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LS-Galleries, the path model and MV-cycles

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Abstract

Let G be a complex semisimple algebraic group. We give an interpretation of the path model of a representation [17] in terms of the geometry of the affine Grassmannian for G. In this setting, the paths are replaced by LS–galleries in the affine Coxeter complex associated to the Weyl group of G. The connection with geometry is obtained as follows: consider a Demazure–Hansen–Bott–Samelson desingularization $\hat{\Sigma}(\lambda)$ of the closure of an orbit $G(\mathbb{C}[[t]]).\lambda$ in the affine Grassmannian. The points of this variety can be viewed as galleries of a fixed type in the affine Tits building associated to G. The retraction with center $-\infty$ of the Tits building onto the affine Coxeter complex induces, in this way, a stratification of the $G(\mathbb{C}[[t]])$–orbit (identified with an open subset of $\hat{\Sigma}(\lambda)$), indexed by certain folded galleries in the Coxeter complex. Each strata can be viewed as an open subset of a Białynicki–Birula cell of $\hat{\Sigma}(\lambda)$. The connection with representation theory is given by the fact that the closures of the strata associated to LS-galleries are the MV–cycles [23].

Introduction

The aim of the present article is to provide a connection between the path model [17] for finite dimensional representations of complex semisimple algebraic groups and the geometry of the affine Grassmann variety. As a byproduct of this interpretation, we associate in a canonical way to each path (or rather gallery) in a path model of a representation one of the algebraic cycles occurring in the work of Mirkovic and Vilonen [23]. Recall that these cycles form, by [23] and the work of Vasserot [25], a canonical basis of the finite dimensional complex representations of the Langland's dual group $G^\vee$.

An important tool in our construction is the theory of Tits buildings. This theory provides a retraction, with center $-\infty$, of the affine Tits building onto the affine Coxeter

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complex. We embed the affine Grassmannian $\mathcal{G}$ into the affine building and show that the retraction above induces a stratification of the $G(\mathbb{C}[[t]])$–orbits in $\mathcal{G}$, the strata being indexed by certain folded galleries in the Coxeter complex. We show that the MV–cycles [23] are exactly the closures of the strata associated to LS-galleries. So this construction links the path model theory to another construction of bases for finite representations (for the connection with SMT see [19], for the connection with crystal bases [13], [14]).

To be more precise, let G be a connected semisimple complex algebraic group and let $G^\vee$ be its Langland's dual group. So if T is a maximal torus of G, then the group of co-characters $X^\vee = \mathrm{Mor}\,(\mathbb{C}^*, T)$ of T is the character group of the corresponding maximal torus $T^\vee$ of $G^\vee$. Let λ be a dominant co-character and let $V(\lambda)$ denote the irreducible $G^\vee$-module of highest weight λ. The quotient space $\mathcal{G} = G\big(\mathbb{C}((t))\big)/G\big(\mathbb{C}[[t]]\big)$ is called the affine Grassmannian, it is an infinite dimensional projective variety. We view the co-characters as points in $\mathcal{G}$ and we denote by $\mathcal{G}_\lambda = G\big(\mathbb{C}[[t]]\big)\lambda$ the orbit of $G\big(\mathbb{C}[[t]]\big)$ through λ. Its (Zariski) closure $X_\lambda = \overline{\mathcal{G}_\lambda}$ is a finite dimensional projective variety.

To connect this construction with the combinatorics of the path model, we view the affine Grassmannian in a canonical way as a subset of the affine Tits building associated to G [12]. Having fixed a maximal torus $T \subset G$, we have a corresponding apartment $\mathcal{A}$ in the building. As a simplicial complex, the apartment is isomorphic to the affine Coxeter complex associated to the Weyl group of G. The retraction with center $-\infty$ is a map of simplicial complexes of the entire building onto the fixed apartment $\mathcal{A}$. We want to study the fibres of the restriction of this map to the $G\big(\mathbb{C}[[t]]\big)$–orbits in the affine Grassmannian.

To do so, we develop first the necessary combinatorial tools. Assume for simplicity that λ is a regular co-character and an element of the coroot lattice. By a gallery joining the origin and λ we mean a sequence of alcoves $(\Delta_i)_{i=0,\dots,r}$ in the affine Coxeter complex such that $0 \in \Delta_0$, $\lambda \in \Delta_r$, and two consecutive alcoves have at least a codimension one face in common. Fix a minimal such gallery γ_λ and let $\Gamma^+(\gamma_\lambda)$ denote the set of all galleries of the same type starting in the origin and which are positively folded (see section 4). We define a dimension function for each gallery, which roughly speaking counts the number of hyperplanes crossed by the gallery from the negative to the positive half space (see section 5). Let $e(\delta)$ be the endpoint (or target) of a gallery δ and let ρ denote half the sum of the positive roots. The dimension of a gallery δ is bounded above by $\langle \lambda + e(\delta), \rho\rangle$. We call a gallery in $\Gamma^+(\gamma_\lambda)$ an LS–gallery if its dimension is equal to this upper bound.

We construct "folding operators" e_α, f_α on $\Gamma^+(\gamma_\lambda)$ for all simple roots and show: (see Proposition 4 and Theorem 2)

Theorem A. *The set of LS–galleries in $\Gamma^+(\gamma_\lambda)$ is stable under the folding operators. Let $B(\gamma_\lambda)$ be the directed colored graph having as vertices the set of LS-galleries in $\Gamma^+(\gamma_\lambda)$, and put an arrow $\delta \xrightarrow{\alpha} \delta'$ with color α between two galleries if $f_\alpha(\delta) = \delta'$. Then this graph is connected, and it is isomorphic to the crystal graph of the irreducible representation $V(\lambda)$ of $G^\vee$ of highest weight λ. In particular,* $\mathrm{Char}\, V(\lambda) = \sum_{\delta\in\Gamma^+(\gamma_\lambda),\, \delta\,\mathrm{LS-gallery}} \exp(e(\delta))$.

To connect the combinatorics with the geometry, recall that in terms of the affine Kac-Moody group associated to the extended Dynkin diagram of G, the affine Grassmannian is a generalized flag variety and X_λ is a Schubert variety. The affine Tits–building is the union of its apartments, all being isomorphic to the fixed apartment $\mathcal{A}$. Fixing a minimal gallery in $\mathcal{A}$ joining the origin and λ is equivalent to fix a reduced decomposition of λ in $W^{\mathfrak{a}}/W$, where $W^{\mathfrak{a}}$ is the affine Weyl group. In geometric terms, such a choice is equivalent to fix a Demazure–Hansen–Bott–Samelson desingularization $\pi : \hat{\Sigma}(\gamma_\lambda) \to X_\lambda$.

Contou–Carrère [9] has shown that the points of an affine Bott–Samelson variety can be viewed as the set of all galleries in the affine building starting at the origin ($= G(\mathbb{C}[[t]])$) and of the same type as γ_λ. We show that the birational desingularization map identifies the open orbit $\mathcal{G}_\lambda$ in X_λ with the open set of points $\hat{\mathcal{G}}_\lambda$ in the affine Bott–Samelson variety $\hat{\Sigma}(\gamma_\lambda)$ corresponding to all minimal galleries of type γ_λ in the Tits–building. Having the isomorphism $\hat{\mathcal{G}}_\lambda \simeq \mathcal{G}_\lambda$ in mind for the application, we show that: (see Theorem 3 and 4, Corollary 4)

Theorem B. *The retraction $r_{-\infty}$ centered at infinity of the anti-dominant Weyl chamber induces a map $r_{\gamma_\lambda} : \hat{\mathcal{G}}_\lambda \to \Gamma^+(\gamma_\lambda)$ onto the set of all positively folded galleries of the same type as γ_λ. For such a gallery δ, the fibre $r_{\gamma_\lambda}^{-1}(\delta)$ is naturally equipped with the structure of an irreducible quasi-affine variety, it is the intersection of a Białynicki–Birula cell of $\hat{\Sigma}(\gamma_\lambda)$ with $\hat{\mathcal{G}}_\lambda$. The dimension of the fibre is equal to the combinatorially defined dimension $\dim\delta$, and $r_{\gamma_\lambda}^{-1}(\delta)$ admits a finite decomposition into a union of subvarieties, each being a product of $\mathbb{C}$'s and $\mathbb{C}^*$'s. In particular, the fibre admits a canonical open and dense subvariety isomorphic to $\mathbb{C}^a \times (\mathbb{C}^*)^b$, where $b = \sharp J^-_{-\infty}(\delta)$ (section 10) and $a + b = \dim\delta$.*

The set $\Gamma^+(\gamma_\lambda)$ provides hence via the isomorphism $\hat{\mathcal{G}}_\lambda \simeq \mathcal{G}_\lambda$ a stratification of the orbit $\mathcal{G}_\lambda$. For $\delta \in \Gamma^+(\gamma_\lambda)$ we write $Z(\delta)$ for the closure $\overline{r_{\gamma_\lambda}^{-1}(\delta)}$ of the fibre in X_λ, and let $X_\lambda(\mu)$ be the union of all $Z(\delta)$ for $\delta \in \Gamma^+(\gamma_\lambda)$ having μ as target. In group theoretic terms, we have $X_\lambda(\mu) = \overline{U^-(\mathcal{K}).\nu \cap \mathcal{G}_\lambda}$, where $U^- \subset B^-$ is the unipotent radical. The special rôle played by the strata corresponding to LS-galleries is that their closures correspond to the irreducible components of $X_\lambda(\mu)$ (see Theorem 4).

Theorem C. *The irreducible components of $X_\lambda(\mu)$ are given by the $Z(\delta)$ for δ a LS–gallery, i.e., $X_\lambda(\mu) = \bigcup_\delta Z(\delta)$, where δ runs over all LS–galleries in $\Gamma^+(\gamma_\lambda)$ having as target $e(\delta) = \mu$. These irreducible components are precisely the MV–cycles.*

As a consequence, the algebraic cycles in

$$MV(\lambda) = \{Z(\delta) = \overline{r_{\gamma_\lambda}^{-1}(\delta)} \mid \delta \text{ a LS–gallery in } \Gamma^+(\gamma_\lambda)\},$$

form by [23] and [25] a canonical basis of $V(\lambda)$, realized as the intersection homology of X_λ. So, Theorem C provides the representation theoretic interpretation of the combinatorial character formula in Theorem A.

It should be very interesting to connect directly the methods presented in this paper with the methods (using the moment map) developed by Anderson and Kogan in [1], [2].

To make the paper as self contained as possible, we recall in section 1 the definition of the affine Grassmannian and we explain how we restrict ourselves to the case of a simply connected semisimple group. The next section deals with the affine Kac-Moody group associated to G and various incarnations of the affine Grassmannian. The third section concerns the *building theory*, we recall the definitions of the three buildings one can associate to G: the *spherical building*, the *affine building* and the *building at infinity*, and we introduce the *retraction centered at* $-\infty$.

In section 4, we define our main objects, the *combinatorial galleries*. In section 5, we associate a dimension to a positively folded gallery and we introduce the *LS-galleries*.

To prove the majoration of the dimension of a positively folded gallery, we introduce in section 6 the folding operators, which are appropriate analogues of the root operators for the path model of a representation. We also give a combinatorial characterization of the LS-galleries, and we give the character formula for $V(\lambda)$ in terms of LS-galleries.

In section 7, we discuss the connection between a Demazure–Hansen–Bott–Samelson resolution of the Schubert variety X_λ and galleries of a fixed type in the Tits–building. The last sections are devoted to the construction of the stratification of the orbit $\mathcal{G}_\lambda$ indexed by the positively folded galleries and the proof of the connection between MV-cycles and strata corresponding to LS–galleries.

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1 The Affine Grassmannian

Let G be a connected semisimple complex algebraic group. For a $\mathbb{C}$-algebra $\mathcal{R}$ let $G(\mathcal{R})$ be the set of $\mathcal{R}$–rational points of G, i.e., the set of algebra homomorphisms from the coordinate ring $\mathbb{C}[G] \to \mathcal{R}$. Then $G(\mathcal{R})$ comes naturally equipped again with a group structure, for example for $G = SL_n(\mathbb{C})$, we can identify $SL_n(\mathcal{R})$ with the set of $n \times n$–matrices with entries in $\mathcal{R}$ and determinant 1. Similarly, by embedding $G \hookrightarrow SL_n(\mathbb{C})$, we can identify $G(\mathcal{R})$ naturally with a subgroup of $SL_n(\mathcal{R})$.

Denote $\mathcal{O} = \mathbb{C}[[t]]$ the ring of formal power series in one variable and let $\mathcal{K} = \mathbb{C}((t))$ be its fraction field, the field of formal Laurent series. Denote $v : \mathcal{K}^* \to \mathbb{Z}$ the standard valuation on $\mathcal{K}$ such that $\mathcal{O} = \{f \in \mathcal{K} \mid v(f) \geq 0\}$. The *loop group* $G(\mathcal{K})$ is the set of $\mathcal{K}$–valued points of G, we denote by $G(\mathcal{O})$ its subgroup of $\mathcal{O}$–valued points. The latter has a decomposition as a semi-direct product $G \ltimes G^1(\mathcal{O})$, where we view $G \subset G(\mathcal{O})$ as the subgroup of constant loops and $G^1(\mathcal{O})$ is the subgroup of elements congruent to the identity modulo t. Note that we can describe $G^1(\mathcal{O})$ also as the image of $(\mathrm{Lie}\, G)\otimes_{\mathbb{C}} t\mathbb{C}[[t]]$ via the exponential map. As a set, the *affine grassmannian* $\mathcal{G}$ is the quotient

$$\mathcal{G} = G(\mathcal{K})/G(\mathcal{O}).$$

Note that $G(\mathcal{K})$ and $\mathcal{G}$ are *ind*–schemes and $G(\mathcal{O})$ is a group scheme (see [4], [15], [16], [22]). There is a model of $\mathcal{G}$ due to Lusztig which describes $\mathcal{G}$ as an increasing union of finite dimensional complex projective varieties $\mathcal{G}^{(n)}$, where each $\mathcal{G}^{(n)}$ is a subvariety of some finite dimensional Grassmann variety.

Fix a *maximal torus* $T \subset G$ and *Borel subgroups* $B, B^- \subset G$ such that $B \cap B^- = T$. We denote $\langle\cdot,\cdot\rangle$ the non–degenerate pairing between the *character group* $X := \mathrm{Mor}\,(T, \mathbb{C}^*)$ of T and the group $X^\vee := \mathrm{Mor}\,(\mathbb{C}^*, T)$ of *co-characters*. We identify the lattice $X^\vee$ with the quotient $T(\mathcal{K})/T(\mathcal{O})$, so we use the same symbol λ for the co-character and the point in $\mathcal{G}$. Let now $p : G' \to G$ be an isogeny with G' being simply connected. Then $G'(\mathcal{O}) \simeq G' \ltimes (G')^1(\mathcal{O})$, and since $(G')^1(\mathcal{O})$ and $G^1(\mathcal{O})$ are naturally isomorphic, we see that the natural map $p_{\mathcal{O}} : G'(\mathcal{O}) \to G(\mathcal{O})$ is surjective and has the same kernel as p. Let X' and $X'^\vee$ be the character group respectively group of co-characters of G' for a maximal torus $T' \subset G'$ such that $p(T') = T$, then $p : T' \to T$ induces an inclusion $X'^\vee \hookrightarrow X^\vee$.

The quotient $X^\vee/X'^\vee$ measures the difference between $\mathcal{G}$ and the affine grassmannian $\mathcal{G}' = G'(\mathcal{K})/G'(\mathcal{O})$. In fact, $\mathcal{G}'$ is connected, and the connected components of $\mathcal{G}$ are indexed by $X^\vee/X'^\vee$. The natural maps $p_{\mathcal{K}} : G'(\mathcal{K}) \to G(\mathcal{K})$ and $p_{\mathcal{O}} : G'(\mathcal{O}) \to G(\mathcal{O})$ induce a $G'(\mathcal{K})$–equivariant inclusion $\mathcal{G}' \hookrightarrow \mathcal{G}$, which is an isomorphism onto the component of $\mathcal{G}$ containing the class of 1. Now $G'(\mathcal{K})$ acts via $p_{\mathcal{K}}$ on all of $\mathcal{G}$, and each connected component is a homogeneous space for $G'(\mathcal{K})$, isomorphic to $G'(\mathcal{K})/\mathcal{Q}$ for some parahoric subgroup $\mathcal{Q}$ of $G'(\mathcal{K})$ which is conjugate to $G(\mathcal{O})$ by an outer automorphism (see 3.3 below, for a definition of parahoric subgroups).

So to study $G(\mathcal{O})$–orbits on $G(\mathcal{K})/G(\mathcal{O})$ for G semisimple, without loss of generality we may assume that G is simply connected, but we have to investigate more generally $G(\mathcal{O})$–orbits on $G(\mathcal{K})/\mathcal{Q}$ for all parahoric subgroups $\mathcal{Q} \subset G(\mathcal{K})$ conjugate to $G(\mathcal{O})$ by an outer automorphism.

2 The affine Kac-Moody group

In the following let G be a simply connected semisimple algebraic group. Let again $\mathcal{O} = \mathbb{C}[[t]]$ denote the ring of formal power series in one variable and let $\mathcal{K} = \mathbb{C}((t))$ be its fraction field.

The *rotation operation* $\gamma : \mathbb{C}^* \to \mathrm{Aut}\,(\mathcal{K})$, $\gamma(z)\big(f(t)\big) = f(zt)$ gives rise to group automorphisms $\gamma_G : \mathbb{C}^* \to \mathrm{Aut}\,(G(\mathcal{K}))$, we denote $\mathcal{L}(G(\mathcal{K}))$ the semidirect product $\mathbb{C}^* \ltimes G(\mathcal{K})$. The rotation operation on $\mathcal{K}$ restricts to an operation of $\mathcal{O}$ and hence we have a natural subgroup $\mathcal{L}(G(\mathcal{O})) := \mathbb{C}^* \ltimes G(\mathcal{O})$ (for this and the following see [15], Chapter 13).

Let $\hat{\mathcal{L}}(G)$ be the affine Kac-Moody group associated to the affine Kac–Moody algebra

$$\hat{\mathcal{L}}(\mathfrak{g}) = \mathfrak{g} \otimes \mathcal{K} \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where $0 \to \mathbb{C}c \to \mathfrak{g} \otimes \mathcal{K} \oplus \mathbb{C}c \to \mathfrak{g} \otimes \mathcal{K} \to 0$ is the universal central extension of the *loop algebra* $\mathfrak{g} \otimes \mathcal{K}$ and d denotes the scaling element. We have corresponding exact sequences also on the level of groups, i.e., $\hat{\mathcal{L}}(G)$ is a central extension of $\mathcal{L}(G(\mathcal{K}))$

$$1 \to \mathbb{C}^* \to \hat{\mathcal{L}}(G) \xrightarrow{\pi} \mathcal{L}(G(\mathcal{K})) \to 1$$

(see [15], Chapter 13). Denote $\mathcal{P} \subset \hat{\mathcal{L}}(G)$ the "parabolic" subgroup $\pi^{-1}(\mathcal{L}(G(\mathcal{O})))$. We have four incarnations of the affine grassmannian:

$$\mathcal{G} = G(\mathcal{K})/G(\mathcal{O}) = \mathcal{L}(G(\mathcal{K}))/\mathcal{L}(G(\mathcal{O})) = \hat{\mathcal{L}}(G)/\mathcal{P} = G(\mathbb{C}[t, t^{-1}])/G(\mathbb{C}[t]). \qquad (1)$$

We consider now the various maximal tori and Weyl groups. Let $N = N_G(T)$ be the normalizer in G of the fixed maximal torus $T \subset G$, we denote by W the *Weyl group* N/T of G. Let $N_\mathcal{K}$ be the subgroup of $G(\mathcal{K})$ generated by N and $T(\mathcal{K})$, let $\overline{T}$ be the *standard maximal torus* $\mathbb{C}^* \times T$ in $\mathcal{L}(G(\mathcal{K}))$ and denote by $\overline{N} \subset \mathcal{L}(G(\mathcal{K}))$ the extension of $N_\mathcal{K}$. Finally, let $\mathcal{T} \subset \hat{\mathcal{L}}(G)$ be the standard maximal torus (such that $\pi(\mathcal{T}) \subset \overline{T}$), and let $\mathcal{N}$ be its normalizer in $\hat{\mathcal{L}}(G)$, then we get 3 incarnations of the affine Weyl group:

$$W^\mathfrak{a} = N_\mathcal{K}/T \simeq \overline{N}/\overline{T} \simeq \mathcal{N}/\mathcal{T}$$

Let $ev : G(\mathcal{O}) \to G$ and $ev_\mathcal{L} : \mathcal{L}(G(\mathcal{O})) \to \mathbb{C}^* \times G$ be the evaluation maps at $t = 0$, and let $\mathcal{B}_\mathcal{O} = ev^{-1}(B)$ respectively $\mathcal{B}_\mathcal{L} = ev_\mathcal{L}^{-1}(\mathbb{C}^* \times B)$ be the corresponding Iwahori subgroups, and let $\mathcal{B} = \pi^{-1}(B_\mathcal{L}) \subset \hat{\mathcal{L}}(G)$ be the "Borel" subgroup. We have the corresponding Bruhat decompositions, and note that the pair $(\mathcal{B}_\mathcal{O}, N_\mathcal{K})$ is a BN–pair in the loop group $G(\mathcal{K})$, and $(\mathcal{B}, \mathcal{N})$ is a BN–pair in the affine Kac–Moody group $\hat{\mathcal{L}}(G)$.

$$G(\mathcal{K}) = \bigcup_{w \in W^\mathfrak{a}} \mathcal{B}_\mathcal{O} w \mathcal{B}_\mathcal{O}, \quad \mathcal{L}(G(\mathcal{K})) = \bigcup_{w \in W^\mathfrak{a}} \mathcal{B}_\mathcal{L} w \mathcal{B}_\mathcal{L}, \quad \hat{\mathcal{L}}(G) = \bigcup_{w \in W^\mathfrak{a}} \mathcal{B} w \mathcal{B}.$$

Using similar identifications as in (1), whenever appropriate we may replace the study of $G(\mathcal{O})$–orbits on $G(\mathcal{K})/\mathcal{Q}$, where $\mathcal{Q}$ is a parahoric subgroup of $G(\mathcal{K})$ conjugate to $G(\mathcal{O})$ by an outer automorphism of $G(\mathcal{K})$, by the study of $\mathcal{P}$–orbits on $\hat{\mathcal{L}}(G)/\hat{\mathcal{Q}}$, where $\hat{\mathcal{Q}} \subset \hat{\mathcal{L}}(G)$ is a parabolic subgroup of $\hat{\mathcal{L}}(G)$, conjugate to $\mathcal{P}$ by a diagram automorphism of $\hat{\mathcal{L}}(G)$. These orbits correspond exactly to each other because the kernel of π acts trivially on $\hat{\mathcal{L}}(G)/\hat{\mathcal{Q}} = \mathcal{L}(G(\mathcal{K}))/\pi(\hat{\mathcal{Q}})$, so $\mathcal{L}(G(\mathcal{O}))$ acts naturally on $\hat{\mathcal{L}}(G)/\mathcal{Q}$, and the $\mathcal{L}(G(\mathcal{O}))$–orbits and the $G(\mathcal{O})$–orbits in $\hat{\mathcal{L}}(G)/\hat{\mathcal{Q}}$ obviously coincide since by the Bruhat decomposition the orbits are parameterized by the same T– respectively $\overline{T}$– respectively $\mathcal{T}$–fixed points in $G(\mathcal{K})/\mathcal{Q} = \hat{\mathcal{L}}(G)/\hat{\mathcal{Q}}$.

3 Buildings, roots and characters

From an abstract point of view, a building $\mathcal{J}$ is a simplicial complex covered by some subcomplexes, called apartments, such that, each apartment is a Coxeter complex, any two simplices of $\mathcal{J}$ are always contained in an apartment, and given two apartments A and A' with a common maximal simplex, then there is an isomorphism $A \to A'$ fixing every simplex in $A \cap A'$ (see for example [7] or [28]).

These properties can be used to define retractions $r : \mathcal{J} \to A$. We will use the building theory later because one can identify a $G(\mathcal{O})$–orbit in $G(\mathcal{K})/\mathcal{Q}$ with a subset of the affine building (see 3.3) associated to G. In particular, the retraction of the building onto an apartment will be the main tool to identify certain strata of a $G(\mathcal{O})$–orbit with combinatorial objects in the apartment. Note that the apartment is in our case nothing else then the Coxeter complex of the affine Weyl group.

In this section we fix some notation and recall for the convenience of the reader some general facts on building theory. As references we suggest [7], [8], [24] and/or [28].

3.1 Roots and characters

We denote $\langle\cdot,\cdot\rangle$ the non–degenerate pairing between the character group $X := \mathrm{Mor}\,(T, \mathbb{C}^*)$ of T and its group $X^\vee := \mathrm{Mor}\,(\mathbb{C}^*, T)$ of co-characters. Let $\Phi \subset X$ be the root system of the pair (G, T), and, corresponding to the choice of B, denote Φ^+ the set of positive roots, let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be the set of simple roots, and let ρ be half the sum of the positive roots. Let $\Phi^\vee \subset X^\vee$ be the dual root system, together with a bijection $\Delta \to \Delta^\vee$, $\alpha \mapsto \alpha^\vee$. We denote $R^\vee_+$ the submonoide of the coroot lattice $R^\vee$ generated by the positive coroots $\Phi^\vee_+$. We define on $X^\vee$ a partial order by setting $\lambda \succ \nu \Leftrightarrow \lambda - \nu \in R^\vee_+$, and let $X^\vee_+$ be the cone of dominant co-characters.

$$X^\vee_+ := \{\lambda \in X^\vee \mid \langle\lambda, \alpha\rangle \geq 0 \,\forall\, \alpha \in \Phi^+\}$$

In the following, we will deal with three buildings associated to the simply connected complex semisimple group G. The first building we consider is the

3.2 Spherical Building

Let us denote by $\mathcal{J}^s$ the set of all the parabolic subgroups of G. The opposite relation of the inclusion between parabolic subgroups endows this set with a structure of simplicial complex. The maximal simplices given by the Borel subgroups will be called "spherical" chambers, the others, simplices or "spherical" faces. We will often drop the term "spherical" when no confusion may arise.

For every maximal torus T in G, we define an apartment in $\mathcal{J}^s$, by letting $\mathbb{A}^s(T)$ be the set of all the parabolic subgroups that contain T. Together with those apartments, the simplicial complex $\mathcal{J}^s$ is a building (see for example [28] or [7]). The apartments are isomorphic to the Coxeter complex $C(W, S)$ defined by the Coxeter presentation of the Weyl group W. The existence of the retractions in the spherical building is equivalent to the Bruhat decompositions of the group G.

Example 1. Denote by $\mathfrak{C}_f$ the chamber corresponding to the Borel subgroup B. Then the retraction $r_{\mathfrak{C}_f} : \mathcal{J}^s \to \mathbb{A}^s$ onto the apartment $\mathbb{A}^s$ of center $\mathfrak{C}_f$ is defined as follows: Given a chamber $F_{B'}$ in the building associated to a Borel subgroup B', by the Bruhat decomposition be can find $b \in B$ and $w \in W$ such that $B' = bwB/B$ in G/B. We set $r_{\mathfrak{C}_f}(F_{B'}) := F_{wB}$. This map is actually defined on any face since there exists only one parabolic subgroup of a given type containing a given Borel subgroup.

3.3 Affine Building

We can identify in a similar way the affine building $\mathcal{J}^{\mathfrak{a}}$, as a set, with the set of all the parahoric subgroups of $G(\mathcal{K})$, that is the set of all the subgroups of $G(\mathcal{K})$ that contain a conjugate of $\mathcal{B}_{\mathcal{O}}$. The construction below differs from the one above, but, with the appropriate changes, it would apply in the same way also to the spherical building.

Set $\mathcal{A} := X^\vee \otimes_{\mathbb{Z}} \mathbb{R}$, the affine Weyl group $W^{\mathfrak{a}}$ acts on $\mathcal{A}$ as an *affine reflection group*. Let $\mathtt{H}^{\mathfrak{a}} \subset \mathcal{A}$ be the set of affine reflection hyperplanes for the action of $W^{\mathfrak{a}}$ (i.e., the affine reflections with respect to these hyperplanes generate $W^{\mathfrak{a}}$ and the set $\mathtt{H}^{\mathfrak{a}}$ is stable under the action of $W^{\mathfrak{a}}$). The connected components of $\mathcal{A} - \mathtt{H}^{\mathfrak{a}}$ are called *open alcoves*, the closure of such a component is called a *closed alcove* or just *alcove*.

The hyperplanes in $\mathtt{H}^{\mathfrak{a}}$ are all of the form $\mathtt{H}_{\beta,m} = \{a \in \mathcal{A} \mid \langle a, \beta\rangle = m\}$ for some positive root $\beta \in \Phi^+$ and $m \in \mathbb{Z}$, we denote $s_{\beta,m}$ the corresponding affine reflection. The associated closed affine halfspaces are denoted $\mathtt{H}^+_{\beta,m} = \{a \in \mathcal{A} \mid \langle a, \beta\rangle \geq m\}$, resp. $\mathtt{H}^-_{\beta,m} = \{a \in \mathcal{A} \mid \langle a, \beta\rangle \leq m\}$. We use the notation $\mathtt{H}^{+,o}_{\beta,m}$, respectively $\mathtt{H}^{-,o}_{\beta,m}$, for the corresponding open affine halfspaces.

Definition 1. By a *face* F we mean a subset of $\mathcal{A}$ obtained as the intersection of closed affine halfspaces and affine hyperplanes, the intersection running over all pairs (β, m), $\beta \in \Phi^+$, $m \in \mathbb{Z}$. By the corresponding *open face* F^o we mean the subset of F obtained when replacing the closed affine halfspaces in the definition of F by the corresponding open affine halfspaces.

The open faces define a partition of $\mathcal{A}$: we call two elements $x, y \in \mathcal{A}$ equivalent if for all pairs (β, m), x and y are both in either $\mathtt{H}_{\beta,m}$, or $\mathtt{H}^{+,o}_{\beta,m}$, or $\mathtt{H}^{-,o}_{\beta,m}$. The equivalence classes of this relation are the open faces F^o, and the closed face F is the closure of F^o in the affine subspace $\langle F^o\rangle_{\text{aff}}$ spanned by F^o.

We call $\langle F^o\rangle_{\text{aff}} = \langle F\rangle_{\text{aff}}$ the *support* of the (open) face, the *dimension* of the face is the dimension of its support. So the alcoves are the faces of maximal dimension. Let F, C be two faces. We say that F is a face of C if F is defined by changing some inequalities in the definition of C into equalities. A *wall of an alcove* is the support of a codimension one face. In general, instead of the term hyperplane we use often the term *wall*, which is more common in the language of buildings. The following is well-known:

Theorem 1. *The affine Weyl group $W^{\mathfrak{a}}$ acts simply transitive on the set of all alcoves. The fundamental alcove $\Delta_f = \{\nu \in \mathcal{A} \mid 0 \leq \langle x, \beta \rangle \leq 1 \,\forall\, \beta \in \Phi^+\}$ is a fundamental domain for the action, and $W^{\mathfrak{a}}$ is generated by the affine reflections*

$$S^{\mathfrak{a}} = \{s_{\beta,m} \mid \mathtt{H}_{\beta,m} \text{ is a wall for } \Delta_f\}$$

Denote by $S \subset S^{\mathfrak{a}}$ the set of reflections $S = \{s_\beta := s_{\beta,0} \mid \mathtt{H}_{\beta,0} \text{ is a wall for } \Delta_f\}$; we can describe S also as $S^{\mathfrak{a}}(0) = \{s_{\beta,m} \in S^{\mathfrak{a}} \mid 0 \subset \mathtt{H}_{\beta,m}\}$. More generally, for a face F of Δ_f let

$$S^{\mathfrak{a}}(F) = \{s_{\beta,m} \in S^{\mathfrak{a}} \mid F \subset \mathtt{H}_{\beta,m}\}.$$

We call $S^{\mathfrak{a}}(F)$ the *type of* F, so $S^{\mathfrak{a}}(0) = S$ and $S^{\mathfrak{a}}(\Delta_f) = \emptyset$. For an *arbitrary face F its type* is defined as the type of F', where F' is the unique face of Δ_f such that $F = w(F')$ for some $w \in W^{\mathfrak{a}}$.

Remark 1. The alcoves are actually the chambers of the *Coxeter complex* $\mathcal{A}^{\mathfrak{a}}$ associated to the Coxeter group $(W^{\mathfrak{a}}, S^{\mathfrak{a}})$, but since we look at the same time also at the spherical complex, to not confuse the "affine" and the spherical chambers, we prefer the term alcove for the faces of maximal dimension.

We view the (spherical) Weyl group $W \subset W^{\mathfrak{a}}$ as the subgroup generated by the reflections in S. By an *open chamber* we mean always an open spherical chamber, i.e., a connected component of $\mathcal{A} - \bigcup_{\beta\in\Phi^+} \mathtt{H}_{\beta,0}$, and a *chamber* is the closure of such a connected component. We have the *dominant chamber* $\mathfrak{C}_f$ corresponding to the choice of the Borel subgroup B and the *anti-dominant chamber* $-\mathfrak{C}_f$.

Definition 2. A *sector* $\mathfrak{s}$ in $\mathcal{A}$ is a $W^{\mathfrak{a}}$–translate of a chamber. Two sectors $\mathfrak{s}, \mathfrak{s}'$ *are called equivalent* if there exists a third sector $\mathfrak{s}''$ in the intersection: $\mathfrak{s}'' \subset \mathfrak{s} \cap \mathfrak{s}'$.

Lemma 1. *The equivalence classes of sectors are in one-to-one correspondence with the spherical chambers, i.e., in every class there exists a unique spherical chamber.*

For a root $\beta \in \Phi$ let $\mathfrak{g}_\beta \subset \mathfrak{g}$ be the root subspace of the complex Lie algebra $\mathfrak{g} = \operatorname{Lie} G$ and fix a generator X_β. Via the exponential map $\exp : \mathbb{C} \to G$, $x \mapsto \exp(xX_\beta)$ we get a one-dimensional unipotent subgroup U_β normalized by T.

Definition 3. For a real number r let $U_{\beta,r} \subset U_\beta(\mathcal{K})$ be the unipotent subgroup

$$U_{\beta,r} = \{1\} \cup \Big\{exp(X_\beta \otimes f) \mid f \in \mathcal{K}^*, v(f) \geq r\Big\}.$$

For a non-empty subset $\Omega \subset \mathcal{A}$ let $\ell_\beta(\Omega) = -\inf_{x\in\Omega} \beta(x)$. We attach to Ω a subgroup of $G(\mathcal{K})$ by setting

$$U_\Omega := \langle U_{\beta,\ell_\beta(\Omega)} \mid \beta \in \Phi\rangle. \tag{2}$$

Example 2. If $\Omega = 0$, then we have $U_0 = G(\mathcal{O})$, and if $w \in W^{\mathfrak{a}}$, then $U_{w(\Omega)} = n_w U_\Omega n_w^{-1}$ for any representative $n_w \in N(\mathcal{K})$ of w.

To define the affine building $\mathcal{J}^{\mathfrak{a}}$, let $\sim$ be the relation on $G(\mathcal{K}) \times \mathcal{A}$ defined by:

$$(g, x) \sim (h, y) \quad \text{if } \exists\, n \in N(\mathcal{K}) \text{ such that } nx = y \text{ and } g^{-1}hn \in U_x.$$

Definition 4. The *affine building* $\mathcal{J}^{\mathfrak{a}} := G(\mathcal{K}) \times \mathcal{A}/\sim$ associated to G is the quotient of $G(\mathcal{K}) \times \mathcal{A}$ by "$\sim$". The building $\mathcal{J}^{\mathfrak{a}}$ comes naturally equipped with a $G(\mathcal{K})$–action $g \cdot (h, y) := (gh, y)$ for $g \in G(\mathcal{K})$ and $(h, y) \in \mathcal{J}^{\mathfrak{a}}$.

The map $\mathcal{A} \to \mathcal{J}^{\mathfrak{a}}$, $x \mapsto (1, x)$ is injective and $N(\mathcal{K})$ equivariant, we will identify in the following $\mathcal{A}$ with its image in $\mathcal{J}^{\mathfrak{a}}$.

The stabilizer $P_{(g,y)}$ of any point is a parahoric subgroup. In fact, for $x \in \mathcal{A}$ it is the parahoric subgroup generated by the stabilizer $N_x = \{n \in N(\mathcal{K}) \mid nx = x\}$ and U_x:

$$P_x = \langle U_x,\ N_x \rangle.$$

Example 3. If $x \in \Delta_f^o$ is in the open face of the fundamental alcove Δ_f, then $P_x = \mathcal{B}_\mathcal{O}$. More generally, let F be a face of the fundamental alcove and denote by $W^{\mathfrak{a}}(F) \subset W^{\mathfrak{a}}$ the subgroup generated by its type $S^{\mathfrak{a}}(F)$. If $x \in F^o$ is an element in the corresponding open face, then $P_x = P_F$ is the corresponding *standard parahoric subgroup of type* F:

$$P_F := \bigcup_{w \in W^{\mathfrak{a}}(F)} \mathcal{B}_\mathcal{O} w \mathcal{B}_\mathcal{O}.$$

Definition 5. The subsets of $\mathcal{J}^{\mathfrak{a}}$ of the form $g\mathcal{A}$ are called *apartments.*

Lemma 2. *For any $g \in G(\mathcal{K})$ and $x \in \mathcal{A} \cap g^{-1}\mathcal{A}$, there exists a $n \in N(\mathcal{K})$ such that $gx = nx$, or, in other words, $G(\mathcal{K})x \cap \mathcal{A} = N(\mathcal{K})x$ for $x \in \mathcal{A}$.*

So we can extend the partition of $\mathcal{A}$ into open faces into a partition of $\mathcal{J}^{\mathfrak{a}}$ by calling a subset F^o of $\mathcal{J}^{\mathfrak{a}}$ an open face if it is of the form gF'^o for some face $F' \in \mathcal{A}$. Similarly, we call a subset of $\mathcal{J}^{\mathfrak{a}}$ a face, an alcove, a sector etc. if it is of the form gF, $g\Delta$, $g\mathfrak{s}$ etc. for some face, alcove, sector etc. in $\mathcal{A}$. Note that for any non-empty set $\Omega \subset \mathcal{A}$ the group U_Ω acts transitively on the set of all apartments which contain Ω.

We have a Bruhat decomposition $G(\mathcal{K}) = U_x N(\mathcal{K}) U_y$ for any pair of elements $x, y \in \mathcal{A}$. Another important fact from the theory of buildings is that any two faces or any two alcoves are contained in a common apartment, and if $\mathfrak{s}, \mathfrak{s}'$ are two sectors, then there exist subsectors $\mathfrak{s}_1 \subset \mathfrak{s}$ and $\mathfrak{s}'_1 \subset \mathfrak{s}'$ such that $\mathfrak{s}_1$ and $\mathfrak{s}'_1$ are contained in a common apartment, and if $\mathfrak{s}$ is a sector and Δ is an alcove, then there exists a subsector $\mathfrak{s}_1 \subset \mathfrak{s}$ such that Δ and $\mathfrak{s}_1$ are in a common apartment.

We see that on the level of sets the affine building $\mathcal{J}^{\mathfrak{a}}$ is the disjoint union of $G(\mathcal{K})/P_F \times F^o$'s, where the F's are running through the set of faces of the fundamental alcove.

3.4 The Building at Infinity and the Retraction $r_{-\infty}$

The last building we introduce is called the *spherical building at infinity*, we will denote it by $\mathcal{J}^\infty$. We refer to [7] or [24] for a precise definition, we recall quickly its construction and the properties which we will need.

The apartments for $\mathcal{J}^\infty$ are the same as for $\mathcal{J}^\mathfrak{a}$, only the structure of the complex is now different. The chambers of the complex in an apartment are the equivalence classes of sectors. So the structure is similar to that of the spherical complex $\mathcal{J}^s$, only that one does not have anymore a preferred vertex for the "Weyl chambers". Still, the apartments of $\mathcal{J}^\infty$ are isomorphic to the Coxeter complex $C(W,S)$ of the Weyl group W. Since the equivalence classes of sectors are in one-to-one correspondence with the spherical chambers, it makes sense to use the following notation:

Definition 6. *The equivalence class of sectors of the anti-dominant Weyl chamber $-\mathfrak{C}_f$ is called the anti-dominant chamber at $-\infty$ and is denoted $\mathfrak{C}_{-\infty}$.*

For any alcove Δ in the apartment $\mathcal{A} \hookrightarrow \mathcal{J}^\mathfrak{a}$, one can define a chamber complex map $r_{\Delta,\mathcal{A}} : \mathcal{J}^\mathfrak{a} \to \mathcal{A}$, called the retraction onto $\mathcal{A}$ of center Δ. The retraction has the following two properties, which in addition characterize the map uniquely: the first property is that for any face F of Δ, $r^{-1}_{\Delta,\mathcal{A}}(F) = \{F\}$.

To recall the second property, we define a distance on the set of alcoves. Two alcoves are called *adjacent* if they have a common codimension one face. A *gallery of alcoves of length* r is a sequence $(\Delta_0,\dots,\Delta_r)$ of alcoves such that Δ_i and Δ_{i-1} are adjacent for $1 \le i \le r$. The distance $d(\Delta,\Delta')$ of two alcoves in $\mathcal{J}^\mathfrak{a}$ is the minimal length of a gallery $(\Delta_0,\dots,\Delta_r)$ joining Δ and Δ' (i.e., $\Delta_0 = \Delta$ and $\Delta_r = \Delta'$).

A gallery of alcoves $(\Delta_0,\dots,\Delta_r)$ is called *minimal* if the length is equal to the distance $d(\Delta_0,\Delta_r)$ between the first and the last alcove. Recall ([8], section 2.3.6), two alcoves lie always in an apartment containing both, and any apartment that contains both also contains all minimal galleries joining the two.

The second important property of the retraction is that for any alcove Δ', $d(\Delta,\Delta') = d\big(\Delta, r_{\Delta,\mathcal{A}}(\Delta')\big)$, so the map preserves the distance from its center. Further, $r_{\Delta,\mathcal{A}}$ restricts to an isomorphism of chamber complexes $g\mathcal{A} \simeq \mathcal{A}$, for any apartment $g\mathcal{A}$, $g \in G(\mathcal{K})$.

The image $r_{\Delta,\mathcal{A}}(\Delta')$ of an alcove Δ' depends of course on the alcove Δ, but the image of Δ' becomes in fact stable if Δ is only "far away" enough from Δ'. To be more precise, consider an alcove $\Delta' \in \mathcal{J}^\mathfrak{a}$ and let $-\mathfrak{C}_f$ be the anti-dominant chamber. Then there exists a sector $\mathfrak{s}$ equivalent to $-\mathfrak{C}_f$ such that $\mathfrak{s}$ and Δ' lie in a common apartment $g\mathcal{A}$. Let Δ be an alcove in the sector and recall that $r_{\Delta,\mathcal{A}}$ restricts to an isomorphism of chamber complexes $g\mathcal{A} \simeq \mathcal{A}$, which fixes of course the common sector $\mathfrak{s}$. It follows (see for example, [24] §9.4) that $r_{\Delta,\mathcal{A}}(\Delta')$ is independent of the choice of $\Delta \subset \mathfrak{s}$.

Definition 7. The map $r_{-\infty} : \mathcal{J}^\mathfrak{a} \to \mathcal{A}$, defined by $r_{-\infty}(\Delta') = r_{\Delta,\mathcal{A}}(\Delta')$ for some alcove $\Delta \subset \mathfrak{s}$, where $\mathfrak{s}$ is a sector, equivalent to $-\mathfrak{C}_f$, contained in a common apartment with Δ', is called the *retraction of center* $-\infty$.

The retraction $r_{\Delta,\mathcal{A}}$ can also be expressed in terms of the action of the group U_Δ (see (2)). Recall that the latter operates transitively on the set of apartments containing Δ, and the retraction is in fact the projection $r_\Delta : u\mathcal{A} \to \mathcal{A}$ that maps a face uF, $u \in U_\Delta$, $F \subset \mathcal{A}$, onto F. So the fibres of the retraction r_Δ are exactly the U_Δ–orbits.

Consider the retraction $r_{-\infty}$ of center $-\infty$. For a face $F \in \mathcal{A}$ and a face F' in the fibre, we can find a subsector $\mathfrak{s} \subset -\mathfrak{C}_f$ such that for all $\Delta \in \mathfrak{s}$ there exists an $u \in U_\Delta$ so that $F' = uF$. This means for all $N \gg 0$ we can choose an alcove in the anti-dominant chamber such that $\Delta \subset \mathfrak{s}$ and $\ell_\beta(\Delta) > N$ for all positive roots. This means (compare [8], §6 and §7) that we can actually choose $u \in U^-(\mathcal{K})$, where $U^- \subset B^-$ is the unipotent radical of the opposite Borel subgroup B^-. Summarizing we have the following description of the fibres of $r_{-\infty}$ in terms of the group operation on $\mathcal{J}^{\mathfrak{a}}$:

Proposition 1. *The fibres of $r_{-\infty} : \mathcal{J}^{\mathfrak{a}} \to \mathcal{A}$ are the $U^-(\mathcal{K})$–orbits on $\mathcal{J}^{\mathfrak{a}}$.*

4 Generalized Galleries

In the following we need a more general version of a gallery as the one in section 3, we will essentially follow [9].

Definition 8. A *generalized gallery in the affine building* is a sequence of faces γ in $\mathcal{J}^{\mathfrak{a}}$

$$\gamma = (\Gamma'_0 \subset \Gamma_0 \supset \Gamma'_1 \subset \cdots \subset \Gamma_{j-1} \supset \Gamma'_j \subset \Gamma_j \supset \cdots \supset \Gamma'_p \subset \Gamma_p \supset \Gamma'_{p+1}),$$

such that

- the first and the last faces, Γ'_0 and Γ'_{p+1}, in other words, the *source* and the *target* of γ, are vertices of $\mathcal{J}^{\mathfrak{a}}$,
- the Γ_j's are faces, all of the same dimension,
- the Γ'_j's, for $j = 1, \dots p$, are faces of two consecutive faces, of relative codimension one.

If such a gallery is contained in the apartment $\mathcal{A}$, it will be called a *combinatorial gallery.*

For any subset Ω and any face F contained in an apartment A of the affine building $\mathcal{J}^{\mathfrak{a}}$, we say that a wall $\mathtt{H}$ *separates* Ω *and* F if Ω is contained in the corresponding closed half space and F is a subset of the opposite open half space. Let E and F be two faces in the building. In any apartment A containing both of them, there exists a finite number of walls that separate E and F. We denote this set by $\mathcal{M}(E, F)$. Note, if E is not an alcove, all the walls containing E but not F belong to $\mathcal{M}(E, F)$.

Let $\Delta \supset E$ be an alcove in A, then, by [28], Proposition 2.29, there exists a unique alcove in A, denoted $proj_F(\Delta)$, such that any face of the convex hull of Δ and F containing F is contained in $proj_F(\Delta)$.

Definition 9. The alcove Δ is said to be *at maximal distance to* F if the length of the minimal alcoves galleries between Δ and $proj_F(\Delta)$ is $\sharp\mathcal{M}(E,F)$.

Note, any of these minimal galleries will cross only once each wall of $\mathcal{M}(E,F)$. Such an alcove $\Delta \supset E$ at maximal distance to F is not uniquely determined.

Lemma 3. *The alcoves $\Delta \supset E$ in A at maximal distance to F are conjugate under the stabilizer of $E \cup F$.*

Proof. Let Δ' be another alcove in A, the fixed apartment which contains the two faces E and F. Suppose $\Delta' \supset E$ is at maximal distance to F and Δ and Δ' are adjacent, then the support $\mathtt{H}$ of the common face contains E. Now for any wall $\mathtt{H}' \neq \mathtt{H}$, the relative position of Δ and Δ' is the same, i.e., they lie in the same closed halfspace. Since they lie on different sides of $\mathtt{H}$, the two can be at maximal distance to F at the same time only if $F \subset \mathtt{H}$. In the general case, choose a minimal gallery $(\Delta_0 = \Delta, \ldots, \Delta_r = \Delta')$. Since $E \subset \Delta, \Delta'$, one concludes by the minimality of the gallery that $E \subset \Delta_j$ for all j. Further, for all walls in $\mathcal{M}(E,F)$, Δ and Δ' lie in the same closed halfspace, and hence, by the minimality of the gallery, so do all Δ_j. But this implies that all the Δ_j are at maximal distance to F, which finishes the proof since the Δ_j are pairwise adjacent •

We have also to generalize the notion of a minimal gallery. Roughly speaking, a generalized gallery is called minimal if it can be embedded in a minimal gallery of alcoves. More precisely:

Definition 10. A generalized gallery

$$\gamma = (F_f = \Gamma'_0 \subset \Gamma_0 \supset \Gamma'_1 \subset \cdots \subset \Gamma_{j-1} \supset \Gamma'_j \subset \Gamma_j \supset \cdots \supset \Gamma'_p \subset \Gamma_p \supset F)$$

in the affine building $\mathcal{J}^{\mathfrak{a}}$ is called *minimal* if the following holds: let $\Delta \supset F_f$ be an alcove at maximal distance to F, then

a) there exists a minimal gallery of alcoves between Δ and $proj_F(\Delta)$

$$\mu = (\Delta = \Delta^0_1, ..., \Delta^0_{q_0}, ..., \Delta^j_1, ..., \Delta^j_{q_j}, ..., \Delta^p_1, ..., \Delta^p_{q_p} = proj_F(\Delta)),$$

and for all j the alcoves $\Delta^j_1, ..., \Delta^j_{q_j}$ contain Γ'_j.

b) let A be an apartment containing γ, and denote $\mathtt{H}^A$ the set of affine hyperplanes in A. The set $\mathcal{M}(F_f, F)$ is the disjoint union of the sets

$$\mathcal{H}_j := \{\mathtt{H} \in \mathtt{H}^A \mid \Gamma'_j \subset \mathtt{H},\ \Gamma_j \not\subset \mathtt{H}\},\ j = 0, \ldots, p.$$

Remark 2. A minimal gallery γ is contained in an apartment by condition *a)*, and an apartment containing Δ and $proj_F(\Delta))$ contains also all minimal galleries between the two alcoves. One checks easily that if *b)* holds for one apartment, then it holds for all apartments $A \supset \gamma$.

Remark 3. We will also need to consider minimal galleries of the form $(E \supset F' \subset F)$ in an appartment A. Such a gallery is called *minimal* if the set of walls $\mathcal{M}(E, F)$ is exactly the set of walls that contain F' but not F.

If γ is a minimal gallery (in the sense of Definition 10), then all $(\Gamma_{j-1} \supset \Gamma'_j \subset \Gamma_j)$, $j = 1, \ldots, p$ are minimal.

Remark 4. Using the alcoves at maximal distance, we can reformulate the condition of $(E \supset F' \subset F)$ to be minimal as follows: Let $\mathcal{H}(F', F)$ be the set of walls that contain F' but not F. Then $(E \supset F' \subset F)$ is a minimal gallery in A if for any alcove $\Delta \supset E$ in A at maximal distance to F, any minimal alcoves gallery between Δ and $proj_F(\Delta)$ has length $\sharp\mathcal{H}(F', F)$.

Our main objects will be minimal galleries having the vertex corresponding to $G(\mathcal{O})$ as source. Since all apartments containing this vertex are conjugate under the action of $G(\mathcal{O})$, the following definitions make sense for arbitrary galleries contained in an apartment (recall, if γ is minimal then it is contained in at least one apartment). Our aim is to express the condition on the minimality in terms of the walls crossed by the gallery.

Let $\lambda \in X^\vee$ be a co-character and denote $\mathtt{H}_\lambda := \bigcap_{\langle\lambda,\alpha\rangle=0} \mathtt{H}_{\alpha,0}$ the intersection of all the hyperplanes corresponding to the positive roots orthogonal to λ. Let F_f be the face (of type S) of $\mathcal{A}$ corresponding to the origin of $\mathcal{A}$ and let F_λ be the face corresponding to λ.

Definition 11. A *combinatorial gallery joining* 0 *with* λ is a generalized gallery γ in $\mathcal{A}$ that starts at F_f and ends in F_λ:

$$\gamma = (F_f = \Gamma'_0 \subset \Gamma_0 \supset \Gamma'_1 \subset \cdots \subset \Gamma_{j-1} \supset \Gamma'_j \subset \Gamma_j \supset \cdots \supset \Gamma'_p \subset \Gamma_p \supset F_\lambda).$$

such that the dimension of the large faces is always equal to $\dim \mathtt{H}_\lambda$. The faces Γ_j's will be called the *large faces* of the gallery γ and the Γ'_j's the *small faces* of γ.

Lemma 4. *A combinatorial gallery* $\gamma = (F_f = \Gamma'_0 \subset \Gamma_0 \supset \cdots \supset \Gamma'_p \subset \Gamma_p \supset F_\lambda)$ *joining* 0 *with* λ *is minimal if and only if* $\gamma \subset \mathtt{H}_\lambda$ *and* p *is minimal in the following sense:*

- *for* $j = 0, 1, ..., p$, *let* $\mathcal{H}_j$ *be the set of all the affine hyperplanes* $\mathtt{H} \in \mathtt{H}^{\mathfrak{a}}$ *such that* $\Gamma'_j \subset \mathtt{H}$ *and* $\Gamma_j \not\subset \mathtt{H}$, *then the sets* $\mathcal{H}_j$ *are pairwise distinct and* $\cup_{j\in\{0,\ldots,p-1\}}\mathcal{H}_j = \mathcal{M}(F_f, F_\lambda)$, *the set of all the affine hyperplanes separating any alcove* $\Delta \supset F_f$ *at maximal distance to* F_λ *and the latter.*

Proof. Suppose γ is minimal, then $0, \lambda \in \mathtt{H}_\lambda$ implies $\gamma \subset \mathtt{H}_\lambda$ by minimality, and the second part of the condition follows from the definition of a minimal gallery.

For the reverse implication, we construct the minimal gallery of alcoves inductively. We start with an alcove $\Delta = \Delta^0_1 \subset F_f$ at maximal distance to Γ_0, set $\Delta^0_{q_0} = proj_{\Gamma_0}(\Delta)$. If $p = 0$, then the corresponding minimal gallery joining Δ^0_1 and $\Delta^0_{q_0}$ has length $\sharp\mathcal{H}_0$ and is already the desired minimal gallery of alcoves. If $p \geq 1$, note that $\Delta^0_{q_0}$ is at maximal distance to Γ_1: a wall separating Δ^0_1 and Γ_1 is either an element of $\mathcal{H}_0$ or $\mathcal{H}_1$, so by construction the walls in $\mathcal{H}_1$ separate $\Delta^0_{q_0}$ and Γ_1. But this implies that the length of a minimal gallery joining $\Delta^0_{q_0}$ and $\Delta^1_{q_1} = proj_{\Gamma_1}(\Delta^0_{q_0})$ is $\sharp\mathcal{H}_1$. By repeating the procedure, we obtain the desired minimal gallery containing γ. •

Let $\lambda \in X^\vee_+$ be a dominant co-character and let γ_λ denote a minimal combinatorial gallery joining F_f with F_λ

$$\gamma_\lambda = (F_f \subset \Upsilon_0 \supset \Upsilon'_1 \subset \cdots \subset \Upsilon_{j-1} \supset \Upsilon'_j \subset \Upsilon_j \supset \cdots \supset \Upsilon'_p \subset \Upsilon_p \supset F_\lambda).$$

Because of the minimality assumption, all the faces of γ_λ are contained in the (spherical) dominant chamber $\mathfrak{C}_f$, and Υ_0 is a face of the fundamental alcove Δ_f.

Definition 12. The *gallery of types* associated to γ_λ is the list of the types of the faces in the gallery above:

$$t_{\gamma_\lambda} = type(\gamma_\lambda) = (S = t'_0 \supset t_0 \subset t'_1 \supset \cdots \supset t_{j-1} \subset t'_j \supset t_j \subset \cdots \subset t'_p \supset t_p \subset t_\lambda),$$

where t_λ is the type of the face F_λ and t_j is the type of the face Υ_j, t'_j the type of Υ'_j.

Remark 5. Fix a minimal combinatorial gallery γ_λ joining F_f with F_λ, and consider the set of all generalized galleries δ in the affine building $\mathcal{J}^{\mathfrak{a}}$ of type t_{γ_λ}. Let $\gamma = (F_f = \Gamma'_0 \subset \Gamma_0 \supset \cdots \supset \Gamma'_p \subset \Gamma_p \supset F_\nu)$ be such a gallery, and assume that all $(\Gamma_{j-1} \supset \Gamma'_j \subset \Gamma_j)$, $j = 1, \ldots, p$ are minimal. Then one sees easily that γ is a minimal gallery, the fixed type comes from a minimal gallery and hence forces the gallery to be minimal.

Let $\Gamma(\gamma_\lambda)$ be the set of all the combinatorial galleries of type t_{γ_λ} and of source F_f, i.e., an element $\delta \in \Gamma(\gamma_\lambda)$ is a gallery starting at F_f:

$$\delta = (F_f \subset \Sigma_0 \supset {\Sigma_1}' \subset \cdots \supset \Sigma'_p \subset \Sigma_p \supset F_\nu),$$

such that Σ_j is a face of type t_j of $\mathcal{A}$ and ${\Sigma_j}'$ is of type t'_j and F_ν is of the same type as F_λ. In addition, such a sequence of types gives rise to a sequence of subgroups of $W^{\mathfrak{a}}$

$$W'_0 \supset W_0 \subset W'_1 \supset \cdots \supset W_{j-1} \subset W'_j \supset W_j \subset \cdots \subset W'_p \supset W_p,$$

where for $j = 0, ..., p$, W'_j respectively W_j is the Coxeter subgroup of $W^{\mathfrak{a}}$ generated by the reflections in $t'_j \subset S^{\mathfrak{a}}$ respectively in $t_j \subset S^{\mathfrak{a}}$.

These groups are also the stabilizers in $W^{\mathfrak{a}}$ of the faces of the fundamental alcove of the corresponding type. This provides us with another way to "encode" a gallery of type λ: The stabilizer of the F_f in $W^{\mathfrak{a}}$ is $W = W'_0$, and the stabilizer of Υ_0 is W_0, so Σ_0 is completely determined by an element in W/W_0. Proceeding inductively, we see [9]:

Proposition 2. *The set of all the combinatorial galleries of type t_{γ_λ} and of source F_f, $\Gamma(\gamma_\lambda)$, is in bijection with the quotient*

$$W \times_{W_0} W'_1 \times_{W_1} \cdots \times_{W_{p-1}} W'_p/W_p,$$

of the group $W \times W'_1 \times \cdots \times W'_p$ by the subgroup $W_0 \times W_1 \times \cdots \times W_p$ under the action defined by $(w'_0 w_0, w_0^{-1} w'_1 w_1, ..., w_{p-1}^{-1} w'_p w_p)$.

The image of a gallery γ will be denoted by $\gamma = [\delta_0, \delta_1, ..., \delta_p]$. Because of the definition of the quotient we can (and will) assume that $\delta_j \in W'_j$ is *the unique representative of minimal length of its class in* W'_j/W_j. We will use freely both of the notation for a gallery.

Example 4. The gallery γ_λ can be written as $\gamma_\lambda = [1, \tau_1^m, ..., \tau_p^m]$, where each $\tau_j^m \in W'_j$ is the *unique minimal representative of the largest class* (in the induced Bruhat ordering) in W'_j/W_j. The product $\tau_1^m \cdots \tau_p^m \in W^{\mathfrak{a}}$ is a reduced decomposition of the element that sends Δ_f to $proj_{F_\lambda}(\Delta_f)$. All galleries of shape $[\gamma_0, \tau_1^m, ..., \tau_j^m]$ for $\gamma_0 \in W$ (of minimal length modulo W_λ) are minimal.

Let $\delta = [\delta_0, \delta_1, ..., \delta_p] = (F_f \subset \Sigma_0 \supset \Sigma_1{}' \subset \cdots \supset \Sigma'_p \subset \Sigma_p \supset F_\nu) \in \Gamma(\gamma_\lambda)$. The gallery δ is called *folded around the small face* Σ'_j if $\delta_j \neq \tau_j^m$. Such a gallery will be obtained from $[\delta_0, \tau_1^m, ..., \tau_p^m]$ by applying at the places j, where $\delta_j \neq \tau_j^m$, some affine reflections with respect to the affine hyperplanes containing the small faces. I.e., associated to δ we have the following sequence of galleries in $\Gamma(\gamma_\lambda)$:

$$\begin{array}{llll} \gamma_0 = [\delta_0, \tau_1^m, \tau_2^m, ..., \tau_p^m], & \gamma_1 = [\delta_0, \delta_1, \tau_2^m, ..., \tau_p^m], & \gamma_2 = [\delta_0, \delta_1, \delta_2, \tau_3^m, ..., \tau_p^m], \\ \ldots & \gamma_{p-1} = [\delta_0, \delta_1, ..., \delta_{p-1}, \tau_p^m], & \gamma_p = \delta. \end{array}$$

Here γ_1 is obtained from the minimal gallery $\gamma_0 = \delta$ by a folding around Σ'_1, γ_2 is obtained by a folding of γ_1 around Σ'_2, etc.

Definition 13. The gallery $\delta \in \Gamma(\gamma_\lambda)$ is called *positively folded* at Σ'_j if for all the affine hyperplanes $\mathtt{H}$ involved in the sequence of reflections for the folding of γ_{j-1} around Σ'_j to get γ_j, the image of the reflection is "separated" by $\mathtt{H}$ from the anti–dominant chamber $\mathfrak{C}_{-\infty}$ at $-\infty$. We say that the gallery is *positively folded* if all foldings are positive.

Here "*separated*" means that there *exists a representative* $\mathfrak{s}$ of the class $\mathfrak{C}_{-\infty}$ such that the *image of the reflection and* $\mathfrak{s}$ *are separated* by the reflection hyperplane.

Example 5. The minimal galleries are automatically positively folded.

Remark 6. The condition for a combinatorial gallery to be *positively folded* is strongly related to the combinatorial description of *inclusions of Verma modules* for the enveloping algebra $U(\mathrm{Lie}\, G^\vee)$. Let be δ a gallery of type t_{γ_λ} where $\gamma_\lambda = [1, \tau_1^m, \tau_2^m, ..., \tau_p^m]$:

$$\delta = [\delta_0, \delta_1, ..., \delta_p] = (F_f \subset \Sigma_0 \supset \Sigma_1{}' \subset \cdots \supset \Sigma'_p \subset \Sigma_p \supset F_\nu) \in \Gamma(\gamma_\lambda).$$

Let $\gamma_0, \gamma_1, \ldots$ be as above, and suppose that γ_j is obtained from γ_{j-1} using successively the affine reflections with respect to the walls $\mathtt{H}_{\beta_1, m_1}, \ldots, \mathtt{H}_{\beta_t, m_t}$, $\beta_i \in \Phi$. So the target ν_j of γ_j is obtained from the target ν_{j-1} of γ_{j-1} using the affine reflections with respect to these walls. If we shift the target $\hat{\nu}_{j,\Sigma'_j} = \nu_j - \mu_j$ by a point in the open small face $\mu_j \in (\Sigma'_j)^o$ (the actual choice is irrelevant), then this means that $\hat{\nu}_{j-1,\Sigma'_j} = \nu_{j-1} - \mu_j$ is obtained using successively the reflections $s_{\beta_1}, \ldots, s_{\beta_t}$. Since the shifted weights are integral for the roots $\beta_1, \ldots, \beta_t$, the condition for the folding to be positive is equivalent to the condition

$$\hat{\nu}_{j-1,\Sigma'_j} \preceq s_{\beta_1}(\hat{\nu}_{j-1,\Sigma'_j}) \preceq s_{\beta_2}s_{\beta_1}(\hat{\nu}_{j-1,\Sigma'_j}) \preceq \ldots \preceq s_{\beta_t}\cdots s_{\beta_1}(\hat{\nu}_{j-1,\Sigma'_j}) = \hat{\nu}_{j,\Sigma'_j}.$$

By [3], this is equivalent to the existence of a monomorphism $M(\hat{\nu}_{j-1,\Sigma'_j}-\rho) \hookrightarrow M(\hat{\nu}_{j,\Sigma'_j}-\rho)$ between the corresponding Verma modules of highest weights $\nu_{j-1,\Sigma'_j} - \rho$ and $\hat{\nu}_{j,\Sigma'_j} - \rho$. (See also [20] for the connection of the path model with such inclusions and related bases for representations.)

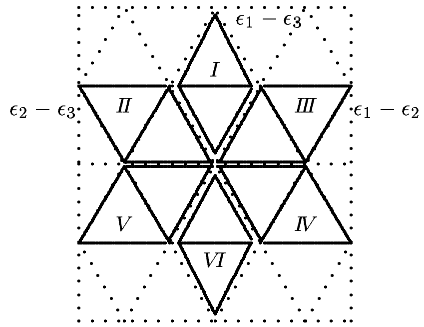
Let $\Gamma^+(\gamma_\lambda)$ denote the subset of $\Gamma(\gamma_\lambda)$ of all *positively folded combinatorial galleries of type* t_{γ_λ}, and for a co-character ν, we denote $\Gamma^+(\gamma_\lambda, \nu)$ the subset of galleries δ in $\Gamma^+(\gamma_\lambda)$ ending in ν, i.e., $\delta = (F_f \subset \Sigma_0 \supset \ldots \subset \Sigma_p \supset F_\nu)$.

Example 6. Consider the group $G = SL_3(\mathbb{C})$, let $S^{\mathfrak{a}} = \{s_0, s_1, s_2\}$ be indexed such that the Weyl group W of G is generated by $\{s_1, s_2\}$. A minimal gallery joining the origin with the highest root β is $\gamma_\beta = [1, s_0]$. The elements in the set $\Gamma^+(\gamma_\beta)$ of all positively folded galleries of the same type and source 0 are:

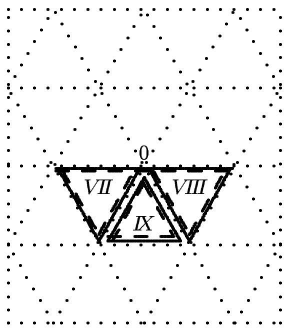
$$I = [1, s_0],\ II = [s_1, s_0],\ III = [s_2, s_0],\ IV = [s_2s_1, s_0],\ V = [s_1s_2, s_0],\ VI = [s_2s_1s_2, s_0]$$

which are the minimal galleries in this set, and

$$VII = [s_1s_2, 1],\ VIII = [s_2s_1, 1],\ IX = [s_2s_1s_2, 1].$$



The minimal galleries



The folded galleries

Note: for each root we have exactly one such gallery ending in this weight (the minimal galleries), and for the zero weight we have three (the non-minimal galleries), which is one too much if one has expected to get the character of the adjoint representation of $SL_3(\mathbb{C})$. We will see in the next section how to refine the choice.

5 Dimension of galleries

We will see that the notion of the dimension of a gallery is related to the dimension of the fibre of the retraction $r_{-\infty}$ with center at $-\infty$ discussed later. Fix a positively folded gallery $\gamma \in \Gamma^+(\gamma_\lambda)$ of type t_{γ_λ}:

$$\gamma = (F_f = \Gamma'_0 \subset \Gamma_0 \supset \Gamma'_1 \subset \cdots \subset \Gamma_{j-1} \supset \Gamma'_j \subset \Gamma_j \supset \cdots \supset \Gamma'_p \subset \Gamma_p \supset F_\nu).$$

For $j = 0, ..., p$, let $\mathcal{H}_j$ be the set of all the affine hyperplanes $\mathtt{H}$ in $\mathtt{H}^{\mathfrak{a}}$ such that $\Gamma'_j \subset \mathtt{H}$ and $\Gamma_j \not\subset \mathtt{H}$. We say that an affine hyperplane $\mathtt{H}$ is a *load-bearing wall* for γ *at* Γ_j if $\mathtt{H} \in \mathcal{H}_j$ and $\mathtt{H}$ separates Γ_j from $\mathfrak{C}_{-\infty}$. Note: for a positively folded gallery all folding hyperplanes are load-bearing walls by definition.

Definition 14. The *dimension* of the gallery $\gamma \in \Gamma^+(\gamma_\lambda)$ is the number of pairs $(\mathtt{H}, \Gamma_j)$ such that $\mathtt{H}$ is a load-bearing wall for γ at Γ_j:

$$\dim \gamma = \sharp\{(\mathtt{H}, \Gamma_j) \mid \mathtt{H} \text{ is a load-bearing wall for } \gamma \text{ at } \Gamma_j\}$$

Example 7. Let λ be a dominant co-character and let γ_λ be a minimal gallery joining 0 with λ. If β is a positive root such that $\langle \lambda, \beta \rangle > 0$, then the affine hyperplanes $\mathtt{H}_{\beta,n}$ for all $0 \leq n < \langle \lambda, \beta \rangle$ are load-bearing walls. It follows that

$$\dim \gamma_\lambda = \sum_{\beta \in \Phi^+} \langle \lambda, \beta \rangle = \langle \lambda, 2\rho \rangle = \langle 2\lambda, \rho \rangle.$$

Next, let $\nu = \tau(\lambda)$ be a Weyl group conjugate of λ (as usual, we identify $\tau \in W/W_\lambda$ with its representative of minimal length), and let γ_τ be the gallery obtained from γ_λ by applying simultaneously τ to all faces of γ_λ. Since $0 = F_f$ is a fixed point for the action of W and $\nu = \tau(\lambda)$, this is a gallery joining 0 with ν. By the minimality of γ_λ, this gallery is also minimal and has no foldings. So, in particular, $\gamma_\tau \in \Gamma^+(\gamma_\lambda)$. One shows easily by decreasing induction on $\langle \nu, \rho \rangle$ that

$$\dim \gamma_\tau = \langle \lambda + \nu, \rho \rangle.$$

Our next aim is to provide such a formula for all $\gamma \in \Gamma^+(\gamma_\lambda)$. Recall that $\Gamma^+(\gamma_\lambda, \nu)$ is the set of all galleries γ in $\Gamma^+(\gamma_\lambda)$ ending in ν, i.e., $\gamma = (F_f \subset \Gamma_0 \supset \cdots \subset \Gamma_{p+1} \supset F_\nu)$.

Proposition 3. *If* $\gamma \in \Gamma^+(\gamma_\lambda, \nu)$*, then* $\dim \gamma \leq \langle \lambda + \nu, \rho \rangle$*.*

Definition 15. A positively folded gallery $\gamma \in \Gamma^+(\gamma_\lambda, \nu)$ is called an *LS-gallery of type* λ if $\dim \gamma = \langle \lambda + \nu, \rho \rangle$.

The proofs of the propositions 3 and 4 will be given in the next section, where we also provide a combinatorial characterization of the LS-galleries. Denote $\Gamma^+_{LS}(\gamma_\lambda, \nu)$ the set of all galleries in $\Gamma^+(\gamma_\lambda, \nu)$ that are LS-galleries. As a consequence of the description we will get the following combinatorial character formula, which is the equivalent in terms of galleries of the path character formula in [17, 18]: Let $V(\lambda)$ be the irreducible complex representation of highest weight λ for the semisimple algebraic group $G^\vee$ (the Langland's dual group of G), and denote $\operatorname{Char} V(\lambda)$ its character:

Proposition 4. $\operatorname{Char} V(\lambda) = \sum_{\nu \in X^\vee} \sharp \Gamma^+_{LS}(\gamma_\lambda, \nu) \exp(\nu)$.

6 Root Operators

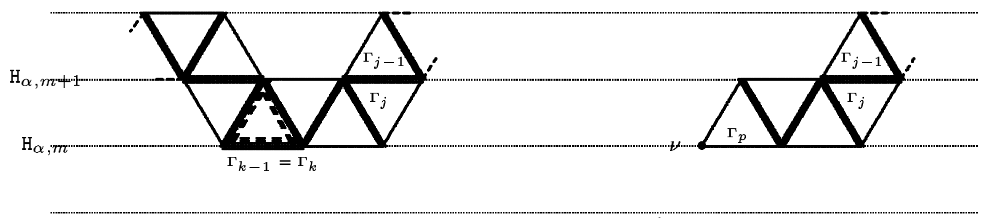
The aim of this section is to prove the dimension formula and to give a combinatorial characterization of the LS-galleries. We define now "folding" operators $f_\alpha, e_\alpha, \widetilde{e}_\alpha$ (for all simple roots) on the set of all combinatorial galleries $\Gamma(\gamma_\lambda)$ of a fixed type. Let λ be a dominant co-character and fix a minimal gallery γ_λ joining the origin with λ.

Let α be a simple root and fix a combinatorial gallery $\gamma \in \Gamma(\gamma_\lambda)$ of type t_{γ_λ}, say

$$\gamma = [\gamma_0, \gamma_1, ..., \gamma_p] = (F_f = \Gamma'_0 \subset \Gamma_0 \supset \Gamma'_1 \subset \cdots \subset \Gamma_p \supset \Gamma'_{p+1} = F_\nu).$$

Let $m \in \mathbb{Z}$ be minimal such that one of the small faces Γ'_k is contained in the hyperplane $\mathtt{H}_{\alpha,m}$, note that $m \leq 0$. The operators e_α and $\widetilde{e}_\alpha$ are different, but the conditions for the operators to be defined are not exclusive, so it might well be that both are defined for a given gallery. Let ν be the target of the gallery.

I) Suppose that $m \leq -1$. Let k be minimal such that $\Gamma'_k \subset \mathtt{H}_{\alpha,m}$, and fix $0 \leq j \leq k$ maximal such that the small face $\Gamma'_j \subset \mathtt{H}_{\alpha,m+1}$ is contained in the hyperplane $\mathtt{H}_{\alpha,m+1}$.



thick lines = small faces Γ'

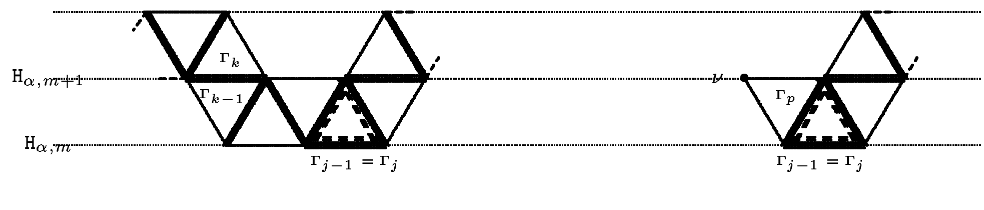
Definition 16. If $\gamma = [\gamma_0, \gamma_1, ..., \gamma_p]$ satisfies I), then let $e_\alpha\gamma$ be the gallery defined by:

$$e_\alpha\gamma = [\delta_0, \delta_1, ..., \delta_p] = (F_f \subset \Delta_0 \supset \Delta'_1 \subset \cdots \supset \Delta'_p \subset \Delta_p \supset F_{\check{\nu}})$$

$$\text{where} \quad \Delta_i = \left\{ \begin{array}{ll} \Gamma_i & \text{for } i \leq j-1, \\ s_{\alpha,m+1}(\Gamma_i) & \text{for } j \leq i \leq k-1, \\ t_{\alpha^\vee}(\Gamma_i) & \text{for } i \geq k. \end{array} \right\} \text{ and } \ t_{\alpha^\vee} = \text{translation by } \alpha^\vee.$$

We define a partial inverse operator f_α to the operator e_α.

II) Suppose that $\langle \nu, \alpha \rangle - m \geq 1$. Let j be maximal such that $\Gamma'_j \subset \mathtt{H}_{\alpha,m}$ and fix $j \leq k \leq p+1$ minimal such that the small face Γ'_k is contained in $\mathtt{H}_{\alpha,m+1}$.



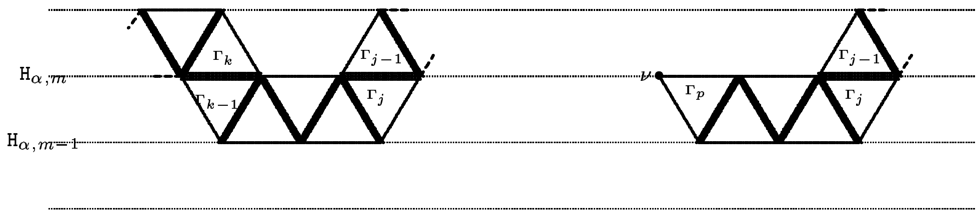
thick lines = small faces Γ'

Definition 17. If γ satisfies II), then let $f_\alpha\gamma$ be the gallery defined by:

$$f_\alpha\gamma = [\gamma_0, \gamma_1, ..., \gamma_p] = (F_f \subset \Delta_0 \supset \Delta'_1 \subset \cdots \supset \Delta'_p \subset \Delta_p \supset F_{\check{\nu}})$$

$$\text{where} \quad \Delta_i = \begin{cases} \Gamma_i & \text{for } i < j, \\ s_{\alpha,m}(\Gamma_i) & \text{for } j \leq i < k, \\ t_{-\alpha^\vee}(\Gamma_i) & \text{for } i \geq k. \end{cases}$$

III) Assume that γ crosses $\mathtt{H}_{\alpha,m}$. Fix j minimal with this property, i.e., $\Gamma'_j \subset \mathtt{H}_{\alpha,m}$ and $\mathtt{H}_{\alpha,m}$ separates the faces Γ_i from $\mathfrak{C}_{-\infty}$ for $i < j$ (in the large sense, i.e., $\Gamma_i \subset \mathtt{H}^+_{\alpha,m}$) but not the face Γ_j (so $\mathtt{H}_{\alpha,m} \in \mathcal{H}_j$, the set of all the affine hyperplanes $\mathtt{H}$ such that $\Gamma'_j \subset \mathtt{H}$ and $\Gamma_j \not\subset \mathtt{H}$). Fix $k > j$ minimal such that $\Gamma'_k \subset \mathtt{H}_{\alpha,m}$.



thick lines = small faces Γ'

Definition 18. If $\gamma = [\gamma_0, \gamma_1, ..., \gamma_p]$ satisfies III), then let $\widetilde{e}_\alpha\gamma$ be the gallery defined by:

$$\widetilde{e}_\alpha\gamma = [\delta_0, \delta_1, ..., \delta_p] = (F_f \subset \Delta_0 \supset \Delta'_1 \subset \cdots \supset \Delta'_p \subset \Delta_p \supset F_{\check{\nu}})$$

$$\text{where} \quad \Delta_i = \begin{cases} \Gamma_i & \text{for } i \leq j-1 \text{ and } i \geq k, \\ s_{\alpha,m+1}(\Gamma_i) & \text{for } j \leq i < k. \end{cases}$$

The proof of the following simple lemma is left to the reader. Let $\gamma \in \Gamma(\gamma_\lambda, \nu)$ be a combinatorial gallery of type t_{γ_λ} ending in ν, let α be a simple root and suppose that m is minimal such that one of the small faces $\Gamma'_k \subset \mathtt{H}_{\alpha,m}$ is contained in the hyperplane $\mathtt{H}_{\alpha,m}$

Lemma 5. *(i) The gallery $e_\alpha\gamma$ is not defined if and only if $m = 0$, and if $e_\alpha\gamma$ is defined, then $e_\alpha\gamma \in \Gamma(\gamma_\lambda, \nu + \alpha)$.*

(ii) The gallery $f_\alpha\gamma$ is not defined if and only if $m = \langle \nu, \alpha \rangle$, and if $f_\alpha\gamma$ is defined, then $f_\alpha\gamma \in \Gamma(\gamma_\lambda, \nu - \alpha)$.

(iii) If $e_\alpha\gamma$ is defined, then $f_\alpha(e_\alpha\gamma)$ is defined and equal to γ. Further, $m+1$ is minimal such that a small face of the gallery $e_\alpha\gamma$ is contained in $\mathtt{H}^+_{\alpha,m+1}$.

(iv) If $f_\alpha\gamma$ is defined, then $e_\alpha(f_\alpha\gamma)$ is defined and equal to γ. Further, $m-1$ is maximal such that a small face of the gallery $f_\alpha\gamma$ is contained in $\mathtt{H}^+_{\alpha,m-1}$.

(v) *Let* p *be maximal such that* $f_\alpha^p\gamma$ *is defined and let* q *be minimal such that* $e_\alpha^q\gamma$ *is defined, then* $p-q=\langle\nu,\alpha\rangle$.

Let $\Gamma(\gamma_\lambda,\mathrm{dom})$ be the set of all combinatorial galleries δ in $\Gamma(\gamma_\lambda)$ such that none of the small faces is contained in one of the hyperplanes $\mathtt{H}_{\alpha,-1}$, or, in other words, $e_\alpha\delta$ is not defined for all simple roots. Set $\mathrm{Char}\,\Gamma(\gamma_\lambda)=\sum\exp(e(\gamma))$, the sum over all $\gamma\in\Gamma(\gamma_\lambda)$. The following character formula can be proved using the same arguments as in [21].

Corollary 1. $\mathrm{Char}\,\Gamma(\gamma_\lambda)=\sum\mathrm{Char}\,V(e(\gamma))$, *the sum running over all* $\gamma\in\Gamma(\gamma_\lambda,\mathrm{dom})$.

Obviously, the character of $V(\lambda)$ occurs in the decomposition above with multiplicity one. We want to show that this character comes from the subset of LS–galleries. A first step is the following lemma.

Lemma 6. *If* $\gamma\in\Gamma^+(\gamma_\lambda)$ *and* $e_\alpha\gamma$ *(respectively* $\widetilde{e}_\alpha\gamma$*) is defined, then this is again a positively folded gallery, and* $\dim e_\alpha\gamma=\dim\gamma+1$ *respectively* $\dim\widetilde{e}_\alpha\gamma=\dim\gamma+1$.

Proof. Translations do not affect the relative position of a large face and a small one $\Gamma'\subset\Gamma$ with respect to $\mathfrak{C}_{-\infty}$, and the affine reflection with respect to a simple root α affects the relative position only if $\Gamma'\subset\mathtt{H}_{\alpha,\ell}$ for some $\ell\in\mathbb{Z}$. So to check whether the new gallery is positively folded and to prove the dimension formula, it suffices to compare the pair $\Gamma'_j\subset\Gamma_j$ and, for $k\neq p+1$, the pair $\Gamma'_k\subset\Gamma_k$, respectively $\Gamma'_p\subset\Gamma_p$, with the corresponding pairs of large and small faces in the new gallery.

By assumption, the gallery γ is positively folded at Γ'_j, but the wall $\mathtt{H}_{\alpha,m+1}$ is not load-bearing for γ at Γ_j. By construction, the gallery $e_\alpha\gamma$, respectively $\widetilde{e}_\alpha\gamma$, is again positively folded at Γ'_j. Indeed, applying the reflection $s_{\alpha,m+1}$ to the part of γ between j and k corresponds to fold this part around $\mathtt{H}_{\alpha,m+1}$ and the image of this folding is separated from $\mathfrak{C}_{-\infty}$. Therefore $\mathtt{H}_{\alpha,m+1}$ is a load-bearing wall for $e_\alpha\gamma$, respectively $\widetilde{e}_\alpha\gamma$, at $\Delta_j=s_{\alpha,m+1}(\Gamma_j)$. This finishes already the proof if $k=p+1$.

If $k\leq p$, then, in the case I), the gallery γ is positively folded at $\Gamma'_k\in\mathtt{H}_{\alpha,m}$, and either this wall is a load-bearing wall, or $\Gamma_k\subset\mathtt{H}_{\alpha,m}$. Using the corresponding existence for inclusions of Verma modules for example (Remark 6), one sees that the folding around $s_{\alpha,m+1}(\Gamma'_k)$ is still positive. Further, the translation by $\alpha^\vee$ does not change the constellation of the corresponding wall containing Γ'_k, i.e., now either $\mathtt{H}_{\alpha,m+2}$ is a load-bearing wall or $t_{\alpha^\vee}(\Gamma_k)\subset\mathtt{H}_{\alpha,m+2}$, which finishes the proof for the case I). The proof for III) is similar and is left to the reader. •

Remark 7. As in the case of the path model, we have a "$*$" operation on the set of galleries starting in the origin. Let $\delta\in\Gamma(\gamma_\lambda)$ be a combinatorial gallery having as target $e(\delta)$ the coweight ν. Let $t_{-\nu}$ be the translation by the coweight $-\nu$. The combinatorial gallery $t_{-\nu}(\delta)$ starts in $-\nu$ and ends in 0. Let δ^* be the gallery having the same faces as $t_{-\nu}(\delta)$ but in reverse order, so the source of δ^* is the origin, and the target is $-\nu$. The type of δ^* is the same as the type of γ_λ^*, which is the same as the type of $w_0(\gamma_\lambda^*)$. The latter is again a minimal gallery, but now joining the origin with the coweight $\lambda^* = -w_0(\lambda)$, the highest weight of the representation dual to $V(\lambda)$. On sees easily that the map $\delta \mapsto \delta^*$ induces a bijection $\Gamma(\gamma_\lambda) \to \Gamma(w_0(\gamma_\lambda^*))$ and preserves the property of being positively folded, i.e., we get a bijection $\Gamma^+(\gamma_\lambda) \to \Gamma^+(w_0(\gamma_\lambda^*))$. Further, $e_\alpha(\delta) = (f_\alpha(\delta^*))^*$ and $f_\alpha(\delta) = (e_\alpha(\delta^*))^*$, so, by Lemma 6, we see that f_α also stabilizes the set of positively folded galleries and $\dim f_\alpha\delta = \dim\delta - 1$ for $\delta \in \Gamma^+(\gamma_\lambda)$.

We recall now Proposition 3 from section 5 and give the proof:

Proposition 3. *If $\delta \in \Gamma^+(\gamma_\lambda, \nu)$, then $\dim\delta \le \langle\lambda + \nu, \rho\rangle$.*

Proof. If $\delta = [id, \delta_1, \ldots, \delta_p]$, then necessarily we have $\delta = \gamma_\lambda$, in which case the dimension formula holds by Example 7. If $\delta = [\delta_0, \delta_1, \ldots, \delta_p]$ for some $\delta_0 \in W/W_\lambda$, $\delta \ne id$, then let α be a simple root such that $s_\alpha\delta_0 < \delta_0$ in the Bruhat order on the set of minimal length representatives of W/W_λ. For such a simple root α either the gallery $e_\alpha\delta$ or the gallery $\widetilde{e}_\alpha\delta$ is defined. Further, applying these operators repeatedly, one gets a gallery of the form $[s_\alpha\delta_0, \delta'_1, \ldots, \delta'_p]$.

This shows that we can find a sequence of simple roots $\alpha_{i_1}, \ldots, \alpha_{i_t}$ such that we obtain the gallery γ_λ successively by applying either $e_{\alpha_{i_j}}$ or $\widetilde{e}_{\alpha_{i_j}}$, $1 \le j \le t$. Let d_δ be the number of times in this procedure that we apply the operator e_α and let c_δ be the number of times we apply in the procedure the operator $\widetilde{e}_\alpha$, so $t = c_\delta + d_\delta$. Now the operator e_α shifts the last face of the gallery by $\alpha^\vee$. So if $\lambda - \nu = \sum a_\alpha\alpha^\vee$, then $t \ge d_\delta = \sum a_\alpha$. It follows:

$$\dim\delta = \dim\gamma_\lambda - t \le \langle 2\lambda, \rho\rangle - d_\delta = \langle 2\lambda, \rho\rangle - \langle\lambda - \nu, \rho\rangle = \langle\lambda + \nu, \rho\rangle. \quad \bullet$$

As an immediate consequence of the proof above, Remark 7, Lemma 5 and using similar arguments as in [21] we see (in particular, a proof of Proposition 4):

Corollary 2. *(i) The dimension $\dim\delta = \langle\lambda+\mu, \rho\rangle$ if and only if there exists a sequence of simple roots such that $e_{\alpha_{i_1}} \cdots e_{\alpha_{i_t}}\delta = \gamma_\lambda$, or, equivalently, $\delta = f_{\alpha_{i_t}} \cdots f_{\alpha_{i_1}}\gamma_\lambda$. In particular, the set of LS–galleries in $\Gamma^+(\gamma_\lambda)$ is the subset generated from γ_λ using the operators f_α.*

(ii) The gallery δ is an LS–gallery if and only if δ^* is an LS–gallery.

(iii) The set of LS–galleries is stable under the operators e_α and f_α and hence $\mathrm{Char}\, V(\lambda) = \sum \exp(e(\gamma))$, *where the sum runs over all LS–galleries in $\Gamma^+(\gamma_\lambda)$.*

We will now describe the connection between the path model and the LS–galleries. Recall that the minimal galleries are of the form $[\gamma_0, \tau_1^m, ..., \tau_p^m]$, where $\gamma_0 \in W$ is a representative of minimal length of a class in W/W_λ, and each $\tau_j^m \in W'_j$ is the unique minimal representative of the class of the longest element in W'_j.

Definition 19. The *companion* $(\sigma_0, \ldots, \sigma_p)$ *of a gallery* γ is a sequence of Weyl group cosets in W/W_λ satisfying the following conditions:

- $\sigma_0 = \gamma_0 \mod W/W_\lambda$
- if there is no folding at Γ'_j, then $\sigma_j = \sigma_{j-1}$
- assume that γ is folded at Γ'_j.

a) If λ is regular, then let s_β be the reflection used for the folding, i.e., $\Gamma_j = s_{\beta,m}\Gamma_{j-1}$ for an appropriate integer m. We set $\sigma_j = s_\beta \sigma_{j-1}$ in W.
b) If λ is not regular, then more than one reflection can occur in a folding. Let $s_{\beta_1}, \dots, s_{\beta_r}$ be the reflections used for the folding, i.e., $\Gamma_j = s_{\beta_1,m_1} \cdots s_{\beta_r,m_r}\Gamma_{j-1}$ for appropriate integers $m_1, \dots, m_r$, then set $\sigma_j = s_{\beta_1} \cdots s_{\beta_r}\sigma_{j-1}$ in W/W_λ.

Remark 8. To compare this notion with the language of the path model, note that one can find a piecewise linear path π starting in 0 and ending in λ, such that all directions are dominant rational weights and have the same stabilizer in W as λ, and the image is contained in $\gamma_\lambda = (F_f \subset \Gamma_0 \supset \Gamma'_1 \subset \Gamma_1 \supset \ldots \subset \Gamma_p \supset F_\lambda)$.

To see an example, assume for simplicity that G is simple. Take the piecewise linear path that joins the origin with the barycenter of Γ'_1, then the barycenter of Γ'_1 with the barycenter of Γ'_2, etc, and finally joins the barycenter of Γ'_p with λ. Note that the segments in Γ_i, $1 \le i \le p-1$, are parallel to all codimension one faces of the Γ_i except for the face Γ'_i and Γ'_{i+1}. So the minimality of the gallery implies that each segment is a positive rational multiple of a dominant weight. Now these may have a larger stabilizer than λ, but at least the first segment has obviously the desired property. Now by moving around the turning point on Γ'_1 (and letting the others fixed), we may still assume that the first segment is dominant rational and has the same stabilizer as λ, and we may assume in addition that this is now also true for the second segment. Continuing the finite procedure, we obtain the desired piecewise linear path.

Now the image of this path in a folded gallery of the same type will be a path such that its directions will be Weyl group conjugates of the directions we started with. The corresponding Weyl group elements (or rather the classes) give the companion.

Definition 20. A gallery is called a *combinatorial LS–gallery* if the following holds: for each pair (σ_{i-1}, σ_i) in the companion, the sequence of folding reflections $s_{\beta_1,m_1}, \dots, s_{\beta_r,m_r}$ can be chosen such that applied to σ_{i-1}, this gives a sequence $\sigma_{i-1} = \tau'_0 > \tau'_1 = s_{\beta_1}\tau'_0 > \ldots > \tau'_r = s_{\beta_r}\tau_{r-1} = \sigma_i$ of Weyl group cosets (modulo W_λ) such that the length is always decreasing by one for each reflection.

Remark 9. The condition that the classes are decreasing in the Bruhat order implies that this is a positive folding: if $\kappa = s_\beta\delta > \delta$, then $\kappa(\lambda) \prec \delta(\lambda)$ in the weight ordering (i.e., $\delta(\lambda) - \kappa(\lambda)$ is a non-negative sum of positive roots), and hence $\langle \kappa(\lambda), \beta \rangle > 0$.

The special rôle played by the combinatorial LS–galleries γ is the following: let α be a simple root and let $m \in \mathbb{Z}$ be minimal such that a small face of γ is contained in $\mathtt{H}_{\alpha,m}$.

Lemma 7. *If γ is a combinatorial LS–gallery and $e_\alpha\gamma$ (respectively $f_\alpha\gamma$) is well defined, then $e_\alpha\gamma$ (respectively $f_\alpha\gamma$) is again a combinatorial LS–gallery of type t_{γ_λ}.*

Proof. We know already that if $e_\alpha\gamma$ is defined, then it is a positively folded gallery. To see that it is in fact a combinatorial LS–gallery, one can now apply the same argument as in [17] on chains of Weyl group cosets to prove that the companion of $e_\alpha\gamma$ has again the desired properties. The proof for f_α follows using the "$*$"–operation. •

Using the operators e_α, we can transform a LS–gallery γ into a LS–gallery γ' having $\sigma_0 = 1$ for the companion, and hence $\gamma' = \gamma_\lambda$. It follows:

Corollary 3. *The set of LS-galleries in* $\Gamma^+(\gamma_\lambda)$ *coincides with the set of combinatorial LS-galleries.*

For the gallery γ_λ let $\pi : [0,1] \to \mathcal{A}$ be a piecewise linear path as in Remark 8. By comparing the definition of the folding operators e_α, f_α on galleries with the the definition of the root operators e_α, f_α on piecewise linear paths, one sees easily that the set of LS–galleries in $\Gamma^+(\gamma_\lambda)$ is exactly the set of galleries obtained in the following way: let $B(\pi)$ be the path model for the representation $V(\lambda)$ obtained by applying the root operators to π. To a path $\eta \in B(\pi)$ we associate the gallery γ_η defined as the sequence of faces gone through by η. Then $\Gamma^+_{LS}(\gamma_\lambda) = \{\gamma_\eta \mid \eta \in B(\pi)\}$, and this identification is equivariant with respect to the operators e_α, f_α. As a consequence we get (by [18], [13], [14]):

Theorem 2. *Let* $B(\gamma_\lambda)$ *be the directed colored graph having as vertices the set of LS-galleries in* $\Gamma^+(\gamma_\lambda)$*, and put an arrow* $\delta \xrightarrow{\alpha} \delta'$ *with color* α *between two galleries if* $f_\alpha(\delta) = \delta'$*. Then this graph is connected, and it is isomorphic to the crystal graph of the irreducible representation* $V(\lambda)$ *of* $G^\vee$ *of highest weight* λ.

7 Variety of Galleries

In this section, we give two equivalent definitions of the Bott-Samelson variety, and we point out the first step of the connection between galleries and MV-cycles. Let $\lambda \in X^\vee_+$ be a dominant co-character and let $t_\lambda = type(F_\lambda)$ be the type of the vertex of $\mathcal{A}$ associated to λ. Let $\mathcal{Q}_\lambda$ be the parahoric subgroup of type t_λ containing $\mathcal{B}$. The Bott-Samelson variety $\hat{\Sigma}(\gamma_\lambda)$ occurs as a natural resolution (see [15]) of the "Schubert variety" $X_\lambda = \overline{\mathcal{G}_\lambda}$, where $\mathcal{G}_\lambda = G(\mathcal{O})\lambda \hookrightarrow G(\mathcal{K})/\mathcal{Q}_\lambda$. By a resolution we mean that $\hat{\Sigma}(\gamma_\lambda)$ is a smooth algebraic variety and the morphism $\pi : \hat{\Sigma}(\gamma_\lambda) \to X_\lambda$ is birational and proper.

Let

$$t_{\gamma_\lambda} = (S = t'_0 \supset t_0 \subset t'_1 \supset \cdots \supset t_{j-1} \subset t'_j \supset t_j \subset \cdots \subset t'_p \supset t_p \subset t_\lambda).$$

be the gallery of types of the fixed minimal combinatorial gallery γ_λ joining F_f with F_λ. Consider first the usual definition of $\hat{\Sigma}(\gamma_\lambda)$ (in [15] §7.1, S. Kumar gives the definition only for the case where the t_j's are trivial, but it makes also sense in our case). For $0 \leq j \leq p$, let us denote by $\mathcal{P}_j$ (resp. $\mathcal{Q}_j$) the parahoric subgroup of type t'_j (resp. t_j) containing $\mathcal{B}$, of course $\mathcal{P}_0 = G(\mathcal{O})$.

Definition 21. *The Bott-Samelson variety* $\hat{\Sigma}(\gamma_\lambda)$ *is defined as*

$$\hat{\Sigma}(\gamma_\lambda) = G(\mathcal{O}) \times_{\mathcal{Q}_0} \mathcal{P}_1 \times_{\mathcal{Q}_1} \cdots \times_{\mathcal{Q}_{p-1}} \mathcal{P}_p/\mathcal{Q}_p,$$

i.e. the algebraic (complex) variety defined as the quotient of the group $G(\mathcal{O}) \times \mathcal{P}_1 \times \cdots \times \mathcal{P}_p$ *by the subgroup* $\mathcal{Q}_0 \times \mathcal{Q}_1 \times \cdots \times \mathcal{Q}_p$ *under the (right) action given by* $g \cdot q = (g_0q_0, q_0^{-1}g_1q_1, ..., q_{p-1}^{-1}g_pq_p)$ *where* $q = (q_0, q_1, ..., q_r) \in \mathcal{Q}_0 \times \mathcal{Q}_1 \times \cdots \times \mathcal{Q}_p$ *and* $g = (g_0, g_1, ..., g_p) \in G(\mathcal{O}) \times \mathcal{P}_1 \times \cdots \times \mathcal{P}_p$.

The Bott-Samelson variety $\hat{\Sigma}(\gamma_\lambda)$ is a smooth, projective (complex) variety of dimension $\dim(\hat{\Sigma}(\gamma_\lambda)) = 2\langle\lambda, \rho\rangle$. We will denote by $g = [g_0, g_1, ..., g_p]$ the points of this variety and we call them *galleries* (the reason will become apparent soon). The Bott-Samelson variety naturally comes also equipped with a proper and birational morphism

$$\pi : \hat{\Sigma}(\gamma_\lambda) \to X_\lambda$$

by associating to a gallery $g = [g_0, g_1, ..., g_p]$ the class of the product $g_0g_1 \cdots g_p$ in the affine Grassmannian $G(\mathcal{O})/\mathcal{Q}_\lambda$. The following Definition–Proposition is a slight generalization of a result of C. Contou-Carrère [9].

Definition-Proposition 1. *The Bott-Samelson variety* $\hat{\Sigma}(\gamma_\lambda)$ *is the closed subvariety of the product*

$$G(\mathcal{K})/\mathcal{Q}_0 \times \Big(\prod_{1\le j\le p} G(\mathcal{K})/\mathcal{P}_j \times G(\mathcal{K})/\mathcal{Q}_j\Big) \times G(\mathcal{K})/\mathcal{Q}_\lambda$$

given by all the sequences of parahoric subgroups of the shape

$$(G(\mathcal{O}) \supset \mathcal{Q}'_0 \subset \mathcal{P}'_1 \supset \mathcal{Q}'_1 \subset \cdots \subset \mathcal{P}'_p \supset \mathcal{Q}'_p \subset \mathcal{Q}'_\lambda),$$

where $type(\mathcal{P}'_i) = t'_i$, $type(\mathcal{Q}'_j) = t_j$ *and* $\mathcal{Q}'_\lambda$ *is a subgroup associated to a vertex of type* t_λ. *In other words, the set of points of* $\hat{\Sigma}(\gamma_\lambda)$ *is given by the set of all the galleries in the affine building* $\mathcal{J}^{\mathfrak{a}}$ *of type* t_{γ_λ} *starting at* F_f *:*

$$g = (F_f \subset F_0 \supset F'_1 \subset F_1 \supset \cdots \supset F'_p \subset F_p \supset F'_\lambda),$$

where the type of each face corresponds to the type of the corresponding parahoric subgroup.

Proof. The proof of the proposition is given in [9] only in the case of a Bott-Samelson variety built of parabolic subgroups of a reductive group scheme, but it applies also to the Kac-Moody case. One has to extend the isomorphism between $G(\mathcal{K})/\mathcal{P}$, for any parahoric subgroup $\mathcal{P}$, and the set of all the parahoric subgroups of the same type as $\mathcal{P}$. The equivalence between the two definitions is then obtained by the isomorphism which associates to a point $[g_0, g_1, ..., g_p]$ in $\hat{\Sigma}(\gamma_\lambda)$ the gallery

$$g = (F_f \subset F_0 \supset F'_1 \subset F_1 \supset \cdots \supset F'_p \subset F_p \supset F'_{p+1})$$

where, for $0 \leq j \leq p$, $F_j = g_0 g_1 \cdots g_j F_{\mathcal{Q}_j}$ ($F_{\mathcal{Q}_j}$ is the only face of the fundamental alcove Δ_f of type t_j). The faces F'_j of the gallery g are uniquely determined by the F_j's and the type t_{γ_λ}. •

Remark 10. The identification of galleries of type t_{γ_λ} and of source F_f with the points in the Bott-Samelson variety allows us to view the combinatorial galleries $\Gamma(\gamma_\lambda)$ of type t_{γ_λ} and of source F_f as points in $\hat{\Sigma}(\gamma_\lambda)$, they are precisely the T–fixed points. Here the action of $G(\mathcal{O})$ on the galleries is the action induced by the operation on the building. Since $G(\mathcal{O})$ operates on $\mathcal{J}^{\mathfrak{a}}$ by simplicial maps, the action preserves the type of a gallery, and, if the gallery is minimal, then so are all galleries in the orbit.

The morphism $\pi : \hat{\Sigma}(\gamma_\lambda) \to X_\lambda$ maps the minimal gallery γ_λ onto the point λ, so π induces a morphism between $G(\mathcal{O})\gamma_\lambda$ and $\mathcal{G}_\lambda$. Now one sees easily that all minimal galleries of type t_{γ_λ} are in one $G(\mathcal{O})$–orbit, so by the birationality of π we get:

Lemma 8. *The desingularization map $\pi : \hat{\Sigma}(\gamma_\lambda) \to X_\lambda$ induces a bijection between the set of all minimal galleries $\hat{\mathcal{G}}_\lambda$ of type t_{γ_λ} in $\hat{\Sigma}(\gamma_\lambda)$ and the open orbit $\mathcal{G}_\lambda \subset X_\lambda$.*

The retraction $r_{-\infty}$ extends naturally to galleries by applying $r_{-\infty}$ simultaneously to all faces in the gallery. Since $r_{-\infty}$ is a map of simplicial complexes, it preserves the type, i.e., the image of a gallery of type t_{γ_λ} is a combinatorial gallery of type t_{γ_λ}. Let $\Gamma(\gamma_\lambda)$ be the set of all combinatorial galleries of type t_{γ_λ}, i.e., the set of T-fixed points (Remark 10) in $\hat{\Sigma}(\gamma_\lambda)$. The retraction map on the galleries can be described geometrically as follows:

Proposition 5. *The retraction with center at $-\infty$ induces a map $\hat{r}_{\gamma_\lambda} : \hat{\Sigma}(\gamma_\lambda) \to \Gamma(\gamma_\lambda)$. The fibres $C(\delta) = \hat{r}_{\gamma_\lambda}^{-1}(\delta)$, $\delta \in \Gamma(\gamma_\lambda)$, are naturally endowed with the structure of a locally closed subvariety, each of them being isomorphic to an affine space. In fact, the $C(\delta)$ are the Bialynicki-Birula cells $\{x \in \hat{\Sigma}(\gamma_\lambda) \mid \lim_{s\to 0} \eta(s)x = \delta\}$ of center δ in $\hat{\Sigma}(\gamma_\lambda)$, associated to a generic one–parameter subgroup η of T in the anti–dominant Weyl chamber.*

Proof. The map $\hat{r}_{\gamma_\lambda}$ is well–defined, it remains to describe the fibres. Let δ be a combinatorial gallery in $\Gamma(\gamma_\lambda)$, $\delta = (F_f = \Delta'_0 \subset \Delta_0 \supset \Delta'_1 \subset \cdots \subset \Delta_p \supset \Delta_{p+1} = F_\nu)$, and suppose that $g = (F_f = F'_0 \subset F_0 \supset \cdots \supset F'_p \subset F_p \supset F'_\lambda)$ is a gallery of type t_{γ_λ} which retracts onto δ. By Proposition 1, we know that there exist hence $u_i \in U^-(\mathcal{K})$, $i = 0, \ldots, p$, such that $F_i = u_i \Delta_i$, and, since g is a gallery (of type t_{γ_λ}), we have necessarily $u_0 \Delta'_0 = \Delta'_0$ and $u_{i-1}\Delta'_i = u_i \Delta'_i$, $i = 1, \ldots, p$. Conversely, given any sequence $(u_0, \ldots, u_p)$ of elements in $U^-(\mathcal{K})$ satisfying $u_{i-1}^{-1} u_i \in \mathrm{Stab}_{G(\mathcal{K})}(\Delta'_i)$, the stabilizer of the face in $G(\mathcal{K})$, for $i = 0, \ldots, p$ (where we set $u_{-1} = 1$), then

$$g = (F_f = \Delta'_0 \subset u_0\Delta_0 \supset u_1\Delta'_1 \subset u_1\Delta_1 \supset \cdots \supset u_p\Delta'_p \subset u_p\Delta_p \supset u_p\Delta_{p+1} = u_pF_\nu) \quad (3)$$

is a gallery of type t_{γ_λ}, which retracts onto δ. The maximal torus T normalizes $U^-(\mathcal{K})$ and $(tu_{i-1}t^{-1})(tu_i^{-1}t^{-1}) \in \mathrm{Stab}_{G(\mathcal{K})}(\Delta'_i)$ for $t \in T$ if and only if this holds for $u_{i-1}u_i^{-1}$, so the fibres of the retraction map $\hat{r}_{\gamma_\lambda}$ are stable under the natural T–action on $\hat{\Sigma}(\gamma_\lambda)$. Let $\eta : \mathbb{C}^* \to T$ be a one–parameter subgroup in generic position in the open anti-dominant Weyl chamber, so $\lim_{s\to 0}(\eta(s)u\eta(s)^{-1}) = 1$ for $u \in U^-(\mathcal{K})$. If g is as in (3), then $\lim_{s\to 0}\eta(s)g = \delta$ because the Δ_i, Δ'_i are T-fixed points in the building, and hence

$$\begin{aligned}\eta(s)g &= (\eta(s)F_f = \eta(s)\Delta'_0 \subset \eta(s)u_0\Delta_0 \supset \eta(s)u_1\Delta'_1 \subset \cdots \supset \eta(s)u_p\Delta_p \supset \eta(s)u_pF_\nu) \\ &= (F_f \subset (\eta(s)u_0\eta(s)^{-1})\Delta_0 \supset (\eta(s)u_1\eta(s)^{-1})\Delta'_1 \subset \cdots \\ &\qquad \cdots \supset (\eta(s)u_p\eta(s)^{-1})\Delta_p \supset (\eta(s)u_p\eta(s)^{-1})F_\nu),\end{aligned}$$

which implies the claim. The T–fixed points in $\hat{\Sigma}(\gamma_\lambda)$ are the combinatorial galleries in $\Gamma(\gamma_\lambda)$. Since η is generic, we can assume that the T–fixed points and the η–fixed points coincide. So, by [5] and [6], the subsets

$$C(\delta) = \{x \in \hat{\Sigma}(\gamma_\lambda) \mid \lim_{s\to 0}\eta(s)x = \delta\}$$

form a decomposition of $\hat{\Sigma}(\gamma_\lambda)$ into locally closed subsets, each of which is isomorphic to an affine space, and the calculation above shows that the $C(\delta)$ are the fibres of $\hat{r}_{\gamma_\lambda}$. •

By Lemma 8, the object of our main interest, the orbit $\mathcal{G}_\lambda$, can be identified with the minimal galleries $\hat{\mathcal{G}}_\lambda$ in $\hat{\Sigma}(\gamma_\lambda)$. Here the special rôle of the positively folded galleries and its connection with the MV-cycles becomes apparent:

Theorem 3. *The restriction of $\hat{r}_{\gamma_\lambda}$ induces a map $r_{\gamma_\lambda} : \mathcal{G}_\lambda \to \Gamma^+(\gamma_\lambda)$. Further, the union $\bigcup_{\delta\in\Gamma^+(\gamma_\lambda,\nu)} r_{\gamma_\lambda}^{-1}(\delta)$ of the fibres over the galleries in $\Gamma^+(\gamma_\lambda)$ with target ν is the (set theoretic) intersection $U^-(\mathcal{K}).\nu \cap \mathcal{G}_\lambda$.*

Proof. By Proposition 1, we know that the fibres of $r_{-\infty} : \mathcal{J}^{\mathfrak{a}} \to \mathcal{A}$ are the $U^-(\mathcal{K})$–orbits. If $\delta \in \hat{\Sigma}(\gamma_\lambda)$ is a point (or rather a gallery), then $r_{-\infty}(\pi(\delta))$ is the target of the gallery $\hat{r}_{\gamma_\lambda}(\delta)$. So the union of the fibres $r_{\gamma_\lambda}^{-1}(\delta)$ with $\delta \in \Gamma(\gamma_\lambda, \nu)$ is precisely the set theoretic intersection of the orbit $U^-(\mathcal{K}).\nu$ with the orbit $\mathcal{G}_\lambda$.

To finish the proof of the proposition, it remains to show that is suffices to consider positively folded galleries, but this follows from the next lemma. •

Lemma 9. *If $\delta = [\delta_0, \delta_1, ..., \delta_p] = (F_f \subset \Delta_0 \supset \Delta'_1 \subset \cdots \Delta_{j-1} \supset \Delta'_j \subset \Delta_j \supset \cdots \supset F_\nu)$ is not a positively folded gallery, then $\hat{r}_{\gamma_\lambda}^{-1}(\delta) \cap \pi^{-1}(\mathcal{G}_\lambda) = \emptyset$.*

Proof. The assumption means that there exists an index $j \in \{1, ..., p\}$ and a wall $\mathtt{H} \in \mathcal{H}_j$ such that $\delta_j \neq \tau_j^m$ and the image of the folding around $\mathtt{H}$ is not separated from $\mathfrak{C}_{-\infty}$ by $\mathtt{H}$. Particularly, the large face Δ_j is not separated from $\mathfrak{C}_{-\infty}$ by $\mathtt{H}$ and, because we fold around $\mathtt{H}$, the two consecutive large faces Δ_{j-1} and Δ_j are on the same side of this wall.

Let $g \in \hat{r}_{\gamma_\lambda}^{-1}(\delta)$ be a gallery that retracts on δ. We can assume that g has already been retracted onto δ up to the index $j-1$, meaning that the already partially retracted gallery g' is of the form

$$g' = [\delta_0, \delta_1, ..., \delta_{j-1}, g_j, ..., g_p] = (F_f \subset \Delta_0 \supset \cdots \subset \Delta_{j-1} \supset \Delta'_j \subset \ F_j \supset F'_{j+1} \cdots \supset F_{p+1}),$$

with $F_j = \delta_0\delta_1 \cdots \delta_{j-1} g_j F_{\mathcal{Q}_j}$, where $g_j \in \mathcal{P}_j/\mathcal{Q}_j$ and $F_{\mathcal{Q}_j}$ is the face of type t_j contained in the fundamental alcove.

Now let us assume that g is minimal, then $g^j = (\Delta_{j-1} \supset \Delta'_j \subset F_j)$ is also minimal. That implies that Δ_{j-1} and F_j are separated by the wall image of $\mathtt{H}$ in any apartment A containing g^j. But, incarnating the retraction $r_{-\infty}$ using a suitable far away alcove $\Sigma \in A \cap \mathcal{A}$ on the same side of $\mathtt{H}$ than Δ_{j-1}, we know that the retraction $r_{-\infty} = r_{\Sigma,\mathcal{A}}$ preserves the distances from Σ and reduces to an isomorphism of complex chambers $A \simeq \mathcal{A}$. So, Δ_{j-1} and $r_{-\infty}(F_j)$ are also separated by the wall $\mathtt{H}$, which is not possible if we want $r_{-\infty}(F_j) = \Delta_j$. Hence, the lemma is proved. •

8 An open covering of $\hat{\Sigma}(\gamma_\lambda)$

The open subsets of the covering of $\hat{\Sigma}(\gamma_\lambda)$ will be centered at the combinatorial galleries $\Gamma(\gamma_\lambda)$. This covering is a slight generalization of the covering considered by J. Tits [27] and already considered in [11]. The open sets will be built using the root subgroups of the Kac-Moody group $\hat{\mathcal{L}}(G)$.

Definition 22. For any *real root* η in the root system of the Kac-Moody group $\hat{\mathcal{L}}(G)$, there exists a one-dimensional root subgroup $\mathcal{U}_\eta$, isomorphic to $(\mathbb{C}, +)$ (see [15], 6.2.7) having as Lie algebra the corresponding root subspace in $\hat{\mathcal{L}}(\mathfrak{g})$. We will denote the elements of this subgroup by $p_\eta(x)$ for $x \in \mathbb{C}$.

The parametrizations can be chosen to have the following property: for any $w \in W^{\mathfrak{a}}$, any (real) root η and any complex number a, $wp_\eta(a)w^{-1} = p_{w(\eta)}(a)$. For any two subgroups $\mathcal{P} \supset \mathcal{Q} \supset \mathcal{B}$ such that ${W^{\mathfrak{a}}}_{\mathcal{P}}$ is finite, let us consider the affine open neighborhood of 1 in $\mathcal{P}/\mathcal{Q}$,

$$\mathcal{U}^-(\mathcal{P}/\mathcal{Q}) = \prod_{\substack{\eta<0, \eta \text{ real} \\ \mathcal{U}_\eta \subset \mathcal{P},\ \mathcal{U}_\eta \not\subset \mathcal{Q}}} \mathcal{U}_\eta.$$

If $w \in {W^{\mathfrak{a}}}_{\mathcal{P}}/{W^{\mathfrak{a}}}_{\mathcal{Q}}$ (again we identify w with a representative of minimal length in its class), let us consider the following sets of real roots

$$R^+(w) = \left\{\eta > 0 \mid \mathcal{U}_{w^{-1}(\eta)} \not\subset \mathcal{Q}\right\} \quad \text{and} \quad R^-(w) = \left\{\theta < 0 \mid w(\theta) < 0,\ \mathcal{U}_\theta \in \mathcal{U}^-(\mathcal{P}/\mathcal{Q})\right\},$$

then $w\mathcal{U}^-(\mathcal{P}/\mathcal{Q}) = \mathcal{U}^+(w)w\mathcal{U}^-(w)$ is an affine open neighborhood of the point $w \in \mathcal{P}/\mathcal{Q}$, where

$$\mathcal{U}^+(w) = \prod_{\eta \in R^+(w)} \mathcal{U}_\eta \quad \text{and} \quad \mathcal{U}^-(w) = \prod_{\theta \in R^-(w)} \mathcal{U}_\theta.$$

Note that if w' is any element in the class of w in $W^{\mathfrak{a}}{}_{\mathcal{P}}/W^{\mathfrak{a}}{}_{\mathcal{Q}}$, then $R^+(w) = R^+(w')$. In addition, the set of positive real roots $R^+(w) \amalg \big(-R^-(w)\big)$ indexes the walls of $\mathcal{A}$ that contain $F_{\mathcal{P}}$, but not $wF_{\mathcal{Q}}$.

Definition 23. Let $\delta \in \Gamma(\gamma_\lambda)$ be a combinatorial gallery of type t_{γ_λ}:

$$\delta = [\delta_0, \delta_1, ..., \delta_r] = (F_f = \Delta'_0 \subset \Delta_0 \supset \cdots \Delta_{j-1} \supset \Delta'_j \subset \Delta_j \supset \cdots \subset \Delta_p \supset F_\nu).$$

We define a subset $\mathcal{U}_\delta$ of $\hat{\Sigma}(\gamma_\lambda)$ as follows,

$$\mathcal{U}_\delta = \Big\{[g_0, g_1, ..., g_p] \in \hat{\Sigma}(\gamma_\lambda) \mid \ g_j \in \mathcal{U}^+(\delta_j)\delta_j\mathcal{U}^-(\delta_j)\Big\}.$$

Now $\hat{\Sigma}(\gamma_\lambda)$ is a sequence of locally Zariski-trivial fibrations, having partial flag varieties as fibres. So $\mathcal{U}_\delta$ is actually an *affine open subset of* $\hat{\Sigma}(\gamma_\lambda)$ *centered at* δ, and $\mathcal{U}_\delta$ is isomorphic to $\mathcal{U}^+(\delta_0)\delta_0\mathcal{U}^-(\delta_0) \times \cdots \times \mathcal{U}^+(\delta_p)\delta_p\mathcal{U}^-(\delta_p)$ (where $\mathcal{P}_0 = G(\mathcal{O})$).

Remark 11. The cells $C(\delta) = \hat{r}_{\gamma_\lambda}^{-1}(\delta)$ (Proposition 5) are subvarieties of the open set $\mathcal{U}_\delta$.

9 Minimal Galleries

Recall that the definition of a minimal gallery is actually valid in any apartment. Further, given any two faces of the building, there exists at least one apartment that contains those two faces, and any minimal gallery between them is contained in this apartment.

To characterize the minimal galleries as a subset of $\hat{\Sigma}(\gamma_\lambda)$, let us begin with the case of a minimal gallery of alcoves. For the following proposition see [8], 2.1.9:

Proposition 6. *Let* $\mu = (\Delta_f, \Delta_1, ..., \Delta_t)$ *be a sequence of alcoves in* $\mathcal{J}^{\mathfrak{a}}$*. Then* μ *is a minimal gallery of alcoves if and only if there exists* $b_j \in \mathcal{B}$ *and* $s_j \in S^{\mathfrak{a}}$*, for* $j = 1, ..., t$*, such that*

$$\Delta_j = b_1 s_1 \cdots b_j s_j \Delta_f$$

and $s_1 s_2 \cdots s_t$ *is a reduced decomposition of an element in* $W^{\mathfrak{a}}$*.*

Remarks 12. 1) By abuse of notation we denote by s_j the simple reflection in $W^{\mathfrak{a}}$ as well as the representative $s_j = p_{\alpha_j}(1)p_{-\alpha_j}(-1)p_{\alpha_j}(1)$ in the Kac-Moody group. Here α_j is the simple root in the root system of $\hat{\mathcal{L}}(G)$ corresponding to s_j.

2) The b_j are elements of the root subgroups, i.e., $b_j = p_{\eta_j}(a_j)$ for some $a_j \in \mathbb{C}$. Note that any element in $\mathcal{U}^+(s_1 \cdots s_j)$ can be written as $p_{\eta_1}(a_1)p_{s_1(\eta_2)}(a_2) \cdots p_{s_1 \cdots s_{j-1}(\eta_j)}(a_j)$ for some complex numbers a_i's.

3) Minimal galleries of alcoves not starting at Δ_f can be obtained using the action of the Kac-Moody group.

4) In a minimal gallery of alcoves (of length t), any two consecutive alcoves are separated by a wall, and all these walls are distinct, so the gallery crosses exactly t distinct walls in any apartment containing it.

We come back to the general case. We cut the gallery of types into smaller blocks, i.e., we consider the following gallery of types $\varepsilon = (\theta \subset \kappa' \supset \kappa) = (t_{j-1} \subset t'_j \supset t_j)$ for some $j \in \{1, ..., p\}$, where $t_{\gamma_\lambda} = (S = t'_0 \supset t_0 \subset \cdots \supset t_{j-1} \subset t'_j \supset t_j \subset \cdots \supset t_p \subset t_\lambda)$. Let $\mathcal{P}$ and $\mathcal{Q}$ be the parahoric subgroups containing $\mathcal{B}$ of type κ' and κ, respectively. Then, we know that $\mathcal{P}/\mathcal{Q} = \bigcup_{w \in W^{\mathfrak{a}}{}_{\mathcal{P}}/W^{\mathfrak{a}}{}_{\mathcal{Q}}} \mathcal{U}^+(w)w\mathcal{U}^-(w)$. We denote by τ^m the smallest representative of the class of the longest element in $W^{\mathfrak{a}}{}_{\mathcal{P}}/W^{\mathfrak{a}}{}_{\mathcal{Q}}$. In addition, as a corollary of the Proposition 6 and the remarks above, we get:

Lemma 10. *Let $g = (E \supset F' \subset F)$ be a gallery of type ε, starting at a face E of the fundamental alcove Δ_f. If $F = xF_{\mathcal{Q}}$, with $x \in \mathcal{U}^+(\tau^m)\tau^m$, then g is a minimal gallery.*

Proof. We have $x \in \mathcal{U}^+(\tau^m)\tau^m$, we want to show that $(E \supset F' \subset xF_{\mathcal{Q}})$ is a minimal gallery. By Proposition 6, we can construct a minimal gallery of alcoves of length $r = \ell(\tau^m)$ between $\Delta_f \supset E$ and the alcove $x\Delta_f = proj_F(\Delta_f)$. This shows that Δ_f is at maximal distance. Since any such minimal gallery is obtained from one of the form above using the operation of the stabilizer of $E \cup F$, the implication follows. •

We will also need the following partial converse statement.

Proposition 7. *If a gallery $g = (E \supset F' \subset F)$ of type ε, starting at a face E of the fundamental alcove Δ_f, is minimal then $F = y'xF_{\mathcal{Q}}$ where $x \in \mathcal{U}^+(\tau^m)\tau^m$ and $y' \in \mathcal{Q}_E = Stab(E)$ (a subgroup of $\mathcal{P}$).*

Proof. Let A be an apartment containing g. By definition, the gallery is minimal in A if, for any alcove $\Delta \supset E$ at maximal distance to F in A, the walls separating Δ from F are exactly the ones that contain F' but not F. Let us denote by $\mathcal{H}(F', F)$ this set of walls and by r its cardinality.

Denote $\Delta \supset E$ an alcove at maximal distance to F in A. Since $g = (E \supset F' \subset F)$ is minimal, any minimal gallery of alcoves between $\Delta \supset E$ and the alcove $proj_F(\Delta)$ is of length r. Let $\mu = (\Delta = \Delta_0, \Delta_1, ..., \Delta_r = proj_F(\Delta))$ be such a gallery. Since $\Delta \supset E \subset \Delta_f$, there exists an element $y \in \mathcal{Q}_E$ such that $y\mu = (\Delta_f = y\Delta_0, y\Delta_1, ..., y\Delta_r)$ is a minimal alcoves' gallery starting at Δ_f. Hence, by Proposition 6, there exists $x \in \mathcal{U}^+(\tau^m)\tau^m$ such that $y\ proj_F(\Delta) = x\Delta_f$. In particular, $F = y^{-1}xF_{\mathcal{Q}}$, hence the proposition is proved. •

Let $w \in W^{\mathfrak{a}}{}_{\mathcal{P}}/W^{\mathfrak{a}}{}_{\mathcal{Q}}$, by abuse of notation we denote by w also the unique minimal representative in $W^{\mathfrak{a}}{}_{\mathcal{P}}$. The following proposition expresses morally the "trace" of minimal galleries on the open set $\mathcal{U}^+(w)w\mathcal{U}^-(w)$ of $\mathcal{P}/\mathcal{Q}$. We use the results of Deodhar [10].

Proposition 8. *The intersection $\mathcal{U}^+(w)w\mathcal{U}^-(w) \cap \mathcal{U}^+(\tau^m)\tau^m$ decomposes into a finite disjoint union of subvarieties, each of which is isomorphic to a product of $\mathbb{C}$ and $\mathbb{C}^*$.*

Proof. Let $w_{\mathcal{P}}$ be the longest element in $W^{\mathfrak{a}}{}_{\mathcal{P}}$, then $\mathcal{U}^+(\tau^m)\tau^m = \mathcal{U}^+(w_{\mathcal{P}})w_{\mathcal{P}}$ modulo $\mathcal{Q}$. Next let $v \in W^{\mathfrak{a}}{}_{\mathcal{P}}$ be such that $w_{\mathcal{P}} = wv$ and $\ell(w) + \ell(v) = \ell(w_{\mathcal{P}})$, then $\mathcal{U}^+(w)\mathcal{U}^+(w_{\mathcal{P}})w_{\mathcal{P}} = \mathcal{U}^+(w_{\mathcal{P}})w_{\mathcal{P}}$. Hence we have:

$$\mathcal{U}^+(w)w\mathcal{U}^-(w) \cap \mathcal{U}^+(\tau^m)\tau^m = \mathcal{U}^+(w)\big(w\mathcal{U}^-(w) \cap \mathcal{U}^+(w_{\mathcal{P}})w_{\mathcal{P}}\big).$$

Let $\mathcal{B}^-$ is the subgroup of the Kac-Moody group $\hat{\mathcal{L}}(G)$ generated by the torus $\mathcal{T}$ and all the root subgroups associated to negative roots, then the latter can be transformed into

$$\begin{aligned} w\mathcal{U}^-(w) \cap \mathcal{U}^+(w_\mathcal{P})w_\mathcal{P} &= w\mathcal{U}^-(w) \cap w_\mathcal{P}\mathcal{U}^-(\mathcal{P}/\mathcal{Q}) \\ &= w_\mathcal{P}\big(w_\mathcal{P}^{-1}w\mathcal{U}^-(w) \cap \mathcal{U}^-(\mathcal{P}/\mathcal{Q})\big) \\ &= w_\mathcal{P}\big(\mathcal{U}^+(v^{-1})v^{-1} \cap \mathcal{U}^-(\mathcal{P}/\mathcal{Q})\big) \\ &= w_\mathcal{P}\big(\mathcal{B}v^{-1} \cap \mathcal{B}^-\mathcal{Q}/\mathcal{Q}\big), \end{aligned}$$

By [10], the intersection $\mathcal{B}v^{-1} \cap \mathcal{B}^-\mathcal{Q}/\mathcal{Q}$ in $\mathcal{P}/\mathcal{Q}$ decomposes into a finite disjoint union of subvarieties isomorphic to products of $\mathbb{C}$ and $\mathbb{C}^*$, which implies the proposition. •

Remark 13. In the Deodhar decomposition of $\mathcal{B}v^{-1} \cap \mathcal{B}^-\mathcal{Q}/\mathcal{Q}$, there is exactly one subvariety of maximal dimension which consists only of copies of $\mathbb{C}^*$ (this is the one which corresponds to the subexpression $(1, 1, ..., 1)$ of v^{-1}). In our situation, this is exactly the open and dense subset where the variables associated to the root subgroups in $\mathcal{U}^-(w)$ are taken to be nonzero:

$$\mathcal{U}^+(w)w\mathcal{U}^-(w)(\mathbb{C}^*) \subsetneq \mathcal{U}^+(w)w\mathcal{U}^-(w) \cap \mathcal{U}^+(\tau^m)\tau^m.$$

10 The Cell Associated to a Positively Folded Gallery

In this section, we describe the fibre $C(\delta) = \hat{r}_{\gamma_\lambda}^{-1}(\delta)$ of the retraction $\hat{r}_{\gamma_\lambda} : \hat{\Sigma}(\gamma_\lambda) \to \Gamma(\gamma_\lambda)$ over a positively folded gallery $\delta \in \Gamma^+(\gamma_\lambda, \nu)$, as a subvariety of the affine open subset $\mathcal{U}_\delta$ of $\hat{\Sigma}(\gamma_\lambda)$. Let us fix

$$\delta = [\delta_0, \delta_1, ..., \delta_p] = (F_f = \Delta_0' \subset \Delta_0 \supset \Delta_1' \subset \cdots \Delta_{j-1} \supset \Delta_j' \subset \Delta_j \supset \cdots \subset \Delta_p \supset F_\nu).$$

For $j \in \{0, ..., p\}$, we denote by τ_j^m the "element of maximal length" in W_j'/W_j, let ℓ_j be the length. We fix a numbering on the set of walls containing $F_{\mathcal{P}_j}$, but not $\delta_j F_{\mathcal{Q}_j}$ (note that this set is equal to $(\delta_0 \cdots \delta_{j-1})^{-1}\mathcal{H}_j$). The numbering is chosen such that the indexing set can be decomposed into

$$I_j = I_j^+ \amalg I_j^- = \left\{j_1, ..., j_{\ell(\delta_j)}\right\} \amalg \left\{j_{\ell(\delta_j)+1}, ..., j_{\ell_j}\right\},$$

so that $R^+(\delta_j) = \left\{\eta_{j_i} \mid i \in I_j^+\right\}$ and $R^-(\delta_j) = \left\{\theta_{j_i} \mid i \in I_j^-\right\}$ (notation as in section 8).

Let $J_{-\infty}(\delta) \subset I_0 \amalg \cdots \amalg I_p$ be the subset of all the indices corresponding to a load-bearing wall of δ. We have a decomposition $J_{-\infty}(\delta) = J_{-\infty}^+(\delta) \amalg J_{-\infty}^-(\delta)$, where we define

$$J_{-\infty}^+(\delta) = J_{-\infty}(\delta) \cap \big(I_0^+ \amalg \cdots \amalg I_p^+\big) = \bigcup_{j=0}^{p} \big(J_{-\infty}(\delta) \cap I_j^+\big),$$

and similarly for $J_{-\infty}^-(\delta)$. Observe that $J_{-\infty}^-(\delta) = I_0^- \amalg \cdots \amalg I_p^-$ since the gallery δ is positively folded, and $\dim \delta = \sharp J_{-\infty}(\delta)$ (see section 5 for the definition of $\dim \delta$).

Lemma 11. *Let* $\delta = [\delta_0, \delta_1, ..., \delta_p] \in \Gamma^+(\gamma_\lambda)$ *be a positively folded gallery. Then* $C(\delta) = \hat{r}_{\gamma_\lambda}^{-1}(\delta) \subset \hat{\Sigma}(\gamma_\lambda)$ *is an affine cell of dimension* $\dim \delta$*. It is the subvariety of* $\mathcal{U}_\delta$ *consisting of all the galleries* $[g_0, g_1, ..., g_p]$ *such that*

$$g_j \in \Big(\prod_{i \in J_{-\infty}(\delta) \cap I_j^+} \mathcal{U}_{\eta_{j_i}} \Big) \delta_j \prod_{i \in I_j^-} \mathcal{U}_{\theta_{j_i}}.$$

The fibre $r_{\gamma_\lambda}^{-1}(\delta) \subset \mathcal{G}_\lambda$ is obtained as the intersection $\hat{r}_{\gamma_\lambda}^{-1}(\delta) \cap \hat{\mathcal{G}}_\lambda$, so we have:

Corollary 4. *The fibre over a positively folded gallery* δ *of the map* $r_{\gamma_\lambda} : \mathcal{G}_\lambda \to \Gamma^+(\gamma_\lambda)$ *is naturally endowed with the structure of an irreducible quasi-affine variety. More precisely, it is an open dense subset of an affine space of dimension* $\dim \delta$.

Before we proceed with the proof of the lemma, we shall discuss the case where λ is a *regular* dominant co-character. In this case, $\delta = [\delta_0, \delta_1, ..., \delta_p]$ is such that $\delta_0 \in W$ and $\delta_j = 1$ or s_{i_j}, a simple reflection associated to the simple (Kac-Moody) root α_{i_j}. Equivalently, the Δ_j's are alcoves and the Δ'_j's are faces of codimension 1 in two consecutive alcoves. Further, the set I_j is reduced to a single element, it corresponds to the wall which is the support of Δ'_j. So, in case λ is regular, the lemma has the following form.

Lemma 12. *Assume* λ *regular. Let* $\delta = [\delta_0, \delta_1, ..., \delta_p] \in \Gamma^+(\gamma_\lambda)$ *be positively folded gallery. Then* $C(\delta)$ *is the subvariety of* $\mathcal{U}_\delta$ *consisting of all the galleries* $[g_0, g_1, ..., g_p]$ *such that*

$$g_0 \in \delta_0 \prod_{\beta<0, \delta_0(\beta)<0} \mathcal{U}_\beta \text{ and } g_j = \begin{cases} \delta_j \text{ if } j \notin J_{-\infty}(\delta) \\ p_{-\alpha_{i_j}}(a_j),\ a_j \in \mathbb{C},\ \text{if } j \in J^-_{-\infty}(\delta) \\ p_{\alpha_{i_j}}(a_j) s_{i_j},\ a_j \in \mathbb{C},\ \text{if } j \in J^+_{-\infty}(\delta). \end{cases}$$

Proof. Consider first the condition on g_0. We want to characterize the alcoves of the form $g_0(\Delta_f)$ containing 0 which retract onto $\delta_0(\Delta_f)$ by $r_{-\infty}$. In this case we can identify the retraction centered at $-\infty$ with the retraction onto the apartment $A^\mathfrak{a}$ centered at $w_0(\Delta_f)$, i.e., $r_{-\infty}\big(g_0(\Delta_f)\big) = r_{w_0(\Delta_f), A^\mathfrak{a}}\big(g_0(\Delta_f)\big)$. So the problem amounts to find all the spherical chambers that retract to the chamber $\delta_0(\mathfrak{C}_f)$ via the retraction $r_{-\mathfrak{C}_f} = r_{w_0(\mathfrak{C}_f), \mathbb{A}^s}$ onto the spherical apartment, centered at $w_0(\mathfrak{C}_f) = -\mathfrak{C}_f$. But we know that $r_{-\mathfrak{C}_f}^{-1}\big(\delta_0(\mathfrak{C}_f)\big) = r_{-\mathfrak{C}_f}^{-1}\big(\delta_0 w_0(-\mathfrak{C}_f)\big) = B^-\delta_0 w_0$ in G/B^- (recall Example 1). In addition,

$$B^-\delta_0 w_0 = \Big(\prod_{\eta<0, (\delta_0 w_0)^{-1}(\eta)>0} \mathcal{U}_\eta \Big) \delta_0 w_0 = \delta_0 \prod_{\beta<0, \delta_0(\beta)<0} \mathcal{U}_\beta \, w_0.$$

Using the isomorphism $G/B^- \simeq G/B$ given by right multiplication by w_0, we see that the alcoves retracting to $\delta_0(\Delta_f)$ via $r_{-\infty}$ are of the form $g_0(\Delta_f)$, $g_0 \in \delta_0 \prod_{\beta<0, \delta_0(\beta)<0} \mathcal{U}_\beta$.

The conditions on g_j for $j \in \{1, ..., p\}$ follow easily from the definition of the retraction $r_{-\infty}$. Let $g = [g_0, ..., g_p]$ be a gallery, we want to know when g retracts onto δ. We can assume that g has been retracted until the index $j-1$, i.e., g retracts to a gallery

$$g' = [\delta_0, \delta_1, ..., \delta_{j-1}, g_j, ..., g_p] = (F_f \subset \Delta_0 \supset \Delta'_1 \subset \cdots \Delta_{j-1} \supset \Delta'_j \subset F_j \supset \cdots \supset F'_\nu),$$

with $F_j = \delta_0\delta_1 \cdots \delta_{j-1} g_j(\Delta_f)$. If $F_j = \Delta_j$, then this corresponds to $a_j = 0$ in the first or the second condition. Otherwise the gallery $(\Delta_{j-1} \supset \Delta'_j \subset F_j)$ (resp. $(s_{\mathtt{H}_j}\Delta_{j-1} \supset \Delta'_j \subset F_j)$) is minimal in any apartment containing it, where $s_{\mathtt{H}_j}$ is the reflection defined by the wall $\mathtt{H}_j$ supporting the face Δ'_j. Let A be an apartment containing F_j, the wall $\mathtt{H}_j$ and a representative sector in the equivalence class of $\mathfrak{C}_{-\infty}$ in $\mathcal{A}$. Let us incarnate the retraction center at $-\infty$ into $r_{-\infty} = r_{\Sigma,\mathcal{A}}$, where $\Sigma \in \mathcal{A} \cap A$ is a sufficiently far away alcove from Δ'_j.

If $j \notin J_{-\infty}(\delta)$, then $\mathtt{H}_j$ is not load-bearing at this place. In particular, Σ and Δ_j are not separated by $\mathtt{H}_j$. But Σ and F_j are separated by $\mathtt{H}_j$ in A. Hence, because of the properties of the retraction, $r_{\Sigma,\mathcal{A}}(F_j) \neq \Delta_j$, unless $g_j = \delta_j$ (and $F_j = \Delta_j$). Now, if $j \in J^-_{-\infty}(\delta)$, the wall $\mathtt{H}_j$ is load-bearing and $\delta_j = 1$. So this time the only condition on $g_j \in \mathcal{P}_j/\mathcal{Q}_j = \mathcal{U}^-(\delta_j)\mathcal{Q}_j/\mathcal{Q}_j \amalg s_{i_j}\mathcal{Q}_j/\mathcal{Q}_j$ is that $g_j \neq s_{i_j}$. Hence, F_j retracts on Δ_j if and only if $g_j = p_{-\alpha_{i_j}}(a_j)$, for some complex number a_j. The case where $j \in J^+_{-\infty}(\delta)$ is treated similarly and corresponds to the decomposition $\mathcal{P}_j/\mathcal{Q}_j = \mathcal{U}^+(\delta_j)s_{i_j}\mathcal{Q}_j/\mathcal{Q}_j \amalg \mathcal{Q}_j/\mathcal{Q}_j$, this finishes the proof of the lemma. •

Proof of Lemma 11. The proof for g_0 is similar to the proof in the regular case, the computation takes now place in the quotients $G(\mathcal{O})/\mathcal{Q}_0 \simeq G/P_\lambda$ and G/P^-_λ, where P^-_λ is the parabolic subgroup of type t_0 containing B^-. So, applying the retraction $r_{-\mathfrak{C}_f} = r_{w_0(\mathfrak{C}_f),\mathbb{A}^s}$ to the spherical faces of type t_0, we obtain that $g_0F_{\mathcal{Q}_0}$ retracts onto $\delta_0 F_{\mathcal{Q}_0}$ if and only if

$$g_0 \in \delta_0 \prod \mathcal{U}_\beta, \text{ where } \beta \in R^-(\delta_0) = \{\beta < 0,\ \delta_0(\beta) < 0, \mathcal{U}_\beta \in \mathcal{U}^-(G/P_\lambda)\}.$$

Note that $J_{-\infty}(\delta) \cap I^+_0 = \emptyset$. Next consider the case $j > 0$. Let $g = [g_0, g_1, ..., g_p] \in \hat{\Sigma}(\gamma_\lambda)$ be a gallery. Again, we can retract step by step, i.e. we can assume that g has been retracted to a gallery g' up to the index $j-1$, so

$$g' = [\delta_0, \delta_1, ..., \delta_{j-1}, g_j, ..., g_p] = (F_f \subset \Delta_0 \supset \Delta'_1 \subset \cdots \Delta_{j-1} \supset \Delta'_j \subset F_j \supset F'_{j+1} \cdots \supset F'_\nu),$$

with $F_j = \delta_0\delta_1 \cdots \delta_{j-1} g_j F_{\mathcal{Q}_j}$ and $g_j \in \mathcal{U}^+(\delta_j)\delta_j\mathcal{U}^-(\delta_j)$. Hence, there exists some complex numbers a_i for $i \in I^+_j$, and b_k for $k \in I^-_j$ such that we can write g_j as follows

$$g_j = p_{\zeta_1}(a_1)s_{\zeta_1} \cdots p_{\zeta_{\ell(\delta_j)}}(a_{\ell(\delta_j)})s_{\zeta_{\ell(\delta_j)}} \prod_{k \in I^-_j} p_{\theta_{j_k}}(b_k), \tag{4}$$

where $s_{\zeta_1} \cdots s_{\zeta_{\ell(\delta_j)}}$ is a reduced decomposition of δ_j and

$$R^+(\delta_j) = \left\{\zeta_1, s_{\zeta_1}(\zeta_2), ..., s_{\zeta_1} \cdots s_{\zeta_{\ell(\delta_j)-1}}(\zeta_{\ell(\delta_j)})\right\}.$$

Assume first $b_k = 0$ for all k. By Proposition 6, the alcove gallery $\mu = (\Gamma_0, \Gamma_1, \Gamma_2, ..., \Gamma_{\ell(\delta_j)})$, where $\Gamma_i = \delta_0\delta_1 \cdots \delta_{j-1} p_{\zeta_1}(a_1)s_{\zeta_1} \cdots p_{\zeta_i}(a_i)s_{\zeta_i}\Delta_f$, is a minimal gallery between $\Gamma_0 \supset \Delta_{j-1}$ and $\Gamma_{\ell(\delta_j)} \supset F_j$. Because of the minimality, F_j retracts on Δ_j if and only if μ retracts onto the minimal gallery $\xi = (\Xi_0, \Xi_1, ..., \Xi_{\ell(\delta_j)})$ in $\mathcal{A}$, where $\Xi_i = \delta_0\delta_1 \cdots \delta_{j-1}s_{\zeta_1} \cdots s_{\zeta_i}\Delta_f$. Again, we can assume that μ has been retracted onto ξ until the index $i-1$, so we have to find out how the alcove $\Gamma'_i = \delta_0\delta_1 \cdots \delta_{j-1}s_{\zeta_1} \cdots s_{\zeta_{i-1}}p_{\zeta_i}(a_i)s_{\zeta_i}\Delta_f$ may retract onto Ξ_i. But we are in the situation where we can apply the discussion concluding the previous proof. Hence, if $i \in J_{-\infty}(\delta) \cap I_j^+$, a_i can be any complex number, otherwise a_i must equal 0. Therefore, we get the expected conditions on the variables associated to indices in I_j^+.

To conclude, the fact that there is no condition on the b_k's in (4) is a consequence of an analogue computation as in the case $j = 0$ and the assumption that the gallery is positively folded. Thus, the lemma is proved. •

Remark 14. The definition of load-bearing wall with respect to $-\infty$ makes sense for any gallery and the previous description of the fibre also extends to arbitrary combinatorial galleries. Therefore, one gets a description in terms of galleries of the Białynicki-Birula cellular decomposition of the Bott-Samelson variety $\hat{\Sigma}(\gamma_\lambda)$ (see also [11] for a cellular decomposition of a Bott-Samelson variety in a slightly different context).

11 LS-galleries and MV-cycles

We describe an affine open subset of $r_{\gamma_\lambda}^{-1}(\delta) = C(\delta) \cap \mathcal{G}_\lambda$, isomorphic to a product of $\mathbb{C}$'s and $\mathbb{C}^*$'s. Thereby, we can associate to each LS-gallery δ an open dense subset of a Mirkovic-Vilonen cycle $Z(\delta)$. We first deal for simplicity first with the *regular* case.

Proposition 9. *Let λ be a regular dominant co-character and let $\delta = [\delta_0, \delta_1, ..., \delta_p] \in \Gamma^+(\gamma_\lambda)$. Then $r_{\gamma_\lambda}^{-1}(\delta)$ is a subvariety of $\mathcal{U}_\delta$ isomorphic to a product of $\mathbb{C}$'s and $\mathbb{C}^*$'s. More precisely, the fibre consists of all the galleries $[g_0, g_1, ..., g_p]$ such that*

$$g_0 \in \delta_0 \prod_{\beta<0, \delta_0(\beta)<0} \mathcal{U}_\beta \text{ and } g_j = \begin{cases} \delta_j \text{ if } j \notin J_{-\infty}(\delta) \\ p_{-\alpha_{i_j}}(a_j),\ a_j \neq 0 \text{ if } j \in J_{-\infty}^-(\delta), \\ p_{\alpha_{i_j}}(a_j)s_{i_j} \text{ if } j \in J_{-\infty}^+(\delta). \end{cases}$$

Proof. Combining the Propositions 6 and 7, we get the following simple criterion in the regular case: a gallery of alcoves $(\Delta_f \supset F' \subset \Delta)$, where F' is a face of codimension 1 associated to the simple root α, is minimal if and only if $\Delta = x\Delta_f$ with $x \neq 1$ in $\mathcal{P}_\alpha/\mathcal{B}$. This applies to each step in the gallery $[g_0, g_1, ..., g_p]$, and the only condition to add in order for the latter to be minimal is the one of the lemma. •

In the general case the situation is slightly more complicated, but we still have:

Theorem 4. *Let λ be a dominant co-character and let $\nu \prec \lambda$ be an arbitrary co-character. For each positively folded gallery $\delta \in \Gamma^+(\gamma_\lambda, \nu)$, the fibre $r_{\gamma_\lambda}^{-1}(\delta)$ admits a decomposition into a union of subvarieties, each of which is isomorphic to a product of $\mathbb{C}$'s and $\mathbb{C}^*$'s. In* *particular, the subvariety* $O(\delta) \simeq \mathbb{C}^a \times (\mathbb{C}^*)^b$ *of* $r_{\gamma_\lambda}^{-1}(\delta)$ *is open and dense, where* $a+b = \dim\delta$*,* $b = \sharp J^-_{-\infty}(\delta)$*, and* $O(\delta)$ *consists of all the galleries* $[g_0, g_1, ..., g_p]$ *such that*

$$g_j \in \Big(\prod_{i\in J_{-\infty}(\delta)\cap I_j^+} \mathcal{U}_{\eta_{j_i}}\Big)\delta_j \prod_{k\in I_j^-} \mathcal{U}_{\theta_{j_k}}(\mathbb{C}^*),$$

By [23], we know that $U^-(\mathcal{K})\nu\cap\mathcal{G}_\lambda$ is equidimensional. It follows that $\overline{U^-(\mathcal{K})\nu\cap\mathcal{G}_\lambda}$ is the union of the $Z(\delta) = \overline{r_{\gamma_\lambda}^{-1}(\delta)}$ for $\delta \in \Gamma^+(\gamma_\lambda, \nu)$ of maximal dimension, i.e., δ a LS–gallery.

Corollary 5. *The MV-cycles corresponding to* ν *(i.e., the irreducible components of* $\overline{U^-(\mathcal{K})\nu\cap\mathcal{G}_\lambda}$*), are given by the closures* $Z(\delta) = \overline{O(\delta)}$ *for* $\delta \in \Gamma^+_{LS}(\gamma_\lambda, \nu)$*.*

Proof of the theorem. The first definition of $\hat{\Sigma}(\gamma_\lambda)$ (Definition 21) allows to iterate the use of the Propositions 7 and 8. The theorem is then obtained using Remark 13 and the same arguments as above in the regular case. •

It is not yet clear how to express the inclusion relations between the closures of the fibres in terms of the combinatorics of the galleries. For simplicity assume λ is regular and fix $w \in W$, then one sees easily for $\delta' = [w', s_{i_1}, ..., s_{i_p}]$ and $\delta = [w, s_{i_1}, ..., s_{i_p}]$:

$$r_{\gamma_\lambda}^{-1}(\delta') \subset \overline{r_{\gamma_\lambda}^{-1}(\delta)} \Leftrightarrow w \le w'.$$

More generally, let δ, γ be two consecutive galleries in the sequence of positive foldings starting with $w\gamma_\lambda = [w, s_{i_1}, ..., s_{i_p}]$, where the s_{i_j}'s are the simple reflections associated to the simple (Kac-Moody) root α_{i_j} (for the sequence of foldings, see Section 4 around Remark 6). More precisely, there exists an index k such that

$$\delta = [w, \delta_1, ..., \delta_{k-1}, s_{i_k}, s_{i_{k+1}}, ..., s_{i_p}] \quad \text{and} \quad \gamma = [w, \delta_1, ..., \delta_{k-1}, 1, s_{i_{k+1}}, ..., s_{i_p}]. \tag{5}$$

Proposition 10. *If* δ, γ *are as in (5), then* $r_{\gamma_\lambda}^{-1}(\delta) \subset \overline{r_{\gamma_\lambda}^{-1}(\gamma)}$*.*

Proof. Let $g = [g_0, g_1, ..., g_{k-1}, p_{-\alpha_{i_k}}(a_k), g_{k+1}, ..., g_p]$ be a point in $r_{\gamma_\lambda}^{-1}(\gamma)$, where $g_j = p_{\alpha_{i_j}}(a_j)s_{i_j}$, for $j \ge i+1$ and some complex number a_j (which might be zero). For any positive (Kac-Moody) root α and any complex number a, one has $p_{-\alpha}(a) = p_\alpha(a^{-1})s_\alpha h_\alpha p_\alpha(a^{-1})$ (see [26]) for some torus element h_α. So, g may be written as

$$g = [g_0, g_1, ..., g_{k-1}, p_{\alpha_{i_k}}(a_k^{-1})s_{i_k}h_{\alpha_{i_k}}p_{\alpha_{i_k}}(a_k^{-1}), g_{k+1}, ..., g_p].$$

But the last part of this gallery is minimal, so it is stable by $p_{\alpha_{i_k}}(a_k^{-1})$, and hence

$$g = [g_0, g_1, ..., g_{k-1}, p_{\alpha_{i_k}}(a_k^{-1})s_{i_k}, g'_{k+1}, ..., g'_p], \quad \text{where } g'_j = p_{\alpha_{i_j}}(a'_j)s_{i_j},\ a'_j = b_j a_j$$

for $j \ge i+1$ and some $b_j \in \mathbb{C}^*$. Therefore, $r_{\gamma_\lambda}^{-1}(\delta) \subset \overline{r_{\gamma_\lambda}^{-1}(\gamma)}$. •

Corollary 6. *If δ, γ are two consecutive LS–galleries in a sequence of foldings, then the associated MV–cycles are contained in each other: $Z(\delta) \subset Z(\gamma)$.*

Remark 15. The same kind of arguments (using the open subvariety $O(\delta) \subset r_{\gamma_\lambda}^{-1}(\delta)$) apply also in the general case, i.e., Proposition 10 and Corollary 6 hold for an arbitrary dominant co-character λ and $\delta \in \Gamma^+(\gamma_\lambda)$.

Example 8. Let us take a simple example to illustrate this section. Recall the setting of Example 6: $G = SL_3(\mathbb{C})$, $S^{\mathfrak{a}} = \{s_0, s_1, s_2\}$ and we let $\lambda = \beta = \alpha_1 + \alpha_2$ be the highest root. We set $\gamma_\beta = [1, s_0]$. The non-minimal positively folded galleries in $\Gamma^+(\gamma_\beta)$ are

$$\delta_{12} = [s_1 s_2, 1],\ \delta_{21} = [s_2 s_1, 1], \delta_{121} = [s_1 s_2 s_1, 1].$$

If we apply the theorem, we get:

$$r_{\gamma_\beta}^{-1}(\delta_{12}) = \left\{[s_1 s_2 p_{-\alpha_1}(a), p_{-\alpha_0}(b)] \in \hat{\Sigma}(t_{\gamma_\beta}),\ b \neq 0\right\},$$

$$r_{\gamma_\beta}^{-1}(\delta_{21}) = \left\{[s_2 s_1 p_{-\alpha_2}(a), p_{-\alpha_0}(b)] \in \hat{\Sigma}(t_{\gamma_\beta}),\ b \neq 0\right\},$$

and

$$r_{\gamma_\beta}^{-1}(\delta_{121}) = \left\{[s_1 s_2 s_1, p_{-\alpha_0}(b)] \in \hat{\Sigma}(t_{\gamma_\beta}),\ b \neq 0\right\}.$$

The two Mirkovic-Vilonen cycles associated to the pair $(\lambda = \beta, \nu = 0)$ are obtained as the closure of the two varieties $r_{\gamma_\beta}^{-1}(\delta_{12})$ and $r_{\gamma_\beta}^{-1}(\delta_{21})$, and one can show that $r_{\gamma_\beta}^{-1}(\delta_{121})$ is in the closure of those two. Using Proposition 10 and, for the inclusions of the $Z(\delta_{ij})$, some arguments similar to the ones above, we get the following inclusion relations (note the very suggestive similarity of the diagram on the left with the crystal graph):

$$\begin{array}{ccccc} Z([s_1 s_2, s_0]) & \subset & Z(\delta_{12}) & \subset & Z([s_1, s_0]) \\ \cup & & & & \cap \\ Z([w_0, s_0]) & & & & Z([1, s_0]) \\ \cap & & & & \cup \\ Z([s_2 s_1, s_0]) & \subset & Z(\delta_{21}) & \subset & Z([s_2, s_0]) \end{array} \quad \text{and} \quad \begin{array}{c} Z([s_2 s_1, s_0]) \subset Z([s_1, s_0]) \\ \\ Z([s_1 s_2, s_0]) \subset Z([s_2, s_0]). \end{array}$$

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