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QUANTUM THEORY OF HUMAN COMMUNICATION

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Quantum theory of human communication

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Contents

1. Introduction

In this paper we use commutative graded algebras to utilize the so-called "occupation number formalism" of the quantum field theory for analysis of metabolisms of information within fixed human communities (cf. [5], [6]). We treat bits of information circulating in such communities as bosons and use the Bose-Fock space structure to investigate their way to form opinions. Questionnaires are used to construct an analogue of the momentum space of finitely many frequencies which in this case are called "attitudes". The propagators of those boson-like bits of information are principally individual respondents which, depending on their "states-of-mind", provide answers yes or no to questions. Organizations can as well get status of respondents and quality of possessing a "state-of-mind". The states-of-mind are functions over the space of attitudes - the procedure often called "quantization". At this early stage of the development of the theory, the only observables we consider are questions directed to respondents. Questions are coupled with orthogonal projections in the space of states-of-mind. Affirmative answers are weighted by the assigned number of energy-bits they carry: electing a Member of Parliament requires many energy-bits in the form of single votes whereas a shareholder's single vote carries the number of energy-bits equal to the number of owned shares.

The subsequent second quantization (cf.[1]) of the space of states-of-mind provides the Bose-Fock space of states-of-opinion which is a commutative graded algebra with an inner product (cf.[4]).

Suppose that within a human community two standpoints are being cultivated, e.g. supporting the government or supporting the opposition in a democratic country. A respondent of a community, where such a difference of opinion occurs, will at any moment of time be inclined to support one or neither of the standpoints and his state-of-mind may change at any time subject to an interaction due to direct conversations, reading newspapers, watching television, election campaigns etc. We shall concentrate on describing the opinions emerging as a fusion (a superposition) of such two different points of view. The statistical approach we introduce goes back to von Neumann [3].

Though the applications we describe are of a very simple nature, they already induce the emergence of two important constants impossible to obtain by use of empirical means.

The first constant called the *interaction coefficient* is a number between 0 and 1 measuring the actual backgrounds for communication between respondents of different fractions: a common language, traditions, religion, interest etc.

The second constant called the *superposition constant* plays a double role - if it is greater or equal to one, it prevents interaction blocking the influence of the interaction coefficient. If negative, it controls regions of high and low frequencies of affirmation to questions asked in the fusioned community.

We show that under high interaction a sudden critical switch of opinion can occur in consequence of a minimal change of the superposition constant (cf. Remark 3). Changes of opinions under a temporary influence of some outside factors (as for instance election campaigns) are investigated. It is shown that such a temporary influence causes diminishing of high amplitudes (cf. Remark 4).

Formal descriptions of the results can be found in Section 5, where also a limited number of applications is presented. Investigation of more refined models goes beyond the reasonable limits of the present paper. In particular, the case of exterior influence which is not simple is being postponed to another paper.

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2. The first quantization

A selected sub-population characterized by a collection of *attitudes* will here be called a *profile*. For example the body of parliament members of a democratic country constitutes a profile. The attitudes will represent different political affiliations. The states-of-mind will then concern actual political problems. Also the government and its members can be considered as a profile. Here the set of attitudes will include different policies. For the profile of workers, the relevant attitudes will be conerned with the unions and the socialistic party.

As already mentioned, the same physical population consists of many different profiles. Profiles connected with professions are easiest revealed by asking a question to which the answer "yes" selects the states-of-mind of the profession. For example, the question "do you have a valid certificate qualifying you as a physician?" automatically extracts the profile of medical doctors. An examination will filter respondents of the profile of a particular profession. The whole population itself constitutes a profile as well.

2.1. Attitudes

Consider a community familiar with subjects which can be presented in a list of statements. The statements can be accepted or rejected by members of the community. In what follows we refer to this list of statements as a *questionnaire.* The term "questionnaire" should not be taken literally. For instance, a questionnaire may consist of a set of examination questions but it can as well be an ordinary questionnaire prepared for a poll.

A copy of a completed questionnaire shall be called an *attitude*. The quantum mechanical counterpart of an attitude is a frequency. Hence the quantum mechanical equivalence of spaces of attitudes are momentum spaces consisting of finitely many points.

2.2. The first quantization: from attitudes to states-of-mind

Let the space of attitudes consist of n attitudes $\{1, 2, ..., n\}$. Consider the real-valued functions x of n real variables $t_1, t_2, ..., t_n$. To the attitude j we attach the function e_j, which is the value of the variable t_j,

$$e_j\left(t_1, t_{2,}, ..., t_n\right) = t_j.$$

We shall consider the real vector space $\mathcal{F}$ of vectors

$$x = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_k e_k,$$

where $\lambda_1, \lambda_2, ..., \lambda_k$ are arbitrary real numbers. Given another vector from $\mathcal{F}$,

$$y = \eta_1 e_1 + \eta_2 e_2 + \cdots + \eta_k e_k,$$

we define the inner product (Hermitian form) setting

$$\langle x, y\rangle = \lambda_1 \eta_1 + \lambda_2 \eta_2 + \cdots + \lambda_n \eta_n$$

so that $e_1, e_2, ..., e_n$ is an orthonormal basis in $\mathcal{F}$ and each vector x from $\mathcal{F}$ can be written in the form

$$x = \langle x, e_1\rangle e_1 + \langle x, e_2\rangle e_2 + \cdots + \langle x, e_n\rangle e_n.$$

Then

$$\langle x, y\rangle = \langle x, e_1\rangle \langle y, e_1\rangle + \langle x, e_2\rangle \langle y, e_2\rangle + \cdots + \langle x, e_n\rangle \langle y, e_n\rangle .$$

We shall write $|x|$ for the length of the vector x,

$$|x| = \langle x, x\rangle^{\frac{1}{2}} .$$

A vector x is called a *state-of-mind* if $|x| = 1$. We do not distinguish between states provided by x and $-x$. We shall write briefly

$$x_/ \overset{def}{=} \frac{x}{|x|}$$

for the state-of-mind corresponding to the vector x.

If a respondent is in the state-of-mind x, his attitude will be j with probability $\langle x, e_j\rangle^2$, i.e. to the question "which is your attitude?" he will name the attitude j with probability $\langle x, e_j\rangle^2$.

The (real) vector space $\mathcal{F}$ shall be called the space of states-of-mind (we have as yet no interpretation for the process of multiplication by the imaginary unit i).

Continuing the analogy with photons, the space $\mathcal{F}$ of states-of-mind is the counterpart of the state-space for photons with fixed finite number of frequencies = attitudes in this paper.

Given states-of-mind x and y, the number $\langle x, y\rangle^2$ is called the *correlation* of x and y. States for which the correlation is equal to zero shall be called *uncorrelated.*

Observe that the space $\mathcal{F}$ can be considered as the space of all real-valued functions x on the set $\{1, 2, ..., n\}$, each such function assigning a real number λ_j to j from the set $\{1, 2, ..., n\}$.

2.3. Questions as observables

The process of assigning an attitude j to a respondent can be "first quantized" to a question directed to a respondent "are you fully accepting the attitude j? " The question itself then becomes an observable taking the form of the projection

$$Q_{e_j} = \langle e_j, \cdot\rangle\, e_j.$$

Now the procedure can be extended over arbitrary states-of-mind by attaching to a state-of-mind x the projection

$$Q_x = \langle x, \cdot\rangle\, x$$

which directed to a respondent runs as follows are you in the state-of-mind x ?

We attach statistics to this question by way of the statement

$$\langle Q_x y, y\rangle = \langle x, y\rangle^2 = \begin{cases} \text{the probability of obtaining the} \\ \text{answer "yes" to the question } Q_x \\ \text{from a respondent in state } y \end{cases}$$

i.e. the probability of the answer "yes" is equal to the correlation of x and y.

More general questions Q are linear combinations of questions of the form Q_x. Then

$$\langle Qy, y\rangle = \begin{cases} \text{the probability of obtaining the} \\ \text{answer "yes" to the question } Q \\ \text{from a respondent in state } y \end{cases} .$$

As an example we consider the projection

$$Qx = \langle e_i, x\rangle\, e_i + \langle e_j, x\rangle\, e_j,$$

where $\mathfrak{i}$ and $\mathfrak{j}$ are different attitudes. The question corresponding to this projection should read "*do you favor precisely the attitudes i and j out of the collection of all possible attitudes?*" Here we have $Qx = x$, exactly for $x = \langle e_i, x\rangle\, e_i + \langle e_j, x\rangle\, e_j$ which means that the answer "yes" comes from the states $x = \lambda e_i + \eta e_j$, with $\lambda^2 + \eta^2 = 1$.

As explained in the Introduction, each affirmative answer to a question carries a number of energy-bits depending on the nature of the corresponding model.

3. The second quantization

The notions of attitude and state-of-mind concern individual respondents. The second quantization amounts to providing a formalism by use of which the parallel notions on the level of profiles can be defined (cf.[1], [4]). The counterpart of the notion of attitude attached to an individual member of a community will be the notion of opinion attached to a group of individuals. Similarly the counterpart of the notion of state-of-mind attached to a respondent will be the notion of state-of-opinion attached to a profile (which can as well be the whole population). As a state-of-mind assigns a number to every possible attitude, a state-of-opinion will assign a number to every possible opinion of a profile. The square of the number assigned to an opinion gives the probability that this is exactly the opinion of the profile.

3.1. Opinions and the Bose Fock space for states-of-opinion

The main goal of this section is to introduce the concept of the state-of-opinion of a profile as an analogue to the concept of state-of-mind of a respondent.

Suppose that from a poll we have gathered information about the actual distribution of attitudes of a profile. It means that we have a collection of attitudes, where the same attitude may appear many times, a single time or not at all. To obtain the precise definition we proceed as follows.

Let $\{1, 2, ..., n\}$ be the set of all attitudes. Then a tuple of positive integers $(k_1, k_2, ..., k_n)$ shall be called an *opinion* in which the attitude j appears k_j times for $j = 1, 2, ..., n$. If a particular attitude, say i, does not appear at all, we write $k_i = 0$. A poll assigns to each attitude the number of respondents that share this attitude i.e. it provides the opinion of the community.

We use the Bargmann version of the Bose Fock space construction starting with the algebra $\widetilde{\mathcal{F}}$of all formal series

$$f = \sum \lambda_{k_1,k_2,...,k_n} e_{k_1,k_2,...,k_n},$$

where $e_{k_1,k_2,...,k_n}$ are products of variables $t_1, t_2, , t_n$:

$$e_{k_1,k_2,...,k_n} (t_1, t_2, , t_n) \overset{def}{=} t_1^{k_1} t_2^{k_2} \cdots t_n^{k_n}.$$

We multiply the series in the standard way setting

$$e_{j_1,j_2,...,j_m} e_{k_1,k_2,...,k_n} = e_{j_1+k_1,j_2+k_2,...,j_m+k_m,k_{m+1},...,k_n}.$$

We postulate that a profile which consists of k_1 members carrying state-of-mind e_1, k_2 members carrying the state-of-mind e_2 etc. up to k_n members carrying the state-of-mind e_n, is in the state-of-opinion $\frac{e_{k_1,k_2,...,k_n}}{\sqrt{k_1!k_2!...,k_n!}}$ and we assume that the set $\left\{ \frac{e_{k_1,k_2,...,k_n}}{\sqrt{k_1!k_2!...,k_n!}} \right\}$, where $(k_1, k_2, ..., k_n)$ runs through all possible opinions, constitutes an orthonormal basis of Bargmann's Hilbert space

$$\Gamma\mathcal{F} = \left\{ \sum \lambda_{k_1,k_2,...,k_n} e_{k_1,k_2,...,k_n} : \sum \lambda^2_{k_1,k_2,...,k_n} k_1!k_2!..., k_n! < \infty \right\}$$

contained in $\widetilde{\mathcal{F}}$ and called the *space of states-of-opinion.* Then the inner product is of the form

$$\left\langle \sum \eta_{j_1,j_2,...,j_m} e_{j_1,j_2,...,j_m}, \sum \lambda_{k_1,k_2,...,k_n} e_{k_1,k_2,...,k_n} \right\rangle$$
$$= \begin{cases} \sum k_1!k_2!..., k_n! \eta_{k_1,k_2,...,k_m} \lambda_{k_1,k_2,...,k_m} & n = m \\ 0 & \text{otherwise} \end{cases}.$$

For $x_1, x_2, \cdots, x_n, y$ from $\mathcal{F}$ we have the following useful formula (cf.[4])

$$\langle x_1 x_2 \cdots x_n, y^m \rangle = \begin{cases} m! \langle x_1, y\rangle \langle x_2, y\rangle \cdots \langle x_m, y\rangle & \text{for } m = n \\ 0 & \text{otherwise.} \end{cases} .$$

A *state-of-opinion* will be a vector f from $\Gamma\mathcal{F}$ such that $\langle f, f\rangle = 1$. This way for a state-of-opinion $f = \sum \lambda_{k_1,k_2,...,k_n} e_{k_1,k_2,...,k_n}$ we have $\sum \lambda^2_{k_1,k_2,...,k_n} = 1$ and for each opinion $(k_1, k_2, ..., k_n)$ the number $\lambda^2_{k_1,k_2,...,k_n}$ represents the probability that the members of the concerned profile share the opinion $(k_1, k_2, ..., k_n)$. We identify states of opinions f and $-f$. If all $k_j = 0$, then we get the vector $\emptyset$, $\emptyset(t_1, t_2, ..., t_n) = 1$, called the *vacuum vector*.

Notice that states-of-opinion can be interpreted as functions defined on the space of opinions, each such function assigning to an opinion $(k_1, k_2, ..., k_n)$ a real number $\lambda_{k_1,k_2,...,k_n}$.

Remark 1. *In this paper there is no need to take for $\lambda_{k_1,k_2,...,k_n}$ the complex numbers. Should such a need occur later, the necessary adjustments are elementary.*

We shall need the notion of the operator $w^*$of *annihilation* by an element w from $\mathcal{F}$ (cf.[4]). We definie w^* first for the basis vectors e_j of $\mathcal{F}$ setting for $f \in -\mathcal{F}$

$$\left(e_j^* f\right)(t_1, t_2, ..., t_n) = \frac{\partial}{\partial t_j} f(t_1, t_2, ..., t_n)$$

and then extend it linearly to include all w from $\mathcal{F}$.

The only infinite sums we will use are the elements of $\Gamma\mathcal{F}$ called *coherent vectors*, which are the exponential functions

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n,$$

of $x \in \mathcal{F}$. It is easy to verify that

$$\langle e^x, e^y \rangle = e^{\langle x,y\rangle}.$$

3.2. Occupation numbers and their statistics

To every orthogonal projection Q in $\mathcal{F}$ and every natural number k we assign a projection Q^k in $\Gamma\mathcal{F}$ which we define as follows:

Take $x_1, ..., x_p, y_1, ..., y_q \in \mathcal{F}$ such that $Qx_j = x_j$ for $j = 1, 2, ..., p$ and $Qy_i = 0$ for $i = 1, 2, ..., q$. Then we define

$$Q^{(k)}\left(x_1 x_2 \cdots x_p y_1 y_2 \cdots y_q\right) \overset{def}{=} \begin{cases} x_1 x_2 \cdots x_p y_1 y_2 \cdots y_q & p = k \\ 0 & \text{otherwise} \end{cases}.$$

It is easy to extend $Q^{(k)}$ to an orthogonal projection of $\Gamma\mathcal{F}$ into itself. The projection $Q^{(k)}$ is an observable in the space of states-of-opinion and corresponds to the question:

Is there exactly k answers"yes"to the question Q?

Consequently, for a state-of-opinion f we have

$$\left\langle Q^{(k)} f, f \right\rangle = \begin{cases} \text{the probability that in the state } f \\ \text{we get precisely } k \text{ answers "yes" to } Q \end{cases}.$$

Let f be a state, i.e let $|f| = 1$. The numbers $\left\langle Q^{(k)} f, f \right\rangle$ are called the *occupation numbers* of affirmation of Q in the state f

We extend Q to a derivation $d\Gamma Q$, i.e. a transformation obeying the Leibniz rule,

$$(d\Gamma Q)\, fg = (d\Gamma Q f)\, g + f\,(d\Gamma Q g)\,.$$

This operation is often called the *second quantization* of Q. It is easy to verify that the spectral decomposition of $d\Gamma Q$ is

$$d\Gamma Q = \Sigma_{k=0}^{\infty} k Q^{(k)}.$$

Hence, if f is a state-of-opinion, then

$$\langle d\Gamma Q f, f \rangle = \Sigma_{k=0}^{\infty} k \left\langle Q^{(k)} f, f \right\rangle = \begin{cases} \text{the expected number of energy-bits} \\ \text{coming from the affirmative} \\ \text{answers to } Q \text{ in the state-of-opinion } f. \end{cases} \tag{3.1}$$

However, it is not the expected number of energy-bits coming from the affirmative answers which is measured by a poll but the expected percentage $\mathcal{R}(Q, f)$ of those energy-bits,

$$\mathcal{R}(Q, f) = \frac{\langle d\Gamma Q f, f \rangle}{\langle d\Gamma I f, f \rangle} = \begin{cases} \text{the expected percentage of energy-} \\ \text{bits coming from the affirmative} \\ \text{answers to } Q \text{ in the state-of-opinion } f. \end{cases}, \tag{3.2}$$

We shall call $\mathcal{R}(Q, f)$ the *relative expectation* for energy of affirmation of Q in the state f. Here the identity operator I corresponds to the question: "How many energy-bits are available"?

Given a state-of-opinion f, we can produce a new one by making a superposition of f with the vacuum

$$(f + \alpha\o)_/ = \frac{f + \alpha\o}{\sqrt{1 + \alpha^2 + 2\langle \o, f\rangle}}.$$

Since

$$\langle d\Gamma Q (f + \alpha\o), f + \alpha\o\rangle = \langle d\Gamma Q f, f\rangle,$$

we get

$$\begin{aligned} &\mathcal{R}\left(Q, (f + \alpha\o)_/\right) \\ = &\frac{\left\langle d\Gamma Q (f + \alpha\o)_/, (f + \alpha\o)_/\right\rangle}{\left\langle d\Gamma I (f + \alpha\o)_/, (f + \alpha\o)_/\right\rangle} = \frac{\langle d\Gamma Q f, f\rangle}{\langle d\Gamma I f, f\rangle} = \mathcal{R}(Q, f) \end{aligned}$$

which means that the superposition with the vacuum does not change the percentage of energy-bits coming from affirmation of Q.

3.3. Coherent states

A coherent state describes respondents with states-of-mind gathered around a special fixed state-of-mind called the mode of coherence, e.g. physicians with their professional curriculum as the mode, members of a political party with their party program as the mode, lawyers with professional know-how as the mode etc.

Take a vector x from the states-of-mind space $\mathcal{F}$. The *coherent state* $c(x)$ generated by x is the normalized coherent vector e^x,

$$\begin{aligned} c(x) &= e^x_/ = e^{-\frac{1}{2}\langle x, x\rangle} e^x \\ c(0) &= \o. \end{aligned} \tag{3.3}$$

Observe that if the number $\langle x - y, x - y\rangle$ is very large, the correlation

$$\langle c(x), c(y)\rangle = e^{-\frac{1}{2}|x-y|^2} \tag{3.4}$$

is almost 0, i.e. $c(x)$ and $c(y)$ are almost uncorrelated. Hence any experiment performed in one of those states has almost no probable relation to an experiment performed in the other state.

The coherent states are "almost" multiplicative; we have

$$c(x+y) = e^{-\langle x,y\rangle} c(x)\, c(y) .$$

The number $|x|^2$, the state-of-mind $x_/$ and the vector x shall be respectively called the *energy*, the *mode* and the *generating vector* of the coherent state $c(x)$. Hence in the background of a given coherent state lies the mode which is the state-of-mind that provides the right frequencies of occurrence of the attitudes from a fixed list. Given a coherent state, we can approximate the mode for this coherent state as follows. We produce a "super-questionnaire" out of all involved attitudes; then count the frequencies of the choice of particular attitudes in a poll and take their square roots as coefficients to the respective attitude.

We can easily compute the relative expectation $\mathcal{R}$ for Q in a coherent state $c(x)$. Since $d\Gamma Q$ is a derivation, we have

$$d\Gamma Q c(x) = (Qx)\, c(x)$$

so that

$$\langle d\Gamma Q c(x), c(x)\rangle = \langle x, Qx\rangle = |Qx|^2$$

and we obtain the number

$$\mathcal{R}(Q, c(x)) = \left\langle x_/, Qx_/ \right\rangle = \left|Qx_/\right|^2$$

which does not depend on the energy $|x|^2$ of $c(x)$.

4. Bicoherence

The concept of bicoherence concerns a community consisting of two coherent fractions, e.g. the government and the opposition in a democratic country, members of two different religious affiliations, a population consisting of natives and immigrants etc. In each of these cases the state-of-opinion of the whole population is a superposition of states-of-opinion of two coherent sub-profiles. The state-of-opinion of the superposition is not any longer coherent and shall be called bicoherent.

One can easily quote important cases involving more than two coherent states but already in the case of three, the amount of necessary computation will double the size of this paper and hence must be postponed to a separate publication.

4.1. Bicoherent states

Take coherent states $c(u)$ and $c(v)$, $u \neq v$, and a number λ. The number

$$\omega = \langle c(u), c(v) \rangle = e^{-\frac{1}{2}|u-v|^2}$$

shall be called the *interaction coefficient.* States of the form

$$c_\lambda(u,v) = \frac{c(u) + \lambda c(v)}{\vartheta(\lambda,\omega)}, \tag{4.1}$$

where

$$\vartheta(\lambda,\omega) = |c(u) + \lambda c(v)| = \sqrt{1+\lambda^2+2\lambda\omega}, \tag{4.2}$$

shall be called *bicoherent states.* For $\lambda \neq 0$ we have

$$c_\lambda(u,v) = c_{\frac{1}{\lambda}}(v,u)$$

so that for λ close to infinity $c_\lambda(u,v)$ behaves exactly as $c_\lambda(v,u)$ behaves for λ close to zero. The coefficient λ will be called the *superposition constant.*

The closer to zero is ω, i.e. the greater is $|u-v|$, the more the states $c(u)$ and $c(v)$ act as uncorrelated, and $c_\lambda(u,v)$ describes a profile split into two groups which hardly communicate with each other.

With fixed u and v, when λ increases to infinity, the state $c_\lambda(u,v)$ converges to the state $c(v)$, and when λ decreases to zero, it converges to the state $c(u)$. Excluding the case of simultaneous $\lambda = -1$ and $u = v$, we get from 3.1 the expected number of energy-bits of the affirmative answers to a question Q :

$$\begin{aligned} &\langle c_\lambda(u,v), (d\Gamma Q)\, c_\lambda(u,v) \rangle \\ = &\frac{\kappa(Q;\lambda,u,v,\omega)}{\vartheta(\lambda,\omega)^2}, \end{aligned} \tag{4.3}$$

where

$$\kappa(Q;\lambda,u,v,\omega) = |Qu|^2 + 2\lambda\omega\langle Qu, v\rangle + \lambda^2 |Qv|^2. \tag{4.4}$$

Applying (3.2) we get

$$\begin{aligned} &\mathcal{R}(Q, c_\lambda(u,v)) \\ = &\left\{ \begin{array}{l} \text{the expected percentage of affirmations} \\ \text{of } Q \text{ in the state-of-opinion } c_\lambda(u,v) \end{array} \right. \\ = &\frac{\kappa(Q;\lambda,u,v,\omega)}{\kappa(I;\lambda,u,v,\omega)}. \end{aligned} \tag{4.5}$$

Since

$$\mathcal{R}\left(Q, c_{\lambda}\left(tu, tv\right)\right)$$
$$\frac{\left|Qu\right|^2 + \lambda^2\left|Qv\right|^2 + 2\lambda\left\langle Qu, v\right\rangle e^{-\frac{1}{2}t^2\left|u-v\right|^2}}{\left|u\right|^2 + \lambda^2\left|v\right|^2 + 2\lambda\left\langle u, v\right\rangle e^{-\frac{1}{2}t^2\left|u-v\right|^2}}$$

by choosing the right t and substituting $e^{-\frac{1}{2}t^2|u-v|^2}$ for ω, we can get every number between 0 and 1 for the interaction coefficient ω. We expose the fact that $\omega \in [0,1]$ can be considered as an independent variable by writing

$$\mathcal{R}_{\omega}\left(Q, \lambda, u, v\right) = \frac{\left|Qu\right|^2 + \lambda^2\left|Qv\right|^2 + 2\lambda\left\langle Qu, v\right\rangle \omega}{\left|u\right|^2 + \lambda^2\left|v\right|^2 + 2\lambda\left\langle u, v\right\rangle \omega} \tag{4.6}$$

and observing that

$$\mathcal{R}_{\omega}\left(Q, \lambda, u, v\right) = \mathcal{R}\left(Q, c_{\lambda}\left(tu, tv\right)\right) \tag{4.7}$$

if $\omega = e^{-\frac{1}{2}t^2|u-v|^2}$.

4.2. Weyl transformations

Suppose that some social forces alter the coherent state $c(x)$ to another coherent state $c(y)$. Then, writing $z = y - x$, we can consider z as the vector altering the generating vector x of the given coherent state to a new generating vector $x + z$ of the new coherent state $c(x+z) = c(y)$. This reduces the process of changing $c(x)$ into $c(y)$ to the application of transformation W_z dependent on a vector z from $\mathcal{F}$. The transformation

$$W_z c(x) = c(x+z)$$

of $c(x)$ into $c(x+z)$ is called the *Weyl transformation.* Given z, the Weyl transformation W_z is uniquely extendable to a linear isometry (states-of-mind preserving transformation) of $\Gamma\mathcal{F}$ onto itself (cf. [4]). The Weyl transformation W_z is fully described by the coherent state $c(z)$ which shall be called the *generator* of W_z.

4.3. The mathematics of bicoherence

In this section we shall prove a series of results necessary for further development of the theory.

Let $y, u \in \mathcal{F}$ and $f, g \in \Gamma\mathcal{F}$ and let Q be an orthogonal projection. In the proofs below we shall freely use the following identities (cf.[4]):

$$\begin{aligned} \langle yf, g\rangle &= \langle f, y^* g\rangle \\ y^*(fg) &= (y^* f)\, g + f\,(y^* g) \\ y^* c(u) &= \langle y, u\rangle\, c(u) \\ \langle d\Gamma Q f, g\rangle &= \langle f, d\Gamma Q g\rangle \\ d\Gamma Q(fg) &= (d\Gamma Q f)\, g + f\,(d\Gamma Q g) \\ d\Gamma Q c(u) &= (Qu)\, c(u)\,. \end{aligned}$$

Given $x, z \in \mathcal{F}$, $|z| = 1$, we briefly write

$$z\#c(x) = \left(\left\langle x, z_{/}\right\rangle \emptyset - z_{/}\right) c(x)$$

Lemma 1. *For every $u, z \in \mathcal{F}$. The vector $z\#c(u)$ is a state-of-opinion.*

Proof. We have

$$\langle c(u), zc(u)\rangle = \langle z^* c(u), c(u)\rangle = \langle z, u\rangle$$

and

$$\begin{aligned} & \langle zc(u), zc(u)\rangle \\ = {} & \langle c(u), z^*(zc(u))\rangle = |z|^2 + \langle z, u\rangle \langle c(u), zc(u)\rangle = |z|^2 + \langle z, u\rangle^2 \end{aligned}$$

so that

$$\begin{aligned} & \langle(\langle u, z\rangle \emptyset - z)\, c(u), (\langle u, z\rangle \emptyset - z)\, c(u)\rangle \\ = {} & \langle\langle u, z\rangle c(u) - zc(u), \langle u, z\rangle c(u) - zc(u)\rangle \\ = {} & \langle u, z\rangle^2 - 2\langle u, z\rangle \langle c(u), zc(u)\rangle + \langle zc(u), zc(u)\rangle \\ = {} & \langle u, z\rangle^2 - 2\langle u, z\rangle^2 + |z|^2 + \langle z, u\rangle^2 = |z|^2 \end{aligned}$$

■

Now we can verify the following

Proposition 2. *We have*

$$\lim_{\alpha \to 0} \left| \frac{c(u + \alpha z) - c(u)}{\sqrt{2}\sqrt{1 - e^{-\frac{1}{2}(\alpha|z|)^2}}} - z\#c(u) \right| = 0$$

i.e. the bicoherent states $c_{-1}(u + \alpha u, u)$ converge strongly to the state $z\#c(u)$.

Proof. Take an arbitrary $y \in \mathcal{F}$. Using l'Hospital Theorem, we get

$$\lim_{\alpha \to 0} \left\langle \frac{c(u+\alpha z) - c(u)}{\sqrt{2}\sqrt{1 - e^{-\frac{1}{2}(\alpha|z|)^2}}} - \left(\left\langle u, z_/ \right\rangle \emptyset - z_/\right) c(u), c(y) \right\rangle = 0.$$

But $z_/ \# c(u)$ lies on the unit sphere and $\{e^x : x \in \mathcal{F}\}$ is total so that the Proposition holds. ■

Proposition 3. *We have*

$$\lim_{\alpha \longrightarrow 0} \langle c_{-1}(u+\alpha z, u), (d\Gamma Q) c_{-1}(u+\alpha z, u) \rangle = |Qu|^2 + \left|Qz_/\right|^2.$$

Proof. We have

$$\begin{aligned} & \langle c(u+\alpha z) - c(u), d\Gamma Q (c(u+\alpha z) - c(u)) \rangle \\ = & \langle c(u+\alpha z), (Q(u+\alpha z)) c(u+\alpha z) \rangle - \langle c(u+\alpha z), (Qu) c(u) \rangle \\ & - \langle c(u), (Q(u+\alpha z)) c(u+\alpha z) \rangle + \langle c(u), (Qu) c(u) \rangle \\ = & \langle Q(u+\alpha z), u+\alpha z \rangle - 2 \langle Qu, u+\alpha z \rangle e^{-\frac{1}{2}\alpha^2|z|^2} + \langle Qu, u \rangle \end{aligned}$$

and using l'Hospital Theorem, we get

$$\lim_{\alpha \to 0} \frac{|Q(u+\alpha z)|^2 - 2\langle Qu, u+\alpha z \rangle e^{-\frac{1}{2}\alpha^2|z|^2} + |Qu|^2}{2\left(1 - e^{-\frac{1}{2}(\alpha|z|)^2}\right)} = |Qu|^2 + \left|Qz_/\right|^2.$$

■

Proposition 4. *We have*

$$\lim (c_\lambda(u,v) - W_{\alpha z} c_\lambda(u,v))_/ = \left(z_/ \# c(u) + \lambda z_/ \# c(v)\right)_/$$

Proof. We have

$$W_{\alpha z} c_\lambda(u,v) = \frac{c(u+\alpha z) + \lambda c(v+\alpha z)}{\vartheta(\lambda, \omega)},$$

where ϑ is given by 4.2. Let

$$\begin{aligned} U_\alpha &= c(u) - c(u+\alpha z) \\ V_\alpha &= c(v) - c(v+\alpha z) \\ |U_\alpha|^2 &= 2\left(1 - e^{-\frac{1}{2}|\alpha z|^2}\right) = |V_\alpha|^2. \end{aligned}$$

Then

$$\begin{aligned}\frac{U_\alpha}{|U_\alpha|} &\to U = \left(\left\langle u, z_/\right\rangle \o - z_/\right) c(u), \\ \frac{V_\alpha}{|V_\alpha|} &\to V = \left(\left\langle v, z_/\right\rangle \o - z_/\right) c(v)\end{aligned}$$

$$\begin{aligned}& c_\lambda(u,v) - W_{\alpha z} c_\lambda(u,v) \\ = \; & \frac{(c(u) - c(u+\alpha z)) + \lambda(c(v) - c(v+\alpha z))}{\vartheta(\lambda,\omega)} \\ = \; & \frac{U_\alpha + \lambda V_\alpha}{\vartheta(\lambda,\omega)}\end{aligned}$$

$$\begin{aligned}& \frac{c_\lambda(u,v) - W_{\alpha z} c_\lambda(u,v)}{|c_\lambda(u,v) - W_{\alpha z} c_\lambda(u,v)|} \\ = \; & \frac{U_\alpha + \lambda V_\alpha}{|U_\alpha + \lambda V_\alpha|} = \frac{\frac{U_\alpha}{|U_\alpha|} + \lambda \frac{V_\alpha}{|V_\alpha|}}{\left|\frac{U_\alpha}{|U_\alpha|} + \lambda \frac{V_\alpha}{|V_\alpha|}\right|} \to \frac{U + \lambda V}{|U + \lambda V|} \\ = \; & \frac{\left(\left\langle u, z_/\right\rangle \o - z_/\right) c(u) + \lambda\left(\left\langle v, z_/\right\rangle \o - z_/\right) c(v)}{\left|\left(\left\langle u, z_/\right\rangle \o - z_/\right) c(u) + \lambda\left(\left\langle v, z_/\right\rangle \o - z_/\right) c(v)\right|}.\end{aligned}$$

■

Given a real number λ and $u, v, z \in \mathcal{F}$ we define

$$\iota(Q,u,v,z) = \langle v-u, z\rangle \left(\langle Qu, v\rangle \langle v-u, z\rangle + \langle (v-u), Qz\rangle\right).$$

Proposition 5. *Take $u, v, z \in \mathcal{F}$, where $|z| = 1$. Then*

$$\begin{aligned}& \langle d\Gamma Q\left(z\#c(u) + \lambda z\#c(v)\right), z\#c(u) + \lambda z\#c(v)\rangle \\ = \; & \vartheta^2(\lambda,\omega)|Qz|^2 + \kappa(Q;\lambda,u,v,\omega) - 2\lambda\omega\iota(Q,u,v,z).\end{aligned}$$

The Proposition is an immediate consequence of the following

Lemma 6. *Take vectors u, v and z from $\mathcal{F}$ and a projection Q. Then the following identity holds*

$$\begin{aligned}& \frac{1}{\omega}\langle d\Gamma Q z\#c(u), z\#c(v)\rangle \\ = \; & \langle z, Qz\rangle + \langle z, z\rangle \langle Qu, v\rangle + \iota(Q,u,v,z).\end{aligned}$$

Proof.

$$
\begin{aligned}
& \langle d\Gamma Q\left(\langle u,z\rangle \emptyset - z\right) c(u), \left(\langle v,z\rangle \emptyset - z\right) c(v)\rangle \\
= \; & \langle d\Gamma Q\left(\langle u,z\rangle \emptyset\right) c(u), \left(\langle v,z\rangle \emptyset\right) c(v)\rangle + \langle d\Gamma Q\left(\langle u,z\rangle \emptyset\right) c(u), (-z)\, c(v)\rangle \\
& + \langle d\Gamma Q\,(-z)\, c(u), \left(\langle v,z\rangle \emptyset\right) c(v)\rangle + \langle d\Gamma Q\,(-z)\, c(u), (-z)\, c(v)\rangle \\
= \; & \langle u,z\rangle \langle v,z\rangle \langle Qu,v\rangle \langle c(u), c(v)\rangle - \langle u,z\rangle \langle Qu,z\rangle \langle c(u), c(v)\rangle \\
& - \langle u,z\rangle^2 \langle Qu,v\rangle \langle c(u), c(v)\rangle - \langle v,z\rangle \langle Qz,v\rangle \langle c(u), c(v)\rangle \\
& - \langle v,z\rangle^2 \langle Qu,v\rangle \langle c(u), c(v)\rangle + \langle Qz,z\rangle \langle c(u), c(v)\rangle \\
& + \langle Qz,v\rangle \langle z,u\rangle \langle c(u), c(v)\rangle + \langle z,z\rangle \langle Qu,v\rangle \langle c(u), c(v)\rangle \\
& + \langle z,Qu\rangle \langle z.v\rangle \langle c(u), c(v)\rangle + \langle z,u\rangle \langle Qu,v\rangle \langle z,v\rangle \langle c(u), c(v)\rangle
\end{aligned}
$$

■

4.4. Consequences of temporary external influence

Consider a profile in a bicoherent state $c_\lambda(u,v)$ and an element $z \in \mathcal{F}$. Let for $\alpha > 0$ an external influence caused by $W_{\alpha z}$,

$$c_\lambda(u,v) \to W_{\alpha z} c_\lambda(u,v),$$

induce a new state $c_\lambda(u+\alpha z, v+\alpha z)$. As the result of the enforcement, the population falls into the superposition state

$$\left(c_\lambda(u+\alpha z, v+\alpha z) - c_\lambda(u,v)\right)_/$$

of the original contra the enforced state-of-opinion $W_{\alpha z} c_\lambda(u,v)$. In Proposition 4 it is proved that when the enforcement fades away, i.e. when $\alpha \to 0$, the state-of-opinion tends to the limit state-of-opinion

$$\left(z\#c(u) + \lambda z\#c(v)\right)_/ .$$

By Proposition 5 the expected percentage of affirmative answers to Q in this state is

$$
\begin{aligned}
& \mathcal{R}\left(Q, \left(z\#c(u) + \lambda z\#c(v)\right)_/\right) \\
= \; & \frac{\langle d\Gamma Q\left(z\#c(u) + \lambda z\#c(v)\right), z\#c(u) + \lambda z\#c(v)\rangle}{\langle d\Gamma I\left(z\#c(u) + \lambda z\#c(v)\right), z\#c(u) + \lambda z\#c(v)\rangle} \\
= \; & \frac{\vartheta^2(\lambda,\omega)\,|Qz|^2 + \kappa(Q;\lambda,u,v,\omega) - 2\lambda\omega\iota(Q,u,v,z)}{\vartheta^2(\lambda,\omega) + \kappa(I;\lambda,u,v,\omega) - 2\lambda\omega\iota(I,u,v,z)},
\end{aligned}
\tag{4.8}
$$

where $\omega = e^{-\frac{1}{2}|u-v|^2}$. Due to the lack of homogeneity relative to u, v, we cannot make ω in 4.8 an independent variable as in 4.7.

The term $\iota(Q, u, v, z)$ measures the balance of the influence of $c(z)$ on components $c(u)$ and $c(v)$ of $c_\lambda(u, v)$. If the influence of $c(z)$ on $c_\lambda(u, v)$ is equally distributed between $c(u)$ and $c(v)$, the term $\iota(Q, u, v, z)$ vanishes.

5. Analysis of a model

Let consider a special case. Suppose that in a community two complementary coherent profiles manifest. Fix two positive numbers a and b and consider states-of-mind

$$u = \begin{pmatrix} a \\ b \end{pmatrix} \text{ and } v = \begin{pmatrix} b \\ a \end{pmatrix}.$$

Suppose further that the community we investigate is polarized into two profiles - one in the state $c_1 = c(u)$ and the other in the state $c_2 = c(v)$.

Let $100\frac{a^2}{a^2+b^2}\%$ of the members of the profile in state $c(u)$ support an attitude X while $100\frac{a^2}{a^2+b^2}\%$ of the members of the profile in state $c(v)$ will reject X. We consider the bicoherent state $c_\lambda\left(\binom{a}{b}, \binom{b}{a}\right)$.

Let the question we ask correspond to the projection

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

with the eigenvector $\binom{1}{0}$ corresponding to the answer "yes" to the question

"Do you support the attitude X?"

We shall now analyze the expected relative frequencies of affirmative answers to Q before and after the exertion of an influence generated by $c(z)$. Analysis of unequal balance of the influence of $c(z)$ manifesting in non-zero ι is too complicated for the first approach we are making here and it will be postponed to a separate paper. Hence we shall take $z = \binom{s}{t}$ such that $\langle v-u, z\rangle = (b-a)(s-t) = 0$ which requires $s = t$, and since z is a unit vector we must have

$$z = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We have the following set of values

$$\begin{aligned} |Qu| &= a\ ,\ |Qv| = b,\ \langle u, v\rangle = 2ab,\ \langle v-u, z\rangle = 0 \\ ab &= \langle Qu, v\rangle\ ,\ |u-v|^2 = 2\,(a-b)^2\ ,\ |Qz|^2 = \frac{1}{2} \end{aligned}$$

yielding

$$\begin{aligned} \kappa\,(Q;\lambda,u,v,\omega) &= a^2+2\lambda\omega ab+\lambda^2 b^2 \\ \kappa\,(I;\lambda,u,v,\omega) &= \left(1+\lambda^2\right)\left(a^2+b^2\right)+4\lambda\omega ab \end{aligned}$$

and

$$\iota\,(Q,u,v,z) = \iota\,(I,u,v,z) = 0.$$

Using 4.5 and 4.8 we get

$$\mathcal{R}_\omega\,(Q,\lambda,u,v) = \frac{a^2+2\lambda\omega ab+\lambda^2 b^2}{\left(1+\lambda^2\right)(a^2+b^2)+4\lambda\omega ab} \tag{5.1}$$

and

$$\begin{aligned} & \mathcal{R}\left(Q,(z\#c\,(u)+\lambda z\#c\,(v))_{/}\right) \\ = & \frac{\frac{1}{2}\left(1+\lambda^2+2\lambda\omega\right)+a^2+2\lambda\omega ab+\lambda^2 b^2}{\left(1+\lambda^2+2\lambda\omega\right)+\left(1+\lambda^2\right)(a^2+b^2)+4\lambda\omega ab} \end{aligned}$$

where $\omega = e^{-\frac{1}{2}|u-v|^2} = e^{-(a-b)^2}$.

5.1. The meanings of the interaction coefficient and the superposition constant

Let us take $a^2+b^2=1$. For $u=\binom{a}{b}$ and $v=\binom{b}{a}$ we get

$$\mathcal{R}_\omega\,(Q,\lambda,u,v) = \frac{a^2+2\lambda\omega a\sqrt{1-a^2}+\lambda^2\left(1-a^2\right)}{\left(1+\lambda^2\right)+4\lambda\omega a\sqrt{1-a^2}}. \tag{5.2}$$

Taking $u=\binom{a}{b}$, $v=\binom{b}{a}$ and $0<\omega<\rho<1$, we get

$$\begin{aligned} & \mathcal{R}_\omega\,(u,v,\lambda)-\mathcal{R}_\rho\,(u,v,\lambda) \\ = & \frac{2\lambda a\sqrt{(1-a^2)}\,(\rho-\omega)\,(1-2a^2)\left(\lambda^2-1\right)}{\left(1+\lambda^2+4\lambda\omega a\sqrt{(1-a^2)}\right)\left(1+\lambda^2+4\lambda\rho a\sqrt{(1-a^2)}\right)}. \end{aligned}$$

For $a^2 < \frac{1}{2}$ and $\lambda \geq 1$ the above difference is non-negative and we can estimate,

$$\begin{aligned} & \frac{2\lambda a\sqrt{(1-a^2)}\,(\rho-\omega)\,(1-2a^2)\left(\lambda^2-1\right)}{\left(1+\lambda^2\right)4\lambda\rho a\sqrt{(1-a^2)}+\left(1+\lambda^2\right)4\lambda\omega a\sqrt{(1-a^2)}} \\ \leq \; & \frac{2\lambda a\sqrt{(1-a^2)}\,(\rho-\omega)\,(1-2a^2)\left(\lambda^2-1\right)}{\left(1+\lambda^2\right)4\lambda\,(\rho+\omega)\,a\sqrt{(1-a^2)}} \\ \leq \; & \frac{2\,(\rho+\omega)\,(1-2a^2)\left(\lambda^2+1\right)}{4\left(1+\lambda^2\right)(\omega+\rho)} \leq \frac{1}{2}\left(1-2a^2\right) \end{aligned}$$

which shows that there is almost no influence of the interaction coefficient if a is close to $\frac{1}{\sqrt{2}}$ and $\lambda \geq 1$.

The interaction coefficient measures the ability for interaction (as for instance speaking the same everyday language, being a citizen of a democratic country, having the same cultural background etc.).

The superposition constant plays two different roles. It points out which percentage of influence on the superposition state has each of the two coherent components and it marks the existence of wish to enter the interaction at all: Catholics and Protestants of Northern Ireland are fully capable of interacting on an arbitrarily high social level but they will not enter the interaction due to some special reasons.

Let us look at the diagrams at the end of the paper. For $\lambda > 1$ there are no significant differences in the forms of the diagrams. In all cases maximum is not attained for $\lambda = 0$ but first for a negative λ. Movement of λ from zero in the negative direction makes $\mathcal{R}_\omega$ increase. Consequently we have

Remark 2. *The increase of the influence of c_2 acts as a buster for c_1 providing more affirmative answers.*

Say the interaction coefficient ω is close to one. Starting at $\lambda = 0$ and moving in the negative direction, we observe a rapid increase of $\mathcal{R}_\omega$. Then, continuing moving λ in the same direction, the situation reverses - now the maximum decreases towards the minimum.

Remark 3. *The closer to one is the interaction coefficient, the shorter is the interval within which R_ω attains first the maximum and then the minimum.*

Then the situation stabilizes and with further decrease of λ, $\mathcal{R}_\omega$ converges to its limit $\mathcal{R}(Q, c_2)$ in $-\infty$ so that the role of c_1 is eliminated.

As an example we can take the government and the opposition of a democratic country both in coherent states respectively. Assume that there is an intensive interaction between c_1 and c_2. Say the government has majority: $\mathcal{R}(Q, c_1) > \mathcal{R}(Q, c_2)$. A small negative weight λ attached to c_2 yields not much contribution itself but by way of the interaction it provokes the other fraction to vote. Similarly if $\mathcal{R}(Q, c_1) < \mathcal{R}(Q, c_2)$, then the increase of negative answers is provoked. Still higher negative weight yields the reversed status - the respondents from c_2 take over and in the first case cause a decrease and in the second case an increase of affirmations. These unusual variations can happen only in the presence of high interaction and in small intervals of negative λ and hence will seldom occur in real life. However, such jumps in the distribution of votes have been observed in the past (cf.[2]).

5.2. Consequences of an equidistributed temporary exterior influence

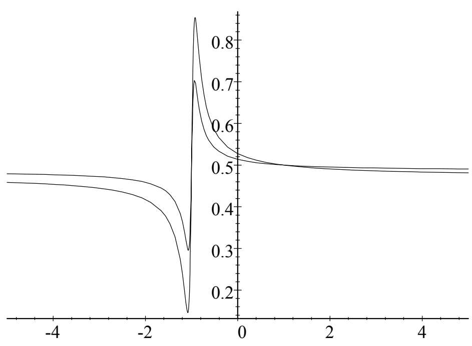
Take

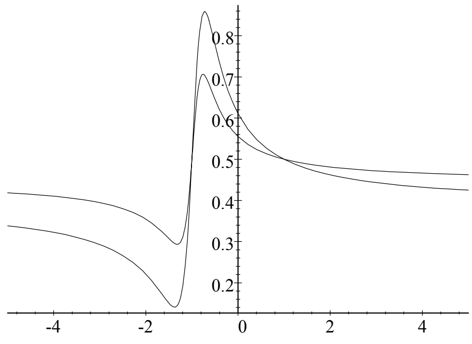
$$w = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}\#c\left(\begin{pmatrix}a\\b\end{pmatrix}\right) + \lambda\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}\#c\left(\begin{pmatrix}b\\a\end{pmatrix}\right)_{/}$$

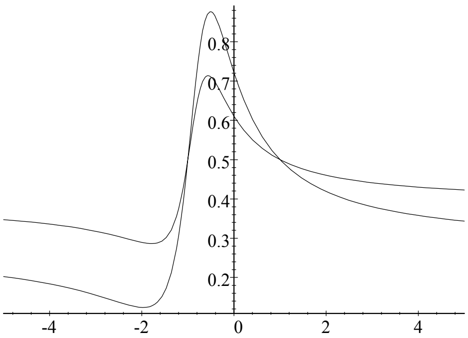
and consider

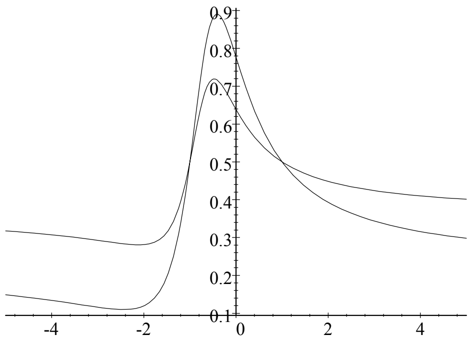
$$\mathcal{R}\left(Q, w_{/}\right) = \frac{\frac{1}{2}\left(1+\lambda^2+2\lambda\omega\right)+a^2+2\lambda\omega a\sqrt{1-a^2}+\lambda^2\left(1-a^2\right)}{\left(1+\lambda^2+2\lambda\omega\right)+\left(1+\lambda^2\right)+4\lambda\omega a\sqrt{1-a^2}}. \tag{5.3}$$

In order to observe the consequences of the influence we shall draw the graphs of 5.3 imposed onto the graphs of 5.2 for $a^2+b^2=1$, $a^2 = \frac{19}{36}, \frac{22}{36}, \frac{26}{36}, \frac{28}{36},$ with respective interaction coefficients $\omega = 0.998\,46,\ 0.975\,31,\ 0.901\,05,\ 0.844\,91$.









It is clearly visible that equally distributed influence of W_z makes the extreme values of $\mathcal{R}_\omega$ diminish.

Remark 4. *The influence of W_z tempers the extreme reactions.*

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