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# MOD $p$ HOMOLOGY OF THE STABLE MAPPING CLASS GROUP

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# MOD $p$ HOMOLOGY OF THE STABLE MAPPING CLASS GROUP

SØREN GALATIUS

Abstract. By a recent result of Madsen and Weiss, the classifying space $B\Gamma_\infty$ of the stable mapping class group is homology equivalent to a component of the space $\Omega^\infty \mathbb{C}P^\infty_{-1}$. In this paper, we compute the homology algebra $H_*(\Omega^\infty \mathbb{C}P^\infty_{-1}; \mathbb{F}_p)$ and hence the group homology $H_*(\Gamma_\infty; \mathbb{F}_p)$.

Contents



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## 1. Introduction

Let $F_{g,1+1}$ be a surface of genus $g$ with two boundary components, and $\mathrm{Diff}^+(F_{g,1+1})$ the topological group of boundary and orientation preserving diffeomorphisms of $F_{g,1+1}$. The space

$$\coprod_{g\geq 0} B\mathrm{Diff}(F_{g,1+1},\partial)$$

classifies fibre bundles of surfaces with trivial boundary. Gluing along the boundary makes this space a topological monoid, and we can form the group completion

$$\mathbb{Z}\times B\Gamma_\infty^+ := \Omega B\Big(\coprod_{g\geq 0} B\mathrm{Diff}(F_{g,1+1},\partial)\Big)$$

By a recent result of Madsen and Weiss ([MW]), this space is homotopy equivalent to another space $\Omega^\infty\mathbb{C}P^\infty_{-1}$ which we proceed to define.

Let $L_n^\perp = \{(v,L)\in\mathbb{C}^{n+1}\times\mathbb{C}P^n \mid v\in L^\perp\}$ be the orthogonal complement of the canonical line bundle and let $\mathrm{Th}(L_n^\perp)$ be its Thom space. Since $L_{n+1}^\perp|_{\mathbb{C}P^n} = L_n^\perp\oplus\underline{\mathbb{C}}$, there are stabilisation maps

$$S^2\wedge\mathrm{Th}(L_n^\perp)\to\mathrm{Th}(L_{n+1}^\perp)$$

defining a (pre-)spectrum $\mathbb{C}P^\infty_{-1}$ with

$$(\mathbb{C}P^\infty_{-1})_{2n+2} = \mathrm{Th}(L_n^\perp)$$

Thus if $L$ denotes the canonical line bundle over $\mathbb{C}P^\infty$ and $-L$ its inverse virtual bundle, then the Thom class $\lambda_{-L}$ sits in degree $-2$.

The space

$$\Omega^\infty\mathbb{C}P^\infty_{-1} = \operatorname{colim}\Omega^{2n+2}(\mathbb{C}P^\infty_{-1})_{2n+2}$$

is the associated infinite loop space and

$$\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1} = \operatorname{colim}\Omega^{2n+1}(\mathbb{C}P^\infty_{-1})_{2n+2}$$

is its first deloop.

The precise statement of the Madsen-Weiss theorem is that a certain map $\mathbb{Z}\times B\Gamma_\infty^+\to\Omega^\infty\mathbb{C}P^\infty_{-1}$ is a homotopy equivalence. By a theorem of Harer and Ivanov, the homology in a degree of the unstable groups $\Gamma_{g,b}=\pi_0\mathrm{Diff}(F_{g,b})$ is independent of $g$ and $b$ for $g$ sufficiently large. Thus $H^*(B\Gamma_\infty;\Lambda) = H^*(\Omega_0^\infty\mathbb{C}P^\infty_{-1};\Lambda)$ classifies *stable* characteristic classes of surface bundles. In this paper we calculate the homology algebra $H_*(\Omega_0^\infty\mathbb{C}P^\infty_{-1};\mathbb{F}_p)$ for any prime $p$.

Along the way, we compute the homology Hopf algebra $H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1};\mathbb{F}_p)$ of the first deloop. This space enters the theory of high-dimensional manifolds via the cyclotomic trace

$$\mathrm{trc}: A(*)\to TC(*)$$

in which the codomain $TC(*)$ is homotopy equivalent to $QS^0\times\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}$ after profinite completion.

1.1. **Outline.** The methods used for the calculation are very classical. The starting point is a cofibration sequence

$$S^{-2} \to \mathbb{C}P^{\infty}_{-1} \to \Sigma^{\infty}\mathbb{C}P^{\infty}_{+}$$

of spectra, in which the map $S^{-2} \to \mathbb{C}P^{\infty}_{-1}$ is induced by the inclusion of a fibre $\mathbb{C}^n \to L_n^{\perp}$ and the map $\mathbb{C}P^{\infty}_{-1} \to \Sigma^{\infty}\mathbb{C}P^{\infty}_{+}$ is induced by the zero section of $L_n$: $\mathrm{Th}(L_n^{\perp}) \to \mathrm{Th}(L_n^{\perp} \oplus L_n) = \mathbb{C}P^n_+ \wedge S^{2n+2}$.

This cofibration sequence induces a fibration sequence of the associated infinite loop spaces

$$\Omega^{\infty}\Sigma\mathbb{C}P^{\infty}_{-1} \xrightarrow{\omega} Q(\Sigma\mathbb{C}P^{\infty}_{+}) \xrightarrow{\partial} QS^0 \tag{1.1}$$

where $Q(X) = \Omega^{\infty}\Sigma^{\infty}(X)$ for a pointed space $X$. Both $H_*(Q_0S^0)$ and $H_*(Q\Sigma\mathbb{C}P^{\infty}_{+})$ are known, as is the induced map $\partial_*$ in homology.

In section 2, we recall the needed results about the Eilenberg-Moore spectral sequence and the functor Cotor. In section 3, we recall the definition of the Dyer-Lashof algebra $R$ and the category of unstable $R$-modules and the free functor $D$ from vectorspaces to unstable $R$-modules. We recall the expression of $H_*(QX)$ as a functor of $H_*(X)$, needed for the cases $X = \Sigma\mathbb{C}P^{\infty}_{+}$ and $X = S^0$. Finally, we introduce a new algebra $\mathscr{R}$ projecting to $R$ and a corresponding category of unstable $\mathscr{R}$-modules. This algebra is a "free" version of $R$, and is used to keep track of relations in $R$.

Section 4 contains the core of the calculation, namely the algebraic computations in $R$ where the map $\partial_*$ is studied from a homological algebraical viewpoint. In the sections 5 and 6 these results are applied to compute the Eilenberg-Moore spectral sequences converging to homology of $\Omega^{\infty}\mathbb{C}P^{\infty}_{-1}$ and $\Omega^{\infty}\Sigma\mathbb{C}P^{\infty}_{-1}$.

Finally in section 7 we carry out the details of the computation at the prime 2.

1.2. **Acknowledgements.** This calculation is part of my phd-project at the University of Aarhus. It is a great pleasure to thank my thesis advisor Ib Madsen for his help and encouragement during my years as a graduate student.

## 2. Recollections

In this introductory section we collect the results we need later in the paper. We start by recalling some important results on the structure of Hopf algebras from [MM] and proceed to review the functor Cotor and the closely related Eilenberg-Moore spectral sequence, cf. [EM], [MS].

2.1. **Hopf algebras.** Here and elsewhere, the field $\mathbb{F}_p$ with $p$ elements is the ground field, and $\otimes = \otimes_{\mathbb{F}_p}$. Until further notice, $p$ is assumed odd. Algebras and coalgebras are as in [MM] and in particular they always have units resp. counits.

**Definition 2.1.** When $A$ is a coalgebra and $M_A$, ${}_AN$ are $A$-comodules with structure maps $\Delta_M : M \to M \otimes A$ and $\Delta_N : N \to A \otimes N$, the cotensor product is defined by the exact sequence

$$0 \longrightarrow M\Box_A N \longrightarrow M \otimes N \longrightarrow M \otimes A \otimes N$$

where the right-hand morphism is $\Delta_M \otimes N - M \otimes \Delta_N$. The functors $M\Box_A-$ and $-\Box_A N$ are left exact functors from $A$-comodules to $k$-vectorspaces in general, and to $A$-comodules when $A$ is cocommutative.

**Definition 2.2.** For a morphism $f : A \to B$ of Hopf algebras, define the *kernel* and *cokernel*

$$A\backslash\!\!\backslash f = A\Box_B k, \quad B/\!\!/ f = B \otimes_A k$$

A priori, the kernel and cokernel are vectorspaces, but when $A$ and $B$ are commutative and cocommutative, they become Hopf algebras and are the kernel and cokernel in the categorical sense. Hopf algebras that are both commutative and cocommutative are called *abelian*, and the category of those is an abelian category (this essentially follows from [MM, Prop. 4.9]).

All Hopf algebras appearing in this paper will be abelian, of finite type and *connected*, i.e. in degree zero $A_0 = k$ is a copy of the ground field (except for $\mathscr{R}$ and $R$ that are not commutative and not connected). We cite results for this class of Hopf algebras, although some of the results are valid for a larger class of Hopf algebras.

**Definition 2.3.** For an augmented algebra $A$, $IA = \mathrm{Ker}(\varepsilon : A \to k)$ and dually for an augmented coalgebra $A$, $JA = \mathrm{Cok}(\eta : k \to A)$. Let $Q$ and $P$ be the functors defined by the exact sequences

$$IA \otimes IA \xrightarrow{\varphi} IA \longrightarrow QA \longrightarrow 0$$

and

$$0 \longrightarrow PA \longrightarrow JA \xrightarrow{\Delta} JA \otimes JA$$

As functors from abelian Hopf algebras to vectorspaces, $Q$ is right exact and $P$ is left exact ([MM, Prop 4.10]).

When $A$ is connected, $PA \subseteq A$ is the subset of elements $x$ satisfying $\Delta x = x \otimes 1 + 1 \otimes x$.

The functors $P$ and $Q$ are related by the short exact sequence of [MM, Thm. 4.23]:

**Theorem 2.4.** *For an abelian Hopf algebra $A$, let $\xi : A \to A$ be the Frobenius map $x \mapsto x^p$ and let $\lambda : A \to A$ be the dual of $\xi : A^* \to A^*$. Let $\xi A \subseteq A$ be the image of $\xi$ and let $A \to \lambda A$ be the coimage of $\lambda$. Then there is the following natural exact sequence*

$$0 \longrightarrow P\xi A \longrightarrow PA \longrightarrow QA \longrightarrow Q\lambda A \longrightarrow 0 \tag{2.1}$$

*In particular* $PA \to QA$ *is an isomorphism except possibly in degrees* $\equiv 0 \pmod{2p}$.

Finally, we recall Borel's structure theorem ([MM, Theorem 7.11])

**Theorem 2.5.** *Any Hopf algebra $A$ is isomorphic as an algebra to a tensor product of Hopf algebras of the form $E[x]$, $\mathbb{F}_p[x]$ and $\mathbb{F}_p[x]/(x^{p^n})$, with $x$ primitive.*

**Corollary 2.6.** *$A$ is isomorphic as an algebra to a polynomial algebra if and only if $\xi : A \to A$ is injective. Dually, $A^*$ is polynomial if and only if $\lambda : A \to A$ is surjective.* $\square$

2.2. **The functor** Cotor**.** When $A$ is a coalgebra and $B$ and $C$ are left resp. right $A$-comodules, the functor

$$\mathrm{Cotor}^A(B, C)$$

is defined as the right derived functor of the cotensor product $\square_A$. To be explicit (and to fix grading conventions), choose an injective resolution $0 \to B \to I_0 \to I_{-1} \to \dots$ of $B$ in the category of right $A$-comodules and set

$$\mathrm{Cotor}^A_n(B, C) = H_n(I_* \square_A C)$$

When $A$, $B$ and $C$ are in the graded category, Cotor gets an inner grading and is thus bigraded with $\mathrm{Cotor}^A_{n,m}(B, C) = (\mathrm{Cotor}^A_n(B, C))_m$. When $A, B, C$ are all positively graded, Cotor is concentrated in the second quadrant.

When $A$, $B$ and $C$ are of finite type over a field, this functor is dual to the more common Tor:

$$\mathrm{Cotor}^A(B, C) = \left(\mathrm{Tor}^{A^*}(B^*, C^*)\right)^*$$

This follows immediately from the duality between $\square_A$ and $\otimes_{A^*}$.

We shall consider Cotor as a functor from diagrams of cocommutative coalgebras

$$\mathscr{S} = \left\{ \begin{array}{ccc} & & B \\ & & \downarrow \\ C & \longrightarrow & A \end{array} \right\}$$

to coalgebras. The external product is an isomorphism (see [CE, Theorem 3.1, p. 209])

$$\mathrm{Cotor}^A(B, C) \otimes \mathrm{Cotor}^{A'}(B', C') \to \mathrm{Cotor}^{A \otimes A'}(B \otimes B', C \otimes C')$$

and under this isomorphism the comultiplication in $\mathrm{Cotor}^A(B, C)$ is given by the comultiplication $\Delta : \mathscr{S} \to \mathscr{S} \otimes \mathscr{S}$ in the diagram $\mathscr{S}$.

Dually, when $\mathscr{S}$ is a diagram of Hopf algebras, $\mathrm{Cotor}^A(B, C)$ is a Hopf algebra with multiplication induced by the multiplication $\varphi : \mathscr{S} \otimes \mathscr{S} \to \mathscr{S}$ of the diagram $\mathscr{S}$.

Later we will need the structure of $\mathrm{Cotor}^A(B, k)$ where $k = \mathbb{F}_p$ is the trivial Hopf algebra and $f : B \to A$ is a morphism of Hopf algebras. From the change of rings spectral sequence and [MM, Theorem 4.9] we get

**Proposition 2.7.** *For a map $f : B \to A$ of Hopf algebras, there is a natural isomorphism of Hopf algebras*

$$\mathrm{Cotor}^A(B,k) \xrightarrow{\cong} B\backslash\!\!\backslash f \otimes \mathrm{Cotor}^{A/\!\!/f}(k,k)$$

□

To complete the description of $\mathrm{Cotor}^A(B,k)$ we need to compute $\mathrm{Cotor}^A(k,k)$. This is easily done by applying Borel's structure theorem to the dual algebra $A^*$ and using Lemma 2.8 below. The Hopf algebra $\Gamma[x]$ is dual to a polynomial algebra: $\Gamma[x] = (k[x^*])^*$ and $s^{-\nu}$ denotes bigraded desuspension: $(s^{-\nu}V)_{-\nu,n} = V_n$ for a singly graded object $V$.

**Lemma 2.8.** *The following isomorphisms hold as Hopf algebras*

$$\begin{aligned}\mathrm{Tor}^{E[x]}(k,k) &= \Gamma[s^{-1}x]\\ \mathrm{Tor}^{k[x]}(k,k) &= E[s^{-1}x]\\ \mathrm{Tor}^{k[x]/(x^{p^n})}(k,k) &= E[s^{-1}x]\otimes\Gamma[s^{-2}x^{p^n}]\end{aligned}$$

*Proof.* Write down resolutions. □

By the duality between Tor and Cotor we obtain the Hopf algebra structure of $\mathrm{Cotor}^A(k,k)$ in terms of a set of generators of the dual algebra $A^*$.

**Corollary 2.9.** *For any Hopf algebra $A$, $\mathrm{Cotor}^A(k,k)$ is a free commutative, primitively generated Hopf algebra. The generators of $\mathrm{Cotor}^A(k,k)$ are in bidegrees*

$$\begin{aligned}(-1,k) &\quad \text{for } x\in A^*_k \text{ an odd generator}\\ (-1,k) &\quad \text{for } x\in A^*_k \text{ an even generator}\\ (-2,p^m k) &\quad \text{for } x\in A^*_k \text{ an even generator of height } p^m\end{aligned}$$

*The primitive elements of $\mathrm{Cotor}^A(k,k)$ are in bidegrees*

$$\begin{aligned}p^n(-1,k) &\quad \text{for } x\in A^*_k \text{ an odd generator}\\ (-1,k) &\quad \text{for } x\in A^*_k \text{ an even generator}\\ p^n(-2,p^m k) &\quad \text{for } x\in A^*_k \text{ an even generator of height } p^m\end{aligned}$$

□

In a more functorial formulation, $\mathrm{Cotor}^A(k,k)$ is the free commutative algebra on the vectorspace $P_{-1}A \oplus P_{-2}A$, where $P_\nu$ denotes the primitive elements of bidegree $(\nu,*)$ in $\mathrm{Cotor}^A(k,k)$.

In particular, the only primitive elements of odd total degree are in bidegrees $(-1,k)$ for even generators $x \in A^*_k$.

*Remark* 2.10. The functorial formulation of the above theorem, and the definition of $P_{-1}$ and $P_{-2}$ is due to [MS]. The functor $P_{-1}$ is naturally isomorphic to the functor $P$, and the functor $P_{-2}$ measures truncations in the dual algebra. $P_{-1}$ and $P_{-2}$ constitute a delta-functor from abelian Hopf algebras to vectorspaces ([MS, Cor. 4.11]).

Finally, we shall need a criterion for left exactness of the functor $Q$, namely

**Proposition 2.11.** *Let*

$$k \to A \to B \to C \to k$$

*be a short exact sequence of abelian Hopf algebras. If $C$ is a free commutative algebra, then the sequence*

$$0 \to QA \to QB \to QC \to 0$$

*is short exact.*

*Proof.* Since $C$ is free, we may split $B \to C$ with a map of algebras. Thus $B \cong A \otimes C$ as an algebra, and $Q(B)$ depends only on the algebra structure of $B$. $\square$

A peculiar consequence of Corollary 2.6 is that if $A$ is a Hopf algebra that is free as an algebra, then any Hopf subalgebra of $A$ is also free as an algebra.

2.3. **The spectral sequence.** In this section, we recall the spectral sequence of [EM] and some of its properties.

We consider *homotopy cartesian squares*

$$\mathscr{C} = \left\{ \begin{array}{ccc} F & \longrightarrow & E \\ \downarrow & & \downarrow \\ X & \longrightarrow & B \end{array} \right\}$$

of connected spaces, and with $B$ simply connected (homotopy cartesian means that $F = \operatorname{holim}(X \to B \leftarrow E)$. One can always find a model that is a *fibre square*, i.e. where $E \to B$ is a fibration, and $F \to X$ is the pullback fibration).

**Definition 2.12.** The Eilenberg-Moore spectral sequence $E^r$ is a functor from fibre squares $\mathscr{C}$ as above to spectral sequences of coalgebras. It has

$$E^2 = \operatorname{Cotor}^{H_*(B)}(H_*(E), H_*(X))$$

and converges as coalgebra to $H_*F$.

**Theorem 2.13** ([EM, Proposition 16.4])**.** *The external product induces an isomorphism*

$$E^r(\mathscr{C}) \otimes E^r(\mathscr{C}') \to E^r(\mathscr{C} \times \mathscr{C}')$$

*Under this isomorphism, the coalgebra structure is induced by the diagonal $\Delta : \mathscr{C} \to \mathscr{C} \times \mathscr{C}$.* $\square$

Dually, when $\mathscr{C}$ is a diagram of $H$-spaces and $H$-maps (here meaning maps commuting *strictly* with the multiplication such as loop spaces and loop maps), there is a multiplication $m : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ inducing a multiplication $\varphi = m_* : E^r(\mathscr{C}) \otimes E^r(\mathscr{C}) \to E^r(\mathscr{C})$. In this case, the spectral sequence is one of Hopf algebras. Furthermore it is clear that on the $E^2$-term the Hopf algebra structure is the same as the one on Cotor described above.

2.4. **The loop suspension.** We shall use the spectral sequence only in the case when $X$ is a point. This corresponds to a fibration

$$F \to E \to B$$

and the spectral sequence computes homology of the fibre. When $E$ is also a point, we have the path-loop fibration

$$\Omega X \to * \to X$$

In this case, the fibre line

$$E^2_{0,*} = \mathrm{Cotor}^{H_*(X)}_{0,*}(k,k) = k \square_{H_*(X)} k = k$$

is concentrated in degree 0 and hence there is a "secondary edge homomorphism"

$$H_*(\Omega X) \to E^\infty_{-1,*} \hookrightarrow E^2_{-1,*} \cong PH_*X \tag{2.2}$$

**Proposition 2.14** ([S, Proposition 4.5])**.** *The morphism in* (2.2) *is the loop suspension*

$$\sigma_* : QH_*(\Omega X) \to PH_*X$$

□

We shall also need

**Lemma 2.15.** *Let $C_*$ be a connected differential graded Hopf algebra. If $x$ is an element of minimal degree with $dx \neq 0$, then $x$ is indecomposable and $dx$ is primitive.*

*Proof.* Immediate from the Leibniz rules for product and coproduct. □

**Corollary 2.16.** *Minimal differentials in the spectral sequence of a path-loop fibration correspond to minimal elements in the cokernel of $\sigma_*$.*

*Proof.* Since $dx$ is primitive and not in $E^2_{-1,*}$ it is of even total degree. Hence $x$ is of odd. By Lemma 2.8, the only odd dimensional indecomposable elements are in $E^2_{-1,*}$ and the result follows. □

## 3. Unstable $R$-modules

In this section, we define several categories of graded vectorspaces with a set of linear transformations $\{\beta^\varepsilon Q^s \mid \varepsilon \in \{0,1\}, s \in \mathbb{Z}_{\geq\varepsilon}\}$ of degree $2(p-1)-\varepsilon$. Most of the material can be found in [CLM] or [DL]. These categories and some forgetful functors fit in a diagram:

$$\begin{tikzcd}
Q\text{-unstable } R\text{-modules} \arrow[r] \arrow[d] & Q\text{-unstable } \mathscr{R}\text{-modules} \arrow[d] \\
\text{unstable } R\text{-modules} \arrow[r] & \text{unstable } \mathscr{R}\text{-modules} \arrow[d] \\
 & \mathscr{R}\text{-modules} \arrow[d] \\
 & \text{graded vectorspaces}
\end{tikzcd} \tag{3.1}$$

Here, $\mathscr{R}$ is the free non-commutative algebra on the set $\{\beta^\varepsilon Q^s \mid \varepsilon \in \{0,1\}, s \in \mathbb{Z}_{\geq\varepsilon}\}$, and the various entries in (3.1) differ in what relations the action of the operations $\beta^\varepsilon Q^s$ are assumed to satisfy. It is the left part of the diagram that is geometrically relevant, since the homology of an infinite loop space $X$ is naturally an unstable $R$-modules, and so is the space of primitive elements $PH_*(X)$. The space of indecomposable elements $QH_*(X)$ is naturally a $Q$-unstable $R$-module.

All of the above forgetful functors to graded vectorspaces have left adjoint "free" functors. From $\mathscr{R}$-modules it is the functor $V \mapsto \mathscr{R} \otimes V$, and the other four are quotients thereof.

In 3.2, we define the algebras $\mathscr{R}$ and $R$ and the four categories of unstable modules. In 3.3 we construct the four adjoint functors $\mathscr{D}$, $\mathscr{D}'$, $D$ and $D'$. Finally, in 3.4 we recall the computation of $H_*(QX)$ in terms of $H_*(X)$. It should be noted that the algebra $\mathscr{R}$ and the related categories are needed only in the proof of Theorem 4.4. It is $R$ that is geometrically relevant but it is also more complicated than $\mathscr{R}$.

3.1. **Araki-Kudo-Dyer-Lashof operations.** Recall that an *infinite loop space* is a sequence $E_0, E_1, \ldots$ of spaces and homotopy equivalences $\Omega E_{i+1} \to E_i$. One thinks of $E_0$ as the "underlying space" of the infinite loop space. In particular, $E_0 = \Omega^2 E_2$ is a homotopy commutative $H$-space. Thus $H_*(E_0)$ is a commutative algebra under the Pontrjagin product. Furthermore $H_*(E_0)$ naturally carries a set of linear transformations $Q^s$, $s \geq 0$. These linear transformations are commonly called *Dyer-Lashof operations* (or *Araki-Kudo* operations) and are operations

$$Q^s : H_n(E_0) \to H_{n+2s(p-1)}(E_0)$$

natural with respect to infinite loop maps. They measure the failure of chain level commutativity of the Pontrjagin product.

They satisfy a number of relations that makes $H_*(E_0)$ an *unstable $R$-module*, the notion of which is defined below.

### 3.2. The algebras $\mathscr{R}$ and $R$ and categories of unstable modules.

**Definition 3.1.** Let $\mathscr{R}$ be the free (non-commutative) algebra generated by symbols

$$\beta^\varepsilon Q^s, \quad \varepsilon \in \{0,1\}, s \in \mathbb{Z}_{\geq \varepsilon}.$$

and write $\beta Q^s = \beta^1 Q^s$ and $Q^s = \beta^0 Q^s$. $\mathscr{R}$ is a graded algebra with

$$\deg(\beta^\varepsilon Q^s) = 2s(p-1) - \varepsilon$$

It will occasionally be convenient to consider $\mathscr{R}$ as a *bigraded* algebra with gradings

$$\deg_Q(\beta^\varepsilon Q^s) = 2s(p-1), \quad \deg_\beta(\beta^\varepsilon Q^s) = -\varepsilon$$

$\mathscr{R}$ is a cocommutative Hopf algebra with comultiplication

$$\Delta(\beta^\varepsilon Q^s) = \sum_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon \\ s_1+s_2=s}} \beta^{\varepsilon_1} Q^{s_1} \otimes \beta^{\varepsilon_2} Q^{s_2}$$

**Definition 3.2.** An $\mathscr{R}$-module is called *unstable*, if

$$\beta^\varepsilon Q^s x = 0 \quad \text{whenever } 2s - \varepsilon < \deg(x) \tag{3.2}$$

It is called *$Q$-unstable* if furthermore

$$Q^s x = 0 \quad \text{whenever } 2s = \deg(x) \tag{3.3}$$

For an infinite loop space $X$, $H_*(X)$ is naturally an unstable $\mathscr{R}$-module. However, the ideal in $\mathscr{R}$ of elements with universally trivial action is nonzero, and hence the action of $\mathscr{R}$ on $H_*X$ factors through a quotient of $\mathscr{R}$. This quotient is the *Dyer-Lashof algebra $R$*.

**Definition 3.3.** For each $r, s \in \mathbb{N}$ and $\varepsilon \in \{0,1\}$ with $r > ps$, define elements in $\mathscr{R}$

$$\mathscr{A}^{(\varepsilon,r,0,s)} = \beta^\varepsilon Q^r Q^s - \left(\sum_{i=0}^{r+s} (-1)^{r+i}(pi - r, r-(p-1)s - i - 1)\beta^\varepsilon Q^{r+s-i} Q^i\right)$$

For $r \geq ps$ define elements

$$\begin{aligned}\mathscr{A}^{(0,r,1,s)} = Q^r \beta Q^s - \Bigg(&\sum_{i=0}^{r+s} (-1)^{r+i}(pi - r, r-(p-1)s - i)\beta Q^{r+s-i} Q^i \\ &- \sum_{i=0}^{r+s} (-1)^{r+i}(pi - r - 1, r-(p-1)s - i) Q^{r+s-i}\beta Q^i\Bigg)\end{aligned}$$

and

$$\mathscr{A}^{(1,r,1,s)} = \beta Q^r \beta Q^s - \left(-\sum_{i=0}^{r+s} (-1)^{r+i}(pi - r - 1, r-(p-1)s - i)\beta Q^{r+s-i}\beta Q^i\right)$$

where, $(i,j) = (i+j)!/(i!j!)$. These elements are the socalled *Adem relations.*

Let $\mathscr{A} \subseteq \mathscr{R}$ be the $k$-span of all Adem elements. This is a bigraded subspace of $\mathscr{R}$. Let $\langle \mathscr{A} \rangle \subseteq \mathscr{R}$ be the two-sided ideal generated by $\mathscr{A}$. Let $\mathscr{J} \subseteq \mathscr{R}$ be the two-sided ideal (or equivalently the left ideal) generated by the relations (3.2) (for $x \in \mathscr{R}$). $\mathscr{J}$ is the smallest ideal such that $\mathscr{R}/\mathscr{J}$ is unstable as a left $\mathscr{R}$-module.

**Definition 3.4.** The *Dyer-Lashof algebra* is the quotient

$$R = \mathscr{R}/(\langle \mathscr{A} \rangle + \mathscr{J})$$

The action of $\mathscr{A}$ and hence $\langle \mathscr{A} \rangle$ on homology of infinite loop spaces is trivial by results from [CLM], dual to Adem's result for the Steenrod algebra. So is the action of $\mathscr{J}$, by unstability. Hence $H_*(X)$ is an $R$-module when $X$ is an infinite loop space. Conversely, a theorem of Dyer and Lashof states that the map $R \to H_*(QS^0)$ induced by acting on the zero-dimensional class [1] represented by the identity map of $S^n, n \to \infty$ is an injection, so there are no further relations.

The set of all products of generators form a vector space basis of $\mathscr{R}$. To have an explicit basis for $R$, we recall the notion of admissible monomials, [CLM, p. 16].

A sequence

$$I = (\varepsilon_1, s_1, \dots, \varepsilon_k, s_k)$$

of integers $\varepsilon_i \in \{0,1\}$ and $s_i \in \mathbb{Z}_{\geq \varepsilon_i}$ determines the iterated homology operation

$$Q^I = \beta^{\varepsilon_1} Q^{s_1} \dots \beta^{\varepsilon_k} Q^{s_k} \in \mathscr{R}$$

This sequence is called *admissible* if for all $i = 2, \dots, k$,

$$s_i \leq p s_{i-1} - \varepsilon_{i-1} \tag{3.4}$$

The corresponding iterated homology operations $Q^I \in \mathscr{R}$ are called *admissible monomials.* The *length* and *excess* of $I$ are

$$\ell(I) = k, \quad e(I) = 2s_1 - \varepsilon_1 - \sum_{j=2}^{k} [2s_j(p-1) - \varepsilon_j]$$

Furthermore, define

$$b(I) = \varepsilon_1$$

Using the Adem relations one may rewrite an arbitrary element of $\mathscr{R}$ as a linear combination of admissible monomials in $R$. Applying Adem relations does not raise the excess.

There is a natural quotient map $\mathscr{R} \to R$. Thus $R$-modules are also $\mathscr{R}$-modules.

**Definition 3.5.** An $R$-module is called *unstable*, respectively *Q-unstable*, if it is so as an $\mathscr{R}$-module.

3.3. **Free functors.**

**Definition 3.6.** For a graded vectorspace $V$ we define $\mathscr{D}V$ to be the quotient of $\mathscr{R} \otimes V$ by the relations (3.2) and $\mathscr{D}'V$ to be the quotient of $\mathscr{D}V$ by the relations (3.3). Define also

$$DV = R \otimes_{\mathscr{R}} \mathscr{D}V, \quad D'V = R \otimes_{\mathscr{R}} \mathscr{D}'V$$

The functor $\mathscr{D}$ is left adjoint to the forgetful functor from unstable $\mathscr{R}$-modules to vectorspaces. Thus $\mathscr{D}V$ is the "free unstable $\mathscr{R}$-module" generated by $V$. Similarly, $D$ is left adjoint to the forgetful functor from unstable $R$-modules to graded vectorspaces. Analogous remarks apply to $\mathscr{D}'$ and $D'$. The functors appear in the following exact sequences, natural in $V$

$$\langle \mathscr{A} \rangle \otimes_{\mathscr{R}} \mathscr{D}V \to \mathscr{D}V \to DV \to 0 \tag{3.5}$$

$$\langle \mathscr{A} \rangle \otimes_{\mathscr{R}} \mathscr{D}'V \to \mathscr{D}'V \to D'V \to 0 \tag{3.6}$$

When $V = k\iota$ for a homogeneous element $\iota$, $DV$ has basis

$$\{Q^I \iota \mid I \text{ admissible}, \, e(I) \geq \deg(\iota)\}$$

Together with additivity of $D$, this describes $DV$ as a $k$-vectorspace. Since $R \cong Dk$ as a left $R$-module, we also have a basis of $R$ over $k$.

3.4. **Homology of $QX$.** Here, we recall the computation of $H_*(QX)$. We shall need only the cases $X = \Sigma\mathbb{C}P^\infty$ and $X = S^0$. We shall give a nonfunctorial description of $H_*(Q_0X)$ in terms of a basis of $JH_*(X)$. A slicker formulation is given in [CLM, Theorem 4.2], where $H_*(QX)$ is expressed as a functor of $H_*(X)$.

For a general (possibly non-connected) pointed space $X$, write $\tilde{\pi}_0(X) = \pi_0(X) - \{0\}$ where $0 \in \pi_0(X)$ denotes the component of the basepoint. Then $\pi_0(QX) = \mathbb{Z}[\tilde{\pi}_0(X)]$, and in particular $QX$ is connected if and only if $X$ is connected. Write $Q_0X \subseteq QX$ for the basepoint component of $QX$. Define the "translation" map $\tau : QX \to Q_0X$ as the map that on the component $Q_iX$, $i \in \pi_0(QX)$ multiplies by a point in the component $Q_{-i}X$. This defines a unique homotopy class $\tau : Q_iX \to Q_0X$.

**Theorem 3.7.** *Let $B \subseteq JH_*(X)$ be a basis consisting of homogeneous elements. Then, $H_*(Q_0X)$ is the free commutative algebra on the set*

$$\{\tau_*(Q^I x) \mid x \in B, \, I \text{ admissible}, \, e(I) + b(I) > \deg x, \, \deg(Q^I x) > 0\}$$

$\square$

If $x$ is in the component $ax \in \pi_0 X \subseteq \pi_0 QX$ then $Q^I x$ will be in the component $p^{\ell(I)} ax$. Hence the element $\tau_*(Q^I x) = Q^I x * [-p^{\ell(I)} ax]$ will be in homology of the identity component $Q_0X$.

**Corollary 3.8.** *The map*

$$\varphi_Q : D'JH_*(X) \to QH_*(Q_0X) \tag{3.7}$$

*sending $Q^I x$ to $\tau_*(Q^I x)$ is an isomorphism of $Q$-unstable $R$-modules.*

*If $X$ is connected and $H_*(X)$ has trivial comultiplication (e.g. if $X$ is a suspension), the natural map*

$$\varphi_P : DJH_*(X) \to PH_*(QX) \tag{3.8}$$

*is an isomorphism of unstable $R$-modules.*

*Proof.* $\varphi_Q$ is an isomorphism by Theorem 3.7. It is $R$-linear because the Cartan formula ([CLM, Thm. 1.1,(6)]) implies that the translation $\tau_*$ given by multiplication by zero-dimensional classes commutes modulo products in $H_*(Q_0X)$ with $Q^s$.

The Cartan formula for the coproduct implies that the $Q^s$ preserves primitives. Thus $\varphi_P$ has image in the primitive elements. It is injective by Theorem 3.7 and surjective by Theorem 2.4. □

## 4. Homological algebra of unstable modules

The map

$$Q(\partial_*) : QH_*(Q\Sigma\mathbb{C}P^\infty_+) \to QH_*(Q_0S^0)$$

was computed in [MMM, Theorem 4.5]. The left hand side is $D'JH_*(\Sigma\mathbb{C}P^\infty_+)$ and the right hand side is $D'k$. The starting point of our theorems is

**Theorem 4.1** ([MMM])**.** *Let $a_s \in H_s(\Sigma\mathbb{C}P^\infty_+)$ be the generator, $s$ odd. Then*

$$Q(\partial_*)(a_s) = \begin{cases} \beta Q^r[1] * [-p] & s = 2r(p-1)-1 \\ 0 & \textit{otherwise} \end{cases}$$

*Proof.* The map $\partial : \Sigma\mathbb{C}P^\infty_+ \to QS^0$ coincides with the universal $S^1$-transfer denoted $t_0$ in [MMM]. The formula for $Q(\partial_*)(a_s)$ in the theorem now follows from ignoring all decomposable terms in [MMM, Theorem 4.5]. □

4.1. **Main technical theorems.** To state the theorems, recall from subsection 3.2 that $\mathscr{R}$ may be bigraded by $\deg = \deg_Q + \deg_\beta$. Since the Adem relations are homogeneous with respect to deg and $\deg_\beta$, there is an induced bigrading of $R$. If $V$ is bigraded, $\mathscr{R} \otimes V$ is a bigraded left $\mathscr{R}$-module. Since the unstability relations (3.2) can be chosen homogeneous, there is an induced bigrading of $\mathscr{D}V$. Similarly for $\mathscr{D}'V$, $DV$ and $D'V$. Thus by Corollary 3.8 a bigrading of $JH_*(X)$ will induce a bigrading of $QH_*(Q_0X)$ and, for $X$ a suspension, a bigrading of $PH_*(QX)$.

For bigraded modules $V$ with $\deg = \deg_Q + \deg_\beta$ as above, we shall write $V^{i,j} = \{x \in V \mid \deg_Q(x) = i, \deg_\beta(x) = j\}$ and $V^n = \oplus_{i+j=n} V^{i,j}$ and $V^{(n)} = \oplus_i V^{i,n}$. We will only consider gradings in the fourth quadrant, i.e. $V^{i,j} = 0$ unless $i \geq 0$ and $j \leq 0$. Write $V^{(-)} = \oplus_{n<0} V^{(n)}$.

**Theorem 4.2.** $\mathrm{Im}(Q(\partial_*)) = QH_*(Q_0S^0)^{(-)}$

*Proof.* The inclusion $\mathrm{Im}(Q\partial_*) \subseteq QH_*(Q_0S^0)^{(0)}$ is immediate from Theorem 4.1. The other inclusion follows from Lemma 4.3 below. Indeed, the two-sided ideal in $R$ generated by the set $\{\beta Q^s \mid s \geq 1\}$ is spanned by operations $Q^I$ with at least one $\beta$. By Lemma 4.3 below, any such operation is also in the left ideal with the same generators, i.e. is a linear combination of elements of the form $Q^J\beta Q^s$. In particular, any element in $QH_*(Q_0S^0)^{(-)}$ is also in $\mathrm{Im}(\partial_*)$. □

**Lemma 4.3.** *The left ideal in $R$ generated by the set $\{\beta Q^s \mid s \geq 1\}$ is also a right ideal.*

*Proof.* Write $R' \subseteq R$ for the left ideal generated by $\{\beta Q^s \mid s \geq 1\}$.

For $r \leq ps$, consider the Adem relation $\mathscr{A}^{(0,ps,1,r-(p-1)s)}$:

$$\begin{aligned} Q^{ps}\beta Q^{r-(p-1)s} = {} & \beta Q^r Q^s \\ & + \sum_{i>s} \lambda_i \beta Q^{r+s-i} Q^i \\ & + \text{terms of form } Q^{r+s-i}\beta Q^i \end{aligned}$$

where we have singled out the term in the Adem relation corresponding to $i = s$, and where the $\lambda_i \in k$ are certain binomial coefficients. This shows that in the left $R$-module $R/R'$ we can write $\beta Q^r Q^s$ as a linear combination of $\beta Q^a Q^b$ with $a < r$. In particular, $\beta Q^1 Q^s = 0 \in R/R'$ and by induction $\beta Q^r Q^s = 0 \in R/R'$.

Thus we have $\beta Q^r Q^s \in R'$ whenever $\beta Q^r Q^s$ is admissible. Since a nonadmissible $\beta Q^r Q^s$ is a linear combination of admissible ones, we have $\beta Q^r Q^s \in R'$ for any $r, s$. This shows that $R'$ is invariant under right multiplication with $Q^s$. Since it is obviously invariant under right multiplication with $\beta Q^s$ it follows that $R'$ is a right ideal. □

The kernel of $Q\partial_*$ is harder to determine explicitly. The partial information contained in Theorem 4.4 below suffices for the calculation.

Notice that for any $\mathscr{R}$-module $V$, the augmentation of $\mathscr{R}$ gives a natural quotient map $V \to k \otimes_{\mathscr{R}} V$ identifying $k \otimes_{\mathscr{R}} V$ with the quotient of $V$ by the relations $\beta^\varepsilon Q^s x = 0$ for $x \in V, \varepsilon \in \{0,1\}, s \geq 1$. The functor $k \otimes_{\mathscr{R}} -$ agrees with the functor $k \otimes_R -$ on $R$-modules. Thus the vectorspace $k \otimes_R V$ measures the dimensions of a minimal set of $R$-module generators of an unstable $R$-module $V$.

**Theorem 4.4.** *Bigrade $JH_*(\Sigma\mathbb{C}P^\infty_+)$ by concentrating it in $\deg_\beta = -1$ and give $QH_*(Q\Sigma\mathbb{C}P^\infty_+)$ the induced bigrading. Then the bigraded vectorspace*

$$k \otimes_R \mathrm{Ker}(Q\partial_*) = k \otimes_R Q(H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\backslash\partial_*)$$

*is concentrated in bidegrees $\deg_\beta = -1$ and $\deg_\beta = -2$. In particular $\mathrm{Ker}(Q\partial_*)$ is generated as an $R$-module by the elements $a_s \in \mathrm{Ker}(Q\partial_*)$ with $s \not\equiv -1 \pmod{2(p-1)}$ together with elements of degree $\equiv -1$ and $\equiv -2 \pmod{2(p-1)}$.*

*Proof.* The equality $\mathrm{Ker}(Q\partial_*) = Q(H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\backslash\partial_*)$ in the theorem follows from Proposition 2.11 because $H_*(Q_0S^0)$ is a free algebra.

The last statement of the theorem follows from the first. Indeed the elements $Q^I a_s$ are all in the kernel of $Q(\partial_*)$ when $s \not\equiv -1 \pmod{2(p-1)}$ because $a_s$ is in the kernel. These elements give rise to one "trivial" element $a_s \in k \otimes_R \operatorname{Ker}(Q\partial_*)$. On the span of the $Q^I a_s$ with $s \equiv -1 \pmod{2(p-1)}$ the claim about degrees of generators follows since on these elements $\deg \equiv \deg_\beta \pmod{2(p-1)}$. Thus we need only prove the first statement of the theorem.

We have the short exact sequence of $Q$-unstable $R$-modules

$$0 \longrightarrow \operatorname{Ker}(Q\partial_*) \longrightarrow QH_*(Q\Sigma\mathbb{C}P^\infty_+) \xrightarrow{Q\partial_*} QH_*(Q_0S^0)^{(-)} \longrightarrow 0 \tag{4.1}$$

If we applied the functor $k \otimes_R -$ from $R$-modules to vectorspaces, we would get a long exact sequence involving $\operatorname{Tor}^R_*(k,-)$, and a determination of the map induced by $Q\partial_*$ in $\operatorname{Tor}_1$ would give the result. This is more or less what we do, except that it is technically more convenient to replace the functor $k \otimes_R -$ by $k \otimes_{\mathscr{R}} -$ and to replace Tor by a suitable functor taking unstability into account. We proceed to make these ideas precise.

The category of $Q$-unstable $\mathscr{R}$-modules is abelian and has enough projectives. The functor $k \otimes_{\mathscr{R}} -$ from $Q$-unstable $\mathscr{R}$-modules is right exact, hence the left derived functors $L_r(k \otimes_{\mathscr{R}} -)$ are defined. These are unstable versions of $\operatorname{Tor}^{\mathscr{R}}_r(k,-)$. For brevity, let us write $T_1^{\mathscr{R}}(k,-) = L_1(k \otimes_{\mathscr{R}} -)$.

With these definitions, applying the functor $k \otimes_{\mathscr{R}} -$ to the sequence (4.1) induces the exact sequence

$$0 \longrightarrow \operatorname{Cok}(T_1^{\mathscr{R}}(k, Q\partial_*)) \longrightarrow k \otimes_R \operatorname{Ker}(Q\partial_*) \longrightarrow \operatorname{Ker}(k \otimes_R Q\partial_*) \longrightarrow 0 \tag{4.2}$$

*Claim 1*: The elements $a_s \in \operatorname{Ker}(Q\partial_*)$ with $s \not\equiv -1 \pmod{2(p-1)}$ maps in (4.2) to a generating set in $\operatorname{Ker}(k \otimes_R Q\partial_*)$.

*Proof of Claim 1.* This is the kernel of the map

$$k \otimes_R Q\partial_* : k \otimes_R QH_*(Q\Sigma\mathbb{C}P^\infty_+) \to k \otimes_R QH_*(Q_0S^0)^{(-)}$$

Clearly, the natural map $JH_*(Q\Sigma\mathbb{C}P^\infty_+) \to k \otimes_R QH_*(Q\Sigma\mathbb{C}P^\infty_+)$ is an isomorphism, and by Lemma 4.3 we get that $k \otimes_R QH_*(Q_0S^0)^{(-)}$ is spanned by $\{\beta Q^s[1] * [-p] \mid s \geq 1\}$. Thus Claim 1 follows from Theorem 4.1. $\square$

*Claim 2*: $\operatorname{Cok}(T_1^{\mathscr{R}}(k, Q\partial_*))$ is concentrated in $\deg_\beta = -1$ and $\deg_\beta = -2$.

*Proof of Claim 2.* We will compute $T_1^{\mathscr{R}}(k, Q\partial_*)$ using suitable free resolutions. For brevity, write $V = JH_*(\Sigma\mathbb{C}P^\infty_+)$. By Corollary 3.8 we may consider $Q\partial_*$ as a map from $D'V$ onto $D'k^{(-)}$. Let $W \subseteq (\mathscr{D}'k)^{(-)}$ denote the subspace with basis $\{\beta Q^{s_1} \dots Q^{s_k} \mid s_1 \geq 1, s_2, \dots, s_k \geq 0\}$. In the diagram

$$\begin{tikzcd} & & W \arrow[d] & D'V \arrow[d, "Q\partial_*"] & \\ 0 \arrow[r] & (\langle \mathscr{A} \rangle \cdot \mathscr{D}'k)^{(-)} \arrow[r] & \mathscr{D}'k^{(-)} \arrow[r] & D'k^{(-)} \arrow[r] & 0 \end{tikzcd} \tag{4.3}$$

in which the lower exact sequence is an instance of (3.6), we may choose a lifting $\rho : W \to D'V$ since $Q\partial_*$ is surjective. Writing $V = V_0 \oplus V_1$ where $V_0 = \text{span}\{a_s \mid x \equiv -1 \pmod{2(p-1)}\}$ and $V_1 = \text{span}\{a_s \mid x \not\equiv -1 \pmod{2(p-1)}\}$, we may choose the lifting $\rho$ to have $\rho(W) \subseteq D'V_0$ since $D'V = D'V_0 \oplus D'V_1$ and since $Q\partial_*$ vanishes on $D'V_1$. We may also choose the lifting to have $\rho(\beta Q^s) = a_{2s(p-1)-1}$ and extend (4.3) to the following exact diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}(\rho) \arrow[r, "j"] \arrow[d, "\sigma"] & \mathscr{D}'W \arrow[r, "\rho"] \arrow[d] & D'V_0 \arrow[r] \arrow[d, "Q\partial_*"] & 0 \\ 0 \arrow[r] & (\langle \mathscr{A} \rangle \cdot \mathscr{D}'k)^{(-)} \arrow[r, "i"] & \mathscr{D}'k^{(-)} \arrow[r] & D'k^{(-)} \arrow[r] & 0 \end{tikzcd} \tag{4.4}$$

Note that the middle map in (4.4) is an isomorphism.

Next we apply the functor $k \otimes_{\mathscr{R}} -$ to (4.4). This gives a diagram involving the left derived functor $T_1^{\mathscr{R}}(k, -) = L_1(k \otimes_{\mathscr{R}} -)$. This functor vanishes on the middle part of (4.4) since these (isomorphic) objects are free. Thus, a part of the induced diagram looks like this

$$\begin{tikzcd} 0 \arrow[r] & T_1^{\mathscr{R}}(k, D'V_0) \arrow[r] \arrow[d, "T_1^{\mathscr{R}}(k, Q\partial_*)"] & k \otimes_{\mathscr{R}} \mathrm{Ker}\rho \arrow[r, "j_*"] \arrow[d, "\sigma_*"] & k \otimes_{\mathscr{R}} \mathscr{D}'W \arrow[d, "\cong"] \\ 0 \arrow[r] & T_1^{\mathscr{R}}(k, D'k^{(-)}) \arrow[r] & k \otimes_{\mathscr{R}} (\langle \mathscr{A} \rangle \cdot \mathscr{D}'k)^{(-)} \arrow[r, "i_*"] & k \otimes_{\mathscr{R}} (\mathscr{D}'k)^{(-)} \end{tikzcd} \tag{4.5}$$

where a star in subscript is shorthand for $k \otimes_{\mathscr{R}} -$ on morphisms. Thus we have represented $T_1^{\mathscr{R}}(k, D'V_0)$ and $T_1^{\mathscr{R}}(k, \mathscr{D}'k^{(-)})$ as the kernels of $j_*$ and $i_*$, and the map $T_1^{\mathscr{R}}(k, Q\partial_*)$ as the restriction of $\sigma_*$.

To calculate the cokernel of $T_1^{\mathscr{R}}(k, Q\partial_*)$ and to prove Claim 2, note that

$$(\langle \mathscr{A} \rangle \cdot \mathscr{D}'k)^{(-)} = \mathscr{R}^{(-)} \cdot \mathscr{A}^{(0)} \cdot \mathscr{D}'k^{(0)} + \mathscr{R} \cdot \mathscr{A}^{(-)} \cdot \mathscr{D}'k^{(0)} + \mathscr{R} \cdot \mathscr{A} \cdot \mathscr{D}'k^{(-)}$$

This is generated over $\mathscr{R}$ by the subspace

$$\mathscr{R}^{(-1)} \cdot \mathscr{A}^{(0)} \cdot \mathscr{D}'k^{(0)} + \cdot \mathscr{A}^{(-)} \cdot \mathscr{D}'k^{(0)} + \mathscr{A} \cdot \mathscr{D}'k^{(-)} \tag{4.6}$$

The corresponding $\mathscr{R}$-indecomposable classes will span $k \otimes_{\mathscr{R}} (\langle \mathscr{A} \rangle \cdot \mathscr{D}'k)^{(-)}$ as a vectorspace, and since the first and the second term in (4.6) has $\deg_\beta \in \{-1, -2\}$, it suffices to prove that the last term $\mathscr{A} \cdot \mathscr{D}'k^{(-)}$ does not contribute to the cokernel of $T_1^{\mathscr{R}}(k, Q\partial_*)$.

To this end, notice that $\mathscr{A} \cdot \mathscr{D}'k^{(-)}$ corresponds to $\mathscr{A} \cdot \mathscr{D}'W$ under the middle isomorphism in (4.4), and that $\mathscr{A} \cdot \mathscr{D}'W$ is in the kernel of $\rho$ since the action of $\mathscr{A}$ is trivial in $D'V$. Notice also that $\mathscr{A} \cdot \mathscr{D}'W$ vanishes under the projection $\mathscr{D}'W \to k \otimes_{\mathscr{R}} \mathscr{D}'W$ and thus by exactness of (4.4) and (4.5) the classes corresponding to $\mathscr{A} \cdot \mathscr{D}'k^{(-)}$ in $k \otimes_{\mathscr{R}} (\langle \mathscr{A} \rangle \cdot \mathscr{D}'k)^{(-)}$ lifts all the way to $T_1^{\mathscr{R}}(k, D'V_0)$ and therefore does not contribute to the cokernel of $T_1^{\mathscr{R}}(k, Q\partial_*)$. $\square$

Now Theorem 4.4 follows from the exact sequence (4.2) and the Claims above. $\square$

## 5. Homology of $\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}$

The spectral sequence associated to the fibration (1.1) has

$$E^2 = \mathrm{Cotor}^{H_*(Q_0S^0)}(k,k) \Rightarrow H_*(\Omega^\infty\mathbb{C}P^\infty_{-1}) \tag{5.1}$$

By Proposition 2.7 the $E^2$-term splits as

$$E^2 \cong \mathrm{Cotor}^{H_*(Q_0S^0)/\!/\partial_*}(k,k)\otimes H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\backslash\partial_* \tag{5.2}$$

In this section, $p$ is odd so after localising the fibration (1.1), the base-space is simply connected and the spectral sequence converges.

To proceed, we need to compute the coalgebra structure of $H_*(Q_0S^0)/\!/\partial_*$ or, equivalently, the algebra structure of $H^*(Q_0S^0)\backslash\!\backslash\partial^*$.

5.1. **The Hopf algebra cokernel of $\partial_*$.** To state the results, let us introduce a bigrading of $H_*(Q_0S^0)$. Recall that $H_*(Q_0S^0)$ is the free commutative algebra on the set

$$\{Q^I[1]*[-p^{\ell(I)}] \mid I \text{ admissible}, \ e(I)+b(I)>0\}$$

Make it a bigraded algebra by setting $\deg_\beta(Q^I[1]*[-p^{\ell(I)}]) = \deg_\beta(Q^I)$. By the Cartan formula for the coproduct we get that the subalgebra $H_*(Q_0S^0)^{(0)}$ is a Hopf subalgebra, but notice that $H_*(Q_0S^0)$ is not a bigraded $R$-module because of the relation (3.3).

**Theorem 5.1.** *Consider $H_*(Q_0S^0)$ as a bigraded algebra as above. Then the composition*

$$H_*(Q_0S^0)^{(0)} \to H_*(Q_0S^0) \to H_*(Q_0S^0)/\!/\partial_* \tag{5.3}$$

*is an isomorphism of Hopf algebras. Thus $H_*(Q_0S^0)/\!/\partial_*$ is a polynomial algebra on the set*

$$\{Q^I[1]*[-p^{\ell(I)}] \mid I \text{ admissible}, \ \deg_\beta(Q^I)=0, \ e(I)>0\}$$

*Proof.* With the bigrading introduced above, we have $H_*(Q_0S^0) = H_*(Q_0S^0)^{(0)} \oplus H_*(Q_0S^0)^{(-)}$ where the first summand is a subalgebra and the second is an ideal. Since $\mathrm{Im}(\partial_*) \subseteq k \oplus H_*(Q_0S^0)^{(-)}$, the composition (5.3) is injective.

To see surjectivity, note that $Q(H_*(Q_0S^0)/\!/\partial_*) = \mathrm{Cok}(Q\partial_*)$ since $Q$ is right exact. By Theorem 4.2 we have $\mathrm{Im}(Q\partial_*) = QH_*(Q_0S^0)^{(-)}$ and hence $\mathrm{Cok}(Q\partial_*) = (QH_*(Q_0S^0))^{(0)} = Q(H_*(Q_0S^0)^{(0)})$. $\square$

**Theorem 5.2.** *$H_*(Q_0S^0)^{(0)}$ is dual to a polynomial algebra.*

*Proof.* It suffices to prove that $\lambda : H_*(Q_0S^0)^{(0)} \to H_*(Q_0S^0)^{(0)}$ is surjective. $\lambda$ is given by the dual Steenrod operations: If $\deg(x) = 2ps$, $\lambda x = \mathcal{P}^s_* x$. By the Nishida relations ([CLM, Theorem 1.1 (9)]), one gets $\lambda Q^{ps} = Q^s\lambda$ and thus

$$\lambda(Q^{ps_1}Q^{ps_2}\dots Q^{ps_k}[1]*[-p^k]) = Q^{s_1}Q^{s_2}\dots Q^{s_k}[1]*[-p^k]$$

Thus $\lambda$ hits the generators of $H_*(Q_0S^0)$ and since it is a map of algebras, it is surjective. $\square$

5.2. **The spectral sequence.** We are now ready to compute the $E^2$-term of the spectral sequence (5.1) and to prove that it collapses at the $E^2$-term, $E^2 = E^\infty$.

**Theorem 5.3.** *The spectral sequence collapses at the $E^2$-term. The $E^2$-term is given by*

$$E^2 = H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\!\backslash\partial_* \otimes E[s^{-1}P(H_*(Q_0S^0)/\!\!/\partial_*)]$$

*as a Hopf algebra.*

*Proof.* We need to identify the factor $\mathrm{Cotor}^{H_*(Q_0S^0)/\!\!/\partial_*}(k,k)$ in the splitting (5.2) of the $E^2$-term. By Theorem 5.2, the dual algebra $H^*(Q_0S^0)\backslash\!\!\backslash\partial^*$ is polynomial and hence by Corollary 2.9 we get

$$\mathrm{Cotor}^{H_*(Q_0S^0)/\!\!/\partial_*}(k,k) \cong E[s^{-1}P(H_*(Q_0S^0)/\!\!/\partial_*)]$$

as claimed.

In this $E^2$-term, primitives and generators are concentrated in bidegrees $(0,*)$ and $(-1,*)$ and hence by Lemma 2.15 there can be no non-zero differentials in the spectral sequence. $\square$

**Corollary 5.4.** *There is an isomorphism of algebras*

$$H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}) \cong H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\!\backslash\partial_* \otimes E[s^{-1}P(H_*(Q_0S^0)/\!\!/\partial_*)]$$

*Proof.* This is a formal consequence of the $E^\infty$-term in Theorem 5.3 being free. Explicitly, the spectral sequence gives a filtration $F_0 \supseteq F_{-1} \supseteq \dots$ on $H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$ and an isomorphism $E^2_{-k,*} = E^\infty_{-k,*} \cong F_{-k}/F_{-k-1}$. In particular

$$F_0/F_{-1} \cong H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\!\backslash\partial_*$$

via the natural map $\omega_*$ of (1.1), and there is a natural map

$$s^{-1}P(H_*(Q_0S^0)/\!\!/\partial_*) \to F_{-1}/F_{-2} = E^\infty_{-1,*}$$

that is an isomorphism onto the primitive elements in $E^\infty_{-1,*}$.

Choosing liftings

$$H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\!\backslash\partial_* \to F_0 = H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$$

and

$$s^{-1}P(H_*(Q_0S^0)/\!\!/\partial_*) \to F_{-1} \subseteq H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$$

of these gives a map of algebras

$$E[s^{-1}P(H_*(Q_0S^0)/\!\!/\partial_*)] \otimes H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\!\backslash\partial_* \to H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$$

that is surjective by induction on the filtration degrees and injective for dimensional reasons. $\square$

Note that under this isomorphism, projection on the first factor corresponds to the natural map $\omega_*$ induced from the fibration (1.1).

**Corollary 5.5.** *Let* $\omega : \Omega^\infty\Sigma\mathbb{C}P^\infty_{-1} \to Q\Sigma\mathbb{C}P^\infty_+$ *be the map from* (1.1)*. Then*

$$H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\!\backslash\omega_* \cong E[s^{-1}P(H_*(Q_0S^0)/\!\!/\partial_*)]$$

*as a Hopf algebra. In particular, the vectorspace*

$$\operatorname{Ker}(P\omega_*) = P(H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\!\backslash\omega_*) = Q(H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\!\backslash\omega_*)$$

*is concentrated in degrees* $\equiv -1$ *and* $\equiv -2 \pmod{2(p-1)}$*.*

*Proof.* The vectorspace $s^{-1}P(H_*(Q_0S^0)/\!\!/\partial_*)$ corresponds to odd-dimensional algebra generators of $H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$. By Theorem 2.4 these have unique primitive representatives. Thus the map

$$s^{-1}P(H_*(Q_0S^0)/\!\!/\partial_*) \to F_{-1} \subseteq H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$$

from the proof of Corollary 5.4 may be rechosen to map into the primitive elements. Then it will map into $\operatorname{Ker}(P\partial_*) = P(H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\!\backslash\omega_*)$ and there is a well-defined injective map of Hopf algebras

$$E[s^{-1}P(H_*(Q_0S^0)/\!\!/\partial_*)] \to H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\!\backslash\omega_*$$

This map is surjective for dimensional reasons. □

Notice that the map $s^{-1}P(H_*(Q_0S^0)/\!\!/\partial_*) \to H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$ in the proof of Corollary 5.5 is independent of previous choices, so the isomorphism in Corollary 5.5 is actually canonical.

The isomorphism in Corollary 5.4 is not canonical. The sequence

$$k \longrightarrow H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\!\backslash\omega_* \longrightarrow H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}) \xrightarrow{\omega} H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\!\backslash\partial_* \longrightarrow k \tag{5.4}$$

is canonical, with a noncanonical splitting of $\omega_*$. A priori, this splitting is a map of algebras, so we have not yet determined the coalgebra structure of $H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$. This is the topic of the next subsection since the coalgebra structure of $H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$ is necessary for the computation of $H_*(\Omega^\infty\mathbb{C}P^\infty_{-1})$.

### 5.3. Coalgebra structure of $H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1};\mathbb{F}_p)$.

It turns out that the splitting in the exact sequence (5.4) can be chosen as a map of Hopf algebras.

**Lemma 5.6.** *Let*

$$k \longrightarrow A \longrightarrow B \xrightarrow{\pi} C \longrightarrow k$$

*be a short exact sequence of Hopf algebras. If either $A$ or $C$ is exterior, the sequence is split exact in the category of Hopf algebras.*

*Proof.* Assume $C$ is exterior. Then by Theorem 2.4 we have that $PC \cong QC$ and the diagram

$$\begin{tikzcd} PB \arrow[r, "P\pi"] \arrow[d] & PC \arrow[r] \arrow[d, "\cong"] & 0 \\ QB \arrow[r] & QC \arrow[r] & 0 \end{tikzcd}$$

is exact since $Q(-)$ is right exact. Thus $PB \to PC$ is surjective and a choice of splitting $PC \to PB$ of $P\pi$ induces a splitting $C \cong E[PC] \to B$ of $\pi$.

The case where $A$ is exterior follows by duality. □

The following theorem summarises our computation of $H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$:

**Theorem 5.7.** *The sequence* (5.4) *is split exact. Hence*

$$H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}) \cong H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\!\backslash\omega_* \otimes H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\!\backslash\partial_* \tag{5.5}$$

*as a Hopf algebra. In particular, it is primitively generated and free as an algebra.*

*Proof.* This follows from Lemma 5.6 applied to the short exact sequence (5.4), since $H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$ is exterior by Corollary 5.5. □

## 6. Homology of $\Omega^\infty\mathbb{C}P^\infty_{-1}$

Here, the method is to consider the path-loop fibration over $\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}$. From the fibration (1.1) one easily gets $\pi_1(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}) = \mathbb{Z}$ and therefore we have an equivalence

$$\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1} \simeq S^1 \times \tilde{\Omega}^\infty\Sigma\mathbb{C}P^\infty_{-1}$$

where $\tilde{\Omega}^\infty\Sigma\mathbb{C}P^\infty_{-1} \to \Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}$ is the universal covering map. Furthermore we have $\Omega(\tilde{\Omega}^\infty\Sigma\mathbb{C}P^\infty_{-1}) = \Omega^\infty_0\mathbb{C}P^\infty_{-1}$, the basepoint component of $\Omega^\infty\mathbb{C}P^\infty_{-1}$. Similarly $Q\Sigma\mathbb{C}P^\infty_+ \simeq S^1 \times \tilde{Q}\Sigma\mathbb{C}P^\infty_+$ and under these splittings the map $\omega$ in the fibration (1.1) restricts to a map $S^1 \to S^1$ of degree 2. Since $p$ is odd we see that replacing $\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}$ by its universal covering space just remoes a one-dimensional vectorspace from the left factor in Theorem 5.7.

The Eilenberg-Moore spectral sequence associated to the path-loop fibration over $\tilde{\Omega}^\infty\Sigma\mathbb{C}P^\infty_{-1}$ is

$$E^2 = \mathrm{Cotor}^{H_*(\tilde{\Omega}^\infty\Sigma\mathbb{C}P^\infty_{-1})}(k,k) \Rightarrow H_*(\Omega^\infty_0\mathbb{C}P^\infty_{-1}) \tag{6.1}$$

and by Theorem 5.7, the $E^2$-term splits as

$$E^2 \cong \mathrm{Cotor}^{H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\!\backslash\omega_*}(k,k) \otimes \mathrm{Cotor}^{H_*(\tilde{Q}\Sigma\mathbb{C}P^\infty_+)\backslash\!\!\backslash\partial_*}(k,k) \tag{6.2}$$

I claim it must collapse. As before, we consider a possibly nonzero differential $dx = y \neq 0$ with $\deg(x)$ minimal. We will reach a contradiction in a number of steps. The argument is based on Theorem 4.4 and a careful analysis of degrees modulo $2(p-1)$ in the spectral sequence.

By Corollary 5.5, the first factor in (6.2) is an exterior algebra on generators of total degree $\equiv -2 \pmod{2(p-1)}$. To gain information about the second factor, we map the spectral sequence (6.1) into the spectral sequence of the path-loop fibration over $\tilde{Q}\Sigma\mathbb{C}P^\infty_+$ via the map $\omega : \tilde{\Omega}^\infty\Sigma\mathbb{C}P^\infty_{-1} \to \tilde{Q}\Sigma\mathbb{C}P^\infty_+$. This is a map $E^r(\omega)$ of spectral sequences whose restriction to the first factor in the splitting (6.2) is zero, and whose restriction to the second factor in (6.2) is induced by the inclusion $H_*(\tilde{Q}\Sigma\mathbb{C}P^\infty_+)\backslash\!\!\backslash\partial_* \to H_*(\tilde{Q}\Sigma\mathbb{C}P^\infty_+)$. The next lemma says that this second factor in (6.2) injects under $E^2(\omega)$.

**Lemma 6.1.** *Let $f : A \to B$ be an injection of primitively generated Hopf algebras. Then* $\mathrm{Cotor}^f(k,k) : \mathrm{Cotor}^A(k,k) \to \mathrm{Cotor}^B(k,k)$ *is also injective.*

*Proof.* By Theorem 2.4, $A^*$ and $B^*$ are tensor products of exterior algebras and polynomial algebras truncated at height $p$. Thus we can split $f^* : B^* \to A^*$ in the category of algebras (since a splitting can be chosen on the generators of $A^*$). Dually, $f : A \to B$ is split injective as a map of coalgebras and thus $\mathrm{Cotor}^f(k,k)$ is injective. □

**Corollary 6.2.** *Relative to the splitting* (6.2)*, a minimal differential $dx = y \neq 0$ will have $x$ in the right factor and $y$ in the left.*

*Proof.* Recall that $P$ and $Q$ are logarithmic: $P(A \otimes B) = PA \oplus PB$ and $Q(A \otimes B) = QA \oplus QB$. Thus $x$ and $y$ does not contain products between the two factors in (6.2).

Since $y$ is primitive and in bidegree $(\leq -3, *)$, it must be of even total degree by Corollary 2.9, and thus $x$ is of odd total degree. By Corollary 5.5 this is only possible if $x$ is in the right factor.

By Lemma 6.1, the right factor injects into the spectral sequence of $Q\Sigma\mathbb{C}P^\infty_+$, and since all differentials vanish in this spectral sequence, $y$ must map to 0 there, and hence $y$ is in the left factor. □

The remaining part of the collapse proof is to eliminate the possibility of differentials from the right factor to the left. This is the hardest part of the proof, the main ingredient of which is Theorem 4.4.

**Theorem 6.3.** *The spectral sequence* (6.1) *collapses.*

*Proof.* Consider a minimal differential $dx = y \neq 0$. Then $y$ is a primitive element in $\mathrm{Cotor}^{H_*(\tilde{\Omega}^\infty \Sigma \mathbb{C}P^\infty_{-1})}(k,k)$. By Corollary 6.2 and Corollary 2.9 it is of the form

$$y = (s^{-1}z)^{p^k}$$

for a $z \in P(H_*(\tilde{\Omega}^\infty \Sigma \mathbb{C}P^\infty_{-1}) \backslash\!\backslash \omega_*)$. By Corollary 5.5 we must have $\deg(z) \equiv -1 \pmod{2(p-1)}$. Write

$$\deg(z) = 2n(p-1) - 1$$

Then

$$\deg y = p^k(2n(p-1) - 2) = 2p^k(n(p-1) - 1) \equiv -2 \pmod{2(p-1)}$$

and thus $\deg x \equiv -1 \pmod{2(p-1)}$ because the differential has degree $-1$. By Proposition 2.15 we get that $x$ corresponds to a minimal element in the cokernel of $\sigma_* : QH_*(\Omega^\infty_0 \mathbb{C}P^\infty_{-1}) \to PH_*(\tilde{\Omega}^\infty \Sigma \mathbb{C}P^\infty_{-1})$, of degree $\equiv 0 \pmod{2(p-1)}$. By Corollary 6.2, $x$ is also a minimal element in the cokernel of the composition

$$QH_*(\Omega^\infty_0 \mathbb{C}P^\infty_{-1}) \xrightarrow{\sigma_*} PH_*(\tilde{\Omega}^\infty \Sigma \mathbb{C}P^\infty_{-1}) \xrightarrow{P\omega_*} P(H_*(\tilde{Q}\Sigma\mathbb{C}P^\infty_+) \backslash\!\backslash \partial_*)$$

By minimality this element is not a $p$th power and hence is not zero in $Q(H_*(\tilde{Q}\Sigma\mathbb{C}P^\infty_+) \backslash\!\backslash \partial_*)$. Again by minimality, and because the loop suspension $\sigma_*$ is $R$-linear, this element is $R$-indecomposable and hence since $\sigma_*$ has degree 1, $x$ will map to a nonzero element of degree $\equiv 0 \pmod{2(p-1)}$ in

$$k \otimes_R Q(H_*(\tilde{Q}\Sigma\mathbb{C}P^\infty_+)\backslash\!\backslash\partial_*)$$

in contradiction with Theorem 4.3. $\square$

**Corollary 6.4.** *As an algebra,*

$$H_*(\Omega_0^\infty \mathbb{C}P^\infty_{-1}) \cong k[s^{-2}P(H_*(Q_0S^0)/\!/\partial_*)] \otimes \mathrm{Cotor}^{H_*(\tilde{Q}\Sigma\mathbb{C}P^\infty_+)\backslash\!\backslash\partial_*}(k,k)$$

*Proof.* This is precisely analogous to Corollary 5.4, using only that the $E^\infty$-term is a free algebra. $\square$

## 7. The case $p = 2$

At the prime 2, the calculation of $H_*(\Omega^\infty\mathbb{C}P^\infty_{-1})$ and $H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$ can also be made. Some details are quite different however. In particular, we will use the looped fibration

$$\Omega^\infty\mathbb{C}P^\infty_{-1} \to Q(\mathbb{C}P^\infty_+) \to \Omega QS^0 \tag{7.1}$$

to compute $H_*(\Omega^\infty\mathbb{C}P^\infty_{-1})$, instead of the path-loop fibration over $\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}$. At $p = 2$ our base spaces in the fibrations are no longer simply connected. The following lemma deals with this

**Lemma 7.1.** *As spaces we have*

$$QS^0 \simeq \mathbb{Z} \times \mathbb{R}P^\infty \times \tilde{Q}_0S^0$$
$$\Omega QS^0 \simeq \mathbb{Z}/2 \times \mathbb{R}P^\infty \times \tilde{\Omega}_0QS^0$$

*where $\tilde{X} \to X$ denotes the universal covering.*

*Proof.* Let $X$ be an $(n-1)$-connected $H$-space with $\pi_n(X) = G$. There is an $H$-map $X \to K(G,n)$ inducing an isomorphism in $\pi_n$ and with fibre the $n$-connected cover $X\langle n\rangle$. If one can find a map $K(G,n) \to X$ inducing an isomorphism in $\pi_n$, this map will give a splitting $X \simeq X\langle n\rangle \times K(G,n)$.

For $n = 0$ this is automatic.

For $X = Q_1S^0 \simeq Q_0S^0$, $\pi_1(X) = \mathbb{Z}/2$ and the definition of the Dyer-Lashof operation $Q^1[1] \in H_1(Q_1S^0;\mathbb{F}_2)$ gives a map

$$\mathbb{R}P^\infty = B\mathbb{Z}/2 \to Q_0S^0$$

inducing an isomorphism in $H_1$ and thus by the Hurewicz theorem an isomorphism in $\pi_1$ and the splitting of $QS^0$ follows.

For $X = \Omega_0Q_0S^0$, $\pi_1(X) = \mathbb{Z}/2$. The Hopf map gives an infinite loop map $\eta : Q(S^1) \to Q_0S^0$. I claim it is nonzero in $\pi_2$. To see this it suffices to show that $(\eta\langle 1\rangle)_*$ is nonzero in $H_2$ which can be seen as follows. Let $\sigma \in H_1(QS^1)$ be the fundamental class. Since $QS^1 \approx S^1 \times QS^1\langle 1\rangle$, the element $Q^1\sigma \in H_2(QS^1)$ must be in the image from $H_*(QS^1\langle 1\rangle)$. Since $\eta_*(Q^1\sigma) = Q^1(Q^1[1] * [-2]) \neq 0$, $\eta\langle 1\rangle_*$ is indeed nonzero in $H_2$.

Hence, $\Omega_0\eta : Q_0S^0 \to \Omega_0 Q_0 S^0$ is nonzero in $\pi_1$ and thus the composition

$$\mathbb{R}P^\infty \to Q_0S^0 \to \Omega_0 Q^0 S^0$$

is nonzero in $\pi_1$ and the splitting of $\Omega QS^0$ follows. □

Lemma 7.1 ensures that our spectral sequences has trivial local coefficients and hence that the spectral sequences converges.

7.1. **Recollections.** The structural results about Hopf-algebras from Section 2 hold with the following remarks: The Frobenius map $\xi : A \to A, x \mapsto x^2$ is no longer automatically 0 in odd dimensions. Thus, in Theorem 2.4, we can only conclude that $PA \to QA$ is an isomorphism in odd degrees. Borel's structure theorem 2.5 holds for $p = 2$ with the remark that there are no restrictions on the parity of the generators, and that polynomial algebras truncated at height $2 = p^1$ is the same thing as exterior algebras. In particular, it still holds that $A$ is polynomial if $\xi : A \to A$ is injective and that $A^*$ is polynomial if $\lambda : A \to A$ is surjective.

The Dyer-Lashof algebra is also quite different. For $p = 2$, we let $\mathscr{R}$ be the free non-commutative algebra on the set $\{Q^s \mid s \geq 0\}$ with $\deg(Q^s) = s$. The Adem relation $\mathscr{A}^{(0,r,0,s)}$ in Definition 3.3 still makes sense, and we let $\mathscr{A} \subseteq \mathscr{R}$ be the span of the $\mathscr{A}^{(0,r,0,s)}$. The unstability relations at $p = 2$ are

$$Q^s x = \begin{cases} x^2 & \text{if } \deg x = s \\ 0 & \text{if } \deg x > s \end{cases}$$

and the algebra $R$ is defined from these data as before. Corresponding to $I = (s_1, s_2, \ldots, s_k)$ there is an iterated operation $Q^I = Q^{s_1} \ldots Q^{s_k}$, and this operation is called admissible if $s_i \leq 2s_i$ for all $i$. The definition of excess at $p = 2$ is

$$e(I) = s_1 - \sum_{j=2}^{k} s_j$$

Given a basis $B \subseteq JH_*(X)$, then $H_*(Q_0X)$ is the polynomial algebra on the set

$$\{\tau_*(Q^I x) \mid x \in B,\, I \text{ admissible},\, e(I) > \deg(x),\, \deg(Q^I x) > 0\}$$

where $\tau : QX \to Q_0X$ is the "translation" map from Section 3.4.

One pleasant feature of $p = 2$ is the following

**Lemma 7.2.** *The cohomology algebra $H^*(Q_0X)$ is polynomial if $H^*(X)$ is polynomial.*

*Proof.* This is because the Nishida relation $\lambda Q^{2s} = Q^s\lambda$ makes $\lambda : H_*(Q_0X) \to H_*(Q_0X)$ surjective if $\lambda : H_*(X) \to H_*(X)$ is surjective. □

In particular, $H_*(Q_0S^0)$ and $H_*(Q_0\mathbb{C}P^\infty_+)$ are both polynomial.

The calculation in Theorem 2.8 is valid with the remark that $k[x]/(x^2)$ must be interpreted as $E[x]$ and thus it does not produce generators of Cotor in bidegree $(-2, *)$. Only truncations at height $p^n, n \geq 2$ does that.

An important difference is that for odd primes, $\mathrm{Cotor}^A(k, k)$ is automatically a free algebra. This is no longer true for $p = 2$, since $\mathrm{Tor}^{k[x]}(k, k) = E[s^{-1}x]$, and exterior algebras are not free in characteristic 2.

One consequence of the above remarks is the following

**Proposition 7.3.** *Let $X$ be a simply connected space with $H^*(X)$ polynomial. Then $H_*(\Omega X)$ is an exterior algebra and the suspension*

$$\sigma_* : QH_*(\Omega X) \to PH_*(X)$$

*is an isomorphism. The spectral sequence*

$$\mathrm{Cotor}^{H_*(X)}(k, k) \Rightarrow H_*(\Omega X)$$

*collapses.*

*Proof.* This is because

$$\mathrm{Cotor}^{H_*(X)}(k, k) \cong E[s^{-1}PH_*(X)]$$

has generators and primitives in bidegrees $(-1, *)$. Together with Lemma 2.15, this proves the claims. □

Similarly, we have

**Proposition 7.4.** *For any space $X$, the spectral sequence*

$$\mathrm{Cotor}^{H_*(\tilde{Q}\Sigma X)}(k, k) \Rightarrow H_*(Q_0X)$$

*collapses and the suspension*

$$\sigma_* : QH_*(Q_0X) \to PH_*(\tilde{Q}\Sigma X)$$

*is an isomorphism.*

*Proof.* $\sigma_*$ is surjective since it hits $JH_*(\Sigma X)$ and since it is $R$-linear. Thus by Corollary 2.16, the spectral sequence must collapse. Now $H_*(Q\Sigma X)$ is primitively generated, so by Theorem 2.4 we get that $H^*(Q\Sigma X)$ is exterior and hence the spectral sequence has

$$E^2 = \mathrm{Cotor}^{H_*(\tilde{Q}\Sigma X)}(k, k) \cong k[s^{-1}PH_*(\tilde{Q}\Sigma X)]$$

Since this is free as an algebra, there are no extension problems in homology, and since $QH_*(Q_0X)$ is in linear bijection with $E^{\infty}_{-1,*}$, we get that $\sigma_*$ is injective. □

7.2. **Homology of $\Omega^\infty\mathbb{C}P^\infty_{-1}$.** The lemmas in subsection 7.1 imply the following diagram

$$\begin{tikzcd} QH_*(Q_0\mathbb{C}P^\infty_+) \arrow[r, "Q(\Omega_0\partial)_*"] \arrow[d, "\cong"'] & QH_*(\Omega_0 QS^0) \arrow[d, "\cong"] \\ PH_*(\tilde{Q}\Sigma\mathbb{C}P^\infty_+) \arrow[r, "P\partial_*"] & PH_*(\tilde{Q}_0S^0) \end{tikzcd} \tag{7.2}$$

in which the vertical isomorphisms are the suspensions.

The formula for $\partial_*$ has an extra term because of the Hopf map $\eta$. We quote the result from [MMM, Theorem 4.4]:

**Theorem 7.5** ([MMM])**.** *Let $a_s \in H_*(\mathbb{C}P^\infty_+)$ be the generator, $s$ odd. Then*

$$Q(\partial_*)(a_s) = Q^{2s+1}[1] * [-2] + Q^{s+1}Q^s[1] * [-4]$$

□

We shall need a lemma analogous to Lemma 4.3

**Lemma 7.6.** *The left ideal in $R$ generated by $\{Q^{2s+1} \mid s \geq 0\}$ is also a right ideal.*

*Proof.* This is completely analogous to the proof of Lemma 4.3. One uses the Adem relation

$$Q^{2s}Q^{r-s} = Q^rQ^s + \sum_{i>s} \lambda_i Q^{r+s-i}Q^i$$

valid for $r \leq 2s$, for $r$ odd and $s$ even. □

**Lemma 7.7.** *Let $b_{2s+1} \in PH_*(Q_0S^0)$ be the unique primitive element with $b_{2s+1} - Q^{2s+1}[1] * [-2]$ decomposable. Then $PH_*(Q_0S^0)$ is generated over $R$ by the set $\{b_{2s+1} \mid s \geq 0\}$.*

*Proof.* Let $\lambda : QH_*(Q_0S^0) \to QH_*(Q_0S^0)$ be the dual of the squaring. By the Nishida relation $\lambda Q^{2s} = Q^s\lambda$, the coimage of $\lambda$ has basis

$$\{Q^I[1] * [-2^{\ell(I)}] \mid I \text{ admissible, } e(I) > 0,\ 2|I\}$$

where $2|I$ means that all entries of $I$ are even. Thus Theorem 2.4 implies that the image of $PH_*(Q_0S^0) \to QH_*(Q_0S^0)$ has basis

$$\{Q^I[1] * [-2^{\ell(I)}] \mid I \text{ admissible, } e(I) > 0,\ 2 \nmid I\}$$

and by Lemma 7.6, this is generated over $R$ by the subset

$$\{Q^{2s+1}[1] * [-2] \mid s \geq 0\}$$

Thus the subspace of $PH_*(Q_0S^0)$ generated over $R$ by $\{b_{2s+1} \mid s \geq 0\}$ contains all indecomposable primitives. But this generated subspace is clearly preserved by the Frobenius map $\xi : x \mapsto x^2$, so the claim follows from Theorem 2.4. □

We are now ready to prove the mod 2 analogue of Theorem 5.1. The result is much simpler, and the extra term in Theorem 7.5 does not give much trouble.

**Theorem 7.8.** *The map*

$$P\partial_* : PH_*(Q\Sigma\mathbb{C}P^\infty_+) \to PH_*(Q_0S^0)$$

*is surjective.*

*Proof.* By the previous lemma, it suffices to prove that $Q\partial_*$ hits the classes $Q^{2s+1}[1] * [-2]$. Indeed, any indecomposable class mapping to $Q^{2s+1}[1] * [-2]$ is odd-dimensional and thus by 2.4 has a unique primitive representative that will map to $b_{2s+1}$.

For $s = 0$, this is immediate, since $\partial_*(a_1) = Q^1[1] * [-2]$. For general $s$ we use the Adem relation $Q^{2s}Q^1 = Q^{s+1}Q^s$ to get

$$\begin{aligned} Q(\partial_*)(a_{2s+1}) &= Q^{2s+1}[1] * [-2] + Q^{s+1}(Q^s[1] * [-2]) \\ &= Q^{2s+1}[1] * [-2] + Q^{2s}(Q^1[1] * [-2]) \end{aligned}$$

Thus we have

$$Q(\partial_*)(a_s - Q^{2s}a_1) = Q^{2s+1}[1] * [-2]$$

□

*Remark* 7.9. The claim of [MMM, Cor. 7.5] that $\partial_*$ and thus $P(\partial_*)$ is *injective* is incorrect. The $Q^IQ^{2r+1}$ of [MMM, Cor. 7.4] is not necessarily admissible, and in fact an application of the Adem relations shows that

$$\partial_*(Q^3a_1 - Q^2Q^1a_1) = 0$$

Together with the diagram (7.2), Theorem 7.8 makes the spectral sequence

$$\mathrm{Cotor}^{H_*(\tilde{\Omega}QS^0)}(H_*(\tilde{Q}\mathbb{C}P^\infty_+), k) \Rightarrow H_*(\Omega_0^\infty\mathbb{C}P^\infty_{-1}) \tag{7.3}$$

very simple. We have

**Theorem 7.10.** *The spectral sequence* (7.3) *collapses and the map*

$$(\Omega_0\omega)_* : H_*(\Omega_0^\infty\mathbb{C}P^\infty_{-1}) \to H_*(Q_0\mathbb{C}P^\infty_+)\backslash\!\!\backslash(\Omega_0\partial)_*$$

*is an isomorphism.*

*Proof.* From Theorem 7.8 and the diagram (7.2) we get that $Q(\Omega_0\partial_*)$ and hence $\Omega_0\partial_*$ itself are surjective maps. Hence in the splitting of the $E^2$-term

$$E^2 \cong \mathrm{Cotor}^{H_*(\tilde{\Omega}_0QS^0)/\!\!/(\Omega\partial)_*}(k,k) \otimes H_*(\tilde{Q}_0\mathbb{C}P^\infty_+)\backslash\!\!\backslash(\Omega\partial)_*$$

the Cotor-factor vanishes, and the spectral sequence is concentrated on the fibre line $E^2_{0,*} \subseteq E^2_{*,*}$. □

7.3. **Homology of $\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}$.** This part of the calculation is similar to the odd primary case. We consider again the spectral sequence (5.1) with the splitting (5.2). Notice that the fibration (1.1) splits off the fibration $S^1 \to S^1 \to \mathbb{R}P^\infty$ and hence it has trivial local coefficients. As for odd primes, we need to determine the coalgebra structure on $H_*(Q_0S^0)/\!\!/\partial_*$. The following theorem is analogous to Theorem 5.1.

**Theorem 7.11.** *Let $H_*(Q_0S^0)^{(0)} \subseteq H_*(Q_0S^0)$ denote the subalgebra generated by the set*

$$\{Q^I[q] * [2^{\ell(I)}] \mid I \text{ admissible, } e(I) > 0,\ 2|I\}$$

*Then the composition*

$$H_*(Q_0S^0)^{(0)} \to H_*(Q_0S^0) \to H_*(Q_0S^0)/\!\!/\partial_*$$

*is an isomorphism of algebras.*

*Proof.* Since $Q$ is right exact we have $Q(H_*(Q_0S^0)/\!\!/\partial_*) = \mathrm{Cok}(Q\partial_*)$, and from the calculation in the proof of Lemma 7.7 follows that the composition is surjective.

To prove injectivity, consider again the dual squaring $\lambda : H_*(Q_0S^0) \to H_*(Q_0S^0)$. It is a map of Hopf algebras, and since $\lambda Q^{2s} = Q^s\lambda$ and $\lambda Q^{2s+1} = 0$ we get that

$$\lambda : H_*(Q_0S^0)^{(0)} \to H_*(Q_0S^0)$$

is an isomorphism. Hence

$$H_*(Q_0S^0) = H_*(Q_0S^0)^{(0)} \oplus \mathrm{Ker}(\lambda)$$

where the first summand is a subalgebra and the second is an ideal. Now the injectivity of the map in the theorem follows from the fact that $\mathrm{Ker}(\lambda)$ is an ideal and that $\mathrm{Im}(\partial_*) \subseteq k \oplus \mathrm{Ker}(\lambda)$. □

**Theorem 7.12.** *$H_*(Q_0S^0)/\!\!/\partial_*$ is dual to a polynomial algebra.*

*Proof.* This follows since $\lambda : H_*(Q_0S^0) \to H_*(Q_0S^0)$ is surjective. □

Notice that $H^*(Q_0S^0)$ itself is polynomial. This is in contrast to the odd primary case, where only the subalgebra $H^*(Q_0S^0)\backslash\!\!\backslash\partial^* \subseteq H^*(Q_0S^0)$ is polynomial.

As for odd primes, Theorem 7.12 makes the spectral sequence collapse. The spectral sequence is

$$E^2 = \mathrm{Cotor}^{H_*(Q_0S^0)}(H_*(Q\Sigma\mathbb{C}P^\infty_+), k) \Rightarrow H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}) \tag{7.4}$$

and the $E^2$-term splits as

$$\begin{aligned} E^2 &\cong \mathrm{Cotor}^{H_*(Q_0S^0)/\!\!/\partial_*}(k,k) \otimes H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\!\backslash\partial_* \\ &\cong E[s^{-1}P(H_*(Q_0S^0)/\!\!/\partial_*)] \otimes H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\!\backslash\partial_* \end{aligned}$$

Again primitives and generators are concentrated in $E^2_{-1,*}$ and $E^2_{0,*}$ and hence we have

**Theorem 7.13.** *The spectral sequence* (7.4) *collapses and* $E^2 = E^\infty$ *is given by*

$$E^2 \cong E[s^{-1}P(H_*(Q_0S^0)/\!/\partial_*)] \otimes H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\backslash\partial_*$$

□

*Remark* 7.14. Since the $E^\infty$-term is not a free algebra as it was for $p$ odd, we cannot immediately get the algebra structure of $H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$ up to isomorphism. This is the topic of the next section.

7.4. **Hopf algebra structure of** $H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$**.** ¿From the spectral sequence (7.4) and Theorem 7.13 we get that

$$k \longrightarrow H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\backslash\omega_* \longrightarrow H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1}) \xrightarrow{\omega_*} H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\backslash\partial_* \longrightarrow k \tag{7.5}$$

is a short exact sequence of Hopf algebras. We proceed to identify the kernel $H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\backslash\omega_*$ and to prove a splitting result analogous to Theorem 5.7.

**Proposition 7.15.** $H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\backslash\omega_*$ *is an exterior algebra primitively generated by the vectorspace* $s^{-1}P(H_*(Q_0S^0)/\!/\partial_*)$*.*

*Proof.* Since $H_*(Q\Sigma\mathbb{C}P^\infty_+)$ is a free algebra, so is $H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\backslash\partial_*$, and so by Proposition 2.11 we get an isomorphism

$$Q(H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\backslash\omega_*) \xrightarrow{\cong} \mathrm{Ker}(Q\omega_*)$$

The spectral sequence (7.4) and Theorem 7.13 gives a map

$$s^{-1}P(H_*(Q_0S^0)/\!/\partial_*) \to E^\infty_{-1,*} = F_{-1}/F_{-2}$$

for a filtration $F_0 \supseteq F_{-1} \supseteq \dots$ of $H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$. By choosing a lift to $F_{-1} \subseteq H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$ we get a map of algebras

$$\varphi : k[s^{-1}P(H_*(Q_0S^0)/\!/\partial_*)] \to H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$$

and since the image of $\varphi$ generates the ideal $F_{-1} \subseteq H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})$ we get

$$H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})/\!/\varphi \cong H_*(Q\Sigma\mathbb{C}P^\infty_+)\backslash\!\backslash\partial_*$$

By right exactness of $Q$ we get an induced isomorphism

$$s^{-1}P(H_*(Q_0S^0)\backslash\!\backslash\partial_*) \to \mathrm{Ker}(Q\partial_*) = Q(H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\backslash\omega_*)$$

Thus $H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\backslash\omega_*$ is generated by odd-dimensional classes that may be assumed primitive by Theorem 2.4.

For dimensional reasons, these generators must have height 2, and hence $H_*(\Omega^\infty\Sigma\mathbb{C}P^\infty_{-1})\backslash\!\backslash\omega_*$ is exterior. □

As for odd primes, Lemma 5.6 gives the following

**Theorem 7.16.** *The sequence* (7.5) *is split exact in the category of abelian Hopf algebras. Hence as Hopf algebras we have*

$$H_*(\Omega^\infty \Sigma \mathbb{C}P^\infty_{-1}) \cong H_*(\Omega^\infty \Sigma \mathbb{C}P^\infty_{-1}) \backslash\!\backslash \omega_* \otimes H_*(Q\Sigma CP^\infty_+) \backslash\!\backslash \partial_*$$

□

## References

[AK] S. Araki, T. Kudo: *Topology of $H_n$-spaces and $H_n$-squaring Operations*, Mem. Fac. Sci. Kyushu Univ. Ser. A, **10** (1956), 85–120.

[CLM] F. R. Cohen, T. J. Lada, J. P. May: *The Homology of Iterated Loop Spaces*, Lecture Notes in Mathematics **533**, Springer-Verlag, 1976.

[DL] E. Dyer, R. Lashof: *Homology of Iterated Loop Spaces*, Amer. J. Math. **84** (1962), 35–88.

[EM] S. Eilenberg, J. C. Moore: *Homology and Fibrations I Coalgebras, cotensor products and its derived functors*, Comm. Math. Helv. **40** (1965), 199–236.

[MM] J. Milnor, J. C. Moore: *On the Structure of Hopf algebras*, Ann. Math. **81** (1965), 211–264.

[MMM] B. M. Mann, E. Y. Miller, H. R. Miller: *$S^1$-equivariant function spaces*, Trans. Amer. Math. Soc. **295** (1989), 233–256.

[MT] I. Madsen, U. Tillmann: *The Stable Mapping Class Group and* $Q(\mathbb{C}P^\infty_+)$, Invent. Math. **145** (2001), 509–544.

[MS] J. C. Moore, L. Smith: *Hopf Algebras and Multiplicative Fibrations II*, Amer. J. Math. **90** (1968), 1113–1150.

[MW] I. Madsen, M. Weiss: *Cohomology of the Stable Mapping Class Group*, in preparation.

[CE] H. Cartan, S. Eilenberg: *Homological algebra*, Princeton University Press, 1956.

[S] L. Smith: *Homological algebra and the Eilenberg-Moore spectral sequence*, Trans. Amer. Math. Soc. **129** (1967), 58–93.

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