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Abstract. Given an automorphism τ of a smooth complex algebraic curve X, there is an induced action on the moduli space M of semi-stable rank 2 holomorphic bundles with fixed determinant. We give a complete description of the fixed variety in terms of moduli spaces of parabolic bundles on the quotient curve $X/\langle\tau\rangle$.

Contents

1. Introduction

Throughout this paper X will be a smooth complex algebraic curve, and τ an automorphism of X, say of order n. Let $Y = X/\langle\tau\rangle$ be the quotient curve, which is of course again a smooth complex algebraic curve. Let $\pi : X \to Y$ be the corresponding possibly ramified covering. We will assume that Y is connected. However, any of our statements may be easily generalised by considering one component of Y at a time.

Let M be the moduli space of semi-stable holomorphic bundles of rank 2 and fixed determinant on X. We get an induced action of $\langle\tau\rangle$ on M, provided that this fixed determinant line bundle is preserved up to isomorphism under pullback by τ. Let us denote the fixed variety of this group action by $|M|$.

In the present paper we give a construction of the fixed variety $|M|$ from a certain finite set of admissible moduli spaces of semi-stable parabolic bundles on Y with the parabolic structures concentrated at the ramification points of $\pi\colon X \to Y$.

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The construction in outline is as follows: First we establish that any fixed point in the moduli space M, can be represented by an equivariant (semi-stable) bundle, i.e. by a bundle with an action of the group $\langle\tau\rangle$ covering the action on X. By fixing the determinant line bundle equivariantly, to the extend this can be done, we reduce the ambiguity in the choice of the corresponding equivariant bundle as much as possible (see Lemma 2.3 and the paragraphs following it).

This prompts us to study equivariant bundles in general up to equivariant isomorphism. Suppose $(E, \tilde{\tau})$ is an equivariant bundle. In the case where $\pi : X \to Y$ is ramified, there are obvious numeric invariants associated to the equivariant bundle $(E, \tilde{\tau})$ obtained in the following manner: At a special orbit y of τ, say of length $k(y)$, $\tilde{\tau}^{k(y)}$ acts on the fiber E_x over each point x in the orbit. Thus, its eigenvalues, which are $\frac{n}{k(y)}$-roots of unity, will be invariants of the equivariant isomorphism class of $(E, \tilde{\tau})$. These form discrete invariants of the equivariant bundle. To determine the equivariant bundle up to equivariant isomorphism, we devise a scheme for performing a number of equivariant elementary modifications at the special orbits, so as to obtain a quasi-parabolic bundle, which is seen to be equivariantly isomorphic to the pullback of a quasi-parabolic bundle $(\bar{E}, \bar{F})$ on Y. We specify weights w for this quasi-parabolic bundle $(\bar{E}, \bar{F})$ on Y as functions of the discrete eigenvalue invariants of $(E, \tilde{\tau})$.

The next step is to specify a set $\tilde{P}_a$ of admissible parabolic bundles (see Definition 2.22 and 2.23), and prove that these are exactly the parabolic bundles obtained as describe above. This is done in Theorem 2.25, where we establish that there is a bijective correspondence between this set of admissible parabolic bundles and the set of equivariant bundles as specified above. We view this bijection as a construction of all such equivariant bundles.

This construction behaves very well with respect to semi-stability. Namely, an admissible parabolic bundle is parabolically semi-stable if and only if the corresponding equivariant bundle is semi-stable as a vector bundle as stated in Proposition 3.1. Furthermore, S-equivalent parabolic bundles are taken to S-equivalent bundles, hence we get a well defined set map $\mathcal{F}$ from the moduli space of admissible semi-stable parabolic bundles P_a on Y to the fixed variety $|M|$. — Since any fixed point can be represented by an equivariant bundle, this map is clearly surjective. The way we have defined the set of admissible parabolic bundles, implies that this map has finite fibers and we give a complete description of all fibers of $\mathcal{F}$ in the main set-theoretic Theorem 3.4. The content of this theorem is in words:

There is a specific equivalence relation on admissible semi-stable, but not stable, parabolic bundles, induced from a finite group action on parabolic line bundles, which describes the fibers of $\mathcal{F}$. If n is odd, $\mathcal{F}$ is a bijection between stable admissible parabolic bundles and stable fixed points and it induces a bijection between equivalence classes of semi-stable non-stable parabolic bundles and semi-stable non-stable fixed points.

If n is even, there is an involution on the moduli space of admissible parabolic bundles under which $\mathcal{F}$ is invariant. If the greatest common divisor r of the orbit lengths for τ is odd, $\mathcal{F}$ gives a bijection between the quotient of the moduli space of stable admissible parabolic bundles and the stable fixed points, and again there is a bijection between equivalence classes of semi-stable non-stable parabolic bundles and semi-stable non-stable fixed points.

If r is even, $\mathcal{F}$ induces a bijection between the quotient of the stable admissible parabolic bundles which are not fixed by the involution and stable fixed points. The stable admissible parabolic bundles, fixed by the involution together with the equivalence classes of the semi-stable but not stable admissible parabolic bundles, goes bijectively under $\mathcal{F}$ into the semi-stable but non-stable points in the fixed variety.

We describe in details how these equivalence relations affect the discrete invariants, i.e. the parabolic weight and how they are induced by taking elementary modifications and tensoring with certain line bundles (see point 1. and 2. following Theorem 3.4).

Using the geometric invariant theory construction of these moduli spaces, we analyse our set-theoretic map $\mathcal{F}$ and show that it is a morphism of varieties in Proposition 4.2. Furthermore, Theorem 4.3 establishes that $\mathcal{F}$ (or in the case of n even, the induced map on the $\mathbb{Z}/2$-quotient mentioned before) is a birational equivalence, which, when restricted to components of the moduli space of admissible parabolic bundles (resp. its quotient), is the normalising map onto the corresponding irreducible components of the fixed variety.

As examples, the un-ramified case and ramified hyper-elliptic case are treated in details.

The motivation for an explicit construction of this fixed variety in terms of parabolic bundles stems from the gauge theoretic approach to $2+1$-dimensional topological quantum field theories, as initially outlined by Witten in [27], Axelrod, Della Pietra and Witten in [4] and Atiyah in [3]. From the algebraic geometric viewpoint, this approach has received considerable attention, e.g. Hitchin [13], Faltings [8] and Thaddeus [24], Beauville & Laszlo [7], Narasimhan and Ramanan [21] to mention a few. The program of proving all the axioms of a full TQFT, has however not been complete from this purely gauge-theoretic point of view. On the conformal field theory side, further progress was made in the direction of providing a full construction of a modular functor, which in turn will provide the basis for another combinatorial construction of these TQFTs, namely the construction of TQFT from Modular functors, see [26] and [9]. Here the works of Segal [23] and Tsuchiya, Ueno and Yamada [25] stand out. Work is in progress to provide a full construction of a modular functor based solely on the techniques of [25] (see [2]). Once this is complete it can be combined with the results of Laszlo [17], to establish that the association of the vector space $H^0(M,\Theta)$, where $\Theta \in \mathrm{Pic}(M)$ is ample, to the curve X, can be extended to a full modular functor.

From this point of view it is therefore an interesting problem to use the geometry of the moduli space M to study the character of the resulting representation of the automorphism group of X. As explained in [1], this character can be expressed as a certain cohomological/homological pairing on the fixed variety $|M|$, by an application of the Lefschetz-Riemann Rock formula due to Baum, Fulton, Quart and MacPherson, [5], [6] and [22]. This gave a proof of the Asymptotic Expansion Conjecture for mapping tori of automorphisms of curves. — To further study this character, a complete description of the fixed variety is needed. Such a description is given in this paper as explained above in terms of the moduli space of admissible parabolic bundles P_a. It is therefore an interesting problem to pull back this pairing to the moduli space of admissible parabolic bundles bundles P_a on Y and seek an evaluation there of the pairing, e.g. by means of the results of Jeffrey, Kiem, Kirwan and Woolf, [14], on the intersection cohomology of these moduli spaces.

We remark, that although the techniques presented here all generalise to higher rank, only rank 2 is treated. This choice was made in order to eliminate the more involved bookkeeping required to treat the general rank case. Furthermore, we point out that it would be possible to achieve the *set theoretic* identification of the fixed point set by studying flat connections. However, that approach does not appear to be any easier than the algebraic technique given here, and in no way can it provide the algebraic results of Proposition 4.2 and Theorem 4.3.

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2. Equivariant Bundles

Recall that X is a smooth complex algebraic curve, $\tau\colon X \to X$ an automorphism of order n, which gives a possibly ramified covering $\pi\colon X \to Y = X/\langle\tau\rangle$ of curves.

In the sequel we employ the convention that the set of special orbits for τ is denoted $P \subset Y$. For every orbit $y \in P$ we denote its length by $k = k(y)$, so the ramification number is $n' = n'(y) = \frac{n}{k(y)}$.

Let $\xi = \xi_y$ be the n'-th root of unity given by $\xi_y = d_x\tau^k : T_xX \to T_xX$ for any $x \in \pi^{-1}(y)$. It is not hard to see that there exists a neighbourhood $U = U_x$ of x in which there are local coordinates z centered around x such that τ^k can be expressed as $z \circ \tau^{-k} = \xi_y^{-1} \cdot z$.

Before we go on, observe the following important remark:

Remark 2.1. For any divisor d of n, the length of the orbit of τ^d through a point $x \in X$ is $k/\gcd(d,k)$. This is because the length of the orbit is the smallest $l > 0$ such that k divides dl, which is the factor of k that is not in d: $\operatorname{lcm}(d,k)/d = k/\gcd(d,k)$. This means that the length of the fiber $\pi_d^{-1}(\pi(x))$ of the projection $\pi_d \colon X/\langle\tau^d\rangle \to Y$ is $\gcd(d,k)$.

The projection $\pi_d \colon X/\langle\tau^d\rangle \to Y$ is unramified precisely when the lengths of its fibers are constantly equal to the generic length d. That in turn means that $\gcd(d,k) = d$ for all x, in particular, that d divides all k, i.e. that d divides the greatest common divisor $r = \gcd\{\, k(y) \mid y \in Y \,\}$ of the orbit lengths.

This shows that any ramified cover coming from the action of a single automorphism τ of X factors into a ramified part $\pi_r \colon X \to \hat{X} = X/\langle\tau^r\rangle$ for which the fiber lengths are co-prime, and an unramified part $\pi_u \colon \hat{X} \to Y$ whose degree is the greatest common divisor r of the orbits lengths of τ.

Let now M be the moduli space of S-equivalence classes of semi-stable bundles on X of rank 2 and determinant isomorphic to some fixed line bundle. Provided that this line bundle is invariant under pullback by τ, we see that pullback by τ induces an action of $\langle\tau\rangle$ on the set of S-equivalence classes of such bundles. It is easily seen that this action is indeed well defined, for if W_i, $i = 1,2$, are two semi-stable holomorphic bundles, such that $\operatorname{Gr}(W_1) \cong \operatorname{Gr}(W_2)$ then

$$\operatorname{Gr}(\tau^* W_1) \cong \tau^* \operatorname{Gr}(W_1) \cong \tau^* \operatorname{Gr}(W_2) \cong \operatorname{Gr}(\tau^* W_2).$$

As pullback by morphisms between smooth complex algebraic curves gives morphisms between the moduli spaces, we actually get an induced algebraic action of $\langle\tau\rangle$ on M (see section 4) and we denote the fixed variety by $|M|$. Note that a bundle W represents a point in $|M|$ if and only if $\operatorname{Gr}(\tau^*(W)) \cong \operatorname{Gr}(W)$. However, this same point in $|M|$ is also represented by $\operatorname{Gr}(W)$ which satisfies that $\tau^*(\operatorname{Gr}(W)) \cong \operatorname{Gr}(W)$. Hence, any fixed point in M can be represented by a semi-stable bundle which is preserved under pullback by τ.

Consider now a holomorphic vector bundle V over X of any rank, and suppose that τ^*V is isomorphic to V. Then there exists a bundle map $\tilde{\tau} : V \to V$ covering τ. Two such pairs $(V_\nu, \tilde{\tau}_\nu)$ are said to be isomorphic, if there is an isomorphism of bundles $\Phi : V_1 \to V_2$ intertwining $\tilde{\tau}_1$ and $\tilde{\tau}_2$ and we write $(V_1, \tilde{\tau}_1) \cong (V_2, \tilde{\tau}_2)$ or just $V_1 \cong_e V_2$.

Definition 2.2. *A pair $(V, \tilde{\tau})$ consisting of a holomorphic vector bundle V and a bundle map $\tilde{\tau}$ covering τ is called a lift, if*

$$\tilde{\tau}^n = \operatorname{Id}_V .$$

We shall also refer to such a pair $(V, \tilde{\tau})$ as an *equivariant holomorphic bundle*. We note that the group of n-th roots of unity ζ_n acts freely on the set of isomorphism classes of equivariant holomorphic bundles.

If we are given a divisor on X which is τ-invariant, then there is a naturally induced lift to the bundle associated to the divisor, just by inducing the action from the $\langle\tau\rangle$-action on the meromorphic functions $\mathcal{M}(X)$ on X. Here we use the convention for $\tau^* : \mathcal{M}(X) \to \mathcal{M}(X)$ that $\tau^* f = f \circ \tau^{-1}$.

If V is a simple bundle which is preserved by τ, i.e. $\tau^*V \cong V$, then we can always find a lift of τ to V and ζ_n acts simply transitively on the set of such lifts. In particular, this is the case for all preserved stable bundles and in particular for all preserved line bundles.[1]

This may not always be the case for preserved semi-stable vector bundles. However, if we only consider them modulo S-equivalence, then, as we shall see now, this can always be achieved: Assume that a semi-stable bundle W of rank 2 on X represents a fixed point in $|M|$, i.e. it satisfies that its S-equivalence class is preserved by τ

$$\mathrm{Gr}(\tau^* W) \cong \mathrm{Gr}(W).$$

It is in this case easily shown that there is a lift $\tilde{\tau}$ of τ to $\mathrm{Gr}(W)$:

The graded object of W is $\mathrm{Gr}(W) = L_1 \oplus L_2$, where L_ν is a line bundle of degree $\frac{1}{2}\deg E$ for $\nu = 1, 2$. So $\tau^*(L_1 \oplus L_2) \cong L_1 \oplus L_2$ and since any non-zero homomorphism between line bundles of the same degree is an isomorphism, we either get that $\tau^* L_\nu \cong L_\nu$ (which we refer to as the *invariant* case) or $\tau^* L_1 \cong L_2$ and vice versa, but $L_1 \ncong L_2$ (which we refer to as the *degenerate* case). In the latter case we observe that $(\tau^*)^2 L_\nu \cong L_\nu$, which in particular means that n must be even, for this case to occur.

In the invariant case, it follows from the above argument, that we may choose $(L_\nu, \tilde{\tau}_\nu)$ so that $\tilde{\tau}_\nu^n = \mathrm{Id}_{L_\nu}$. For the degenerate case suppose $\tilde{\tau} : L_1 \oplus L_2 \to L_1 \oplus L_2$ is a bundle isomorphism covering τ. Then, since $L_1 \ncong L_2$, $\tilde{\tau}$ must be off-diagonal. However $\tilde{\tau}^n$ has to be diagonal with respect to the splitting, since $H^0\big(\mathrm{End}(L_1 \oplus L_2)\big) = \mathbb{C} \oplus \mathbb{C}$, and a moments thought shows that one can find a diagonal matrix Λ so that $(\Lambda\tilde{\tau})^n = \mathrm{Id}_{L_1 \oplus L_2}$.

Hence, we have now seen that any fixed point in M can be represented by an equivariant bundle $(E, \tilde{\tau})$.

Now observe that if we change a lift $\tilde{\tau}$ by a $\mu \in \zeta_n$, then the induced lift to the determinant $\det(E)$ is changed by μ^2. We can therefore always change the lift $\tilde{\tau}$ so that the induced lift on the determinant is a given one in the case n is odd and one of two possibilities in case n is even. Fix therefore a set $\mathcal{D}$ of such equivariant line bundles whose underlying bundles is the determinant fixed above. I.e. in the case n is odd, $\mathcal{D}$ contains exactly one lift to the determinant line bundle and in case n is even exactly two lifts to the determinant line bundle, which are not equivalent under the action of $\{\, \mu^2 \mid \mu \in \zeta_n \,\}$. We have thus arrived at

Lemma 2.3. *Every fixed point in M can be represented by a semi-stable equivariant bundle $(E, \tilde{\tau})$ with* $\det(E, \tilde{\tau}) \in \mathcal{D}$. □

We observe, in the case of stable bundles, that when n is odd, we have by these means a unique way to represent such fixed points by equivariant bundles. When n is even there are exactly two such equivariant bundles, say $(E, \tilde{\tau})$ and then $(E, -\tilde{\tau})$. For semi-stable, but not stable bundles, the situation is of course more involved. — We shall see the significance of fixing the determinant equivariantly in the following section.

Let $\mathcal{L}_\mathcal{D}$ be the set of isomorphism classes of equivariant bundles $(E, \tilde{\tau})$ with $\det(E, \tilde{\tau}) \in \mathcal{D}$, $\mathcal{L}'_\mathcal{D}$ the subset consisting of isomorphism classes of equivariant bundles, where the underlying bundle is semi-stable and let $\Pi_e : \mathcal{L}'_\mathcal{D} \to |M|$ be the projection map, which forgets the lift and takes the S-equivalence class of the underlying bundle. The content of Lemma 2.3 is exactly that Π_e is surjective.

Let now $(E, \tilde{\tau})$ be an equivariant bundle. For each $y \in P$ we now define two integers $0 \leq d_1 = d_1(y) \leq d_2 = d_2(y) < n'$ by requiring that

$$\theta_\nu = \xi^{d_\nu}, \qquad \nu = 1, 2,$$

[1] We shall actually see shortly that any preserved line bundle can be represented by a τ-invariant divisor.

where $\theta_\nu = \theta_\nu(y)$, $\nu = 1, 2$, are the eigenvalues of $\tilde{\tau}^k$ acting on fibers of E over $\pi^{-1}(y)$. We note that the ordered pair (d_1, d_2) is an invariant of the isomorphism class of the equivariant bundle $(E, \tilde{\tau})$. We call this pair the *numeric data* or *numeric invariant* of $(E, \tilde{\tau})$, and it is a map

$$(d_1, d_2) : \mathcal{L}_\mathcal{D} \longrightarrow \prod_{y \in P} T_{n'(y)},$$

where

$$T_n = \{(d_1, d_2) \in \mathbb{Z} \mid 0 \leq d_1 \leq d_2 < n\}.$$

The numeric data is a discrete invariant of an equivariant bundle, but we need of course much more than this discrete invariant to determine the equivariant bundle. We proceed as follows in our further analysis of $(E, \tilde{\tau})$.

Let $\mathcal{E}$ be the sheaf of sections in E. Let $\mu \in \zeta_n$ and define the eigensubsheaf $\mathcal{E}_\mu$ of $\pi_*\mathcal{E}$ corresponding to the eigenvalue μ by

$$\mathcal{E}_\mu(U) = \left\{\, s \in \mathcal{E}\big(\pi^{-1}(U)\big) \,\middle|\, \tilde{\tau}s = \mu s \,\right\}$$

for any open subset U of Y, where $(\tilde{\tau}s)(x) = \tilde{\tau}\big(s(\tau^{-1}(x))\big)$, $x \in X$.

Let us now consider the sheaf morphisms $\pi_\mu \colon \pi_*\mathcal{E} \to \pi_*\mathcal{E}$ by letting

$$\pi_\mu = \frac{1}{n}\sum_{l=0}^{n-1} \mu^{-l}\tilde{\tau}^l. \tag{2.4}$$

It is easy to see that π_μ is a projection:

$$\pi_\mu \circ \pi_\mu = \frac{1}{n^2}\sum_{j,l=0}^{n-1} \mu^{-j-l}\tilde{\tau}^{j+l} = \frac{1}{n}\sum_{l=0}^{n-1} \mu^{-l}\tilde{\tau}^l = \pi_\mu,$$

with $\operatorname{Im}\pi_\mu \subset \mathcal{E}_\mu$:

$$\tilde{\tau} \circ \pi_\mu = \frac{1}{n}\sum_{l=0}^{n-1} \mu^{-l}\tilde{\tau}^{l+1} = \mu \cdot \frac{1}{n}\sum_{l=0}^{n-1} \mu^{-l+1}\tilde{\tau}^{l+1} = \mu \cdot \pi_\mu.$$

We calculate that

$$\sum_{\mu \in \zeta_n} \pi_\mu = \frac{1}{n}\sum_{\mu \in \zeta_n}\sum_{l=0}^{n-1} \mu^{-l}\tilde{\tau}^l = \frac{1}{n}\sum_{l=0}^{n-1}\Big(\sum_{\mu \in \zeta_n} \mu^{-l}\Big)\tilde{\tau}^l = \mathrm{Id}_{\pi_*\mathcal{E}}\,.$$

It is clear from these properties of π_μ that

$$\pi_*\mathcal{E} = \bigoplus_{\mu \in \zeta_n} \mathcal{E}_\mu. \tag{2.5}$$

as sheaves of $\mathcal{O}_Y$-modules.

Let us now give an elementary argument for the fact that $\pi_*\mathcal{E}$ is locally free:

It is clear that $\mathcal{E}\big(\pi^{-1}(U)\big)$ for small enough U is a locally free $\mathcal{O}_X\big(\pi^{-1}(U)\big)$-module. Hence, to establish that $\mathcal{E}\big(\pi^{-1}(U)\big)$ is a free $\mathcal{O}_Y(U)$-module for small enough U, we just need to prove that $\mathcal{O}_X\big(\pi^{-1}(U)\big)$ is a locally free $\mathcal{O}_Y(U)$-module for small enough U. Away from the ramification points of π, this is completely trivial. At a point $x \in \pi^{-1}(y)$ of a special orbit $y \in P$, we consider a centered holomorphic coordinate say z such that $\tau^k z = \xi \cdot z$. It is easily seen that $\pi_\mu(z^j)$, $j = 0, \dots, n'-1$, and $\mu \in \zeta_n$ such that $\xi^j = \mu^k$ provides a basis for $\mathcal{O}_X(\pi^{-1}(U))$ as an $\mathcal{O}_Y(U)$-module for small enough U around y. Here π_μ is defined just like in (2.4), except we use τ in place of $\tilde{\tau}$.

Using the fact that $\pi_*\mathcal{E}$ is locally free, equation (2.5) and the following exact sequence

$$\pi_*\mathcal{E} \xrightarrow{\oplus_{\mu'\neq\mu}\pi_{\mu'}} \pi_*\mathcal{E} \xrightarrow{\pi_\mu} \mathcal{E}_\mu \longrightarrow 0 \ ,$$

we see that $\mathcal{E}_\mu$ is coherent and torsion free. Then $\mathcal{E}_\mu$ is locally free, since $\dim_{\mathbb{C}}(Y) = 1$ (see Corollary (5.15) in [15]). Let E_μ be the underlying holomorphic bundle of $\mathcal{E}_\mu$. Then of course

$$\pi_* E \cong \bigoplus_{\mu\in\zeta_n} E_\mu.$$

It is clear that E_μ has rank 2 for all $\mu \in \zeta_n$.

Later we shall need the following technical lemma, which can be derived directly from the above.

Lemma 2.6. *Near each special point $x \in X$ we can find a local frame (s_1, s_2) for E such that*

$$\tilde{\tau}^k(s_i) = \theta_i \cdot s_i,$$

where $\theta_i \in \zeta_{n'}$ are the eigenvalues of $\tilde{\tau}^k$ acting on E_x.

Proof. We observe, that if μ^k is different from θ_1 and θ_2 then any s in the stalk $(\mathcal{E}_\mu)_x$ must vanish at x. Now choose (s^1_μ, s^2_μ) local frame for $\mathcal{E}_\mu$ around x for each $\mu \in \zeta_n$. Then $\{s^i_\mu\}$, $\mu \in \zeta_n$ and $i = 1, 2$ is a local frame for π_*E. If $\theta_1 = \theta_2$ then this implies that $\big(s^1_\mu(x), s^2_\mu(x)\big)$, where $\mu^k = \theta_1$, is a basis of E_x, hence $(s^1_{\theta_1}, s^2_{\theta_1})$ is the local frame we want in this case. If $\theta_1 \neq \theta_2$ then there will be an i_1 and an i_2 such that $\big(s^{i_1}_{\theta_1}(x), s^{i_2}_{\theta_2}(x)\big)$ is a basis of E_x, hence $(s^{i_1}_{\theta_1}, s^{i_2}_{\theta_2})$ is the local frame we want in this case. □

Our description of equivariant bundles and in turn the fixed point set $|M|$ is based on associating to an equivariant bundle $(E, \tilde{\tau})$ the bundle E_1 over Y together with the numeric data (d_1, d_2), but endowed also with some extra structure, namely a parabolic structure. The reason for this is, that the bundle E_1 itself and numeric data is not enough to determine the fixed point in general. This parabolic structure is of course closely related to the eigenspace decomposition of the fiber of E over $\pi^{-1}(P)$. To see how this goes, it is convenient to describe how the bundle E_1 can be constructed using elementary modifications, so let us here briefly recall the basics of elementary modifications in this equivariant setting. — In fact we shall give a complete analysis of equivariant bundles, just using these (inverse) elementary modifications and only after that, shall we as an aside return to the bundle E_1.

Let $y \in P$ be a special orbit, $(E, \tilde{\tau})$ an equivariant bundle of rank 2 and $F_y = \oplus_{x\in\pi^{-1}(y)} F_x$, $F_x \subset E_x$, be a $\tilde{\tau}$ invariant set of one-dimensional flags over the orbit y. We then see that F_x is an eigensubspace for $\tilde{\tau}^k_x$. Let θ_1 be the eigenvalue of $\tilde{\tau}^k_x$ corresponding to the eigensubspace F_x and let θ_2 be the other eigenvalue.

Let $S_x = E_x/F_x$ and let S_y be the skyscraper sheaf on X with support at $\pi^{-1}(y)$ and fiber S_x at $x \in \pi^{-1}(y)$.

Lemma 2.7. *There is a unique lift $(E', \tilde{\tau}')$ fitting into the following short exact equivariant sequence*

$$0 \longrightarrow E' \xrightarrow{\ \iota\ } E \xrightarrow{\ \lambda\ } S_y \longrightarrow 0 \ ,$$

such that $F_y = \oplus_{x\in\pi^{-1}(y)} \ker(\lambda_x) = \oplus_{x\in\pi^{-1}(y)} \mathrm{Im}(\iota_x)$. *There is a set of flags* $\oplus_{x\in\pi^{-1}(y)} \ker(\iota_x)$ *in* E' *which consist of eigensubspaces for* $(\tilde{\tau}')^k_x$ *whose corresponding eigenvalue is* $\xi^{-1}\theta_2$. *The other eigenvalue of* $(\tilde{\tau}')^k_x$ *is* θ_1. *The determinants are related by* $\det E' \cong_e \det E \otimes [-\pi^{-1}(y)]$.

In this case we say that the equivariant bundle $(E', \tilde{\tau}')$ is obtained from the equivariant bundle $(E, \tilde{\tau})$ by *elementary modifications* in the direction F_y.

Proof. Consider the sheaf kernel

$$\mathcal{E}' = \ker\{\lambda : \mathcal{E} \longrightarrow S_y\}.$$

We observe, that $\mathcal{E}'$ is $\tilde{\tau}$-invariant. By using a local frame adapted to the flag F_x in E_x one easily sees that $\mathcal{E}'$ is locally free. The associated holomorphic vector bundle E' is uniquely determined, since $\mathcal{E}'$ as a subsheaf of $\mathcal{E}$ is uniquely determined by F_y. Furthermore, since $\mathcal{E}'$ is $\tilde{\tau}$-invariant, $\tilde{\tau}$ induces a lift $\tilde{\tau}'$ of τ to E'.

The determinant is clearly $\det E' \cong \det E \otimes (\det S_y)^{-1}$, and $\det S_y$ can be calculated through the defining sequence

$$0 \longrightarrow \mathcal{O}[-\pi^{-1}(y)] \xrightarrow{\ i\ } \mathcal{O}_X \xrightarrow{\ p\ } S_y \longrightarrow 0. \tag{2.8}$$

Again, using a frame of E near x, which at x gives an eigenbasis for $\tilde{\tau}^k_x$ acting on E_x, one sees immediately, that $\ker(\iota_x) \subset E'_x$ is an eigenspace of $(\tilde{\tau}')^k_x$ corresponding to the eigensubspace $\xi^{-1}\theta_1$ and the other eigenvalue of $(\tilde{\tau}')^k_x$ is θ_2. Using this it is easy to verify that the stated isomorphism relation for the determinants also holds equivariantly. □

We observe, that the eigenvalue corresponding to F_y is left unchanged, whereas the other eigenvalue is changed.

Remark 2.9. We notice that if L is an equivariant line sub-bundle of E then

$$L' = L \otimes \bigotimes_{\{x \mid L_x \neq F_x\}} [-x]$$

is an equivariant line sub-bundle of E'. Conversely, if L' is a line sub bundle of E' then

$$L = L' \otimes \bigotimes_{\{x \mid L'_x = \ker(\iota_x)\}} [x]$$

is a line sub-bundle of E also equivariantly. This sets up a one to one correspondence between line sub-bundles of E and E'. By considering the following commutative diagram of short exact sequences for such a pair of corresponding equivariant line sub-bundles (L, L')

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L' & \longrightarrow & L & \longrightarrow & S' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E' & \xrightarrow{\ \iota\ } & E & \xrightarrow{\ \lambda\ } & S_y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q' & \longrightarrow & Q & \longrightarrow & S'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $S' = \bigoplus_{\{x|L_x \neq F_x\}} S_x$ and $S'' = \bigoplus_{\{x|L_x = F_x\}} S_x$, we see there is a similar equivariant correspondence between quotients and

$$Q' = Q \otimes \bigotimes_{\{x|L_x=F_x\}} [-x].$$

Going in the opposite direction we let $S'_x = T_x X \otimes F_x$ and S'_y be the skyscraper sheaf on X with support at $\pi^{-1}(y)$ and fiber S'_x at $x \in \pi^{-1}(y)$.

Lemma 2.10. *There is a unique lift $(E', \tilde{\tau}')$ fitting into the following short exact equivariant sequence*

$$0 \longrightarrow E \xrightarrow{\ \iota\ } E' \xrightarrow{\ \lambda\ } S'_y \longrightarrow 0\ ,$$

such that $F_y = \oplus_{x\in\pi^{-1}(y)} \ker(\iota_x)$. There is a set of flags $\oplus_{x\in\pi^{-1}(y)} \mathrm{Im}(\iota_x)$ in E' consisting of eigenspaces of $(\tilde{\tau}')^k_x$ whose corresponding eigenvalue is θ_2. The other eigenvalue of $(\tilde{\tau}')^k_x$ is $\xi\theta_1$.

In this case we say that the equivariant bundle $(E', \tilde{\tau}')$ is obtained from the equivariant bundle $(E, \tilde{\tau})$ by *inverse elementary modifications* in the direction F_y.

Proof. Let $\tilde{\mathcal{E}}$ be the sheaf of meromorphic sections of E which are holomorphic everywhere except at $\pi^{-1}(y)$, where the sections have a pole of order at most 1. Consider now the residue morphism

$$\mathrm{Res} : \tilde{\mathcal{E}} \longrightarrow T_y X \otimes E_y,$$

and compose it with the natural quotient map to $T_y X \otimes S_y$ to obtain the composite morphism Res_{F_y}. Now simply consider the sheaf kernel

$$\mathcal{E}' = \ker\{\mathrm{Res}_{F_y} : \tilde{\mathcal{E}} \longrightarrow T_y X \otimes S_y\}.$$

By the very construction of $\mathcal{E}'$, we see that it is $\tilde{\tau}$-invariant. Again by considering a local frame adapted to F_x it is easy to see that $\mathcal{E}'$ is locally free. It clearly fits into the above exact sequence and is uniquely determined by F_y as a subsheaf of $\tilde{\mathcal{E}}$. By the very construction we have that $\ker(\iota_x) = F_x$ and by further assuming that the local frame is an eigenbasis over x one gets the statement about the eigenvalues. □

The statement we made about the determinants, line sub-bundles and quotients in the case of elementary modification of course also applies suitably adapted to inverse elementary modifications.

Let us now specify the iterations of (inverse) elementary modifications, we shall use. Let $(E, \tilde{\tau})$ an equivariant rank 2 bundle. We are interested in devising iterations of elementary modification which at each of the special orbits $y \in P$ changes one of the eigenvalues by ξ^{-1} to the power, say $m(y) \in \mathbb{Z}$ while keeping the other eigenvalue fixed. So, given the *multiplicities* $m(y) \in \mathbb{Z}$ we proceed as follows (under the convention, that if $m(y)$ is positive we will iterate elementary modification at the orbit y and if $m(y)$ is negative we will iterate inverse elementary modification at y):

Suppose we are given a set of $\tilde{\tau}$-equivariant flags F of $E|_{\pi^{-1}(P)}$. These flags and the signs of the multiplicities m then *determines* eigenvalues $\theta(y)$ at each orbit y of non-zero multiplicity by the following rule:

At orbits y, where $\tilde{\tau}_x^{k(y)}$ has two distinct eigenvalues and $m(y)$ is positive, F_y corresponds to the eigenvalue $\theta(y)$ and where $m(y)$ is negative, F_y is complementary to the set of eigenspaces corresponding to the eigenvalue $\theta(y)$.

We note that at the points $y \in P$, where the eigenvalues are distinct, $\theta(y)$ and $\operatorname{sign}(m(y))$ uniquely determines $F_y = \oplus_{x\in\pi^{-1}(y)} F_x$. Applying either elementary or inverse elementary modification, according to the sign of $m(y)$, once to $(E,\tilde{\tau})$ at F_y for a $y \in P$, results in a new equivariant bundle $(E',\tilde{\tau}')$. We can define a set of flags F' by the following assignments:

(1) If $y' \in P - \{y\}$ then let $F'_{y'} = F_{y'}$.
(2) If $(\tilde{\tau}')_x^{k(y)}$ has two distinct eigenvalues for $x \in \pi^{-1}(y)$, then
 (a) if $m(y)$ is positive then let F'_y be the set of flags given by the eigenspaces of $(\tilde{\tau}')_x^{k(y)}$, $x \in \pi^{-1}(y)$, corresponding to the eigenvalue $\theta(y)$.
 (b) if $m(y)$ is negative then let F'_y be the set of flags given by the eigenspaces of $(\tilde{\tau}')_x^{k(y)}$, $x \in \pi^{-1}(y)$, corresponding to the eigenvalue different from $\theta(y)$.
(3) If $(\tilde{\tau}')_x^{k(y)}$ has only one eigenvalue for $x \in \pi^{-1}(y)$, then let F'_y be the set of flags specified by the (inverse) elementary modification construction (see Lemma 2.7 and 2.10).

By the construction of F', we see that it determines the same eigenvalues $\theta(y)$, $y \in P$ as F did. — Note that under (2), F'_y is set to be complementary to the set of flags specified by the (inverse) elementary modification construction.

It is the operation of taking $(E,\tilde{\tau},F)$ to $(E',\tilde{\tau}',F')$ through the (inverse) elementary modification construction specified as above, that we shall iterate $|m(y)|$-times at each $y \in P$.

Definition 2.11. Let $(E,\tilde{\tau})$ be an equivariant rank 2 bundle. Let $m : P \to \mathbb{Z}$ be a multiplicity and F be a set of $\tilde{\tau}$-invariant flags for E over $y \in P$ such that $m(y) \neq 0$. We then define $\Gamma_{(m,F)}(E,\tilde{\tau})$ to be the equivariant rank 2 holomorphic bundle obtained by iterating elementary modification at $\{\, y \in P \mid m(y) \geq 0 \,\}$ $m(y)$-times and inverse elementary modification at $\{\, y \in P \mid m(y) < 0 \,\}$ $|m(y)|$-times as described above.

Note that there is a naturally induced set of flags in $\Gamma_{(m,F)}(E,\tilde{\tau})$ by its very construction.

Remark 2.12. We observe, that the equivariant bundle $\Gamma_{(m,F)}(E,\tilde{\tau})$ together with the induced set of flags, say F', uniquely determines $(E,\tilde{\tau})$, since

$$(E,\tilde{\tau}) = \Gamma_{(-m,F')}\big(\Gamma_{(m,F)}(E,\tilde{\tau})\big).$$

Remark 2.13. Suppose $(E,\tilde{\tau})$ is an equivariant bundle with eigenvalues $\theta_1 = \theta_1(y)$ and $\theta_2 = \theta_2(y)$ at the special orbits $y \in P$. Let m be a multiplicity and let F be a set of flags which determines θ_1 as described above. Then the eigenvalues of the equivariant bundle $(E',\tilde{\tau}') = \Gamma_{(m,F)}(E,\tilde{\tau})$ are $\theta_1(y)$ and $\xi^{-m(y)}\theta_2(y)$.

This iteration will shortly be applied to equivariant bundles with eigenvalues different from 1, in order to associate other equivariant bundles, which have all eigenvalues equal to 1. The significance of this is clear from the following lemma.

Lemma 2.14. *An equivariant bundle $(E,\tilde{\tau})$ is equivariantly isomorphic to a pull back bundle from Y if and only if its numeric data vanishes.*

Proof. It is clear that a pullback has this property. Conversely, suppose $(E,\tilde{\tau})$ has this property. We then claim that the natural bundle map from $\pi^*(E_1)$ to E is an equivariant isomorphims. This is easily seen using the eigenframe for E provided by Lemma 2.6. □

We will further need the following lemma concerning equivariant line bundles.

Lemma 2.15. *For any equivariant line bundle L, there exists a τ-invariant divisor D such that $[D]$ is equivariantly isomorphic to L. The difference between any two such divisors is the pullback of a principal divisor on Y.*

Proof. Let $\tilde{\tau}$ be a lift of τ to L and $\theta(y)$ be the eigenvalues of $\tilde{\tau}^{k(y)}$ acting on L_x, $x \in \pi^{-1}(y)$, $y \in P$. Define $0 \leq d(y) < n'(y)$, $y \in P$ by

$$\theta(y) = \xi(y)^{d(y)}$$

and set $D = \sum_{y \in P} d(y) \cdot \pi^{-1}(y)$. The induced lift $\tilde{\tau}'$ on the bundle $L' = L \otimes [-D]$ has all eigenvalues equal to 1. By arguing just as in the proof of the Lemma 2.14, it is easily seen that there exists a line bundle $\bar{L}'$ on Y, such that $L' \cong_e \pi^*(\bar{L}')$. Let $\bar{D}'$ be a divisor on Y representing $\bar{L}'$. Then $D_L = \pi^*(\bar{D}') + D$ is clearly a τ-invariant divisor and $L \cong_e [D_L]$.

Now note that for any τ-invariant divisor D'_L such that $L \cong_e [D'_L]$, we must have

$$D'_L(y) = d(y) \mod n'(y), \quad \text{for all } y \in P,$$

since the eigenvalues must agree. But then $D'_L - D_L$ is clearly a pullback of a divisor from Y. If $f \in \mathcal{M}(X)$ such that $[D'_L - D_L] \cong_e (f)$, then f has to be τ-invariant, since $[D'_L - D_L] \cong_e \mathcal{O}_X$. Hence, there exists $g \in \mathcal{M}(Y)$ such that $(\pi^* g) = (f) = [D'_L - D_L]$. □

Corollary 2.16. *For a τ-invariant divisor D, the numeric data $D(y) \mod n'(y)$, $y \in P$, is an invariant of the isomorphism class of the equivariant bundle $[D]$.* □

If L is an equivariant line bundle, we will write $0 \leq L(y) < n'(y)$ for the numeric data $D(y) \mod n'(y)$, $y \in P$, where D is any τ-invariant divisor representing L equivariantly. Clearly, this numeric data of a line bundle is related to the eigenvalues of the lift just like for rank 2 bundles, but for line bundles there is of course only one eigenvalue per point in P.

Corollary 2.17. *An equivariant line bundle L is equivariantly isomorphic to a pull back bundle from Y if and only if its numeric data $L(y)$ vanishes at every $y \in P$.* □

Let us now return to the rank 2 situation, so let $(E, \tilde{\tau})$ be an equivariant rank 2 bundle with numeric data (d_1, d_2) and corresponding eigenvalues (θ_1, θ_2). Let $m(y) = d_2(y) - d_1(y)$ and define a τ-invariant divisor D_2 on X by $D_2 = \sum_{y \in P} d_2(y) \cdot \pi^{-1}(y)$. Consider the holomorphic bundle $E' = E \otimes [-D_2]$ with the induced lift $\tilde{\tau}'$. The eigenvalues of $(\tilde{\tau}')^k$ are 1 and $\theta(y) = \theta_1(y)\theta_2(y)^{-1}$.

For $x \in \pi^{-1}(P)$, such that $m\big(\pi(x)\big) \neq 0$, we let $F'_x \subset E'_x$ be the eigenspace of $(\tilde{\tau}')^k_x$ corresponding to the eigenvalue $\theta(y)$ and let $F'_y = \oplus_{x \in \pi^{-1}(y)} F'_x$.

Observe, that the equivariant bundle $\Gamma_{(-m,F')}(E', \tilde{\tau}')$ induces the identity on fibers over every $x \in \pi^{-1}(P)$, hence there is a unique quasi-parabolic bundle $(\bar{E}, \bar{F})$ on Y such that $\big(\pi^*\bar{E}, \pi^*\bar{F}\big)$ with the naturally induced pullback lift is equivariantly isomorphic to $\Gamma_{(-m,F')}(E', \tilde{\tau}')$, and such that the flags $\pi^*\bar{F}$ get identified with the induced flags of $\Gamma_{(-m,F')}(E', \tilde{\tau}')$.

We give $\big(\bar{E}, \bar{F}\big)$ the structure of a parabolic bundle by letting the weights $w(y)$ associated to the parabolic points $y \in P$, $m(y) \neq 0$, be given by

$$w(y) = \frac{m(y)}{n'(y)}. \tag{2.18}$$

Denote the resulting parabolic bundle by $(\bar{E}, \bar{F}, w)$. In the notation of [23], $w(y) = a_{y2} - a_{y1}$.

We observe, that the determinants of E and of $\bar{E}$ are equivariantly related by

$$\pi^* \det(\bar{E}) \cong \det(E) \otimes [D - 2D_2],$$

where $D = \sum_{y\in P} m(y) \cdot \pi^{-1}(y)$.

Let $\Delta \in \mathcal{D}$ be the equivariant determinant of $(E, \tilde{\tau})$. Then from Corollary 2.17 we get the following conditions on the numeric invariants of $(E, \tilde{\tau})$:

$$\Delta(y) = d_1(y) + d_2(y) \mod n'(y), \tag{2.19}$$

for all $y \in P$. Define a line bundle $\bar{\Delta}$ on Y by requiring

$$\pi^* \bar{\Delta} \cong_e \Delta \otimes \big[- \sum_{y\in P} \big(d_1(y) + d_2(y)\big) \cdot \pi^{-1}(y)\big]. \tag{2.20}$$

Clearly, there is a unique such line bundle on Y up to isomorphism and

$$\det(\bar{E}) \cong \bar{\Delta}.$$

Lemma 2.21. *Given $\mathcal{D}$, the isomorphism class of the parabolic bundle $(\bar{E}, \bar{F}, w)$ together with $d_2(y)$, $y \in P$ uniquely determines the equivariant bundle $(E, \tilde{\tau})$. In the case n is odd, the parabolic bundle $(\bar{E}, \bar{F}, w)$ alone determines the equivariant bundle $(E, \tilde{\tau})$. In case n is even, the parabolic bundle $(\bar{E}, \bar{F}, w)$ uniquely determines $\Delta(y) - 2d_2(y) \mod n'(y)$ for all $y \in P$.*

Whether $(\bar{E}, \bar{F}, w)$ actually also determines $(E, \tilde{\tau})$ or not in case n is even, comes down to some delicate questions about existence of meromorphic functions with divisors of certain special kind with support contained in $\pi^{-1}(P)$. We leave it to the reader to check the general fact, that if $n'(y)$ is even, for one of the $y \in P$, then $\Delta \in \mathcal{D}$ is also determined by $(\bar{E}, \bar{F}, w)$.

Proof. That $(\bar{E}, \bar{F}, w)$ together with $d_1(y)$, $y \in P$ determines $(E, \tilde{\tau})$ is immediate from Remark 2.12. The determinant relation, which holds for some $\Delta \in \mathcal{D}$ states that,

$$\pi^* \det \bar{E} \cong \Delta \otimes [D - 2D_2].$$

Hence,

$$0 = \Delta(y) + n'(y)w(y) - 2d_2(y) \mod n'(y).$$

In case n is odd, $\Delta(y)$ is uniquely determined from the outset and since $n'(y)$ must be odd in this case, we see d_2 is uniquely determined from this.

In case n is even, the wanted conclusion follows immediately from this equation. □

Let us now reverse the process and provide a construction of all equivariant bundles from a certain set of admissible parabolic bundles. Clearly the numeric data need to satisfy equation (2.19), so we make the following definition.

Definition 2.22. For each $\Delta \in \mathcal{D}$, define the subset

$$\Lambda_\Delta \subset \prod_{y\in P} T_{n'(y)}$$

by $(d_1, d_2) \in \Lambda_\Delta$ if and only if

$$\Delta(y) = d_1(y) + d_2(y) \mod n'(y),$$

for all $y \in P$. In case $P = \emptyset$, we let $\Lambda_\Delta = \{(0, 0)\}$. For each pair $(d_1, d_2) \in \Lambda_\Delta$ we let the line bundle $\bar{\Delta}$ be given by formula (2.20) and weights $w : P \to [0, 1)$ given by

$$w(y) = \frac{d_2(y) - d_1(y)}{n'(y)},$$

for all $y \in P$.

We note that $\bar{\Delta}$ and w are uniquely determined by Δ and the pair $(d_1, d_2) \in \Lambda_\Delta$.

Let $\tilde{P}(\bar{\Delta}, w)$ be the set of isomorphism classes of rank 2 parabolic bundles on (Y, P) with the parabolic weight $w(y)$ at $y \in P$ and determinant isomorphic to $\bar{\Delta}$ and let $P(\bar{\Delta}, w)$ be the moduli space of semi-stable parabolic bundles on (Y, P) with the parabolic weight $w(y)$ at $y \in P$ and determinant isomorphic to $\bar{\Delta}$. We note here that no quasi-parabolic structure is present at a point y, if $w(y) = 0$.

Definition 2.23. The *set of admissible parabolic bundles* on Y is defined to be

$$\tilde{P}_a = \coprod_{\Delta \in \mathcal{D}} \coprod_{(d_1, d_2) \in \Lambda_\Delta} \tilde{P}(\bar{\Delta}, w),$$

and the *moduli space of admissible parabolic bundles* on Y is defined to be

$$P_a = \coprod_{\Delta \in \mathcal{D}} \coprod_{(d_1, d_2) \in \Lambda_\Delta} P(\bar{\Delta}, w).$$

Now let $(d_1, d_2) \in \Lambda_\Delta$, for $\Delta \in \mathcal{D}$ and suppose $(\bar{E}, \bar{F}, w)$ is a parabolic bundle representing an element in $\tilde{P}(\bar{\Delta}, w)$. Then we define

$$(E, \tilde{\tau}) = \Gamma_{(wn', \pi^* \bar{F})}(\pi^* \bar{E}) \otimes [D_2], \tag{2.24}$$

where $D_2 = \sum_{y \in P} d_2(y) \cdot \pi^{-1}(y)$ as before and $\pi^*(\bar{E})$ is given the pullback lift.

Theorem 2.25. *Let $\tilde{\mathcal{F}} : \tilde{P}_a \to \mathcal{L}_\mathcal{D}$ be the map given by (2.24) above. Then $\tilde{\mathcal{F}}$ is a bijection and the inverse map is given by the construction described just after Corollary 2.17.*

Proof. Immediate from Remark 2.12 ☐

We will denote the inverse map of $\tilde{\mathcal{F}}$ by $\tilde{\mathcal{P}} : \mathcal{L}_\mathcal{D} \to P_a$.

Let us end this section by establishing the relation between the underlying bundle of $\tilde{\mathcal{P}}(E, \tilde{\tau})$ and E_1, for any equivariant bundle $(E, \tilde{\tau})$.

Lemma 2.26. *Suppose $(E, \tilde{\tau})$ is an equivariant bundle and $(\bar{E}, \bar{F}, w) = \tilde{\mathcal{P}}(E, \tilde{\tau})$. Then $\bar{E} \cong E_1$.*

Proof. Let (d_1, d_2) be the numeric data of $(E, \tilde{\tau})$. Then it is rather easy to see that

$$\pi^* E_1 \cong_e \Gamma_{(m_2, F_1)} \circ \Gamma_{(m_1, F_2)}(E, \tilde{\tau}) = \Gamma_{(m, F_1)}(E, \tilde{\tau}) \otimes [-D_1],$$

where $m_\nu(y) = d_\nu(y)$, $y \in P$, F_ν is the set of eigenspaces corresponding to the eigenvalue θ_ν over P, and $D_1 = \sum_{y \in P} d_1(y) \cdot \pi^{-1}(y)$. From the construction of $\bar{E}$, we have that

$$\begin{aligned} \pi^* \bar{E} &= \Gamma_{(-m, F')}\big((E, \tilde{\tau}) \otimes [-D_2]\big) = \Gamma_{(-m, F')} \circ \Gamma_{(m, G_2)} \circ \Gamma_{(m, G_1)}\big((E, \tilde{\tau}) \otimes [-D_1]\big) \\ &= \Gamma_{(m, G_1)}\big((E, \tilde{\tau}) \otimes [-D_1]\big) = \Gamma_{(m, F_1)}(E, \tilde{\tau}) \otimes [-D_1], \end{aligned}$$

where F' corresponds to the eigenvalue $\theta_1 \theta_2^{-1}$, G_1 to 1 and G_2 is induced in $\Gamma_{(m, G_1)}\big((E, \tilde{\tau}) \otimes [-D_1]\big)$ in the usual fashion. This implies the lemma. ☐

3. The fixed Points in the Moduli Space.

Recall that by Lemma 2.3 any fixed point in M can be represented by an equivariant rank 2 bundle $(E, \tilde{\tau}) \in \mathcal{L}_{\mathcal{D}}$ and that Theorem 2.25 gives a complete classification of equivariant bundles in terms of admissible parabolic bundles.

Once we have established that $E = \Pi_e \tilde{\mathcal{F}}(\bar{E}, \bar{F}, w)$ is semi-stable provided $(\bar{E}, \bar{F}, w)$ is parabolically semi-stable and that S-equivalent semi-stable parabolic bundles go to S-equivalent bundles under $\Pi_e \tilde{\mathcal{F}}$, we get an induced map from P_a to $|M|$. Let us first address the semi-stability issue.

Proposition 3.1. *The bundle E is semi-stable if and only if $(\bar{E}, \bar{F}, w)$ is parabolically semi-stable.*

Based on this proposition, we see that if $\tilde{P}'_a$ is defined to be the subset of $\tilde{P}_a$ consisting of isomorphism class of those parabolic bundles, which are semi-stable, then we get an induced bijection $\tilde{\mathcal{F}} : \tilde{P}'_a \to \mathcal{L}'_{\mathcal{D}}$.

We need the following lemma to prove the Proposition.

Lemma 3.2. *Suppose L is a $\tilde{\tau}$-invariant subbundle of E. Then there is an induced parabolic subbundle $\bar{L}$ of $\bar{E}$ such that*

$$\operatorname{par}\mu(\bar{E}) - \operatorname{par}\mu(\bar{L}) = \tfrac{1}{n}\big(\mu(E) - \mu(L)\big).$$

Conversely, a parabolic subbundle $\bar{L}$ of $\bar{E}$ induces an invariant subbundle L of E, and the same equality holds.

Proof. From equation (2.24) relating E and $\bar{E}$ it is clear that there is a one to one correspondence between line sub-bundles of $\bar{E}$ and $\tilde{\tau}$-invariant sub-bundles of E. Let $(\bar{L}, L)$ be such a pair of corresponding line sub-bundles.

Recall that when $\bar{L}$ is given the induced parabolic structure from $(\bar{E}, \bar{F}, w)$, then we have that

$$\operatorname{par}\mu(\bar{E}) - \operatorname{par}\mu(\bar{L}) = \tfrac{1}{2}\deg \bar{E} - \deg \bar{L} + \tfrac{1}{2}\sum_{\bar{L}_y \neq \bar{F}_y} w(y) - \tfrac{1}{2}\sum_{\bar{L}_y = \bar{F}_y} w(y).$$

Now

$$\deg E = n \deg \bar{E} - \deg D + 2\deg D_2,$$

and

$$\deg L = n \deg \bar{L} - \sum_{\bar{L}_y \neq \bar{F}_y} kn'(y)w(y) + \deg D_2.$$

Computing $\frac{1}{2}\deg E - \deg L$ from the equations for $\deg E$ and $\deg L$ and comparing with the above expression for $\operatorname{par}\mu(\bar{E}) - \operatorname{par}\mu(\bar{L})$, we get the formula we wanted. □

Proof of Proposition 3.1. Suppose $(\bar{E}, \bar{F}, w)$ is parabolically semi-stable and let W be the unique maximal semi-stable sub-bundle of E. As $\tilde{\tau}(W)$ is a bundle with the same properties, $\tilde{\tau}(W) = W$ by uniqueness. Assume that $W \neq E$ so that E is not semi-stable. Then W is an invariant line bundle in E and employing the previous lemma we get: $\mu(W) \leq \mu(E)$ by semi-stability of $(\bar{E}, \bar{F}, w)$, thus contradicting the assumption on E. Hence, we have proved that E is semi-stable whenever $(\bar{E}, \bar{F}, w)$ is parabolically semi-stable.

If on the other hand E is stable (resp. semi-stable), then it follows immediately from Lemma 3.2 that $(\bar{E}, \bar{F}, w)$ is parabolically stable (resp. semi-stable). □

Let us now check that parabolic S-equivalence classes are taken to S-equivalence classes of semi-stable bundles: $\mathrm{Gr}\left(\Gamma_{(wn',\pi^*\bar{F})}(\bar{E},\bar{F},w)\right) = \Gamma_{(wn',\pi^*\bar{F})}\left(\mathrm{Gr}(\bar{E},\bar{F},w)\right)$. What remains to be shown is that this is the case for semi-stable, but non-stable bundles. So suppose that

$$0 \longrightarrow (\bar{L}_1, w_1) \longrightarrow (\bar{E},\bar{F},w) \longrightarrow (\bar{L}_2, w_2) \longrightarrow 0 ,$$

is an exact sequence of semi-stable parabolic bundles such that $\mathrm{par}\,\mu(\bar{L}_\nu, w_\nu) = \mathrm{par}\,\mu(\bar{E},\bar{F},w)$. Then

$$\Gamma_{(wn',\pi^*\bar{F})}\left(\pi^*(\bar{L}_1,w_1)\oplus\pi^*(\bar{L}_2,w_2)\right) = (L_1,\tilde{\tau}_1)\oplus(L_2,\tilde{\tau}_2),$$

where according to Remark 2.9 $(L_\nu,\tilde{\tau}_\nu) = \pi^*\bar{L}_\nu \otimes [-\sum_{y\in P} w_\nu(y)n'(y)\cdot\pi^{-1}(y)]$.

On the other hand, $\bar{L}_1$ gives a $\tilde{\tau}$-invariant destabilising line subbundle E which is clearly equivariantly isomorphic to $(L_1,\tilde{\tau}_1)$. Put $(L_2',\tilde{\tau}_2') = (E,\tilde{\tau})/L_1$ so that $\mathrm{Gr}(E,\tilde{\tau}) = (L_1,\tilde{\tau}_1)\oplus(L_2',\tilde{\tau}_2')$. Then it follows from Remark 2.9 that $(L_2',\tilde{\tau}_2') \cong_e (L_2,\tilde{\tau}_2)$ and we get that $\mathrm{Gr}(\bar{E},\bar{F},w) = (\bar{L}_1,w_1)\oplus(\bar{L}_2,w_2)$.

We have now established that the following map is well-defined:

Definition 3.3. We define a map

$$\mathcal{F} : P_a \longrightarrow |M|$$

by requiring commutativity of

$$\begin{array}{ccc} \tilde{P}_a' & \xrightarrow{\tilde{\mathcal{F}}} & \mathcal{L}_{\mathcal{D}}' \\ \downarrow & & \downarrow \\ P_a & \xrightarrow{\mathcal{F}} & |M|. \end{array}$$

From the above discussion, it is clear that $\mathcal{F}$ is surjective and we shall now discuss the fibers of $\mathcal{F}$. First let

$$P_a = P_a^{ss} \sqcup P_a^s,$$

where P_a^s is the set of point in P_a represented by stable parabolic bundles and P_a^{ss} is the complement of these in P_a.

Let T be the sub-group generated by the involutions in $\ker\pi^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$. If r is even (which is not the case if n is odd), then T is the unique sub-group of this kernel of order two. Otherwise, it is the trivial group $\{1\}$. In case r is even, T will act on P_a^s and we decompose

$$P_a^s = P_a^{s,i} \sqcup P_a^{s,g},$$

where $P_a^{s,i} = (P_a^s)^T$ is the fixed point set under the T-action. If r is odd, we let $P_a^{s,g} = P_a^s$ and $P_a^{s,i} = \emptyset$.

Likewise, let $M^s \subseteq M$ be the subset represented by stable bundles and $M^{ss} = M - M^s$.

Our main set-theoretic theorem states:

Theorem 3.4. *The map $\mathcal{F} : P_a \to |M|$ is onto and has finite fibers.*

In case n is odd

$$\mathcal{F} : P_a^s \longrightarrow |M^s|$$

is a bijection and

$$\mathcal{F} : P_a^{ss} \longrightarrow |M^{ss}|$$

is invariant under the finite[2] *equivalence relation* $\sim_o$ *described below under point 1. and it induces a bijection between* $P_a^{ss}/\sim_o$ *and* $|M^{ss}|$.

In case n is even, $\mathcal{F}$ *is invariant under the* ζ_2*-action described below under point 2. and it descends to*

$$\mathcal{F} : \bar{P}_a = P_a/\zeta_2 \longrightarrow |M|.$$

This ζ_2 *action preserves stability and coincides with the* T*-action whenever* r *is even and we get an induced decomposition* $\bar{P}_a^s = \bar{P}_a^{s,i} \coprod \bar{P}_a^{s,g}$. *The restriction*

$$\mathcal{F} : \bar{P}_a^{s,g} \longrightarrow |M^s|$$

is a bijection and

$$\mathcal{F} : \bar{P}_a^{s,i} \sqcup \bar{P}_a^{ss} \longrightarrow |M^{ss}|$$

is invariant under the finite equivalence relation $\sim_e$ *on* $\bar{P}_a^{ss}$ *described under point 1. below and induces a bijection between* $\bar{P}_a^{s,i} \sqcup (\bar{P}_a^{ss}/\sim_e)$ *and* $|M^{ss}|$.

1. The equivalence relations $\sim_o$ **and** $\sim_e$**.** Using the ζ_n-action on the set of equivariant line bundles we introduce the following equivalence relation.

Definition 3.5. We define the equivalence relation $\sim_o$ on P_a^{ss} by declaring that

$$\mathrm{Gr}(\bar{E}, \bar{F}, w) \sim_o \mathrm{Gr}(\bar{E}', \bar{F}', w')$$

if and only if there exists $\mu \in \zeta_n$ and equivariant line bundles $(L_\nu, \tilde{\tau}_\nu)$, $\nu = 1, 2$, such that

$$\begin{aligned} \tilde{\mathcal{F}}\big(\mathrm{Gr}(\bar{E}, \bar{F}, w)\big) &\cong (L_1, \tilde{\tau}_1) \oplus (L_2, \tilde{\tau}_2), \\ \tilde{\mathcal{F}}\big(\mathrm{Gr}(\bar{E}', \bar{F}', w')\big) &\cong (L_1, \mu \cdot \tilde{\tau}_1) \oplus (L_2, \mu^{-1} \cdot \tilde{\tau}_2). \end{aligned}$$

Let us now examine how the numeric data (d_1, d_2) of $\tilde{\mathcal{F}}\big(\mathrm{Gr}(\bar{E}, \bar{F}, w)\big)$ is related to the numeric data (d_1', d_2') of $\tilde{\mathcal{F}}\big(\mathrm{Gr}(\bar{E}', \bar{F}', w')\big)$. Let f_μ be a meromorphic function on X, such that $f_\mu \circ \tau^{-1} = \mu \cdot f_\mu$ and let $d_\mu(y) = (f_\mu)(y)$, $y \in P$. Further let $\mathrm{Gr}(\bar{E}, \bar{F}, w) = (\bar{L}_1, w_1) \oplus (\bar{L}_2, w_2)$ and $\mathrm{Gr}(\bar{E}', \bar{F}', w') = (\bar{L}_1', w_1') \oplus (\bar{L}_2', w_2')$, where

$$w_\nu(y) = \begin{cases} 0, & \text{for } \bar{L}_{\nu,y} \neq \bar{F}_y, \\ w(y), & \text{for } \bar{L}_{\nu,y} = \bar{F}_y \end{cases}$$

and w_ν' similarly. Set $0 \leq d^\nu(y) < n'(y)$, $y \in P$, to be

$$\begin{aligned} d^1(y) &= d_2(y) - n'(y) w_1(y) + d_\mu(y) \mod n'(y), \\ d^2(y) &= d_2(y) - n'(y) w_2(y) - d_\mu(y) \mod n'(y). \end{aligned}$$

Then if $P_\times = \{\, y \in P \mid d^1(y) > d^2(y) \,\}$, we have that

$$\big(d_1', d_2'\big)(y) = \begin{cases} \big(d^1(y), d^2(y)\big), & \text{for } y \in P - P_\times, \\ \big(d^2(y), d^1(y)\big), & \text{for } y \in P_\times. \end{cases} \tag{3.6}$$

Of course $w'(y) = \big(d_2'(y) - d_1'(y)\big)/n'(y)$, $w_\nu'(y) = d^\nu(y)/n'(y)$ and $\bar{F}_y' = \bar{L}_{\nu,y}'$ if and only if $\bar{F}_y = \bar{L}_{\nu,y}$ for $y \in P - P_\times$ and for $y \in P_\times$ the flag is switched: $\bar{F}_y' = \bar{L}_{\nu+1,y}'$ if and only if $\bar{F}_y = \bar{L}_{\nu,y}$.

[2] We call an equivalence relation *finite*, if it has finite equivalence classes.

Lemma 3.7. *The numeric data (d_1, d_2) together with the subset $P_1 = \{ y \in P \mid w_1(y) \neq 0 \} \subseteq P$ and μ determines a line bundle L such that*

$$\bar{L}_1' \cong \bar{L}_1 \otimes L, \quad \text{and} \quad \bar{L}_2' \cong \bar{L}_2 \otimes L^{-1}.$$

Proof. Let $\bar{D}_\nu$ be divisors on Y such that $\bar{L}_\nu = [\bar{D}_\nu]$. Then let

$$D^1 = \pi^* \bar{D}_1 + D_2 + (f_\mu) - \sum_{y \in P} n'(y) w_1(y) \cdot \pi^{-1}(y).$$

By the formula, we see that $[D^1] \cong (L_1, \mu \cdot \tilde{\tau}_1)$ as an equivariant bundle. We now observe that there is a divisor $\bar{D}_1'$ on Y uniquely determined by

$$\pi^* \bar{D}_1' = D^1 - D_2' + \sum_{y \in P} n'(y) w_1'(y) \cdot \pi^{-1}(y),$$

where $D_2' = \sum_{y \in P} d_2'(y) \cdot \pi^{-1}(y)$. From this we see that $\pi^* \bar{D}_1' - \pi^* \bar{D}_1 = \pi^* \bar{D}$ and $\bar{D}$ is determined completely by (d_1, d_2) and P_1. By calculating the determinants, one can check that $L = [\bar{D}]$ has the required properties. □

In case n is *even*, we define an equivalence relation $\sim_e$ on P_a^{ss} generated by $\sim_o$ just described and then the following equivalence relation $\sim$: Let $\mu \in \zeta_n$ be such that $\mu \cdot \Delta \cong \Delta'$ as equivariant bundles, where $\mathcal{D} = \{\Delta, \Delta'\}$.

Definition 3.8. The $\mathrm{Gr}(\bar{E}, \bar{F}, w) \in P_a^{ss}(\Delta)$ is $\sim$-equivalent to $\mathrm{Gr}(\bar{E}', \bar{F}', w') \in P_a^{ss}(\Delta')$ (written $\mathrm{Gr}(\bar{E}', \bar{F}', w') \sim \mathrm{Gr}(\bar{E}, \bar{F}, w)$) if and only if there exists equivariant line bundles $(L_\nu, \tilde{\tau}_\nu)$, $\nu = 1, 2$, such that

$$\tilde{\mathcal{F}}\big(\mathrm{Gr}(\bar{E}, \bar{F}, w)\big) \cong (L_1, \tilde{\tau}_1) \oplus (L_2, \tilde{\tau}_2),$$
$$\tilde{\mathcal{F}}\big(\mathrm{Gr}(\bar{E}', \bar{F}', w')\big) \cong (L_1, \mu \cdot \tilde{\tau}_1) \oplus (L_2, \tilde{\tau}_2).$$

Again, we let $\mathrm{Gr}(\bar{E}, \bar{F}, w) = (\bar{L}_1, w_1) \oplus (\bar{L}_2, w_2)$ and $\mathrm{Gr}(\bar{E}', \bar{F}', w') = (\bar{L}_1', w_1') \oplus (\bar{L}_2', w_2')$. Set $0 \leq d^\nu(y) < n'(y)$, $y \in P$, to be

$$d^1(y) = d_2(y) - n'(y) w_1(y) + d_\mu(y) \mod n'(y),$$
$$d^2(y) = d_2(y) - n'(y) w_2(y) \mod n'(y).$$

Then d_ν' and w_ν' are given by (3.6) and the lines following it.

Lemma 3.9. *The indexing of the line bundles $\bar{L}_\nu$ and $\bar{L}_\nu'$, $\nu = 1, 2$, can be chosen such that*

$$\bar{L}_1' \cong \bar{L}_1 \otimes \bar{\Delta}_c, \quad \text{and} \quad \bar{L}_2' \cong \bar{L}_2,$$

where $\bar{\Delta}_c = \bar{\Delta}' \bar{\Delta}^{-1}$.

Proof. Obvious. □

2. The action of ζ_2 on P_a. In the case where n is even, there is an natural ζ_2-action on $\mathcal{L}_{\mathcal{D}}$, simply gotten by letting $-1 \in \zeta_2$ map an equivariant bundle $(E, \tilde{\tau})$ to the equivariant bundle $(E, -\tilde{\tau})$, which of course maps $\mathcal{L}'_{\mathcal{D}}$ to it self. Using $\tilde{\mathcal{F}}$, we get an induced ζ_2 action on $\tilde{P}'_a$, which by the properties of $\tilde{\mathcal{F}}$ induces an action on P_a. Let us now explicitly describe this action on P_a.

If (d_1, d_2) is the numeric data of $\tilde{\mathcal{F}}(\bar{E}, \bar{F}, w) = (E, \tilde{\tau})$ and (d'_1, d'_2) that of $\tilde{\mathcal{F}}(\bar{E}', \bar{F}', w') = (E, -\tilde{\tau})$. Then we observe, that the numeric data are related by

$$(d'_1, d'_2)(y) = \begin{cases} \big(d_1(y), d_2(y)\big), & \text{for } y \in P - P_o, \\ \big(d_1(y) + \frac{n'(y)}{2}, d_2(y) + \frac{n'(y)}{2}\big), & \text{for } y \in P_+, \\ \big(d_1(y) - \frac{n'(y)}{2}, d_2(y) - \frac{n'(y)}{2}\big), & \text{for } y \in P_-, \\ \big(d_2(y) - \frac{n'(y)}{2}, d_1(y) + \frac{n'(y)}{2}\big), & \text{for } y \in P_\times, \end{cases} \tag{3.10}$$

where

$$P_o = \{\, y \in P \mid k(y) \text{ odd} \,\}, \qquad P_+ = \{\, y \in P_o \mid d_2(y) < \tfrac{n'(y)}{2} \,\},$$

$$P_- = \{\, y \in P_o \mid d_1(y) \geq \tfrac{n'(y)}{2} \,\}, \qquad P_\times = \{\, y \in P_o \mid d_1(y) < \tfrac{n'(y)}{2} \leq d_2(y) \,\}.$$

Let $m_\times(y) = 1$ for $y \in P_\times$ and zero otherwise. Clearly, both $m_\times$ and w' are determined just by (d_1, d_2).

Lemma 3.11. *The numeric data (d_1, d_2) determines a line bundle $\bar{L}$ on Y, such that*

$$(\bar{E}', \bar{F}') = \Gamma_{(-m_\times, \bar{F})}(\bar{E}, \bar{F}) \otimes \bar{L}. \tag{3.12}$$

Observe, that if r is even then all $k(y)$ are even and therefore $m_\times(y) = 0$ for all $y \in P$, so $(\bar{E}', \bar{F}') = (\bar{E}, \bar{F}) \otimes \bar{L}$. But then $\bar{L} \in \ker \pi^*$ and in fact $\pi^*(\bar{L}) \cong_e [(f_{-1})]$, which means that $T = \langle \bar{L} \rangle$.

Proof. Let $D_- = (f_{-1})$. Then we have an isomorphism of equivariant bundles

$$\Gamma_{(wn', \pi^*\bar{F})}(\pi^*\bar{E}) \otimes [D_2] \otimes [D_-] \cong_e (E, -\tilde{\tau}) \cong_e \Gamma_{(w'n', \pi^*\bar{F}')}(\pi^*\bar{E}') \otimes [D'_2].$$

Let F' be the flag induced in E from this isomorphism. Then

$$\pi^*\bar{E}' \cong_e \Gamma_{(-w'n', F')} \circ \Gamma_{wn', \pi^*\bar{F}}(\pi^*\bar{E}) \otimes [D_- - D'_2 + D_2].$$

Now

$$w'(y) = \begin{cases} w(y), & \text{for } y \in P - P_\times \\ 1 - w(y), & \text{for } y \in P_\times, \end{cases}$$

so at $y \in P - P_\times$ no modification is done to $\pi^*\bar{E}$, but at each $y \in P_\times$ we do $-n'(y)$ elementary modifications in the direction $\pi^*\bar{F}_y$, followed by $n'(y)w(y)$ elementary modifications in two complementary directions, which just corresponds to tensoring with $[-n'(y)w(y) \cdot \pi^{-1}(y)]$. Hence, we see that

$$\pi^*\bar{E}' \cong_e \Gamma_{(-n'm_\times, \pi^*\bar{F})}(\pi^*\bar{E}) \otimes [-D_\times + D_- - D'_2 + D_2],$$

where $D_\times = \sum_{y \in P_\times} w(y)n'(y) \cdot \pi^{-1}(y)$. But now observe that $(-D_\times + D_- - D'_2 + D_2)(y)$ is zero mod $n'(y)$ for all $y \in P$, hence $(-D_\times + D_- - D'_2 + D_2)$ is a pullback of a divisor, say $\bar{D}$, from Y and we get that

$$\bar{E}' \cong \Gamma_{(-m_\times, \bar{F})}(\bar{E}) \otimes [\bar{D}].$$

□

Note that this ζ_2 action on P_a^{ss} is compatible with the equivalence relation $\sim_e$, thus we get an induced equivalence relation of $\bar{P}_a^{ss}$, which we also denote $\sim_e$.

In order to prove Theorem 3.4, we need to understand which stable parabolic bundles in P_a^s are mapped to non-stable (but of course semi-stable) bundles under $\mathcal{F}$:

Lemma 3.13. *Let* $(\bar{E},\bar{F},w)\in P_a^s$. *Then* $\mathcal{F}(\bar{E},\bar{F},w)\in|M^{ss}|$ *if and only if* $(\bar{E},\bar{F},w)\in P_a^{s,i}$, *i.e.* r *is even and* $(\bar{E},\bar{F},w)\cong(\bar{E},\bar{F},w)\otimes\bar{L}$, *where* $T=\langle\bar{L}\rangle$.

Proof. Assume that $\tilde{\mathcal{F}}(\bar{E},\bar{F},w)=(E,\tilde{\tau})$ and $\mathrm{Gr}(E)=L_1\oplus L_2\in|M^{ss}|$, where $L_1\subset E$. Since $\tau^*\mathrm{Gr}(E)\cong\mathrm{Gr}(E)$, we have either that $\tau^*L_1\cong L_1$ and $\tau^*L_2\cong L_2$ or $\tau^*L_1\cong L_2\not\cong L_1$.

First the invariant case, $\tau^*L_\nu\cong L_\nu$. Consider the sub-case, where $\tilde{\tau}(L_1)=L_1$. Then this $\tilde{\tau}$-invariant subbundle L_1 induces a parabolic subbundle $(\bar{L}_1,w_1)\subset(\bar{E},\bar{F},w)$ with $\operatorname{par}\mu(\bar{L}_1,w_1)=\operatorname{par}\mu(\bar{E},\bar{F},w)$ simply by following L_1 through the construction of $(\bar{E},\bar{F},w)$ from $(E,\tilde{\tau})$. But this contradicts the stability of $(\bar{E},\bar{F},w)$. Now, in the sub-case where $\tilde{\tau}(L_1)\neq L_1$ then $L_1\cap\tilde{\tau}(L_1)=\{0\}$ and $\tilde{\tau}(L_1)\cong L_2$, so $E\cong L_1\oplus\tilde{\tau}(L_1)$ and $L_1\cong L_2$, i.e. $E\cong L_1\oplus L_1$. It is then easy to see that there is an inclusion of $L_1\hookrightarrow E$, which is preserved by $\tilde{\tau}$, and we are in fact the in sub-case just discussed.

Now consider the case where $\tau^*(L_1)\cong L_2\not\cong L_1$. Then we get an isomorphism $\tilde{\tau}(L_1)\cong L_2$ and $\tilde{\tau}^2(L_\nu)=L_\nu$, $\nu=1,2$. So n has to be even, $E\cong L_1\oplus L_2$ and $\tilde{\tau}$ acts off diagonally. If we let $\check{X}=X/\langle\tau^2\rangle$ and $\check{\pi}:\check{X}\to Y$ be the projection, then by applying the above, we get an equivariant isomorphism $(\check{\pi}^*\bar{E},\check{\pi}^*\bar{F},w)\cong(\check{L},w_1)\oplus(\tau^*(\check{L}),w_2)$. Note in particular that $\check{\pi}^*\bar{F}_{\check{x}}$ must be either $\check{L}_{\check{x}}$ or $\tau^*\check{L}_{\check{x}}$ inside $\check{L}_{\check{x}}\oplus\tau^*\check{L}_{\check{x}}$ for $\check{x}\in\check{\pi}^*(P)$. Also note that $\check{\pi}:\check{X}\to Y$ cannot be ramified, for else $\tau:\check{X}\to\check{X}$ would have fixed points, over which the action on $\check{L}\oplus\tau^*\check{L}$ has eigenvalues 1 and -1, contradicting the fact that $\check{L}\oplus\tau^*\check{L}\cong_e\check{\pi}^*\bar{E}$. However, $\check{\pi}:\check{X}\to Y$ is unramified if and only if r is even.

Since

$$\bar{E}=\check{L}\oplus\tau^*\check{L}/\langle\tau\rangle,$$

we get by Lemma 2.1 in [21] that if $\langle\bar{L}\rangle=\ker\{\check{\pi}^*:\mathrm{Pic}_0(Y)\to\mathrm{Pic}_0(\check{X})\}$, then $\bar{E}\cong\bar{E}\otimes\bar{L}$. In fact it is easy to write this isomorphism on $\check{L}\oplus\tau^*\check{L}$ explicitly as $\check{\Phi}=\mathrm{Id}\oplus(-\mathrm{Id})$ which commutes with the τ action. But then we see that $\check{\Phi}$ takes $\check{\pi}^*\bar{F}$ to it self, hence we get that $(\bar{E},\bar{F},w)\cong(\bar{E},\bar{F},w)\otimes\bar{L}$.

Conversely, suppose that r is even and $(\bar{E},\bar{F},w)\in P_a^{s,i}$, i.e. $(\bar{E},\bar{F},w)\cong(\bar{E},\bar{F},w)\otimes\bar{L}$. Then $\check{\pi}:\check{X}\to Y$ is a 2-fold unramified cover and $\check{\pi}^*L\cong_e[(\check{f}_{-1})]$, where $\check{f}_-\in\mathcal{M}(\check{X})$ is anti invariant under τ. The isomorphism $(\bar{E},\bar{F},w)\cong(\bar{E},\bar{F},w)\otimes\bar{L}$ then lifts to a $\check{\Phi}\in\mathrm{Aut}(\check{\pi}^*\bar{E},\check{\pi}^*\bar{F})$ such that

$$\begin{array}{ccc}\check{\pi}^*\bar{E} & \xrightarrow{\check{\Phi}} & \check{\pi}^*\bar{E}\\ {\scriptstyle\tau^*}\downarrow & & \downarrow{\scriptstyle -\tau^*}\\ \check{\pi}^*\bar{E} & \xrightarrow{\check{\Phi}} & \check{\pi}^*\bar{E}.\end{array}$$

Now $\check{\Phi}$ can be normalised such that $\check{\Phi}^2=1$, since $\check{\Phi}^2\in\check{\pi}^*\mathrm{Aut}(\bar{E},\bar{F})\cong\mathbb{C}^*$. By the relation $-\tau^*=\check{\Phi}\tau^*\check{\Phi}$, we conclude that $\check{\Phi}\neq\pm\mathrm{Id}$. Hence, $\check{\Phi}$ gives an eigen decomposition $\check{\pi}^*\bar{E}\cong\check{L}\oplus\tau^*\check{L}$ and $\check{\pi}^*\bar{F}$ has to be compatible with this direct sum decomposition, because of $\check{\Phi}$-invariance. But then it is clear that $\tilde{\mathcal{F}}(\bar{E},\bar{F},w)\in|M^{ss}|$. □

Proof of Theorem 3.4. We recall that $\mathcal{F}:P_a\to|M|$ is surjective, since $\tilde{\mathcal{F}}:P_a'\to\mathcal{L}_{\mathcal{D}}'$ is a bijection and $\Pi_e:\mathcal{L}_{\mathcal{D}}'\to|M|$ is a surjection. Let us then prove the statement about the fibers of $\mathcal{F}$. In case n is odd, $\Pi_e:\mathcal{L}_{\mathcal{D}}^s\to|M^s|$ is a bijection, hence so is $\mathcal{F}:P_a^s\to|M^s|$. In case n is even, Π_e is invariant under the ζ_2 action on $\mathcal{L}_{\mathcal{D}}'$ and $\Pi_e:\mathcal{L}_{\mathcal{D}}^s/\zeta_2\to|M^s|$ is a bijection. Thus, by Lemma 3.13, we get that $\mathcal{F}:\bar{P}_a^{s,g}\to|M^s|$ is a bijection.

Let us now consider the fibers of $\mathcal{F}$ over $|M^{ss}|$. Assume we have $(\bar{E},\bar{F},w)\in P_a^{s,i}$ and $(\bar{E}',\bar{F}',w')\in P_a^{ss}\sqcup P_a^{s,i}$. As we saw in the proof of Lemma 3.13, we must have that $\tilde{\mathcal{F}}(\bar{E},\bar{F},w)\cong(L\oplus\tau^*L,\tilde{\tau})$, where $L\not\cong\tau^*L$ and $\tilde{\tau}(L)=\tau^*(L)$. If $(\bar{E}',\bar{F}',w')\in P_a^{s,i}$, then there is a line bundle L' which satisfies the same as L. But, then since L', L and τ^*L all have the same degree, we see that either $L'\cong L$ or $L'\cong\tau^*L$. In both cases we get an equivariant isomorphism $(L\oplus\tau^*L,\tilde{\tau})\cong(L'\oplus\tau^*L',\tilde{\tau}')$, which gives that the numeric invariants are the same and that there is an isomorphism $(\bar{E},\bar{F},w)\cong(\bar{E}',\bar{F}',w')$. If $(\bar{E}',\bar{F}',w')\in P_a^{ss}$, we can assume that $(\bar{E}',\bar{F}',w')\cong \mathrm{Gr}(\bar{E}',\bar{F}',w')$ and so $\tilde{\mathcal{F}}(\bar{E}',\bar{F}',w')\cong (L_1,\tilde{\tau}_1)\oplus(L_2,\tilde{\tau}_2)$ is a direct sum of equivariant line bundles. From this we get that $L\cong L_1$ or $L\cong L_2$, because these line bundles have the same degree. This gives a contradiction, since $L\not\cong\tau^*L$. — Summing up, we have that $\mathcal{F}:P_a^{s,i}\to\mathcal{F}(P_a^{s,i})\subset|M^{ss}|$ is a bijection and $\mathcal{F}(P_a^{ss})=|M^{ss}|-\mathcal{F}(P_a^{s,i})$.

Let us now examine the fibers of $\mathcal{F}:P_a^{ss}\to|M^{ss}|$, so let

$$\tilde{\mathcal{F}}\big((\bar{L}_1,w_1)\oplus(\bar{L}_1,w_1)\big)\cong(L_1,\tilde{\tau}_1)\oplus(L_2,\tilde{\tau}_2)$$

and

$$\tilde{\mathcal{F}}\big((\bar{L}_1',w_1')\oplus(\bar{L}_1',w_1')\big)\cong(L_1',\tilde{\tau}_1')\oplus(L_2',\tilde{\tau}_2')$$

and suppose that $L_1\oplus L_2\cong L_1'\oplus L_2'$. By re-indexing if necessary, we can assume that $L_1\cong L_1'$ and $L_2\cong L_2'$. It now follows immediately that in case n is odd, $[(\bar{L}_1,w_1)\oplus(L_2,w_2)]\sim_o[(\bar{L}_1',w_1')\oplus(L_2',w_2')]$ and in case n is even, that $[(\bar{L}_1,w_1)\oplus(L_2,w_2)]\sim_e[(\bar{L}_1',w_1')\oplus(L_2',w_2')]$ by the very construction of $\sim_o$ and $\sim_e$. □

Example: The unramified case. Assume that $\pi:X\to Y$ is unramified, i.e. $P=\emptyset$. Then $\Lambda_\Delta=\{(0,0)\}$ and for each $\Delta\in\mathcal{D}$, $P(\bar{\Delta},w)=\bar{M}(\bar{\Delta})$ the moduli space of semi-stable holomorphic bundles of rank 2 on Y with determinant isomorphic to $\bar{\Delta}$, so

$$P_a=\begin{cases}\bar{M}(\bar{\Delta}), & n \text{ odd}\\ \bar{M}(\bar{\Delta}')\sqcup\bar{M}(\bar{\Delta}''), & n\text{ even,}\end{cases}$$

where $\{\pi^*\bar{\Delta}',\pi^*\bar{\Delta}''\}=\mathcal{D}$ when n is even. The map $\mathcal{F}:P_a\to|M|$ is just $\pi^*:P_a\to|M|$ and if n is even, the ζ_2-action is just given by tensoring with the line bundle $\bar{L}=\bar{L}_\pi^{n/2}$, where $\langle\bar{L}_\pi\rangle=\ker\{\pi^*:\mathrm{Pic}_0(Y)\to\mathrm{Pic}_0(X)\}$. Thus, we get that

$$|M^s|\cong\begin{cases}\bar{M}^s(\bar{\Delta}), & n\text{ odd}\\ \bar{M}^{s,g}(\bar{\Delta}')/\langle\bar{L}\rangle\sqcup\bar{M}^{s,g}(\bar{\Delta}'')/\langle\bar{L}\rangle, & n\text{ even.}\end{cases}$$

Let us consider the case n is *odd* first. If $\deg\Delta$ is odd, then so is $\deg\bar{\Delta}$ and we simply get that $|M|\cong\bar{M}(\bar{\Delta})$. If $\deg\Delta$ is even, then so is $\deg\bar{\Delta}$. Let $\bar{\Delta}^{1/2}$ be a square root of $\bar{\Delta}$. Then the map $\Xi:\mathrm{Pic}_0(Y)\to\bar{M}(\bar{\Delta})^{ss}$ given by $\Xi(\bar{H})=\bar{H}\bar{\Delta}^{1/2}\oplus\bar{H}^{-1}\bar{\Delta}^{1/2}$ is surjective and invariant under the ζ_2-action on $\mathrm{Pic}_0(Y)$ given by $\bar{H}\mapsto\bar{H}^{-1}$ and $\Xi:\mathrm{Pic}_0(Y)/\zeta_2\to\bar{M}(\bar{\Delta})^{ss}$ is a bijection. The action of $\mathrm{Pic}_0(Y)$ on itself restricts to an action of $\langle\bar{L}_\pi\rangle$ on $\mathrm{Pic}_0(Y)$ and if $G=\langle\bar{L}_\pi\rangle\ltimes\zeta_2$, then

$$\pi^*\circ\Xi:\mathrm{Pic}_0(Y)/G\longrightarrow|M^{ss}|$$

is a bijection.

Let us now consider the case where n is even. Then there are two subcases, the first being $\deg\Delta/n$ odd. We get then immediately that

$$|M^{ss}|\cong\bar{M}(\bar{\Delta}')^{s,i}\sqcup\bar{M}(\bar{\Delta}'')^{s,i},$$

i.e. by Corollary 3.6 in [21] a disjoint union of two copies of the Prym variety $\check{P}$ of the double cover $\check{\pi}:\check{X}\to X$, where $\check{X}=X/\langle\tau^{n/2}\rangle$.

The other subcase is $\deg\Delta/n$ even. As before we have bijections

$$\Xi':\mathrm{Pic}_0(Y)/G\longrightarrow\bar{M}(\bar{\Delta}')^{ss}/\sim_o$$

and

$$\Xi'' : \mathrm{Pic}_0(Y)/G \longrightarrow \bar{M}(\bar{\Delta}'')^{ss}/\sim_o .$$

But $\sim$ gives a bijection between $\bar{M}(\bar{\Delta}')^{ss}/\sim_o$ and $\bar{M}(\bar{\Delta}'')^{ss}/\sim_o$, so we get that

$$|M^{ss}| \cong \mathrm{Pic}_0(Y)/G \cup \big(\bar{M}(\bar{\Delta}')^{s,i} \sqcup \bar{M}(\bar{\Delta}'')^{s,i}\big).$$

Hence, the irreducible components of $|M^{ss}|$ are the four Kumar-varieties $\bar{M}(\bar{\Delta}')^{s,i} \sqcup \bar{M}(\bar{\Delta}'')^{s,i}$, all isomorphic to $\check{P}/\{\pm 1\}$ and the quotient $\mathrm{Pic}_0(Y)/G$, intersecting each of the Kumar's in finitely many points.

4. Geometric Invariant Theory Analysis of the Morphism

In this section we will prove that $\mathcal{F}\colon P_a \to |M|$ is a *morphism* of varieties using geometric invariant theory (GIT), and we will study the structure of this morphism and of the fixed point variety. We refer the reader to [18], [23], [19] and [11] for the details about GIT. Here we review just the minimum to fix the notation.

Let Z be a smooth complex algebraic curve of genus $g = g(Z)$. Suppose that $\mathbb{E}$ is the trivial bundle of rank $p = d - 2(g-1)$ over Z. Fix a Hilbert polynomial $\rho_0(T) = p + 2T$, and assume that $d > 2(2g-1)$. Let $Q_Z = \mathrm{Quot}^{\rho_0}_{\mathbb{E}/Z/\mathbb{C}}$ be Grothendieck's Quot scheme, [11], [23], and $\mathcal{U}_Z$ the universal quotient sheaf over $Q_Z \times_{\mathbb{C}} Z$. Then there is an open sub-scheme R_Z of Q_Z, characterised by the property that R_Z is exactly the points $q \in Q_Z$ for which $\mathcal{U}_{Z,q}$ is locally free over $Z_q = \{q\} \times_{\mathbb{C}} Z$ and the homomorphism $H^0(Z_q, \mathbb{E}) \to H^0(Z_q, \mathcal{U}_{Z,q})$ given by the quotient morphism is an isomorphism. Then $\mathcal{U}_{R_Z} = \mathcal{U}_Z|_{R_Z \times_{\mathbb{C}} Z}$ is locally free and the sub-scheme R_Z satisfies *local universality*, [23, Proposition 1.III.21].

The moduli space M consisting of strong equivalence classes of semi-stable holomorphic bundles is a *good quotient* (in the sence of GIT) of the semi-stable part R^{ss}_X with respect to action of the group $\mathrm{PGL}(p)$, [23, p. 34].

For the parabolic case let $P \subset Z$ be a finite set of points and denote by

$$\mathbb{P}(\mathcal{U}_{R_Z})_P = \prod_{z \in P} \mathbb{P}(\mathcal{U}_Z|_{R_Z \times_{\mathbb{C}} \{z\}}).$$

Let $\bar{R}_Z$ be the closed subvariety of $\mathbb{P}(\mathcal{U}_{R_Z})_P$ whose points $\{F_z\}$ satisfy that for any pair of points (z, z') in P, the two projections to R_Z from $\mathbb{P}(|_{R_Z \times_{\mathbb{C}} \{z\}})$ respectively $\mathbb{P}(\mathcal{U}_Z|_{R_Z \times_{\mathbb{C}} \{z'\}})$ agree. This is precisely what ensures that we get a (canonical) projection

$$\Pi\colon \bar{R}_Z \longrightarrow R_Z.$$

Notice that a $q \in \bar{R}_Z$ defines a quasi parabolic structure on the vector bundle $\mathcal{U}_{R_Z, \Pi(q)}$ over Z. We shall write $\bar{\mathcal{U}}_{\bar{R}_Z, q}$ for this quasi parabolic bundle and $\bar{\mathcal{U}}_{\bar{R}_Z}$ for the corresponding bundle over $\bar{R}_Z \times_{\mathbb{C}} Z$. Given a set of weights w over P, we denote the part of $\bar{R}_Z$ which is semi-stable with respect to w by $\bar{R}^{ss}_Z(w)$, and the part whose points has a fixed determinant L by $\bar{R}^{ss}_{Z,L}(w)$.

The moduli space $P(Z; L, w)$ of semi-stable parabolic bundles over Z with weights w and determinant L is a good quotient of $\bar{R}^{ss}_{Z,L}(w)$ by $\mathrm{PGL}(p)$, [23, p. 84]. When $g(Z) \geq 2$, $\bar{R}^{ss}_{Z,L}(w)$ is an open subset of an irreducible variety, [23, p. 84]. Thus, $P(Z; L, w)$ is irreducible. For low genus fixing, the weights and the determinant is not necessarily enough to ensure irreducibility. See e.g. the following section.

Lemma 4.1. *Suppose a set-theoretic map*

$$f\colon \bar{R}^{ss}_{Z,L}(w) \longrightarrow P(X;L',w'),$$

is presented by a locally free sheaf $\mathcal{V}$ over $\bar{R}^{ss}_{Z,L}(w) \times_{\mathbb{C}} X$; i.e. that $f(q) = [\mathcal{V}_q]$ for all $q \in \bar{R}^{ss}_{Z,L}(w)$, where $[\cdot]$ means strong equivalence class. Then f is a morphism provided $\mathcal{V}$ satisfies local universality. Moreover, if f is $\mathrm{PGL}(p)$*-invariant it induces a morphism*

$$\hat{f}\colon P(Z;L,w) \longrightarrow P(X;L',w').$$

Of course the above statement holds for semi-stable bundles without parabolic structures as well.

Proof. By assumption, $\mathcal{V}$ conforms to the requirements of local universality so for every element $q \in \bar{R}^{ss}_{Z,L}(w)$ there is a neighbourhood V_q of q in $\bar{R}^{ss}_{Z,L}(w)$ and a morphism $f_q\colon V_q \to \bar{R}_X$ so that $\mathcal{V}|_{V_q \times_{\mathbb{C}} X} \cong \check{f}^* \bar{\mathcal{U}}_{\bar{R}_{X L'}}$, where $\check{f}\colon V_q \times_{\mathbb{C}} X \to \bar{R}_X \times_{\mathbb{C}} X$ is the induced morphism. Clearly $\operatorname{Im} f_q \subseteq \bar{R}^{ss}_{X,L'}(w')$.

Let $g_X : \bar{R}^{ss}_{X,L'}(w') \to P(X;L',w')$ be the good quotient. Then $f|_{V_q} = g_X \circ f_q \colon V_q \to P(X;L',w')$ is a composition of morphisms. That means that f is a morphism locally around every point $q \in \bar{R}^{ss}_{Z,L}(w)$, hence, it is a morphism.

Now if the morphism $f\colon \bar{R}^{ss}_{Z,L}(w) \to P(X;L',w')$ is $\mathrm{GL}(p)$-invariant, it gives by universality of good quotients a morphism $\hat{f}$ so that

$$\begin{array}{ccc} \bar{R}^{ss}_{Z,L}(w) & & \\ {\scriptstyle g_X}\downarrow & \searrow^{f} & \\ P(Z;L,w) & \xrightarrow[\hat{f}]{\cdots\cdots} & P(X;L',w') \end{array}$$

commutes. ☐

By choosing $Z = X$ and $\mathcal{V} = \bar{\mathcal{U}}_X|_{\bar{R}^{ss}_{X,L}} \otimes p_X^* L'$ we verify that tensoring with a line bundle L' induces a morphism $P(X;L,w) \to P(X;L\otimes L'^2,w)$. Similarly, we see that pullback with respect to morphisms between curves induces morphisms between corresponding moduli spaces. In particular we get that the action of $\langle\tau\rangle$ on M is algebraic.

Proposition 4.2. *The map $\mathcal{F}\colon P_a \to |M|$ is a projective morphism of varieties.*

Note that we in this proposition really are considering the restriction of this map to each irreducible component of P_a mapping to the corresponding target component in $|M|$.

Proof. Let $\Delta \in \mathcal{D}$ and consider the corresponding $\bar{\Delta} \in \mathrm{Pic}_0(Y)$. By altering $\bar{\Delta}$ (and therefore also Δ correspondingly) by a sufficiently high power of an ample line bundle over Y, we may assume we are in the realm of the GIT-construction of $P(\bar{\Delta},w)$ and $M(\Delta)$. Formula (2.24) gives an explicit expression for the lift $\bar{\mathcal{F}} : \bar{R}^{ss}_{Y,\bar{\Delta}} \to |M(\Delta)|$ of $\mathcal{F} : P(\bar{\Delta},w) \to |M(\Delta)|$. Tensoring by $[D_2]$ clearly induces morphisms as remarked before, so we only need to concentrate on the iterated elementary modification. In order to use Lemma 4.1, we need to construct a locally free sheaf $\mathcal{V}$ representing these iterated elementary modifications $\Gamma_{(wn',\cdot)}$. Let $\mathcal{G}_0$ be the sheaf over $\bar{R}^{ss}_{Y,\bar{\Delta}}(w) \times_{\mathbb{C}} X$ defined by the flag such that $\mathcal{G}_{0,((\bar{E},\bar{F},w),x)} = \pi^*\bar{F}_x \subseteq \pi^*\bar{E}_x$ for $x \in \pi^{-1}(P_w)$, $P_w = \{\, y \in P \mid w(y) \neq 0 \,\}$, and

0 elsewhere. Put $\mathcal{V}_0 = (\mathrm{Id}_{\bar{R}_{Y,\bar{\Delta}}(w)} \times \pi)^* \bar{\mathcal{U}}_{\bar{R}_{Y,\bar{\Delta}}(w)}$, and consider the "universal skyscraper sheaf" $\mathcal{S}_0$ with stalks

$$\mathcal{S}_{0,((\bar{E},\bar{F},a),x)} = \begin{cases} \mathcal{V}_{0,((\bar{E},\bar{F},w),x)}/\mathcal{G}_{0,((\bar{E},\bar{F},w),x)}, & \text{for } x \in \pi^{-1}(P_w) \\ 0, & \text{elsewhere.} \end{cases}$$

There is a natural surjection

$$\lambda_0 \colon \mathcal{V}_0 \longrightarrow \mathcal{S}_0 \to 0.$$

Put $\mathcal{V}_1 = \ker \lambda_0$, let and $\iota_0 \colon \mathcal{V}_1 \to \mathcal{V}_0$ be the canonical sheaf inclusion. As in section 2, $\ker \lambda_0$ is locally free, the natural lift $\tilde{\tau}$ to $\mathcal{V}_0$ induces a lift $\tilde{\tau}$ to $\mathcal{V}_1$ and this gives a sheaf $\mathcal{G}_1$ defined as the eigenspaces for $\tilde{\tau}_x$, $x \in \pi^{-1}(P_w)$, that are not annihilated by the morphism on fibers, induced by ι_0. Then define $\mathcal{S}_1$, $\lambda_1 \colon \mathcal{V}_1 \twoheadrightarrow \mathcal{S}_1$, and $\mathcal{V}_2 = \ker \lambda_1$ in the same way and continue the process through to $\mathcal{S}_M$, M being the number of steps necessary to do $\Gamma_{(wn',\cdot)}$, and define

$$\mathcal{V} = \mathcal{V}_{M+1} = \ker \lambda_M.$$

By the alteration of $\bar{\Delta}$, the degree is constrained such that $\mathcal{V}$ satisfies local universality.

The morphism represented by $\mathcal{V}$ is $\mathrm{GL}(\bar{p})$-invariant (because $\bar{\mathcal{U}}_{\bar{R}_{Y,\bar{\Delta}}(w),q} \cong \bar{\mathcal{U}}_{\bar{R}_{Y,\bar{\Delta}}(w),A\cdot q}$ for all $A \in \mathrm{GL}(\bar{p})$). Therefore, there is an induced morphism from the moduli space $P(\bar{\Delta}, w)$ to $|M(\Delta)|$ represented by $\mathcal{V}$. But clearly this morphism is $\mathcal{F}$.

That $\mathcal{F}$ is actually projective in the sense of Grothendieck, follows immediately since $|M|$ is separated over $\mathrm{Spec}\,\mathbb{C}$ and both P_a and $|M|$ are projective (over $\mathrm{Spec}\,\mathbb{C}$). ☐

Our main statement about the algebraic geometric properties of $\mathcal{F}$ reads as follows:

Theorem 4.3. *When n is odd, the morphism*

$$\mathcal{F} \colon P_a \longrightarrow |M|,$$

is a birational equivalence which is the normalisation morphism of each of the irreducible components of $|M|$. When n is even, the morphism $\mathcal{F}$ factors through the ζ_2-quotient and gives a projective morphism

$$\mathcal{F} \colon P_a/\zeta_2 \longrightarrow |M|,$$

which is a birational equivalence. When restricted to each of the irreducible components of P_a/ζ_2 it is likewise the normalisation morphism for the corresponding irreducible component of $|M|$.

Proof. It follows from Lemma 3.11 and 4.1 and arguments similar to what we saw in the proof of Proposition 4.2, that the ζ_2-action on P_a is algebraic. General theory gives that the quotient of a projective variety under the action of a finite group acting algebraically is again a projective variety. Moreover, if the variety is normal, then so is the quotient. Hence, in the case n is even, we see that P_a/ζ_2 is a projective variety whose irreducible components are normal. It is a consequence of Zariski's Main Theorem, [10, Corollary (4.4.9)], that the restriction of $\mathcal{F} \colon P_a/\zeta_2 \to |M|$ to each of these components is the normalisation morphisms. This is because it is projective and therefore proper, and since it is generically a bijection, it is both a birational equivalence and a dominant morphism. The result then follows from the universal properties of the normalisation of a variety. The same arguments applies to $\mathcal{F} \colon P_a \to |M|$ when n is odd. ☐

Remark 4.4. We may split up the equivalence relations $\sim_o$ resp. $\sim_e$ into an equivalence relation $\sim_n$ acting only within each component, and an equivalence relation acting strictly between components. We remark that a component of $|M|$ is itself normal if and only $\sim_n$ is trivial on the corresponding component of P_a resp. P_a/ζ_2.

5. The Hyperelliptic Involution

Let X be a compact, hyperelliptic curve of genus $g \geq 2$ with a hyperelliptic involution J. Then $\pi\colon X \to X/\langle J\rangle = \mathbb{P}^1$ and the set of fixed points of J in X are exactly the Weierstrass points $x_1, \ldots x_{2g+2}$. We denote $\pi(x_j) = z_j$ and let $P = \{z_1, \ldots z_{2g+2}\}$. So in this case $n = 2$ and for each $y \in P$ we see that $k = 1$, $n' = 2$ and $\xi = -1$. Let us now apply Theorem 3.4 to describe the fixed variety $|M|$ of J in M, the moduli space of semi-stable holomorphic bundles with trivial determinant.

In this case $\mathcal{D} = \{\Delta_0, \Delta_1\}$, where $\Delta_0 = 0$ and $\Delta_1 = (f_-)$ with f_- a meromorphic function such that $f_- = -f_- \circ J$. If we choose an identification of $\mathbb{P}^1 = \mathbb{C} \cup \infty$, such that $\infty \notin P$, then we have explicitly that f_- is the anti-invariant meromorphic function on X determined by the multivalued meromorphic function

$$z \longmapsto \sqrt{\prod_{i=1}^{2g+2} (z - z_i)}$$

on $\mathbb{P}^1$, whose associated ramified cover exactly is $\pi : X \to \mathbb{P}^1$. Thus,

$$\Delta_1 = \sum_{i=1}^{2g+2} x_i - (g+1) \cdot \pi^{-1}(\infty).$$

Let us examine the case $\Delta_0 \in \mathcal{D}$ first. In this case

$$\Lambda_{\Delta_0} = \{(0,0), (1,1)\}^{\times(2g+2)}.$$

Hence, $d_1(y) = d_2(y)$ so $w(y) = 0$ for all $y \in P$. An element of Λ_{Δ_0} is just determined by the subset $Q \subseteq P$ given by $Q = \{\, y \in P \mid d_1(y) = d_2(y) = 1 \,\}$ and the corresponding line bundle on $\mathbb{P}^1$ given by formula (2.20) is

$$\bar{\Delta}_Q = [-Q] = \mathcal{O}(-d_Q)$$

where $d_Q = |Q|$. Since we only allow quasi-parabolic structures at the points, where the weights w are non-zero, there are no quasi-parabolic structures to consider in this case and

$$P_a|_{\Delta_0} = \coprod_{Q \subseteq P} M\big(\mathcal{O}(-d_Q)\big),$$

where $M\big(\mathcal{O}(-d_Q)\big)$ is the moduli space of semi-stable bundles on $\mathbb{P}^1$ with determinant $\mathcal{O}(-d_Q)$. Grothendieck's classification of vector bundles on $\mathbb{P}^1$ combined with semi-stability implies that

$$M\big(\mathcal{O}(-d_Q)\big) = \begin{cases} \emptyset & \text{if } d_Q \text{ is odd,} \\ \{\mathcal{O}\big(-\frac{d_Q}{2}\big) \oplus \mathcal{O}\big(-\frac{d_Q}{2}\big)\} & \text{if } d_Q \text{ is even.} \end{cases}$$

Denote the point in P_a corresponding to $Q \subseteq P$ by $\langle\!\langle Q \rangle\!\rangle$, then

$$P_a|_{\Delta_0} = \coprod_{\substack{Q \subseteq P \\ d_Q \text{ even}}} \langle\!\langle Q \rangle\!\rangle.$$

Let us now examine the case $\Delta_1 \in \mathcal{D}$. Then Λ_{Δ_1} consist only of one element, namely

$$\Lambda_{\Delta_1} = \{(0,1)\}^{\times(2g+2)},$$

and we have that $w(y) = \frac{1}{2}$ for all $y \in P$. We see that $\bar{\Delta}_1 = \mathcal{O}\big(-(g+1)\big)$, so in this case

$$P_a|_{\Delta_1} = P\left(\mathcal{O}\big(-(g+1)\big), \tfrac{1}{2}\right),$$

where $\frac{1}{2}$ means weight $\frac{1}{2}$ at all points of P.

Let $d = -(g+1)$ be the degree of the underlying bundle and let us further analyse this parabolic moduli space $\mathcal{P} = P\left(\mathcal{O}(d), \frac{1}{2}\right) = P_a|_{\Delta_1}$. If $(\bar{E}, \bar{F}, \frac{1}{2}) \in \mathcal{P}$, then there is an integer c such that $\bar{E} \cong \mathcal{O}(c) \oplus \mathcal{O}(d-c)$. We see that $\operatorname{par}\mu(\bar{E}) = 0$. By symmetry, we can assume that $d - c \le \frac{d}{2} \le c$.

If $\bar{L}$ is a parabolic subbundle of $\bar{E}$ then $\operatorname{par}\deg(\bar{L}) = \deg(\bar{L}) + g + 1 - \frac{1}{2}|P(\bar{L})|$, where $P(\bar{L}) = \{\, y \in P \mid \bar{L}_y \ne F_y \,\}$, so the semi-stability condition reads

$$\deg(\bar{L}) + g + 1 \le \frac{|P(\bar{L})|}{2}$$

with "$<$" substituted for "$\le$" if we want stability. In particular, we notice that $\deg(\bar{L}) \le 0$, since $|P(\bar{L})| \le 2(g+1)$. If we apply this to the subbundle $\mathcal{O}(c)$, we get that $c \le 0$. Let now $\mathcal{P}^s_c$ be the moduli space of stable parabolic bundles over $\mathbb{P}^1$ with parabolic weights $\frac{1}{2}$ over each point in P, such that the underlying vector bundle is isomorphic to $\mathcal{O}(c) \oplus \mathcal{O}(d-c)$. Then clearly $\mathcal{P}^s_0 = \emptyset$ and we have that

$$\mathcal{P}^s = \coprod_{\frac{d}{2} \le c \le 0} \mathcal{P}^s_c.$$

The subsets $\mathcal{P}^s_c$ are of course disjoint, however for $c > 0$ their closures in $\mathcal{P}$, which we will denote $\mathcal{P}_c$, do intersect in the subsets represented by only semi-stable bundles, as we shall now see. For each subset $Q \subseteq P$, such that d_Q is even, there is a semi-stable parabolic structure on $\mathcal{O}\big(\frac{-d_Q}{2}\big) \oplus \mathcal{O}\big(d + \frac{d_Q}{2}\big)$ with all weights $\frac{1}{2}$, namely over points in Q, let the flag be the fiber of the first summand and over the rest of the points, choose the fiber of the second summand. Let us denote this parabolic bundle $\langle Q \rangle$. It is clearly semi-stable, since it is the direct sum of two semi-stable parabolic bundles with parabolic slope 0. We define $\mathcal{P}_0$ to be the subset of $\mathcal{P}$ whose underlying bundle can be represented by $\mathcal{O}(0) \oplus \mathcal{O}(d)$. Thus, $\mathcal{P}_0 = \{\, \langle \emptyset \rangle \,\}$.

Proposition 5.1. *When $c > 0$, each of the moduli spaces $\mathcal{P}^s_c$ is a non-empty connected quasi-projective variety whose closure $\mathcal{P}_c$ is an irreducible component of $\mathcal{P}$. The set of S-equivalence classes of semi-stable, but not stable, parabolic bundles in $\mathcal{P}$ are in bijective correspondence with the even cardinality subsets of P by the above construction. Moreover, the closure $\mathcal{P}_c$ is explicitly*

$$\mathcal{P}_c = \mathcal{P}^s_c \sqcup \coprod_{\substack{Q \subseteq P \\ \text{even } d_Q \le -2c}} \langle Q \rangle.$$

We see in particular from this proposition that the intersection of all the irreducible components consists of exactly one element

$$\langle \emptyset \rangle = \bigcap_{\frac{d}{2} \le c \le 0} \mathcal{P}_c = \mathcal{P}_0$$

and hence, that $\mathcal{P}$ is in fact connected.

Proof. First we shall establish that for each $\frac{d}{2} \le c < 0$, there exist stable parabolic vector bundles whose underlying vector bundle is $\mathcal{O}(c) \oplus \mathcal{O}(d-c)$. The possible set of flags F is parameterised by $(\mathbb{P}^1)^{2g+2}$ corresponding to the $2g+2$ parabolic points. Stability means that for every inclusion $i : \mathcal{O}(e) \to \mathcal{O}(c) \oplus \mathcal{O}(d-c)$, we must have that $e + g + 1 < \frac{1}{2}\big|P\big(i(\mathcal{O}(e))\big)\big|$. We see immediately that for such an inclusion to exist, we must have $e \leq c$, and if $d-c < e \leq c$, then $e = c$ and the inclusion is forced to be $i : \mathcal{O}(c) \to \mathcal{O}(c) \oplus \{0\}$. For $e < d$ the stability condition is trivially satisfied. Hence, the only interesting range for e is $d \leq e \leq d - c$. For a given e in this range let I^c_e be the set of inclusions of $\mathcal{O}(e)$ into $\mathcal{O}(c) \oplus \mathcal{O}(d-c)$, modulo automorphisms of $\mathcal{O}(e)$. We note that I^c_e is an open dense subset of

$$\mathbb{P}\Big(\operatorname{Hom}\big(\mathcal{O}(e), \mathcal{O}(c)\big) \oplus \operatorname{Hom}\big(\mathcal{O}(e), \mathcal{O}(d-c)\big)\Big) \cong \mathbb{P}^{-g-2e},$$

each of which are linearly mapped into the set of flag $(\mathbb{P}^1)^{2g+2}$. Hence, we see that $\cup_{d \leq e \leq d-c} I^c_e$ has codimension g. Since $g \geq 1$, this means there are always flag configurations, which are not contained in the subset $\cup_{d \leq e \leq d-c} I^c_e$. Therefore, the set of flag configurations, which makes $\mathcal{O}(c) \oplus \mathcal{O}(d-c)$ stable is a non-empty, open, dense subset of $(\mathbb{P}^1)^{2g+2}$. Since $\mathcal{P}^s_c$ is the $\operatorname{Aut}\big(\mathcal{O}(c) \oplus \mathcal{O}(d-c)\big)$-quotient of this non-empty dense subset, we get the desired conclusion about the parabolic moduli spaces $\mathcal{P}^s_c$.

Suppose we have two parabolic line bundles (L_i, w_i), $i = 1, 2$, of zero parabolic degree, such that $(L_1, w_1) \oplus (L_2, w_2)$ represents the S-equivalence class of a semi-stable parabolic bundle in $\mathcal{P}$. Then we can assume there is an $\frac{d}{2} \leq e \leq 0$ integer such that $L_1 \cong \mathcal{O}(e)$ and $L_2 \cong \mathcal{O}(d-e)$. Since (L_i, w_i), $i = 1, 2$, have zero parabolic degree, there is a subset $Q \subseteq P$ of cardinality $2e$, such that $z \in Q$ if and only if $w_1(z) = \frac{1}{2}$ and $z \in P - Q$ if and only if $w_2(z) = \frac{1}{2}$. Hence, $(L_1, w_1) \oplus (L_2, w_2) \cong \langle Q \rangle$. In order for $\langle Q \rangle$ to be contained in $\mathcal{P}_c$ we just need that $\mathcal{O}\big(d + \frac{d_Q}{2}\big)$ can be included in $\mathcal{O}(c) \oplus \mathcal{O}(d-c)$, which is the case if an only $d + \frac{d_Q}{2} \leq d - c$, i.e. if and only if $d_Q \leq -2c$. □

By analyzing how the automorphism group of the bundle $\mathcal{O}(c) \oplus \mathcal{O}(d-c)$ acts on the flags of the bundle, we arrive at the following dimension formula

$$\dim \mathcal{P}_c = \begin{cases} g - 2c - 1, & \text{if } 0 > c > \frac{d}{2} = -\frac{g+1}{2} \\ 2g - 1, & \text{if } c = \frac{d}{2} = -\frac{g+1}{2}. \end{cases} \tag{5.2}$$

Now, let us consider the ζ_2-action and equivalence relations $\sim_o$ and $\sim$ on P_a:

A moments examining of Definition 3.8 and Lemma 3.9 gives that $\sim$ bijectively relates $P^{ss}_a|_{\Delta_0} = \coprod_{Q \subseteq P, d_Q \text{ even}} \langle\!\langle Q \rangle\!\rangle$ and $P^{ss}_a|_{\Delta_1} = \coprod_{Q \subseteq P, d_Q \text{ even}} \langle Q \rangle$. Namely $\langle\!\langle Q_1 \rangle\!\rangle \sim \langle Q_2 \rangle$ if and only if $Q_1 = Q_2$. Since $P_a|_{\Delta_0} = P^{ss}_a|_{\Delta_0}$, we see that $P_a|_{\Delta_0}$ gets identified with $P^{ss}_a|_{\Delta_1}$. As $n = 2$ the ζ_2-action coincides with the relation $\sim_o$ on P^{ss}_a, the induced relation on P^{ss}_a/ζ_2 is trivial from which it follows that the components of $|M|$ will be normal. In fact, it is not difficult to see that the action is trivial on P^{ss}_a. Since $\sim_o$ and $\sim$ are compatible, we obtain that

$$P_a/(\sim_e, \zeta_2) = \mathcal{P}/\zeta_2,$$

where we note that the ζ_2-action preserves each of the irreducible components $\mathcal{P}_c$.

We conclude:

Proposition 5.3. *Let X be a hyperelliptic curve of genus $g \geq 1$ and let J be a hyperelliptic involution. The fixed point set $|M|$ is decomposed into irreducible components*

$$|M| = \mathcal{P}/\zeta_2 = \bigcup_{\frac{d}{2} \leq c < 0} \mathcal{P}_c/\zeta_2.$$

The irreducible components $\mathcal{P}_c/\zeta_2$, $c > 0$, are normal sub-varieties of $|M|$ of the dimensions stated in formula (5.2)*. They only intersect within the finite subset $|M^{ss}|$, and their intersections are given by Proposition 5.1.* □

We note that the component of $|M|$ of maximal dimension is $\mathcal{P}_c$, with $c = -\frac{g+1}{2}$ in case g is odd and $c = -\frac{g}{2}$ in case g is even. The dimension of this component is $2g - 1$.

Appendix A. The Kernel of $\pi^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$

In this section we calculate the kernel $\ker\{\,\pi^*\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(X)\,\}$ for the covering map $\pi\colon X \to Y = X/\langle\tau\rangle$. We will refer to this kernel as $\ker\pi^*$.

Assume that $D \in \mathrm{Div}(Y)$ is a divisor for which the associated line bundle pulls back to the trivial bundle, $\mathcal{O}_X$, on X, i.e. $\pi^*D = (f)$ for some meromorphic function $f \in \mathcal{M}(X)$. Then using the relation between the pullback π^* and the norm map Nm,

$$n \cdot D = \mathrm{Nm} \circ \pi^*(D) = \mathrm{Nm}\,\big((f)\big) = \big(\mathrm{Nm}(f)\big),$$

we see that every element of the kernel has order at most n; hence, the kernel is a subgroup of the n-torsion points $\mathrm{Pic}_0^{(n)}(Y)$ in $\mathrm{Pic}(Y)$. The analysis of which subgroup, is divided into two cases.

Before we proceed we shall need to prove the following general statement:

Lemma A.1. *Let $\tau\colon X \to X$ be an automorphism of order n of a curve X. Then for any n'th root of unity, μ, there exists a meromorphic function $h \in \mathcal{M}(X)$ such that*

$$h \circ \tau = \mu \cdot h.$$

Proof. The space of meromorphic functions $\mathcal{M}(Y)$ on Y is a subfield of $\mathcal{M}(X)$. In fact as $\pi^*\mathcal{M}(Y) = \mathcal{M}(X)^{\langle\tau\rangle}$ are the τ-invariant meromorphic functions, it follows from a theorem of Artin, [16, Theorem VIII.1.8], that $\mathcal{M}(X)$ is a cyclic Galois extension of $\mathcal{M}(Y)$ of degree n. It is then a consequence of Hilbert's Theorem 90, [16, Theorem VIII.6.1], that there is a function $h \in \mathcal{M}(X) - \{0\}$ such that the element $\varepsilon \in \zeta_n \subset \mathcal{M}(X)$ satisfies that $\varepsilon = h/(h \circ \tau)$. I.e. $h \circ \tau = \mu \cdot h$, if $\mu = \varepsilon^{-1}$. □

Using Remark 2.1 we split up the calculation of the kernel into two; one regarding the unramified case and the other the *completely* ramified case:

Lemma A.2. *Let $\tau\colon X \to X$ be an automorphism of order n without any special orbits, so that the covering projection $\pi\colon X \to Y = X/\langle\tau\rangle$ is unramified. Then there is a line bundle $\bar{L}_\pi \in \mathrm{Pic}(Y)$ of order n such that*

$$\ker\pi^* = \langle \bar{L}_\pi \rangle.$$

Proof. Let $\mu \in \zeta_n$ be a prime root of unity and $h \in \mathcal{M}(X)$ the function from Lemma A.1 such that $h \circ \tau = \mu \cdot h$. Then the norm map of h is given by

$$\mathrm{Nm}(h)(y) = \prod_{i=0}^{n-1} h\big(\tau^i(x)\big) = \mu^{\sum_{i=0}^{n-1} i} \cdot h(x)^n = \mu^{\frac{n(n-1)}{2}} \cdot h(x)^n = (-1)^{n-1} h(x)^n,$$

for any $x \in \pi^{-1}(y)$. We see that the divisor $\big(\mathrm{Nm}(h)\big)$ is divisible by n. Notice also that the pullback is given by $\pi^*\,\mathrm{Nm}(h)(x) = \mathrm{Nm}(h)\big(\pi(x)\big) = (-1)^{n-1}h(x)^n$. Let $D \in \mathrm{Div}(Y)$ be such that $nD = \big(\mathrm{Nm}(h)\big)$, then

$$n\pi^*D = \big(\pi^*\,\mathrm{Nm}(h)\big) = \big((-1)^{n-1}h^n\big) = (h^n) = n(h),$$

and thus π^*D is a principal divisor with $\pi^*D = (h)$; i.e. $[D] \in \ker\pi^*$.

Suppose that $E \in \mathrm{Div}(Y)$ with $[E] \in \ker\pi^*$. Then $\pi^*E = (f)$ for some $f \in \mathcal{M}(X)$. As π^*E is τ-invariant, there is a root of unity $\mu' = \mu^j$, $0 \le j \le n-1$, so that $f \circ \tau = \mu' \cdot f$, and $\pi^*(jD - E) = \left(\frac{h^j}{f}\right)$. But

$$\frac{h^j}{f} \circ \tau = \frac{\mu' \cdot h^j}{\mu' \cdot f} = \frac{h^j}{f},$$

so there must be a $g \in \mathcal{M}(Y)$ with $\pi^* g = \frac{h^j}{f}$, and in that case

$$jD - E = (g).$$

This means that $\ker \pi^* = \langle [D] \rangle$.

We know that $[D]$ has order at most n since $nD = \big(\mathrm{Nm}(h)\big)$, so suppose that $jD = (g)$ for some $g \in \mathcal{M}(Y)$. Then $(\pi^* g) = (h^j)$ in which case there must exist a $\mu' \in \mathcal{O}^*(X) = \mathbb{C}^*$ such that $\pi^* g = \mu' \cdot h^j$. But $\pi^* g$ is τ-invariant while $h^j \circ \tau = \mu^j \cdot h^j$, so $\pi^* g = \mu' \cdot h^j$ if and only if $j = 0$ mod n. This means $\bar{L}_\pi = [D]$ is indeed of order n. □

Notice that the meromorphic function h in the proof is not unique, since we can modify it by the pullback of any meromorphic function on Y. Hence, D isn't unique either, but there is a unique class of linearly equivalent divisors, and thus, $\bar{L}_\pi$ is in fact unique given $\mu \in \zeta_n$.

As a final remark to this we observe that if $n = n_1 \cdot n_2$ then an unramified π factors through $\pi_1 \colon X \to X_1 = X/\langle \tau^{n_2} \rangle$ and the induced $\pi_2 \colon X_1 \to Y$. Then there are divisors D, $D_2 \in \mathrm{Div}(Y)$ and $D_1 \in \mathrm{Div}(X_1)$ as above, such that $\ker \pi^* = \langle [D] \rangle$, $\ker \pi_2^* = \langle [D_2] \rangle$ and $\ker \pi_1^* = \langle [D_1] \rangle$, where $D_2 = n_1 D$ and $\pi_2^* D = D_1$. This is because the meromorphic functions h, $h_1 \in \mathcal{M}(X)$ and $h_2 \in \mathcal{M}(X_1)$ can be chosen as $h_1 = h$ and $h_2 = \mathrm{Nm}_{\pi_1}(h)$.

Lemma A.3. *Let $\tau \colon X \to X$ be an automorphism such that the projection $\pi \colon X \to Y = X/\langle \tau \rangle$ is a ramified covering and such that the orbit lengths $\{ k(y) \mid y \in Y \}$ are co-prime. Then the pullback*

$$\pi^* \colon \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(X)$$

is injective.

Proof. The proof of this statement is essentially due to Mumford in [20]. Let $q \colon L \to Y$ be a line bundle which is a n-torsion point in $\mathrm{Pic}(Y)$, and define the curve

$$X_L = \{ s \in L \mid s^n = 1 \in L^n \cong \mathcal{O}_Y \}$$

and the unramified covering

$$\pi_L \colon X_L \longrightarrow Y.$$

Notice that a global non-zero section s of L, if it exists, can be scaled to satisfy $s^n = 1$, so that it gives a global section of X_L. On the other hand, a global section of X_L is a global section of L. Hence, L is a trivial bundle if and only if X_L is a trivial covering. Since

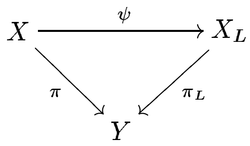
$$\pi^* X_L = \{ (x, \hat{x}) \in X \times X_L \mid \pi(x) = \pi_L(\hat{x}) \} = \{ \hat{s} \in \pi^* L \mid \hat{s}^n = 1 \} = X_{\pi^* L}$$

it follows by the same argument that $\pi^* L$ is a trivial line bundle if and only if $\pi^* X_L$ is a trivial covering of X.

Suppose now that $L \in \ker \pi^*$ so that $\pi^* X_L$ is trivial and $\varphi \colon X \times \{ 1, \dots n \} \to \pi^* X_L$ is a trivialisation. That gives rise to a commutative diagram

$$\begin{array}{ccccc}
X \times [n] & \xrightarrow[\cong]{\varphi} & \pi^* X_L & \xrightarrow{\mathrm{pr}_2} & X_l \\
\Big\downarrow{\scriptstyle \mathrm{pr}} & & \Big\downarrow{\scriptstyle \mathrm{pr}_1} & & \Big\downarrow{\scriptstyle \pi_L} \\
X & = \!= \!= & X & \xrightarrow{\pi} & Y
\end{array}$$

where $[n] = \{1, \ldots, n\}$. The projection pr has obvious sections $\sigma\colon X \to X \times [n]$, and the composition $\psi = \mathrm{pr}_2 \circ \varphi \circ \sigma$ is a morphism of coverings



Now for every $x \in X$ there is an n'th root of unity $\mu(x)$ so that $\psi \circ \tau(x) = \mu(x) \cdot \psi(x)$. If $k(x) = k \circ \pi(x)$ denotes the length of the orbit through x, $\tau^{k(x)}(x) = x$ so $\mu^{k(x)} = 1$ for all x which means that $\operatorname{ord} \mu(x) \mid k(x)$. By the basic assumption that the action of τ does not split into components, it follows that for each of the finitely many $k(x) \in \{ k(y) \mid y \in Y \}$ and any component X_α of X there is an $x' \in X_\alpha$ through which there is an orbit of length $k(x') = k(x)$. By continuity, μ is constant on each component of X and thus $\operatorname{ord} \mu \mid k(x)$ for all x, so $\operatorname{ord} \mu \mid \gcd\{ k(y) \mid y \in Y \} = 1$. Hence, $\mu = 1$ which in turn means that $|\operatorname{Im} \psi \cap \pi_L^{-1}(x)| = 1$ for every $x \in X$. But

$$n = \deg \pi = \deg(\pi_L|_{\operatorname{Im} \psi}) \cdot \deg \psi = \deg(\pi_L|_{\operatorname{Im} \psi}) \cdot n,$$

so $\deg(\pi_L|_{\operatorname{Im} \psi}) = 1$ and

$$(\pi_L|_{\operatorname{Im} \psi})^{-1}\colon Y \longrightarrow \operatorname{Im} \psi \subseteq X_L$$

is a global section. Hence, $L \cong \mathcal{O}_Y$. □

Combining Lemma A.2 and Lemma A.3 we get the general statement:

Proposition A.4. *Let X be a smooth algebraic curve and $\tau\colon X \to X$ an automorphism of order n with possible special orbits and let $\pi\colon X \to Y = X/\langle \tau \rangle$ be the induced covering. Then there is a line bundle $\bar{L}_\pi$ over Y such that*

$$\ker \pi^* = \langle \bar{L}_\pi \rangle,$$

and the order of $\bar{L}_\pi$ is the greatest common divisor, $r = \gcd\{ k(y) \mid y \in Y \}$, of the orbit lengths. □

References

[1] J.E. Andersen, *The Witten invariant of finite order mapping tori I* (to appear in J. Reine und angew. Math.).

[2] J.E. Andersen and K. Ueno, *Constructing topological quantum field theories from conformal field theories*, in preparation.

[3] M.F. Atiyah, *Topological quantum field theories*, Publ. Math. Inst. Hautes Etudes Sci. Paris **68** (1989), 175–186.

[4] S. Axelrod, S. Della Pietra and E. Witten, *Geometric quantization of Chern Simons gauge theory*, J. Diff. Geom. **33** (1991), 787–902.

[5] P. Baum, W. Fulton and R. MacPherson, *Riemann-Roch and topological K-theory for singular varieties*, Acta Mathematica **143** (1979), 155–192.

[6] P. Baum, W. Fulton, G. Quart, *Lefschetz-Riemann-Roch for singular varieties*, Acta Mathematica **143** (1979), 193–211.

[7] A. Beauville and Y. Laszlo, *Conformal blocks and generalized theta functions*, Comm. Math. Phys. **164** (1994), no. 2, 385–419.

[8] G. Faltings, *Stable G-bundles and projective connctions*, J. Alg. Geom. **2** (1993), 507–568.

[9] J. Grove, *Constructing TQFTs from Modular Functors*, J. Knot Th. Ram. **10** (2001), no. 8, 1085–1131.

[10] A. Grothendieck, *Élément de géométrie algébrique III, Étude cohomologique des faisceaux cohérents*, Publications Mathématiques, No 11, Institut des Hautes Études Scientifiques, 5, Rond-Point Bugeaud - Paris (XVIe), 1961.

[11] ———, *Fondements de la géométrie algébrique, exposé 221*, Extraits du Séminaire Bourbaki 1957–1962, Secrétariat mathématique, Paris, 1962.

[12] R. Hartshorne, *Algebraic geometry*, GTM 52, Springer-Verlag, New York, 1977.

[13] N. Hitchin, *Flat connections and geometric quantization*, Comm. Math. Phys. **131** (1990), 347–380.

[14] L.C. Jeffrey, Y-H. Kiem, F. Kirwan and J. Woolf, *Cohomology pairings on singular quotients in geometric invariant theory*, Preprint, math.AG/0101079 (2001).

[15] S. Kobayashi, *Differential geometry of complex vector bundles*, Publications of the Mathematical Society of Japan, 15. Kanô Memorial Lectures, 5, Princeton University Press, Princeton, NJ; Iwanami Shoten, Tokyo, 1987.

[16] S. Lang, *Algebra*, 2nd edition, Addison Wesley, 1984.

[17] Y. Laszlo, *Hitchin's and WZW connections are the same*, J. Diff. Geom. **49** (1998), no. 3, 547–576.

[18] V.B. Mehta and C.S. Seshadri, *Moduli of vector bundles on curves with parabolic structures*, Math. Ann. **248** (1980), 205–239.

[19] D. Mumford, *Geometric invariant theory*, Ergebnisse der Mathematik, Band 34, Springer Verlag, Berlin-Heidelberg, 1965.

[20] ———, *Prym varieties I*, Contribut. to Analysis, Collect. of Papers dedicated to Lipman Bers (L.V. Ahlfors, I. Kra, B. Maskit and L. Nirenberg, eds.), Academic Press, New York and London, 1974, pp. 325–350.

[21] M.S. Narasimhan and S. Ramanan, *Generalised Prym varieties as fixed points*, J. Indian Math. Soc. (N.S.) **39** (1975), 1–19.

[22] G. Quart, *Localization theorem in Ktheory for singular varieties*, Acta Mathematica **143** (1979), 213–217.

[23] C.S. Seshadri, *Fibrés vectoriels sur les courbes algébriques*, Notes by J.-M. Drezet, Astérisque **96** (1982), 209p.

[24] M. Thaddeus, *Conformal field theory and the cohomology of the moduli spaces of stable bundles*, J. of Diff. Geom. **35** (1992), 131–150.

[25] A. Tsuchiya, K. Ueno and Y. Yamada, *Conformal field theory on universal families of stable curves with gauge symmetries*, Advanced Studies in Pure Math. **19** (1989), 459–566.

[26] K. Walker, *On Witten's 3-manifold invariants, Preliminary version # 2*, Preprint (1991).

[27] E. Witten, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989), 351–399.

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