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Cells in affine Weyl groups and tilting modules

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Abstract.

Let G be a reductive algebraic group over a field of positive characteristic. In this paper we explore the relations between the behaviour of tilting modules for G and certain Kazhdan-Lusztig cells for the affine Weyl group associated with G. In the corresponding quantum case at a complex root of unity V. Ostrik has shown that the weight cells defined in terms of tilting modules coincide with right Kazhdan-Lusztig cells. Our method consists in comparing our modules for G with quantized modules for which we can appeal to Ostrik's results. We show that the minimal Kazhdan-Lusztig cell breaks up into infinitely many "modular cells" which in turn are determined by bigger cells. At the opposite end we call attention to recent results by T. Rasmussen on tilting modules corresponding to the cell next to the maximal one. Our techniques also allow us to make comparisons with the mixed quantum case where the quantum parameter is a root of unity in a field of positive characteristic.

§1. Introduction

Let $\mathfrak{g}$ be a semisimple complex Lie algebra and denote by U_q the associated quantum group at a primitive l'th root of unity $q \in \mathbb{C}$. If $A = \mathbb{Z}[v, v^{-1}]$ then U_q is obtain by specializing Lusztig's A-form of the quantized enveloping algebra of $\mathfrak{g}$ at q.

Following Ostrik [18] the set of dominant weights for $\mathfrak{g}$ is divided into weight cells. Two weights are in the same cell if the two corresponding indecomposable tilting modules have the property that each of them occurs as a summand of the tensor product of the other by some third tilting module. Ostrik proved [17] that these weight cells coincide with the Kazhdan-Lusztig cells in the affine Weyl group W_l associated with U_q.

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Denote by k an algebraically closed field of characteristic $p > 0$ and let G be a connected reductive algebraic group over k with the same root system as $\mathfrak{g}$. In an effort to gain some better understanding of the tilting modules for G we lift them to U_{A_p} and then study the specializations at q. Here $A_p = A_{(v-1,p)}$ is the localization of A at the maximal ideal $(v-1,p)$ and $\mathbb{C}$ is made into an A_p-algebra by specializing v at q (q now being a p'th root of 1). The (wide open) problem of finding the characters of the indecomposable tilting modules for G is then equivalent to the problem of decomposing these specializations into indecomposable U_q-modules.

When λ is a dominant weight we denote by $T(\lambda)$, respectively $T_q(\lambda)$ the indecomposable tilting module for G, respectively for U_q with highest weight λ. The U_q-module obtained from $T(\lambda)$ by the "quantization" described above is denoted $T(\lambda)_q$. With this notation the conjecture stated in [3, 5.1] says that $T(\lambda)_q = T_q(\lambda)$ as long as λ belongs to the lowest p^2-alcove. We shall demonstrate that this conjecture combined with the work of Ostrik on weight cells allow us to carry over some of Ostrik's results on tensor ideals in the category of tilting modules. This relies on some of the known properties of Kazhdan-Lusztig cells.

In a different direction we also report on some recent work by T. E. Rasmussen [20] concerning second cell tilting modules. By the second cell we understand the one next to the cell consisting of weights in the first alcove. Rasmussen's main result (so far only valid in type ADE) can be phrased as saying that $T(\lambda)_q \equiv T_q(\lambda)$ modulo tilting modules with highest weights belonging to smaller cells. Recall that Soergel [22] has determined the characters of all $T_q(\lambda)$'s. Hence this result of Rasmussen gives for a given tilting module with known character a way of finding its indecomposable summands with highest weights in the second cell.

Let now $r \in \mathbb{N}$ and denote by $H_r(q)$ the Hecke algebra over $\mathbb{C}$ of the symmetric group Σ_r with parameter q. Denote for a partition λ the simple $k\Sigma_r$-module, respectively $H_r(q)$-module by D^λ, respectively D_q^λ. Then taking $G = GL(V)$ in Rasmussen's theorem he obtains via Schur-Weyl duality that $\dim_k D^\lambda = \dim_{\mathbb{C}} D_q^\lambda$ for all λ in the second cell. Again the right hand side is known (e.g. via Soergel's results mentioned above). This proves parts of Mathieu's conjecture [16].

Finally, we describe how the above technique for comparing tilting modules for G with those for U_q can similarly be used to compare also with tilting modules for the quantum group U_ζ at an l'th root of unity $\zeta \in k$. In [4, Section 4] we conjectured that if λ belongs to the lowest lp-alcove then the character of the indecomposable tilting module $T_\zeta(\lambda)$ for U_ζ coincides with the character for $T_q(\lambda)$. As above this implies that

for p large we can locate some of the tensor ideals in the category of tilting modules for U_ζ.

Rasmussen's results about second cell tilting modules also generalize to this case. They lead to an equality of dimensions for certain simple modules of $H_r(\zeta)$ and $H_r(q)$. Here $H_r(\zeta)$ is the Hecke algebra over k with parameter ζ.

§2. Notation and recollection

In addition to the notation already introduced above we shall need the following.

The root system for $\mathfrak{g}$ (with respect to a Cartan subalgebra $\mathfrak{h}$) and for G (with respect to a maximal torus T) will be denoted R. We shall fix a set of positive roots R^+ in R. The set of characters $X = X(T)$ of T (which may be identified with the integral weights of $\mathfrak{h}$) then contains a cone X^+ consisting of the dominant weights, namely

$$X^+ = \{\lambda \in X | \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in R^+\}.$$

Here $\alpha^\vee$ is the coroot of $\alpha \in R$. This set is a fundamental domain for the action of the Weyl group W on X. We shall often use the 'dot' action of W on X given by $w \cdot \lambda = w(\lambda + \rho) - \rho, w \in W, \lambda \in X$. As usual ρ is half the sum of the positive roots.

If $l \in \mathbb{N}$ then the affine Weyl group W_l is the group generated by the affine reflections $s_{\alpha,n}, \alpha \in R^+, n \in \mathbb{Z}$ given by

$$s_{\alpha,n} \cdot \lambda = s_\alpha \cdot \lambda + nl\alpha, \quad \lambda \in X.$$

Here s_α is the reflection in W attached to α. As fundamental domain for this action we choose the closure $\bar{C}_l$ (obtained by replacing all $<$ by $\leq$ below) of the bottom alcove C_l in X^+ determined by

$$C_l = \{\lambda \in X \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle < l, \quad \alpha \in R^+\}.$$

Consider now $\lambda \in X^+$. We have four G-modules associated with λ (all having λ as their unique highest weight)

- the simple module $L(\lambda)$,
- the Weyl module $\Delta(\lambda)$,
- the dual Weyl module $\nabla(\lambda)$,

and

- the indecomposable tilting module $T(\lambda)$.

These modules may be constructed as follows. Choose $B \subset T$ to be the Borel subgroup corresponding to the negative roots $-R^+$. Let then $\nabla(\lambda)$ be the G-module induced from the 1-dimensional B-module determined by λ and set $\Delta(\lambda) = \nabla(\lambda)^*$ (for a G-module M we let G act on the dual module $M^* = \mathrm{Hom}_k(M, k)$ via the Chevalley antiautomorphism on G so that the set of weights of M^* coincides with the set of weights on M). It is an easy observation to see that $L(\lambda)$ is the image of the natural homomorphism $\Delta(\lambda) \to \nabla(\lambda)$. Finally, following Ringel [21] we may construct $T(\lambda)$ by first setting $E_1 = \Delta(\lambda)$, then choose $\lambda_1 \in X^+$

maximal with $d_1 = \dim_k \mathrm{Ext}^1_G(\Delta(\lambda_1), \Delta(\lambda)) > 0$ and set E_2 equal to the corresponding extension $0 \to E_1 \to E_2 \to \Delta(\lambda_1)^{\oplus d_1} \to 0$. Now we repeat this process after having replaced E_1 by E_2. After a finite number of such steps we arrive at a G-module E_r which has $\mathrm{Ext}^1_G(\Delta(\mu), E_r) = 0$ for all $\mu \in X^+$. Then $T(\lambda) = E_r$ (see [3] for details).

The above constructions work equally well in the quantum case (using the concept of induction from a "Borel subalgebra" of U_q introduced in [7]). We shall denote the corresponding modules for U_q by $L_q(\lambda), \Delta_q(\lambda), \nabla_q(\lambda)$, and $T_q(\lambda)$, respectively.

Moreover, we have A-forms $\Delta_A(\lambda)$ and $\nabla_A(\lambda)$ of the Weyl module and its dual. These are U_A-modules which are free over A and satisfy

$$\Delta_A(\lambda) \otimes_A \mathbb{C} \simeq \Delta_q(\lambda), \quad \nabla_A(\lambda) \otimes_A \mathbb{C} \simeq \nabla_q(\lambda),$$

and

$$\Delta_A(\lambda) \otimes_A k \simeq \Delta(\lambda), \quad \nabla_A(\lambda) \otimes_A k \simeq \nabla(\lambda).$$

Here $\mathbb{C}$, respectively k is considered an A-algebra by specializing v to $q \in \mathbb{C}$, respectively to $1 \in k$. In the last case we have also used the identification between G-modules and modules for the hyperalgebra $U_k = U_A \otimes_A k$, see [7].

Recall that $A_p = A_{(v-1,p)}$. If in the above construction of $T(\lambda)$ we replace the appropriate G-modules by the corresponding U_A-modules and use 'minimal number of generators' instead of '$\dim_k$' then we obtain a tilting module $T_{A_p}(\lambda)$ for U_{A_p}, which satisfies $T_{A_p}(\lambda) \otimes_{A_p} k \simeq T(\lambda)$. The "quantization" $T(\lambda)_q$ referred to in the introduction is then the specialization of this module at q, i.e.

$$T(\lambda)_q = T_{A_p}(\lambda) \otimes_{A_p} \mathbb{C}.$$

(Now q is a p'th root of unity in $\mathbb{C}$).

The indecomposable tilting modules for U_q have the form $T_q(\mu), \mu \in X^+$. Hence there are unique (fusion numbers) $a_{\mu\lambda} \in \mathbb{N}$ such that

$$T(\lambda)_q = \bigoplus_{\mu \in X^+} a_{\mu\lambda} T_q(\mu).$$

Clearly, $a_{\lambda\lambda} = 1$ for all λ and $a_{\mu,\lambda} = 0$ unless $\mu \leq \lambda$.

Problem 1. Determine $a_{\mu\lambda}$ for all $\mu, \lambda \in X^+$.

Remark 2. i) This problem is wide open. Since by [22] the characters of $T_q(\mu), \mu \in X^+$ are known (with the exception of certain small values of p for some types) this problem is equivalent to the problem of finding the characters of $T(\lambda), \lambda \in X^+$.

ii) Set $X_p = \{\lambda \in X^+ \mid \langle \lambda, \alpha^\vee \rangle < p \text{ for all simple roots } \alpha\}$. If $p \geq 2h-2$ then knowledge of the finite set of fusion numbers $\{a_{\mu\lambda} \mid \lambda \in (p-1)\rho + X_p\}$ is equivalent to knowledge of the characters of all $L(\lambda), \lambda \in X^+$ (see [3]).

iv) The only group for which the $a_{\mu\lambda}$'s have been found for all $\mu, \lambda \in X^+$ is $G = SL_2(k)$, see e.g. [11]. For $G = SL_3(k)$ some partial results can be found in [13]. Even in that case a complete solution seems to be far away.

Conjecture 3. ([3, Conjecture 5.4]). If $\lambda \in C_{p^2}$ then $T(\lambda)_q = T_q(\lambda)$.

If $p \geq 2h-2$ then $(p-1)\rho + X_p \subseteq C_{p^2}$. The strongest evidence for this conjecture is that it holds in this subset if and only if Lusztig's conjecture on simple G-modules (known for $p \gg 0$ by [5] and reported proved recently by Bezrukavnikov et al [9]) holds. Conjecture 3 is also known to hold (for all $p > 2$) for types A_1 and A_2.

It should be noted that the conjecture is also of interest for $2 < p < 2h-2$. In this case its verification will not give the characters of all irreducible G-modules. However, for small primes we still have no general conjecture which covers all irreducible characters for G.

§3. Cells and tensor ideals for quantum groups

In this section $q \in \mathbb{C}$ will be a primitive l'th root of 1 with l odd (and prime to 3 if R contains a component of type G_2).

Let $\mathcal{T}_q$ denote the category consisting of all tilting modules for U_q. Recall that the tensor product of two tilting modules is again tilting [19], i.e. $\mathcal{T}_q$ is a tensor category. Clearly, if Q_1, Q_2 are two U_q-modules then $Q_1 \oplus Q_2 \in \mathcal{T}_q$ if and only if $Q_1, Q_2 \in \mathcal{T}_q$.

Definition 4. Let $\lambda, \mu \in X^+$. We write $\lambda \underset{\mathcal{T}_q}{\leq} \mu$ if there exists $Q \in \mathcal{T}_q$ such that $T_q(\lambda)$ is a summand of $T_q(\mu) \otimes_{\mathbb{C}} Q$. If both $\lambda \underset{\mathcal{T}_q}{\leq} \mu$ and $\mu \underset{\mathcal{T}_q}{\leq} \lambda$ then we write $\lambda \underset{\mathcal{T}_q}{\sim} \mu$. The equivalence classes for $\underset{\mathcal{T}_q}{\sim}$ are called weight cells (after Ostrik [18]).

Example 5. Let $\lambda \in X^+$. Then $\lambda + \nu \underset{\mathcal{T}_q}{\leq} \lambda$ for all $\nu \in X^+$. In fact, $T_q(\lambda + \nu)$ is clearly a summand of $T_q(\lambda) \otimes_{\mathbb{C}} T_q(\nu)$.

Consider the special weight $(l-1)\rho$. In this case we have

$$L_q((l-1)\rho) = \Delta_q((l-1)\rho) = \nabla_q((l-1)\rho) = T_q((l-1)\rho)$$

and this module is both simple and injective in the category of finite dimensional U_q-modules. It is denoted St_q and called the Steinberg module for U_q.

Proposition 6. The set $(l-1)\rho + X^+$ is a weight cell.

Proof. By Example 5 we have $(l-1)\rho + \nu \underset{\mathcal{T}_q}{\leq} (l-1)\rho$ for all $\nu \in X^+$. To see that also $(l-1)\rho \underset{\mathcal{T}_q}{\leq} (l-1)\rho + \nu$ it is (by the above properties of St_q) enough to check that there is a non-zero homomorphism $St_q \to T_q((l-1)\rho + \nu) \otimes_{\mathbb{C}} T_q(\nu)$. But since $T_q(\nu) \simeq T_q(\nu)^*$ we have $\mathrm{Hom}_{U_q}(St_q, T_q((l-1)\rho + \nu) \otimes_{\mathbb{C}} T_q(\nu)) \simeq \mathrm{Hom}_{U_q}(St_q \otimes_{\mathbb{C}} T_q(\nu), T_q((l-1)\rho + \nu))$, which is clearly non-zero.

To finish the proof we check that if $\nu \underset{\mathcal{T}_q}{\leq} (l-1)\rho$ then $\nu \in (l-1)\rho + X^+$. But if $\nu \underset{\mathcal{T}_q}{\leq} (l-1)\rho$ then $T_q(\nu)$ is injective and the only indecomposable tilting modules, which are injective, are those with highest weight in $(l-1)\rho + X^+$, see [5].

Remark 7. i) The partial order $\underset{\mathcal{T}_q}{\leq}$ on X^+ induces a partial order (denoted in the same way) on the set of weight cells.

The observation in Example 5 shows that the weight cell $(l-1)\rho+X^+$ from Proposition 6 is the unique smallest cell in this ordering.

ii) The proof of Proposition 6 shows that $(l-1)\rho+X^+$ parametrizes the injective indecomposable tilting modules (the PIM's in the category of finite dimensional U_q-modules).

The linkage principle for U_q [7] implies that $\mathcal{T}_q$ divides into blocks corresponding to the orbits of W_l. Moreover, for each $\lambda, \mu \in \bar{C}_l$ we have a translation functor T_λ^μ from the λ-block in $\mathcal{T}_q$ to the μ-block.

Assume for the rest of this section that $l \geq h$. This is equivalent to C_l being non-empty. If $\lambda, \mu \in C_l$ then T_λ^μ is an equivalence of categories. This means then that if for $w \in W_l$ the alcove $C = w.\ C_l \subset X^+$ contains one weight in a weight cell $\underline{c}$ then all weights in C belong to $\underline{c}$. A little more elaborate argument (see [4]) shows that also the intersection of X^+ with the lower closure of C is in fact contained in $\underline{c}$. Hence

Proposition 8. Each weight cell is the union of lower closures of alcoves (intersected with X^+).

Note that the intersection of X^+ with the lower closure of C_l equals C_l. The result in [2, Theorem 3.4] says

Proposition 9. C_l is a weight cell.

Clearly C_l is the unique maximal cell in the ordering $\underset{\mathcal{T}_q}{\leq}$.

When $\underline{c}$ is a weight cell in X^+ we denote by $\mathcal{T}_q(\leq \underline{c})$ the subcategory in $\mathcal{T}_q$ whose objects are direct sums of $T_q(\lambda)$ with λ in a cell $\underline{c}'$ which satisfies $\underline{c}' \underset{\mathcal{T}_q}{\leq} \underline{c}$. Clearly, $\mathcal{T}_q(\leq c)$ is a tensor ideal in $\mathcal{T}_q$. The following result allows us to determine completely all such ideals.

We identify W_l with the set of alcoves in X by matching $w \in W_l$ with $w \cdot C_l$. Then the above division of X^+ into weight cells gives a corresponding division of $W_l^+ = \{w \in W_l \mid w \cdot C_l \subset X^+\}$. For this we have

Theorem 10 (Ostrik [17]). *The weight cells in X^+ correspond to the right Kazhdan-Lusztig cells in W_l^+.*

For later use we record the following consequences of this theorem.

Corollary 11. Let $\underline{c}$ be a weight cell in X^+. Then there exist finitely many alcoves $A_1, \cdots, A_r$ in $\underline{c}$ such that any alcove $C \subset \underline{c}$ can be reached from some A_i via a sequence $A_i = C_1 < C_2 < \cdots < C_m = C$ of alcoves C_j in $\underline{c}$ where C_j and C_{j+1} share a common wall, $j = 1, \cdots, m-1$

Proof. Use Theorem 10 and the arguments in [24], Section 3.

§4. Modular tensor ideals

Let $\mathcal{T}$ denote the category of tilting modules for G. We can then define $\underset{\mathcal{T}}{\leq}$ and $\underset{\mathcal{T}}{\sim}$ in analogy with the corresponding quantum case studied in Section 3. The equivalence classes of $\underset{\mathcal{T}}{\sim}$ are called modular weight cells.

The same arguments as in Section 3 give that (for $p \geq h$) each modular weight cell is a union of lower closures of alcoves intersected with X^+. Also we have in analogy with Proposition 9 (see [6], [12]).

Proposition 12. *C_p is a modular weight cell.*

On the other hand the modular analogue of Proposition 6 fails: For each $r \in \mathbb{N}$ we have a Steinberg module St_r in $\mathcal{T}$ given by

$$\mathrm{St}_r = L((p^r-1)\rho) = \Delta((p^r-1)\rho) = \nabla((p^r-1)\rho) = T((p^r-1)\rho).$$

However, St_r is not injective (in fact, there are no injective finite dimensional G-modules at all). Moreover, it is clear (e.g. because St_{r+1} is injective for the $(r+1)$'th Frobenius kernel in G whereas St_r is not) that St_r is not a summand of $\mathrm{St}_{r+1} \otimes_k Q$ for any $Q \in \mathcal{T}$. Hence $(p^r-1)\rho \not\underset{\mathcal{T}}{\leq} (p^{r+1}-1)\rho$.

This observation shows (in contrast to the case considered in Section 3) that there are infinitely many modular weight cells (as pointed out by Ostrik in [18]).

Set now $Y_r = (p^r-1)\rho + X^+$. If $\lambda \in Y_r$ then we can write uniquely $\lambda = \lambda_0^r + p^r\lambda_1^r$ with $\lambda_0^r \in (p^r-1)\rho + X_{p^r}$ and $\lambda_1^r \in X^+$. At least for $p \geq 2h-2$ (see [10]) we have

$$T(\lambda) \cong T(\lambda_0^r) \otimes_k T(\lambda_1^r)^{(r)}. \tag{1}$$

Here $^{(r)}$ denotes twist by the r'th Frobenius homomorphism on G.

Lemma 13. *Suppose $p \geq 2h-2$ and let $\lambda, \mu \in Y_r$. Then $\lambda \underset{\mathcal{T}}{\leq} \mu$ if and only if $\lambda_1^r \underset{\mathcal{T}}{\leq} \mu_1^r$.*

Proof. Arguing as in the proof of Proposition 6 we see that $\lambda_0^r \underset{\mathcal{T}}{\sim} (p^r-1)\rho$ (the assumption $p \geq 2h-2$ gives that St_r is injective among G-modules whose dominant weights are in $C_{2p^r(h-1)}$). When we combine this with (1) we see that $\lambda \underset{\mathcal{T}}{\sim} (p^r-1)\rho + p^r\lambda_1^r$. Hence we may assume $\lambda_0^r = (p^r-1)\rho = \mu_0^r$.

Suppose $T(\lambda_1^r)$ is a summand of $T(\mu_1^r) \otimes_k Q$ for some $Q \in \mathcal{T}$. Then $\mathrm{St}_r \otimes_k T(\lambda_1^r)^{(r)}$ is a summand of $\mathrm{St}_r \otimes_k T(\mu_1^r)^{(r)} \otimes_k Q^{(r)}$. Now St_r is a

summand of $\mathrm{St}_r \otimes_k \mathrm{St}_r \otimes_k \mathrm{St}_r$ and hence the latter module is a summand of $\mathrm{St}_r \otimes_k T(\mu_1^r)^{(r)} \otimes_k (\mathrm{St}_r \otimes_k \mathrm{St}_r \otimes Q^{(r)})$. Noting that $\mathrm{St}_r \otimes_k Q^{(r)}$ and hence also $Q^1 = \mathrm{St}_r \otimes_k \mathrm{St}_r \otimes Q^{(r)}$ are in $\mathcal{T}$ we conclude that $T(\lambda)$ is a summand of $T(\mu) \otimes_k Q^1$, i.e. $\lambda \underset{\mathcal{T}}{\leq} \mu$.

Conversely, suppose $T(\lambda)$ is a summand of $T(\mu) \otimes_k Q$ for some $Q \in \mathcal{T}$. Recalling that $T(\mu) = \mathrm{St}_r \otimes_k T(\mu_1^r)^{(r)}$ (by our assumption on μ_0^r and (1)) we consider first $\mathrm{St}_r \otimes_k Q \in \mathcal{T}$. The summands of this module all have the form $T(\nu)$ with $\nu \in Y_r$. Hence

$$T(\mu) \otimes_k Q \simeq \bigoplus_{\nu \in Y_r} c_\nu T(\nu_0^r) \otimes_k (T(\mu_1^r) \otimes_k T(\nu_1^r))^{(r)}$$

for some $c_\nu \in \mathbb{N}$. We see that since $T(\lambda) = \mathrm{St}_r \otimes_k T(\lambda_0^r)^{(r)}$ occurs in this sum there must exist $\nu \in Y_r$ such that $\nu_0^r = (p^r - 1)\rho$ and $T(\lambda_1^r)$ is a summand of $T(\mu_1^r) \otimes_k T(\nu_1^r)$. In particular $\lambda_1^r \underset{\mathcal{T}}{\leq} \mu_1^r$ as desired.

Combining Proposition 12 with this lemma we get

Proposition 14. For each $r \in \mathbb{N}$ the set

$$\underline{c}_1^r = (p^r - 1)\rho + X_{p^r} + p^r C_p$$

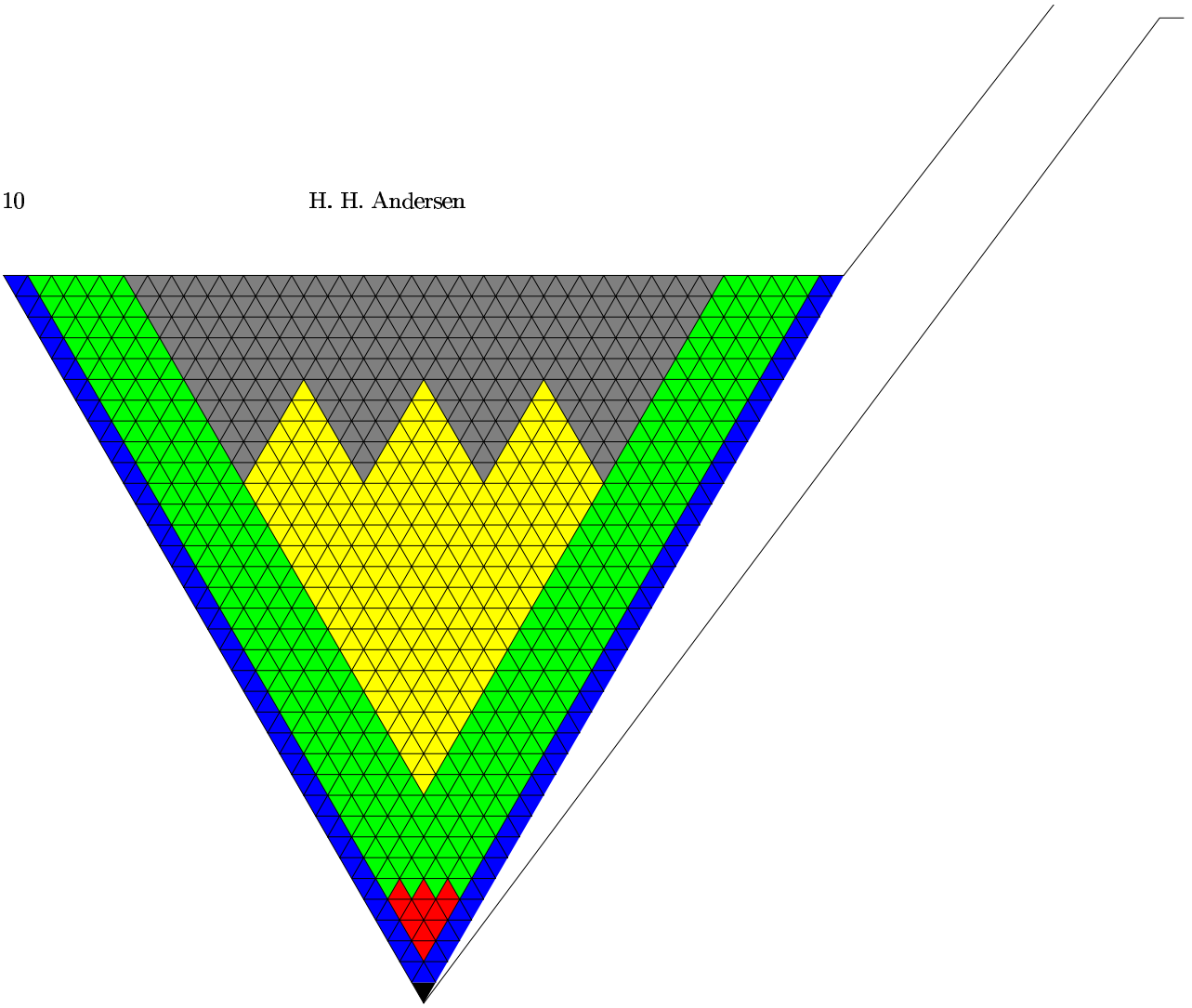
is a modular weight cell.

Note that for $r = 0$ we have $\underline{c}_1^0 = C_p$. For $r > 0$ the cell $\underline{c}_1^r$ contains the Steinberg weight $(p^r - 1)\rho$. All $\underline{c}_1^r$ are finite.

Example 15. Consider the case where R has type A_2 and $p > 3$. Then it follows from [14] (see also Theorem 19 below) that the set $\underline{c}_2 = X \backslash (C_p \cup ((p-1)\rho + X^+))$ is a modular weight cell. Using the notation from Proposition 14 we set

$$\underline{c}_2^r = Y_r \backslash (Y_{r+1} \cup \underline{c}_1^r).$$

Then the modular weight cells in X^+ are $\{\underline{c}_i^r \mid i = 1, 2; r \in \mathbb{N}\}$. We illustrate this for the case $p = 5$ in Figure 1. As a consequence the modular tensor ideals (i.e. the tensor ideals in $\mathcal{T}$) can be described as $\mathcal{T}(\leq \underline{c}_i^r)$ for some $i \in \{1, 2\}, r \in \mathbb{N}$.



c_1^0 black, c_2^0 blue, c_1^1 red, c_2^1 green, c_1^2 yellow, $\underline{c}_2^2$ grey.
Figure 1: Type $A_2, p = 5$

In the above example we see that each weight cell breaks up into a union of modular weight cells. We expect this to be true in general. The following result presents some evidence for this.

Proposition 16. Assume Conjecture 3 and let $p \gg 0$. If $a_{\mu\lambda} \neq 0$ for some $\mu, \lambda \in X^+$ then $\mu \underset{\overline{T}_q}{\leq} \lambda$.

Proof. We assume (as we may, see Proposition 8) that λ is p-regular. Let C be the alcove which contains λ and let $\underline{c}$ denote the weight cell containing C. Then by Corollary 11 there exists a sequence $A_i = C_1 < C_2 < \cdots < C_m = C$ of alcoves in $\underline{c}$ such that C_j and C_{j+1} have a common wall. Denote by Θ_j the wallcrossing functor associated to this wall. Then $T(\lambda)$ is a summand of $\Theta_{m-1} \cdots \Theta_2 \Theta_1 T(\lambda^1)$ where

$\lambda^1 \in A_i$ is in $W_p.\lambda$. Since $p \gg 0$ we have $A_i \subset C_{p^2}$ and hence by Conjecture 3 we get $T(\lambda^1)_q = T_q(\lambda^1)$. If therefore $T_q(\mu)$ is a summand of $T(\lambda)_q$ then it is also a summand of $\Theta_{m-1} \cdots \Theta_1 T_q(\lambda^1)$, i.e. $\mu \underset{\overline{\mathcal{T}}_q}{\leq} \lambda^1 \underset{\overline{\mathcal{T}}_q}{\sim} \lambda$.

Corollary 17. Let $\lambda, \mu \in X^+$. With the same assumptions as in Proposition 16 we have that if $\lambda \underset{\overline{\mathcal{T}}}{\leq} \mu$ then also $\lambda \underset{\overline{\mathcal{T}}_q}{\leq} \mu$. In particular, each weight cell is a union of modular weight cells.

Proof. Suppose $T(\lambda)$ is a summand of $T(\mu) \otimes_k Q$ for some $Q \in \mathcal{T}$. Then $T(\lambda)_q$ and in particular $T_q(\lambda)$ is a summand of $T(\mu)_q \otimes_{\mathbb{C}} Q_q$. It follows that there exists $\nu \in X^+$ with $a_{\nu\mu} \neq 0$ such that $T_q(\lambda)$ is a summand of $T_q(\nu) \otimes_{\mathbb{C}} Q_q$, i.e. $\lambda \underset{\overline{\mathcal{T}}_q}{\leq} \nu$. By Proposition 16 we also have $\nu \underset{\overline{\mathcal{T}}_q}{\leq} \mu$ and the Corollary follows.

As an immediate consequence of this corollary we get

Corollary 18. With assumptions as in Proposition 16 we have that $\mathcal{T}(\underset{\overline{\mathcal{T}}_q}{\leq} \underline{c})$ is a tensor ideal of $\mathcal{T}$ for any weight cell $\underline{c}$ in X^+.

§5. The second cell

In this section we report on some recent work by T. E. Rasmussen [20].

Assume throughout this section that R is irreducible and simply laced of rank > 1. We shall denote by $\underline{c}_2$ the weight cell which contains the upper closure of C_p. We refer to $\underline{c}_2$ as the second cell ($\underline{c}_1 = C_p$ being the first cell).

Recall that if $Q \in \mathcal{T}$ then we denote by $Q_q \in \mathcal{T}_q$ the "quantization" of Q.

Theorem 19 ([20]). *Let $Q \in \mathcal{T}$. Then we have for all $\lambda \in \underline{c}_2$*

$$[Q : T(\lambda)] = [Q_q : T_q(\lambda)]$$

This may be thought of as a generalization of [12] and [6] which show that the equality in Theorem 19 holds for $\lambda \in \underline{c}_1$. More explicitly, these results give the following explicit formula

Theorem 20 ([6], [12]). *If $Q \in \mathcal{T}$ (respectively $\mathcal{T}_q$) then we have for all $\lambda \in \underline{c}_1$*

$$[Q : T(\lambda)] = \sum_{\substack{w \in W_p \\ w \cdot \lambda \in X^+}} (-1)^{l(w}[Q : \Delta(w \cdot \lambda)]$$

(respectively

$$[Q : T_q(\lambda)] = \sum_{\substack{w \in W_p \\ w \cdot \lambda \in X^+}} (-1)^{l(w)}[Q : \Delta_q(w \cdot \lambda)]).$$

Consider now $r \in \mathbb{N}$ and let λ be a partition of r. Associated to λ we have a simple $k[\Sigma_r]$-module D^λ, respectively, a simple $H_r(q)$-module D^λ_q. Here $H_r(q)$ denotes the Hecke algebra over $\mathbb{C}$ for Σ_r with parameter q. These modules are 0 unless λ is p-regular (i.e. no p lines in λ are equal). Then

Corollary 21 ([20]). Suppose the partition $\lambda = (\lambda_1, \cdots, \lambda_n)$ satisfies either $\lambda_1 - \lambda_{n-1} < p - n + 2$ or $\lambda_2 - \lambda_n < p - n + 2$. Then

$$\dim_k D^\lambda = \dim_{\mathbb{C}} D^\lambda_q.$$

Proof. This is a consequence of Theorem 19 for the case $G = GL_n(k)$ via Schur-Weyl duality. In fact, this gives us (for any partition/dominant weight λ)

$$[V^{\otimes r} : T(\lambda)] = \dim_k D^\lambda.$$

Here V is the natural n-dimensional module for $GL_n(k)$. This is a tilting module and hence so is $V^{\otimes r}$. There is a completely analogous equality in the quantum case. Hence the corollary follows once it is checked that the weight λ belongs to $\underline{c}_1 \cup \underline{c}_2$ if and only if the corresponding partition λ satisfies one of the equalities stated. See [20] for details.

Remark 22.

i) For $n = 3$ the corollary was proved by Jensen and Mathieu [14]

ii) If λ satisfies $\lambda_1 - \lambda_n < p - n$ then λ belongs to the first weight cell $\underline{c}_1$ (the bottom alcove). In this case Mathieu has given an explicit algorithm for $\dim_k D^\lambda$ (and for $\dim_{\mathbb{C}} D_q^\lambda$), see [15].

iii) The corollary verifies in part a conjecture by Mathieu [16, Conj. 15.4] (In my lecture at the conference I hinted that there seemed to be some evidence of inconsistency between this conjecture and our Conjecture 3. This turned out to rely on a misunderstanding).

§6. The mixed case

In this section we fix $l \in \mathbb{N}$ with $(p,l) = 1$. As in Section 3 we shall also assume l odd (and prime to 3 if R has a component of type G_2).

We choose a primitive l'th root of 1 in k which we denote ζ. In this section we shall consider tilting modules for $U_\zeta = U_A \otimes_A k$ where the A-algebra structure of k is given by specializing v to $\zeta \in k$. Let m be the kernel of the structure homomorphism $A \to k$ and set $A_{l,p} = A_m$. Then k is also an $A_{l,p}$-algebra and so is $\mathbb{C}$ via $v \mapsto q$ (where $q \in \mathbb{C}$ now again is a primitive l'th root of 1).

The theory in Section 3 may again be carried over to this situation. We denote the simple module, Weyl module, dual Weyl module and indecomposable tilting module for U_ζ with highest weight $\lambda \in X^+$ by $L_\zeta(\lambda), \Delta_\zeta(\lambda), \nabla_\zeta(\lambda)$ and $T_\zeta(\lambda)$, respectively. We have

$$\Delta_\zeta(\lambda) \simeq \Delta_A(\lambda) \otimes_A k \text{ and } \nabla_\zeta(\lambda) \simeq \nabla_A(\lambda) \otimes_A k.$$

Again, $L_\zeta(\lambda)$ is the image of the natural homomorphism $\Delta_\zeta(\lambda) \to \nabla_\zeta(\lambda)$. Moreover, there exists a tilting module $T_{p,l}(\lambda)$ for $U_{A_{p,l}}$ which satisfies $T_{p,l}(\lambda) \otimes_{A_{p,l}} k = T_\zeta(\lambda)$.

In analogy with Section 2 we then define $T_\zeta(\lambda)_q$ by

$$T_\zeta(\lambda)_q = T_{p,l}(\lambda) \otimes_{A_{p,l}} \mathbb{C}.$$

This U_q-module decomposes into a direct sum of $T_q(\mu)$'s and we define the corresponding (fusion) multiplicities $a^\zeta_{\mu\lambda} \in \mathbb{N}$ by

$$T_\zeta(\lambda)_q = \bigoplus_{\mu \in X^+} a^\zeta_{\mu\lambda} T_q(\mu)$$

These numbers again satisfy

$$a^\zeta_{\lambda\lambda} = 1 \text{ and } a^\zeta_{\mu\lambda} = 0 \text{ unless } \mu \leq \lambda.$$

Problem 23. Determine $a^\zeta_{\mu\lambda}$ for all $\mu, \lambda \in X^+$.

Remark 24. Just as in the case $\zeta = 1$ treated in Section 2 (see Problem 1) this problem is wide open. Only the case where R is of type A_1 is known. The problem is equivalent to the problem of finding the characters of all $T_\zeta(\lambda)$'s.

We have also a conjecture analogous to Conjecture 3. It was discussed in [4, Section 4] (see in particular Proposition 4.6 and Remark 4.7) as a consequence of a stronger conjecture.

Conjecture 25. If $\lambda \in C_{lp}$ then $T_\zeta(\lambda)_q = T_q(\lambda)$.

Remark 26. This conjecture is known to hold only for types A_1 and A_2 (see [4], 4.5). If $p \geq 2h-2$ it implies (as pointed out in loc. cit Remark 4.7 (v)) that $\operatorname{ch} L_\zeta(\lambda) = \operatorname{ch} L_q(\lambda)$ for all $\lambda \in X_l$. This last identity is conjectured in [1, 4c.3 (iv)]. It is known to hold for p large. For arbitrary p it was verified for rank 2 and for type A_3 by Thams [23].

Let now $\mathcal{T}_\zeta$ denote the category of tilting modules for U_ζ. We then have relations $\underset{\mathcal{T}_\zeta}{\leq}$ and $\underset{\mathcal{T}_\zeta}{\sim}$ on X^+ analogous to those studied in Section 3-4. The equivalence classes for $\underset{\mathcal{T}_\zeta}{\sim}$ are called mixed weight cells.

As before we have

(2) A mixed weight cell is the union of lower closures of alcoves (for W_l) intersected with X^+.

(3) C_l is a mixed cell.

Consider now the weight cell $(l-1)\rho + X^+$, see Proposition 6. If $\nu \in (l-1)\rho + X^+$ we write $\nu = \nu_0 + l\nu_1$ with $\nu_0 \in (l-1)\rho + X_l$ and $\nu_1 \in X^+$. The same argument as in the modular case shows that for $p \geq 2h-2$ the module $T_\zeta(\nu_0)$ restricts to a PIM for the small quantum group $u_\zeta \subset U_\zeta$. Then we can also reuse Donkin's argument from [10] to obtain

$$T_\zeta(\nu) \simeq T_\zeta(\nu_0) \otimes_k T(\nu_1)^{[l]}. \tag{4}$$

Here $^{[l]}$ denotes twist by the quantum Frobenius homomorphism from U_ζ to the hyperalgebra of G_1, see [8].

If $\operatorname{ch} L_\zeta(\lambda) = \operatorname{ch} L_q(\lambda)$ for all $\lambda \in X_l$ (which we know is true for $p \gg 0$, see Remark 26) then we have $T_\zeta(\nu_0)_q = T_q(\nu_0)$ for all $\nu_0 \in (l-1)\rho + X^+$. Hence (4) shows that the determination of $T_\zeta(\nu)$ for $\nu \in (l-1)\rho + X^+$ is equivalent to the determination of all modular indecomposable tilting modules. On the other hand a solution of the modular problem (Problem 1) is not enough to give the mixed tilting modules with highest weights outside this smallest weight cell.

As in Section 4 we can combine Conjecture 25 and Corollary 11 to see that weight cells divide into mixed weight cells when p is large. Here we only state the analogue of Corollary 18 giving some of the tensor ideals in $\mathcal{T}_\zeta$.

Corollary 27. Assume Conjecture 25 holds and let $p \gg 0$. Then for each weight cell $\underline{c}$ in X^+ the subcategory $\mathcal{T}_\zeta(\underset{\mathcal{T}_q}{\leq} \underline{c})$ is a tensor ideal in $\mathcal{T}_\zeta$.

Finally assume R is irreducible and simply laced of rank > 1. Consider the second cell $\underline{c}_2$ as in Section 5. The same arguments as used by Rasmussen in [20] give

Proposition 28. Let $Q = \mathcal{T}_\zeta$. Then we have for all $\lambda \in \underline{c}_1 \cup \underline{c}_2$

$$[Q : T_\zeta(\lambda)] = [Q_q : T_q(\lambda)].$$

It follows that $\underline{c}_2$ is also a mixed weight cell.

Consider type A and let $r \in \mathbb{N}$. Set $H_r(\zeta)$ equal to the Hecke algebra over k corresponding to Σ_r and with parameter ζ. Then Schur-Weyl duality gives the following application of Proposition 28, cf. Corollary 21.

Corollary 29. If $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ is a partition of r with either $\lambda_1 - \lambda_{n-1} < p - n + 2$ or $\lambda_2 - \lambda_n < p - n + 2$ then $\dim_k D_\zeta^\lambda = \dim_{\mathbb{C}} D_q^\lambda$.

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