \cap G_0$, and ϕ_1 and f_0 agree on $H_1 \cap G_0$, we get using (4) that

$$\begin{aligned} &\phi_1(x_1) - \phi_1(y_1) + f_0(x_0) - f_0(y_0) = \phi_1\big(y_1^{-1}x_1\big) + f_0(x_0) - f_0(y_0) \\ &= f_0\big(y_0x_0^{-1}\big) + f_0(x_0) - f_0(y_0) \\ &= f_0(y_0) - f_0(x_0) + \frac{1}{2}f_0\big(\big[y_0,x_0^{-1}\big]\big) + f_0(x_0) - f_0(y_0) = \frac{1}{2}f_0\big(\big[y_0,x_0^{-1}\big]\big), \end{aligned} \quad (17)$$

so $2\Delta = f_0\big([y_0, x_0^{-1}]\big) + f_0([x_1, x_0]) - f_0([y_1, y_0])$. Using Lemma 2.3(c) we see that $2\Delta = f_0\Big([y_0, x_0^{-1}][x_1, x_0][y_1, y_0]^{-1}\Big)$, so to show that $\Delta = 0$ it suffices to prove that $[y_0, x_0^{-1}][x_1, x_0][y_1, y_0]^{-1} \in [G, [G, G]]$. Computing modulo $[G, [G, G]]$ we find by Lemma 2.1 that

$$
\begin{aligned}
&[y_0, x_0^{-1}][x_1, x_0][y_1, y_0]^{-1} = [x_0, y_0][x_1, x_0][y_0, y_1] \\
&= [x_0, y_1^{-1}x_1x_0][x_1, x_0][y_1^{-1}x_1x_0, y_1] \\
&= [x_0, y_1^{-1}][x_0, x_1][x_0, x_0][x_1, x_0][y_1^{-1}, y_1][x_1, y_1][x_0, y_1] = [x_1, y_1] = e,
\end{aligned}
\tag{18}
$$

where the last equality sign is justified because H_1 is abelian modulo $[G, [G, G]]$. Thus f_1, given by (14), is a well-defined function on G_1.

Obviously $f_1 = f_0$ on G_0 and $f_1 = \phi_1$ on H_1. It is left to show that $f_1 \in S(G_1, \mathbf{C})$, so we let $x_1, y_1 \in H_1$ and $x_0, y_0 \in G_0$ and compute

$$
\begin{aligned}
&L := f_1(x_1x_0y_1y_0) + f_1\Big(x_1x_0(y_1y_0)^{-1}\Big) \\
&= f_1\big(x_1y_1x_0[x_0^{-1}, y_1^{-1}]y_0\big) + f_1\big(x_1y_1^{-1}x_0y_0^{-1}[y_0x_0^{-1}, y_1]\big) \\
&= \phi_1(x_1y_1) + f_0\big(x_0[x_0^{-1}, y_1^{-1}]y_0\big) + \frac{1}{2}f_0\big([x_1y_1, x_0[x_0^{-1}, y_1^{-1}]y_0]\big) \\
&+ \phi_1(x_1y_1^{-1}) + f_0\big(x_0y_0^{-1}[y_0x_0^{-1}, y_1]\big) + \frac{1}{2}f_0\big([x_1y_1^{-1}, x_0y_0^{-1}[y_0x_0^{-1}, y_1]]\big).
\end{aligned}
\tag{19}
$$

Since the commutators $[x_0^{-1}, y_1^{-1}]$ and $[y_0x_0^{-1}, y_1]$ belong to $Z(G)$ modulo $[G, [G, G]]$ they disappear from the two terms with the factor 1/2 in front, so we get that

$$
\begin{aligned}
&L := \phi_1(x_1) + \phi_1(y_1) + f_0(x_0y_0) + f_0\big([x_0^{-1}, y_1^{-1}]\big) + \frac{1}{2}f_0([x_1y_1, x_0y_0]) \\
&+ \phi_1(x_1) - \phi_1(y_1) + f_0\big(x_0y_0^{-1}\big) + f_0\big([y_0x_0^{-1}, y_1]\big) + \frac{1}{2}f_0\big([x_1y_1^{-1}, x_0y_0^{-1}]\big) \\
&= 2\phi_1(x_1) + 2f_0(x_0) \\
&+ \frac{1}{2}f_0\Big([x_0^{-1}, y_1^{-1}]^2[x_1y_1, x_0y_0][y_0x_0^{-1}, y_1]^2[x_1y_1^{-1}, x_0y_0^{-1}]\Big) \\
&= 2f_1(x_1x_0) - f_0([x_1, x_0]) \\
&+ \frac{1}{2}f_0\Big([x_0^{-1}, y_1^{-1}]^2[x_1y_1, x_0y_0][y_0x_0^{-1}, y_1]^2[x_1y_1^{-1}, x_0y_0^{-1}]\Big) \\
&= 2f_1(x_1x_0) \\
&+ \frac{1}{2}f_0\Big([x_1, x_0]^{-2}[x_0^{-1}, y_1^{-1}]^2[x_1y_1, x_0y_0][y_0x_0^{-1}, y_1]^2[x_1y_1^{-1}, x_0y_0^{-1}]\Big),
\end{aligned}
\tag{20}
$$

so it suffices to show that the argument of f_0 in the last term, i.e.

$$
[x_1, x_0]^{-2}[x_0^{-1}, y_1^{-1}]^2[x_1y_1, x_0y_0][y_0x_0^{-1}, y_1]^2[x_1y_1^{-1}, x_0y_0^{-1}] \tag{21}
$$

is e modulo $[G, [G, G]]$. And that is elementary in view of Lemma 2.1. $\square$

We have in Theorem 2.2 seen that functions in $S(G, \mathbf{C})$ restrict to homomorphisms of the quotient subgroup $[G, G]/[G, [G, G]]$ into $\mathbf{C}$. This is the basis of Theorem 3.2.

Theorem 3.2

The restriction map $f \to f|_{[G,G]}$ *of* $S(G, \mathbf{C})$ *into* *Hom* $([G,G]/[G,[G,G]], \mathbf{C})$ *is a linear, surjective map, and its kernel is Hom*$(G, \mathbf{C})$.

Proof: The linearity is obvious, and the statement about the kernel follows from Lemma 6.2 below, which is proved independently of the present theorem.

In proving the surjectivity we may assume that G is metabelian. Given a homomorphism $\phi : [G,G] \to \mathbf{C}$ we shall produce $f_0 \in S(G, \mathbf{C})$ such that $f_0 = \phi$ on $[G,G]$.

Consider the set $\mathcal{P}$ of all pairs $\{f, H\}$ where H is a subgroup of G containing $[G,G]$, and where $f \in S(H, \mathbf{C})$ is such that $f = \phi$ on $[G,G]$. We introduce a partial ordering in $\mathcal{P}$ by writing $\{f_1, H_1\} \geq \{f_2, H_2\}$ if and only if $H_1 \supseteq H_2$ and $f_1 = f_2$ on H_2. Obviously each totally ordered subset of $\mathcal{P}$ has a majorant in $\mathcal{P}$ (the subgroup is the union of the subgroups occurring in the subset), so by Zorn's lemma there is a maximal element $\{f_0, G_0\}$ in $\mathcal{P}$. It is left to show that $G_0 = G$. This we do by contradiction, so we assume the existence of an element $a \in G \setminus G_0$.

Taking $H_1 := \langle a \rangle$ in Lemma 3.1 we see that it to derive a contradiction to the maximality of $\{f_0, G_0\}$, suffices to find a $\phi_1 \in$ Hom $(\langle a \rangle, \mathbf{C})$ such that $\phi_1 = f_0$ on $\langle a \rangle \cap G_0$.

(a) If $a^p \notin G_0$ for each $p \in \{1, 2, \cdots\}$ then we take $\phi_1 = 0$.

(b) If $a^p \in G_0$ for some $p \in \{2, 3, \cdots\}$ then let p be the smallest such positive integer. There are two possibilities:

  a. $\langle a \rangle$ is a finite group. Here $f_0(\langle a \rangle \cap G_0) = \{0\}$ by Lemma 2.4, so we may take $\phi_1 = 0$.

  b. $\langle a \rangle$ is an infinite group. Here we may define ϕ_1 by $\phi_1(a) := f_0(a^p)/p$. □

It is known (see for example Friis [6]) that even in the simple case of the Heisenberg group there are odd solutions of Jensen's functional equation that are not homomorphisms (see Example 5.1 below). Ng [9; Corollary 8] earlier observed the same for the free group on two generators. Theorem 3.2 tells us that this phenomenon is a common trait of Jensen's functional equation on non-abelian groups: For any homomorphism $\phi \in$ Hom $([G,G]/[G,[G,G]], \mathbf{C})$ there exist odd solutions f of Jensen's functional equation on G such that $f = \phi$ on $[G,G]$, and the solutions f are only homomorphisms when $\phi = 0$.

A natural question is for which groups the parameter space degenerates to $\{0\}$, so that $S(G, \mathbf{C}) =$ Hom $(G, \mathbf{C})$. Proposition 3.3 characterizes these groups in a theoretical way, while Section 6 contains sufficient conditions on G implying that $S(G, \mathbf{C}) =$ Hom $(G, \mathbf{C})$.

Proposition 3.3

The following three statements are equivalent

(a) $S(G, \mathbf{C}) =$ *Hom* $(G, \mathbf{C})$.

(b) *Hom* $([G,G]/[G,[G,G]],\mathbf{C}) = \{0\}$.

(c) $[G,G]/[G,[G,G]]$ *is a torsion group, i.e. an abelian group in which each element has finite order.*

Proof. (a)⇒(b): Assume that $S(G,\mathbf{C}) = \text{Hom}\,(G,\mathbf{C})$. Any $\phi \in \text{Hom}\,([G,G]/[G,[G,G]],\mathbf{C})$ extends according to Theorem 3.2 to a solution $f_\phi \in S(G,\mathbf{C})$. By assumption $f_\phi \in \text{Hom}\,(G,\mathbf{C})$, so ϕ is the restriction of a homomorphism of G. But then it vanishes on commutators and hence on the commutator subgroup $[G,G]$.

(b)⇒(c): If $[G,G]/[G,[G,G]]$ is not a torsion group, then there exists an $x \in [G,G]/[G,[G,G]]$ such that $\langle x\rangle$ is an infinite cyclic group. We may define a homomorphism $\phi : \langle x\rangle \to \mathbf{C}$ by $\phi(x^n) = n$, $n \in \mathbf{Z}$. Now ϕ extends to a homomorphism $\phi : [G,G]/[G,[G,G]] \to \mathbf{C}$, because $\mathbf{C}$ is divisible (see Theorem A.7 of [8]).

(c)⇒(a): If $[G,G]/[G,[G,G]]$ is a torsion group, then $\text{Hom}\,([G,G]/[G,[G,G]],\mathbf{C}) = \{0\}$ by Lemma 2.4. If $f \in S(G,\mathbf{C})$, then $f \in \text{Hom}\,([G,G]/[G,[G,G]],\mathbf{C})$ by Theorem 2.2, so $f|_{[G,G]} = 0$. And then $f \in \text{Hom}\,(G,\mathbf{C})$ by the formula (4). □

4. Explicit solution formulas

Theorem 3.2 says that any homomorphism $\phi : [G,G] \to \mathbf{C}$ that vanishes on $[G,[G,G]]$ extends to a solution of Jensen's functional equation on all of G, but Theorem 3.2 does not construct an extension. The extensions are only determined modulo the homomorphisms of G into $\mathbf{C}$, so unless there is some information on G it is not possible to single out which extension to choose for a given ϕ.

Nevertheless, we proceed by constructing extensions, but of course with extra information on G, namely on the structure of G. We even give explicit formulas for the extensions.

Theorem 4.1

Let $\{H_\alpha \mid \alpha \in A\}$ be a family of abelian subgroups of G, such that the mapping $\{x_\alpha\}_{\alpha\in A} \to \prod_{\alpha\in A} x_\alpha[G,G]$ is a bijection of $\prod_{\alpha\in A} H_\alpha$ onto $G/[G,G]$. Then for any $\phi \in$ Hom $([G,G]/[G,[G,G]],\mathbf{C})$ there exists exactly one function $f_\phi \in S(G,\mathbf{C})$ such that $f_\phi = \phi$ on $[G,G]$ and $f_\phi = 0$ on each of the subgroups H_α, $\alpha \in A$, of G. If $x = x_1x_2\cdots x_Nc$, where $x_i \in H_{\alpha_i}$ for some $\alpha_i \in A$ for $i = 1,2,\cdots,N$, and $c \in [G,G]$, then

$$f_\phi(x_1x_2\cdots x_Nc) = \frac{1}{2}\phi\left(\prod_{1\le i<j\le N}[x_i,x_j]\right) + \phi(c). \tag{22}$$

Modulo Hom $(G,\mathbf{C})$ all functions in $S(G,\mathbf{C})$ are obtained in this way for varying ϕ.

Proof: The uniqueness of f_ϕ and the formula (22) are obvious from the formula (5). According to Theorem 3.2 there exists an $f \in S(G,\mathbf{C})$ such that $f = \phi$ on $[G,G]$.

We define $l : G \to \mathbf{C}$ by

$$l(h_1 h_2 \cdots h_N c) := f(h_1) + f(h_2) + \cdots + f(h_N) \tag{23}$$

for $h_i \in H_{\alpha_i}$, $i = 1, 2, \cdots, N$, and $c \in [G, G]$. Here $\alpha_i \neq \alpha_j$ for $i \neq j$. If $x = h_1 h_2 \cdots h_N c$ and $x' = h'_1 h'_2 \cdots h'_N c'$, where $h_i, h'_i \in H_{\alpha_i}$ for $i = 1, 2, \cdots, N$, and $c, c' \in [G, G]$, then $xy = (h_1 h'_1)(h_2 h'_2) \cdots \left(h_N h'_N\right) c''$ for some $c'' \in [G, G]$. Noting that $f(x_i x'_i) = f(x_i) + f(x'_i)$ for each $i = 1, 2, \cdots, N$, because H_{α_i} is abelian so that $S(H_{\alpha_i}, \mathbf{C}) = \text{Hom}\,(H_{\alpha_i}, \mathbf{C})$, we find that

$$\begin{aligned} l(xy) &= l\Big(\big(h_1 h'_1\big)\big(h_2 h'_2\big) \cdots \big(h_N h'_N\big) c''\Big) \\ &= f\big(h_1 h'_1\big) + f\big(h_2 h'_2\big) + \cdots + f\big(h_N h'_N\big) \\ &= f(h_1) + f\big(h'_1\big) + f(h_2) + f\big(h'_2\big) + \cdots + f(h_N) + f\big(h'_N\big) \\ &= f(h_1) + f(h_2) + \cdots + f(h_N) + f\big(h'_1\big) + f\big(h'_2\big) + \cdots + f\big(h'_N\big) \\ &= l(x) + l(y), \end{aligned} \tag{24}$$

so $l \in \text{Hom}\,(G, \mathbf{C})$. Since $l = f$ on each H_α, $\alpha \in A$, we see that $f_\phi := f - l$ is a solution with the desired properties. □

Corollary 4.2

Let the assumptions be as in Theorem 4.1. To any $\phi \in$ *Hom* $([G, G]/[G, [G, G]], \mathbf{C})$ *and any family* $\phi_\alpha \in$ *Hom* $(H_\alpha, \mathbf{C})$, $\alpha \in A$, *of homomorphisms, there exists exactly one* $f \in S(G, \mathbf{C})$ *such that* $f = \phi$ *on* $[G, G]$ *and* $f = \phi_\alpha$ *on* H_α *for each* $\alpha \in A$.

All functions in $S(G, \mathbf{C})$ *are obtained in this way for varying* ϕ *and* ϕ_α, $\alpha \in A$.

The case of only two factors in Corollary 4.2 is particularly interesting for our examples, so we single it out:

Theorem 4.3

Let H_1 *and* H_2 *be abelian subgroups of* G, *and assume that each element* $x \in G$ *in exactly one way may be written in the form* $x = x_1 x_2 c$ *where* $c \in [G, G]$ *and* $x_i \in H_i$ *for* $i = 1, 2$, *modulo* $[G, [G, G]]$. *Let* $\phi \in$ *Hom* $([G, G], \mathbf{C})$ *be such that* $\phi([x, [y, z]]) = 0$ *for all* $x, y, z \in G$. *Let* $\phi_i \in$ *Hom* $(H_i, \mathbf{C})$ *for* $i = 1, 2$.

Then there is exactly one function $f_\phi \in S(G, \mathbf{C})$ *such that* $f_\phi = \phi$ *on* $[G, G]$ *and* $f_\phi = \phi_i$ *on* H_i *for* $i = 1, 2$. *It is given by the formula*

$$\begin{gathered} f_\phi(x_1 x_2 c) = \phi_1(x_1) + \phi_2(x_2) + \frac{1}{2}\phi([x_1, x_2]) + \phi(c), \\ c \in [G, G],\ x_i \in H_i \text{ for } i = 1, 2. \end{gathered} \tag{25}$$

All functions in $S(G, \mathbf{C})$ *are obtained in this way for varying* ϕ, ϕ_1 *and* ϕ_2.

5. Examples

In this section we discuss some groups to which we can apply Theorem 4.3.

Example 5.1

The $(2n+1)$–dimensional Heisenberg group, $n = 1, 2, \cdots$, is the matrix group $G := \{(x, y, z) \in \mathbf{R}^{2n+1} \,|\, x \in \mathbf{R}^n, y \in \mathbf{R}^n, z \in \mathbf{R}\}$, where we use the abbreviation $(x, y, z) = ((x_1, x_2, \cdots, x_n), (y_1, y_2, \cdots, y_n), z)$ for the $(n+2) \times (n+2)$ matrix

$$(x,y,z) = \begin{Bmatrix} 1 & x_1 & x_2 & \cdots & x_n & z \\ 0 & 1 & 0 & \cdots & 0 & y_1 \\ \vdots & \vdots & \ddots & & 0 & y_2 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & y_n \\ 0 & 0 & \cdots & \cdots & 0 & 1 \end{Bmatrix}. \tag{26}$$

The 3–dimensional Heisenberg group, corresponding to $n = 1$, is usually just called the Heisenberg group.

The composition rule of G is $(x, y, z)(u, v, w) = (x + u, y + v, z + w + x \cdot v)$ for $(x, y, z), (u, v, w) \in G$, where $x \cdot v = (x_1, \cdots, x_n) \cdot (v_1, \cdots, v_n) := \sum_{i=1}^n x_i v_i$. The neutral element of G is the identity matrix $I = (0, 0, 0)$.

We find by elementary computations that $[G, G] = Z(G) = \{(0, 0, z) \in G \,|\, z \in \mathbf{R}\} \simeq \mathbf{R}$ and $[G, [G, G]] = \{I\}$, so G is metabelian. The pair $H_1 := \{(x, 0, 0) \in G \,|\, x \in \mathbf{R}^n\}$, $H_2 := \{(0, y, 0) \in G \,|\, y \in \mathbf{R}^n\}$ satisfies the conditions of Theorem 4.3, as the formula $(x, 0, 0)(0, y, 0)(0, 0, z) = (x, y, z + x \cdot y)$ $x, y \in \mathbf{R}^n$, $z \in \mathbf{R}$, shows.

Let $\phi : [G, G] \simeq \mathbf{R} \to \mathbf{C}$ be a homomorphism. The function f_ϕ from Theorem 4.3 is here

$$\begin{aligned} & f_\phi(x,y,z) = f_\phi((x,0,0)(0,y,0)(0,0,z - x\cdot y)) \\ & = \frac{1}{2}\phi([(x,0,0),(0,y,0)]) + \phi(z - x\cdot y) \\ & = \frac{1}{2}\phi(0,0,x\cdot y) + \phi(z - x\cdot y) = \frac{1}{2}\phi(2z - x\cdot y). \end{aligned} \tag{27}$$

The same arguments as above work for the Heisenberg group with integer entries, i.e. for the subgroup $\mathbb{H}_{2n+1}(\mathbf{Z}) = \{(x, y, z) \in G \,|\, x, y, z \in \mathbf{Z}\}$ of the $(2n+1)$–dimensional Heisenberg group.

Jensen's functional equation on the Heisenberg group was solved in [6; Section 4], but by other methods than ours.

Example 5.2 This example was thoroughly discussed in [9] by other methods than ours.

Let $G = \langle a, b \rangle$ be the free group on the two generators a and b. We claim that given $\alpha, \beta, \gamma \in \mathbf{C}$ there exists exactly one function $f \in S(G, \mathbf{C})$ such that $f(a) = \alpha$,

$f(b) = \beta$ and $f([a,b]) = \gamma$. The solution f can be given by the explicit formula

$$\begin{aligned} & f(a^{m_1}b^{n_1}\cdots a^{m_l}b^{n_l}) \\ &= \left(\sum_{i=1}^{l} m_i\right)\alpha + \left(\sum_{i=1}^{l} n_i\right)\beta + \left(\sum_{i\le j\le l} m_i n_j - \sum_{j<i\le l} m_i n_j\right)\gamma/2 \qquad (28) \end{aligned}$$

for $m_1, n_1, \cdots, m_l, n_l \in \mathbf{Z},\ l = 0, 1, \cdots$.

Proof of these claims: Let us note that $[G,G]$ modulo $[G,[G,G]]$ is the infinite cyclic group generated by the commutator $[a,b]$, and that G modulo $[G,[G,G]]$ decomposes into $G = \langle a\rangle\langle b\rangle\langle[a,b]\rangle$.

The uniqueness and the existence of a solution f is immediate from Theorem 4.3 with $H_1 = \langle a\rangle$ and $H_2 = \langle b\rangle$.

We shall only sketch how to derive (28). It suffices to verify that

$$f_\phi(a^{m_1}b^{n_1}\cdots a^{m_l}b^{n_l}) = \left(\sum_{i\le j\le l} m_i n_j - \sum_{j<i\le l} m_i n_j\right)\gamma/2, \qquad (29)$$

which can be done by induction on l. For $l = 1$ it is the formula (25). □

6. Conditions ensuring all odd solutions are homomorphisms

We will in this section find sufficient conditions of a general nature on a group G to ensure that $S(G,\mathbf{C}) = \text{Hom}\,(G,\mathbf{C})$.

It is a well known fact that $S(G,\mathbf{C}) = \text{Hom}\,(G,\mathbf{C})$ on any abelian group G. A slight improvement (Lemma 6.1) of this fact was proved by Aczél, Chung and Ng as Lemma 1 of [1]:

Lemma 6.1

The general solution $f : G \to \mathbf{C}$ of Jensen's functional equation (1) satisfying also $f(xy) = f(yx)$ for all $x, y \in G$, is given by $f(x) = \phi(x) + \alpha$, $x \in G$, where $\alpha \in \mathbf{C}$ is an arbitrary constant and ϕ an arbitrary homomorphism of G into the additive group $(\mathbf{C}, +)$ of $\mathbf{C}$, i.e. $\phi(xy) = \phi(x) + \phi(y)$, $x, y \in G$.

Lemma 6.2

The following 4 statements (i) – (iv) are equivalent for a function $f \in S(G,\mathbf{C})$:

(i) $f \in$ *Hom*$\,(G,\mathbf{C})$.
(ii) $f(xy) = f(yx)$ *for all* $x, y \in G$.
(iii) $f([x,y]) = 0$ *for all* $x, y \in G$, *i.e.* f *vanishes on each commutator.*
(iv) f *vanishes on the commutator subgroup* $[G,G]$.

<u>Proof:</u> Clearly (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii). The identity (4) shows that (iii) $\Rightarrow$ (ii). Finally Lemma 6.1 shows that (ii) $\Rightarrow$ (i). $\square$

Corovei [4] proved that $S(G,\mathbf{C}) = \text{Hom}\,(G,\mathbf{C})$ on any P_3–group G, i.e. a group in which all elements in the commutator subgroup have order 1 or 2 (also called a 3–rewritable group). In Theorem 6 of [5] he relaxed the assumptions: It suffices that each element of the commutator group has finite order. This is point (c) of Proposition 6.3 below.

Proposition 6.3

(a) *If the quotient group* $G/[G,[G,G]]$ *is abelian, then* $S(G,\mathbf{C}) = \textit{Hom}\,(G,\mathbf{C})$.
(b) *If* $[G,[G,G]] = [G,G]$, *then* $S(G,\mathbf{C}) = \textit{Hom}\,(G,\mathbf{C})$.
(c) *If each commutator has finite order then* $S(G,\mathbf{C}) = \textit{Hom}\,(G,\mathbf{C})$.

<u>Proof:</u> (a) is a consequence of Theorem 2.2(d) and the fact that $S(H,\mathbf{C}) = \text{Hom}\,(H,\mathbf{C})$ for any abelian group H.

(b) follows from (a), because the quotient group $G/[G,G]$ always is abelian.

(c): Any $f \in S(G,\mathbf{C})$ satisfies $f(x^n) = nf(x)$ for all $x \in G$ and all $n \in \mathbf{Z}$ (Theorem 2.2(a)). Since $f(x^n)$ is bounded for any $x \in [G,G]$, hence so is $nf(x)$. But then $f(x) = 0$, so f vanishes on each commutator. We then refer to Lemma 6.2. $\square$

Lemma 6.4

Let G_0 *be a normal subgroup of a group* G *such that* G/G_0 *is finite.*

(a) *If* $S(G_0,\mathbf{C}) = \textit{Hom}\,(G_0,\mathbf{C})$, *then* $S(G,\mathbf{C}) = \textit{Hom}\,(G,\mathbf{C})$.
(b) *If* $S(G_0,\mathbf{C}) = \{0\}$, *then* $S(G,\mathbf{C}) = \{0\}$.

<u>Proof:</u> (a) Let $f \in S(G,\mathbf{C})$. Let $x, y \in G$ be arbitrary and let m and n denote the orders of xG_0 and yG_0 in G/G_0. So $x^m \in G_0$ and $y^n \in G_0$. Now $f = 0$ on $[G_0, G_0]$ because $S(G_0,\mathbf{C}) = \text{Hom}\,(G_0,\mathbf{C})$ (Lemma 6.2), so that $f([x^m, y^n]) = 0$. We get by Lemma 2.1 and Theorem 2.2(a) that $0 = f([x^m, y^n]) = f\big([x,y]^{mn}\big) = mnf([x,y])$, so $f([x,y]) = 0$. We infer from Lemma 6.2 that $f \in \text{Hom}\,(G,\mathbf{C})$.

(b) is proved in a similar way. $\square$

Proposition 6.5

(a) *If* H *is a real or complex semisimple Lie group with at most finitely many connected components, then* $S(H,\mathbf{C}) = \{0\}$.
(b) *Let* $G = N \times_s H$ *be a semidirect product of an abelian group* N *(the normal part) and a semisimple Lie group* H, *where* H *has at most finitely many connected components. Then* $S(G,\mathbf{C}) = \textit{Hom}\,(G,\mathbf{C})$.

Proof: (a) The connected component H_e of the identity $e \in H$ is a connected semisimple Lie group, and so $H_e = [H_e, H_e]$ (see Corollary 3.18.10 of [11]). In particular $H_e = [H_e, [H_e, H_e]]$, so we get from Theorem 2.2 that $S(H_e, \mathbf{C}) = \{0\}$. (a) now follows from Lemma 6.4(b), because H_e is normal (Theorem 7.1 of [8]).

(b) Let us consider the subgroup $G_0 := N \times_s H_e$, Using that $H_e = [H_e, H_e]$ we see that $[G_0, [G_0, G_0]] \supseteq [H_e, [H_e, H_e]] = H_e$ so that the restriction to N of the natural projection map $\pi : G_0 \to G_0/[G_0, [G_0, G_0]]$ is surjective. Since N is abelian by assumption, so is $G_0/[G_0, [G_0, G_0]]$. We may thus refer to Proposition 6.3(a) to see that $S(G_0, \mathbf{C}) =$ Hom$(G_0, \mathbf{C})$. But $G_0 = N \times_s H_e$ is a normal subgroup of G of finite index, so $S(G, \mathbf{C}) =$ Hom$(G, \mathbf{C})$ by Lemma 6.4(a). □

7. Examples

In this section we collect some examples, mainly from geometry, in which the results of Section 6 are used.

Examples 7.1

Many examples of semisimple Lie groups are listed in Ch. X, §2 of [7]. We mention the special linear groups $SL(n, \mathbf{R})$ and $SL(n, \mathbf{C})$ for $n \geq 2$, and the orthogonal groups $O(n)$ and $SO(n)$ for $n \geq 3$. $SL(n, \mathbf{R})$, $SL(n, \mathbf{C})$ and $SO(n)$ are connected, while $O(n)$ has two connected components. The Lorentz group $O(3, 1)$ has 4 connected components.

The group of unit quaternions, which occurs as an example in some works on functional equations, e.g. in [1], is isomorphic to the connected semisimple Lie group $SU(2)$ (see section 1.9 of [3]).

If G is any one of these groups, then $S(G, \mathbf{C}) = \{0\}$ according to Proposition 6.5.

Example 7.2 This example was also treated in [10; Theorem 4], but by other methods than ours.

Consider the general linear group $G = GL(n, \mathbf{R})$ for $n \geq 2$. Since $[SL(n, \mathbf{R}), SL(n, \mathbf{R})] = SL(n, \mathbf{R})$ (Example 7.1) and since any commutator has determinant 1 and hence belongs to $SL(n, \mathbf{R})$, we see that $[G, G] = SL(n, \mathbf{R})$, so that $[G, [G, G]] = [G, G]$. According to Proposition 6.3(b) we then have $S(G, \mathbf{C}) =$ Hom$(G, \mathbf{C})$. We may write any $A = \{a_{ij}\} \in GL(n, \mathbf{R})$ in the form

$$\left\{ \begin{matrix} \det A & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{matrix} \right\} \left\{ \begin{matrix} a_{11}(\det A)^{-1} & a_{12}(\det A)^{-1} & \cdots & a_{1n}(\det A)^{-1} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix} \right\} \tag{30}$$

where the last factor has determinant 1 and hence belongs to $SL(n, \mathbf{R})$. From Example 7.1 we know that any $f \in S(G, \mathbf{C}) =$ Hom$(G, \mathbf{C})$ takes the value 0 on the last factor of (30), so

$$f(A) = f\left(\left\{\begin{matrix} \det A & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{matrix}\right\}\right) = \phi(\det A) \tag{31}$$

for some homomorphism $\phi : \mathbf{R}_\times \to \mathbf{C}$. An example of such a homomorphism is $\phi(t) = \log|t|,\ t \in \mathbf{R}_\times$.

The same discussion applies to $GL(n, \mathbf{C})$ for $n \geq 2$.

Remark 7.3

Example 7.2 can be generalized for $n \geq 3$. For any ring R with unit the subgroup $E_n(R)$ of $GL(n, R)$ generated by the elementary matrices satisfies $[E_n(R), E_n(R)] = E_n(R)$ (see [2]). If R is a field, or just a commutative Euclidean ring, then $E_n(R) = SL(n, R)$, and in that case we find like in Example 7.2 that $S(GL(n, R), \mathbf{C}) = \mathrm{Hom}\,(GL(n, R), \mathbf{C})$, and that any $f \in S(GL(n, R), \mathbf{C})$ has the form $f(A) = \phi(\det A)$ for some homomorphism ϕ of the group of units of R into $\mathbf{C}$.

The remaining examples of this Section discuss groups G that are semidirect products of two groups N and H, where N is the normal part and H acts on N. A typical example is the group of rigid motions of $\mathbf{R}^3$, where the rotation group $H = O(3)$ acts on Euclidean space $N = \mathbf{R}^3$. With Example 7.1 in mind the solutions of Jensen's functional equation on such groups can be computed by the formulas in [6; Section 4]. However, our methods are different from those of [6; Section 4], that only deals with semidirect products, and we want to illustrate how our results of Section 6 can be applied.

Let $G = N \times_s H$ be a semidirect product with N as the normal component. We let $h \cdot n$ denote the action by $h \in H$ on $n \in N$. For any $f \in \mathrm{Hom}(G, \mathbf{C})$, we have $f_{|N} \in \mathrm{Hom}(N, \mathbf{C})$ and a small computation reveals that

$$f_{|N}(h \cdot n) = f_{|N}(n) \ \text{ for } h \in H,\ n \in N. \tag{32}$$

Example 7.4

Let $G = N \times_s H$ be a semidirect product of an abelian group N and a semisimple Lie group H with at most finitely many connected components. Here $S(G, \mathbf{C}) = \mathrm{Hom}\,(G, \mathbf{C})$ according to Proposition 6.5(b).

Let $f \in S(G, \mathbf{C}) = \mathrm{Hom}\,(G, \mathbf{C})$. Since f vanishes on H (Proposition 6.5(a)), we get from the decomposition $(n, h) = (n, e)(0, h)$ that $f(n, h) = f(n, e) = \phi(n)$, where $\phi \in \mathrm{Hom}(N, \mathbf{C})$. Taking (32) into account we find that $\mathrm{Hom}\,(G, \mathbf{C})$ consists of the functions of the form $f(n, h) = f(n, e) = \phi(n)$, where $\phi \in \mathrm{Hom}(N, \mathbf{C})$ satisfies that $\phi(h \cdot n) = \phi(n)$ for all $h \in H$ and $n \in N$.

If the action of H on G is trivial, so that the semidirect product is a direct product, then $\mathrm{Hom}(G, \mathbf{C})$ consists of the functions $f : G \to \mathbf{C}$ of the form $f(n, h) = \phi(n)$, where ϕ ranges over $\mathrm{Hom}(N, \mathbf{C})$.

But if there exists an $h \in H$ such that $h \cdot n = -n$ for all $n \in N$, then $\phi = 0$ and hence also $f = 0$. Examples of this are the groups $G = \mathbf{R}^n \times_s O(n)$ of rigid motions of $\mathbf{R}^n$, $n \geq 3$, and the inhomogeneous Lorentz group $G = \mathbf{R}^4 \times_s SO(3,1)$. For these groups $S(G, \mathbf{C}) = \{0\}$.

Example 7.5

The $(ax + b)$–group, i.e. the group of affine motions of the real line

$$G := \left\{ (b, a) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in]0, \infty[,\ b \in \mathbf{R} \right\}, \tag{33}$$

is the semidirect product $G = \mathbf{R} \times_s \mathbf{R}^+$. We find by elementary computations that $[G, G] = [G, [G, G]] = \{(1, x) \mid x \in \mathbf{R}\}$, so $S(G, \mathbf{C}) = \mathrm{Hom}(G, \mathbf{C})$ by Proposition 6.3(b). Using the decomposition $(b, a) = (b, 1)(0, a)$ we find by help of (32) that $f = 0$ on the first factor. So the odd solutions of Jensen's functional equation on G are the functions of the form $f(b, a) = \phi(a)$, $(b, a) \in G$, where $\phi : \mathbf{R}^+_\times \to \mathbf{C}$ is a homomorphism.

This example was discussed in [6] by other methods than ours.

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