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D'ALEMBERT'S AND WILSON'S FUNCTIONAL EQUATIONS ON STEP 2 NILPOTENT GROUP

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d'Alembert's and Wilson's functional equations on step 2 nilpotent groups

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Abstract

We describe the set of solutions of Wilson's functional equation on any step 2 nilpotent group and how the set of classical solutions in certain cases must be supplemented by 4-dimensional spaces of solutions.

1 Introduction

The subject of the present paper is the theory of functional equations on groups. More precisely the study of Wilson's functional equation on step 2 nilpotent groups (this notion is defined below).

By *Wilson's functional equation* on a group G we will here understand the functional equation

$$f(xy) + f(xy^{-1}) = 2f(x)g(y), \quad x, y \in G, \tag{1.1}$$

where $f, g : G \to \mathbb{C}$ are two unknown functions to be determined.

Special cases of Wilson's functional equation are *d'Alembert's functional equation*

$$g(xy) + g(xy^{-1}) = 2g(x)g(y), \quad x, y \in G. \tag{1.2}$$

and *Jensen's functional equation*

$$f(xy) + f(xy^{-1}) = 2f(x), \quad x, y \in G, \tag{1.3}$$

It is our purpose to characterize the set of complex valued solutions of Wilson's and d'Alembert's functional equations on any step 2 nilpotent group. We single out the step 2 nilpotent groups, because they represent a first step away from the abelian groups, where the situation has been completely cleared up. We do not impose conditions like 2-divisibility. In fact, that condition would deprive us of some of the most interesting examples. Our main interest is to find out what kinds of new phenomena that emerge on non-abelian groups, and not just on step 2 nilpotent groups. So on our way we will derive results about solutions of d'Alembert's and Wilson's functional equations that are valid on any group.

2 Notation etc.

Throughout the paper we let G denote a group with neutral element e and center $Z(G)$. Its *commutator subgroup*, i.e. the subgroup generated by the commutators $[x,y] = xyx^{-1}y^{-1}$, $x,y \in G$, will be denoted by $[G,G]$. The group G is said to be *step 2 nilpotent*, if $[G,G] \subseteq Z(G)$.

In the paper [16] we used the word metabelian instead of 2 step nilpotent. However, the word metabelian has another technical meaning, so we will in the present paper use the terminology step 2 nilpotent.

Lemma 2.1. *For any group G the subgroup generated by its squares contains the commutator group $[G,G]$.*

Proof. This follows from the identity $[x,y] = (xy)^2 y^{-2}(yx^{-1}y^{-1})^2$. □

G is said to be *2-divisible*, if $G = \{x^2 \,|\, x \in G\}$.

An *additive function* on G is a homomorphism of G into the additive group $(\mathbb{C},+)$.

If f is a function on G we let $\check{f}$ be the function $\check{f}(x) = f(x^{-1})$, $x \in G$. We say that f is *even*, if $\check{f} = f$, and that f is *odd* if $\check{f} = -f$.

$$Z(f) = \{u \in G \,|\, f(xyu) = f(xyu) \text{ for all } x,y \in G\} \tag{2.1}$$

is a normal subgroup of G. That $u \in Z(f)$ means that u behaves as if it were in the center of G, when it occurs inside an argument of f, so it can be moved around at will inside. Of course $Z(G) \subseteq Z(f)$. *Kannappan's condition* on f is that $Z(f) = G$.

Let H be a subgroup of G. A function f on the coset space G/H will be identified with the function $f \circ q$ on G, where $q : G \to G/H$ denotes the canonical quotient map.

We let $\mathbb{C}^* = (\mathbb{C} \setminus \{0\}, \cdot)$ be the multiplicative group of non-zero complex numbers.

3 The classical solutions

Let G be any group. For any homomorphism $M : G \to \mathbb{C}^*$ and any $\alpha, \beta \in \mathbb{C}$ the pair $\{f, g\}$, where

$$f = \beta \frac{M + \check{M}}{2} + \alpha \frac{M - \check{M}}{2}, \quad g = \frac{M + \check{M}}{2}, \tag{3.1}$$

is a solution of Wilson's functional equation (1.1).

In the degenerate case of $M = \check{M}$ the pair $\{\beta M + aM, M\}$ where $a : G \to \mathbb{C}$ is an additive function, is also a solution. The degeneracy is inherent in Jensen's functional equation (1.3) where $M = \check{M} = 1$. Thus a solution of Jensen's functional equation is $f = \beta + a$, where a is an additive map a and $\beta \in \mathbb{C}$.

$g = 1$ is always a solution of d'Alembert's functional equation (1.2). More generally $g = (M + \check{M})/2$ is a non-zero solution of d'Alembert's functional equation for any homomorphism $M : G \to \mathbb{C}^*$.

Let us also mention the trivial solutions of Wilson's functional equation of the form $\{f = 0, g$ any complex valued function$\}$, and the trivial solution $g = 0$ of d'Alembert's functional equation.

We will for the sake of brevity call the solutions described above in this section *the classical solutions* of Wilson's, Jensen's and d'Alembert's functional equations.

As we shall see, there are groups G on which the classical solutions do not exhaust the set of all solutions. On the other hand, it is known that all solutions are classical if G is abelian or more generally if all functions occuring satisfy Kannappan's condition.

4 Examples of step 2 nilpotent groups

As mentioned above the set of solutions of d'Alembert's and Wilson's functional equations are known on abelian groups, so the next step in the study of them would naturally be to discuss groups that are close to being abelian. That G is abelian means that $[G, G] = \{e\}$, so that G is close to being abelian could be interpreted as G having a small commutator subgroup $[G, G]$. One way of expressing this is to require that the commutator group is contained in the center of G, i.e. that G is step 2 nilpotent.

Of course, any abelian group is step 2 nilpotent, but there are other step 2 nilpotent groups than the abelian ones. For example the 3-dimensional

Heisenberg group

$$H_3 = \{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R}\} \tag{4.1}$$

Indeed, writing for brevity

$$(x, y, z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \tag{4.2}$$

we find that $Z(H_3) = [H_3, H_3] = \{(0, 0, z) \mid z \in \mathbb{R}\}$.

Also the *Heisenberg group with integer entries*

$$H_3(\mathbb{Z}) = \{(x, y, z) \mid x, y, z \in \mathbb{Z}\} \tag{4.3}$$

is step 2 nilpotent, because we by the same computations as for H_3 find that $Z(H_3(\mathbb{Z})) = [H_3(\mathbb{Z}), H_3(\mathbb{Z})] = \{(0, 0, z) \mid z \in \mathbb{Z}\}$.

More generally we could have considered the groups of Heisenberg type that were introduced in [14, Section 1.1].

By Lemma 2.1, if $x^2 \in Z(G)$ for all $x \in G$ then G is step 2 nilpotent. This holds in particular for the *quaternion group* $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, so Q_8 is step 2 nilpotent.

Any P_3-group, i.e. a group G for which $[G, G]$ has order 1 or 2, is step 2 nilpotent (This follows from Lemma 2.1, because a property of a P_3-group is that $x^2 \in Z(G)$ for any $x \in G$). The quaternion group is a P_3-group.

5 Background and results

In this section we describe earlier works in the literature on complex valued solutions of d'Alembert's and Wilson's functional equations on groups and how our results relate to them.

The continuous solutions of Wilson's functional equation on the group $G = \mathbb{R}$ are derived in Aczél's book [1, Section 3.2.1], where references to works in the area prior to 1960 can be found.

d'Alembert's functional equation was solved on arbitrary abelian groups by Kannappan [10] in 1968. He actually proved a stronger statement, namely that a solution g on any group G is classical if $Z(g) = G$.

Explicit formulas for the solutions of Wilson's functional equation on any abelian group can for example be found in [2, Lemma 3], [11, Lemma 4.2] and [15, Theorem 2.2]. All the solutions are classical. Thus the abelian case is completely solved. The assumption about the group G being abelian may also here be weakened to the assumption that any function occuring satisfies Kannappan's condition.

Our results are based on the results just mentioned.

For certain nilpotent and more generally certain solvable groups the classical solutions of Wilson's functional equation are the only ones: This was proved for (a) connected nilpotent Lie groups (except for Jensen's functional equation) by Friis [8], (b) generalized nilpotent groups in which any element has odd order by Corovei [5], (c) the $(ax + b)$-group by Friis [8]. The groups are not in general step 2 nilpotent, so the results are not covered by the results here.

Jensen's functional equation has its own features. The investigations [12, 13] by Ng, and the ones later by Friis [8], on Jensen's functional equation revealed that other solutions than the classical ones sometimes occur. Stetkær [17] showed that any solution of Jensen's functional equation on any group G is a solution on the quotient group $G/[G, [G, G]]$. This quotient group is always step 2 nilpotent, so the study of Jensen's functional equation really boils down to a study of it on step 2 nilpotent groups. Since Jensen's functional equation is studied in [17] we shall here concentrate on Wilson's functional equation outside the cases where it becomes Jensen's functional equation.

Corovei [6, 7] studied Wilson's functional equation on P_3-groups and more generally on step 2 nilpotent groups in which each commutator has finite order. He obtained rather general expressions for the solutions. Our results contain his and are more explicit.

Basic results on solutions of d'Alembert's functional equation on non-abelian groups were derived by Corovei in [4].

Our previous paper [16] gave a description of the solutions of d'Alembert's functional equation on any step 2 nilpotent group. The present paper relies heavily on the main result of [16] and it can be viewed as a continuation [16].

The main contribution of the present paper to the theory of functional equations on groups is the following description of the set of solutions of d'Alembert's and Wilson's functional equations on any step 2 nilpotent group G:

For any solution $\{f, g\}$ of Wilson's functional equation, g is a solution of d'Alembert's functional equation at least if $f \neq 0$ (Lemma 9.1 + Lemma 7.1(a)). Let g be a solution of d'Alembert's functional equation on G. Then

If $Z(g) = G$, then g has the classical form $g = (M + \check{M})/2$ where $M : G \to \mathbb{C}^*$ is a homomorphism. The corresponding set of solutions f of Wilson's functional equation consists of the classical ones if $M \neq \check{M}$, and if $M = \check{M}$ of the solutions of Jensen's functional equation, multiplied by M (Theorem

10.1).

The condition $Z(g) = G$ is satisfied if G is generated by its squares, so it holds in particular if G is 2-divisible.

If $Z(g) \neq G$, then g is not of the classical form, but of another special form: There exists a surjective homomorphism $\Phi : G \to Q_8$ such that $f = f_0 \circ \Phi$ and $g = g_0 \circ \Phi$, where the pair $\{f_0, g_0\}$ is a solution of Wilson's functional equation on Q_8 (Theorem 11.4). For each fixed non-classical g the corresponding vector space of solutions f of Wilson's functional equation on G has dimension 4, of which the odd solutions form a 3-dimensional subspace, and the even ones a 1-dimensional subspace (Theorem 11.5).

All solutions g of d'Alembert's and $\{f, g\}$ of Wilson's functional equations such that $Z(g) \neq G$ are written down explicitely for the quaternion group Q_8 (See Example 11.3) and for the Heisenberg group with integer entries $H_3(\mathbb{Z})$ (See Example 11.6).

Remark 5.1. The quaternion group Q_8 is a subgroup of the group of unit quaternions $G = \{\alpha + \beta i + \gamma j + \delta k \,|\, \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1\}$. The latter group is not nilpotent. By the contrary, $[G, G] = G$, because G is isomorphic to a connected semisimple Lie group, viz. $SU(2)$ (see Section 1.9 of [3]). Aczél, Chung and Ng exhibited in [2, Remark 5] a non-classical solution of d'Alembert's functional equation on G. Actually $g = 1$ is the only classical solution on this group because $[G, G] = G$. This example is in complete contrast to the abelian case: All solutions are non-classical, apart from the trivial functions $g = 0$ and $g = 1$, that are solutions on any group. G is not a step 2 nilpotent group, so our results do not apply to it.

6 d'Alembert's functional equations on general groups

We will encounter *d'Alembert's long functional equation*

$$g(xy) + g(yx) + g(xy^{-1}) + g(y^{-1}x) = 4g(x)g(y), \quad x, y \in G. \tag{6.1}$$

in our studies of Wilson's functional equation on a general group (Lemma 9.1). That is our reason for including this section about d'Alembert's long functional equation.

The first result implies that the results in this section hold not just for d'Alembert's long functional equation, but also for d'Alembert's functional equation (1.2).

Lemma 6.1. *Let G be any group.*

(a) *Any solution g of d'Alembert's functional equation (1.2) is invariant under inner automorphisms, i.e. $g(xy) = g(yx)$ for all $x, y \in G$.*

(b) *Any solution of d'Alembert's functional equation is also a solution of the long one.*

Proof. Theorem IV.1 and Lemma V.1 of [16]. □

Let us note the following properties of the solutions of d'Alembert's long functional equation.

Lemma 6.2. *Let G be any group, and let g be a non-zero solution of d'Alembert's long functional equation on G.*

(a) *g is even, so that $g = \check{g}$, $g(e) = 1$ and $g(x^2) + 1 = 2g(x)^2$ for all $x \in G$. In particular $g(x^2) = 1$ if and only if $g(x)^2 = 1$.*

(b) *If there exists a $u_0 \in Z(g)$ such that $g(u_0)^2 \neq 1$, then g has the form $g = (M + \check{M})/2$ for some homomorphism $M : G \to \mathbb{C}^*$.*

(c) *If $g(u_0)^2 = 1$ for some $u_0 \in Z(g)$ then $g(xu_0) = g(x)g(u_0)$ for all $x \in G$.*

(d) *If $Z(g) \neq G$, then $g(xu) = g(x)g(u)$ for all $x \in G$ and $u \in Z(g)$, and g is a homomorphism of $Z(g)$ into the multiplicative group $\{\pm 1\}$ with kernel $H_1 = \{x \in Z(g) \mid g(x) = 1\}$.*

(e) *If $|g(x)| = 1$ for all $x \in G$, then g is a homomorphism of G into the multiplicative group $\{\pm 1\}$.*

Proof. (a) comes from [16, Lemma III.2], and (b) from [4, Theorem 1] or [16, Theorem III.2]. (c) comes from the formula (22) of Lemma III.5 of [16].
(d) Assume that $Z(g) \neq G$. Combining (b) and (c) we get that $g(xu) = g(x)g(u)$ for all $x \in G$ and $u \in Z(g)$. It follows that g is a homomorphism on $Z(g)$. Being an even homomorphism g must take its values in $\{\pm 1\}$.
(e) By the assumption of this case g only takes values on the unit circle, so the left hand side of

$$\frac{g(xy) + g(xy^{-1}) + g(yx) + g(y^{-1}x)}{4} = g(x)g(y), \quad \text{for all } x, y \in G, \tag{6.2}$$

is a convex combination of four numbers, each of length 1. Also the right hand side has length 1. Due to the strict convexity of the unit ball all four numbers in the numerator on the left hand side equals the right hand side. In particular we get that $g(xy) = g(x)g(y)$ which shows that g is a homomorphism of G into the unit circle. By (a) g is even, so for any $x \in G$ we get that $g(x)^2 = g(x)g(x) = g(x)g(x^{-1}) = g(xx^{-1}) = g(e) = 1$, from which we infer that $g(x) = \pm 1$. □

The next result means that a solution of d'Alembert's long functional equation is a function on a smaller group than G, viz. on a quotient group G/H_1. In special cases later on we will find that G/H_1 is isomorphic to the quaternion group Q_8.

Proposition 6.3. *Let G be any group, and let $g \neq 0$ be a solution of d'Alembert's long functional equation on G. Let $H_1 = \{x \in Z(g) \mid g(x) = 1\}$.*

(a) H_1 is a normal subgroup of G.

(b) $Z(G/H_1) = Z(g)/H_1$.

(c) g is a function on G/H_1.

Proof. (a) and (c) If there exists a $u_0 \in Z(g)$ such that $g(u_0)^2 \neq 1$ then g has according to Lemma 6.2(b) the form $g = (M + \check{M})/2$ for some homomorphism $M : G \to \mathbb{C}^*$. In particular $Z(g) = G$. Since we for given $x \in G$ have that $g(x) = 1$ if and only if $M(x) = 1$ we see that $H_1 = \{x \in G \mid M(x) = 1\} = \ker M$. As the kernel of a homomorphism H_1 is a normal subgroup.

To prove that g is a function on G/H_1 we let $x \in G$ and $u \in H_1$ be arbitrary. Then $2g(xu) = M(xu) + \check{M}(xu) = M(x)M(u) + M(u^{-1}x^{-1}) = M(x) + M(u)^{-1}M(x^{-1}) = M(x) + M(x^{-1}) = 2g(x)$ shows that g is a function on G/H_1.

If $g(u)^2 = 1$ for all $u \in Z(g)$ then we know by Lemma 6.2(c) that $g(xu) = g(x)g(u)$ for all $x \in G$ and $u \in Z(g)$. The remainder of this part of the proof is obvious, so we don't give it.

(b) If g has the form $g = (M + \check{M})/2$ then $Z(g) = G$ and $H_1 = \{x \in G \mid g(x) = 1\} = \{x \in G \mid M(x) = 1\}$, so the identity here in (b) reduces to $Z(G/\ker M) = G/\ker M$. But this identity is true, because M is a group character.

If $g(u)^2 = 1$ for all $u \in Z(g)$ we have that $g(xu) = g(x)g(u)$ for all $x \in G$ and $u \in Z(g)$. We will first prove that $Z(G/H_1) \subseteq Z(g)/H_1$, so let $x_0H_1 \in Z(G/H_1)$. Then $x_0^{-1}H_1 \in Z(G/H_1)$, so $[x_0^{-1}, y^{-1}] \in H_1 \subseteq Z(g)$ and hence $g([x_0^{-1}, y^{-1}]) = 1$ for all $y \in G$. Now we find for all $x, y \in G$ that $g(xx_0y) = g(xyx_0[x_0^{-1}, y^{-1}]) = g(xyx_0)g([x_0^{-1}, y^{-1}]) = g(xyx_0)$, which shows that $x_0 \in Z(g)$, and so that $x_0H_1 \in Z(g)/H_1$ as desired.

Consider conversely $u_0H_1 \in Z(g)/H_1$. We shall show that $u_0x = xu_0$ modulo H_1 for each $x \in G$. In other words that $[u_0, x] \in H_1$. But $Z(g)$ is a normal subgroup of G, so $[u_0, x] \in Z(g)$. Finally, since $u_0 \in Z(g)$, we get that $g([u_0, x]) = g(u_0xu_0^{-1}x^{-1}) = g(xu_0u_0^{-1}x^{-1}) = g(e) = 1$, so $[u_0, x] \in H_1$ as desired. $\square$

We end this section by stating a result that connects solutions of d'Alembert's functional equation on different groups.

Lemma 6.4. *Let G and G_0 be two groups, and let $\Phi : G \to G_0$ be a homomorphism of G onto G_0. Let $g_0 : G_0 \to \mathbb{C}$ be a function on G_0, and define $g = g_0 \circ \Phi : G \to \mathbb{C}$.*

(a) g is a non-zero solution of d'Alembert's functional equation on G, if and only if g_0 is a non-zero solution of d'Alembert's functional equation on G_0.

(b) g has the form $g = (M + \check{M})/2$, where $M : G \to \mathbb{C}^*$ is a homomorphism, if and only if g_0 has the form $g_0 = (M_0 + \check{M}_0)/2$, where $M_0 : G_0 \to \mathbb{C}^*$ is a homomorphism.

Proof. The only statement which is not trivial to prove, is the only if statement in (b), so we shall content ourselves with writing down a proof of that. So let g be of the form $g = (M + \check{M})/2$, where $M : G \to \mathbb{C}^*$ is a homomorphism. Now, $g_0(\Phi(x) = g(x) = (M(x) + \check{M}(x))/2$ for all $x \in G$. Taking $x \in \ker\Phi$ we get that $1 = (M(x) + \check{M}(x))/2$ which implies that $M(x) = 1$. So $M(x) = 1$ for all $x \in \ker\Phi$. Thus M is a homomorphism of $G/\ker\Phi$ into $\mathbb{C}^*$. Since $G/\ker\Phi \cong G_0$, Φ being surjective, we may define $M_0 : G_0 \to \mathbb{C}^*$ by $M_0(\Phi(x)) = M(x)$ for all $x \in G$. Then M_0 is a homomorphism and

$$g_0(\Phi(x)) = g(x) = (M(x) + \check{M}(x))/2 = (M_0(\Phi(x) + \check{M}_0(\Phi(x))/2,$$

which shows that $g_0 = (M_0 + \check{M}_0)/2$ as desired. □

7 d'Alembert's functional equation on step 2 nilpotent groups

In this section we specialize to step 2 nilpotent groups. In view of the next result (Lemma 7.1) we may for such groups concentrate on d'Alembert's functional equation in the short form.

Lemma 7.1. *If G is step 2 nilpotent, then*

(a) the solutions of d'Alembert's functional equation (1.2) and d'Alembert's long functional equation (6.1) are the same.

(b) If g is a solution of d'Alembert's functional equation (1.2) on G, then $\{x \in Z(g) \mid g(x) = 1\} = \{x \in G \mid g(x) = 1\}$.

Proof. (a) Theorem IV.1 of [16]. (b) If there exists a $u_0 \in Z(g)$ such that $g(u_0)^2 \neq 1$ then $Z(g) = G$ (Lemma 6.2), so the statement is true. If $g(u)^2 = 1$ for all $u \in Z(g)$ then $g(xu) = g(x)g(u)$ for all $x \in G$ and all $u \in Z(g)$, again by Lemma 6.2. We are through by Proposition II.2(d) of [16]. □

The following fundamental result on d'Alembert's functional equation was derived in [16, Theorem V.4 and Theorem III.2] for a more general functional equation. We choose to formulate it in slightly other way than in [16]. The reason is that, in contrast to the corresponding result of [16], the two cases of Theorem 7.2 below exclude one another: In Case I the function g is identically 1 on $[G, G]$, while g in Case II attains the value -1 somewhere on $[G, G]$.

Theorem 7.2. *Let G be a step 2 nilpotent group. The non-zero solutions $g : G \to \mathbb{C}$ of d'Alembert's functional equation (1.2) are the following:*

(I) There exists a homomorphism $M : G \to \mathbb{C}^*$ such that $g = (M + \check{M})/2$.

Here $Z(g) = G$.

(II) There exists a normal subgroup H of G with the property that $x^2 \in H$ for all $x \in G$, and there exists a homomorphism $m : H \to \{\pm 1\}$ with the properties $m(x^2) = -1$ for all $x \in G \setminus H$ and $m(u) = -1$ for some $u \in [G, G]$, such that $g = m$ on H and $g = 0$ on $G \setminus H$.

Here $Z(g) = H \neq G$.

Proof. Comparing with [16, Theorem V.4] we see that it suffices to show the following: Let H be a subgroup of G containing $[G, G]$ and let $m : H \to \{\pm 1\}$ be a homomorphism. Then m extends to a homomorphism $M : G \to \mathbb{C}^*$, if and only if $m([G, G]) = \{1\}$. Assume first that m extends to a homomorphism $M : G \to \mathbb{C}^*$. Any homomorphism M on all of G is 1 on $[G, G]$, hence so is m. Assume conversely that $m \equiv 1$ on $[G, G]$. Then m is a homomorphism of $H/[G, G]$ into $\{\pm 1\}$, so m is a character on the subgroup $H/[G, G]$ of the abelian group $G/[G, G]$. Equipping $G/[G, G]$ with the discrete topology the theory of locally compact abelian groups tells us that m extends to a character on all of $G/[G, G]$ (See [9, Corollary 24.12]). $\square$

Case I is the only one that occurs, when G is generated by its squares, because H contains all squares, so in that case all solutions of d'Alembert's functional equation are classical. Thus a sufficient condition for all solutions of d'Alembert's functional equation on a step 2 nilpotent group G to be classical is that G is generated by its squares x^2, $x \in G$. The example $G = \mathbb{Z}$ shows that the condition is not necessary. Being 2-divisible, the Heisenberg group H_3 is generated by its squares, so all solutions of d'Alembert's functional equation on the Heisenberg group are classical.

Example 7.3. A class of groups similar to the Heisenberg group H_3 comes about as follows. The groups are special instances of the ones constructed by Reiter [14, Section 1.1].

Let A be an abelian, locally compact, Hausdorff, topological group with dual group $\widehat{A}$. Let $\mathbb{T} = \{t \in \mathbb{C} \,|\, |t| = 1\}$ denote the circle group. The composition rule $(x_1, \gamma_1, t_1)(x_2, \gamma_2, t_2) = (x_1 + x_2, \gamma_1\gamma_2, t_1t_2\gamma_2(x_1))$, where $(x_1, \gamma_1, t_1), (x_2, \gamma_2, t_2) \in A \times \widehat{A} \times \mathbb{T}$, makes $G = A \times \widehat{A} \times \mathbb{T}$ a step 2 nilpotent group. It is not abelian, unless A is a 1-element group.

If $m : H \to \{\pm 1\}$ is a homomorphism as in Case II of Theorem 7.2, then $m|Z(G) = 1$, because the center $Z(G) = \{(0, 1, t) \in G \,|\, t \in \mathbb{T}\}$ is 2-divisible. In particular we get that $m|[G, G] = 1$, so Case II does not occur for G. Thus all solutions of d'Alembert's functional equation on G are classical.

As mentioned above, the group G is not abelian. Furthermore it is not necessarily generated by its squares as the example $A = \mathbb{Z}$ shows.

Case II does occur in concrete instances, for example for the quaternion group Q_8 and for the Heisenberg group with integer entries $H_3(\mathbb{Z})$. This is the contents of the next two examples. The first example is also discussed in [6, Example].

Example 7.4. A non-classical solution g_0 of d'Alembert's functional equation on the quaternion group Q_8 is $g_0(\pm 1) = \pm 1$, $g_0(\pm i) = g_0(\pm j) = g_0(\pm k) = 0$. This solution is in Case II of Theorem 7.2, because $g_0(-1) = -1$ and $-1 \in [Q_8, Q_8]$. In the set up of Theorem 7.2 it corresponds to the subgroup $H = \{\pm 1\}$, and the homomorphism $m : H \to \{\pm 1\}$ given by $m(\pm 1) = \pm 1$.

Actually g_0 is the only non-classical solution on Q_8. We proceed by showing that statement.

Let H' and $m' : H' \to \{\pm 1\}$ be respectively a subgroup of Q_8 and a homomorphism corresponding to a solution for which the conditions of Theorem 7.2(II) are satisfied. We will first prove that $H' = H (= \{\pm 1\})$. Since H' contains the squares we have that $H' \supseteq \{\pm 1\}$. If H' contains more than the two elements ± 1, say $i \in H'$, then also $-i \in H'$, because H' is a group. However, H' cannot contain more, because that would imply that $H' = Q_8$, which is not the case for a non-classical solution. Now, since $j \notin H'$, we get that $-1 = m'(j^2) = m'(-1) = m'(i^2) = m'(i)^2 = (\pm 1)^2 = 1$, which is a contradiction. So $H' = H$.

We next show that $m' = m$. The homomorphism m' is uniquely determined by H, because H is generated by squares. Indeed, from Theorem 7.2 we have that $m'(x^2) = m'(x)^2 = 1$ if $x \in H$ and $m'(x^2) = -1$ if $x \notin H$. But the same holds for m. Hence $m' = m$.

The next theorem, which is one of the main results of this paper, reveals the role of the quaternion group Q_8 for the existence and for the form of the non-classical solutions of d'Alembert's functional equation. We shall later (Theorem 11.4) see that Q_8 plays the same role for Wilson's functional equation as it does for d'Alembert's functional equation.

Theorem 7.5. *Let G be a step 2 nilpotent group. Let $g_0 : Q_8 \to \mathbb{C}$ denote the function from Example 7.4.*

The non-classical solutions of d'Alembert's functional equation (1.2) on G are the functions of the form $g_0 \circ \Phi$, where Φ ranges over the surjective homomorphisms of G onto the quaternion group Q_8.

Proof. It follows from Lemma 6.4 that any function of the form $g_0 \circ \Phi$, where Φ is a surjective homomorphism of G onto Q_8, is a non-classical solution of d'Alembert's functional equation on G. Thus it is left to show any non-classical solution of d'Alembert's functional equation on G has this form. The proof of that constitutes Section 8. $\square$

Example 7.6. Let us consider the Heisenberg group with integer entries $H_3(\mathbb{Z}) = \{(x, y, z) \mid x, y, z \in \mathbb{Z}\}$ from Section 4.

The map $\Phi : H_3(\mathbb{Z}) \to Q_8$ given by $\Phi(x, y, z) = (-1)^z k^y j^x$ for $(x, y, z) \in H_3(\mathbb{Z})$ is a surjective homomorphism as is easy to verify. Theorem 7.5 tells us that $g = g_0 \circ \Phi$, where g_0 is the function from Example 7.4, is a non-classical solution of d'Alembert's functional equation on $H_3(\mathbb{Z})$.

Calculations show that $g^{-1}(\mathbb{C}\setminus\{0\}) = \Phi^{-1}(\{\pm 1\}) = \{(2x, 2y, z) \mid x, y, z \in \mathbb{Z}\}$, so the H from Theorem 7.2(II) is here $H = \{(2x, 2y, z) \mid x, y, z \in \mathbb{Z}\}$. We know that $g(x, y, z) = 0$ if $(x, y, z) \notin H$, and we find the following explicit expression for g on H:

$$g(2x, 2y, z) = (-1)^{x+y+z} \text{ for } x, y, z \in \mathbb{Z}. \tag{7.1}$$

To complete the discussion of d'Alembert's functional equation on $H_3(\mathbb{Z})$ we note that the g above is the only non-classical solution.

To prove this uniqueness we start by noting that H is the subgroup of $H_3(\mathbb{Z})$ generated by the squares: By the general theory (Theorem 7.2) we know that H contains the subgroup generated by the squares. To obtain the converse inclusion we note that the group generated by the squares contains the commutator subgroup (Lemma 2.1). In particular it contains the element $(0, 0, 1) = [(1, 0, 0), (0, 1, 0)]$ and hence each element of the form $(0, 0, z)$, $z \in \mathbb{Z}$. From the identities

$$(2x, 2y, z) = (0, 2y, 0)(2x, 0, 0)(0, 0, z) = (0, y, 0)^2 (x, 0, 0)^2 (0, 0, z)$$

we see that each element of H belongs to the group generated by the squares.

Let H' and $m' : H' \to \{\pm 1\}$ be respectively a subgroup of $H_3(\mathbb{Z})$ and a homomorphism satisfying the conditions of Theorem 7.2(II). We will first show that $H^{'} = H$ and then later that $m' = m$. Now $H' \supseteq H$, since H is generated by the squares. Note that $m'(0,0,1) = -1$, because m' otherwise is identically 1 on the commutator subgroup $[H_3(\mathbb{Z}), H_3(\mathbb{Z})] = \{(0,0,z) \,|\, z \in \mathbb{Z}\}$, contradicting that we are in Case II.

We will show that $H' = H$ by contradiction, so we assume that there exists an element $(x_0, y_0, z_0) \in H' \setminus H$. Since $(x_0, y_0, z_0) \notin H'$ we have that x_0 or y_0 or both x_0 and y_0 are odd.

To get on we use the formula $(x,0,0)(a,b,c)(0,y,z-xb-(x+a)y) = (a+x, b+y, c+z)$ which is valid for all $x,y,z,a,b,c \in \mathbb{Z}$. The formula implies that if $(a,b,c) \in H'$ then $(a+2l, b+2m, c+n) \in H'$ for any $l,m,n \in \mathbb{Z}$. So we can add arbitrary even numbers to the two first coordinates of any element of H' and remain in H'. This we use below without explicit mentioning.

Let us assume that x_0 is odd and y_0 is even. The other possibilities can be treated similarly, so we do not write them down. Here we get that $(1,0,0) \in H'$ and so that $H' \supseteq \{(x, 2y, z) \,|\, x,y,z \in \mathbb{Z}\}$. If the inclusion is strict we get that $H' = H_3(\mathbb{Z})$, contradicting that we are in Case II. Thus $H' = \{(x,2y,z) \,|\, x,y,z \in \mathbb{Z}\}$.

Since $(1,1,0) \notin H'$, $(0,1,0) \notin H'$ and $(1,0,0) \in H'$ we find that

$$\begin{aligned}
&-1 = m'((1,1,0)^2) = m'(2,2,1) = m'((0,2,0)(2,0,0)(0,0,1)) \\
&= m'(0,2,0)m'(2,0,0)m'(0,0,1) = m'((0,1,0)^2)m'((1,0,0)^2)(-1) \\
&= (-1)(m'(1,0,0))^2(-1) = (-1)(\pm 1)^2(-1) = 1,
\end{aligned}$$

which is the desired contradiction. Thus $H' = H$.

That $m' = m$ can be proved in exactly the same way as it was done for the quaternion group in Example 7.4.

Remark 7.7. On each of the groups Q_8 and $H_3(\mathbb{Z})$ (Examples 7.4 and 7.6) there is only one non-classical solution of d'Alembert's functional equation. However, the uniqueness is not a general phenomenon: Let $G = G_1 \times G_2$ be the product of two step 2 nilpotent groups G_1 and G_2. Let g_1 and g_2 be non-classical solutions of d'Alembert's functional equation on G_1 and G_2 respectively. Then G is a step 2 nilpotent group, and $g_1 \otimes 1$ and $1 \otimes g_2$ are two different non-classical solutions of d'Alembert's functional equation on G.

8 Proof of Theorem 7.5

Thoughout this section G denotes a step 2 nilpotent group, and $g_0 : Q_8 \to \mathbb{C}$ the function from Example 7.4. Furthermore g denotes a fixed non-classical solution of d'Alembert's functional equation on G such that the conditions of

Theorem 7.2(II) hold. We will complete the proof of Theorem 7.5 by showing that there exists a surjective homomorphisms Φ of G onto the quaternion group Q_8, such that $g = g_0 \circ \Phi$.

Since g is a non-classical solution we have that $H = Z(g) \neq G$. From Proposition 6.3(a) we get that $H_1 = \{x \in Z(g) \,|\, g(x) = 1\}$ is a normal subgroup of G, so $\widetilde{G} = G/H_1$ is a group with neutral element $\widetilde{e} = eH_1$.

Lemma 8.1. *Let $\widetilde{G} = G/H_1$ and write $\widetilde{x} = xH_1 \in \widetilde{G}$ for any $x \in G$.*

(a) $\widetilde{G}$ is a step 2 nilpotent group, but it is not abelian.

(b) The map $uH_1 \mapsto g(u)$ is an isomorphism of $Z(g)/H_1$ onto the multiplicative group $\{\pm 1\}$.

(c) $Z(\widetilde{G}) = \{\widetilde{e}, \widetilde{u_0}\}$, where u_0 is any element in $Z(g)$ for which $g(u_0) = -1$.

(d) If $\widetilde{x} \in Z(\widetilde{G})$ then $\widetilde{x}^2 = \widetilde{e}$. If $\widetilde{x} \in \widetilde{G} \setminus Z(\widetilde{G})$ then $\widetilde{x}^2 = \widetilde{u_0}$. Thus $\widetilde{x}^2 \in Z(\widetilde{G})$ for all $x \in G$.

Proof. (a) $\widetilde{G}$ is step 2 nilpotent as the homomorphic image of a step 2 nilpotent group.
Since $Z(g) \neq G$, we can find an $x_0 \in G \setminus Z(g)$. Using that $Z(\widetilde{G}) = Z(g)/H_1$ (Proposition 6.3) we observe that $x_0 H_1 \notin Z(g)/H_1 = Z(\widetilde{G})$, from which we infer that $\widetilde{G}$ is not abelian.
(b) According to Lemma 6.2(d) the map $uH_1 \mapsto g(u)$ is an injective homomorphism of $Z(g)/H_1 = H/H_1$ into $\{\pm 1\}$, so it is left to prove that it is onto. We do this by contradiction, so we assume that $g(H) = \{1\}$. Since H contains the commutator group we get that g is identically 1 on $[G, G]$, which by [11, Lemma 4.1] means that g is a classical solution. But this contradicts our assumption about g.
(c) Combining the characterization of the center in Proposition 6.3 with (b) we get that $Z(\widetilde{G}) = \{\widetilde{e}, \widetilde{u_0}\}$.
(d) If $x \in G \setminus H$ then $\widetilde{x}^2 \in H/H_1 = Z(\widetilde{G})$ and $g(x^2) = -1$ in this Case II.
$x^2 \in H = Z(g)$ for all $x \in G$, so $\widetilde{x}^2 \in Z(\widetilde{G})$ for all $x \in G$. ☐

During the rest of the proof of Theorem 7.5 we let u_0 denote a fixed element in $Z(g)$ for which $g(u_0) = -1$.

Lemma 8.2. *There exist $x_1, x_2 \in G$ such that $[\widetilde{x_2}, \widetilde{x_1}] = \widetilde{u_0}$. Any such two elements satisfy that $\widetilde{x_1}^2 = \widetilde{x_2}^2 = (\widetilde{x_1}\widetilde{x_2})^2 = \widetilde{u_0}$.*

Proof. If no such two elements exist we get from $[\widetilde{G}, \widetilde{G}] \subseteq Z(\widetilde{G}) = \{\widetilde{e}, \widetilde{u_0}\}$ that $[\widetilde{G}, \widetilde{G}] = \{\widetilde{e}\}$, which means that $\widetilde{G}$ is abelian. But it is not, according to the Lemma just proved. For the last statement we note that if $\widetilde{x_1}^2 = \widetilde{e}$, then $\widetilde{x_1} = \widetilde{e}$ or $\widetilde{x_1} = \widetilde{u_0}$, so that $\widetilde{x_1} \in Z(\widetilde{G})$. We get the contradiction that $[\widetilde{x_2}, \widetilde{x_1}] = \widetilde{e}$. The remaining two elements can be treated similarly. □

Lemma 8.3. *Choose any $x_1, x_2 \in G$ such that $[\widetilde{x_2}, \widetilde{x_1}] = \widetilde{u_0}$. Then each element $\widetilde{x} \in \widetilde{G}$ has the form $\widetilde{x} = \widetilde{x_1}^a \widetilde{x_2}^b \widetilde{u_0}^c$, where $a, b, c \in \{0, 1\}$ are uniquely determined.*

Proof. To avoid excessive notation we will in the proof of this Lemma 8.3 delete the tildes and so write x instead of $\widetilde{x}$ etc.

First the existence of the decomposition of $x \in \widetilde{G}$. If $x \in Z(\widetilde{G})$, then the decomposition is clear, so we may assume that $x \in \widetilde{G} \setminus Z(\widetilde{G})$. In this case $x^2 = u_0$.

If $[x, x_1] = e$, i.e. if x and x_1 commute, then $(x_1^{-1}x)^2 = x_1^{-1}xx_1^{-1}x = x_1^{-2}x^2 = u_0u_0 = e$, so $x_1^{-1}x = e$ or $x_1^{-1}x = u_0$. In any of these two cases the decomposition is trivial. The same argument works of course if $[x, x_2] = e$ and if $[x, x_1x_2] = e$. Left is the case of $[x, x_1] = [x, x_2] = [x, x_1x_2] = u_0$. But that case does not occur, because $[x, x_1x_2] = [x, x_1]x_1[x, x_2]x_1^{-1} = [x, x_1][x, x_2]$, where we used that commutators belong to the center of $\widetilde{G}$.

Next the uniqueness of the decomposition. Let $x_1^a x_2^b u_0^c = x_1^d x_2^e u_0^f$, where $a, b, c, d, e, f \in \{0, 1\}$. We shall prove that $a = d$, $b = e$ and $c + f$. If $a \neq d$, then we get that x_1 has the form $x_1 = x_2^g u_0^h$ for some integers g and h, implying that x_1 commutes with x_2. This is a contradiction, so $a = d$. Cancelling x_1^a we find that $x_2^b u_0^c = x_2^e u_0^f$. If $b \neq e$, then it follows that $x_2 = e$ or $x_2 = u_0$, which again is a contradiction. □

Lemma 8.4. *Choose any $x_1, x_2 \in G$ such that $[\widetilde{x_2}, \widetilde{x_1}] = \widetilde{u_0}$. Then*

(a) the map $\widetilde{\Phi} : \widetilde{G} \to Q_8$ given by $\widetilde{\Phi}(x_1^a x_2^b u_0^c H_1) = (-1)^c k^a j^b$ for $a, b, c \in \{0, 1\}$ is an isomorphism of $\widetilde{G}$ onto the quaternion group Q_8.

(b) Let $a, b, c \in \{0, 1\}$. Then $x_1^a x_2^b u_0^c \in H$ if and only if $a = b = 0$.

Proof. The proof of the first statement is just a straightforward case-by-case checking. For the second one we note that $x_1^a x_2^b u_0^c \in H$ if and only if $x_1^a x_2^b u_0^c H_1 \in H/H_1 = Z(g)/H_1 = Z(\widetilde{G})$. Using that $\widetilde{\Phi}$ is an isomorphism this happens if and only if $(-1)^c k^a j^b \in Z(Q_8)$. And this is the case if and only if $a = b = 0$. □

Let $\Phi = \widetilde{\Phi} \circ q$, where $q : G \to G/H_1$ is the canonical quotient map. We continue with the proof of Theorem 7.5 by showing that $g = g_0 \circ \Phi$. Letting δ_a denote Kronecker's delta we find for any $a, b, c \in \{0, 1\}$ and $h_1 \in H_1$ that

$$g_0(\Phi(x_1^a x_2^b u_0^c h_1)) = g_0((-1)^c k^a j^b) = (-1)^c \delta_a \delta_b.$$

It follows from Lemma 8.4 that $x_1^a x_2^b u_0^c \in H$ if and only if $a = b = 0$. Using that $g(x) \neq 0$ if and only if $x \in H$ we find that $g(x_1^a x_2^b u_0^c h_1) = g(u_0^c h_1)\delta_a \delta_b = g(u_0)^c g(h_1)\delta_a \delta_b = (-1)^c \delta_a \delta_b$.

Thus $g_0(\Phi(x_1^a x_2^b u_0^c h_1)) = g(x_1^a x_2^b u_0^c h_1)$, which finishes the proof of Theorem 7.5.

9 Wilson's functional equation on a general group

Let the pair $\{f, g\}$ be a solution of Wilson's functional equation. Lemma 9.1 below says, when $f \neq 0$, that g is a non-zero solution of d'Alembert's long functional equation. From Lemma 6.1 we get in the step 2 nilpotent case that this means that g is a solution of (the short) d'Alembert functional equation, which was solved in Theorem 7.2. Our strategy to solve Wilson's functional equation is therefore to examine each of the 2 cases of Theorem 7.2 separately.

As prelude, we shall in this Section 9 discuss properties of solutions of Wilson's functional equation

$$f(xy) + f(xy^{-1}) = 2f(x)g(y), \quad x, y \in G, \tag{9.1}$$

on an arbitrary group G which need not be 2 step nilpotent.

Wilson's functional equation and d'Alembert's long functional equation are closely related. The following lemma shows how.

Lemma 9.1. *Let G be any group. If the pair $\{f, g\}$ is a solution of Wilson's functional equation, and $f \neq 0$, then g is a non-zero solution of d'Alembert's long functional equation*

$$g(xy) + g(yx) + g(xy^{-1}) + g(y^{-1}x) = 4g(x)g(y), \quad x, y \in G. \tag{9.2}$$

Proof. This is [5, Lemma 1]. □

Lemma 9.2. *Let G be any group. Let the pair $\{f, g\}$ be a solution of Wilson's functional equation such that $f \neq 0$. Then $Z(f) \subseteq Z(g)$, so that g satisfies Kannappan's condition, if f does.*

Lemma 9.3. *Let G be any group. Let the pair $\{f, g\}$ be a solution of Wilson's functional equation. Let f_e denote the even part of f. Then $f_e = f(e)g$. In particular we see that f is odd, if and only if $f(e) = 0$.*

Proof. Put $x = e$ in Wilson's functional equation. □

Lemma 9.4. *Let G be any group. Let the pair $\{f, g\}$ be a solution of Wilson's functional equation. Let f_e and f_o denote the even and odd parts of f respectively. If g is a solution of d'Alembert's functional equation, then*

$$f_e(xy) + f_e(xy^{-1}) = 2f_e(x)g(y), \quad x, y \in G, \tag{9.3}$$
$$f_o(xy) + f_o(xy^{-1}) = 2f_o(x)g(y), \quad x, y \in G, \tag{9.4}$$

i.e. f_e and f_o both satisfy Wilson's functional equation with g unchanged.

Proof. Use that f_e is proportional to g by Lemma 9.3. □

Lemma 9.5. *Let G be any group. Let the pair $\{f, g\}$ in which f is odd, be a solution of Wilson's functional equation. Then*

$$\frac{f(xy) + f(yx)}{2} = f(x)g(y) + f(y)g(x) \text{ for all } x, y \in G. \tag{9.5}$$

Proof. Using Wilson's functional equation we reformulate the right hand side of (9.5) as follows:

$$\begin{aligned}
&2\{f(x)g(y) + f(y)g(x)\} = [f(xy) + f(xy^{-1})] + [f(yx) + f(yx^{-1})] \\
&= [f(xy) + f(yx)] + [f(xy^{-1}) + f(yx^{-1})] \\
&= [f(xy) + f(yx)] + [f(xy^{-1}) + f((xy^{-1})^{-1})].
\end{aligned}$$

The last term vanishes, f being odd, so we get the identity (9.5). □

Lemma 9.6. *Let G be any group. Let the pair $\{f, g\}$ be a solution of Wilson's functional equation on G, such that $f \neq 0$. Assume furthermore that there exists a $z_0 \in Z(f)$ such that $g(z_0)^2 \neq 1$. Then there exists a homomorphism $M : G \to \mathbb{C}^*$, such that*

(a) $g = (M + \check{M})/2$.

(b) $f = \beta g + \alpha(M - \check{M})$ *for some constants* $\alpha, \beta \in \mathbb{C}$.

Proof. (a) g has the stated form according to Lemma 9.1 and Lemma 6.2. (b) Noting that $\{M, g\}$ is a solution of Wilson's functional equation we see that we may replace f by $f - f(e)M$ and still get a solution of Wilson's functional equation with g unchanged. This has the advantage that it makes $f(e) = 0$, which by Lemma 9.3 means that f is odd.

Putting $y = z_0$ in the identity (9.5) we get using that $z_0 \in Z(f)$ that

$$f(xz_0) = f(x)g(z_0) + f(z_0)g(x) \text{ for all } x \in G. \tag{9.6}$$

When we here replace x by xz_0^{-1} we find from Wilson's functional equation that

$$\begin{aligned} f(x) &= f(xz_0^{-1})g(z_0) + f(z_0)g(xz_0^{-1}) \\ &= [2f(x)g(z_0) - f(xz_0)]g(z_0) + f(z_0)g(xz_0^{-1}), \end{aligned}$$

in which we substitute our expression (9.6) for $f(xz_0)$ to get

$$\begin{aligned} f(x) &= [2f(x)g(z_0) - f(x)g(z_0) - f(z_0)g(x)]g(z_0) + f(z_0)g(xz_0^{-1}) \\ &= [f(x)g(z_0) - f(z_0)g(x)]g(z_0) + f(z_0)g(xz_0^{-1}) \\ &= f(x)g(z_0)^2 - f(z_0)g(z_0)g(x) + f(z_0)g(xz_0^{-1}). \end{aligned}$$

Isolating the terms involving $f(x)$ we find that

$$f(x)[1 - g(z_0)^2] = f(z_0)[g(xz_0^{-1}) - g(z_0)g(x)].$$

Since $g(z_0)^2 \neq 1$ we may divide by $1 - g(z_0)^2$ and get

$$f(x) = \frac{f(z_0)[g(xz_0^{-1}) - g(z_0)g(x)]}{1 - g(z_0)^2},$$

which upon substitution of $g = (M + \check{M})/2$ makes us end up with

$$f(x) = \frac{f(z_0)}{1 - g(z_0)^2} \frac{\check{M}(z_0) - M(z_0)}{2} \frac{M(x) - \check{M}(x)}{2} = \alpha \frac{M(x) - \check{M}(x)}{2},$$

where α is a complex constant. $\square$

Lemma 9.7. *Let G be any group. Let the pair $\{f, g\}$ be a solution of Wilson's functional equation on G. Let $u_0 \in Z(f)$. Assume that*

(a) f is odd,

(b) $g(u_0)^2 = 1$, and

(c) there exists an $x_0 \in G$ for which $g(x_0)^2 \neq 1$.

Then $f(xu_0) = f(x)g(u_0)$ for all $x \in G$.

Proof. Let us fix $x_0 \in G$. We shall prove that $f(x_0u_0) = f(x_0)g(u_0)$.

Consider the subgroup G_0 of G generated by x_0 and u_0. If $f|_{G_0} = 0$, then the result is trivial, so we may assume that $f|_{G_0} \neq 0$. Clearly $Z(f|_{G_0}) = G_0$, so Kannappan's condition is satisfied on G_0. We infer from the abelian case that $g|_{G_0} = (M + \check{M})/2$, where $M : G_0 \to \mathbb{C}^*$ is a homomorphism. Since $g(x_0)^2 \neq 1$ we get that $M \neq \check{M}$, so that (again by the classical case) $f|_{G_0}$ has the form $f|_{G_0} = \alpha(M - \check{M}) + \beta g$, where α and β are complex constants. f being odd this expression reduces to $f|_{G_0} = \alpha(M - \check{M})$. Using the assumption $g(u_0)^2 = 1$ we get that $M(u_0) = \check{M}(u_0) = g(u_0)$, so finally

$$\begin{aligned} f(x_0u_0) &= \alpha[M(x_0u_0) - \check{M}(x_0u_0)] = \alpha[M(x_0)M(u_0) - \check{M}(x_0)\check{M}(u_0)] \\ &= \alpha[M(x_0) - \check{M}(x_0)]g(u_0) = f(x_0)g(u_0), \end{aligned}$$

as desired. □

10 Wilson's functional equation in Case I

In this Section 10 we shall discuss Case I of Theorem 7.2 of Wilson's functional equation, i.e. we shall consider the functional equation

$$f(xy) + f(xy^{-1}) = 2f(x)g(y), \quad x, y \in G, \tag{10.1}$$

in which g has the special form $g = (M + \check{M})/2$, where $M : G \to \mathbb{C}^*$ is a homomorphism.

Theorem 10.1. *Let G be a step 2 nilpotent group. Let the pair $\{f, g\}$ be a solution of Wilson's functional equation (10.1) in which g has the special form $g = (M + \check{M})/2$, $M : G \to \mathbb{C}^*$ being a homomorphism.*

(a) If $M \neq \check{M}$ then f has the form $f = \beta g + \alpha(M - \check{M})$ for some constants $\alpha, \beta \in \mathbb{C}$.

(b) If $M = \check{M}$ then $f = FM$, where F is a solution of Jensen's functional equation on G, i.e.

$$F(xy) + F(xy^{-1}) = 2F(x) \quad \text{for all } x, y \in G. \tag{10.2}$$

Proof. (a) We claim that the result is true, if the assumptions of Lemma 9.6 or the assumptions of Lemma 9.7 are satisfied. For Lemma 9.6 this is incorporated in its very conclusion. To see it for Lemma 9.7 we assume that its hypotheses are satisfied, so that we have its conclusion $f(xz) = f(x)g(z)$ for all $x \in G$ and $z \in Z(G)$. We note that $g([x, y]) = 1$ for all $x, y \in G$ due to the special form of g. Since $[G, G] \subseteq Z(G)$, G being step 2 nilpotent, we find for any $x, y, z \in G$ that $f(xyz) = f(xzy[y^{-1}, z^{-1}]) = f(xzy)g([y^{-1}, z^{-1}]) = f(xzy)$, which says that f satisfies Kannappan's condition. So does g by its very form. The conclusion (a) holds, because it holds when Kannappan's condition is fulfilled (See for example [2, Lemma 2.3]).

If the assumptions of neither Lemma 9.6 nor Lemma 9.7 are satisfied, then $g(x)^2 = 1$ for all $x \in G$. In this case we get by Lemma 6.2(e) that g is a homomorphism of G into the multiplicative group $\{\pm 1\}$ so that $M = \check{M}$. But then we are not in case (a), where $M \neq \check{M}$.

(b) If $M = \check{M}$ we get by dividing through by $M(x)M(y)$ in Wilson's functional equation that

$$\frac{f}{M}(xy) + \frac{f}{M}(xy^{-1}) = 2\frac{f}{M}(x) \quad \text{for all } x, y \in G,$$

proving (b). $\square$

The result of Proposition 10.1 looks very much like it does in the abelian case. There is, however, a catch in it: The solutions of Jensen's functional equation in Case (b) need not all be classical, even on a step 2 nilpotent group. The Heisenberg group provides a counterexample (See [8] or [17]).

11 Wilson's functional equation in Case II

We shall in this Section 11 discuss Wilson's functional equation

$$f(xy) + f(xy^{-1}) = 2f(x)g(y), \quad x, y \in G, \tag{11.1}$$

where G is a step 2 nilpotent group, and where g is in Case II of Theorem 7.2.

We know that g may be written in the form $g = g_0 \circ \Phi$, where Φ is a surjective homomorphisms of G onto the quaternion group Q_8 (Theorem 7.5). In the earlier notation we have that $H = Z(g) \neq G$ and $H_1 = \ker \Phi$.

Lemma 11.1. *Let G be a step 2 nilpotent group, and let g be a solution of d'Alembert's functional equation on G in Case II of Theorem 7.2. If $\{f, g\}$ is a solution of Wilson's functional equation, such that $f \neq 0$ and f is odd, then $Z(f) = Z(g)$.*

Proof. $Z(f) \subseteq Z(g)$ by Lemma 9.2. To prove the converse inclusion we note from Lemma 9.7 that $f(xu) = f(x)g(u)$ for all $x \in G$ and all $u \in Z(G)$. Let $u \in Z(g)$ be arbitrary. We get for any $x, y \in G$ since $[G, G] \subseteq Z(G)$ from the just derived formula that $f(xyu) = f(xuy[y^{-1}, u^{-1}]) = f(xuy)g([y^{-1}, u^{-1}]) = f(xuy)g(e) = f(xuy)$, which shows that $u \in Z(f)$. $\square$

The following result will make it easy to verify whether a prescribed function is a solution of Wilson's functional equation, and it gives information about the form of a solution.

Lemma 11.2. *Let G be a step 2 nilpotent group, and let g be a solution of d'Alembert's functional equation on G in Case II of Theorem 7.2. Let $f : G \to \mathbb{C}$. Then the pair $\{f, g\}$ is a solution of Wilson's functional equation, if and only if*

$$f(xu) = f(x)g(u) \text{ for all } x \in G \text{ and } u \in H. \tag{11.2}$$

Proof. We may assume that $f \neq 0$, because the theorem is trivially true, if $f = 0$. Since $H = Z(g) = Z(f)$ here (Lemma 11.1) we get from Lemma 9.7 that (11.2) holds, if $\{f, g\}$ is a solution and f is odd. It also holds for f even by the formula $g(xu) = g(x)g(u)$ (Lemma 6.2(d)). It then holds for all solutions, because any solution f is a sum of an even and an odd solution (Lemma 9.4 + Lemma 9.1 + Lemma 7.1).

Assume conversely that (11.2) holds. Using that $1 + g(y^2) = 2g(y)^2$ for all $x \in G$ (Lemma 6.2(a)), we look at Wilson's functional equation, where we with the notation LHS = left hand side and RHS = right hand side find that

$$\begin{aligned} LHS &= f(xy) + f(xy^{-1}) = f(xy) + f(xyy^{-2}) = f(xy) + f(xy)g(y^{-2}) \\ &= f(xy)[1 + g(y^{-2})] = f(xy)[1 + g(y^2)] = 2f(xy)g(y)^2, \end{aligned}$$

while $RHS = 2f(x)g(y)$.

If $y \in G \setminus H$, then $g(y) = 0$, so $LHS = 0 = RHS$. If $y \in H$, then we get by (11.2) that $f(xy)g(y)^2 = f(x)g(y)g(y)^2 = f(x)g(y^{-1})g(y)^2 = f(x)g(y^{-1}y^2) = f(x)g(y)$ as desired. $\square$

Example 11.3. Let us return to the quaternion group Q_8. We saw in Example 7.4 that the there is exactly one non-classical solution g_0 of d'Alembert's functional equation on Q_8, viz. $g_0(\pm 1) = \pm 1$, $g_0(\pm i) = g_0(\pm j) = g_0(\pm k) = 0$. Any odd solution f of Wilson's functional equation corresponding to g_0 vanishes on $\{\pm 1\}$ because $(\pm 1)^{-1} = \pm 1$. Furthermore $f(-i) = f(i^{-1}) = -f(i)$ and similarly $f(-j) = -f(j)$ and $f(-k) = -f(k)$. It is now straightforward (use for example Lemma 11.2) to verify that the vector space of odd solutions of Wilson's functional equation corresponding to g_0 is 3-dimensional and spanned by the 3 functions $\{f_i, f_j, f_k\}$ defined by

$$\begin{aligned} &f_i(\pm 1) = 0, \quad f_i(\pm i) = \pm 1, \quad f_i(\pm j) = 0, \quad f_i(\pm k) = 0, \\ &f_j(\pm 1) = 0, \quad f_j(\pm i) = 0, \quad f_j(\pm j) = \pm 1, \quad f_j(\pm k) = 0, \\ &f_k(\pm 1) = 0, \quad f_k(\pm i) = 0, \quad f_k(\pm j) = 0, \quad f_k(\pm k) = \pm 1. \end{aligned} \tag{11.3}$$

As we know (Lemma 9.3), the even solutions are proportional to g_0, so the total solution space is 4-dimensional.

This example was also worked out in [6, Example].

Let the pair $\{f, g\}$ be a solution of Wilson's functional equation on G. It follows immediately from the formula (11.2) that f is a function on G/H_1, so that f may be written in the form $f = f_0 \circ \Phi$, where $f_0 : Q_8 \to \mathbb{C}$. As is easy to see, the pair $\{f_0, g_0\}$ is a solution of Wilson's functional equation on Q_8 if and only the pair $\{f_0 \circ \Phi, g_0 \circ \Phi\}$ is a solution of Wilson's functional equation on G. We have thus proved the following theorem, that once again points out that the quaternion group Q_8 plays a central role:

Theorem 11.4. *Let G be a step 2 nilpotent group. Consider the solutions $\{f, g\}$ of Wilson's functional equation on G in which g is in Case II of Theorem 7.2. They are the pairs of the form $\{f_0 \circ \Phi, g_0 \circ \Phi\}$ in which $\Phi : G \to Q_8$ ranges over the surjective homomorphisms, $g_0 : Q_8 \to \mathbb{C}$ is defined by $g_0(\pm 1) = \pm 1$, $g_0(\pm i) = g_0(\pm j) = g_0(\pm k) = 0$, and $f_0 \in \operatorname{span}\{g_0, f_i, f_j, f_k\}$, where $f_i, f_j, f_k : Q_8 \to \mathbb{C}$ are defined in Example 11.3.*

Corollary 11.5. *Let G be a step 2 nilpotent group, and let g be a solution of d'Alembert's functional equation on G in Case II of Theorem 7.2.*

The vector space of solutions f of Wilson's functional equation with the given g has dimension 4. The odd solutions form a 3-dimensional subspace, while the even ones form the 1-dimensional subspace $\mathbb{C}g$.

Example 11.6. Let us consider the Heisenberg group with integer entries $H_3(\mathbb{Z}) = \{(x, y, z) \mid x, y, z \in \mathbb{Z}\}$.

We saw in Example 7.6 that the map $\Phi : H_3(\mathbb{Z}) \to Q_8$ given by $\Phi(x, y, z) = (-1)^z k^y j^x$ for $(x, y, z) \in H_3(\mathbb{Z})$ is a surjective homomorphism, and that the function $g = g_0 \circ \Phi$, where g_0 is the function from Example 7.4, is the only non-classical solution of d'Alembert's functional equation on $H_3(\mathbb{Z})$.

According to Theorem 11.4 a basis for the 3-dimensional vector space of all odd solutions f to Wilson's functional equation on $H_3(\mathbb{Z})$, corresponding to the given g, is $\{f_1 = f_i \circ \Phi, f_2 = f_j \circ \Phi, f_3 = f_k \circ \Phi\}$. Concrete calculations show that

$$\begin{aligned}
&f_1(2m+1, 2n, z) = (-1)^{m+n+z},\\
&f_1(2m+1, 2n+1, z) = f_1(2m, n, z) = 0, \text{ and}\\
&f_2(2m, 2n+1, z) = (-1)^{m+n+z},\\
&f_2(2m, 2n, z) = f_2(2m+1, n, z) = 0, \text{ and}\\
&f_3(2m+1, 2n+1, z) = (-1)^{m+n+z},\\
&f_3(2m, n, z) = f_3(2m+1, 2n, z) = 0.
\end{aligned} \tag{11.4}$$

References

[1] J. Aczél: "*Vorlesungen über Funktionalgleichungen und ihre Anwendungen.*" Birkhäuser. Basel-Stuttgart. 1961.

[2] J. Aczél, J.K. Chung and C.T. Ng, *Symmetric second differences in product form on groups.* Topics in mathematical analysis (pp. 1-22) edited by Th.M. Rassias. Ser. Pure Math., 11, World Scientific Publ. Co., Teaneck, NJ, 1989.

[3] T. Bröckner and T. Dieck: *Representations of Compact Lie Groups.* Springer-Verlag. New York, Berlin, Heidelberg, Tokyo, 1985.

[4] I. Corovei, *The functional equation* $f(xy) + f(yx) + f(xy^{-1}) + f(y^{-1}x) = 4f(x)f(y)$ *for nilpotent groups.* (Romanian, English summary) Bul. Ştiinţ. Instit. Politehn. Cluj-Napoca Ser. Mat.-Fiz.-Mec. Apl. **20** (1977), 25-28.

[5] I. Corovei, *The functional equation* $f(xy) + f(xy^{-1}) = 2f(x)g(y)$ *for nilpotent groups.* Mathematica (Cluj) **22** (45) (1980), 33-41.

[6] I. Corovei, *Wilson's functional equation on* P_3*-groups.* Aequationes Math. **61** (2001), 212-220.

[7] I. Corovei, *Wilson's functional equation on metabelian groups.* Manuscript of September 2000. pp. 1-9. To appear in Mathematica (Cluj).

[8] P. de Place Friis, *d'Alembert's and Wilson's Equations on Lie Groups.* Preprint Series 2000 No 8, Department of Mathematics, Aarhus University, Denmark. pp. 1–12. Accepted for publication by Aequationes Math.

[9] E. Hewitt and K.A. Ross, *Abstract Harmonic Analysis I.* Springer-Verlag. Berlin, Göttingen, Heidelberg. 1963.

[10] Pl. Kannappan, *The functional equation* $f(xy)+f(xy^{-1}) = 2f(x)f(y)$ *for groups.* Proc. Amer. Math. Soc. **19** (1968), 69-74.

[11] Pl. Kannappan, *Cauchy equations and some of their applications.* Topics in mathematical analysis (pp. 518-538) edited by Th.M. Rassias. Ser. Pure Math., 11, World Scientific Publ. Co., Teaneck, NJ, 1989.

[12] C.T. Ng, *Jensen's functional equation on groups.* Aequationes Math. **39** (1990), 85–99.

[13] C.T. Ng, *Jensen's functional equation on groups, II.* Aequationes Math. **58** (1999), 311–320.

[14] H. Reiter, *Über den Satz von Wiener und lokalkompakte Gruppen.* Comment. Math. Helv. **49** (1974), 333-364.

[15] P. Sinopoulos, *Generalized sine equations, I.* Aequationes Math. **48** (1994), 171-193.

[16] H. Stetkær, *d'Alembert's functional equations on metabelian groups.* Aequationes Math. **59** (2000), 306-320.

[17] H. Stetkær, *On Jensen's functional equation on groups.* Preprint series no. 3. Department of Mathematics, Aarhus University. June 2001. Accepted for publication by Aequationes Math.

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