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By Morten Skarsholm Risager

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Ny Munkegade, Bldg. 530
DK-8000 Aarhus C, Denmark

http://www.imf.au.dk
institut@imf.au.dk

ASYMPTOTIC DENSITIES OF NEWFORMS

MORTEN SKARSHOLM RISAGER

Abstract. We define the counting function for non-analytic (Maass) newforms of Hecke congruence groups $\Gamma_0(M)$. We then calculate the three main terms of this counting function and give necessary and sufficient conditions on M for this counting function to have the same shape as if it were counting eigenvalues related to a co-compact group.

1. Introduction

Let Γ be a congruence subgroup of the full modular group. It is well known that the selfadjoint automorphic Laplacian, Δ_Γ, has infinitely many eigenvalues,

$$0 = \lambda_0 \leq \lambda_1^\Gamma \leq \ldots \leq \lambda_i^\Gamma \leq \ldots,$$

listed with their multiplicities which are finite. Selberg has proved that the counting function

$$N_\Gamma(\lambda) = \#\{i | \lambda_i^\Gamma \leq \lambda\}$$

satisfies a Weyl law namely

$$N_\Gamma(\lambda) = \frac{|F_\Gamma|}{4\pi}\lambda + O(\sqrt{\lambda}\log\lambda), \tag{1}$$

where $|F_\Gamma|$ is the area of a fundamental domain of Γ. There are various refinements of this result (see e.g. theorem 4.3).

In this paper we investigate what happens if we only count the eigenvalues corresponding to newforms.

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2. Newforms and oldforms

The theory of newforms was originally developed by Atkin & Lehner [1970] for holomorphic forms. Their theory can be translated into a similar theory of Maass forms which are the ones we are studying. This has been done independently by various people and details may be found in e.g. [Strömbergsson 2001]. We shall only need one result (Lemma 3.1 below) and shall hence only sketch enough of the theory for this result to make sense.

For any $\lambda > 0$, $M \in \mathbb{N}$ we denote by $A(\lambda, M)$ the λ-eigenspace for $\Delta_{\Gamma_0(M)}$, where $\Gamma_0(M)$ is the Hecke congruence group of level M i.e.

$$\Gamma_0(M) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \middle| \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} c \equiv 0 \mod M \right\}.$$

Then it is obvious that

$$N_{\Gamma_0(M)}(\lambda) = 1 + \sum_{0 < \tilde{\lambda} \leq \lambda} \dim A(\tilde{\lambda}, M), \tag{2}$$

where the sum is certainly finite.

We define the λ-oldspace to be

$$A_{\text{old}}(\lambda, M) := \text{span}\{ f(dz) | f \in A(\lambda, K) \quad Kd|M \quad K \neq M \}.$$

This is contained in $A(\lambda, M)$ by the $\mathrm{SL}_2(\mathbb{R})$-invariance of Δ_Γ, and the fact that $f(dz)$ is $\Gamma_0(M)$-invariant when $f(z)$ is $\Gamma_0(K)$-invariant and $Kd|M$. We then define the λ-newspace to be the orthogonal complement in $A(\lambda, M)$ with respect to the inner product

$$(f, g) = \int_{F_{\Gamma_0(M)}} f(z) \overline{g(z)} d\mu(z),$$

i.e.

$$A_{\text{new}}(\lambda, M) := A(\lambda, M) \ominus A_{\text{old}}(\lambda, M).$$

We then define new spectral counting functions

$$\begin{aligned} N^{\text{old}}_{\Gamma_0(M)}(\lambda) &:= 1 + \sum_{0 < \tilde{\lambda} \leq \lambda} \dim A_{\text{old}}(\tilde{\lambda}, M) \qquad M > 0 \\ N^{\text{new}}_{\Gamma_0(M)}(\lambda) &:= \sum_{0 < \tilde{\lambda} \leq \lambda} \dim A_{\text{new}}(\tilde{\lambda}, M) \qquad M > 0. \end{aligned}$$

For $M = 1$ we of course define $N^{\text{old}}_{\Gamma_0(1)}(\lambda) = 0$ and $N^{\text{new}}_{\Gamma_0(1)}(\lambda) = N_{\Gamma_0(1)}(\lambda)$.

3. Calculation of asymptotic densities

In order to calculate the main terms of $N^{\text{new}}_{\Gamma_0(M)}$ we remind about some well known structure theory of arithmetical functions. When $f, g : \mathbb{N} \to \mathbb{C}$ are arithmetical functions we define the *Dirichlet convolution*, $f * g : \mathbb{N} \to \mathbb{C}$ to be the arithmetical function

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

We say that f is multiplicative if $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$. The structure theory we shall use is the following:

Theorem 3.1. *The arithmetical functions form a commutative group under Dirichlet convolution. The identity element is the function*

$$\begin{array}{rcl} I: \mathbb{N} & \to & \mathbb{C} \\ n & \mapsto & \left[\frac{1}{n}\right] = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

The multiplicative arithmetical functions form a subgroup.

Proof. This follows from [Apostol 1976, Theorems 2.6,2.8,2.14, 2.16] □

Example 3.1. (See [Apostol 1976, §2.13] for details.) Consider the arithmetical function

$$\sigma_\alpha(n) = \sum_{d|n} d^\alpha.$$

Then this in a *multiplicative* arithmetical function whose inverse may be calculated to be

$$\sigma_\alpha^{-1}(n) = \sum_{d|n} d^\alpha \mu(d) \mu\left(\frac{n}{d}\right), \tag{3}$$

where μ is the Möbius function, i.e.

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n = p_1 \cdots p_k \\ 0 & \text{otherwise.} \end{cases}$$

The Mangoldt Λ-function

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{where } p \text{ is a prime and } m \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

is an example of a non-multiplicative function. Another multiplicative arithmetical function we will use is Eulers totient function

$$\Phi(n) = \#\{d \in \mathbb{N} | 1 \leq d \leq n \wedge (d,n) = 1\}.$$

We can now begin to calculate asymptotic densities of newforms. We cite a result from [Strömbergsson 2001].

Lemma 3.1.

$$\dim A(\lambda, \cdot) = \sigma_0 * \dim A_{\text{new}}(\lambda, \cdot).$$

Proof. This is Theorem 4.6.c) in Chapter III of [Strömbergsson 2001]. □

Let now f_i, $i = 1 \dots n$ be real positive functions of decreasing order i.e

$$f_{i+1} = o(f_i) \text{ for } i = 1 \dots n-1.$$

Proposition 3.1. *Assume that for any $M \in \mathbb{N}$*

$$N_{\Gamma_0(M)}(\lambda) = \sum_{i=1}^{n-1} c_i(M) f_i(\lambda) \quad + O(f_n(\lambda)).$$

Then

$$N^{\text{new}}_{\Gamma_0(M)}(\lambda) = \sum_{i=1}^{n-1} c_i^{\text{new}}(M) f_i(\lambda) \quad + O(f_n(\lambda)).$$

where $c_i^{\text{new}} = c_i * \sigma_0^{-1}$.

Proof. The $M = 1$ case is clear by the definitions of $N^{\text{new}}_{\Gamma_0(1)}(\lambda)$ and $c_i^{\text{new}}(1)$. We observe that by lemma 3.1 we have

$$\begin{aligned} N_{\Gamma_0(M)}(\lambda) &= 1 + \sum_{K|M} \sigma_0\left(\frac{M}{K}\right) \sum_{0<\tilde{\lambda}\leq\lambda} \dim A_{\text{new}}(\tilde{\lambda}, K) \\ &= \sum_{K|M} \sigma_0\left(\frac{M}{K}\right) N^{\text{new}}_{\Gamma_0(K)}(\lambda). \end{aligned}$$

By the definition of c_i^{new} we have

$$c_i(M) = \sum_{K|M} \sigma_0\left(\frac{M}{K}\right) c_i^{\text{new}}(K)$$

and therefore

$$\left|N^{\text{new}}_{\Gamma_0(M)}(\lambda) - \sum_{i=1}^{n-1} c_i^{\text{new}}(M) f_i(\lambda)\right| \leq \left|N_{\Gamma_0(M)}(\lambda) - \sum_{i=1}^{n-1} c_i(M) f_i(\lambda)\right| + \sum_{\substack{K|M \\ K\neq M}} \sigma_0\left(\frac{M}{K}\right)\left|N^{\text{new}}_{\Gamma_0(K)}(\lambda) - \sum_{i=1}^{n-1} c_i^{\text{new}}(K) f_i(\lambda)\right|.$$

Induction in M now gives that this is $\leq C f_n(\lambda)$ which is the desired result. □

The above proposition together with Theorem 3.1 shows that $c_i^{\text{new}}(N)$ is multiplicative if and only if $c_i(N)$ is multiplicative. It also shows that if we know the expansion of the counting function for eigenvalues of $\Delta_{\Gamma_0(M)}$ for any $M \in \mathbb{N}$ and if these expansions are of the same type then it is easy to find the expansion of the corresponding counting function for newforms. Finding the expansion of $N_{\Gamma_0(M)}$ is the objective of the next section.

4. The refined Weyl law

We start by citing a result by Venkov [1982](Theorem 5.2.1) which is the basis of our calculations. We have adjusted the theorem to our situation, and corrected the obvious misprint in the O-term.

Theorem 4.1. *The following asymptotic formula holds:*

$$N_{\Gamma_0(M)}(\lambda) - \frac{1}{4\pi}\int_{-T}^{T} \frac{\phi_M'}{\phi_M}\left(\frac{1}{2}+ir\right)dr = \frac{|F_M|}{4\pi}\lambda - \frac{k_M}{\pi}\sqrt{\lambda}\ln\sqrt{\lambda} + \frac{k_M(1-\ln 2)}{\pi}\sqrt{\lambda} + O(\sqrt{\lambda}/\ln\sqrt{\lambda})$$

where ϕ_M is the determinant of the scattering matrix, $\lambda = 1/4+T^2$, k_M is the number of cusps of $\Gamma_0(M)$ and $|F_M|$ is the area of the fundamental domain of $\Gamma_0(M)$.

For definitions of cusps and the scattering matrix we refer to [Iwaniec 1995] or [Kubota 1973].

In order to apply Proposition 3.1 to the estimate obtained in Theorem 4.1 we need to estimate the integral and therefore also the logarithmic derivative of the scattering matrix. We can do this by using the following theorem which was proved by Huxley [1984].

Theorem 4.2. *Let $\phi_M(s)$ be the determinant of the scattering matrix for the Hecke congruence group of level M, $\Gamma_0(M)$, and let Λ_χ be the* *completed L-function of an Dirichlet character mod K ,χ , i.e.*

$$\Lambda_\chi(s) = \Gamma\left(\frac{s}{2}\right)\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s} \qquad \text{when } \Re(s) > 1.$$

Then

$$\phi_M(s) = (-1)^l\left(\frac{A(M)}{\pi^{k_M}}\right)^{1-2s}\prod_{i=1}^{k_M}\frac{\Lambda(2-2s,\overline{\chi_i})}{\Lambda(2s,\chi_i)}$$

where $l \in \mathbb{N}$, the χ_i's are some Dirichlet characters mod K where $K|M$, and

$$A(M) = \prod_{\substack{\chi \text{ primitive } \bmod q \\ q|m,\, mq|M}} \frac{qM}{(m, M/m)}.$$

The set $\{\chi_i | i = 1, \ldots, k_M\}$ is closed under complex conjugation.

We now use this to evaluate the integral in theorem 4.1. We let $B(M) = \frac{A(M)}{\pi^{k_M}}$. From the above we conclude that

$$\frac{\phi_M'}{\phi_M}\left(\frac{1}{2}+ir\right) = -2\left(\ln B(M) + \sum_{i=1}^{k_M}\frac{\Lambda_{\chi_i}'}{\Lambda_{\chi_i}}(1-2it) + \frac{\Lambda_{\chi_i}'}{\Lambda_{\chi_i}}(1+2it)\right).$$

An easy consideration then shows that

$$-\frac{1}{4\pi}\int_{-T}^{T}\frac{\phi_M'}{\phi_M}\left(\frac{1}{2}+ir\right)dr = \frac{T}{\pi}\ln B(M) + \sum_{i=1}^{k_M}\frac{1}{\pi}\int_{-T}^{T}\frac{\Lambda_{\chi_i}'}{\Lambda_{\chi_i}}(1+2ir)dr.$$

We must therefore evaluate

$$\int_{-T}^{T}\frac{\Lambda_{\chi_i}'}{\Lambda_{\chi_i}}(1+2ir)dr,$$

and we observe that

$$\int_{-T}^{T}\frac{\Lambda_{\chi_i}'}{\Lambda_{\chi_i}}(1+2ir)d = \frac{1}{2}\int_{-T}^{T}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)dr + \int_{-T}^{T}\frac{L_\chi'}{L_\chi}\left(1+i2r\right)dr.$$

We shall address each term separately. To evaluate the first term we use Stirling's approximation formula i.e.

$$\frac{\Gamma'}{\Gamma}(s) = \log(s) - \frac{1}{2s} + O(|s|^{-2}),$$

valid for $|\arg(s) - \pi| > \epsilon$. We see that for $|r| > \epsilon$ we have

$$\begin{aligned}
&\left|\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right) - \left(\log|r| + i\arg\left(\frac{1}{2}+ir\right) - (1+i2r)^{-1}\right)\right| \\
&\qquad \leq \left|\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right) - \left(\log\left|\frac{1}{2}+ir\right| + i\arg\left(\frac{1}{2}+ir\right) - (1+i2r)^{-1}\right)\right| \\
&\qquad + \left|\log\left|\frac{1}{2}+ir\right| - \log|r|\right|.
\end{aligned}$$

It is easy to see check that the last summand is $O((|r|\log|r|)^{-1})$ while the first is $O(|r|^{-2})$ by Stirling's approximation formula. Hence

$$\frac{1}{2}\int_{-T}^{T}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)dr$$
$$=\frac{1}{2}\int_{\substack{-T\\|r|>\epsilon}}^{T}\log|r|+i\arg\left(\frac{1}{2}+ir\right)-(1+i2r)^{-1}\,dr+\quad O\left(\int_{\epsilon}^{T}\frac{1}{r\log r}\right)$$

The integral over $(1+i2r)^{-1}$ is bounded and the integral over $i\arg(1/2+ir)$ vanishes. We conclude that

$$\frac{1}{2}\int_{-T}^{T}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)dr=T\log T-T+O(\log(\log T))$$

To evaluate the integral over the logarithmic derivative of $L_\chi(1+2ir)$ we note that

$$\int_{\epsilon}^{T}\frac{L'_\chi}{L_\chi}(1+2ir)dr=-i(\log L_\chi(1+2iT))+C$$

where C is a constant and that the first term is $O(\log T)$ by [Apostol 1976, Theorem 12.24]. We conclude that

$$-\frac{1}{4\pi}\int_{-T}^{T}\frac{\phi'_M}{\phi_M}\left(\frac{1}{2}+ir\right)dr=\frac{T}{\pi}\log B(M)+\frac{k_M}{\pi}(T\log T-T)+O_M(\log(T))$$

We have hence proven the following

Theorem 4.3. *The counting function $N_{\Gamma_0(M)}(\lambda)$ satisfies the following asymptotic formula*

$$N_{\Gamma_0(M)}(\lambda)=\frac{|F_M|}{4\pi}\lambda-\frac{2k_M}{\pi}\sqrt{\lambda}\log\sqrt{\lambda}$$
$$+\frac{1}{\pi}\left[(2-\log 2+\log\pi)k_M)-\log(A(M))\right]\sqrt{\lambda}+O(\sqrt{\lambda}/\log\sqrt{\lambda})$$

This theorem puts us in a situation where proposition 3.1 can be applied with

$$\begin{aligned}f_1(\lambda)&=\lambda\\ f_2(\lambda)&=\sqrt{\lambda}\log\sqrt{\lambda}\\ f_3(\lambda)&=\sqrt{\lambda}\\ f_4(\lambda)&=\sqrt{\lambda}/\log\sqrt{\lambda}.\end{aligned}$$

From [Shimura 1971, Theorem 1.43] we conclude that

$$k_M = \sum_{d|M} \Phi((d, M/d)) \tag{4}$$

$$|F_M| = \frac{\pi}{3} M \prod_{\substack{p|M \\ p \text{ prime}}} (1 + p^{-1}). \tag{5}$$

This means that we have explicit expressions for all the terms in theorem 4.3 except $A(M)$. We need to know the number of primitive Dirichlet characters mod K. We hence define

$$D(K) = \#\{\chi \text{ primitive Dirichlet character mod } K\}.$$

Then we have

Lemma 4.1. *The arithmetical function $D(K)$ is multiplicative and satisfies*

$$D(K) = (\Phi * \mu)(K)$$

Proof. From [Apostol 1976] theorem 6.15 and theorem 8.18 we conclude that $\Phi(K) = \sum_{d|K} D(d) = (u * D)(K)$ where $u(n) = 1$ for $n \in \mathbb{N}$. Since Φ and u are multiplicative we use theorem 3.1 to conclude that D is multiplicative. Theorem 2.1 in [Apostol 1976] proves that $u^{-1} = \mu$ so

$$\Phi * \mu = u * D * \mu = u * u^{-1} * D = D$$

which concludes the proof □

5. Coefficients related to newforms

In this section we calculate c_1^{new},c_2^{new} and c_3^{new}. The calculations are basically corollaries of proposition 3.1 and theorem 4.3. We are particularly interested in the case when $c_2^{\text{new}}(M) = c_3^{\text{new}}(M) = 0$. If this is the case we say that $N_{\Gamma_0(M)}^{\text{new}}$ *is of cocompact type.* To see why this is sensible we remind of the following

Theorem 5.1. *Assume Γ is a cocompact Fuchsian group and let $N_\Gamma(\lambda)$ be the counting function for the eigenvalues of Δ_Γ. Then*

$$N_\Gamma(\lambda) = \frac{|F_\Gamma|}{4\pi}\lambda + O(\sqrt{\lambda}/\log\sqrt{\lambda}),$$

where $|F_\Gamma|$ is the area of an fundamental domain of Γ.

Proof. This is a special case of [Venkov 1982](Theorem 5.2.1). Notice again that we have corrected the obvious misprint. □

Hence $N^{\text{new}}_{\Gamma_0(M)}$ is of cocompact type if and only if it 'has the same shape' as if it where the counting function of the eigenvalues related to a cocompact group.

5.1. **The first coefficient.** We start by calculating $c_1^{\text{new}}(M)$. This is the simplest of the three coefficients.

Proposition 5.1. *The arithmetical function $v(M) = 12c_1^{\text{new}}(M)$ is multiplicative and satisfies*

$$v(p^n) = \begin{cases} 1 & \text{if } n = 0 \\ p-1 & \text{if } n = 1 \\ p^2-p-1 & \text{if } n = 2 \\ (p^3-p^2-p+1)p^{n-3} & \text{if } n \geq 3 \end{cases} \tag{6}$$

when p is a prime. We furthermore have

$$L_v(s) := \sum_{n=1}^{\infty} \frac{v(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(2s)\zeta(s)},$$

where $\zeta(s)$ is Riemann's zeta function.

Proof. By using proposition 3.1, theorem 4.3 and 5 we conclude that

$$M \prod_{\substack{p|M \\ p \text{ prime}}} (1+p^{-1}) = (\sigma_0 * v)(M).$$

Since the left hand side and σ_0 are multiplicative theorem 3.1 says that v is multiplicative. By considering the case where $M = p^m$ we see that

$$p^m + p^{m-1} = \sum_{d|p^m} \sigma_0(d) v\left(\frac{p^m}{d}\right) = \sum_{i=0}^{n} (i+1) v(p^{n-i}).$$

By applying the theory of generating functions to this relation we find that if

$$f_p(c) = \sum_{n=0}^{\infty} v(p^n) x^n \quad \text{then} \quad f_p(x) = \frac{(1-x^2)(1-x)}{1-px}.$$

By making formal expansion we get (6). Since v is multiplicative the claim about L_v follows. □

5.2. **The second coefficient.** We now calculate $c_2^{\text{new}}(M)$. We remind that by proposition 3.1 and theorem 4.3 we have

$$c_2^{\text{new}}(M) = -\frac{2}{\pi}(k_{(\cdot)} * \sigma_0^{-1})(M)$$

We hence need to have more information about the number of cusps of $\Gamma_0(M)$

Lemma 5.1. *The number of cusps, k_M, of $\Gamma_0(M)$ is a multiplicative arithmetical function and satisfies*

$$k_{p^m}\begin{cases} 1 & \text{if } m=0\\ 2 & \text{if } m=1\\ 2p^n & \text{if } m=2n+1 \text{ where } n>1\\ (p+1)p^{n-1} & \text{if } m=2n \text{ where } n>1\end{cases} \tag{7}$$

Proof. We noted earlier in (4) that

$$k_M=\sum_{d|M}\Phi((d,M/d)).$$

Let $M_1,M_2\in\mathbb{N}$ and assume $(M_1,M_2)=1$. Then

$$\begin{aligned}k_{M_1M_2}&=\sum_{d|M_1M_2}\Phi((d,(M_1M_2)/d))\\&=\sum_{d_1|M_1}\sum_{d_2|M_2}\Phi((d_1d_2,(M_1M_2)/(d_1d_2)))\\&=\sum_{d_1|M_1}\sum_{d_2|M_2}\Phi((d_1,M_1/d_1)(d_2,M_2/d_2))\\&=\sum_{d_1|M_1}\Phi((d_1,M_1/d_1))\sum_{d_2|M_2}\Phi((d_2,M_2/d_2))\\&=k_{M_1}k_{M_2}\end{aligned}$$

Hence k_M is multiplicative. The claim about k_{p^m} is clear for $m=0$ and $m=1$. Assume $m\geq 2$. We then have

$$\begin{aligned}k_{p^m}&=\sum_{i=0}^{m}\Phi((p^i,p^{m-i}))\\&=\sum_{i=0}^{m}\Phi(p^{\min(i,m-i)})\\&=2+\sum_{i=1}^{m-1}(p-1)p^{\min(i,m-i)-1}\end{aligned}$$

We now assume $m = 2n+1$.

$$\begin{aligned}
&= 2 + (p-1)\left(\sum_{i=1}^{n} p^{i-1} + \sum_{i=n+1}^{2n} p^{2n-i}\right) \\
&= 2 + 2(p-1)\sum_{i=0}^{n-1} p^{i} \\
&= 2 + 2(p-1)\frac{1-p^n}{1-p} = 2p^n.
\end{aligned}$$

The even case is similar. □

From the above we can now prove the following

Proposition 5.2. *The second coefficients ,$c_2^{\text{new}}(M)$, is a multiplicative arithmetical function and satisfies*

$$(8) \qquad -\frac{\pi}{2}c_2^{\text{new}}(p^m) = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m = 2n+1 \\ p-2 & \text{if } m = 2 \\ (p+1)^2 p^{n-1} & \text{if } m = 2n \text{ where } n > 1 \end{cases}$$

Proof. From lemma 5.1 and theorem 3.1 follows that $c_2^{\text{new}}(M)$ is multiplicative. From (3) it is easy to see that

$$\sigma_0^{-1}(p^m) = \begin{cases} 1 & \text{if } m = 0 \\ -2 & \text{if } m = 1 \\ 1 & \text{if } m = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$c_2^{\text{new}}(p^m) = -\frac{2}{\pi}(k_{p^m} - 2k_{p^{m-1}} + k_{p^m}), \text{ when } m \geq 2.$$

Using lemma 5.1 it is now easy to check the claim. We omit the details. □

As an easy corollary we get the following

Corollary 5.1. *The second coefficient, $c_2^{\text{new}}(M)$, is non-zero if and only if $M = t^2$ where $t \in \mathbb{N}$ is not of the form $t = 2t'$ with $(2, t') = 1$.*

5.3. **The third coefficient.** We finally calculate $c_3^{\text{new}}(M)$. This is the most difficult of the three coefficients.

We start by observing that by proposition 3.1 and theorem 4.3

$$c_3^{\text{new}}(M) = \frac{1}{\pi}\left((2 - \log 2 + \log \pi)\left(-\frac{\pi}{2}c_2^{\text{new}}(M)\right) - L(M)\right)$$

where

$$L(M) = \left(\log A(\cdot) * \sigma_0^{-1}\right)(M).$$

We hence direct our attention to $L(M)$.

Lemma 5.2. *Assume* $(M_1, M_2) = 1$. *Then*

$$L(M_1M_2) = U(M_1)L(M_2) + U(M_2)L(M_1)$$

where

$$U(M) = \sum_{d|M} \sum_{m|d} \sum_{q|(m,\frac{d}{m})} D(q)\sigma_0^{-1}\left(\frac{M}{d}\right).$$

Proof. We have

$$L(M_1M_2) = \sum_{d|M_1M_2} \log A(d)\sigma_0^{-1}\left(\frac{M_1M_2}{d}\right)$$

$$= \sum_{d|M_1M_2} \sum_{\substack{q|m \\ mq|d}} D(q) \log\left(\frac{qd}{(m,\frac{d}{m})}\right) \sigma_0^{-1}\left(\frac{M_1M_2}{d}\right)$$

$$= \sum_{d|M_1M_2} \sum_{m|d} \sum_{q|(m,d/m)} D(q) \log\left(\frac{qd}{(m,\frac{d}{m})}\right) \sigma_0^{-1}\left(\frac{M_1M_2}{d}\right)$$

$$= \sum_{d_1|M_1} \sum_{d_2|M_2} \sum_{m_1|d_1} \sum_{m_2|d_2} \sum_{q_1|(m_1,\frac{d_1}{m_1})} \sum_{q_2|(m_2,\frac{d_2}{m_2})} D(q_1q_2) \log\left(\frac{q_1q_2d_1d_2}{(m_1m_2,\frac{d_1d_2}{m_1m_2})}\right) \sigma_0^{-1}\left(\frac{M_1M_2}{d_1d_2}\right)$$

The summand is clearly

$$D(q_1)D(q_2)\sigma_0^{-1}\left(\frac{M_1}{d_1}\right)\sigma_0^{-1}\left(\frac{M_2}{d_2}\right)\left(\log\left(\frac{q_1d_1}{(m_1,\frac{d_1}{m_1})}\right) + \log\left(\frac{q_2d_2}{(m_2,\frac{d_2}{m_2})}\right)\right).$$

We have

$$\sum_{d_1|M_1} \sum_{d_2|M_2} \sum_{m_1|d_1} \sum_{m_2|d_2} \sum_{q_1|(m_1,\frac{d_1}{m_1})} \sum_{q_2|(m_2,\frac{d_2}{m_2})} D(q_1)D(q_2)\sigma_0^{-1}\left(\frac{M_1}{d_1}\right)\sigma_0^{-1}\left(\frac{M_2}{d_2}\right)\left(\log\left(\frac{q_1d_1}{(m_1,\frac{d_1}{m_1})}\right)\right)$$

$$= U(M_2)L(M_1),$$

from which the identity easily follows. □

It turns out that U is a very nice arithmetical function. In fact we have the following.

Lemma 5.3. *The function $U(M)$, is a multiplicative arithmetical function and satisfies*

$$U(p^m) = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m = 2n+1 \\ p-2 & \text{if } m = 2 \\ (p^2-2p+1)p^{n-2} & \text{if } m = 2n \text{ where } n > 1 \end{cases} \tag{9}$$

Proof. Let $M_1, M_2 \in \mathbb{N}$ be coprime. Then

$$\begin{aligned} U(M_1M_2) &= \sum_{d|M_1M_2} \sum_{m|d} \sum_{q|(m,\frac{d}{m})} D(q)\sigma_0^{-1}\left(\frac{M_1M_2}{d}\right) \\ &= \sum_{d_1|M_1} \sum_{d_2|M_2} \sum_{m_1|d_1} \sum_{m_2|d_2} \sum_{q_1|(m_1,\frac{d_1}{m_1})} \sum_{q_2|(m_2,\frac{d_2}{m_2})} D(q_1)D(q_2)\sigma_0^{-1}\left(\frac{M_1}{d_1}\right)\sigma_0^{-1}\left(\frac{M_2}{d_2}\right) \\ &= U(M_1)U(M_2). \end{aligned}$$

Hence U is multiplicative.

Let p be a prime and $m \in \mathbb{N}$. We assume $m \geq 2$ Then

$$\begin{aligned} L(p^m) &= \sum_{i=0}^{m} \sum_{j=0}^{i} \sum_{l=0}^{\min(j,i-j)} D(p^l)\sigma_0^{-1}\left(p^{m-i}\right) \\ &= \sum_{j=0}^{m-2} \sum_{l=0}^{\min(j,m-2-j)} D(p^l) - 2\sum_{j=0}^{m-1} \sum_{l=0}^{\min(j,m-1-j)} D(p^l) + \sum_{j=0}^{m} \sum_{l=0}^{\min(j,m-j)} D(p^l) \end{aligned}$$

Assume $j \leq n-2-j$. Then $j \leq n-1-j \leq n-j$ and we have that all minimum values are j. Hence these terms cancels out. We now assume $m = 2n+1$. Hence we may sum from $j \geq (2n+1)/2 - 1 = n - 1/2$.

$$\begin{aligned} &= \sum_{j=n}^{m-2} \sum_{l=0}^{\min(j,m-2-j)} D(p^l) - 2\sum_{j=n}^{m-1} \sum_{l=0}^{\min(j,m-1-j)} D(p^l) + \sum_{j=n}^{m} \sum_{l=0}^{\min(j,m-j)} D(p^l) \\ &= \sum_{j=n}^{m-2} \sum_{l=0}^{m-2-j} D(p^l) - 2\sum_{j=n}^{m-1} \sum_{l=0}^{m-1-j} D(p^l) + \sum_{j=n+1}^{m} \sum_{l=0}^{m-j} D(p^l) + \sum_{l=0}^{m} D(p^l) \\ &= \sum_{l=0}^{m-2-n} D(p^l) - 2\sum_{l=0}^{m-1-n} D(p^l) + \sum_{l=0}^{n} D(p^l) \\ &\quad + \sum_{j=n+1}^{m-2} \left(\sum_{l=0}^{m-2-j} D(p^l) - 2\sum_{l=0}^{m-1-j} D(p^l) + \sum_{l=0}^{m-j} D(p^l) \right) \end{aligned}$$

$$
\begin{aligned}
&-2\sum_{l=0}^{m-1-(m-1)} D(p^l)+\sum_{l=0}^{m-(m-1)} D(p^l)+\sum_{l=0}^{m-m} D(p^l)\\
=&-D(p^n)+\sum_{j=n+1}^{m-2}(-2D(p^{m-1-j})+D(p^{m-1-j})+D(p^{m-j}))+D(p)\\
=&-D(p^n)-\sum_{j=1}^{n-1}D(p^j)+\sum_{j=2}^{n}D(p^j)+D(p)=0
\end{aligned}
$$

The even case is similar but slightly easier. The $m=1$ case is also similar. □

Remark 5.1. By successive use of the two lemmas above we find that

$$
L(p_1^{n_1}\dots p_k^{n_k})=\sum_{i=1}^{k}\left(\prod_{j\in\{1,\dots,k\}\setminus\{i\}} U(p_j^{n_j})\right)L(p_i^{n_i}),
$$

when $p_1,\dots,p_k$ are different primes. Notice that $L(p_i^{n_i})$ is of the form $\tilde{m}_i\log p_i$ where $\tilde{m}_i\in\mathbb{Z}$. We also note that $U(M)\in\mathbb{Z}$. Hence L is on the form

$$
m_1\log p_1+\dots, m_k\log p_k \text{ where } m_i\in\mathbb{Z}.
$$

By unique factorization in $\mathbb{N}$ this is zero if and only if $m_i=0$ for all i's. We would therefore like to know when $L(p^m)$ is zero.

Lemma 5.4. *The function $L(p^m)$ satisfies*

$$
L(p^m)=\begin{cases} 2\left(\sum_{j=0}^{n}D(p^j)\right)\log p & \text{if } m=2n+1\\ \left(\sum_{j=0}^{n-1}D(p^j)+mD(p^n)\right)\log p & \text{if } m=2n.\end{cases}\tag{10}
$$

In particular $L(p^m)$ is never zero.

Proof. This follows by a lengthy but elementary calculation similar to that in the proof of lemma 5.3. □

From the above lemma and the preceding remark we conclude that

$$
L(p_1^{n_1}\dots p_k^{n_k})=0
$$

if and only if $U(p_i^{n_i})=0$ for at least two different primes. Since $c_2^{\mathrm{new}}(M)=c_3^{\mathrm{new}}(M)=0$ if and only if $c_2^{\mathrm{new}}(M)=L(M)=0$ we have proved the following which settles the question of when $N_{\Gamma_0(M)}^{\mathrm{new}}(\lambda)$ is of cocompact type.

Theorem 5.2. *Let $M \in \mathbb{N}$ and let $n, t \in \mathbb{N}$ be the integers defined uniquely by the requirements that n should be squarefree and $M = t^2 n$. Then $N^{\text{new}}_{\Gamma_0(M)}(\lambda)$ is of cocompact type if and only if n, t satisfies one of the following:*

(1) *n contains more than one prime.*
(2) *$n > 1$ and $4|M$ and $(2, M/4) = 1$.*

6. Concluding remarks

Remark 6.1. From proposition 5.1 we conclude that

$$N^{\text{new}}_{\Gamma_0(M)}(\lambda) = \frac{1}{12}\lambda + O(\sqrt{\lambda}\log\sqrt{\lambda})$$

if and only if $M \in \{1, 2, 4\}$. This shows that theorem 2 of [Balslev & Venkov 1998] cannot be generalized to more general Hecke congruence groups by simply choosing another character.

Remark 6.2. We wish to draw attention to a particular case of theorem 5.2 namely the case when $M > 1$ is squarefree with an even number of primes. Hence, by Theorem 5.2 (1) $N^{\text{new}}_{\Gamma_0(N)}(\lambda)$ has the same form as if it were the counting function for the eigenvalues related to a co-compact group with invariant area $4\pi c_1^{\text{new}}(M)$. We can give an alternative and much more sophisticated proof of this by referring to the Jacquet-Langlands correspondence. A part of this correspondence is described classically in [Strömbergsson 2001] where the following is proven:

> Let $\mathcal{O}$ be a maximal order in an indefinite rational quaternion division algebra over $\mathbb{Q}$, and let $d = d(\mathcal{O})$ be its (reduced) discriminant. This is always a *squarefree integer with an even number of prime factors.* The norm one unit group $\mathcal{O}^1$ can be viewed as a Fuchsian group *which is cocompact.* Then:
>
> *The eigenvalues of the Laplacian on $\mathcal{O}^1 \setminus \mathcal{H}$ are exactly the same (with multiplicities) as the eigenvalues corresponding to the newspace on $\Gamma_0(d) \setminus \mathcal{H}$.*

Hence $N^{\text{new}}_{\Gamma_0(N)}(\lambda)$ *is* the counting function for the eigenvalues related to a cocompact group, and hence obviously has the corresponding type as predicted by Theorem 5.1. This has our theorem 5.2 as an easy corollary (compare with theorem 5.1). We note that any squarefree d with an even number of primes may be constructed in this way.

Our calculation indicates that there might be a similar correspondence in a lot of other cases. We hope to address this on a later occasion.

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Department of Mathematical Sciences, University of Aarhus Ny Munkegade Building 530, 8000 Århus, Denmark

E-mail address: `risager@imf.au.dk`