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NIELS LAURITZEN, JESPER FUNCH THOMSEN

1. Introduction

Let G be a semisimple, simply connected linear algebraic group over an algebraically closed field k and B be a Borel subgroup in G. If $w = (P_1, \ldots, P_n)$ is a sequence of minimal parabolic subgroups containing B, we may form the quotient $Z_w = P_w/B^n$, where $P_w = P_1 \times \cdots \times P_n$ and B^n acts on P_w from the right via

$$(p_1, \ldots, p_n)(b_1, \ldots, b_n) = (p_1 b_1, b_1^{-1} p_2 b_2, b_2^{-1} p_3 b_3, \ldots, b_{n-1}^{-1} p_n b_n).$$

The quotient Z_w is inductively a sequence of $\mathbb{P}^1$-bundles with natural sections starting with the $\mathbb{P}^1$-bundle P_1/B (over a point). The product map $P_w \to G$ induces a proper morphism $\varphi_w : Z_w \to G/B$ whose image is a Schubert variety in G/B. For "reduced" sequences w the morphism φ_w is birational and equal to the celebrated Demazure desingularization of the Schubert variety $\varphi_w(Z_w)$. In general we call Z_w the *Bott-Samelson* variety associated with w .The construction of Z_w originates in the papers [1][2][3] of Bott-Samelson, Demazure and Hansen. See also the master's thesis (speciale) [4] by Hansen.

We characterize the globally generated, ample and very ample line bundles on Z_w. The generators of the ample cone are naturally defined $\mathcal{O}(1)$-bundles for successive $\mathbb{P}^1$-bundles. They form a basis of $\mathrm{Pic}\,(Z_w)$. Proving that they account for all ample line bundles originally lead us to some quite involved computer calculations in the Chow ring of Z_w. It later turned out that the key point is Lemma 2.1.

Using Frobenius splitting [7] and our description of globally generated line bundles we prove the vanishing theorem

$$\mathrm{H}^i(Z_w, \mathcal{L}(-D)) = 0, i > 0$$

where $\mathcal{L}$ is any globally generated line bundle on Z_w and D a subdivisor of the boundary of Z_w corresponding to a reduced subexpression of w (cf. Theorem 7.4 for a precise description).

A special case ($D = 0$) of this vanishing theorem has been proved in [6] (with no details on the involved line bundles). The vanishing theorem above is a generalization of the crucial vanishing theorem for pull backs of globally generated line bundles on G/B in Kumar's proof [5] of the Borel-Bott-Weil theorem in the Kac-Moody case. Kumar relied heavily on the Grauert-Riemenschneider vanishing theorem available only in characteristic zero. Our approach shows that one may give a characteristic free generalization using only the theory of Frobenius splitting.

2. Notation

Fix a semisimple algebraic group G over an algebraically closed field k and let B be a Borel subgroup in G containing the maximal torus T. A simple reflection s (wrt. B) in the Weyl group $W = N_G(T)/T$ determines the minimal parabolic subgroup $P_s = BsB \cup B \supseteq B$. We let w denote a sequence $(s_1, s_2, \cdots, s_n)$ of simple reflections, $w[j]$ the truncated sequence $(s_1, \dots, s_{n-j})$, $P_w = P_1 \times \cdots \times P_n$ and $Z_w = P_w/B^n$ the associated Bott-Samelson variety, where B^n acts from the right on P_w as

$$(p_1, \dots, p_n)(b_1, \dots, b_n) = (p_1 b_1, b_1^{-1} p_2 b_2, b_2^{-1} p_3 b_3, \dots, b_{n-1}^{-1} p_n b_n).$$

By convention $Z_{w[n]}$ will denote a 1-point space. We may also write Z_w as

$$P_1 \times^B P_2 \times^B \cdots \times^B P_n/B$$

where $X \times^B Y$ is the quotient $X \times Y/B$ with B acting as $(x, y)b = (xb, b^{-1}y)$. This shows that Z_w comes as the sequence

$$Z_w \to Z_{w[1]} \to \cdots \to Z_{w[n-2]} = P_1 \times^B P_2/B \to Z_{w[n-1]} = P_1/B$$

of successive $\mathbb{P}^1$-fibrations. In general we let $\pi_{w[j]}$ denote the natural morphism

$$Z_w \to Z_{w[j]}$$

in the sequence of $\mathbb{P}^1$-bundles above and use π_w to denote the morphism $\pi_{w[1]} : Z_w \to Z_{w[1]}$. Let $w(j) = (s_1, \cdots, \widehat{s_j}, \cdots, s_n)$. The natural embedding $P_{w(j)} \to P_w$ induces a closed embedding $\sigma_{w,j} : Z_{w(j)} \to Z_w$ which makes $Z_{w(j)}$ into a divisor in Z_w. The divisor $\partial Z_w = Z_{w(1)} \cup \cdots \cup Z_{w(n)}$ in Z_w has normal crossing. When $A \subseteq \{1, 2, \dots, n\}$ we define

$$Z_{w(A)} = \cap_{j \in A} Z_{w(j)},$$

and let $\sigma_{w,A} : Z_{w,A} \to Z_w$ denote the closed embedding given by $\sigma_{w,j}$ for $j \in A$. Finally we let $\pi : Z_w \to G/B$ denote the natural proper morphism coming from the product map $P_w \to G$.

2.1. **Induced bundles on Z_w.** We let $\mathcal{L}_w(V)$ denote the locally free sheaf of sections of the associated vector bundle $P_w \times^{B^n} V$ on Z_w, where V is a finite dimensional B^n-representation. We view a B-representation V as a B^n-representation by letting B^n act on $v \in V$ as $(b_1, b_2, \cdots, b_n).v = b_n.v$. With this convention we get for a B-character $\lambda \in X^*(B)$ that $\mathcal{L}_w(\lambda) = \mathcal{L}_w(0, \cdots, 0, \lambda)$.

2.2. **Induced bundles on G/B.** Let V be a finite dimensional B representation. We let $\mathcal{L}_{G/B}(V)$ denote the locally free sheaf on G/B associated with V. This is the sheaf of sections of the vector bundle $G \times^B V \to G/B$, $(g, v) \mapsto gB$. When V has dimension 1, associated to a B-character λ, the sheaf $\mathcal{L}_{G/B}(\lambda) = \mathcal{L}_{G/B}(V)$ is a line bundle. This gives a bijection between B-characters and line bundles on G/B. It is well known, that $\mathcal{L}_{G/B}(\lambda)$ is globally generated (resp. ample) exactly when λ is dominant (resp. regular) wrt. the Borel subgroup opposite to B.

$\langle \lambda, \alpha^\vee \rangle \geq 0$ (resp. $\langle \lambda, \alpha^\vee \rangle > 0$) for all simple roots $\alpha \in S$. The pull back of $\mathcal{L}_{G/B}(V)$ to Z_w under the morphism $\pi : Z_w \to G/B$ is given by the formula

$$\pi^*(\mathcal{L}_{G/B}(V)) \simeq \mathcal{L}_{Z_w}(V).$$

Lemma 2.1. *Let $1 \leq i \leq n$ and $0 \leq j \leq n-1$ be integers and let λ denote a B-character. Then*

$$\sigma_{w,i}^* \pi_{w[j]}^* \mathcal{L}_{Z_{w[j]}}(\lambda) \simeq \begin{cases} \pi_{w(i)[j-1]}^* \mathcal{L}_{w(i)[j-1]}(\lambda) & \text{if } i > n-j, \\ \pi_{w(i)[j]}^* \mathcal{L}_{w(i)[j]}(\lambda) & \text{if } i \leq n-j. \end{cases}$$

Proof. When $i > n - j$ the claim follows by the commutativity of the diagram

$$\begin{tikzcd} Z_{w(i)} \arrow[r, "\pi_{w(i)[j-1]}"] \arrow[d, "\sigma_{w,i}"'] & Z_{w(i)[j-1]} \arrow[d, equal] \arrow[dr] & \\ Z_w \arrow[r, "\pi_{w[j]}"] & Z_{w[j]} \arrow[r, "\pi"] & G/B. \end{tikzcd}$$

Similarly, the case $i \leq n - j$ follows from the commutative diagram

$$\begin{tikzcd} Z_{w(i)} \arrow[r, "\pi_{w(i)[j]}"] \arrow[d, "\sigma_{w,i}"'] & Z_{w(i)[j]} \arrow[r, equal] & Z_{w[j](i)} \arrow[d, "\sigma_{w[j],i}"'] \arrow[dr] & \\ Z_w \arrow[rr, "\pi_{w[j]}"] & & Z_{w[j]} \arrow[r, "\pi"] & G/B. \end{tikzcd}$$

□

3. Line bundles on Z_w

The Picard group $\mathrm{Pic}(Z_w)$ is a free abelian group of rank n. This follows easily by induction using the $\mathbb{P}^1$-fibration $\pi_w : Z_w \to Z_{w[1]}$. In fact we have a decomposition $\mathrm{Pic}(Z_w) = \mathrm{Pic}(Z_{w[1]}) \oplus \mathbb{Z}\mathcal{L}$, where $\mathcal{L}$ is any line bundle on Z_w with degree one along the fibers of π_w.

3.1. **The $\mathcal{O}(1)$-basis.** Recall our notation $w = (s_1, \ldots, s_n)$ for a sequence of simple reflections defining Z_w. Suppose that s_n is a reflection in the simple root α. Then we let

$$\mathcal{O}_w(1) = \mathcal{L}_w(\omega_\alpha),$$

where ω_α denotes the fundamental dominant weight corresponding to α. Then $\mathcal{O}_w(1)$ has degree one along the fibres of π_w. It is globally generated since it is the pull back of the globally generated line bundle $\mathcal{L}_{G/B}(\omega_\alpha)$ on G/B. This gives inductively a basis for $\mathrm{Pic}\,(Z_w)$ which we call the $\mathcal{O}(1)$-basis. Thus

$$\mathrm{Pic}\,(Z_w) = \mathbb{Z}\mathcal{O}_w(1) \oplus \mathbb{Z}\mathcal{O}_{w[1]}(1) \oplus \cdots \oplus \mathbb{Z}\mathcal{O}_{w[n-1]}(1),$$

where we write $\mathcal{O}_{w[j]}(1)$ instead of the pull back $\pi^*_{w[j]}\mathcal{O}_{w[j]}(1)$. The line bundle $m_1\mathcal{O}_w(1) + \cdots + m_n\mathcal{O}_{w[n-1]}(1) \in \mathrm{Pic}\,(Z_w)$ is denoted

$$\mathcal{O}_w(m_1, \ldots, m_n),$$

where $m_1, \ldots, m_n \in \mathbb{Z}$.

Theorem 3.1. *A line bundle $\mathcal{L} = \mathcal{O}_w(m_1, \ldots, m_n)$ is very ample on Z_w if and only if $m_1, \ldots, m_n > 0$.*

Proof If $n = 1$ it is well known that $\mathcal{L}$ is ample and very ample if and only if $m_1 > 0$. We proceed using induction on n. In general the $\mathbb{P}^1$-bundle $\pi_w : Z_w \to Z_{w[1]}$ may by identified with the projective bundle $\mathbb{P}(V) \to Z_{w[1]}$, where V is the rank two bundle $\mathcal{L}_{w[1]}(\mathrm{H}^0(P_\alpha/B, \omega_\alpha))$ on $Z_{w[1]}$ and $\mathcal{O}_{\mathbb{P}(V)}(1) \cong \mathcal{O}_w(1)$. Since V is the pull back of a globally generated vector bundle on G/B it is globally generated. This implies that we have a commutative diagram (for some $N \in \mathbb{N}$)

$$\begin{tikzcd} \mathbb{P}(V) \arrow[r, "\varphi"] \arrow[d] & \mathbb{P}^N \times Z_{w[1]} \arrow[dl] \\ Z_{w[1]} & \end{tikzcd},$$

where φ is a closed embedding. Since $\varphi^*(\mathcal{O}_{\mathbb{P}^N}(1) \times \mathcal{O}_{Z_w}) \cong \mathcal{O}_{\mathbb{P}(V)}(1)$ it follows that $\mathcal{O}_{\mathbb{P}(V)}(n) \otimes \pi_w^*\mathcal{L}'$ is very ample if $n > 0$ and $\mathcal{L}'$ is very ample on $Z_{w[1]}$. By induction $\mathcal{L}$ is very ample if $m_1, \ldots, m_n > 0$.

Suppose on the other hand that $\mathcal{L}$ is very ample. By induction we get that $m_2, \ldots, m_n > 0$, since

$$\sigma^*_{w,1}\mathcal{L} \simeq \mathcal{O}_{w(1)}(m_2, \ldots, m_n)$$

by Lemma 2.1. Furthermore, Lemma 2.1 also gives

$$\sigma^*_{w,2}\mathcal{L} \simeq \mathcal{O}_{w(2)}(m_1, m_3, \ldots, m_n) \otimes \pi^*_{w(2)[n-2]}(\mathcal{L}_{w(2)[n-2]}(\omega_\beta))$$

where β is the simple root corresponding to s_2. Suppose s_1 is a reflection in the simple root α. Using that $w(2)[n-2] = (s_1)$ and hence $Z_{w(2)[n-2]} = P_\alpha/B \simeq \mathbb{P}^1$, we identify $\mathcal{L}_{w(2)[n-2]}(\omega_\beta)$ with $\mathcal{O}_{\mathbb{P}^1}(\langle \omega_\beta, \alpha^\vee \rangle)$. When $s_1 \neq s_2$ this means that the line bundle $\pi^*_{w(2)[n-2]}(\mathcal{L}_{w(2)[n-2]}(\omega_\beta))$ is trivial, and $m_1 > 0$ by induction.

If $s_1 = s_2$ and s_1 is a reflection in the simple root α, then

$$Z_w \cong P_\alpha/B \times Z_{w(1)} \simeq \mathbb{P}^1 \times Z_{w(1)}.$$

Under this isomorphism $\mathcal{L}$ identifies with $\mathcal{O}_{\mathbb{P}^1}(m_1)\times\mathcal{L}_{w(1)}(m_2,\ldots,m_n)$. This proves that $m_1>0$. $\square$

We obtain the following two corollaries as immediate consequences.

Corollary 3.2. *Ample line bundles on Z_w are very ample.*

Corollary 3.3. *A line bundle $\mathcal{L}=\mathcal{O}_w(m_1,\ldots,m_n)$ is globally generated on Z_w if and only if $m_1,\ldots,m_n\geq 0$.*

Proof If $m_1,\ldots,m_n\geq 0$ then $\mathcal{L}$ is globally generated being a tensor product of globally generated line bundles. Assume there is a globally generated line bundle $\mathcal{L}=\mathcal{O}_w(m_1,\ldots,m_n)$ with some $m_i<0$. Since ample tensor globally generated is ample this contradicts Theorem 3.1. $\square$

3.2. **The divisor basis.** The $\mathbb{P}^1$-bundle $\pi_w: Z_w\to Z_{w[1]}$ comes with a natural section $\sigma_{w,n}: Z_{w[1]}=Z_{w(n)}\to Z_w$. So the line bundle $\mathcal{O}(Z_{w(n)})$ defined by the divisor $Z_{w(n)}$ has degree one along the fibres of π_w. Inductively this shows that the line bundles $\mathcal{O}_{Z_w}(Z_{w(j)})$, $j=1,2,\ldots,n$, form a basis of the Picard group of Z_w. We call this basis the Z-basis. The line bundle $m_1\mathcal{O}_{Z_w}(Z_{w(1)})+\cdots+m_n\mathcal{O}_{Z_w}(Z_{w(n)})\in \mathrm{Pic}(Z_w)$ in the Z-basis is denoted

$$\mathcal{O}_{Z_w}(m_1,\ldots,m_n),$$

where $m_1,\ldots,m_n\in\mathbb{Z}$.

3.3. **Effective line bundles.** The line bundles $\mathcal{O}(Z_{w(j)})$ are effective. They do not necessarily generate the cone of effective line bundles unless the expression w is reduced as shown by the following example.

Example 3.4. *Consider $w=(s_\alpha,s_\alpha)$, where α is a simple reflection. Then the corresponding Bott-Samelson variety Z_w is isomorphic to $P_\alpha/B\times P_\alpha/B$ by the map $(p_1:p_2)\mapsto(p_1B,p_1p_2B)$. Under this isomorphism the 2 divisors $Z_{w(1)}$ and $Z_{w(2)}$ corresponds to the diagonal $\Delta_{P_\alpha/B}$ and $\{eB\}\times P_\alpha/B$. From this we conclude that the effective line bundle corresponding to the divisor $P_\alpha/B\times\{eB\}$ is not contained in the cone generated by $\mathcal{O}(Z_{w(1)})$ and $\mathcal{O}(Z_{w(2)})$.*

Proposition 3.5. *Let $w=(s_1,s_2,\ldots,s_n)$ be a reduced sequence and $\mathcal{L}=\mathcal{O}(\sum_{j=1}^n m_jZ_{w(j)})$ a line bundle on Z_w. Then $\mathcal{L}$ is effective if and only if $m_j\geq 0$ for all j.*

Proof. If $m_j\geq 0$ then $\mathcal{L}$ is clearly effective. We are hence left with proving that $m_j\geq 0$ if $\mathcal{L}$ is effective. So assume that $\mathcal{L}$ is effective. Clearly $\mathcal{L}$ is B linearized as Z_w and all $Z_{w(j)}$ are compatible B-spaces. Hence the global sections $\mathcal{L}(Z_w)$ is a non-zero finite dimensional (as Z_w is projective) B-representation. This allows us to pick a B-semiinvariant (i.e. invariant up to constants) global section s of $\mathcal{L}$. The zero-scheme

$Z(s)$ of s is then a B-invariant divisor of Z_w. As w is reduced the morphism

$$\psi : Z_w \to X(s_1 \cdots s_n),$$

is known to be birational. In fact, it is known (essentially by the Bruhat decomposition) that ψ is an isomorphism above the dense Bruhat cell $C(s_1 \cdots s_n)$ of $X(s_1 \cdots s_n)$. This shows that $Z_w \setminus \cup_{j=1}^n Z_{w(j)}$ is a dense B-orbit. Hence, $Z(s) \subseteq \cup_{j=1}^n Z_{w(j)}$ and

$$\mathcal{L} \simeq \mathcal{O}(Z(s)) \simeq \mathcal{O}(\sum_{j=1}^{n} m'_j Z_{w(j)}),$$

where m'_j are positive integers. As $\mathcal{O}(Z_{w(j)})$, $j = 1, 2, \ldots, n$ is a basis for the Picard group of Z_w, we conclude $m_j = m'_j \geq 0$. $\square$

4. Frobenius splitting

Let $\pi : X \to \mathrm{Spec}\,(k)$ be a scheme defined over an algebraically closed field k of positive characteristic p. The *absolute Frobenius morphism* on X is the identity on point spaces and raising to the p-th power locally on functions. The absolute Frobenius morphism is *not* a morphism of k-schemes. Let X' be the scheme obtained from X by base change with the absolute Frobenius morphism on $\mathrm{Spec}\,(k)$, i.e., the underlying topological space of X' is that of X with the same structure sheaf $\mathcal{O}_X$ of rings, only the underlying k-algebra structure on $\mathcal{O}_{X'}$ is twisted as $\lambda \odot f = \lambda^{1/p} f$, for $\lambda \in k$ and $f \in \mathcal{O}_{X'}$. Using this description of X', *the relative Frobenius morphism* $F : X \to X'$ is defined in the same way as the absolute Frobenius morphism and it is a morphism of k-schemes.

4.1. Definition and results.

Recall that a variety X is called *Frobenius split* [7] if the homomorphism $\mathcal{O}_{X'} \to F_*\mathcal{O}_X$ of $\mathcal{O}_{X'}$-modules is split. A homomorphism $\sigma : F_*\mathcal{O}_X \to \mathcal{O}_{X'}$ is a splitting of $\mathcal{O}_{X'} \to F_*\mathcal{O}_X$ (called a *Frobenius splitting*) if and only if $\sigma(1) = 1$.

A Frobenius splitting σ is said to *compatibly split* a subvariety Z of X if $\sigma(\mathcal{I}_Z) \subseteq \mathcal{I}_Z$, where $\mathcal{I}_Z$ is the ideal sheaf of Z in X. In this case, σ induces a Frobenius splitting of Z.

When X is a smooth variety with canonical bundle ω_X, there is a natural isomorphism of $\mathcal{O}_{X'}$-modules :

$$F_*(\omega_X^{1-p}) \cong Hom_{\mathcal{O}_{X'}}(F_*\mathcal{O}_X, \mathcal{O}_{X'}).$$

In this way global sections of ω_X^{1-p} correspond to homomorphisms $F_*\mathcal{O}_X \to \mathcal{O}_{X'}$. A section of ω_X^{1-p} which, up to a non-zero constant, corresponds to a Frobenius splitting in this way, is called a *splitting section*. One of the first known criteria ([7], Prop. 8) for a smooth projective variety to be Frobenius split was the following

Proposition 4.1. *Let X be a smooth projective variety over k of dimension n. Assume that there exists a global section s of the anti canonical bundle ω_X^{-1}, with divisor of zeroes of the form*

$$\operatorname{div}(s) = Z_1 + Z_2 + \cdots + Z_n + D,$$

satisfying :

1. *The scheme theoretic intersection $\cap_i Z_i$ is a point $P \in X$.*
2. *D is an effective divisor not containing $P = \cap_j Z_j$.*

Then s is a splitting section of X which compatibly splits the subvarieties $Z_1, Z_2, \ldots, Z_n$.

This criterion was taylormade to suit the Bott-Samelson varieties.

Proposition 4.2. *Let X be a smooth variety over k and let s be a global section of the anti-canonical bundle ω_X^{-1} such that s^{p-1} is a Frobenius splitting section of X. Write the zero divisor* $\operatorname{div}(s)$ *of s as*

$$\operatorname{div}(s) = \sum_{j=1}^{r} Z_j,$$

with Z_j, $j = 1, \ldots, m$, being irreducible subvarieties of X of codimension 1. Then for any choice of integers $0 \leq m_j < p$, $j = 1, \ldots, r$, and any line bundle $\mathcal{L}$ on X, we have, for all integers i, an embedding of cohomology

$$\mathrm{H}^i(X, \mathcal{L}) \hookrightarrow \mathrm{H}^i(X, \mathcal{L}^p \otimes \mathcal{O}(\sum_{j=1}^{r} m_j Z_j)).$$

This result is well known, but we have not been able to find an explicit reference. We therefore include a proof.

Proof. Let $\phi : F_*\mathcal{O}_X \to F_*\omega_X^{1-p}$ denote the map induced by the global section s^{p-1} of ω_X^{1-p}. Composing this map with the twisted Cartier operator $\mathrm{C} : F_*{\omega_X}^{1-p} \to \mathcal{O}_{X'}$ we get, by the identifications above, the splitting section $\psi : F_*\mathcal{O}_X \to \mathcal{O}_{X'}$ defined by s^{p-1}. In particular, the morphism $\mathcal{O}_{X'} \to F_*\omega_X^{1-p}$ defined by s^{p-1} is split. As a consequence, if $\omega^{1-p} \simeq \mathcal{M} \otimes \mathcal{M}'$ and $s^{p-1} = t \otimes t'$ with t and t' global sections of line bundles $\mathcal{M}$ and $\mathcal{M}'$, then the morphism $\mathcal{O}_{X'} \to F_*\mathcal{M}$ defined by t is split. Tensoring with the line bundle $\mathcal{L}'$ on X' corresponding to $\mathcal{L}$ we find, using the projection formula and $F^*\mathcal{L}' \simeq \mathcal{L}^p$, that the morphism $\mathcal{L}' \to F_*(\mathcal{L}^p \otimes \mathcal{M})$ defined by t is split. Now use this on $\mathcal{M} = \mathcal{O}(\sum_{j=1}^{r} m_i Z_i)$. $\square$

As an immediate consequence of Proposition 4.2 we find

Corollary 4.3. *Let X be a Frobenius split projective variety and $\mathcal{L}$ an ample line bundle on X. Then, for each integer $j > 0$,*

$$\mathrm{H}^j(X, \mathcal{L}) = 0.$$

We also have the following result which will be important later.

Proposition 4.4. *Let X be a variety which is Frobenius split compatibly with subvarieties $Z_1, \dots, Z_n$ of codimension 1, and let $a_1, \dots, a_n$ be a collection of positive integers. Let $1 \leq r \leq n$ be an integer. Then there exists an integer M such that for each line bundle $\mathcal{L}$, each integer i and each integer $m \geq M$ we have an embedding*

$$\mathrm{H}^i(X, \mathcal{L} \otimes \mathcal{O}(-\sum_{j=1}^r Z_j)) \hookrightarrow \mathrm{H}^i(X, \mathcal{L}^{p^m} \otimes \mathcal{O}(-\sum_{j=1}^r (a_j+1) Z_j + \sum_{j=r+1}^n a_j Z_j)).$$

Proof. Notice first of all that if $a_j = 0$ for all $j = 1, \dots, n$, then the result follows by successive use of Proposition 4.2 (with $m_j = p-1$, $j = 1 \dots, r$). Assume hence that not all a_j are zero and write $a_j = pa'_j + a''_j$ with $a'_j \geq 0$ and $0 \leq a''_j < p$. By induction we may assume that there exists an integer M' and an embedding

$$\mathrm{H}^i(X, \mathcal{L} \otimes \mathcal{O}(-\sum_{j=1}^r Z_j)) \hookrightarrow \mathrm{H}^i(X, \mathcal{L}^{p^m} \otimes \mathcal{O}(-\sum_{j=1}^r (a'_j+1) Z_j + \sum_{j=r+1}^n a'_j Z_j)).$$

for each $m \geq M'$. Using Proposition 4.2, with values $m_j = p - 1 - a''_j$ when $j = 1, \dots, t$ and $m_j = a''_j$ when $j = t+1, \dots, n$, we furthermore find an embedding of the right hand side into

$$\mathrm{H}^i(X, \mathcal{L}^{p^{m+1}} \otimes \mathcal{O}(-\sum_{j=1}^r (a_j + 1) Z_j + \sum_{j=r+1}^n a_j Z_j)).$$

The claim (with $M = M' + 1$) now follows by composing the two embeddings above. □

5. Frobenius splitting of Bott-Samelson varieties

Let us now return to the situation of a Bott-Samelson variety Z_w associated to a sequence w of simple reflections. Recall the following lemma ([8], Proposition 2).

Lemma 5.1. *The anti-canonical bundle on Z_w is isomorphic to the line bundle*

$$\omega_{Z_w}^{-1} = \mathcal{O}(\sum_{j=1}^n Z_{w(j)}) \otimes \mathcal{L}_w(\rho).$$

The divisors $Z_{w(j)}$, $j = 1, 2, \dots, n$, intersect transversally and as $\mathcal{L}_w(\rho)$ is globally generated, and hence base point free, Proposition 4.1 and Lemma 5.1 implies

Theorem 5.2 ([7], Thm. 1)**.** *The Bott-Samelson variety Z_w is Frobenius split compatibly with the divisors $Z_{w(j)}$, $j = 1, 2, \dots, n$.*

The following proposition is a consequence of Proposition 4.2.

Proposition 5.3. *Let $\mathcal{L}$ be a line bundle on Z_w. Then for any choice of integers $0 \leq m_j < p$, $j = 1, 2 \ldots, n$ and $0 \leq m < p$, we have, for each integer i, an embedding of cohomology*

$$\mathrm{H}^i(X, \mathcal{L}) \hookrightarrow \mathrm{H}^i(X, \mathcal{L}^p \otimes \mathcal{O}(\sum_{j=1}^{n} m_j Z_{w(j)})) \otimes \mathcal{L}_w(m\rho)).$$

Proof. Let s_i be the global section of $\mathcal{O}(Z_i)$ with divisor of zeroes equal to Z_i, and let t be a global section of $\mathcal{L}_w(\rho)$ with divisor of zeroes $\sum_j D_j$ with D_j irreducible divisors not containing the point $\cap Z_i$. As above $s = t \cdot \prod s_i$ is a global section of $\omega_{Z_w}^{-1}$ such that s^{p-1} is a Frobenius splitting section of Z_w. By Propostion 4.2 applied to s, with coefficients m_i corresponding to Z_i and m corresponding to D_j, the result now follows by identifying $\mathcal{L}_w(\rho)$ with the line bundle $\mathcal{O}(\sum_j D_j)$. □

6. Cohomology vanishing of globally generated line bundles

In this section we will prove that the higher cohomology of globally generated line bundles on Bott-Samelson varieties vanishes.

Lemma 6.1. *There exists integers $m_1, \ldots, m_n > 0$ such that the line bundle $\mathcal{O}_{Z_w}(m_1, \ldots, m_n)$ is ample.*

Proof. Inductively we know there exists positive integers $m_1 \ldots, m_{n-1}$ such that $\mathcal{L} = \mathcal{O}_{Z_{w[1]}}(m_1, \ldots, m_{n-1})$ is ample on $Z_{w[1]}$. As $\mathcal{O}(Z_{w(n)})$ has degree one along the fibers of the $\mathbb{P}^1$-bundle $\sigma_w : Z_w \to Z_{w[1]}$ this means that $\pi_w^*(\mathcal{L}^m) \otimes \mathcal{O}(Z_{w(n)})$ is ample for m sufficiently large. □

Theorem 6.2. *Let $\mathcal{L}$ be a globally generated line bundle on a Bott-Samelson variety Z_w. Then*

$$\mathrm{H}^i(X, \mathcal{L}) = 0, \;\; i > 0.$$

Proof. Choose $m_1, \ldots, m_n$ according to Lemma 6.1 such that the line bundle $\mathcal{O}(\sum_{j=1}^{n} m_j Z_{w(j)})$ is ample. By Proposition 4.4 there exists an embedding

$$\mathrm{H}^i(Z_w, \mathcal{L}) \hookrightarrow \mathrm{H}^i(Z_w, \mathcal{L}^{p^m} \otimes \mathcal{O}(\sum_{j=1}^{n} m_j Z_{w(j)})),$$

for some m. Now the result follows from Lemma 4.3 and the fact that a tensorproduct of an ample line bundle with a globally generated line bundle is ample. □

7. Kumar vanishing

In his proof [5] of the Demazure character formula in the Kac-Moody case S. Kumar in a crucial way uses the following cohomological vanishing result.

Theorem 7.1. *Let $w = (s_1, \ldots, s_n)$ be an ordered collection of simple reflections and let Z_w be the associated Bott-Samelson variety over a field of characteristic zero. Assume that the subexpression $(s_t, \cdots, s_r)$ of w is reduced for integers $1 \leq t \leq r \leq n$. Then*

$$\mathrm{H}^i(Z_w, \mathcal{L}_w(\lambda) \otimes \mathcal{O}(-\sum_{j=t}^{r} Z_{w(j)})) = 0, \ i > 0,$$

whenever λ is a dominant weight.

Using Frobenius splitting techniques one may extend this result to arbitrary globally generated line bundles and positive characteristic (notice $\mathcal{L}_w(\lambda)$ is globally generated when λ is a dominant weight).

Lemma 7.2. *The line bundle $\mathcal{L} = \mathcal{L}_w(\rho)^m \otimes \mathcal{O}(Z_{w(n)}) \otimes \pi_w^*(\mathcal{L}_{w[1]}(-\rho))$ is globally generated when m is sufficiently large.*

Proof. Using Lemma 5.1 we may express the relative anti-canonical sheaf of the $\mathbb{P}^1$-bundle $\pi_w : Z_w \to Z_{w[1]}$ as

$$\omega^{-1}_{Z_w/Z_{w[1]}} = \omega^{-1}_{Z_w} \otimes \pi_w^*(\omega_{Z_{w[1]}}) = \mathcal{L}_w(\rho) \otimes \mathcal{O}(Z_{w(n)}) \otimes \pi_w^*(\mathcal{L}_{w[1]}(-\rho))$$

Suppose that s_n is a reflection in the simple root α. Consider then the fiber product diagram

$$\begin{array}{ccc} Z_w & \xrightarrow{\pi} & G/B \\ \downarrow{\scriptstyle \pi_w} & & \downarrow \\ Z_{w[1]} & \longrightarrow & G/P_\alpha \end{array}$$

where the lower horizontal morphism is the map induced by the product map $P_{w[1]} \to G$ and the morphism $G/B \to G/P_\alpha$ is the natural projection map. From this we conclude that $\omega^{-1}_{Z_w/Z_{w[1]}}$ is the pull back through π of the anti-canonical sheaf of the $\mathbb{P}^1$-bundle $G/B \to G/P_\alpha$. The latter is isomorphic to $\mathcal{L}_{G/B}(\alpha)$, and hence $\omega^{-1}_{Z_w/Z_{w[1]}} \simeq \mathcal{L}_w(\alpha)$. In particular, $\mathcal{L} \simeq \mathcal{L}_w((m-1)\rho + \alpha)$. Using that $(m-1)\rho + \alpha$ is dominant for m sufficiently large, the result now follows. ☐

Lemma 7.3. *Let $w = (s_1, \ldots, s_n)$ be an ordered collection of simple reflections and Z_w the associated Bott-Samelson variety. Assume that the subexpression $(s_t, \cdots, s_r)$ of w is reduced for integers $1 \leq t \leq r \leq n$. Then there exists integers $m_1, \ldots, m_n$ and $m \geq 0$ satisfying all of the following properties*

1. $m_j \geq 0$, $j \notin \{t, \ldots, r\}$.
2. $m_j \leq 0$, $j \in \{t, \ldots, r\}$ *and* $m_r = -1$.
3. *The line bundle* $\mathcal{O}(\sum_{j=1}^{n} m_j Z_{w(j)}) \otimes \mathcal{L}_w(\rho)^m$ *is globally generated.*

Proof. Assume first of all that $r < n$. By induction (in n) we find integers $m_1, \ldots, m_{n-1}$ and $m \geq 0$ satisfying the equivalent proporties for $Z_{w[1]}$. In particular, the line bundle

$$\mathcal{O}(\sum_{j=1}^{n-1} m_j Z_{w(j)}) \otimes \pi_w^*(\mathcal{L}_{w[1]}(\rho)^m),$$

is globally generated. Now choose a positive integer m_n such that the line bundle $\mathcal{L} = \mathcal{L}_w(\rho)^{m_n} \otimes \mathcal{O}(Z_{w(n)}) \otimes \pi_w^*(\mathcal{L}_{w[1]}(-\rho))$ is globally generated (Lemma 7.2). Then the line bundle

$$\mathcal{O}(\sum_{j=1}^{n-1} m_j Z_{w(j)}) \otimes \mathcal{O}(m_n Z_{w(n)}) \otimes \mathcal{L}_w(\rho)^{m_n m},$$

is globally generated. We are left with the case $r = n$.

Write $\mathcal{O}_{w[1]}(1)$ in the Z-basis as

$$\mathcal{O}_{w[1]}(1) = \mathcal{O}_{Z_w}(a_1, \ldots, a_{n-1}, 1).$$

By Proposition 3.5 we know that $a_t, \ldots, a_{n-1}$ are positive integers. Now

$$\mathcal{L}_w(\rho) \otimes \mathcal{O}(-Z_{w(n)}) \otimes \mathcal{O}(\sum_{j=t}^{n-1} -a_j Z_{w(j)}) = \mathcal{L} \otimes \mathcal{O}(\sum_{j=1}^{t-1} a_j Z_{w(j)}),$$

where $\mathcal{L} = \mathcal{L}_w(\rho) \otimes \mathcal{O}_{w[1]}(-1)$ is a globally generated line bundle. It remains to find positive integers $m_1, \ldots, m_{t-1}$ such that the line bundle $\mathcal{O}(\sum_{j=1}^{t-1}(m_j + a_j)Z_{w(j)})$ is globally generated. That this is possible follows from Lemma 6.1. $\square$

Theorem 7.4. *Let $w = (s_1, \ldots, s_n)$ be an ordered collection of simple reflections and Z_w the associated Bott-Samelson. Assume that the subexpression $(s_t, \cdots, s_r)$ of w is reduced for integers $1 \leq t \leq r \leq n$. Then*

$$\mathrm{H}^i(Z_w, \mathcal{L} \otimes \mathcal{O}(-\sum_{j=t}^{r} Z_{w(j)})) = 0, \ i > 0,$$

whenever $\mathcal{L}$ is a globally generated line bundle.

Proof. Choose integers $m_1, \ldots, m_n$ and $m \geq 0$ according to the conditions in Lemma 7.3. By successive use of Proposition 5.3 we may assume that $\mathcal{L} \otimes \mathcal{L}_w(-\rho)^m$ is globally generated. By Proposition 4.4 (with values $a_j = |m_j|$, $j \neq r$ and $a_r = 0$) the cohomology group $\mathrm{H}^i(Z_w, \mathcal{L} \otimes \mathcal{O}(-\sum_{j=t}^{r} Z_{w(j)}))$ embeds into (for some m') :

$$\mathrm{H}^i(Z_w, \mathcal{L}^{p^{m'}} \otimes \mathcal{O}(\sum_{j=1}^{n} m_j Z_{w(j)} - \sum_{j=t}^{r-1} Z_{w(j)})),$$

which we may rewrite as

$$\mathrm{H}^i(Z_w, \mathcal{L}' \otimes \mathcal{O}(-\sum_{j=t}^{r-1} Z_{w(j)})),$$

where $\mathcal{L}' = \mathcal{L}^{p^{m'}} \otimes \mathcal{O}(\sum_{j=1}^n m_j Z_{w(j)})$ is globally generated by choice of $m_1, \dots, m_n$. The claim now follows by induction in $r - t$. $\square$

As already noted by S. Kumar it is crucial that the subexpression $(s_t, \cdots, s_r)$ is reduced.

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