UNIVERSITY OF AARHUS
DEPARTMENT OF MATHEMATICS

ISSN: 1397–4076

CATEGORIFICATION OF THE TEMPERLEY CATEGORY, TANGLES AND COBORDISMS VIA PROJECTIVE FUNCTORS

By Catharina Stroppel

Preprint Series No.: 11

October 2003

Ny Munkegade, Bldg. 530
DK-8000 Aarhus C, Denmark

http://www.imf.au.dk
institut@imf.au.dk

Categorification of the Temperley category, tangles and cobordisms via projective functors

Catharina Stroppel*

Abstract

To each generic tangle projection from the three dimensional real vector space onto the plane, we associate a derived endofunctor on a graded parabolic version of the Bernstein-Gelfand category $\mathcal{O}$. We show that this assignment is (up to shifts) invariant under tangle isotopies and Reidemeister moves and defines therefore invariants of tangles. The occurring functors are defined via so-called projective functors. The first part of the paper deals with the indecomposability of such functors and their connection with generalised Temperley-Lieb algebras which are known to have a realisation via decorated tangles. The second part of the paper describes a categorification of the Temperley-Lieb category and proves the main conjectures of [BFK99]. Moreover, we describe a functor from the category of 2-cobordisms into the category of projective functors.

*Partially supported by CAALT and EPSRC.

Introduction

On the way of finding topological invariants for knots and links, recently some new ideas concerning a connection to representation theory appeared (see e.g. [Kho00], [FK97]). Our paper was mainly motivated by [BFK99] and contains a proof of the main conjectures therein. Bernstein, Frenkel and Khovanov constructed a realisation of the Temperley-Lieb algebra via projective functors on parabolic versions of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$. The category $\mathcal{O}$ is given by representations (with certain finiteness conditions) of a complex semisimple Lie algebra $\mathfrak{g}$. It is stable under tensoring with a finite dimensional $\mathfrak{g}$-module E. A direct summand of $\bullet \otimes E$ is called a *projective functor*, since it preserves projectivity. Such functors play a crucial role in representation theory. The indecomposable projective functors on $\mathcal{O}$ were classified by Bernstein and Gelfand ([BG80]). When restricting to the main block $\mathcal{O}_0$, their isomorphism classes are in bijection to the Weyl group. The famous Kazhdan-Lusztig-theory is based on the fact that the Grothendieck ring of projective functors is described by the corresponding (specialised) (Iwahori-)Hecke algebra. In other words, this algebra has a 'functorial realisation', i.e. there is a ring homomorphism from the specialised (Iwahori-)Hecke algebra into the Grothendieck ring of projective functors on a regular integral block of $\mathcal{O}$. In type A, there is a well-known quotient of the Iwahori-Hecke algebra which is called *Temperley-Lieb algebra*. Because of its diagrammatical description it is directly linked with knot theory and has several applications in physics and science (see e.g. [Kau01]). In [BFK99], the authors considered the action of the specialised Iwahori-Hecke algebra induced via projective functors on the direct sum over all maximal parabolic subcategories of $\mathcal{O}_0$. They proved that it factors through the specialised Temperley-Lieb algebra. On the level of the Grothendieck group the resulting representation coincides with the natural representation on the n-fold tensor product of $\mathbb{C}^2$ given by place permutations.
The following questions appeared in this context (and are the content of our paper):

(I) ([BFK99]) Is there a 'functorial realisation' of the Temperley-Lieb algebra where the deformation variable comes into the picture?

(II) ([Bac01]) Is there a classification of indecomposable projective functors in the parabolic setup?

(III) Is it possible to generalise the results of [BFK99] to other types?

(IV) ([BFK99]) Is there a 'functorial realisation' of the Temperley-Lieb 2-category and of arbitrary tangles?

The first problem can be solved using the graded version of category $\mathcal{O}$ introduced in [BGS96]. In [Str03], a graded version of translation functors is defined such that one can easily get the required 'functorial realisation' (Theorem 4.1). In this context we also obtain a 'functorial realisation' of the Temperley-Lieb algebras of Type B, C and D. This might be interesting, since these algebras can be realised via *decorated* tangles (see e.g. [Gre98]). The classification problem, however, seems to be much more complicated. We are far away from a reasonable answer. Nevertheless, we give a combinatorial formula (Proposition 3.6, Theorem 5.7) for the number of isomorphism classes of indecomposable functors. This formula was motivated by discussions with W. Soergel, who conjectured that it should determine the number of indecomposable projective functors. However, in non-simply-laced cases we do not have equality in general (see Examples 3.7). In Proposition 3.8 we prove, that it is sufficient to study the case of simple Lie algebras.
A very nice (and helpful!) result is given by Theorem 5.1, where we prove that an indecomposable projective functor on $\mathcal{O}_0(\mathfrak{sl}_n)$ stays either indecomposable or becomes zero after restricting to a maximal parabolic subcategory. (This was conjectured in [BFK99]. Note that it is *not* true for other types; see Examples 3.7). Moreover, the indecomposable functors corresponding to a non-braid avoiding Weyl group elements becomes always trivial after restriction (Lemma 5.2).
Our main results are the proofs of the Conjectures 1 to 4 of [BFK99]: Let $\mathcal{O}_n^{\max}$ be the direct sum of all parabolic subcategories of the main block of $\mathcal{O}(\mathfrak{sl}_n)$ given by parabolic subgroups of the form $S_k \times S_{n-k}$. We associate to each morphism f of the Temperley-Lieb 2 category, i.e. to each (m, n)-tangle projection without crossings, a projective functor $F(f) : \mathcal{O}_m^{\max} \to \mathcal{O}_n^{\max}$. Theorem 6.2 can be read as

If $f \simeq g$ via planar isotopies then $F(f) \cong F(g)$ as functors

and

To each 2-morphism in the Temperley 2-category we can assign a natural transformation between the functors given by the corresponding 1-morphisms.

We extend this 'functorial realisation' to tangles with crossings as follows: Let $\mathcal{D}^b(\mathcal{O}_n^{\max})$ denote the bounded derived category of $\mathcal{O}_n^{\max}$. To each (m, n)-tangle projection t we associate a functor $\mathcal{T}(t) : \mathcal{D}^b(\mathcal{O}_m^{\max}) \to \mathcal{D}^b(\mathcal{O}_n^{\max})$.

The functors assigned to a right or left-curl are given as mapping cones of the adjunction morphisms between the identity functor and translation functors through the wall. That means they coincide with the derived functors of Irving's shuffling functors ([Irv93]). We prove in Theorem 7.1:

$$\textit{If } t \simeq t' \textit{ via ambient isotopies then } \mathcal{T}(t) \cong \mathcal{T}(t')$$

up to a grading shift and a shift in the derived category. These results prove Conjecture 3 and Conjecture 4 from [BFK99]. Using the fact that projective functors are Koszul dual to Zuckerman's functors (as proved in [RH]), a 'functorial realisation' of tangles via singular blocks of category $\mathcal{O}$ follows ([BFK99, Conjecture 1 and 2]).
Therefore, we get functor invariants for tangles. In particular, we can assign to a disjoint union of closed oriented 1-manifolds a certain endofunctor on a parabolic version of category $\mathcal{O}(\mathfrak{sl}_n)$. Our final result (Theorem 8.1) is a 'functorial realisation' of the category of 2-cobordisms. In other words, we assign to each cobordism a natural transformation between the corresponding functors and prove that this assignment is invariant under isomorphisms of cobordisms. Since all the occurring functors can be lifted to a $\mathbb{Z}$-graded version (as explained in [Str03]), the natural transformations corresponding to cobordisms can be interpreted as (homogeneous) transformations between $\mathbb{Z}$-functors. It turns out that the Euler characteristic of the cobordism surface coincides with the degree of the assigned natural transformation. How to realise the 2-category of tangle cobordisms in terms of projective functors will be explained in a subsequent paper.

The paper is organised as follow: In the first section we recall the main results on Category $\mathcal{O}$, its parabolic version and its combinatorics. In section 2 we explain how the deformation variable of the (Iwahori-)Hecke algebra can be interpreted as grading shifts. In the third section we define the categories of projective functors and prove some basic and general results. The problem about indecomposability of projective functors is worked out; including a description of how to define graded lifts of projective functors. In section 4 we describe 'functorial realisations' of generalised Temperley-Lieb algebras. Section 5 considers the maximal parabolic situation of type A. It includes the theorem on indecomposability of indecomposable projective functors after restriction to the parabolic category. Since some proofs rely on explicit calculations, we attached an appendix containing the description of distinguished coset representatives for maximal parabolic subgroups. The following two sections contain (the proof of) the two 'functorial realisation' theorems for tangles. In the last section we finally describe 'functorial realisations' of the 2-cobordisms category and mention how the Euler characteristic of cobordism surfaces can be realised as degrees of natural transformations between $\mathbb{Z}$-functors.

Acknowledgement: I wish to thank Robert Marsh for helpful discussion related to the combinatorics in this paper and for his invitation to Paris. I am in particular grateful to Henning Haahr Andersen for answering many questions and for helpful comments. Thanks to Wolfgang Soergel who initiated this work. I thank Mikhail Khovanov for several comments on a previous version and for pointing out an inaccuracy in a proof.

1 Category $\mathcal{O}$ and its Combinatorics

Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ be a semisimple complex Lie algebra with a fixed Borel and Cartan subalgebra. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{b} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$ be the corresponding Cartan decomposition. The universal enveloping algebras are denoted by $\mathcal{U} = \mathcal{U}(\mathfrak{g})$, $\mathcal{U}(\mathfrak{b})$ etc. Let $\mathcal{Z} \subset \mathcal{U}$ be the centre.
We consider the category $\mathcal{O}$ of Bernstein and Gelfand ([BG80]) which is the full subcategory of the category of all $\mathcal{U}$-modules given by the following set of objects

$$\mathrm{Ob}(\mathcal{O}) := \left\{ M \in \mathfrak{g} - \mathrm{mod} \;\middle|\; \begin{array}{l} M \text{ is finitely generated as a } \mathcal{U}(\mathfrak{g})\text{-module} \\ M \text{ is locally finite for } \mathfrak{b} \\ \mathfrak{h} \text{ acts diagonally on } M \end{array} \right\}$$

where the second condition means that $\dim_{\mathbb{C}} \mathcal{U}(\mathfrak{n}) \cdot m < \infty$ for all $m \in M$ and the last says that $M = \oplus_{\mu \in \mathfrak{h}^*} M_\mu$, where $M_\mu = \{m \in M \mid h \cdot m = \mu(h)m \text{ for all } h \in \mathfrak{h}\}$ is the μ-weight space of M. Many results about this category can be found for example in [BGG76, Jan79, Jan83].
For a given weight $\lambda \in \mathfrak{h}^*$, let $M(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda$ denote the Verma module with highest weight λ and simple head $L(\lambda)$. Let $P(\lambda) \in \mathcal{O}$ be the projective cover of $L(\lambda)$.
Let $\pi \subset R$ be the set of simple roots inside the set of all roots. For $\alpha \in R$ let $\mathfrak{g}_\alpha$ be the α-weight space of $\mathfrak{g}$ under the adjoint action. The coroot of α is denoted by $\check{\alpha}$. We use the letter W for the Weyl group with unit element e and denote by $\mathcal{S}$ the set of simple reflections. The length of $w \in W$ is denoted by $l(w)$. For $w, a_1, \ldots, a_r \in W$ we call an expression $w = a_1 a_2 \cdots a_r$ *minimal*, if $\sum_{i=1}^r l(a_i) = l(w)$. In particular, any reduced expression is minimal. The Weyl group acts in a natural way on $\mathfrak{h}^*$ (with

fix point 0); for any $\lambda \in \mathfrak{h}^*$ we denote by $w \cdot \lambda = w(\lambda + \rho) - \rho$ the image of λ under the 'translated' action of W with fix point $-\rho$, where ρ is the half-sum of positive roots.
Let W_λ denote the stabiliser of λ under this action. We denote by $\mathcal{O}_\lambda$ the full subcategory of $\mathcal{O}$ having as objects all modules annihilated by a large enough power of the maximal ideal $\ker \chi_\lambda = \mathrm{Ann}_{\mathcal{Z}} M(\lambda)$ in the centre of $\mathcal{U}$. We will call $\lambda \in \mathfrak{h}^*$ *dominant* (with respect to $-\rho$) if $\langle \lambda + \rho, \check{\alpha} \rangle \geq 0$ and label the subcategories $\mathcal{O}_\lambda$ always with dominant weights.

1.1 The parabolic category $\mathcal{O}^{\mathfrak{p}}$

Let $S \subseteq \pi$ be a subset of the simple roots with corresponding root system $R_S = R \cap \mathbb{Z}S$. We define the Lie algebra $\mathfrak{g}_S \subseteq \mathfrak{g}$ as

$$\mathfrak{g}_S = \mathfrak{n}_S^- \oplus \mathfrak{h}_S \oplus \mathfrak{n}_S^+,$$

where $\mathfrak{n}_S^{\mp} = \bigoplus_{\alpha \in \mp R \cap R_S} \mathfrak{g}_\alpha$. Then $\mathfrak{g}_S$ is semisimple with Cartan subalgebra $\mathfrak{h}_S = \oplus_{\alpha \in S} \mathbb{C}\check{\alpha}$ and root system R_S. Let us denote the corresponding Weyl group by W_S and let W^S be the set of minimal length coset representatives for $W_S \backslash W$, i.e. $W^S = \{w \in W \mid \forall s \in \mathcal{S} \cap W_S : l(sw) > l(w)\}$. The parabolic subalgebras (containing $\mathfrak{b}$) of $\mathfrak{g}$ are parametrised by the elements of the power set of π, in the way that $S \subseteq \pi$ corresponds to

$$\mathfrak{p}_S = (\mathfrak{g}_S \oplus \mathfrak{h}^S) + \mathfrak{n},$$

where $\mathfrak{h}^S = \bigcap_{\alpha \in S} \ker \alpha$. Using this bijection we identify $W_{\mathfrak{p}_S} = W_S$, $W^S = W^{\mathfrak{p}_S}$ etc.
Let $\mathfrak{p} = \mathfrak{p}_S$ be a parabolic subalgebra (containing $\mathfrak{b}$) of $\mathfrak{g}$ with universal enveloping algebra $\mathcal{U}(\mathfrak{p})$. The category $\mathcal{O}^S = \mathcal{O}^{\mathfrak{p}}$ is the full subcategory of $\mathcal{O}$ whose objects are exactly the locally $\mathfrak{p}$-finite modules of $\mathcal{O}$, i.e. $M \in \mathcal{O}^{\mathfrak{p}}$ if and only if $\dim_{\mathbb{C}} \mathcal{U}(\mathfrak{p})m < \infty$ for all $m \in M$. This category is called the *parabolic category* $\mathcal{O}$ (with respect to $\mathfrak{p}$ or S respectively).
Let $P_{\mathfrak{p}}^+ = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \check{\alpha} \rangle \in \mathbb{N},\ \forall \alpha \in S\}$ be the strict dominant integral weights with respect to S. The map which sends a simple $\mathcal{U}(\mathfrak{p})$-module to its highest weight gives (see [RC80]) a bijection

$$\{\text{iso-classes of finite dimensional simple } \mathcal{U}(\mathfrak{p})\text{-modules}\} \xleftrightarrow{1:1} P_{\mathfrak{p}}^+. \quad (1.1)$$

We denote the (unique up to isomorphism) simple $\mathfrak{p}$-module of highest weight λ by $E(\lambda)$. The *parabolic Verma module* (with respect to S or $\mathfrak{p}$ respectively) of highest weight λ is defined as

$$M^{\mathfrak{p}}(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} E(\lambda).$$

It has a unique simple quotient $L^{\mathfrak{p}}(\lambda) \cong L(\lambda)$ (see [RC80, Proposition 3.3]). Note that if $S = \emptyset$, then $\mathfrak{p}_S = \mathfrak{b}$ and $M^{\mathfrak{p}}(\lambda) = M(\lambda)$ is the 'ordinary' Verma module. In the other extreme case where $S = \pi$, we have $M^{\mathfrak{g}}(\lambda) \cong L(\lambda)$.
There is a bijection between the isomorphism classes of simple modules in $\mathcal{O}^{\mathfrak{p}}$ and the elements of $P_{\mathfrak{p}}^{+}$ by mapping a module to its highest weight. The category $\mathcal{O}^{\mathfrak{p}}$ has enough projectives. We denote the projective cover of the simple module $L(\lambda)$ corresponding to $\lambda \in P_{\mathfrak{p}}^{+}$ by $P^{\mathfrak{p}}(\lambda)$. (for details see [RC80, Proposition 3.3, Corollaries 4.2 and 4.4]). A categorical characterisation of the parabolic Verma modules is given by the following fact:

Lemma 1.1 (Parabolic Verma Modules as Projective Objects). *Let $\lambda \in P_{\mathfrak{p}}^{+}$. The module $M^{\mathfrak{p}}(\lambda)$ is projective in the full subcategory $\mathcal{O}^{\mathfrak{p}}_{\lambda\geq}$ of $\mathcal{O}^{\mathfrak{p}}$, objects of which have only composition factors of the form $L(\mu)$ with $\mu \not> \lambda$.*

Proof. For $M \in \mathcal{O}^{\mathfrak{p}}_{\lambda\geq}$ we have by Frobenius' reciprocity

$$\mathrm{Hom}_{\mathfrak{g}}(M^{\mathfrak{p}}(\lambda), M) = \mathrm{Hom}_{\mathfrak{g}}(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} E(\lambda), M) \cong \mathrm{Hom}_{\mathfrak{p}}(E(\lambda), M) \cong M_{\lambda}.$$

Therefore, $\mathrm{Hom}_{\mathfrak{g}}(M^{\mathfrak{p}}(\lambda), \bullet)$ is exact and $M^{\mathfrak{p}}(\lambda)$ is projective. (Note that the last isomorphism follows from (1.1) and the fact that λ is by assumption a maximal possible weight.) $\square$

The following Proposition describes how to construct the projective covers in $\mathcal{O}^{\mathfrak{p}}$ given a projective cover in $\mathcal{O}$:

Proposition 1.2 (Projective Covers in $\mathcal{O}^{\mathfrak{p}}$). *Let $Q \in \mathcal{O}^{\mathfrak{p}}$ with projective cover $P \in \mathcal{O}$. Then the projective cover of Q in $\mathcal{O}^{\mathfrak{p}}$ is (up to isomorphism) the quotient P/M, where M is the smallest submodule of P containing all composition factors of P not contained in $\mathcal{O}^{\mathfrak{p}}$.*

Proof. First of all, it is clear from the definition of $\mathcal{O}^{\mathfrak{p}}$ that $P/M \in \mathcal{O}^{\mathfrak{p}}$. Since $\mathrm{Hom}_{\mathcal{O}}(P, \bullet) = \mathrm{Hom}_{\mathcal{O}}(P/M, \bullet) = \mathrm{Hom}_{\mathcal{O}^{\mathfrak{p}}}(P/M, \bullet)$ on $\mathcal{O}^{\mathfrak{p}}$, the projectivity of P/M follows. If a submodule of P/M surjects onto Q, then its preimage under the canonical map $P \twoheadrightarrow P/M$ maps surjectively onto Q as well. Hence, P/M is a projective cover by the minimality of P. $\square$

Restriction to the subcategory $\mathcal{O}^{\mathfrak{p}}_{\lambda\geq}$ gives the following

Corollary 1.3. *For $\lambda \in \mathfrak{h}^*$, there is an isomorphism $M^{\mathfrak{p}}(\lambda) \cong M(\lambda)/M$, where M denotes the smallest submodule containing all composition factors not contained in $\mathcal{O}^{\mathfrak{p}}$.*

1.2 The parabolic Hecke module $\mathcal{N}$

We recall some facts on the Kazhdan-Lusztig combinatorics developed in [KL79], [Deo87]. We use the notation of [Soe97].
Let $\mathbb{Z}[v, v^{-1}]$ be the ring of Laurent polynomials in one variable v. Let $\mathcal{H} = \mathcal{H}(W, \mathcal{S})$ denote the Hecke algebra of $(W, \mathcal{S})$, i.e. the free $\mathbb{Z}[v, v^{-1}]$-module with basis $\{H_x \mid x \in W\}$ and relations

$$H_s^2 = H_e + (v^{-1} - v)H_s \text{ for } s \in \mathcal{S} \text{ and} \tag{1.2}$$

$$H_x H_y = H_{xy}, \text{ if } l(x) + l(y) = l(xy). \tag{1.3}$$

We denote by $H \mapsto \overline{H}$ the *duality* on $\mathcal{H}$, i.e. the ring homomorphism given by $H_x \mapsto H_{x^{-1}}^{-1}$ and $v \mapsto v^{-1}$. The *Kazhdan-Lusztig basis* is given by elements $\underline{H}_x$ (for $x \in W$) such that $\underline{H}_x$ is self-dual (i.e. $\overline{\underline{H}_x} = \underline{H}_x$) and $\underline{H}_x \in H_x + \sum_{y \in W} v\mathbb{Z}[v]H_y$. In particular $C_s := \underline{H}_s = H_s + v$ is a Kazhdan-Lusztig basis element for each simple reflection $s \in W$. For $S \subseteq \pi$, a subset of the simple roots, let $\mathcal{H}_S = \mathcal{H}(W_S, W_S \cap \mathcal{S})$ be the corresponding Hecke algebra. We consider $\mathbb{Z}[v, v^{-1}]$ as a right $\mathcal{H}_S$-module where H_s for $s \in \mathcal{S}$ acts by multiplication with $-v$. On the other hand, the Hecke algebra $\mathcal{H}$ is in a natural way, via restriction, a left $\mathcal{H}_S$-module. Therefore, the following definition of the *parabolic Hecke module* (with respect to S or $\mathfrak{p}$) makes sense:

$$\mathcal{N}^{\mathfrak{p}} := \mathbb{Z}[v, v^{-1}] \otimes_{\mathcal{H}_S} \mathcal{H}.$$

Hence, the parabolic Hecke module $\mathcal{N}^{\mathfrak{p}}$ is a right $\mathcal{H}$-module and a free $\mathbb{Z}[v, v^{-1}]$-module with basis $\{N_x^{\mathfrak{p}} := 1 \otimes H_x \mid x \in W^{\mathfrak{p}}\}$. The structure as a right $\mathcal{H}$-module is given by the following

Lemma 1.4. (see [Soe97])

$$N_x^{\mathfrak{p}} C_s = \begin{cases} N_{xs}^{\mathfrak{p}} + vN_x^{\mathfrak{p}} & \text{if } xs > x \text{ and } xs \in W^{\mathfrak{p}} \\ N_{xs}^{\mathfrak{p}} + v^{-1}N_x^{\mathfrak{p}} & \text{if } xs < x \text{ and } xs \in W^{\mathfrak{p}} \\ 0 & \text{if } xs \notin W^{\mathfrak{p}} \end{cases}$$

1.3 Translation through the wall and the parabolic Hecke module

For $\lambda \in \mathfrak{h}^*$ dominant and integral, let $\theta_0^\lambda : \mathcal{O}_0 \to \mathcal{O}_\lambda$ (and $\theta_\lambda^0 : \mathcal{O}_\lambda \to \mathcal{O}_0$ respectively) be the corresponding translation functors. For $W_\lambda = \{e, s\}$, $s \in \mathcal{S}$, we denote by $\theta_s = \theta_\lambda^0 \theta_0^\lambda$ the translation functor through the wall (for more details see e. g. [Jan79], [Jan83]).

Since for a module M being $\mathfrak{p}$-locally finite the tensor product $M \otimes E$ is also $\mathfrak{p}$-locally finite for any finite dimensional $\mathfrak{g}$-modules E, the functor θ_s restricts to a functor $\mathcal{O}_0^{\mathfrak{p}} \longrightarrow \mathcal{O}_0^{\mathfrak{p}}$ for any parabolic subalgebra containing $\mathfrak{b}$. Let $[\mathcal{O}_0^{\mathfrak{p}}]$ denote the Grothendieck group of $\mathcal{O}_0^{\mathfrak{p}}$. Since θ_s is exact, it induces a group homomorphism on $[\mathcal{O}_0^{\mathfrak{p}}]$ which is denoted by $\big[[\theta_s]\big]$. Each of the following sets is a basis of $[\mathcal{O}_0^{\mathfrak{p}}]$: $\{[P^{\mathfrak{p}}(x \cdot 0)] \mid x \in W^{\mathfrak{p}}\}$, $\{[M^{\mathfrak{p}}(x \cdot 0)] \mid x \in W^{\mathfrak{p}}\}$ and $\{[L(x \cdot 0)] \mid x \in W^{\mathfrak{p}}\}$. In the following we use the abbreviations $P^{\mathfrak{p}}(x) = P^{\mathfrak{p}}(x \cdot 0)$, $M^{\mathfrak{p}}(x) = M^{\mathfrak{p}}(x \cdot 0)$ and $L(x) = L(x \cdot 0)$ for any $x \in W$.

We state the following well-known results

Proposition 1.5. *Let $x \in W^{\mathfrak{p}}$. Let s be a simple reflection.*

1.) If $xs \in W^{\mathfrak{p}}$ and $x < xs$ then $\theta_s M^{\mathfrak{p}}(x) \cong \theta_s M^{\mathfrak{p}}(xs)$ and there is a short exact sequence of the form

$$0 \to M^{\mathfrak{p}}(x) \longrightarrow \theta_s M^{\mathfrak{p}}(x) \longrightarrow M^{\mathfrak{p}}(xs) \to 0$$

2.) If $xs \notin W^{\mathfrak{p}}$ then $\theta_s M^{\mathfrak{p}}(x) = 0$.

3.) The following diagram commutes

$$\begin{tikzcd}
\mathcal{N}^{\mathfrak{p}} \arrow[r, "N_x^{\mathfrak{p}} \mapsto [M^{\mathfrak{p}}(x)]", "v=1"'] \arrow[d, "\cdot C_s"'] & {[\mathcal{O}_0^{\mathfrak{p}}]} \arrow[d, "{\big[[\theta_s]\big]}"] \\
\mathcal{N}^{\mathfrak{p}} \arrow[r, "N_x^{\mathfrak{p}} \mapsto [M^{\mathfrak{p}}(x)]", "v=1"'] & {[\mathcal{O}_0^{\mathfrak{p}}]}.
\end{tikzcd}$$

Proof. The first part of the theorem is [Irv85, Proposition v]. For the second part we assume $xs \notin W^{\mathfrak{p}}$, hence $xs > x$ (Otherwise, choose $t \in W_{\mathfrak{p}} \cap \mathcal{S}$ such that $txs < xs$. Hence $l(txs) = l(xs) - 1 = l(x) - 2 = l(tx) - 3$. This is a contradiction.) Any non-zero quotient of $\theta_s M(x)$ contains $L(xs)$ as a composition factor; hence there is no nontrivial quotient which is $\mathfrak{p}$-locally finite. In particular, $\theta_s M^{\mathfrak{p}}(x) = 0$. The commutativity of the diagram is then clear by Lemma 1.4. □

2 Gradable Modules and graded translation

In the following we consider an integral regular block (say $\mathcal{O}_0$) of category $\mathcal{O}$ with its parabolic subcategory. Let $P = \oplus_{x \in W} P(x)$ be the sum over all indecomposable projectives in this block. This is a minimal projective generator. How its endomorphism ring becomes a $\mathbb{Z}$-graded ring is explained in [BGS96] and [Str03]. In the following, let $A = \operatorname{End}_{\mathfrak{g}}(P)$ be equipped with this ($\mathbb{Z}$-)grading. By Morita equivalence we can consider $\mathcal{O}_0$ as a category of finitely generated (non-graded!) right modules over a graded ring A. If we denote by $\operatorname{mof} -A$ the category of finitely generated right A-modules this means

$$\mathcal{O}_0 \cong \operatorname{mof} -A.$$

We denote by $(\mathrm{g})\operatorname{mof} -A$ the category of finitely generated *graded* right A-modules. As in [Str03] we call a module $M \in \mathcal{O}_0$ *gradable*, if $\operatorname{Hom}_{\mathfrak{g}}(P, M)$ is gradable, i.e. if there exists a graded right A-module $\tilde{M}$ such that $\mathrm{f}(\tilde{M}) \cong \operatorname{Hom}_{\mathfrak{g}}(P, M)$. (Here f denotes the grading forgetting functor $\operatorname{gmof} -A \longrightarrow \operatorname{mof} -A$.) In this case, $\tilde{M}$ is called *a lift* of M. We call $M \in \mathcal{O}_0^{\mathfrak{p}}$ *gradable*, if it is gradable considered as an object of $\mathcal{O}_0$.
In [Str03] and [BGS96] it is shown that all 'important' objects of $\mathcal{O}_0$, like projective modules, simple modules and Verma modules etc. are gradable. We generalise this result to the parabolic situation:

Theorem 2.1. *Let $M \in \mathcal{O}_0^{\mathfrak{p}}$ be either a simple object, a projective object, or a parabolic Verma module. Then M is gradable (considered as objects in $\mathcal{O}_0$).*

Proof. Since the simple modules in the parabolic subcategory are also simple in $\mathcal{O}_0$, the statement for simple objects is proved in [Str03]. By Lemma 1.2, there is for each $x \in W^{\mathfrak{p}}$ an isomorphism $P^{\mathfrak{p}}(x) \cong P(x)/M$, where M is the smallest submodule containing all simple composition factors of the form $L(y)$ with $y \notin W^{\mathfrak{p}}$. We consider the graded lift $\tilde{P}(x)$ of $P(x)$, which is defined in [Str03]. Let M be its smallest submodule which is generated by the collection of one-dimensional subspaces corresponding to simple composition factors of the form $L(y)$ of $P(x)$ with $y \notin W^{\mathfrak{p}}$. Therefore, M is by definition generated by homogeneous elements; hence the module $\tilde{P}(x)/M$ is a lift of $P^{\mathfrak{p}}(x)$. For the parabolic Verma modules we can do (by Lemma 1.1 and Lemma 1.2) an analogous construction. The theorem follows. $\square$

The proof of the last theorem gives the following

Corollary 2.2. *Let $P^{\mathfrak{p}} := \bigoplus_{x \in W^{\mathfrak{p}}} P^{\mathfrak{p}}(x)$ be a minimal projective generator of $\mathcal{O}_0^{\mathfrak{p}}$. Then $\operatorname{End}_{\mathfrak{g}}(P^{\mathfrak{p}})$ is a quotient of $\operatorname{End}_{\mathfrak{g}}(P)$ even as a graded ring.*

Remark 2.3. 1.) By construction, the graded rings $A = \operatorname{End}_{\mathfrak{g}}(P)$ and $A^{\mathfrak{p}} = \operatorname{End}_{\mathfrak{g}}(P^{\mathfrak{p}})$ coincide with the ones introduced in [BGS96].

2.) The graded lifts of $P^{\mathfrak{p}}(x)$ (with $x \in W^{\mathfrak{p}}$) are unique up to isomorphism and a shift of the grading (see [Str03, Lemma 1.5]). We defined the lifts in a way such that the simple head is concentrated in degree 0. The same is true for the lifts of the parabolic Verma modules and of the simple objects.

3.) It follows directly from the construction that a module $M \in \mathcal{O}_0^{\mathfrak{p}}$ is gradable if and only if it is gradable as an object of $\mathcal{O}_0^{\mathfrak{p}}$, i.e. there exists a graded right $A^{\mathfrak{p}}$-module $\tilde{M}$ such that $\mathrm{f}(\tilde{M}) \cong \mathrm{Hom}_{\mathfrak{g}}(P^{\mathfrak{p}}, M)$ (as non-graded right $A^{\mathfrak{p}}$-modules).

4.) In [Bac99] is proved that $A^{\mathfrak{p}}$ (even for singular blocks) becomes a Koszul ring, generalising the results of [BGS96].

2.1 Combinatorics of graded translation functors

Let us from now on denote by $\tilde{P}^{\mathfrak{p}}(x)$, $\tilde{M}^{\mathfrak{p}}(x)$, and $\tilde{L}(x)$ the graded lifts of $P^{\mathfrak{p}}(x)$, $M^{\mathfrak{p}}(x)$ and $L(x)$, respectively, as defined in the proof of Theorem 2.1; i.e. with head concentrated in degree zero. We consider $A^{\mathfrak{p}} := \mathrm{End}_{\mathfrak{g}}(P^{\mathfrak{p}})$, the endomorphism ring of the minimal projective generator $P^{\mathfrak{p}} = \bigoplus_{x \in W^{\mathfrak{p}}} P^{\mathfrak{p}}(x)$ of $\mathcal{O}_0^{\mathfrak{p}}$ as a graded ring. For $m \in \mathbb{Z}$ let $M\langle m\rangle$ be the graded module defined by $M\langle m\rangle_n := M_{n-m}$ with the same module structure as M, i.e. $\mathrm{f}(M\langle m\rangle) = \mathrm{f}(M)$. Let $[\mathrm{gmof}-A^{\mathfrak{p}}]$ be the Grothendieck group of the category of all finitely generated right $A^{\mathfrak{p}}$-modules. Each of the following three sets is a basis of $[\mathrm{gmof}-A^{\mathfrak{p}}]$: $\{[\tilde{L}(x)\langle i\rangle] \mid x \in W^{\mathfrak{p}}, i \in \mathbb{Z}\}$, $\{[\tilde{M}^{\mathfrak{p}}(x)\langle i\rangle] \mid x \in W^{\mathfrak{p}}, i \in \mathbb{Z}\}$, $\{[\tilde{P}^{\mathfrak{p}}(x)\langle i\rangle \mid x \in W^{\mathfrak{p}}, i \in \mathbb{Z}\}$. Let $\tilde{\theta}_s : \mathrm{gmof}-A \longrightarrow \mathrm{gmof}-A$ denote the graded version of θ_s with the graded adjunction morphisms $\mathrm{ID}\langle 1\rangle \to \tilde{\theta}_s$ and $\tilde{\theta}_s \to \mathrm{ID}\langle -1\rangle$ as defined in [Str03]. We get the following generalisation of [Str03, Theorems 3.6 and 5.3]:

Theorem 2.4. *Let $s \in W$ be a simple reflection.*

1.) Let x, $xs \in W^{\mathfrak{p}}$ such that $x < xs$. The graded lifts of the parabolic Verma modules fit into the following short exact sequences of graded modules

$$\begin{array}{ccccccccc} 0 & \to & \tilde{M}^{\mathfrak{p}}(x\cdot 0)\langle 1\rangle & \to & \tilde{\theta}_s\tilde{M}^{\mathfrak{p}}(x\cdot 0) & \to & \tilde{M}^{\mathfrak{p}}(xs\cdot 0) & \to & 0 \\ 0 & \to & \tilde{M}^{\mathfrak{p}}(x\cdot 0) & \to & \tilde{\theta}_s\tilde{M}^{\mathfrak{p}}(xs\cdot 0) & \to & \tilde{M}^{\mathfrak{p}}(xs\cdot 0)\langle -1\rangle & \to & 0. \end{array}$$

2.) Let $x \in W^{\mathfrak{p}}$ such that $xs \notin W^{\mathfrak{p}}$ then $\tilde{\theta}_s\tilde{M}^{\mathfrak{p}}(x) = 0$

Proof. Note that the maps have to be (up to a scalar) the adjunction morphisms, since the homomorphism spaces in question are all one-dimensional.

Hence, the upper inclusion and the lower surjection is clear. On the other hand, the canonical surjection $\tilde{M}(xs) \twoheadrightarrow \tilde{M}^{\mathfrak{p}}(xs)$ is homogeneous of degree 0 by definition. The surjection of graded modules (see [Str03, Theorem 3.6]) $\tilde{\theta}_s\tilde{M}(x) \twoheadrightarrow \tilde{M}(xs)$ has kernel $\tilde{M}(x)\langle 1\rangle$. Therefore the surjection in the first row has to be homogenous of degree 0. For the injection in the second row, we consider the inclusion of graded modules $\tilde{M}(x) \hookrightarrow \tilde{\theta}_s\tilde{M}(xs)$ (see [Str03, Theorem 5.3]). This induces the injection in the second sequence.
For the second part, we know already the statement when forgetting the grading (Proposition 1.5), hence there is nothing to do. □

We get a combinatorial description of the graded translation functors:

Corollary 2.5. *The following diagram commutes*

$$\begin{tikzcd} \mathcal{N}^{\mathfrak{p}} \arrow[r, "v^n N_x \mapsto [\tilde{M}^{\mathfrak{p}}(x)\langle n\rangle]"] \arrow[d, "\cdot C_s"'] & {[\mathrm{gmof}-A^{\mathfrak{p}}]} \arrow[d, "{[[\tilde{\theta}_s]]}"] \\ \mathcal{N}^{\mathfrak{p}} \arrow[r, "v^n N_x \mapsto [\tilde{M}^{\mathfrak{p}}(x)\langle n\rangle]"] & {[\mathrm{gmof}-A^{\mathfrak{p}}].} \end{tikzcd}$$

Proof. This follows from the previous theorem using Lemma 1.4. □

Remark 2.6. The horizontal maps in the corollary are in fact isomorphisms of $\mathbb{Z}[v, v^{-1}]$-modules where the action on $[\mathrm{gmof}-A]$ is given by $v^i[M] = [M\langle i\rangle]$.

3 The categories of projective functors

In this section we define the additive categories of projective functors.

For each dominant weight $\mathfrak{h}^*$ we denote by $\mathrm{p}_\lambda^{\mathfrak{p}} : \mathcal{O}^{\mathfrak{p}} \to \mathcal{O}_\lambda^{\mathfrak{p}}$ the canonical projection. An endofunctor on $\mathcal{O}_\lambda^{\mathfrak{p}}$ is called *projective*, if it is a (nonzero) direct summand of $\mathrm{p}_\lambda^{\mathfrak{p}}(\bullet \otimes E)$ for some finite dimensional $\mathfrak{g}$-module E. Note that the direct sum of two such functors is again projective. Together with the zero functor, these functors form an additive category $\mathcal{P}_\lambda^{\mathfrak{p}}$ with the usual morphisms (i.e. natural transformation between functors) and the usual notation of (finite) direct sums.
A (projective) functor F on $\mathcal{O}_\lambda^{\mathfrak{p}}$ is *indecomposable*, if $F \cong F_1 \oplus F_2$ for some endofunctors F_1, F_2 on $\mathcal{O}_\lambda^{\mathfrak{p}}$ implies $F_1 = 0$ or $F_2 = 0$. In particular, a projective functor on $\mathcal{O}_\lambda^{\mathfrak{p}}$ is indecomposable if and only if it is an indecomposable object in $\mathcal{P}_\lambda^{\mathfrak{p}}$; hence there is no obstacle to call such functors *indecomposable projective functors* (on $\mathcal{O}_\lambda^{\mathfrak{p}}$).

In the following we write just $\mathcal{P}^{\mathfrak{p}}$ instead of $\mathcal{P}_0^{\mathfrak{p}}$ and $\mathcal{P}_\lambda = \mathcal{P}_\lambda^{\emptyset}$. We denote by $\mathrm{IndP}(\mathfrak{g}, \mathfrak{p})$ the set of isomorphism classes of indecomposable objects in $\mathcal{P}^{\mathfrak{p}}$ and by $\#\,\mathrm{IndP}(\mathfrak{g}, \mathfrak{p})$ the order of this set. The indecomposable functors on $\mathcal{O}_\lambda$ are classified by the following

Theorem 3.1. ([BG80], Theorem 3.3 and Theorem 3.5)

1. *Let λ be an integral dominant weight. Let F, $G \in \mathcal{P}_\lambda$. Then*
$$F \cong G \iff F(M(\lambda)) \cong G(M(\lambda)).$$

2. *The assignment $F \mapsto F(M(\lambda))$ defines a bijection between* $\mathrm{IndP}(\mathfrak{g}, \mathfrak{b})$ *and the set of isomorphism classes of indecomposable projective objects in $\mathcal{O}_\lambda$.*

For $\lambda = 0$ let $F_w \in \mathcal{P}$ such that $F_w M(e) \cong P(w)$. We denote by $\big[[F]\big]$ the induced homomorphism on the Grothendieck group.

Remark 3.2. Since for $F \in \mathcal{P}$ the module $F(M(e))$ is projective, we can reformulate the first part of the Theorem as follows

$$F \cong G \quad \iff \quad [F(M(e))] = [G(M(e))] \quad \iff \quad \big[[F]\big] = \big[[G]\big].$$

Unfortunately, the obvious generalisation of Theorem 3.1 to the parabolic situation is no longer true: Let s be a simple reflection. Choose $\mathfrak{p}$ such that $L(s \cdot 0) \notin \mathcal{O}_0^{\mathfrak{p}}$. We have $\theta_s \neq 0$ in general, but $\theta_s(M^{\mathfrak{p}}(e)) = 0$. Nevertheless, we conjecture a generalisation of Remark 3.2

Conjecture 3.3. *Let F, $G \in \mathcal{P}^{\mathfrak{p}}$ then*

$$F \cong G \quad \iff \quad \big[[F]\big] = \big[[G]\big].$$

3.1 Indecomposable projective functors

In the following section, we state some characterisations of indecomposable (projective) functors in a general setup and define graded lifts.

Proposition 3.4. *Let $\lambda \in \mathfrak{h}^*$ be dominant and integral. Let $F \in \mathcal{P}_\lambda^{\mathfrak{p}}$ be indecomposable. The following are equivalent:*

(i) F is indecomposable.

(ii) The only idempotents in $\mathrm{End}(F)$ *are* 0 *and* 1.

(iii) $\operatorname{End}(F)$ *is a local ring.*

Proof. (ii) $\Rightarrow$ (i): Assume F is decomposable; $F \cong F_1 \oplus F_2$. The natural transformation given by projection onto the first factor defines obviously a nontrivial idempotent.
(i) $\Rightarrow$ (ii): Let $\pi \in \operatorname{End}(F)$ be a nontrivial idempotent. This defines an endofunctor F_π on $\mathcal{O}_\lambda^\mathfrak{p}$ by $F_\pi(M) = \pi(F(M))$ on objects and $F_\pi(f) = \pi_N \circ F(f)|_{\pi_M(F(M))}$ on morphisms $f \in \operatorname{Hom}(M,N)$: Since π is idempotent it is $F_\pi(\mathrm{ID}_M) = \mathrm{ID}_{\pi(M)}$ for any object M. Let $f \in \operatorname{Hom}(M,N)$ and $g \in \operatorname{Hom}(Q,M)$. We get $\pi_N \circ F(f) \circ \pi_M \circ F(g)|_{\pi_Q(F(Q))} = \pi_N \circ F(f) \circ F(g) \circ \pi_Q|_{\pi_Q(F(Q))} = \pi_N \circ F(f \circ g)|_{\pi_Q(F(Q))}$, since π is idempotent. Hence F_π is indeed a functor. On the other hand $(\mathrm{ID} - \pi) \in \operatorname{End}(F)$ is also idempotent and we have $F \cong F_\pi \oplus F_{\mathrm{ID} - \pi}$.
(ii) $\Leftrightarrow$ (iii): Note that $\dim \operatorname{End}(F) \leq \dim \operatorname{End}_\mathfrak{g}(F(P))$ for some (minimal) projective generator P of $\mathcal{O}_0^\mathfrak{p}$, hence $\dim \operatorname{End}(F) < \infty$. Then the equivalence is well-known (see e.g. [Lam91]). □

We get a generalisation of the classical Krull-Remak-Azumaya-Schmidt-Theorem (see e.g. [Lam91]).

Corollary 3.5. *Let $\lambda \in \mathfrak{h}^*$ be dominant and integral. Let $F \in \mathcal{P}_\lambda^\mathfrak{p}$. Then F is isomorphic to a finite direct sum of indecomposable projective functors on $\mathcal{O}_\lambda^\mathfrak{p}$. Moreover, this decomposition is unique up to isomorphism and order of the summands.*

Proof. Let $\mathrm{ll}(F)$ denote the length of a Jordan-Hölder series of $F(P^\mathfrak{p})$. Of course, $\mathrm{ll}(F_1) < \mathrm{ll}(F)$ when F_1 is a direct summand of F. This shows that the desired decomposition exists. The uniqueness follows then by standard arguments (see e.g. [Lam91, 19.23]) using Lemma 3.4. □

3.2 The image of the Hecke algebra

The action of the Hecke algebra on the parabolic Hecke module (see Lemma 1.4) induces a homomorphism

$$\Phi^\mathfrak{p} : \mathcal{H} \longrightarrow \operatorname{End}_\mathbb{Z}(\mathcal{N}^\mathfrak{p}_{v=1}) = \operatorname{End}_\mathbb{Z}([\mathcal{O}_0^\mathfrak{p}]), \tag{3.1}$$

where $\mathcal{N}^\mathfrak{p}_{v=1}$ denotes the specialisation $v \rightsquigarrow 1$ of $\mathcal{N}^\mathfrak{p}$. The $\mathbb{Z}$-rank of the image of $\Phi^\mathfrak{p}$ is denoted by $\mathrm{R}(\mathfrak{g},\mathfrak{p})$. The following Lemma gives a lower bound for the number of indecomposable projective functors:

Proposition 3.6. *For any parabolic subalgebra $\mathfrak{p}$ containing $\mathfrak{b}$ the following holds:*

$$\#\operatorname{IndP}(\mathfrak{g},\mathfrak{p}) \geq \operatorname{R}(\mathfrak{g},\mathfrak{p}). \tag{3.2}$$

Proof. The image of $\Phi^{\mathfrak{p}}$ is generated by the multiplications $\cdot\underline{H}_x$ with $x \in W$. Let $\{\cdot\underline{H}_x \mid x \in I\}$ be a maximal linear independent subset. Let $\{F_x \mid x \in I\}$ be the corresponding projective functors on $\mathcal{O}_0$. Let $\{G_i \mid 1 \leq i \leq m\}$ be a system of representatives for $\operatorname{IndP}(\mathfrak{g},\mathfrak{p})$. For $x \in I$ we have therefore $F_{x|\mathcal{O}_0^{\mathfrak{p}}} \cong \oplus_{i=1}^m G_i^{\alpha_i}$ for some non-negative integers α_i. (Here, $G_i^{\alpha_i}$ denotes the direct sum of α_i copies of G_i.) Hence $\big[[F_{x|\mathcal{O}_0^{\mathfrak{p}}}]\big] = \sum_{i=1}^m \alpha_i \big[[G_i]\big]$. That means, the $\big[[G_i]\big]$ generate the image of $\Phi^{\mathfrak{p}}$ and the claim follows. □

We list some examples including some where we have strict inequality in formula (3.2).

Examples 3.7. (a) Let $\mathfrak{g}$ be arbitrary semisimple. For $\mathfrak{p} = \mathfrak{b}$ both sides of formula (3.2) are equal to the order of the Weyl group. The left hand side by Theorem 3.1; and the right hand side, since the self-dual elements $\underline{H}_x = H_e\underline{H}_x$, with $x \in W$ constitute a $\mathbb{Z}[v, v^{-1}]$-basis of $\mathcal{H}$ giving rise to a $\mathbb{Z}$-basis after specialisation.

(b) In the other extremal case, we have $\operatorname{IndP}(\mathfrak{g},\mathfrak{g}) = 1$. The isomorphism $\operatorname{End}_{\mathbb{Z}}(\mathcal{N}^{\mathfrak{p}}_{v=1}) \cong \operatorname{End}_{\mathbb{Z}}(\mathbb{Z})$ implies $\operatorname{R}(\mathfrak{g},\mathfrak{g}) = 1$.

(c) Let $\mathfrak{g}$ be of type B_2 or G_2 with $\mathfrak{p}$ a maximal parabolic subalgebra. Then

$$\begin{aligned} 10 = \#\operatorname{IndP}(\mathfrak{so}_3,\mathfrak{p}) &> \operatorname{R}(\mathfrak{g},\mathfrak{p}) = 6. \\ 26 = \#\operatorname{IndP}(\mathfrak{g} \text{ of type } G_2,\mathfrak{p}) &> \operatorname{R}(\mathfrak{g},\mathfrak{p}) = 10. \end{aligned}$$

Consider $\mathfrak{g} = \mathfrak{so}_3$. Let $W_{\mathfrak{p}} = \langle t\rangle \subseteq \langle s,t\rangle$, hence $W^{\mathfrak{p}} = \langle e, s, st, sts\rangle$. By Theorem 3.9, θ_t is indecomposable (with $X = \{L(t)\}$). For $\lambda \in \mathfrak{h}^*$ dominant and integral such that $W_\lambda = \{e, s\}$, the category $\mathcal{O}^{\mathfrak{p}}_\lambda$ is semisimple. (Note that $\theta_0^\lambda P^{\mathfrak{p}}(e) \cong M^{\mathfrak{p}}(\lambda)$ and $[\theta_0^\lambda P^{\mathfrak{p}}(st)] = [\theta_0^\lambda(M^{\mathfrak{p}}(st) \oplus M^{\mathfrak{p}}(sts))] = [M^{\mathfrak{p}}(st\cdot\lambda) \oplus M^{\mathfrak{p}}(st\cdot\lambda)]$. Therefore, $P^{\mathfrak{p}}(x\cdot\lambda) = M^{\mathfrak{p}}(x\cdot\lambda) = L(x\cdot\lambda)$ for $x \in \{e, st\}$.)
We define (exact) endofunctors G_e, G_{st} on $\mathcal{O}^{\mathfrak{p}}_\lambda$ by

$$G_w L(x\cdot\lambda) = \begin{cases} L(x\cdot\lambda) & \text{if } w = x \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

In particular $\mathrm{ID} \cong G_e \oplus G_{st}$. Set $\mathcal{G}_w = \theta^0_\lambda G_w \theta^\lambda_0$. Then $\theta_s \cong \mathcal{G}_e \oplus \mathcal{G}_{st}$, hence decomposable. If $x \notin \{e, s\}$ then $G_e\theta^\lambda_0 M^{\mathfrak{p}}(x) = 0$. Otherwise,

$$\begin{aligned}[][G_e\theta^\lambda_0 M^{\mathfrak{p}}(x)] &= [G_e M^{\mathfrak{p}}(\lambda)] = [M^{\mathfrak{p}}(\lambda)] = [G_e(M^{\mathfrak{p}}(\lambda) \oplus M^{\mathfrak{p}}(st\cdot\lambda))] \\ &= [G_e\theta^\lambda_0(M^{\mathfrak{p}}(s) \oplus M^{\mathfrak{p}}(st))] = [G_e\theta^\lambda_0\theta_t(M^{\mathfrak{p}}(e) \oplus M^{\mathfrak{p}}(s))] \\ &= [G_e\theta^\lambda_0\theta_t\mathcal{G}_e M^{\mathfrak{p}}(x)]\end{aligned}$$

By the semi-simplicity of $\mathcal{O}^{\mathfrak{p}}_\lambda$ we get $G_e\theta^\lambda_0\theta_t\mathcal{G}_e \cong G_e\theta^\lambda_0$, hence $\mathcal{G}_e\theta_t\mathcal{G}_e \cong \mathcal{G}_e$. Analogously it is $\mathcal{G}_{st}\theta_t\mathcal{G}_{st} \cong \mathcal{G}_{st}$. One can easily check that for w, $z \in \{e, st\}$ with $w \neq z$ the functors

$$\mathrm{ID}, \theta_t, \mathcal{G}_w, \mathcal{G}_w\theta_t, \theta_t\mathcal{G}_w, \theta_w\theta_t\mathcal{G}_z$$

induce pairwise distinct morphisms on the Grothendieck group $[\mathcal{O}^{\mathfrak{p}}_0]$. The criterion of Theorem 3.9 shows that the functors are all indecomposable. Hence (by Theorem 3.1 and Corollary 3.5) they represent $\mathrm{Ind}(\mathfrak{g}, \mathfrak{p})$. (We remark that the induced representation on the Grothendieck group is isomorphic to the one obtained by taking by $\mathfrak{g} = \mathfrak{sl}_4$ where $W_{\mathfrak{p}}$ is generated by non-commuting simple reflections s_2, s_3.)
The case G_2 can be done in an analogous way.

3.3 Restriction to the simple case

To get a description of indecomposable projective functors it is enough to consider simple Lie algebras because of the following

Proposition 3.8. *Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ be two semisimple complex Lie algebras with parabolic subalgebras $\mathfrak{p}_i$ containing the fixed borel subalgebra $\mathfrak{b}_i \subset \mathfrak{g}_i$ for i=1, 2. Then*

$$\#\,\mathrm{IndP}(\mathfrak{g}_1 \times \mathfrak{g}_2, \mathfrak{p}_1 \times \mathfrak{p}_2) = \#\,\mathrm{IndP}(\mathfrak{g}_1, \mathfrak{p}_1) \cdot \#\,\mathrm{IndP}(\mathfrak{g}_2, \mathfrak{p}_2).$$

Proof. There is a triangular decomposition $\mathfrak{g}_1 \times \mathfrak{g}_2 = (\mathfrak{n}_1^- \times \mathfrak{n}_2^-) \oplus (\mathfrak{h}_1 \times \mathfrak{h}_2) \oplus (\mathfrak{n}_1 \times \mathfrak{n}_2)$ arising from the corresponding triangular decompositions of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively. The identification $\mathcal{U}(\mathfrak{g}_1 \times \mathfrak{g}_2) = \mathcal{U}(\mathfrak{g}_1) \boxtimes \mathcal{U}(\mathfrak{g}_2)$ induces $\mathcal{Z}(\mathcal{U}(\mathfrak{g}_1 \times \mathfrak{g}_2)) = \mathcal{Z}(\mathcal{U}(\mathfrak{g}_1)) \boxtimes \mathcal{Z}(\mathcal{U}(\mathfrak{g}_2))$ (where $\boxtimes$ denotes the outer tensor product over $\mathbb{C}$). This corresponds to an identification $\mathfrak{h}_1^* \times \mathfrak{h}_2^* = (\mathfrak{h}_1 \times \mathfrak{h}_2)^*$ and an isomorphism between the Weyl group of $\mathfrak{g}_1 \times \mathfrak{g}_2$ and the product $W_1 \times W_2$ of the single Weyl groups. Then the outer tensor product defines a functor

$$\boxtimes : \mathcal{O}_0(\mathfrak{g}_1) \times \mathcal{O}_0(\mathfrak{g}_2) \longrightarrow \mathcal{O}_{(0,0)}(\mathfrak{g}_1 \times \mathfrak{g}_2).$$

The simple objects in $\mathcal{O}_{(0,0)}(\mathfrak{g}_1 \times \mathfrak{g}_2)$ are given as tensor products of simple objects; in formulas $L((x,y)) \cong L(x) \boxtimes L(y)$ for $(x,y) \in W_1 \times W_2$ with projective cover

$$P((x,y)) \cong P(x) \boxtimes P(y). \tag{3.4}$$

(For more details concerning this see [Bac01, section 2]). On the other hand, the outer tensor product defines a map between the sets of projective functors

$$\boxtimes : \mathcal{P}(\mathfrak{g}_1) \times \mathcal{P}(\mathfrak{g}_1) \longrightarrow \mathcal{P}(\mathfrak{g}_1 \times \mathfrak{g}_2).$$

The isomorphisms (3.4) together with the Classification Theorem 3.1 imply that the map is in fact a bijection. This proves the Theorem for $\mathfrak{p}_i = \mathfrak{b}_i$, $i = 1, 2$.
On the other hand, for any F, $G \in \mathcal{P}$, there is an isomorphism of rings

$$\Gamma : \mathrm{End}(F) \boxtimes \mathrm{End}(G) \longrightarrow \mathrm{End}(F \boxtimes G) \tag{3.5}$$

given by $\Gamma(\phi \otimes \psi)_{(P,Q)}(p,q) = \phi_P(p) \otimes \psi_Q(q)$ on projective generators $P \in \mathcal{O}_0(\mathfrak{g}_1)$, $Q \in \mathcal{O}_0(\mathfrak{g}_2)$ and $p \in P$, $q \in Q$. (It is not difficult to see that this defines in fact a homomorphism of functors. For the bijectivity see e.g. [Bac01, Lemma 2.1]).
It induces an isomorphism $\Gamma^{\mathfrak{p}} : \mathrm{End}(F_{|\mathcal{O}_0^{\mathfrak{p}_1}}) \boxtimes \mathrm{End}(G_{|\mathcal{O}_0^{\mathfrak{p}_2}}) \longrightarrow \mathrm{End}(F \boxtimes G_{|\mathcal{O}_0(\mathfrak{p}_1 \times \mathfrak{p}_2)})$. Note that $\Gamma^{\mathfrak{p}}(\phi \otimes \psi)$ is an idempotent if and only if ϕ, ψ are idempotents. Moreover, $\Gamma^{\mathfrak{p}}(\phi \otimes \psi)$ is a trivial idempotent if and only if so are ϕ and ψ. Hence, pairs of indecomposable projective functors corresponds to indecomposable projective functors. This proves the proposition.

□

3.4 A criterion for indecomposability

For $F \in \mathcal{P}^{\mathfrak{p}}$ and $X \subseteq W^{\mathfrak{p}}$, we consider the sets of **simple objects** in $\mathcal{O}_0^{\mathfrak{p}}$:

$$\begin{aligned} \mathrm{supp}(F) &= \{L \mid F(L) \neq 0\}, \\ \mathrm{Comp}(X) &= \{L \mid [M^{\mathfrak{p}}(x) : L] \neq 0 \text{ for some } x \in X\}. \end{aligned}$$

We get a criterion for a projective functor to be indecomposable:

Theorem 3.9. *Let $0 \neq F \in \mathcal{P}^{\mathfrak{p}}$. Assume, there exists $X \subseteq W$ with $\mathrm{supp}(F) \subseteq \mathrm{Comp}(X)$ such that*

(a) $F(M^{\mathfrak{p}}(x))$ is indecomposable for any $x \in X$.

(b) For any nontrivial decomposition $X = X_1 \cup X_2$ we have

$$\mathrm{Comp}(X_1) \cap \mathrm{Comp}(\mathrm{X}_2) \cap \mathrm{supp}(F) \neq \emptyset.$$

Then F is indecomposable.

Remark 3.10.

- The functor $\mathrm{ID} \in \mathcal{P}^{\mathfrak{p}}$ is indecomposable, therefore $\mathrm{IndP}(\mathfrak{g}, \mathfrak{g}) = \{\mathrm{ID}\}$.
- Let $\mathfrak{p} = \mathfrak{b}$. The theorem gives an alternative proof of the fact that the indecomposability of $F(M(e))$ implies the indecomposability of F: Let $X = \{e\}$. Since every simple module occurs as a composition factor in $M(e)$, the assumptions are satisfied if and only if $F(M(e))$ is indecomposable.
  On the other hand we could also choose $X = W$. Note that $F(M(e))$ is indecomposable if and only if so is $F(M(x))$ for all $x \in W$ by [AS03, Theorem 2.2 and Corollary 4.2]. Since $F(L(w_o)) \neq 0$ and $L(w_o)$ occurs in the socle of each Verma module, the assumption (b) is satisfied.
- Let us consider the situation $\mathfrak{g} = \mathfrak{so}_3$ of Example 3.7 for $F = \theta_s$. Condition (a) is always satisfied. Let us assume the existence of a set X as in the theorem. For $i \in \{1, 2\}$, set $X_i = X \cap A_i$, where $A_1 = \{e, s\}$ and $A_2 = \{st, sts\}$. Hence, condition (b) is not satisfied. It turned out that θ_s is indeed decomposable.

Proof of Theorem 3.9. We assume the existence of X. Let $\pi \in \mathrm{End}(F)$ be an idempotent. By assumption (a) it is $\pi_{M^{\mathfrak{p}}(x)} \in \{\mathrm{ID}, 0\}$ for all $x \in X$. Choose $x_1 \in X$ such that $FM^{\mathfrak{p}}(x_1) \neq 0$. Set $X_1 = \{x_1\}$ and $X_2 = X \backslash X_1$. Let L be an element of the intersection given in (b) occurring in say $\mathrm{Comp}(\{x_2\})$, $x_2 \in X_2$. If $\pi_{M^{\mathfrak{p}}(x_1)} = \mathrm{ID}$ then $\pi_L = \mathrm{ID}$ and therefore $\pi_{M^{\mathfrak{p}}(x_2)} = \mathrm{ID}$. Going on with $X_1 := \{x_1, x_2\}$ etc. in the same way gives finally $\pi_{M^{\mathfrak{p}}(x)} = \mathrm{ID}$ for any $x \in X$. The same arguments work if $\pi_{M^{\mathfrak{p}}(x_1)} = 0$. Hence π is on all simple objects simultaneously either the identity or zero.
We first consider the case where π is the identity on simple objects. We prove that $\pi_M = \mathrm{ID}$ for any M by induction on the length. Let $M_1 \hookrightarrow M \twoheadrightarrow M_2$ be a short exact sequence. Then $F(M_1) \overset{i}{\hookrightarrow} F(M) \overset{p}{\twoheadrightarrow} F(M_2)$ is exact. Let $x \in F(M)$. If $x = i(y)$ for some $y \in F(M_1)$ then $\pi_M(x) = \pi_M(i(y)) = i(y) = x$. Otherwise $0 \neq p(x) = \pi_{M_2}(p(x)) = p(\pi_M(x))$. Hence $x - \pi_M(x) = i(y)$ for some $y \in F(M_1)$. Since π is idempotent, we have $0 = \pi_M(\pi_M(x) - x) = \pi_M(i(y)) = i(y)$. Therefore, $y = 0$ and $\pi_M = \mathrm{ID}$. Let now $\pi_{M_i} = 0$ for $i = 1$, 2. For $x \in F(M)$ we get $p(\pi_M(x)) = \pi_{M_2}(p(x)) = 0$,

hence $\pi_M(\pi_M(x)) = \pi_M(i(y)) = i(\pi_{M_1}(y)) = 0$ for some $y \in M_1$. The theorem follows.

$\square$

3.5 Lifts of projective functors

Let F be an exact endofunctor on $\mathcal{O}_0^{\mathfrak{p}}$. We call $\tilde{F} : \mathrm{gmof}-A^{\mathfrak{p}} \longrightarrow \mathrm{gmof}-A^{\mathfrak{p}}$ a *graded lift* of F, if $\tilde{F}$ is a $\mathbb{Z}$-functor and induces F after forgetting the grading and applying the equivalence $\mathrm{mof}-A^{\mathfrak{p}} \to \mathcal{O}_0^{\mathfrak{p}}$ (for details see [Str03] or [AJS94, section E3]). If such a lift exists, we call F *gradable.*

Proposition 3.11. *Let $F \in \mathcal{P}^{\mathfrak{p}}$ be indecomposable. A lift $\tilde{F}$ of F (if it exists) is unique up to isomorphism and grading shift.*

Proof. Under the equivalence $\mathcal{O}_0^{\mathfrak{p}} \cong \mathrm{gmof}-A^{\mathfrak{p}}$, the functor F corresponds to $\bullet \otimes_{A^{\mathfrak{p}}} X$ for some $A^{\mathfrak{p}}$-bimodule X (see [Bas68]). Moreover, F is indecomposable if and only if so is X (as an $A^{\mathfrak{p}}$-bimodule). A graded lift $\tilde{F}$ of F is therefore given as tensoring with some graded $A^{\mathfrak{p}}$-bimodule $\tilde{X}$ such that $\tilde{X} \cong X$ after forgetting the grading. By the indecomposability of X, a lift is unique up to isomorphism and grading shift (use [Str03, Lemma 1.5] for the graded ring $A^{\mathfrak{p}} \otimes (A^{\mathfrak{p}})^{\mathrm{opp}}$.) $\square$

Corollary 3.12. *Let $F \in \mathcal{P}$ be indecomposable. Then F is gradable. A lift of F is unique up to isomorphism and grading shift.*

Proof. The translation functors through a wall are gradable ([Str03]), hence, their compositions as well. Theorem 3.1 shows that there is a decomposition of functors

$$\theta_{s_r}\theta_{s_{r-1}} \cdot \ldots \cdot \theta_{s_1} \cong F_x \oplus \bigoplus_{y<x} F_y^{\alpha_y} \tag{3.6}$$

for some $\alpha_y \in \mathbb{N}$ and $x = s_1 \cdot \ldots \cdot s_r$ a reduced expression of x. By induction hypothesis, the F_y's are gradable for $y < x$. (Note that $F_e = \mathrm{ID}$ is gradable.) Therefore, F_x is gradable (see [Str03, Lemma 1.4]). The statement on the uniqueness is the previous proposition. $\square$

We fix a lift $\tilde{F}_w$ of F_w such that $\tilde{F}_w\tilde{M}(e) \cong \tilde{P}(w)$.

Remark 3.13. Let $[w] = s_1 s_2 \cdot \ldots \cdot s_r$ be a reduced expression of $w \in W$. With the conventions on the lift $\tilde{F}_w$ and Theorem 2.5 we get

$$\tilde{\theta}_{s_r}\tilde{\theta}_{s_{r-1}} \cdot \ldots \cdot \tilde{\theta}_{s_1} \cong \bigoplus_{y\in W} (\tilde{F}_y\langle i\rangle)^{\alpha_{[w],y,i}},$$

where the $\alpha_{[w],y,i}$ are defined as $C_{s_1}C_{s_2}\cdot\ldots\cdot C_{s_r} = \sum_{y\in W, i\in\mathbb{Z}} \alpha_{w,y,i}\, v^i \underline{H}_y$. Note that $\alpha_{w,y,i}$ does not depend on the reduced expression of w provided w is braid-avoiding.

4 Generalised Temperley-Lieb algebras

In this section we describe 'functorial realisations' of generalised Temperley-Lieb algebras. Let W be a Weyl group of type A, B, C or D with corresponding Hecke algebra $\mathcal{H}$. We consider the (generalised) Temperley-Lieb algebra $\mathcal{H}/(\mathrm{TL})$; i.e. TL is generated by all $\sum_{w\in W_{adj}} v^{-l(w)}H_w$ such that $W_{adj}\subset W$ is generated by two simple reflections where the corresponding vertices in the Dynkin diagram are connected. These algebras were introduced by Temperley and Lieb [TL71] for type A and Dieck [Die98] for other types. Alternatively, they can be defined by the following relations (with s, $t\in\mathcal{S}$):

$$C_s^2 = (v+v^{-1})C_s \tag{4.1}$$

$$C_sC_t = C_tC_s \quad \text{if } ts=st. \tag{4.2}$$

Additionally for type A B, C and D:

$$C_sC_tC_s = C_s \quad \text{if } ts\neq st, \tag{4.3}$$

and for type B and C:

$$C_tC_sC_tC_s = C_tC_s + C_tC_s \quad \text{if } ts\neq st \text{ and } sts\neq tst. \tag{4.4}$$

Theorem 4.1 (TL-algebras and projective functors). *Let $\mathfrak{g}$ be a simple Lie algebra of type A, B, C or D. Let $\mathfrak{p}\subset\mathfrak{g}$ be maximal parabolic. With the interpretation of v^i as grading shift $\langle i\rangle$, the graded translation functors satisfy the relations* (4.1), (4.2) *and the corresponding Temperley-Lieb algebra relations.*

Proof. Let s, t be commuting simple reflections. By Theorem 3.1, $\theta_s\theta_t\cong\theta_t\theta_s$. The functors are indecomposable and therefore $\tilde\theta_s\tilde\theta_t\cong\tilde\theta_t\tilde\theta_s\langle i\rangle$ for some $i\in\mathbb{Z}$ (Corollary 3.12). Since $\tilde M(e)\langle 2\rangle$ occurs in both $\tilde\theta_s\tilde\theta_t\tilde M(e)$ and $\tilde\theta_t\tilde\theta_s\tilde M(e)$ as a submodule (Theorem 2.4), it follows $i=0$. Therefore, relation (4.2) is satisfied. Since there is an isomorphism $\theta_s^2\cong\theta_s\oplus\theta_s$, we get $\tilde\theta_s^2\cong\theta_s\langle i\rangle\oplus\tilde\theta_s\langle j\rangle$ for some i, j (again using Corollary 3.12). On the other hand, Corollary 2.5 shows $[\tilde\theta_s^2\tilde M(e)] = [\tilde\theta_s\tilde M(e)\langle 1\rangle]+[\tilde\theta_s\tilde M(e)\langle -1\rangle]$. The relation (4.1) is satisfied. Let now $st\neq ts$ but $sts=tst$. We just recall the arguments of [BFK99]:

We have $\theta_s\theta_t\theta_s \cong F_{sts} \oplus \theta_s$ by Theorem 3.1. Let $\eta \in \mathfrak{h}^*$ be dominant and integral such that $W_\eta = \langle s, t\rangle$. Since $F_{sts}M(e) \cong P(sts) \cong \theta_\eta^0\theta_0^\eta M(e)$ (see e.g. [Jan83, 14.13 (1)] and [Soe97, Proposition 2.9]) we get $F_{sts} \cong \theta_\eta^0\theta_0^\eta$ by Theorem 3.1. In particular, $F_{sts} = 0$ when restricting to $\mathcal{O}_0^{\mathfrak{p}}$ (for $\mathfrak{p}$ maximal parabolic!). In the graded picture we have $\tilde{\theta}_s\tilde{\theta}_t\tilde{\theta}_s \cong \tilde{F}_{sts}\langle i\rangle \oplus \tilde{\theta}_s\langle j\rangle$ for some i, $j \in \mathbb{Z}$. Since we did not prove Remark 3.13 we determine i and j directly: By Theorem 2.4 $\tilde{\theta}_s\tilde{\theta}_t\tilde{\theta}_s\tilde{M}(e)$ surjects onto $\tilde{M}(sts)$ and therefore, $i = 0$. Corollary 2.5 shows that $\tilde{M}(e)\langle k\rangle$ occurs as a submodule in $\tilde{\theta}_s\tilde{\theta}_t\tilde{\theta}_s\tilde{M}(e)$ for $k = 3,\ 1$. By Theorem 2.4 $\tilde{M}(e)\langle 3\rangle$ is a submodule of $\tilde{\theta}_t\tilde{\theta}_s\tilde{M}(e)\langle 1\rangle$. The latter is contained in $\tilde{\theta}_s\tilde{\theta}_t\tilde{\theta}_s\tilde{M}(e) \cong \tilde{P}(sts) \oplus \tilde{P}(s)\langle j\rangle$, hence it must be a submodule of $\tilde{P}(sts)$. Therefore, $\tilde{M}(e)\langle 1\rangle$ is a submodule of $\tilde{\theta}_s\tilde{M}(e)\langle j\rangle$. Theorem 2.4 implies $j = 0$. Formula (4.3) follows.
If $st \neq ts$ and $sts \neq tst$ then $\theta_s\theta_t\theta_s\theta_t \cong F_{tsts} \oplus \theta_{ts} \oplus \theta_{ts}$ by Theorem 3.1. The same arguments as above show that $F_{tsts} = 0$ when restricting to $\mathcal{O}_0^{\mathfrak{p}}$ and $\tilde{\theta}_t\tilde{\theta}_s\tilde{\theta}_t\tilde{\theta}_s \cong \tilde{F}_{stst} \oplus \tilde{\theta}_{ts}\langle i\rangle \oplus \theta_{ts}\langle j\rangle$ for some i, $j \in \mathbb{Z}$. Corollary 2.5 implies that $\tilde{M}(e)\langle 4\rangle \oplus \tilde{M}(e)\langle 2\rangle \oplus \tilde{M}(e)\langle 2\rangle$ occurs as submodule in $\tilde{\theta}_t\tilde{\theta}_s\tilde{\theta}_t\tilde{\theta}_s\tilde{M}(e)$. Since $\tilde{P}(tst)\langle 1\rangle$ is a submodule of $\tilde{\theta}_s P(tst)$ (hence of $\tilde{P}(tsts)$), the module $\tilde{M}(e)\langle 4\rangle$ is a submodule of $P(tsts)$. Theorem 2.4 shows that $\tilde{M}(e)\langle 2\rangle$ is a submodule of $\tilde{\theta}_t\tilde{\theta}_s\tilde{M}(e)$. We get $i = j = 0$. The theorem follows. □

For W of type A and $N \in \mathbb{N}_{>0}$, the N-generalised Temperley-Lieb algebra $\mathcal{H}_N$ is defined as $\mathcal{H}/I_N$, where I_N is the $\mathbb{Z}[v, v^{-1}]$-span of all (Kazhdan-Lusztig) basis elements indexed by tableaux with more than N columns (see [Här99], [Lip94], [BK95]). In particular, $\mathcal{H}_1 \cong \mathbb{Z}[v, v^{-1}]$ and $\mathcal{H}_2$ is the ordinary Temperley-Lieb algebra.

Proposition 4.2. *Let $n > 1$. Let $\mathfrak{p} = \mathfrak{p}_S \subset \mathfrak{sl}_{n+1}$ be a parabolic subalgebra and $N \geq n - |S| + 1$. The corresponding $\Phi^{\mathfrak{p}}$ from* (3.1) *factors through $\mathcal{H}_N$.*

Proof. This is just a reformulation of say [Här99, 3.1], [Mar92], or [BK95, Theorem 3.1]. □

Remark 4.3.

- Via (3.1), the previous proposition provides an *injection* $\mathcal{H}_N \to \operatorname{End}\mathbb{Z}[v, v^{-1}](\oplus\mathcal{N}^{\mathfrak{p}_S})$ where the sum runs over all $S \subset \pi$ satisfying $|S| \geq n - N + 1$ ([Mar92] or [Här99]).
- The description of I_N in terms of Kazhdan-Lusztig basis elements (see [Här99], [Lip94]) provides many $F \in \mathcal{P}$ which become zero after restricting to $\mathcal{O}_0^{\mathfrak{p}}$. In particular, the case $I_2 = (\{\underline{H}_{sts}), st \neq ts\})$ together with the relation $C_sC_tC_s = \underline{H}_{sts} + C_s$ implies Theorem 4.1 for type A.

- It is not clear, whether the $\mathcal{H}_N$ have a diagrammatical/topological interpretation. However, the generalised Temperley-Lieb algebras of type B and D are known to have a description via *decorated* tangles (see e.g. [Gre98]). This might be of topological interest.

5 Type A: Maximal Parabolic Subalgebras

In general, an indecomposable projective functor does not stay indecomposable when restricting to parabolic subcategories (see Example 3.7 (c)). However, this section is devoted to a proof of the following result (conjectured in [BFK99]):

Theorem 5.1 (Indecomposability). *Let $n > 1$. Let $\mathfrak{p} \subset \mathfrak{g} = \mathfrak{sl}_n$ be a maximal parabolic subalgebra. Let $F \in \mathcal{P}$ be indecomposable, then its restriction to $\mathcal{O}_0^{\mathfrak{p}}$ is indecomposable or zero.*

For $w \in W$ with a reduced expression $[w] = s_{i_1}s_{i_2}\cdot\ldots\cdot s_{i_r}$ let $\theta_{[w]} = \theta_{s_{i_r}}\theta_{s_{i_{r-1}}}\cdot\ldots\cdot\theta_{s_{i_1}}$. If $\mathfrak{g} = \mathfrak{sl}_n$ then $w \in W = S_n$ is *braid-avoiding* if some (=any) reduced expression does not contain a substring of the form sts with non-commuting simple reflections s and t. In this case $\theta_{[w]} \in \mathcal{P}$ is indecomposable (see [BW01, Theorem1]), hence isomorphic to F_w. In particular it is independent of the chosen reduced expression.
In the following we study the case $\mathfrak{g} = \mathfrak{sl}_n$ with corresponding category $\mathcal{O}(\mathfrak{sl}_n)$. We always consider the Weyl group of $\mathfrak{sl}_n$ as generated by $s_i = s_{\alpha_i}$, $1 \leq i \leq n$ such that $s_is_j = s_js_i$ iff $|i-j| > 1$. To simplify notation set $\mathcal{O}_i(\mathfrak{sl}_n) = \mathcal{O}(\mathfrak{sl}_n)_\lambda$, where $\lambda \in \mathfrak{h}^*$ is dominant and integral such that $W_\lambda = \{e, s_i\}$. For $1 \leq k \leq n$ let $S_k = \pi\backslash\{\alpha_k\}$ and set $S_0 = S_{n+1} = \pi$. We denote by $\mathcal{O}^k(\mathfrak{sl}_n)$ the main block of the corresponding parabolic category $\mathcal{O}^{S_k}$. To make formulas easier, $\mathcal{O}^k(\mathfrak{sl}_n)$ denotes the zero category if $k < 0$ or $k > n+1$. We also use the notation $\mathcal{O}_i^k(\mathfrak{sl}_n)$ for the full subcategory of $\mathcal{O}_i(\mathfrak{sl}_n)$ defined by all locally $\mathfrak{p}_{S_k}$-finite modules. Let $\theta_0^i : \mathcal{O}_0(\mathfrak{sl}_n) \longrightarrow \mathcal{O}_i(\mathfrak{sl}_n)$ (and $\theta_i^0 : \mathcal{O}^i(\mathfrak{sl}_n) \longrightarrow \mathcal{O}_0(\mathfrak{sl}_n)$ respectively) denote the translation onto/out of the i-th wall and let $\theta_i = \theta_{s_i}$ denote the translation through the i-th wall.

The following observation simplifies the proof of Theorem 5.1:

Lemma 5.2. *Let $n > 1$ and let $\mathfrak{p} \subset \mathfrak{g} = \mathfrak{sl}_n$ be a maximal parabolic subalgebra. Let $F = F_w \in \mathcal{P}$ be indecomposable with w not braid-avoiding. Restriction to $\mathcal{O}_0^{\mathfrak{p}}$ gives $F_w = 0$.*

Proof. Let $w = b(sts)a$ be minimal with simple reflections s, t. Then, $F_w \in \mathcal{P}$ is a direct summand of $\theta_{[a]}\theta_{[sts]}\theta_{[b]} \cong \theta_{[a]}F_{sts}\theta_{[b]} \oplus \theta_{[a]}\theta_s\theta_{[b]}$. If F_w occurs in the first summand, then $F_w = 0$ when restricting to $\mathcal{O}_0^{\mathfrak{p}}$, since F_{sts} becomes zero after restriction (see proof of Theorem 4.1). On the other hand, by construction it cannot occur in the second summand (see formula (3.6)), because $l(asb) \leq l(w) - 2$. The lemma follows. ☐

The following lemma describes Supp(F) for certain $F \in \mathcal{P}^{\mathfrak{p}}$:

Lemma 5.3. *Let $n > 1$ and $\mathfrak{p} \subset \mathfrak{g} = \mathfrak{sl}_n$ be a maximal parabolic subalgebra. Let $x \in W^{\mathfrak{p}}$ and $w \in W$ with reduced expression $[w] = s_{i_1}s_{i_2} \cdot \ldots \cdot s_{i_r}$.*

1. *If $i_{j+1} = i_j + 1$ for all $1 \leq j < r$ or if $i_{j+1} = i_j - 1$ for all $1 \leq j < r$ then the following holds:*

$$\theta_{[w]}L(x) \neq 0 \iff \theta_{s_{i_1}}L(x) \neq 0 \iff x > xs_{i_1}.$$

2. *If $s_{i_j}s_{i_k} = s_{i_k}s_{i_j}$ for all $1 \leq j, k \leq r$ then the following holds:*

$$\theta_{[w]}L(x) \neq 0 \iff (xs_{i_j} < x \text{ for } 1 \leq j \leq r).$$

Proof. 1. Let $F = \theta_{i_{r-1}}\theta_{s_{i_{r-2}}} \cdot \ldots \cdot \theta_{s_{i_1}}$. The definitions and Proposition 4.1 give the following implications

$$0 \neq \theta_{[w]}L(x) \quad \Rightarrow \quad 0 \neq FL(x) = \theta_{i_{r-1}}\theta_{i_r}FL(x) \qquad \Rightarrow \quad \theta_{[w]}L(x) \neq 0.$$

Inductively the first equivalence follows. The second is well-known ([Jan83, 4.12 (3), 4.13 (3)]).

2. We already verified the implication from the left hand side to the right. If x satisfies the condition on the right hand side it is $[\theta_{[w]}L(x)] = [\theta_{s_{i_r}} \cdot \ldots \cdot \theta_{s_{i_2}}L(x)] + [\theta_{s_{i_r}} \cdot \ldots \cdot \theta_{s_{i_2}}M]$ for some module M (see [Jan83, 4.12 (3), 4.13 (3')]. By induction hypothesis the first summand is non-trivial and the statement follows. ☐

Proposition 5.4. *Let $n > 1$. Let $\mathfrak{p}_m \subset \mathfrak{sl}_n$ be a maximal parabolic subalgebra ($1 \leq m \leq n$). Let $w \in W$ be of the form described in Lemma 5.3* (2). *The following holds:*

a.) $F_w(M^{\mathfrak{p}}(x))$ is indecomposable or zero for any $x \in W^{\mathfrak{p}}$.

b.) The restriction $F_w = \theta_{[w]}$ to $\mathcal{O}_0^{\mathfrak{p}}$ is indecomposable.

Proof. a.) If $xs_{i_j} \notin W^{\mathfrak{p}}$ for some $j \in \{1, 2, \dots, r\}$ then $F_w M^{\mathfrak{p}}(x) = 0$. By Proposition 1.5, we may assume $xs_{i_j} > x$ and $xs_{i_j} \in W^{\mathfrak{p}}$ for $1 \leq j \leq r$. In particular, xs_{i_j} is braid-avoiding (see Proposition A-2). Since all the s_{i_j} are pairwise commuting, the expression xw is minimal and braid-avoiding as well ([BW01, Lateral Convexity]). Hence $F_w M^{\mathfrak{p}}(x)$ is a homomorphic image of $F_w P(x) \cong \theta_{[w]}\theta_{[x]}M(e) \cong \theta_{[xw]}M(e) \cong P(xw)$. (We used [BW01, Theorem 1] and Theorem 3.1.) In particular, $F_w M^{\mathfrak{p}}(x)$ is indecomposable.

b.) It remains to check that $X = \{x \in W^{\mathfrak{p}} \mid L(x) \in \mathrm{Supp}(F_w)\}$ satisfies property (b) of Theorem 3.9. Assume, there is a decomposition $X = X_1 \cup X_2$ such that $\mathrm{Comp}(X_1) \cap \mathrm{Comp}(X_2) \cap X = \emptyset$. We first consider the special case $r = 1$. Let $i_1 = i$; hence $X = \{x \in W^{\mathfrak{p}}, xs_i < x\}$. With the notation of Proposition A-2 the elements of X are exactly those containing i but not $i-1$. Let $x = k_1 \triangleright k_2 \triangleright \dots \triangleright k_m \in X$ with $k_j = i$. We consider $x_o^j = (i+j-1) \triangleright (i+j-2) \triangleright \dots \triangleright (i+1) \triangleright i$. It is not difficult to see that there exists a chain x_o^j, x_1^j,…, $x_p^j = x$ where $x_l^j \in X$ and $l(x_{l+1}^j) = l(x_l^j) + 1$ for $0 \leq l < p$. Therefore, $x_o^j \in X_a \Rightarrow x \in X_a$ for $a = 1, 2$.
We choose j minimal such that $x_o^j \neq e$. Let $x_o^j \in X_1$, say. We show that $X \subset X_1$, hence $X_2 = \emptyset$. If $i + j - 1 = n$, then j is also maximal such that $x_o^j \neq e$ and we are done. Otherwise let $y = (i+j) \triangleright (i+j-1) \triangleright \dots \triangleright (i+2) \triangleright i$, that is $y \in X_1$ and $x_o^{j+1} = ys_{i+1}s_i$. On the other hand we have $[\theta_i\theta_{i+1}M^{\mathfrak{p}}(y)] = [M^{\mathfrak{p}}(x_o^{j+1})] + [M^{\mathfrak{p}}(ys_{i+1})] + [M^{\mathfrak{p}}(y)] + [M(ys_i)]$. In particular $0 \neq [\theta_{i+i}\theta_i P^{\mathfrak{p}}(y) : M^{\mathfrak{p}}(y)] = [P^{\mathfrak{p}}(x_o^{j+1}) : M^{\mathfrak{p}}(y)] = [M^{\mathfrak{p}}(y) : L(x_o^{j+1})]$. (Note that the first equality uses the fact that x_o^{j+1} is braid-avoiding and [BW01, Theorem 1]). Therefore, $x_o^{j+1} \in X_1$, because $y \in X_1$. Inductively, all x_o^j are contained in X_1. Hence, $X \subseteq X_1$. That means the assumptions of Theorem 3.9 are satisfied, and $F_w = \theta_i$ is indecomposable.
Let us now consider the general case. By Lemma 5.3 $L(x) \in X$ if and only if $xs_k < x$ for all simple reflections s_k occurring in a reduced expression of w. In the notation of Proposition A-2, the expression for x contains all such k's but none of the $k-1$. Similar arguments as above show again that a nontrivial decomposition $X = X_1 \cup X_2$ such that $\mathrm{Comp}(X_1) \cap \mathrm{Comb}(X_2) \cap X = \emptyset$ does not exist. We omit the details.
(Or perhaps not completely: We assume $i_j > i_{j'}$ if $j < j'$. For J a sequence of numbers $n \geq j_1 > j_2 > \dots > j_k \geq m$ let $X^J = \{x = x_1 \triangleright x_2 \triangleright \dots \triangleright x_m \in X \mid x_{j_k} = i_k, 1 \leq k \leq r\}$ be the elements of X, where the 'important' numbers occur exactly at the places given by J. Again,

it is easy to see from Proposition A-2 that X^J is a finite set of the form $x_0^J < x_1^J < \cdots < x_{|X^J|}^J$ such that $l(x_{b+1}^J) = l(x_b^J) + 1$ for $0 \leq b < |X^J|$. This implies in particular $x_0^J \in X_1 \iff x_b^J \in X_1$ for any b. We fix now some J, such that $X^J \neq \emptyset$. Assume $j_{l+1} \neq j_l + 1$ for some l. Without loss of generality let $x_{j_l+1} = i_l - 2$. We consider the following two cases

(I) Assume it exists $x \in X^J$ of the form $x = x_1 \triangleright \cdots \triangleright x_m$ such that $x_{j_l-1} > i_l+1$. Then $y = xs_{i_l+1}s_{i_l-1}s_{i_l} \in X^{J'}$, where $j'_l = j_l+1$ and $j'_i = j_i$ otherwise. On the other hand $[\theta_k M^{\mathfrak{p}}(x) : M^{\mathfrak{p}}(x)] \neq 0$ for $k = i_{l+1}, i_{l-1}, i_l$. Hence $[\theta_{i_l}\theta_{i_{l-1}}\theta_{i_l+1}M^{\mathfrak{p}}(x) : M^{\mathfrak{p}}(x)] \neq 0$. Since x, y are braid-avoiding we get ([BW01]) $P^{\mathfrak{p}}(y) \cong \theta_{i_r}\theta_{i_{r-1}}\theta_{i_r+1}P^{\mathfrak{p}}(x)$. In particular, $0 \neq [P^{\mathfrak{p}}(y) : M^{\mathfrak{p}}(x)] = [M^{\mathfrak{p}}(x) : L(y)]$. We proved that $x \in X^J$, $y \in X^{J'}$, then $x \in X_1 \iff y \in X_1$.

(II) Let z be minimal such that $x_i = x_{i+1} + 1$ for $z \leq i \leq j_l$. Let $x' = s_{j_z+1}s_{j_z} \cdot \ldots \cdot s_{j_l+2}$. Note that $xx' \in X^J$ and $y = xx's_{j_l+1}s_{j_l-1}s_{j_l} \in X^{J'}$ as above. Direct calculations give $[\theta_{[x']}M^{\mathfrak{p}}(x) : M^{\mathfrak{p}}(xx')] \neq 0$. Applying the earlier arguments to xx' gives $[P^{\mathfrak{p}}(y) : M^{\mathfrak{p}}(xx')] = [M^{\mathfrak{p}}(xx') : L(y)] \neq 0$.

Inductively, it follows that $X_i = \emptyset$ for some $i \in \{1, 2\}$. Therefore, the assumptions of Theorem 3.9 are satisfied. The Proposition follows.) $\square$

Lemma 5.5. *Let $\mathfrak{g} = \mathfrak{sl}_n$ and $e \neq w \in W$ be braid-avoiding. Then there exists a reduced expression $w = s_{i_1}s_{i_2} \cdot \ldots \cdot s_{i_r}$ such that (at least) one of the following properties is satisfied*

(i) $s_{i_j}s_{i_k} = s_{i_k}s_{i_j}$ *for* $1 \leq j, k \leq r$.

(ii) $s_{i_1}s_{i_2} \neq s_{i_2}s_{i_1}$.

(iii) $s_{i_r}s_{i_{r-1}} \neq s_{i_{r-1}}s_{i_r}$.

Proof. Write $w = d_1 d_2 \cdots d_n$ minimal such that $d_1 d_2 \cdots d_m \in \langle s_1, s_2, \ldots, s_m \rangle$ and $d_m \in \langle s_1, s_2, \ldots, s_{m-1} \rangle \backslash \langle s_1, s_2, \ldots s_m \rangle$ is a distinguished coset representative of minimal length for any $m \in \{1, 2, \ldots, n\}$. By Proposition A-2 $d_m = s_m s_{m-1} \cdot \cdots \cdot s_{m-k}$ for some k or $d_m = e$. Pick (if it exists) $j \in \{1, 2, \ldots, n\}$ minimal such that $d_j, d_{j+1} \neq e$. By assumption $d_{j-1} = e$, hence we get a minimal expression $w = ds_j s_{j+1} w' d_{j+2} \cdots d_n$ for some $w' \in W$ and $d \in \langle s_1, s_2, \ldots, s_{j-2} \rangle$. Therefore, $w = s_j s_{j+1} x$ for some $x \in W$, and w satisfies (ii).

If j as above does not exist we proceed by induction on the length of w. Without loss of generality let $d_n \neq e$. If $l(d_n) > 1$, then obviously (iii) holds; otherwise $d_n = s_n$ (and $d_{n-1} = e$)), and therefore, $w = xs_n = s_n x$ for some $x \in \langle s_1, s_2 \ldots, s_{n-2}\rangle$. Certainly, x is braid-avoiding. The lemma follows from the induction hypothesis. $\square$

Proof of Theorem 5.1: By Lemma 5.2, we may assume $w \in W$ to be braid-avoiding. If $w = e$ or if w satisfies the assumptions of Proposition 5.4 we are done. Otherwise we prove the statement by induction on the length of w. Let us assume that w has a minimal expression of the form $w = w'ts$ with non-commuting $s,\, t \in \mathcal{S}$; in particular $F_w \cong \theta_s\theta_t F_{w'}$. Let $F_w \cong G_1 \oplus G_2$ for some nontrivial G_i when restricting to $\mathcal{O}_0^{\mathfrak{p}}$. Considered as functor on $\mathcal{O}_0^{\mathfrak{p}}$ we have $\theta_t F_w \cong \theta_t\theta_s\theta_t F_{w'} \cong \theta_t F_{w'} \cong F_{w't}$; hence it is indecomposable by induction hypothesis. This implies $\theta_t G_i = 0$ for $i = 1$, say. Note that $\theta_s F_w \cong F_w \oplus F_w$. We claim that

$$\theta_s G_1 \cong G_1 \oplus H \tag{5.1}$$

for some $H \in \mathcal{P}^{\mathfrak{p}}$ such that $\big[[H]\big] = \big[[G_1]\big]$. Let us believe this for a moment, then $\theta_s\theta_t\theta_s G_1 \cong \theta_s G_1 \cong G_1 \oplus H \neq 0$. Hence, $\theta_t\theta_s G_1 \neq 0$. On the other hand $\big[[\theta_t\theta_s G_1]\big] = \big[[\theta_t(G_1 \oplus G_1)]\big]$. Therefore, $\theta_t G_1 \neq 0$. This gives a contradiction. To prove (5.1) we fix an embedding $i : G_1 \to F_w$ together with a split i' and consider the following diagram

$$\begin{tikzcd}
G_1 \arrow[r, shift right, "i"'] \arrow[d, "\alpha"] & F_w \arrow[l, shift right, bend right=10, "j"'] \arrow[d, shift right, "\beta"'] \\
\theta_s G_1 \arrow[r, shift left, "\theta_s i"] & \theta_s F_w \arrow[l, shift left, bend left=10, "\theta_s j"] \arrow[u, shift right, "\beta'"']
\end{tikzcd}$$

The vertical maps α and β are the adjunction morphisms, so the inner diagram commutes. The isomorphism $\theta_s^2 \cong \theta_s \oplus \theta_s$ provides a split β' of β. The composition $\phi := j \circ \beta' \circ \theta_s i$ is then a split of α, because $\phi \circ \alpha = j \circ \beta' \circ \theta_s i \circ \alpha = j \circ \beta' \circ \beta \circ i = j \circ i = \mathrm{id}$. This gives an isomorphism as in (5.1) for some $H \in \mathcal{P}^{\mathfrak{p}}$. Let $Q \in \mathcal{O}_0^{\mathfrak{p}}$ be projective, hence $G_1(Q) \cong \oplus_{x \in W^{\mathfrak{p}}} P(x)^{\alpha_x}$ for some $\alpha_x \in \mathbb{N}$. Moreover, $\alpha_x = 0$ if $xs > x$. If $x,\, xs \in W^{\mathfrak{p}}$ such that $x > xs$ then $\theta_s P^{\mathfrak{p}}(x) \cong P^{\mathfrak{p}}(x) \oplus P^{\mathfrak{p}}(x)$. (Note that $\theta_s P(x)$ is projective and its head is isomorphic to $L(x) \oplus L(x)$.) We get $\big[\theta_s G_1(Q)] = [(G_1 \oplus G_1)(Q)\big]$ for any projective object $Q \in \mathcal{O}_0^{\mathfrak{p}}$ hence $\big[[\theta_s G_1]\big] = \big[[G_1 \oplus G_1]\big]$. The claim follows.

By Lemma 5.5 we are left with the case where $w = tsw'$ with non-commuting $s,\, t \in \mathcal{S}$. Similar arguments as above prove the indecomposability. $\square$

Remark 5.6. Applying the same induction arguments as in the proof of Theorem 5.1 shows the indecomposability of $F(M^{\mathfrak{p}}(x))$ for any indecomposable $F \in \mathcal{P}$ and $x \in W^{\mathfrak{p}}$. Moreover, just using the description from Proposition A-2 one can easily deduce that, with the assumptions of Lemma 5.3(1), $\theta_{[w]} M^{\mathfrak{p}}(x) \cong \theta_{s_{i_1}} M^{\mathfrak{p}}(y)$ for some $y \in W^{\mathfrak{p}}$. So, the indecomposability of F_w in this case follows directly from the proof of Proposition 5.4 using Lemma 5.3 (1).

Theorem 5.7. *Let $\mathfrak{p} \subset \mathfrak{sl}_n$ be maximal parabolic. Then Conjecture 3.3 holds and we have equality in* (3.2).

Proof. Let $F = \oplus_{w \in W} (F_{w|\mathcal{O}_o^{\mathfrak{p}}})^{\alpha_w}$ and $G = \oplus_{w \in W} (F_{w|\mathcal{O}_o^{\mathfrak{p}}})^{\beta_w}$ such that $\big[[F]\big] = \big[[G]\big]$. By Lemma 5.2 we may assume $\alpha_w = 0 = \beta_w$ for non-braid avoiding w. The specialisation of $\mathcal{H}/(TL)$ at $v = 1$ is semisimple (see e.g. [Wes95]) hence $\mathcal{N}^{\mathfrak{p}}_{v=1} \cong \oplus_{i=1}^{r} L_i$ for some simple $\mathcal{H}/(TL)_{v=1}-$modules L_i. Since (see [Wes95]) $\mathrm{Ann}_{\mathcal{H}/(TL)_{v=1}} L_i = \mathbb{C}\{\underline{H}_x \mid x \in W[i]\}$ for some $W[i] \subset W$ we get $\alpha_w = \beta_w$ for all $w \notin I := \cap_{i=1}^{r} W[i]$. Hence, $F \cong \oplus_{w \notin I} (F_{|\mathcal{O}_o^{\mathfrak{p}}})^{\alpha_w} \cong G$. On the other hand, it also shows that $\mathrm{Ind}(\mathfrak{g}, \mathfrak{p}) = |\{\underline{H}_w \mid w \notin I\} = \mathrm{R}(\mathfrak{g}, \mathfrak{p})$. The theorem follows. □

6 The Temperley-Lieb 2-category

In this section we describe a functor from the Temperley-Lieb 2-category into a 2-category given by projective functors with their natural transformations. Let $\mathcal{O}(\mathfrak{sl}_n)^{\max} = \bigoplus_{k=0}^{n+1} \mathcal{O}^k(\mathfrak{sl}_n)$. In [BFK99], the authors consider functors

$$\begin{aligned} \cap_{i,n} : \mathcal{O}(\mathfrak{sl}_n)^{\max} &\longrightarrow \mathcal{O}(\mathfrak{sl}_{n-2})^{\max} \\ \cup_{i,n} : \mathcal{O}(\mathfrak{sl}_n)^{\max} &\longrightarrow \mathcal{O}(\mathfrak{sl}_{n+2})^{\max} \end{aligned}$$

which are given on each summand as follows: For any $0 < k \leq n+1$ let

$$\begin{aligned} \cap_{i,n}^k : \mathcal{O}^k(\mathfrak{sl}_n) &\longrightarrow \mathcal{O}^{k-1}(\mathfrak{sl}_{n-2}) \\ \cup_{i,n}^k : \mathcal{O}^k(\mathfrak{sl}_n) &\longrightarrow \mathcal{O}^{k+1}(\mathfrak{sl}_{n+2}) \end{aligned}$$

defined as

$$\begin{aligned} \cap_{i,n}^k &= \zeta_{n,k} \theta_0^1 \theta_2 \theta_3 \cdots \theta_i \\ \cup_{i,n}^k &= \theta_i \theta_{i-1} \cdots \theta_2 \theta_1^0 \zeta_{n+2,k}^{-1}, \end{aligned}$$

for $1 \leq k \leq n$ and zero otherwise. Here, $\zeta_{n,k} : \mathcal{O}_1^k(\mathfrak{sl}_n) \tilde{\rightarrow} \mathcal{O}^{k-1}(\mathfrak{sl}_{n-2})$ denotes the *Enright-Shelton-equivalence* (see [ES87, chapter 11]). Let $\zeta_n = \oplus_k \zeta_{n,k}$. The following statement follows directly from the definitions

Lemma 6.1. *The functors* $\cap_{i,n}$ *and* $\cup_{i,n-2}$ *are biadjoint.*

We prove the following main result ([BFK99, Conjecture 3]):

Theorem 6.2. *Let* $j \geq i$. *There are isomorphisms of functors*

$$\begin{aligned}
\cap_{i+1,n+2} \cup_{i,n} &\cong \mathrm{ID} && (6.1)\\
\cap_{i,n+2} \cup_{i+1,n} &\cong \mathrm{ID} && (6.2)\\
\cap_{j,n} \cap_{i,n+2} &\cong \cap_{i,n} \cap_{j+2,n+2} && (6.3)\\
\cup_{j,n-2} \cap_{i,n} &\cong \cap_{i,n+2} \cup_{j+2,n} && (6.4)\\
\cup_{i,n-2} \cap_{j,n} &\cong \cap_{j+2,n+2} \cup_{i,n} && (6.5)\\
\cup_{i,n+2} \cup_{j,n} &\cong \cup_{j+2,n+2} \cup_{i,n} && (6.6)\\
\cap_{i,n+2} \cup_{i,n} &\cong \mathrm{ID} \oplus \mathrm{ID} && (6.7)
\end{aligned}$$

Proof. By adjointness (Lemma 6.1), it is enough to prove (6.1), (6.2), (6.3), (6.4), and (6.7). As already mentioned in [BFK99], the isomorphisms (6.1) and (6.2) follow from the definitions and [BFK99, Lemma 4]. The formula (6.7) can be verified as follows

$$\begin{aligned}
\cap_{i,n+2} \cup_{i,n} &= \zeta_{n+2} \theta_0^1 \theta_2 \cdots \theta_{i-2} (\theta_{i-1} (\theta_i \theta_i) \theta_{i-1}) \theta_{i-2} \cdots \theta_2 \theta_1^0 \zeta_{n+2}^{-1}\\
&\cong \zeta_{n+2} \theta_0^1 \theta_2 \theta_1^0 \oplus \zeta_{n+2} \theta_0^1 \theta_2 \theta_1^0 \zeta_{n+2}^{-1}\\
&\cong \mathrm{ID} \oplus \mathrm{ID}\,.
\end{aligned}$$

The first isomorphism follows from $\theta_i^2 \cong \theta_i \oplus \theta_i$ and the relation (4.3). The second isomorphism follows from [BFK99, Lemma 4]. The rest of the section is devoted to proving formulas (6.3) and (6.4) (Propositions 6.6 and 6.4). $\square$

Lemma 6.3. *Let* $j \geq i$ *then*

$$\cup_{j,n-2} \cap_{i,n} \cong \theta_j \theta_{j-1} \cdots \theta_i \quad \textit{and} \quad \cup_{i,n-2} \cap_{j,n} \cong \theta_i \theta_{i+1} \cdots \theta_j .$$

Proof. Using again Theorem 4.1 we get

$$\begin{aligned}
\cup_{j,n-2} \cap_{i,n} &\cong \theta_j \theta_{j-1} \cdots \theta_2 \theta_1^0 \zeta_n^{-1} \zeta_n \theta_0^1 \theta_2 \theta_3 \cdots \theta_i\\
&\cong \theta_j \theta_{j-1} \cdots \theta_2 \theta_1 \theta_2 \cdots \theta_i\\
&\cong \theta_j \theta_{j-1} \cdots \theta_i;\\
\cup_{i,n-2} \cap_{j,n} &\cong \theta_i \theta_{i-1} \cdots \theta_2 \theta_1^0 \zeta_n^{-1} \zeta_n \theta_0^1 \theta_2 \theta_3 \cdots \theta_j\\
&\cong \theta_i \theta_{i-1} \cdots \theta_2 \theta_1 \theta_2 \cdots \theta_j\\
&\cong \theta_i \theta_{i+1} \cdots \theta_j .
\end{aligned}$$

$\square$

Proposition 6.4. *Let $j \geq i$. There exists an isomorphism*

$$\cup_{j,n-2}\cap_{i,n} \cong \cap_{i,n+2} \cup_{j+2,n} .$$

Proof. First we claim

$$\cap_{j,n+2} \cup_{j+2,n} \cap_{i,n+2}\cup_{j+1,n} \cong \cap_{i,n+2} \cup_{j+2,n} . \tag{6.8}$$

By Lemma 6.3 and Theorem 4.1 we are able to handle the left hand side of the formula:

$$\begin{aligned} LHS &\cong \zeta_{n+2}\theta_0^1\theta_2\theta_3\cdots\theta_j(\theta_{j+2}\theta_{j+1}\cdots\theta_i)\cup_{j+1,n} \\ &\cong \zeta_{n+2}\theta_0^1\theta_{j+2}\theta_2\theta_3\cdots\theta_{i-1}(\theta_i\theta_{i+1}\cdots\theta_j\theta_{j+1}\theta_j\cdots\theta_i)\cup_{j+1,n} \\ &\cong \zeta_{n+2}\theta_0^1\theta_{j+2}\theta_2\theta_3\cdots\theta_{i-1}(\theta_i)(\theta_{j+1}\theta_j\cdots\theta_2\theta_1^0\zeta_{n+2}^{-1}) \\ &\cong \zeta_{n+2}\theta_0^1\theta_2\theta_3\cdots\theta_{i-1}(\theta_i)(\theta_{j+2}\theta_{j+1}\theta_j\cdots\theta_2\theta_1^0\zeta_{n+2}^{-1}). \end{aligned}$$

The latter is by definition the RHS of the formula (6.8). We prove now the statement by induction on $a = j - i$. By induction hypothesis and Lemma 6.3 we get

$$\begin{aligned} \cap_{i,n+2}\cup_{j+2,n} &\cong \cap_{j,n+2}\cup_{j+2,n}\cap_{i,n+2}\cup_{j+1,n} \\ &\cong \cup_{j,n-2}\cap_{j,n}\cup_{j-1,n-2}\cap_{i,n} \\ &\cong \theta_j(\theta_{j-1}\cdots\theta_i) \\ &\cong \cup_{j,n-2}\cap_{i,n} . \end{aligned}$$

It remains to check the starting point of the induction, i.e. $\cap_{i,n+2}\cup_{i+2,n} \cong \theta_i$. We first note that the functor in question is a projective functor. To see this one has to consider the construction of $\zeta_{n,k}$ (see [ES87]). It is a composite of four functors Λ_i. Two functors ($i = 1, 3$) are given by tensoring with a finite dimensional representation and two functors are given by compositions of parabolic induction and Zuckerman's functor. In particular, if E is a finite dimensional $\mathfrak{g}$-modules then $\zeta_n(\bullet \otimes E) \cong (\bullet \otimes E')\zeta_n$ for some finite dimensional module E'. That means projective functors are sent to projective functors via the equivalence.

Direct calculations (using the explicit formula [ES87, section 11]) show that $[\cap_{i,n+2}\cup_{j+2,n}(M^{\mathfrak{p}}(e))] = [M^{\mathfrak{p}}(e) \oplus M^{\mathfrak{p}}(s_i)]$ if $\mathfrak{p} = \mathfrak{p}_i$ and zero otherwise. Hence the projective functor $\cap_{i,n+2}\cup_{j+2,n}$ contains θ_i as a direct summand. Hence, it is sufficient to show that $\cap_{i,n+2}\cup_{j+2,n}(M^{\mathfrak{p}}(x))$ has a generalised Verma flag whose length is equal to the length of a generalised Verma flag of $\theta_i(M^{\mathfrak{p}}(x))$ for any $x \in W^{\mathfrak{p}}$. We claim that $\cap_{i,n+2}\cup_{i+2,n} M^{\mathfrak{p}}(\lambda)$ has a

generalised Verma flag of length 2 or 0 for any $M^{\mathfrak{p}} \in \mathcal{O}(\mathfrak{sl}_n)^{\max}$; equivalently: $\theta_0^1\theta_{i+2}\theta_2\theta_1^0 M^{\mathfrak{p}}(\lambda)$ has such a flag for any $M^{\mathfrak{p}}(\lambda) \in \mathcal{O}_1(\mathfrak{sl}_{n+2})^{\max}$. Since $\theta_1^0\theta_0^1\theta_{i+2}\theta_2\theta_1^0\theta_0^1 M^{\mathfrak{p}}(x) \cong \theta_{i+2}\theta_1\theta_2\theta_1 M^{\mathfrak{p}}(x) \cong \theta_{i+2}\theta_1 M^{\mathfrak{p}}(x)$ has always a Verma flag of length 4 or 0, the claim follows (see [Jan83, 4.12 (2); 4.13 (1)]). To get an isomorphism $\cap_{i,n+2}\cup_{j+2,n} \cong \theta_i$ it is therefore enough to show that $\cap_{i,n+2}\cup_{j+2,n} M^{\mathfrak{p}}(\lambda) \neq 0$ implies $\theta_i M^{\mathfrak{p}}(\lambda) \neq 0$ for any parabolic Verma module $M^{\mathfrak{p}}(\lambda)$.
Since $\zeta_{n,k}^{-1}$ is an equivalence, it induces a natural map ϕ such that $\zeta_{n,k}^{-1}L(x) \cong L(\phi(x))$. There is an explicit formula in [ES87, Proposition 11.2], namely $\phi(x) = wx^{\Rsh}r$ for a certain $w \in W$ (depending on k) and $r = s_n s_{n-1} \cdot \ldots \cdot s_2$. (The symbol $x^{\Rsh}$ means that we have to renumbering the indices $i \rightsquigarrow i+1$ in a reduced expression of x). In particular, xs_i is a distinguished coset representative if and only if $w((xs_i)^{\Rsh})r = (wx^{\Rsh}r)r^{-1}(s_i)^{\Rsh}r = wx^{\Rsh}r(s_{i+1})^{\Rsh} = wx^{\Rsh}rs_{i+2}$ is so. On the other hand $\cap_{i,n+2}\cup_{j+2,n}(M^{\mathfrak{p}}(x)) \cong \zeta_n\theta_0^1\theta_{i+2}\theta_2\theta_1^0\zeta_{n+2}^{-1}(M^{\mathfrak{p}}(x)) \neq 0$ implies $\theta_{i+2}M(\phi(x)) \neq 0$. Therefore, we get $\cap_{i,n+2}\cup_{j+2,n} M^{\mathfrak{p}}(\lambda) \neq 0 \Rightarrow \theta_i M^{\mathfrak{p}}(\lambda) \neq 0$. The theorem follows. □

Lemma 6.5. *Let $j \geq i$. There are isomorphisms of functors*

$$\cup_{i,n-2}\cap_{j,n} \cong \cap_{j+2,n+2}\cup_{i,n} \tag{6.9}$$
$$\cup_{j+2,n}\cup_{i,n-2}\cap_{i,n}\cap_{j+2,n+2} \cong \theta_i\theta_{j+2} \tag{6.10}$$
$$\cup_{j+2,n}\cup_{i,n-2}\cap_{j,n}\cap_{i,n+2} \cong \theta_i\theta_{j+2} \tag{6.11}$$

Proof. Formula (6.9) is clear, since the adjoint functors are isomorphic (Lemma 6.1 and Proposition 6.4). Therefore, we get with Lemma 6.3

$$\cup_{j+2,n}(\cup_{i,n-2}\cap_{j,n})\cap_{i,n+2} \cong \cup_{j+2,n}\cap_{j+2,n+2}\cup_{i,n}\cap_{i,n+2} \cong \theta_{j+2}\theta_i.$$

This shows formula (6.11). Proposition 6.4, Lemma 6.3 and Theorem 4.1 imply

$$\begin{aligned}
\cup_{j+2,n}(\cup_{i,n-2}\cap_{i,n})\cap_{j+2,n+2} &\cong \cup_{j+2,n}\cap_{i,n+2}\cup_{i+2,n}\cap_{j+2,n+2}\\
&\cong (\theta_{j+2}\theta_{j+1}\cdots\theta_i)(\theta_{i+2}\theta_{i+3}\cdots\theta_{j+2})\\
&\cong \theta_{j+2}\theta_{j+1}\cdots\theta_{i+2}\theta_{i+1}\theta_{i+2}\cdots\theta_{j+2}\theta_i\\
&\cong \theta_{j+2}\theta_i.
\end{aligned}$$

This proves formula (6.10). □

Finally we can do the last step of proving Theorem 6.2:

Proposition 6.6. *Let $j \geq i$. There exist an isomorphism of functors*

$$\cap_{j,n}\cap_{i,n+2} \cong \cap_{i,n}\cap_{j+2,n+2}$$

Proof. Let $F = \cap_{i,n} \cap_{j+2,n+2} \cup_{j+2,n} \cup_{i,n-2}$. Applying relation (6.7) twice we get $F \cong \oplus_{m=1}^{2} \cap_{i,n} \cup_{i,n-2} \cong \oplus_{m=1}^{4} \mathrm{ID}$. Lemma 6.5 implies $F \cap_{i,n} \cap_{j+2,n+2} \cong \cap_{i,n} \cap_{j+2,n+2} \theta_i \theta_{j+2} \cong F \cap_{j,n} \cap_{i,n+2}$. In other words

$$\bigoplus_{l=1}^{4} \cap_{i,n} \cap_{j+2,n+2} \quad \cong \quad \bigoplus_{l=1}^{4} \cap_{j,n} \ \cap_{i,n+2} .$$

The proposition follows from Corollary 3.5. □

As a preparation for the next section we prove the following result: (It contains in fact a refinement of formula (6.7).)

Proposition 6.7. *1. The functors $\cap_{i,n}$ and $\cup_{i,n}$ are gradable.*

2. There are graded lifts $\tilde{\cap}_{i,n}$, $\tilde{\cup}_{i,n}$ with isomorphisms of graded functors ($j \geq i$)

$$\begin{aligned} \tilde{\cup}_{j,n-2} \tilde{\cap}_{i,n} &\cong \tilde{\theta}_j \tilde{\theta}_{j-1} \cdots \tilde{\theta}_i \\ \tilde{\cup}_{i,n-2} \tilde{\cap}_{j,n} &\cong \tilde{\theta}_i \tilde{\theta}_{i+1} \cdots \tilde{\theta}_j \\ \tilde{\cap}_{i,n+2} \tilde{\cup}_{i,n} &\cong \mathrm{ID}\langle 1 \rangle \oplus \mathrm{ID}\langle -1 \rangle. \end{aligned}$$

With these choices, the remaining isomorphisms of Theorem 6.2 are compatible with the grading.

Proof. 1. Let G be one of the functors in question. In [RH], it is proved that the Enright-Shelton-equivalence is compatible with the grading. All the other functors occurring in the definition of G are gradable by the results of [Str03]. This defines graded lifts $\tilde{\cap}_{i,n}$ and $\tilde{\cup}_{i,n}$. In the following, concerning the notation, we will not distinguish between the Enright-Shelton equivalence and its graded lift.

2. The first two isomorphisms follow from the proof of Lemma 6.3 and Theorem 4.1, since we have canonically $\tilde{\theta}_1^0 \tilde{\theta}_0^1 \cong \tilde{\theta}_1$ (see [Str03, Corollary 8.3]). To get the third isomorphism, we first note that $\tilde{\cap}_{i,n} \tilde{\cup}_{i,n-2} \cong \mathrm{ID}\langle j \rangle \oplus \mathrm{ID}\langle k \rangle$ for certain j, $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} \tilde{\theta}_i \langle j \rangle \oplus \tilde{\theta}_i \langle k \rangle &\cong \tilde{\cup}_{i,n-2} \tilde{\cap}_{i,n} \langle j \rangle \oplus \tilde{\cup}_{i,n-2} \tilde{\cap}_{i,n} \langle k \rangle \\ &\cong \tilde{\cup}_{i,n-2} (\mathrm{ID}\langle j \rangle \oplus \mathrm{ID}\langle k \rangle) \tilde{\cap}_{i,n} \\ &\cong \tilde{\cup}_{i,n-2} \tilde{\cap}_{i,n} \tilde{\cup}_{i,n-2} \tilde{\cap}_{i,n} \\ &\cong (\tilde{\theta}_i)^2 \cong \tilde{\theta}_i \langle 1 \rangle \oplus \tilde{\theta}_i \langle -1 \rangle. \end{aligned}$$

(The last isomorphism is given by Theorem 4.1.) This implies $\{j,k\} = \{-1,1\}$ and provides the third isomorphism. We have to check the compatibility of (6.1). Since a graded lift of an indecomposable exact functor is unique up to isomorphism and grading shift (see [Str03, Lemma 1.5]) we may assume $F = (\zeta_{n,k}\tilde{\theta}_0^1\tilde{\theta}_2\cdots\tilde{\theta}_{i+1})(\tilde{\theta}_i\cdots\tilde{\theta}_2\tilde{\theta}_1^0\zeta_{n,k}^{-1}) \cong \mathrm{ID}\langle l\rangle$ for some $l \in \mathbb{Z}$. Using again Theorem 4.1 and the isomorphism $\tilde{\theta}_1^0\tilde{\theta}_0^1 \cong \tilde{\theta}_1$ we get

$$\begin{aligned}
\tilde{\theta}_1\langle l\rangle &\cong \tilde{\theta}_1^0\tilde{\theta}_0^1\langle l\rangle \\
&\cong \tilde{\theta}_0^1\zeta_{n,k}^{-1}F\zeta_{n,k}\tilde{\theta}_0^1 \\
&\cong \tilde{\theta}_1\tilde{\theta}_2\cdots\tilde{\theta}_{i+1}\tilde{\theta}_i\cdots\tilde{\theta}_2\tilde{\theta}_1 \\
&\cong \tilde{\theta}_1\tilde{\theta}_2\tilde{\theta}_1 \cong \tilde{\theta}_1.
\end{aligned}$$

The indecomposability of θ_1 implies therefore $l = 0$. The compatibility with the grading for the isomorphism (6.2) can be proved in an analogous way. To see that the isomorphism (6.4) is compatible with the grading, it is by induction sufficient to consider the case $i = j$ (see formula (6.8) and its proof). Then $\tilde{\cup}_{i,n-2}\tilde{\cap}_{i,n} \cong \tilde{\theta}_i$ is self-adjoint ([Str03, Corollary 8.5]). Assume $\tilde{\theta}_i \cong \tilde{\cap}_{i,n+2}\tilde{\cup}_{i+2,n}\langle l\rangle$ for some $l \in \mathbb{Z}$. The adjoint pairs $(\tilde{\theta}_0^1, \tilde{\theta}_1^0\langle -1\rangle)$ and $(\tilde{\theta}_1^0, \tilde{\theta}_0^1\langle 1\rangle)$ (see [Str03, Corollary 8.3]) directly imply that $\tilde{\cap}_{i,n+2}\tilde{\cup}_{i+2,n}$ is self-adjoint; hence $l = 0$.

Let us consider (6.6). We fix $0 \leq k \leq n$. We first claim that the restriction, call it R, of $F = \cup_{i,n+2}\cup_{j,n}$ to $\mathcal{O}^k(\mathfrak{sl}_n)$ is indecomposable. Assume $R = F_1 \oplus F_2$. There are isomorphisms of functors

$$\begin{aligned}
F\cap_{j,n+2}\cap_{i,n+4} &\cong \cup_{i,n+2}\cup_{j,n}\cap_{j,n+2}\cap_{i,n+4} \\
&\cong \cup_{i,n+2}\cap_{j,n+4}\cup_{j+2,n+2}\cap_{i,n+4} \\
&\cong (\theta_i\theta_{i+1}\cdots\theta_j)(\theta_{j+2}\theta_{j+1}\cdots\theta_i) \\
&\cong (\theta_i\theta_{i+1}\cdots\theta_j)(\theta_{j+1}\cdots\theta_i)\theta_{j+2} \\
&\cong \theta_i\theta_{j+2}.
\end{aligned}$$

In particular, its restriction to $\mathcal{O}^{k+2}(\mathfrak{sl}_{n+4})$ is indecomposable, hence $F_m\cap_{j,n+2}^{k+1}\cap_{i,n+4}^{k+2} = 0$ for $m = 1$, say. This implies $F_1 \cong F_1\cap_{j,n+2}^{k+1}\cap_{i,n+4}^{k+2}\cup_{i+1,n+2}^{k+1}\cup_{j+1,n}^{k} = 0$. Hence, R is indecomposable, therefore it exists $l \in \mathbb{Z}$ such that

$$\tilde{\cup}_{i,n+2}\tilde{\cup}_{j,n} \cong \tilde{\cup}_{j+2,n+2}\tilde{\cup}_{i,n}\langle l\rangle.$$

Using the graded versions of (6.1) and (6.4) we get isomorphisms

$$\begin{aligned}\tilde{\cup}_{j,n} &\cong \tilde{\cap}_{i+1,n+4}\tilde{\cup}_{j+2,n+2}\,\tilde{\cup}_{i,n}\langle l\rangle \\ &\cong \tilde{\cup}_{j,n}\tilde{\cap}_{i+1,n+2}\tilde{\cup}_{i,n}\langle l\rangle \\ &\cong \tilde{\cup}_{j,n}\langle l\rangle.\end{aligned}$$

Hence $l = 0$. The compatibility with the grading of the remaining two isomorphisms (6.3) and (6.5) follows then easily by adjointness properties.

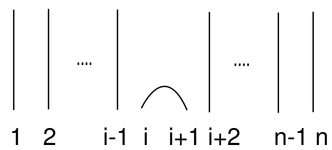
□

7 Tangles and knot invariants

Any tangle in $\mathbb{R}^3$ has a generic plane projection which is isomorphic to a concatenation of elementary tangles t_i^1, t_i^2, t_i^3 as depicted below and the right twisted curls t_i^4. We associate now to each tangle a certain projective functor and prove that this assignment is compatible with concatenation/composition and well-defined up to isomorphism.
We consider $\mathrm{D}^b\left(\tilde{\mathcal{O}}(\mathfrak{sl}_n)^{\max}\right)$, the bounded derived category of the graded version of $\mathcal{O}(\mathfrak{sl}_n)^{\max}$. (More precisely: for $0 \leq k \leq n+1$ let P_n^k be a minimal projective generator of $\mathcal{O}^k(\mathfrak{sl}_n)$ with endomorphism ring A_n^k equipped with the grading from [BGS96] or [Bac99]. Then $\bigoplus_k \mathcal{O}^k(\mathfrak{sl}_n) \cong \bigoplus_k \text{mof} - A_n^k$, and $\mathrm{D}^b\left(\tilde{\mathcal{O}}(\mathfrak{sl}_n)^{\max}\right)$ denotes the bounded derived category of $\bigoplus_k \text{gmof} - A_n^k$.)

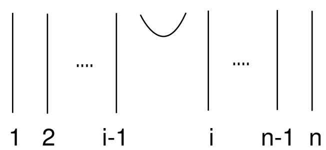
For an exact endofunctor F of $\mathcal{O}(\mathfrak{sl}_n)^{\max}$ we denote by F also its extension to $\mathrm{D}^b\left(\mathcal{O}(\mathfrak{sl}_n)^{\max}\right)$. As suggested in [BFK99], we associate functors to elementary tangles as follows:

t_i^1 :



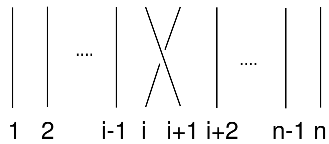
$$\rightsquigarrow \qquad \tilde{\cap}_{i,n} : \ \mathcal{D}^b\big(\tilde{\mathcal{O}}(\mathfrak{sl}_n)^{\max}\big) \longrightarrow \mathcal{D}^b\big(\tilde{\mathcal{O}}(\mathfrak{sl}_{n-2})^{\max}\big).$$

t_i^2 :



$$\rightsquigarrow \qquad \tilde{\cup}_{i,n}: \quad \mathcal{D}^b\big(\tilde{\mathcal{O}}(\mathfrak{sl}_n)^{\max}\big) \longrightarrow \mathcal{D}^b\big(\tilde{\mathcal{O}}(\mathfrak{sl}_{n+2})^{\max}\big).$$

t_i^3 :



$$\rightsquigarrow \qquad \mathcal{C}_i := \text{Cone}\,(\text{ID}\langle 1\rangle \overset{\text{adj}}{\to} \tilde{\theta}_i)\langle 1\rangle: \quad \mathcal{D}^b\big(\tilde{\mathcal{O}}(\mathfrak{sl}_n)^{\max}\big) \longrightarrow \mathcal{D}^b\big(\tilde{\mathcal{O}}(\mathfrak{sl}_n)^{\max}\big).$$

Note that $\mathcal{C}_i$ is the left derived functor of the graded version of the shuffling functor studied by Irving ([Irv93]). These derived shuffling functors also occur in the context of tilting complexes ([Ric94]). Let K_i be the adjoint functor of $\mathcal{C}_i$. The main properties of these functors are (see [MS03])

(P1) They define auto-equivalences of derived categories, i.e. $\mathcal{C}_iK_i \cong \text{ID} \cong K_i\mathcal{C}_i$.

(P2) Let $w = s_1 s_2 \cdot \ldots \cdot s_r$ be a reduced expression. Up to isomorphism, the composition $\mathcal{C}_w = \mathcal{C}_{s_1}\mathcal{C}_{s_2} \cdot \cdots \cdot \mathcal{C}_{i_r}$ is independent of the choice of the reduced expression.

We associate to the right-twisted curl t_i^4 the functor $\mathcal{K}_i$. We call a tangle with m bottom and n top points an (m,n)-tangle. Given a presentation t_α of a tangle t as a composition of elementary tangles, we associate $\mathcal{T}(t_\alpha)$, the corresponding composition of functors. (If t' is an (m,n)-tangle and t is an (n,n')-tangle, the composition tt' is given by putting t above t'.) We state the main result (see [BFK99, Conjecture 4]):

Theorem 7.1. *Let t be an (m,n)-tangle with representations t_α, t_β and corresponding functors $\mathcal{T}(t_\alpha)$, $\mathcal{T}(t_\beta)$. Then*

$$\mathcal{T}(t_\alpha) \cong \mathcal{T}(t_\beta)\langle r\rangle[s] : \mathcal{D}^b\big(\tilde{\mathcal{O}}(\mathfrak{sl}_m)^{\max}\big) \longrightarrow \mathcal{D}^b\big(\tilde{\mathcal{O}}(\mathfrak{sl}_n)^{\max}\big)$$

for certain r, $s \in \mathbb{Z}$. In particular, up to grading and degree shifts, $\mathcal{T}(t_\alpha t'_{\alpha'}) \cong \mathcal{T}(t_\alpha)\mathcal{T}(t'_{\alpha'})$ for any two tangles t, t' with representations t_α and $t_{\alpha'}$ respectively, such that the concatenation is defined.

Proof. In Theorem 6.2 we already proved that for tangles without crossings $\mathcal{T}(t_\alpha) \cong \mathcal{T}(t_\beta)$ if $\alpha \cong \beta$ via plane diagram automorphisms. It remains to check the compatibility with the isotopies depicted in Figure 1, its vertical flip, and that the assignment is stable under Reidemeister moves (see e.g [Kau01], [Tur94]).

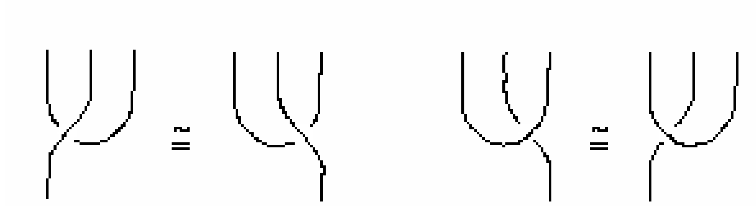


Figure 1: Tangle isotopies

(I) *Addition/removal of a left-twisted curl:* Using Proposition 6.7 we get isomorphisms

$$\begin{aligned}
\mathcal{T}(t^1_{i,n}t^3_{i,n}) &= \tilde{\cap}_{i,n} \circ \mathcal{C}_{i,n} \cong \mathrm{Cone}\left(\tilde{\cap}_{i,n}\langle 1\rangle \xrightarrow{adj} \tilde{\cap}_{i,n}\tilde{\theta}_i\right)\langle 1\rangle \\
&\cong \mathrm{Cone}\left(\tilde{\cap}_{i,n}\langle 1\rangle \xrightarrow{adj} \tilde{\cap}_{i,n}\langle 1\rangle \oplus \tilde{\cap}_{i,n}\langle -1\rangle\right)\langle 1\rangle \\
&\cong \tilde{\cap}_{i,n}\langle -1\rangle\langle 1\rangle \\
&\cong \mathcal{T}(t^1_{i,n}).
\end{aligned}$$

(II) *Addition/removal of a right-twisted curl:*

$$\begin{aligned}
\mathcal{T}(t^1_{i,n}t^4_{i,n}) &= \tilde{\cap}_{i,n} \circ \mathcal{K}_{i,n} \cong \mathrm{Cone}\left(\tilde{\cap}_{i,n}\tilde{\theta}_i \xrightarrow{adj} \tilde{\cap}_{i,n}\langle -1\rangle\right)\langle -1\rangle \\
&\cong \tilde{\cap}_{i,n}\langle 1\rangle\langle -1\rangle \\
&\cong \mathcal{T}(t^1_{i,n}).
\end{aligned}$$

(III) *Tangency moves:* $\mathcal{T}(t^3_{i,n}t^4_{i,n}) = \mathcal{C}_i\mathcal{K}_i \cong \mathrm{ID}$ and $\mathcal{T}(t^4_{i,n}t^3_{i,n}) = \mathcal{K}_i\mathcal{C}_i \cong \mathrm{ID}$ by property (P1).

(IV) *Triple point move:* $\mathcal{T}(t^3_{i,n}t^3_{i+1,n}t^3_{i,n}) \cong \mathcal{T}(t^3_{i+1,n}t^3_{i,n}t^3_{i+1,n})$ by property (P2), therefore also $\mathcal{T}(t^4_{i,n}t^4_{i+1,n}t^4_{i,n}) \cong \mathcal{T}(t^4_{i+1,n}t^4_{i,n}t^4_{i+1,n})$ by property (P1).

(V) *Height shifting:* The property (P2) implies $\mathcal{C}_j\mathcal{C}_i \cong \mathcal{C}_i\mathcal{C}_j$ and then also isomorphisms like $\mathcal{K}_j\mathcal{C}_i \cong \mathcal{C}_i\mathcal{K}_j$ if $|i-j| \geq 2$ by property (P1).

To see the compatibility with Figure 1, we recall the equivalence

$$\tilde{\mathcal{O}}(\mathfrak{sl}_n)^{\max} = \bigoplus_k \mathrm{gmof}-A^k_n.$$

Set $B := \bigoplus_k \mathrm{gmof}-A^k_n$. Via the equivalence, $\mathcal{C}_i$ and $\mathcal{K}_i$ are given by tensoring with the tilting complex $\left(B\langle 1\rangle \xrightarrow{adj} \theta_i B\right)\langle 1\rangle$ and $\left(\theta_i B \xrightarrow{adj} B\langle -1\rangle\right)\langle -1\rangle$ respectively (see [Ric94], [MS03]). Let us consider the first picture from

Figure 1. By (6.1) and (6.2), it is sufficient to verify $\mathcal{K}_i\tilde{\theta}_{i+1} \cong \mathcal{C}_{i+1}\tilde{\theta}_i\tilde{\theta}_{i+1}$ up to shifts. The LHS is described by tensoring with the tilting complex $\mathbf{T}_i$ given as

$$\big(\mathbf{T}_0 = \tilde{\theta}_i\tilde{\theta}_{i+1} \stackrel{f:=adj_{\tilde{\theta}_{i+1}}}{\longrightarrow} \tilde{\theta}_{i+1}\langle -1\rangle = \mathbf{T}_{-1}\big)\langle -1\rangle,$$

whereas the RHS is given by tensoring with $\mathbf{G}$ defined as

$$(\mathbf{G}_1 = \theta_i\theta_{i+1}\langle 1\rangle \stackrel{g:=adj_{\tilde{\theta}_i\tilde{\theta}_{i+1}}}{\longrightarrow} \mathbf{G}_0 = \tilde{\theta}_{i+1}\tilde{\theta}_i\tilde{\theta}_{i+1})\langle 1\rangle.$$

We claim that (with Theorem 4.1) $\mathbf{T}\langle 3\rangle[1] \cong \mathbf{G}$. To avoid explicit calculations let us for a moment consider the translation functors as endofunctors of gmof $-A$, the graded version of $\mathcal{O}_0$. The isomorphism $\tilde{\theta}_{i+1}\tilde{\theta}_i\tilde{\theta}_{i+1} \cong \tilde{F}_{s_{i+1}s_is_{i+1}} \oplus \tilde{\theta}_{i+1}$ gives a natural transformation $p : \tilde{\theta}_{i+1}\tilde{\theta}_i\tilde{\theta}_{i+1} \cong \tilde{\theta}_{i+1}$, homogeneous of degree zero. Using Corollary 8.8 which is proved later we get that $R = \operatorname{Hom}(\tilde{\theta}_i\tilde{\theta}_{i+1}, \tilde{\theta}_{i+1}) \cong \operatorname{Hom}(\tilde{\theta}_{i+1}\tilde{\theta}_i\tilde{\theta}_{i+1}, \mathrm{ID})$ is strictly positively graded and $R_1 \cong \mathbb{C}$. Hence $g \circ p = \lambda \cdot f$ for some $\lambda \in \mathbb{C}$. Restricting to the parabolic categories, p becomes an isomorphism (Theorem 4.1) and the maps $\lambda^{-1}\cdot\mathrm{id}$ and p define the required isomorphism. The compatibility with the second picture in Figure 1 and the vertically flipped ones is proved in an analogous way. (The compatibility with the vertically flipped follows also by adjointness properties.) □

Remark 7.2 (Kauffman bracket and Jones polynomial). If we renormalise by taking $\mathcal{C}_i := \text{Cone }(\mathrm{ID}\langle 1\rangle \stackrel{\mathrm{adj}}{\to} \tilde{\theta}_i)$ and $\mathcal{K}_i := \mathcal{C}_i^{-1}[1]\langle 1\rangle$ then we have the following equalities in the Grothendieck group $[\tilde{\cap}_{i,n+2}\tilde{\cup}_{i,n}] = (v + v^{-1})[\mathrm{ID}]$, $[\mathcal{C}_i] = [\tilde{\theta}_i] - v[\mathrm{ID}]$ and $[\mathcal{K}_i] = [\mathrm{ID}] - v[\tilde{\theta}_i]$. This can be considered as a Kauffman bracket in the normalisation as in [Kho00]. Given a tangle t with c_j crossings of type t_i^j for $j = 3, 4$ we can define $K(t) := \big[\mathcal{T}(t)\langle c_4 - 2c_3\rangle[c_3]\big]$. Then $K(t)$ satisfies the defining relation for the scaled Kauffman bracket (as in [Kho00]) which is up to a normalisation the Jones polynomial.

Remark 7.3. Using the main result of [RH] that translation and Zuckerman's functors are Koszul dual to each other, Conjecture 1 of [BFK99] follows directly. On the other hand, it is not clear, if one really needs Ryom-Hansen's result to prove the Conjecture. All our arguments can easily be transfered to the singular case with one exception: It doesn't seem to be obvious how to translate the starting point for the induction in the proof of Proposition 6.4.

8 Cobordisms and natural transformations

To each tangle, hence in particular to a closed loop/circle, we assigned a functor. The goal of this section now is to describe a functor from the category $\mathcal{COB}$ of 2-cobordisms into a 2-category given by projective functors. The objects of $\mathcal{COB}$ are disjoint unions of labelled oriented closed 1-dimensional manifolds, that is a disjoint union of labelled oriented circles. (We also allow the emptyset, i.e. no circle.) A *surface between* n oriented circles $\mathbf{n}$ and m oriented circles $\mathbf{m}$ is a surface S with an orientation preserving isomorphism ϕ_S between the boundary δS of S and the union $\mathbf{n}^r \sqcup \mathbf{m}$, where $\mathbf{n}^r$ denotes the manifold $\mathbf{n}$ but with reversed orientation. Two surfaces S and T between $\mathbf{n}$ and $\mathbf{m}$ are *equivalent* if there is an isomorphism of surfaces (=a diffeomorphism) $\psi : S \tilde{\to} T$ such that the following diagram commutes

$$\begin{tikzcd}
\delta S \arrow[r, "\phi_S"] \arrow[d, "\psi_{|\delta S}"] & \mathrm{n}^r \sqcup \mathrm{m} \\
\delta T \arrow[ur, "\phi_T"'] &
\end{tikzcd}$$

A *morphism* $\Sigma : \mathbf{n} \to \mathbf{m}$ in $\mathcal{COB}$ is an equivalence class of surfaces $S : \mathbf{n} \to \mathbf{m}$ between $\mathbf{n}$ and $\mathbf{m}$. The morphisms in $\mathcal{COB}$ are generated by glueing copies of the six basic surfaces depicted in Figure 2 subject to certain relations. For details we refer to [Abr96, section 4].

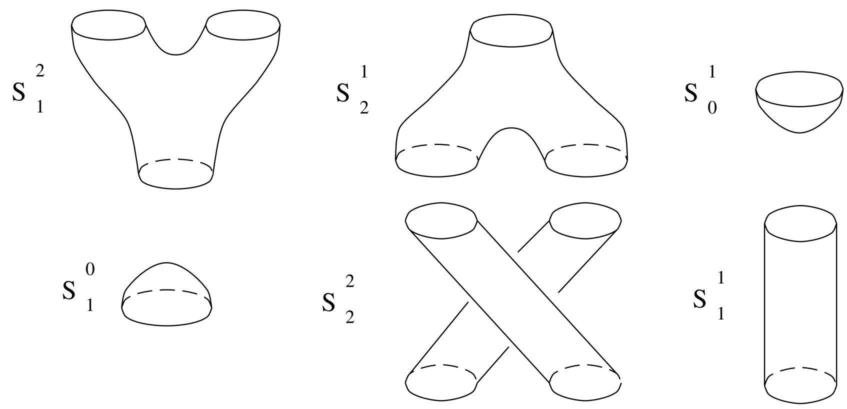


Figure 2: Basic cobordisms

8.1 Basic cobordisms

We fix $n \in \mathbb{N}$ and write $\zeta_{n+2} = \zeta$. Recall that we assigned to an occurring circle the composition

$$\cap_{i,n+2}\cup_{i,n} = (\zeta\theta_0^1\theta_2\cdots\theta_i)(\theta_i\theta_{i-1}\cdots\theta_2\theta_1^0\zeta^{-1}). \tag{8.1}$$

Since this is (up to isomorphism) independent of i we choose $i = 2$ and set $G = \zeta\theta_0^1$, $F = \theta_1^0\zeta^{-1}$. Note that $G\theta_2 F \cong \mathrm{ID}$ and $\theta_2 FG\theta_2 \cong \theta_2\theta_1^0\zeta\zeta^{-1}\theta_0^1\theta_2 \cong \theta_2\theta_1\theta_2 \cong \theta_2$ as endofunctors on $\mathcal{O}^{\max}(\mathfrak{sl}_n)$. Let $\mathrm{adj} : \mathrm{ID} \to \theta_2$ and $\overline{\mathrm{adj}} : \theta_2 \to \mathrm{ID}$ denote the adjunction morphisms. Since $\tilde\theta_2\tilde\theta_2 \cong \tilde\theta_2\langle 1\rangle \oplus \tilde\theta_2\langle -1\rangle$, there is a monomorphism $\tilde\alpha' : \tilde\theta_2 \to \tilde\theta_2\tilde\theta_2$ and also an epimorphism $\tilde\beta' : \tilde\theta_2\tilde\theta_2 \to \tilde\theta_2$ of degree -1. Let α' and β' denote the corresponding morphisms of functors after forgetting the grading. There is an isomorphism of graded functors $\tilde\sigma' : \tilde\theta_2\tilde\theta_2\tilde\theta_2 \cong \tilde\theta_2\langle 2\rangle \oplus \tilde\theta_2 \oplus \tilde\theta_2 \oplus \tilde\theta_2\langle -2\rangle$ by switching the two middle summands. Let σ' denote the corresponding isomorphism after forgetting the grading. To each basic cobordism we assign a natural transformation as follows

$$\begin{array}{rclccc}
\Phi(S_1^2) & = & \Delta : & \theta_2\theta_2 & \xrightarrow{\theta_2\,\mathrm{adj}_{\theta_2(\bullet)}} & \theta_2\theta_2\theta_2, \\
\Phi(S_2^1) & = & \mu : & \theta_2\theta_2\theta_2 & \xrightarrow{\theta_2\overline{\mathrm{adj}}_{\theta_2(\bullet)}} & \theta_2\theta_2, \\
\Phi(S_0^1) & = & i : & \theta_2 & \xrightarrow{\alpha'} & \theta_2\theta_2, \\
\Phi(S_1^0) & = & \epsilon : & \theta_2\theta_2 & \xrightarrow{\beta'} & \theta_2, \\
\Phi(S_2^2) & = & \sigma' : & \theta_2\theta_2\theta_2 & \xrightarrow{\sigma'} & \theta_2\theta_2\theta_2, \\
\Phi(S_1^1) & = & \mathrm{id} : & \theta_2\theta_2 & \xrightarrow{\mathrm{id}} & \theta_2\theta_2.
\end{array} \tag{8.2}$$

If $S = S_1 \sqcup S_2 \sqcup \cdots \sqcup S_r$ is a disjoint union of basic cobordisms $S_i : \mathbf{m_i} \to \mathbf{n_i}$, then $\Phi(S) : (\theta_2)^{m_1+m_2+\cdots+m_r+1} \longrightarrow (\theta_2)^{n_1+n_2+\cdots+n_r+1}$ is inductively defined as the composite

$$(\theta_2)^{n_1}(\Phi(S_2 \sqcup \cdots \sqcup S_r)) \circ \Phi(S_1)_{(\theta_2)^{m_2+m_3+\cdots+m_r+1}}.$$

Let $\mathcal{P}_n^{\max}$ denote the category of projective functors on $\mathcal{O}^{\max}(\mathfrak{sl}_n)$.

8.2 The Functor from Cobordisms into Algebraics

With the notations above we get the following main result:

Theorem 8.1 (Cobordisms as natural transformations). *There is a functor $\mathcal{CAT} = \mathcal{CAT}_n : \mathcal{COB} \to \mathcal{P}_n^{\max}$ given by*

$$\mathbf{m} \longmapsto G(\theta_2)^{m+1}F$$

on objects and on disjoint unions of basic morphisms as

$$\mathcal{CAT}(S) = G\Phi(S)_{F(\bullet)}$$

To make computations easier we use Soergel's functor $\mathbb{V} : \mathcal{O}_0 \longrightarrow C-$mof, where $C-$ mof denotes the category of finitely generated modules over the coinvariant algebra $C = S(\mathfrak{h})/(S(\mathfrak{h})_+)^{W\cdot}$ for the 'translated' W-action. We recall its main properties, give explicit formulas and then prove Theorem 8.1. For details see [Soe90], [Soe00]. The functor $\mathbb{V}$ is exact and fully faithful on projectives. For a simple reflection s, there is a natural isomorphism $\mathbb{V}\theta_s \cong C \otimes_{C^s} \mathbb{V}$, where C^s denotes the invariants of C under s. Note that C is a free C^s-module. A basis is given by 1 and X, the coroot corresponding to s.

Lemma 8.2. *The adjunction morphisms corresponds under $\mathbb{V}$ to the natural transformations given by the following morphisms of C-modules:*

$$\begin{array}{rrcl} m_N: & C \otimes_{C^s} N & \longrightarrow & N, \\ & c \otimes n & \longmapsto & cn \end{array} \qquad \begin{array}{rrcl} \delta_N: & N & \longrightarrow & C \otimes_{C^s} N \\ & 1 \otimes n & \longmapsto & 1 \otimes Xn + X \otimes n \end{array}$$

for $c \in C$, $n \in N \in C-\mathrm{mof}$.

Proof. The first adjunction morphism is given as the preimage of the identity under the canonical isomorphism

$$\begin{array}{rcl} \mathrm{Hom}_C(C \otimes_{C^s} N, N) & \longrightarrow & \mathrm{Hom}_{C^s}(N, N) \\ f & \longmapsto & \big(n \mapsto f(1 \otimes n)\big) \\ \big(c \otimes n \mapsto cf(n)\big) & \longleftarrow & f \end{array} \tag{8.3}$$

Hence $m_N(c \otimes n) = cn$. To prove the second statement we use the isomorphisms

$$\begin{array}{rcccl} \mathrm{Hom}_{C^s}(N, N) & \cong & \mathrm{Hom}_{C^s}(N^*, N^*) & \cong & \mathrm{Hom}_C(C \otimes_{C^s} N^*, N^*) \\ & \overset{\Psi}{\cong} & \mathrm{Hom}_C((C \otimes_{C^s} N)^*, N^*) & \cong & \mathrm{Hom}_C(N, C \otimes_{C^s} N). \end{array}$$

The second isomorphism is given by (8.3). According to [Soe00, Lemma 2.9.2], there is an isomorphism $\psi : C \otimes_{C^s} N^* \tilde{\rightarrow} (C \otimes_{C^s} N)^*$ of C-modules given by $\psi(1 \otimes f)(1 \otimes n) = 0$ and $\psi(1 \otimes f)(X \otimes n) = f(n)$ for $f \in N^*$, $n \in N$. This defines Ψ. All the other maps are given by duality. Since $m_{N^*}(1 \otimes f) = f$, we get $\Psi(m_{N^*})(\psi(1 \otimes f)) = f$. On the other hand $\delta^*(\psi(1 \otimes f)(n) = \psi(1 \otimes f)(\delta(n)) = \psi(1 \otimes f)(1 \otimes Xn + X \otimes n) = \psi(1 \otimes f)(X \otimes n) = f(n)$. This proves in fact that δ is the adjunction morphism. $\square$

The isomorphism $\theta_s\theta_s \cong \theta_s \oplus \theta_s$ becomes via $\mathbb{V}$ the following

Lemma 8.3. *There are natural isomorphisms of C-modules*

$$\begin{array}{rrcl} Q_N: & C\otimes_{C^s} C\otimes_{C^s} N & \longrightarrow & C\otimes_{C^s} N \oplus C\otimes_{C^s} N \\ & 1\otimes 1\otimes n & \longmapsto & (1\otimes n, 0) \\ & X\otimes 1\otimes n & \longmapsto & (X\otimes n, 0) \\ & 1\otimes X\otimes n & \longmapsto & (-X\otimes n, 1\otimes n) \\ & X\otimes X\otimes n & \longmapsto & (-X^2\otimes n, X\otimes n) \end{array}$$

Hence, β' corresponds to $\beta = p_1 \circ Q$ where p_2 denotes the projection onto the second summand, α' corresponds to $\alpha = Q^{-1}\circ i_1$ where i_1 denotes the inclusion of the first summand.

Proof. The inverse map is defined by $(1\otimes n, 0) \mapsto 1\otimes 1\otimes n$ and $(0, 1\otimes n) \mapsto 1\otimes X\otimes n + X\otimes 1\otimes n$. □

The permutation morphism σ becomes under $\mathbb{V}$ the following isomorphism

Lemma 8.4. *There is an isomorphism of functors*

$$\sigma : C\otimes_{C^s} C\otimes_{C^s} (C\otimes_{C^s} \bullet) \longrightarrow C\otimes_{C^s} C\otimes_{C^s} (C\otimes_{C^s} \bullet)$$

given by $\sigma = Q^{-1}_{C\otimes_{C^s}(\bullet)} \circ (Q^{-1}\oplus Q^{-1}) \circ (\mathrm{id}\oplus\overline{\sigma}\oplus\mathrm{id}) \circ (Q\oplus Q)\circ Q_{C\otimes_{C^s}(\bullet)}$, *where* $\overline{\sigma} : (C\otimes_{C^s} N)\oplus(C\otimes_{C^s} N)\tilde{\to}(C\otimes_{C^s} N)\oplus(C\otimes_{C^s} N)$, $\overline{\sigma}(x,y) = (y,x)$.

Proof. This follows directly from the previous lemma. □

Proof of Theorem 8.1. By [Abr96, Proposition 12] we first have to check that Δ, μ, ϵ, i and σ satisfy formally the properties of a (co-)associative and (co-)commutative, '(co-)multiplication', a '(co-)unit' and a 'permutation map'. Secondly, we have to show that $\theta_2\mu\circ\Delta_{\theta_2} = \Delta\circ\mu : \theta_2\theta_2\theta_2 \longrightarrow \theta_2\theta_2\theta_2$.

- *Associativity*, that is $\mathcal{CAT}\big(S^1_2\circ(S^1_2\sqcup S^1_1)\big) = \mathcal{CAT}\big(S^1_2\circ(S^1_1\sqcup S^1_2)\big)$. It is enough to verify

$$\overline{\mathrm{adj}}\;\overline{\mathrm{adj}}_{\theta_2(\bullet)} = \overline{\mathrm{adj}}\;\theta_2\;\overline{\mathrm{adj}}(\bullet) : (\theta_2)^2 \to \mathrm{ID}\,.$$

  We claim that this holds even on $\mathcal{O}_0$. Let $N \in C-\mathrm{mof}$. Let $c\otimes d\otimes n \in C\otimes_{C^s} C\otimes_{C^s} N$. We calculate $m\circ(m\otimes\mathrm{id})(c\otimes d\otimes n) = m(cd\otimes n) = cdn$, on the other hand $m\circ(\mathrm{id}\otimes m)(c\otimes d\otimes N) = m(c\otimes dn) = cdn$. The associativity follows.

- *Coassociativity.* Since

$$\begin{aligned} & (\delta\otimes \mathrm{id})\circ\delta(1\otimes n) = \delta\otimes \mathrm{id}(1\otimes(X\otimes n)+X\otimes(1\otimes n)) \\ = \quad & 1\otimes X\otimes X\otimes n + X\otimes 1\otimes X\otimes n + 1\otimes X^2\otimes 1\otimes n + X\otimes X\otimes 1\otimes n \\ = \quad & 1\otimes 1\otimes X^2\otimes n + 1\otimes X\otimes X\otimes n + X\otimes 1\otimes X\otimes n + X\otimes X\otimes 1\otimes n \\ = \quad & \mathrm{id}\otimes\delta(1\otimes X\otimes n + X\otimes 1\otimes n) \\ = \quad & (\mathrm{id}\otimes\delta)\circ\delta(1\otimes n), \end{aligned}$$

  it follows $\mathcal{CAT}\big((S_1^2\sqcup S_1^1)\circ S_1^2\big) = \mathcal{CAT}\big((S_1^1\sqcup S_1^2)\circ S_1^2\big)$.

- *Unit*, i.e. $\mathcal{CAT}\big(S_2^1\circ(S_1^1\sqcup S_0^1)\big) = \mathcal{CAT}(S_1^1) = \mathcal{CAT}\big(S_2^1\circ(S_1^1\circ S_0^1)\big)$. For the first equality it is enough to check $\overline{\mathrm{adj}}_{\theta_2}\circ\alpha' = \mathrm{id} : \theta_2\to\theta_2$. This, however, is true, since $(m\otimes\mathrm{id})\circ\alpha(1\otimes n) = (m\otimes\mathrm{id})(1\otimes 1\otimes n) = 1\otimes n$ by Lemma 8.3. Similarly, $(\mathrm{id}\otimes m)\circ\alpha(1\otimes n) = \mathrm{id}\otimes m(1\otimes 1\otimes n) = 1\otimes n$ proving the second equality.

- *Counit.* We omit to show explicitly the dual statement for the counit.

- The *commutativity* $\mu\circ\sigma' = \mu$ follows from the commutativity of the diagram

$$\begin{tikzcd} \theta_2\theta_2\theta_2 \arrow[r, "\sigma'"] \arrow[dr, "\theta_2\overline{adj}_{\theta_2}"'] & \theta_2\theta_2\theta_2 \arrow[d, "\theta_2\overline{adj}_{\theta_2}"] \\ & \theta_2\theta_2 = \theta_2\,\mathrm{ID}\,\theta_2. \end{tikzcd}$$

  Or more general from the commutativity of

$$\begin{tikzcd} C\otimes_{C^s} C\otimes_{C^s}(C\otimes_{C^s} N) \arrow[r, "\sigma"] \arrow[dr, "\mathrm{id}\otimes m\otimes\mathrm{id}"'] & C\otimes_{C^s} C\otimes_{C^s}(C\otimes_{C^s} N) \arrow[d, "\mathrm{id}\otimes m\otimes\mathrm{id}"] \\ & C\otimes_{C^s} C\otimes_{C^s}\otimes N. \end{tikzcd}$$

  Direct calculations using Lemma 8.3 and Lemma 8.4 show that $\sigma(c_1\otimes c_2\otimes c_3\otimes n) = c_1\otimes c_3\otimes c_2\otimes n$ for $c_i\in\{1,X\}$, $n\in N$. Therefore, $(\mathrm{id}\otimes m\otimes\mathrm{id})\circ\sigma(c_1\otimes c_2\otimes c_3\otimes) = c_1\otimes c_3c_2\otimes n = c_1\otimes c_2c_3\otimes n = \mathrm{id}\otimes m\otimes\mathrm{id}(c_1\otimes c_2\otimes c_3\otimes n)$. The commutativity follows.

- The *cocommutativity* follows from the following calculations: $\sigma\circ(\mathrm{id}\otimes\delta\otimes\mathrm{id})(c\otimes 1\otimes n) = \sigma(c\otimes 1\otimes X\otimes n + c\otimes X\otimes 1\otimes n) = c\otimes X\otimes 1\otimes n + c\otimes 1\otimes X\otimes n = (\mathrm{id}\otimes\delta\otimes\mathrm{id})(c\otimes 1\otimes n)$ and $\sigma\circ(\mathrm{id}\otimes\delta\otimes\mathrm{id})(c\otimes X\otimes n) = \sigma(c\otimes X\otimes X\otimes n) = c\otimes X\otimes X\otimes n = (\mathrm{id}\otimes\delta\otimes\mathrm{id})(c\otimes 1\otimes n)$.

- To prove the remaining relation it is enough to check the commutativity of

$$\begin{tikzcd} \theta_2\theta_2\theta_2 \arrow[r, "\mu=\theta_2\overline{\mathrm{adj}}_{\theta_2}"] \arrow[d, "\Delta_{\theta_2}=\theta_2\,\mathrm{adj}_{\theta_2\theta_2}"'] & \theta_2\theta_2 \arrow[d, "\Delta=\theta_2\,\mathrm{adj}_{\theta_2}"] \\ \theta_2\theta_2\theta_2\theta_2 \arrow[r, "\theta_2\mu=\theta_2\theta_2\overline{\mathrm{adj}}_{\theta_2}"] & \theta_2\theta_2\theta_2 \end{tikzcd}$$

  or just the commutativity of one of the following diagrams (with arbitrary $N \in C-\mathrm{mof}$):

$$\begin{tikzcd} \theta_2 \arrow[r, "\overline{\mathrm{adj}}"] \arrow[d, "\mathrm{adj}_{\theta_2}"] & \mathrm{ID} \arrow[d, "\mathrm{adj}"] \\ \theta_2\theta_2\theta_2\overline{\mathrm{adj}} \arrow[r, "\theta_2\overline{\mathrm{adj}}"] & \theta_2 \end{tikzcd} \qquad \begin{tikzcd} C\otimes_{C^s} N \arrow[r, "m"] \arrow[d, "\delta_{C\otimes_{C^s}}"] & N \arrow[d, "\delta"] \\ C\otimes_{C^s} C\otimes_{C^s} N \arrow[r, "\mathrm{id}\otimes m"] & C\otimes_{C^s} N. \end{tikzcd}$$

  Let $1\otimes n \in C\otimes_{C^s} N$. Since $\delta\circ m(1\otimes n) = \delta(n) = 1\otimes Xn + X\otimes n$ and $(\mathrm{id}\otimes m)\circ\delta)(1\otimes n) = (\mathrm{id}\otimes m)(1\otimes X\otimes n + X\otimes 1\otimes n) = 1\otimes Xn + X\otimes n$, the last diagram above commutes. This proves $\theta_2\mu\circ\Delta_{\theta_2(\bullet)} = \Delta\circ\mu$. Therefore, the assignment of the theorem is well-defined and defines a functor as described. □

Remark 8.5 (Gradings and Euler characteristic). All the occurring functors assigned to closed oriented labelled 1-manifolds are gradable. Choosing the standard lifts, by construction, the natural transformations assigned to the basic cobordisms become homogeneous with the following degrees: $\deg\tilde{\Delta} = \deg\tilde{\mu} = 1$, $\deg(\tilde{i}) = \deg(\tilde{\epsilon}) = -1$, $\deg\tilde{\sigma}' = \deg\mathrm{id} = 0$. Let $S = S_1S_2\cdots S_r : \mathrm{n}\to\mathrm{m}$ be a surface between two disjoint unions of labelled oriented closed 1-manifolds given as a product of disjoint unions of surfaces S_i $(1\leq i\leq r)$ from Figure 2 with corresponding natural transformations $\Phi(S_i)$. Set $\deg(S) = \sum_{i=1}^r \deg\Phi(S_i)$. The relations [Abr96] directly imply that $\deg(S)$ is well-defined, i.e. constant on equivalence classes. If $\chi(S)$ denotes the Euler characteristic of S. Then we get

$$\chi(S) = -\deg(S).$$

Remark 8.6. If a surface between two closed oriented 1-manifolds contains a punctured genus $> l(w_o)$ surface, where w_o is the longest element in the Weyl group corresponding to $\mathfrak{sl}_n$, then $\mathcal{CAT}_n(S) = 0$. To verify this one has to consider the composition $g = (m\circ\delta)^{l(w_o)}$. Since $\mathbb{V}P$ for any projective module $P \in \mathcal{O}_0(\mathfrak{sl}_n)$ has a natural grading (see [Soe90]), g induces a homogeneous endomorphism of degree $(l(w_o) \cdot \deg(X))$ on $C \otimes_{C^s} \mathbb{V}P$ for any $P \in \mathcal{O}_0(\mathfrak{sl}_n)$. On the other hand, however, $\mathbb{V}P_i \neq 0 \Rightarrow l \leq i \leq l + l(w_0) \cdot (\deg(X))$ for some $l \in \mathbb{Z}$ (again by e.g. [Soe90]).

We finish with a small result describing homomorphisms between translation functors on the graded version of the main block of $\mathcal{O}$ via bimodules over the coinvariant algebra (see also [Bac01]). Let C be given the *even* grading induced from $S(\mathfrak{h})$, where $S(\mathfrak{h})^2 = \mathfrak{h}$. Let $x \in W$ with a reduced expression $[x] = s_1 s_2 \cdot \ldots \cdot s_r$. Let $\tilde{\theta}_{[x]} = \tilde{\theta}_{s_r} \cdots \tilde{\theta}_{s_2} \tilde{\theta}_{s_1}$ considered as endofunctor of gmof $-A$. We denote $\mathbf{C}_{[x]} = C \otimes_{C^{s_r}} C \otimes_{C^{s_{r-1}}} \cdots \otimes_{C^{s_2}} C \otimes_{C^{s_1}} \bullet$ as functor on graded C-modules.

Proposition 8.7. *Let x, $w \in W$ with fixed reduced expressions $[x]$ and $[w]$ respectively. There is a natural isomorphism of graded vector spaces*

$$\operatorname{Hom}(\tilde{\theta}_{[x]}, \tilde{\theta}_{[w]}) \cong \operatorname{Hom}_{C-\text{gmof}-C}\big(\mathbf{C}_{[x]}(C)\langle -l(x)\rangle, \mathbf{C}_{[w]}(C)\langle -l(w)\rangle\big).$$

Proof. The results of [Soe90] give a natural map

$$\begin{array}{rcl} \Phi : \operatorname{Hom}(\tilde{\theta}_{[x]}, \tilde{\theta}_{[w]}) & \to & \operatorname{Hom}_{C-\text{gmof}}\big(\mathbf{C}_{[x]}\langle -l(x)\rangle, \mathbf{C}_{[w]}\langle -l(w)\rangle\big) \\ f & \mapsto & \hat{\mathbb{V}} f_{\mathbb{V}\tilde{P}(w_0 \cdot 0)}, \end{array}$$

where $\hat{\mathbb{V}}$ denotes the functor $\operatorname{Hom}_{\text{gmof}-A}(\tilde{P}(w_0 \cdot 0), \cdot) : \text{gmof}-A \to \text{gmof}-C$. Since f is a natural transformation, we have $f \circ \mathbf{C}_{[x]}(g) = \mathbf{C}_{[w]}(g) \circ f$ for any endomorphism $g \in \operatorname{End}_{\text{gmof}-A}(\tilde{P}(w_o \cdot 0)) = C$. Hence $\Phi(f)$ is a morphism of graded C-*bi*modules. Φ is injective, since any projective object $Q \in \text{gmof}-A$ has a copresentation of the form

$$Q \hookrightarrow \bigoplus_{i \in I_1} \tilde{P}(w_0 \cdot 0)\langle i\rangle \to \bigoplus_{i \in I_2} \tilde{P}(w_0 \cdot 0)\langle i\rangle$$

for some finite multisets I_1, I_2. Any homomorphism in the target space of Φ defines a natural transformation between functors $\mathbf{C}_{[x]}\langle -l(x)\rangle$ and $\mathbf{C}_{[w]}\langle -l(w)\rangle$ on the category of graded C-modules. By Soergel's structure theorem [Soe90] we therefore get a natural transformation g between the functors $\tilde{\theta}_{[x]}$ and $\tilde{\theta}_{[w]}$ restricted to projective objects. For arbitrary $N \in \text{gmof}-A$ we choose a projective resolution $P^\bullet$. Since g is a natural transformation, it provides a morphism of resolutions $\tilde{\theta}_{[x]}P^\bullet \to \tilde{\theta}_{[w]}P^\bullet$ inducing a unique morphism $g_N : \tilde{\theta}_{[x]}N \to \tilde{\theta}_{[w]}N$. By standard arguments g_N does not depend on the actual choice of the projective resolution and these maps define a natural transformation of functors. Hence Φ is surjective and the statement of the proposition follows. $\square$

We give the following example needed in the proof of Theorem 7.1:

Corollary 8.8. *Let $x = sts = tst$ for non-commuting simple reflections s and t. Fix $[x] = tst$. Then $R = \mathrm{Hom}(\tilde{\theta}_{[x]}, \mathrm{ID})$ is strictly positively graded (i.e. $R_i = 0$ for $i \leq 0$) and $R_1 = \mathbb{C}$.*

Proof. Direct calculations show that $\mathbf{C}_{[x]}(C)$ is generated as a C-bimodule by $1 \otimes 1 \otimes 1 \otimes 1$ and $1 \otimes X \otimes 1 \otimes 1$, where X denotes the coroot corresponding to s. Hence, $\mathbf{C}_{[x]}(C)\langle -3 \rangle$ is generated in degrees -3 and -1. Since C is positively graded with $C_0 = \mathbb{C}$ the statement follows, because there is a nontrivial transformation of degree 1 (namely $p \circ \mathrm{adj}_{\tilde{\theta}_s \tilde{\theta}_t}$ occurring in the proof of Theorem 7.1). $\square$

Appendix: Explicit calculations in Type A

We consider the special example, where $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{p} = \mathfrak{p}_m \cong \mathfrak{sl}_m \times \mathfrak{sl}_{n-m}$ is a maximal parabolic subalgebra.

Distinguished coset representatives

We first explicitly describe distinguished coset representatives. Let $W(n) = \langle s_1, \ldots, s_n \rangle$ be the Weyl group of type A_n.

Lemma A-1. *Let $n \leq 1$. Then*

$$W(n)^{\mathfrak{p}_1} = \{e, s_1, s_1 s_2, \ldots, s_1 s_2 \cdots s_n\}; \tag{A-1}$$

and all the expressions are reduced.

Proof. The expressions in (A-1) are obviously reduced since no braid relation or commutator relation can be applied. For $n = 1$ or $n = 2$ the assertion is true. Let us assume the lemma to be true for type A_{n-1}. For $2 < j \leq n$ we get

$$\begin{aligned} l(s_j(s_1 s_2 \cdots s_k) &= l(s_1 s_j s_2 \cdots s_k) = 1 + l(s_j s_2 \cdots s_k) \\ &> 1 + l(s_2 \cdots s_k) = l(s_1 s_2 \cdots s_k) \end{aligned}$$

by the induction hypothesis. On the other hand $l(s_2(s_1 s_2 \cdots s_k)) = l(s_2 s_1 s_2) + l(s_3 s_4 \cdots s_k) = 3 + l(s_3 \cdots s_k) = 1 + l(s_1 s_2 s_3 \cdots s_k)$. Hence, the elements of the set (A-1) are distinguished coset representatives. Since $\frac{|W|}{|W_S|} = n + 1$ the lemma follows. $\square$

Let $S(n+1,m)$ be the set of all subsets of order m of $\{0,1,\dots,n\}$. We write $i_1 \triangleright i_2 \triangleright \cdots \triangleright i_k$ to denote the element $\{i_1,\dots i_k\} \in S(n,k)$ with $i_1 > i_2 > \cdots > i_k$.

Proposition A-2. *Let $m \in \{1,\dots,n\}$. There is a bijection of sets*

$$\begin{array}{rcll} \Psi(n,m): S(n+1,m) & \longrightarrow & W(n)^{\mathfrak{p}_m} & \\ i_1 \triangleright i_2 \triangleright \dots \triangleright i_k & \longmapsto & (s_m s_{m+1} \cdots s_{i_1})(s_{m-1} s_m \cdots s_{i_2}) \cdots (s_1 s_2 \cdots s_{i_m}), \end{array} \tag{A-2}$$

where, by definition, $s_j s_{j+1} \cdot \ldots \cdot s_r = e$ if $r < j$. All the expressions occurring in the image of this map are reduced.

We just write $w = i_1 \triangleright i_2 \triangleright \cdots \triangleright i_k$ if they correspond via the bijection above. Moreover we abuse notation and write just $i_1 \triangleright i_2 \triangleright \cdots \triangleright i_l$ with $l < m$ if $s_j s_{j-1} \cdot \ldots \cdot s_r = e$ for $j > l$.

Proof. For $n = 2$, or for n arbitrary but $m = 1$, the Proposition holds by Lemma A-1. Let now $1 \leq m < n$. We assume that the claim holds for $\Psi(n', m')$, if either $n' < n$ or if $n' = n$ and $m' < m$. Lemma A-1 successively shows that the occurring expressions are reduced.
Let $w = (s_m s_{m+1} \cdots s_{i_1})(s_{m-1} s_m \cdots s_{i_2}) \cdots (s_1 s_2 \cdots s_{i_m}) = (s_m s_{m+1} \cdots s_{i_1}) y = w'(s_1 s_2 \cdots s_{i_m})$ To show, that $w \in W^{\mathfrak{p}_m}$ we consider two cases:

- $j \in \{2,3,\dots n\} \backslash \{m\}$: Then $l(s_j w) = l(s_j w') + i_k = l(w') + 1 + i_k = l(w) + 1$ by induction hypothesis.
- $j = 1$: By induction hypothesis, $l(s_1 w) = l(s_1 s_m s_{m+1} \cdots s_{i_1}) + l(y) = 1 + l(s_m s_{m+1} \cdots s_{i_1}) + l(y) = 1 + l(w)$.

Hence, all the elements occurring in the image of $\Psi(n,m)$ are distinguished coset representatives. The remaining thing, we have to prove is the injectivity of the map. Let us assume $\Psi(n,m)(i_1 \triangleright \dots, \triangleright i_m\}) = \Psi(n,m)(j_1 \triangleright \dots \triangleright j_m)$. Since $s_{\max\{i_1,j_1\}}$ has to occur on both sides, we conclude $i_1 = j_1$, hence

$$(s_{m-1} s_m \cdots s_{i_2}) \cdots (s_1 s_{n-1} \cdots s_{i_k}) = (s_{m-1} s_m \cdots s_{i_2}) \cdots (s_1 s_{n-1} \cdots s_{i_k})$$

The same argumentation gives successively $i_2 = j_2$, …, $i_k = j_k$. The theorem follows. $\square$

References

[Abr96] L. Abrams. Two-dimensional topological quantum field theories and Frobenius algebras. *J. Knot Theory Ramifications*, 5(5):569–587, 1996.

[AJS94] H. H. Andersen, J. C. Jantzen, and W. Soergel. Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p. *Astérisque*, (220), 1994.

[AS03] H.H. Andersen and C. Stroppel. Twisting functors on $\mathcal{O}_0$. Preprint. University of Aarhus (Denmark), 2003.

[Bac99] E. Backelin. Koszul duality for parabolic and singular category $\mathcal{O}$. *Represent. Theory*, 3:139–152 (electronic), 1999.

[Bac01] E. Backelin. The Hom-spaces between projective functors. *Represent. Theory*, 5:267–283 (electronic), 2001.

[Bas68] H. Bass. *Algebraic K-theory.* W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[BFK99] J. Bernstein, I. Frenkel, and M. Khovanov. A categorification of the Temperley-Lieb algebra and Schur quotients of $U(\mathfrak{sl}_2)$ via projective and Zuckerman functors. *Selecta Math. (N.S.)*, 5(2):199–241, 1999.

[BG80] J. N. Bernstein and S. I. Gel′fand. Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras. *Compositio Math.*, 41(2):245–285, 1980.

[BGG76] I. N. Bernstein, I. M. Gel′fand, and S. I. Gel′fand. A certain category of $\mathfrak{g}$-modules. *Funkcional. Anal. i Priložen.*, 10(2):1–8, 1976.

[BGS96] A. Beilinson, V. Ginzburg, and W. Soergel. Koszul duality patterns in representation theory. *J. Amer. Math. Soc.*, 9(2):473–527, 1996.

[BK95] T. Brzeziński and J. Katriel. Representation-theoretic derivation of the Temperley-Lieb-Martin algebras. *J. Phys. A*, 28(18):5305–5312, 1995.

[BW01] S. C. Billey and G. S. Warrington. Kazhdan-Lusztig polynomials for 321-hexagon-avoiding permutations. *J. Algebraic Combin.*, 13(2):111–136, 2001.

[Deo87] V. Deodhar. On some geometric aspects of Bruhat orderings. II. The parabolic analogue of Kazhdan-Lusztig polynomials. *J. Algebra*, 111(2):483–506, 1987.

[Die98] T. tom Dieck. Temperley-Lieb algebras associated to the root system D. *Arch. Math. (Basel)*, 71(5):407–416, 1998.

[ES87] T. J. Enright and B. Shelton. Categories of highest weight modules: applications to classical Hermitian symmetric pairs. *Mem. Amer. Math. Soc.*, 67(367), 1987.

[FK97] I. B. Frenkel and M. G. Khovanov. Canonical bases in tensor products and graphical calculus for $U_q(\mathfrak{sl}_2)$. *Duke Math. J.*, 87(3):409–480, 1997.

[Gre98] R. M. Green. Generalized Temperley-Lieb algebras and decorated tangles. *J. Knot Theory Ramifications*, 7(2):155–171, 1998.

[Här99] M. Härterich. Murphy bases of generalized Temperley-Lieb algebras. *Arch. Math. (Basel)*, 72(5):337–345, 1999.

[Irv85] R. Irving. Projective modules in the category $\mathcal{O}_S$: self-duality. *Trans. Amer. Math. Soc.*, 291(2):701–732, 1985.

[Irv93] R. Irving. Shuffled Verma modules and principal series modules over complex semisimple Lie algebras. *J. London Math. Soc. (2)*, 48(2):263–277, 1993.

[Jan79] J. C. Jantzen. *Moduln mit einem höchsten Gewicht*, volume 750 of *Lecture Notes in Mathematics.* Springer, Berlin, 1979.

[Jan83] J. C. Jantzen. *Einhüllende Algebren halbeinfacher Lie-Algebren*, volume 3 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3).* Springer-Verlag, Berlin, 1983.

[Kau01] L. H. Kauffman. *Knots and physics*, volume 1 of *Series on Knots and Everything.* World Scientific Publishing Co. Inc., River Edge, NJ, third edition, 2001.

[Kho00] M. Khovanov. A categorification of the Jones polynomial. *Duke Math. J.*, 101(3):359–426, 2000.

[KL79] D. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.*, 53(2):165–184, 1979.

[Lam91] T. Y. Lam. *A first course in noncommutative rings*, volume 131 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1991.

[Lip94] J. Lipp. Vergleich von Basen von Hecke-Algebren vom Typ A. Diplomarbeit Universität Stuttgart, 1994.

[Mar92] P. Martin. On Schur-Weyl duality, A_n Hecke algebras and quantum sl(N) on $\bigotimes^{n+1} \mathbf{C}^N$. In *Infinite analysis, Part A, B (Kyoto, 1991)*, volume 16 of *Adv. Ser. Math. Phys.*, pages 645–673. World Sci. Publishing, 1992.

[MS03] V. Mazorchuk and C. Stroppel. Translation and shuffling of projectively presentable modules and a categorification of a parabolic Hecke module. Preprint. University of Uppsala (Sweden), 2003.

[RC80] A. Rocha-Caridi. Splitting criteria for $\mathfrak{g}$-modules induced from a parabolic and the Berňsteĭn-Gel′fand-Gel′fand resolution of a finite-dimensional, irreducible $\mathfrak{g}$-module. *Trans. Amer. Math. Soc.*, 262(2):335–366, 1980.

[RH] S. Ryom-Hansen. Koszul duality of Translation and Zuckerman functors. *to appear in Journal of Lie theory.*

[Ric94] J. Rickard. Translation functors and equivalences of derived categories for blocks of algebraic groups. In *Finite-dimensional algebras and related topics (Ottawa, ON, 1992)*, volume 424 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 255–264. Kluwer Acad. Publ., Dordrecht, 1994.

[Soe90] W. Soergel. Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe. *J. Amer. Math. Soc.*, 3(2):421–445, 1990.

[Soe97] W. Soergel. Kazhdan-Lusztig polynomials and a combinatorics for tilting modules. *Represent. Theory*, 1:83–114 (electronic), 1997.

[Soe00] W. Soergel. On the relation between intersection cohomology and representation theory in positive characteristic. *J. Pure Appl. Algebra*, 152(1-3):311–335, 2000.

[Str03] C. Stroppel. Category $\mathcal{O}$: Gradings and Translation functors. *Journal of Algebra*, 268(1):301–326, 2003.

[TL71] H. N. V. Temperley and E. H. Lieb. Relations between the “percolation” and “colouring” problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the "percolation" problem. *Proc. Roy. Soc. London Ser. A*, 322:251–280, 1971.

[Tur94] V. G. Turaev. *Quantum invariants of knots and 3-manifolds*, volume 18 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1994.

[Wes95] B. W. Westbury. The representation theory of the Temperley-Lieb algebras. *Math. Z.*, 219(4):539–565, 1995.

Catharina Stroppel
University of Aarhus (Denmark)
email: stroppel@imf.au.dk