UNIVERSITY OF AARHUS
DEPARTMENT OF MATHEMATICS

ISSN: 1397–4076

THE STABLE MODULI SPACE OF RIEMANN SURFACES: MUMFORD'S CONJECTURE

By Ib Madsen and Michael Weiss

Preprint Series No.: 2

April 2003

Ny Munkegade, Bldg. 530
DK-8000 Aarhus C, Denmark

http://www.imf.au.dk
institut@imf.au.dk

Version 1
23 December 2002

The stable moduli space of Riemann surfaces: Mumford's conjecture

Ib Madsen*
Institute for the Mathematical Sciences
Aarhus University
8000 Aarhus C, Denmark
email: imadsen@imf.au.dk

Michael Weiss
Department of Mathematics
Aberdeen University
Aberdeen AB24 3UE, United Kingdom
email: m.weiss@maths.abdn.ac.uk

*Supported in part by American Institute of Mathematics.

Contents

1 Introduction: Results and methods

1.1 Results

The main result of this paper amounts to a complete evaluation of the integral cohomological structure of the stable mapping class group. In particular it verifies the conjecture of D. Mumford about the rational cohomology of the mapping class group:

$$H^*(B\mathit{\Gamma}_{g,b};\mathbb{Q}) = \mathbb{Q}[\kappa_1,\kappa_2,\dots] \qquad \text{for } 2* < g-1$$

where $\mathit{\Gamma}_{g,b}$ is the mapping class group of an oriented surface $F_{g,b}$ of genus g with b boundary circles (and no punctures). The κ_i are the Miller-Morita-Mumford "tautological" classes of degree $2i$.

For $b > 0$, the standard homomorphisms

$$\begin{array}{ccc} \mathit{\Gamma}_{g,b} & \to & \mathit{\Gamma}_{g+1,b}, \\ \mathit{\Gamma}_{g,b} & \to & \mathit{\Gamma}_{g,b-1} \end{array} \tag{1.1}$$

yield maps of classifying spaces that induce isomorphisms in integral cohomology in degrees less than $g/2-1$ by the stability theorems of Harer [17] and Ivanov [21]. The colimit of the maps

$$B\mathit{\Gamma}_{g,b} \longrightarrow B\mathit{\Gamma}_{g+1,b} \longrightarrow B\mathit{\Gamma}_{g+2,b} \longrightarrow \cdots$$

will be denoted $B\mathit{\Gamma}_{\infty,b}$,

$$B\mathit{\Gamma}_{\infty,b} = \operatorname*{colim}_g \, B\mathit{\Gamma}_{g,b} \simeq \operatorname*{hocolim}_g \, B\mathit{\Gamma}_{g,b}.$$

The groups $\mathit{\Gamma}_{g,b}$ are perfect for $g > 1$, so $B\mathit{\Gamma}_{\infty,b}$ has a perfect fundamental group and one may apply Quillen's plus construction to it. The result is independent of b up to homotopy equivalence, so we denote it by $B\mathit{\Gamma}_\infty^+$. A celebrated result from [40] asserts that $\mathbb{Z}\times B\mathit{\Gamma}_\infty^+$ and $B\mathit{\Gamma}_\infty^+$ are infinite loop spaces, so that homotopy classes of maps to either of these spaces form the degree 0 part of a generalized cohomology theory.

Next we review a completely different infinite loop space, one which is rather well known to homotopy theorists. Let us write $\mathrm{Gr}_2(\mathbb{R}^{2+n})$ for the Grassmann manifold of oriented 2-dimensional subspaces of $\mathbb{R}^{2+n}$. There are two canonical bundles over $\mathrm{Gr}_2(\mathbb{R}^{2+n})$, namely, the tautological 2-plane bundle L_n and its n-dimensional orthogonal complement $L_n^\perp$. The restriction

$$L_{n+1}^\perp \,\big|\, \mathrm{Gr}_2(\mathbb{R}^{2+n})$$

is the direct sum of $L_n^\perp$ and a trivialized real line bundle. This yields an inclusion of associated Thom spaces,

$$S^1 \wedge \mathrm{Th}\,(L_n^\perp) \longrightarrow \mathrm{Th}\,(L_{n+1}^\perp)$$

and hence a sequence of maps (in fact cofibrations)

$$\cdots \longrightarrow \Omega^{n+1}\mathrm{Th}\,(L_{n-1}^\perp) \longrightarrow \Omega^{n+2}\mathrm{Th}\,(L_n^\perp) \longrightarrow \Omega^{n+3}\mathrm{Th}\,(L_{n+1}^\perp) \longrightarrow \cdots$$

whose colimit is traditionally denoted

$$\Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1} = \operatorname*{colim}_n \; \Omega^{n+2}\mathrm{Th}\,(L_n^\perp).$$

There is a map

$$\alpha_\infty \colon \mathbb{Z} \times B\Gamma_\infty^+ \longrightarrow \Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1}$$

constructed and examined in considerable detail in [25]. Our main result is the following theorem conjectured in [25]:

Theorem 1.1.1 *The map* $\alpha_\infty \colon \mathbb{Z} \times B\Gamma_\infty^+ \longrightarrow \Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1}$ *is a homotopy equivalence.*

The cohomological structure of $\Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1}$ is completely known both with $\mathbb{Q}$ coefficients and with $\mathbb{F}_p$ coefficients for all p, so the theorem gives the cohomology of $B\Gamma_\infty^+$ and hence of $B\Gamma_{\infty,b}$ with these coefficients.

The space $\Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1}$ fits into the homotopy fibration sequence of [33],

$$\Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1} \xrightarrow{\ \omega\ } \Omega^\infty S^\infty(\mathbb{C}P^\infty_+) \xrightarrow{\ \partial\ } \Omega^{\infty+1}S^\infty \tag{1.2}$$

where the subscript $+$ denotes an added disjoint base point. The structure of the cohomology $H^*(\Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1};\mathbb{F}_p)$ was recently determined in [11]. It is rather involved and we refrain from listing the result. The rational structure is much easier to describe.
The homotopy groups of $\Omega^{\infty+1}S^\infty$ are equal to the stable homotopy groups of spheres, up to a shift of one, and are therefore finite. Thus $H^*(\omega;\mathbb{Q})$ is an isomorphism. The canonical complex line bundle over $\mathbb{C}P^\infty$, considered as a map from $\mathbb{C}P^\infty$ to $\{1\} \times B\mathrm{U}$ induces via Bott periodicity a map

$$L \colon \Omega^\infty S^\infty(\mathbb{C}P^\infty_+) \longrightarrow \mathbb{Z} \times B\mathrm{U},$$

and L is a rational equivalence. It follows that the rational cohomology of a component, say $\Omega^\infty_0 \mathbb{C}P^\infty_{-1}$, is equal to the rational cohomology of $B\mathrm{U}$, and hence by theorem 1.1.1 that

$$H^*(B\Gamma_{\infty,b};\mathbb{Q}) \cong H^*(B\mathrm{U};\mathbb{Q}).$$

This yields Mumford's conjecture.

1.2 A geometric formulation

Fix an integer $d \geq 0$. Let $\pi \colon M \to X$ be a smooth fiber bundle with oriented d-dimensional fibers. We assume that the fibers are closed. There are two canonical vector bundles on M, namely the vertical tangent bundle $T^\pi M$ and the stable vertical normal bundle $N^\pi M$. The latter is defined to be the normal bundle of a fiberwise embedding of M into $X \times \mathbb{R}^{d+n}$ for large n. Let $\mathrm{Gr}_d(\mathbb{R}^{d+n})$ be the Grassmann manifold of oriented d-dimensional subspaces of $\mathbb{R}^{d+n}$, and let $U_{d,n}$ and $U_{d,n}^\perp$ be the two standard vector bundles over it of dimension d and

n, respectively. The vertical tangent bundle and the vertical normal bundle are classified by bundle maps

$$T^\pi M \longrightarrow U_{d,n}\,, \qquad N^\pi M \longrightarrow U_{d,n}^\perp\,,$$

respectively, for large n. We can view $N^\pi M$ as a tubular neighborhood of M in $X\times\mathbb{R}^{d+n}$ and obtain the fiberwise Thom-Pontryagin map

$$X_+\wedge S^{d+n}\cong \mathrm{Th}\,(X\times\mathbb{R}^{d+n}) \longrightarrow \mathrm{Th}\,(N^\pi M) \longrightarrow \mathrm{Th}\,(U_{d,n}^\perp),$$

by mapping the complement of $N^\pi M$ in $\mathrm{Th}\,(X\times\mathbb{R}^{d+n})$ to $\infty\in\mathrm{Th}\,(N^\pi M)$. Generalizing the description of $\Omega^\infty\mathbb{C}\mathbf{P}^\infty_{-1}$ given earlier, but switching to different notation, we put

$$\Omega^\infty\mathbf{hV} = \mathrm{colim}_n\ \Omega^{d+n}\mathrm{Th}\,(U_{d,n}^\perp).$$

The adjoint of the above composition gives a homotopy class of maps from X to $\Omega^\infty\mathbf{hV}$. The universal case is $X=\coprod B\mathrm{Diff}(F)$, where F runs over over a set of representatives of the diffeomorphism classes of closed, smooth and oriented d-manifolds. In this case we obtain

$$\alpha\colon \coprod B\mathrm{Diff}(F)\longrightarrow \Omega^\infty\mathbf{hV}\,. \tag{1.3}$$

In the case $d=2$, it is convenient to make extra assumptions. For example, we may wish to consider only *connected* oriented surfaces F with an embedded copy of $S^0\times D^2$, and diffeomorphisms $F\to F$ relative to the embedded $S^0\times D^2$. With these conventions, $\coprod B\mathrm{Diff}(F)$ becomes an A_∞-monoid under connected sum, and the map α can be shown to factor over the group completion of $\coprod B\mathrm{Diff}(F)$. Each $\mathrm{Diff}(F)$ has contractible components by [9], [10]. The group of components is the mapping class group $\varGamma_{g,2}$ where g is the genus of F. Hence in this case $\coprod B\mathrm{Diff}(F)$ becomes homotopy equivalent to $\coprod_g B\varGamma_{g,2}$. The group completion is

$$\Omega B\big(\coprod_g B\varGamma_{g,2}\big) \ \simeq\ B\varGamma_\infty^+\times\mathbb{Z}\,, \tag{1.4}$$

cf. [25], [40], and α of (1.3) induces the map α_∞ of theorem 1.1.1.

Let us return to the general case, with fixed $d\in\mathbb{N}$. We give a geometric interpretation of homotopy classes of maps from a smooth manifold X without boundary into $\Omega^\infty\mathbf{hV}$. We represent such a homotopy class by a pointed map

$$X_+\wedge S^{d+n} \longrightarrow \mathrm{Th}\,(U_{d,n}^\perp)$$

for some large n, transverse to the zero section of $U_{d,n}^\perp$. The resulting inverse image of the zero section is a submanifold $M\subset X\times\mathbb{R}^{d+n}$, of dimension $\dim(X)+d$, with a map $v_M\colon M\to \mathrm{Gr}_d(\mathbb{R}^{d+n})$. The projection $\pi_M\colon M\to X$ is proper, since M is closed in $\mathrm{Th}\,(X\times\mathbb{R}^{d+n})$. The normal bundle of M in $X\times\mathbb{R}^{d+n}$ is identified with $v_M^*U_{d,n}^\perp$, so

$$TM\times\mathbb{R}^{d+n} \cong TM\oplus v_M^*U_{d,n}^\perp\oplus v_M^*U_{d,n}\cong(\pi_M^*TX\times\mathbb{R}^{d+n})\oplus v_M^*U_{d,n}\,. \tag{1.5}$$

Standard obstruction theory now implies that

$$TM\times\mathbb{R} \cong (\pi_M^*TX\times\mathbb{R})\oplus v_M^*U_{d,n}\,. \tag{1.6}$$

We set $E = M \times \mathbb{R}$, write π_E for the composition $E \to M \to X$ and v_E for the composition $E \to M \to \mathrm{Gr}_d(\mathbb{R}^{d+n})$, and obtain from (1.6) a surjective bundle map

$$\hat{\pi}_E \colon TE \to \pi_E^* TX .$$

Since E is open, the submersion theorem of Phillips [30], [16], [15] applies, showing that the pair $(\pi_E, \hat{\pi}_E)$ is homotopic through vector bundle surjections to a pair consisting of a submersion $\pi \colon E \to X$ and its differential $d\pi \colon TE \to \pi^* TX$. For us it is also important to ensure that the underlying homotopy $E \times [0,1] \to X$ combines with the projection $f \colon E \to \mathbb{R}$ to give a proper map $E \times [0,1] \to X \times \mathbb{R}$. This is trivially the case when X is closed, in particular when X is a sphere, because then the projection $f \colon E \to \mathbb{R}$ is proper. In the general case a more careful application of the submersion theorem is required; we omit the details.

There is an additional feature in this situation. Namely, the vertical tangent bundle of the submersion $\pi \colon E \to X$ is identified with $v_E^* U_{d,n} \times \mathbb{R}$ and therefore projects to a trivial line bundle. In terms of the vertical 1-jet bundle

$$p_\pi^1 \colon J_\pi^1(E, \mathbb{R}) \longrightarrow E$$

whose fiber at $z \in E$ consists of all affine maps from the vertical tangent space $(T^\pi E)_z$ to $\mathbb{R}$, this feature together with $f \colon E \to \mathbb{R}$ amounts to a section $\hat{f}$ of p_π^1 such that $\hat{f}(z) \colon (T^\pi E)_z \to \mathbb{R}$ is surjective for every $z \in E$.

We introduce the notation $h\mathcal{V}(X)$ for the set of pairs $(\pi, \hat{f})$, where $\pi \colon E \to X$ is a smooth submersion with $(d+1)$-dimensional oriented fibers and $\hat{f} \colon E \to J_\pi^1(E, \mathbb{R})$ is a section of p_π^1 with underlying map $f \colon E \to \mathbb{R}$, subject to two conditions: for each $z \in E$ the affine map $\hat{f}(z) \colon (T^\pi E)_z \to \mathbb{R}$ is surjective, and $(\pi, f) \colon E \to X \times \mathbb{R}$ is proper.

Concordance defines an equivalence relation on $h\mathcal{V}(X)$. Let $h\mathcal{V}[X]$ be the set of equivalence classes. Then we have a natural bijection

$$h\mathcal{V}[X] \cong [X, \Omega^\infty \mathbf{hV}] . \tag{1.7}$$

There is a similar but easier interpretation of homotopy classes of maps from X to the source of (1.3). Namely, let $\mathcal{V}(X)$ be the set of pairs (π, f) with π as before and $f \colon E \to \mathbb{R}$ a smooth function, subject to two conditions: the restriction of f to any fiber of the submersion $\pi \colon E \to X$ is regular (= nonsingular), and $(\pi, f) \colon E \to X \times \mathbb{R}$ is proper. Let $\mathcal{V}[X]$ be the set of concordance classes of elements in $\mathcal{V}(X)$. Then

$$\mathcal{V}[X] \cong [X, \coprod B\mathrm{Diff}(F)] \tag{1.8}$$

with $\coprod B\mathrm{Diff}(F)$ as in (1.3). Indeed, an element of $\mathcal{V}(X)$ is a proper submersion with target $X \times \mathbb{R}$, hence a smooth fiber bundle on $X \times \mathbb{R}$ by Ehresmann's fibration theorem. An element $(\pi, f) \in \mathcal{V}(X)$, with $\pi \colon E \to X$, determines a section $j_\pi^1 f$ of the projection $J_\pi^1(E, \mathbb{R}) \to E$ by fiberwise 1-jet prolongation. The map

$$\mathcal{V}(X) \longrightarrow h\mathcal{V}(X); \quad (\pi, f) \mapsto (\pi, j_\pi^1 f) \tag{1.9}$$

respects the concordance relation and so induces a map $\mathcal{V}[X] \to h\mathcal{V}[X]$, which corresponds to α in (1.3) under the isomorphisms (1.7) and 1.8). When $d = 2$, it is a good idea to modify the source of (1.9); we will return to this point in a little while.

The new description of α reformulates theorem 1.1.1 as a statement about integrability of certain jet bundle sections, up to homotopy or concordance. Statements of this type are called h-principles [14].

1.3 Outline of proof

The celebrated "first main theorem" of V.A.Vassiliev [41], see also [42], is a wonderful source of (established) h-principles. One of these Vassiliev h-principles, slightly modified, turns out to be a rather close approximation to the one we are after in connection with the Mumford conjecture. We describe this approximation.

Fix $d \geq 0$ as before. For smooth X without boundary, let $\mathcal{W}(X)$ be the set of all pairs (π, f) where $\pi\colon\ E \to X$ is a smooth submersion with oriented fibers of dimension $d+1$, and $f\colon\ E \to \mathbb{R}$ is a smooth function subject to two conditions: the restriction of f to each fiber of π is a Morse function, and $(\pi, f)\colon\ E \to (X, \mathbb{R})$ is proper.
Let $h\mathcal{W}(X)$ be the set of all pairs $(\pi, \hat{f})$ where $\pi\colon\ E \to X$ is a smooth submersion with fibers of dimension $d+1$ as before, and $\hat{f}$ is a section of the vertical 2-jet bundle

$$p^2_\pi\colon\ J^2_\pi(E, \mathbb{R}) \longrightarrow E\,,$$

subject to two conditions. Namely, for $x \in X$ and $z \in E_x = \pi^{-1}(x)$, the value $\hat{f}(z)$ can be represented by a germ near z of *Morse* functions $E_x \to \mathbb{R}$; and $(\pi, f)\colon\ E \to X \times \mathbb{R}$ is proper, where $f\colon\ E \to \mathbb{R}$ is the underlying map of $\hat{f}$.
An element $(\pi, f) \in \mathcal{W}(X)$, with $\pi\colon\ E \to X$, determines a section $j^2_\pi f$ of the projection $J^2_\pi(E, \mathbb{R}) \to E$ by fiberwise 2-jet prolongation. The map

$$\mathcal{W}(X) \quad\longrightarrow\quad h\mathcal{W}(X)\,; \quad (\pi, f) \quad\mapsto\quad (\pi, j^2_\pi f) \tag{1.10}$$

respects the concordance relation and so induces a map between the sets of concordance classes, $\mathcal{W}[X] \to h\mathcal{W}[X]$. This is obviously very similar to (1.9). The contravariant functors $X \mapsto \mathcal{W}[X]$ and $X \mapsto h\mathcal{W}[X]$ are representable, so that elements of $\mathcal{W}[X]$ and $h\mathcal{W}[X]$ are in bijective natural correspondence with homotopy classes of maps

$$X \quad\longrightarrow\quad |\mathcal{W}|\,, \qquad\qquad X \quad\longrightarrow\quad |h\mathcal{W}|\,,$$

respectively, for certain spaces $|\mathcal{W}|$ and $|h\mathcal{W}|$. The natural map $\mathcal{W}[X] \to h\mathcal{W}[X]$ given by 2-jet prolongation corresponds a map between the representing spaces,

$$|\mathcal{W}| \longrightarrow |h\mathcal{W}|\,. \tag{1.11}$$

Vassiliev's first main theorem is the main ingredient in our proof of:

Theorem 1.3.1 *The map (1.11) is a homotopy equivalence.*

We now relate (1.11) to (1.9). The functors $X \mapsto \mathcal{V}[X]$ and $X \mapsto h\mathcal{V}[X]$ also have representing spaces $|\mathcal{V}|$ and $|h\mathcal{V}|$, respectively. The inclusions $\mathcal{V}(X) \to \mathcal{W}(X)$ respect the concordance relation and so induce a map $|\mathcal{V}| \to |\mathcal{W}|$. To have a similar inclusion $h\mathcal{V}(X) \to h\mathcal{W}(X)$, we need to adjust the definition of $h\mathcal{V}(X)$ by insisting on pairs $(\pi, \hat{f})$, with $\pi\colon E \to X$ etc., where $\hat{f}$ is a section of the fiberwise 2-jet bundle

$$j^2_\pi\colon\ J^2_\pi(E, \mathbb{R}) \longrightarrow E\,,$$

but the conditions are, as before, on the induced section of the fiberwise 1-jet bundle j^1_π. Then there is a commutative square

$$\begin{tikzcd} |\mathcal{V}| \arrow[r] \arrow[d] & |h\mathcal{V}| \arrow[d] \\ |\mathcal{W}| \arrow[r] & |h\mathcal{W}| \end{tikzcd} \tag{1.12}$$

where the vertical arrows are inclusion-induced and the horizontal ones are given by jet prolongation. In order to make an efficient comparison between the two rows of diagram (1.12), we introduce a "localized" version of the second row.

For a smooth X without boundary, let $\mathcal{W}_{\mathrm{loc}}(X)$ consist of pairs (π, f), with $\pi\colon E \to X$ etc., as in the definition of $\mathcal{W}(X)$, except for one change. We no longer require that $(\pi, f)\colon E \to X \times \mathbb{R}$ be proper; instead we require that the restriction of (π, f) to the fiberwise singularity set

$$\Sigma(\pi, f) = \{ z \in E \mid df = 0 \text{ on } (T^\pi E)_z \}$$

be a proper map $\Sigma(\pi, f) \to X \times \mathbb{R}$. There is an h-version $h\mathcal{W}_{\mathrm{loc}}(X)$, consisting of pairs $(\pi, \hat{f})$ as in the definition of $h\mathcal{W}(X)$, except for a weakening of the properness condition. Jet prolongation defines a map between the representing spaces, $|\mathcal{W}_{\mathrm{loc}}| \to |h\mathcal{W}_{\mathrm{loc}}|$.

Theorem 1.3.2 *The jet prolongation map $|\mathcal{W}_{\mathrm{loc}}| \to |h\mathcal{W}_{\mathrm{loc}}|$ is a homotopy equivalence.*

This is much easier than 1.3.1. We briefly describe the ideas involved. For an element $(\pi, f) \in \mathcal{W}_{\mathrm{loc}}(X)$, the set $\Sigma = \Sigma(\pi, f) \subset E$ consists of all $z \in E$ where f restricted to the fiber of π through z has a singularity. Since these singularities are nondegenerate by assumption, Σ is a smooth submanifold of E which is everywhere transverse to the fibers of π. Hence the projection $\Sigma \to X$ is an étale map, alias codimension zero submersion. Because of the weakened properness condition, knowledge of the étale map $\Sigma \to X$, the normal bundle of Σ in E and the Morse index map $\Sigma \to \{0, 1, 2, 3, \ldots, d+1\}$ turns out to be sufficient to reconstruct the concordance class of (π, f). This makes it easy to give a simple description of $\mathcal{W}_{\mathrm{loc}}[X]$. There is a similar description of $h\mathcal{W}_{\mathrm{loc}}[X]$, and theorem 1.3.2 is an easy consequence of these simplified descriptions.

The sets $h\mathcal{V}(X) \subset h\mathcal{W}(X) \subset h\mathcal{W}_{\mathrm{loc}}(X)$ consist of smooth maps $(\pi, f)\colon E \to X \times \mathbb{R}$ with extra tangential structure. It is always a relatively easy matter to classify tangential structures, and in fact we are able to determine the homotopy type of all three spaces. The space $|h\mathcal{V}|$ was implicitly determined in the previous subsection:

$$|h\mathcal{V}| \;\simeq\; \Omega^\infty \mathbf{hV}\,.$$

A very similar analysis gives

$$|h\mathcal{W}| \;\simeq\; \Omega^\infty \mathbf{hW}$$

for another (twice looped down) Thom spectrum $\mathbf{hW}$. Finally,

$$|h\mathcal{W}_{\mathrm{loc}}| \;\simeq\; \prod_{\substack{p,q\\ p+q=d+1}} \Omega^\infty S^\infty(S^1 \wedge B\mathrm{SO}(p,q)_+)$$

where $S\mathrm{O}(p,q) \subset \mathrm{GL}(p+q)$ is the subgroup which stabilizes the standard quadratic form

$$(x_1,\dots,x_{d+1}) \mapsto -(x_1^2+\cdots+x_p^2)+(x_{p+1}^2+\cdots+x_{d+1}^2)\,.$$

Given the homotopy types, one observes the following

Theorem 1.3.3 *The maps $|h\mathcal{V}| \to |h\mathcal{W}| \to |h\mathcal{W}_{\mathrm{loc}}|$ define a homotopy fibration sequence of infinite loop spaces.*

Theorems 1.3.1, 1.3.2, 1.3.3 are valid for any choice of $d \geq 0$. This is not the case for the final result that goes into the proof of theorem 1.1.1, although substantial parts of it are valid for all d. For the moment we take $d=2$. In this case we have a modified version $\mathcal{V}_c(X)$ of $\mathcal{V}(X)$. Namely, an element of $\mathcal{V}_c(X)$ is a proper submersion, or equivalently, a bundle $(\pi,f)\colon\; E \to X\times\mathbb{R}$ whose fibers F are *connected*, closed, smooth, 2-dimensional; in addition we assume that each fiber comes equipped with an orientation preserving embedding of $S^0 \times D^2$. The functor $X \mapsto \mathcal{V}_c[X]$ has a representing space $|\mathcal{V}_c|$. By the discussion leading up to 1.4, we have $|\mathcal{V}_c| \simeq \coprod_g B\Gamma_{g,2}$ and therefore

$$\Omega B|\mathcal{V}_c| \;\simeq\; B\Gamma_\infty^+ \times \mathbb{Z}\,.$$

It is a consequence of theorems 1.3.1 and 1.3.2 that the spaces $|\mathcal{W}|$ and $|\mathcal{W}_{\mathrm{loc}}|$ are group complete, so that

$$|\mathcal{W}| \;\simeq\; \Omega B|\mathcal{W}|\,, \quad |\mathcal{W}_{\mathrm{loc}}| \;\simeq\; \Omega B|\mathcal{W}_{\mathrm{loc}}|\,.$$

Theorem 1.3.4 *With $d=2$, the sequence $\Omega B|\mathcal{V}_c| \longrightarrow |\mathcal{W}| \longrightarrow |\mathcal{W}_{\mathrm{loc}}|$ is a homotopy fibration sequence.*

Using this in conjunction with 1.3.1 and 1.3.2, we have another homotopy fibration sequence $\Omega B|\mathcal{V}_c| \to |h\mathcal{W}| \longrightarrow |h\mathcal{W}_{\mathrm{loc}}|$, and therefore by theorem 1.3.3 the conclusion

$$B\Gamma_\infty^+ \times \mathbb{Z} \;\simeq\; \Omega B|\mathcal{V}_c| \;\simeq\; |h\mathcal{V}| \;\simeq\; \Omega^\infty\mathbf{hV} \;=\; \Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1}.$$

The proof of theorem 1.3.4 is technically the most demanding part of the paper. It rests on compatible stratifications of $|\mathcal{W}|$ and $|\mathcal{W}_{\mathrm{loc}}|$, where the strata are certain bundle theories. This part of the analysis is valid for all $d \geq 0$. But the Harer stability theorem is also used in an essential way, if only as a black box. This leads to the condition $d = 2$.

We end the paragraph with an explanation of where the stratifications of $|\mathcal{W}|$ and $|\mathcal{W}_{\mathrm{loc}}|$ come from, again for arbitrary but fixed $d \geq 0$.
Let (π, f) be an element of $\mathcal{W}(X)$, with $\pi\colon\ E \to X$ and $f\colon\ E \to \mathbb{R}$. We can then associate to each $x \in X$ a finite set T_x. This is the set of critical points with critical value 0 of the Morse function $f|E_x$, where $E_x = \pi^{-1}(x)$. It comes with a map

$$T_x \to \{0, 1, 2, \ldots, d+1\}\,,$$

the Morse index map. Therefore (π, f) determines a partition of X into locally closed subsets $X_{\langle T\rangle}$, indexed by the isomorphism classes $\langle T\rangle$ of finite sets over $\{0, 1, \ldots, d+1\}$. Namely, $X_{\langle T\rangle} \subset X$ consists of all $x \in X$ such that $T_x \cong T$.
If the partition has only one nonempty part corresponding to a single isomorphism class $\langle T\rangle$, then we say that (π, f) is *pure of class* $\langle T\rangle$. At the other extreme, we have the case where (π, f) is "generic". Then the partition of X determined by (π, f) is a stratification. Each stratum $X_{\langle T\rangle}$ is a smooth submanifold of X of codimension $|T|$. The image of (π, f) in $\mathcal{W}(X_{\langle T\rangle})$ is pure of class $\langle T\rangle$.
Let $\mathcal{W}_{\langle T\rangle}(X) \subset \mathcal{W}(X)$ consist of the elements which are pure of class $\langle T\rangle$. Dividing by the concordance relation, we have a contravariant functor $X \mapsto \mathcal{W}_{\langle T\rangle}[X]$. The above observations suggest that the representing space $|\mathcal{W}|$ of $X \mapsto \mathcal{W}[X]$ has a stratified model whose strata are indexed by isomorphism classes $\langle T\rangle$ of finite sets over $\{0, 1, \ldots, d+1\}$, and such that the stratum corresponding to $\langle T\rangle$ is a representing space for $X \mapsto \mathcal{W}_{\langle T\rangle}[X]$. We confirm this in section 5. There is a compatibly stratified model of $|\mathcal{W}_{\mathrm{loc}}|$.

The usefulness of the stratifications of $|\mathcal{W}|$ and $|\mathcal{W}_{\mathrm{loc}}|$ comes from the fact that the strata $|\mathcal{W}_{\langle T\rangle}|$ and $|\mathcal{W}_{\mathrm{loc},\langle T\rangle}|$ represent *genuine* bundle theories. To make this more precise in the case of $|\mathcal{W}|$, let (π, f) be an element of $\mathcal{W}_{\langle T\rangle}(X)$. By definition, the projection from

$$\Sigma_0(\pi, f) \;=\; \Sigma(\pi, f) \,\cap\, f^{-1}(0)$$

to X is then a $|T|$-sheeted covering. Consequently $\Sigma_0(\pi, f)$ is a codimension 0 submanifold of $\Sigma(\pi, f)$, hence a union of connected components of $\Sigma(\pi, f)$. It turns out that the remaining components of $\Sigma(\pi, f)$ are "removable". That is, every class in $\mathcal{W}_{\langle T\rangle}[X]$ has a representative (π, f) with

$$\Sigma_0(\pi, f) = \Sigma(\pi, f)\,.$$

In this situation, $\pi\colon\ E \to X$ is automatically a bundle of $(d+1)$-manifolds. Moreover, for every nonzero $c \in \mathbb{R}$, the restriction of π to $f^{-1}(c)$ is a bundle of closed d-manifolds. When $d = 2$, this brings us back to surface bundles and leads (with the Harer stability theorem) to a proof of theorem 1.3.4.

* * *

For the rest of this paper we consider only the surface case $d = 2$, with additional boundary data in the definitions of $\mathcal{V}$, $\mathcal{W}$ etc., cf. section 2. The reader can easily adapt the arguments of sections 2–5 to the general case $d \geq 2$.

2 Some generalized bundle theories

This section defines the six generalized bundle theories sketched out in the introduction. They are considered to be sheaves on the category $\mathscr{X}$ of smooth manifolds without boundary (of arbitrary dimension) and with smooth maps as morphisms.

2.1 The basic sheaves

Let E be a smooth manifold with boundary and $\pi\colon E \to X$ a smooth map to an object of $\mathscr{X}$. The map π is a *submersion* if its differentials

$$\begin{aligned} d\pi\colon \quad & (TE)_z \to TX_{\pi(z)}\,, \quad z \in E \smallsetminus \partial E \\ d(\pi|\partial E)\colon \quad & (T\partial E)_z \to TX_{\pi(z)}\,, \quad z \in \partial E \end{aligned}$$

are all surjective. In all cases considered below $\pi\colon E \to X$ is assumed to be a product bundle near ∂E. If the submersion π is also proper, then π is a smooth fiber bundle by Ehresmann's fibration theorem [2, thm. 8.12].

Pull-back or base change of bundles or submersions is not strictly associative. To get around this we shall assume that π is a *graphic* map. The rule which to an X in $\mathscr{X}$ associates the set of *graphic* submersions $\pi\colon E \to X$ is a contravariant functor on $\mathscr{X}$.

Definition 2.1.1 Let $f\colon S \to T$ be a map of sets. We say that f is *graphic* if f has a factorization $S \to U \times T \to T$ where the first arrow is an inclusion (not just an injection) and the second arrow is the projection from $U \times T$ to T.

An arbitrary map $f\colon S \to T$ has a *graphic replacement* $\bar{f}\colon \bar{S} \to T$ where $\bar{S} \subset S \times T$ is the graph of f.

Let $f\colon S \to T_2$ be a graphic map and let $g\colon T_1 \to T_2$ be any map. We make a pullback square

$$\begin{tikzcd} g^*S \arrow[r] \arrow[d] & S \arrow[d, "f"] \\ T_1 \arrow[r, "g"] & T_2 \end{tikzcd} \tag{2.1}$$

by letting g^*S consist of all ordered pairs (u,t) such that $t \in T_1$ and $(u, g(t_1)) \in S$. The left hand vertical arrow is given by $(u,t) \mapsto t$ and it is again a graphic map. Moreover, if g is an identity, then $g^*S = S$; and if g is a composition, $g = g_2 g_1$, then $g^*S = {g_1}^* {g_2}^* S$. Thus, with the above definitions, *base change* is associative.

Let E be a smooth manifold and $p^k\colon\ J^k(E,\mathbb{R}) \to E$ the k-jet bundle, where $k \geq 0$. Its fiber at $z \in E$ consists of equivalence classes of smooth map germs $f\colon\ (E,z) \to \mathbb{R}$, with f equivalent to g if the k-th Taylor expansions of f and g agree at z (in local coordinates near z). The elements of $J^k(E,\mathbb{R})$ are called *k-jets* of maps from E to $\mathbb{R}$. The k-jet bundle $p^k\colon\ J^k(E,\mathbb{R}) \to E$ is a vector bundle.

A smooth function $f\colon\ E \to \mathbb{R}$ induces a smooth section $j^k f$ of p^k, which we call the *k-jet prolongation* of f, following e.g. Hirsch [19]. (Some writers choose to call it the *k-jet* of f, which can be confusing.) Not every smooth section of p^k has this form. Sections of the form $j^k f$ are called *integrable.* Thus a smooth section of p^k is integrable if and only if it agrees with the k-jet prolongation of its underlying smooth map $f\colon\ E \to \mathbb{R}$.

We need a fiberwise version $J^k_\pi(E,\mathbb{R})$ of $J^k(E,\mathbb{R})$, fiberwise with respect to a submersion $\pi\colon\ E^{j+r} \to X^j$ with fibers E_x for $x \in X$. In a neighborhood of any $z \in E$ we may choose local coordinates $\mathbb{R}^j \times \mathbb{R}^r$ so that π becomes the projection onto $\mathbb{R}^j$ and $z = (0,0)$. Two smooth map germs $f, g\colon\ (E,z) \to \mathbb{R}$ define the same element of $J^k_\pi(E,\mathbb{R})_z$ if their k-th Taylor expansions in the $\mathbb{R}^r$ coordinates agree at $(0,0)$. Thus $J^k_\pi(E,\mathbb{R})_z$ is a quotient of $J^k(E,\mathbb{R})_z$ and $J^k_\pi(E,\mathbb{R})_z = J^k(E_{\pi(z)},\mathbb{R})$. There is a surjection of vector bundles on E,

$$J^k(E,\mathbb{R}) \longrightarrow J^k_\pi(E,\mathbb{R}).$$

Sections of the bundle projection $p^k_\pi\colon\ J^k_\pi(E,\mathbb{R}) \to E$ will be denoted $\hat{f}$, $\hat{g}$ and their underlying smooth functions from E to $\mathbb{R}$ by f, g, etc.

A smooth function $f\colon\ E \to \mathbb{R}$ induces a section $j^k_\pi f$ of p^k_π, which we call the *fiberwise k-jet prolongation* of f. The sections of the form $j^k_\pi f$ are called *integrable.*

We now take $k = 2$ and introduce the following (standard)

Notation 2.1.2 **(i)** A section $\hat{f}$ of p^2_π is *fiberwise nonsingular* if $\hat{f}(z) \in J^2(E_{\pi(z)},\mathbb{R})$ has a non-vanishing linear part, for each $z \in E$.

(ii) A section $\hat{f}$ of p^2_π is *fiberwise Morse* if each value $\hat{f}(z)$ is either nonsingular or, when singular, has a non-degenerate quadratic part.

(iii) A smooth map $f\colon\ E \to \mathbb{R}$ is *fiberwise nonsingular*, resp. *fiberwise Morse*, if $j^2_\pi f$ is fiberwise nonsingular, resp. Morse.

(iv) The singularity set $\Sigma(\pi, \hat{f}) \subset E$ is the set of points z with $\hat{f}(z)$ singular. If $\hat{f} = j^2_\pi f$, then we write $\Sigma(\pi, f)$ instead of $\Sigma(\pi, \hat{f})$.

Let $\Sigma_\pi(E,\mathbb{R}) \subset J^2_\pi(E,\mathbb{R})$ be the submanifold consisting of the singular jets, i.e., those with vanishing linear part. Then for a section $\hat{f}$ of p^2_π, we have

$$\Sigma(\pi, \hat{f}) = \hat{f}^{-1}(\Sigma_\pi(E,\mathbb{R}))\,,$$

so that $\hat{f}$ is fiberwise nonsingular if and only if it misses $\Sigma_\pi(E,\mathbb{R})$. For integrable $\hat{f}$, we can also say that $\hat{f}$ is fiberwise Morse if and only if it is fiberwise transverse to $\Sigma_\pi(E,\mathbb{R})$. See [12, II.6.1-4]. This has the following consequence.

Lemma 2.1.3 *Suppose that $f\colon E \to \mathbb{R}$ is fiberwise Morse. Then the restriction of π to $\Sigma(\pi,f)$ is a local diffeomorphism $\Sigma(\pi,f) \to X$.*

Proof The assumption implies that the fiberwise differential $d_\pi f$ viewed as a section of the vertical cotangent bundle $T^*_\pi E \to E$ is transverse to the zero section. In particular $\Sigma = \Sigma(\pi,f)$ is a submanifold of E, of the same dimension as E. But moreover, the fiberwise Morse condition implies that for each $z \in \Sigma$, the tangent space $(T\Sigma)_z$ has trivial intersection in $(TE)_z$ with the vertical tangent space $(T^\pi E)_z$. This means that Σ is transverse to each fiber of π, and also that $\pi|\Sigma$ is a local diffeomorphism.

It is customary to call local diffeomorphisms for *étale* maps. We will follow this tradition:

Definition 2.1.4 A smooth map $p\colon Y \to X$ between smooth manifolds of the same dimension is called *étale* if its differential at every point $y \in Y$ is a linear isomorphism from $(TY)_y$ to $TX_{\pi(z)}$.

Recall from Ehresmann's fibration lemma, [2, 8.12], that a proper submersion is a fiber bundle, and in particular that a proper étale map is a covering projection.

We now define a number of sheaves on the category $\mathscr{X}$, that is, contravariant functors $\mathcal{F}$ from $\mathscr{X}$ to the category of sets, which satisfy the following condition: for every X in $\mathscr{X}$ and open cover $\{Y_i\}_{i\in I}$ of X, the sequence

$$\star \longrightarrow \mathcal{F}(X) \longrightarrow \textstyle\prod_{i\in I} \mathcal{F}(Y_i) \rightrightarrows \textstyle\prod_{i,j\in I\times I} \mathcal{F}(Y_i \cap Y_j)$$

is exact. In other words, given $s_i \in \mathcal{F}(Y_i)$ for $i \in I$ such that $s_i|Y_i \cap Y_j = s_j|Y_i \cap Y_j$, there exists a unique $s \in \mathcal{F}(X)$ with $s|Y_i = s_i$.

Definition 2.1.5 For an object X in $\mathscr{X}$, let $h\mathcal{V}(X)$ be the set of pairs $(\pi,\hat{f})$ where $\pi\colon E \to X$ is a graphic submersion with oriented 3-dimensional fibers and $\hat{f}$ is a section of $p^2_\pi\colon J^2_\pi(E,\mathbb{R}) \to E$, subject to the following three conditions:

(i) $(\pi,f)\colon E \to X \times \mathbb{R}$ is proper.

(ii) $\hat{f}$ is nonsingular.

(iii) near their respective boundaries, E and $(S^1 \times \mathbb{R} \times [0,1]) \times X$ agree. Moreover π is the standard projection to X and $\hat{f}$ is the jet prolongation of the projection to $\mathbb{R}$.

Analogously we have an integrable version $\mathcal{V}(X) \subset h\mathcal{V}(X)$, consisting of pairs (π, f) where $f\colon E \to \mathbb{R}$ is a smooth function such that $(\pi, j^2_\pi f) \in h\mathcal{V}(X))$. This means that for each $x \in X$ the restriction of f to the fiber E_x is nonsingular. Thus $(\pi, f)\colon E \to X \times \mathbb{R}$ is a proper submersion, hence a smooth fiber bundle. Each fiber is an oriented surface with two boundary circles, but it need not be connected.

Definition 2.1.6 Let $\mathcal{V}_c(X) \subset \mathcal{V}(X)$ denote the subset of pairs (π, f) where the fibers of $(\pi, f)\colon E \to X \times \mathbb{R}$ are connected.

Definition 2.1.7 For X in $\mathscr{X}$ let $h\mathcal{W}(X)$ be the set of pairs $(\pi, \hat{f})$, as in definition 2.1.5, which satisfy conditions (i) and (iii), but where condition (ii) is replaced by the weaker condition

(iia) $\hat{f}$ is fiberwise Morse.

Again we have an integrable version $\mathcal{W}(X) \subset h\mathcal{W}(X)$ consisting of pairs (π, f) where $f\colon E \to \mathbb{R}$ is a smooth function such that $(\pi, j^2_\pi f) \in h\mathcal{W}(X)$. The integrability condition means that, for each $x \in X$, the restriction of f to the fiber E_x is a Morse function.

Definition 2.1.8 For X in $\mathscr{X}$ let $h\mathcal{W}_{\mathrm{loc}}(X)$ be the set of pairs $(\pi, \hat{f})$, as in definition 2.1.5, which satisfy condition (iii), but where conditions (i) and (ii) are replaced by

(ia) the map $\Sigma(\pi, \hat{f}) \to X \times \mathbb{R}$; $z \mapsto (\pi(z), f(z))$ is proper,

(iia) $\hat{f}$ is fiberwise Morse.

The integrable version of $h\mathcal{W}_{\mathrm{loc}}(X)$ is denoted $\mathcal{W}_{\mathrm{loc}}(X)$. Making X into a variable, we have contravariant functors $h\mathcal{V}$, $h\mathcal{W}$, $h\mathcal{W}_{\mathrm{loc}}$ on $\mathscr{X}$ and their integrable versions $\mathcal{V}$, $\mathcal{W}$, $\mathcal{W}_{\mathrm{loc}}$. This uses base change for graphic maps to X (here smooth submersions $\pi\colon E \to X$) as defined in 2.1.1. All six functors have the sheaf property. They fit together into the diagram of sheaves

$$\begin{tikzcd} \mathcal{V} \arrow[r] \arrow[d, "j^2_\pi"'] & \mathcal{W} \arrow[r] \arrow[d, "j^2_\pi"'] & \mathcal{W}_{\mathrm{loc}} \arrow[d, "j^2_\pi"'] \\ h\mathcal{V} \arrow[r] & h\mathcal{W} \arrow[r] & h\mathcal{W}_{\mathrm{loc}}. \end{tikzcd} \tag{2.2}$$

2.2 Homotopy theory of sheaves

Given a sheaf $\mathcal{F}$ on $\mathscr{X}$ we want to consider for each X in $\mathscr{X}$ the concordance classes of elements of $\mathcal{F}(X)$. This requires that we extend $\mathcal{F}$ to be defined on manifolds with boundary. It suffices to consider manifolds with collared boundary. For such X we have a canonical projection $p\colon Y \smallsetminus \partial X \to \partial X$ for a sufficiently small open neighborhood Y of ∂X in X.

Definition 2.2.1 Let X be smooth with collared boundary. Let $\mathcal{F}(X)$ be the set of all pairs $(r,s) \in \mathcal{F}(X \smallsetminus \partial X) \times \mathcal{F}(\partial X)$ such that r and $p^*(s)$ agree on $Y \smallsetminus \partial X$, for a sufficiently small open neighbourhood Y of ∂X in X. (Here p is the collar projection.)

Definition 2.2.2 Let $\mathcal{F}$ be a sheaf on $\mathscr{X}$, let X be an object of $\mathscr{X}$ and let $s_0, s_1 \in \mathcal{F}(X)$. A *concordance* from s_0 to s_1 is an element (h,s) of $\mathcal{F}(X \times [0,1])$ such that $s \mapsto (s_0, s_1)$ under the canonical bijection $\mathcal{F}(X \times \partial[0,1]) \to \mathcal{F}(X) \times \mathcal{F}(X)$.

We also say that $h \in \mathcal{F}(X \times]0,1[\,)$ is a concordance from s_0 to s_1. If such a concordance exists, then s_0 and s_1 are said to be *concordant* and we write $s_0 \simeq s_1$ or $h\colon s_0 \simeq s_1$. Concordance is an equivalence relation on $\mathcal{F}(X)$. The set of equivalence classes will be denoted by $\mathcal{F}[X]$.

It is necessary to have a relative version of $\mathcal{F}[X]$. Suppose that $A \subset X$ is a closed subset, where X is in $\mathscr{X}$. Let $s \in \operatorname{colim}_U \mathcal{F}(U)$ where U ranges over the open neighborhoods of A in $\mathscr{X}$. Note for example that any $z \in \mathcal{F}(\star)$ gives rise to such an element, namely $s = \{p_U^*(z)\}$ where $p_U\colon U \to \star$. In this case we often write z instead of s.

Definition 2.2.3 Let $\mathcal{F}(X, A; s) \subset \mathcal{F}(X)$ consist of the elements $t \in \mathcal{F}(X)$ whose germ near A is equal to s. Two such elements t_0 and t_1 are concordant *relative to* A if they are concordant by a concordance whose germ near A is the constant concordance from s to s. The equivalence classes are denoted $\mathcal{F}[X, A; s]$.

We now construct the representing space $|\mathcal{F}|$ of $\mathcal{F}$ and list its most important properties.

Let $\mathbf{\Delta}$ be the category whose objects are the ordered sets $\underline{n} := \{0, 1, 2, \ldots, n\}$ for $n \geq 0$, with order preserving maps as morphisms. For $n \geq 0$ let $\Delta_e^n \subset \mathbb{R}^{n+1}$ be the extended standard n-simplex,

$$\Delta_e^n := \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \Sigma x_i = 1\}.$$

An order-preserving map $\underline{m} \to \underline{n}$ induces a map of affine spaces $\Delta_e^m \to \Delta_e^n$. This makes $\underline{n} \mapsto \Delta_e^n$ into a covariant functor from Δ to $\mathscr{X}$.

Definition 2.2.4 The *representing space* $|\mathcal{F}|$ of a sheaf $\mathcal{F}$ on $\mathscr{X}$ is the geometric realization of the simplicial set $\underline{n} \mapsto \mathcal{F}(\Delta_e^n)$.

An element $z \in \mathcal{F}(\star)$ gives a point $z \in |\mathcal{F}|$ and $\mathcal{F}[\star] = \pi_0|\mathcal{F}|$. In appendix A we prove that $|\mathcal{F}|$ represents the contravariant functor $X \mapsto \mathcal{F}[X]$. Indeed we prove the following slightly more general

Proposition 2.2.5 *For X in $\mathscr{X}$, let $A \subset X$ be closed and $z \in \mathcal{F}(\star)$. There is a natural bijection ϑ from the set of homotopy classes of maps $(X, A) \to (\,|\mathcal{F}|, z)$ to the set $\mathcal{F}[X, A; z]$.*

Taking $X = S^n$ and A equal to the base point, we see that the homotopy group $\pi_n(|\mathcal{F}|, z)$ is identified with the set of concordance classes $\mathcal{F}[S^n, \star; z]$. We introduce the notation

$$\pi_n(\mathcal{F}, z) := \mathcal{F}[S^n, \star; z]\,.$$

A map $v\colon\ \mathcal{E} \to \mathcal{F}$ of sheaves induces a map $|v|\colon\ |\mathcal{E}| \to |\mathcal{F}|$ of representing spaces. We call v a *weak equivalence* if $|v|$ is a homotopy equivalence.

Proposition 2.2.6 *Let $v\colon\ \mathcal{E} \to \mathcal{F}$ be a map of sheaves on $\mathscr{X}$. Suppose that v induces a surjective map*

$$\mathcal{E}[X, A; s] \longrightarrow \mathcal{F}[X, A; v(s)]$$

for every X in $\mathscr{X}$ with a closed subset $A \subset X$ and a germ $s \in \mathrm{colim}_U \mathcal{E}(U)$, where U ranges over the neighborhoods of A in X. Then v is a weak equivalence.

Proof The hypothesis implies easily that the induced map $\pi_0\mathcal{E} \to \pi_0\mathcal{F}$ is onto and that, for any choice of base point $z \in \mathcal{E}(\star)$, the map of concordance sets $\pi_n(\mathcal{E}, z) \to \pi_n(\mathcal{F}, v(z))$ induced by v is bijective. □

Applying the representing space construction to the sheaves displayed in diagram (2.2), but using $\mathcal{V}_c$ instead of $\mathcal{V}$, we get a commutative diagram of representing spaces

$$\begin{tikzcd}
|\mathcal{V}_c| \arrow[r] \arrow[d, "j^2_\pi"'] & |\mathcal{W}| \arrow[r] \arrow[d, "j^2_\pi"'] & |\mathcal{W}_{\mathrm{loc}}| \arrow[d, "j^2_\pi"'] \\
|h\mathcal{V}| \arrow[r] & |h\mathcal{W}| \arrow[r] & |h\mathcal{W}_{\mathrm{loc}}|.
\end{tikzcd} \tag{2.3}$$

Note that $|\mathcal{V}_c| \simeq \coprod_{g=0}^{\infty} B\Gamma_{g,2}$.

2.3 Different models and monoid structures

Let $\mathcal{F}$ be one of the sheaves from section 2.1. Concatenation along a boundary component defines a composition law

$$\mathcal{F}[X] \times \mathcal{F}[X] \longrightarrow \mathcal{F}[X]$$

so that each of the spaces in diagram (2.3) comes equipped with a multiplication which is homotopy associative and with a homotopy unit. Our first task is to give an upgraded version of the sheaves that turns their values into monoids, and therefore makes the representing spaces into topological monoids (without unit). We describe this in detail for $\mathcal{W}$ and leave the other cases to the reader.

Let $t\colon X \to]0, \infty[$ be a smooth function. We define

$$[0,t] \times X := \{(s,x) \in \mathbb{R} \times X \mid 0 \le s \le t(x)\}.$$

Definition 2.3.1 For X in $\mathscr{X}$ let $\mathcal{W}'(X)$ be the set of quadruples (t, u, π, f) where t is a function as above, $(u,\pi)\colon E \to [0,t] \times X$ is a smooth graphic map whose X–coordinate is a submersion $\pi\colon E \to X$ with 3-dimensional fibers, and $f\colon E \to \mathbb{R}$ is a smooth map, subject to the following conditions.

(i) $(\pi, f)\colon E \to X \times \mathbb{R}$ is proper.

(ii) For each $x \in X$ the restriction $f_x\colon E_x \to \mathbb{R}$ is Morse.

(iii) Near their respective boundaries, the manifolds E and $(S^1 \times \mathbb{R}) \times ([0,t] \times X)$ agree, and u, π, f agree there with the obvious projections.

There is a monoid structure on $\mathcal{W}'(X)$. Indeed, for (t,u,π,f) and (t',u',π',f') in $\mathcal{W}'(X)$ one defines

$$(t,u,\pi,f) \circ (t',u',\pi',f') = (t+t', u'', \pi'', f''). \tag{2.4}$$

Here the source of π'' is the union (concatenation) of E and σ^*E', where σ^* denotes the base change, as in (2.1), along the translation homeomorphism

$$\sigma\colon [t, t+t'] \times X \longrightarrow [0,t'] \times X\,; \qquad (s,x) \mapsto (s - t(x), x)\,.$$

The maps π'' and f'' are defined by

$$\pi'' = \pi \cup \sigma^*\pi'\,, \qquad f'' = f \cup \sigma^* f'\,.$$

There is no unit for the product in (2.4), since we assumed $t > 0$ in definition 2.3.1. As a result the representing space $|\mathcal{W}'|$ becomes a topological monoid without a strict unit. However, the classifying space construction $B|\mathcal{W}'|$ and hence the group completion $\Omega B|\mathcal{W}'|$ make perfectly good sense. We also describe a way to attach an artificial unit to monoids without unit in appendix C.

Lemma 2.3.2 *The sheaves $\mathcal{W}'$ and $\mathcal{W}$ are homotopy equivalent.*

Proof There is a subsheaf $\mathcal{W}''$ of $\mathcal{W}'$ which we obtain by allowing only the constant function t with value 1 in definition 2.3.1. The inclusion $\mathcal{W}'' \to \mathcal{W}'$ is a weak equivalence, and so is the forgetful map from $\mathcal{W}''$ to $\mathcal{W}$. □

There are similar enlarged models for the other sheaves of section 2.1, so diagram (2.3) is equivalent to a diagram of topological monoids and monoid maps. In the rest of the paper, we will not make explicit use of these larger models with monoid structures: it is usually enough to know that they exist.

We next introduce sheaves $\mathcal{W}^0$ and $h\mathcal{W}^0$ on $\mathscr{X}$. They are weakly equivalent to $\mathcal{W}$ and $h\mathcal{W}$, respectively, but are better related to Vassiliev's h-principle, see [42, Thm.0.A] and [41, III, 1.1], than $\mathcal{W}$ and $h\mathcal{W}$.

Definition 2.3.3 For X in $\mathscr{X}$ let $h\mathcal{W}^0(X)$ be the set of all pairs $(\pi, \hat{f})$ as in definition 2.1.7, replacing however condition (iia) by the weaker

(iib) $\hat{f}$ is fiberwise Morse in some neighborhood of $f^{-1}(0)$.

From the definition, there is an inclusion $h\mathcal{W} \to h\mathcal{W}^0$. There are also an integrable version $\mathcal{W}^0$ and an inclusion $\mathcal{W} \to \mathcal{W}^0$.

Lemma 2.3.4 *The inclusions $\mathcal{W} \to \mathcal{W}^0$ and $h\mathcal{W} \to h\mathcal{W}^0$ are homotopy equivalences.*

Proof We will concentrate on the first of the two inclusions, $\mathcal{W} \to \mathcal{W}^0$. Fix (π, f) in $\mathcal{W}^0(X)$, with $\pi\colon E \to X$ and $f\colon E \to \mathbb{R}$. We will subject (π, f) to a concordance ending in $\mathcal{W}(X)$. Choose an open neighborhood U of $f^{-1}(0)$ in E such that, for each $x \in X$, the critical points of $f_x = f|E_x$ on $E_x \cap U$ are all nondegenerate. Since $E \smallsetminus U$ is closed in E and the map $(\pi, f)\colon E \to X \times \mathbb{R}$ is proper, the image of $E \smallsetminus U$ under that map is a closed subset of $X \times \mathbb{R}$ which has empty intersection with $X \times 0$. (Proper maps between locally compact spaces are closed maps.) We can therefore choose a smooth function $\varphi\colon X \to]0, 1]$ such that U contains all $z \in E$ for which $|f(z)| < \varphi(\pi(z))$. And we can choose a smooth isotopy of embeddings $\iota_t\colon X \times \mathbb{R} \to X \times \mathbb{R}$, where $0 \le t \le 1$, such that

(i) each ι_t is a map over X,

(ii) $\iota_0 = \mathrm{id}$ and $\iota_1(X \times \mathbb{R}) \quad \subset \quad \{(x, t) \mid -\varphi(x) < t < \varphi(x)\}$,

(iii) $\iota_t = \iota_0$ for t close to 0 and $\iota_t = \iota_1$ for t close to 1.

Then let $E^{(t)}$ be the inverse image of $\iota_t(X \times \mathbb{R})$ under the map $(\pi, f)\colon E \to X \times \mathbb{R}$. Let $f^{(t)}\colon E^{(t)} \to \mathbb{R}$ be the second coordinate of ${\iota_t}^{-1}$ following on $(\pi, f)\colon E \to X \times \mathbb{R}$. Let $\pi^{(t)}$ be the restriction of π to $E^{(t)}$. Now

$$t \mapsto (\pi^{(t)}, f^{(t)})$$

defines a concordance from $(\pi, f) \in \mathcal{W}^0(X)$ to an element in $\mathcal{W}(X)$. (Strictly speaking, some renaming of some of the elements of $E^{(t)}$ for $0 \le t \le 1$ is required because of the boundary conditions in the definitions.) If the restriction of (π, f) to an open neighborhood Y_1 of a closed $A \subset X$ belongs to $\mathcal{W}(Y_1)$, then the concordance can be made relative to Y_0, where Y_0 is a smaller open neighborhood of A in X. $\square$

For later use we list

Lemma 2.3.5 *If X in $\mathscr{X}$ is compact, then every class in $\mathcal{W}[X]$ or $h\mathcal{W}[X]$ has a representative (π, f), resp. $(\pi, \hat{f})$, in which $f\colon E \to \mathbb{R}$ is a bundle projection, so that*

$$E \cong f^{-1}(0) \times \mathbb{R}\,.$$

Proof We concentrate on the first case. First pick a representative $(\pi, f) \in \mathcal{W}(X)$ such that $f\colon E \to \mathbb{R}$ has 0 as a regular value. Since $(\pi, f)\colon E \to X \times \mathbb{R}$ is proper and X is compact, f itself is proper and therefore closed. It follows that there exists an $\varepsilon > 0$ such that all $s \in [-\varepsilon, \varepsilon]$ are regular values for f. Now choose an isotopy of embeddings $\iota_t\colon \mathbb{R} \to \mathbb{R}$, where $0 \le t \le 1$, such that ι_0 is the identity and $\iota_1(\mathbb{R}) \subset [-\varepsilon, \varepsilon]$. Here t runs from 0 to 1 and the isotopy is stationary near $t = 0$ and $t = 1$. The shrinking argument of the previous lemma gives a concordance from (π, f) to an element $(\pi^{(1)}, f^{(1)})$, where the source $E^{(1)}$ of $\pi^{(1)}$ is $f^{-1}(\iota_1(\mathbb{R}))$ and $f^{(1)}$ equals $f|E^{(1)}$ followed by the inverse of ι_1. Since $f|E^{(1)}$ is regular, $f^{(1)}$ is regular. Since $f^{(1)}$ is also proper, $f^{(1)}$ is a proper submersion, i.e., a bundle projection. □

3 The spaces of diagram (2.3)

This section determines the homotopy types of the spaces of (2.3), save the space $|\mathcal{W}|$ which is deferred to section 4.

3.1 A cofiber sequence of Thom spectra

Let $\mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+n})$ be the manifold of triples (V, ℓ, q) consisting of an oriented 3-dimensional linear subspace $V \subset \mathbb{R}^{3+n}$, a linear map $\ell\colon V \to \mathbb{R}$ and a quadratic form $q\colon V \to \mathbb{R}$, subject to the condition that if $\ell = 0$, then q is nondegenerate. $\mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+n})$ classifies 3-dimensional oriented vector bundles whose fibers have the above extra structure, i.e., each fiber V comes equipped with a Morse type map $\ell + q\colon V \to \mathbb{R}$ and with a linear embedding into $\mathbb{R}^{3+n}$.

Let $S_3(\mathbb{R})$ be the vector space of quadratic forms on $\mathbb{R}^3$ (or equivalently, symmetric 3×3 matrices) and $\Delta \subset S_3(\mathbb{R})$ the subspace of the degenerate forms (not a linear subspace). The complement $Q(\mathbb{R}^3) = S_3(\mathbb{R}) \smallsetminus \Delta$ is the space of non-degenerate quadratic forms on $\mathbb{R}^3$, and

$$A^2(\mathbb{R}^3) = (\mathbb{R}^3)^* \times S_3(\mathbb{R}) \smallsetminus (0 \times \Delta)$$

is precisely the space of pairs (ℓ, q) as above, where $\ell + q\colon \mathbb{R}^3 \to \mathbb{R}$ is a Morse type map. The group $\mathrm{GL}(\mathbb{R}^3)$ acts on the right of $A^2(\mathbb{R}^3)$ by $(\ell, q) \cdot g = (\ell g, qg)$. Restricting this action to $\mathrm{SO}(3)$ we have

$$\mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+n}) \cong \big(\mathrm{O}(3+n)/\mathrm{O}(n) \times A^2(\mathbb{R}^3)\big)/\mathrm{SO}(3)\,. \tag{3.1}$$

We turn to a description of the homotopy type of $A^2(\mathbb{R}^3)$ in more familiar terms. Since quadratic forms can be diagonalized,

$$Q(\mathbb{R}^3) = \coprod_{i=0}^{3} Q(i, 3-i)$$

where $Q(i, 3-i)$ is the connected component containing the form q_i given by

$$q_i(x_1, x_2, x_3) = -(x_1^2 + \cdots + x_i^2) + (x_{i+1}^2 + \cdots + x_3^2).$$

The stabilizer $\mathrm{O}(i, 3-i)$ of q_i for the (transitive) action of $\mathrm{GL}_3(\mathbb{R})$ on $Q(i, 3-i)$ has $\mathrm{O}(i) \times \mathrm{O}(3-i)$ as a maximal compact subgroup and $\mathrm{GL}_3(\mathbb{R})$ has $\mathrm{O}(3)$ as a maximal compact subgroup. Hence the inclusion

$$(\mathrm{O}(i) \times \mathrm{O}(3-i))\backslash\mathrm{O}(3) \longrightarrow Q(i, 3-i)\,; \qquad \text{coset of } g \;\mapsto\; q_i g$$

is a homotopy equivalence, and therefore the subspace

$$\begin{aligned} Q^0(\mathbb{R}^3) &= \{q_0, q_1, q_2, q_3\} \cdot \mathrm{O}(3) \\ &\cong \textstyle\coprod_{i=0}^{3} (\mathrm{O}(i) \times \mathrm{O}(3-i))\backslash\mathrm{O}(3) \\ &\cong \star \amalg \mathbb{R}P^2 \amalg \mathbb{R}P^2 \amalg \star \end{aligned} \tag{3.2}$$

of $Q(\mathbb{R}^3)$ is a deformation retract, $Q(\mathbb{R}^3) \simeq Q^0(\mathbb{R}^3)$.

Lemma 3.1.1 *There is a homotopy equivalence from the join $S^2 * Q^0(\mathbb{R}^3)$ to $A^2(\mathbb{R}^3)$ which is equivariant for the actions of $\mathrm{O}(3)$.*

Proof The space $A^2(\mathbb{R}^3)$ is the union of $((\mathbb{R}^3)^* - 0) \times S_3(\mathbb{R})$ and $(\mathbb{R}^3)^* \times Q(\mathbb{R}^3)$ with intersection $((\mathbb{R}^3)^* - 0) \times Q(\mathbb{R}^3)$. Since $S_3(\mathbb{R})$ and $(\mathbb{R}^3)^*$ have canonical contractions and since the inclusion $Q^0(\mathbb{R}^3) \to Q(\mathbb{R}^3)$ is a homotopy equivalence, we get a canonical homotopy equivalence from the double mapping cylinder (alias homotopy colimit) of the diagram

$$S^2 \longrightarrow S^2 \times Q^0(\mathbb{R}^3) \longrightarrow Q^0(\mathbb{R}^3)$$

to $A^2(\mathbb{R}^3)$. The homotopy colimit is precisely the join $S^2 * Q^0(\mathbb{R}^3)$ and the map respects the $\mathrm{O}(3)$-actions. □

The tautological 3-dimensional vector bundle $U_{\mathcal{W},n}$ on $\mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+n})$ is canonically embedded in a trivial bundle $\mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+n}) \times \mathbb{R}^{3+n}$. Let

$$U_{\mathcal{W},n}^{\perp} \;\subset\; \mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+n}) \times \mathbb{R}^{3+n}$$

be the orthogonal complement, an n-dimensional vector bundle on $\mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+n})$.

The tautological bundle $U_{\mathcal{W},n}$ comes equipped with the extra structure consisting of a map from (the total space of) $U_{\mathcal{W},n}$ to $\mathbb{R}$ which, on each fiber of $U_{\mathcal{W},n}$, is a Morse type map. (The fiber of $U_{\mathcal{W},n}$ over a point $(V, q, \ell) \in \mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+n})$ is identified with the 3-dimensional vector space V and the map can then be described as $\ell + q$.)

For the submanifold $\Sigma_{\mathcal{W},n} \subset \mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+n})$ consisting of the triples (V, ℓ, q) with $\ell = 0$ we have

$$\Sigma_{\mathcal{W},n} \;\cong\; \big(\mathrm{O}(3+n)/\mathrm{O}(n) \times Q(\mathbb{R}^3)\big)/\mathrm{SO}(3)\,. \tag{3.3}$$

The restriction of $U_{\mathcal{W},n}$ to $\Sigma_{\mathcal{W},n}$ comes equipped with the extra structure of a fiberwise nondegenerate quadratic form. There is a canonical normal bundle for $\Sigma_{\mathcal{W},n}$ in $\mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+n})$ which is identified with $U^*_{\mathcal{W},n}|\Sigma_{\mathcal{W},n}$. Hence there is a homotopy cofiber sequence

$$\mathrm{Gr}_{\mathcal{V}}(\mathbb{R}^{3+n}) \hookrightarrow \mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+n}) \longrightarrow \mathrm{Th}\,(U^*_{\mathcal{W},n}|\Sigma_{\mathcal{W},n})$$

where $\mathrm{Gr}_{\mathcal{V}}(\mathbb{R}^{3+n}) = \mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+n}) \smallsetminus \Sigma_{\mathcal{W},n}$ and $\mathrm{Th}\,(\dots)$ denotes the Thom space. This leads to a homotopy cofiber sequence of Thom spaces

$$\mathrm{Th}\,(U^{\perp}_{\mathcal{W},n}|\mathrm{Gr}_{\mathcal{V}}(\mathbb{R}^{3+n})) \longrightarrow \mathrm{Th}\,(U^{\perp}_{\mathcal{W},n}) \longrightarrow \mathrm{Th}\,(U^{\perp}_{\mathcal{W},n} \oplus U^*_{\mathcal{W},n}|\Sigma_{\mathcal{W},n})$$

which, as n varies, becomes a homotopy cofiber sequence of spectra

$$\mathbf{hV} \longrightarrow \mathbf{hW} \longrightarrow \mathbf{hW}_{\mathrm{loc}}.$$

Here we view $\mathrm{Th}\,(U^{\perp}_{\mathcal{W},n})$ as the $(2+n)$-th space of the spectrum $\mathbf{hW}$, and similarly for the other two spectra. We then have the corresponding infinite loop spaces

$$\begin{aligned} \Omega^{\infty}\mathbf{hV} &= \operatorname{colim} \Omega^{2+n}\mathrm{Th}\,(U^{\perp}_{\mathcal{W},n}|\mathrm{Gr}_{\mathcal{V}}(\mathbb{R}^{3+n}))\,, \\ \Omega^{\infty}\mathbf{hW} &= \operatorname{colim} \Omega^{2+n}\mathrm{Th}\,(U^{\perp}_{\mathcal{W},n})\,, \\ \Omega^{\infty}\mathbf{hW}_{\mathrm{loc}} &= \operatorname{colim} \Omega^{2+n}\mathrm{Th}\,(U^{\perp}_{\mathcal{W},n} \oplus U^*_{\mathcal{W},n}|\Sigma_{\mathcal{W},n}). \end{aligned}$$

The homotopy cofiber sequence of spectra above yields a homotopy fiber sequence of infinite loop spaces

$$\Omega^{\infty}\mathbf{hV} \longrightarrow \Omega^{\infty}\mathbf{hW} \longrightarrow \Omega^{\infty}\mathbf{hW}_{\mathrm{loc}}\ , \tag{3.4}$$

that is, $\Omega^{\infty}\mathbf{hV}$ is homotopy equivalent to the homotopy fiber of the right-hand map. In particular there is a long exact sequence of homotopy groups associated with diagram (3.4) and a Serre-Leray spectral sequence of homology groups.

Lemma 3.1.2 *There is a homotopy equivalence of infinite loop spaces*

$$\Omega^{\infty}\mathbf{hW}_{\mathrm{loc}} \simeq \Omega^{\infty}S^{1+\infty}(\Sigma_{\mathcal{W},\infty})_+$$

where $\Sigma_{\mathcal{W},\infty} \simeq B\mathrm{SO}(3) \amalg B\mathrm{O}(2) \amalg B\mathrm{O}(2) \amalg B\mathrm{SO}(3)$.

Proof Since $U_{\mathcal{W},n}|\Sigma_{\mathcal{W},n}$ comes equipped with a fiberwise nondegenerate quadratic form, $U^*_{\mathcal{W},n}|\Sigma_{\mathcal{W},n}$ is canonically identified with $U_{\mathcal{W},n}|\Sigma_{\mathcal{W},n}$. Consequently the restriction

$$U^{\perp}_{\mathcal{W},n} \oplus U^*_{\mathcal{W},n} \,\big|\, \Sigma_{\mathcal{W},n}$$

is trivialized, so that $\mathrm{Th}\,(U^{\perp}_{\mathcal{W},n} \oplus U^*_{\mathcal{W},n} \,\big|\, \Sigma_{\mathcal{W},n}) \simeq S^{3+n}(\Sigma_{\mathcal{W},n})_+$. Hence

$$\Omega^{\infty}\mathbf{hW}_{\mathrm{loc}} \simeq \Omega^{\infty}S^{1+\infty}(\Sigma_{\mathcal{W},\infty})_+$$

where $\Sigma_{\mathcal{W},\infty} = \bigcup \Sigma_{\mathcal{W},n}$. Using the description (3.3) of $\Sigma_{\mathcal{W},n}$ and the homotopy equivalence $Q(\mathbb{R}^3) \simeq Q^0(\mathbb{R}^3)$, see (3.2), we get

$$\Sigma_{\mathcal{W},n} \simeq \big(\mathrm{O}(3+n)/\mathrm{O}(n)) \times Q^0(\mathbb{R}^3)\big)/\mathrm{SO}(3).$$

The union $\bigcup_n \mathrm{O}(3{+}n)/\mathrm{O}(n)$ is a contractible free $\mathrm{SO}(3)$-space, so that $\Sigma_{\mathcal{W},\infty}$ is homotopy equivalent to the homotopy orbit space of the canonical right action of $\mathrm{SO}(3)$ on

$$Q^0(\mathbb{R}^3) \;\cong\; \coprod_{i=0}^{3} (\mathrm{O}(i) \times \mathrm{O}(3-i))\backslash \mathrm{O}(3)\,. \qquad \square$$

The Grassmann manifold $\mathrm{Gr}_2(\mathbb{R}^{2+n})$ of oriented 2-planes P in $\mathbb{R}^{2+n}$ can be identified with a subspace of $\mathrm{Gr}_{\mathcal{V}}(\mathbb{R}^{3+n}) = \mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+n}) \smallsetminus \Sigma_{\mathcal{W},n}$, by $P \mapsto (\mathbb{R} \oplus P, \mathrm{pr}_{\mathbb{R}}, 0)$. The injection is covered by a monomorphism of vector bundles

$$L_n^{\perp} \to U_{\mathcal{W},n}^{\perp} \,|\, \mathrm{Gr}_{\mathcal{V}}(\mathbb{R}^{3+n})$$

where $L_n^{\perp}$ is the standard n-plane bundle on $\mathrm{Gr}_2(\mathbb{R}^{2+n})$.

Lemma 3.1.3 *The space* $\mathrm{Gr}_{\mathcal{V}}(\mathbb{R}^{3+n})$ *is homotopy equivalent to* $\mathrm{SO}(3{+}n)/\mathrm{SO}(2){\times}\mathrm{SO}(n)$, *and the map* $\mathrm{Th}\,(L_n^{\perp}) \longrightarrow \mathrm{Th}\,(U_{\mathcal{W},n}^{\perp} \,|\, \mathrm{Gr}_{\mathcal{V}}(\mathbb{R}^{3+n}))$ *just constructed is* $(2n+1)$*-connected.*

Proof From (3.1) and lemma 3.1.1 we have an embedding and a homotopy equivalence

$$\big(\mathrm{O}(3+n)/\mathrm{O}(n) \times S^2\big)/\mathrm{SO}(3) \quad\longrightarrow\quad \mathrm{Gr}_{\mathcal{V}}(\mathbb{R}^{3+n})$$

where $\big(\mathrm{O}(3+n)/\mathrm{O}(n) \times S^2\big)/\mathrm{SO}(3) \cong \mathrm{O}(3+n)/(\mathrm{SO}(2) \times \mathrm{O}(n))$. Using this as an identification, we may identify the above embedding $\mathrm{Gr}_2(\mathbb{R}^{2+n}) \to \mathrm{Gr}_{\mathcal{V}}(\mathbb{R}^{3+n})$ with the inclusion

$$\mathrm{O}(2+n)/(\mathrm{SO}(2) \times \mathrm{O}(n)) \quad\longrightarrow\quad \mathrm{O}(3+n)/(\mathrm{SO}(2) \times \mathrm{O}(n))\,.$$

This is $(n+1)$-connected. Passing to the corresponding map between Thom spaces raises the connectivity by n. $\square$

We collect the results of this section, 3.1, in

Proposition 3.1.4 *The homotopy fiber sequence (3.4) is homotopy equivalent to*

$$\Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1} \longrightarrow \Omega^\infty \mathbf{hW} \longrightarrow \Omega^\infty S^{1+\infty}(\Sigma_{\mathcal{W},\infty})_+ . \quad \square$$

3.2 The spaces $|h\mathcal{W}|$ and $|h\mathcal{V}|$

In section 2.1 we described the jet bundle $J^2(E,\mathbb{R})$ and its fiberwise version as certain spaces of smooth map germs $(E,z) \to \mathbb{R}$, modulo equivalence. For our use in this section and the next it is better to view it as a construction on the tangent bundle. For a vector space V, let $J^2(V)$ denote the vector space of maps

$$\hat{f}\colon \quad V \to \;\mathbb{R}\,, \qquad \hat{f}(v) \;=\; c + \ell(v) + q(v)$$

where $c \in \mathbb{R}$ is a constant, $\ell \in V^*$ and $q\colon V \to \mathbb{R}$ is a quadratic map. This is a contravariant continuous functor on vector spaces, so extends to a functor on vector bundles with $J^2(F)_z = J^2(F_z)$ when F is a vector bundle over E.

When $F = TE$ is the tangent bundle of a manifold E, then there is an isomorphism of vector bundles

$$J^2(TE) \cong J^2(E, \mathbb{R}).$$

Indeed after choice of a spray [2] on E, the associated exponential map induces a diffeomorphism germ

$$\exp_z\colon\ (TE_z, 0) \to (E, z)$$

and $\hat{f}(z) \in J^2(TE_z)$ gives an element of $J^2(E, \mathbb{R})_z$. The resulting vector bundle map $J^2(TE) \to J^2(E, \mathbb{R})$ is the required isomorphism. (To see that it is smooth one may use that the exponential map takes a neighborhood of the zero section in TE diffeomorphically to a neighborhood of the diagonal in $E \times E$.)

Given a submersion $\pi\colon\ E \to X$ with vertical tangent bundle $T^\pi E$, we similarly have an isomorphism of vector bundles

$$J^2(T^\pi E) \cong J^2_\pi(E, \mathbb{R})\,. \tag{3.5}$$

This time we need an exponential map $TE \to E$ for which the restricted map $T^\pi E \to E$ is a map over X, i.e., such that $((T^\pi E)_z, 0) \to (E_{\pi(z)}, z)$ is a diffeomorphism germ for each $z \in E$.

Our object now is to construct a natural map

$$\tau\colon\ h\mathcal{W}[X] \longrightarrow [X, \Omega^\infty \mathbf{hW}] \tag{3.6}$$

compatible with the concatenation in the source and loop sum in the target. Here $[\ ,\ ]$ in the right-hand side denotes a set of homotopy classes of maps.

We assume familiarity with the Thom-Pontryagin relationship between Thom spectra and their infinite loop spaces on the one hand, and bordism theory on the other. See [39] and especially [31]. Applied to our situation this identifies $[X, \Omega^\infty \mathbf{hW}]$ with a group of bordism classes of certain triples $(M, g, \hat{g})$. Here M is smooth without boundary, $\dim(M) = \dim(X) + 2$, and g, $\hat{g}$ together constitute a vector bundle pullback square

$$\begin{array}{ccc} TM \times \mathbb{R}^{1+j} & \xrightarrow{\hat{g}} & TX \times U_{\mathcal{W},\infty} \times \mathbb{R}^j \\ \big\downarrow & & \big\downarrow \\ M & \xrightarrow{g} & X \times \mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+\infty}). \end{array} \tag{3.7}$$

The X–coordinate of g is required to be a proper map $M \to X$. (We write $U_{\mathcal{W},\infty}$ for the tautological 3-dimensional vector bundle on $\mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+\infty}) = \bigcup_r \mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+r})$.) The sum of bordism classes is given by disjoint union of representatives.

For our purposes a slightly different description is preferable. For this we fix a triple $(S^1, g_0, \hat{g}_0)$ in which $g_0\colon\ S^1 \to \mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^3)$ is the constant map to the base point $\star = (\mathbb{R}^3, \ell, 0)$ with $\ell(t_1, t_2, t_3) = t_1$, and $\hat{g}_0$ is the composite map

$$(TS^1 \times \mathbb{R}) \times \mathbb{R} \xrightarrow{\text{standard framing}\times\text{id}} \mathbb{R}^2 \times \mathbb{R} \xrightarrow{\text{switch}} \mathbb{R} \times \mathbb{R}^2.$$

We can then describe $[X, \Omega^\infty \mathbf{hW}]$ as the bordism group of triples $(M, g, \hat{g})$ as above but with $\partial M = S^1 \times \partial[0,1] \times X$. (The restrictions of g and $\hat{g}$ to ∂M and $TM|\partial M$, respectively, are also prescribed: they must agree with the pullbacks of g_0 and $\hat{g}_0$ under the projection from $S^1 \times \partial[0,1] \times X$ to S^1.) With this description, the group structure is given by concatenation, much as in section 2.3. The isomorphism from the standard description to the modified one is given by taking disjoint union with $S^1 \times [0,1] \times X$.

Let now $(\pi, \hat{f}) \in h\mathcal{W}(X)$, where $\pi\colon E \to X$ is a submersion with 3-dimensional fibers and $\hat{f}$ is a section of $J^2(T^\pi E) \to E$ with underlying map $f\colon E \to \mathbb{R}$. See definition 2.1.7. After a small deformation which does not affect the concordance class of $(\pi, \hat{f})$, we may assume that f is transverse to $0 \in \mathbb{R}$ (not necessarily fiberwise) and get a manifold $M = f^{-1}(0)$ with $\dim(M) = \dim(X) + 2$. The boundary ∂M is identified with $S^1 \times \partial[0,1] \times X$ and the restriction of π to M is a proper map $M \to X$, by the definition of $h\mathcal{W}(X)$. The section $\hat{f}$ yields for each $z \in E$ a map

$$\hat{f}(z) \quad = \quad f(z) + \ell_z + q_z\colon \quad (T^\pi E)_z \to \mathbb{R}$$

with the property that the quadratic term q_z is nondegenerate when the linear term ℓ_z is zero. For $z \in M$ the constant $f(z)$ is zero, so the restriction $T^\pi E|M$ is a 3-dimensional oriented vector bundle on M with the extra structure considered in subsection 3.1. Thus $T^\pi E|M$ is classified by a map from M to the space $\mathrm{Gr}_\mathcal{W}(\mathbb{R}^{3+\infty})$: there is a bundle diagram

$$\begin{tikzcd} T^\pi E|M \arrow[r] \arrow[d] & U_{\mathcal{W},\infty} \arrow[d] \\ M \arrow[r, "\kappa"] & \mathrm{Gr}_\mathcal{W}(\mathbb{R}^{3+\infty}). \end{tikzcd}$$

Let $g\colon M \longrightarrow X \times \mathrm{Gr}_\mathcal{W}(\mathbb{R}^{3+\infty})$ be the map $z \mapsto (\pi(z), \kappa(z))$. We now have a canonical vector bundle map

$$\hat{g}\colon TM \times \mathbb{R} \;\cong\; TE|M \;\cong\; \pi^* TX|M \oplus T^\pi E|M \;\longrightarrow\; TX \times U_{\mathcal{W},\infty}$$

and we get a triple $(M, g, \hat{g})$ which represents an element of $[X, \Omega^\infty \mathbf{hW}]$ in the (modified) bordism-theoretic description. It is easily verified that the bordism class of $(M, g, \hat{g})$ depends only on the concordance class of the pair $(\pi, \hat{f})$. Thus we have defined the map τ of 3.6.

Theorem 3.2.1 *The natural map $\tau\colon h\mathcal{W}[X] \to [X, \Omega^\infty \mathbf{hW}]$ is a bijection when X is a closed manifold.*

Proof We define a map σ in the other direction by running the construction τ backwards. Again we view $[X, \Omega^\infty \mathbf{hW}]$ as a bordism group. Let $(M, g, \hat{g})$ be a representative, with $g\colon M \to X \times \mathrm{Gr}_\mathcal{W}(\mathbb{R}^{3+\infty})$ and

$$\hat{g}\colon TM \times \mathbb{R}^{1+j} \;\longrightarrow\; TX \times U_{\mathcal{W},\infty} \times \mathbb{R}^j.$$

We also assume $\partial M = S^1 \times \partial[0,1] \times X$. By obstruction theory, see lemma 3.2.2 below, we can suppose that $j = 0$. Writing $E = M \times \mathbb{R}$ we obtain a map $\pi_E\colon E \to X$ by composing the projection $E \to M$ with the first component of g. Similarly the map $\hat{g}$, now with $j = 0$ and $TM \times \mathbb{R} = TE$, has a first component which is a map of vector bundles

$$\hat{\pi}_E\colon\ TE \longrightarrow TX,$$

covering π_E and epimorphic in the fibers. Since E is an open manifold, Phillips' submersion theorem [30], [15], [16] applies to show that $(\pi_E, \hat{\pi}_E)$ is homotopic through fiberwise surjective bundle maps to a pair $(\pi, d\pi)$ where $\pi\colon E \to X$ is a submersion and $d\pi\colon TE \to TX$ is its differential.

This homotopy lifts to a homotopy of (fiberwise isomorphic) vector bundle maps, starting with $\hat{g}\colon TE \to TX \times U_{\mathcal{W},\infty}$ and ending with a map $TE \to TX \times U_{\mathcal{W},\infty}$ which refines the differential $d\pi\colon TE \to TX$. Its restriction to $T^\pi E \subset TE$ is a vector bundle map $T^\pi E \to U_{\mathcal{W},\infty}$ which equips each fiber $(T^\pi E)_z$ of $T^\pi E$ with a Morse type map

$$\ell_z + q_z\colon\ (T^\pi E)_z \to \mathbb{R}.$$

Let $f\colon E \to \mathbb{R}$ be the projection onto the $\mathbb{R}$ factor, and let

$$\hat{f}(z) = f(z) + \ell_z + q_z \in J^2(T^\pi E) \cong J^2_\pi(E, \mathbb{R}).$$

The map f is proper, since X and hence M are compact. Consequently the pair $(\pi, \hat{f})$ represents an element in $h\mathcal{W}[X]$. Its concordance class depends only on the bordism class of $(M, g, \hat{g})$. This describes a map

$$\sigma\colon\ [X, \Omega^\infty \mathbf{hW}] \longrightarrow h\mathcal{W}[X].$$

It is obvious from the constructions that $\tau \circ \sigma = \mathrm{id}$. In order to evaluate the composition $\sigma \circ \tau$, it suffices by lemma 2.3.5 to evaluate it on an element $(\pi, \hat{f})$ where $f\colon E \to \mathbb{R}$ is regular, so that $E \cong M \times \mathbb{R}$ with $M = f^{-1}(0)$. For $(y, r) \in M \times \mathbb{R}$, the map

$$\hat{f}(y, r)\colon\ (T^\pi(M \times \mathbb{R}))_{(y,r)} \quad\longrightarrow\quad \mathbb{R}$$

is a second degree polynomial of Morse type. The homotopy

$$\hat{f}_{(t)}(y, r) = \hat{f}(y, tr) + (1 - t)r$$

shows that $(\pi, \hat{f})$ is concordant to $(\pi, \hat{f}_{(0)})$, which represents the image of $(\pi, \hat{f})$ under $\sigma \circ \tau$. Therefore $\sigma \circ \tau = \mathrm{id}$. □

Lemma 3.2.2 *Let T and U be k-dimensional vector bundles over a manifold M. Let $[T, U]_{\mathrm{iso}}$ be the set of homotopy classes of isomorphisms $\gamma\colon T \to U$. The stabilization map $[T, U]_{\mathrm{iso}} \longrightarrow [T \times \mathbb{R}, U \times \mathbb{R}]_{\mathrm{iso}}$ is bijective for $k > \dim(M) + 1$ and surjective for $k = \dim(M) + 1$.*

Proof Let $\mathrm{iso}(T, U) \to M$ be the fiber bundle over M whose fiber at $x \in M$ is the space of linear isomorphisms from T_x to U_x. Then $[T, U]_{\mathrm{iso}}$ is the set of homotopy classes of sections of $\mathrm{iso}(T, U) \to M$. The fibers of $\mathrm{iso}(T, U) \to M$ are homotopy equivalent to $\mathrm{O}(k)$, and $\pi_j(\mathrm{O}(k + 1), \mathrm{O}(k)) = 0$ for $j < k$. Induction over the skeletons in a (smooth) triangulation of M completes the proof. □

Finally we give a short description of a map $|h\mathcal{W}| \to \Omega^\infty \mathbf{hW}$ which induces 3.6. We allow ourselves some flexibility with the models for $|h\mathcal{W}|$ and $\Omega^\infty \mathbf{hW}$.

Fix an integer $r > 0$ and X in $\mathscr{X}$. To the data $(\pi, \hat{f})$ in definition 2.1.7, with $\pi\colon E \to X$ and $f\colon E \to \mathbb{R}$, we add the following: a smooth embedding

$$w\colon E \quad \longrightarrow \quad X \times \mathbb{R} \times [0,1] \times \mathbb{R}^{1+r}$$

which covers $(\pi, f)\colon E \to X \times \mathbb{R}$, and a vertical tubular neighborhood N for the submanifold $w(E)$ of $X \times \mathbb{R} \times [0,1] \times \mathbb{R}^{1+r}$, so that the projection $N \to w(E)$ is a map over $X \times \mathbb{R}$. Near ∂E both w and N are assumed to be standard. In particular, near ∂E the embedding w must agree with the standard embedding of

$$(S^1 \times \mathbb{R} \times [0,1]\,) \times X \quad \cong \quad X \times \mathbb{R} \times [0,1] \times S^1$$

in $X \times \mathbb{R} \times \mathbb{R}^{1+r} \times [0,1]$. Making X into a variable now, we can interpret the forgetful map taking $(\pi, \hat{f}, w, N)$ to $(\pi, \hat{f})$ as a map of sheaves

$$h\mathcal{W}^{(r)} \longrightarrow h\mathcal{W}$$

on $\mathscr{X}$. This map is highly connected if r is large, by Whitney's embedding theorem, so that the resulting map from $\operatorname{colim}_r h\mathcal{W}^{(r)}$ to $h\mathcal{W}$ is a weak equivalence of sheaves.
Let $\mathcal{Z}^{(r)}$ be the sheaf taking an X in $\mathscr{X}$ to the set of pointed maps

$$S^1 \wedge (X \times \mathbb{R})_+ \longrightarrow \Omega^{1+r}\mathrm{Th}\,(U^\perp_{\mathcal{W},r})$$

where we use an exotic base point in the target, to be specified below. Then the representing space of $\mathcal{Z}^{(r)}$ is a good approximation to $\Omega^\infty \mathbf{hW}$, that is, $\operatorname{colim}_r |\mathcal{Z}^{(r)}| \simeq \Omega^\infty \mathbf{hW}$. The Thom-Pontryagin collapse construction gives us a map of sheaves

$$\tau^{(r)}\colon\ h\mathcal{W}^{(r)} \longrightarrow \mathcal{Z}^{(r)}. \tag{3.8}$$

In detail: let $(\pi, \hat{f}, w, N)$ be an element of $h\mathcal{W}^{(r)}(X)$. We assume that $\hat{f}$ is a section of $J^2(T^\pi E) \to E$, see (3.5). Now the differential dw promotes each fiber $(T^\pi E)_z$ of the vector bundle $T^\pi E$ to a triple $(V_z, \ell_z, q_z) \in \mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+r})$. Here $V_z = dw((T^\pi E)_z)$, viewed as a subspace of the vertical tangent space at $w(z)$ of the projection

$$X \times \mathbb{R} \times [0,1] \times \mathbb{R}^{1+r} \quad \longrightarrow \quad X\,,$$

which we in turn may identify with $\mathbb{R}^{3+n}$, and $\ell_z + q_z$ is the non-constant part of $\hat{f}(z)$. In particular $z \mapsto (V_z, \ell_z, q_z)$ defines a map $\kappa\colon E \to \mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+r})$. This extends canonically to a pointed map

$$\mathrm{Th}\,(N) \longrightarrow \mathrm{Th}\,(U^\perp_{\mathcal{W},r})$$

because N is identified with $\kappa^* U^\perp_{\mathcal{W},r}$. But $\mathrm{Th}\,(N)$ is a quotient of $X \times \mathbb{R} \times [0,1] \times S^{1+r}$ where we regard S^{1+r} as the one-point compactification of $\mathbb{R}^{1+r}$. Thus we have constructed a map

$$X \times \mathbb{R} \times [0,1] \times S^{1+r} \longrightarrow \mathrm{Th}\,(U^\perp_{\mathcal{W},r})$$

or equivalently, $X \times \mathbb{R} \times [0,1] \longrightarrow \Omega^{1+r}\mathrm{Th}\,(U^{\perp}_{\mathcal{W},r})$. This is constant on $X \times \mathbb{R} \times \partial[0,1]$ by inspection, the constant value being the exotic base point. Therefore our map can also be written in the form

$$S^1 \wedge (X \times \mathbb{R})_+ \longrightarrow \Omega^{1+r}\mathrm{Th}\,(U^{\perp}_{\mathcal{W},r})$$

and so is an element of $\mathcal{Z}^{(r)}(X)$. This element is the image of $(\pi, \hat{f}, w, N)$ under $\tau^{(r)}$ in (3.8). Taking colimits over r, we therefore have a diagram

$$|h\mathcal{W}| \xleftarrow{\simeq} \mathrm{colim}_r\,|h\mathcal{W}^{(r)}| \longrightarrow \mathrm{colim}_r\,|\mathcal{Z}^{(r)}| \xrightarrow{\simeq} \Omega^\infty \mathbf{hW}$$

which we informally describe as a map $\tau\colon\ |h\mathcal{W}| \to \Omega^\infty\mathbf{hW}$.

Theorem 3.2.3 *The map $\tau\colon\ |h\mathcal{W}| \to \Omega^\infty\mathbf{hW}$ is a homotopy equivalence.*

Proof This follows easily from theorem 3.2.1 and the fact that τ can be taken to be a map of topological monoids (cf. section 2.3). First, theorem 3.2.1 with $X = \star$ implies that the map τ induces a bijection

$$\pi_0|h\mathcal{W}| \longrightarrow \pi_0(\Omega^\infty\mathbf{hW})$$

and consequently that $\pi_0|h\mathcal{W}|$ is a group, like $\pi_0(\Omega^\infty\mathbf{hW})$. Next, we use theorem 3.2.1 with $X = S^n$, noting that the grouplike monoid structures imply

$$\pi_n|h\mathcal{W}| \;\cong\; [S^n, |h\mathcal{W}|\,]/[\star, |h\mathcal{W}|\,], \qquad \pi_n(\Omega^\infty\mathbf{hW}) \;\cong\; [S^n, \Omega^\infty\mathbf{hW}]/[\star, \Omega^\infty\mathbf{hW}]$$

for arbitrary choices of base points. Thus the map τ induces an isomorphism of homotopy groups, and Whitehead's theorem implies that it is a homotopy equivalence. □

The arguments above work in a completely similar fashion to identify $|h\mathcal{V}|$. In fact the map τ in theorem 3.2.3 restricts to a map from $|h\mathcal{V}|$ to $\Omega^\infty\mathbf{hV}$ and the analogue of theorem 3.2.1 holds. Keeping the letter τ for this restriction, we therefore have

Theorem 3.2.4 *The map $\tau\colon\ |h\mathcal{V}| \to \Omega^\infty\mathbf{hV}$ is a homotopy equivalence.* □

3.3 The space $|h\mathcal{W}_{\mathrm{loc}}|$

We start with a description of $[X, \Omega^\infty\mathbf{hW}_{\mathrm{loc}}]$ as a bordism group. This is very similar to the description of $[X, \Omega^\infty\mathbf{hW}]$ used in the construction of the map (3.6).

Lemma 3.3.1 *For X in $\mathscr{X}$, the group $[X, \Omega^\infty\mathbf{hW}_{\mathrm{loc}}]$ can be identified with the group of bordism classes of triples $(M, g, \hat{g})$ consisting of a smooth M without boundary,* $\dim(M) = \dim(X) + 2$, *and a vector bundle pullback square*

$$\begin{array}{ccc} TM \times \mathbb{R}^{1+j} & \xrightarrow{\hat{g}} & TX \times U_{\mathcal{W},\infty} \times \mathbb{R}^j \\ \downarrow & & \downarrow \\ M & \xrightarrow{g} & X \times \mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+\infty}) \end{array}$$

where the restriction of the X-coordinate of g to $g^{-1}(X \times \Sigma_{\mathcal{W},\infty})$ is proper.

Proof We first identify $U_{\mathcal{W},\infty}|\Sigma_{\mathcal{W},\infty}$ with its dual using the canonical quadratic form q, and then with the normal bundle N of $\Sigma_{\mathcal{W},\infty}$ in $\mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+\infty})$. Let $(M, g, \hat{g})$ be a triple as above, but with g transverse to $X \times \Sigma_{\mathcal{W},\infty}$. Then $Z = g^{-1}(X \times \Sigma_{\mathcal{W},\infty})$ is a smooth $(n-1)$-dimensional submanifold of M, with normal bundle N_Z. Restriction of g and $\hat{g}$ yields a vector bundle pullback square

$$\begin{tikzcd}
(TZ \oplus N_Z) \times \mathbb{R}^{1+j} \arrow[r] \arrow[d] & TX \times N \times \mathbb{R}^j \arrow[d] \\
Z \arrow[r] & X \times \Sigma_{\mathcal{W},\infty} .
\end{tikzcd}$$

But since N_Z is also identified with the pullback of N, this amounts to a vector bundle pullback square

$$\begin{tikzcd}
TZ \times \mathbb{R}^{1+k} \arrow[r, "\hat{g}_Z"] \arrow[d] & TX \times \Sigma_{\mathcal{W},\infty} \times \mathbb{R}^k \arrow[d] \\
Z \arrow[r, "g_Z"] & X \times \Sigma_{\mathcal{W},\infty}
\end{tikzcd} \tag{3.9}$$

for some $k \gg 0$. Here the X–coordinate $Z \to X$ of g_Z is still a proper map.
Conversely, given data Z, g_Z and $\hat{g}_Z$ as in (3.9), let M be the (total space of the) pullback of N to Z. There is a canonical map from M to $N \subset \mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+\infty})$, and another from M to X, hence a map $g\colon M \to X \times \mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+\infty})$. Moreover $\hat{g}_Z$ determines the $\hat{g}$ in a triple $(M, g, \hat{g})$ as above. In this way, the bordism group in 3.3.1 is isomorphic to the bordism group of triples $(Z, g_Z, \hat{g}_Z)$ as in (3.9). But this is the standard bordism group description of $[X, \Omega^\infty \mathbf{hW}_{\mathrm{loc}}]$; see lemma 3.1.2. □

We now turn to the construction of a localized version of (3.6), namely, a natural map

$$\tau_{\mathrm{loc}}\colon\ h\mathcal{W}_{\mathrm{loc}}[X] \longrightarrow [X, \Omega^\infty \mathbf{hW}_{\mathrm{loc}}]. \tag{3.10}$$

First we modify the bordism group description in 3.3.1 by requiring $\partial M = S^1 \times \partial[0,1] \times X$ instead of $\partial M = \emptyset$. The group structure is then given by concatenation.
Let now $(\pi, \hat{f}) \in h\mathcal{W}_{\mathrm{loc}}(X)$, where $\pi\colon E \to X$ is a submersion with 3-dimensional fibers and $\hat{f}$ is a section of $J^2(T^\pi E) \to E$ with underlying map $f\colon E \to \mathbb{R}$. See definition 2.1.8 and (3.5). We may assume that f is transverse to 0 and get a manifold $M = f^{-1}(0)$. Proceeding exactly as in the construction of the map (3.6), we can promote this to a triple $(M, g, \hat{g})$ where $(g, \hat{g})$ is a vector bundle pullback square

$$\begin{tikzcd}
TM \times \mathbb{R}^{1+j} \arrow[r, "\hat{g}"] \arrow[d] & TX \times U_{\mathcal{W},\infty} \times \mathbb{R}^j \arrow[d] \\
M \arrow[r, "g"] & X \times \mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+\infty}) .
\end{tikzcd}$$

This time, we cannot expect that the X-component of g, in other words $\pi|M$, is proper. But its restriction to

$$g^{-1}(X \times \Sigma_{\mathcal{W},\infty}) = \Sigma(\pi, \hat{f}) \cap M$$

is proper, thanks to condition (ia) in definition 2.1.8. Therefore $(M, g, \hat{g})$ represents an element in $[X, \Omega^\infty \mathbf{hW}_{\mathrm{loc}}]$. This is the image of $(\pi, \hat{f})$ under τ_{loc}. □

Theorem 3.3.2 *The natural map $\tau_{\mathrm{loc}}\colon\ h\mathcal{W}_{\mathrm{loc}}[X] \to [X, \Omega^\infty \mathbf{hW}_{\mathrm{loc}}]$ is a bijection.*

Proof There is a map σ_{loc} in the other direction. The construction of σ_{loc} is analogous to that of σ in the proof of theorem 3.2.1. It is clear that $\tau_{\mathrm{loc}} \circ \sigma_{\mathrm{loc}}$ is the identity. The verification of $\sigma_{\mathrm{loc}} \circ \tau_{\mathrm{loc}} = \mathrm{id}$ uses lemma 3.3.3 below. □

Lemma 3.3.3 *Let $(\pi, \hat{f}) \in h\mathcal{W}_{\mathrm{loc}}(X)$, with $\pi\colon\ E \to X$. Let U be an open neighborhood of $\partial E \cup \Sigma(\pi, \hat{f})$ in E. Then $(\pi|U, \hat{f}|U) \in h\mathcal{W}_{\mathrm{loc}}(X)$ is concordant to $(\pi, \hat{f})$.*

Proof The concordance that we need is an element $(\pi^\sharp, \hat{f}^\sharp)$ in $h\mathcal{W}_{\mathrm{loc}}(X \times]0,1[)$. Let $E^\sharp$ be the union of $E \times]0, 1/2[$ and $U \times]0,1[$. Let $\pi^\sharp(z,t) = (\pi(z), t)$ and $\hat{f}^\sharp(z,t) = (\hat{f}(z), t)$ for $(z,t) \in E^\sharp$. Some renaming of the elements of $E^\sharp$ is required to ensure that $\pi^\sharp$ is graphic. □

Next we give a short description of a map $|h\mathcal{W}_{\mathrm{loc}}| \to \Omega^\infty \mathbf{hW}_{\mathrm{loc}}$ which induces (3.10). This is analogous to the construction of the map named τ in theorem 3.2.3.

Fix an integer $r > 0$ and X in $\mathscr{X}$. To the data $(\pi, \hat{f})$ in definition 2.1.8, with $\pi\colon\ E \to X$ and $f\colon\ E \to \mathbb{R}$, we add the following: a smooth embedding

$$w\colon\ E \quad \longrightarrow \quad X \times \mathbb{R} \times [0,1] \times \mathbb{R}^{1+r}$$

which covers $(\pi, f)\colon\ E \to X \times \mathbb{R}$, a vertical tubular neighborhood N for the submanifold $w(E)$ of $X \times \mathbb{R} \times [0,1] \times \mathbb{R}^{1+r}$, and a smooth function $\psi\colon\ E \to [0,1]$ such that $\psi(z) = 1$ for all $z \in \Sigma(\pi, \hat{f})$. We require that the restriction of $(\pi, f)\colon\ E \to X \times \mathbb{R}$ to the support of ψ be proper. Near ∂E, the function ψ is assumed to vanish and both w and N are assumed to be standard.

Making X into a variable now, we can interpret the forgetful map taking $(\pi, \hat{f}, w, N, \psi)$ to $(\pi, \hat{f})$ as a map of sheaves

$$h\mathcal{W}^{(r)}_{\mathrm{loc}} \longrightarrow h\mathcal{W}_{\mathrm{loc}}$$

on $\mathscr{X}$. This map is highly connected if r is large. Let $\mathcal{Z}^{(r)}_{\mathrm{loc}}$ be the sheaf taking an X in $\mathscr{X}$ to the set of pointed maps

$$S^1 \wedge (X \times \mathbb{R})_+ \longrightarrow \Omega^{1+r} \mathrm{cone}\big(\mathrm{Th}\,(U^\perp_{\mathcal{W},r} | \mathrm{Gr}_{\mathcal{V}}(\mathbb{R}^{3+r})) \hookrightarrow \mathrm{Th}\,(U^\perp_{\mathcal{W},r})\big).$$

Here the cone is a reduced mapping cone, regarded as a quotient of a subspace of

$$\mathrm{Th}\,(U^\perp_{\mathcal{W},r}) \times [0,1]\,,$$

with $\mathrm{Th}\,(U^\perp_{\mathcal{W},r}) \times 1$ corresponding to the base of the cone. The Thom-Pontryagin collapse construction gives us a map of sheaves

$$\tau^{(r)}_{\mathrm{loc}}\colon\ h\mathcal{W}^{(r)}_{\mathrm{loc}} \longrightarrow \mathcal{Z}^{(r)}_{\mathrm{loc}}. \tag{3.11}$$

In detail: let $(\pi, \hat{f}, w, N, \psi)$ be an element of $h\mathcal{W}^{(r)}_{\mathrm{loc}}(X)$. We assume that $\hat{f}$ is a section of $J^2(T^\pi E) \to E$, see (3.5). The differential dw promotes each fiber $(T^\pi E)_z$ of the vector bundle $T^\pi E$ to a triple $(V_z, \ell_z, q_z) \in \mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+r})$, as in the proof of theorem (3.2.3). In particular the formula $z \mapsto ((V_z, \ell_z, q_z), \psi(z))$ defines a map $\kappa\colon E \to \mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+r}) \times [0,1]$. This fits into a vector bundle pullback square

$$\begin{tikzcd}
N \arrow[r, "\hat{\kappa}"] \arrow[d] & U^{\perp}_{\mathcal{W},r} \times [0,1] \arrow[d] \\
E \arrow[r, "\kappa"] & \mathrm{Gr}_{\mathcal{W}}(\mathbb{R}^{3+r}) \times [0,1]
\end{tikzcd}$$

because N is identified with $\kappa^* U^{\perp}_{\mathcal{W},r}$. Now we obtain a map from $X \times \mathbb{R} \times [0,1] \times S^{1+r}$ to the mapping cone

$$\mathrm{cone}\big(\mathrm{Th}\,(U^{\perp}_{\mathcal{W},r}|\mathrm{Gr}_{\mathcal{V}}(\mathbb{R}^{3+r})) \hookrightarrow \mathrm{Th}\,(U^{\perp}_{\mathcal{W},r})\big),$$

viewed as a subquotient of $\mathrm{Th}\,(U^{\perp}_{\mathcal{W},r}) \times [0,1]$, by $z \mapsto \hat{\kappa}(z)$ for $z \in N$ and $z \mapsto \star$ for $z \notin N$. It can also be written in the form

$$S^1 \wedge (X \times \mathbb{R})_+ \longrightarrow \Omega^{1+r}\mathrm{cone}\big(\mathrm{Th}\,(U^{\perp}_{\mathcal{W},r}|\mathrm{Gr}_{\mathcal{V}}(\mathbb{R}^{3+r})) \hookrightarrow \mathrm{Th}\,(U^{\perp}_{\mathcal{W},r})\big)$$

so that it is an element of $\mathcal{Z}^{(r)}_{\mathrm{loc}}(X)$. This defines the map $\tau^{(r)}_{\mathrm{loc}}$. Taking colimits over r, we therefore have a diagram

$$|h\mathcal{W}_{\mathrm{loc}}| \xleftarrow{\simeq} \mathrm{colim}_r\, |h\mathcal{W}^{(r)}_{\mathrm{loc}}| \longrightarrow \mathrm{colim}_r\, |\mathcal{Z}^{(r)}_{\mathrm{loc}}| \xrightarrow{\simeq} \Omega^\infty \mathbf{hW}_{\mathrm{loc}}$$

which we informally describe as a map $\tau_{\mathrm{loc}}\colon |h\mathcal{W}_{\mathrm{loc}}| \to \Omega^\infty \mathbf{hW}_{\mathrm{loc}}$. The following is a straightforward consequence of theorem 3.3.2 (cf. the proof of theorem 3.2.3):

Theorem 3.3.4 *The map $\tau_{\mathrm{loc}}\colon |h\mathcal{W}_{\mathrm{loc}}| \to \Omega^\infty \mathbf{hW}_{\mathrm{loc}}$ is a homotopy equivalence.* □

The combination of theorems 3.3.4, 3.2.3, 3.2.4 and proposition 3.1.4 amounts to a proof of theorem 1.3.3 from the introduction.

Remark 3.3.5 It must be understood that the expression "homotopy fiber sequence" used in theorem 1.3.3 is short for a commutative *square* of pointed spaces and maps

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\
C \arrow[r] & X_2
\end{tikzcd}$$

which is homotopy cartesian and has a contractible lower left-hand term C. This leaves us with the task of saying exactly how the lower row of diagram (2.3) should be completed to a commutative square in which the added term is contractible.
Define a sheaf $h\mathcal{V}_{\mathrm{loc}}$ on $\mathscr{X}$ by copying definition 2.1.5, the definition of $h\mathcal{V}$, but leaving

out condition (i). Then $|h\mathcal{V}_{\mathrm{loc}}|$ is contractible by an application of proposition 2.2.5. There is a commutative square of inclusion maps of pointed CW-spaces

$$\begin{tikzcd} |h\mathcal{V}| \arrow[r] \arrow[d] & |h\mathcal{W}| \arrow[d] \\ |h\mathcal{V}_{\mathrm{loc}}| \arrow[r] & |h\mathcal{W}_{\mathrm{loc}}|. \end{tikzcd} \tag{3.12}$$

The precise meaning of theorem 1.3.3, apart from the statement $|h\mathcal{V}| \simeq \Omega^\infty \mathbb{C}\mathbf{P}^\infty_{-1}$, is that (3.12) is homotopy cartesian. That is also what we have proved.

3.4 The space $|\mathcal{W}_{\mathrm{loc}}|$

The goal is to show that the inclusion of $\mathcal{W}_{\mathrm{loc}}$ in $h\mathcal{W}_{\mathrm{loc}}$ is a weak equivalence. We begin with the observation that the analogue of lemma 3.3.3 holds for $\mathcal{W}_{\mathrm{loc}}$:

Lemma 3.4.1 *Let $(\pi, f) \in \mathcal{W}_{\mathrm{loc}}(X)$, with $\pi\colon E \to X$. Let U be an open neighborhood of $\partial E \cup \Sigma(\pi, f)$ in E. Then $(\pi|U, f|U) \in \mathcal{W}_{\mathrm{loc}}(X)$ is concordant to (π, f).* □

Corollary 3.4.2 *For X in $\mathscr{X}$, there are natural bijections between $\mathcal{W}_{\mathrm{loc}}[X]$ and*

(i) *the set of bordism classes of triples (Σ, p, g), where Σ is a smooth manifold without boundary, $p\colon \Sigma \to X \times \mathbb{R}$ is a proper smooth map whose X-coordinate $\Sigma \to X$ is an étale map (= codimension zero immersion), and g is a map from Σ to $\Sigma_{\mathcal{W},\infty}$;*

(ii) *the set of bordism classes of triples (Σ_0, v, c) where Σ_0 is a smooth manifold without boundary, $v\colon \Sigma_0 \to X$ is a proper smooth codimension 1 immersion with oriented normal bundle and c is a map from Σ_0 to $\Sigma_{\mathcal{W},\infty}$.*

Proof An element (π, f) of $\mathcal{W}_{\mathrm{loc}}(X)$ determines by lemma 2.1.3 a triple (Σ, p, g) as in (i), where Σ is $\Sigma(\pi, f)$ and $p(z) = (\pi(z), f(z))$ for $z \in \Sigma \subset E$. The map $g\colon \Sigma \to \Sigma_{\mathcal{W},\infty}$ classifies the vector bundle $T^\pi E|\Sigma$ with the nondegenerate quadratic form determined by (one-half) the fiberwise Hessian of f. Conversely, given a triple (Σ, p, g) we can make an element (π, f) in $\mathcal{W}_{\mathrm{loc}}(X)$. Namely, let U_Σ be the 3-dimensional vector bundle on Σ classified by g, with the canonical quadratic form $q\colon U_\Sigma \to \mathbb{R}$. Let E be the disjoint union of U_Σ and $(S^1 \times \mathbb{R} \times [0,1]) \times X$. Let $(\pi, f)\colon E \to X \times \mathbb{R}$ agree with $q + \bar{p}$ on U_Σ, where $\bar{p}$ denotes the composition of the vector bundle projection $U_\Sigma \to \Sigma$ with $p\colon \Sigma \to X \times \mathbb{R}$. The resulting maps from $\mathcal{W}_{\mathrm{loc}}[X]$ to the bordism set in (i), and from the bordism set in (i) to $\mathcal{W}_{\mathrm{loc}}[X]$, are inverses of one another: One of the compositions is obviously an identity, the other is an identity by lemma 3.4.1.
Next we relate the bordism set in (i) to that in (ii). A triple (Σ, p, g) as in (i) gives rise to a triple (Σ_0, v, c) as in (ii) provided p is transverse to $X \times 0$. In that case we

set $\Sigma_0 = p^{-1}(X \times 0)$ and define v and c as the restrictions of p and g, respectively. Conversely, a triple (Σ_0, v, c) as in (ii) does of course determine a triple (Σ, p, g) as in (i) with $\Sigma = \Sigma_0 \times \mathbb{R}$. The resulting maps from the bordism set in (i) to that in (ii), and vice versa, are inverses of one another: One of the compositions is obviously an identity, the other is an identity by a shrinking lemma analogous to (but easier than) lemma 2.3.5. $\square$

It is well known that the bordism set (ii) in corollary 3.4.2 is in natural bijection with

$$[X, \Omega^\infty S^{1+\infty}(\Sigma_{\mathcal{W},\infty})_+] \quad \cong \quad [X, \Omega^\infty \mathbf{hW}_{\mathrm{loc}}].$$

Namely, Thom-Pontryagin theory allows us to represent elements of $[X, \Omega^\infty S^{1+\infty}(\Sigma_{\mathcal{W},\infty})_+]$ by quadruples $(\Sigma_0, v, \hat{v}, c)$ where Σ_0 is smooth without boundary, v and $\hat{v}$ constitute a vector bundle pullback square

$$\begin{tikzcd} T\Sigma_0 \times \mathbb{R}^{1+j} \arrow[r, "\hat{v}"] \arrow[d] & TX \times \mathbb{R}^j \arrow[d] \\ \Sigma_0 \arrow[r, "v"] & X \end{tikzcd}$$

(for some $j \gg 0$) with proper v, and c is any map from Σ_0 to $\Sigma_{\mathcal{W},\infty}$. By lemma 3.2.2 we can take $j = 0$ and by immersion theory we can assume $\hat{v} = dv$, that is, v is an immersion and $\hat{v}$ is its (total) differential.

Consequently $\mathcal{W}_{\mathrm{loc}}[X]$ is in natural bijection with $[X, \Omega^\infty \mathbf{hW}_{\mathrm{loc}}]$. It is easy to verify that this natural bijection is induced by the composition

$$|\mathcal{W}_{\mathrm{loc}}| \hookrightarrow |h\mathcal{W}_{\mathrm{loc}}| \xrightarrow{\tau_{\mathrm{loc}}} \Omega^\infty \mathbf{hW}_{\mathrm{loc}}$$

where τ_{loc} is the map of (3.11), 3.10 and theorem 3.3.4. We conclude that the composition is a homotopy equivalence (cf. the proof of theorem 3.2.3). Since τ_{loc} itself is a homotopy equivalence, it follows that the inclusion $|\mathcal{W}_{\mathrm{loc}}| \hookrightarrow |h\mathcal{W}_{\mathrm{loc}}|$ is a homotopy equivalence. This is theorem 1.3.2 from the introduction.

4 Application of Vassiliev's h-principle

This section contains the proof of theorem 1.3.1. It is based upon a special case of Vassiliev's *first main theorem*, [41, ch.III] and [42].

Let $\mathfrak{A} \subset J^2(\mathbb{R}^r, \mathbb{R})$ denote the space of 2-jets represented by $f\colon (\mathbb{R}^r, z) \to \mathbb{R}$ with $f(z) = 0$, $df(z) = 0$ and $\det(d^2 f(z)) = 0$, where $d^2 f(z)$ denotes the Hessian. This set has codimension $r + 2$ and is invariant under diffeomorphisms $\mathbb{R}^r \to \mathbb{R}^r$.

Let N^r be a smooth compact manifold with boundary and let $\psi\colon N \to \mathbb{R}$ be a fixed smooth function with $j^2\psi(z) \notin \mathfrak{A}$ for z in a neighborhood of the boundary. (Use local coordinates near z; the condition means that near ∂N, all singularities of ψ with value 0 are of Morse type, i.e., nondegenerate.) Define spaces

$$\begin{aligned}\Phi(N,\mathfrak{A},\psi) &= \{f \in C^\infty(N,\mathbb{R}) \mid f=\psi \text{ near } \partial N, \quad j^2f(z)\notin\mathfrak{A} \text{ for } z\in N\},\\ h\Phi(N,\mathfrak{A},\psi) &= \{\hat f \in \Gamma J^2(N,\mathbb{R}) \mid \hat f = j^2\psi \text{ near } \partial N, \quad \hat f(z)\notin\mathfrak{A} \text{ for } z\in N\},\end{aligned}$$

where $\Gamma J^2(N,\mathbb{R})$ denotes the space of smooth sections of the jet bundle $J^2(N,\mathbb{R})\to N$. Both are equipped with the standard C^∞ topology. The special case of Vassiliev's theorem that we need is the statement that the map

$$j^2\colon\ \Phi(N,\mathfrak{A},\psi)\longrightarrow h\Phi(N,\mathfrak{A},\psi) \tag{4.1}$$

induces an isomorphism on integral homology. We use this when $\dim(N)=3$.

4.1 Sheaves with category structure

Let $\mathcal{F}\colon\ \mathscr{X}\to\mathscr{C}at$ be a sheaf with values in small categories. Taking nerves defines a sheaf with values in the category of simplicial sets,

$$N_\bullet\mathcal{F}\colon\ \mathscr{X}\to\mathscr{S}ets_\bullet$$

with $N_0\mathcal{F}$ the sheaf of objects, $N_0\mathcal{F}=\mathrm{ob}(\mathcal{F})$, and $N_1\mathcal{F}$ the sheaf of morphisms. We have the associated bisimplicial set $N_\bullet\mathcal{F}(\Delta_e^\bullet)$ and recall [32] that the realization of its diagonal is homeomorphic to either of its double realizations,

$$|\,\underline{k}\mapsto N_k\mathcal{F}(\Delta_e^k)\,| \;\cong\; \big|\,\underline{\ell}\mapsto|\,\underline{k}\mapsto N_k\mathcal{F}(\Delta_e^\ell)\,|\big| \;\cong\; \big|\,\underline{k}\mapsto|\,\underline{\ell}\mapsto N_k\mathcal{F}(\Delta_e^\ell)\,|\big|. \tag{4.2}$$

Since $|\,\underline{\ell}\mapsto N_k\mathcal{F}(\Delta_e^\ell)\,| = N_k|\mathcal{F}|$ by A.2.1, the right hand side of (4.2) is the classifying space $B|\mathcal{F}|$ of the topological category $|\mathcal{F}|$.

We next give another construction of $B|\mathcal{F}|$ related to Steenrod's view of principal bundles as 1-cocycles. We shall consider locally finite open covers $Y_? = (Y_j)_{j\in J}$ of spaces X in $\mathscr{X}$ indexed by a fixed uncountable set J. For each finite nonempty subset $S\subset J$ we write

$$Y_S = \bigcap_{j\in S} Y_j\,.$$

Associated to the cover $Y_?$ there is a topological category, denoted $X_{Y_?}$ in [37, §4], with

$$\mathrm{ob}(X_{Y_?}) = \coprod_S Y_S\,, \qquad \mathrm{mor}(X_{Y_?}) = \coprod_{\substack{R,S\\ R\subset S}} Y_S\,.$$

A continuous functor from $X_{Y_?}$ to a topological group G, viewed as a topological category with one object, is equivalent to a collection of maps

$$\varphi_{RS}\colon\ Y_S\longrightarrow G\,,$$

one for each pair $R\subset S$ of finite subsets of J, subject to the cocycle conditions listed in definition 4.1.1 below. More general versions can be found in [29].

Definition 4.1.1 For X in $\mathscr{X}$ an element of $\beta\mathcal{F}(X)$ is a pair $(Y_?, \varphi_?)$ where $Y_?$ is a locally finite open covering of X, indexed by J, and $\varphi_?$ associates to each pair of finite, nonempty subsets $R \subset S$ of J a morphism $\varphi_{RS} \in N_1\mathcal{F}(Y_S)$ subject to the following cocycle conditions:

(i) every φ_{RR} is an identity morphism;

(ii) for $R \subset S \subset T$, we have $\varphi_{RT} = (\varphi_{RS}|Y_T) \circ \varphi_{ST}$.

Condition (ii) includes the condition that the right-hand composition is *defined*; in particular, taking $S = T$ one finds that the source of φ_{RS} is the object φ_{SS}, and taking $R = S$ one finds that the target of φ_{ST} is $\varphi_{SS}|Y_T$.

Remark 4.1.2 By an open cover of X indexed by J we mean a *map* $j \mapsto Y_j$ from J to the set of open subsets of X such that $\bigcup_j Y_j = X$. This map is not required to be injective, and cannot always be injective, as the case $X = \emptyset$ shows. In definition 4.1.1, we use open covers indexed by a fixed set J to ensure that $\beta\mathcal{F}$ has the sheaf property. In appendix B, definition B.1.1, we give a variant of definition 4.1.1 which does not mention an indexing set, but uses surjective étale maps to X rather than open covers of X.

The sets $\beta\mathcal{F}(X)$ define a sheaf $\beta\mathcal{F}\colon \mathscr{X} \to \mathscr{S}ets$ and hence a space $|\beta\mathcal{F}|$. The following key theorem is one of our main tools used in the proof of both theorem 1.3.1 and theorem 1.3.4. It may be viewed as a generalization of the result that isomorphism classes of Steenrod's principal coordinate bundles (over X) are in bijective correspondence with homotopy classes of maps from X to BG. See also [29]. Its proof is deferred to appendix B.

Theorem 4.1.3 *The spaces $|\beta\mathcal{F}|$ and $B|\mathcal{F}|$ are homotopy equivalent.*

Definition 4.1.4 Let $\mathcal{E}, \mathcal{F}\colon \mathscr{X} \to \mathscr{C}at$ be sheaves and $g\colon \mathcal{E} \to \mathcal{F}$ a map between them. We say that g is a *transport projection* if the following square is a pullback square of sheaves on $\mathscr{X}$:

$$\begin{tikzcd} N_1\mathcal{E} \arrow[r, "d_0"] \arrow[d, "g"] & N_0\mathcal{E} \arrow[d, "g"] \\ N_1\mathcal{F} \arrow[r, "d_0"] & N_0\mathcal{F} \end{tikzcd}$$

where d_0 is the source operator.

Proposition 4.1.5 *Let $g\colon \mathcal{E} \to \mathcal{F}$ and $g'\colon \mathcal{E}' \to \mathcal{F}$ be transport projections as in definition 4.1.4. Let $u\colon \mathcal{E} \to \mathcal{E}'$ be a map of sheaves over $\mathcal{F}$ which respects the category structures. Suppose that the maps $N_0\mathcal{E} \to N_0\mathcal{F}$ and $N_0\mathcal{E}' \to N_0\mathcal{F}$ obtained from g and g' have the concordance lifting property, cf. definition A.2.4. Suppose also that, for each object a of $\mathcal{F}(\star)$, the restriction $\mathcal{E}_a \to \mathcal{E}'_a$ of u to the fibers over a is a weak equivalence (resp. induces an integral homology equivalence of the representing spaces). Then $\beta u\colon \beta\mathcal{E} \to \beta\mathcal{E}'$ is a weak equivalence (resp. induces an integral homology equivalence of the representing spaces).*

Proof According to theorem 4.1.3 it suffices to prove that u induces a homotopy (homology) equivalence from $B|\mathcal{E}|$ to $B|\mathcal{E}'|$. By (4.2) it is then also enough to show that

$$N_k(u)\colon\ N_k\mathcal{E} \longrightarrow N_k\mathcal{E}'$$

becomes a homotopy equivalence (homology equivalence) after passage to representing spaces, for each $k \geq 0$.
Since g and g' are transport projections, an obvious inductive argument shows that the diagrams

$$\begin{tikzcd} N_k\mathcal{E} \arrow[r] \arrow[d, "g"] & N_0\mathcal{E} \arrow[d, "g"] \\ N_k\mathcal{F} \arrow[r] & N_0\mathcal{F} \end{tikzcd} \qquad \begin{tikzcd} N_k\mathcal{E}' \arrow[r] \arrow[d, "g'"] & N_0\mathcal{E}' \arrow[d, "g'"] \\ N_k\mathcal{F} \arrow[r] & N_0\mathcal{F} \end{tikzcd}$$

are pullback squares for all $k \geq 0$. They are therefore homotopy cartesian by A.2.6 and by our assumptions. Hence it suffices to consider the case $k = 0$,

$$N_0u\colon\ N_0\mathcal{E} \longrightarrow N_0\mathcal{E}'.$$

By assumptions again, $N_0\mathcal{E} \to N_0\mathcal{F}$ and $N_0\mathcal{E}' \to N_0\mathcal{F}$ have the concordance lifting property and N_0u induces a weak equivalence (homology equivalence) of the fibers. By A.2.6, the fibers turn into homotopy fibers upon passage to representing spaces. Consequently $N_0u\colon\ N_0\mathcal{E} \to N_0\mathcal{E}'$ is a homotopy equivalence (homology equivalence). $\square$

In our applications of theorem 4.1.3, the categories $\mathcal{F}(X)$ for X in $\mathscr{X}$ will typically be partially ordered sets or will have been obtained from a functor

$$\mathcal{F}_\bullet\colon\ \mathscr{C}^{\mathrm{op}} \longrightarrow \ \text{sheaves on } \mathscr{X}$$

where $\mathscr{C}$ is a small category. Recall that given such a functor one can define a category valued sheaf $\mathscr{C}^{\mathrm{op}}\!\int\!\mathcal{F}_\bullet$ on $\mathscr{X}$. Its value on a manifold X is the category whose objects are pairs (c, ω) with $c \in \mathrm{ob}(\mathscr{C})$ and $\omega \in \mathcal{F}_c(X)$ and whose morphisms are pairs (f, ω) with $f\colon\ b \to c$ in $\mathrm{mor}(\mathscr{C})$ and $\omega \in \mathcal{F}_c(X)$. Then

$$|\beta(\mathscr{C}^{\mathrm{op}}\!\int\!\mathcal{F}_\bullet)\,| \ \simeq\ B|\mathscr{C}^{\mathrm{op}}\!\int\!\mathcal{F}_\bullet| \ \simeq\ \operatorname*{hocolim}_{c\in\mathscr{C}}\ |\mathcal{F}_c|$$

(see appendix D for details).

Definition 4.1.6 The sheaf $\beta(\mathscr{C}^{\mathrm{op}}\!\int\!\mathcal{F}_\bullet)\colon\ \mathscr{X} \longrightarrow \mathscr{C}\!at$ will be written $\operatorname*{hocolim}_{c\in\mathscr{C}}\ \mathcal{F}_c$.

It is in order to spell out that an element of $(\mathrm{hocolim}_c\,\mathcal{F}_c)(X)$ consists of a covering $Y_?$ of X indexed by the elements of J, a functor $S \mapsto \theta(S)$ from the poset of finite nonempty subsets of J to $\mathcal{C}$, and elements $\omega_S \in \mathcal{F}_{\theta(S)}(Y_S)$ connected to each other via the maps

$$\mathcal{F}_{\theta(T)}(Y_T) \longrightarrow \mathcal{F}_{\theta(S)}(Y(T)) \longleftarrow \mathcal{F}_{\theta(S)}(Y(S))$$

for each $S \subset T$.

4.2 Armlets

We begin by defining sheaves $\mathcal{W}^{\mathscr{A}}$ and $h\mathcal{W}^{\mathscr{A}}$ on $\mathscr{X}$ with values in partially ordered sets, and natural transformations

$$
\begin{tikzcd}
& \mathscr{P}\!osets \arrow[dd, "\text{forget}"] \\
\mathscr{X} \arrow[ur, bend left, "\mathcal{W}^{\mathscr{A}}"] \arrow[dr, bend right, "\mathcal{W}^0"'] \arrow[r, phantom, "\Downarrow"] & \\
& \mathscr{S}\!ets
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& \mathscr{P}\!osets \arrow[dd, "\text{forget}"] \\
\mathscr{X} \arrow[ur, bend left, "h\mathcal{W}^{\mathscr{A}}"] \arrow[dr, bend right, "h\mathcal{W}^0"'] \arrow[r, phantom, "\Downarrow"] & \\
& \mathscr{S}\!ets
\end{tikzcd}
$$

where $\mathcal{W}^0$ and $h\mathcal{W}^0$ are the sheaves introduced in section 2.3, weakly equivalent to $\mathcal{W}$ and $h\mathcal{W}$, respectively.

Definition 4.2.1 An *armlet* for an element $(\pi, f) \in \mathcal{W}^0(X)$ is a compact interval $A \subset \mathbb{R}$ such that $0 \in \mathrm{int}(A)$ and f is fiberwise transverse to the endpoints of A.

Definition 4.2.2 An *armlet* for an element $(\pi, \hat{f}) \in h\mathcal{W}^0(X)$ is a compact interval $A \subset \mathbb{R}$ such that $0 \in \mathrm{int}(A)$ and

(i) f is fiberwise transverse to the endpoints of A;

(ii) $\hat{f}$ is integrable on an open neighborhood of $f^{-1}(\mathbb{R} \smallsetminus \mathrm{int}(A))$.

We introduce a partial ordering on elements of $\mathcal{W}^0(X)$ or $h\mathcal{W}^0(X)$ equipped with armlets, namely for elements of $\mathcal{W}^0(X)$:

$$(\pi, f, A) \leq (\pi', f', A) \quad \text{if} \quad (\pi, f) = (\pi', f') \quad \text{and} \quad A \subset A'$$

and similarly for elements of $h\mathcal{W}^0(X)$.

Definition 4.2.3 For a connected X in $\mathscr{X}$ we let $\mathcal{W}^{\mathscr{A}}(X)$ denote the partially ordered set of elements (π, f, A) with A an armlet for $(\pi, f) \in \mathcal{W}^0(X)$. Similarly, $h\mathcal{W}^{\mathscr{A}}(X)$ is the partially ordered set of elements $(\pi, \hat{f}, A)$ where $(\pi, \hat{f}) \in h\mathcal{W}^0(X)$ and A and armlet for $(\pi, \hat{f})$. If X is not connected we (must) define

$$\mathcal{W}^{\mathscr{A}}(X) = \textstyle\prod_i \mathcal{W}^{\mathscr{A}}(X_i), \qquad h\mathcal{W}^{\mathscr{A}}(X) = \textstyle\prod_i h\mathcal{W}^{\mathscr{A}}(X_i)$$

where the X_i are the path components of X.

Any sheaf $\mathcal{F}\colon \mathscr{X} \to \mathscr{S}ets$ can be considered to be a sheaf with category structure, namely $\mathcal{F}(X)$ is the object set, and only identity morphisms are allowed. In this case an element of $\beta\mathcal{F}(X)$ is a pair $(Y_?, s)$ consisting of a locally finite open covering $Y_? = \{Y_j \mid j \in J\}$ and a single element $s \in \mathcal{F}(X)$. Thus $\beta\mathcal{F} \cong \mathcal{F} \times \beta\star$ where $\star$ denotes the terminal sheaf, viewed as a sheaf with category values. In particular there is a forgetful projection $\beta\mathcal{F} \longrightarrow \mathcal{F}$ which is a weak equivalence.

Proposition 4.2.4 *The forgetful maps $\beta\mathcal{W}^{\mathscr{A}} \to \mathcal{W}^0$ and $\beta h\mathcal{W}^{\mathscr{A}} \to h\mathcal{W}^0$ are weak equivalences of sheaves.*

The proof of proposition 4.2.4 will be broken up into the following three lemmas.

Lemma 4.2.5 *Let X be in $\mathscr{X}$ and $(\pi, f) \in \mathcal{W}^0(X)$. Every $x \in X$ has an open neighborhood U in X such that the image of (π, f) in $\mathcal{W}^0(U)$ admits an armlet.*

Proof Write $\pi\colon E \to X$ and $E_x = \pi^{-1}(x)$. By Sard's theorem, we can find numbers $a < 0$ and $b > 0$ such that $f_x\colon E_x \to \mathbb{R}$ is transverse to a and b (in other words, a and b are regular values of f_x). Let $A = [a, b]$. Let $C \subset E$ be the closed subset consisting of all $z \in E$ where f has a fiberwise singularity and $f(z) = a$ or $f(z) = b$. Then $\pi|C$ is proper and so $\pi(C)$ is a closed subset of X. Let $U = X \smallsetminus \pi(C)$. □

Lemma 4.2.6 *With the assumptions of lemma 4.2.5, there exists an element of $\beta\mathcal{W}^{\mathscr{A}}(X)$ mapping to (π, f) under the forgetful transformation $\beta\mathcal{W}^{\mathscr{A}} \to \mathcal{W}^0$.*

Proof Choose a locally finite covering of X by open subsets Y_j, where $j \in J$, such that the restriction of (π, f) to each Y_j admits an armlet $A_j \subset \mathbb{R}$. For a finite nonempty subset $S \subset J$ with nonempty Y_S let $A_S = \bigcap_{j \in S} A_j$. Then A_S is an armlet for the restriction of (π, f) to Y_S. Therefore, given nonempty finite $R, S \subset J$ with $R \subset S$ and $Y_S \neq \emptyset$, we can define $\varphi_{RS} \in N_1\mathcal{W}^{\mathscr{A}}(Y_S)$ to be the relation

$$(\pi, f, A_S)|Y_S \leq (\pi, f, A_R)|Y_S.$$

If Y_S is empty, there is only one element in $\mathcal{W}^{\mathscr{A}}(Y_S)$ and so we have only one choice for φ_{RS}. The data φ_{RS} for all finite nonempty $R, S \subset J$ with $R \subset S$ then constitute an element of $\beta\mathcal{W}^{\mathscr{A}}(X)$ which clearly projects to $(\pi, f) \in \mathcal{W}(X)$. □

It follows from the two previous lemmas that the forgetful map $\beta\mathcal{W}^{\mathscr{A}}[X] \to \mathcal{W}^0[X]$ is surjective for any X in $\mathscr{X}$. What we really need in order to prove the first half of proposition 4.2.4 is the relative surjectivity as in proposition 2.2.6. This comes from the next lemma.

Lemma 4.2.7 *Let X in $\mathscr{X}$ and let $(\pi, f) \in \mathcal{W}^0(X)$. Let $C \subset X$ be closed and suppose that a germ of lifts of (π, f) across $\beta\mathcal{W}^{\mathscr{A}} \longrightarrow \mathcal{W}^0$ has been specified near C. Then there exists an element in $\beta\mathcal{W}^{\mathscr{A}}(X)$ which lifts $(\pi, f) \in \mathcal{W}(X)$ and extends the prescribed germ of lifts near C.*

Proof Let U be a sufficiently small open neighborhood of C in X so that the prescribed germ of lifts is represented by an actual lift of $(\pi, f)|U$ across $\beta\mathcal{W}^{\mathscr{A}}(U) \longrightarrow \mathcal{W}^0(U)$. This gives us a locally finite covering $Y'_?$ of U, and for each nonempty finite $S \subset J$ and each $z \in \pi_0(Y'_S)$, a compact interval $A'_{S,z} \subset \mathbb{R}$ such that $0 \in \mathrm{int}(A'_{S,z})$. We have $A'_{S,z} \subset A'_{R,\bar{z}}$ if

$R \subset S$ and $\bar{z}$ is the image of z under $\pi_0(Y'_S) \to \pi_0(Y'_R)$. Making U smaller if necessary, we can assume that the covering $Y'_?$ is locally finite in the strong sense that every $x \in X$ has an open neighborhood on X which intersects only finitely many of the Y'_j.
Now we make a locally finite covering of X by open subsets Y_j as follows. For $j \in J$ such that Y'_j is nonempty, let $Y_j = Y'_j$. For all other $j \in J$ define Y_j in such a way that Y_j avoids a fixed neighborhood of C and the restriction of (π, f) to each path component $z \in \pi_0(Y_j)$ admits an armlet $A_{j,z}$.
It remains to find enough armlets. We need one armlet $A_{S,z} \subset \mathbb{R}$ for each nonempty finite $S \subset J$ and every component $z \in \pi_0(Y_S)$. These armlets must satisfy $A_{S,z} \subset A_{R,\bar{z}}$ if $R \subset S$ and $\bar{z}$ is the image of z under $\pi_0(Y_S) \to \pi_0(Y_R)$. But, reasoning as in the proof of lemma 4.2.6, we find that it is enough to say what $A_{S,z} = A_{j,z}$ should be when S is a singleton $\{j\}$. We have already said it in the cases where $Y_j \neq Y'_j$; in the other cases we say $A_{j,z} := A'_{j,z}$. □

The proof of the second half of proposition 4.2.4 goes like the proof of the first half, except for an additional observation which is as follows. For X in $\mathscr{X}$ let $h_c\mathcal{W}^0(X)$ consist of all $(\pi, \hat{f}) \in h\mathcal{W}^0(X)$, with $\pi\colon E \to X$ etc., such that $\hat{f}$ is integrable on some open $U \subset E$ and π restricted to $E \smallsetminus U$ is proper.

Lemma 4.2.8 *The inclusion of sheaves $h_c\mathcal{W}^0 \hookrightarrow h\mathcal{W}^0$ is a weak equivalence.*

Proof Let $(\pi, \hat{f}) \in h\mathcal{W}^0(X)$, with $\pi\colon E \to X$. Choose an open $U \subset E$ such that π restricted to $E \smallsetminus U$ is proper and such that the closure of U has empty intersection with $f^{-1}(0)$. Using the convexity of the fibers of $J^2_\pi(E, \mathbb{R}) \to E$, especially over points $z \in U$, deform $\hat{f}$ (leaving f unchanged) in such a way that it becomes integrable on U. This shows that $h_c\mathcal{W}^0[X] \to h\mathcal{W}^0[X]$ is surjective. The argument can easily be refined to prove a relative statement as in the hypothesis of proposition 2.2.6. □

4.3 Proof of theorem 1.3.1

According to lemma 2.3.4 and proposition 4.2.4 it remains to show that

$$j^2_\pi\colon \beta\mathcal{W}^{\mathscr{A}} \to \beta h\mathcal{W}^{\mathscr{A}}$$

is a weak equivalence. To this end we introduce a new sheaf

$$\mathcal{T}^{\mathscr{A}}\colon \mathscr{X} \longrightarrow \mathscr{P}osets.$$

Suppose given a submersion $\pi\colon E \to X$ with 3-dimensional fibers and standard behavior near the boundary as in condition (iii) of definitions 2.1.5 and 2.1.7. We consider pairs (ψ, A) where $\psi\colon E \to \mathbb{R}$ is a smooth function such that $(\pi, \psi)\colon E \to X \times \mathbb{R}$ is proper, $A \subset \mathbb{R}$ is a compact interval with $0 \in \mathrm{int}(A)$, and ψ is fiberwise transverse to ∂A (and prescribed near ∂E in the usual way). There is no restriction on the fiberwise singularities that ψ might have.

Definition 4.3.1 For connected X in $\mathscr{X}$, the set $\mathcal{T}^{\mathscr{A}}(X)$ consists of equivalence classes of triples (π, ψ, A) as above, where $(\pi, \psi, A) \sim (\pi', \psi', A')$ if $\pi = \pi'$, $A = A'$ and the support of $\psi - \psi'$ is contained in the interior of $\psi^{-1}(A)$.

As for $\mathcal{W}^{\mathscr{A}}$, we get $\mathcal{T}^{\mathscr{A}}\colon \mathscr{X} \to \mathscr{P}osets$. Moreover there is an obvious commutative diagram of sheaves

$$\begin{tikzcd} \mathcal{W}^{\mathscr{A}} \arrow[rr, "j^2_\pi"] \arrow[dr, "p"'] & & h\mathcal{W}^{\mathscr{A}} \arrow[dl, "q"] \\ & \mathcal{T}^{\mathscr{A}} & \end{tikzcd} \tag{4.3}$$

where $p(\pi, f, A)$ and $q(\pi, \hat{f}, A)$ are the equivalence classes of (π, f, A); in the second case f is the underlying function of $\hat{f}$.

Let (π, ψ, A) be a representative of an element of $\mathcal{T}^{\mathscr{A}}(X)$ with $\pi\colon E \to X$, $\psi\colon E \to \mathbb{R}$ and $A \subset \mathbb{R}$. The manifold $\psi^{-1}(A)$ is independent of the choice of representative for the equivalence class, and $\pi|\psi^{-1}(A)$ is a proper submersion, hence a smooth fiber bundle by Ehresmann's fibration theorem [2]. Moreover, near the boundary of $\psi^{-1}(A)$, the function ψ is independent of the choice of representative.

Lemma 4.3.2 *The maps p and q in (4.3) have the concordance lifting property.*

Proof We give the proof for p, since the proof for q is much the same. Write $I = [0, 1]$. Given an element $[\pi, \psi, A] \in \mathcal{T}^{\mathscr{A}}(X \times I)$ with a lift to $\mathcal{W}^{\mathscr{A}}(X \times 0)$ of its restriction to $X \times 0$, the projection

$$\psi^{-1}(A) \xrightarrow{\pi} X \times I \tag{4.4}$$

is a smooth manifold bundle. Hence there exists a diffeomorphism $N \times I \cong \psi^{-1}(A)$ over $X \times I$, where $N = \psi^{-1}(A) \cap \pi^{-1}(X \times 0)$. But what we need here is a diffeomorphism

$$u\colon N \times I \longrightarrow \psi^{-1}(A)$$

over $X \times I$ such that $\psi(u(z,t)) = \psi(u(0,t))$ for all $t \in I$ and all z near ∂N, and of course $u(z, 0) = z$ for all $z \in N$.
Let $\partial_h \psi^{-1}(A)$ and $\partial_v \psi^{-1}(A)$ be the parts of $\partial\psi^{-1}(A)$ which are mapped to $X \times (I \smallsetminus \partial I)$ and $X \times \partial I$, respectively, by π. Constructing u with the properties above is equivalent to constructing a smooth vector field $\xi = du/dt$ on $\psi^{-1}(A)$ which

(i) covers the vector field $(x,t) \mapsto (0,1) \in TX_x \times T\mathbb{R}_t$ on $X \times I$,

(ii) is parallel to $\partial_h \psi^{-1}(A)$,

(iii) satisfies $\langle d\psi, \xi \rangle \equiv 0$ near $\partial_h \psi^{-1}(A)$.

(It should be added that $\xi = du/dt$ is also prescribed near $\partial_v \psi^{-1}(A)$ due to the details in definition 2.2.2.) This problem has local solutions which can be pieced together by means of a partition of unity on $\psi^{-1}(A)$. Hence u with the required properties exists.
Now we define the lifted concordance $(\pi, f, A) \in \mathcal{W}^{\mathscr{A}}(X \times I)$ in such a way that $f(u(z,t)) = f(u(z,0))$ for $(z,t) \in N \times I$, bearing in mind that $f(u(z,0))$ is prescribed for all $z \in N$ and f must equal ψ outside $u(N \times I) = \psi^{-1}(A)$. □

Proposition 4.3.3 *The fiberwise jet prolongation map*

$$j^2_\pi \colon |\beta\mathcal{W}^{\mathscr{A}}| \longrightarrow |\beta h\mathcal{W}^{\mathscr{A}}|$$

induces an isomorphism on integral homology.

Proof This will be deduced from proposition 4.1.5 and diagram (4.3). By inspection, both maps p and q in (4.3) are transport projections. We must determine the fibers of p and q and check that j^2_π induces a homology equivalence between fibers over the same point.
We first determine the fiber $p^{-1}(\tau)$ of

$$p\colon \mathcal{W}^{\mathscr{A}} \longrightarrow \mathcal{T}^{\mathscr{A}}$$

over an element $\tau = [F^3, \psi, A] \in \mathcal{T}^{\mathscr{A}}(\star)$. That is, for each X in $\mathscr{X}$ we are interested in the subset of $\mathcal{W}^{\mathscr{A}}(X)$ which maps to the element $[\pi, \psi \circ \mathrm{pr}_F, A] \in \mathcal{T}^{\mathscr{A}}(X)$ where π and pr_F are the projections $F \times X \to X$ and $F \times X \to F$, respectively. This subset consists of $(\pi, f, A) \in \mathcal{W}^{\mathscr{A}}(X)$ with

$$\mathrm{supp}(f - \psi \circ \mathrm{pr}_F) \quad \subset \quad \mathrm{int}(\psi^{-1}(A)) \times X.$$

Thus in the notation of (4.1), the fiber of p over τ is the sheaf taking X in $\mathscr{X}$ to the set of smooth maps from X to

$$\Phi(\psi^{-1}(A), \mathfrak{A}, \psi).$$

Similarly, the fiber $q^{-1}(\tau)$ of q in (4.3) over the same element $\tau \in \mathcal{T}^{\mathscr{A}}(\star)$ is the sheaf taking X in $\mathscr{X}$ to the set of smooth maps from X to

$$h\Phi(\psi^{-1}(A), \mathfrak{A}, \psi).$$

Thus the representing spaces $|p^{-1}(\tau)|$ and $|q^{-1}(\tau)|$ have canonical comparison maps to $\Phi(\psi^{-1}(A), \mathfrak{A}, \psi)$ and $h\Phi(\psi^{-1}(A), \mathfrak{A}, \psi)$, respectively, which are homotopy equivalences. With these as identifications, the jet prolongation map from $|p^{-1}(\tau)|$ to $|q^{-1}(\tau)|$ turns into a special case of (4.1), and so is a homology equivalence by Vassiliev's first main theorem. □

Combining lemma 2.3.4, proposition 4.2.4 and proposition 4.3.3, we get that

$$j^2_\pi \colon |\mathcal{W}| \longrightarrow |h\mathcal{W}|$$

induces an isomorphism in homology. Both $|\mathcal{W}|$ and $|h\mathcal{W}|$ are topological monoids (cf. sections 2.3 and C.2) and j^2_π is a map of monoids. The target $|h\mathcal{W}|$ is an infinite loop space by theorem 3.2.3, hence it is group complete. (That is, the monoid $\pi_0|h\mathcal{W}|$ is a group, or equivalently, the canonical map $|h\mathcal{W}| \to \Omega B|h\mathcal{W}|$ is a homotopy equivalence). Since $H_*(j^2_\pi;\mathbb{Z})$ is an isomorphism, especially when $*=0$, the source $|\mathcal{W}|$ is also group complete. It is well known that a homomorphism between group complete topological monoids is a homology equivalence if and only if it is a homotopy equivalence. (The statement is easily reduced to the case where both monoids are connected, so that their classifying spaces are simply connected. One verifies that the induced map of classifying spaces is a homology equivalence, hence a homotopy equivalence, and deduces by applying Ω that the original homomorphism is a homotopy equivalence.) This completes the proof of theorem 1.3.1. □

5 Some homotopy colimit decompositions

5.1 Description of main results

The organization and the main results of this section can be summarized in a commutative diagram of sheaves on $\mathscr{X}$ and maps of sheaves

$$\begin{tikzcd}
\mathcal{W} \arrow[r] & \mathcal{W}_{\mathrm{loc}} \\
\mathcal{L} \arrow[u, "\simeq"'] \arrow[r] & \mathcal{L}_{\mathrm{loc}} \arrow[u, "\simeq"'] \\
\operatorname*{hocolim}_{T \text{ in } \mathscr{K}} \mathcal{L}_T \arrow[u, "\simeq"'] \arrow[r] \arrow[d, "\simeq"] & \operatorname*{hocolim}_{T \text{ in } \mathscr{K}} \mathcal{L}_{\mathrm{loc},T} \arrow[u, "\simeq"'] \arrow[d, "\simeq"] \\
\operatorname*{hocolim}_{T \text{ in } \mathscr{K}} \mathcal{W}_T \arrow[r] & \operatorname*{hocolim}_{T \text{ in } \mathscr{K}} \mathcal{W}_{\mathrm{loc},T} \,.
\end{tikzcd} \tag{5.1}$$

The symbol $\simeq$ indicates weak equivalences. The homotopy colimits in the diagram are homotopy colimits in the category of sheaves on $\mathscr{X}$, as in definition 4.1.6. But their representing spaces can be regarded as homotopy colimits in the category of spaces according to lemma D.1.5.

The top row of diagram (5.1) is the inclusion map $\mathcal{W} \to \mathcal{W}_{\mathrm{loc}}$. The bottom row is what we eventually want to substitute for the top row in order to prove theorem 1.3.4. We now give a detailed description of the bottom row. This must begin with a definition of the category $\mathscr{K}$ by which the homotopy colimits are indexed.

Definition 5.1.1 An object of $\mathscr{K}$ is a finite set S equipped with a map to $\underline{3} = \{0,1,2,3\}$. A morphism from S to T is a pair (k,ε) where k is an injective map (over $\underline{3}$) from S

to T and ε is a function $T \smallsetminus k(S) \to \{-1,+1\}$. The composition of two morphisms $(k_1,\varepsilon_1)\colon S \to T$ and $(k_2,\varepsilon_2)\colon T \to U$ is $(k_2k_1,\varepsilon_3)\colon S \to U$ where ε_3 agrees with ε_2 outside $k_2(T)$ and with $\varepsilon_1 \circ {k_2}^{-1}$ on $k_2(T \smallsetminus k_1(S))$.

Definition 5.1.2 Let T be an object of $\mathscr{K}$. For X in $\mathscr{X}$, let $\mathcal{W}_{\mathrm{loc},T}(X)$ be the set of oriented, smooth, riemannian 3-dimensional vector bundles V on $T \times X$ equipped with a fiberwise linear isometric involution ϱ and subject to the following conditions.

(i) For $(t,x) \in T \times X$, the dimension of the fixed point space of $-\varrho$ acting on the fiber $V_{(t,x)}$ is equal to the label of t in $\underline{3}$;

(ii) The composition $V \to T \times X \to X$ is a graphic map.

A smooth map $g\colon X \to Y$ induces a map $\mathcal{W}_{\mathrm{loc},T}(Y) \to \mathcal{W}_{\mathrm{loc},T}(X)$, given by pullback of vector bundles along $\mathrm{id} \times g\colon T \times X \to T \times Y$. This makes $\mathcal{W}_{\mathrm{loc},T}$ into a sheaf on $\mathscr{X}$.

In definition 5.1.2, the involution on V leads to an orthogonal vector bundle splitting $V = V^{\varrho} \oplus V^{-\varrho}$, where V^{ϱ} consists of the vectors fixed by ϱ and $V^{-\varrho}$ consists of the vectors fixed by $-\varrho$. In the next definition, $D(V^{\varrho})$ and $S(V^{-\varrho})$ denote the disk and sphere bundles associated with V^{ϱ} and $V^{-\varrho}$, respectively.

Definition 5.1.3 For T in $\mathscr{K}$, a sheaf $\mathcal{W}_T$ on $\mathscr{X}$ is defined as follows. For X in $\mathscr{X}$, an element of $\mathcal{W}_T(X)$ consists of

(i) a smooth graphic bundle $q\colon M \to X$ of compact oriented surfaces;

(ii) an element (V,ϱ) of $\mathcal{W}_{\mathrm{loc},T}(X)$;

(iii) a smooth and fiberwise orientation preserving embedding over X,

$$e\colon D(V^{\varrho}) \times_{T\times X} S(V^{-\varrho}) \quad\longrightarrow\quad M \smallsetminus \partial M\,.$$

Boundary condition: Near their respective boundaries, the manifolds M and $(S^1\times[0,1])\times X$ agree, and there q agrees as an oriented map with the projection to X.

The sheaves defined in 5.1.2 and 5.1.3 depend contravariantly on the variable T in $\mathscr{K}$. This is clear in the case of 5.1.2: A morphism $(k,\varepsilon)\colon S \to T$ in $\mathscr{K}$ induces a map from $\mathcal{W}_{\mathrm{loc},T}(X)$ to $\mathcal{W}_{\mathrm{loc},S}(X)$ given by pullback of vector bundles along $k\times\mathrm{id}\colon S\times X \to T\times X$. The case 5.1.3 is much more interesting. Let $(k,\varepsilon)\colon S \to T$ be a morphism in $\mathscr{K}$. If k is bijective, there is an obvious identification $\mathcal{W}_T \cong \mathcal{W}_S$ and this is the induced map. Therefore we may assume that k is an inclusion $S \hookrightarrow T$. Then we can reduce to the case where $T \smallsetminus S$ has exactly one element, a. This case has two subcases: $\varepsilon(a) = +1$ and $\varepsilon(a) = -1$.

Definition 5.1.4 Let $(k,\varepsilon)\colon S \to T$ be a morphism in $\mathscr{K}$ where k is an inclusion and $T \smallsetminus S = \{a\}$ with $\varepsilon(a) = +1$. We describe the induced map

$$\mathcal{W}_T(X) \longrightarrow \mathcal{W}_S(X).$$

Let (q, V, ϱ, e) be an element of $\mathcal{W}_T(X)$, with $q\colon M \to X$. Map this to an element of $\mathcal{W}_S$ by keeping $q\colon M \to X$, restricting V to $S \times X$ and restricting ϱ and e accordingly.

Definition 5.1.5 Let $(k,\varepsilon)\colon S \to T$ be a morphism in $\mathscr{K}$ where k is an inclusion and $T \smallsetminus S = \{a\}$ with $\varepsilon(a) = -1$. For X in $\mathscr{X}$, the induced map

$$\mathcal{W}_T(X) \longrightarrow \mathcal{W}_S(X)$$

is defined as follows. Let (q, V, ϱ, e) be an element of $\mathcal{W}_T(X)$, with $q\colon M \to X$. Map this to the element (q', V', ϱ', e') of $\mathcal{W}_S$ where

(1) $q'\colon M' \to X$ is the surface bundle obtained from $q\colon M \to X$ by fiberwise surgery on the embedded bundle of thickened spheres $e\big(D(V^{\varrho}|X_a) \times_{X_a} S(V^{-\varrho}|X_a)\big)$, where X_a means $a \times X$;

(2) (V', ϱ') is the restriction of (V, ϱ) to $S \times X$;

(3) e' is obtained from e by restriction.

Remark 5.1.6 The fiberwise surgery in (i) amounts to removing the interior of the embedded thickened sphere bundle and gluing in a copy of $D(V^{-\varrho}|X_a) \times_{X_a} S(V^{\varrho}|X_a)$ instead. Note in particular that when $V^{-\varrho} = 0$, the embedded thickened sphere bundle whose interior we have to remove is empty. In this case the fiberwise surgery consist in adding a disjoint copy of the sphere bundle $S(V)|X_a$ to M.

Remark 5.1.7 To ensure that $T \mapsto \mathcal{W}_T$ really is a contravariant functor on $\mathscr{K}$, it is wise to add two conditions of a set-theoretic nature to definition 5.1.3. Namely, e should be an inclusion and $M \smallsetminus \mathrm{im}(e)$ should have no elements in common with V. Surgeries as in definition 5.1.5 should then be performed by removing something from M which also happens to be a subset of V, and gluing in another subset of V. (All thickened sphere bundles in sight can be thought of as subbundles of V.)

There is a forgetful map of sheaves $\mathcal{W}_T \to \mathcal{W}_{\mathrm{loc},T}$. It has the concordance lifting property, so that the representing spaces of its fibers are the homotopy fibers of the induced map or representing spaces

$$|\mathcal{W}_T| \to |\mathcal{W}_{\mathrm{loc},T}|.$$

It is easy to see that the representing space of any fiber of $\mathcal{W}_T \to \mathcal{W}_{\mathrm{loc},T}$ is a classifying space for certain bundles of compact oriented surfaces with fixed boundary.

5.2 Morse singularities, Hessians and surgeries

We begin by recalling some well known facts about elementary and multi-elementary Morse functions. The reader is referred to [27, ch.I] and [28] for more details in the non-parametrized situation. By an *elementary* Morse function we shall mean a proper smooth map $E \to \mathbb{R}$ which is regular on ∂E and has exactly one critical point in $E \smallsetminus \partial E$, that one nondegenerate. By a *multi-elementary* Morse function we mean a proper smooth map $E \to \mathbb{R}$ which is regular on ∂E and has finitely many critical points in $E \smallsetminus \partial E$, all nondegenerate and all with the same critical value.

Fix a finite dimensional real vector space V with an inner product (i.e., a positive definite bilinear form) and a linear isometric involution $\varrho\colon V \to V$. Then the function $f_V\colon V \to \mathbb{R}$ given by

$$f_V(v) = \langle v, \varrho v\rangle \tag{5.2}$$

is a Morse function on V with exactly one critical point. If we write $V = V^{\varrho} \oplus V^{-\varrho}$, then the fomula for f_V becomes

$$f_V(v) = \|v_+\|^2 - \|v_-\|^2$$

where v_+ and v_- are the components of v in V^{ϱ} and $V^{-\varrho}$, respectively. The gradient of f_V on V is everywhere perpendicular to the gradient of $v \mapsto \|v_+\|^2\|v_-\|^2$, so that the latter function is constant on the trajectories of the gradient flow of f_V. This motivates the following definition.

Definition 5.2.1 $\mathrm{saddle}(V,\varrho) = \{v \in V \mid \|v_+\|^2\|v_-\|^2 \le 1\}$.

If $V^{\varrho} = 0$ or $V^{-\varrho} = 0$, then $\mathrm{saddle}(V,\varrho) = V$. In the remaining cases, the formula

$$v \mapsto (\|v_-\|v_+,\ \|v_-\|^{-1}v_-,\ \langle v, \varrho v\rangle) \tag{5.3}$$

defines a smooth embedding of $\mathrm{saddle}(V,\varrho) \smallsetminus V^{\varrho}$ in $D(V^{\varrho}) \times S(V^{-\varrho}) \times \mathbb{R}$, with complement $0 \times S(V^{-\varrho}) \times [0,\infty[$. It respects boundaries and takes the gradient vectors of $v \mapsto \langle v, \varrho v\rangle$ to tangent vectors which are parallel to the $\mathbb{R}$ factor. It is a map over $\mathbb{R}$, where we use the restriction of f_V on the source and the function $(x,y,t) \mapsto t$ on the target.
Dually, the formula

$$v \mapsto (\|v_+\|v_-,\ \|v_+\|^{-1}v_+,\ \langle v, \varrho v\rangle) \tag{5.4}$$

defines a smooth embedding of $\mathrm{saddle}(V,\varrho) \smallsetminus V^{-\varrho}$ in $D(V^{-\varrho}) \times S(V^{\varrho}) \times \mathbb{R}$, with complement $0 \times S(V^{\varrho}) \times]-\infty, 0]$. It respects boundaries and takes the gradient vectors of $v \mapsto \langle v, \varrho v\rangle$ to tangent vectors which are parallel to the $\mathbb{R}$ factor. It is also a map over $\mathbb{R}$.

The map f_V in (5.2) restricted to $\mathrm{saddle}(V,\varrho)$ is a good local model for elementary Morse functions. Let M be any smooth compact manifold and let

$$e\colon\ D(V^{\varrho}) \times S(V^{-\varrho}) \to M \smallsetminus \partial M \tag{5.5}$$

be a codimension zero embedding ("surgery data"), assuming $\dim(V)=\dim(M)+1$. Then in $M\times\mathbb{R}$ we have an embedded copy of $D(V^{\varrho})\times S(V^{-\varrho})\times\mathbb{R}$. We can remove its interior and glue in $\mathrm{saddle}(V,\varrho)$ instead, using formula (5.3) to identify the boundary of $\mathrm{saddle}(V,\varrho)$ with the boundary of $D(V^{\varrho})\times S(V^{-\varrho})\times\mathbb{R}$. The result is a smooth manifold $\mathrm{trace}(e)$ of dimension $\dim(M)+1$. More precisely:

Definition 5.2.2 The *long trace* of e, denoted $\mathrm{trace}(e)$, is the pushout of the two smooth codimension zero embeddings

$$
\begin{array}{ccc}
\mathrm{saddle}(V,\varrho)\smallsetminus V^{\varrho} & \xrightarrow{(e\times\mathrm{id})\circ(5.3)} & (M\times\mathbb{R})\smallsetminus e(0\times S(V^{-\varrho}))\times[0,\infty[\,, \\
\mathrm{saddle}(V,\varrho)\smallsetminus V^{\varrho} & \hookrightarrow & \mathrm{saddle}(V,\varrho).
\end{array}
\tag{5.6}
$$

For example, if $V^{-\varrho}=0$, then $\mathrm{saddle}(V,\varrho)=V$ and $\mathrm{saddle}(V,\varrho)\smallsetminus V^{\varrho}$ is empty, so that $\mathrm{trace}(e)$ becomes the disjoint union of $M\times V$ and $V=V^{\varrho}$. If $V^{\varrho}=0$, then M contains a codimension zero copy of $S(V)$. The long trace is obtained by removing $S(V)\times[0,\infty[$ from the copy of $S(V)\times\mathbb{R}$ in $M\times\mathbb{R}$ and adding a single point instead, so that $\mathrm{trace}(e)$ becomes the disjoint union of $(M\smallsetminus\mathrm{im}(e))\times\mathbb{R}$ and $V=V^{-\varrho}$.

The description 5.2.2 determines a structure of smooth manifold on $\mathrm{trace}(e)$ and shows that $\mathrm{trace}(e)$ comes with a (smooth) elementary Morse function, the *height function*, which is the projection to $\mathbb{R}$ on the complement of V^{ϱ} and equal to $v\mapsto\langle v,\varrho v\rangle$ on the glued-in copy of $\mathrm{saddle}(V,\varrho)$. The unique critical point is the origin of $V^{\varrho}\subset\mathrm{trace}(e)$. The corresponding critical value is 0.

Conversely, suppose that N is any smooth manifold with boundary and $g\colon N\to\mathbb{R}$ is an elementary Morse function, with critical value 0 and unique critical point $z\in N\smallsetminus\partial N$. Let $V=TN_z$. Choose an exponential map $h\colon V\to N\smallsetminus\partial N$, an inner product $\langle\,,\,\rangle$ on V, a linear isometric involution ϱ on V and $\delta>0$ such that $gh(v)=\langle v,\varrho v\rangle$ for all $v\in V$ with $\langle v,v\rangle<\delta$. This is possible by the Morse-Palais lemma; see for example [23]. At the price of replacing g by $3\delta^{-1}g$, we can assume $\delta=3$. Now choose a smooth vector field ξ on N which extends $h_*(\mathrm{grad}(gh))$ on $h(D(V^{\varrho})\times D(V^{-\varrho}))$, is tangential to ∂N and satisfies $\langle dg,\xi\rangle>0$ on $N\smallsetminus z$. For the function $f\colon V\to\mathbb{R}$ given by $v\mapsto\langle v,\varrho v\rangle$, we then have a unique smooth embedding $\bar{h}\colon\mathrm{saddle}(V,\varrho)\to N$ which extends h on $D(V^{\varrho})\times D(V^{-\varrho})$, maps gradient flow trajectories of f to flow trajectories of ξ and satisfies $g\circ\bar{h}=f$ on $\mathrm{saddle}(V,\varrho)$. This identifies N with a long trace.

The long trace construction has some obvious generalizations. For example, we can allow simultaneous surgeries on a finite number of pairwise disjoint thickened spheres. In this case the surgery data consist of a finite set T, a riemannian vector bundle V on T with an isometric involution ϱ, where $\dim(V)=\dim(M)+1$, and a smooth embedding

$$e\colon\ D(V^{\varrho})\times_T S(V^{\varrho})\longrightarrow M\smallsetminus\partial M\,.$$

Then trace(e) is defined as the manifold obtained from $M \times \mathbb{R}$ by deleting the embedded copy of

$$D(V_t^{\varrho}) \times S(V_t^{-\varrho}) \times \mathbb{R}$$

for each $t \in T$, and substituting saddle(V_t, ϱ) for it using formula (5.3) to do the gluing. There is a canonical height function on trace(e). It is a Morse function with one critical point for each $t \in T$. The only critical value is 0 (if $T \neq \emptyset$).

Then there is a parametrized version of the previous construction. Let $q\colon M \to X$ be a bundle of smooth compact n-manifolds, let $V \to T \times X$ be a riemannian vector bundle of fiber dimension $n+1$ with isometric involution ϱ, and let

$$e\colon D(V^{\varrho}) \times_{T \times X} S(V^{-\varrho}) \longrightarrow M \smallsetminus \partial M$$

be a smooth embeding over X. We can regard e as a family of embeddings e_x for $x \in X$, each from a disjoint union of finitely many thickened spheres to a fiber M_x of q. The manifolds trace(e_x) for $x \in X$ are the fibers of a smooth bundle

$$\mathrm{trace}(e) \longrightarrow X\,. \tag{5.7}$$

It comes equipped with a smooth height function $f\colon \mathrm{trace}(e) \longrightarrow \mathbb{R}$ which is fiberwise Morse; if $T \neq \emptyset$, then the unique critical value is 0.

For a useful naturality property of saddle(V, ϱ), suppose given a smooth orientation preserving embedding $e\colon \mathbb{R} \to \mathbb{R}$ such that $e(0) = 0$. Let $f_V\colon V \to \mathbb{R}$ be the canonical quadratic function, $f_V(v) = \langle v, \varrho v \rangle$.

Proposition 5.2.3 *There is a diffeomorphism*

$$\lambda\colon \mathrm{saddle}(V,\varrho) \quad\longrightarrow\quad f_V^{-1}(e(\mathbb{R})) \cap \mathrm{saddle}(V,\varrho)$$

such that $f_V \lambda = e f_V$ *on* saddle(V, ϱ) *and* $\lambda'(0) = e'(0) \times \mathrm{id}_V$.

Proof The Morse-Palais lemma as presented in [23], for example, gives us the germ of λ near $0 \in \mathrm{saddle}(V, \varrho) \subset V$. We may assume that this is defined on a neighborhood of a subset of saddle(V, ϱ) of the form

$$K_{a,b} = \{ v \in V \mid \|v_+\|^2 \|v_-\|^2 \le a \text{ and } |f_V(v)| \le b \}$$

where a and b are small positive numbers (to begin with). The boundary $\partial K_{a,b}$ is the union of a vertical and a horizontal part,

$$\begin{aligned} \partial_0 K_{a,b} &= \{ v \in V \mid \|v_+\|^2 \|v_-\|^2 = a \text{ and } |f_V(v)| \le b \}, \\ \partial_1 K_{a,b} &= \{ v \in V \mid \|v_+\|^2 \|v_-\|^2 \le a \text{ and } |f_V(v)| = b \}. \end{aligned}$$

Since f_V and $e f_V$ are both regular away from 0, in particular outside $K_{a,b}$, it is easy to construct an extension to all of saddle(V, ϱ) having the required properties. The extension can be made in two steps. In the first step, note that $K_{1,b}$ is the union of $K_{a,b}$ and a closed collar on $\partial_0 K_{a,b}$. Use this to define λ on all of $K_{1,b}$, in such a way that $K_{1,b}$ is mapped diffeomorphically to

$$\{v \in \mathrm{saddle}(V,\varrho) \mid |e^{-1} f_V(v)| \le b\}.$$

Then note that $\mathrm{saddle}(V,\varrho) = K_{1,\infty}$ is the union of $K_{1,b}$ and an open collar on $\partial_1 K_{1,b}$. Use this to define λ on all of $\mathrm{saddle}(V,\varrho)$. $\square$

5.3 Second row

As the subsection title indicates, we are going to concentrate on the second row of diagram (5.1), but we will also explain how it is related to the first row.

Definition 5.3.1 Let $\mathcal{L}_{\mathrm{loc}}$ be the following sheaf on $\mathscr{X}$. For X in $\mathscr{X}$, an element of $\mathcal{L}_{\mathrm{loc}}(X)$ is a triple (p,g,V) where

(i) p is a graphic and étale map from some smooth Y to X ;

(ii) g is a smooth function $Y \to \mathbb{R}$;

(iii) V is an oriented, smooth, riemannian 3-dimensional vector bundle on Y equipped with a linear isometric involution $\varrho\colon V \to V$ over Y.

Conditions: The map $(p,g)\colon Y \to X \times \mathbb{R}$ is proper and the composition of the vector bundle projection $V \to Y$ with $p\colon Y \to X$ is a graphic map.

Remark 5.3.2 Let (π, f) be an element of $\mathcal{W}_{\mathrm{loc}}(X)$, with $\pi\colon E \to X$ and $f\colon X \to \mathbb{R}$. Let $\Sigma = \Sigma(\pi, f)$ be the fiberwise singularity set of f. We showed in lemma 2.1.3 that $\pi|\Sigma$ is étale. By definition of $\mathcal{W}_{\mathrm{loc}}(X)$, the map $(\pi|\Sigma, f|\Sigma)\colon \Sigma \to X \times \mathbb{R}$ is proper. The restriction V of the vertical tangent bundle $T^\pi E$ to Σ comes with an everywhere nondegenerate symmetric bilinear form $\frac{1}{2}H$, where H is the vertical Hessian of f. See [27, I,§2]. We can choose an orthogonal splitting $V = V^+ \oplus V^-$ of $T^\pi E|\Sigma$ into a positive definite subbundle and a negative definite subbundle. By changing the sign of $\frac{1}{2}H$ on V^-, we make V into a riemannian vector bundle, with an isometric involution which is $-\mathrm{id}$ on V^- and $+\mathrm{id}$ on V^+. In this way, the element (π, f) of $\mathcal{W}_{\mathrm{loc}}(X)$ determines an element $(\pi|\Sigma,\ f|\Sigma,\ V) \in \mathcal{L}_{\mathrm{loc}}(X)$.

We come to the construction of the map from $\mathcal{L}_{\mathrm{loc}}$ to $\mathcal{W}_{\mathrm{loc}}$ which appears in diagram (5.1). Fix X in $\mathscr{X}$ and let (p,g,V) be an element of $\mathcal{L}_{\mathrm{loc}}(X)$, with $p\colon Y \to X$ and isometric involution $\varrho\colon V \to V$. Let E be the disjoint union of V and $(S^1 \times \mathbb{R} \times [0,1]) \times X$. Let $\pi\colon E \to \mathbb{R}$ agree with the composition $V \to Y \to X$ on the summand V, and with the projection to X on the other summand. Let $f\colon E \to \mathbb{R}$ be given by

$$f(v) = g(y) + \langle v, \varrho v\rangle$$

for $y \in Y$ and v in the fiber of V over y, and let f agree with the projection to $\mathbb{R}$ on the other summand. Then (π, f) is an element of $\mathcal{W}_{\mathrm{loc}}(X)$. The rule $(p,g,V) \mapsto (\pi, f)$ is our map.

Proposition 5.3.3 *The map $\mathcal{L}_{\text{loc}} \to \mathcal{W}_{\text{loc}}$ so defined is a weak homotopy equivalence.*

Proof We are going to use the relative surjectivity criterion of proposition 2.2.6. To deal with the absolute case first, we assume given X in $\mathscr{X}$ and $(\pi, f) \in \mathcal{W}_{\text{loc}}(X)$, with $\pi\colon E \to \mathbb{R}$ and $f\colon E \to \mathbb{R}$. Let $\Sigma = \Sigma(\pi, f)$ be the fiberwise singularity set of f. Choose a tubular neighborhood V of Σ in E such that the vector bundle projection $V \to \Sigma$ is over X. This is possible by lemma 2.1.3. Choose an open neighborhood E' of ∂E in E which, as a space over $X \times \mathbb{R}$, agrees with an open neighborhood of the boundary in $(S^1 \times \mathbb{R} \times [0,1]) \times X$. This is possible because of the boundary condition in the definition of $\mathcal{W}_{\text{loc}}$ (which is identical with the boundary condition in definition 2.1.6). By proposition 3.4.1, the element (π, f) in $\mathcal{W}_{\text{loc}}(X)$ is concordant to $(\pi^{(1)}, f^{(1)})$ where $\pi^{(1)}$ and $f^{(1)}$ are the restrictions of π and f to $V \cup E'$, respectively. Another application of proposition 3.4.1 gives us that $(\pi^{(1)}, f^{(1)})$ is concordant to $(\pi^{(2)}, f^{(2)})$, where $\pi^{(2)}$ and $f^{(2)}$ are the canonical extensions of $\pi^{(1)}$ and $f^{(1)}$ to

$$V \amalg (S^1 \times \mathbb{R} \times [0,1]) \times X .$$

In particular $f^{(2)}$ is still equal to f on V and is the projection to $\mathbb{R}$ on the summand $(S^1 \times \mathbb{R} \times [0,1]) \times X$. The next step is to improve $f^{(2)}|V = f|V$.
Let $\psi\colon [2,3] \to [0,1]$ be a smooth non-increasing function such that $\psi(t) = 1$ for t close to 2 and $\psi(t) = 0$ for t close to 3. For $t \in [2,3]$ let $f^{(t)}$ be given by

$$v \mapsto \begin{cases} fp(v) + \psi(t)^{-2}(f(\psi(t)v) - fp(v)) & \text{for } \psi(t) > 0 \text{ and } v \in V \\ fp(v) + \frac{1}{2}H(pv)(v,v) & \text{for } \psi(t) = 0 \text{ and } v \in V \end{cases}$$

where $H(pv)$ denotes the vertical Hessian of f at $p(v)$, alias second derivative in the fiber direction. Let $f^{(t)}$ agree with $f^{(2)}$ on $(S^1 \times \mathbb{R} \times [0,1]) \times X$ and let $\pi^{(t)} = \pi^{(2)}$ for convenience. Then $t \mapsto (\pi^{(t)}, f^{(t)})$ defines a concordance, parametrized by the interval $[2,3]$, from $(\pi^{(2)}, f^{(2)})$ to $(\pi^{(3)}, f^{(3)})$. To lift $(\pi^{(3)}, f^{(3)})$ to an element of $\mathcal{L}_{\text{loc}}(X)$ we only need to choose a maximal negative definite subbundle of V for (half) the vertical Hessian. Compare remark 5.3.2. We have now established the absolute case of the relative surjectivity condition of 2.2.6 for our map $\mathcal{L}_{\text{loc}} \to \mathcal{W}_{\text{loc}}$. The relative case is not much more difficult and we leave it to the reader. ◻

For later use we note the following:

Lemma 5.3.4 *Let $(p, g, V) \in \mathcal{L}_{\text{loc}}(X)$, with $p\colon Y \to X$. For every $x \in X$ and every $b > 0$ there exist a neighborhood U of x in X such that, on every component of $p^{-1}(U)$, the function g is either bounded below by $-b$ or bounded above by b.*

Proof Chose a descending sequence of open balls U_i for $i = 0, 1, 2, 3, \ldots$ forming a neighborhood basis for x in X. If the statement is false, then there exists $b > 0$ and connected subsets $K_i \subset Y$ for $i = 0, 1, 2, 3, \ldots$ such that $p(K_i) \subset U_i$ and $g(K_i) \supset [-b, b]$ for all i. Choose $z_i \in K_i$ such that $g(z_i) = 0$. The sequence $z_0, z_1, z_2, \ldots$ in Y must have a convergent (infinite) subsequence, because $(p, g)\colon Y \to X \times \mathbb{R}$ is proper and the two image sequences in X and $\mathbb{R}$ converge. Let $z_\infty \in Y$ be the point which the subsequence converges to. Then $p(z_\infty) = x$ and $g(z_\infty) = 0$. Now $p\colon Y \to X$ is étale. Hence, for sufficiently large i, there are unique neighborhoods U'_i of z_∞ in Y such that p maps U'_i diffeomorphically to U_i. It follows that $z_i \in U'_i$ for infinitely many i and hence $K_i \subset U'_i$ for infinitely many i. But it is also clear that the diameter of $g(U'_i)$ tends to zero as i tends to infinity; hence the lim inf of the diameters of the intervals $g(K_i)$ is zero, which contradicts our assumption. ◻

We turn to the left hand side of the second row in diagram (5.1).

Definition 5.3.5 For X in $\mathscr{X}$, an element of $\mathcal{L}(X)$ shall consist of

(i) an element $(\pi, f) \in \mathcal{W}(X)$, with $\pi\colon E \to X$ etc.,

(ii) an element $(p, g, V) \in \mathcal{L}_{\mathrm{loc}}(X)$, with $p\colon Y \to X$ etc.,

(iii) a smooth, orientation preserving embedding $\lambda\colon \mathrm{saddle}(V, \varrho) \longrightarrow E \smallsetminus \partial E$ over $X \times \mathbb{R}$ such that $\mathrm{im}(\lambda)$ contains $\Sigma(\pi, f)$.

Here $\mathrm{saddle}(V, \varrho)$ is a subbundle of V, defined as $\{v \in V \mid \|v_+\|^2 \|v_-\|^2 \le 1\}$. We have a canonical map $V \to \mathbb{R}$ defined by $v \mapsto g(y) + \langle v, \varrho v\rangle$ for v in a fiber V_y of the vector bundle V. (This was also used in the construction of the map $\mathcal{L}_{\mathrm{loc}} \to \mathcal{W}_{\mathrm{loc}}$.) In this way, $\mathrm{saddle}(V, \varrho)$ becomes a space over $X \times \mathbb{R}$. The conditions in (iii) then imply that λ identifies the zero section of V with the fiberwise singularity set $\Sigma(\pi, f)$.

Remark 5.3.6 The embedding $\lambda\colon \mathrm{saddle}(V) \to E \smallsetminus \partial E$ need not have a closed image, because the étale map $Y \to X$ need not be a closed map. But $\mathrm{im}(\lambda)$ is locally compact, therefore locally closed in E.

Proposition 5.3.7 *The forgetful map $\mathcal{L} \to \mathcal{W}$ is a weak homotopy equivalence.*

Proof Again we use the relative surjectivity criterion of proposition 2.2.6 and again we begin with the absolute case. Fix X in $\mathscr{X}$ and $(\pi, f) \in \mathcal{W}(X)$, with $\pi\colon E \to X$. We want to lift the concordance class of (π, f) to a class in $\mathcal{L}[X]$. As in the proof of proposition 5.3.3, we begin by choosing a vertical tubular neighborhood $V \subset E$ of $\Sigma = \Sigma(\pi, f)$, with vector bundle projection

$$\omega\colon V \to \Sigma$$

over X. Then V is canonically identified with $T^\pi E|\Sigma$, the restriction of the vertical tangent bundle of E to Σ. Using this identification, we define $f_V\colon V \to \mathbb{R}$ by

$$f_V(v) = \tfrac{1}{2} H(\omega(v))(v, v)\,,$$

where $H(\omega(v))$ is the second derivative of f, at $\omega(v)$, in the vertical direction. (This second derivative is a symmetric bilinear form on the vertical tangent space at $\omega(v)$, well defined because the vertical part of the first derivative of f at $\omega(v)$ vanishes.) By the Morse-Palais lemma [23], we can choose the vector bundle structure on V in such a way that $f(v) = f_V(v) + f\omega(v)$ holds in a neighborhood U of the zero section of V. Next we choose a positive definite inner product on V such that $f_V(v) = \langle v, \varrho v\rangle$ for a (unique) linear isometric involution $\varrho\colon V \to V$. Without loss of generality, the neighborhood U contains all $v \in \mathrm{saddle}(V, \varrho)$ for which $|f\omega(v)| \le 1$ and $|f_V(v) + f\omega(v)| \le 1$. If not, replace f by ψf where $\psi\colon E \to [1, \infty[$ is a suitable smooth function which factors through $\pi\colon E \to X$. Multiply the inner product on V by ψ, too. The pairs (π, f) and $(\pi, \psi f)$ are concordant. Summarizing, we can arrange

$$f(v) = f_V(v) + f\omega(v) \qquad\qquad \text{for } v \in V \text{ with } |f\omega(v)| \le 1 \text{ and } |f_V(v) + f\omega(v)| \le 1.$$

Now choose a smooth embedding $e\colon \mathbb{R} \to \mathbb{R}$ with $\mathrm{im}(e) =]-1, 1[$, equal to the identity near $0 \in \mathbb{R}$. Then (π, f) is concordant to $(\pi^\sharp, f^\sharp)$, where $\pi^\sharp$ is the restriction of π to $E^\sharp = f^{-1}(\mathrm{im}(e))$ and $f^\sharp$ is $e^{-1}f$ on $E^\sharp$. Let $\Sigma^\sharp = \Sigma \cap E^\sharp$ and $V^\sharp = V|\Sigma^\sharp$. Let

$$K = \Big\{ v \in \mathrm{saddle}(V^\sharp, \varrho) \;\Big|\; |f_V(v) + f\omega(v)| < 1 \Big\}.$$

Then $f|K = f_V|K + f\omega|K$ by our assumptions, so $K \subset E^\sharp$. Using proposition 5.2.3, we can construct an orientation preserving diffeomorphism

$$\lambda\colon \mathrm{saddle}(V^\sharp, \varrho) \longrightarrow K$$

relative to and over $\Sigma^\sharp$, such that $(f_V + f\omega)\lambda = e(f_V + f\omega)$. This can also be viewed as an embedding of $\mathrm{saddle}(V^\sharp, \varrho)$ in $E^\sharp$. Then

$$f^\sharp\lambda = e^{-1}f\lambda = e^{-1}(f_V + f\omega)\lambda = e^{-1}e(f_V + f\omega) = f_V + f\omega$$

on $\mathrm{saddle}(V^\sharp, \varrho)$. That is, λ promotes the pair $(\pi^\sharp, f^\sharp)$ to an element of $\mathcal{L}(X)$. This establishes the absolute case of the relative surjectivity condition.

The truly relative case is slightly more difficult. We sketch it. Again fix X in $\mathscr{X}$ and $(\pi, f) \in \mathcal{W}(X)$, with $\pi\colon E \to X$. Let $C \subset X$ be closed. We want to find an element in $\mathcal{L}(X)$ whose image in $\mathcal{W}(X)$ is concordant rel C to (π, f). This can be constructed essentially as in the absolute case, except for one change which consists in replacing the embedding $e\colon \mathbb{R} \to \mathbb{R}$ above by a smooth family of smooth embeddings $e_x\colon \mathbb{R} \to \mathbb{R}$, depending on $x \in X$. Then we have the option to choose $e_x = \mathrm{id}_{\mathbb{R}}$ for x in a very small neighborhood of C, while having $\mathrm{im}(e_x) \subset [-1, +1]$ for x outside a slightly larger neighborhood of C. $\square$

5.4 Third row

Definition 5.4.1 Fix S in $\mathscr{K}$ and X in $\mathscr{X}$. We define a sheaf $\mathcal{L}_{\mathrm{loc},S}$ on $\mathscr{X}$. For X in $\mathscr{X}$, an element of $\mathcal{L}_{\mathrm{loc},S}(X)$ is an element (p, g, V) of $\mathcal{L}_{\mathrm{loc}}(X)$, where p has source Y, together with

(i) an embedding $h\colon S \times X \to Y$ over $\underline{3} \times X$,

(ii) a continuous function $\delta\colon\ Y \smallsetminus \mathrm{im}(h) \longrightarrow \{-1,+1\}$.

Condition: Every $x \in X$ has a neighborhood U in X such that g admits a lower bound on $p^{-1}(U) \cap \delta^{-1}(+1)$ and an upper bound on $p^{-1}(U) \cap \delta^{-1}(-1)$.

In definition 5.4.1, the function δ has to be constant on each component of $Y \smallsetminus \mathrm{im}(h)$. Using partitions of unity, one can reformulate the local bound condition by a global one, as follows: there exists a continuous function $b\colon\ X \to \mathbb{R}$ such that $-bp \le g$ on $\delta^{-1}(+1)$ and $bp \ge g$ on $\delta^{-1}(-1)$.

A morphism $(k,\varepsilon)\colon\ R \to S$ in $\mathscr{K}$ induces a map $\mathcal{L}_{\mathrm{loc},S} \to \mathcal{L}_{\mathrm{loc},R}$ taking an element (p,g,V,h,δ) of $\mathcal{L}_{\mathrm{loc},S}(X)$ to (p,g,V',h',δ') where V' is obtained from V by pulling back, $h'(r,x) = h(k(r),x)$ for $(r,x) \in R \times X$ and

$$\delta'(y) = \begin{cases} \delta(y) & \text{if } \delta(y) \text{ is defined} \\ \varepsilon(s) & \text{if } y = h(s,x) \text{ for some } (s,x) \in (S \smallsetminus k(R)) \times X. \end{cases}$$

This makes the rule $T \mapsto \mathcal{L}_{\mathrm{loc},T}$ into a contravariant functor from $\mathscr{K}$ to the category of sheaves on $\mathscr{X}$. Moreover, for each T in $\mathscr{K}$ there is a forgetful map

$$\mathcal{L}_{\mathrm{loc},T} \to \mathcal{L}_{\mathrm{loc}}\,.$$

The maps $\mathcal{L}_{\mathrm{loc},T} \to \mathcal{L}_{\mathrm{loc},S}$ induced by morphisms $S \to T$ in $\mathscr{K}$ are over $\mathcal{L}_{\mathrm{loc}}$. This leads to a canonical map of sheaves

$$v\colon\ \underset{T \text{ in } \mathscr{K}}{\mathrm{hocolim}}\ \mathcal{L}_{\mathrm{loc},T} \quad\longrightarrow\quad \mathcal{L}_{\mathrm{loc}}\,. \tag{5.8}$$

Proposition 5.4.2 *The map v in (5.8) is a weak equivalence.*

Proof Let $\mathcal{L}^{\delta}_{\mathrm{loc}}$ be the following sheaf on $\mathscr{X}$ with category structure. An object of $\mathcal{L}^{\delta}_{\mathrm{loc}}(X)$ is an element (p,g,V) of $\mathcal{L}_{\mathrm{loc}}(X)$, with $p\colon\ Y \to X$, together with a continuous function $\delta\colon\ Y \to \{-1,0,+1\}$ subject to the following condition:

> Every $x \in X$ has a neighborhood U in X such that g admits a lower bound on $p^{-1}(U) \cap \delta^{-1}(+1)$, an upper bound on $p^{-1}(U) \cap \delta^{-1}(-1)$, and both an upper and a lower bound on $p^{-1}(U) \cap \delta^{-1}(0)$.

Given two such objects, (p,g,V,δ_a) and (p,g,V,δ_b) with the same underlying (p,g,V), we write $(p,g,V,\delta_a) \le (p,g,V,\delta_b)$ if $\delta_a^{-1}(+1) \subset \delta_b^{-1}(+1)$ and $\delta_a^{-1}(-1) \subset \delta_b^{-1}(-1)$. In this situation there is a unique morphism from (p,g,V,δ_a) to (p,g,V,δ_b), otherwise there is none. Thus the category $\mathcal{L}^{\delta}_{\mathrm{loc}}(X)$ is a poset.
The map v in (5.8) can now be factorized as follows:

$$\underset{T \text{ in } \mathscr{K}}{\mathrm{hocolim}}\ \mathcal{L}_{\mathrm{loc},T} \xrightarrow{v_1} \beta\mathcal{L}^{\delta}_{\mathrm{loc}} \xrightarrow{v_2} \mathcal{L}_{\mathrm{loc}} \tag{5.9}$$

Here v_2 is induced by the forgetful map $\mathcal{L}^\delta_{\mathrm{loc}} \to \mathcal{L}_{\mathrm{loc}}$. (Compare proposition 4.2.4.) To describe v_1 we recall that $\mathrm{hocolim}_T\, \mathcal{L}_{\mathrm{loc},T}$ was defined as

$$\beta(\mathscr{K}^{\mathrm{op}} \textstyle\int \mathcal{L}_{\mathrm{loc},\bullet}).$$

An object in $(\mathscr{K}^{\mathrm{op}} \int \mathcal{L}_{\mathrm{loc},\bullet})(X)$ consists of an object T in $\mathscr{K}$ and an element a in $\mathcal{L}_{\mathrm{loc},T}(X)$. A morphism from (T,a) to (S,b) is a morphism $S \to T$ in $\mathscr{K}$ taking a to b. An object (T,a) in $(\mathscr{K}^{\mathrm{op}} \int \mathcal{L}_{\mathrm{loc},\bullet})(X)$ with $a = (p,g,V,\delta,h)$ determines an object

$$(p,g,V,\bar{\delta})$$

in $\mathcal{L}^\delta_{\mathrm{loc}}(X)$, where $\bar{\delta}(z) = \delta(z)$ if $\delta(z)$ is defined and $\bar{\delta}(z) = 0$ otherwise. This canonical association is a functor, for each X, and as such induces v_1.
For simply connected X, the functor so defined, from $(\mathscr{K}^{\mathrm{op}} \int \mathcal{L}_{\mathrm{loc},\bullet})(X)$ to $\mathcal{L}^\delta_{\mathrm{loc}}(X)$, is clearly an equivalence of categories. (The point is that, by Ehresmann's fibration theorem, a proper étale map to a simply connected manifold is always a trivial bundle with finite fiber.) In particular, it is an equivalence of categories for the extended simplices, $X = \Delta^k_e$ where $k \geq 0$. Consequently v_1 in (5.9) is a weak equivalence. Compare section 4.1.
With a view to showing that v_2 is also a weak equivalence, we make the following observation. Given objects (p,g,V,δ_1) and (p,g,V,δ_2) in $\mathcal{L}^\delta_{\mathrm{loc}}(X)$, with the same underlying $(p,g,V) \in \mathcal{L}_{\mathrm{loc}}(X)$, there always exists an object (p,g,V,δ_3) in $\mathcal{L}^\delta_{\mathrm{loc}}(X)$ such that

$$\begin{array}{rcl} (p,g,V,\delta_3) & \leq & (p,g,V,\delta_1) \\ (p,g,V,\delta_3) & \leq & (p,g,V,\delta_2). \end{array}$$

Namely, let $\delta_3(z) = +1$ if and only if $\delta_1(z) = +1 = \delta_2(z)$; let $\delta_3(z) = -1$ if and only if $\delta_1(z) = -1 = \delta_2(z)$, and let $\delta_3(z) = 0$ in the remaining cases.
Now we apply proposition 2.2.6 to v_2. Given $(p,g,V) \in \mathcal{L}_{\mathrm{loc}}(X)$, we can by lemma 5.3.4 find a locally finite covering of X by open subsets U_j, where $j \in J$, such that $(p,g,V)\,|U$ has a lift φ_{jj} to $\mathrm{ob}(\mathcal{L}^\delta_{\mathrm{loc}})(U_j)$ for all j. With the observation just above, it is easy to extend the collection of the φ_{jj} to a collection of elements $\varphi_{RR} \in \mathrm{ob}(\mathcal{L}^\delta_{\mathrm{loc}})(U_R)$, in such a way that $\varphi_{RR} \leq \varphi_{QQ}|U_R$ whenever $Q \subset R$. The collection of these φ_{RR} is then an element of $\beta\mathcal{L}^\delta_{\mathrm{loc}}(X)$. This establishes the absolute case of the hypothesis in 2.2.6, and the verification is much the same in the relative case. $\square$

Definition 5.4.3 For T in $\mathscr{K}$, we define a sheaf $\mathcal{L}_T$ as the pullback of

$$\mathcal{L} \xrightarrow{\text{forget}} \mathcal{L}_{\mathrm{loc}} \xleftarrow{\text{forget}} \mathcal{L}_{\mathrm{loc},T}\,.$$

Remark 5.4.4 An element in $\mathcal{L}_T(X)$ consists of $(\pi,f) \in \mathcal{W}(X)$ with $\pi\colon E \to X$, an element (p,g,V,h,δ) in $\mathcal{L}_{\mathrm{loc},T}(X)$, with $p\colon Y \to X$ and $h\colon T \times X \to Y$, and a smooth embedding $\lambda\colon \mathrm{saddle}(V,\varrho) \to E$ over $X \times \mathbb{R}$ satisfying condition (iii) in definition 5.3.5. Here E denotes the source of π and f.

Definition 5.4.3 leads to a canonical map u from $\operatorname{hocolim}_T \mathcal{L}_T$ to $\mathcal{L}$.

Proposition 5.4.5 *The map* $u\colon \operatorname*{hocolim}_{T \text{ in } \mathscr{K}} \mathcal{L}_T \longrightarrow \mathcal{L}$ *is a weak equivalence.*

Proof The proof is completely analogous to that of proposition 5.4.2. There is a factorization of u having the form

$$\operatorname*{hocolim}_{T \text{ in } \mathscr{K}} \mathcal{L}_T \xrightarrow{u_1} \beta\mathcal{L}^\delta \xrightarrow{u_2} \mathcal{L} \tag{5.10}$$

where $\mathcal{L}^\delta$ is defined as the pullback of $\mathcal{L} \longrightarrow \mathcal{L}_{\text{loc}} \longleftarrow \mathcal{L}^\delta_{\text{loc}}$. One shows that u_1 and u_2 are weak equivalences. ◻

5.5 Fourth row, right hand column

Definition 5.5.1 Fix T in $\mathscr{K}$. We define a map from $\mathcal{L}_{\text{loc},T}$ to $\mathcal{W}_{\text{loc},T}$ by

$$\mathcal{L}_{\text{loc},T}(X) \ni (p,g,V,h,\delta) \quad \mapsto \quad h^*(V) \in \mathcal{W}_{\text{loc},T}(X).$$

To make sense of this formula, recall that $p\colon Y \to X$ denotes an étale map, V denotes a riemannian vector bundle with involution on Y and $h\colon T \times X \to Y$ is an embedding over X. Therefore $h^*(V)$ is a riemannian vector bundle with involution on $T \times X$.
There is an equally simple map in the other direction, $\mathcal{W}_{\text{loc},T} \to \mathcal{L}_{\text{loc},T}$. Namely, for X in $\mathscr{X}$ we can identify $\mathcal{W}_{\text{loc},T}(X)$ with a subset of $\mathcal{L}_{\text{loc},T}(X)$, consisting of the elements $(p,g,V,h,\delta) \in \mathcal{L}_{\text{loc},T}(X)$ which have $h = \mathrm{id}_{T\times X}$ and $g \equiv 0$.

Lemma 5.5.2 *The inclusion* $\mathcal{W}_{\text{loc},T} \to \mathcal{L}_{\text{loc},T}$ *is a weak equivalence.*

Proof We use proposition 2.2.6. Given $(p,g,V,h,\delta) \in \mathcal{L}_{\text{loc},T}(X)$ with $p\colon Y \to X$, choose a smooth $\psi\colon [0,1/2[\to [0,\infty[$ such that $\psi(s) = 0$ for s close to 0 and $\psi(s)$ tends to $+\infty$ for $s \to 1/2$. Choose another smooth $\varphi\colon [0,1] \to [0,1]$ such that $\varphi(s) = 1$ for s close to 0 and $\varphi(s) = 0$ for s close to 1. Then define

$$(\bar p, \bar g, \bar V, \bar h, \bar\delta) \in \mathcal{L}_{\text{loc},T}(X \times]0,1[)$$

in the following way. The source of $\bar p$ is the union of $Y \times]0,1/2[$ and $h(T \times X) \times]0,1[$. The formula for $\bar p$ is $\bar p(y,s) = (p(y),s)$. (To ensure that $\bar p$ is graphic, we should define the source of $\bar p$ and $\bar g$ as a subset of the pullback of $p\colon Y \to X$ along the projection $X \times]0,1[\longrightarrow X$. See definition 2.1.1.) The formula for $\bar g$ is $\bar g(y,s) := g(y)\cdot\varphi(s)$ if y is in $h(T \times X)$ and $\bar g(y,s) := g(y) + \delta(y)\psi(s)$ otherwise. The vector bundle $\bar V$ is the pullback of V under the projection. The formula for $\bar h$ is $\bar h(t,x,s) := (h(t,x),s)$ and the formula for $\bar\delta$ is $\bar\delta(y,s) = \delta(y)$. By inspection, $(\bar p, \bar g, \bar V, \bar h, \bar\delta)$ is a concordance from $(p,g,V,h,\delta) \in \mathcal{L}_{\text{loc},T}(X)$ to an element $(p^\flat, g^\flat, V^\flat, h^\flat, \delta^\flat) \in \mathcal{L}_{\text{loc},T}(X)$ where $h^\flat$ is a homeomorphism and $g^\flat \equiv 0$. With some renaming we can arrange $h^\flat$ to be an identity map, so that $(p^\flat, g^\flat, V^\flat, h^\flat, \delta^\flat) \in \mathcal{W}_{\mathrm{loc},T}(X)$. If a closed subset C of X is given, and the restriction of (p, g, V, h, δ) to some open neighborhood U of C is already in $\mathcal{W}_{\mathrm{loc},T}(U)$, then the concordance just constructed is constant on U, giving the relative surjectivity condition in proposition 2.2.6. $\square$

Corollary 5.5.3 *The map $\mathcal{L}_{\mathrm{loc},T} \to \mathcal{W}_{\mathrm{loc},T}$ of definition 5.5.1 is a weak equivalence.*

Proof The composite map, from $\mathcal{W}_{\mathrm{loc},T}$ to $\mathcal{L}_{\mathrm{loc},T}$ and back to $\mathcal{W}_{\mathrm{loc},T}$, is clearly a weak equivalence. $\square$

5.6 Fourth row, left hand column

The goal is to write down a map $\mathcal{L}_T \to \mathcal{W}_T$, depending naturally on T in $\mathscr{K}$, and to show that it is a weak equivalence.

We organise the information contained in a single element of $\mathcal{L}_T(X)$ as in remark 5.4.4 and use the same notation. Write $\omega\colon V \to Y$ for the vector bundle projection, $\langle\,,\,\rangle$ for the inner product on V and $\varrho\colon V \to V$ for the isometric involution, as usual. In addition let

$$\begin{aligned} C_+ &= \{v \in \mathrm{saddle}(V, \varrho) \mid \delta(\omega(v)) = +1 \text{ and } \langle v, \varrho v\rangle \geq -1\}, \\ C_0 &= \{v \in \mathrm{saddle}(V, \varrho) \mid \delta(\omega(v)) \text{ undefined }\}, \\ C_- &= \{v \in \mathrm{saddle}(V, \varrho) \mid \delta(\omega(v)) = -1 \text{ and } \langle v, \varrho v\rangle \leq +1\}. \end{aligned}$$

The images $\lambda(C_+)$, $\lambda(C_0)$ and $\lambda(C_-)$ are closed subsets of E, despite remark 5.3.6. They are also codimension zero submanifolds of E, with corners in the case of C_+ and C_-. Writing U for the various connected components of V, let

$$\begin{aligned} C_+^{\mathrm{rg}} &= \coprod_{U \text{ with } \delta\omega|U \equiv +1} D(U^\varrho) \times_Y S(U^{-\varrho}) \times [-1, \infty[\,, \\ C_0^{\mathrm{rg}} &= \coprod_{U \text{ with } \delta\omega|U \text{ undef.}} D(U^\varrho) \times_Y S(U^{-\varrho}) \times \mathbb{R}\,, \\ C_-^{\mathrm{rg}} &= \coprod_{U \text{ with } \delta\omega|U \equiv -1} D(U^{-\varrho}) \times_Y S(U^{\varrho}) \times\,]-\infty, +1]\,. \end{aligned}$$

Then we have identifications

$$\begin{aligned} \partial C_+ &\cong \partial C_+^{\mathrm{rg}} && \text{by (5.3)}, \\ \partial C_0 &\cong \partial C_0^{\mathrm{rg}} && \text{by (5.3)}, \\ \partial C_- &\cong \partial C_-^{\mathrm{rg}} && \text{by (5.4)}. \end{aligned}$$

Definition 5.6.1 The *regularization* E^{rg} of E above is obtained by removing the interior of the closed codimension zero submanifold $\lambda(C_+ \cup C_0 \cup C_-)$ and gluing in a copy of $C_+^{\mathrm{rg}} \cup C_0^{\mathrm{rg}} \cup C_-^{\mathrm{rg}}$ instead, using the boundary identifications $\partial C_+ \cong \partial C_+^{\mathrm{rg}}$, $\partial C_0 \cong \partial C_0^{\mathrm{rg}}$ and $\partial C_- \cong \partial C_-^{\mathrm{rg}}$ just defined.

Remark 5.6.2 More precisely, E^{rg} is defined in two steps. First, remove $\lambda(C_+ \cap V^{\varrho})$, $\lambda(C_0 \cap V^{\varrho})$ and $\lambda(C_- \cap V^{-\varrho})$ from E. The result is a manifold with disjoint, properly embedded codimension zero copies of $C_+ \smallsetminus V^{\varrho}$, $C_0 \smallsetminus V^{\varrho}$ and $C_- \smallsetminus V^{-\varrho}$. Then make a (triple) cobase change along the codimension zero embeddings

$$\begin{array}{ccc} C_+ \smallsetminus V^{\varrho} & \longrightarrow & C_+^{\mathrm{rg}}, \\ C_0 \smallsetminus V^{\varrho} & \longrightarrow & C_0^{\mathrm{rg}}, \\ C_- \smallsetminus V^{-\varrho} & \longrightarrow & C_-^{\mathrm{rg}} \end{array}$$

determined by (5.3) and (5.4). This description gives a preferred structure of smooth manifold on E^{rg}.

The manifolds C_+^{rg}, C_0^{rg} and C_-^{rg} come with a canonical map to X, via the projections to Y. They also come with a canonical map to $\mathbb{R}$, given by $(v,t) \mapsto t + \langle v, \varrho v\rangle$. Under the identifications in 5.6.2, these maps match $\pi\colon E \to X$ and $f\colon E \to \mathbb{R}$, respectively, which leads to well defined and smooth maps

$$\begin{array}{rcl} \pi^{\mathrm{rg}}\colon E^{\mathrm{rg}} & \to & X, \\ f^{\mathrm{rg}}\colon E^{\mathrm{rg}} & \to & \mathbb{R}. \end{array}$$

By construction, π^{rg} is still a submersion and the product map $(\pi^{\mathrm{rg}}, f^{\mathrm{rg}})\colon E^{\mathrm{rg}} \to X \times \mathbb{R}$ is still proper. But in addition f^{rg} is regular when restricted to any fiber of π^{rg}. Therefore and by Ehresmann's fibration theorem we have proved

Proposition 5.6.3 *The map $(\pi^{\mathrm{rg}}, f^{\mathrm{rg}})\colon E^{\mathrm{rg}} \to X \times \mathbb{R}$ is a bundle of smooth compact surfaces.* $\square$

Keeping the above notation, let $M = \{z \in E^{\mathrm{rg}} \mid f^{\mathrm{rg}}(z) = 0\}$ and let $q = \pi^{\mathrm{rg}}|M$. Then $q\colon M \to X$ is a bundle of smooth compact surfaces by proposition 5.6.3. The intersection of M with the embedded copy of C_0^{rg} in E^{rg} is identified with

$$D(h^*V^{\varrho}) \times_{T \times X} S(h^*V^{-\varrho}). \tag{5.11}$$

The surface bundle $q\colon M \to X$, the riemannian vector bundle $h^*V \to T \times X$ and the canonical embedding e of (5.11) in M now constitute an element of $\mathcal{W}_T(X)$.

Definition 5.6.4 The map $\mathcal{L}_T \to \mathcal{W}_T$ promised in diagram (5.1) takes the element in remark 5.4.4 to $(q, h^*V, e) \in \mathcal{W}_T(X)$, where $q = \pi^{\mathrm{rg}}|M$ and $M = \{z \in E^{\mathrm{rg}} \mid f^{\mathrm{rg}}(z) = 0\}$.

By inspection, the map 5.6.4 is natural in the variable X and in the variable T, where T runs through the objects of $\mathscr{K}$. It makes diagram (5.1) commutative.

Remark 5.6.5 There is a slightly different way to describe the regularization process; it will help us in proving that the map $\mathcal{L}_T \to \mathcal{W}_T$ defined in 5.6.4 is a weak equivalence. Keeping the notation of remark 5.4.4, let

$$\begin{aligned} K_+ &= \{v \in \mathrm{saddle}(V,\varrho) \mid \delta(\omega(v)) = +1\}, \\ K_0 = C_0 &= \{v \in \mathrm{saddle}(V,\varrho) \mid \delta(\omega(v)) \text{ undefined } \}, \\ K_- &= \{v \in \mathrm{saddle}(V,\varrho) \mid \delta(\omega(v)) = -1\}, \end{aligned}$$

so that $\mathrm{saddle}(V,\varrho) = K_+ \cup K_0 \cup K_-$. The regularized versions are

$$\begin{aligned} K_+^{\mathrm{rg}} &= \coprod_{U \text{ with } \delta\omega|U\equiv+1} D(U^\varrho) \times_Y S(U^{-\varrho}) \times \mathbb{R}\,, \\ K_0^{\mathrm{rg}} &= \coprod_{U \text{ with } \delta\omega|U \text{ undef.}} D(U^\varrho) \times_Y S(U^{-\varrho}) \times \mathbb{R}\,, \\ K_-^{\mathrm{rg}} &= \coprod_{U \text{ with } \delta\omega|U\equiv-1} D(U^{-\varrho}) \times_Y S(U^{\varrho}) \times \mathbb{R}. \end{aligned}$$

We can re-define E^{rg} as follows: First, remove $\lambda(K_+ \cap V^\varrho)$, $\lambda(K_0 \cap V^\varrho)$ and $\lambda(K_- \cap V^{-\varrho})$ from E. The result is a manifold with disjointly embedded codimension zero copies of $K_+ \smallsetminus V^\varrho$, $K_0 \smallsetminus V^\varrho$ and $K_- \smallsetminus V^{-\varrho}$. Then make a (triple) cobase change along the codimension zero embeddings

$$\begin{aligned} K_+ \smallsetminus V^\varrho &\longrightarrow K_+^{\mathrm{rg}}, \\ K_0 \smallsetminus V^\varrho &\longrightarrow K_0^{\mathrm{rg}}, \\ K_- \smallsetminus V^{-\varrho} &\longrightarrow K_-^{\mathrm{rg}} \end{aligned}$$

determined by (5.3) and (5.4).

Comparison with remark 5.6.2 shows that this new description of E^{rg} agrees with the old one up to a canonical diffeomorphism (over $X \times \mathbb{R}$). The new description has the advantage of giving us a canonical (in general non-closed) codimension zero embedding

$$K_+^{\mathrm{rg}} \cup K_0^{\mathrm{rg}} \cup K_-^{\mathrm{rg}} \longrightarrow E^{\mathrm{rg}}.$$

Intersecting its image with $M = \{z \in E^{\mathrm{rg}} \mid f^{\mathrm{rg}}(z) = 0\}$, we get a canonical (and in general non-closed) embedding

$$D(V^{\bar\varrho}) \times_Y S(V^{-\bar\varrho}) \longrightarrow M \tag{5.12}$$

where $\bar\varrho\colon V \to V$ is equal to $-\varrho$ on connected components of V having $\delta\omega \equiv -1$, and $\bar\varrho = \varrho$ elsewhere on V. This extends the embedding of (5.11) into M and leads to a factorization of the map $\mathcal{L}_T \to \mathcal{W}_T$ from definition 5.6.4, which we now make explicit.

Definition 5.6.6 Let $\hat{\mathcal{W}}_T$ be the sheaf on $\mathscr{X}$ defined as follows. For X in $\mathscr{X}$, an element of $\hat{\mathcal{W}}_T(X)$ consists of

(i) a smooth graphic bundle $q\colon M \to X$ of compact oriented surfaces;

(ii) an element (p, g, V, h, δ) of $\mathcal{L}_{\mathrm{loc},T}(X)$, with $p\colon Y \to X$;

(iii) a smooth and fiberwise orientation preserving embedding over X,

$$e\colon D(V^{\bar{\varrho}}) \times_Y S(V^{-\bar{\varrho}}) \quad\longrightarrow\quad M \smallsetminus \partial M\,.$$

(It is understood that V in (ii) comes with an involution ϱ; as in (5.12), this determines $\bar{\varrho}\colon V \to V$.) *Boundary condition*: as in 5.1.3.

There is a forgetful map $\hat{\mathcal{W}}_T \to \mathcal{W}_T$ obtained by passing from V in (ii) of definition 5.6.6 to h^*V, and making the corresponding changes in (iii). By the observations leading up to definition 5.6.6, the map $\mathcal{L}_T \to \mathcal{W}_T$ in definition 5.6.4 has a factorization

$$\mathcal{L}_T \longrightarrow \hat{\mathcal{W}}_T \xrightarrow{\text{forget}} \mathcal{W}_T\,. \tag{5.13}$$

Lemma 5.6.7 *The map $\mathcal{L}_T \to \hat{\mathcal{W}}_T$ in (5.13) is a weak equivalence.*

Proof An map which is inverse to $\mathcal{L}_T \to \hat{\mathcal{W}}_T$ up to canonical concordances can be defined as follows. Given a surface bundle $q\colon M \to X$, an element $(p, g, V, h, \delta) \in \mathcal{L}_{\mathrm{loc},T}(X)$ and an embedding e as in definition 5.6.6, let $E^{\mathrm{rg}} = M \times \mathbb{R}$ and define C_+^{rg}, C_0^{rg}, C_-^{rg}, C_+, C_0 and C_- exactly as in definition 5.6.1. Make an embedding

$$C_+^{\mathrm{rg}} \cup C_0^{\mathrm{rg}} \cup C_-^{\mathrm{rg}} \quad\longrightarrow\quad E^{\mathrm{rg}}$$

by $(v, t) \mapsto (e(v), t + g(\omega(v)))$, where $\omega\colon V \to Y$ is the vector bundle projection. Remove from $M \times \mathbb{R}$ the interior of the image of this embedding, and glue in $C_+ \cup C_0 \cup C_-$. Call the result E. This comes with a submersion $\pi\colon E \to X$ and a smooth $f\colon E \to \mathbb{R}$ which is fiberwise Morse. There is also a canonical embedding $\lambda\colon \mathrm{saddle}(V, \varrho) \to E$; to see this more clearly, reason as in remark 5.6.5. The result is therefore an element of $\mathcal{L}_T(X)$, consisting of (π, f), the element $(p, g, V, h, \delta) \in \mathcal{L}_{\mathrm{loc},T}(X)$ and the embedding λ. □

Lemma 5.6.8 *The forgetful map $\hat{\mathcal{W}}_T \to \mathcal{W}_T$ in (5.13) is a weak equivalence.*

Proof The map has a section $\mathcal{W}_T \to \hat{\mathcal{W}}_T$. This identifies each set $\mathcal{W}_T(X)$ with the subset of $\hat{\mathcal{W}}_T(X)$ obtained by adding the conditions $h = \mathrm{id}_{T\times X}$ and $g \equiv 0$ in definition 5.6.6(ii). It suffices to show that the section $\mathcal{W}_T \to \hat{\mathcal{W}}_T$ satisfies the relative surjectivity criterion of proposition 2.2.6.
Let a surface bundle $q\colon M \to X$, an element $(p, g, V, h, \delta) \in \mathcal{L}_{\mathrm{loc},T}(X)$ and an embedding e as in definition 5.6.6 be given. The proof of lemma 5.5.2 gives us an explicit concordance

$$(\bar{p}, \bar{g}, \bar{V}, \bar{h}, \bar{\delta}) \in \mathcal{L}_{\mathrm{loc},T}(X \times \,]0,1[\,) \tag{5.14}$$

from $(p, g, V, h, \delta) \in \mathcal{L}_{\mathrm{loc},T}(X)$ to an element in the image of $\mathcal{W}_{\mathrm{loc},T}(X) \to \mathcal{L}_{\mathrm{loc},T}(X)$. Here $\bar{p}$ is obtained from $p \times \mathrm{id}\colon Y \times]0,1[\longrightarrow X \times]0,1[$ by restriction to an open subset, and similarly the vector bundle $\bar{V}$ is obtained from $V \times]0,1[$ by restriction. It is therefore clear that (5.14) lifts to a concordance between elements of $\hat{\mathcal{W}}_T(X)$, in such a way that the underlying surface bundle of the concordance is

$$\bar{q} = q \times \mathrm{id}_{]0,1[}\colon M \times]0,1[\longrightarrow X \times]0,1[$$

and the underlying embedding $\bar{e}$ is obtained from $e \times \mathrm{id}_{]0,1[}$ by restriction. If a closed subset C of X is given, and the restriction of (p, g, V, h, δ) to some open neighborhood U of C is already in $\mathcal{W}_{\mathrm{loc},T}(U)$, then the concordance so constructed is constant on U. □

Corollary 5.6.9 *The map $\mathcal{L}_T \to \mathcal{W}_T$ in definition 5.6.4 is a weak equivalence.* □

This completes the construction of diagram (5.1) and the verification that all the vertical arrows in it are weak equivalences.

5.7 Using the concordance lifting property

Lemma 5.7.1 *For fixed T in $\mathscr{X}$, the forgetful map $\mathcal{W}_T \to \mathcal{W}_{\mathrm{loc},T}$ has the concordance lifting property.*

Proof Let X be a smooth manifold. Any riemannian vector bundle on $T \times X \times [0,1]$ with isometric involution is isomorphic to the pullback of a riemannian vector bundle on $T \times X$ (with isometric involution) along the projection $T \times X \times [0,1] \to T \times X$. Consequently, any concordance starting at an element z of $\mathcal{W}_{\mathrm{loc},T}(X)$ is trivial up to an isomorphism of vector bundles. A choice of such a trivializing isomorphism determines, for each $y \in \mathcal{W}_T(X)$ which lifts z, a lifted concordance starting at y. □

Now fix an element (V, ϱ) in $\mathcal{W}_{\mathrm{loc},T}(\star)$. That is, V is an oriented 3-dimensional riemannian vector bundle on T, with a fiberwise isometric involution ϱ. For each $t \in T$, the dimension of the eigenspace $V_t^{-\varrho}$ is equal to the label of t in $\underline{3}$. The following is true by definition.

Lemma 5.7.2 *The fiber of the forgetful map $\mathcal{W}_T \to \mathcal{W}_{\mathrm{loc},T}$ over $(V, \varrho) \in \mathcal{W}_{\mathrm{loc},T}(\star)$ is the sheaf which takes an X in $\mathscr{X}$ to the set of all pairs (q, e) where*

(i) *q denotes a smooth graphic bundle $M \to X$ of compact oriented surfaces, subject to a boundary condition as in definition 5.1.3;*

(ii) *$e\colon D(V^{\varrho}) \times_T S(V^{-\varrho}) \times X \longrightarrow M \smallsetminus \partial M$ is a smooth embedding over X which is fiberwise orientation preserving.* □

Corollary 5.7.3 *The fiber of the forgetful map $\mathcal{W}_T \to \mathcal{W}_{\mathrm{loc},T}$ over $(V,\varrho) \in \mathcal{W}_{\mathrm{loc},T}(\star)$ is weakly equivalent to the sheaf which takes an X in $\mathscr{X}$ to the set of all smooth graphic bundles $q\colon M \to X$ of oriented compact surfaces, where each fiber has its (oriented) boundary identified with*

$$\partial(S^1 \times [0,1]\,)\ \amalg\ -S(V^{\varrho}) \times_T S(V^{-\varrho}).$$

Proof To get from data (q,e) as in lemma 5.7.2 to the kind of bundle described in corollary 5.7.3, delete the interior of $\mathrm{im}(e)$ from the total space of the surface bundle q. To get from a surface bundle $M \to X$ as in corollary 5.7.3 to the data described in lemma 5.7.2, form the union of M and $(D(V^{\varrho}) \times_T S(V^{-\varrho})) \times X$ along $(S(V^{\varrho}) \times_T S(V^{-\varrho})) \times X$. □

Remark 5.7.4 The description of the (homotopy) fiber in corollary 5.7.3 uses only the part of T lying over $\{1,2\} \subset \underline{3}$, since spheres of dimension -1 are empty.

6 The connectivity problem

6.1 Overview and definitions

The previous section gave us decompositions of $\mathcal{W}$ and $\mathcal{W}_{\mathrm{loc}}$ into pieces $\mathcal{W}_S$ and $\mathcal{W}_{\mathrm{loc},S}$, respectively, and a description of the homotopy fibers of the forgetful maps

$$\mathcal{W}_S \longrightarrow \mathcal{W}_{\mathrm{loc},S}$$

as certain surface bundle theories, cf. corollary 5.7.3. For a given S in $\mathscr{K}$, the surfaces involved are typically not connected, so that the representing space of the fiber theory is not directly related to the moduli space whose group completion we are studying. In this section we remedy this by showing that upon taking the homotopy colimit over S, we can in fact assume that the relevant surfaces are connected.

Definition 6.1.1 For X in $\mathscr{X}$ let $\mathcal{W}_{c,S}(X) \subset \mathcal{W}_S(X)$ consist of the triples (q,V,e) as in definition 5.1.3, with $q\colon M \to X$ etc., such that the surface bundle $M \smallsetminus \mathrm{im}(e) \longrightarrow X$ has connected fibers.

Then $\mathcal{W}_{c,S}$ is a subsheaf of $\mathcal{W}_S$ and $|\mathcal{W}_{c,S}|$ is a union of connected components of $|\mathcal{W}_S|$. The forgetful map from $\mathcal{W}_{c,S}$ to $\mathcal{W}_{\mathrm{loc},S}$ still has the concordance lifting property. By analogy with corollary 5.7.3, we have the following analysis of its fibers.

Corollary 6.1.2 *The fiber of the forgetful map $\mathcal{W}_{c,S} \to \mathcal{W}_{\mathrm{loc},S}$ over $V \in \mathcal{W}_{\mathrm{loc},S}(\star)$ is weakly equivalent to the sheaf which takes an X in $\mathscr{X}$ to the set of all smooth graphic bundles $q\colon M \to X$ of oriented compact connected surfaces, where the boundary of each fiber M_x is identified with*

$$\partial(S^1 \times [0,1]\,)\ \amalg\ -(S(V^{\varrho}) \times_S S(V^{-\varrho})).$$

It would therefore be nice to have a statement saying that the inclusion of $\operatorname{hocolim}_S \mathcal{W}_{c,S}$ in $\operatorname{hocolim}_S \mathcal{W}_S$ is a weak equivalence. Unfortunately such a statement is nonsensical if we insist on letting S run through the entire category $\mathscr{K}$. We have a contravariant functor $S \mapsto \mathcal{W}_S$ from $\mathscr{K}$ to the category of sheaves on $\mathscr{X}$, but we do *not* have a subfunctor $S \mapsto \mathcal{W}_{c,S}$. It is not the case that the map

$$(k,\varepsilon)^*\colon\ \mathcal{W}_T \to \mathcal{W}_S$$

induced by a morphism $(k,\varepsilon)\colon\ S \to T$ in $\mathscr{K}$ will always map the subsheaf $\mathcal{W}_{c,T}$ to the subsheaf $\mathcal{W}_{c,S}$. Let us take a more careful look at this phenomenon.
We may assume that k is an inclusion and that $T \smallsetminus S$ has exactly one element t, with label $\lambda(t) \in \underline{3}$ and sign $\varepsilon(t) \in \{\pm 1\}$. Fix (q,V,e) in $\mathcal{W}_T(X)$, with $q\colon\ M \to X$ and let (q',V',e') be the image of (q,V,e) in $\mathcal{W}_S(X)$, with $q'\colon\ M' \to X$. For each $x \in X$ there is a canonical embedding of surfaces

$$M_x \smallsetminus \operatorname{im}(e_x) \longrightarrow M'_x \smallsetminus \operatorname{im}(e'_x).$$

The complement of its image is identified with

$$\begin{array}{lll} D(V^{\varrho}_{(t,x)}) \times S(V^{-\varrho}_{(t,x)}) & \text{if} \quad \varepsilon(t) = +1, & \text{and} \\ S(V^{\varrho}_{(t,x)}) \times D(V^{-\varrho}_{(t,x)}) & \text{if} \quad \varepsilon(t) = -1, & \end{array}$$

where $V_{(t,x)}$ is the fiber of V over $(t,x) \in T \times X$. We have a problem when the complement is nonempty but has empty boundary, because then it will contribute an additional connected component. This happens precisely when $(\lambda(t),\varepsilon(t)) = (3,+1)$ and when $(\lambda(t),\varepsilon(t)) = (0,-1)$. In all other cases, there is no problem.

Now our indexing category $\mathscr{K}$ is equivalent to a product $\mathscr{K}_{03} \times \mathscr{K}_{12}$. The categories $\mathscr{K}_{03}$ and $\mathscr{K}_{12}$ can be described as full subcategories of $\mathscr{K}$: namely, $\mathscr{K}_{03}$ is spanned by the objects S whose reference map $S \to \underline{3}$ has image contained in $\{0,3\}$ and $\mathscr{K}_{12}$ is spanned by the objects S whose reference map $S \to \underline{3}$ has image contained in $\{1,2\}$.
For homotopy colimits of functors from a product category to spaces (or to sheaves on $\mathscr{X}$) there is a Fubini principle. In our case it states that

$$\operatorname*{hocolim}_{T \text{ in } \mathscr{K}} \mathcal{W}_T \simeq \operatorname*{hocolim}_{Q \text{ in } \mathscr{K}_{03}} \ \operatorname*{hocolim}_{S \text{ in } \mathscr{K}_{12}} \mathcal{W}_{Q \amalg S}. \tag{6.1}$$

Lemma 6.1.3 *For any morphism $(k,\varepsilon)\colon\ P \to Q$ in $\mathscr{K}_{03}$, the commutative square*

$$\begin{tikzcd} \operatorname*{hocolim}_{S \text{ in } \mathscr{K}_{12}} \mathcal{W}_{Q\amalg S} \arrow[r, "(k,\varepsilon)^*"] \arrow[d] & \operatorname*{hocolim}_{S \text{ in } \mathscr{K}_{12}} \mathcal{W}_{P\amalg S} \arrow[d] \\ \operatorname*{hocolim}_{S \text{ in } \mathscr{K}_{12}} \mathcal{W}_{\mathrm{loc},Q\amalg S} \arrow[r, "(k,\varepsilon)^*"] & \operatorname*{hocolim}_{S \text{ in } \mathscr{K}_{12}} \mathcal{W}_{\mathrm{loc},P\amalg S} \end{tikzcd}$$

is homotopy cartesian (after passage to representing spaces).

Theorem 6.1.4 *The inclusion*

$$\operatorname*{hocolim}_{S \text{ in } \mathscr{K}_{12}} \mathcal{W}_{c,S} \longrightarrow \operatorname*{hocolim}_{S \text{ in } \mathscr{K}_{12}} \mathcal{W}_S$$

is a weak equivalence.

Theorem 6.1.4 is the main result of the section. We develop a surgery method to prove it. The idea is to make nonconnected surfaces connected by means of multiple surgeries on embedded (thickened) 0-spheres. Then we need to know that such multiple 0-surgeries on a surface are essentially unique. In order to state the uniqueness, we view them as the objects of a category.

6.2 Categories of multiple surgeries

Definition 6.2.1 Let M be a compact, smooth, nonempty surface. Let $\mathscr{C}_M$ be the topological category defined as follows. An object consists of a finite set T and a smooth orientation preserving embedding e_T of $D^2 \times S^0 \times T$ in $M \smallsetminus \partial M$, subject to the condition that surgery on e_T results in a *connected* surface. A morphism from (S, e_S) to (T, e_T) is an injective map $k\colon S \to T$ such that $k^* e_T = e_S$.
The set of objects $\mathrm{ob}(\mathscr{C}_M)$ is topologized as a disjoint union, over all T, of spaces of smooth embeddings from $D^2 \times S^0 \times T$ to $M \smallsetminus \partial M$, with the compact-open C^∞ topology. The total morphism set $\mathrm{mor}(\mathscr{C}_M)$ is topologized as a closed subset of $\mathrm{ob}(\mathscr{C}_M) \times \mathrm{ob}(\mathscr{C}_M)$ via the map (source,target).

Proposition 6.2.2 *The space $B\mathscr{C}_M$ is contractible.*

The proof requires a lemma.

Lemma 6.2.3 *Let $\sigma\colon N \to X$ be a submersion of smooth manifolds without boundary, $\dim(N) > \dim(X)$. Suppose that for each $x \in X$ there exists a contractible open neighborhood V of x in X, a finite set Q and a map $Q \times V \to N$ over X inducing a surjection from $Q \cong \pi_0(Q \times V)$ to $\pi_0(N_y)$ for every $y \in V$. Then there exists a locally finite covering of X by contractible open sets V_j, where $j \in J$, and finite sets Q_j, and a smooth embedding*

$$a\colon \coprod_j Q_j \times V_j \longrightarrow N$$

over X, such that the restriction of a to $Q_j \times V_j$ induces surjections $Q_j \to \pi_0(N_x)$, for each $j \in J$ and $x \in V_j$.

Example 6.2.4 The submersion $\mathbb{R}^2 \smallsetminus (0,0) \longrightarrow \mathbb{R}$; $(x,y) \mapsto x$ satisfies the hypothesis of lemma 6.2.3. The submersion $\mathbb{R} \smallsetminus 0 \to \mathbb{R}$; $x \mapsto x$ does not. Surjectivity is not directly related to the issue; the projection from $(\mathbb{R} \times \{0,1\}) \smallsetminus (0,0)$ to $\mathbb{R}$ is a surjective submersion which also fails to satisfy the hypothesis of lemma 6.2.3.

Proof of lemma 6.2.3. Note first that the statement is not completely trivial. Using the hypothesis, we could start with a locally finite covering of X by contractible open sets V_j, and choose finite sets Q_j and maps $a_j\colon Q_j \times V_j \to N$ over X inducing surjections $Q_j \to \pi_0(N_y)$ for every $y \in V_j$. This would give us a map

$$a\colon \coprod_j Q_j \times V_j \longrightarrow N$$

which is an immersion. Unfortunately there is no guarantee that it is an embedding. To solve this problem we will partition a "large", dense open subset U of N into "levels" indexed by the real numbers, and arrange that a maps distinct connected components of $\coprod Q_j \times V_j$ to distinct levels of U. Then a is an embedding.
The jet transversality theorem, applied to sections of the vertical tangent bundle of N, implies that we can find a $k \gg 0$ and a smooth $f\colon N \to \mathbb{R}$ such that the fiberwise k-jet prolongation $j^k_\sigma f\colon N \to J^k_\sigma(N,\mathbb{R})$ is nowhere 0. Let $U \subset N$ consist of all $z \in N$ such that $f|N_{\sigma(z)}$ is regular at z. Then U is open in N and $U_x := U \cap N_x$ is dense in N_x, for each $x \in X$. Hence the inclusions $U_x \to N_x$ induce surjections $\pi_0(U_x) \to \pi_0(N_x)$. The hypotheses on σ now give us a a covering of X by contractible open subsets V_j, and for each V_j a finite set Q_j and a map $a_j\colon Q_j \times V_j \to U$ over X such that the induced composite map $Q_j \to \pi_0(U_x) \to \pi_0(N_x)$ is onto for every $x \in V_j$. We can assume that the V_j are the open stars of the vertices in a sufficiently fine triangulation of X, in which case the covering is locally finite. But in addition we can easily arrange that fa_j is constant on $q \times V_j$ for each $q \in Q_j$, and that the resulting map $\coprod_j Q_j \to \mathbb{R}$ is injective. Then the map a which equals a_j on $Q_j \times V_j$ satisfies all our requirements. □

In the proof of theorem 6.1.4, we will use a sheaf version $\mathcal{C}_M$ of $\mathscr{C}_M$. For connected X in $\mathscr{X}$ let $\mathcal{C}_M(X)$ be the (discrete) category whose objects are the pairs (T, e_T) where T is a finite set and

$$e_T\colon D^2 \times S^0 \times T \times X \quad\longrightarrow\quad (M \smallsetminus \partial M) \times X$$

is a smooth embedding over X, fiberwise orientation preserving and subject to the condition that fiberwise surgery on e_T results in a bundle of connected surfaces. A morphism from (S, e_S) to (T, e_T) is an injective map $k\colon S \to T$ such that $k^* e_T = e_S$.
Since $\mathrm{ob}(\mathcal{C}_M(\Delta^k_e))$ is the set of smooth maps from Δ^k_e to the embedding space $\mathrm{ob}(\mathscr{C}_M)$, one gets a functor of topological categories $|\mathcal{C}_M| \to \mathscr{C}_M$ which induces a degreewise homotopy equivalence of the nerves and therefore a homotopy equivalence $B|\mathcal{C}^{\mathrm{op}}_M| \cong B|\mathcal{C}_M| \to B\mathscr{C}_M$. (Here it is best to define $B\mathscr{C}_M$ as the fat realization [35] of the nerve of $\mathscr{C}_M$, ignoring the degeneracy operators.)

Proof of proposition 6.2.2. We show that $\beta\mathcal{C}^{\mathrm{op}}_M$ is weakly equivalent to the terminal sheaf taking every X in $\mathscr{X}$ to a singleton. By proposition 2.2.6, this reduces to the following

> *Claim.* Let X in $\mathscr{X}$ be given with a closed subset A and a germ $s \in \mathrm{colim}_U\, \beta\mathcal{C}^{\mathrm{op}}_M(U)$, where U ranges over the neighborhoods of A in X. Then s extends to an element of $\beta\mathcal{C}^{\mathrm{op}}_M(X)$.

To verify this, choose an open neighborhood U of A in X such that the germ s can be represented by some $s_0 \in \beta\mathcal{C}_M^{\mathrm{op}}(U)$. The information contained in s_0 includes a locally finite covering of U by open subsets U_j for $j \in J$. (Making U smaller if necessary, we can assume that this is locally finite in the strong sense that every $x \in X$ has a neighborhood which meets only finitely many U_j.) It also includes a choice of object $\psi_{RR} \in \mathrm{ob}(\mathcal{C}_M(U_R))$ for each finite nonempty subset R of J. (There are also morphisms $\psi_{RS} \in \mathrm{mor}(\mathcal{C}_M(U_S))$, but they are of course determined by their sources $\psi_{RR}|U_S$ and targets ψ_{SS}.) Next, choose an open $X_0 \subset X$ such that $U \cup X_0 = X$ and the closure of X_0 in X avoids A.
Let N be the open subset of $(M \smallsetminus \partial M) \times X_0$ obtained by removing from $(M \smallsetminus \partial M) \times X_0$ the closures of the embedded 2-disk bundles determined by the various $\varphi_{RR}|U_R \cap X_0$. By making U and X_0 and the U_j smaller if necessary, but taking care that the U_j remain the same near A, we can arrange that the projection $N \to X_0$ satisfies the hypothesis of lemma 6.2.3.
By the lemma, there exists a locally finite covering of X_0 by contractible open sets V_j, and finite sets Q_j and an embedding a of $\coprod_j Q_j \times V_j$ in N, over X_0 , such that a induces surjections $Q_j \to \pi_0(N_x)$ for each j and $x \in V_j$. (Again, making X_0 smaller if necessary, we can assume that this is locally finite in the strong sense that every $x \in X$ has a neighborhood which meets only finitely many V_j.) We can also choose a smooth embedding b of $\coprod_j Q_j \times V_j$ in N, over X_0, inducing constant maps $Q_j \to \pi_0(N_x)$ for each j and $x \in V_j$, and such that $\mathrm{im}(a) \cap \mathrm{im}(b) = \emptyset$. (For example, the distinct sheets of $b\big|Q_j \times V_j$ can be chosen very close to a selected sheet of $a\big|Q_j \times V_j$.) Since the V_j are contractible, the normal bundles of a and b can be trivialized (as oriented 2-dimensional vector bundles), and so the "union" of a and b extends to a smooth and fiberwise orientation preserving embedding

$$c\colon\ D^2 \times S^0 \times \textstyle\coprod_j (Q_j \times V_j) \quad \longrightarrow N$$

over X_0. For each j with nonempty V_j, the restriction of c to $D^2 \times S^0 \times Q_j \times V_j$ is an object φ_{jj} of $\mathcal{C}_M(V_j)$. Finally we can arrange that V_j is empty whenever U_j is nonempty.
We are now ready to define an explicit element in $\beta\mathcal{C}_M^{\mathrm{op}}(X)$ which extends the germ s. Let $Y_j = U_j$ if U_j is nonempty, $Y_j = V_j$ if V_j is nonempty, and $Y_j = \emptyset$ for all other $j \in J$. Then the Y_j form a locally finite open covering of X. For finite $R \subset J$ with nonempty Y_R, we can write $Y_R = U_S \cap V_T$ for disjoint subsets S, T of R with $S \cup T = R$. Let $\varphi_{RR} \in \mathrm{ob}(\mathcal{C}_M(Y_R))$ be the coproduct (which exists by construction) of $\psi_{SS}|Y_R$ and the $\varphi_{jj}|Y_R$ for $j \in T$. The covering $j \mapsto Y_j$ together with the data φ_{RR} for finite nonempty $R \subset J$ is an element in $\beta\mathcal{C}_M^{\mathrm{op}}(X)$ which extends the germ s. ☐

6.3 Parametrized multiple surgeries

We reformulate proposition 6.2.2 in a parametrized setting and deduce theorem 6.1.4 from the reformulation. First we remind the reader of Segal's edgewise subdivision of a category.

Remark 6.3.1 For any category $\mathscr{D}$, the edgewise subdivision $\mathrm{es}(\mathscr{D})$ of $\mathscr{D}$ is another category defined as follows. An object of $\mathrm{es}(\mathscr{D})$ is a morphism $f\colon c_0 \to c_1$ in $\mathscr{D}$. A morphism in $\mathrm{es}(\mathscr{D})$ from an object $f\colon c_0 \to d_0$ to an object $g\colon d_0 \to d_1$ is a commutative square

$$\begin{tikzcd} c_0 \arrow[r, "f"] & c_1 \arrow[d] \\ d_0 \arrow[u] \arrow[r, "g"] & d_1 \end{tikzcd}$$

in $\mathscr{D}$. It is well known that $B(\mathrm{es}(\mathscr{D}))$ is homeomorphic to $B\mathscr{D}$, if $\mathscr{D}$ is a discrete category. More precisely, by [13, Lm.2.4] the nerve of $\mathrm{es}(\mathscr{D})$ is isomorphic as a simplicial set to the edgewise subdivision of the nerve of $\mathscr{D}$, and this implies by [36] that the realizations are homeomorphic. In the case of a simplicial category $\mathscr{D}$ one can argue degreewise. The general case of a topological category can in most cases be reduced to the case of a simplicial category.

Definition 6.3.2 Fix an object S in $\mathscr{K}_{12}$. Let (T,U) be a pair of finite sets with $U \subset T$ and $T \cap S = \emptyset$. We introduce a sheaf $\mathcal{W}_{S;(T,U)}$ on $\mathscr{X}$ with a forgetful map $\mathcal{W}_{S;(T,U)} \to \mathcal{W}_S$. For X in $\mathscr{X}$, an element in $\mathcal{W}_{S;(T,U)}(X)$ is an element (q,V,e) of $\mathcal{W}_S(X)$ with $q\colon M \to X$ etc., together with a smooth embedding

$$e_T\colon D^2 \times S^0 \times T \times X \longrightarrow M \smallsetminus \partial M$$

over X, avoiding $\mathrm{im}(e)$. *Condition:* Fiberwise surgery on e_U results in a bundle of connected surfaces; here e_U denotes the restriction of e_T to $D^2 \times S^0 \times U \times X$.

Let $\mathscr{P}$ be the category whose objects are pairs of finite sets (T,U) with $U \subset T$, where a morphism from (Q,R) to (T,U) is an injective map $h\colon Q \to T$ with $h(R) \supset U$. Such a morphism $(Q,R) \to (T,U)$ induces a map of sheaves $\mathcal{W}_{S;(T,U)} \longrightarrow \mathcal{W}_{S;(Q,R)}$, so that there is a contravariant functor from $\mathscr{P}$ to sheaves on $\mathscr{X}$ given by

$$(T,U) \mapsto \mathcal{W}_{S;(T,U)}.$$

Corollary 6.3.3 *The forgetful maps $\mathcal{W}_{S;(T,U)} \to \mathcal{W}_S$ induce a homotopy equivalence*

$$\operatorname*{hocolim}_{(T,U)} |\mathcal{W}_{S;(T,U)}| \quad \simeq \quad |\mathcal{W}_S|\,.$$

Proof Fix an element in $\mathcal{W}_S(\star)$, consisting of an oriented surface M and a smooth orientation preserving embedding

$$e\colon D(V^{\varrho}) \times_S S(V^{-\varrho}) \longrightarrow M \smallsetminus \partial M.$$

where V denotes a 3-dimensional oriented riemannian vector bundle over S with involution. It is enough to show that the homotopy fiber of

$$\operatorname*{hocolim}_{(T,U)} |\mathcal{W}_{S;(T,U)}| \longrightarrow |\mathcal{W}_S|$$

over the point corresponding to M is contractible. In this situation the processes of forming homotopy colimits and homotopy fibers commute. Moreover each of the forgetful maps $\mathcal{W}_{S;(T,U)} \to \mathcal{W}_S$ has the concordance lifting property, so by proposition A.2.6, the homotopy fiber which we are interested in is weakly equivalent to

$$\operatorname*{hocolim}_{(T,U)} |\operatorname{fiber}_M(\mathcal{W}_{S;(T,U)} \to \mathcal{W}_S)|. \tag{6.2}$$

Let M_S be the compact surface obtained from M by deleting $\operatorname{int}(\operatorname{im}(e))$. It is clear that each expression $|\operatorname{fiber}_M(\mathcal{W}_{S;(T,U)} \to \mathcal{W}_S)|$ in (6.2) can be replaced by the naturally homotopy equivalent

$$\operatorname{mor}_{(T,U)}\mathscr{C}_{M_S},$$

the space of morphisms in $\mathscr{C}_{M_S}$ of definition 6.2.1 which induce the inclusion $U \to T$ of finite sets. The homotopy colimit now becomes the classifying space of the transport category

$$\mathscr{P}^{\mathrm{op}} \textstyle\int \operatorname{mor}_\bullet \mathscr{C}_{M_S},$$

cf. section D.1, where the bullet stands for objects (T,U) of $\mathscr{P}$. This is a category whose objects are the morphisms $(U, e_U) \to (T, e_T)$ in $\mathscr{C}_{M_S}$ where the underlying map $U \to T$ is an inclusion. The morphisms correspond to certain commutative squares in $\mathscr{C}_{M_S}$. What we have here is a category equivalent to the edgewise subdivision (see remark 6.3.1 above) of $\mathscr{C}_{M_S}$. Its classifying space is therefore homotopy equivalent to $B\mathscr{C}_{M_S}$, hence contractible by proposition 6.2.2. □

Proof of theorem 6.1.4. Using the homotopy invariance property of homotopy direct limits, we obtain from corollary 6.3.3 a homotopy equivalence of spaces

$$\eta_+ : \operatorname*{hocolim}_{S \text{ in } \mathscr{K}_{12}} \operatorname*{hocolim}_{(T,U) \text{ in } \mathscr{P}} |\mathcal{W}_{S;(T,U)}| \longrightarrow \operatorname*{hocolim}_{S \text{ in } \mathscr{K}_{12}} |\mathcal{W}_S|.$$

We compare this with the map

$$\eta_- : \operatorname*{hocolim}_{S \text{ in } \mathscr{K}_{12}} \operatorname*{hocolim}_{(T,U) \text{ in } \mathscr{P}} |\mathcal{W}_{S;(T,U)}| \longrightarrow \operatorname*{hocolim}_{R \text{ in } \mathscr{K}_{12}} |\mathcal{W}_{c,R}| \tag{6.3}$$

induced by the composite maps

$$\mathcal{W}_{S;(T,U)} \longrightarrow \mathcal{W}_{S\cup T} \xrightarrow{(-)^*} \mathcal{W}_{S\cup(T\smallsetminus U)} \tag{6.4}$$

and renaming, $S \cup (T \smallsetminus U) \rightsquigarrow R$. Here the first arrow in (6.4) is self-explanatory. The second is induced by the inclusion $S \cup (T \smallsetminus U) \to S \cup T$, with the sign function on U which is $\equiv -1$. Thus the first arrow amounts to adding the surgery data corresponding to labels in T (but not performing any surgeries), while the second amounts to performing the surgeries corresponding to labels in $U \subset T$. It follows that the composite map in (6.4) lands in the subsheaf $\mathcal{W}_{c,S\cup(T\smallsetminus U)}$, as required in (6.3). The map η_- in (6.3) is clearly a retraction, with a canonical section ζ which identifies each $\mathcal{W}_{c,R}$ with $\mathcal{W}_{R;(\emptyset,\emptyset)}$. The target of η_- is contained in the target of η_+, so we may ask whether η_- and η_+ are homotopic as maps to $\mathrm{hocolim}_S\,|\mathcal{W}_S|$. This is indeed the case, by remark D.1.3 and the fact that each $\mathcal{W}_{S;(T,U)}$ fits into a natural commutative diagram

$$\begin{tikzcd} & \mathcal{W}_{S;(T,U)} \arrow[dl, "\text{forget}"'] \arrow[d] \arrow[dr, "(6.4)"] & \\ \mathcal{W}_S & \mathcal{W}_{S\cup T} \arrow[l, "(+)^*"'] \arrow[r, "(-)^*"] & \mathcal{W}_{S\cup(T\smallsetminus U)}\,. \end{tikzcd}$$

The homotopy restricts to a constant homotopy from $\eta_+\zeta$ to $\eta_-\zeta$. Consequently, it is a deformation retraction of $\mathrm{hocolim}_S\,|\mathcal{W}_S|$ to $\mathrm{hocolim}_S\,|\mathcal{W}_{c,S}|$. ◻

6.4 Annihiliation of 2-spheres

The goal is to prove lemma 6.1.3. Most of the proof is based on some elementary product decompositions.

Lemma 6.4.1 *Let $T = T_1 \cup T_2$ be a disjoint union, where T_1 is an object of $\mathscr{K}_{03}$ and T_2 is an object of $\mathscr{K}$. There are weak equivalences, natural in T_2 for fixed T_1,*

$$\mathcal{W}_T \longrightarrow \mathcal{W}_{\mathrm{loc},T_1} \times \mathcal{W}_{T_2}\,, \qquad \mathcal{W}_{\mathrm{loc},T} \longrightarrow \mathcal{W}_{\mathrm{loc},T_1} \times \mathcal{W}_{\mathrm{loc},T_2}.$$

Proof The second map is induced by the inclusions $T_1 \to T$ and $T_2 \to T$. It should be clear that it is a weak equivalence. Note that sign functions on T_2 and T_1 are not needed. The first coordinate of the first map is again induced by the inclusion $T_1 \to T$. The second coordinate of the first map,

$$\mathcal{W}_T \longrightarrow \mathcal{W}_{T_2}\,,$$

is defined as follows. Let (q, V, e) be an element of $\mathcal{W}_T(X)$ as in definition 5.1.3, with $q\colon M \to X$. For $a \in T_1$, the bundle

$$D(V_a^{\varrho}) \times_{X_a} S(V_a^{-\varrho})$$

(where $X_a = a \times X$ and $V_a = V|X_a$) is either empty or a bundle of 2-spheres. In any case it has empty boundary and its image under e is a union of connected components of M. Let M' be obtained from M by deleting these components, for all $a \in T_1$. Let V' be the restriction of V to $T_2 \times X$ and let e' be the restriction of e to

$$\coprod_{b\in T_2} D(V_b^{\varrho}) \times_{X_b} S(V_b^{-\varrho})\,.$$

Then $(q', V', e') \in \mathcal{W}_{T_2}(X)$. This determines the map $\mathcal{W}_T \longrightarrow \mathcal{W}_{T_2}$. Again it should be clear that the resulting map

$$\mathcal{W}_{T_2} \longrightarrow \mathcal{W}_{\mathrm{loc},T_1} \times \mathcal{W}_{T_2}$$

is a weak equivalence: it is easy to write down an inverse for the induced map on homotopy groups. ☐

Proof of lemma 6.1.3. We may assume that the complement of P in Q has exactly one element. Using lemma 6.4.1, we can isolate a common factor $\mathcal{W}_{\mathrm{loc},P}$ in each of the four terms of the square, then remove it. In other words, we can also assume that $P=\emptyset$, which implies that Q has exactly one element. (We still have a morphism in $\mathscr{K}_{03}$ from $\emptyset$ to Q and we are not going to throw it away.) By the same reasoning we can now isolate a common factor

$$\operatorname*{hocolim}_{S \text{ in } \mathscr{K}_{12}} \mathcal{W}_{\mathrm{loc},S}$$

in the two terms of the lower row of the square, then remove it. Next we can isolate a common factor $\mathcal{W}_{\mathrm{loc},Q}$ in the two terms of the left-hand column. We can again remove it because $\mathcal{W}_{\mathrm{loc},Q}$ is connected: $\mathcal{W}_{\mathrm{loc},Q}[\star]$ is a singleton. This leaves us with a square of the form

$$\begin{tikzcd}
\operatorname*{hocolim}_{S \text{ in } \mathscr{K}_{12}} \mathcal{W}_S \arrow[r, "u"] \arrow[d] & \operatorname*{hocolim}_{S \text{ in } \mathscr{K}_{12}} \mathcal{W}_S \arrow[d] \\
\star \arrow[r] & \star
\end{tikzcd}$$

where the map u can be described as follows. Choose a base point $z \in \mathcal{W}_{\mathrm{loc},Q}(\star)$. Then $\mathcal{W}_{\mathrm{loc},Q}$ becomes a sheaf on $\mathscr{X}$ with values in the category of pointed sets and gives us a canonical inclusion of $\mathcal{W}_S$ in

$$\mathcal{W}_{\mathrm{loc},Q} \times \mathcal{W}_S \simeq \mathcal{W}_{Q\amalg S}$$

for each S in $\mathscr{K}_{12}$. Compose that with the map $\mathcal{W}_{Q\amalg S} \to \mathcal{W}_S$ induced by our morphism $\emptyset \to Q$, regard S as a variable and apply $\operatorname{hocolim}_S$.
We have to show that u is a weak equivalence. In order to do that we make a case distinction. The reference map $Q \to \{0,3\} \subset \underline{3}$ amounts to a choice of an element ℓ from $\{0,3\}$ and the morphism $\emptyset \to Q$ amounts to a choice of an element $m \in \{-1,+1\}$.
Case 1 is the case where $(\ell,m)=(0,+1)$ or $(\ell,m)=(3,-1)$. By inspection, u is the identity map in that case.
Case 2 is the case where $(\ell,m)=(3,+1)$ or $(\ell,m)=(0,-1)$. Here we note that our choice of $z \in \mathcal{W}_{\mathrm{loc},Q}(\star)$ determines an oriented 3-dimensional vector space V with inner product. We can assume $V=\mathbb{R}^3$. The map u is given by disjoint union of all surfaces in sight with S^2. More precisely, for each S in $\mathscr{K}_{12}$ and X in $\mathscr{X}$, we have a map

$$u_{S,X}\colon\ \mathcal{W}_S(X) \to \mathcal{W}_S(X)$$

given by $(q,V,e) \mapsto (q^\sharp,V,e)$ where $q\colon\ M \to X$ is a surface bundle etc., and $q^\sharp$ is obtained from q by disjoint union with a trivial sphere bundle $S^2 \times X \to X$. This is natural in the variables X and S and so induces u above.

Lemma 6.4.2 *The map*

$$u\colon \operatorname*{hocolim}_{S \text{ in } \mathscr{K}_{12}} \mathcal{W}_S \longrightarrow \operatorname*{hocolim}_{S \text{ in } \mathscr{K}_{12}} \mathcal{W}_S$$

given by disjoint union of all surfaces in sight with S^2 is a weak equivalence.

Proof Using concatenation of surfaces, we can put a monoid structure on $\operatorname{hocolim}_S \mathcal{W}_S$. More precisely, we have for each S and T in $\mathscr{K}_{12}$ a map

$$\text{concatenation}\colon \mathcal{W}_S \times \mathcal{W}_T \to \mathcal{W}_{S \amalg T}$$

and this induces a multiplication

$$\big(\operatorname*{hocolim}_{S \text{ in } \mathscr{K}_{12}} \mathcal{W}_S\big) \times \big(\operatorname*{hocolim}_{T \text{ in } \mathscr{K}_{12}} \mathcal{W}_T\big) \longrightarrow \big(\operatorname*{hocolim}_{U \text{ in } \mathscr{K}_{12}} \mathcal{W}_U\big)\,.$$

Now let y and y' be the elements of $\mathcal{W}_\emptyset(\star)$ determined by the surfaces $(S^1 \times [0,1]\,)\ \amalg\ S^2$ and $S^1 \times [0,1]$, respectively. The map u under investigation is simply given by concatenation with y, where we use the inclusion

$$\mathcal{W}_\emptyset \quad\subset\quad \operatorname*{hocolim}_{S \text{ in } \mathscr{K}_{12}} \mathcal{W}_S\,.$$

But the homotopy class of u (after passage to representing spaces) depends only on the component of y in

$$\big|\operatorname*{hocolim}_{S \text{ in } \mathscr{K}_{12}} \mathcal{W}_S\big|\,.$$

The surgery methods of the previous subsection show immediately that this agrees with the component of y'. Hence u is homotopic, after passage to representing spaces, to the map given by concatenation with y'. □

7 Stabilization

7.1 Stabilization

Choose $z \in \mathcal{V}_c(\star)$ of genus 1. For every X in $\mathscr{X}$, the unique map $X \to \star$ induces $\mathcal{V}_c(\star) \to \mathcal{V}_c(X)$ and so allows us to think of z as an element of $\mathcal{V}_c(X)$. Let $z^{-1}\mathcal{V}_c$ be the sheaf on $\mathscr{X}$ obtained by sheafifying the contravariant functor (alias presheaf)

$$X \quad\longrightarrow\quad \operatorname{colim}\Big(\mathcal{V}_c(X) \xrightarrow{z\cdot} \mathcal{V}_c(X) \xrightarrow{z\cdot} \mathcal{V}_c(X) \xrightarrow{z\cdot} \mathcal{V}_c(X) \xrightarrow{z\cdot} \cdots\Big)\,,$$

where $z\cdot$ denotes concatenation with z. The sheafification process is very mild in this case. In particular, the presheaf and its sheafification agree on compact objects of $\mathscr{X}$, such as spheres. Hence the canonical map from

$$z^{-1}|\mathcal{V}_c| = \operatorname{colim}\Big(|\mathcal{V}_c| \xrightarrow{z\cdot} |\mathcal{V}_c| \xrightarrow{z\cdot} |\mathcal{V}_c| \xrightarrow{z\cdot} |\mathcal{V}_c| \xrightarrow{z\cdot} \cdots\Big)$$

to $|z^{-1}\mathcal{V}_c|$ is a homotopy equivalence.
Similarly, for each S in $\mathscr{K}$, define $z^{-1}\mathcal{W}_S$ and $z^{-1}\mathcal{W}_{c,S}$ as the colimits, in the category of sheaves on $\mathscr{X}$, of the diagrams

$$\mathcal{W}_S \xrightarrow{z\cdot} \mathcal{W}_S \xrightarrow{z\cdot} \mathcal{W}_S \xrightarrow{z\cdot} \mathcal{W}_S \xrightarrow{z\cdot} \cdots,$$
$$\mathcal{W}_{c,S} \xrightarrow{z\cdot} \mathcal{W}_{c,S} \xrightarrow{z\cdot} \mathcal{W}_{c,S} \xrightarrow{z\cdot} \mathcal{W}_{c,S} \xrightarrow{z\cdot} \cdots,$$

respectively. Then again we have homotopy equivalences

$$|z^{-1}\mathcal{W}_S| \simeq z^{-1}|\mathcal{W}_S|, \qquad |z^{-1}\mathcal{W}_{c,S}| \simeq z^{-1}|\mathcal{W}_{c,S}|.$$

Moreover, since $z^{-1}|\mathcal{V}_c|$, $z^{-1}|\mathcal{W}_S|$ and $z^{-1}|\mathcal{W}_{c,S}|$ have been defined as sequential colimits of CW–spaces, they can also be regarded as homotopy colimits: for example,

$$z^{-1}|\mathcal{W}_S| \;\simeq\; \operatorname{hocolim}\Big(|\mathcal{W}_S| \xrightarrow{z\cdot} |\mathcal{W}_S| \xrightarrow{z\cdot} |\mathcal{W}_S| \xrightarrow{z\cdot} |\mathcal{W}_S| \xrightarrow{z\cdot} \cdots\Big).$$

Proposition 7.1.1 $\Omega B|\mathcal{V}_c| \simeq \mathbb{Z} \times B\Gamma^{+}_{\infty,2}$.

Proof We noted in section 1 that $|\mathcal{V}_c| \simeq \coprod_g B\Gamma_{g,2}$. It follows that

$$|z^{-1}\mathcal{V}_c| \;\simeq\; \mathbb{Z} \times B\Gamma_{\infty,2}.$$

On the other hand the bar construction gives us a simplicial space $E_\bullet$ with

$$E_k = |z^{-1}\mathcal{V}_c| \times |\mathcal{V}_c|^k$$

and a simplicial map from it to

$$\underline{k} \quad\mapsto\quad \star \times |\mathcal{V}_c|^k.$$

The Harer stability theorem implies that this simplicial map satisfies the hypotheses of corollary C.1.2, so that we have a homology fibration sequence

$$|z^{-1}\mathcal{V}_c| \longrightarrow |E_\bullet| \longrightarrow B|\mathcal{V}_c|.$$

It only remains to show that $|E_\bullet| \simeq \star$. To this end observe that $|E_\bullet|$ is homotopy equivalent to the realization of a monotone union of simplicial spaces of the form $\underline{k} \mapsto |\mathcal{V}_c|^{k+1}$. Each of these has a contractible realization. □

For an object T in $\mathscr{K}_{12}$, corollary 6.1.2 implies that the homotopy fiber of the localization map $|\mathcal{W}_{c,T}| \longrightarrow |\mathcal{W}_{\mathrm{loc},T}|$ over any base point is homotopy equivalent to $\coprod_g B\Gamma_{g,\,2+2|T|}$.

Lemma 7.1.2 *For T in $\mathscr{K}_{12}$, any homotopy fiber of $|z^{-1}\mathcal{W}_{c,T}| \longrightarrow |\mathcal{W}_{\mathrm{loc},T}|$ is homotopy equivalent to $\mathbb{Z} \times B\Gamma_{\infty,\,2+2|T|}$.* □

Finally we have the stabilized versions of lemma 6.1.3 and theorem 6.1.4:

Corollary 7.1.3 *For any morphism $(k,\varepsilon)\colon P \to Q$ in $\mathscr{K}_{03}$, the commutative square*

$$\begin{array}{ccc} \underset{S \text{ in } \mathscr{K}_{12}}{\operatorname{hocolim}} \ |z^{-1}\mathcal{W}_{Q\amalg S}| & \xrightarrow{(k,\varepsilon)^*} & \underset{S \text{ in } \mathscr{K}_{12}}{\operatorname{hocolim}} \ |z^{-1}\mathcal{W}_{P\amalg S}| \\ \downarrow & & \downarrow \\ \underset{S \text{ in } \mathscr{K}_{12}}{\operatorname{hocolim}} \ |\mathcal{W}_{\mathrm{loc},Q\amalg S}| & \xrightarrow{(k,\varepsilon)^*} & \underset{S \text{ in } \mathscr{K}_{12}}{\operatorname{hocolim}} \ |\mathcal{W}_{\mathrm{loc},P\amalg S}| \end{array}$$

is homotopy cartesian.

Corollary 7.1.4 *The inclusion*

$$\underset{T \text{ in } \mathscr{K}_{12}}{\operatorname{hocolim}} \ |z^{-1}\mathcal{W}_{c,T}| \longrightarrow \underset{T \text{ in } \mathscr{K}_{12}}{\operatorname{hocolim}} \ |z^{-1}\mathcal{W}_{T}|$$

is a homotopy equivalence.

Corollaries 7.1.3 and 7.1.4 are about a new homotopy colimit decomposition of $|\mathcal{W}|$:

Lemma 7.1.5 $|\mathcal{W}| \ \simeq \ |\, z^{-1}\mathcal{W}| \ \simeq \ \underset{T \text{ in } \mathscr{K}}{\operatorname{hocolim}} \ |z^{-1}\mathcal{W}_T|\,.$

Proof Since $|\mathcal{W}|$ is group complete, the inclusion $|\mathcal{W}| \to z^{-1}|\mathcal{W}| \simeq |z^{-1}\mathcal{W}|$ is a homotopy equivalence. The second homotopy equivalence in the chain follows from $|z^{-1}\mathcal{W}_T| \simeq z^{-1}|\mathcal{W}_T|$ and

$$\underset{T \text{ in } \mathscr{K}}{\operatorname{hocolim}} \ z^{-1}|\mathcal{W}_T| \ \simeq \ z^{-1}\big(\underset{T \text{ in } \mathscr{K}}{\operatorname{hocolim}} \ |\mathcal{W}_T|\big). \qquad \square$$

7.2 Using the Harer-Ivanov stability theorem

Lemma 7.2.1 *The canonical map from $\mathbb{Z} \times B\Gamma_{\infty,2}$ to the homotopy fiber (over the base point) of the forgetful map*

$$\underset{S \text{ in } \mathscr{K}_{12}}{\operatorname{hocolim}} \ |z^{-1}\mathcal{W}_{c,S}| \quad \longrightarrow \quad \underset{S \text{ in } \mathscr{K}_{12}}{\operatorname{hocolim}} \ |\mathcal{W}_{\mathrm{loc},S}|$$

induces an isomorphism in homology with integer coefficients.

Proof For the object $S = \emptyset$ of $\mathscr{K}_{12}$, we have $|z^{-1}\mathcal{W}_{c,S}| \simeq \mathbb{Z} \times B\Gamma_{\infty,2}$ and $|\mathcal{W}_{\mathrm{loc},S}| = \star$, so that there is indeed a canonical map from $\mathbb{Z} \times B\Gamma_{\infty,2}$ to the homotopy fiber of

$$\underset{S \text{ in } \mathscr{K}_{12}}{\operatorname{hocolim}} \ |z^{-1}\mathcal{W}_{c,S}| \quad \longrightarrow \quad \underset{S \text{ in } \mathscr{K}_{12}}{\operatorname{hocolim}} \ |\mathcal{W}_{\mathrm{loc},S}|.$$

We now check that the hypothesis of corollary C.1.3 is satisfied. Let $(k,\varepsilon)\colon\ S \to T$ be a morphism in $\mathscr{K}_{12}$. We have to verify that, in the commutative square of spaces

$$\begin{array}{ccc} |z^{-1}\mathcal{W}_{c,T}| & \longrightarrow & |\mathcal{W}_{\mathrm{loc},T}| \\ \big\downarrow{\scriptstyle (k,\varepsilon)^*} & & \big\downarrow{\scriptstyle (k,\varepsilon)^*} \\ |z^{-1}\mathcal{W}_{c,S}| & \longrightarrow & |\mathcal{W}_{\mathrm{loc},S}|, \end{array}$$

the induced map from any of the homotopy fibers in the upper row to the corresponding homotopy fiber in the lower row induces an isomorphism in homology. The homotopy fibers in question are related by a map

$$\mathbb{Z} \times B\Gamma_{\infty,\,2+2|T|} \longrightarrow \mathbb{Z} \times B\Gamma_{\infty,\,2+2|S|}$$

given geometrically by attaching cylinders $D^1 \times S^1$ or double disks $D^2 \times S^0$ to those pairs of boundary circles which correspond to elements of $T \smallsetminus k(S)$. This map is an integral homology equivalence by the Harer-Ivanov stability theorem. Apply corollary C.1.3. □

Corollary 7.2.2 *The canonical map from $\mathbb{Z} \times B\Gamma_{\infty,2}$ to the homotopy fiber (over the base point) of the forgetful map*

$$\operatorname*{hocolim}_{S \text{ in } \mathscr{K}} |z^{-1}\mathcal{W}_S| \quad \longrightarrow \quad \operatorname*{hocolim}_{S \text{ in } \mathscr{K}} |\mathcal{W}_{\mathrm{loc},S}|$$

induces an isomorphism in homology with integer coefficients.

Proof Combine lemma 7.2.1 with corollaries 7.1.4 and 7.1.3. □

Proof of theorem 1.3.4. By lemma 7.1.5 and diagram 5.1, we have

$$\operatorname*{hocolim}_{S \text{ in } \mathscr{K}} |z^{-1}\mathcal{W}_S| \simeq |\mathcal{W}|, \qquad \operatorname*{hocolim}_{S \text{ in } \mathscr{K}} |\mathcal{W}_{\mathrm{loc},S}| \simeq |\mathcal{W}_{\mathrm{loc}}|.$$

Therefore corollary 7.2.2 implies that the homotopy fiber of $|\mathcal{W}| \to |\mathcal{W}_{\mathrm{loc}}|$ is homology equivalent to $\mathbb{Z} \times B\Gamma_{\infty,2}$. On the other hand, $|\mathcal{W}|$ and $|\mathcal{W}_{\mathrm{loc}}|$ are infinite loop spaces by theorems 1.3.1 and 1.3.2, and the map $|\mathcal{W}| \to |\mathcal{W}_{\mathrm{loc}}|$ is an infinite loop map. Hence its homotopy fiber is an infinite loop space, hence group complete. It follows that the homotopy fiber is $\mathbb{Z} \times B\Gamma^+_{\infty,2} \simeq \Omega B|\mathcal{V}_c|$. □

A More about sheaves

A.1 Concordance and the representing space

Let $\mathcal{F}$ be a sheaf on $\mathscr{X}$. We shall construct a natural transformation $\vartheta\colon\ [X, |\mathcal{F}|\,] \longrightarrow \mathcal{F}[X]$, and an inverse $\xi\colon\ \mathcal{F}[X] \to [X, |\mathcal{F}|\,]$ for ϑ.

We start with the construction of ξ. Fix X in $\mathscr{X}$ and an element $u \in \mathcal{F}(X)$. Choose a smooth triangulation of X, with vertex set T. Suppose that $S \subset T$ is a distinguished subset (the vertex set of a simplex in the triangulation). Let

$$\begin{aligned}\Delta_e(S) &= \{w\colon S \to \mathbb{R} \mid \Sigma_s w(s) = 1\} \\ \Delta(S) &= \{w \in \Delta_e(S) \mid w \geq 0\}.\end{aligned}$$

The triangulation gives us characteristic embeddings $c_S\colon \Delta(S) \to X$, one for each distinguished $S \subset T$. By induction on S, we can choose smooth embeddings

$$c_{e,S}\colon \Delta_e(S) \to X ,$$

extending the c_S , which are compatible: i.e., if S is distinguished and $R \subset S$ is nonempty, then $c_{e,S}$ agrees with $c_{e,R}$ on $\Delta_e R) \subset \Delta_e(S)$. Let $u_S = c_{e,S}{}^*(u) \in \mathcal{F}(\Delta_e(S))$.
Finally choose a total ordering of T. This leads to an identification of each $\Delta_e(S)$ with a standard extended simplex. Consequently it promotes each u_S to a simplex of the simplicial set $\underline{n} \mapsto \mathcal{F}(\Delta^n_e)$. We then have a unique map $\xi(u)\colon X \to |\mathcal{F}|$ such that, for each S as above with $|S| = n+1$, the diagram

$$\begin{tikzcd} \Delta(S) \arrow[r, "\cong"] \arrow[d, "c_S"] & \Delta^n \arrow[d, "\text{char. map for } u"] \\ X \arrow[r, "\xi(u)"] & |\mathcal{F}| \end{tikzcd}$$

commutes. It is straightforward to show that the resulting homotopy class of maps $X \to |\mathcal{F}|$ depends only on the concordance class of $u \in \mathcal{F}(X)$.

Next we construct $\vartheta\colon [X, |\mathcal{F}|\,] \longrightarrow \mathcal{F}[X]$. We may replace $|\mathcal{F}|$ by the "fat" realization $\|\mathcal{F}\|$, which is obtained by forgetting the degeneracy operators in

$$\underline{n} \mapsto \mathcal{F}(\Delta^n_e)$$

and realizing the resulting incomplete simplicial set. So we start with a choice of map $g\colon X \to \|\mathcal{F}\|$. Without loss of generality, we may assume that g is simplicial for a smooth triangulation of X with totally ordered vertex set T. That is, for each $n \geq 0$ and each distingushed $S \subset T$, with characteristic map $c_S\colon \Delta(S) \to X$, the composition

$$\Delta^n \cong \Delta(S) \xrightarrow{c_S} X \xrightarrow{g} \|\mathcal{F}\|$$

is the characteristic map associated with some $u_S \in \mathcal{F}(\Delta^n_e)$.
Choose a smooth homotopy of smooth maps $h_t\colon X \to X$, where $0 \leq t \leq 1$, such that

(1) the map h_0 is the identity,

(2) for every t, the map h_t maps each simplex of the triangulation to itself and

(3) each simplex of the triangulation has a neighbourhood in X which is mapped to the simplex by h_1.

Then for each $n \geq 0$ and each distinguished S with $|S|-1=n$ and a sufficiently small neighborhood V_S of $c_S(\Delta(S))$ in X, we obtain a smooth map $V_S \to \Delta_e(S) \cong \Delta_e^n$ by composing $h_1|V_S$ with the inclusion of $\Delta(S)$ in $\Delta_e(S) \cong \Delta_e^n$. Using this map to pull back $u_S \in \mathcal{F}(\Delta_e^n)$, we obtain compatible elements $u'_S \in \mathcal{F}(V_S)$ which, by the sheaf property of $\mathcal{F}$, determine a unique element $\vartheta(g)$ of $\mathcal{F}(X)$. Again, it is straightforward to verify that the concordance class of $\vartheta(g)$ depends only on the homotopy class of g.

Proposition A.1.1 *The maps ξ and ϑ are reciprocal inverses.*

Proof Let $u \in \mathcal{F}(X)$. We want to show that $\vartheta\xi(u)$ is concordant to u. With suitable choices in the constructions above, we have $V_S \supset \operatorname{im}(c_{e,S})$ for all distinguished S, and then $\vartheta\xi(u)$ equals ${h_1}^*(u)$, where $(h_t\colon X \to X)_{0\leq t\leq 1}$ is the homotopy which appears in the definition of ϑ. Since h_1 is smoothly homotopic to $h_0 = \mathrm{id}_X$, this implies that $\vartheta\xi(u)$ is indeed concordant to u. Therefore

$$\vartheta\xi = \mathrm{id}\colon \mathcal{F}[X] \longrightarrow \mathcal{F}[X].$$

In order to show that $\xi\vartheta$ is the identity on $[X, |\mathcal{F}|\,]$, we introduce a simplicial monoid $Q_\bullet$ whose realization acts on $|\mathcal{F}|$. Namely, Q_n is the monoid of smooth maps $f\colon \Delta_e^n \to \Delta_e^n$ taking each (extended) face of Δ_e^n to itself. Then Q_n acts on the right of $\mathcal{F}(\Delta_e^n)$ by

$$s \cdot f = f^*(s),$$

and so $|Q_\bullet|$ acts on $|\mathcal{F}|$. Consequently the monoid $[X, |Q_\bullet|\,]$ acts on the right of $[X, |\mathcal{F}|\,]$. The effect of $\xi\vartheta$ on an element $[g] \in [X, |\mathcal{F}|\,]$ can be described in terms of this action. Indeed,

$$\xi\vartheta[g] = [g] \cdot [w]$$

for some $w\colon X \to |Q_\bullet|$. The map w is determined by $h_1\colon X \to X$ constructed above as part of a homotopy of maps from X to X. Since we are assuming that h_1 maps the image of each $c_{e,S}$ to itself, ${c_{e,S}}^{-1}h_1c_{e,S}$ is defined. This gives us for each $n \geq 0$ and each n-simplex in the triangulation of X an element in Q_n, hence a map from X to the realization of $Q_\bullet$. Since $|Q_\bullet|$ is contractible, $[w] \in [X, |Q_\bullet|\,]$ is always the neutral element, so that $[g] \cdot [w] = [g]$. □

The discussion above has a compact support version as follows. Fix $z \in \mathcal{F}(*)$, so that $\mathcal{F}$ becomes a functor from $\mathscr{X}^{\mathrm{op}}$ to pointed sets. For X in $\mathscr{X}$, we will say that an element $s \in \mathcal{F}(X)$ has *compact support* if its image in $\mathcal{F}(X \smallsetminus K)$ is the base point, for some compact $K \subset X$. A concordance between elements s_0, s_1 of $\mathcal{F}(X)$ with compact support is said to have compact support if it restricts to a constant concordance between elements of $\mathcal{F}(X \smallsetminus K)$, for some compact $K \subset X$. The set of compactly supported elements in $\mathcal{F}(X)$ modulo compactly supported concordance is denoted $\mathcal{F}_c[X]$. Similarly, a map $X \to |\mathcal{F}|$ is said to have compact support if its restriction to $X \smallsetminus K$ is constant with value z, for some compact $K \subset X$. We let $\mathrm{map}_c(X, |\mathcal{F}|\,)$ be the set of such maps. Then we have an obvious extension of the above proof:

Proposition A.1.2 *There is a bijection $\mathcal{F}_c[X] \cong \pi_0 \operatorname{map}_c(X, |\mathcal{F}|)$.* □

This proves proposition 2.2.5 in the special case where $A = \{z\}$. The general case is very similar. □

A.2 Categorical properties

Proposition A.2.1 *The construction $\mathcal{F} \mapsto |\mathcal{F}|$ takes pullback squares of sheaves to pullback squares of compactly generated Hausdorff spaces. In particular it respects products.*

Proof This is obvious from the definition of $|\mathcal{F}|$. □

Definition A.2.2 The categorical coproduct $\mathcal{F}_1 \amalg \mathcal{F}_2$ of two sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathscr{X}$ can be defined by $(\mathcal{F}_1 \amalg \mathcal{F}_2)(X) = \prod_i \mathcal{F}_1(X_i) \amalg \mathcal{F}_2(X_i)$ where X_i denotes the path component of X corresponding to an $i \in \pi_0(X)$.

Proposition A.2.3 $|\mathcal{F}_1 \amalg \mathcal{F}_2| \cong |\mathcal{F}_1| \amalg |\mathcal{F}_2|$.

Proof Note that Δ^n_e is path-connected for $n \geq 0$. □

Definition A.2.4 A natural transformation $u\colon \mathcal{F} \to \mathcal{G}$ of sheaves on $\mathscr{X}$ has the *concordance lifting property* if, for X in $\mathscr{X}$ and $s \in \mathcal{F}(X)$, any concordance $h \in \mathcal{G}(X \times]0,1[)$ starting at $u(s)$ lifts to a concordance $H \in \mathcal{F}(X \times]0,1[)$ starting at s.

Example A.2.5 Given a natural transformation $u\colon \mathcal{F} \to \mathcal{G}$ of sheaves on $\mathscr{X}$, make a new sheaf $\mathcal{F}^\sharp$ as follows. An element of $\mathcal{F}^\sharp(X)$ is a triple (h, s_0, s_1) where $s_0 \in \mathcal{F}(X)$, $s_1 \in \mathcal{G}(X)$ and h is a concordance from $u(s_0)$ to s_1. Then it is not hard to show that the forgetful transformations $\mathcal{F}^\sharp \to \mathcal{F}$ and $\mathcal{F}^\sharp \to \mathcal{G}$ given by $(h, s_0, s_1) \mapsto s_0$ and $(h, s_0, s_1) \mapsto s_1$, respectively, have the concordance lifting property. It is also clear from proposition 2.2.5 that the forgetful map $\mathcal{F}^\sharp \to \mathcal{F}$ defined by $(h, s_0, s_1) \mapsto s_0$ is a weak homotopy equivalence.

Proposition A.2.6 *Suppose given sheaves $\mathcal{E}, \mathcal{F}, \mathcal{G}$ on $\mathscr{X}$ and morphisms (alias natural transformations) $u\colon \mathcal{E} \to \mathcal{G}$, $v\colon \mathcal{F} \to \mathcal{G}$. Let $\mathcal{E} \times_{\mathcal{G}} \mathcal{F}$ be the fiber product (pullback) of u and v. If u has the concordance lifting property, then the projection $\mathcal{E} \times_{\mathcal{G}} \mathcal{F} \to \mathcal{F}$ has the concordance lifting property and the following square is homotopy cartesian:*

$$\begin{tikzcd}
|\mathcal{E} \times_{\mathcal{G}} \mathcal{F}| \arrow[r] \arrow[d] & |\mathcal{F}| \arrow[d, "v"] \\
|\mathcal{E}| \arrow[r, "u"] & |\mathcal{G}|.
\end{tikzcd}$$

A.3 Relative homotopy and fibrations

We begin with a special case of proposition A.2.6. Given a natural transformation $u\colon \mathcal{E} \to \mathcal{G}$ of sheaves on $\mathscr{X}$ with the concordance lifting property, let z be a point in $\mathcal{G}(\star)$ and let $\mathcal{E}_z$ be the fiber of u over z (in the category of sheaves). Let $\mathrm{hofiber}_z\,|u|$ denote the homotopy fiber of $|u|\colon |\mathcal{E}| \to |\mathcal{G}|$ over the point z.

Lemma A.3.1 *For any $y \in \mathcal{E}_z(\star)$, the homotopy set $\pi_n(\mathcal{E}_z, y)$ is in canonical bijection with $\pi_n(\mathrm{hofiber}_z\,|u|, y)$.*

Proof (sketch): Because of the concordance lifting property, $\pi_n(\mathcal{E}_z, y)$ can be identified with an appropriate relative homotopy group (or homotopy set) of the map of sheaves $u\colon \mathcal{E} \to \mathcal{G}$. Representatives of the latter are elements

$$(r, s) \in \mathcal{G}(B^{n+1}) \times \mathcal{F}(S^n),$$

where $B^{n+1} = D^{n+1} \smallsetminus S^n$, such that $s \in \mathcal{F}(S^n)$ is based at y and $(r, u(s))$ belongs to $\mathcal{G}(D^{n+1}) \subset \mathcal{G}(B^{n+1}) \times \mathcal{G}(S^n)$. See definition 2.2.1. We can identify this relative homotopy group (set) with a relative homotopy group (set) of the map of spaces $|u|\colon |\mathcal{E}| \to |\mathcal{G}|$, which can then be identified with a homotopy group (set) of the homotopy fiber of $|u|$ over z. ◻

Corollary A.3.2 *In the situation of lemma A.3.1, the sequence*

$$|\mathcal{E}_z| \hookrightarrow |\mathcal{E}| \xrightarrow{|u|} |\mathcal{G}|$$

is a homotopy fiber sequence.

Proof The composite map from $|\mathcal{E}_z|$ to $|\mathcal{G}|$ is constant. This leads to a canonical map from $|\mathcal{E}_z|$ to the homotopy fiber of $|u|\colon |cE| \to |\mathcal{G}|$ over z. It is easy to verify directly that this induces a surjection on π_0. For each $y \in \mathcal{E}_z(\star)$, the induced map of homotopy sets

$$\pi_n(\mathcal{E}_z, y) \longrightarrow \pi_n(\mathrm{hofiber}_z\,|u|, y)$$

is the one from lemma A.3.1. It is therefore always a bijection. ◻

Proof of proposition A.2.6. We fix $z \in \mathcal{F}(\star)$ and obtain $v(z) \in \mathcal{G}(\star)$. The fiber of

$$\mathcal{E} \times_{\mathcal{G}} \mathcal{F} \longrightarrow \mathcal{F}$$

over z is identified with the fiber of $u\colon \mathcal{E} \to \mathcal{G}$ over $v(z)$. Using corollary A.3.2 we can conclude that the homotopy fiber of $|\mathcal{E} \times_{\mathcal{G}} \mathcal{F}| \longrightarrow |\mathcal{F}|$ over z maps to the homotopy fiber of $|u|\colon |\mathcal{E}| \to |\mathcal{G}|$ over $v(z)$ by a homotopy equivalence. ◻

B Sheaves with a category structure

This section contains the proof of theorem 4.1.3.

B.1 Cocycle sheaves without indices

We begin with the definition of a close relative $\beta'\mathcal{F}$ of $\beta\mathcal{F}$. In the definition of $\beta'\mathcal{F}$ we trade the open coverings in the definition of $\beta\mathcal{F}$ for surjective étale maps. Recall a smooth map is *étale* if it is locally diffeomorphic, i.e., if its differential at any point of the source is invertible. A covering of a smooth manifold X by open subsets Y_j, for $j \in J$, gives rise to such a map $Y \to X$, where $Y \subset J \times X$ is the set of all (j, x) with $x \in Y_j$.

Definition B.1.1 For X in $\mathscr{X}$, an element in $\beta'\mathcal{F}(X)$ consists of the following data:

(i) A smooth manifold Y and a graphic, surjective and étale map $Y \to X$. (We will write $Y^{(\underline{n})}$ for the manifold of all maps $\underline{n} \to Y$ such that the composition $\underline{n} \to Y \to X$ is constant.)

(ii) For $m, n \geq 0$ and each injective $g\colon \underline{m} \to \underline{n}$ (which need not be order-preserving), a morphism φ_g in the category $\mathcal{F}(Y^{(\underline{n})})$.

The morphisms φ_g are subject to a 1-cocycle condition, which comes in two parts. The first part says that φ_g is an identity morphism if g is bijective. The second part says

$$\varphi_{gf} = g_*(\varphi_f) \circ \varphi_g.$$

Here f and g can have the form $f\colon \underline{\ell} \to \underline{m}$ and $g\colon \underline{m} \to \underline{n}$, so that gf is defined. We have written $g_*\colon \mathcal{F}(Y^{(\underline{m})}) \to \mathcal{F}(Y^{(\underline{n})})$ for the map induced by $g^*\colon Y^{(\underline{n})} \longrightarrow Y^{(\underline{m})}$.

Suppose that $(Y, \varphi_?)$ is an element of $\beta'\mathcal{F}$. Suppose also that Y is an open subset of $J \times X$ and the surjective étale map $Y \to X$ which is part of the data has been obtained by restricting the projection $J \times X \to X$. Then Y determines an open covering of X by open subsets Y_j. Namely, Y_j can be defined as the image of $Y \cap (j \times X)$ under the projection $J \times X \to X$. Each $Y^{(\underline{n})}$ can be identified with a disjoint union of copies of open subsets $Y_S \subset X$, where S is a nonempty subset of J with at most $n+1$ elements, and $Y_S = \bigcap_{j \in S} Y_j$ as usual. In this way, $\varphi_?$ breaks up into data $\varphi_{RS} \in \mathcal{F}(Y_S)$, one for each pair of finite nonempty $R, S \subset J$ with $R \subset S$. We leave the detailed verification to the reader. The conclusion is that $\beta\mathcal{F}$ is a subsheaf of $\beta'\mathcal{F}$.

Proposition B.1.2 *The inclusion of $\beta\mathcal{F}$ in $\beta'\mathcal{F}$ is a weak homotopy equivalence.*

For the proof we need two lemmas, mostly about étale maps.

Lemma B.1.3 *Let $Y \to X$ be a smooth, étale and surjective map. There exists an open subset $Y^\flat \subset J \times X$ such that the projection $Y^\flat \to X$ is locally finite and surjective, and a map $a\colon Y^\flat \to Y$ over X.*
In addition, suppose given a closed $C \subset X$, an open $Y^\flat_C \subset J \times C$ such that the projection $Y^\flat_C \to C$ is locally finite and surjective, and a map $a_C\colon Y^\flat_C \to Y$ over X. Then we can construct $Y^\flat$ and $a\colon Y^\flat \to Y$ above in such a way that $Y^\flat|C = Y^\flat_C$ and $a_C = a|Y^\flat_C$.

Proof For the absolute case, select a locally finite open covering of X by open subsets $Y^\flat_j$, where $j \in J$, such that $Y \to X$ admits a section s_j over each $Y^\flat_j$. Let

$$Y^\flat = \{(j,x) \in J \times X \mid x \in Y^\flat_j\}.$$

The s_j together define a map $Y^\flat \to Y$ over X.
In the relative case we make a case distinction. If $j \in J$ is such that $Y^\flat_C$ has nonempty intersection with $j \times X$, select an open $Y^\flat_j \subset X$ in such a way that $Y \to X$ admits a section s_j over $Y^\flat_j$ and $Y^\flat_j \cap C$ equals the image of $(Y^\flat_C) \cap (j \times X)$ in X. For all other $j \in J$, select an open $Y^\flat_j \subset X \smallsetminus C$ in such a way that $Y \to X$ admits a section s_j over $Y^\flat_j$. This is to be done in such a way that the $Y^\flat_j$ constitute a covering of X. Then let $Y^\flat = \{(j,x) \in J \times X \mid x \in Y^\flat_j\}$ as before. □

Lemma B.1.4 *Let $(Y^\flat, \varphi^\flat_?)$ and $(Y, \varphi_?)$ be elements of $\beta'\mathcal{F}(X)$. Suppose also that there exists a map $g\colon Y^\flat \to Y$ over X such that $g^*\varphi_? = \varphi^\flat_?$. Then $(Y, \varphi_?)$ and $(Y^\flat, \varphi^\flat_?)$ are concordant.*
Moreover, if $C \subset X$ is closed and g is an identity map over a neighborhood of C, then the concordance can be constructed so as to be constant over a neighborhood of C.

Proof *Absolute case*: We ignore minor set-theoretic issues related to definition 2.1.1. — We want to make a concordance of the form

$$(Z, \psi_?) \in \beta'\mathcal{F}(X \times]0,1[)$$

where Z is the disjoint union of $Y^\flat \times]0,2/3[$ and $Y \times]1/3,1[$. This comes with an obvious surjective étale map to $X \times]0,1[$. There is also a map $q\colon Z \to Y$ over X defined by

$$q(y,t) := \begin{cases} g(y) & \text{for } (y,t) \in Y^\flat \times]0,2/3[, \\ y & \text{for } (y,t) \in Y \times]1/3,1[. \end{cases}$$

We let $\psi_? = q^*\varphi_?$.
Relative case: We proceed somewhat differently. We are assuming $Y^\flat|U = Y|U$ where U is an open subset of X containing C. Also, g equals id on $Y^\flat|U$. We make a concordance of the form $(Z, \psi) \in \beta'\mathcal{F}(X \times]0,1[)$ where Z is the disjoint union of

$$(Y^\flat \times]0,1/2[) \cup (Y|U \times]0,1[) \cup (Y \times]1/2,1[)$$

and $(Y^\flat|(X \smallsetminus C)) \times]1/4,3/4[$. This comes with an obvious surjective étale map to $X \times]0,1[$. There is also a map $q\colon Z \to Y$ over X given by

$$q(y,t) := \begin{cases} g(y) & \text{for } (y,t) \in Y^\flat \times]0,1/2[\\ y & \text{for } (y,t) \in (Y|U) \times]0,1[, \\ y & \text{for } (y,t) \in Y \times]1/2,1[, \\ g(y) & \text{for } (y,t) \in (Y^\flat|(X \smallsetminus C)) \times]1/4,3/4[. \end{cases}$$

Again we let $\psi_? = q^*\varphi_?$. □

Proof of B.1.2. This is now a direct consequence of the relative surjectivity criterion in proposition 2.2.6 and the two lemmas just above. □

B.2 A variation on the nerve construction

For $n \geq 0$ let $\mathscr{D}\underline{n}$ be the set of nonempty subsets of $\{0,1,2,\ldots,n\}$. This is partially ordered by inclusion. There are functors $v_n \colon \mathscr{D}\underline{n} \to \underline{n}$ given by $v_n(S) = \max(S) \in \underline{n}$.

Lemma B.2.1 *Let $\mathscr{C}$ be a small category. Then the map of simplicial sets*

$$(n \mapsto \hom(\underline{n}^{\mathrm{op}}, \mathscr{C})) \quad \longrightarrow \quad (n \mapsto \hom(\mathscr{D}\underline{n}^{\mathrm{op}}, \mathscr{C}))$$

given by composition with $v_\bullet$ induces a homotopy equivalence of the geometric realizations.

Proof The simplicial set $(n \mapsto \hom(\mathscr{D}\underline{n}^{\mathrm{op}}, \mathscr{C}))$ is obtained by applying Kan's functor *ex*, the right adjoint of the barycentric subdivision, to $(n \mapsto \hom(\underline{n}^{\mathrm{op}}, \mathscr{C}))$. The statement is therefore a special case of [22, 3.7]. □

We note that the simplicial set $(n \mapsto \hom(\underline{n}^{\mathrm{op}}, \mathscr{C}))$ is precisely the nerve of $\mathscr{C}$, denoted $N_\bullet\mathscr{C}$ in section 4.

Corollary B.2.2 *Let $m \mapsto \mathscr{C}_m$ be a simplicial category. The map of bisimplicial sets*

$$\begin{array}{c} (m,n) \longmapsto \hom(\underline{n}^{\mathrm{op}}, \mathscr{C}_m) \\ \Big\Downarrow \\ (m,n) \longmapsto \hom(\mathscr{D}\underline{n}^{\mathrm{op}}, \mathscr{C}_m) \end{array}$$

given by composition with the functors $v_n \colon \mathscr{D}\underline{n} \to \underline{n}$ induces a homotopy equivalence of the geometric realizations. □

B.3 Completion of the proof

Continuing with the proof of 4.1.3, we come to a user–friendly description of the classifying space $B|\mathcal{F}|$. Recall that $\Delta^n_e = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \Sigma_i x_i = 1\}$.

Lemma B.3.1 *The classifying space $B|\mathcal{F}|$ is homotopy equivalent to the geometric realization of the simplicial set given by*

$$n \mapsto \hom(\mathscr{D}\underline{n}^{\mathrm{op}}, \mathcal{F}(\Delta^n_e)).$$

Proof We defined $B|\mathcal{F}|$ in section 4 as the geometric realization of a simplicial space given by $\underline{n} \mapsto |\hom(\underline{n}^{\mathrm{op}}, \mathcal{F})|$, where $\hom(\underline{n}^{\mathrm{op}}, \mathcal{F})$ is viewed as a sheaf on $\mathscr{X}$ and the vertical bars around it indicate the representing space construction. This means that we have defined $B|\mathcal{F}|$ as the geometric realization of a bisimplicial set

$$(m,n) \mapsto \hom(\underline{n}^{\mathrm{op}}, \mathcal{F}(\Delta^m_e)).$$

By corollary B.2.2, we may instead use the geometric realization of the bisimplicial set

$$(m,n) \mapsto \hom(\mathscr{D}\underline{n}^{\mathrm{op}}, \mathcal{F}(\Delta_e^m)).$$

It is well known that the geometric realization of a bisimplicial set is homeomorphic to the geometric realization of its diagonal. In our situation this is the simplicial set given by

$$n \;\mapsto\; \hom(\mathscr{D}\underline{n}^{\mathrm{op}}, \mathcal{F}(\Delta_e^n)). \qquad \square$$

Lemma B.3.2 *Let S be an infinite set and let $Z_\bullet$ be a simplicial set. The geometric realization $|Z_\bullet|$ is homotopy equivalent to the geometric realization of the incomplete simplicial set $n \mapsto Z_n \times \mathrm{emb}(\underline{n}, S)$, where $\mathrm{emb}(\underline{n}, S)$ is the set of injective maps from $\underline{n}$ to S.*

Proof There is a projection map from the realization of $n \mapsto Z_n \times \mathrm{emb}(\underline{n}, S)$ to $|Z_\bullet|$. We will show that it has contractible fibers. Let y be a point in a k-cell of $|Z_\bullet|$, corresponding to some nondegenerate simplex in Z_k. The fiber over y is homeomorphic to the classifying space of the poset $\mathscr{P}$ whose elements are the nonempty finite subsets of S equipped with a total ordering and a surjection to $\underline{k}$. For each finite subset $\mathscr{P}'$ of $\mathscr{P}$, there exists $T \in \mathscr{P}$ such that $\sup(T, T')$ exists for all $T' \in \mathscr{P}'$. This implies that the inclusion of $|\mathscr{P}'|$ in $|\mathscr{P}|$ is homotopic to a constant map (with value equal to the vertex determined by T). Therefore $\mathscr{P}$ is contractible, i.e., the fiber in question is contractible. $\square$

Corollary B.3.3 *Let J be the (fixed) infinite set from definition 4.1.1. The classifying space $B|\mathcal{F}|$ is homotopy equivalent to the geometric realization of the incomplete simplicial set $Z_\bullet$ given by*

$$n \quad \mapsto \quad \hom(\mathscr{D}\underline{n}^{\mathrm{op}}, \mathcal{F}(\Delta_e^n)) \times \mathrm{emb}(\underline{n}, J).$$

We come to the construction of a comparison map Ψ from the incomplete simplicial set in corollary B.3.3 to the simplicial set $\underline{n} \mapsto \beta\mathcal{F}(\Delta_e^n)$. The idea is simple. An n-simplex in the incomplete simplicial set of corollary B.3.3 consists of a functor

$$\varphi\colon\ \mathscr{D}\underline{n}^{\mathrm{op}} \longrightarrow \mathcal{F}(\Delta_e^n)$$

and an injective map $\lambda\colon\ \underline{n} \to J$. The functor φ carries exactly the same information as an element in $\beta\mathcal{F}(\Delta_e^n)$ whose underlying J-indexed open covering is given by $j \mapsto \Delta_e^n$ if $j = \lambda(t)$ for some $t \in \underline{n}$ and $j \mapsto \emptyset$ otherwise. To make this information more functorial, i.e., compatible with face operators, we replace the nonempty open sets in the open covering by smaller ones, according to the rule

$$j = \lambda(t) \quad \mapsto \quad \{\, (x_0, x_1, \dots, x_n) \in \Delta_e^n \mid x_t > 0\}. \tag{B.1}$$

The remaining data can be restricted and we now have an element $\Psi(\varphi, \lambda) \in \beta\mathcal{F}(\Delta_e^n)$. The construction Ψ respects the face operators. We now restate theorem 4.1.3 as

Lemma B.3.4 *The map Ψ induces a homotopy equivalence from $B|\mathcal{F}| \simeq |Z_\bullet|$ to $|\beta\mathcal{F}|$.*

Proof Let z be a vertex of $|Z_\bullet|$. For each $n \geq 0$, the map Ψ induces a map

$$\pi_n(\,|Z_\bullet|, z) \longrightarrow \pi_n(\beta\mathcal{F}, z) \cong \pi_n(\beta'\mathcal{F}, z).$$

We will show that this is bijective by constructing the inverse map. As in section A.1, we can represent elements of $\pi_n(\beta'\mathcal{F}, z)$ by elements of $\beta'\mathcal{F}(\mathbb{R}^n)$ with compact support. Let $(Y, \varphi_?)$ be such an element of $\beta'\mathcal{F}(\mathbb{R}^n)$, with notation as in definition B.1.1. There exists a smooth triangulation of $\mathbb{R}^n$, with vertex set T, which for each $v \in T$ allows an embedding $g_v\colon \mathrm{st}^2(v) \to Y$ over $\mathbb{R}^n$. Here $\mathrm{st}^2(v)$ is the union of the open stars $\mathrm{st}(w)$ of all vertices w adjacent to v. Choose such a triangulation and such embeddings g_v. Also, for each finite nonempty subset S of T spanning a simplex of the triangulation, choose a smooth map $c_{e,S}\colon \Delta_e(S) \to \mathbb{R}^n$ extending the characteristic inclusion $c_S\colon \Delta(S) \to \mathbb{R}^n$. This is to be done in such a way that $c_{e,S}$ agrees with $c_{e,R}$ on a face $\Delta_e(R) \subset \Delta_e(S)$ and

$$c_{e,S}(\Delta_e(S)) \subset \mathrm{st}^2(v)$$

whenever $v \in S$. We then have, for each S as above, a commutative square

$$\begin{tikzcd} S \times \Delta_e(S) \arrow[r] \arrow[d] & Y \arrow[d] \\ \Delta_e(S) \arrow[r, "c_{e,S}"] & \mathbb{R}^n \end{tikzcd}$$

where the top row is given by $(v, x) \mapsto g_v(c_{e,S}(x))$ and the vertical arrows are étale and surjective. Using this to pull back the data $\varphi_?$, we obtain for each S an element

$$x_S \in \hom(\mathscr{D}(S)^{\mathrm{op}}, \mathcal{F}(\Delta_e(S)))$$

where $\mathscr{D}(S)$ is the poset of nonempty subsets of S. Finally we choose a total ordering on T and an injection $T \to J$. This promotes each x_S to an element of $\hom(\mathscr{D}\underline{n}^{\mathrm{op}}, \mathcal{F}(\Delta^n_e))$ where $n = |S| - 1$. For each S we also get a canonical injection u_S from $\underline{n} \cong S$ to $T \subset J$, so that the pair (x_S, u_S) can be regarded as an n-simplex of $Z_\bullet$. Now we have a unique map from $\mathbb{R}^n$ to $|Z_\bullet|$ which, on $\Delta(S) \subset \mathbb{R}^n$, is the characteristic map for the simplex (x_S, u_S). It has compact support. Its compactly supported homotopy class depends only on the compactly supported concordance class of $(Y, \varphi_?)$. This gives us the map

$$\Lambda\colon \pi_n(\beta'\mathcal{F}, z) \longrightarrow \pi_n(\,|Z_\bullet|, z) \cong \pi_n(B|\mathcal{F}|, z)$$

which we need.

The composition $\Psi_*\Lambda\colon \pi_n(\beta'\mathcal{F}, z) \to \pi_n(\beta'\mathcal{F}, z)$ is the identity. Namely, $\vartheta^{-1}\Psi_*\Lambda = \vartheta^{-1}$ by construction, where ϑ is the bijective map of propositions 2.2.5 and section A.

To show that $\Lambda\Psi_*$ is the identity on $\pi_n(\,|Z_\bullet|, z)$, we resurrect the simplicial monoid $Q_\bullet$ which was introduced in section A.1. We may replace $|Z_\bullet|$ by the geometric realization of the (complete) simplicial set

$$n \quad \mapsto \quad \hom(\mathscr{D}\underline{n}^{\mathrm{op}}, \mathcal{F}(\Delta^n_e))$$

on which $|Q_\bullet|$ acts (from the right). In this way we get a right action of the monoid $\pi_n|Q_\bullet|$ on $\pi_n(\,|Z_\bullet|, z)$, with monoid structure coming from that on $|Q_\bullet|$. For every $[g] \in \pi_n(\,|Z_\bullet|, z)$ we have

$$\Lambda\Psi_*[g] = [g] \cdot [h_1]$$

for some $[h_1] \in \pi_n|Q_\bullet|$ which may depend on $[g]$. But $|Q_\bullet|$ is contractible, so $[h_1]$ will always be the neutral element and $[g] \cdot [h_1] = [g]$. □

C Geometric realizations and the bar construction

C.1 Realization, quasifibrations and homology fibrations

Lemma C.1.1 *Let $u_\bullet \colon E_\bullet \longrightarrow B_\bullet$ be a map between incomplete simplicial spaces (or good simplicial spaces). Suppose that the squares*

$$\begin{array}{ccc} E_k & \xrightarrow{u_k} & B_k \\ \downarrow{\scriptstyle d_i} & & \downarrow{\scriptstyle d_i} \\ E_{k-1} & \xrightarrow{u_{k-1}} & B_{k-1} \end{array}$$

are all homotopy cartesian ($k \geq i \geq 0$). Then the following is also homotopy cartesian:

$$\begin{array}{ccc} E_0 & \xrightarrow{u_0} & B_0 \\ \downarrow{\scriptstyle \text{incl.}} & & \downarrow{\scriptstyle \text{incl.}} \\ |E_\bullet| & \xrightarrow{|u_\bullet|} & |B_\bullet|. \end{array}$$

Proof It suffices to prove the statement for incomplete simplicial spaces. (The realization of a good simplicial space is homotopy equivalent to the realization of the underlying incomplete simplicial space.) Without loss of generality all the maps $u_k \colon E_k \to B_k$ are fibrations. Then, by inspection, $|u_\bullet|$ from $|E_\bullet|$ to $|B_\bullet|$ is a quasifibration in the sense of [6]. By [6], [7], this implies that each fiber of $|u_\bullet|$ maps by a weak homotopy equivalence to the corresponding homotopy fiber. Hence the canonical map from E_0 to the homotopy pullback of

$$|E_\bullet| \xrightarrow{|u_\bullet|} |B_\bullet| \xleftarrow{\text{incl.}} B_0$$

is a weak homotopy equivalence. □

Corollary C.1.2 *Let $u_\bullet \colon E_\bullet \longrightarrow B_\bullet$ be a map between incomplete simplicial spaces (or good simplicial spaces). Suppose that, in each square*

$$\begin{array}{ccc} E_k & \xrightarrow{u_k} & B_k \\ \downarrow{\scriptstyle d_i} & & \downarrow{\scriptstyle d_i} \\ E_{k-1} & \xrightarrow{u_{k-1}} & B_{k-1} \end{array}$$

the canonical map from any homotopy fiber of u_k to the corresponding homotopy fiber of u_{k-1} induces an isomorphism in integer homology. Then in the square

$$\begin{tikzcd} E_0 \arrow[r, "u_0"] \arrow[d, "\text{incl.}"] & B_0 \arrow[d, "\text{incl.}"] \\ {|E_\bullet|} \arrow[r, "|u_\bullet|"] & {|B_\bullet|}, \end{tikzcd}$$

the canonical map from any homotopy fiber of u_0 to the corresponding homotopy fiber of $|u_\bullet|$ induces an isomorphism in integer homology.

Proof It suffices to prove the statement for incomplete simplicial spaces. Again we may assume that each $u_k\colon E_k \to B_k$ is a fibration. Let $\mathcal{D}$ be the functor $X \mapsto S^1 \wedge X_+$ from spaces to pointed spaces. Let $\mathcal{D}(E_k; u_k)$ be the result of applying $\mathcal{D}$ to each fiber of $u_k\colon E_k \to B_k$. We still have quasifibrations

$$\mathcal{D}(E_k; u_k) \longrightarrow B_k$$

and we are in a situation where the previous lemma can be applied; so we get a homotopy cartesian square

$$\begin{tikzcd} \mathcal{D}(E_0; u_0) \arrow[r] \arrow[d, "\text{incl.}"] & B_0 \arrow[d, "\text{incl.}"] \\ {|\mathcal{D}(E_\bullet; u_\bullet)|} \arrow[r] & {|B_\bullet|} \end{tikzcd}$$

where the horizontal arrows are quasifibrations. But the lower left hand term is homeomorphic to $\mathcal{D}(|E_\bullet|; |u_\bullet|)$, the space obtained by applying $\mathcal{D}$ fiberwise to the fibers of $|u_\bullet|$. Hence we may "undo" the $\mathcal{D}$ operation in the left–hand column, replacing $\mathcal{D}(E_0; u_0)$ by E_0 and $|\mathcal{D}(E_\bullet; u_\bullet)| \cong \mathcal{D}(|E_\bullet|; |u_\bullet|)$ by $|E_\bullet|$, without changing the homology of the horizontal fibers except for a degree shift. □

Corollary C.1.3 *Let $\mathscr{C}$ be a small category and let $u\colon \mathcal{G}_1 \to \mathcal{G}_2$ be a natural transformation between functors from $\mathscr{C}$ to spaces. Suppose that, for each morphism $f\colon a \to b$ in $\mathscr{C}$, the map f_* from any homotopy fiber of u_a to the corresponding homotopy fiber of u_b induces an isomorphism in integer homology. Then for each object a of $\mathscr{C}$, the inclusion of any homotopy fiber of u_a in the corresponding homotopy fiber of $u_*\colon \operatorname{hocolim} \mathcal{G}_1 \to \operatorname{hocolim} \mathcal{G}_2$ induces an isomorphism in integer homology.*

Proof Apply corollary C.1.2 with $E_k := \coprod \mathcal{G}_1(D(k))$ and $B_k = \coprod \mathcal{G}_2(D(k))$, where both coproducts run over the set of contravariant functors D from the poset $\underline{k}$ to $\mathscr{C}$. Then $|E_\bullet|$ is $\operatorname{hocolim} \mathcal{G}_1$ and $|B_\bullet|$ is $\operatorname{hocolim} \mathcal{G}_2$. □

C.2 The bar construction for monoids without unit

Let A be a topological monoid, not necessarily with unit. This determines an incomplete simplicial space $X_\bullet$ where $X_k = A^k$ and the face operators $d_i\colon X_k \to X_{k-1}$ are given by

$$\begin{array}{lcll} (a_1, a_2, \dots, a_k) & \mapsto & (a_1, a_2, \dots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \dots, a_k) & \text{if } i \neq 0, k \\ (a_1, a_2, \dots, a_k) & \mapsto & (a_2, a_3, \dots, a_k) & \text{if } i = 0 \\ (a_1, a_2, \dots, a_k) & \mapsto & (a_1, a_2, \dots, a_{k-1}) & \text{if } i = k. \end{array}$$

Of course, $X_\bullet$ is known as the bar construction on A and $|X_\bullet|$ is known as the classifying space of A.

If A has a unit (neutral element), we can use it to define degeneracy operators in $X_\bullet$, making $X_\bullet$ into a simplicial space. If this is a *good* simplicial space [35], then its realization as a simplicial space is homotopy equivalent to the realization of the underlying incomplete simplicial space. Either of these two realizations can therefore be regarded as the classifying space of A.

A topological monoid A, with or without unit, determines another topological monoid A_+ which, as a space, is the disjoint union of A with a singleton. The added point serves as the neutral element (unit) in a topological monoid structure on A_+ which extends the one on A. (If A did have a unit to begin with, then that will no longer be the unit in A_+, but of course it will be a central idempotent in A_+.)

Lemma C.2.1 *The simplicial space $k \mapsto (A_+)^k$ is good and its realization (as a complete simplicial space) is homeomorphic to the realization of the incomplete simplicial space $k \mapsto A^k$.*

Proof It is well known that the forgetful functor from complete simplicial spaces to incomplete simplicial spaces has a left adjoint. We denote it by $X_\bullet \mapsto X_\bullet^\sim$. The most common description of $X_\bullet^\sim$ is as follows:

$$X_k^\sim := \coprod_{m=0}^{k} \coprod_{f\colon \underline{k} \twoheadrightarrow \underline{m}} X_m$$

where f runs through all surjective order preserving maps from $\underline{k}$ to $\underline{m}$. This makes it fairly clear how $X_\bullet^\sim$ is a simplicial space. Namely, suppose given an order preserving $g\colon \underline{j} \to \underline{k}$ and a triple (m, f, x) in $X_k^\sim$, so that $f\colon \underline{k} \to \underline{m}$ is order preserving and $x \in X_m$. Then we let

$$g^*(m, f, x) := (m', v, u^* x)$$

where $fg = uv$ is the unique decomposition of $fg\colon \underline{j} \to \underline{m}$ into an order preserving surjection $v\colon \underline{j} \to \underline{m'}$ and an order preserving injection $u\colon \underline{m'} \to \underline{m}$.
The simplicial space $X_\bullet^\sim$ is good and its geometric realization as a complete simplicial space is homeomorphic to the geometric realization of the incomplete simplicial space $X_\bullet$.

If $X_\bullet$ is the bar construction on A, that is, $X_k = A^k$, then $X_\bullet^\sim$ becomes the bar construction on A_+ , that is, $X_k^\sim = (A_+)^k$. Therefore $k \mapsto (A_+)^k$ is a good simplicial space and its realization as such is homeomorphic to the realization of the incomplete simplicial space $k \mapsto A^k$. □

The observations above concerning topological monoids with or without unit can be generalized to topological categories with or without identity morphisms. (Monoids are categories with only one object.) The category version of lemma C.2.1 is implicit in [25, §2.1].

D Generalities about homotopy colimits and stratifications

D.1 Homotopy colimits

Any functor $\mathcal{D}$ from a small (discrete) category $\mathscr{C}$ to the category of spaces has a *colimit*, $\operatorname{colim}\mathcal{D}$. This is the quotient space of the coproduct

$$\coprod_{a \text{ in } \mathscr{C}} \mathcal{D}(a)$$

obtained by identifying $x \in \mathcal{D}(a)$ with $f_*(x) \in \mathcal{D}(b)$ for any morphisms $f\colon a \to b$ in $\mathscr{C}$ and elements $x \in \mathcal{D}(a)$. It is well known that the colimit construction is not well behaved from a homotopy theoretic point of view. Namely, suppose that $w\colon \mathcal{D}_1 \to \mathcal{D}_2$ is a natural transformation between functors from $\mathscr{C}$ to spaces and that $w_a : \mathcal{D}_1(a) \to \mathcal{D}_2(a)$ is a homotopy equivalence for any object a in $\mathscr{C}$. Then this does not in general imply that the map induced by w from $\operatorname{colim}\mathcal{D}_1$ to $\operatorname{colim}\mathcal{D}_2$ is again a homotopy equivalence.

Example D.1.1 Let $\mathscr{C}$ be the poset of proper subsets of $\{0,1\}$, ordered by inclusion. The diagrams

$$[0,1] \hookleftarrow \partial[0,1] \hookrightarrow [0,1], \qquad\qquad \star \leftarrow \partial[0,1] \hookrightarrow \star$$

can be regarded as functors from $\mathscr{C}$ to spaces. There is a natural transformation w from the first to the second such that w_a is a homotopy equivalence for each object a in $\mathscr{C}$. The colimit of the first diagram is homeomorphic to S^1. The colimit of the second diagram is a single point.

Call a functor $\mathcal{D}$ from $\mathscr{C}$ to spaces *cofibrant* if, for any diagram of functors (from $\mathscr{C}$ to spaces) and natural transformations

$$\mathcal{D} \xrightarrow{v} \mathcal{E} \xleftarrow{w} \mathcal{F}$$

where $w_a\colon \mathcal{F}(a) \to \mathcal{E}(a)$ is a homotopy equivalence for all $a \in \mathscr{C}$, there exists a natural transformation $v'\colon \mathcal{D} \to \mathcal{F}$ and a *natural* homotopy $\mathcal{D}(a) \times [0,1] \to \mathcal{E}(a)$ (for all a) connecting wv' and v. It is not hard to show the following. If $v\colon \mathcal{D}_1 \to \mathcal{D}_2$ is a natural transformation between cofibrant functors such that $v_a\colon \mathcal{D}_1(a) \to \mathcal{D}_2(a)$ is a homotopy equivalence for each $a \in \mathscr{C}$, then v has a natural homotopy inverse (with *natural* homotopies) and therefore the induced map $\operatorname{colim}\mathcal{D}_1 \to \operatorname{colim}\mathcal{D}_2$ is a homotopy equivalence.

This suggests the following procedure for making colimits homotopy invariant. Suppose that $\mathcal{D}$ from $\mathscr{C}$ to spaces is any functor. Try to find a natural transformation $\mathcal{D}' \to \mathcal{D}$ specializing to homotopy equivalences $\mathcal{D}'(a) \to \mathcal{D}(a)$ for all a in $\mathscr{C}$, where $\mathcal{D}'$ is cofibrant. Then define the *homotopy colimit* of $\mathcal{D}$ to be $\operatorname{colim}\mathcal{D}'$. If it can be done, $\operatorname{hocolim}\mathcal{D}$ is at least well defined up to homotopy equivalence. (If $\mathcal{D}$ is the second diagram in example D.1.1, then the first diagram in the same example can serve as $\mathcal{D}'$ because it happens to be cofibrant. This gives $\operatorname{hocolim}\mathcal{D} \cong S^1$.)

This point of view is carefully presented in [8]. Some of the ideas go back to [26]. As we will see in a moment, there is a canonical construction for $\mathcal{D}'$ which depends naturally on $\mathcal{D}$.

The standard foundational reference for homotopy colimits and homotopy limits is the book [3] by Bousfield and Kan. But the first explicit construction of homotopy colimits in general appears to be due to Segal [37].

Again let $\mathcal{D}$ be a functor from a discrete small category $\mathscr{C}$ to the category of spaces. Following Segal we introduce a topological category denoted $\mathscr{C}\!\int\!\mathcal{D}$, the *transport category* of $\mathcal{D}$:

$$\operatorname{ob}(\mathscr{C}\!\textstyle\int\!\mathcal{D}) = \displaystyle\coprod_{a\in\operatorname{ob}(\mathscr{C})} \mathcal{D}(a)\,, \qquad \operatorname{mor}(\mathscr{C}\!\textstyle\int\!\mathcal{D}) = \displaystyle\coprod_{f\in\operatorname{mor}(\mathscr{C})} \mathcal{D}(\sigma(f))\,.$$

Here $\sigma(f)$ denotes the source of a morphism f in $\mathscr{C}$. We will write morphisms in $\mathscr{C}\!\int\!\mathcal{D}$ as pairs (f,x) where $f \in \operatorname{mor}(\mathscr{C})$ and $x \in \mathcal{D}(\sigma(f))$. The composition $(g,y)\circ(f,x)$ of two such morphisms is defined if an only if $g\circ f$ is defined in $\mathscr{C}$ and $f_*(x) = y$, in which case $(g,y)\circ(f,x) = (g\circ f, x)$. The classifying space $B(\mathscr{C}\!\int\!\mathcal{D})$ is a model for the homotopy colimit of $\mathcal{D}$.

To relate $B(\mathscr{C}\!\int\!\mathcal{D})$ to our earlier discussion we define a functor $\mathcal{D}'$ from $\mathscr{C}$ to spaces as follows. For $a \in \operatorname{ob}(\mathscr{C})$ let $\mathscr{C}\!\downarrow\! a$ be the category of $\mathscr{C}$-objects over a, [24, II.6]. Let

$$\mathcal{D}'(a) := B\left((\mathscr{C}\!\downarrow\! a)\!\textstyle\int\!\mathcal{D}\right)$$

for objects a in $\mathscr{C}$, where we view $\mathcal{D}$ as a functor on $\mathscr{C}\!\downarrow\! a$. Then $\mathcal{D}'$ is cofibrant and the canonical map $\mathcal{D}'(a) \to \mathcal{D}(a)$ is a homotopy equivalence for every a in $\mathscr{C}$. Moreover,

$$B(\mathscr{C}\!\textstyle\int\!\mathcal{D}) \cong \operatorname{colim}\mathcal{D}'.$$

Note in passing that if $\mathcal{D}(a)$ is a singleton for each a in $\mathcal{C}$, then the transport category $\mathscr{C}\!\int\!\mathcal{D}$ is identified with $\mathscr{C}$ and so $\operatorname{hocolim}\mathcal{D} = B\mathscr{C}$.

To make a homotopy colimit, we need a pair $(\mathscr{C},\mathcal{D})$ consisting of a small category $\mathscr{C}$ and a functor $\mathcal{D}$ from $\mathscr{C}$ to spaces. By a *morphism* from one such pair $(\mathscr{C}^s,\mathcal{D}^s)$ to another, $(\mathscr{C}^t,\mathcal{D}^t)$, we understand a pair $(\mathcal{F},\nu)$ consisting of a functor $\mathcal{F}\colon \mathscr{C}^s \to \mathscr{C}^t$ and a natural transformation ν from $\mathcal{D}^s$ to $\mathcal{D}^t\mathcal{F}$.

Remark D.1.2 *Such a morphism induces a map $(\mathcal{F},\nu)_*$ from* $\operatorname{hocolim}\mathcal{D}^s$ *to* $\operatorname{hocolim}\mathcal{D}^t$.

Suppose that $(\mathcal{F}_0,\nu_0)$ and $(\mathcal{F}_1,\nu_1)$ are morphisms from $(\mathscr{C}^s,\mathcal{D}^s)$ to $(\mathscr{C}^t,\mathcal{D}^t)$. Let θ be a natural transformation from $\mathcal{F}_0$ to $\mathcal{F}_1$ such that $\nu_1 = \mathcal{D}^t(\theta)\circ\nu_0$.

Remark D.1.3 *Such a θ induces a homotopy θ_* from $(\mathcal{F}_0,\nu_0)_*$ to $(\mathcal{F}_1,\nu_1)_*$.*

Proof Let $\mathscr{I} = \{0,1\}$, viewed as an ordered set with the usual order and then as a category. Then $B\mathscr{I} \cong [0,1]$. Let $p\colon \mathscr{C}\times\mathscr{I}\to\mathscr{C}$ be the projection. The data $(\mathcal{F}_0,\nu_0)$, $(\mathcal{F}_1,\nu_1)$ and θ taken together define a morphism from $(\mathscr{C}^s\times\mathscr{I},\mathcal{D}^s\circ p)$ to $(\mathscr{C}^t,\mathcal{D}^t)$. By remark D.1.2, this leads to a map from $\operatorname{hocolim}(\mathcal{D}^s\circ p)\cong(\operatorname{hocolim}\mathcal{D}^s)\times B\mathscr{I}$ to $\operatorname{hocolim}\mathcal{D}^t$. $\square$

Let $\mathscr{C}$ be a small category and let $a\mapsto\mathcal{F}_a$ be a covariant functor from $\mathscr{C}$ to the category of sheaves on $\mathscr{X}$. Define a sheaf $\mathscr{C}\!\int\!\mathcal{F}$ with category structure as follows. For 0-connected X in $\mathscr{X}$, let $(\mathscr{C}\!\int\!\mathcal{F})(X) = \mathscr{C}\!\int\!\mathcal{F}_\bullet(X)$, where each $\mathcal{F}_a(X)$ for $a\in\operatorname{ob}(\mathscr{C})$ is regarded as a discrete space. For X which is not 0-connected we define $(\mathscr{C}\!\int\!\mathcal{F})(X) = \prod_i(\mathscr{C}\!\int\!\mathcal{F})(X_i)$ where the X_i are the connected components of X. In section 4.1 we used the following notation.

Definition D.1.4 In the situation above, we let $\operatorname{hocolim}_a\mathcal{F}_a := \beta(\mathscr{C}\!\int\!\mathcal{F})$.

Lemma D.1.5 $|\operatorname{hocolim}_a\mathcal{F}_a| \simeq \operatorname{hocolim}_a|\mathcal{F}_a|$.

Proof Theorem 4.1.3, proved in appendix B above, gives $|\operatorname{hocolim}_a\mathcal{F}_a| \simeq B|\mathscr{C}\!\int\!\mathcal{F}|$ and propositions A.2.1, A.2.3 imply $B|\mathscr{C}\!\int\!\mathcal{F}| \cong B(\mathscr{C}\!\int|\mathcal{F}_?|)$, where $|\mathcal{F}_?|$ denotes the functor $a\mapsto|\mathcal{F}_a|$ from $\mathscr{C}$ to spaces. $\square$

Corollary D.1.6 *Let $\mathscr{C}$ be a small category and let $a\mapsto\mathcal{E}_a$ and $\mapsto\mathcal{E}'_a$ be covariant functors from $\mathscr{C}$ to the category of sheaves on $\mathscr{X}$. Let $\nu=\{\nu_a\colon\mathcal{E}_a\to\mathcal{E}'_a\}$ be a natural transformation such that every $\nu_a\colon\mathcal{E}_a\to\mathcal{E}'_a$ is a weak equivalence. Then the induced map* $\operatorname{hocolim}_a\mathcal{E}_a\to\operatorname{hocolim}_a\mathcal{E}'_a$ *is a weak equivalence (between sheaves on $\mathscr{X}$).* $\square$

D.2 Stratifications and homotopy colimit decompositions

Here we describe a relationship between stratifications and homotopy colimit decompositions. The point which we want to make, without proving anything definite in that direction, is that a stratification of a space often comes from a homotopy colimit decomposition of the space where the indexing category is an EI-category (a category in which all endomorphisms are isomorphisms). Compare [38].

Definition D.2.1 A *stratification* of a topological space X is a partition of X into locally closed nonempty subsets, the strata, such that the closure of each stratum is a union of strata.

The strata X_i of a stratified space X form a poset S where $X_i \leq X_j$ if the closure of X_i contains X_j. (This is the reverse of the obvious ordering.) The tautological map $X \to S$ does of course completely describe the stratification. Hence a stratification of X can always be described by a *stratification function*, a map from X to a poset S. (We will not discuss the question which maps from X to a poset give rise to stratifications.)

Definition D.2.2 A stratification of a CW-space X is a *CW-stratification* if the closure of each stratum is a CW-subspace.

Definition D.2.3 Let $\mathscr{C}$ be a small EI-category. Let $\iota(\mathscr{C})$ be the poset of isomorphism classes of objects in $\mathscr{C}$, ordered in such a way that $[C_0] \leq [C_1]$ iff there exists a morphism $C_0 \to C_1$. Define

$$f\colon\ B\mathcal{C} \longrightarrow \iota(\mathcal{C})$$

in such a way that $f(x) = [C_k]$ if the unique open cell of $B\mathcal{C}$ containing x corresponds to a k-simplex of the form $C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_k$. Then f is the stratification function for a CW-stratification of $B\mathscr{C}$.

The following example of an EI-category is closely related to the category $\mathscr{K}$ in section 5.

Definition D.2.4 We make an EI-category $\mathscr{J}$ as follows. The objects are the finite subsets of a fixed universe. A morphism from S_1 to S_2 consists of an injection $k\colon\ S_1 \to S_2$ and a sign function ε from $S_2 \smallsetminus \mathrm{im}(f)$ to $\{-1,+1\}$. The composition of $(k_1,\varepsilon_1)\colon\ S_1 \to S_2$ and $(k_2,\varepsilon_2)\colon\ S_2 \to S_3$ is (k_2k_1,ε_3) where ε_3 agrees with ε_2 outside $k_2(S_2)$ and with $\varepsilon_1 \circ {k_2}^{-1}$ on $k_2(S_2 \smallsetminus k_1(S_1))$. Then $B\mathscr{J}$ is stratified as above.

Remark D.2.5 We have $B\mathscr{J} = \Omega^\infty S^{\infty+1}$ by [35, 3.2].

Definition D.2.6 Let $\mathcal{D}$ be a contravariant functor from a small EI-category $\mathscr{C}$ to spaces (i.e., a covariant functor from $\mathscr{C}^{\mathrm{op}}$ to spaces). Then we have the projection map

$$\mathrm{hocolim}\ \mathcal{D} \longrightarrow B\mathscr{C}$$

and so we get a stratification on $\mathrm{hocolim}\,\mathcal{D}$, the pullback of the stratification of $B\mathscr{C}$ just defined. (It is true but not completely trivial that this does give a stratification on $\mathrm{hocolim}\,\mathcal{D}$.)

The homotopy theoretic relationship between the values of $\mathcal{D}$ and the strata of $\mathrm{hocolim}\,\mathcal{D}$ is as follows. For an object C in $\mathscr{C}$, the stratum with label $[C]$ has the homotopy type of a homotopy orbit space $\mathcal{D}(C)_{hA}$ where A is the automorphism group of C in $\mathscr{C}$. (In more detail, the stratum has a deformation retraction to the homotopy colimit of $\mathcal{D}|\mathscr{A}$ where $\mathscr{A}$ is the full subcategory of $\mathscr{C}$ spanned by the objects isomorphic to C.)

References

[1] **J F Adams** *Infinite loop spaces*, Annals of Math. Studies 90, Princeton Univ. Press (1978)

[2] **T Bröcker, K Jänich** *Introduction to Differential Topology*, Engl. edition Cambridge Univ. Press (1982); German edition Springer-Verlag (1973)

[3] **A K Bousfield, D Kan** *Homotopy limits, completions and localizations*, Springer Lecture Notes in Math., vol. 304, Springer (1972)

[4] **T. Broecker** *Differentiable germs and catastrophes*, Lond. Math. Soc. Lecture Note ser., vol. 17 (1975)

[5] **S Buoncristiano, C Rourke, B Sanderson** *A geometric approach to homology theory*, Lond. Math. Soc. Lecture Note ser., vol. 18 (1976)

[6] **A Dold, R Thom** *Une généralisation de la notion d'espace fibré. Application aux produits symétriques infinis*, C. R. Acad. Sci. Paris 242 (1956) 1680–1682

[7] **A Dold, R Thom** *Quasifaserungen und unendliche symmetrische Produkte*, Ann. of Math. 67 (1958) 239–281

[8] **E Dror** *Homotopy and homology of diagrams of spaces*, in: *Algebraic Topology, Seattle Wash. 1985*, Springer Lec. Notes in Math. 1286 (1987), 93–134

[9] **C J Earle, J Eells** *A fibre bundle description of Teichmueller theory*, J. Differential Geom. 3 (1969), 19–43

[10] **C J Earle, A Schatz** *Teichmueller theory for surfaces with boundary*, J. Differential Geom. 4 (1970), 169–185

[11] **S Galatius** *Mod p homology of the stable mapping class group*, to appear

[12] **M Golubitsky, V Guillemin** *Stable mappings and their singularities*, Springer Grad. Texts (1973)

[13] **T Goodwillie, J Klein, M Weiss**, *A Haefliger style description of the embedding calculus tower*, Topology, to appear

[14] **M Gromov** *Partial Differential Relations*, Springer-Verlag, Ergebnisse series (1984)

[15] **M Gromov** *A topological technique for the construction of solutions of differential equations and inequalities*, Actes du Congrès International des Mathematiciens, Nice 1970, Gauthier–Villars (1971)

[16] **A Haefliger** *Lectures on the theorem of Gromov*, in Proc. of 1969/70 Liverpool Singularities Symp., Lecture Notes in Math. vol. 209, Springer (1971) 128-141

[17] **J L Harer** *Stability of the homology of the mapping class groups of oriented surfaces*, Ann. of Math. 121 (1985) 215–249

[18] **M Hirsch** *Immersions of manifolds*, Trans. Amer. Math. Soc. 93 (1959) 242–276

[19] **M Hirsch** *Differential Topology*, Springer–Verlag (1976)

[20] **K Igusa** *Higher singularities of smooth functions are unnecessary*, Ann. of Math. 119 (1984) 1–58

[21] **N V Ivanov** *Stabilization of the homology of the Teichmueller modular groups*, Algebra i Analiz 1 (1989) 120–126; translation in: Leningrad Math. J. 1 (1990) 675–691

[22] **D Kan** *On c.s.s. complexes*, Amer. J. Math. 79 (1957), 449–476.

[23] **S Lang** *Differential manifolds*, Addison-Wesley (1972)

[24] **S MacLane** *Categories for the working mathematician*, Grad. texts in Math., Springer-Verlag (1971)

[25] **I Madsen, U Tillmann** *The stable mapping class group and* $Q(\mathbb{C}P^\infty)$, Invent. Math. 145 (2001) 509–544

[26] **J Milnor** *On axiomatic homology theory*, Pacific J. Math. 12 (1962) 337–341

[27] **J Milnor** *Morse Theory*, Ann. of Math. Studies 51, Princeton University Press (1963, 1969)

[28] **J Milnor** *Lectures on the h-cobordism theorem*, Mathematical notes 1, Princeton University Press (1965)

[29] **I Moerdijk** *Classifying spaces and classifying topoi*, Lecture Notes in Math. 1616, Springer-Verlag (1995)

[30] **A Phillips** *Submersions of open manifolds*, Topology 6 (1967) 170–206

[31] **D Quillen** *Elementary proofs of some results of cobordism theory using Steenrod operations*, Advances in Math. 7 (1971) 29–56

[32] **D Quillen** *Higher algebraic* K*-theory. I*, in Algebraic K-theory, I, Proc. of 1972 Battelle Memorial Inst. conf., Springer Lecture Notes in Math. 341 (1973), 85–147

[33] **D Ravenel** *The Segal conjecture for cyclic groups and its consequences*, Amer. J. Math. 106 (1984) 415–446

[34] **C P Rourke, B Sanderson** Δ*-sets. I. Homotopy theory*, Quart. J. Math. Oxford 22 (1971) 321–338

[35] **G Segal** *Categories and cohomology theories*, Topology 13 (1974) 293–312

[36] **G Segal** *Configuration-spaces and infinite loop-spaces*, Invent. Math. 21 (1973) 213–221

[37] **G Segal** *Classifying spaces and spectral sequences*, Inst. Hautes Etudes Sci. Publ Math. 34 (1968) 105–112

[38] **J Slominska** *Homotopy colimits on E-I-categories*, in Alg. topology Poznan 1989, Springer Lect. Notes in Math. vol. 1474 (1991) 273–294

[39] **R E Stong** *Notes on cobordism theory*, Mathematical Notes, Princeton University Press (1968)

[40] **U Tillmann** *On the homotopy of the stable mapping class group*, Invent. Math. 130 (1997) 257–275

[41] **V Vassiliev** *Complements of Discriminants of Smooth Maps: Topology and Applications*, Transl. of Math. Monographs Vol. 98, revised edition, Amer. Math. Soc. (1994)

[42] **V Vassiliev** *Topology of spaces of functions without complicated singularities*, Funktsional Anal. i Prilozhen 93 no. 4 (1989), 24–36; Engl. translation in Funct. Analysis Appl. 23 (1989) 266–286.