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THE MOTION OF A NONLINEAR OSCILLATOR ATTACHED TO A ROTATING RIGID BODY

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ABSTRACT. We develop the differential equations of the motion of a nonlinear oscillator with three degrees of freedom that is attached to a rotating rigid body. The usual Hamiltonian analysis does not apply to rigid body dynamics. We present an alternative that retains the desirable properties of a Hamiltonian system. A perturbation analysis is applied to the alternative system. The solution of the perturbed system exhibits a homoclinic tangle which for three degrees of freedom results in Arnold diffusion. A real-time display of trajectories of the motion of the oscillator can be found at the website *http://www.csulb.edu/~fatt.*

INTRODUCTION

We analyze the motion of a nonlinear spring-mass oscillator with three degrees of freedom that is attached to a freely rotating rigid body. Because it is attached to the rotating rigid body, the oscillator can transfer momentum in each of the three directions; the three spatial modes of vibration then interact with one another by way of the rotating rigid body. In one direction the springs are nonlinear and in the remaining two they are linear. The rigid body mediates the interaction. The effect of the interaction is a perturbation of the motion. The resulting perturbed motion of the mass is an example of Arnold diffusion. Arnold diffusion is characteristic of many physical phenomena involving nonlinear motion and periodic perturbation, both of which are experienced by our oscillator.

The analysis splits into developing the mechanical model and into analyzing the effect of the perturbation. The mechanical model is developed by the introduction of a Lie algebra of real functions on a momentum space. The Lie algebra is derived from the Poisson bracket on a phase space and is then carried over to the momentum space by a momentum map. This derived Lie algebra contains the dynamics of the model. The model including the attachment is suggested by the work of Marsden and Krishnaprasad, [8]. One would expect that the usual Hamiltonian analysis would suffice to study the motion of the system, but this is not the case because spacial coordinates for the rigid body can not be fixed in space for which Hamilton's equations are invariant. To consider the perturbation, we look at a Poincare section of the phase space of the nonlinear oscillator and at the Poincare map of the section. When the interaction of the oscillator with the rigid body is taken into consideration, we get a perturbation of the orbits in the section. If the full dimension of the phase space is taken into account, it becomes apparent that the wild motion of the oscillator, which in the case of two dimensional phase space results in chaotic motion, in this case has additional spatial freedom and can thus move about unrestricted. It

is this unrestricted motion that is termed Arnold diffusion. The motion has the property that each trajectory will eventually connect any two open sets in a constant energy surface.

1 THE DYNAMICS OF A RIGID BODY

In this section we discuss the general rotational motion of a rigid body about its center of mass. This motion is characterized by an instantaneous axis of rotation passing through the center of mass; the axis, which is along the angular velocity vector, is allowed to vary its orientation with time together with the rotation of the rigid body. To study this motion, one is tempted to introduce a fixed set of axes in $\mathbf{R}^3$ and then to coordinatize the rotational displacement of the rigid body with respect to these axes by Euler angles. The Euler angles also parameterize the group of rotations $SO(3)$ on $\mathbf{R}^3$ so that it is possible to think of $SO(3)$ coordinatizing the rotational displacement of the rigid body. However, if we focus on a Hamiltonian analysis, we see that the expression for the kinetic energy must involve the inertia tensor and that the corresponding products of inertia vary with the motion of the rigid body. Consequently any computation in terms of the resulting Hamiltonian would be most awkward. Though we dare not take $T^*SO(3)$ as our state space, it does play a role in finding a suitable candidate for the state space.

Our method of attack is first to follow our temptation and to fix a set of inertial axes in $\mathbf{R}^3$. Coordinates with respect to this fixed set of axes are refered to as *space coordinates*. We measure the displacment of the rigid body by noting the Euler angles subtended by the principal axes of the body with respect to these fixed axes. Take $T^*SO(3)$, the cotangent space to $SO(3)$, as a phase space. We then "reduce" our analysis from $T^*SO(3)$ to $so(3)^*$ which is invariant with respet to rotation, i.e., with respect to the action of $SO(3)$ on $so(3)^*$. It is on $so(3)^*$ that we study the motion of the rigid body.

1.1 REDUCTION OF $T^*SO(3)$ AND THE MOMENTUM MAP

We take coordinate $(\phi, \theta, \psi, \dot{\phi}, \dot{\theta}, \dot{\psi})$ on $TSO(3)$, the tangent space to $SO(3)$, as induced by the Euler angles. In order to carry out a *psuedo* Hamiltonian analysis we must first transform $TSO(3)$ to $T^*SO(3)$. This transformation is carried out by the Legendre map Fl:$TSO(3) \rightarrow T^*SO(3)$. Once we are on the space $T^*SO(3)$ our aim is to reduce it to $so(3)^*$, the tangent space at the identity, which is appropriate for our analysis of the rigid body. The reduction of $T^*SO(3)$ to $so(3)^*$ legitimatizes our analysis of the rigid body because a Lie algebra is always abelian as a vector space. The angular momentum is then well defined. See [4], chapter four for a complete discussion of infinitesimal rotations. We must somehow carry the natural Lie algebra on $T^*SO(3)$ given by the Poisson bracket over to $so(3)^*$. We shall see that with this induced Lie

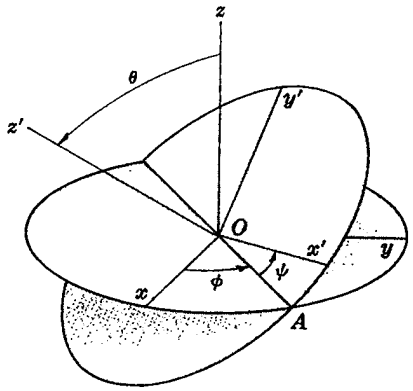


Figure 1:

algebra on $so(3)^*$ we can define our dynamics of the rigid body. It should be noted that the Legendre map can not be defined until the rigid body is specified. Every tangent vector on $SO(3)$ can be inteperated as a velocity vector, but the corresponding statement does not hold for cotangent vectors on $SO(3)$. Every cotangent vector on $SO(3)$ can not be said to be a momentum vector. It is only after we have introduced the rigid body that momenta can be considered. With the rigid body in hand and its corresponding inertia tensor we can define an inner product of tangent vectors on $SO(3)$ providing us with a correspondence between the vectors (velocities) and covectors (momenta) on $SO(3)$. We should be mindful of the fact that momenta are dynamical coordinates whereas velocities are not. The Legendre map carries the velocity vectors specifically to the momenta.

Since the consequences of a fixed set of inertial axes is not acceptable, we instead take the principal axes of the rigid body as the reference frame so that the reference frame is moving along with the body. In this moving frame the matrix of the inertia tensor reduces to the diagonal form with the moments of inertia (the diagonal elements) remaining constant throughout the motion. Of course we can no longer refer to Euler angles and in fact we need no longer refer to the *psuedo* Hamiltonian analysis on $T^*SO(3)$. The dynamics on $T^*SO(3)$ is reduced to dynamics on momentum space which we will see is $so(3)^*$. The key to the reduction, and thus the dynamics on $so(3)^*$, is the "momentum map" which we now proceed to define.

In anticipation of a dicussion of the momentum map, we look at the Lie algebras $so(3)$ and $so(3)^*$. Let $R(t)$ be curve on $SO(3)$. Then $\dot{R}(t\ )$ is a tangent vector on $SO(3)$. A straight forward calculation yields $R(\dot{R})^T = -\dot{R}R^T$ so that the matrix $\widehat{\Omega} = R^{-1}\dot{R}$ is skew symmetric. By the symbol Ω we understand the 3-vector resulting from $\widehat{\Omega}$. From the expression $R^{-1}\dot{R}$ we see that R^{-1} acts on the

tangent vector $\dot{R}$ on $SO(3)$ and translates it from the point R to the identity, so that $R^{-1}\dot{R}$ must be an element of the Lie algebra of $SO(3)$. A more direct though lengthier explanation that $R^{-1}\dot{R}$ is a tangent vector to $SO(3)$ at the identity is the following. We write

.

$$R^{-1}(t)\dot{R}(t) = R^{-1}(t)\lim_{\Delta t\to 0}\frac{R(t+\Delta t)-R(t)}{\Delta t}$$

$$= \lim_{\Delta t\to 0}\frac{R^{-1}(t)R(t+\Delta t)-R^{-1}(t)R(t)}{\Delta t}$$

$$= \lim_{\Delta t\to 0}\frac{R(-t)R(t+\Delta t)-R(-t)R(t)}{\Delta t}$$

$$= \lim_{\Delta t\to 0}\frac{R(-t+t+\Delta t)-R(-t+t)}{\Delta t}$$

$$= \lim_{\Delta t\to 0}\frac{R(\Delta t)-R(0)}{\Delta t} = \dot{R}(0).$$

From the expression $R^{-1}(t)\dot{R}(t) = \dot{R}(0)$ it follows that $R^{-1}(t)\dot{R}(t)$ is a tangent vector to $SO(3)$ at the identity, thus an element of $so(3)$. We can take either of the expressions, $\widehat{\Omega}$ or Ω, as our representation of the element of $so(3)$. A straight forward but tedious calculation shows that

$$\Omega = \begin{pmatrix} \dot{\theta}\cos\psi + \dot{\phi}\sin\psi\sin\theta \\ -\dot{\theta}\sin\psi + \dot{\phi}\cos\psi\sin\theta \\ \dot{\phi}\cos\theta + \dot{\psi} \end{pmatrix}. \tag{1}$$

This expression is standard for the angular velocity of a rigid body with respect to the coordinate system fixed to the principal axes and rotating with the body. Therefore we can take $so(3)$ as the space of angular velocities of the rigid body. Evidently (1) defines the map $\Omega : TSO(3) \to so(3)$. It is noteworthy that the angular velocity of a rigid body can not be interperated as a time derivative.

Since it is desirable to have $so(3)^*$ as a momentum space, we introduce the quadratic form $<\alpha,\beta>:=\alpha^T I\beta$ where α, $\beta \in so(3)$ and I is the matrix of the moment of inertia tensor with respect to the principal axes. Then we have that $<\ ,\Omega>$ is a linear real function on $so(3)$ and can be interperated as the momentum $\Pi \in so(3)^*$ that corresponds to the angular velocity Ω. In fact we can now write $I\Omega = \Pi$ and $<\Omega,\Omega> = \Omega^T I\Omega = 2KE$, noting that Ω and consequently Π are functions of $\phi,\theta,\psi,\dot{\phi},\dot{\theta},\dot{\psi}$. We can also interperate the phase space $T^*SO(3)$

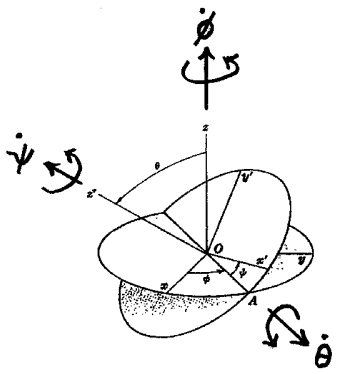


Figure 2:

as a momentum space by way of the Legendre map. The kinetic energy can be written as

$$KE = \frac{1}{2}\left[\frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{\Pi_3^2}{I_3}\right].$$

From the expressions $I\Omega = \Pi$ and (1) it follows that KE is a funtion of $\phi, \theta, \psi, \dot{\phi}, \dot{\theta}, \dot{\psi}$. We then define the functions p_ϕ,p_θ, p_ψ as follows. Let

$$p_\phi := \frac{\partial KE}{\partial \dot{\phi}}, \quad p_\theta := \frac{\partial KE}{\partial \dot{\theta}}, \quad p_\psi := \frac{\partial KE}{\partial \dot{\psi}}.$$

The three expressions for p_ϕ, p_θ, p_ψ are linear funtions of $\dot{\phi}$, $\dot{\theta}, \dot{\psi}$ so it is consequently easy to solve for $\dot{\phi}$, $\dot{\theta}, \dot{\psi}$. For $\left(\phi, \theta, \psi, \dot{\phi}, \dot{\theta}, \dot{\psi}\right) \in TSO(3)$ we make the substitution $\left(\dot{\phi}, \dot{\theta}, \dot{\psi}\right) \rightarrow (p_\phi$,$p_\theta, p_\psi)$. This substitution defines the Legendre map

$$Fl : TSO(3) \rightarrow T^*SO(3).$$

That is, by introducing the rigid body we are able to define the canonical coordinates $(\phi, \theta, \psi, p_\phi$,$p_\theta, p_\psi)$ on $T^*SO(3)$. Actually the funcion Fl is defined only on the parameters $(\dot{\phi}, \dot{\theta}, \dot{\psi})$ so that it is defined on the states. When we make the above substitution in the expression for Π we get

$$\Pi = \begin{pmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \end{pmatrix} = \begin{pmatrix} \frac{(p_\phi - p_\psi \cos\theta)\sin\psi + p_\theta \sin\theta \cos\psi}{\sin\theta} \\ \frac{(p_\phi - p_\psi \cos\theta)\cos\psi - p_\theta \sin\theta \sin\psi}{\sin\theta} \\ p_\psi \end{pmatrix} \tag{2}$$

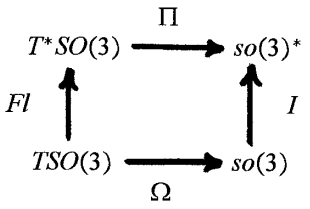


Figure 3:

so that the map $\Pi : T^*SO(3) \to so(3)^*$ is defined. We can assemble expressions (1) and (2) into the commutative diagram Figure 3.

Before we continue with the definition of the momentum map it is instructive to look at why we are interested in it. We have a well recognized and useful Lie algebra on $T^*SO(3)$ given by the Poisson bracket of real valued functions on $T^*SO(3)$. To continue our investigations, we need a similar Lie algebra of real valued functions on $so(3)^*$ which is induced by Π. For F and G, two such functions on $so(3)^*$, one is tempted to look at $\{F \circ \Pi, G \circ \Pi\}_P$ where $\{,\}_P$ is the Poisson bracket on $T^*SO(3)$, change variables to Π_1,Π_2,Π_3 and then calculate the partial derivative with respect to Π_1,Π_2,Π_3 by means of the chain rule and (2) with the hope that the desired Lie algebra on $so(3)^*$ will result. One soon learns that this procedure is fruitless. The purpose of the momentum map is to provide us with a procedure which does bear fruit.

It is most expedient to approach the problem of producing the desired Lie algebra "in abstracto". To this end take G to be a Lie group with a finite dimensional Lie algebra $\mathbf{g}$. It follows that G acts on T^*G symplectically, that is, the action of G on T^*G preserves the natural Poisson bracket on T^*G. Such an action is given by a Hamiltonian vector field on T^*G for each element ξ of $\mathbf{g}$. For convenience we write $P := T^*G$ with coordinates (q,p) on P. The Hamiltonian vector field is referred to as the *infinitesimal generator* corresponding to ξ. The Hamiltonian for this vector field is written as $J(\xi)$, so we now have the map $J : \mathbf{g} \to \mathcal{F}(P)$ where $\mathcal{F}(P)$ is the set of all differentiable functions on P. The *momentum map* $\mathbf{J}$: $P \to \mathbf{g}^*$ is defined by

$$< \mathbf{J}(q,p), \xi >= J(\xi)(q,p) \tag{3}$$

where $<,>$ is the pairing of $\mathbf{g}^*$ and $\mathbf{g}$ and J is linear. For our application we take $\mathbf{g}$ as $so(3)=\mathbf{R}^3$ so that we have $\xi = (\xi^1, \xi^2, \xi^3)$. It follows then that $J(\xi)$ can be written as

$$J(\xi) = \xi^1\Pi_1 + \xi^2\Pi_2 + \xi^3\Pi_3 \tag{4}$$

so that $J(\xi)(q,p) = \xi^1\Pi_1(q,p) + \xi^2\Pi_2(q,p) + \xi^3\Pi_3(q,p)$. The relation (2) is an example of a momentum map. There are other examples, another of which will be used in what follows to establish relation (18). Keep in mind that for our application where we take $G := SO(3)$, we understand that $(q,p) := (\phi,\theta,\psi,p_\phi,p_\theta,p_\psi)$.

It is the action of G on T^*G that makes the usual definition of angular momentum possible. In fact it is exactly the Hamiltonian $J(\xi)$ given by (4) that is the angular momentum.

1.2 CONSTRUCTION OF THE LIE-POISSON ALGEBRA ON $so(3)^*$

We are now in the position to construct the appropriate Lie algebra of funtions on $so(3)^*$. The construction is accomplished by mapping the Poisson algebra on $T^*SO(3)$, which we write as $\{\ ,\ \}_P$, over to $so(3)^*$ via a momentum map. Let $F, G : so(3)^* \to \mathbf{R}$ be two real functions on $so(3)^*$. Then the desired Lie algebra on $so(3)^*$, called the *Lie-Poisson algebra on* $so(3)^*$, is given by $\{F\circ\mathbf{J},G\circ\mathbf{J}\}_P := \{F,G\}_{L-P}$. In order to calculate $\{F\circ\mathbf{J},G\circ\mathbf{J}\}_P$ we must first make a number of observations. First, for F a real valued function on $so(3)^* = \mathbf{R}^3$ we write $\nabla_\Pi F$ as the gradient of F. As a differential $\nabla_\Pi F$ maps elements of $so(3)^* = \mathbf{R}^3$ linearly to $\mathbf{R}$ so that $\nabla_\Pi F$ is an element of $so(3)$ and the quantity $J(\nabla_\Pi F)$ is well defined. Second, if X is a vector field on a manifold P, then $X[h] = \nabla_P h \cdot X$ where h is a real function on P and ∇_P is the gradient operator with respect to the coordinates (q,p) on P. Then if X_f is the Hamiltonian vector field on $T^*SO(3)$ with Hamiltonian f, and g is a real function on P, we get

$$X_f[g] = \nabla_P g \cdot X_f = \frac{\partial g}{\partial q}\frac{\partial f}{\partial p} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} := -\{f,g\}_P = \{g,f\}_P.$$

Keep in mind that the Hamiltonian vector field X_f can be written in components as $(\frac{\partial f}{\partial p}, -\frac{\partial f}{\partial q})^T$. As mentioned above, for our application we take $P = T^*SO(3)$ and $\mathbf{g}^* = so(3)^*$ so that for the momentum map we have

$$\mathbf{J} := \begin{pmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \end{pmatrix} : P \to \mathbf{g}^*.$$

where the Π's are now taken as the generic coordinates of $so(3)^*$. Now let $H : P \to R$ and $F : \mathbf{g}^* \to R$ be real functions and write $\mathbf{T}_P\mathbf{J}$ as the Jacobian of $\mathbf{J}$ so that we can make the following calculations

$$X_{F\circ\mathbf{J}}[H] = \nabla_P H \cdot X_{F\circ\mathbf{J}} = \{H, F\circ\mathbf{J}\}_P = -\{F\circ\mathbf{J},H\}_P$$

$$= -X_H[F\circ\mathbf{J}] = -\boldsymbol{\nabla}_P(F\circ\mathbf{J})\cdot X_H = -(\nabla_\Pi F)(\mathbf{T}_P\mathbf{J})X_H$$

$$= -(\frac{\partial F}{\partial \Pi_1}, \frac{\partial F}{\partial \Pi_2}, \frac{\partial F}{\partial \Pi_3}) \begin{pmatrix} \frac{\partial \Pi_1}{\partial q} & \frac{\partial \Pi_1}{\partial p} \\ \frac{\partial \Pi_2}{\partial q} & \frac{\partial \Pi_2}{\partial p} \\ \frac{\partial \Pi_3}{\partial q} & \frac{\partial \Pi_3}{\partial p} \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{pmatrix}$$

$$= -(\frac{\partial F}{\partial \Pi_1}\frac{\partial \Pi_1}{\partial q} + \frac{\partial F}{\partial \Pi_2}\frac{\partial \Pi_2}{\partial q} + \frac{\partial F}{\partial \Pi_3}\frac{\partial \Pi_3}{\partial q}, \frac{\partial F}{\partial \Pi_1}\frac{\partial \Pi_1}{\partial p} + \frac{\partial F}{\partial \Pi_2}\frac{\partial \Pi_2}{\partial p} + \frac{\partial F}{\partial \Pi_3}\frac{\partial \Pi_3}{\partial p}) \cdot X_H$$

$$= -\nabla_P(\frac{\partial F}{\partial \Pi_1}\Pi_1 + \frac{\partial F}{\partial \Pi_2}\Pi_2 + \frac{\partial F}{\partial \Pi_3}\Pi_3) \cdot X_H$$

$$= -\boldsymbol{\nabla}_P(J(\nabla_\Pi F)) \cdot X_H = -X_H[J(\nabla_\Pi F)] = \{H, J(\nabla_\Pi F)\}_P$$

$$= -\{J(\nabla_\Pi F), H\}_P = X_{J(\nabla_\Pi F)}[H]$$

In summary we have

$$X_{F\circ\mathbf{J}}[H] = X_{J(\nabla_\Pi F)}[H]. \tag{5}$$

A similar calculation yields

$$X_F[G \circ \mathbf{J}] = X_F[J(\nabla_\Pi G)]. \tag{6}$$

Applying (4) to the calculation of $\{\ ,\ \}_P$ yields

$$\begin{aligned} \{J(\nabla_\Pi F), J(\nabla_\Pi G)\}_P = \Pi_1(&\tfrac{\partial F}{\partial \Pi_3}\tfrac{\partial G}{\partial \Pi_2} - \tfrac{\partial F}{\partial \Pi_2}\tfrac{\partial G}{\partial \Pi_3}) \\ &+ \Pi_2(\tfrac{\partial F}{\partial \Pi_1}\tfrac{\partial G}{\partial \Pi_3} - \tfrac{\partial F}{\partial \Pi_3}\tfrac{\partial G}{\partial \Pi_1}) \\ &+ \Pi_3(\tfrac{\partial F}{\partial \Pi_2}\tfrac{\partial G}{\partial \Pi_1} - \tfrac{\partial F}{\partial \Pi_1}\tfrac{\partial G}{\partial \Pi_2}) \\ &= -\Pi \cdot (\nabla_\Pi F \times \nabla_\Pi G) \end{aligned} \tag{7}$$

Here we take advantage of the fact that $\{\Pi_1, \Pi_3\}_P = \Pi_2$, $\{\Pi_3, \Pi_2\}_P = \Pi_1$, $\{\Pi_2, \Pi_1\}_P = \Pi_3$ which is a consequence of [11], Execise 11.2-2, that is, *equivariance* of the momentum map, namely $\{<\mathbf{J}, \boldsymbol{\xi}>, <\mathbf{J}, \boldsymbol{\eta}>\} = <\mathbf{J}, [\boldsymbol{\xi}, \boldsymbol{\eta}]>$. Combining (5) and (6) yields

$$\begin{aligned} \{F \circ \mathbf{J}, G \circ \mathbf{J}\}_P &= -X_{F\circ\mathbf{J}}[G \circ \mathbf{J}] = -X_{J(\nabla_\Pi F)}[G \circ \mathbf{J}] = -X_{J(\nabla_\Pi F)}[\ J(\nabla_\Pi G)] \\ &= \{J(\nabla_\Pi F), J(\nabla_\Pi G)\}_P \end{aligned} \tag{8}$$

For the functions $F, G : \mathbf{g}^* \to \mathbf{R}$ we define the $Lie - Poisson\ A\lg ebra$ on $\mathbf{g}^*$as

$$\{F, G\}_{L-P} := -\Pi \cdot (\nabla_\Pi F \times \nabla_\Pi G) \tag{9}$$

From (7) and (8) we also have

$$\{F \circ \mathbf{J}, G \circ \mathbf{J}\}_P = -\Pi \cdot (\nabla_\Pi F \times \nabla_\Pi G) \tag{10}$$

The left side of (10) is written as a function of (q,p) while the right side can be understood to be a function of (Π_1, Π_2, Π_3). The $\Pi' s$ in relation (10) are taken from (2), where they are given as functions of (q,p), and are used to define a momentum map **J.** It is therefore valid to write

$$\{F, G\}_{L-P} \circ \mathbf{J} = \{F \circ \mathbf{J}, G \circ \mathbf{J}\}_P \tag{11}$$

for any momentum map **J.** If it is attempted to calculate $\{F \circ \mathbf{J}, G \circ \mathbf{J}\}_P$ directly from (2) by use of the chain rule, it is soon found that the calculations are intractable. In such an attempt no use is made of the fact that $J(\xi)$ is the Hamiltonian of an infinitesimal generator. But it is just this fact that we used in our derivations of (5) and (6). It is worth noting that (7) holds for the momentum map **J** with respect to any coordinates that are *canonically* related so that its derivation is independent of the canonical coordinates that are used. Relation (2) defines the momentum map in terms of the coordinate system given by the Euler angles and their conjugate momenta. It can be said that $\{\ ,\ \}_{L-P}$ is the natural bracket on $so(3)^*$ just as $\{\ ,\ \}_P$ is the natural bracket on $T^*SO(3)$. The momentum map **J** is defined in terms of Hamiltonian fields, but the fields are all with respect to $\{\ ,\ \}_P$ on $T^*SO(3)$. We say that **J** carries $\{\ ,\ \}_P$ over to $\{\ ,\ \}_{L-P}$.

1.3 DYNAMICS OF A RIGID BODY ON $so(3)^*$

We start by fixing the general concept of a dynamical system on a phase space as given by a set of Hamilton's equations. We consider a phase space consisting of a manifold M of dimension $2n$ and write Hamiltons equations, with Hamiltonian H, in terms of generalized coordinates (also called canonical coordinates) as

$$\begin{array}{l} \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \\ \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad i = 1...n. \end{array}$$

It is these eqations that define the Hamiltonian vector field on M. Now take a real function F on M and evaluate its time derivative along a solution curve of the above Hamiltonian system. From the chain rule we have

$$\frac{dF}{dt} = \sum \left(\frac{\partial F}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial F}{\partial p_i}\frac{dp_i}{dt} \right) = \sum \left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i} \right) = \{F, H\}_P$$

For $M = T^*SO(3)$ we transfer the dynamics on $T^*SO(3)$, given by a Hamiltonian system of equations, to dynamics on $so(3)^*$, by way of the momentum map $\mathbf{J}: T^*SO(3) \rightarrow so(3)^*$. In order to carry out the transfer we look at the above general equation

$$\frac{dF}{dt} = \{F, H\}_P$$

for F and G real functions on $T^*SO(3)$ and where both sides of the equation are evaluated along integral curves of the Hamiltonian vector field X_H on $T^*SO(3)$. In effect we transfer this eqation over to $so(3)^*$ via the map **J.** Let f and h be real functions on $so(3)^*$ given by $F = f \circ \mathbf{J}$ and $H = h \circ \mathbf{J}$ so that

$$\frac{df \circ \mathbf{J}}{dt} = \{f \circ \mathbf{J}, h \circ \mathbf{J}\}_P = \{f, h\}_{L-P} \circ \mathbf{J}$$

The the above expression is evaluated along the integral curve of X_H on $T^*SO(3)$. Now by applying $\mathbf{J}$ to these integral curves we have image curves in $so(3)^*$ so that it is valid to write

$$\frac{df}{dt} = \{f, h\}_{L-P} \tag{12}$$

along the image curves.

Remark. We can arrive at equation (12) by a more direct consideration. Let f and h be functions in $\mathcal{F}(so(3)^*)$ where h is fixed and f varies. Define the vector field X_h on $so(3)^*$ by the following relation

$$\nabla_\Pi f \cdot X_h = -\Pi \cdot (\nabla_\Pi f \times \nabla_\Pi h)$$

Because this equatiion is an identity in f, it can be solved for X_h. In fact we see that $X_h = \Pi \times \nabla_\Pi h$. By the chain rule the left side of the equation is the derivative of f along the integral curves of X_h. Consequently we retain equation (12). It is of interest to note that the above procedure can be generalized to any differentiable manifold M that is a *Poisson manifold*, that is, a manifold with a Lie algebra of functions in $\mathcal{F}(M)$ with the condition that the Lie algebra fulfills Leibniz's rule. For fixed h in $\mathcal{F}(M)$ we can then define the vector $X_h = \{\ , h\}$. This definition of a vector is valid because X_h is a differentiation according to Leibniz's rule for $\{\ , h\}$. Now we have $X_h[f] = \{f,\ h\}$ so that

$$\frac{df}{dt} = \{f, h\}$$

We can conclude that any Poisson manifold whose Lie algebra of functions obeys Leibniz's rule is a phase space for a dynamical system.

The dynamics on the cotagent bundle $P = T^*SO(3)$ is given by the Hamiltonian system

$$\begin{aligned} \tfrac{dq}{dt} &= \tfrac{\partial H}{\partial p} = \{q, H\}_P \\ \tfrac{dp}{dt} &= -\tfrac{\partial H}{\partial q} = \{p, H\}_P \end{aligned}$$

It is natural then to take (12) for our dynamics on $so(3)^*$ so that we have

$$\frac{d\Pi_i}{dt} = \{\Pi_i, h\}_{L-P}, \quad \text{for} \ \ i = 1, 2, 3 \tag{13}$$

where

$$h = \frac{1}{2}\left[\frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{\Pi_3^2}{I_3}\right]$$

By computation of (12) with the given expression for h the usual *Euler Equations* for the rigid body result

$$\begin{array}{l} \frac{d\Pi_1}{dt} = \frac{I_2 - I_3}{I_2 I_3}\Pi_2\Pi_3 \\ \frac{d\Pi_2}{dt} = \frac{I_3 - I_1}{I_1 I_3}\Pi_1\Pi_3 \\ \frac{d\Pi_3}{dt} = \frac{I_1 - I_2}{I_1 I_2}\Pi_1\Pi_2 \end{array} \tag{14}$$

There are two immediate integrals of system (14). One is the energy integral

$$h = \frac{1}{2}\left[\frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{\Pi_3^2}{I_3}\right]. \tag{15}$$

The other integral is the *total* angular momentum integral

$$m = \frac{1}{2}[\Pi_1^2 + \Pi_2^2 + \Pi_3^2]. \tag{16}$$

It should be noted that the angular momentum components Π_1, Π_2, Π_3 themselves are not integrals of system (13), that is to say, that the angular momentum $\Pi = (\Pi_1, \Pi_2, \Pi_3)$ of a rigid body is not conserved when viewed with respect to the *body coordinates,* that is, when the coordinate axes are fixed to the principal axes of the rigid body. Equation (14) is valid only for body coordinates which are the natural coordinates for $so(3)^*$. Also to be noted is that the system (14) can not be decoupled. Consequently the components of angular momentum interact with one another.

1.4 DYNAMICS OF A RIGID BODY WITH NONLINEAR ATTACHMENT ON $so(3)^* \times T^*\mathbf{R}^3$

We now introduce a nonlinear oscillator attached to the rigid body. The center of oscillation is the center of gravity of the rigid body and the right handed coordinate system is assumed to coincide with the body coordinates. Imagine the nonlinear oscillator to be a system of three degrees of freedom (three modes) consisting of a mass suspended by six springs, pairwise colinear and coincident with each of the coordinate axes. The pair of springs parallel to the x-axis will be nonlinear soft springs while the remaining two pairs will be the usual linear Hooke's springs. To avoid mathematical complications, we say that each pair of springs moves transversely with the mass.

The phase space of the spring-mass system in isolation can be taken as $T^*\mathbf{R}^3$. We attach this oscillator to the rigid body and look for the interaction between the rigid body and the oscillator. In space coordinates the phase space of the rigid body together with the attachment is taken as $T^*SO(3) \times T^*\mathbf{R}^3_S$ where we apply the subscript S to remind us that we are referring to space coordinates.

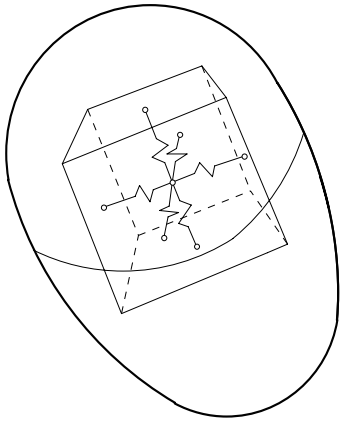


Figure 4:

To study the interaction between the rigid body and the oscillator, we must look at the dynamics of the system. We already know that the phase space $T^*SO(3) \times T^*\mathbf{R}^3_S$ will not accommodate the dynamics. We must rather use the phase space $so(3)^* \times T^*\mathbf{R}^3_B$, where the subscript B refers to body coordinates. The dynamics of the system on this phase space will be determined by an appropriate Lie algebra on $so(3)^* \times T^*\mathbf{R}^3_B$ just as the dynamics of the bare rigid body was developed from the Lie-Poisson algebra on $so(3)^*$. In order to obtain this desired Lie algebra we define the map

$$\phi : T^*SO(3) \times T^*\mathbf{R}^3_S \to so(3)^* \times T^*\mathbf{R}^3_B$$

by

$$\phi\left((q,p),\mathbf{x}\right) := \left(\mathbf{J}(q,p), R^{-1}\mathbf{x}\right) \tag{17}$$

where $\mathbf{x}$ is carried from $T^*\mathbf{R}^3_S$ to $T^*\mathbf{R}^3_B$ by the matrix R^{-1} which is the element of $SO(3)$ that corresponds to q. The term $R^{-1}\mathbf{x}$ is to be interperated as a *convection action* of the point $\mathbf{x}$ which fixes it to the rigid body. Part of our desired Lie algebra will be the Lie-Poisson algebra carried over by the momentum map $\mathbf{J}$ of the left translation

$$TL^*_g : T^*SO(3) \to so(3)^*$$

that is, TL^*_g translates $\alpha_g \in T^*_gSO(3)$ to $T^*_eSO(3)$ by multiplication on the left by g^{-1}.

Remark. If we take (q,p) to be the element of the parameter space corresponding to α_g,then $(0,p)$ is the element of the parameter space corresponding to $TL^*_e\alpha_g$, that is we can say $\mathbf{J}(q,p) = (0,p)$ for this momentum map.

We must now find the remainder of our Lie algerbra which is induced by the map $\mathbf{x} \to R^{-1}\mathbf{x}$. Since R^{-1} and $\mathbf{x}$ appear in this map, we can expect interaction

terms to occur in the remainder. In order to introduce this Lie algebra remainder we first define a *Casimir function with respect to a Lie algebra*. Let C be a function on a manifold with a Lie algebra of real functions $\{,\}$ such that $\{C, F\} = 0$ for any other function F on the manifold. Then C is said to be a *Casimir function*. From equation (16) we see that $\Pi_1^2 + \Pi_2^2 + \Pi_3^2 := |\Pi|^2$ is an integral of the Euler equation and as such suggests a Casimir function for the L-P algebra on $so(3)^*$. We take the bold step and say that for our desired Lie algebra on $so(3)^* \times T^*\mathbf{R}^3_B$ we must have that $|\Pi + \mathbf{r} \times \mathbf{p}|^2$ is a Casimir funtion, where **(r,p)** is a generic element of $T^*\mathbf{R}^3_B$. That is, we demand that the total angular momentum of both the rigid body and the nonlinear attachment is conserved. This conservation should characterize the interaction of the two systems. A calculation shows that

$$\begin{aligned}\left\{|\Pi + \mathbf{r} \times \mathbf{p}|^2, G\right\}_{PB} = &-\nabla_{\mathbf{r}} G \cdot (\nabla_{\Pi} |\Pi + \mathbf{r} \times \mathbf{p}|^2 \times \mathbf{r}) \\ &- \nabla_{\mathbf{p}} G \cdot (\nabla_{\Pi} |\Pi + \mathbf{r} \times \mathbf{p}|^2 \times \mathbf{p}).\end{aligned}$$

where $\{,\}_{PB}$ is the usual Poisson bracket on $T^*\mathbf{R}^3_B$. In order that the function $|\Pi + \mathbf{r} \times \mathbf{p}|^2$ is Casimir in the desired bracket, the desired bracket must take the form

$$\begin{aligned}\left\{|\Pi + \mathbf{r} \times \mathbf{p}|^2, G\right\} = \left\{|\Pi + \mathbf{r} \times \mathbf{p}|^2, G\right\}_{PB} &+ \nabla_{\mathbf{r}} G(\nabla_{\Pi} |\Pi + \mathbf{r} \times \mathbf{p}|^2 \times \mathbf{r}) \\ &+ \nabla_{\mathbf{p}} G \cdot (\nabla_{\Pi} |\Pi + \mathbf{r} \times \mathbf{p}|^2 \times \mathbf{p}).\end{aligned}$$

Similarly we should have

$$\begin{aligned}\left\{F, |\Pi + \mathbf{r} \times \mathbf{p}|^2\right\} = \left\{F, |\Pi + \mathbf{r} \times \mathbf{p}|^2\right\}_{PB} &- \nabla_{\mathbf{r}} F \cdot (\nabla_{\Pi} |\Pi + \mathbf{r} \times \mathbf{p}|^2 \times \mathbf{r}) \\ &- \nabla_{\mathbf{p}} F \cdot (\nabla_{\Pi} |\Pi + \mathbf{r} \times \mathbf{p}|^2 \times \mathbf{p}).\end{aligned}$$

The obvious choice for the desired Lie algebra of functions F and G on $so(3)^* \times T^*\mathbf{R}^3_B$ is then

$$\begin{aligned}\{F, G\} := \{F, G\}_{L-P} + \{F, G\}_{PB} &\\ - \nabla_{\mathbf{r}} F \cdot (\nabla_{\Pi} G \times \mathbf{r}) - \nabla_{\mathbf{p}} F \cdot (\nabla_{\Pi} G \times \mathbf{p}) &\\ + \nabla_{\mathbf{r}} G \cdot (\nabla_{\Pi} F \times \mathbf{r}) + \nabla_{\mathbf{p}} G \cdot (\nabla_{\Pi} F \times \mathbf{p}) &\end{aligned} \tag{18}$$

so that $|\Pi + \mathbf{r} \times \mathbf{p}|^2$ is a Casimir function with respect to this Lie algebra. A calculation shows that the function ϕ as given above carries the Lie algebra $\{F, G\}_{L-P} + \{F, G\}_{PB}$ on $T^*SO(3) \times T^*\mathbf{R}^3_S$ to our newly defined Lie algebra $\{F, G\}$ on $so(3)^* \times T^*\mathbf{R}^3_B$.

A direct derivation of (18) comes about by carrying the Lie algebra $\{\ ,\ \}_P$ on $T^*SO(3) \times T^*\mathbf{R}^3_S$ over to a Lie algebra on $so(3)^* \times T^*\mathbf{R}^3_B$ induced by the map ϕ given by (17). For this purpose we take $F, G : \mathbf{g}^* \times T^*\mathbf{R}^3_B \to \mathbf{R}$. If we

hold the variable $\mathbf{x}$ constant, then we understand the bracket $\{F\circ\phi, G\circ\phi\}_P$ to be the canonical bracket on $T^*SO(3)$ where the differentiation is with respect to the real variables q and p. For our present application we keep $\mathbf{x}$ constant and write $\{F\circ\phi, G\circ\phi\}_{T^*SO(3)}$ as the canonical bracket on $T^*SO(3)$ where the differentation is defined as usual with respect to the elements q and p of the parameter space of $T^*SO(3)$. We have

$$\{F\circ\phi, G\circ\phi\}_{T^*SO(3)} = \frac{\partial F\circ\phi}{\partial q}\frac{\partial G\circ\phi}{\partial p} - \frac{\partial G\circ\phi}{\partial q}\frac{F\circ\phi}{\partial p}$$

The momentum map $\mathbf{J}$ as used in (17) is defined on $T^*SO(3)$ so we can take it as a function of (q,p). Applying the chain rule where we write $\mathbf{z} = R^{-1}\mathbf{x}$, yields

$$\begin{aligned}\{F\circ\phi, G\circ\phi\}_{T^*SO(3)} = &\left(\tfrac{\partial F}{\partial \mathbf{J}}\tfrac{\partial \mathbf{J}}{\partial q} + \tfrac{\partial F}{\partial \mathbf{z}}\tfrac{\partial \mathbf{z}}{\partial q}\right)\tfrac{\partial G}{\partial \mathbf{J}}\tfrac{\partial \mathbf{J}}{\partial p}\\ &-\left(\tfrac{\partial G}{\partial \mathbf{J}}\tfrac{\partial \mathbf{J}}{\partial q} + \tfrac{\partial G}{\partial \mathbf{z}}\tfrac{\partial \mathbf{z}}{\partial q}\right)\tfrac{\partial F}{\partial \mathbf{J}}\tfrac{\partial \mathbf{J}}{\partial p}\end{aligned}$$

On multiplying out the the right side of the above expression we get

$$\{F\circ\phi, G\circ\phi\}_{T^*SO(3)} = \{F,G\}_{L-P}\circ\phi - \left(\frac{\partial F}{\partial \mathbf{J}}\frac{\partial G}{\partial \mathbf{z}}\frac{\partial \mathbf{z}}{\partial q} - \frac{\partial G}{\partial \mathbf{J}}\frac{\partial F}{\partial \mathbf{z}}\frac{\partial \mathbf{z}}{\partial q}\right)\frac{\partial \mathbf{J}}{\partial p}.$$

The derivative $\frac{\partial \mathbf{z}}{\partial q}$ gives us $\frac{\partial \mathbf{z}}{\partial q} = -\dot{R}(0)R^{-1}\mathbf{x}$. Combine $\dot{R}(0)$ with $\frac{\partial F}{\partial \mathbf{J}}$, $\frac{\partial G}{\partial \mathbf{J}}$,and $\mathbf{x}$ and note the remark on page (12) so that $\frac{\partial \mathbf{J}}{\partial p} = 1$. The term in parenthesis then yields

$$\begin{aligned}&(-\nabla_{\mathbf{r}}F\cdot(\nabla_\Pi G\times\mathbf{r}) - \nabla_{\mathbf{p}}F\cdot(\nabla_\Pi G\times\mathbf{p})\\ &\quad+\nabla_{\mathbf{r}}G\cdot(\nabla_\Pi F\times\mathbf{r}) + \nabla_{\mathbf{p}}G\cdot(\nabla_\Pi F\times\mathbf{p}))\circ\phi\end{aligned}$$

The above epression is composed with ϕ in order to reconcile with the variables in $T^*SO(3)$ and is identical to the terms on the second line of equation (18). We take the dynamics describing the motion of the rigid body-spring-mass system by means of (18) to be the desired dynamics. Let F and G be real functions defined on $so(3)^*\times T^*\mathbf{R}^3_B$ and take the Lie algebra to be given by

$$\begin{aligned}\{F,G\} := \{F,G\}_{L-P} + \{F,G\}_{PB} - \nabla_{\mathbf{r}}F\cdot(\nabla_\Pi G\times\mathbf{r}) - \nabla_{\mathbf{p}}F\cdot(\nabla_\Pi G\times\mathbf{p})\\ +\nabla_{\mathbf{r}}G\cdot(\nabla_\Pi F\times\mathbf{r}) + \nabla_{\mathbf{p}}G\cdot(\nabla_\Pi F\times\mathbf{p}).\end{aligned}$$

In the previous derivation of this formula we had to invoke the conservation of angular momentum. This time we used the map ϕ to carry the dynamics over to $so(3)^*\times T^*\mathbf{R}^3_B$. The conservation of angular momentum is included in the dynamics.

In the *Casimir* derivation no use was made of the map ϕ. The derivation rested entirely on requiring that total angular momentum is conserved. In the

direct derivation the map ϕ is introduced and is required to carry the established Lie-Poisson algebra from $T^*SO(3)\times T^*\mathbf{R}^3_S$ over to $so(3)^*\times T^*\mathbf{R}^3_B$. Interestingly the bracket on $so(3)^*\times T^*\mathbf{R}^3_B$ that is obtained this way coincides with the one obtained from our more empirical derivation using the Casimir functions and the conservation principle.

1.5 EQUATIONS OF MOTION ON $so(3)^*\times T^*\mathbf{R}^3_B$

From the remark following equation (18) we write the general equation

$$\frac{dF}{dt} = \{F, H\} \tag{18.4}$$

where the bracket on the right is taken from (18). This general eqaution completely defines the dynamics for the motion of our system. Take the Hamiltonian as

$$H(\Pi, \mathbf{r}, \mathbf{p}) = 1/2\left(\frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{\Pi_3^2}{I_3}\right) + \frac{1}{2m}\mathbf{p}^2 + V(\mathbf{r}) \tag{18.5}$$

where m is the mass of the attachment and V is the potential of the attachment's spring system. It is noteworthy that the Hamiltonian gives no indication of the coupling (that is, interaction) of the spring-mass system and the rigid body. It is the bracket $\{,\}$ that manifests the coupling.

For our application we take the potential function as

$$V = \frac{1}{2}kx^2 - \frac{1}{4}ax^4 + \frac{1}{6}bx^6 + \frac{1}{2}k_1y^2 + \frac{1}{2}k_2z^2. \tag{18.6}$$

The terms involving x describe the force of a *soft* spring. The y and z terms describe springs with the usual Hooke's forces. For our perturbation analysis in the following section we shall take the perturbation parameters as $\varepsilon_1 = \frac{1}{I_1}$, $\varepsilon_2 = \frac{1}{I_2}$, and $\varepsilon_3 = \frac{1}{I_3}$. For a massive body the ε's are small. The phase plane for the ε's equal to zero is given by Figure 5. The above general equation (18.4) then yields for the equations of motion of the coupled system

$$\begin{aligned}
\dot{\Pi}_1 &= \{\Pi_1, H\} = y(k_2z) - z(k_1y) \quad - \ \varepsilon_2(\Pi_2\Pi_3) + \varepsilon_3(\Pi_2\Pi_3)\\
\dot{\Pi}_2 &= \{\Pi_2, H\} = -x(k_2z) + z(kx - ax^3 + bx^5) + \varepsilon_1(\Pi_1\Pi_3 \ - \varepsilon_3(\Pi_1\Pi_3)\\
\dot{\Pi}_3 &= \{\Pi_3, H\} = x(k_1y) - y(kx - ax^3 + bx^5) \quad - \varepsilon_1(\Pi_1\Pi_2) + \varepsilon_2(\Pi_1\Pi_2)\\
\dot{x} &= \{x, H\} = \tfrac{p_x}{m} \ - \ \varepsilon_2(\Pi_2 z) + \varepsilon_3(\Pi_3 y)\\
\dot{y} &= \{y, H\} = \tfrac{p_y}{m} + \varepsilon_1(\Pi_1 z) \quad - \quad \varepsilon_3(\Pi_3 x)\\
\dot{z} &= \{z, H\} = \tfrac{p_z}{m} - \varepsilon_1(\Pi_1 y) + \varepsilon_2(\Pi_2 x)\\
\dot{p}_x &= \{p_x, H\} = -(kx - ax^3 + bx^5) \quad - \ \varepsilon_2(\Pi_2 p_z) + \varepsilon_3(\Pi_3 p_y)\\
\dot{p}_y &= \{p_y, H\} = -k_1 y \ + \varepsilon_1(\Pi_1 p_z) \quad - \quad \varepsilon_3(\Pi_3 p_x)\\
\dot{p}_z &= \{p_z, H\} = -k_2 z \ - \varepsilon_1(\Pi_1 p_y) + \varepsilon_2(\Pi_2 p_x)
\end{aligned} \tag{19}$$

The last six of the equations (the equations describing the motion of the spring-mass system) we call the *restricted system.* They can be rewritten as

$$\begin{aligned} D\mathbf{r} &:= (\tfrac{d}{dt}+\Omega\times)\mathbf{r} = \tfrac{\mathbf{p}}{m} \\ D\mathbf{p} &:= (\tfrac{d}{dt}+\Omega\times)\mathbf{p} = -\tfrac{\partial V}{\partial \mathbf{r}} \end{aligned} \tag{19.5}$$

in terms of the covariant derivative $D := \frac{d}{dt}+\Omega\times$. Thus we refer the motion of the oscillating mass to the rotating coordinate system of the rigid body.

2 PERTURBATION ANALYSIS

Now we want to analyze the solutions of (19). For all the $\varepsilon_i = 0$ the system is integrable by elementary mehods and is called *the nonperturbed system.* For this case the $x - p_x$ phase plane is illustrated by Figure 5. When the ε_i are not zero, we are unable to solve the system by elementary methods and must resort to a perturbation analysis to study the behavior of the solutions. In a perturbation analysis a system of the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) + \boldsymbol{\varepsilon}\mathbf{g}(\mathbf{x},t) \tag{20}$$

is considered where the function $\mathbf{g}$ is periodic in t. The bold characters all represent vector quantities of dimension n. The idea is first to analyze the unperturbed system and then to look at the influence of the of the small pertubation term $\boldsymbol{\varepsilon}\mathbf{g}(\mathbf{x},t)$. In this way we shall be able to anticipate the behavior of the perturbed system, at least qualitatively, when it is finally displayed by numerical methods as a solution curve.

In particular we are interested in the last six equations of the system (19), that is, the equations describing the motion of the spring-mass system. We note that this system is Hamiltonian with the perturbation Hamiltonian containing the parameters Π_1, Π_2, Π_3. The complete Hamiltonian for these equations is given by

$$H(\Pi, \mathbf{r}, \mathbf{p}) = \frac{1}{2m}\mathbf{p}^2 + V(\mathbf{r}) - \varepsilon\Pi\cdot(\mathbf{r}\times\mathbf{p}) := H_0(\mathbf{r},\mathbf{p}) + \boldsymbol{\varepsilon}H_1(\Pi,\mathbf{r},\mathbf{p}) \tag{20.5}$$

where $\varepsilon\Pi = (\varepsilon_1\prod_1, \varepsilon_2\prod_2, \varepsilon_3\prod_3)$. It is to the equations of the restricted system that we apply our perturbation analysis. The term $\varepsilon\Pi\cdot(\mathbf{r}\times\mathbf{p})$ reflects the interaction between the oscillating mass and the rotating rigid body. In the system (19.5) it is the covariant derivative that reflects the interaction.

2.1 ACTION-ANGLE COORDINATES

In order to carry out the analysis of our restricted system we must turn to action-angle coordinates. To this end we first consider the unpertubed system. From (18.5) and (18.6) the Hamiltonian of the unperturbed system can be written as

$$H_0(x,p_x,y,p_y,z,p_z) = \frac{1}{2m}\mathbf{p}^2 + V(\mathbf{r}) = H_x(x,p_x) + H_y(y,p_y) + H_z(z,p_z).$$

That is, the Hamiltonian *separates*. This separability tells us that the solution manifold can be viewed as a cartesian product of the solution manifolds in each degree of freedom. Since the unperturbed form of the restriced system has periodic solution, in each degree of freedom we can make the action-angle transformation $(x, p_x, y, p_y, z, p_z) \rightarrow (x, p_x, y, p_y, J_z, \theta_z)$. The Hamiltonian H_0 then reduces to the form

$$H_0(x, p_x, y, p_y, J_z, \theta_z) = H_x(x, p_x) + H_y(y, p_y) + H_z(J_z)$$

where

$$z = \sqrt{\frac{2J_z}{\omega}} \sin\theta_z, \;\; p_z = \sqrt{2\omega J_z} \cos\theta_z \quad for \; \omega = \sqrt{\frac{k_2}{m}}. \tag{21}$$

Note that the angle θ_z appears nowhere in the above expression for the Hamiltonian H_z. This property is characteristic for action-angle variables, namely, that the Hamiltonian, when written in action-angle variables, is independent of the angle. The independence of angle defines *circular symmetry*

For the time being it is only the coordinate pair (z, p_z) that we convert to action angle-variables (J_z, θ_z). We can then write the Hamiltonian for the restricted system with perturbation as

$$\begin{aligned} H(\Pi, x, p_x, y, p_y, J_z, \theta_z) \;\; &= \;\; H_x(x, p_x) + H_y(y, p_y) + H_z(J_z) \\ &\quad +\varepsilon H_1(\Pi, x, p_x, y, p_y, J_z, \theta_z). \end{aligned} \tag{22}$$

To continue with the perturbation analysis, we must convert the restricted system to a nonautonomous system of differential equations. It is for this conversion that we resorted to the variables (J_z, θ_z). The idea is to use θ_z as the time variable. We use substitutions (21) in $\varepsilon H_1(\Pi, x, p_x, y, p_y, J_z, \theta_z)$ so that we get

$$\varepsilon H_1(\Pi, x, p_x, y, p_y, J_z, \theta_z) : = \varepsilon H_1(\Pi, x, p_x, y, p_y, J_z, \sin\theta_z, \cos\theta_z) \tag{22$'$}$$

Clearly the Hamiltonian is periodic in the angle θ_z so that it also must be periodic in the restricted system. From the conservation of energy we have

$$H(\Pi, x, p_x, y, p_y, J_z, \theta_z) = \alpha$$

and solving for J_z

$$J_z = L(\Pi, x, p_x, y, p_y, \theta_z, \alpha).$$

From [5, Eq.4.8.23] we arrive at the periodic system

$$\begin{aligned} \dot{x} &= -\frac{\partial L^0(x,p_x,y,p_y,\alpha)}{\partial p_x} - \varepsilon \frac{\partial L^1(\Pi,x,p_x,y,p_y,t,\alpha)}{\partial p_x} \\ \dot{p}_x &= \frac{\partial L^0(x,p_x,y,p_y,\alpha)}{\partial x} + \varepsilon \frac{\partial L^1(\Pi,x,p_x,y,p_y,t,\alpha)}{\partial x} \\ \dot{y} &= -\frac{\partial L^0(x,p_x,y,p_y,\alpha)}{\partial p_y} - \varepsilon \frac{\partial L^1(\Pi,x,p_x,y,p_y,t,\alpha)}{\partial p_y} \\ \dot{p}_y &= \frac{\partial L^0(x,p_x,y,p_y,\alpha)}{\partial y} + \varepsilon \frac{\partial L^1(\Pi,x,p_x,y,p_y,t,\alpha)}{\partial y} \end{aligned} \tag{23}$$

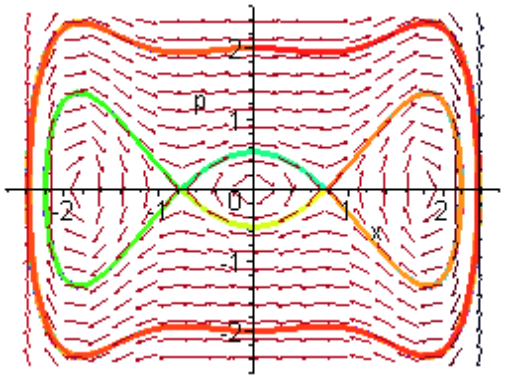


Figure 5:

The derivation of L^0 and L^1 from H_0 and H_1 can be found in the above reference where we note that the angle variable θ_z has been converted to the time variable t. The process of eliminating a pair of phase space variables such as (z, p_z) and replacing them by the time variable and a new Hamiltonian L is called *reduction.* It is the circular symmetry defined above that enables the reduction.

2.2 THE POINCARE SECTION AND THE POINCARE MAP

We consider the phase space of the unperturbed equations (23). As mentioned above, this phase space is the cartesian product of the individual phase spaces (x, p_x) and (y, p_y) where the phase portrait consists of periodic solutions. First we look at the phase space (x, p_x) of the nonlinear oscillator.

We shall subsequently write bold $\mathbf{p}$ for a point of phases space and bold $\mathbf{x}$ for an orbit in phase space. Bold $\mathbf{x}(t)$ is the point on the orbit $\mathbf{x}$ at time t. We confine our attention to the portion of the phase portrait containing the hyperbolic point $\mathbf{p}_h$ and the elliptic point $\mathbf{p}_e$. That is, we are interested only in the orbits centered at $\mathbf{p}_e$. The Hamiltonian L^0 also separates so that L^0 can be written as

$$L^0(x, p_x, y, p_y, \alpha) = L^0_x(x, p_x, \alpha) + L^0_y(y, p_y, \alpha)$$

Hamilton's equations for the unperturbed motion in the (x, p_x) phase plane are then given by

$$\dot{x} = -\frac{\partial L^0_x}{\partial p_x}$$
$$\dot{p}_x = \frac{\partial L^0_x}{\partial p_x}$$

This nonlinear system can also be converted to action-angle coordinates

(J, θ),though not by (21), and therefore can be written as

$$\begin{aligned} \dot{J} &= 0 \\ \dot{\theta} &= \frac{\partial K}{\partial J} = \omega(J) \end{aligned} \tag{24}$$

where K is the Hamiltonian into which H_x transforms and again does not contain θ. These equations are autonomous so that we can consider K as a periodic function of t for arbitrary period. In particular we take the period as 2π. The extended phase space (or the suspended phase space) is then a family of tori where each torus has coordinates (θ, t) and the family is parameterized by J which we can take as the radius of each torus in the family.

From (24) we see that each orbit given by the parameter J has a constant angular velocity $\omega(J)$. The *homoclinic* orbit $\mathbf{x}_h$ passing through the hyperbolic point $\mathbf{p}_1$ evidently has angular velocity $\omega = 0$. This fact follows from $\lim_{t \to \pm\infty} \mathbf{x}_h(t) = \mathbf{p}_h$ and from the fact that at a singular point the time derivtive is zero. It is instructive to look at a *Poincare section* $\sum_0$, that is, a section of the torus for a fixed value of t, say $t = 0$. We select a point on an arbitrary closed orbit $\mathbf{x}_J$ of the Poincare section, say $\mathbf{x}_J(0)$ and follow the trace of this point on the torus given by J as t goes to 2π. That is, we take the point $\mathbf{x}_J(0)$ as initial data and solve the system (24), finally setting $t = 2\pi$. The point returns to the starting Poincare section $\sum_0$. We define the *Poincare map* $\mathbf{P}$**:**$\sum_0 \to \sum_0 = \sum_{2\pi}$ by

$$\mathbf{P}(\mathbf{x}_J(0)) = \mathbf{x}_J(2\pi).$$

If the underlying system of differential equation is autonomous, as is the system (24), the Poincare map actually returns the point to the orbit $\mathbf{x}_J$. We now face the question: where on the orbit does the point return?

We look at the path traced out by $\mathbf{x}_J(0)$ on the orbit $\mathbf{x}_J$ as t goes from 0 to 2π. The possible values of $\omega(J)$ must be considered. For $\omega(J) = m$, m an integer, the point $\mathbf{x}_J(0)$ will be traced out exactly m times on $\mathbf{x}_J$ so that $\mathbf{P}(\mathbf{x}_J(0)) = \mathbf{x}_J(0)$ for any point on $\mathbf{x}_J$. For $\omega(J) = \frac{m}{n}$, $\frac{m}{n}$ a rational number, we get $\mathbf{P}^n(\mathbf{x}_J(0)) = \mathbf{x}_J(0)$ where $\mathbf{P}^n$ is the $n - th$ iterate of $\mathbf{P}$**.** Finally for $\omega(J) = irrational\ number$, $\mathbf{P}^k(\mathbf{x}_J(0))$ will be distinct for all iterates $\mathbf{P}^k$ and in fact $\mathbf{P}^k(\mathbf{x}_J(0))$ will cover $\mathbf{x}_J$ ergodically as the iteration continues.

We keep in mind that the system (24) is Hamiltonian and as such it preserves area. Thus if we apply $\mathbf{P}$ to any area in the section $\sum_0$, it will be preserved. This conservation of area is of greatest importance in what follows.

2.3 THE PERTURBED POINCARE MAP AND STOCHASTICITY

If we include the perturbation, the system (24) is written as

$$\begin{aligned} \dot{J} &= \varepsilon f(\Pi, J, \theta, y, p_y, t) \\ \dot{\theta} &= \omega(J) + \varepsilon g(\Pi, J, \theta, y, p_y, t). \end{aligned} \tag{24_ε}$$

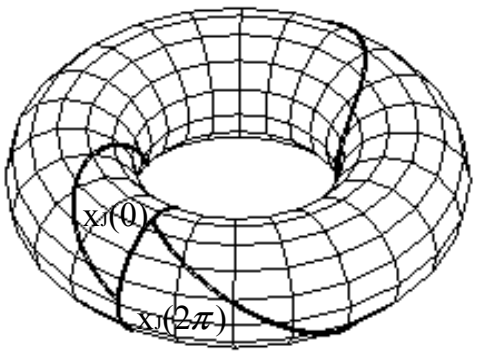


Figure 6:

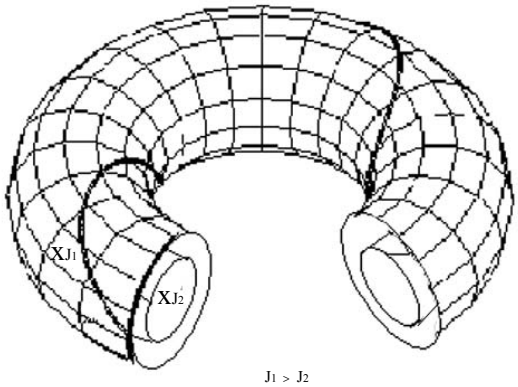


Figure 7:

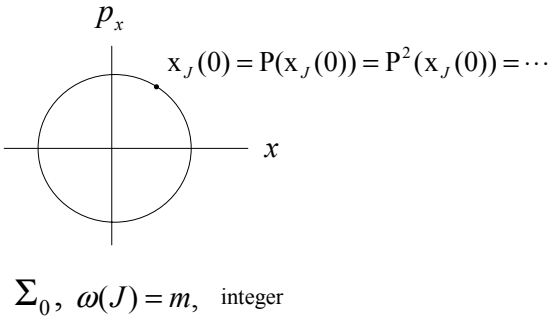
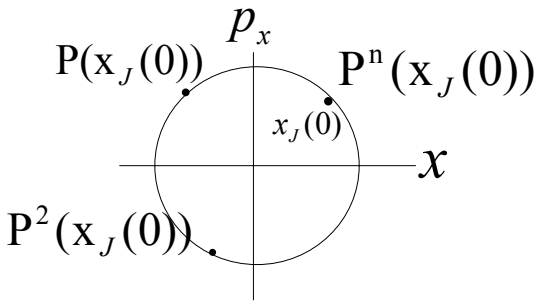
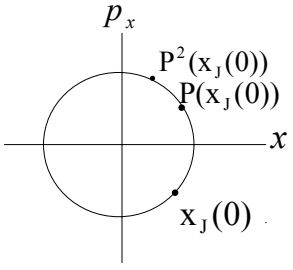


Figure 8:



$\Sigma_0,\ \omega(J) = m/n,$ rational

Figure 9:



$\Sigma_0, \quad \omega(J) = irrational\ \ number$

Figure 10:

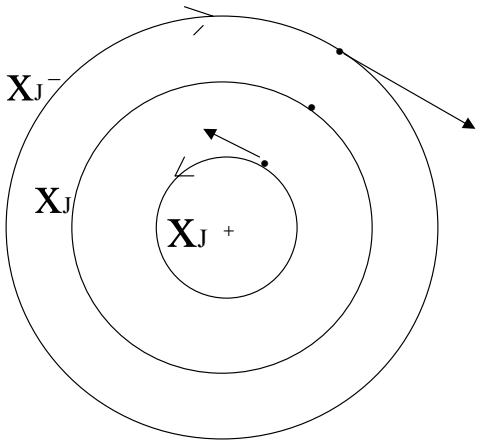


Figure 11:

where f and g are periodic functions of t with period 2π resulting from the change to action-angle variables, and Π, y, p_y are considered as parameters for the time being. The fact that f and g are periodic in t means that these functions have a fixed definition on the section $\sum_0$,that is, $\sum_0$ is invariant with respect to the system $(24)_\varepsilon$. The *perturbed Poincare map* $\mathbf{P}_\varepsilon : \sum_0 \to \sum_0$ is defined by solving the system $(24)_\varepsilon$ rather than (24). Technically the phase space should also accommodate the variaables $\Pi = (\Pi_1, \Pi_2, \Pi_3)$. Since it is only in the perturbation that these variables appear, we do not incllude them in the discussion of the Poincare map or the Poincare section.

Next we look at the effect of a perturbation on the Poincare map Since $\omega(J)$ approaches zero as J approaches the homoclinic orbit, we assume that ω is a decreasing function of J. It is due to the nonlinearity of the restoring force of the x springs that we have that ω is not identically constant. The fact that ω is not identically constant is the substance of what follows. Again take $\omega(J) = \frac{m}{n}$ so that for $J^- < J < J^+$ we have $\omega(J^-) > \omega(J) > \omega(J^+)$. Then the unperturbed Poincare map $\mathbf{P}^n$ holds each point on the orbit $\mathbf{x}_J$ fixed, rotates each point on the orbit $\mathbf{x}_{J^+}$ counterclockwise, and rotates each point on the orbit $\mathbf{x}_{J^-}$ clockwise. Along each ray $\theta = cons \tan t$ $\mathbf{P}^n$ generates a tangent vector to each of the orbits $\mathbf{x}_{J^+}$ and $\mathbf{x}_{J^-}$. This property of rotating some orbits and holding others fixed is called the *stroboscopic property* of the Poincare map. It is just this stroboscopic property that makes the Poincare map so useful. See Figure 11

The perturbed Poincare map $\mathbf{P}^n_\varepsilon$ generates vectors that are no longer tangent to the orbits of the nonperturbed system, though they are roughly in the same direction Since the perturbed vectors along a ray must swing form "roughly" counter clockwise on the orbit $\mathbf{x}_{J^-}$ to "roughly" clockwise on the orbit $\mathbf{x}_{J^+}$, at some point $\mathbf{p}(\theta)$ between the two there is a vector based on the ray and directed

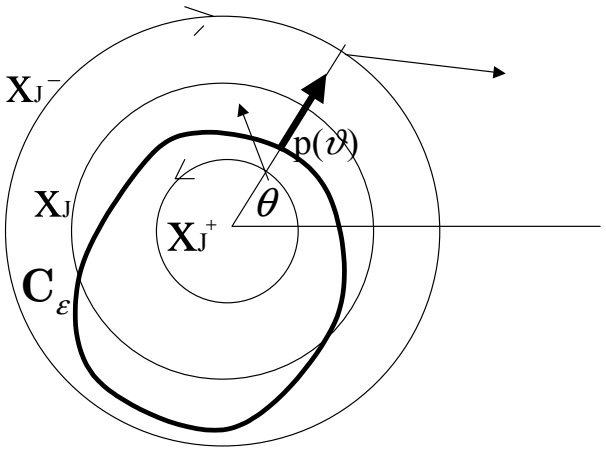


Figure 12:

along it. See Figure12. The set of all such base points is the closed curve $\mathbf{C}_\varepsilon$ in the phase space of the perturbed system. The set of based vectors along the rays form a field $\mathbf{V}_\varepsilon(\mathbf{C}_\varepsilon)$ on $\mathbf{C}_\varepsilon$ and direct the action of $\mathbf{P}_\varepsilon^n$ on $\mathbf{C}_\varepsilon$. We write the result of this directed action as $\mathbf{P}_\varepsilon^n(\mathbf{C}_\varepsilon)$. Since the system (24_ε) is Hamiltonian and $\mathbf{P}_\varepsilon^n$ must preserve area, the image curve $\mathbf{P}_\varepsilon^n(\mathbf{C}_\varepsilon)$ cannot lie entirely within nor entirely without the curve $\mathbf{C}_\varepsilon$. The field $\mathbf{V}_\varepsilon(\mathbf{C}_\varepsilon)$ is part of the vector field $\mathbf{V}_\varepsilon$ of the system $(24)_\varepsilon$. In Figure 13 we show the vector field $\mathbf{V}_\varepsilon(\mathbf{C}_\varepsilon)$ augmented to the entire vector field $\mathbf{V}_\varepsilon$ together with the singular points A, B, C, and D. The points B and D are elliptic and the points A and C are hyperbolic. An orbit that connects hyperbolic points (that even connects a hyperbolic point to itself) is called a *separatrix* because it separates orbits of different topological types. In Figure 14 we elaborate on the vector field $\mathbf{V}_\varepsilon$ at the time $2\pi n$ and sketch in the *heteroclinic* solutions, that is, solutions connecting saddle points; we also show solutions in the neighborhood of elliptic points. These heteroclinic solutions are suggested solutions in that they are integral curves of an autonomous system given by the field $\mathbf{V}_\varepsilon$ that does not differ by much from the field of the perturbed system (24_ε) when t is in a neighborhood of $2\pi n$. For each application of $\mathbf{P}_\varepsilon$ the increment of perturbation is the same because of the periodicity of H_1. This process of generating a phase portrait by perturbation of the Poincare map $\mathbf{P}^n$ was proposed by *Poincare and Birkhoff.*

In addition to the creation of heteroclinic paths due to perturbation there is another phenomenon that occurs on perturbation. This phenomenon is the *splitting* of the heteroclinic solution. We superimpose the phase spaces for the autonomous field $\mathbf{V}_\varepsilon$ at $t = 2\pi n$ and for the perturbed system $(24)_\varepsilon$ for $t > 2\pi n$. See Figure 15. For our illustration of splitting we use a *homoclinic* setting where the separatrix is an orbit that begins and ends at the same hyperbolic point **q.** The perturbed hyperbolic point is $\mathbf{q}_\varepsilon$. Look at a point $\mathbf{p}_\varepsilon$ on the perturbed

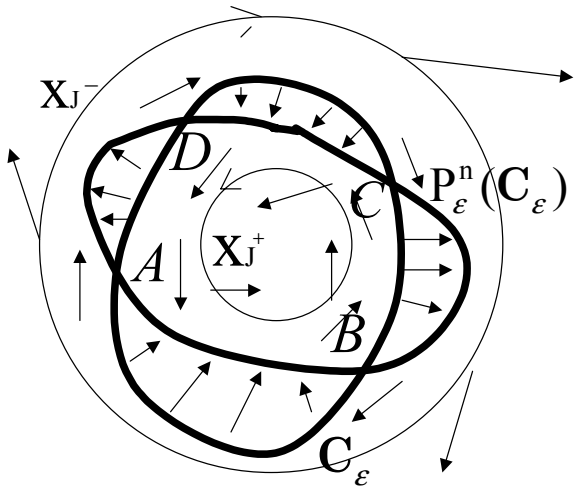


Figure 13:

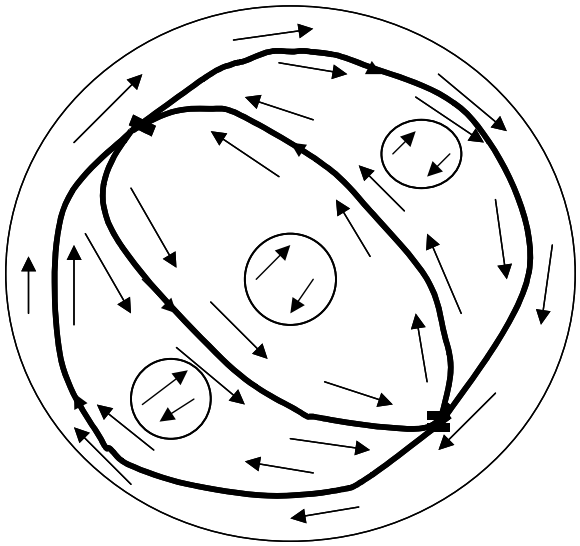


Figure 14:

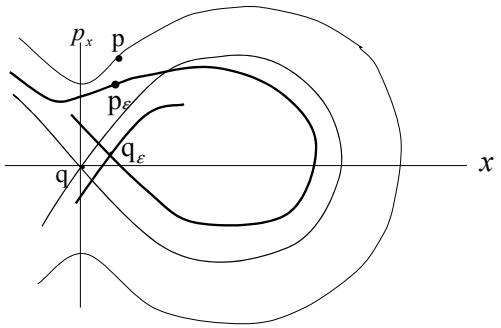


Figure 15:

separatrix $\mathbf{x}_\varepsilon$ that neighbors a point $\mathbf{p}$ lying on an orbit of the unperturbed system and also lying close to the unpertubed separatrix. Both of these points are imagined enclosed in a rectangular box which is allowed to flow with $\mathbf{V}_\varepsilon$. Since $\mathbf{p}_\varepsilon$ is on an orbit of a perturbation, we would expect $\mathbf{p}_\varepsilon$ and $\mathbf{p}$ to remain close. But we know that $\mathbf{p}$ has a meandering path once it distances itself from the homoclinc point. But then $\mathbf{p}_\varepsilon$ must follow the same sort of meandering course as $\mathbf{p}$ if it is to remain close to **p.** We conclude that the homoclinic solution of (24_ε), of which $\mathbf{p}_\varepsilon$ is a point, could possibly follow an erratic path.

The task now is to analyze the motion of the meandering perturbed homoclinic orbit. If the split does indeed occur, the orbit has two branches. From the above paragraph we know that for one branch $\lim_{t\to-\infty} \mathbf{x}_\varepsilon = \mathbf{q}_\varepsilon$ and for the other $\lim_{t\to\infty} \mathbf{x}_\varepsilon =$? The question is did the two branches actually split or did they not split and thus remain as a single continuous orbit based at $\mathbf{q}_\varepsilon$. The answer is contained in *Melnikov's method* which essentially calculates the distance separating the two branches of the possibly split homoclinic orbit as viewed from the unperturbed homoclinic orbit. If the distance function as defined by Melnikov has a simple zero, then the two branches intersect transversely.

We return to the heteroclnic setting and study the system given by the abridged Hamiltonian

$$L_x(\Pi, x, p_x, y, p_y, J, t, \alpha) = L_x(x, p_x, \alpha) + \varepsilon L_1(\Pi, x, p_x, y, p_y, J, t, \alpha)$$

where we have separated the Hamiltonian L that appears in equation (23). For a discussion of how Melnikov's method is applied to this system see references [5], [9]. The anaysis of the particular application of Melnikov's method to the Hamiltonian appearing in the system (19) is found in [14, page 52]. It is shown there that in our case the two branches do indeed intersect transversely. It is possible to have a nonlinear restoring force that according to Melnikov's method does not yield a distance function with simple zeros, but for us the branches actually experience multiple intersections, called *heteroclinic points*, the points of intersection approaching the respective hyperbolic points $\mathbf{p}_{h1}$ and $\mathbf{p}_{h2}$. The

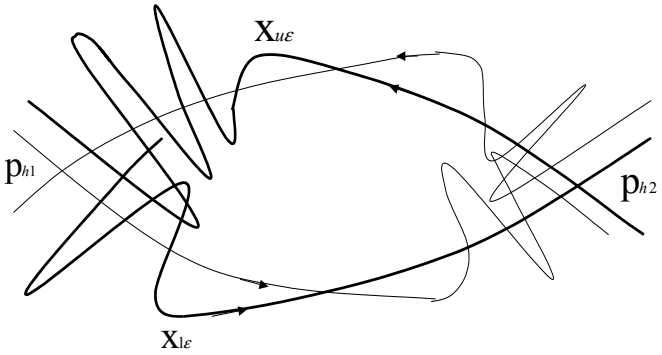


Figure 16:

area enclosed in the loops formed by the successive intersections of the two branches is constant because Hamiltonian systems preserve area. Since successive heteroclinic points accumulate on the hyperbolic points, the lengths of the bases of the loops approach zero so that their heights must increase in order to maintain constant area. By the KAM theorem (see below) the resulting loops must not intersect any of the stable orbits; they are confined and so display an erratic disposition. The two branches of the heteroclinic orbits are said to form a *heteroclinic tangle.* See Figure 15.

The structure of the two dimensional phase portrait as shown in Figure 16 is dictated by the *Kolmogorov, Arnold, Moser theorem.* For our application the theorem can be stated as follows: *(KAM theorem) Consider an autonomous Hamiltonian system whose periodic solutions are enclosed within seperatices (homoclinic orbits, heteroclinic orbits) and let these periodic solutions have periods that are an increasing function of the action variable J and, further, let the autonomous system experience a Hamiltonian perturbation that is time dependent with period 2π. Then those periodic solutions of the unperturbed autonomous system with ω equal to a rational number wll split and form a tangle when perturbed. Those periodic solutions of the autonomous system with ω sufficiently close to a rational number, depending on the magnitude of the perturbation, will also split on being perturbed. The split solutions can not intersect the surviving periodic solutions. Most of the periodic solution will retain their topology, that is, survive.* From the statement of the theorem it is apparent that the tangled solutions will appear in layers. These layers are called *stochastic layers* and the tangled solutions are said to be *chaotic.* We could repeat the same process by applying $\mathbf{P}_\varepsilon^n$ wherever we find a surviving orbit with rational angular velocity of the form $\omega = \frac{m'}{n}$. We could also apply the process to periodic orbits that appear in the neighbohoods of the elliptic points that are generated. In any event most of the peiodic orbits survive the process. Also, all of the surviving paths are invariant and so can not intersect any other paths, and in fact they

form a Cantor set. From the statement of the theorem it follow that the tangled solutions are dense in the space of the surviving solutions. Illustrations of the tangled solutions can be found in [6].

We see that the x and z modes of the oscillator are coupled by the time variable t in equations (24_ε). The rigid body in mediating the interaction essentially allows the the x and y modes to be only weakly coupled in the case of the surviving periodic solutions. On the other hand the chaotic solutions are strongly coupled by the rigid body. This fact is the essence of the *KAM* theorem.

2.4 THE TOROIDAL MOUSTACHE AND ARNOLD DIFFUSION

Finally in our perturbation analysis we account for the harmonic oscillator that defines the phase space (y, p_y). In order to be consistent with accepted terminology, we shall refer to the periodic orbits of this phase portrait as *tori,* keeping in mind that they are 1-tori. Once again we write this family of tori in action-angle coordinates as (J_y, θ_y). We take J_y as the parameter specifying the member of the family and θ_y as the coordinate of a point on a particular torus from the family. Extend the perturbed system (24_ε) to

$$\begin{aligned} \dot{J} &= \varepsilon f(\Pi, J, \theta, J_y, \theta_y, t) \\ \dot{\theta} &= \omega(J) + \varepsilon g(\Pi, J, \theta, J_y, \theta_y, t) \\ \dot{J}_y &= \varepsilon h(\Pi, J, \theta, J_y, \theta_y, t) \\ \dot{\theta}_y &= \omega + \varepsilon k(\Pi, J, \theta, J_y, \theta_y, t) \end{aligned} \qquad (25_\varepsilon)$$

where again h and k are 2π periodic functions of t, and the ω in the θ_y equation is independent of J_y because we have an harmonic oscillator. The system (25_ε) defines a four dimensional phase space. We define $\mathbf{x}_{u\varepsilon}$ as the *upper heteroclinic solution* of (24_ε) and $\mathbf{x}_{l\varepsilon}$ as the *lower heteroclinic solution* of (24_ε), both solutions emminating from the same hyperbolic point on the right as shown in Figure 16. The solution of the second set of equations in (25_ε) is a family of circles each of which is slightly misshapen and parameterized by J_y. The phase space of the system is filled with orbits of the form $(\mathbf{x}_J, \mathbf{x}_{J_y})$ where $\mathbf{x}_J$ is a solution of the (J, θ) system and $\mathbf{x}_{J_y}$ is a solution of the (J_y, θ_y) system. Note that $\mathbf{x}_{l\varepsilon}$ and $\mathbf{x}_{u\varepsilon}$ are members of the $\mathbf{x}_J$ family of solutions. All orbits of the form $(\mathbf{x}_{l\varepsilon}, \mathbf{x}_{J_y})$ is a submanifold of the phase space called the *arriving moustache* and it is written as $\mathbf{M}^+(J_y)$. Likewise we have the *departing moustache* written as $\mathbf{M}^-(J_y)$ which consists of the submanifold of orbits of the form $(\mathbf{x}_{u\varepsilon}, \mathbf{x}_{J_y})$. Then it follows that $\mathbf{M}^+(J_y) \cap \mathbf{M}^-(J_y) = \mathbf{T}(J_y)$, which is the support of the orbit $\mathbf{x}_{J_y}$ and which we referred to as a *torus.* We might refer to the orbits in the moustaches as *whiskers* with $\mathbf{w}^+_{J_y}$ a whisker in $\mathbf{M}^+(J_y)$ and $\mathbf{w}^-_{J_y}$ a whisker in $\mathbf{M}^-(J_y)$. From Melnikov's theorem we know that any whisker $\mathbf{w}^-_{J_y}$ must penetrate $\mathbf{M}^+(J_y)$ transversly. But then the whisker will accumulate on the opposite branch of the lower heteroclinic orbit that has split off from

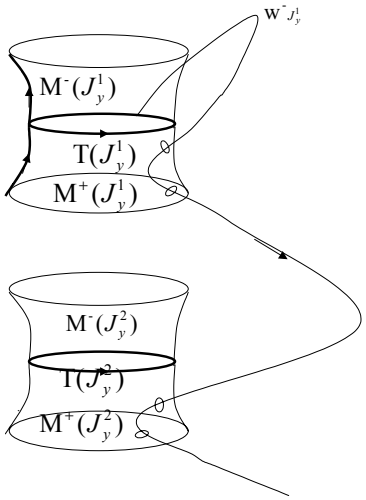


Figure 17:

$\mathbf{x}_{l\varepsilon}$ close to where it departs from its torus and experience wild oscillation [13, $\lambda - lemma$], much as the curve $y = \frac{1}{x}\sin\frac{1}{x}$ behaves close to the y-axis. The wildly oscillating $\mathbf{w}^-_{J_y}$ has multiple transverse intersections with $\mathbf{M}^+(J_y)$ As a result of the wild oscillations any other departing moustache that is sufficiently nearby will also be tansversly penetrated by $\mathbf{w}^-_{J_y}$. Designate the first moustache penetrated by $\mathbf{w}^-_{J_y}$ as $\mathbf{M}^+(J^1_y)$ and the second one as $\mathbf{M}^+(J^2_y)$. The process repeats with $\mathbf{w}^-_{J_y}$ accumulating on itself in $\mathbf{M}^+(J^2_y)$. We can index the penetrated moustaches: $\mathbf{M}^+(J^1_y)$,$\mathbf{M}^+(J^2_y)$,$\mathbf{M}^+(J^3_y), \ldots$. By this sequence of transverse penetrations $\mathbf{w}^-_{J^1_y}$ wanders far from its initial muostache $\mathbf{M}^+(J^1_y)$. This wandering process is termed *Arnold diffusion.* The whisker $\mathbf{w}^-_{J^1_y}$ in Figure 17 is undersood to lie entirely in the moustache $\mathbf{M}^-(J^1_y)$ and so drags the moustache along as it penetrates the stable moustaches.From the KAM theorem we know that the *tangled* solutions, that is the split solutions, are confined to bands that are bounded by the periodic solutions. One would think that the solutions of the system (25_ε) would be confined to the tori in θ, J, θ_y coordinates. However such a consideration does not account for the J_y variable which gives us a four dimensional phase space. In a four dimensional phase space a one dimensional curve can avoid intersecting two dimensional tori. For this reason we do not have a truly chaotic path. A good discussion connecting the concepts of stochastic layers, chaotic motion and Arnold diffusion can be found in [6, section 2.7]

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