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1 Introduction

To make this note accessible to a broad spectrum of readers, we briefly recall the necessary analytic and algebraic background for the concept of vertex operators (cf. [3], Chapter 14).

Let $\Gamma_0\mathcal{D}$ be the algebra of all polynomials of finitely many variables ξ_j from the set $\{\xi_1, \xi_2, ..., \xi_n, ...\}$ and $\mathcal{D}$ the space of all linear forms belonging to $\Gamma_0\mathcal{D}$. Finally, let $\widetilde{\Gamma}\mathcal{D} \supset \Gamma_0\mathcal{D}$ be the algebraic adjoint of $\Gamma_0\mathcal{D}$ identified with the algebra of all formal series of variables ξ_j. An operator which is the sum of compositions of operators of multiplication by variables ξ_j composed with derivations $\frac{\partial}{\partial\xi_k}$ is said to be of *normal form* if the operators of multiplication always precede the operators of derivation. $\Gamma_0\mathcal{D}$ is provided with a scalar product $\langle , \rangle$ which makes $\frac{\partial}{\partial\xi_k}$ the adjoint to the operator of multiplication by ξ_k. $k = 1, 2, ...$ respectively.

Select two sequences of complex numbers $\{t_n\}$ and $\{s_n\}$. Operators of the form

$$e^{\sum_{n=1}^{\infty} t_n z^n \xi_n} e^{\sum_{n=1}^{\infty} s_n z^{-n} \frac{\partial}{\partial\xi_n}},$$

where $z \in \mathbb{C}$, acting on $\Gamma_0\mathcal{D}$, are called *vertex operators.* The operators $\mathcal{S}_m$ of the expansion

$$e^{\sum_{n=1}^{\infty} t_n z^n \xi_n} e^{\sum_{n=1}^{\infty} s_n z^{-n} \frac{\partial}{\partial\xi_n}} = \sum_{n\in\mathbb{Z}} \mathcal{S}_n z^n \tag{1}$$

are called Schur polynomials. In [3], chapter 14, there are produced recursive formulas for $\mathcal{S}_m$ and it is shown that for $s_n = \frac{1}{n}$ and $t_n = 1$, the operators $\mathcal{S}_m$, $m \geq 1$, anti-commute, i.e. represent fermions. More details concerning vertex operators and vertex operators algebras can found in the book [1] and in several papers, in particular [2].

In this paper we provide an explicit formula for S_m in their normal form, for any given vertex operator written in the mathematical frame of a general Bose algebra (recall that all Bose algebras with infinite dimensional separable one-particle space are all canonically isomorphic cf.[4]).

2 Preliminaries

Let $\Gamma_0\mathcal{D}$ be a Bose algebra (cf. [4]) i.e. a commutative graded algebra generated by a pre-Hilbert space $\mathcal{D}, \langle , \rangle$ (the so-called one-particle space) and the unity ϕ (the vacuum) provided with the extension $\langle , \rangle$ of the scalar product of $\mathcal{D}$ making ϕ a unit vector and fulfilling the property that for every $x \in \mathcal{D}$, the adjoint x^* to the operator of multiplication by x is defined on the whole $\Gamma_0\mathcal{D}$ and constitutes a derivation (i.e. fulfils the Leibniz rule). We make the space $\widetilde{\Gamma\mathcal{D}}$ of all antilinear functionals on $\Gamma_0\mathcal{D}$ the extension of $\Gamma_0\mathcal{D}$ by identifying $f \in \Gamma_0\mathcal{D}$ with the antilinear functional $\langle \cdot, f \rangle$. The space $\widetilde{\Gamma\mathcal{D}}$ can be naturally made into an algebra containing $\Gamma_0\mathcal{D}$ as a subalgebra. We consider $\widetilde{\Gamma\mathcal{D}}$ as a locally convex space with the weak topology $\sigma\big(\widetilde{\Gamma\mathcal{D}}, \Gamma_0\mathcal{D}\big)$. The weak closure $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ is a subspace of $\widetilde{\Gamma\mathcal{D}}$. It is easy to show that $\Gamma_0\mathcal{D}, \langle \, , \, \rangle$ admits the completion $\Gamma\overline{\mathcal{D}}$ within $\widetilde{\Gamma\mathcal{D}}$.

We shall use the exponentials of elements $w \in \mathcal{D}$,

$$e^w = \sum_{n=0}^{\infty} \frac{1}{n!} w^n \in \Gamma\overline{\mathcal{D}},$$

which are called *coherent vectors*. In [4] the following relations are verified:

$$\langle a, b \rangle^j = \frac{1}{j!} \left\langle a^j, b^j \right\rangle \tag{2}$$

$$(x^n)^* \, e^w = \langle x, w \rangle^n \, e^w \tag{3}$$

$$\langle e^u, fg \rangle = \langle e^u, f \rangle \, \langle e^u, g \rangle \tag{4}$$

$$e^{\mathbf{a}(w)} e^v = e^{\langle w, v \rangle} e^v. \tag{5}$$

Also a proof that the set $\{e^x : x \in \mathcal{D}\}$ of coherent vectors is total in $\Gamma\overline{\mathcal{D}}$ can be found in [4].

3 The Laurent Expansion for a Vertex Operator

Let $\mathcal{D}$ be spanned by an orthonormal system $\{f_n\}$ and by an orthonormal system $\{g_n\}$ as well. The operator valued functions of z

$$V(z) = e^{\sum_{n=1}^{\infty} z^n f_n} e^{\sum_{n=1}^{\infty} z^{-n} g_n^*} : \Gamma_0\mathcal{D} \to \widetilde{\Gamma\mathcal{D}},$$

shall be called a *vertex operator.*

Write $(\mathfrak{p}, \mathfrak{q})$ for tuples of non-negative integers

$$(\mathfrak{p}, \mathfrak{q}) = (p_1, q_1, p_2, q_2, \ldots, p_k, q_k, \ldots)$$

and define

$$\mathfrak{N}_m = \left\{ (\mathfrak{p}, \mathfrak{q}) : \sum_{k=1}^{\infty} (p_k + q_k) = m \right\}$$

and

$$\mathfrak{N}^w = \left\{ (\mathfrak{p}, \mathfrak{q}) : \sum_{k=1}^{\infty} (p_k + q_k) < \infty, \; \sum_{j=1}^{\infty} j \, (p_j - q_j) = w \right\}.$$

For $\mathfrak{s} = (s_1, s_2, ...)$, write

$$\mathfrak{s}! = \prod_{k=1}^{\infty} s_k!.$$

Our main result is the following

Theorem. *Vertex operators admit the weak evaluation on $\Gamma_0\mathcal{D}$ and the weak convergent Laurent expansion*

$$V(z) = e^{\sum_{n=1}^{\infty} z^n f_n} e^{\sum_{n=1}^{\infty} z^{-n} g_n^*} = \sum_{w\in\mathbb{Z}} \mathcal{S}_w \{f_n, g_n^*\} z^w$$

with coefficients

$$\mathcal{S}_w \{f_n, g_n^*\} = \sum_{m=0}^{\infty} \sum_{(\mathfrak{p},\mathfrak{q})\in\mathfrak{N}_m\cap\mathfrak{N}^w} \frac{1}{\mathfrak{p}!\mathfrak{q}!} \left(\prod_{k=1}^{\infty} f_k^{p_k}\right) \left(\prod_{k=1}^{\infty} g_k^{q_k}\right)^*$$

called the Schur polynomials (cf.[3]).

To prove the Theorem we shall need the following

Lemma. *Take any pair of elements $u, v \in \mathcal{D}$. Then the element $V(z)e^u$ is well defined in $\widetilde{\Gamma}\mathcal{D}$ and we have*

$$\langle e^u, V(z) e^v\rangle = \left\langle e^u, \left(\sum_{w\in\mathbb{Z}} \mathcal{S}_w \{f_n, g_n^*\} z^w\right) e^v \right\rangle, \tag{6}$$

where

$$\mathcal{S}_w \{f_n, g_n^*\} = \sum_{m=0}^{\infty} \sum_{(\mathfrak{p},\mathfrak{q})\in\mathfrak{N}_m\cap\mathfrak{N}^w} \frac{1}{\mathfrak{p}!\mathfrak{q}!} \left(\prod_{k=1}^{\infty} f_k^{p_k}\right)\left(\prod_{k=1}^{\infty} g_k^{q_k}\right)^*.$$

Proof. Take u,$v \in \mathcal{D}$. Since by (5)

$$\left\langle e^u, e^{\mathbf{a}^+(x)} e^{\mathbf{a}(y)} e^v \right\rangle = e^{\langle u,v\rangle} e^{\langle u,x\rangle + \langle y,v\rangle},$$

we obtain

$$\langle e^u, V(z) e^v\rangle = e^{\langle u,v\rangle} e^{\sum_{n=1}^{\infty}\left(\langle f_n,,u\rangle z^n + \langle v,g_n\rangle z^{-n}\right)}.$$

Since u and v are linear combinations of f_k and g_k respectively, $\langle f_{n,}, u\rangle z^n = \langle v, g_n\rangle z^{-n} = 0$ for large n. Due to (3) we get

$$\left\langle e^u, \left(\prod_{k=1}^{\infty} f_k^{p_k}\right)\left(\prod_{k=1}^{\infty} g_k^{q_k}\right)^* e^v \right\rangle$$
$$= \left\langle \left(\prod_{k=1}^{\infty} f_k^{p_k}\right)^* e^u, \left(\prod_{k=1}^{\infty} g_k^{q_k}\right)^* e^v \right\rangle = \left(\prod_{k=1}^{\infty} \langle f_k, u\rangle^{p_k} \langle v, g_k\rangle^{q_k}\right) e^{\langle u,v\rangle},$$

where all the products are finite and they are non-zero only when p_k and q_k are zeros for f_k and g_k orthogonal to v and u respectively. Consequently

$$\frac{1}{m!}\left(\sum_{n=1}^{\infty} \langle f_{n,}, u\rangle z^n + \sum_{n=1}^{\infty} \langle v, g_n\rangle z^{-n}\right)^m$$
$$= \sum_{(\mathfrak{p},\mathfrak{q})\in\mathfrak{N}_m} \frac{1}{\mathfrak{p}!\mathfrak{q}!} \prod_{k=1}^{\infty} \left(\langle f_k, u\rangle^{p_k} \langle v, g_k\rangle^{q_k} z^{k(p_k-q_k)}\right)$$
$$= \sum_{w\in\mathbb{Z}} \sum_{(\mathfrak{p},\mathfrak{q})\in\mathfrak{N}^w\cap\mathfrak{N}_m} \frac{1}{\mathfrak{p}!\mathfrak{q}!} \left(\prod_{k=1}^{\infty} \langle f_k, u\rangle^{p_k} \langle v, g_k\rangle^{q_k}\right) z^w$$
$$\left\langle e^u, \sum_{w\in\mathbb{Z}} \left(\sum_{(\mathfrak{p},\mathfrak{q})\in\mathfrak{N}^w\cap\mathfrak{N}_m} \frac{1}{\mathfrak{p}!\mathfrak{q}!} \left(\prod_{k=1}^{\infty} f_k^{p_k}\right)\left(\prod_{k=1}^{\infty} g_k^{q_k}\right)^* \right) z^w e^v \right\rangle e^{-\langle u,v\rangle}.$$

Consequently

$$\frac{1}{m!}\left(\sum_{n=1}^{\infty}\langle f_{n,},u\rangle z^{n}+\sum_{n=1}^{\infty}\langle v,g_{n}\rangle z^{-n}\right)^{m}$$
$$=\left\langle e^{u},\sum_{w\in\mathbb{Z}}\left(\sum_{(\mathfrak{p},\mathfrak{q})\in\mathfrak{N}_{m}\cap\mathfrak{N}^{w}}\frac{1}{\mathfrak{p}!\mathfrak{q}!}\left(\prod_{k=1}^{\infty}f_{k}^{p_{k}}\right)\left(\prod_{k=1}^{\infty}g_{k}^{q_{k}}\right)^{*}\right)z^{w}e^{v}\right\rangle e^{-\langle u,v\rangle},$$

and finally

$$\langle e^{u},V(z)e^{v}\rangle=e^{\langle u,v\rangle}e^{\sum_{n=1}^{\infty}\left(\langle f_{n,},u\rangle z^{n}+\langle v,g_{n}\rangle z^{-n}\right)}$$
$$=\left\langle e^{u},\sum_{m=0}^{\infty}\sum_{w\in\mathbb{Z}}\left(\sum_{(\mathfrak{p},\mathfrak{q})\in\mathfrak{N}_{m,w}}\frac{1}{\mathfrak{p}!\mathfrak{q}!}\left(\prod_{k=1}^{\infty}f_{k}^{p_{k}}\right)\left(\prod_{k=1}^{\infty}g_{k}^{q_{k}}\right)^{*}\right)z^{w}e^{v}\right\rangle$$

which concludes the proof of the Lemma. □

Proof of the Theorem

Since $\Gamma_0\mathcal{D}$ is the linear span of the set $\left\{x^k : x\in\mathcal{D},\ k=1,2,..\right\}$ ([4]), it is sufficient to show that for any $u,v\in\mathcal{D}$ and any natural numbers k,j we have

$$\left\langle u^{k},V(z)v^{j}\right\rangle=\left\langle u^{k},\left(\sum_{w\in\mathbb{Z}}\mathcal{S}_{w}\left\{f_{n},g_{n}^{*}\right\}z^{w}\right)v^{j}\right\rangle$$

which follows by differentiating respectively k and j times at 0 the variables t and s of the identity 6 with tu and sv substituted for u and v. □

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