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GERBES, SIMPLICIAL FORMS AND INVARIANTS FOR FAMILIES OF FOLIATED BUNDLES

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Abstract. The notion of smooth Deligne cohomology is conveniently reformulated in terms of the simplicial deRham complex. In particular the usual Chern-Weil and Chern-Simons theory is well adapted to this framework and rather easily gives rise to characteristic Deligne cohomology classes associated to families of bundles and connections. In turn this gives invariants for families of foliated bundles. The construction provides representing cocycles in the usual Čech-deRham model for smooth Deligne cohomology called 'gerbes with connection' as they generalize usual Hermitian line bundles with connection. A special case is the Quillen line bundle associated to families of flat SU(2)-bundles.

Contents

1. Introduction

The determinant line bundle was constructed by Quillen [32] for families of Riemann surfaces and generalized to higher dimension by Bismut and Freed (see e.g. [1], [18], [19]). It also admits a 'geometric' construction (and further generalization) in terms of families of principal G-bundles with connection for G any Lie group (see e.g. Bonora et.al. [2], Brylinski [5], [6], Dupont–Johansen [14]).

In this situation the construction in the present paper more generally provides 'ℓ–gerbes with connection' for suitable $\ell = 0, 1, 2, \ldots$ depending on curvature conditions on the fibre connections in the family. We use the phrase '(Hermitian line)

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gerbe' (respectively '(Hermitian line) gerbe with connection') as an abreviation for the notion of a representing cocycle (with a shift in degree) in the Čech (respectively Čech–deRham) model for the usual (respectively Deligne) cohomology associated to the sheaf $\underline{U(1)}$ of smooth functions with values in the circle group $U(1) \subseteq \mathbb{C}$. We are aware that the word 'gerbe' originally was used for a rather different kind of object which however, in the abelian case, is closely related to our '2–gerbe' in the same way as a '1–gerbe' corresponds to a Hermitian line bundle. Similarly our notion of a '2–gerbe with connection' is in accordance with Hitchin [25] and is in line with a widespread use of the word 'gerbe' in mathematical physics (see e.g. Carey–Mickelsson [8].) We refer to Brylinski [5], [7] for more information about Deligne cohomology and its relation to the original notion of 'gerbes' (see also Breen–Messing [3]). However, we are using the word 'gerbe' only in the restricted sense described in section 2.

Let us now describe our main results. In the following, X will be a compact oriented smooth manifold and G a Lie group with finitely many components.

Definition 1.1. A *family* of principal G-bundles over X with connections consists of the following:

(i) A smooth fibre bundle $\pi\colon Y \to Z$ with fibre X and structure group $\mathrm{Diff}^+(X)$ of orientation preserving diffeomorphisms.

(ii) A principal G-bundle $p\colon E \to Y$.

(iii) A smooth family $A = \{A_z \mid z \in Z\}$ of connections in the G-bundles $P_z = E\mid X_z\ ,\ X_z = \pi^{-1}(z)$.

Notice that the family of connections in (iii) can always be obtained (using a partition of unity) from some 'global' connection B in the G-bundle E such that $A_z = B \mid TP_z$ for all $z \in Z$. But this *global extension* is not part of the structure. Furthermore let $I^{n+1}_{\mathbb{Z}}(G) \subseteq I^{n+1}(G)$ denote the set of invariant homogeneous polynomials of degree $n+1$ on the Lie algebra $\mathfrak{g}$ such that the Chern-Weil image is an *integral* class. That is, $Q \in I^{n+1}_{\mathbb{Z}}(G)$ corresponds in the cohomology $H^{2n+2}(BG, \mathbb{R})$, BG the classifying space, to the image of a class $u \in H^{2n+2}(BG, \mathbb{Z})$ by the map induced by the natural inclusion $\mathbb{Z} \subseteq \mathbb{R}$. We shall distinguish between two cases: In case I (the 'Godbillon–Vey' case) we have $Q \in \ker(I^*(G) \to I^*(K))$, $K \subseteq G$ a maximal compact subgroup, and u can be chosen to be 0. Otherwise in case II we have $u \neq 0$ (the 'Cheeger–Chern–Simons case'). With this notation we shall prove the following in case I:

Theorem 1.2. *Consider $Q \in I^{n+1}(G)$ as in case* I *above and let $E \to Y$ be a family of G-bundles with connections $\{A_z \mid z \in Z\}$ as in definition* 1.1. *Let* $\dim X = 2n+1-\ell$ *with* $0 \leq \ell \leq 2n+1$.

(*i*) *For B a global extension of the family there is associated a natural class of ℓ-forms $[\Lambda_{Y/Z}(Q, B)] \in \Omega^\ell(Z)/d\Omega^{\ell-1}(Z)$.*

(*ii*) *This class is independent of the choice of extension provided $F^{n+1-\ell}_{A_z} = 0$ for all $z \in Z$, where F_{A_z} is the curvature form in the fibre P_z.*

(*iii*) *Curvature formula* :

$$d\Lambda_{Y/Z}(Q, B) = (-1)^{\ell-1} \int_{Y/Z} Q(F^{n+1}_B),$$

where $Q(F^{n+1}_B) \in \Omega^{2n+2}(Y)$ is the characteristic form associated to Q.

(iv) If $F_{A_z}^{n-\ell} = 0$ *for all* $z \in Z$ *then* $[\Lambda_{Y/Z}(Q,B)]$ *lies in* $H^\ell(Z,\mathbb{R})$.

Here $\int_{Y/Z}$ denotes integration over the fibre in the bundle $\pi\colon Y \to Z$. Also the *curvature* F_A of a connection A in a principal G-bundle $P \to X$ is defined as usual by $F_A = dA + \frac{1}{2}[A,A]$.

For $Q \in I_{\mathbb{Z}}^{n+1}(G)$ as in case II above we shall prove (section 6) a result analogous to Theorem 1.2 only the integral class $u \in H^{2n+2}(BG,\mathbb{Z})$ has to be taken into account, and the deRham complex $\Omega^*(Z)$ is going to be replaced by the *simplicial* deRham complex (as in Dupont [11] or [12]) for the nerve of an open covering of Z. In terms of the above mentioned notion of gerbes with connections (see section 2 below) we shall prove the following:

Theorem 1.3. *Consider* $Q \in I^{n+1}(G)$ *and* $u \in H^{2n+2}(BG,\mathbb{Z})$ *as in case* II *above, and let* $E \to Y$ *be a family of* G*–bundles with connections* $\{A_z \mid z \in Z\}$ *as in definition* 1.1. *Let* $\dim X = 2n+1-\ell$, $0 \le \ell \le 2n+1$.

(i) For B *a global extension of the family there is associated a natural equivalence class of* ℓ*–gerbes* $\theta = \theta(Q,u,B)$ *with connection* $\omega = (\omega^0,\dots,\omega^\ell)$ *for a suitable open covering* $\mathcal{U} = \{U_i \mid i \in I\}$.

(ii) This class $[\theta,\omega]$ *is independent of the choice of extension provided* $F_{A_z}^{n+1-\ell} = 0$ *for all* $z \in Z$*, where* F_{A_z} *is the curvature form in the fibre* P_z.

(iii) Curvature formula :

$$d\omega^0 = (-1)^{\ell-1}\,\varepsilon^* \int_{Y/Z} Q(F_B^{n+1}) \qquad \textit{and} \qquad \delta_*[\theta] = (-1)^{\ell-1}\,\pi_!(u(E)). \tag{1.1}$$

(iv) If $F_{A_z}^{n-\ell} = 0$ *then* $d\omega^0 = 0$ *and the invariant* $[\theta,\omega]$ *lies in* $H^\ell(Z,\mathbb{R}/\mathbb{Z})$.

In (1.1) $\varepsilon^*\colon \Omega^*(Z) \to \check{C}^0(\mathcal{U},\underline{\Omega}^*)$ is the natural inclusion of the deRham complex into the Čech bicomplex. Furthermore $u(E) \in H^{2n+2}(Y,\mathbb{Z})$ is the associated characteristic class for the G–bundle $E \to Y$ and $\pi_!\colon H^{2n+2}(Y,\mathbb{Z}) \to H^{\ell+1}(Z,\mathbb{Z})$ is the usual transfer map. Finally

$$\delta_*\colon H^\ell(Z,\underline{U(1)}) \xrightarrow{\cong} H^{\ell+1}(Z,\mathbb{Z})$$

is the usual isomorphism in Čech–cohomology.

The above theorems contain the classical secondary characteristic classes by taking $X = \{\mathrm{pt}\}$ and $\ell = 2n+1$; but in this case the invariants may depend on the extension B (see section 5). We are more concerned with the case $\ell \le n$ where this does not happen. In particular we shall apply the Theorems 1.2 and 1.3 to families of foliated G–bundles of codimension q in the sense of Kamber–Tondeur [30]. These have *adapted* connections A whose curvature F_A satisfy $F_A^{q+1} = 0$. Hence we obtain invariants for families of such foliations provided $n - \ell \ge q$. We refer to section 6 for a precise statement.

In the case $\ell = 1$ Theorem 1.3 includes the construction of the generalized Quillen line bundles considered in [14] which was our motivating example. In section 6 we shall also consider a relative version of our construction generalizing the notion of a 'Chern-Simons section' considered in [14].

Our Theorems 1.2 and 1.3 overlap with the results of Freed [20] but the methods are rather different. In fact we take advantage of the reformulation of 'gerbes with connection' and smooth Deligne cohomology in terms of simplicial differential forms as explained in section 3. In particular the notion of *integration along the fibres* which we are going to use, is fairly straight forward in this formulation (see section 4 below or Dupont–Ljungmann [17]). Also, as we shall see in section 5, the Cheeger–Chern–Simons characters are represented by simplicial differential forms. There are by now several ways of looking at gerbes with connection (see e.g. Hitchin [25]), but we hope to demonstrate that the representation as a simplicial differential form is both an attractive and a convenient point of view.

The results of the paper go back a few years but the presentation follows a talk given by the first author in November 2002 during the program 'Aspects of Foliation Theory' at the Erwin Schrödinger Institute in Vienna. Both authors gratefully acknowledge the hospitality and support of the Erwin Schrödinger Institute. The second author visited Århus on several occasions during the preparation of this work and would like to thank the Department of Mathematics at Aarhus University for its hospitality and support. Finally we want to thank the referee for some very useful comments in particular on the terminology of 'gerbes' and 'Deligne cohomology'.

2. Gerbes with connection

In this section we briefly recall the notion of a 'gerbe with connection' and smooth 'Deligne cohomology'. We refer to [5] for more information. We shall only consider *Hermitian line gerbes* which are by definition Čech cocycles for the sheaf $\underline{U(1)}$ of smooth functions with values in the circle group $U(1) \subseteq \mathbb{C}$. For convenience we shall identify this group with $\mathbb{R}/\mathbb{Z}$ via the map $z \leftrightarrow \frac{1}{2\pi i}\log z$, $z \in U(1)$. Hence a (Hermitian line) *p–gerbe* on a smooth manifold X is a p-cocycle in the Čech complex

$$\check{C}^p(\mathcal{U}, \underline{\mathbb{R}/\mathbb{Z}}) = \prod_{(i_0,\dots,i_p)} C^\infty(U_{(i_0,\dots,i_p)}, \mathbb{R}/\mathbb{Z}),$$

with the usual coboundary

$$(\check{\delta}\theta)_{i_0,\dots,i_p} = \sum_{\nu=0}^{p+1}(-1)^i \theta_{i_0,\dots,\widehat{i_\nu},\dots,i_p}. \tag{2.1}$$

Here $\mathcal{U} = \{U_i \mid i \in I\}$ is an open covering of X. For convenience we assume that $\mathcal{U}$ is 'good' in the sense that all non-empty intersections $U_{i_0,\dots,i_p} = U_{i_0} \cap \dots \cap U_{i_p}$ are contractible. Is is well-known that every open covering has a good refinement and that for such covering we have

$$H^p(\check{C}^*(\mathcal{U}, \mathbb{R}/\mathbb{Z})) \cong H^p(X, \underline{\mathbb{R}/\mathbb{Z}}).$$

Notice also that every cochain is the reduction of a cochain in $\check{C}^*(\mathcal{U}, \underline{\mathbb{R}})$ and that the isomorphism

$$\delta_* \colon H^p(X, \underline{\mathbb{R}/\mathbb{Z}}) \xrightarrow{\cong} H^{p+1}(X, \mathbb{Z}) \tag{2.2}$$

is indeed induced by $\check{\delta}$ in (2.1) applied to such a lift.

In general consider the Čech–deRham bicomplex

$$\check{\Omega}^{p,q}_{\mathbb{R}}(\mathcal{U}) = \check{C}^p(\mathcal{U}, \underline{\Omega}^q) \tag{2.3}$$

with differential in the total complex $\check{\Omega}^*_{\mathbb{R}}(\mathcal{U})$ given on $\check{\Omega}^{p,*}_{\mathbb{R}}$ by $D = \check{\delta} + (-1)^p d$. Notice that there are natural inclusions of chain complexes

$$\check{C}^*(\mathcal{U}, \mathbb{Z}) \subseteq \check{C}^*(\mathcal{U}, \underline{\Omega}^0) \subseteq \check{\Omega}^*_{\mathbb{R}}(\mathcal{U}), \tag{2.4}$$

and

$$\varepsilon^*\colon \Omega^*(X) \xrightarrow{\subseteq} \check{C}^0(\mathcal{U}, \underline{\Omega}^*) \subseteq \check{\Omega}^*_{\mathbb{R}}(\mathcal{U}), \tag{2.5}$$

where ε^* is induced by the natural map

$$\varepsilon\colon \bigsqcup_i U_i \to X.$$

Since $\mathcal{U}$ is good we have

$$\check{C}^*(\mathcal{U}, \underline{\mathbb{R}/\mathbb{Z}}) = \check{C}^*(\mathcal{U}, \underline{\Omega}^0)/\check{C}^*(\mathcal{U}, \mathbb{Z})$$

and we put

$$\check{\Omega}^*_{\mathbb{R}/\mathbb{Z}}(\mathcal{U}) = \check{\Omega}^*_{\mathbb{R}}(\mathcal{U})/\check{C}^*(\mathcal{U}, \mathbb{Z}). \tag{2.6}$$

Notice that the canonical map

$$\varepsilon^*\colon \Omega^*(X) \to \check{\Omega}^*_{\mathbb{R}}(\mathcal{U}) \to \check{\Omega}^*_{\mathbb{R}/\mathbb{Z}}(\mathcal{U})$$

is injective in degrees > 0. We now have the following:

Lemma 2.1. *Let $\mathcal{U}$ be a good covering of X. Then*
(i) $H^*(\check{\Omega}^*_{\mathbb{R}}(\mathcal{U})/\varepsilon^*\Omega^*(X)) = 0$.
(ii) $H^*(\check{\Omega}^*_{\mathbb{R}/\mathbb{Z}}(\mathcal{U})) \cong H^*(X, \mathbb{R}/\mathbb{Z})$ for $\mathbb{R}/\mathbb{Z}$ the constant sheaf.
(iii) There is a natural isomorphism

$$D_*\colon H^\ell(\check{\Omega}^*_{\mathbb{R}/\mathbb{Z}}(\mathcal{U})/\varepsilon^*\Omega^*(X)) \cong H^{\ell+1}(X, \mathbb{Z})$$

for $\ell \geq 0$.

Proof. (i) follows since $\varepsilon^*\colon \Omega^*(X) \xrightarrow{\subseteq} \check{\Omega}^*_{\mathbb{R}}(\mathcal{U})$ is a homology isomorphism.
(ii) follows since, for $\mathbb{R}$ the constant sheaf, the inclusion $\check{C}^*(\mathcal{U}, \mathbb{R}) \xrightarrow{\subseteq} \check{\Omega}^*_{\mathbb{R}}(\mathcal{U})$ is a homology isomorphism.
(iii) Now D_* is just the connecting homomorphism for the exact sequence

$$0 \to \check{C}^*(\mathcal{U}, \mathbb{Z}) \to \check{\Omega}^*_{\mathbb{R}}(\mathcal{U})/\varepsilon^*\Omega^*(X) \to \check{\Omega}^*_{\mathbb{R}/\mathbb{Z}}(\mathcal{U})/\varepsilon^*\Omega^*(X) \to 0.$$

□

We can now define a gerbe with connection as follows:

Definition 2.2. Let $\mathcal{U}$ be a good covering for X.
(i) A *connection* ω in an ℓ–gerbe $\theta \in \check{\Omega}^{\ell,0}_{\mathbb{R}/\mathbb{Z}}(\mathcal{U})$, $\check{\delta}\theta = 0$, is given by $\omega \in \check{\Omega}^{\ell}_{\mathbb{R}}(\mathcal{U})$, that is a sequence $\omega = (\omega^0, \ldots, \omega^\ell)$, $\omega^\nu \in \check{\Omega}^{\nu,\ell-\nu}_{\mathbb{R}}(\mathcal{U})$, $\nu = 0, \ldots, \ell$, with $\omega^\ell \equiv -\theta$ mod $\mathbb{Z}$, such that ω is a cycle in $\check{\Omega}^*_{\mathbb{R}/\mathbb{Z}}(\mathcal{U})/\varepsilon^*\Omega^*(X)$.
(ii) The *curvature form* for ω is the unique closed $(\ell+1)$–form F_ω such that

$$\varepsilon^* F_\omega = d\omega^0 \in \check{\Omega}^{0,\ell+1}_{\mathbb{R}}(\mathcal{U})$$

The connection is called *flat* if $F_\omega = 0$.
(iii) Two ℓ–gerbes θ_1, θ_2 with connections ω_1, ω_2 are *equivalent* if $\omega_1 - \omega_2$ is a coboundary in $\check{\Omega}^*_{\mathbb{R}/\mathbb{Z}}(\mathcal{U})$. The set of equivalence classes $[\theta, \omega]$ is denoted $H^{\ell+1}_{\mathcal{D}}(X, \mathbb{Z})$ and is called the (smooth) *Deligne cohomology* in degree $\ell + 1$ (note the shift in degree).

Remarks 2.3. 1. Thus $H^{\ell+1}_{\mathcal{D}}(X,\mathbb{Z})$ is the homology of the sequence

$$\check{\Omega}^{\ell-1}_{\mathbb{R}/\mathbb{Z}}(\mathcal{U}) \xrightarrow{d} \check{\Omega}^{\ell}_{\mathbb{R}/\mathbb{Z}}(\mathcal{U}) \xrightarrow{d} \check{\Omega}^{\ell+1}_{\mathbb{R}/\mathbb{Z}}(\mathcal{U})/\varepsilon^*\Omega^{\ell+1}(X). \tag{2.7}$$

2. The set of equivalence classes of ℓ–gerbes with *flat* connections is isomorphic to $H^\ell(X,\mathbb{R}/\mathbb{Z})$ by Lemma 2.1.

3. It follows also using Lemma 2.1, that there is a natural exact sequence

$$0 \to H^\ell(X,\mathbb{R}/\mathbb{Z}) \longrightarrow H^{\ell+1}_{\mathcal{D}}(X,\mathbb{Z}) \xrightarrow{d_*} \Omega^{\ell+1}_{\mathrm{cl}}(X,\mathbb{Z}) \longrightarrow 0. \tag{2.8}$$

Here $\Omega^{\ell+1}_{\mathrm{cl}}(X,\mathbb{Z}) \subseteq \Omega^*(X)$ denotes the set of closed forms with integral periods, and d_* is induced by the map sending ω to the curvature form F_ω. In particular, as the notation indicates, $H^{\ell+1}_{\mathcal{D}}(X,\mathbb{Z})$ does not depend on the choice of a (good) covering $\mathcal{U}$.

4. Notice the natural commutative diagram

$$\begin{array}{ccc} H^{\ell+1}_{\mathcal{D}}(X,\mathbb{Z}) & \longrightarrow & H^\ell(X,\underline{\mathbb{R}/\mathbb{Z}}) \\ \downarrow & & \cong\downarrow\delta_* \\ H^\ell(\check{\Omega}^*_{\mathbb{R}/\mathbb{Z}}(\mathcal{U})/\varepsilon^*\Omega^*(X)) & \xrightarrow[\cong]{D_*} & H^{\ell+1}(X,\mathbb{Z}) \end{array} \tag{2.9}$$

where the top horizontal map is induced by the map forgetting the connection and where D_* is given by Lemma 2.1.

5. The explicit description of an ℓ–gerbe θ with connection ω is as follows. Let ω be a sequence $(\omega^0,\dots,\omega^\ell)$ of cochains $\omega^\nu \in \check{\Omega}^{\nu,\ell-\nu}_{\mathbb{R}}(\mathcal{U})$, $\nu = 0,\dots,\ell$, satisfying

$$\begin{aligned} &\check{\delta}\omega^{\nu-1} + (-1)^\nu d\omega^\nu = 0\ , \qquad \nu = 1,\dots,\ell, \\ &\check{\delta}\omega^\ell \equiv 0 \mod \mathbb{Z}. \end{aligned} \tag{2.10}$$

The first equation for $\nu = 1$ in (2.10) implies that $\check{\delta}d\omega^0 = 0$, and $d\omega^0$ defines a global closed $(\ell+1)$–form F_ω, that is $\varepsilon^*F_\omega = d\omega^0$. The last equation in (2.10) says that $-\check{\delta}\omega^\ell$ is an integral $(\ell+1)$–cycle $z \in \check{Z}^{\ell+1}(\mathcal{U},\mathbb{Z})$, that is $-\omega^\ell \in \check{\Omega}^{\ell,0}_{\mathbb{R}}(\mathcal{U})$ is the lift of a unique ℓ–cycle $\theta \in \check{\Omega}^{\ell,0}_{\mathbb{R}/\mathbb{Z}}(\mathcal{U})$. Thus from (2.2) we have $\delta_*[\theta] = [z]$. Moreover by construction, the integral class $[z]$ determines the class $[F_\omega]$ under the canonical homomorphism $r : H^{\ell+1}(X,\mathbb{Z}) \to H^{\ell+1}(X,\mathbb{R})$. Then ω is a connection for the ℓ–gerbe θ.

6. In terms of the notation in [5] our smooth Deligne cohomology group $H^{\ell+1}_{\mathcal{D}}(X,\mathbb{Z})$ is canonically isomorphic to the group $H^{\ell+1}_{\mathcal{D},\infty}(X,\mathbb{Z}(\ell+1))$, that is the hypercohomology group of X in degree $\ell+1$ with values in the sheaf complex

$$\mathbb{Z} \to \underline{\Omega}^0 \to \underline{\Omega}^1 \to \dots \to \underline{\Omega}^\ell\ .$$

Since in the smooth case $H^k_{\mathcal{D},\infty}(X,\mathbb{Z}(\ell+1))$ is ordinary cohomology with coefficients $\mathbb{R}/\mathbb{Z}$ for $k < \ell+1$, respectively $\mathbb{Z}$ for $k > \ell+1$, $k = \ell+1$ is the only degree which needs a special name and we have therefore deleted the extra index from the notation. This is of course in contrast to the holomorphic Deligne cohomology for an algebraic variety.

Finally let us mention the interpretation of $H^*_{\mathcal{D}}(X,\mathbb{Z})$ as the group of *differential characters* in the sense of Cheeger-Simons [9] (see also Dupont et.al. [13]). Let $C^{\mathrm{Sing}}_*(X)$ denote the chain complex of (smooth) singular chains in X and let

$$\mathcal{I}\colon \Omega^*(X) \to C^*_{\mathrm{Sing}}(X,\mathbb{R}) = \mathrm{Hom}_{\mathbb{Z}}(C^{\mathrm{Sing}}_*(X),\mathbb{R})$$

be the deRham integration map.

Definition 2.4. The group of *differential characters* (mod $\mathbb{Z}$) in degree $\ell+1$ is

$$\widehat{H}^{\ell+1}(X,\mathbb{Z}) = \{(f,\alpha) \in \mathrm{Hom}_{\mathbb{Z}}(Z^{\mathrm{Sing}}_\ell(X),\mathbb{R}/\mathbb{Z}) \oplus \Omega^{\ell+1}(X) \mid \delta f = \mathcal{I}(\alpha) \text{ and } d\alpha = 0\}.$$

Here $Z^{\mathrm{Sing}}_\ell(X) \subseteq C^{\mathrm{Sing}}_\ell(X)$ is the set of cycles. The following is well-known (cf. [13]) but is included for completeness:

Proposition 2.5. *There is a natural isomorphism* $H^*_{\mathcal{D}}(X,\mathbb{Z}) \cong \widehat{H}^*(X,\mathbb{Z})$.

Proof. Choose a good open covering $\mathcal{U} = \{U_i \mid i \in I\}$ of X and let $i\colon C^{\mathrm{Sing}}_*(X,\mathcal{U}) \subseteq C^{\mathrm{Sing}}_*(X)$ be the inclusion of the subcomplex generated by $\bigcup_{i\in I} C^{\mathrm{Sing}}_*(U_i)$. Since i is a chain equivalence we can choose a chain map

$$p\colon C^{\mathrm{Sing}}_*(X) \to C^{\mathrm{Sing}}_*(X,\mathcal{U})$$

such that $p\circ i = \mathrm{id}$ and $i\circ p$ is chain homotopic to the identity with chain homotopy s. Then for $(f,\alpha) \in \widehat{H}^{\ell+1}(X,\mathbb{Z})$ and $\xi \in Z^{\mathrm{Sing}}_\ell(X)$ we have

$$\langle f,\xi\rangle - \langle f, i\circ p(\xi)\rangle = \langle \delta f, s(\xi)\rangle = \langle \mathcal{I}(\alpha), s(\xi)\rangle$$

so that f is determined by its restriction to the set of cycles $Z^{\mathrm{Sing}}_\ell(X,\mathcal{U})$ in the chain complex $C^{\mathrm{Sing}}_*(X,\mathcal{U})$. Hence we can replace $Z^{\mathrm{Sing}}_\ell(X)$ by $Z^{\mathrm{Sing}}_\ell(X,\mathcal{U})$ in Definition 2.4. Now we consider the Čech bicomplex of singular chains

$$\check{C}^{\mathrm{Sing}}_{p,q}(\mathcal{U}) = \bigoplus_{(i_0,\dots,i_p)} C^{\mathrm{Sing}}_q(U_{i_0\cdots i_p})$$

with associated total complex $\check{C}^{\mathrm{Sing}}_*(\mathcal{U})$. Then again the natural chain map

$$\begin{array}{ccc} \check{C}^{\mathrm{Sing}}_*(\mathcal{U}) & \xrightarrow{\varepsilon_*} & C^{\mathrm{Sing}}_*(X,\mathcal{U}) \\ & \searrow \quad \nearrow & \\ & \check{C}^{\mathrm{Sing}}_{0,*}(\mathcal{U}) & \end{array}$$

induced by $\varepsilon\colon \bigsqcup_{i\in I} U_i \to X$, has an 'inverse' chain map j such that $\varepsilon_*\circ j = \mathrm{id}$ and $j\circ\varepsilon_*$ is chain homotopic to the identity. Now we can define a map

$$j_*\colon H^*_{\mathcal{D}}(X,\mathbb{Z}) \to \widehat{H}^*(X,\mathbb{Z})$$

by $j_*[\omega,\theta] = (f,\alpha)$ where $f(\xi) = \langle \mathcal{I}(\omega), j(\xi)\rangle$, $\xi \in Z^{\mathrm{Sing}}_\ell(X,\mathcal{U})$, and $\alpha = (\varepsilon^*)^{-1} d\omega^0$. In fact for $x \in C^{\mathrm{Sing}}_{\ell+1}(X,\mathcal{U})$ we have

$$\begin{aligned} \langle \delta f, x\rangle &= \langle \mathcal{I}(\omega), \partial j(x)\rangle = \langle \mathcal{I}(D(\omega)), j(x)\rangle \\ &= \langle \mathcal{I}(d\omega^0), j(x)\rangle = \langle \mathcal{I}(\alpha), \varepsilon_* j_*(x)\rangle = \langle \mathcal{I}(\alpha), x\rangle \end{aligned}$$

so that $(f,\alpha) \in \widehat{H}^{\ell+1}(X,\mathbb{Z})$. Since any two choices of j are chain homotopic, it is also straight forward to see that j_* does not depend on the particular choice.

Finally, in order to show that j_* is an isomorphism one just observes that there is a natural exact sequence similar to the one in (2.8):

$$0 \to H^\ell(X, \mathbb{R}/\mathbb{Z}) \to \widehat{H}^{\ell+1}(X, \mathbb{Z}) \to \Omega^{\ell+1}_{\mathrm{cl}}(X, \mathbb{Z}) \to 0 \tag{2.11}$$

where the second map is the one sending (f, α) to α. □

3. Gerbes and simplicial forms

In this section we shall reformulate the smooth Deligne cohomology in terms of simplicial deRham cohomology as in [11], [12] and [15]. As before let X be a smooth manifold and let $\mathcal{U} = \{U_i\}_{i\in I}$ be a good covering of X. For convenience we choose a linear ordering of the index set I. The *nerve* $N\mathcal{U}$ of $\mathcal{U}$ is the simplicial manifold $N\mathcal{U} = \{N\mathcal{U}(p)\}_{p\geq 0}$, given by

$$N\mathcal{U}(p) = \coprod_{i_0 \leq \cdots \leq i_p} U_{i_0\cdots i_p} \tag{3.1}$$

and with face and degeneracy operators $\varepsilon_i \colon N\mathcal{U}(p) \to N\mathcal{U}(p-1)$, $i = 0, \ldots, p$, $\eta_j \colon N\mathcal{U}(p) \to N\mathcal{U}(p+1)$, $j = 1, \ldots, p$, given by the obvious inclusion maps corresponding to deletion of the i-th index respectively repeating the j-th index. Also let $\Delta^p \subseteq \mathbb{R}^{p+1}$ be the *standard* p-simplex

$$\Delta^p = \Big\{ t = (t_0, \ldots, t_p) \;\Big|\; t_i \geq 0, \sum_{i=0}^{p} t_i = 1 \Big\}$$

with the corresponding face and degeneracy maps $\varepsilon^i \colon \Delta^{p-1} \to \Delta^p$, $i = 0, \ldots, p$, respectively $\eta^j \colon \Delta^{p+1} \to \Delta^p$, $j = 0, \ldots, p$.

Definition 3.1. (i) A *simplicial* k-form ω on $N\mathcal{U}$ is a sequence of k-forms $\omega^{(p)}$ on $\Delta^p \times N\mathcal{U}(p)$ satisfying

$$(\varepsilon^i \times \mathrm{id})^* \omega^{(p)} = (\mathrm{id} \times \varepsilon_i)^* \omega^{(p-1)}, \qquad i = 0, \ldots, p, \quad p = 1, 2, \ldots.$$

(ii) ω is called *normal* if it furthermore satisfies

$$(\eta^i \times \mathrm{id})^* \omega^{(p)} = (\mathrm{id} \times \eta_i) \omega^{(p+1)}, \qquad i = 1, \ldots, p, \quad p = 1, 2, \ldots.$$

We shall denote the set of simplicial k-forms (respectively normal k-forms) by $\Omega^k(||N\mathcal{U}||)$ (respectively $\Omega^k(|N\mathcal{U}|)$) corresponding to the 'fat' (respectively 'thin') realizations $||N\mathcal{U}||$ (respectively $|N\mathcal{U}|$). Clearly $\Omega^*(||N\mathcal{U}||)$ is a differential graded algebra and $\Omega^*(|N\mathcal{U}|) \subseteq \Omega^*(||N\mathcal{U}||)$ is a DGA-subalgebra. Notice that the inclusions $U_i \subseteq X$ induce a natural simplicial map $\varepsilon \colon N\mathcal{U} \to N\{X\}$ and this in turn induces a DGA-map

$$\varepsilon^* \colon \Omega^*(X) \to \Omega^*(|N\mathcal{U}|) \subseteq \Omega^*(||N\mathcal{U}||) \tag{3.2}$$

where $\Omega^*(X) = \Omega^*(|N\{X\}|)$ is the usual deRham complex. It follows from [11] that ε^* induces homology isomorphisms

$$\varepsilon^* \colon H(\Omega^*(X)) \xrightarrow[\cong]{} H(\Omega^*(|N\mathcal{U}|)) \xrightarrow[\cong]{} H(\Omega^*(||N\mathcal{U}||)). \tag{3.3}$$

The relation with the Čech–deRham complex in section 2 is given by the integration map

$$\mathcal{I}_\Delta \colon \Omega^{p,q}(||N\mathcal{U}||) \to \check{\Omega}^{p,q}_{\mathbb{R}}(\mathcal{U}), \quad \mathcal{I}_\Delta(\omega) = \int_{\Delta^p} \omega^{(p)}, \tag{3.4}$$

where ω lies in $\Omega^{p,q}$ if it has degree p as a form in the variables of Δ^n, $n \geq p$. This is a map of bicomplexes and again by [11] the corresponding map of total complexes induces an isomorphism

$$\mathcal{I}_\Delta \colon H(\Omega^*(||N\mathcal{U}||)) \xrightarrow[\cong]{} H(\check{\Omega}^*_{\mathbb{R}}(\mathcal{U})). \tag{3.5}$$

Also $\mathcal{I}_\Delta$ clearly commutes with ε^* given by (2.5) and (3.2).

For the representation of the integral cohomology we also consider the discrete simplicial set $N_d\mathcal{U}$ where a p-simplex is a point $(i_0, \dots, i_p)$ for each non-empty intersection $U_{i_0} \cap \dots \cap U_{i_p}$, $i_0 \leq i_1 \leq \ldots \leq i_p$, and we let $\eta \colon N\mathcal{U} \to N_d\mathcal{U}$ denote the simplicial map sending $U_{i_0} \cap \dots \cap U_{i_p}$ to $(i_0, \dots, i_p)$. Notice that for $\mathcal{U}$ a good covering we have a commutative diagram of homotopy equvalences

$$\begin{array}{ccc} ||N\mathcal{U}|| & \xrightarrow[\simeq]{} & |N\mathcal{U}| \\ \Big\downarrow{\scriptstyle ||\eta||} & & \Big\downarrow{\scriptstyle |\eta|} \\ ||N_d\mathcal{U}|| & \xrightarrow[\simeq]{} & |N_d\mathcal{U}| \end{array} \tag{3.6}$$

and a similar diagram of isomorphisms

$$\begin{array}{ccc} H(\Omega^*(||N\mathcal{U}||)) & \xleftarrow[\cong]{} & H(\Omega^*(|N\mathcal{U}|)) \\ {\scriptstyle \eta^*}\Big\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\Big\uparrow{\scriptstyle \eta^*} \\ H(\Omega^*(||N_d\mathcal{U}||)) & \xleftarrow[\cong]{} & H(\Omega^*(|N_d\mathcal{U}|)) \end{array} \tag{3.7}$$

Also notice that η^* maps

$$\Omega^*(||N_d\mathcal{U}||) = \Omega^{*,0}(||N_d\mathcal{U}||)$$

injectively into $\Omega^{*,0}(||N\mathcal{U}||) \subseteq \Omega^*(||N\mathcal{U}||)$ and that $\omega \in \Omega^*(||N\mathcal{U}||)$ lies in the image if and only if it only involves the variables of Δ^p.

Definition 3.2. (i) A k-form $\omega \in \Omega^*(||N\mathcal{U}||)$ is called *discrete* if $\omega \in \eta^*(\Omega^*(||N_d\mathcal{U}||)$.
(ii) $\omega \in \Omega^*(||N\mathcal{U}||)$ is called *integral* if it is discrete and if furthermore

$$\mathcal{I}_\Delta(\omega) \in \check{C}^*(\mathcal{U}, \mathbb{Z}) \subseteq \check{\Omega}^{*,0}(\mathcal{U}).$$

We let $\Omega^*_{\mathbb{Z}}(||N\mathcal{U}||) \subseteq \Omega^*(||N\mathcal{U}||)$ (respectively $\Omega^*_{\mathbb{Z}}(|N\mathcal{U}|) \subseteq \Omega^*(|N\mathcal{U}|)$) denote the chain complex of integral forms (respectively integral normal forms) and we also put

$$\Omega^*_{\mathbb{R}/\mathbb{Z}}(||N\mathcal{U}||) = \Omega^*(||N\mathcal{U}||)/\Omega^*_{\mathbb{Z}}(||N\mathcal{U}||) \tag{3.8}$$

respectively

$$\Omega^*_{\mathbb{R}/\mathbb{Z}}(|N\mathcal{U}|) = \Omega^*(|N\mathcal{U}|)/\Omega^*_{\mathbb{Z}}(|N\mathcal{U}|). \tag{3.9}$$

We now have the following:

Proposition 3.3. *Let $\mathcal{U}$ be a good covering. Then there are natural isomorphisms*

(i) $H(\Omega^*_{\mathbb{Z}}(||N_d\mathcal{U}||)) \overset{\eta^*}{\cong} H(\Omega^*_{\mathbb{Z}}(||N\mathcal{U}||)) \overset{\mathcal{I}_{\Delta *}}{\cong} H(\check{C}^*(\mathcal{U}, \mathbb{Z})) = H^*(X, \mathbb{Z})$,

(ii) $H(\Omega^*_{\mathbb{R}/\mathbb{Z}}(||N\mathcal{U}||)) \overset{\mathcal{I}_{\Delta *}}{\cong} H(\check{\Omega}^*_{\mathbb{R}/\mathbb{Z}}(\mathcal{U})) \cong H^*(X, \mathbb{R}/\mathbb{Z})$,

(iii) $H^\ell(\Omega^*(||N\mathcal{U}||)/(\Omega^*_{\mathbb{Z}}(||N\mathcal{U}||) + \varepsilon^*\Omega^*(X))) \overset{d_*}{\cong} H^{\ell+1}(\Omega^*_{\mathbb{Z}}(||N\mathcal{U}||)) \cong H^{\ell+1}(X, \mathbb{Z})$.

(iv) Furthermore $\mathcal{I}_\Delta$ induces a natural isomorphism to $H^{\ell+1}_{\mathcal{D}}(X,\mathbb{Z})$ from the homology of the sequence

$$\Omega^{\ell-1}_{\mathbb{R}/\mathbb{Z}}(||N\mathcal{U}||) \xrightarrow{d} \Omega^{\ell}_{\mathbb{R}/\mathbb{Z}}(||N\mathcal{U}||) \xrightarrow{d} \Omega^{\ell+1}_{\mathbb{R}/\mathbb{Z}}(||N\mathcal{U}||)/\varepsilon^*\Omega^{\ell+1}(X). \tag{3.10}$$

(v) In (i)–(iv) above $||N\mathcal{U}||$ can be replaced by $|N\mathcal{U}|$.

Proof. (i) In the commutative diagram

$$\begin{array}{ccc} \Omega^*_{\mathbb{Z}}(||N_d\mathcal{U}||) & \xrightarrow{\mathcal{I}_\Delta} & \\ \downarrow \eta^* & & \check{C}^*(\mathcal{U},\mathbb{Z}) \\ \Omega^*_{\mathbb{Z}}(||N\mathcal{U}||) & \xrightarrow{\mathcal{I}_\Delta} & \end{array}$$

η^* is an isomorphism and $\mathcal{I}_\Delta$ for $N_d\mathcal{U}$ is a homology isomorphism since it is surjective and the kernel has vanishing homology by the simplicial deRham theorem. Hence also $\mathcal{I}_\Delta$ for $N\mathcal{U}$ is a homology isomorphism.

(ii) now follows from (i) and (3.5) together with Lemma 2.1 (ii).

(iii) is similar to lemma 2.1, (iii).

(iv) follows from the five–lemma applied to the sequence in (2.8) and the corresponding sequence for the homology group in (3.10).

(v) follows similarly. □

Corollary 3.4. *Every class in $H^{\ell+1}_{\mathcal{D}}(X,\mathbb{Z})$ can be represented by an ℓ–gerbe θ with connection ω of the form $\omega = \mathcal{I}_\Delta(\Lambda)$ for some simplicial ℓ-form $\Lambda \in \Omega^\ell(||N\mathcal{U}||)$ satisfying*

$$d\Lambda = \varepsilon^*\alpha - \eta^*\beta, \qquad \alpha \in \Omega^{\ell+1}(X), \quad \beta \in \Omega^{\ell+1}_{\mathbb{Z}}(||N_d\mathcal{U}||). \tag{3.11}$$

Furthermore Λ and β can be chosen to be normal in the sense of Definition 3.1.

Remarks 3.5. 1. We shall call a (normal) simplicial ℓ-form Λ a (*normal*) *simplicial ℓ–gerbe* if it satisfies (3.11).

2. Continuing with the previous notation, we write

$$\Lambda = \Lambda^0 + \cdots + \Lambda^\ell \in \bigoplus_{\nu=0}^{\ell} \Omega^{\nu,\ell-\nu}(||N\mathcal{U}||)$$

and we put

$$\theta = -\int_{\Delta^\ell} \Lambda^\ell, \qquad \omega^\nu = \int_{\Delta^\nu} \Lambda^\nu, \qquad \nu = 0,\dots,\ell. \tag{3.12}$$

Then (3.11) corresponds to the condition (2.10) for the ℓ–gerbe θ with connection $\omega = (\omega^0,\dots,\omega^\ell \equiv -\theta)$.

3. Note that α and β in (3.11) are uniquely determined by Λ and that α is the curvature form of ω. We shall refer to it as the *curvature form* for Λ.

4. By (3.11) and (3.12) we have

$$\mathcal{I}_\Delta(\beta) = -\int_{\Delta^{\ell+1}} d\Lambda^\ell = \check{\delta}\theta \in \check{C}^{\ell+1}(\mathcal{U},\mathbb{Z}). \tag{3.13}$$

Hence β represents the characteristic class

$$z = \check{\delta}_*[\theta] \in H^{\ell+1}(X,\mathbb{Z}) = H^{\ell+1}(\Omega^*_{\mathbb{Z}}(||N_d\mathcal{U}||)).$$

5. The simplicial deRham complexes $\Omega^*(||N\mathcal{U}||)$ and $\Omega^*(|N\mathcal{U}|)$ as well as the corresponding subcomplexes of integral forms are clearly functorial with respect to smooth maps $f\colon X' \to X$ and *compatible coverings*. By this we mean coverings $\mathcal{U}' = \{U'_{i'}\}_{i'\in I'}$ of X' and $\mathcal{U} = \{U_i\}_{i\in I}$ of X together with an order preserving map $\nu\colon I' \to I$ such that $f(U'_{i'}) \subseteq U_{\nu(i')}$ for all $i' \in I'$; that is, $\mathcal{U}'$ is a refinement of $f^{-1}(\mathcal{U})$. The induced maps in the deRham complexes do depend on ν but the induced map in Deligne cohomology does not. Notice that this is the case also for $f = \mathrm{id}\colon X \to X$, that is, when $\mathcal{U}'$ is a refinement of $\mathcal{U}$.

6. If X has dimension m then it also has covering dimension m (see e.g. [29], chap. II). Hence by taking a suitable refinement we obtain a covering $\mathcal{U}'$ for which $N\mathcal{U}'$ has only *non-degenerate* simplices of dimension $\leq m$. In particular for such a covering we have

$$\Omega^{k,\ell}(|N\mathcal{U}'|) = 0 \qquad \text{and} \qquad \Omega^k(|N_d\mathcal{U}'|) = 0 \qquad \text{for } k > m. \tag{3.14}$$

4. Fibre integration of simplicial forms

Fibre integration in smooth Deligne cohomology can be done in various ways, see e.g. Freed [20], Gomi–Terashima [22] or Hopkins–Singer [26]. In this section we sketch how to define it in terms of simplicial forms. We refer to Dupont–Ljungmann [17] for the details.

In the following X denotes an oriented compact manifold of dimension m possibly with boundary and $\pi\colon Y \to Z$ is a smooth fibre bundle with fibre X and structure group $\mathrm{Diff}^+(X)$ of orientation perserving diffeomorphisms. Also let $\mathcal{V} = \{V_j\}_{j\in J}$ and $\mathcal{U} = \{U_i\}_{i\in I}$ be good open coverings of Y respectively Z (not necessarily compatible). We shall define *integration along the fibre* for a *normal* simplical $(k+m)$-form $\omega \in \Omega^{k+m}(|N\mathcal{V}|)$ as a simplicial k-form $\int_{Y/Z}\omega \in \Omega^k(||N\mathcal{U}||)$ defined by usual fibre integration in the bundle $\Delta^p \times N(\pi^{-1}\mathcal{U})(p) \to \Delta^p \times N\mathcal{U}(p)$, $p = 0, 1, 2, \ldots$ with fibre X:

$$\int_{Y/Z} \omega|_{\Delta^p\times N(\pi^{-1}\mathcal{U})(p)} = \int_{(\Delta^p\times N(\pi^{-1}\mathcal{U})(p))/(\Delta^p\times N\mathcal{U}(p))} \tilde{\phi}^*\omega, \tag{4.1}$$

where $\pi^{-1}\mathcal{U} = \{\pi^{-1}U_i\}_{i\in I}$ is the obvious covering of Y and $\tilde{\phi}\colon ||N(\pi^{-1}\mathcal{U})|| \to |N\mathcal{V}|$ denotes a 'piecewise smooth' map associated to a choice of partition of unity for the coverings $\{\pi^{-1}U_i \cap V_j\}_{j\in J}$ for each $i \in I$. For the construction of $\tilde{\phi}$ let us assume for simplicity that $\pi\colon Y \to Z$ is the product fibration $X \times Z \to Z$. For the case of a general fibration we refer to [17]. By remark 3.5, 5 we can assume that $\mathcal{V} = \mathcal{U}' \times \mathcal{U} = \{V_{ij} = U'_j \times U_i\}_{i\in I,\, j\in J}$ where $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{U}' = \{U'_j\}_{j\in J}$ are open coverings af Z and X respectively and we order $I \times J$ lexicographically with $i \in I$ before $j \in J$. (Notice the interchange in V_{ij}.) Also as in remark 3.5, 6 we can assume that $N\mathcal{U}'$ has only non-degenerate simplices of dimension $\leq m$ and that $N(\mathcal{U}' \cap \partial X)$ has only non-degenerate simplices of dimension $\leq m-1$ ($m =$ dimension of X). Finally we choose a partition of unity $\{\phi_j\}_{j\in J}$ subordinate $\mathcal{U}'$. Then the natural projection $|N\mathcal{U}'| \to X$ has a right-inverse $\bar{\phi}\colon X \to |N\mathcal{U}'|$ defined by

$$\bar{\phi}(x) = ((\phi_{j_0}(x), \ldots, \phi_{j_q}(x), x)_{j_0\cdots j_q} \in \Delta^q \times N\mathcal{U}'(q) \tag{4.2}$$

for those $x \in U_{j_0\cdots j_q} \subseteq X$ satisfying $\phi_{j_0}(x) + \cdots + \phi_{j_q}(x) = 1$. Now, we would like to define $\tilde{\phi}$ in a similar fashion as the composite in the diagram

$$\begin{array}{ccc} ||N(\pi^{-1}\mathcal{U})|| & & \\ \downarrow & \searrow^{\tilde{\phi}} & \\ |N(\pi^{-1}\mathcal{U})| & & |N\mathcal{U}' \times N\mathcal{U}| = |N(\mathcal{U}' \times \mathcal{U})| \\ \| & & \tau \downarrow \approx \\ X \times |N\mathcal{U}| & \xrightarrow{\bar{\phi}\times \mathrm{id}} & |N\mathcal{U}'| \times |N\mathcal{U}| \end{array} \tag{4.3}$$

where the homeomorphism τ is induced by the Eilenberg-Zilber triangulation map

$$\Delta^n \times (N\mathcal{U}'(n) \times N\mathcal{U}(n)) \to (\Delta^n \times N\mathcal{U}'(n)) \times (\Delta^n \times N\mathcal{U}(n))$$

given by the diagonal $\Delta^n \to \Delta^n \times \Delta^n$. It is well-known that τ^{-1} is given by the triangulation of a prism $\Delta^q \times \Delta^p$ into n-simplices ($n = p + q$) one for each '(q, p)-shuffle' of $(0, \dots, n)$, that is, a pair of non-decreasing functions

$$(\nu, \mu)\colon \{0, \dots, n\} \to \{0, \dots, q\} \times \{0, \dots, p\}$$

satisfying

$$\mu(0) = \nu(0) = 0, \qquad \mu(n) = p, \qquad \nu(n) = q, \qquad \text{and} \tag{4.4}$$

$$\mu(r) - \mu(r-1) + \nu(r) - \nu(r-1) = 1, \qquad r = 1, \dots, n, \tag{4.5}$$

(so that for increasing r the functions μ and ν alternate increasing by 1). It follows that $\tilde{\phi}^*\omega \in \Omega^{k+m}(||N(\pi^{-1}\mathcal{U})||)$ is the simplicial form defined explicitly on $\Delta^p \times (X \times U_{i_0\cdots i_p})$ in a neighborhood of a point (t, x, z) by the sum

$$(\tilde{\phi}^*\omega)_{i_0\cdots i_p} = \sum_{(\nu,\mu)} \tilde{\phi}^*_{(\nu,\mu)}\omega \tag{4.6}$$

with (ν, μ) running through the (q, p)-shuffles as above. Here q is determined such that $\phi_{j_0} + \cdots + \phi_{j_q} = 1$ near x and

$$\tilde{\phi}_{(\nu,\mu)}\colon \Delta^p \times (U'_{j_0\cdots j_q} \times U_{i_0\cdots,i_p}) \to \Delta^n \times (U'_{j_{\nu(0)}} \times U_{i_{\mu(0)}}) \cap \cdots \cap (U'_{j_{\nu(n)}} \times U_{i_{\mu(n)}})$$

is given by the formula

$$\tilde{\phi}_{(\nu,\mu)}(t, x, z) = (\sigma_0, \dots, \sigma_n, x, z) \tag{4.7}$$

where

$$\sigma_r = \sum_{(\nu',\mu')} t_{\mu'}\phi_{j_{\nu'}}(x) \tag{4.8}$$

is a sum over the pairs of integers (ν', μ'), $\mu' = 1, \dots, p$, $\nu' = 0, \dots, q$, satisfying $(\nu(r-1), \mu(r-1)) < (\nu', \mu') \leq (\nu(r), \mu(r))$ in the lexicographical order. That is,

$$\sigma_r = \begin{cases} t_{\mu(r)}\phi_{j_{\nu(r)}}(x) & \text{if } \mu(r-1) = \mu(r),\ \nu(r-1) < \nu(r), \\ t_{\mu(r-1)} \displaystyle\sum_{\nu(r)<\nu'} \phi_{j_{\nu'}}(x) + \\ + \ t_{\mu(r)} \displaystyle\sum_{\nu' \leq \nu(r)} \phi_{j_{\nu'}}(x) & \text{if } \mu(r-1) < \mu(r),\ \nu(r-1) = \nu(r). \end{cases}$$

The form given by (4.6) clearly defines a smooth form in $\Delta^p \times N(\pi^{-1}\mathcal{U})(p)$ so that $\int_{Y/Z} \omega$ is indeed well-defined by the formula (4.1). Also it is easy to see from the construction that it is a simplicial k-form, i.e., that it satisfies Definition 3.1 (i). It is however not necessarily a *normal* simplicial form even though ω was normal to begin with.

We note the following properties of fibre integration. The signs are determined by the convention that we always integrate the variables starting from the left:

Proposition 4.1. *(i) Let* $\omega \in \Omega^{k+m-1}(|N\mathcal{V}|)$, $m = \dim X$. *Then*

$$\int_{Y/Z} d\omega = \int_{\partial Y/Z} \omega + (-1)^m \, d \int_{Y/Z} \omega.$$

(ii) If $\partial X = \emptyset$ *and* $\omega \in \Omega^*_{\mathbb{Z}}(|N\mathcal{V}|)$ *then* $\int_{Y/Z} \omega$ *is also integral.*

(iii) Suppose $\partial X = \emptyset$. *Then*

$$\int_{Y/Z} : \Omega^{k+m}(|N\mathcal{V}|) \to \Omega^k(||N\mathcal{U}||)$$

induces the usual transfer map $\pi_! \colon H^{k+m}(Y) \to H^k(Z)$ *with coefficients in* $\mathbb{R}$, $\mathbb{Z}$ *or* $\mathbb{R}/\mathbb{Z}$. *Also it induces a well-defined map of smooth Deligne cohomology*

$$\pi_! \colon H^{k+m}_{\mathcal{D}}(Y, \mathbb{Z}) \to H^k_{\mathcal{D}}(Z, \mathbb{Z})$$

independent of choices of coverings and partition of unity.

(iv) $\pi_!$ *is functorial with respect to bundle maps and compatible coverings.*

Proof. Again we restrict to the product case, referring to [17] for the general case.

(i) By (4.1) this follows as for usual fibre integration from Stokes' Theorem.

(ii) We shall prove that the Čech cochain $c = \mathcal{I}_\Delta \int_{Y/Z} \omega$ for the covering $\mathcal{U}$ has integral values. For this we observe that $(\tilde{\phi}^*\omega)_{i_0\cdots i_k}$ in (4.6) only involves ω restricted to the $(k+m)$-skeleton of $N_d\mathcal{V}$, hence by (3.14) can be assumed to be a closed integral form. Since $\bar{\phi} \colon X \to |N\mathcal{U}'|$ has degree one, it is straight forward from (4.3) and the Eilenberg–Zilber Theorem that $c_{i_0\cdots i_k}$ is the evaluation of $\mathcal{I}_\Delta(\omega)$ on the chain $[X] \times (i_0, \ldots, i_p)$ and hence is integral. In fact it follows that c represents the slant product $\mathcal{I}_\Delta(\omega)/[X]$ in the integral cohomology.

(iii) Since $\pi_! \colon H^{k+m}(Y) \to H^k(Z)$ is induced by the slant product by $[X]$ the first statement is already contained in the proof of (ii). That $\pi_!$ in Deligne cohomology is independent of choice of partition of unity follows from (i) applied to $Y \times [0,1]$ and the partition of unity $\{(1-t)\phi_j + t\phi'_j\}_{j\in J}$ where $\{\phi_j\}_{j\in J}$ and $\{\phi'_j\}_{j\in J}$ are the two given ones for the covering $\mathcal{V}$. Independence of choice of covering is now straightforward using remark 3.5, 5.

(iv) is also straightforward. □

Remark 4.2. In the case of a product fibration $\pi \colon X \times Z \to Z$ with the covering $\mathcal{V} = \{U'_j \times U_i\}_{(i,j)\in I\times J}$ as above, we can also represent a class in $H^*_{\mathcal{D}}(X \times Z, \mathbb{Z})$ by a normal simplicial form ω in the bisimplicial manifold $N\mathcal{U}' \times N\mathcal{U}$ (cf. [15]), i.e., by a collection of compatible forms on $\Delta^q \times \Delta^p \times N\mathcal{U}'(q) \times N\mathcal{U}(p)$. We can then define $\int_\xi \omega \in \Omega^{n-\ell}(|N\mathcal{U}|)$ for $\xi \in \check{C}^{\mathrm{Sing}}_\ell(N\mathcal{U}')$ any class in the Čech bicomplex of singular chains in the notation at the end of section 2 above. In fact, for a singular r-simplex $\xi = \sigma \colon \Delta^r \to U'_{j_0\cdots j_q} \subseteq X$ with $r + q = \ell$, we just integrate the pull-back of ω to $\Delta^q \times \Delta^p \times \Delta^r \times N\mathcal{U}(p)$ over $\Delta^q \times \Delta^r$. For $\xi = [X]$ a representative for the fundamental cycle of X (in case $\partial X = \emptyset$) we have (with τ being the Eilenberg-Zilber map as in (4.3) above):

$$\int_{[X]} \omega = \int_{Y/Z} \tau^* \omega. \tag{4.9}$$

Also we have the Stokes' formula similar to Proposition 4.1 (i):

$$\int_\xi d\omega = \int_{\partial\xi} \omega + (-1)^\ell \, d\int_\xi \omega \qquad \text{for } \omega \in \Omega^n(|N\mathcal{U}'| \times |N\mathcal{U}|). \tag{4.10}$$

We refer to [17] for further details on fibre integration of simplicial forms.

5. Secondary characteristic classes

In this section we reformulate the classical constructions of secondary characteristic classes and 'characters' for connections on principal G-bundles in terms of simplicial forms. For the classical constructions we refer to Kamber–Tondeur [30], Chern–Simons [10], Cheeger–Simons [9] or Dupont–Kamber [16].

In the following $p\colon P \to X$ is a smooth principal G-bundle, G a Lie-group with only finitely many components and $K \subseteq G$ is the maximal compact subgroup. As in section 1 we fix an invariant homogeneous polynomial $Q \in I^{n+1}(G)$, $n \geq 0$, such that one of the following 2 cases occur:

Case I: $Q \in \ker(I^{n+1}(G) \to I^{n+1}(K))$.

Case II: $Q \in I^{n+1}_{\mathbb{Z}}(G)$, that is, there exists an integral class $u \in H^{2n+2}(BK, \mathbb{Z})$ representing the Chern-Weil image of Q in $H^*(BG, \mathbb{R}) \cong H^*(BK, \mathbb{R})$.

Let us introduce the notation

$$H^{\ell+1}_{\mathcal{D}}(X) = \Omega^\ell(X)/d\Omega^{\ell-1}(X) \tag{5.1}$$

In the notation of [5], $H^{\ell+1}_{\mathcal{D}}(X)$ is canonically isomorphic to the smooth Deligne cohomology group $H^{\ell+1}_{\mathcal{D},\infty}(X, 0(\ell+1))$, with '0' denoting the 0-ring, that is the hypercohomology group $\mathbf{H}^\ell(X, \underline{\Omega}^{(\ell)})$ with values in the truncated sheaf complex

$$\underline{\Omega}^{(\ell)} \, : \underline{\Omega}^0 \to \underline{\Omega}^1 \to \ldots \to \underline{\Omega}^\ell \ .$$

The elements $[\omega] \in H^{\ell+1}_{\mathcal{D}}(X)$ can be interpreted as equivalence classes of connections on the trivial ℓ–gerbe $\theta = 0$ by setting

$$\omega^0 = \varepsilon^*\omega \ , \qquad F_\omega = d\omega \ , \qquad \check{\delta}\omega^0 = 0 \ , \qquad \omega^1 = \ldots = \omega^\ell = 0. \tag{5.2}$$

Clearly the connection is flat if and only if $F_\omega = d\omega = 0$, that is $[\omega] \in H^\ell(X, \mathbb{R})$. Combining this with (2.8), the data in (5.2) determine a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^\ell(X,\mathbb{R}) & \longrightarrow & H^{\ell+1}_{\mathcal{D}}(X) & \longrightarrow & \Omega^\ell(X)/\Omega^\ell_{\mathrm{cl}}(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow d & & \\ 0 & \longrightarrow & H^\ell(X,\mathbb{R}/\mathbb{Z}) & \longrightarrow & H^{\ell+1}_{\mathcal{D}}(X,\mathbb{Z}) & \xrightarrow{d_*} & \Omega^{\ell+1}_{\mathrm{cl}}(X,\mathbb{Z}) & \longrightarrow & 0. \end{array} \tag{5.3}$$

Further, it is easy to see that the center vertical arrow in diagram (5.3) is induced by the exact hypercohomolgy sequence associated to the exact triangle of complexes

$$\begin{array}{ccc} \underline{\Omega}^{(\ell)}[-1] & \xrightarrow{\qquad\qquad} & \{\mathbb{Z} \to \underline{\Omega}^0 \to \underline{\Omega}^1 \to \ldots \to \underline{\Omega}^\ell\} \,. \\ & {}_{+1}\nwarrow \quad \swarrow & \\ & \mathbb{Z} & \end{array}$$

With this notation the *secondary characteristic class* associated to Q (case I) or (Q, u) (case II) for a connection A on $P \to X$ is a class

$$\begin{aligned} &[\Lambda(Q, A)] \in H^{2n+2}_{\mathcal{D}}(X) \qquad \text{in case I,} \\ &[\Lambda(Q, u, A)] \in H^{2n+2}_{\mathcal{D}}(X, \mathbb{Z}) \qquad \text{in case II.} \end{aligned} \tag{5.4}$$

Note that that the characteristic classes in $H^*_{\mathcal{D}}(X)$ are defined by global forms, whereas the classes in $H^*_{\mathcal{D}}(X, \mathbb{Z})$ are defined by simplicial forms.

For the construction we need the following well-known lemma which we include for completeness:

Lemma 5.1. *Given a G-bundle $p\colon P \to X$ with connection A and an integer N, there is a bundle map*

$$\begin{array}{ccc} P & \xrightarrow{\bar\psi} & \bar P \\ \downarrow & & \downarrow \\ X & \xrightarrow{\psi} & \bar X \end{array} \tag{5.5}$$

and a connection $\bar A$ on $\bar P$ such that $\bar P$ is N-connected and such that $A = \bar\psi^* \bar A$.

Proof. By choosing $\bar X$ to be a smooth approximation to the classifying space BG we can clearly establish the bundle map in (5.5) with $\bar P$ N-connected. Furthermore, by multiplying $\bar P$ with a Euclidean space, the classifying map ψ can be assumed to be an embedding. Then the connection A on P clearly extends over a tubular neighborhood of X on $\bar X$ and subsequently over all of $\bar X$ by use of a partition of unity. □

Remarks 5.2. 1. Since the classifying map $\bar X \to BG$ for $\bar P$ is unique up to homotopy, we have a natural identification of the cohomology $H^k(\bar X, \mathbb{Z}) \cong H^k(BG, \mathbb{Z})$ for $k \leq N$.
2. There is a functorial construction of the bundle map in (5.5) using simplicial manifolds which however requires the use of multi-simplicial constructions for the Deligne cohomology (cf. [16], [13]).

The classes in (5.4) are now constructed as follows: Choose a bundle map and connection $\bar A$ as in Lemma 5.1 with $N > 2n + 2$ and choose compatible good coverings $\mathcal{U} = \{U_i\}_{i \in I}$ and $\bar{\mathcal{U}} = \{\bar U_{\bar\imath}\}_{\bar\imath \in \bar I}$ of X respectively $\bar X$. Also in case II choose a representative $\bar\gamma \in \Omega^{2n+2}_{\mathbb{Z}}(|N\mathcal{U}|)$ for the cohomology class $u \in H^{2n+2}(|N\bar{\mathcal{U}}|, \mathbb{Z}) \cong H^{2n+2}(BG, \mathbb{Z})$. Then for F_A and $F_{\bar A}$ the curvature forms for A respectively $\bar A$, we can find (normal simplicial) forms $\Lambda(Q, \bar A)$ respectively $\Lambda(Q, u, \bar A)$ such that

$$\begin{aligned} &Q(F^{n+1}_{\bar A}) = d\Lambda(Q, \bar A) \qquad \text{in case I,} \\ &\varepsilon^* Q(F^{n+1}_{\bar A}) - \bar\gamma = d\Lambda(Q, u, \bar A) \qquad \text{in case II,} \end{aligned} \tag{5.6}$$

and we put

$$(5.7)\qquad \begin{aligned} \Lambda(Q,A) &= \psi^*\Lambda(Q,\bar{A}) \in \Omega^{2n+1}(X) &&\text{in case I,}\\ \Lambda(Q,u,A) &= \psi^*\Lambda(Q,u,\bar{A}) \in \Omega^{2n+1}_{\mathbb{R}/\mathbb{Z}}(|N\mathcal{U}|) &&\text{in case II.}\end{aligned}$$

Proposition 5.3. *(i) The classes* $[\Lambda(Q,A)]$*, respectively* $[\Lambda(Q,u,A)]$ *in* (5.4) *are well-defined given* $\bar{P}$ *and* $\bar{A}$*.*

(ii) They are independent of the choice of $\bar{P}$ *and* $\bar{A}$*.*

(iii) They are natural with respect to bundle maps and compatible coverings.

(iv) Curvature formula :

$$(5.8)\qquad \begin{aligned} d\Lambda(Q,A) &= Q(F_A^{n+1}) &&\textit{in case I}\\ d\Lambda(Q,u,A) &= \varepsilon^*Q(F_A^{n+1}) - \gamma &&\textit{in case II}\end{aligned}$$

where $\gamma = \psi^*\bar{\gamma} \in \Omega_{\mathbb{Z}}(|N\mathcal{U}|)$ *represents the characteristic class* $u(P)$ *associated with* u*.*

(v) If $Q(F_A^{n+1}) = 0$*, then*

$$(5.9)\qquad \begin{aligned} [\ \Lambda(Q,A)\ ] &\in H^{2n+1}(X,\mathbb{R}) &&\textit{in case I}\\ [\ \Lambda(Q,u,A)\ ] &\in H^{2n+1}(X,\mathbb{R}/\mathbb{Z}) &&\textit{in case II,}\end{aligned}$$

and

$$(5.10)\qquad d_*[\Lambda(Q,u,A)] = -u(P)$$

where $d_*\colon H^{2n+1}(X,\mathbb{R}/\mathbb{Z}) \to H^{2n+2}(X,\mathbb{Z})$ *is the Bockstein homomorphism.*

Proof. (i), (iii), (iv), and (v) are obvious from the construction in (5.6) and (5.7). Finally for (ii), let $\bar{\psi}'\colon P \to \bar{P}'$ and $\bar{A}'$ be another choice of bundle map and connection as in Lemma 5.1. Then

$$\begin{array}{ccc} P & \xrightarrow{\bar{\psi}\times\bar{\psi}'} & \bar{P}\times\bar{P}' \\ \downarrow & & \downarrow \\ X & \longrightarrow & (\bar{P}\times\bar{P}')/G \end{array}$$

is also a bundle map of the required form and $A_t = (1-t)\bar{A} + t\bar{A}'$, $t \in [0,1]$ gives a family of connections on $\bar{P}\times\bar{P}'$ pulling back to the constant family A in P. The claim therefore follows from the following more general formula (with $\frac{dA_t}{dt} = 0$). □

Lemma 5.4. *Variational formula : Let* A_t*,* $t \in [0,1]$*, be a smooth family of connections on* $P \to X$ *and let* $\widetilde{A}$ *denote the corresponding connection on* $P \times [0,1]$ *over* $X \times [0,1]$*. Then we have on* $\Omega^{2n+1}(X)$ *respectively* $\Omega^{2n+1}(|N\mathcal{U}|)$ *:*

(5.11)

$$\Lambda(Q,A_1) - \Lambda(Q,A_0) = (n+1)\int_0^1 Q\Big(\frac{dA_t}{dt}\wedge F_{A_t}^n\Big)\,dt + d\int_0^1 i_{\frac{d}{dt}}\Lambda(Q,\widetilde{A})\,dt\ ,$$

$$\Lambda(Q,u,A_1) - \Lambda(Q,u,A_0) = (n+1)\varepsilon^*\int_0^1 Q\Big(\frac{dA_t}{dt}\wedge F_{A_t}^n\Big)dt + d\int_0^1 i_{\frac{d}{dt}}\Lambda(Q,u,\widetilde{A})dt$$

in cases I and II respectively.

Proof. Notice that the connection $\widetilde{A}$ on $P \times I$ satisfies $i_{\frac{d}{dt}} \widetilde{A} = 0$. Hence for the curvature $F_{\widetilde{A}} = d\widetilde{A} + \frac{1}{2}[\widetilde{A}, \widetilde{A}]$ we have

$$i_{\frac{d}{dt}} F_{\widetilde{A}} = i_{\frac{d}{dt}} d\widetilde{A} = \frac{dA_t}{dt}.$$

In case II say, we therefore obtain from (5.6):

$$\begin{aligned} \frac{d}{dt}\Lambda(Q, u, A_t) - d i_{\frac{d}{dt}} \Lambda(Q, u, \widetilde{A}) &= i_{\frac{d}{dt}} d\Lambda(Q, u, \widetilde{A}) \\ &= \varepsilon^* i_{\frac{d}{dt}} Q(F_{\widetilde{A}}^{n+1}) \\ &= (n+1)\ \varepsilon^*\ Q\Big(\frac{dA_t}{dt} \wedge F_{A_t}^n\Big), \end{aligned} \tag{5.12}$$

since we can choose the representing integral form for u independent of t. Formula (5.11) now follows from (5.12) by integration. □

The invariants in (5.7) have certain *multiplicative properties* which we state next.

Proposition 5.5. (*i*) *For Q_1 and Q_2 both satisfying case I, we have*

$$\begin{aligned} [\ \Lambda(Q_1Q_2, A)\ ] &= [\ Q_1(A) \wedge \Lambda(Q_2, A)\ ] \\ &= [\ \Lambda(Q_1, A) \wedge Q_2(A)\ ] \in H^*_{\mathcal{D}}(X). \end{aligned} \tag{5.13}$$

(*ii*) *In case II, let u_1, u_2 and $u_1 \cup u_2 \in H^*(BG, \mathbb{Z})$ be represented by integral forms γ_1, γ_2 and γ_3 respectively, and choose the form μ such that $d\mu = \gamma_1 \wedge \gamma_2 - \gamma_3$. Then we have in $H^*_{\mathcal{D}}(X, \mathbb{Z})$:*

$$\begin{aligned} [\ \Lambda(Q_1Q_2, u_1 \cup u_2, A)\ ] &= [\ \Lambda(Q_1, u_1, A) \wedge \psi^*\gamma_2 + \varepsilon^* Q_1(A) \wedge \Lambda(Q_2, u_2, A) - \psi^*\mu\ ] \\ &= [\ \psi^*\gamma_1 \wedge \Lambda(Q_2, u_2, A) + \Lambda(Q_1, u_1, A) \wedge \varepsilon^* Q_2(A) - \psi^*\mu\ ]. \end{aligned}$$

Proof. This is straightforward from the definitions in (5.7). □

We now apply proposition 5.3 to the case of foliated bundles in the sense of Kamber–Tondeur [30]. We recall that a principal G-bundle $p\colon P \to X$ is *foliated* if there are given two foliations $\overline{\mathcal{F}}$ on P, $\mathcal{F}$ on X such that

(5.14)
(i) $\overline{\mathcal{F}}$ is given by a G–equivariant involutive subbundle $T\overline{\mathcal{F}} \subset TP$, that is the action by G on P permutes the leaves of $\bar{\mathcal{F}}$,
(ii) for each $u \in P$ the differential $p_*\colon T_u\overline{\mathcal{F}} \to T_{p(u)}\mathcal{F}$ is an isomorphism.

Also the *codimension* of the foliated bundle is by definition the codimension of $\mathcal{F}$ in X. It is well-known that a foliated G-bundle $p\colon P \to X$ has an *adapted connection*, i.e., a connection A satisfying $A(v) = 0$ for $v \in T_u\overline{\mathcal{F}}$, $u \in P$. Then it follows that the curvature form F_A satisfies $F_A \in J$, where J is the defining ideal of the foliation $\mathcal{F}$. For the codimension q of $\mathcal{F}$, we have $J^{q+1} = 0$ and the curvature form satisfies $F_A^{q+1} \equiv 0$.

Theorem 5.6. (*i*) *The classes* $[\Lambda(Q, A)]$ *respectively* $[\Lambda(Q, u, A)]$ *in* (5.4) *are well-defined.*

(*ii*) *They are natural with respect to maps of foliated bundles.*

(*iii*) *Curvature formula : We have*

$$\begin{aligned} d\Lambda(Q, A) &= Q(F_A^{n+1}) && \textit{in case I} \\ d\Lambda(Q, u, A) &= \varepsilon^* Q(F_A^{n+1}) - \gamma && \textit{in case II} \end{aligned} \tag{5.15}$$

where $\gamma = \psi^*\bar{\gamma} \in \Omega_{\mathbb{Z}}(|N\mathcal{U}|)$ *represents the characteristic class* $u(P)$ *associated with* $u \in H^{2n+2}(BK, \mathbb{Z})$.
(iv) *If* $n \geq q$, *then* $Q(F_A^{n+1}) \in J^{q+1} = 0$ *and*

$$
\begin{aligned}
&[\ \Lambda(Q,A)\ ] \in H^{2n+1}(X,\mathbb{R}) \qquad \textit{in case I} \\
(5.16)\qquad &[\ \Lambda(Q,u,A)\ ] \in H^{2n+1}(X,\mathbb{R}/\mathbb{Z}) \qquad \textit{in case II.}
\end{aligned}
$$

Moreover these classes are independent of the choice of adapted connection A.
(v) *Rigidity* : *If* $n \geq q+1$, *then the cohomology classes in* (iv) *are rigid under variation of the foliated structure* $(P, \overline{\mathcal{F}}) \to (X, \mathcal{F})$.

Proof. (i) to (iii) follow from the construction in (5.6), (5.7) and from proposition 5.3. The statements in (iv) and (v) essentially follow from the variational formulas in (5.11). (5.16) in (iv) follows directly from (5.15). For the last statement in (iv), let A' be another choice for the adapted connection. Then the family of adapted connections A_t given by the convex combination $A_t = (1-t)A + tA'$, $t \in [0,1]$ satisfies $\frac{dA_t}{dt} = A' - A = \alpha \in J$. Thus we have $Q(\alpha \wedge F_{A_t}^n) \in J^{q+1} = 0$ for $n \geq q$ and the statement follows from (5.11). For (v), let $(\overline{\mathcal{F}}_t, \mathcal{F}_t)$, $t \in [0,1]$, be a smooth family of foliated structures on $P \to X$. Let A_t, $t \in [0,1]$, be a smooth family of $(\overline{\mathcal{F}}_t, \mathcal{F}_t)$–adapted connections on $P \to X$. Then for $n \geq q+1$ we have $Q(\frac{dA_t}{dt} \wedge F_{A_t}^n) \in J_t^{q+1} = 0$ and (v) follows also from (5.11). □

Remarks 5.7. 1. Theorem 5.6 is essentially a reformulation of theorem 2.2 in [16]. The above constructions could of course be extended to define more general characteristic classes associated to elements in (the cohomology of) the relative Weil algebra $F^{2(q+1)}W(G,K)$ as in [16].
2. Following Kamber-Tondeur [30], section 2.24 we call the adapted connection A *basic* if the Lie derivative $L_X A = i_X dA$ vanishes for all $\overline{\mathcal{F}}$-horizontal vector fields X on P or equivalently if $i_X F_A = 0$, that is $F_A \in J^2$. If we can choose the connections in Theorem 5.6 to be basic, then the condition $n \geq q$ in (iv) can be replaced by $2n \geq q$ and the condition $n \geq q+1$ in (v) can be replaced by $2n \geq q+1$. In fact, we have $Q(\alpha \wedge F_{A_t}^n) \in J^{2n+1}$ in (iv), and $Q(\frac{dA_t}{dt} \wedge F_{A_t}^n) \in J_t^{2n}$ in (v).

6. Invariants for families of connections

We now return to the situation of a family of principal G-bundles with connections as in Definition 1.1. That is, (i) $\pi\colon Y \to Z$ is a $\mathrm{Diff}^+(X)$-fibre bundle with fibre X, (ii) $p\colon E \to Y$ is a principal G-bundle, and (iii) $A = \{A_z \mid z \in Z\}$ is a family of connections on $P_z = E\,|X_z$, $z \in Z$, where $X_z = \pi^{-1}(z)$. Also $\mathcal{V} = \{V_j\}_{j\in J}$ and $\mathcal{U} = \{U_i\}_{i \in I}$ are good coverings of Y respectively Z. Finally $Q \in I^{n+1}(G)$ is an invariant polynomial satisfying case I or II as in section 5. Our main result is the following:

Theorem 6.1. *Suppose* $\partial X = \emptyset$ *and* $\dim X = 2n+1-\ell$, $0 \leq \ell \leq 2n+1$. *Also let* B *be a global connection on* E *extending the family* A. *Then the following holds:*
(i) *The* (*simplicial*) ℓ*-form defined by*

$$
\begin{aligned}
&\Lambda_{Y/Z}(Q,B) = \int_{Y/Z} \Lambda(Q,B) \qquad \textit{in case I} \\
(6.1)\qquad &\Lambda_{Y/Z}(Q,u,B) = \int_{Y/Z} \Lambda(Q,u,B) \qquad \textit{in case II}
\end{aligned}
$$

gives well-defined classes in $H^{\ell+1}_{\mathcal{D}}(Z)$*, respectively* $H^{\ell+1}_{\mathcal{D}}(Z,\mathbb{Z})$*, functorial with respect to bundle maps*

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z' & \longrightarrow & Z \end{array} \tag{6.2}$$

and the induced connections.

(ii) *These classes are independent of the choice of the global extension* B *provided that* $F^{n+1-\ell}_{A_z} = 0$ *for all* $z \in Z$.

(iii) *Curvature formula : We have*

$$\begin{aligned} \int_{Y/Z} Q(F^{n+1}_B) &= (-1)^{\ell-1}\ d\Lambda_{Y/Z}(Q,B) && \textit{in case I} \\ \varepsilon^* \int_{Y/Z} Q(F^{n+1}_B) - \int_{Y/Z} \gamma &= (-1)^{\ell-1}\ d\Lambda_{Y/Z}(Q,u,B) && \textit{in case II} \end{aligned} \tag{6.3}$$

where γ *represents* $u(E) \in H^{2n+2}(Y,\mathbb{Z})$.

(iv) *In particular in case II we have in* $H^{\ell+1}(Z,\mathbb{Z})$:

$$d_*\ [\ \Lambda_{Y/Z}(Q,u,B)\ ] = (-1)^{\ell}\ \pi_!(u(E)).$$

(v) *If* $F^{n-\ell}_{A_z} = 0$ *for all* $z \in Z$ *then* $\Lambda_{Y/Z}(Q,B)$*, respectively* $\Lambda_{Y/Z}(Q,u,B)$ *are closed, respectively closed* $\mod \mathbb{Z}$*, and*

$$\begin{aligned} [\ \Lambda_{Y/Z}(Q,B)\ ] &\in H^{\ell}(Z,\mathbb{R}) && \textit{in case I} \\ [\ \Lambda_{Y/Z}(Q,u,B)\ ] &\in H^{\ell}(Z,\mathbb{R}/\mathbb{Z}) && \textit{in case II} \end{aligned} \tag{6.4}$$

are well-defined invariants of the family $\{A_z \mid z \in Z\}$.

Proof. Again (i), (iii), and (iv) follow from the definitions and the properties of fibre integration listed in proposition 4.1. Also (ii) follows from Lemma 5.4 applied to the family $B_t = (1-t)B_0 + tB_1,\ t \in [0,1]$ for B_0, B_1 two choices of global connections extending the family A. The first statements in (v) are a consequence of formula (6.3) and the properties of fibre integration. Finally, the last statement in (v) follows from (ii), since the curvature assumption in (v) is stronger than the assumption in (ii). □

Remarks 6.2. 1. Theorems 1.2 and 1.3 are reformulations of Theorem 6.1. In fact the ℓ–gerbe $\theta = \theta(Q,u,B)$ with connection $\omega = (\omega^0,\dots,\omega^\ell)$ in Theorem 1.3 is given by the formulas in (3.12) for $\Lambda = \Lambda_{Y/Z}(Q,u,B)$.

2. In particular we recover from Theorem 6.1 the construction of the Quillen 'determinant line bundle' and their Hermitian connections as in [14] by taking $\ell = 1$ and specializing Y to a product and $\pi\colon X \times Z \to Z$ the projection on Z (compare also example 7.5).

3. Again in the product situation $\pi\colon X \times Z \to Z$ and the covering $\mathcal{V} = \{U'_j \times U_i\}_{(i,j)\in I\times J}$ as in remark 4.2, we can define more generally for $\xi \in C^{\mathrm{Sing}}_{2n+1-\ell}(X)$ respectively $\xi \in \check{C}^{\mathrm{Sing}}_{2n+1-\ell}(N\mathcal{U}')$ the invariant

$$\begin{aligned}\Lambda_\xi(Q,B) &= \int_\xi \Lambda(Q,B) \qquad \text{in case I}\\ \Lambda_\xi(Q,u,B) &= \int_\xi \Lambda(Q,u,B) \qquad \text{in case II}\end{aligned} \tag{6.5}$$

Then we obtain from (4.10) and (5.8):

$$\begin{aligned}\int_\xi Q(F_B^{n+1}) &= \Lambda_{\partial\xi}(Q,B) + (-1)^{\ell-1}\, d\Lambda_\xi(Q,B) \qquad \text{in case I}\\ \varepsilon^*\int_\xi Q(F_B^{n+1}) - \int_\xi \gamma &= \Lambda_{\partial\xi}(Q,u,B) + (-1)^{\ell-1}\, d\Lambda_\xi(Q,u,B) \qquad \text{in case II.}\end{aligned} \tag{6.6}$$

Here γ represents $u(E)$ in $\Omega^{2n+2}_{\mathbb{Z}}(|N\mathcal{U}|)$, hence in particular $\int_\xi \gamma$ is integral. Thus, under the appropriate vanishing conditions for the fibre curvature the left hand side of (6.6) is going to vanish (mod $\mathbb{Z}$ in case II). Hence $\Lambda_\xi(Q,B)$, respectively $\Lambda_\xi(Q,u,B)$, defines a cycle in the total complex of the bicomplex

$$\begin{aligned}&\mathrm{Hom}(C_*(X),\Omega^*(Z)) \qquad \text{in case I}\\ &\mathrm{Hom}(\check{C}_*(\mathcal{U}'),\Omega^*_{\mathbb{R}/\mathbb{Z}}(||N\mathcal{U}||)) \qquad \text{in case II.}\end{aligned}$$

Notice that for $\ell = 0$, $\Lambda_\xi(Q,u,B)$ is essentially the 'Chern–Simons section' of the line bundle given by $\Lambda_{\partial\xi}(Q,u,B)$ as defined in [14].

We can now apply theorem 6.1 to the general case of families of foliated bundles. By a *family* of *foliated* G-bundles of codimension q we mean the following:

(6.7)

(i) $\pi\colon Y \to Z$ is a $\mathrm{Diff}^+(X)$-fibre bundle with fibre X.

(ii) $p\colon E \to Y$ is a principal G-bundle.

(iii) $\overline{\mathcal{F}}$, $\mathcal{F}$ are foliations of E, respectively Y, such that $T\mathcal{F} \subset T(\pi)$, respectively $T\overline{\mathcal{F}} \subset T(\pi\circ p)$ are involutive (G–equivariant) subbundles, inducing foliated structures $(\overline{\mathcal{F}}_z, \mathcal{F}_z)$ of codimension q in the principal bundles $p_z : P_z \to X_z$ for $z \in Z$.

In this situation $(\overline{\mathcal{F}}, \mathcal{F})$ makes $p\colon E \to Y$ into a foliated G-bundle. By restriction to $T(\pi\circ p) \subset TE$, a global adapted connection B induces a smooth family $A = \{A_z\}$ of adapted connections on the principal bundles $p_z : P_z \to X_z$, $z \in Z$, satisfying the curvature condition $F^{q+1}_{A_z} = 0$. Conversely, any global extension B of a smooth family $A = \{A_z\}$ of adapted connections is adapted to $(\overline{\mathcal{F}}, \mathcal{F})$. Thus by choosing a global adapted connection B, we conclude the following from Theorem 6.1:

Theorem 6.3. *Suppose* $\partial X = \emptyset$ *and* $\dim X = 2n+1-\ell$, $0 \le \ell \le 2n+1$. *Let* B *be an adapted connection for the family of foliated bundles of codimension* q *as above. Then the following holds* :

(*i*) *The classes*

$$\begin{aligned}[\,\Lambda_{Y/Z}(Q,B)\,] &\in H^{\ell+1}_{\mathcal{D}}(Z) \qquad \textit{in case I}\\ [\,\Lambda_{Y/Z}(Q,u,B)\,] &\in H^{\ell+1}_{\mathcal{D}}(Z,\mathbb{Z}) \qquad \textit{in case II}\end{aligned} \tag{6.8}$$

are well-defined and independent of the choice of adapted connection B if $n-\ell \geq q$.
(ii) Suppose that $n-\ell > q$. Then $\Lambda_{Y/Z}(Q,B)$, respectively $\Lambda_{Y/Z}(Q,u,B)$ are closed, respectively closed $\bmod \mathbb{Z}$ *and*

$$\begin{aligned} &[\ \Lambda_{Y/Z}(Q,B)\ ] \in H^\ell(Z,\mathbb{R}) \qquad \textit{in case I} \\ &[\ \Lambda_{Y/Z}(Q,u,B)\ ] \in H^\ell(Z,\mathbb{R}/\mathbb{Z}) \qquad \textit{in case II} \end{aligned} \tag{6.9}$$

are well-defined invariants of the family of foliated bundles.
(iii) Suppose again that $n-\ell > q$. Then the cohomology classes $[\ \Lambda_{Y/Z}(Q,B)\ ]$, respectively $[\ \Lambda_{Y/Z}(Q,u,B)\ ]$ in (6.9) above are rigid, that is they are invariant under (germs of) smooth deformations of the data in 6.7, (iii).

In either case, we call the invariants in (6.8), respectively in (6.9) the *characteristc ℓ–gerbe*, respectively the *characteristc flat ℓ–gerbe* of the family of foliated bundles, associated to the pair (Q,u).

Proof. (i) needs some elaboration, since the family A of adapted connections on $T(\pi\circ p)$ is now not fixed. We want to show that (i) follows from the variational formulas in (5.11). Let A, A' be two families of adapted connections along the fibres and consider corresponding global extensions B, B' of A, A'. Then the convex combination $B_t = (1-t)B + tB'$, $t \in [0,1]$ is an extension of the adapted connection $A_t = (1-t)A + tA'$, $t \in [0,1]$ on the fibres. Further B_t satisfies $\frac{dB_t}{dt} = B' - B = \beta$, where the $\overline{\mathcal{F}}$–transversal 1–form β on Y is of the form $\beta = \alpha + \gamma$, with $\alpha = \alpha^{1,0} = A' - A$ on $T(\pi)$ being fibrewise in the ideal J_z of $\mathcal{F}_z$, that is α vanishes on the subbundle $T\overline{\mathcal{F}} \subset T(\pi)$, and $\gamma = \gamma^{0,1}$ being of type $(0,1)$ on Y, that is γ vanishes on the subbundle $T(\pi) \subset TY$. Thus we have, observing that $F^{2,0}_{B_t} = F_{A_t}$,

$$\begin{aligned} \int_{\Delta^1} Q(F^{n+1}_{\widetilde{B}}) &= (n+1)\int_{\Delta^1} dt \wedge Q(\beta \otimes F^n_{B_t}) \\ &= (n+1)\int_0^1 Q\Big((\alpha+\gamma)\wedge \Big(F^{2,0}_{B_t} + F^{1,1}_{B_t} + F^{0,2}_{B_t}\Big)^n\Big)\, dt \\ &= (n+1)\int_0^1 Q\Big(\alpha^{1.0}\wedge \Big(F_{A_t} + F^{1,1}_{B_t} + F^{0,2}_{B_t}\Big)^n\Big)\, dt \\ &+ (n+1)\int_0^1 Q\Big(\gamma^{0,1}\wedge \Big(F_{A_t} + F^{1,1}_{B_t} + F^{0,2}_{B_t}\Big)^n\Big)\, dt\ . \end{aligned} \tag{6.10}$$

As we will have to integrate over the fibre, only the components of type $(2n+1-\ell, \ell)$ of the $(2n+1)$–form in the integrand can contribute non–trivial terms. Therefore the relevant terms in the first summand of (6.10) must contain $\alpha \wedge F^k_{A_t}$ for $k \geq n-\ell \geq q$, that is $k+1 \geq (n-\ell)+1 \geq q+1$, while the relevant terms in the second summand of (6.10) must contain $F^k_{A_t}$ for $k \geq n-(\ell-1) = (n-\ell)+1 \geq q+1$. Since $\alpha \wedge F^q_{A_t} = 0$ and $F^{q+1}_{A_t} = 0$, it follows that all the relevant terms vanish in either case. Thus (6.10) vanishes under integration over the fibre and (i) follows from (5.11).

(iii) is proved by the same argument, with the following modifications. Let A_σ , $\sigma \in [0,1]$ be families of connections along the fibres, adapted to the data $(\overline{\mathcal{F}}_\sigma,\ \mathcal{F}_\sigma)$ and let B_σ be corresponding global extensions of A_σ. Then we can write again $\frac{dB_\sigma}{d\sigma} = \alpha^{1,0}_\sigma + \gamma^{0,1}_\sigma$ as above, except that the horizontal forms $\alpha^{1,0}_\sigma$ on $T(\pi)$ do not necessarily satisfy the fibrewise condition of being in the ideal $(J_\sigma)_z$ of $(\mathcal{F}_\sigma)_z$. However, the argument in the proof of (ii) remains valid, since the relevant terms $\alpha_\sigma \wedge F^k_{A_\sigma}$ satisfy now $k \geq n-\ell \geq q+1$ and therefore vanish.

Of course, (ii) follows from theoren 6.1 (v). Explicitly, we have to show that the curvature term in (6.3) vanishes under the assumption $n - \ell > q$. Writing $F_B = F_A + F_B^{1,1} + F_B^{0,2}$ as above and expanding

$$Q(F_B^{n+1}) = Q\left((F_A + F_B^{1,1} + F_B^{0,2})^{n+1}\right),$$

the claim follows by a counting argument similar to the one above. □

Remarks 6.4. 1. In applying the variation formula (5.11) in lemma 5.4 in the proof of theorem 6.3 (i), we observe that the adapted connection $A = B \mid T(\pi \circ p)$ is not fixed during a variation B_t of B, but we still have $F_{A_t}^{q+1} = 0\ ,\ t \in [0,1]$.

2. The results of theorem 6.3 apply in particular to families $p\colon E \to Y$ of *flat* bundles, that is $A = \{A_z\}$ is a family of flat connections on the G–principal bundles $P_z = E\ |X_z \to X_z$ for $z \in Z$. In this case we have $q = 0$ and $T\mathcal{F} = T(\pi)$ and the relevant conditions are $n \geq \ell$ in (i) and $n > \ell$ in (ii) and (iii). This case occurs in all examples in section 7 except for the last example 7.6. Families of flat bundles are also considered in [21].

3. The characteristic classes of foliated bundles in theorem 5.6 can be recovered from theorem 6.3 by taking $Z = \{\mathrm{pt}\}$ and $\ell = 0$. In this case, we have $B = A$ and the non–integrated classes $[\ \Lambda(Q,A)\ ] \in H^{2n+1}(X,\mathbb{R})$, respectively $[\ \Lambda(Q,u,A)\ ] \in H^{2n+1}(X,\mathbb{R}/\mathbb{Z})$ are well-defined under the assumption $n \geq q$ as in theorem 5.6, (iv). Moreover, the restriction $\dim X = 2n+1$ is obviously not necessary.

Recall that the family of adapted connections A is basic if the Lie derivative $L_X A = i_X dA$ vanishes for all $\overline{\mathcal{F}}$-horizontal vector fields X on E or equivalently if $i_X F_A = 0$.

Proposition 6.5. *If we can choose A basic in Theorem 6.3, then the conditions $n - \ell \geq q$ in (i), respectively $n - \ell > q$ in (ii), can be replaced by $2(n - \ell) \geq q$, respectively $2(n-\ell) > q$.*

Proof. Again, we restrict attention the first statement (i) in theorem 6.3. Counting powers of J_t instead of curvature terms, we see that the above estimates for the relevant terms in the proof of (i) give $2k+1 \geq 2(n-\ell)+1$ for the first summand of (6.10), and $2k \geq 2(n-(\ell-1))$, that is $2k \geq 2(n-\ell)+2$ for the second summand of (6.10). Thus in either case, the condition $2(n-\ell) \geq q$ implies that the relevant curvature terms are in $J^{q+1} = 0$. □

Remarks 6.6. 1. Observe that the global extensions B_t on E, respectively the connection $\widetilde{B}$ on $E \times [0,1]$ in the proof of theorem 6.3 (i) will in general not be basic for the respective foliated structures, even if A, A' and hence A_t are.

2. One might expect the correct conditions in proposition 6.5 to be $n - \ell \geq q'$ in (i), respectively $n - \ell > q'$ in (i), where $q' = [\frac{q}{2}]$, that is $q = 2q'$ for q even and $q = 2q'+1$ for q odd. Then the basic vanishing property is $F_A^{q'+1} = 0$, since $2(q'+1) = 2q'+2 \geq q+1$. However, for q odd and $n - \ell = q'$, the estimate $2k+1 \geq 2(n-\ell)+1$ gives $2k+1 \geq 2(n-\ell)+1 = 2q'+1 = q$ which is not sufficient.

7. Examples

In this section we give a few examples of increasing complexity.

Example 7.1. We start with a simple example, found together with R. Ljungmann, which gives non–trivial classes in $H^2_{\mathcal{D}}(Z)$. Let $X = T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $Z = \mathbb{R}^2$ and consider the trivial $\mathrm{GL}(1,\mathbb{R})_+ = \mathbb{R}^\times_+$–bundle E over $Y = T^2 \times \mathbb{R}^2$ with coordinates $(x_1, x_2; z_1, z_2; \lambda)$. If ω_0 denotes the Cartan–Maurer form, then $\omega = \omega_0 + B$, $B = z_1\ dx_1 + z_2\ dx_2$, defines a foliated structure on E which is flat along the fibres T^2 of $\pi : Y \to Z$. In fact, the curvature on Y is given by $F = dB = dz_1 \wedge dx_1 + dz_2 \wedge dx_2$, which is clearly of type $(1,1)$ and vanishes on every fibre $T^2_z = \pi^{-1}(z)$, $z = (z_1, z_2)$. The flat structure of $E \mid T^2_z \to T^2_z$ is not trivial; in fact, the holonomy depends on $z \in Z$ and is given by the homomorphism $h_z : \Lambda \to \mathbb{R}^\times_+$, $\Lambda = \pi_1(T^2_z) \cong \mathbb{Z}^2$, where

$$h_z(\lambda_1, \lambda_2) = e^{\lambda_1 z_1 + \lambda_2 z_2}. \tag{7.1}$$

Since the Lie algebra of $\mathbb{R}^\times_+$ is $\mathbb{R}$ we can take the polynomial $Q(\xi) = \xi^2$ to obtain the 3–form

$$\Lambda(Q, B) = B \wedge dB = (dx_1 \wedge dx_2) \wedge (z_2\ dz_1 - z_1\ dz_2),$$

with $d\Lambda(Q,B) = dB^2 = -2\ (dx_1 \wedge dx_2) \wedge (dz_1 \wedge dz_2)$. Thus on Z we have the characteristic form

$$\Lambda_{Y/Z}(Q, B) = \int_{T^2} B \wedge dB = z_2\ dz_1 - z_1\ dz_2. \tag{7.2}$$

which defines a non-zero class in $H^2_{\mathcal{D}}(Z)$, and can be interpreted as a connection in the trivial line bundle on Z with curvature $-2\ dz_1 \wedge dz_2 = -2V$, where V is the volume form on $Z = \mathbb{R}^2$.

Restricting Y and (7.2) to $\mathbb{S}^1 \subset Z = \mathbb{R}^2$ by setting $z_1 = \cos\theta,\ z_2 = \sin\theta$, we obtain on $T^2 \times \mathbb{S}^1$

$$\Lambda(Q, B) = B \wedge dB = -dx_1 \wedge dx_2 \wedge d\theta.$$

Thus on $\mathbb{S}^1$ we have the characteristic form

$$\Lambda_{Y/\mathbb{S}^1}(Q, B) = -\Big(\int_{T^2} dx_1 \wedge dx_2\Big)\, d\theta = -d\theta\ , \tag{7.3}$$

representing a non-zero element in $H^1(\mathbb{S}^1, \mathbb{R}) \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{R}) = \mathbb{R}$. Thus the restriction of the class in (7.2) is closed, that is the above line bundle is flat on $\mathbb{S}^1$ with holonomy determined by (7.3).

Example 7.2. More generally let $X = X_g$ be a surface of genus $g \geq 2$ and let $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ be a set of closed 1–forms representing a symplectic basis for the cup–product pairing in cohomology, that is

$$\int_{X_g} \alpha_i \wedge \alpha_j = 0\ , \qquad \int_{X_g} \alpha_i \wedge \beta_j = \delta_{ij}\ , \qquad \int_{X_g} \beta_i \wedge \beta_j = 0.$$

We let $Z = \mathbb{R}^{2g}$ with coordinates $(z_1, \ldots, z_{2g})$ and again consider the foliated $\mathbb{R}^\times_+$–bundle E with the foliated structure given by the 1–form $\omega = \omega_0 + B$, $B = z_1\ \alpha_1 + z_2\ \beta_1 + \ldots + z_{2g-1}\ \alpha_g + z_{2g}\ \beta_g$, similar to example 7.1. The curvature $F = dB$ on Y is again of type $(1,1)$ and vanishes on every fibre $T^2_z = \pi^{-1}(z)$, $z \in Z$. The holonomy of the flat bundles $E\ |X_{g,z}$ is determined as a homomorphism $h_z : \Gamma \to \mathbb{R}^\times_+$, $\Gamma = H_1(X_g, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, by a formula similar to (7.1), namely

$$h_z(\gamma_1, \ldots, \gamma_{2g}) = e^{\int_{\gamma_1} \alpha_1 + \int_{\gamma_2} \beta_1 + \ldots + \int_{\gamma_{2g-1}} \alpha_g + \int_{\gamma_{2g}} \beta_g}. \tag{7.4}$$

Again we take the polynomial $Q(\xi) = \xi^2$ to obtain the characteristic form 1–form on Z:

(7.5)
$$\Lambda_{Y/Z}(Q,B) = \int_{X_g} B \wedge dB = (z_2\ dz_1 - z_1\ dz_2) + \ldots + (z_{2g}\ dz_{2g-1} - z_{2g-1}\ dz_{2g}),$$

which defines a non-zero class in $H^2_{\mathcal{D}}(Z)$, and can be interpreted as a connection in the trivial line bundle on Z with curvature

$$d\Lambda_{Y/Z}(Q,B) = \int_{X_g} dB^2 = -2\,(dz_1 \wedge dz_2 + \ldots + dz_{2g-1} \wedge dz_{2g})\,, \tag{7.6}$$

Note that in this and the previous example we have $n = \ell = 1$ and $q = 0$.

Example 7.3. This example is like example 7.1, but here we take $X = T^k = \mathbb{R}^k/\mathbb{Z}^k$ and $Z = \mathbb{R}^k$ and consider again the trivial $\mathrm{GL}(1,\mathbb{R})_+ = \mathbb{R}^\times_+$–bundle E over $Y = T^k \times \mathbb{R}^k$ with coordinates $(x_1,\ldots,x_k; z_1,\ldots,z_k; \lambda)$, with the foliated structure given by the 1–form $\omega = \omega_0 + B\ ,\ B = z_1\ dx_1 + \ldots + z_k\ dx_k$. This foliated structure is flat along the fibres T^k of $\pi : Y \to Z$. In fact, we have for the curvature $F = dB = dz_1 \wedge dx_1 + \ldots + dz_k \wedge dx_k$, which is of type $(1,1)$ and vanishes on every fibre $T^k_z = \pi^{-1}(z)\ ,\ z = (z_1,\ldots,z_k)$. As in example 7.1, the holonomy of the flat bundle $E \mid T^k_z \to T^k_z$ depends on $z \in Z$ and is given by the homomorphism $h_z : \Lambda \to \mathbb{R}^\times_+\ ,\ \Lambda = \pi_1(T^k_z) \cong \mathbb{Z}^k$, where

$$h_z(\lambda_1,\ldots,\lambda_k) = e^{\lambda_1 z_1 + \ldots + \lambda_k z_k}\ ,\ (\lambda_1,\ldots,\lambda_k) \in \Lambda\ . \tag{7.7}$$

Now we take the polynomial $Q(\xi) = \xi^{n+1}\ ,\ k = n+1$ to obtain the characteristic $(2n+1)$–form

$$\begin{aligned}\Lambda(Q,B) &= B \wedge dB^n \\ &= (-1)^{\binom{k}{2}}\ (k-1)!\ (dx_1 \wedge \ldots \wedge dx_k) \wedge \\ &\wedge \left(\sum_{j=1}^{k} (-1)^{j-1}\ z_j\ dz_1 \wedge \ldots \wedge \widehat{dz_j} \ldots \wedge dz_k \right),\end{aligned}$$

Thus on $Z = \mathbb{R}^k$ we have the characteristic form

$$\begin{aligned}\Lambda_{Y/Z}(Q,B) &= \int_{T^k} B \wedge dB^{k-1} \\ &= (-1)^{\binom{k}{2}}\ (k-1)!\ \sum_{j=1}^{k} (-1)^{j-1}\ z_j\ dz_1 \wedge \ldots \wedge \widehat{dz_j} \ldots \wedge dz_k\ ,\end{aligned} \tag{7.8}$$

with curvature

$$d\Lambda_{Y/Z}(Q,B) = (-1)^{\binom{k}{2}}\ k!\ dz_1 \wedge \ldots \wedge dz_k = (-1)^{\binom{k}{2}}\ k!\ V\ , \tag{7.9}$$

with V the volume form on $Z = \mathbb{R}^k$. Hence (7.8) defines a non-zero class in $H^k_{\mathcal{D}}(Z)$.

Restricting Y and (7.8) to $\mathbb{S}^{k-1} = \{(z_1,\ldots,z_k) \mid \sum_{i=1}^k\ z_i^2 = 1\} \subset Z = \mathbb{R}^k$, it is easy to see that $\Lambda_{Y/Z}(Q,B)$ is a non–zero multiple of the volume form on $\mathbb{S}^{k-1}$ and is clearly closed. Thus we have $\Lambda_{Y/\mathbb{S}^{k-1}}(Q,B) \neq 0 \in H^{k-1}(\mathbb{S}^{k-1})$. Note that in this example we have $k = n+1\ ,\ n = \ell$ and $q = 0$. We can interprete the invariants $\Lambda_{Y/Z}(Q,B)$, respectively $\Lambda_{Y/\mathbb{S}^{k-1}}(Q,B)$ as (flat) connections on the trivial $n = (k-1)$–gerbe as in (5.2).

So far, the examples have been for case I. The next two examples will be for case II.

Example 7.4. The *Poincaré* $(k-1)$*–gerbe* (cf. [4], [21]) : This example is the case II analogue of example 7.3. Let T be the k–dimensional real torus, that is $T = \mathbb{R}^k/\Lambda$ for the rank k integral lattice $\Lambda \subset \mathbb{R}^k$. The associated dual torus is defined as

$$\widehat{T} = H^1(T,\mathbb{R})/H^1(T,\mathbb{Z}) \overset{\exp}{\cong} \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathrm{U}(1)) \cong \mathrm{U}(1)^k \ , \tag{7.10}$$

that is the points in $\widehat{T}$ parametrize flat unitary connections on the trivial line bundle $\underline{\mathbb{C}} = T \times \mathbb{C} \longrightarrow T$. For $\xi \in \widehat{T}$, $x \in \mathbb{R}^k$, $a \in \Lambda$ and $\lambda \in \mathbb{C}$, consider the equivalence relation

$$\begin{gathered} \mathbb{R}^k \times \widehat{T} \times \mathbb{C} \longrightarrow \mathbb{R}^k \times \widehat{T} \times \mathbb{C}/\!\sim \ , \\ (x+a, \xi, \lambda) \sim (x, \xi, \exp(2\pi\iota\ \xi(a))\lambda) \ . \end{gathered} \tag{7.11}$$

The quotient space under '$\sim$' defines the Poincaré line bundle $\mathcal{P} \longrightarrow T \times \widehat{T}$. Let $\hat{p}$ denote the projections of $T \times \widehat{T} \to \widehat{T}$. From (7.11) we see that the restriction $\mathcal{P} \mid \hat{p}^{-1}(\xi) \cong \mathcal{L}_\xi$, where the latter denotes the flat line bundle parametrized by $\xi \in \widehat{T}$. There exists a canonical unitary connection B on the U(1)–principal bundle $p : E \to T \times \widehat{T}$ associated to $\mathcal{P}$, with curvature F_B given by

$$F_B = 2\pi\iota \sum_{j=1}^{k} d\xi^j \wedge dx_j \ , \tag{7.12}$$

where $\{x_j\}$ are (flat) coordinates on T and $\{\xi^j\}$ are dual (flat) coordinates on $\widehat{T}$. F_B is of type $(1,1)$ and therefore induces a family $A = \{A_\xi\}$ of flat connections on the fibres $T \to T \times \widehat{T} \xrightarrow{\hat{p}} \widehat{T}$, Now we take $Q = C_1^k$ and $u = c_1^k$, where $c_1 \in H^2(B\,\mathrm{U}(1)),\mathbb{Z}) = H^2(\mathbb{CP}^\infty,\mathbb{Z}) \cong \mathbb{Z}[c_1]$ is the generator. Thus we obtain from theorem 6.3 a (simplicial) characteristic $n = (k-1)$–gerbe

$$[\ \Lambda_{T\times\widehat{T}/\widehat{T}}(C_1^k, c_1^k, B)\ ] = \int_T [\ \Lambda(C_1^k, c_1^k, B)\ ] \in H^k_{\mathcal{D}}(\widehat{T},\mathbb{Z}). \tag{7.13}$$

We remark that $T \times \widehat{T}$ has a canonical Kähler structure for which the Poincaré bundle $\mathcal{P}$ becomes a holomorphic line bundle such that $C_1(\mathcal{P}) = \frac{1}{2\pi\iota}F_B = \omega$. Here ω is the Kähler form, so that $T\times\widehat{T}$ has a Hodge structure. It follows that the curvature of the characteristic gerbe in (7.13) is a non–zero multiple of $\int_T \omega^k = V$, the volume form on $\widehat{T}$. Note that in this and the previous example we have $n = \ell = k-1$ and $q = 0$.

Example 7.5. The *Quillen* 1*–gerbe* [32], [33], [14] : This well-known complex line bundle with unitary connection associated to families of flat SU(2)–bundles appears in our setup as a characteristic 1–gerbe. We briefly recall this non–abelian example, referring to Ramadas– Singer–Weitsman [33] for details. Let $X = X_g$ be an oriented surface of genus g , $G = \mathrm{SU}(2)$ and let Z be the smooth part of the representation variety $\mathrm{Hom}(\pi_1(X_g), G)/G$. This is a symplectic manifold of dimension $6(g-1)$ and the symplectic form is in fact the curvature form for the characteristic 1–gerbe constructed below. The family $E \to X_g \times Z$ is the tautological family of flat SU(2)–bundles $P_\rho \to X_g$ determined by $\rho : \pi_1(X_g) \to \mathrm{SU}(2)$, $\rho \in Z$. The pair (Q,u) is taken to be $Q = C_2$, the second Chern polynomial, and $u = c_2 \in H^4(B\,\mathrm{SU}(2),\mathbb{Z}) \cong \mathbb{Z}[c_2]$ is the universal Chern class. Hence choosing a global SU(2)–connection B on

E, extending the family A of flat connections along the fibres $P_\rho \to X_g$, $\rho \in Z$, we obtain from theorem 6.3 the (simplicial) characteristic 1–gerbe

$$[\ \Lambda_{X_g\times Z/Z}(C_2,c_2,B)\ ] = \int_{X_g} [\ \Lambda(C_2,c_2,B)\ ] \in H^2_{\mathcal{D}}(Z,\mathbb{Z}). \tag{7.14}$$

The above examples are all cases where $q = 0$, that is we have $T\mathcal{F} = T(\pi)$ and $A = \{A_z\}$ is a family of *flat* connections on the fibres $P_z \to X_z$, $z \in Z$. We end with a case I example which relates to variations of the Godbillon–Vey invariant [23] and also gives some new classes of Godbillon–Vey type.

Example 7.6. *Godbillon–Vey gerbes* for families of foliations : Let $\mathcal{F}$ be a family of transversally oriented foliations of codimension q on $\pi : Y \to Z$ as in (6.7), that is $T\mathcal{F} \subset T(\pi)$. The relative transversal bundle $Q_{\mathcal{F}} = T(\pi)/T\mathcal{F}$ has a natural foliated structure given by the partial Bott connection. On the oriented frame bundle $E = F_{\mathrm{GL}(q)^+}(Q_{\mathcal{F}}) \to Y$ this determines a foliated structure $\overline{\mathcal{F}}$. We choose a family $A = \{A_z\}$ of torsion–free, hence adapted connections along the fibres and extend it to a global connection B on $E \to Y$. For given $n \geq q$, we consider invariant polynomials of the form $C_1Q \in \ker(I(\mathrm{GL}(q,\mathbb{R})^+) \to I(\mathrm{SO}(q))$, where $Q \in I^n(\mathrm{GL}(q,\mathbb{R})^+)$ and $I(\mathrm{GL}(q,\mathbb{R})^+) \cong \mathbb{R}[C_1,\dots,C_q]$ is generated by the Chern polynomials C_j, that is the coefficients of t^j in $\det(\mathrm{Id}+\frac{t}{2\pi}A)$, $A \in \mathfrak{gl}(q,\mathbb{R})$. We have $C_1 = \frac{1}{2\pi}$ Tr and the kernel of the restriction to $I(\mathrm{SO}(q))$ is generated by the odd Chern polynomials C_{2k+1}. Then $\Lambda(C_1Q,B)$ is given by the $(2n+1)$–form

$$\Lambda(C_1Q,B) = \beta \wedge Q(F_B^n) \tag{7.15}$$

on Y, satisfying

$$d\Lambda(C_1Q,B) = d\beta \wedge Q(F_B^n) = \frac{1}{2\pi}\,\mathrm{Tr}(F_B) \wedge Q(F_B^n) = C_1(F_B) \wedge Q(F_B^n). \tag{7.16}$$

Here $\beta = \frac{1}{2\pi} s^*\,\mathrm{Tr}(B)$ is the pull–back of the trace of the connection form B on $\det(E) = \Lambda^q(Q_{\mathcal{F}})_0$ by a trivializing section $s : Y \to \Lambda^q(Q_{\mathcal{F}})_0$ given by the transverse orientation on the normal bundle $Q_{\mathcal{F}}$. Note that β satisfies $d\beta = C_1(F_B)$ and that the choice $Q = C_1^n$ corresponds to the Godbillon–Vey form proper, that is $\Lambda(C_1^{n+1},B) = \beta \wedge d\beta^n$. For ℓ satisfying $n - \ell \geq q$, (7.15) now gives rise to characteristic ℓ–gerbes on Z according to theorem 6.3.

First of all, the above data determine a family parametrized by $z \in Z$ of secondary characteristic classes of Godbillon–Vey type on the fibres X_z of $\pi : Y \to Z$, according to theorem 5.6, namely

$$[\ \Lambda(C_1Q,A_z)\ ] = [\ \alpha_z \wedge Q(F^n_{A_z})\ ], \tag{7.17}$$

where $\alpha = \beta \mid T(\pi)$, $d\alpha_z = C_1(F_{A_z})$ and $F_A^{q+1} = 0$. However, for $n > q$ and in particular for $\ell > 0$, that is $\dim X = 2n+1-\ell < 2n+1$, these forms on the fibres vanish identically.

Next, we consider the case $n = q$, that is $\ell = 0$ and $\dim X = 2n+1 = 2q+1$ according to our general convention. Then the classes (7.17) actually live on the fibres $X_z = \pi^{-1}(z)$ and we obtain from theorem 6.3 (i) a global 0–gerbe

$$[\ \Lambda_{Y/Z}(C_1Q,B)\ ] = \int_{Y/Z} [\ \beta \wedge Q(F_B^q)\ ] \in H^1_{\mathcal{D}}(Z) = \Omega^0(Z), \tag{7.18}$$

given fibrewise by

$$[\ \Lambda_{Y/Z}(C_1Q,B)\ ]\ (z) = \int_{X_z} \alpha_z \wedge Q(F^q_{A_z}). \tag{7.19}$$

Thus the family of invariants in (7.19) are the integrated fibrewise Godbillon–Vey invariants, which are well–known to be variable and hence non–constant in $\Omega^0(Z)$ for a suitable choice of the family of foliations (compare Heitsch [23], [24] and also the original work of Thurston [34]). A similar result is obtained for $n = q+\ell\ ,\ \ell > 0$ and $\dim X = 2n+1-\ell = 2q+\ell+1$, in which case theorem 6.3 (i) gives rise to (variable) characteristic ℓ–gerbes

$$[\ \Lambda_{Y/Z}(C_1Q,B)\ ] = \int_{Y/Z} [\ \beta\wedge Q(F^n_B)\ ] \in H^{\ell+1}_{\mathcal{D}}(Z) = \Omega^\ell(Z)/d\Omega^{\ell-1}(Z), \tag{7.20}$$

determined by formula (7.15); compare also (5.3).

A more original class of gerbes is obtained in the 'rigid' range $n-\ell > q$, that is $\ell = 0,\dots n-(q+1)$, in which case we still have $2q+1 < \dim X = 2n+1-\ell$. Then we can invoke theorem 6.3 (ii) to obtain well-defined flat characteristic Godbillon–Vey ℓ–gerbes

$$[\ \Lambda_{Y/Z}(C_1Q,B)\ ] = \int_{Y/Z} [\ \beta\wedge Q(F^n_B)\ ] \in H^\ell(Z,\mathbb{R}). \tag{7.21}$$

Note that for $n-\ell \geq q\ ,\ \ell > 0$, we have $2n+1 > \dim X = 2n+1-\ell > 2q+1$. Hence, as already noted, the fibrewise classes vanish identically on the form level, while the forms $\Lambda(C_1Q,B)$ are not necessarily closed on Y unless $n \geq q+\dim Z$.

In contrast, the classes investigated in Kotschick [31], Hoster–Kamber–Kotschick [28], are families of classes on a fixed manifold X, defined with respect to a 1–parameter family $\mathcal{F}_t$ of foliations and foliated bundles and their suspension on the cylinder $X\times I$. Hoster in his thesis [27] considers fibre spaces with flags of foliations along the fibres, but stays essentially in the context of [28].

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